THREE-DIMENSIONAL MANIFOLDS.

by

David Bernard Alper Epstein.

Trinity College,

Cambridge.

A dissertation presented for the Ph.D. Degree at the University of Cambridge.

Easter Term, 1960.

I hereby declare that my dissertation entitled "Three-dimensional Manifolds" is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already, or is being concurrently, submitted for any such degree, diploma, or other qualification.

The material contained in the dissertation is original except where otherwise stated.

27th April, 1960. D. B. A. Epstein.

D.B.A.Epstein.
Trinity College.

CONTENTS.

Introduction. Page 1.

Chapter I. Linking Spheres. 4.

§1. Spun knots. 4.

§2. Construction. 5.

§3. Algebra. 6.

§4. Realizing elements. 9.

§5. Theorems. 10.

Chapter II. The Projective Plane Theorem. 14.

§6. The Projective Plane Theorem. 15.

§7. Generators of $\pi_2(M)$. 19.

Chapter III. Non-orientable 3-Manifolds with Finite Fundamental groups. 22.

§8. The boundary of M. 22.

§9. The dihedral group appears and disappears. 24.

§10. Homotopy equivalence. 29.

Chapter IV. Fundamental Groups of 3-Manifolds. 31.

§11. Essential mappings. 31.

§12. Definitions of various symbols. 33.

§13. Subgroups of $\pi_1(M)$. 36.

§14. Finitely generated abelian subgroups of $\pi_1(M)$. 40.

Chapter V. Generators and Relations. 44.

§15. An upper bound for def P. 44.

§16. The fundamental group of a 3-manifold. 48.

§17. Applications of the 2-efficiency of $\pi_1(M)$. 52.

Chapter VI. 3-Manifolds which are Topological Products. Page 56.

§18. Ends. 56.

§19. Direct products. The orientable case. 57.

§20. Direct products. The non-orientable case. 60.

§21. Summary of results in Chapter VI. 62.

Bibliography. 64.

INTRODUCTION.

In the post-war years, the theory of 3-dimensional manifolds has developed tremendously. On the one hand, Bing and Moise have proved that 3-manifolds can be triangulated, and that the Hauptvermutung (that any two triangulations of the same space are combinatorially equivalent) is true for 3-manifolds. On the other hand, Papakyriakopoulos has proved Dehn's Lemma, and, using ideas of Papakyriakopoulos, Whitehead has proved the Sphere Theorem. As a result of this concerted attack from two different directions, the theory of 3-manifolds has become an extremely interesting and fruitful field of study. It seems as though we are well on the way to solving the two main problems in the field:- the Poincaré Conjecture, and the classification of closed 3-manifolds.

In this thesis, some theorems connected with 3-manifolds are proved. The most important theorem is the Projective Plane Theorem (6.1), in which it is proved that elements of the second homotopy group of a 3-manifold can be represented, in a certain sense, by 2-spheres or projective planes in the manifold. The Projective Plane Theorem is, perhaps, an important tool in the problem of classifying non-orientable 3-manifolds.

The entire thesis depends on the Projective Plane Theorem, except for Chapters I and III. In Chapter I, the linking of n-spheres in $(n+2)$-space is dealt with. In Chapter III, non-orientable compact 3-manifolds, with finite fundamental groups are considered, with the aim of proving that there is essentially only one such 3-manifold.

The reader is warned that a different definition of a 3-manifold is adopted in each chapter. This is in the interest of brevity and clarity. The author hopes that no confusion will arise. The definition appropriate in each chapter is given in the introduction to that chapter. The exact hypotheses about the 3-manifold, required for each theorem, are given just before the statement of the theorem.

The following conventions are used throughout the thesis:-

i) "M" denotes a 3-manifold;

ii) "$\tilde{X}$" denotes some covering space of the topological space X;

iii) "G" denotes a group;

iv) "0" denotes the group with only one element, or the unit element of a group which is definitely abelian, or the integer zero;

v) "1" denotes the unit element of a (possibly) non-abelian group, or the integer one;

vi) "Homotopic to zero" means "homotopic to the constant map";

vii) "The zero map" of one group into another group denotes the map which sends all elements into the trivial element.

The author would like to thank Dr.E.C.Zeeman most warmly for his constant help and encouragement during the writing of this work. He could not have hoped for a better teacher. The author has found Dr.Zeeman's comments, suggestions and keen interest in his work invaluable and inspiring. The author has also had many very interesting and useful conversations with Dr.J.F.Adams, to whom he is most grateful.

Chapter I. LINKING SPHERES.

J.J.Andrews and M.L.Curtis have shown [1] that one can embed two S^n's in E^{n+2} for $n = 2$, in such a way that one sphere cannot be shrunk to a point in the residue space of the other. In this chapter, the result is shown to be true for any $n \geq 1$. (It is trivial for $n = 1$). The method is to calculate the appropriate homotopy group of the residue space of one sphere, and to show that the embedding of the other sphere represents a non-zero element of the group. If the two S^n's are required to be embedded analytically, the same result holds good. (The easy proof of this is omitted). The material in this chapter is in the process of being published [4].

§1. Spun knots.

In this section we describe a method of embedding knotted S^n's in E^{n+2}. Let E^n be n-dimensional Euclidean space with coordinates $x_1, \ldots\ldots x_n$. Let E^n be embedded in E^{n+1} by putting $x_{n+1} = 0$. Let $H = \{x;\ x \in E^3,\ x_3 \geq 0\}$.

Let K be a polygonal arc lying in H. Let the intersection of K with E^2 be its endpoints. Let $Y = H - K$ and let $Y_o = Y \cap E^2$. Let B^n be the n-ball, with boundary S^{n-1}. Let $X = Y \times S^{n-1} \cup Y_o \times B^n$, as a subspace of $Y \times B^n$. E^{n+2} is homeomorphic to $H \times S^{n-1} \cup E^2 \times B^n$ as a subspace of $H \times B^n$. Therefore X is homeomorphic to E^{n+2} with an

S^n removed. This S^n is called a <u>spun knot.</u> When $n = 2$, the process is equivalent to Artin's method of "spinning" H about the axis E^2 to obtain knotted 2-spheres [2].

<u>Lemma (1.1).</u> <u>Y is a $K(\pi,1)$.</u>

We embed Y in $H' = \{x;\ x \in E^3,\ x_3 > -1\}$. We add to K two open linear segments of unit length perpendicular to E^2, obtaining K'. Y is a deformation retract of H'-K'. Compactifying H' with a point P we get an S^3. $S^3-(K' \cup P) = H'-K'$ is aspherical by (26.3) of [9].

<u>§2. Construction.</u>

Let $p: \tilde{Y} \longrightarrow Y$ be the universal covering of Y. We emphasize that <u>$\tilde{Y}$ is contractible,</u> since all its homotopy groups vanish. Let $\tilde{Y}_o = p^{-1}Y_o$. Let $\tilde{X} = \tilde{Y} \times S^{n-1} \cup \tilde{Y}_o \times B^n$ (a subspace of $\tilde{Y} \times B^n$). We then have the commutative diagram

$$\begin{array}{ccc} \tilde{Y} \times S^{n-1} & \longrightarrow & \tilde{X} \\ \downarrow & & \downarrow \\ Y \times S^{n-1} & \longrightarrow & X \end{array}$$

where the vertical maps are induced by p, and the horizontal maps are inclusions. $\tilde{X}$ is a covering space of X, with the covering map as in the diagram. Our aim is to calculate $\pi_n(X)$, which is isomorphic to $\pi_n(\tilde{X})$.

Let Z denote the additive group of integers and let S be a non-empty set. Then we denote by Z(S) the free abelian group on the elements of S.

Lemma (2.1). $H_o(\tilde{Y}_o) \cong Z(\pi_1(Y,Y_o))$.

The set of components of $\tilde{Y}_o$ is in (1-1) correspondence with the set of homotopy classes of paths in Y, beginning at a base-point and ending in Y_o. The correspondence is obtained by lifting each path representing an element of $\pi_1(Y,Y_o)$ to a path with base-point in $\tilde{Y}$, and noting in which component of $\tilde{Y}_o$ the other endpoint lies.

§3. Algebra.

We have

Lemma (3.1).
$$H_r(\tilde{X},\tilde{Y}\times S^{n-1}) \cong H_r(\tilde{Y}_o\times B^n,\tilde{Y}_o\times S^{n-1}) \cong 0 \quad \text{if } 0 \leq r < n$$
$$\cong H_{r-n}(\tilde{Y}_o) \quad \text{if } r \geq n.$$

The first isomorphism is obtained by excision, and the second from the Künneth relations.

For $1 \leq r < n$, the r-skeleton of $\tilde{X}$ is contained in $\tilde{Y}\times S^{n-1}$. Therefore $\pi_r(\tilde{X},\tilde{Y}\times S^{n-1}) = 0$. So, for $1 \leq r < n$, we have the exact sequence

$$\pi_{r+1}(\tilde{X},\tilde{Y}\times S^{n-1}) \longrightarrow \pi_r(\tilde{Y}\times S^{n-1}) \longrightarrow \pi_r(\tilde{X}) \longrightarrow 0.$$

The homomorphism

$$\pi_{r+1}(\tilde{X}, \tilde{Y}\times S^{n-1}) \longrightarrow \pi_r(\tilde{Y}\times S^{n-1})$$

is onto. This can be seen from the commutative diagram

$$\begin{array}{ccc} \pi_{r+1}(\tilde{X}, \tilde{Y}\times S^{n-1}) & \longrightarrow & \pi_r(\tilde{Y}\times S^{n-1}) \\ \downarrow & & \downarrow \\ \pi_{r+1}(B^n, S^{n-1}) & \longrightarrow & \pi_r(S^{n-1}) \end{array}$$

where the vertical maps are induced by the projection $\tilde{X} \longrightarrow B^n$, resulting from the product structure of $\tilde{X}$, and the horizontal maps are boundary homomorphisms. The vertical map on the right is an isomorphism since $\tilde{Y}$ is contractible. The lower horizontal and left vertical maps are onto. Therefore we have:

<u>Lemma (3.2).</u> <u>$\pi_r(\tilde{X}) = 0$ for $1 \leq r < n$.</u>

We have the diagram

$$\begin{array}{ccccccccc} H_n(\tilde{Y}\times S^{n-1}) & \longrightarrow & H_n(\tilde{X}) & \longrightarrow & H_n(\tilde{X}, \tilde{Y}\times S^{n-1}) & \longrightarrow & H_{n-1}(\tilde{Y}\times S^{n-1}) & \longrightarrow & H_{n-1}(\tilde{X}) \\ \parallel & & \wr\!\parallel & & \wr\!\parallel & & \wr\!\parallel & & \parallel \\ 0 & & \pi_n(\tilde{X}) & & H_o(\tilde{Y}_o) & & Z & & 0 \\ & & \wr\!\parallel & & \wr\!\parallel & & & & \\ & & \pi_n(X) & & Z(\pi_1(Y, Y_o)) & & & & \end{array}$$

where the horizontal row is exact. The vertical isomophisms first and fourth from the left result from the contractibility of $\tilde{Y}$.

The second and fifth isomorphisms are Hurewicz isomorphisms. The third vertical isomorphism is given by (3.1) and (2.1).

Lemma (3.3). $\pi_n(X)$ is isomorphic to the free abelian group generated by the non-trivial homotopy classes of paths in Y, beginning at a base-point in Y_o and ending in Y_o.

From the above discussion, we have the exact sequence

$$0 \longrightarrow \pi_n(X) \xrightarrow{f} Z(\pi_1(Y,Y_o)) \longrightarrow Z \longrightarrow 0.$$

We assert that each element of $\pi_1(Y,Y_o)$ maps onto the same element of Z. This element generates Z.

By (3.1),

$$H_n(\tilde{X},\tilde{Y}\times S^{n-1}) \cong H_n(\tilde{Y}_o\times B^n,\tilde{Y}_o\times S^{n-1}) \cong H_o(\tilde{Y}_o).$$

This isomorphism is induced as follows. With every singular 0-cube in $\tilde{Y}_o$, we associate that singular n-cube in $\tilde{Y}_o\times B^n$, which is the product of the degenerate map into the 0-cube in $\tilde{Y}_o$ and a definite homeomorphism h of an n-cube onto B^n. (The isomorphism is the same as that produced by the Künneth relations). Using the isomorphism of (2.1), the assertion follows.

