Topics in symplectic Gromov–Witten theory

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Declaration

I declare that this thesis is the result of my own work, and includes no work done in collaboration, except for Chapter 2, which was done and written jointly with Mohan Swaminathan, Chapter 6, which was done and written jointly with Soham Chanda and Luya Wang, and Chapter 7, which was done and written jointly with Noah Porcelli. A large part of the contents of Chapter 7 have been submitted as part of Noah Porcelli's doctoral dissertation to the University of Cambridge.

It is not substantially the same as any work which has already been submitted before for any degree or other qualification.

Amanda Hirschi
August 2023

To my parents

Abstract

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Amanda Hirschi

The main focus of this thesis is on the Gromov–Witten theory of general symplectic manifolds. Mohan Swaminathan and I construct a framework to define a virtual fundamental class for the moduli space of stable maps to a general closed symplectic manifold. Our construction, inspired by [AMS21], works for all genera and leads to a more straightfoward definition of symplectic Gromov–Witten invariants as was previously available. We prove a formula for the Gromov–Witten invariants of a product of two symplectic manifolds, conjectured in [KM94].

I generalise the product formula to a formula for the Gromov–Witten invariants of a suitable fibre product of symplectic manifolds. Our invariants satisfy the Kontsevich-Manin axioms and are extended to descendent Gromov–Witten invariants. I show that our definition of Gromov–Witten invariants agrees with the classical Gromov–Witten invariants defined by [RT97] for semipositive symplectic manifolds.

Given a Hamiltonian group action on the target manifold, I construct equivariant Gromov–Witten invariants and prove a virtual Atiyah–Bott-type localisation formula, providing a tool for computations.

Together with Soham Chanda and Luya Wang, I construct infinitely many exotic Lagrangian tori in complex projective spaces of complex dimension higher than 2. We lift tori in $\mathbb{P}^2$, constructed by Vianna, and show that these lifts remain non-symplectomorphic, using an invariant derived from pseudoholomorphic disks.

Noah Porcelli and I use Ljusternik-Schnirelmann theory, applied to moduli spaces of pseudoholomorphic curves, and homotopy theory to prove lower bounds on the number of intersection points of two (possibly non-transverse) Lagrangians in terms of the cuplength of the Lagrangian in generalised cohomology theories, improving previous lower bounds by Hofer.

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Chapter 1

Introduction

... a moduli space is a treasure map that is itself a treasure.
anonymous

1.1 GW theory of general symplectic manifolds

Let (X, ω) be a closed symplectic manifold and J be an element of the space $\mathcal{J}_\tau(X, \omega)$ of ω-tame almost complex structures. The fundamental object of study in GW theory, as introduced in [KM94], is the moduli space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ of n-pointed stable maps of genus g representing a class $A \in H_2(M; \mathbb{Z})$. When (X, ω) is semipositive, [RT95] gave a construction of GW invariants using pseudocycles. Due to issues with transversality and smoothness of gluing, constructing the invariants in general requires the use of virtual techniques. The last decades have seen the development of several virtual frameworks in symplectic geometry, see [LT98, FO99, CM07, HWZ17, MW17, Par16, IP19b] for an inexhaustive list. Most of these constructions begin by representing the moduli space locally using Kuranishi charts and then employ delicate local-to-global arguments to extract from this a global invariant called the *virtual fundamental class*. In this thesis we construct a virtual framework based on a global Kuranishi chart introduced in [AMS21]. This allows for a straightforward definition of symplectic GW invariants as it eliminates the need to patch together local information.

Chapter 2 is devoted to the construction of such a global Kuranishi chart, while Chapters 3 to 5 deal with (foundational) applications thereof in symplectic GW theory.

1.1.1 Global Kuranishi charts for GW theory

In [AMS21] the authors achieved a breakthrough by constructing a global Kuranishi chart for the moduli space $\overline{\mathcal{M}}_{0,n}(X, A; J)$, building on ideas from [Sie98]. A *global Kuranishi chart* for a compact Hausdorff space Z consists of a finite-rank vector bundle $\mathcal{E}$ over a manifold $\mathcal{T}$ together with a section $\mathfrak{s}\colon \mathcal{T} \to \mathcal{E}$, an almost free action by a compact Lie group G on $\mathcal{E}$ and $\mathcal{T}$ such that $\mathfrak{s}$ is equivariant, and a homeomorphism $\mathfrak{s}^{-1}(0)/G \cong Z$. It carries a canonical invariant $[Z]^{\mathrm{vir}} \in \check{H}^*(Z;\mathbb{Q})^\vee$ given by the composite