If α is any element of $\pi_1(Y,Y_o)$ and α_o is the element of $\pi_1(Y,Y_o)$ represented by the constant path, we define

$$g: Z(\pi_1(Y,Y_o)) \longrightarrow \pi_n(X)$$

by $g(\alpha) = f^{-1}(\alpha - \alpha_o)$. The map g induces the isomorphism mentioned in Lemma (3.3).

§4. Realizing elements.

We wish to show that the isomorphism in Lemma (3.3), induced by g, can also be induced by spinning a knot. Explicitly, let a be an arc in Y = H-K, whose intersection with E^2 is its endpoints. Let a represent $\alpha \in \pi_1(Y,Y_o)$.

Lemma (4.1). The spun knot associated with a represents the element $g(\alpha) \in \pi_n(X)$.

Let $\tilde{a}$ be a lifting of a from Y to $\tilde{Y}$. Let the endpoints of a and $\tilde{a}$ be a_o, a_1 and $\tilde{a}_o$, $\tilde{a}_1$ respectively. Let Sa denote the spun knot $a{\times}S^{n-1} \cup a_o{\times}B^n \cup a_1{\times}B^n$ and let $\tilde{S}a$ denote $\tilde{a}{\times}S^{n-1} \cup \tilde{a}_o{\times}B^n \cup \tilde{a}_1{\times}B^n$, embedded in X and $\tilde{X}$ respectively. Then the map $\tilde{S}a \longrightarrow Sa$ induced by $p:\tilde{Y} \longrightarrow Y$ is a homeomorphism. Let us follow the image of the element of $\pi_n(X)$ represented by Sa, in the sequence of groups

$$\pi_n(X) \cong \pi_n(\tilde{X}) \cong H_n(\tilde{X}) \longrightarrow H_n(\tilde{X},\tilde{Y}{\times}S^{n-1}) \cong H_n(\tilde{Y}_o{\times}B^n,\tilde{Y}_o{\times}S^{n-1})$$
$$\cong H_o(\tilde{Y}_o) \cong Z(\pi_1(Y,Y_o)).$$

Then $\{Sa\}$ becomes, in $\pi_n(\tilde{X})$, $\{\tilde{S}a\}$; in $H_n(\tilde{X})$, $\langle\tilde{S}a\rangle$, the image of $\{\tilde{S}a\}$ under the Hurewicz homomorphism. In $H_n(\tilde{X},\tilde{Y}{\times}S^{n-1})$ the image of $\{Sa\}$ becomes $\langle\tilde{a}_1{\times}B^n\rangle - \langle\tilde{a}_o{\times}B^n\rangle$, where $\langle\tilde{a}_i{\times}B^n\rangle$ is the homology class of that singular n-cube which is the product of a degenerate cube in $\tilde{a}_i$, with the homeomorphism h. The image of $\{Sa\}$ in $H_o(\tilde{Y}_o)$ is therefore $\langle a_1 \rangle - \langle a_o \rangle$, where $\langle a_i \rangle$ is the obvious 0-dimensional homology class. The image of $\{Sa\}$ in $Z(\pi_1(Y,Y_o))$ is $\alpha - \alpha_o$ by applying Lemma (2.1). Therefore $f(\{Sa\}) = \alpha - \alpha_o$. Therefore $g(\alpha) = f^{-1}(\alpha - \alpha_o) = \{Sa\}$, which proves (4.1).

§5. Theorems.

Theorem (5.1). Two S^n's can be embedded in E^{n+2} $(n \geq 1)$ in such a way that neither can be shrunk to a point in the residue space of the other.

For n = 1, let the two S^1's be circles which link each other. For n > 1, let K be a trefoil knot with a small linear segment removed, as in §1. Let a also be a trefoil knot with a small linear segment removed, such that a represents a non-trivial element of $\pi_1(Y,Y_o)$. We choose a so that K represents a non-trivial element in $\pi_1(H-a, E^2-a)$. (See Diagram 1). It is easy to show that a suitably chosen a represents a non-trivial element, if one bears in mind the fact that if the group of a knot is infinite cyclic, then the knot is unknotted ((28.1) of [9]), and the fact that there is an exact sequence

$$\pi_1(Y_o) \longrightarrow \pi_1(Y) \longrightarrow \pi_1(Y,Y_o) \longrightarrow 0.$$

(5.1) follows from (4.1).

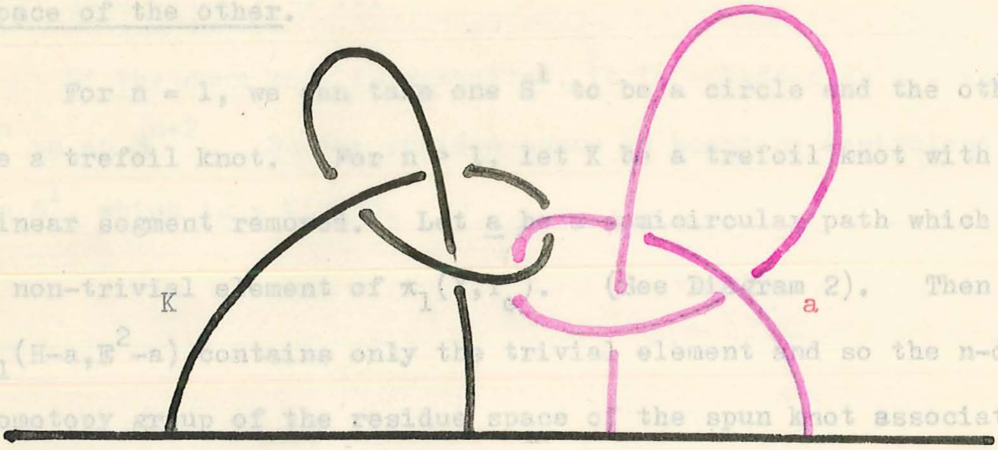

Diagram 1.

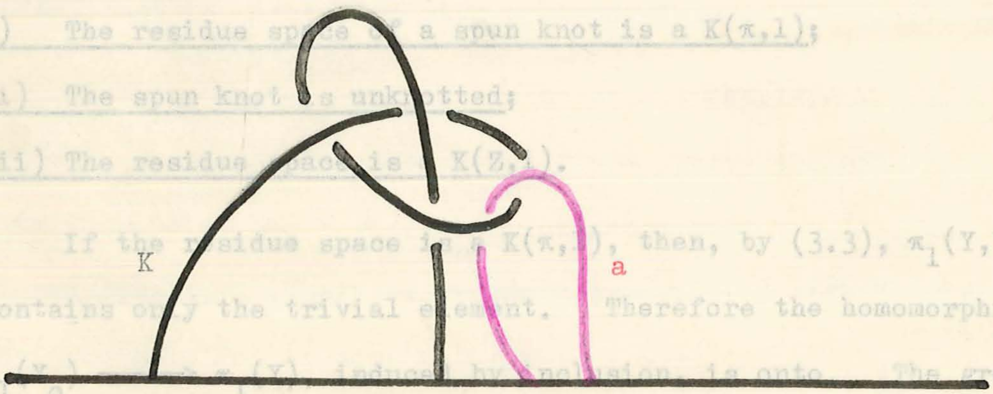

Diagram 2.

Theorem (5.2). Two S^n's can be embedded in E^{n+2} $(n \geq 1)$ so that one S^n can and the other can not be shrunk to a point in the residue space of the other.

For $n = 1$, we can take one S^1 to be a circle and the other to be a trefoil knot. For $n > 1$, let K be a trefoil knot with a small linear segment removed. Let a be a semicircular path which represents a non-trivial element of $\pi_1(Y,Y_o)$. (See Diagram 2). Then $\pi_1(H-a,E^2-a)$ contains only the trivial element and so the n-dimensional homotopy group of the residue space of the spun knot associated with a, is zero by (3.3). (5.2) follows from (4.1).

We now compactify E^{n+2}, so that all our spun knots are embedded in S^{n+2}. This does not affect the homotopy groups in dimensions less than or equal to n.

Theorem (5.3). The following conditions are equivalent:-

i) The residue space of a spun knot is a $K(\pi,1)$;

ii) The spun knot is unknotted;

iii) The residue space is a $K(Z,1)$.

If the residue space is a $K(\pi,1)$, then, by (3.3), $\pi_1(Y,Y_o)$ contains only the trivial element. Therefore the homomorphism $\pi_1(Y_o) \longrightarrow \pi_1(Y)$, induced by inclusion, is onto. The group $\pi_1(Y_o)$ is the free group on two generators, which become the same element in $\pi_1(Y)$. Therefore $\pi_1(Y) \cong Z$. ($\pi_1(Y)$ cannot be finite since $H_1(Y) \cong Z$). Therefore, by (28.1) of [9] and the proof of (1.1), K is unknotted. Unknotting K automatically unknots the spun knot associated with it.

If the spun knot is unknotted, it is embedded as a standard S^n in an S^{n+2}. So the residue space is homotopy equivalent to an S^1, which is a $K(Z,1)$.

Chapter II. THE PROJECTIVE PLANE THEOREM.

In [9] and [15], C.D.Papakyriakopoulos and J.H.C.Whitehead proved the Sphere Theorem:-

Let M be an orientable 3-manifold and $M \subset X$, where X is a topological space. Suppose there is a (singular) 2-sphere in M which is essential in X. Then there is a non-singular 2-sphere in M, which is essential in X.

In this chapter the Sphere Theorem is extended to apply to non-orientable 3-manifolds. The precise statement is given in Theorem (6.1).

The method of proof is to perform standard Dehn cuts on a 2-sphere and its image under the covering translation, in the minimal orientable cover of M. This method of proof was suggested to the author by J.Stallings in connection with a strong form of the Loop Theorem [13].

In this chapter, we shall mean by a 3-manifold M (unless otherwise stated), a connected space with a fixed semilinear structure, such that each point has a simplicial neighbourhood homeomorphic to a closed 3-ball. (That is, we admit 3-manifolds with boundary). The triangulation is combinatorial.

§6. The Projective Plane Theorem.

Let M be a triangulated 3-manifold, possibly non-compact and possibly with boundary. Let $i: M \longrightarrow X$, be a continuous map into a space X. Let $i_*: \pi_2(M) \longrightarrow \pi_2(X)$ be such that $\ker i_* \neq \pi_2(M)$.

Theorem (6.1). There is a semilinear map $g: S^2 \longrightarrow M\text{-}BdM$, whose composition with i is essential in X. The map g is either non-singular or identifies antipodal points. The image of g is a two-sided 2-manifold in M.

Thus the image of g is a two-sided 2-sphere or projective plane (denoted S^2 or P^2 respectively). We note that a projective plane can only be two-sided in a 3-manifold, if the 3-manifold is non-orientable. So Theorem (6.1) implies (1.1) of [15]. (However, [15] is not superceded, since our proof uses [15]).

If M is not paracompact, (and hence not triangulable), and $\ker i_* \neq \pi_2(M)$, then there is a map $h: S^2 \longrightarrow M$, whose composition with i is essential in X. We may take a paracompact neighbourhood of $h(S^2)$ and triangulate it (see [3]) so that it becomes a combinatorial manifold. We may then apply Theorem (6.1). Thus Theorem (6.1) is true for non-paracompact 3-manifolds, if the word "semilinear" is deleted from the statement (see [15]).

Similarly, if we are given a map $h: S^2 \longrightarrow M$, whose composition with i is essential in X we can find a map g, as in the statement of Theorem (6.1), in an arbitrarily small neighbourhood of $h(S^2)$.

We may assume i is an inclusion mapping by using a mapping cylinder. We assume M is non-orientable, since the theorem has been proved if M is orientable (see [15]).

Let $p: \tilde{M} \longrightarrow M$ be the orientable double cover of M. Let Y be the mapping cylinder of $ip: \tilde{M} \longrightarrow X$.

Lemma (6.2). <u>The class of semilinear maps $S^2 \longrightarrow M - BdM$, which are essential in X, whose only singularities are double curves, and whose liftings to $\tilde{M}$ are non-singular, is non-empty.</u> (We use the nomenclature of [9]).

We have the commutative diagram

$$\begin{array}{ccc} \pi_2(\tilde{M}) & \longrightarrow & \pi_2(Y) \\ \downarrow & & \downarrow \\ \pi_2(M) & \longrightarrow & \pi_2(X) \end{array}$$

where the vertical maps are isomorphisms. Therefore

$$\ker(\pi_2(\tilde{M}) \longrightarrow \pi_2(Y)) \neq \pi_2(\tilde{M}).$$

By (1.1) of [15], there is a semilinear embedding of a 2-sphere S in $\tilde{M} - Bd\tilde{M}$, which is essential in Y. Its image under p is essential in X. As pointed out in (3.2) of [11], we can normalize the image under p of S without introducing any singularities in S itself. Then all the singularities of p(S) must be double curves. (6.2) follows.