$$\check{H}^{\mathrm{vdim}+*}(Z;\mathbb{Q}) \xrightarrow{\mathfrak{s}^*\tau(\mathcal{E}/G)} H_c^{\dim(\mathcal{T}/G)+*}(\mathcal{T}/G;\mathbb{Q}) \xrightarrow{\int_{\mathcal{T}/G}} \mathbb{Q}[0], \tag{1.1.1.1}$$

called the *virtual fundamental class* of Z. Here $\mathrm{vdim}(Z) := \dim(\mathcal{T}/G) - \mathrm{rank}(\mathcal{E})$ and $\tau(\mathcal{E}/G)$ is the Thom class of the orbibundle. In Chapter 2, we present the construction of a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A;J)$ for arbitrary genus g, generalising the construction of Abouzaid, McLean and Smith. Recently, they published an independent construction in higher genus, [AMS23].

Theorem 1.1.1 ([HS22])**.** *For integers $g,n \geqslant 0$ and $A \in H_2(X,\mathbb{Z})$, the moduli space of stable maps $\overline{\mathcal{M}}_{g,n}(X,A;J)$ admits a global Kuranishi chart of the expected dimension, depending on certain auxiliary data, but unique in the following sense.*

(i) *Different choices of auxiliary data result in global Kuranishi charts that are related by certain equivalence moves, which do not affect the virtual fundamental class.*

(ii) *Given any other $J' \in \mathcal{J}_\tau(X,\omega)$, there exist auxiliary data such that the associated global Kuranishi charts are cobordant.*

In particular, $\overline{\mathcal{M}}_{g,n}(X,A;J)$ admits a virtual fundamental class $[\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}$. The *Gromov–Witten homomorphism*

$$I^{X,\omega}_{g,n,A} \colon H^*(X^n;\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$$

is defined by

$$I^{X,\omega}_{g,n,A}(\alpha) = \mathrm{PD}(\mathrm{st}_*(\mathrm{ev}^*\alpha \cap [\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}})). \tag{1.1.1.2}$$

where $\mathrm{ev}\colon \overline{\mathcal{M}}_{g,n}(X,A;J) \to X^n$ and $\mathrm{st}_*\colon \overline{\mathcal{M}}_{g,n}(X,A;J) \to \overline{\mathcal{M}}_{g,n}$ are the evaluation and stabilisation map respectively. The *Gromov–Witten invariants* of X are the images of $I^{X,\omega}_{g,n,A}$ evaluated at classes on $\overline{\mathcal{M}}_{g,n}$.

Remark 1.1.2. The thickening $\mathcal{T}$ we construct is not smooth but it admits a topological submersion π to a smooth manifold $\mathcal{M}$ (see Definition 2.1.4) and naturally carries the structure of a *rel–C^∞ manifold* over $\mathcal{M}$. In particular, we can use smoothing theory as in [AMS21, §4.2] to upgrade our construction to a smooth global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A,J)$, allowing for the definition of a Morava K-theory valued virtual fundamental class as in [AMS21, AMS23]. Also, as explained in [BX22], our global chart can be used as an input for the construction of $\mathbb{Z}$-valued GW type invariants in all genera.

As an application of the construction, we prove a product formula for the GW invariants of a product symplectic manifold.

Theorem 1.1.3 ([HS22])**.** *Suppose $(X,\omega) = (X_0,\omega_0) \times (X_1,\omega_1)$ and $A_i \in H_2(X_i;\mathbb{Z})$. Then*

$$\sum_{\mathrm{pr}_{i*}A = A_i} I^{X,\omega}_{g,n,A}(\alpha \times \beta) = I^{X_0,\omega_0}_{g,n,A_0}(\alpha) \cdot I^{X_1,\omega_1}_{g,n,A_1}(\beta)$$

for any $\alpha \in H^*(X_0^n;\mathbb{Q})$ and $\beta \in H^*(X_1^n;\mathbb{Q})$ and $(g,n) \notin \{(1,1),(2,0)\}$.[1]

[1]Using Theorem 1.1.5 we can extend the formula to the remaining two cases.

This result was conjectured in [KM94] and shown in [KM96, Beh99] and [RT95, Hyv12] for projective varieties and semipositive symplectic manifolds respectively. As a consequence of the product formula, we deduce a Künneth formula for quantum cohomology.