Given a finite set of disjoint simple closed curves on a 2-sphere, we define an <u>inner</u> curve to be a curve in the set which divides the 2-sphere into two disks, one of which contains no curves in the set, and one of which contains at least one curve in the set. <u>If the finite set of curves contains more than one curve, there must be an inner curve.</u> If it contains only one curve, there are no inner curves.

From the class of semilinear maps $S^2 \longrightarrow M - BdM$, whose liftings to $\tilde{M}$ are non-singular, whose only singularities are double curves, and which are essential in X, we select one with the minimum possible number of double curves. We denote this map by $h: S^2 \longrightarrow M$. Since h has only double curves as singularities, the inverse image under h of all the double curves forms a finite set of disjoint simple closed curves on S^2. An <u>inner double curve of h</u> will refer to a double curve of $h(S^2)$, such that at least one of its inverse images under h is inner.

<u>Lemma (6.3).</u> <u>h has no orientation preserving inner double curves.</u>

Let τ be the covering translation of $\tilde{M}$ and let $\tilde{h}: S^2 \longrightarrow \tilde{M}$ be a lifting of h. Then all singularities in $h(S^2)$ arise from the fact that $\tilde{h}(S^2) \cap \tau\tilde{h}(S^2) \neq \emptyset$. Suppose $C \subset h(S^2)$ is an orientation preserving inner double curve. Let C_1 and C_2 be the disjoint simple closed curves in $\tilde{h}(S^2)$, such that $C_1 \cup C_2 = p^{-1}(C)$. Let C_1 be inner. Then $\tilde{h}(S^2) = D_1 \cup D_2 \cup E$, where D_1 and D_2 are disks with boundary curves C_1 and C_2 respectively, and E is a cylinder with boundary curves C_1 and C_2. Then $p|D_1$ is non-singular. Now $\tau C_1 = C_2$. So $\tau D_1 \cup D_2$ is a 2-sphere in $\tilde{M}$ without singularities, and

$$p(\tau D_1 \cup D_2) = p(D_1 \cup D_2)$$

has fewer double curves than $h(S^2)$. Therefore $p(D_1 \cup D_2)$ is an inessential 2-sphere in X. In X, we deform $p(D_2)$ into $p(D_1)$, without moving C. Thus we obtain a map $h' : S^2 \longrightarrow M$, such that $h' \simeq h$ in X, and (on normalizing), h' has fewer double curves than h. This contradicts our definition of h. (6.3) follows.

<u>Lemma (6.4).</u> <u>h has no orientation reversing inner double curves.</u>

Suppose $C \subset h(S^2)$ is such a double curve. Then there must be more than one double curve. Let $C_1 = p^{-1}(C)$. Now $\tilde{h}(S^2) = D_1 \cup D_2$, where D_1 and D_2 are disks with boundary C_1. $p|D_1 - C_1$ is non-singular. $D_1 \cup \tau D_1$ is a 2-sphere (non-singular) in $\tilde{M}$. On normalizing $p(D_1 \cup \tau D_1)$, by slightly moving τD_1 and keeping D_1 fixed, we get a singular 2-sphere in M, with only one double curve. Therefore $D_1 \cup \tau D_1$ is contractible in Y, by our choice of h. So we can deform D_1 into τD_1 in Y, keeping C_1 fixed. On normalizing $p(D_2 \cup \tau D_1)$, we obtain a 2-sphere in M which is essential in X, and with fewer double curves than h. This contradicts our choice of h, and (6.4) is proved.

If h has no double curves, (6.1) is proved. There cannot be more than one double curve by (6.3) and (6.4). If there is one double curve $C \subset h(S^2)$, then C must reverse orientation by (6.3). Let $p^{-1}(C) = C_1$ and let C_1 divide $\tilde{h}(S^2)$ into the disks D_1 and D_2. Then $p(D_1 \cup \tau D_1)$ gives a map $g: S^2 \longrightarrow M\text{-}BdM$, which identifies antipodal points. If g were inessential in X, we would get a contradiction as in the proof of (6.4).

Let $\tilde{g}: S^2 \longrightarrow \tilde{M}$ be a lifting of g. Then $\tilde{g}$ is nonsingular. So $\tilde{g}(S^2)$ is two-sided in $\tilde{M}$, since any 2-sphere is two-sided in a 3-manifold. The covering translation τ is orientation reversing on $\tilde{M}$ and on $\tilde{g}(S^2)$. So τ does not interchange the sides of $\tilde{g}(S^2)$. So $g(S^2)$ is a two-sided projective plane in M. The proof of (6.1) is completed.

§7. Generators of $\pi_2(M)$.

In this section, we assume M is a compact, connected, triangulated 3-manifold, possibly with boundary.

Theorem (7.1). There are a finite number of semilinear maps $g_\alpha: S^2 \longrightarrow M\text{-}BdM$, such that:-

i) g_α is either non-singular or identifies antipodal points and $g_\alpha(S^2)$ is a two-sided 2-sphere or projective plane in M;

ii) $g_\alpha(S^2) \cap g_\beta(S^2) = \emptyset$ if $\alpha \neq \beta$;

iii) The maps g_α are a set of $\pi_1(M)$-generators of $\pi_2(M)$.

Condition iii) is elucidated as follows. $\pi_r(M)$ is a $\pi_1(M)$-module for $r \geqslant 2$. By the homotopy extension theorem, any map $g: S^r \longrightarrow M$ is homotopic to a map $h: S^r \longrightarrow M$ which represents an element of $\pi_r(M)$. (That is, h maps base-point into base-point). However, the element of $\pi_r(M)$ represented by h, is only determined up to equivalence under $\pi_1(M)$.

Our proof of (7.1) follows (3.6) of [15].

Let $\rho(M)$ be the minimum number of generators of $\pi_1(\tilde{M})$, where $\tilde{M}$ is the minimal orientable cover of M (the double cover if M is non-orientable, and M itself otherwise). The proof is by induction on $\rho(M)$.

Let Λ be the $\pi_1(M)$-submodule of $\pi_2(M)$ generated by the spherical and projective plane boundary components of M. If $\Lambda = \pi_2(M)$, then by deforming slightly the maps $S^2 \longrightarrow \mathrm{BdM}$, we get a finite number of maps satisfying (7.1).

(7.1) is vacuously true if $\rho(M) < 0$. When $\rho(M) \geqslant 0$, we may assume that $\Lambda \neq \pi_2(M)$, by the previous paragraph. Then by Theorem (6.1) (see [15]), there is a map $g: S^2 \longrightarrow M\text{-}\mathrm{BdM}$, which is essential mod Λ, and which is either non-singular or identifies antipodal points, and whose image is two-sided. We take a regular neighbourhood U of $g(S^2)$ in M, homeomorphic to $g(S^2)\times(0,1)$, where

(0,1) is the open interval. Let the components of M-U be denoted by N_i (there are at most two components). Let $\tilde{N}_i$ be a component of the inverse image of N_i in $\tilde{M}$. Then $\pi_1(\tilde{M}) = \pi_1(\tilde{N}_i) * G$ for some group G, since $\tilde{N}_i$ is attached to $\tilde{M}-\tilde{N}_i$ by 2-spheres. So $\rho(M) = \rho(N_i)+s$, where s is the minimum number of generators of G. If $\rho(M) = \rho(N_i)$ for some i, then $s = 0$, and so $G = 0$. Therefore $\pi_1(\tilde{N}_i) \longrightarrow \pi_1(\tilde{M})$ is an isomorphism. So $\tilde{g}$, a lifting of g to $\tilde{M}$, has an image which is a 2-sphere separating $\tilde{M}$ into two components, one of which is a homotopy 3-sphere with a number of 3-balls removed. (If $\tilde{g}(S^2)$ did not separate $\tilde{M}$, G would have an infinite cyclic free factor - see (11.2)). Therefore g is not essential mod Λ, in contradiction to our choice of g. Thus $\rho(N_i) < \rho(M)$ for each i.

By the inductive hypothesis, we can select maps which satisfy Theorem (7.1) for N_i. Using all such maps for each i, we obtain Theorem (7.1). The fact that the maps are a set of $\pi_1(M)$-generators of $\pi_2(M)$, is deduced by applying the Mayer-Vietoris exact sequence to the universal cover of M.

Chapter III. NON-ORIENTABLE 3-MANIFOLDS WITH FINITE FUNDAMENTAL GROUPS.

In this chapter, we prove that a compact non-orientable 3-manifold M with finite fundamental group is homotopy equivalent to $P^2 \times I$ with a number of 3-balls removed from its interior.

§8. The boundary of M.

Lemma (8.1). Each component of $\mathrm{Bd}M$ is homeomorphic to P^2 or S^2.

Let $\widetilde{M}$ be the orientable double cover of M. Then $\pi_1(\widetilde{M})$ is finite. By Satz IV, §64, [10], each component of $\mathrm{Bd}\widetilde{M}$ is a 2-sphere. (8.1) follows.

Lemma (8.2). $H_2(\widetilde{M};Q) + Q \cong H_2(\mathrm{Bd}\widetilde{M};Q)$ if $\mathrm{Bd}\widetilde{M} \neq \emptyset$, and the projection $H_2(\mathrm{Bd}\widetilde{M};Q) \longrightarrow H_2(\widetilde{M};Q)$ is induced by inclusion. $H_2(\widetilde{M};Q) = 0$ if $\mathrm{Bd}\widetilde{M} = \emptyset$. ($Q$ is the field of rationals).

$H_1(\widetilde{M};Q) = 0$, since $\pi_1(\widetilde{M})$ is finite. Therefore, by duality, $H^2(\widetilde{M},\mathrm{Bd}\widetilde{M};Q) = 0$. By the universal coefficient theorem, $H_2(\widetilde{M},\mathrm{Bd}\widetilde{M};Q) = 0$. (8.2) now follows from the exact homology sequence for $\widetilde{M}$ and $\mathrm{Bd}\widetilde{M}$.

Let T be the 3-manifold obtained from M by filling in the boundary 2-spheres with 3-balls. Then $\pi_1(M) \cong \pi_1(T)$. Let $\mathrm{Bd}T$ have n components (all of which are homeomorphic to P^2 by (8.1)).

<u>Lemma (8.3).</u> <u>$n = 2$.</u>

Let the orientable double cover of T be $\tilde{T}$. Substituting T for M and $\tilde{T}$ for $\tilde{M}$ in (8.2), we see that if $n = 0$, $H_2(\tilde{T};Q) = 0$, and if $n > 0$ then $H_2(\tilde{T};Q) \cong (n-1)Q$. $\tilde{T}$ has a covering translation of period two acting on it. We know the effect of the covering translation on $Bd\tilde{T}$. So we can calculate the Lefschetz index from (8.2). The index of the covering translation turns out to be 1 if $n = 0$, and $1-0+[-(n-1)]$ if $n > 0$. Since the covering translation has no fixed points, the index must be zero. (8.3) follows.

<u>Lemma (8.4).</u> <u>If K is a compact triangulated n-manifold with boundary then $H_{n-1}(K;Z)$ is torsion free.</u>

For every prime p, $H_n(K;Z_p) = 0$. Therefore (8.4) follows from the universal coefficient theorem.

<u>Lemma (8.5).</u> <u>$H_2(\pi_1(T);Z) = 0$.</u> <u>$H_3(\pi_1(T);Z)$ is a quotient group of Z_2+Z_2.</u>

The homology group $H_n(G;Z)$ of a group G is by definition the homology group $H_n(K(G,1);Z)$ of a $K(G,1)$. We shall prove (8.5) by attaching 3-cells and 4-cells to T that will leave π_1 unaltered and kill off π_2.

The universal cover of T is a homotopy 3-sphere with some 3-balls removed. Therefore the two maps $S^2 \longrightarrow BdT$ which identify antipodal points, are $\pi_1(T)$-generators of $\pi_2(T)$. Let the components of Bd T be denoted by R_1 and R_2. We attach one copy of P^4 to T by identifying a standard P^2 in P^4 with R_1. We do the same with another copy of P^4 and R_2. We call the new space A.