Corollary 1.1.4 ([HS22])**.** *Suppose (X_i, ω_i) for $i = 0, 1$ are closed symplectic manifolds and set $(X, \omega) = (X_0, \omega_0) \times (X_1, \omega_1)$. Then the Künneth map induces an isomorphism*

$$\mathrm{QH}(X_0) \otimes_\Lambda \mathrm{QH}(X_1) \to \mathrm{QH}(X)$$

of Λ-algebras, where Λ is the universal Novikov ring.

1.1.2 A fibre-product formula

A natural generalisation of the product formula is a formula describing the GW invariants of a suitable fibre product. This requires the definition of a fibre product of global Kuranishi charts over a third global Kuranishi chart, see also [Joy12]. We show how the virtual fundamental classes relate in §3.1. On the one hand this leads to a general fibre-product formula in Theorem 3.1.15; on the other hand, this section will be used in §4.1.

In §3.2 we show how to adapt the global Kuranishi chart construction of §2 to be able to apply Theorem 3.1.15. We obtain

Theorem 1.1.5. *Let (B, ω_B) be a closed symplectic manifold and $\pi_i \colon (X_i, \omega_i) \to (B, \omega_B)$ a Hamiltonian fibre bundle for $i \in \{0, 1\}$. Then for any $A_B \in H_2(B; \mathbb{Z})$ and $J_B \in \mathcal{J}_\tau(B, \omega_B)$ such that $\overline{\mathcal{M}}_{g,n}(B, A_B; J_B)$ is smooth with obstruction bundle* Ob*, we have*

$$\begin{aligned}\sum_{j_* A = A_0 + A_1} & j_*(\pi^* e(\mathrm{Ob}) \cap [\overline{\mathcal{M}}_{g,n}(X_0 \times_B X_1, A; J)]^{\mathrm{vir}}) \\ &= (\pi_0 \times \pi_1)^* \mathrm{PD}(\Delta_{\overline{\mathcal{M}}_{g,n}(B, A_B; J_B)}) \cap [\overline{\mathcal{M}}_{g,n}(X_0, A_0; J_0) \times \overline{\mathcal{M}}_{g,n}(X_1, A_1; J_1)]^{\mathrm{vir}}.\end{aligned}$$

where $\pi_{i*} A_i = A_B$ and $j \colon X_0 \times_B X_1 \hookrightarrow X_0 \times X_1$ is the inclusion and J is the restriction of $J_0 \times J_1$ to $T(X_0 \times_B X_1)$ for suitable almost complex structures.

Refer Theorem 3.2.1 for the statement in full generality, which requires more notation than we want to introduce here. Furthermore, we show a similar relation in the case where $\pi_0 \colon X_0 \to B$ is a symplectic embedding, generalising the Quantum Lefschetz Hyperplane Theorem of [KKP03]; see also [Man12, Zin11].

Remark 1.1.6. As the last step of the proof of Theorem 1.1.5 relies on the fact that a certain map has degree 1, one should expect that a corresponding formula for GW invariants, valued in generalised cohomology theories, contains correction terms similar to the axioms shown in [AMS23].

Remark 1.1.7. Together with existing computations of the quantum cohomology ring of projective bundles over $\mathbb{P}^k$, [QR98, AM00], Theorem 1.1.5 should allow for the determination of the (small) quantum cohomology of fibre products of projective bundles over $\mathbb{P}^k$.

Remark 1.1.8. A partial motivation for this formula is the question of Pandharipande, [FP05], whether all projective smooth varieties have the property that the GW homomorphism $I^{X,\omega}_{g,n,A}\colon H^*(X^n;\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ take values in $RH^*(\overline{\mathcal{M}}_{g,n})$. The *tautological rings* $\{RH^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ are defined to be the smallest system of $\mathbb{Q}$-subalgebras of $\{H^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ that is closed under exceptional pushforward by the forgetful and gluing maps. These rings are much better understood than the entire cohomology rings and have been intensively studied,[GV01, JP19, PP21]. Except in genus 0, they need not capture the whole cohomology by [GP03]. By the virtual localisation formula [GP99], all homogeneous projective varieties have this property. Janda proved it for surfaces in [Jan17]. By the product formula, if X_0 and X_1 have this property, then so does $X_0 \times X_1$. The fibre-product formula relies on more information than just the fundamental class but might be useful in cases where we have a good knowledge of the thickening.

1.1.3 Kontsevich-Manin axioms and gravitational descendants

In [KM94], Kontsevich and Manin introduce the notion of a stable map and compile a list of properties that invariants based on the moduli space of stable maps are supposed to satisfy. Thereby, they essentially define GW invariants, before such constructions were available for either smooth projective varieties or general symplectic manifolds. The expected properties reflect the rich geometry of the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves and are a formalisation of the seminal paper [Wit91].

Remark 1.1.9. In [AMS23], the authors prove a version of the axioms adapted to GW invariants valued in complex-oriented generalised cohomology theories. Pending a comparison of their construction with [HS22], this is a generalisation of the results below. We hope our more elementary proofs might be of interest nonetheless.

The axioms are listed below; their proof can be found in §4.1.

(Effective) If $\langle[\omega], A\rangle < 0$, then $I^{X,\omega}_{g,n,A} = 0$.

(Homology) $I^{X,\omega}_{g,n,A}$ is induced by a homology class.

(Grading) $I^{X,\omega}_{g,n,A}$ has degree $2((\dim_{\mathbb{C}} X - 3)(1-g) + \langle c_1(T_X), A\rangle + n)$.

(Symmetry) $I^{X,\omega}_{g,n,A}$ is equivariant with respect to the canonical S_n-actions given by permuting the factors, respectively the marked points.

(Mapping to a point) If $\mathbb{E}$ denotes the Hodge bundle, then

$$I^{X,\omega}_{g,n,0}(\alpha_1 \times \cdots \times \alpha_n) = \langle \alpha_1 \cdot \cdots \cdot \alpha_n, c_{\mathrm{top}}(T_X) \cap [X]\rangle\, c_g(\mathbb{E}^*)$$

for all $\alpha_i \in H^*(X;\mathbb{Q})$.

(Fundamental class) If 1_X denotes the unit of $H^*(X;\mathbb{Q})$ and π_n the map forgetting the n^{th} marked point, then

$$I^{X,\omega}_{g,n,A}(\alpha_1 \times \cdots \times \alpha_{n-1} \times 1_X) = \pi_n^* I^{X,\omega}_{g,n-1,A}(\alpha_1 \times \cdots \times \alpha_{n-1}).$$

for any $\alpha_i \in H^*(X;\mathbb{Q})$.

(Divisor) If $|\alpha_n| = 2$, then for any $\alpha_1, \ldots, \alpha_{n-1} \in H^*(X;\mathbb{Q})$

$$\pi_{n!}\, I^{X,\omega}_{g,n,A}(\alpha_1 \times \cdots \times \alpha_n) = \langle \alpha_n, A\rangle\, I^{X,\omega}_{g,n-1,A}(\alpha_1 \times \cdots \times \alpha_{n-1}).$$

(Splitting) Write $\mathrm{PD}(\Delta_X) = \sum_{k\in K} \gamma_k \times \gamma_k'$ with $\gamma_k, \gamma_k' \in H^*(X;\mathbb{Q})$. Given $S \subset \{1, \ldots, n\}$, let $\varphi_S \colon \overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1} \to \overline{\mathcal{M}}_{g,n}$ be the associated clutching map. Then

$$\varphi_S^* I^{X,\omega}_{g,n,A}(\alpha_1\times\cdots\times\alpha_n) = (-1)^{\epsilon(\alpha;S)} \sum_{\substack{A_0+A_1=A\\ k\in K}} I^{X,\omega}_{g_0,n_0+1,A_0}\big(\bigtimes_{i\in S} \alpha_i \times \gamma_k\big)\, I^{X,\omega}_{g_1,n_1+1,A_1}\big(\gamma_k' \times \bigtimes_{j\notin S} \alpha_j\big)$$

where $\epsilon(\alpha; S) = |\{i > j \mid i \in S,\, j \notin S,\, |\alpha_i|, |\alpha_j| \in 1 + 2\mathbb{Z}\}|$ and $\alpha_i \in H^*(X;\mathbb{Q})$

(Genus reduction) If $\psi \colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$ denotes the map which creates a non-separating node by gluing the last two marked points, then

$$\psi^* I^{X,\omega}_{g+1,n-2,A}(\alpha) = I^{X,\omega}_{g,n,A}(\alpha \times \mathrm{PD}(\Delta_X))$$

for any $\alpha \in H^*(X^{n-2};\mathbb{Q})$.