Then $\pi_2(A) = 0$. $\pi_1(A) \cong \pi_1(T)$ by van Kampen's Theorem. A can be converted into a $K(\pi_1(T),1)$ by adding n-cells $(n > 3)$. Therefore $H_2(\pi_1(T);Z) \cong H_2(A;Z)$, which is proved below to be zero, and $H_3(\pi_1(T);Z)$ is a quotient group of $H_3(A;Z)$, which is proved below to be $2Z_2$.

The Mayer-Vietoris exact sequence for the subspaces $P^4 \cup P^4$ and T of A is

$$
\begin{aligned}
&0 \longrightarrow 2Z_2 \longrightarrow H_3(A;Z) \\
&\longrightarrow 0 \longrightarrow H_2(T;Z) \longrightarrow H_2(A;Z) \\
&\longrightarrow 2Z_2 \longrightarrow 2Z_2 + H_1(T;Z) \longrightarrow
\end{aligned}
$$

Therefore $H_3(A;Z) \cong 2Z_2$. $H_2(A;Z) \cong H_2(T;Z)$ which is free abelian by (8.4). Also $H_2(A;Z) \cong H_2(\pi_1(T);Z)$ is finite since it is the homology group of a finite group. So $H_2(\pi_1(T);Z) = 0$.

§9. The dihedral group appears and disappears.

Any compact non-orientable 3-manifold M with finite fundamental group has exactly two projective plane boundaries R_1 and R_2 by (8.3). The map $\pi_1(R_1) \longrightarrow \pi_1(M)$ is a monomorphism. Let the non-trivial element of the image of this map be called α.

<u>Theorem (9.1). Let $\pi_1(M) \not\cong Z_2$. Then $\pi_1(M)$ contains a dihedral subgroup of order $2q$ (q an odd prime) or a subgroup $Z_2 \times Z_2$. In either case the subgroup contains the element α.</u>

Let $\tilde{T}$ be the 3-manifold obtained from M by filling in the boundary 2-spheres with 3-balls. By (8.1), BdT consists only of R_1 and R_2. Let $p: T' \longrightarrow T$ be the proper covering of T, corresponding to the subgroup generated by α. Then $p^{-1}(R_1)$ contains either one or two projective planes, its other components being 2-spheres by (8.1) and (8.3) ($p^{-1}(R_2)$ contains one or no projective planes respectively). Let r be a base-point in R_1.

<u>Lemma (9.2). If $p^{-1}(R_1)$ contains two projective planes, R' and R", then $\pi_1(T)$ contains the subgroup $Z_2 \times Z_2$.</u>

Let $r' \in R'$ and $r'' \in R''$ such that $p(r' \cup r'') = r$. Let $g \in \pi_1(T)$ be represented by a path from r to r in T, which, when lifted to T', runs from r' to r". Then (reading from left to right) $g\alpha g^{-1}\alpha = 1$ or $g\alpha g^{-1}\alpha = \alpha$. Since α is orientation reversing and $g\alpha g^{-1}\alpha$ is not, we must have $g\alpha = \alpha g$.

Therefore $g^2\alpha = \alpha g^2$. On lifting the paths representing $g^2\alpha$ and αg^2 to paths in T' based on r', these paths in T' must have the same endpoint. So, lifting g^2 to a path in T' based on r', it must end at a projective plane in $p^{-1}(R_1)$. This projective plane must be R'. So $g^2 = 1$ or α. As before $g^2 = 1$. so (9.2) is proved.

<u>Lemma (9.3). If $p^{-1}(R_1)$ contains one projective plane only, then $\pi_1(T)$ contains a dihedral subgroup of order $2q$ (q an odd prime).</u>

Let $\tilde{T}$ be the orientable double covering of T. We have the exact sequence

$$0 \longrightarrow \pi_1(\tilde{T}) \longrightarrow \pi_1(T) \longrightarrow Z_2 \longrightarrow 0.$$

The non-trivial element of Z_2 is represented by $\alpha \in \pi_1(T)$. Let $G = \mathrm{Im}(\pi_1(\tilde{T}) \longrightarrow \pi_1(T))$.

<u>Lemma (9.4). Let $p^{-1}(R_1)$ contain only one projective plane. For any $g \in G$, if $\alpha g = g\alpha$ then $g = 1$.</u>

Let $r' \in p^{-1}(r)$ lie in the one projective plane in $p^{-1}(R_1)$. If $\alpha g = g\alpha$ in $\pi_1(T)$, then paths representing αg and $g\alpha$, based on r', must have the same endpoints in T'. This is obviously only true if g lifts to a closed path in T'. So $g = \alpha$ or 1. But $g \neq \alpha$ since $g \in G$.

<u>Lemma (9.5). Let $p^{-1}(R_1)$ contain only one projective plane. For any $g \in G$, $\alpha g \alpha = g^{-1}$.</u>*

Consider the set of elements of G of the form $\alpha h \alpha h^{-1}$ where

---

*The author would like to thank Dr.P.Fong most sincerely for providing the proof of (9.5), which Dr.Fong ascribes to Burnside.

$h \in G$. If $\alpha h \alpha h^{-1} = \alpha k \alpha k^{-1}$, then $h \alpha h^{-1} = k \alpha k^{-1}$ and so $k^{-1} h \alpha = \alpha k^{-1} h$. By (9.4), $k = h$. Therefore the (possibly non-homomorphic) mapping $G \longrightarrow G$ given by $h \longrightarrow \alpha h \alpha h^{-1}$ is a (1-1) map and therefore is onto since G is finite. Given $g \in G$, there is an $h \in G$ such that $\alpha h \alpha h^{-1} = g$. So $\alpha g \alpha = \alpha . \alpha h \alpha h^{-1} . \alpha = h \alpha h^{-1} \alpha = (\alpha h \alpha h^{-1})^{-1} = g^{-1}$.

On selecting an element g of prime order q in G, we see that $q \neq 2$ by (9.4) and (9.5). (9.3) is proved and so (9.1) is proved.

<u>Theorem (9.6).</u> <u>If M is a compact non-orientable 3-manifold with finite fundamental group, then $\pi_1(M) \cong Z_2$.</u>

For if not, (9.1) would give us a compact non-orientable 3-manifold T, with projective planes R_1 and R_2 its only boundary components, and with fundamental group $Z_2 \times Z_2$ or dihedral of order 2q, where q is an odd prime. This is done by simply using the covering spaces associated with these subgroups, and then filling in the boundary 2-spheres with 3-balls.

By the Künneth relations, $H_2(Z_2 \times Z_2; Z) \cong H_1(Z_2; Z) \otimes H_1(Z_2; Z) \cong Z_2$. So, by (8.5), $\pi_1(T) \not\cong Z_2 \times Z_2$.

<u>Lemma (9.7). If D is a dihedral group of order 2q, then $H_3(D;Z) \cong Z_{2q}$.*</u>

Let E^3, Euclidean 3-space, be given cylindrical polar coordinates (r,θ,z), where (r,θ) are polar coordinates in the plane $z = 0$. The lens space $L(q,1)$ is the 3-manifold obtained by taking the unit ball $r \leq 1$, and identifying $(1,\theta,z)$ with $(1,\theta+2\pi/q,-z)$ for $z \geq 0$.

There is a covering map $S^3 \longrightarrow L(q,1)$. So $\pi_2(L(q,1)) = 0$. Since $L(q,1)$ is an orientable 3-manifold, we have

$$\operatorname{Coker}(H_3(S^3;Z) \longrightarrow H_3(L(q,1);Z)) \cong Z_q.$$

If we fill in S^3 with a 4-ball, which we attach to $L(q,1)$ by the covering map, then we kill off $\pi_3(L(q,1))$. In order to convert $L(q,1)$ into a $K(Z_q,1)$, we have to add further n-cells $(n > 4)$. We deduce that $H_1(Z_q;Z) \cong Z_q$, $H_2(Z_q;Z) = 0$, and $H_3(Z_q;Z) \cong Z_q$.

The sign changing automorphism of the fundamental group Z_q, is induced by the homeomorphism $(r,\theta,z) \longrightarrow (r,-\theta,-z)$ of $L(q,1)$. This homeomorphism is orientation preserving. Therefore the sign changing automorphism of Z_q induces the sign changing automorphism of $H_1(Z_q;Z)$ and leaves $H_3(Z_q;Z)$ fixed.

---

*(9.7) can be much more easily deduced from the cohomology ring of Z_q, and the Serre-Hochschild spectral sequence. The author prefers the proof given here.

If D is the dihedral group of order $2q$, we have an exact sequence

$$0 \longrightarrow Z_q \longrightarrow D \longrightarrow Z_2 \longrightarrow 0.$$

Using the previous paragraph, we see that the Serre-Hochschild spectral sequence gives $H_3(D;Z) \cong Z_{2q}$.

Z_{2q} is not a quotient group of Z_2+Z_2. So, by (8.5), $\pi_1(T) \not\cong D$. So (9.6) is true.

§10. Homotopy equivalence.

Let M be a compact non-orientable 3-manifold with finite fundamental group. By (9.6), $\pi_1(M) \cong Z_2$. By (8.1) and (8.3), BdM consists of two projective planes and (say) r 2-spheres. Let T be obtained from M by filling in the boundary 2-spheres with 3-balls. Then BdT consists of two projective planes R_1 and R_2. R_1 has a regular neighbourhood U, homeomorphic to $P^2 \times I$. By using a homeomorphism of T with itself, we can assume that the r 2-spheres in BdM all lie in U. Let P be $P^2 \times I$ with r 3-balls removed. We have the homeomorphism $P \longrightarrow U \cap M$. This gives rise to a map $P \longrightarrow M$, which is a homotopy equivalence, since it induces isomorphisms of all homotopy groups. We have therefore proved:

Theorem (10.1). Given any compact, triangulated, non-orientable 3-manifold M with finite fundamental group, there is a semilinear homeomorphism of $P^2 \times I$ with r 3-balls removed, into M, which is a homotopy equivalence. BdM consists of two projective planes and r 2-spheres.

Let C1 be the conjecture that any closed 3-manifold with trivial fundamental group is a 3-sphere (the Poincaré Conjecture). Let C2 be the conjecture that any orientation reversing involution of S^3 with exactly two fixed points is equivalent to the involution which maps each 2-sphere "of constant latitude" antipodally onto itself. Let C3 be the conjecture that any non-orientable 3-manifold with finite fundamental group is $P^2 \times I$ with r 3-balls removed. Then C1 and C2 together imply C3, and C3 implies both C1 and C2. The author hopes to investigate C2 at some later date.

Chapter IV. FUNDAMENTAL GROUPS OF 3-MANIFOLDS.

In this chapter a 3-manifold may be paracompact or with boundary. M is a 3-manifold if it is a connected Hausdorff space such that each point has a neighbourhood homeomorphic to a closed 3-ball.

We investigate here, under what conditions certain groups can be subgroups of the fundamental group of a 3-manifold. Theorem (13.2) is the logical conclusion of Hopf's Conjecture that there are no elements of finite order in a knot group. Necessary and sufficient conditions are given for $\pi_1(M)$ to have a finite subgroup. (13.7)a) states that any finite subgroup of $\pi_1(M)$ is the fundamental group of a closed orientable 3-manifold. In (14.1) it is proved that the only finitely generated abelian subgroups of $\pi_1(M)$ are Z, $Z+Z$, $Z+Z+Z$, $Z+Z_2$ and Z_r.

§11. Essential mappings.

Let Q and R be two 3-manifolds. We remove, from a closed 3-ball neighbourhood in Q, a small concentric open 3-ball, and similarly in R. We identify the boundary 2-spheres thus created, and obtain a 3-manifold denoted $Q\#R$, with a submanifold 2-sphere S. (The symbol $Q\#R$ is unique up to homeomorphism, but we will not prove this). We say S splits $Q\#R$ into Q and R. We note that $Q\cap(Q\#R) \neq Q$, since there is a 3-ball missing.

Lemma (11.1). $\pi_1(Q\#R) \cong \pi_1(Q) * \pi_1(R)$. If S is not essential in Q#R, then either Q or R is a homotopy 3-sphere (and therefore a compact 3-manifold without boundary).

The isomorphism is a consequence of van Kampen's Theorem.

If S is not essential, it can be deformed to a point in some compact 3-dimensional submanifold of Q#R. (11.1) therefore reduces to the case where Q and R are compact, when the result follows easily by examining the homology of the universal cover.

Lemma (11.2). If S is a submanifold 2-sphere in M-BdM, and S does not separate M, then $\pi_1(M) \cong Z * \pi_1(M-S)$ and S is essential in M.