Remark 1.1.10 (Quantum cohomology). The Splitting axiom together with (Mapping to a point) shows that we can use the genus-0 3-pointed GW invariants to deform the cup product on $H^*(X;\mathbb{Q})$. Due to convergence issues, this deformation is generally defined only on $QH^*(X,\omega) := H^*(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$, where we use the Novikov coefficients[2] associated to the universal Novikov ring

$$\Lambda = \left\{ \sum_{i\in\mathbb{N}} a_i t^{\lambda_i} \mid a_i \in \mathbb{Q},\, \lambda_i \in \mathbb{R},\, \forall c \in \mathbb{R} : \big|\{i \in \mathbb{N} : \lambda_i \leqslant c,\, a_i \neq 0\}\big| < \infty \right\}$$

and $\phi \colon H_2(X;\mathbb{Z}) \to \Lambda$ given by $\phi(A) = t^{\omega(A)}$.

The GW invariants defined above capture only a small part of the cohomology of $\overline{\mathcal{M}}_{g,n}(X, A; J)$. One possibility to obtain more information is to also consider the integrals of natural cohomology classes on the moduli space itself. ψ-classes, first defined in [RT97, KM96] for semipositive manifolds, respectively smooth projective varieties, provide one such collection of cohomology classes. Integrating them corresponds to imposing tangency conditions at the marked points. λ-classes are the Chern class of the Hodge bundle $\mathbb{E}$ and appear naturally as in [GP99]. In §4.2, we define ψ- and λ-classes for general symplectic manifolds. The resulting invariants are called *gravitational descendants*. We prove the analogue of the Fundamental class and the Divisor axiom for gravitational descendants in Propositions 4.2.8 to 4.2.10.

[2]There are several possible Novikov rings and the choice of coefficients influences such properties as grading semisimplicity, see [MS12, Chapter 11], [BM04]. As we will not discuss Frobenius structures in detail, we will simply work with the universal Novikov ring defined here.

1.1.4 Comparison

While a symplectic geometer has a large choice of virtual frameworks by now, it is not inherently clear (although expected) whether the resulting GW invariants agree. In particular, it is desirable to know whether they give the same counts as the invariants defined in [RT97] for the class of semipositive symplectic manifolds since we have relatively many computations of these invariants and their definition requires the least machinery. We prove such a comparison in §5.1.

Theorem 1.1.11. *The GW invariants defined by* (1.1.1.2) *agree with the GW invariants defined via pseudocycles by [RT97] if the latter are defined.*

This has the following consequence, which is not apparent from the global Kuranishi chart construction itself.

Corollary 1.1.12. *If* (X, ω) *is semipositive, then the GW invariants of* X *in genus* 0 *are* $\mathbb{Z}$*-valued.*

It would be interesting to determine whether the invariants defined here agree with the algebraic GW invariants, defined in [BM96, LT98], when X is a smooth projective variety.

1.1.5 Equivariant GW invariants

Equivariant GW invariants were first defined by Kim, [Kim96], for flag manifolds and in [Giv96] for convex symplectic manifolds. They use the localisation formula of [AB84], respectively a generalisation by [GP99], to prove mirror symmetry for toric complete intersections, [Giv96], respectively to determine the quantum cohomology of flag manifolds, [Kim99]. Kontsevich, [Kon95], first proposed to apply the localisation formula in GW theory.

While symplectic toric manifolds are projective, [Del88], there are symplectic manifolds with Hamiltonian torus actions, which are not even Kähler, [Ler96, Tol98, Woo98]. Therefore, an equivariant GW theory in the symplectic setting is needed in order to treat these cases. It could also allow for an extension of the quantum Kirwan map, [GW22], or the equivariant Seidel morphism, [LJ21], to general symplectic manifolds.

In §5.2.3 we define equivariant GW invariants for symplectic manifolds with a Hamiltonian group action by extending (1.1.1.1) to an equivariant virtual fundamental class, see Definition 5.2.2. Equivariant Kuranishi charts are constructed in [Fuk21], but the associated virtual fundamental class is not defined.

Theorem 1.1.13. *Given a Hamiltonian group action* μ *on* (X, ω) *by a compact connected Lie group* K*, there exist equivariant GW homomorphisms*

$$I^{X,\omega,\mu}_{g,n,A} \colon H^*_K(X^n; \mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \otimes_{\mathbb{Q}} H^*_K(\mathrm{pt}; \mathbb{Q})$$

which satisfy properties analogous to the Kontsevich Manin axioms. Moreover, we can recover the ordinary GW invariants from the equivariant ones.

In particular, this allows for the definition of the *equivariant quantum cohomology* $QH^*_K(X,\omega)$ of (X,ω,μ), which is a module over $H^*_K(\mathrm{pt};\mathbb{Q})$. The inclusion $X \hookrightarrow X_K$ induces a surjection $QH^*_K(X,\omega) \to QH^*(X,\omega)$ of rings by Remark 5.2.4.

We prove a virtual localisation formula, analogous to the one of [GP99] in the setting of global Kuranishi charts, see Theorem 5.2.10. A similar formula in the symplectic setting appeared in the preprint [CL06]. Applied to the equivariant GW theory, it shows that any contribution to these invariants comes from the fixed point locus of the moduli space of stable maps, generalising [MT06, Proposition 4.10].

Remark 1.1.14. In [Giv01b], Givental used computations of equivariant GW invariants to express the higher genus GW invariants in terms of invariants in genus 0. His computations rely on the localisation formula of [GP99] and thus require the symplectic manifold to be either a projective variety or convex. Theoerm 5.2.10 might be used to generalise his results to all closed symplectic manifolds with a Hamiltonian torus action whosed fixed points are isolated.

Remark 1.1.15. The localisation formula of Theorem 5.2.10 is phrased purely in terms of global Kuranishi charts with a suitable group action. Thus it can be applied to any other setting for which an equivariant global Kuranishi chart has been constructed.

1.2 Exotic tori in projective spaces

While the previous chapters put their focus squarely on pseudoholomorphic curves and their moduli spaces, the next two chapters show how these curves can be used to investigate symplectic manifolds, in particular, their Lagrangian submanifolds.

From now on, we will avoid the use of virtual techniques and instead require our manifolds to be reasonably nice so that the moduli spaces of pseudoholomorphic curves are sufficiently well-behaved.

In Chapter 6, the result of a collaboration with Soham Chanda and Luya Wang, we study monotone Lagrangian tori in projective spaces. Recall that a Lagrangian $L \subset M$ is *monotone* if the area homomorphism $I_\omega\colon \pi_2(M,L) \to \mathbb{R}$ is positive scalar multiple of the Maslov homomorphism $\mu\colon \pi_2(M,L) \to \mathbb{Z}$ defined in [Arn67]. We call a Lagrangian torus in $\mathbb{P}^n$ (or $\mathbb{C}^n$) *exotic* if it is monotone and not symplectomorphic to the standard Clifford (or product) torus. The earliest example of an exotic torus dates back to [Che96].

Recently, the use of almost toric fibrations has become an important tool in constructing new examples of Lagrangian tori. For example, Vianna has constructed infinitely many exotic tori in $\mathbb{P}^2$ [Via16] and in del Pezzo surfaces [Via17]. For more details on almost toric fibrations, see [Sym01, LS10, Eva22]. For previous constructions of non-Hamiltonian isotopic Lagrangian tori in higher dimensions, see for example [Aur15, PT20, Yua22] and [Bre23].

Our exotic examples are lifts $\overline{T}_{(a,b,c)}$ of the Vianna tori $T_{(a,b,c)}$ in $\mathbb{P}^2$. Here (a,b,c) is a triple of natural numbers satisfying the Markov equation $a^2+b^2+c^2 = 3abc$. These tori are defined and reviewed in §6.1.2. Any exotic torus $T_{(a,b,c)}$ can be obtained from the Clifford torus in $\mathbb{P}^2$ by a sequence of mutations. We show that the lifted Vianna tori can be obtained from the Clifford torus in $\mathbb{P}^n$ by a sequence of *solid mutations*, a generalisation of mutations to higher dimension. We study how the disk potentials change under a solid mutation, using as essential input a wall-crossing formula from [PT20]. Recall that the *disk potential* of a closed monotone oriented spin Lagrangian L^n in a closed symplectic manifold (M^{2n}, ω) is a function

$$W_L \colon \mathrm{Hom}(\pi_1(L), \mathbb{C}^*) \to \mathbb{C}$$

mapping a local system ρ on L to

$$W_L(\rho) = \sum_{\substack{\beta \in \pi_2(M,L) \\ \mu(\beta)=2}} |\mathcal{M}(L,\beta)| \, \rho(\partial\beta), \tag{1.2.0.1}$$

where $p \in L$ and $J \in \mathcal{J}(M, \omega)$ are generic, and $\mathcal{M}(L, \beta)$ is the (zero-dimensional) moduli space of J-holomorphic disks representing β and passing through p.[3] Choosing a basis of $H_1(L;\mathbb{Z})$ we can write the disk potential as a Laurent polynomial; refer to Remark 6.2.1 for more details.