We construct an infinite covering space V of M by cutting at S. Each sheet of the covering is homeomorphic to M-S, and we cross to another sheet whenever we cross S. Let $\widetilde{S}$ be a lifting of S to V. $\widetilde{S}$ separates the covering space into two non-compact components. So $\widetilde{S}$ is essential in V by (11.1). Therefore S is essential in M.

The group of covering translations of V is cyclic infinite. We denote its generator by α.

$$\pi_1(V) \cong \Pi_{n=-\infty}^{n=\infty} \pi_1(\alpha^n(M-S)).$$

We have the exact sequence

$$0 \longrightarrow \pi_1(V) \longrightarrow \pi_1(M) \longrightarrow Z \longrightarrow 0.$$

So $\pi_1(M)$ is determined by giving the automorphism determined by α.

The automorphism is induced by

$$\alpha_*:\pi_1(\alpha^n(M-S)) \longrightarrow \pi_1(\alpha^{n+1}(M-S)).$$

We easily deduce that $\pi_1(M) \cong \pi_1(M-S) * Z$.

Lemma (11.3). If $g: S^2 \longrightarrow M-BdM$ identifies antipodal points and $g(S^2)$ is a two-sided submanifold of M, then g is essential.

Let $\widetilde{M}$ be the orientable double cover of M. Let $\widetilde{g}: S^2 \longrightarrow \widetilde{M}$ be a lifting of g. The map $\widetilde{g}$ is non-singular. If $\widetilde{g}$ is not essential then $\widetilde{g}(S^2)$ splits $\widetilde{M}$ into a homotopy 3-sphere and some other 3-manifold by (11.1) and (11.2). Therefore the closure of one component of $M-g(S^2)$ is a compact non-orientable 3-manifold with finite fundamental group and only one boundary component. This is impossible by (10.1). So (11.3) follows.

§12. Definition of the symbols g_α, p, N^i_G, N^i, N_G, N, V_G, W_G, $M(G;\gamma)$ and U_α.

Let M be a compact triangulated 3-manifold. Let $G \neq 0$ be a subgroup of $\pi_1(M)$. We suppose G has the following property.

Condition (12.1). a) G is not a non-trivial free product;

b) G is not infinite cyclic.

This condition arises from the attempt to prevent the existence of an essential 2-sphere in a certain 3-manifold. (See (11.1) and (11.2)).

Since M is compact, we have a finite set γ of $\pi_1(M)$-generators of $\pi_2(M)$, $g_\alpha: S^2 \longrightarrow M\text{-BdM}$ as given by (7.1). Let $p: V_G \longrightarrow M$ be the covering of M, such that $p_*(\pi_1(V_G)) = G$. We cut V_G at all components of $p^{-1}(g_\alpha(S^2))$ that are 2-spheres, so that V_G falls into a number of components. One and only one of these, W_G, has a non-zero fundamental group, by (12.1) and van Kampen's Theorem. (We note that all 2-spheres separate V_G, by (12.1) and (11.2)). Then $\pi_1(W_G) \cong G$.

Let U_α be an open regular neighbourhood of $g_\alpha(S^2)$, which is homeomorphic to $g_\alpha(S^2)\times(0,1)$ by (7.1), where (0,1) is the open interval. If $W_G - p^{-1}(\bigcup_\alpha U_\alpha)$ is connected, we denote it by N_G. N_G covers some component N of $M - \bigcup_\alpha U_\alpha$. If $W_G - p^{-1}(\bigcup_\alpha U_\alpha)$ is not connected, we denote the components by N_G^i (i = 1, 2,.....). Then, for each i, $\pi_1(N_G^i) \neq 0$, since N_G^i has a projective plane in its boundary. For each i, N_G^i covers some component N^i of $M - \bigcup_\alpha U_\alpha$ (in general, $N^i = N^j$ for some $i \neq j$). If we fill in the boundary 2-spheres of W_G with 3-balls, we get a 3-manifold $M(G;\gamma)$.

We emphasize that the notation above will be retained (at least, whenever M is compact and G satisfies (12.1)) for the remainder of this chapter.

<u>Lemma (12.2).</u> $\underline{\pi_1(M(G;\gamma)) \cong \pi_1(W_G) \cong G}$.

This follows from van Kampen's Theorem.

Lemma (12.3). $H_2(G;Z) \cong H_2(M(G;\gamma);Z)$.

$\pi_2(M(G;\gamma))$ has π_1-generators, all of which have projective planes in $p^{-1}(\bigcup_\alpha g_\alpha(S^2))$ as images. If we attach one copy of P^3 to $M(G;\gamma)$ for each such generator, by identifying the subspaces P^2, we kill off π_2. However, H_2 is unaltered, as can be seen from the Mayer-Vietoris sequence. We now convert the space into a $K(G,1)$ by attaching n-cells ($n > 3$). Obviously H_2 is unaffected. (12.3) follows.

If every component of $p^{-1}(\bigcup_\alpha g_\alpha(S^2)) \cap W_G$ is a 2-sphere, we say (G,γ) is reducible.

Lemma (12.4). If (G,γ) is reducible, then $W_G - p^{-1}(\bigcup_\alpha U_\alpha)$ is connected. (The case where we write N_G and N instead of N_G^i and N^i respectively). N_G is homeomorphic to W_G.

This follows from the definition of W_G.

Lemma (12.5). If (G,γ) is reducible, then $\pi_2(M(G;\gamma)) = 0$.

The maps $\tilde{g}_\alpha : S^2 \longrightarrow W_G$ are π_1-generators of $\pi_2(W_G)$ (using all possible liftings $\tilde{g}_\alpha$ of g_α for each α). On attaching 3-balls to W_G, to form $M(G;\gamma)$, π_2 is killed off.

Lemma (12.6). If $M(G;\gamma)$ is orientable, then (G,γ) is reducible.

For then we could have no two-sided projective planes in W_G.

Lemma (12.7). If all subgroups H of index at most two in G, satisfy

<u>(12.1) and if $G \not\cong Z_2$, then, for some such H, (H, γ) is reducible.</u>

(We assume $G \not\cong Z_2$, since, if $H = 0$, the construction is not defined).

We merely take the minimal orientable cover of V_G. Here we can have no two-sided projective planes.

<u>Lemma (12.8).</u> <u>If G contains no elements of order two, then (G, γ) is reducible.</u>

For then, there could be no two-sided projective planes in W_G.

§13. Subgroups of $\pi_1(M)$.

<u>Lemma (13.1).</u> <u>Let L be a 3-manifold, such that $\pi_2(L) = 0$. Then, if L is not aspherical, it is closed and has a finite fundamental group and is orientable.</u>

This lemma is, the author believes, due to J.H.C. Whitehead.

If L is not aspherical, then its universal cover $\tilde{L}$ has some non-vanishing homology group. Therefore $H_3(\tilde{L}; Z) \neq 0$ (using singular homology since $\tilde{L}$ may not be triangulable). Therefore $H_3(\tilde{L}; Z) \cong Z$. Therefore $\tilde{L}$ is compact and without boundary. Therefore L is compact, $\pi_1(L)$ is finite, and L has no boundary. By (10.1), L is orientable.

We say two loops $f: S^1 \longrightarrow M$ and $g: S^1 \longrightarrow M$ are <u>equivalent</u> under $\pi_1(M)$, if there are maps f' and g' of S^1 with base-point into M with base-point, $f' \simeq f$, $g' \simeq g$ and the elements of $\pi_1(M)$ defined by f' and g' are conjugate.

Theorem (13.2). If M is a possibly non-paracompact 3-manifold, possibly with boundary, and G is a finite subgroup of $\pi_1(M)$, then either

i) $G \cong Z_2$, and the non-zero element of G is equivalent under $\pi_1(M)$ to a loop on a two-sided projective plane submanifold P of M;

or

ii) $M = Q \# R$, where Q is a closed orientable 3-manifold, with finite fundamental group, and G is conjugate to a subgroup of $\pi_1(Q)$. ($\pi_1(M) \cong \pi_1(Q) * \pi_1(R)$, by (11.1), which defines the embedding of $\pi_1(Q)$ in $\pi_1(M)$).

We may assume, without loss of generality, that M is compact. We assume that i) is not true.

Lemma (13.3). If i) is not true and (G, γ) is not reducible then $G \not\cong Z_2$.

For suppose $G \cong Z_2$. Since (G, γ) is not reducible, there is a projective plane component in $p^{-1}(g_\alpha(S^2)) \cap W_G$ for some α. Then $g_\alpha(S^2)$ is a two-sided projective plane in M which satisfies i). But we have assumed that i) is not true.

Lemma (13.4). If π is a group and $K(\pi,1)$ is a finite dimensional aspherical complex, with fundamental group π, then π contains no elements of finite order.

For if there were such elements, the universal cover of $K(\pi,1)$ would be a resolution of a finite cyclic group. This is impossible, because all such groups are infinite dimensional.

Lemma (13.5). If i) is not true then (G, γ) is reducible.

For suppose (G, γ) is not reducible. By (12.7) and (13.3), for some subgroup $H \neq 0$, of index two in G, (H, γ) is reducible. Now $\pi_2(M(H;\gamma)) = 0$ by (12.5), and H is finite. By (13.4), $M(H;\gamma)$ is not aspherical. By (13.1), $M(H;\gamma)$ is closed and orientable. By (12.6), $M(G;\gamma)$ is not orientable. Therefore W_G is a compact non-orientable 3-manifold with finite fundamental group. By (10.1), $G \cong Z_2$, which is impossible by (13.3). So we have a contradiction, if we assume (G, γ) is not reducible.

We assume that i) is not true, so that, by (13.5), (G, γ) is reducible. By (12.5), $\pi_2(M(G;\gamma)) = 0$. By (13.4), $M(G;\gamma)$ is not aspherical. By (13.1), $M(G;\gamma)$ is closed and orientable. Therefore N_G is a finite sheeted cover of N and $\pi_1(N)$ is finite. So N is a compact 3-manifold with finite fundamental group. Since i) is not true, N is orientable by (10.1). So BdN is a finite union of 2-spheres.

Lemma (13.6). If N is a compact 3-manifold, embedded in a 3-manifold L, so that $N \subset L$-BdL, and BdN is a finite union of 2-spheres, then $L = N' \# N''$, where $N' \cap L \subset N$ and the inclusion induces an isomorphism

<u>of fundamental groups, and N' is a closed manifold. $\pi_1(N') \cong \pi_1(N)$.</u>

We merely bore thin tubular tunnels through N, so as to join up all the 2-spheres in BdN, into one 2-sphere.

We have therefore proved (13.2).

<u>Corollaries (13.7) to Theorem (13.2).</u> <u>a) If $\pi_1(M)$ is finite, then $\pi_1(M) \cong \pi_1(Q)$, where Q is a closed orientable 3-manifold.</u>
<u>b) If $G \neq 0$ is a finite subgroup of $\pi_1(M)$ and $\pi_2(M) = 0$, then $\pi_1(M)$ is finite, and M is closed and orientable.</u>

b) is an immediate consequence of (11.3) and (11.1).

<u>Theorem (13.8).</u> <u>Let M be a possibly non-paracompact 3-manifold, possibly with boundary. Let G be isomorphic to the fundamental group of an aspherical closed 3-manifold L. If $G \subset \pi_1(M)$, then either</u>

<u>i) There is a two-sided projective plane P embedded in M;</u>

<u>or</u>

<u>ii) $M = Q \# R$, where R is an aspherical closed 3-manifold, and G is conjugate to a subgroup of finite index in $\pi_1(R)$.</u>

Without loss of generality, we may assume M compact, since G is finitely generated and has a finite number of relations.

By [16], G satisfies (12.1). By (13.4), G is torsion free. By (12.8), (G, γ) is reducible for any set γ of π_1-generators of $\pi_2(M)$, satisfying (7.1). Since G is infinite, we know from (13.1) and (12.5), that $M(G;\gamma)$ is aspherical.

Therefore

$$H_3(M(G;\gamma);Z_2) \cong H_3(G;Z_2) \cong H_3(L;Z_2) \cong Z_2.$$

So $M(G;\gamma)$ is a closed 3-manifold. Therefore N_G is a finite sheeted cover of N and has only spherical boundaries. Therefore N is a compact 3-manifold with only spherical and projective plane boundaries. If we assume i) is not true, then ii) follows from (13.6).