Similar to [Via16], we do not compute the disk potential explicitly. Instead, we show that the associated Newton polytope, defined in §6.3.1, is uniquely determined by (a, b, c). As the Markov tree is infinite, we find

Theorem 1.2.1 ([CHW23]). *$\mathbb{P}^n$ admits infinitely many distinct exotic Lagrangian tori for any $n \geqslant 3$.*

1.3 Cuplengths and the degenerate Arnol'd conjecture

It is a classical problem in symplectic topology to find lower bounds on the number of intersection points of two Lagrangian submanifolds L and L' in a symplectic manifold (X, ω). In Chapter 7, the result of a collaboration with Noah Porcelli, we use generalised cohomology theories to find stronger lower bounds.

This problem has been intensively studied, under various assumptions. When the Lagrangians are assumed to intersect transversely, Floer homology provides lower bounds, see [Flo88, Oh93, FOOO09] for an incomplete list of references.

The classical Arnol'd conjecture concerns a special case of this question, where (X, ω) is the product symplectic manifold $(Y \times Y, \sigma \oplus -\sigma)$ for some compact symplectic manifold (Y, σ), L is the diagonal and L' is the graph of a Hamiltonian diffeomorphism of Y. This case has been the subject of much study, both with and without the additional assumption of transverse intersection. See [FO99, Rud99, Par16, AB21, Rez22, BX22] and the references therein.

[3] As L is spin, each $\mathcal{M}(L, \beta)$ carries a canonical orientation and $|\cdot|$ means that we count the points with signs.

We will not assume that L and L' are transverse, but require X to be either closed or a Liouville manifold, L and L' to be Hamiltonian isotopic, and L to be *relatively exact*, i.e., that $\omega \cdot \pi_2(X, L) = 0$.

Under these assumptions, Floer proved

Theorem 1.3.1 ([Flo88]). *If L and L' intersect transversely, there is a lower bound*

$$\# L \cap L' \geqslant \sum_i \operatorname{Rank}(H_i(L; \mathbb{Z}/2)).$$

Without the transversality assumption, a version of the Arnol'd conjecture states that

Conjecture 1.3.2. *If L is relatively exact and L' is Hamiltonian isotopic to L, then*

$$\# L \cap L' \geqslant \operatorname{Crit}(L)$$

where $\operatorname{Crit}(L)$ *is the minimal number of critical points of any smooth map $L \to \mathbb{R}$.*

A standard application of the Weinstein neighbourhood theorem implies that if true, this bound must be sharp.

Lusternik-Schnirelmann theory is a powerful tool for studying (numbers of) critical points or intersection points without any transversality assumptions, in contrast to Morse theory. It has been used in many fields other than symplectic geometry. For example, Klingenberg proved in [Kli78, Theorem 5.1.1] that any metric on S^2 admits at least three closed geodesics. Another application is to show that any (not necessarily Morse) function on a closed smooth manifold M has at least $c_{\mathbb{Z}}(M)$ critical points, where $c_{\mathbb{Z}}(M)$ is the cuplength of M in singular cohomology with integer coefficients. Lusternik-Schnirelmann theory has also been used in contact topology, e.g. by Ginzburg and Gürel in [GG20] to find lower bounds for numbers of Reeb orbits. For a more comprehensive discussion and further applications we refer to [CLOT03] or Chapter 11 in [MS17].

We use this technique to study Conjecture 1.3.2, generalising results of [Hof88]. Fix a ring spectrum R, representing a multiplicative generalised cohomology theory R^*. Instead of the rank of the cohomology groups we will use the cuplength as a lower bound for the number of intersection points. Given a compact convex domain $G \subset \mathbb{C}$ with smooth boundary, denote by $\mathcal{M}(G)$ the moduli space of (parametrised) pseudoholomorphic maps $(G, \partial G) \to (X, L)$.

Theorem 1.3.3 ([HP22]). *Suppose L is relatively exact and the index bundle over $\mathcal{M}(G)$ is R-orientable for any G. Then the number of intersection points between L and L' satisfies*

$$\# L \cap L' \geqslant c_R(L).$$

This shows that refining standard techniques via stable homotopy theory can result in stronger estimates.

Remark 1.3.4. Similar results have been obtained in the monotone setting by Lê-Ono [LO96] and Gong [Gon21a].

Remark 1.3.5. As we do not assume the transversality of L and L', to our knowledge there is no analogue of our strategy of proof using the setup in [CJS95, Coh09]. However, it may be possible to use their setup to prove Theorem 1.3.3 using the strategy of [Gon21b] instead.