Corollary (13.9) to Theorem (13.8). If the hypotheses of (13.8) hold and if $\pi_2(M) = 0$, then M is an aspherical closed 3-manifold, and G is of finite index in $\pi_1(M)$.

This follows from (11.3) and (11.1).

§14. Finitely generated abelian subgroups of $\pi_1(M)$.

Theorem (14.1). Let M be a possibly non-paracompact 3-manifold, possibly with boundary. The following is a complete list of finitely generated abelian subgroups, which can occur in $\pi_1(M)$. Examples of their realisations are also given.

Z	$Z+Z$	$Z+Z_2$	$Z+Z+Z$	Z_r
$S^1\times I\times I$, $S^1\times S^2$	$S^1\times S^1\times I$	$S^1\times P^2$	$S^1\times S^1\times S^1$	Lens space

We note that all abelian groups except Z satisfy (12.1). We may assume, without loss of generality, that M is a compact 3-manifold.

Lemma (14.2). *$G \cong Z+Z+Z+Z$ cannot be a subgroup of $\pi_1(M)$, where M is a compact 3-manifold.*

For suppose (14.2) is not true. By (12.8) (G, γ) is reducible. Therefore $M(G;\gamma)$ is aspherical by (12.5) and (13.1). Therefore

$$H_3(M(G;\gamma);Z) \cong H_3(G;Z) \cong 4Z,$$

which is impossible, since $M(G;\gamma)$ is a 3-manifold.

Lemma (14.3). *$G \cong Z+Z+Z_2$ cannot be a subgroup of $\pi_1(M)$, where M is a compact 3-manifold.*

For suppose (14.3) is not true. By (12.3),

$$H_2(M(G;\gamma);Z) \cong Z+Z_2+Z_2.$$

By the universal coefficient theorem, $H_3(M(G;\gamma);Z_2)$ contains a subgroup Z_2+Z_2, which is impossible.

Lemma (14.4). *$G \cong Z_r+Z_s$ cannot be a subgroup of $\pi_1(M)$, where M is a compact 3-manifold and $r|s$.*

For suppose (14.4) is not true. From (13.7)a), we see that $G \cong \pi_1(Q)$, where Q is a closed orientable 3-manifold. Therefore Q has a finite covering space, which is a homotopy 3-sphere. Therefore $\pi_2(Q) = 0$. Therefore $H_2(Q;Z) \cong H_2(G;Z) \cong Z_r$. By Poincaré Duality, $H^1(Q;Z) \cong Z_r$. But $H^1(Q;Z)$ is torsion free, by the universal coefficient theorem. So we have a contradiction.

By (13.2), the group $Z+Z_r$ is impossible when $r > 2$. From (14.2), (14.3) and (14.4), we easily deduce (14.1).

Theorem (14.5). If M is a possibly non-paracompact 3-manifold, possibly with boundary, and $\pi_1(M)$ contains a subgroup $G \cong Z+Z_2$, then $M = Q \# R$, where R is a closed 3-manifold and $\pi_1(R) \cong Z+Z_2$, and G is conjugate to a subgroup of $\pi_1(R)$.

We assume, without loss of generality, that M is compact.

By (12.3), $H_2(M(G;\gamma);Z) \cong H_2(Z+Z_2;Z) \cong Z_2$. Therefore $H_3(M(G;\gamma);Z_2) \neq 0$ by the universal coefficient theorem. So we have:

Lemma (14.6). $M(G;\gamma)$ is a closed 3-manifold.

W_G is therefore a compact 3-manifold with only spherical boundaries. Suppose (G,γ) is reducible. Then $\pi_2(M(G;\gamma)) = 0$ by (12.5). So, by (13.1), $M(G;\gamma)$ is aspherical, which is impossible by (13.4).

So $W_G - BdW_G$ contains at least one two-sided projective plane P. Any such projective plane cannot separate W_G, since we would then have a compact 3-manifold, whose boundary had an odd Euler-Poincaré characteristic (see [10], §64, Satz III).

We construct a covering space W of W_G, whose sheets are homeomorphic to $W_G - P$, and such that we cross to a different sheet every time we cross P. We have the exact sequence

$$0 \longrightarrow \pi_1(W) \longrightarrow \pi_1(W_G) \longrightarrow Z \longrightarrow 0$$

where $\pi_1(W_G) \cong G \cong Z + Z_2$ by (12.2). We easily see that $\pi_1(W) \cong Z_2$ from the exact sequence.

From van Kampen's Theorem, we easily see that $\pi_1(W_G - P) \cong Z_2$. Therefore $\pi_1(N_G^i) \cong Z_2$ for each i (it is not zero, since N_G^i contains a two-sided projective plane). Since W_G is compact, N_G^i is compact for each i. Therefore N_G^i is a finite sheeted cover of N^i. Therefore N^i is a compact non-orientable 3-manifold with finite fundamental group. By (10.1), $\pi_1(N^i) \cong Z_2$ and BdN^i consists of a number of 2-spheres and two projective planes. Therefore N_G^i is not a proper cover of N^i. The projective planes in BdN^i do not separate M by the paragraph before the last. By glueing the N^i together along boundary projective planes (using the sets U_α as glue), and using (13.6), we obtain (14.5).

Chapter V. GENERATORS AND RELATIONS.

Let a group G have a finite presentation

$$P = \{x_1, \ldots\ldots x_n / r_1, \ldots\ldots r_m\}.$$

Then we define deficiency of P = def P = n-m.

We define deficiency of G = def G = maximum of def P over all possible presentations P of G.

In this chapter we obtain an upper bound for def P in terms of invariants of the group (its homology groups). This upper bound is actually attained for abelian groups, groups given by presentations with only one relation, and fundamental groups of compact 3-manifolds. We deduce from this some theorems about fundamental groups of 3-manifolds. This chapter is completely independent of Chapters III and IV.

In this chapter a 3-manifold will be assumed compact, connected and possibly with boundary.

§15. An upper bound for def P.

This section contains two lemmas ((15.1) and (15.2)), which are due to Professor P.Hall, for whose assistance I am extremely grateful.

Suppose we are given a presentation P of a group G as above. Let F be the free group on the n generators $x_1, \ldots\ldots x_n$. Let R be the normal subgroup of F, generated by $r_1, \ldots\ldots r_m$. Then $G \cong F/R$.

Let $s(G)$ = minimum number of generators of G.

If A and B are two subgroups of a group H, then $[A,B]$ is the group generated by the set of all commutators $a^{-1}b^{-1}ab$ ($a\in A$, $b\in B$).

Lemma (15.1). $R/[F,R]$ is an abelian group generated by the m generators $r_1[F,R],\ldots,r_m[F,R]$.

Since any element of R is of the form

$$f_1^{-1}r_{i_1}^{\varepsilon_1}f_1\cdots\cdots f_s^{-1}r_{i_s}^{\varepsilon_s}f_s$$

where $f_j\in F$ and $\varepsilon_j = \pm 1$ $(1\leq j\leq s)$, (15.1) is obvious.

Lemma (15.2). $\operatorname{def} P \leq \operatorname{rank}H_1(G;Z) - s(H_2(G;Z))$.

We have the following diagram of abelian groups and homomorphisms

$$\begin{array}{ccccccccc} 0 & \longrightarrow & [F,F]\cap R/[F,R] & \longrightarrow & R/[F,R] & \longrightarrow & R/[F,F]\cap R & \longrightarrow & 0 \\ & & & & & & \wr\Vert & & \\ 0 & \longleftarrow & F/[F,F]R & \longleftarrow & F/[F,F] & \longleftarrow & [F,F]R/[F,F] & \longleftarrow & 0. \end{array}$$

where the horizontal rows are exact. By a well-known result of Hopf (see [6]), $H_2(G;Z)\cong [F,F]\cap R/[F,R]$. Also

$$H_1(G;Z)\cong G/[G,G]\cong F/[F,F]R.$$

$F/[F,F]$ is free abelian. Therefore $[F,F]R/[F,F]$ is free abelian. Therefore $R/[F,R]\cong [F,F]\cap R/[F,R] + R/[F,F]\cap R$. Since the summand on the right is free abelian, we have

$$s(R/[F,R]) = s(H_2(G;Z)) + s(R/[F,F]\cap R) = s(H_2(G;Z)) + n - \operatorname{rank}H_1(G;Z),$$

by the exactness of the second row. Therefore, by (15.1),

$$n - \operatorname{rank} H_1(G;Z) + s(H_2(G;Z)) \leq m.$$

(15.2) follows.

If G is a group such that equality is attained in (15.2), we say G is efficient. The author suspects that $(Z\times Z_2) * (Z\times Z_3)$ is not efficient.

Lemma (15.3). If G is a finitely generated abelian group, it is efficient.

We simply use the canonical presentation of G.

Lemma (15.4). If X is a free complex, then

$$\operatorname{rank}_p H_1(X;Z_p) - \operatorname{rank}_p H_2(X;Z_p) \geq \operatorname{rank} H_1(X;Z) - s(H_2(X;Z)),$$

where p is a prime and rank_p denotes the dimension of the vector space over the field Z_p. We have equality if and only if the canonical presentation of $H_2(X;Z)$

$$H_2(X;Z) \cong Z+\ldots\ldots+Z+Z_a+Z_b+Z_c+\ldots\ldots+Z_j+Z_k, \ (a|b,\ b|c\ldots\ldots \text{ and } j|k),$$

is such that $p|a$.

Let $H = \operatorname{Tor}(H_1(X;Z),Z_p)$. Then

$$H_1(X;Z_p) \cong H_1(X;Z)\otimes Z_p \cong (\operatorname{rank} H_1(X;Z))Z_p + H, \text{ and}$$

$$H_2(X;Z_p) \cong H_2(X;Z)\otimes Z_p + H \cong rZ_p + H,$$

where $r < s(H_2(X;Z))$ if $p \nmid a$ and $r = s(H_2(X;Z))$ if $p|a$. Therefore

$$\operatorname{rank}_p H_1(X;Z_p) = \operatorname{rank} H_1(X;Z) + \operatorname{rank}_p H, \text{ and}$$

$$\operatorname{rank}_p H_2(X;Z_p) = r + \operatorname{rank}_p H.$$

Therefore $\mathrm{rank}_p H_1(X;Z_p) - \mathrm{rank}_p H_2(X;Z_p) = \mathrm{rank} H_1(X;Z) - r$

$\geqslant \mathrm{rank} H_1(X;Z) - s(H_2(X;Z))$.

(15.4) follows.

Lemma (15.5). $\mathrm{rank}_p H_1(G;Z_p) - \mathrm{rank}_p H_2(G;Z_p) \geqslant \mathrm{def}\ G$.

This follows from (15.4) and (15.2).

We say G is p-efficient if there is equality in (15.5). If G is p-efficient, it is efficient. If G is efficient, then, by (15.4), it is p-efficient for some p.

Lemma (15.6). If A and B are p-efficient groups, then $A * B$ is p-efficient and $\mathrm{def}(A*B) = \mathrm{def}\ A + \mathrm{def}\ B$.

$$\mathrm{rank}_p H_1(A;Z_p) + \mathrm{rank}_p H_1(B;Z_p) - \mathrm{rank}_p H_2(A;Z_p) - \mathrm{rank}_p H_2(B;Z_p)$$
$$= \mathrm{rank}_p H_1(A*B;Z_p) - \mathrm{rank}_p H_2(A*B;Z_p)$$
$$\geqslant \mathrm{def}(A*B) \geqslant \mathrm{def}\ A + \mathrm{def}\ B$$ (this follows by adjoining a presentation of A to a presentation of B)
$$= \mathrm{rank}_p H_1(A;Z_p) + \mathrm{rank}_p H_1(B;Z_p) - \mathrm{rank}_p H_2(A;Z_p) - \mathrm{rank}_p H_2(B;Z_p).$$

So (15.6) follows.

Lemma (15.7). If G has a presentation with only one relation $\{x_1,\ldots\ldots x_n/r\}$, G is efficient and def G = n-1, unless r = 1, when def G = n.

In the proof of (15.2), the only place where equality between (n-1) and $\mathrm{rank} H_1(G;Z) - s(H_2(G;Z))$ could break down, is in (15.1).

That is, we have equality in (15.2), unless $R/[R,F]$ has 0 generators. If $R/[R,F] = 0$, then $R = [R,F]$. By induction,

$$R = [[\ldots[[R,F],F],\ldots F],F] \leqslant [[\ldots[[F,F],F],\ldots F],F].$$

So $R \leqslant \bigcap [[\ldots[[F,F],F],\ldots F],F]$, where we take the intersection over all possible subgroups of the form given. By [8], page 38, $R = 0$. Therefore $r = 1$. So G is free, and is efficient by (15.3) and (15.6).

§16. The fundamental group of a 3-manifold.

Lemma (16.1). If M is a compact 3-manifold, there is a cell decomposition of M with only one 3-cell, and with a simplicial 2-skeleton.

We simply triangulate M and abolish interiors of 2-simplexes one by one.

Lemma (16.2). If M is a compact 3-manifold with boundary, then $\operatorname{def} \pi_1(M) \geqslant 1 - \chi(M)$.

We perform a deformation retraction of M by pushing in the interior of a 2-simplex of $\mathrm{Bd}M$, until the 3-cell of (16.1) is entirely retracted onto the 2-skeleton of M. Let there now be p_n n-simplexes. A maximal tree in M_1 contains $(p_o - 1)$ 1-simplexes. There is therefore a presentation of $\pi_1(M)$ with $(p_1 - p_o + 1)$ generators and p_2 relations. Therefore $\operatorname{def} \pi_1(M) \geqslant p_1 - p_o + 1 - p_2 = 1 - \chi(M)$.

Lemma (16.3). If M is a closed 3-manifold, then def $\pi_1(M) \geq 0$.

We remove a small open 3-ball from M, thus changing $\chi(M)$ from 0 to 1. (16.3) now follows from (16.2). (For a closed 3-manifold, $\chi(M) = 0$ by Poincaré Duality).

Lemma (16.4). All torsion elements in $H_2(M;Z)$ are of order two, if M is a compact 3-manifold.

If $H_2(M;Z)$ has a direct summand isomorphic to Z_r, where $r \neq 2$, then $H_3(M;Z_r)$ has a direct summand isomorphic to Z_r, by the universal coefficient theorem. Therefore $H_3(M;Z_r) \cong Z_r$, and $H_3(M;Z) = 0$. (The first isomorphism is the only non-zero possibility, and the second follows from the universal coefficient theorem). These two isomorphisms are contradictory, since the first implies that M is orientable and closed.

For any compact 3-manifold M, there exists a finite set γ of maps $g_\alpha : S^2 \longrightarrow M$, as given by (7.1), which form a set of $\pi_1(M)$-generators of $\pi_2(M)$. Let $n(M, \gamma)$ be the number of α such that g_α is non-singular.

Theorem (16.5). $\pi_1(M)$ is 2-efficient, if M is a compact 3-manifold.

We prove (16.5) by induction on $n(M, \gamma)$.

Lemma (16.6). If $n(M, \gamma) = 0$, then $H_2(M;Z) \cong H_2(\pi_1(M);Z)$. (Compare (12.3)).

We attach one copy of P^3 to M for each α, by identifying the subspace P^2 with $g_\alpha(S^2)$. This does not affect $H_2(M;Z)$. By attaching n-cells ($n > 3$), we obtain a $K(\pi_1(M),1)$. (16.6) follows.

Lemma (16.7). If $n(M, \gamma) = 0$ and M is closed, then (16.5) is true.

$$
\begin{aligned}
0 &= \operatorname{rank}_2 H_1(M;Z_2) - \operatorname{rank}_2 H_2(M;Z_2) && \text{by Poincaré Duality}\\
&= \operatorname{rank}_2 H_1(\pi_1(M);Z_2) - \operatorname{rank}_2 H_2(\pi_1(M);Z_2) && \text{by (16.6)}\\
&\geqslant \operatorname{def} \pi_1(M) && \text{by (15.5)}\\
&\geqslant 0 && \text{by (16.3).}
\end{aligned}
$$

(16.7) follows.

Lemma (16.8). If $n(M, \gamma) = 0$ and M has boundary, (16.5) is true.

$$
\begin{aligned}
1 - \chi(M) &= \operatorname{rank}_2 H_1(M;Z_2) - \operatorname{rank}_2 H_2(M;Z_2)\\
&= \operatorname{rank}_2 H_1(\pi_1(M);Z_2) - \operatorname{rank}_2 H_2(\pi_1(M);Z_2) && \text{by (16.6)}\\
&\geqslant \operatorname{def} \pi_1(M) && \text{by (15.5)}\\
&\geqslant 1 - \chi(M) && \text{by (16.2).}
\end{aligned}
$$

(16.8) follows.

We can now proceed with the proof by induction on $n(M, \gamma)$. If $n(M, \gamma) > 0$, there is at least one 2-sphere $g_\alpha(S^2)$ (denoted by S).

<u>Lemma (16.9). If S separates M, then (16.5) follows from the induction hypothesis.</u>

If we cut M at S and fill in the boundary 2-spheres thus created with 3-balls, we get two 3-manifolds M_1 and M_2. Let γ^1 denote the set of maps g_α (except for the one giving rise to S), whose images lie in M_1, and similarly for γ^2. Then γ^1 is a set of $\pi_1(M_1)$-generators of $\pi_2(M_1)$ and similarly for γ^2. This follows by attaching 3-balls to M by means of the maps g_α and then applying the Mayer-Vietoris sequence to the universal cover of this space. We see that attaching 3-balls to M_1 by the maps in γ^1 kills off $H_2(\tilde{M}_1;Z)$, where $\tilde{M}_1$ is the universal cover of M_1, and similarly for M_2.

Now $n(M, \gamma^1) < n(M, \gamma)$ and $n(M, \gamma^2) < n(M, \gamma)$, since

$$n(M, \gamma) = n(M, \gamma^1)+n(M, \gamma^2)+1.$$

Therefore (16.5) is true for M_1 and M_2. Now $\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2)$. Therefore $\pi_1(M)$ is 2-efficient by (15.6). (16.9) follows.

<u>Lemma (16.10). If S does not separate M, then (16.5) follows from the induction hypothesis.</u>

If we cut M at S and fill in the boundary 2-spheres thus created, we obtain a 3-manifold M'. Let γ' be the set of maps g_α other than the one giving rise to S. $\pi_2(M')$ is $\pi_1(M')$-generated by the set γ'. This can be proved by going to the covering space of M, each sheet of which is homeomorphic to M-S, and in which we cross to a new sheet whenever we cross S. In this space we apply a similar argument to that in the proof of (16.9). Now $n(M', \gamma') < n(M, \gamma)$, since $n(M, \gamma) = n(M', \gamma')+1$. Therefore (16.5) is true for M'.

Now $\pi_1(M) \cong \pi_1(M') * Z$ by (11.2). (16.10) follows from (15.6), since Z is 2-efficient.

The proof of (16.5) is now completed.

§17. Applications of the 2-efficiency of $\pi_1(M)$.

Let M be a closed 3-manifold. Let γ be a set of $\pi_1(M)$-generators of $\pi_2(M)$ as given by (7.1). The image of the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M;Z_2)$ is generated by the maps g_α. So, in determining the image, we can neglect those g_α which identify antipodal points, and those g_α such that $g_\alpha(S^2)$ separates M. Let $\nu(M)$ be the number of times that we can cut M at a 2-sphere $g_\alpha(S^2)$ for some α, so that M remains connected. Let δ be a subset of γ containing $\nu(M)$ such 2-spheres denoted $g_\beta(S^2)$.

Lemma (17.1). If M is closed,

$$\text{rank}_2(\text{Im}(\pi_2(M) \longrightarrow H_2(M;Z_2)) = \nu(M) = \text{def}\,\pi_1(M).$$

Suppose M is triangulated so that $g_\alpha(S^2)$ is simplicial for each α. There is obviously no relation between the elements represented by $g_\beta(S^2)$ in $H_2(M;Z_2)$, since any union of 3-simplexes whose boundary consists only of the 2-spheres $g_\beta(S^2)$, must have a boundary in which each term $g_\beta(S^2)$ occurs twice. Therefore

$$\nu(M) \leq \text{rank}(\text{Im}(\pi_2(M) \longrightarrow H_2(M;Z_2)).$$

On the other hand, each 2-sphere $g_\alpha(S^2)$ is homologous to a sum of the 2-spheres $g_\beta(S^2)$. The first equality of (17.1) follows.

We can make M into a $K(\pi_1(M),1)$ by attaching 3-balls to M by means of the maps g_α, and then adding n-cells (n > 3). Therefore

$$H_2(\pi_1(M);Z_2) + \text{Im}(\pi_2(M) \longrightarrow H_2(M;Z_2)) \cong H_2(M;Z_2).$$

Therefore

$$\text{rank}_2 H_2(\pi_1(M);Z_2) + \nu(M) = \text{rank}_2 H_2(M;Z_2)$$

by the equality just proved. By (16.5),

$$\begin{aligned} \text{def}\ \pi_1(M) &= \text{rank}_2 H_1(\pi_1(M);Z_2) - \text{rank}_2 H_2(\pi_1(M);Z_2) \\ &= \text{rank}_2 H_1(M;Z_2) - \text{rank}_2 H_2(M;Z_2) + \nu(M) \\ &= \nu(M) \qquad \text{by Poincaré Duality.} \end{aligned}$$

If we cut M at a 2-sphere S which does not separate M, and fill in the boundary 2-spheres thus created with 3-balls, we get a closed 3-manifold M'. $\pi_1(M) \cong \pi_1(M') * Z$. Therefore we have:

<u>Lemma (17.2).</u> <u>For some closed 3-manifold N, $\pi_1(M) \cong \pi_1(N) * Z * \ldots * Z$, where there are $\nu(M)$ cyclic infinite free factors.</u>

It is possible to prove that $\pi_1(N)$ contains no further cyclic infinite free factors, but we will not concern ourselves with this point.

Theorem (17.3). The following is a complete list of abelian groups, which can be fundamental groups of closed 3-manifolds: Z, $Z+Z+Z$, $Z+Z_2$ and Z_r. (The proof given here is completely independent of Chapter IV. See (14.1)).

Theorem (17.3) was first proved by Reidemeister.

Since a free product is not abelian, we see from (17.2) that $\nu(M) = 0$, unless $\pi_1(M) \cong Z$. Excluding the case $\pi_1(M) \cong Z$, we see from (17.1) that def $\pi_1(M) = 0$. The only abelian groups, whose deficiencies are zero, are those listed and also $Z+Z_r$ $(r > 2)$.

Since $Z+Z_r$ is not a free product, we see from (11.2) and van Kampen's Theorem, that there is no essential 2-sphere in M. (We recall that a compact 3-manifold with only one boundary component and fundamental group zero is a homotopy 3-ball). By (16.6), $H_2(M;Z) \cong H_2(\pi_1(M);Z)$. By (16.4), all torsion elements of $H_2(\pi_1(M);Z)$ are of order two. Since $H_2(Z+Z_r;Z) \cong Z_r$, we see that $r = 2$. (17.3) follows.

Theorem (17.4). If M is a closed 3-manifold, and $\pi_1(M)$ has a presentation with only one (non-trivial) relation, then $\pi_1(M) \cong Z * \ldots * Z * Z_r$ or $Z * \ldots * Z$.

Let the presentation of $\pi_1(M)$ with only one relation, have n generators. By (15.7), def $\pi_1(M) = n-1$. By (17.1) def $\pi_1(M) = \nu(M)$. By (17.2), $\pi_1(M) \cong A * Z * \ldots * Z$ with n-1 cyclic infinite free factors.

By Grusko's Theorem, ([8], page 58), using the n generators of $\pi_1(M)$, we easily see that A is isomorphic to some quotient group of Z.

Chapter VI. 3-MANIFOLDS WHICH ARE TOPOLOGICAL PRODUCTS.

In this chapter, we consider closed 3-manifolds whose fundamental groups are infinite, and are also non-trivial direct products. The author has conjectured that such a manifold is the topological product of a circle and a closed 2-manifold. Here, we prove only that one of the direct factors of the fundamental group is cyclic infinite. Considerable progress has been made towards proving the conjecture by J. Stallings ([5] and [14]).

§18. Ends.

We recall some facts about ends. If K is a locally finite complex, let $C^*(K;Z)$ and $C^*_f(K;Z)$ be the groups of ordinary and finite cochains with coefficients in Z. We have the exact sequence

$$0 \longrightarrow C^*_f(K;Z) \longrightarrow C^*(K;Z) \longrightarrow C^*_e(K;Z) \longrightarrow 0,$$

which defines the term on the right. We get an exact sequence of homology groups of these chain complexes

$$(18.1) \qquad 0 \longrightarrow H^0_f(K;Z) \longrightarrow H^0(K;Z) \longrightarrow H^0_e(K;Z)$$
$$\longrightarrow H^1_f(K;Z) \longrightarrow H^1(K;Z) \ldots\ldots$$

From results of Specker ([12], Satz III), we easily see that $H^0_e(K;Z)$ is free abelian. Its rank is equal to the number of ends of K ([12], Satz IV).

Let G be a finitely generated group, and let L be a finite, connected simplicial complex. Let K be a regular covering space of L and let G be isomorphic to the group of covering translations. Then the number of ends of K is independent of the particular complexes K and L and depends only on G. The number of ends of G is defined to be the number of ends of K [7].

Lemma (18.2). A necessary and sufficient condition for G to have two ends is that it should have an infinite cyclic subgroup of finite index.

(18.2) is proved in [7].

§19. Direct products. The orientable case.

In this section, we assume M is a closed, orientable 3-manifold, such that $\pi_1(M) = C\times D$, where C is infinite and $D \neq 0$.

Lemma (19.1). M is aspherical.

If $\pi_2(M) \neq 0$, then, by the Sphere Theorem (see [15]), there exists a non-contractible 2-sphere in M. By (11.1) and (11.2), $\pi_1(M) \cong A * B$ or Z, where $A \neq 0$ and $B \neq 0$. Both of these possibilities are excluded since $\pi_1(M)$ is a direct product. (By [8], page 28, a direct product cannot be a free product). Therefore $\pi_2(M) = 0$. By (13.1), M is aspherical.

We have the covering maps $p_C: M_C \longrightarrow M$ and $p_D: M_D \longrightarrow M$ associated with the subgroups C and D respectively. Since M is aspherical, C×D is torsion free by (13.4). Therefore C and D are torsion free. Therefore D is infinite. Therefore M_C and M_D are non-compact 3-manifolds and so $H_3(M_C;Z) = 0$ and $H_3(M_D;Z) = 0$.

Lemma (19.2). If G = C×D is finitely generated, then so are C and D.

This follows immediately by projecting the generators of G into C and D respectively.

Lemma (19.3). $H_1(M_C;Z) \cong Z$ and $H_2(M_D;Z) \cong Z$. (C and D are both infinite, and are therefore in a symmetrical position in the hypotheses of (19.3). A conclusion in which C and D are interchanged in (19.3) is also possible, but will be suppressed for convenience).

$M \simeq M_C \times M_D$ since all three are aspherical. By the Künneth relations

$$Z \cong H_3(M;Z) \cong H_2(M_C;Z) \otimes H_1(M_D;Z) + H_1(M_C;Z) \otimes H_2(M_D;Z) + \mathrm{Tor}(H_1(M_C;Z), H_1(M_D;Z)).$$

$H_1(M_C;Z) \cong C/[C,C]$ is a finitely generated abelian group by (19.2). Similarly for D. Therefore the third summand is finite, and, being a subgroup of Z, vanishes. Without loss of generality, we assume $H_1(M_C;Z) \otimes H_2(M_D;Z) \cong Z$. Writing $H_1(M_C;Z) \cong Z+\ldots+Z+F$, where F is a finite group, we have $H_2(M_D;Z)+\ldots+H_2(M_D;Z)+H \cong Z$, where H is a torsion group. Therefore H = 0, since it is a subgroup of Z.

Therefore $H_2(M_D;Z) \cong Z$. Therefore

$$Z \cong H_1(M_C;Z) \otimes H_2(M_D;Z) \cong H_1(M_C;Z).$$

Lemma (19.4). C has two ends.

Since M_D is a regular covering space of M with C the group of covering translations, the number of ends of M_D is equal to the number of ends of C (see §18). Since M_D is non-compact, $H_f^0(M_D;Z) = 0$. Therefore (18.1) becomes

$$0 \longrightarrow Z \longrightarrow H_e^0(M_D;Z) \longrightarrow H_f^1(M_D;Z)$$

By [12], last line of Satz V,

$$H_f^1(M_D;Z)/\mathrm{Im}(H_e^0(M_D;Z) \longrightarrow H_f^1(M_D;Z))$$

is either free abelian of infinite rank or zero. But $Z \cong H_2(M_D;Z) \cong H_f^1(M_D;Z)$ by (19.3) and Poincaré Duality. Therefore $H_e^0(M_D;Z) \longrightarrow H_f^1(M_D;Z)$ is an epimorphism. From the exact sequence, we deduce $H_e^0(M_D;Z) \cong Z+Z$. Therefore M_D has two ends.

Therefore, by (18.2), C has an infinite cyclic subgroup C' of finite index. We have an epimorphism $C \longrightarrow C/[C,C] \cong Z$ by (19.3). Finally C is torsion free by (19.1) and (13.4).

Lemma (19.5). $C \cong Z$.*

The composition

$$Z \cong C' \longrightarrow C \longrightarrow C/[C,C] \cong Z$$

is either a monomorphism or zero. If it were zero, then, since C' has finite index in C, the image of $C \longrightarrow C/[C,C]$ would be finite. But this map is an epimorphism. Therefore the composition is a monomorphism. Since C' has finite index in C, the kernel of $C \longrightarrow C/[C,C] \cong Z$ is finite. But C is torsion free, and so the kernel is zero. Therefore $C \longrightarrow C/[C,C] \cong Z$ is an isomorphism.

§20. Direct products. The non-orientable case.

In this section, we assume M is a non-orientable, closed 3-manifold, and that $\pi_1(M) = C \times D$, where C is infinite and $D \neq 0$. We shall prove that one of the groups, say C, is cyclic infinite.

Let $p: \tilde{M} \longrightarrow M$ be the orientable double covering of M. Then $p_*(\pi_1(\tilde{M}))$ has index two in $\pi_1(M)$. Let $C \cap p_*(\pi_1(\tilde{M})) = C'$ and let $D \cap p_*(\pi_1(\tilde{M})) = D'$. C' has index at most two in C, and D' has index at most two in D. Let $\bar{M}$ be the covering of M associated with the subgroup $C' \times D'$ of $C \times D$. $\bar{M}$ is a closed, orientable 3-manifold, such that $\pi_1(\bar{M}) \cong C' \times D'$, where C' is infinite.

---

*The author wishes to thank Dr. J. Stallings for providing this proof. The author's original proof was much longer.

<u>Lemma (20.1). The following three statements are equivalent:</u>
<u>$D' \neq 0$; M is aspherical; $D \not\cong Z_2$.</u>

If $D' \neq 0$, we may apply (19.1) to $\overline{M}$, to show that $\overline{M}$ is aspherical. Since $\overline{M}$ covers M, M is also aspherical. If M is aspherical, $C \times D$ is torsion free by (13.4), and so $D \not\cong Z_2$. If $D \not\cong Z_2$, then $D' \neq 0$.

<u>Lemma (20.2). If M is aspherical, then (without loss of generality) $C \cong Z$.</u>

$\pi_1(\overline{M}) \cong C' \times D'$, where C' is infinite and $D' \neq 0$. By (19.5), $C' \cong Z$. Either $C = C'$ or C' has index two in C. Moreover, since M is aspherical, C is torsion free. Therefore, if $C' \neq C$, C has a presentation

$$\{\alpha, \beta / \beta^{-1}\alpha\beta = \alpha^{\varepsilon},\ \beta^2 = \alpha^n\},$$

where α is a generator of C', $\beta \notin C'$ and $\varepsilon = \pm 1$. Then

$$\beta^2 = \beta^{-1}\alpha^n\beta = \alpha^{n\varepsilon} = \beta^{2\varepsilon}.$$

Therefore $\varepsilon = 1$, since C is torsion free. Therefore C is abelian. If $n = 2m$, for some integer m, then $(\beta^{-1}\alpha^m)^2 = 1$, and so $\beta = \alpha^m$, since C is torsion free. This is impossible, since $\beta \notin C'$. Therefore $n = 2m+1$, and C is generated by $(\beta^{-1}\alpha^m)$. Therefore $C \cong Z$.

<u>Lemma (20.3). If M is not aspherical, then either $C \cong Z$ or $C \cong A * B$.</u>

By (20.1), $D' = 0$, for otherwise M is aspherical. Therefore $D \cong Z_2$, and the non-zero element of D is orientation reversing. By

(13.2), there is a two-sided projective plane P in M. Let S be the inverse image of P in $\widetilde{M}$. S is essential in $\widetilde{M}$ by (11.3). Therefore by (11.1) and (11.2), $\pi_1(\widetilde{M}) \cong Z$ or $A * B$, where $A \neq 0$ and $B \neq 0$. Since the non-zero element of D commutes with all elements of $\pi_1(M)$, and $p_*(\pi_1(\widetilde{M})) \cap D = D' = 0$, we have

$$p_*(\pi_1(\widetilde{M})) \times D = \pi_1(M) = C \times D.$$

Therefore

$$C \cong p_*(\pi_1(\widetilde{M})) \cong \pi_1(\widetilde{M}) \cong Z \text{ or } A * B.$$

<u>Lemma (20.4).</u> <u>If M is not aspherical, then $\pi_1(M) \cong Z \times Z_2$.</u>*

If $C \cong A * B$, there is an element of infinite order in C. Therefore $\pi_1(M)$ contains a subgroup isomorphic to $Z \times Z_2$. By (14.5), $\pi_1(M) \cong (Z \times Z_2) * H$, for some group H. Since $C \times D$ is not a free product, $H = 0$. Therefore $\pi_1(M) \cong Z \times Z_2$.

§21. Summary of results in Chapter VI.

<u>Theorem (21.1).</u> <u>If M is a closed 3-manifold, and $\pi_1(M) = C \times D$, where C is infinite and $D \neq 0$, then one of the following three possibilities occurs:</u>

---

*(20.4) was first proved by J. Stallings using the theory of ends, before the Projective Plane Theorem (6.1) was discovered. That proof was very much longer than the proof given above.

i) M is orientable and aspherical and, say $C \cong Z$;

ii) M is non-orientable and aspherical and, say, $C \cong Z$;

iii) M is non-orientable, $\pi_2(M) \neq 0$, and $\pi_1(M) \cong Z \times Z_2$.

i) follows from (19.1) and (19.5). ii) follows from (20.2).
iii) follows from (20.4) and (11.3).

BIBLIOGRAPHY.

[1] J.J.Andrews and M.L.Curtis, "Knotted 2-spheres in the 4-sphere", Annals of Maths., 70 (1959) pp. 565-571.

[2] E.Artin, Abh. Math. Sem. Hamburg Univ., IV (1925) pp. 174-177.

[3] R.H.Bing, "Locally tame sets are tame", Annals of Maths., 59 (1954) pp. 145-158.

[4] D.B.A.Epstein, "Linking spheres", to appear in Proc. Camb. Phil. Soc.

[5] D.B.A.Epstein and J.Stallings, "3-manifolds which are topological products", to appear.

[6] H.Hopf, "Fundamentalgruppe und zweite Bettische Gruppe", Comment. Math. Helv., 14 (1941-42) pp. 257-309.

[7] H.Hopf, "Enden offener Raume und unendliche diskontinuierliche Gruppen", Comment. Math. Helv., 16 (1943-44) pp. 81-100.

[8] A.G.Kurosh, "The Theory of Groups", Chelsea, New York, (1956), Vol. II.

[9] C.D.Papakyriakopoulos, "On Dehn's Lemma and the asphericity of knots", Annals of Maths., 66 (1957) pp. 1-26.

[10] H.Seifert und W.Threllfall, "Lehrbuch der Topologie", Leipzig, (1934).

[11] A.Shapiro and J.H.C.Whitehead, "A proof and extension of Dehn's Lemma", Bull. Am. Math. Soc., 64 (1958) pp. 174-178.

[12] E.Specker, "Die erste Cohomologiegruppe von uberlagerungen und Homotopie-Eigenschaften dreidimensionaler Mannigfaltigkeiten", Cmment. Math. Helv., 23 (1949) pp.303-333.

[13] J.Stallings, "On the Loop Theorem", to appear.

[14] J.Stallings, "An application of the Shapiro-Whitehead Theorem", to appear.

[15] J.H.C.Whitehead, "On 2-spheres in 3-manifolds", Bull. Am. Math. Soc., 64 (1958) pp.161-166.

[16] J.H.C.Whitehead, "On finite cocycles and the Sphere Theorem", Colloquium Mathematicum, VI (1958) pp.271-281.