Chapter 2

Construction of a global Kuranishi chart

2.1 Main construction

Convention 2.1.1. In this and the subsequent three chapters, we consistently use the symbol $\otimes$ to mean tensor product over $\mathbb{C}$ unless explicitly indicated otherwise.

Let (X, ω) be a closed symplectic manifold, $A \in H_2(X, \mathbb{Z})$ and $g, n \geqslant 0$ be integers. Given any $J \in \mathcal{J}_\tau(X, \omega)$, we will construct a rel–C^∞ global Kuranishi chart (in the sense of Definition 2.1.2 below) for the Gromov–Witten moduli space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ using the choice of an auxiliary datum (see Definition 2.1.11 for more details). The construction is independent of this choice in a sense made precise in Theorem 2.1.18 below.

Definition 2.1.2 (Rel–C^∞ global Kuranishi charts)**.** A *rel–C^∞ global Kuranishi chart* $\mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$ consists of

(i) a rel–C^∞ manifold $\mathcal{T} \to \mathcal{M}$, called the *thickening*, where the *base space* $\mathcal{M}$ is a smooth manifold;

(ii) a rel–C^∞ vector bundle $\mathcal{E}$ on $\mathcal{T}/\mathcal{M}$, the *obstruction bundle*, and a rel–C^∞ section $\mathfrak{s}$ of $\mathcal{E}$, the *obstruction section*;

(iii) a compact Lie group G, called the *symmetry group*, which acts on $\mathcal{T}/\mathcal{M}$ and $\mathcal{E}$ with finite stabilizers so that $\mathfrak{s}$ is G-equivariant. The action map

$$(G \times \mathcal{T})/(G \times \mathcal{M}) \to \mathcal{T}/\mathcal{M} \tag{2.1.0.1}$$

and the analogous map for $\mathcal{E}$ are both required to be rel–C^∞ maps.

If, in addition, we are given a Hausdorff space Z and a homeomorphism $\mathfrak{s}^{-1}(0)/G \xrightarrow{\sim} Z$, then we say $\mathcal{K}$ is a *rel–C^∞ global Kuranishi chart for Z*. The rel–C^∞ global Kuranishi chart $\mathcal{K}$ is *oriented* if we are provided with the data of orientations on $\mathcal{T}$ and $\mathcal{E}$ which are preserved by the G-action. We say that $\mathcal{K}$ is *stably complex* if we are given the data of a G-invariant almost complex structure on $\mathcal{M}$ and a G-invariant stably complex lift of the virtual vector bundle $T_{\mathcal{T}/\mathcal{M}} - (\mathcal{E} \oplus \underline{\mathfrak{g}})$. Here, $T_{\mathcal{T}/\mathcal{M}}$ denotes the vertical tangent bundle of $\mathcal{T}/\mathcal{M}$ and $\underline{\mathfrak{g}}$ is the trivial bundle with fibre $\mathfrak{g} = \mathrm{Lie}(G)$.

Remark 2.1.3. See Definition **??** for the explicit definition of a rel–C^∞ manifold and [Swa21] for a discussion of their properties. In §2.3.1 we will show that our thickening is canonically a rel–C^∞ manifold. This relative smoothness will be used throughout the later chapters as it allows for many differential-geometric operations.

We begin by introducing the smooth quasi-projective varieties which play the role of $\mathcal{M}$ in Definition 2.1.2 in our construction of global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X, A; J)$.

Definition 2.1.4 (Algebraic base space)**.** Given integers $N \geqslant 2$ and $m \geqslant 1$, we define the moduli space $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ to consist of all $C \subset \mathbb{P}^N$ with the following properties.

(i) $C \subset \mathbb{P}^N$ is an embedded algebraic prestable genus g curve of degree m.

(ii) The restriction $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \to H^0(C, \mathcal{O}_C(1))$ is an isomorphism and we have $H^1(C, \mathcal{O}_C(1)) = 0$.

In addition, when $\ell \geqslant 0$ is an integer, we define the moduli space $\overline{\mathcal{M}}_{g,\ell}^*(\mathbb{P}^N, m)$ to be the preimage of $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ under the forgetful map $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m) \to \overline{\mathcal{M}}_g(\mathbb{P}^N, m)$.

Remark 2.1.5. The moduli stack $\overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^N, m)$ of stable maps carries an obvious action of the Lie group $\mathrm{PGL}(N+1, \mathbb{C}) = \mathrm{Aut}(\mathbb{P}^N)$ and $\overline{\mathcal{M}}_{g,\ell}^*(\mathbb{P}^N, m) \subset \overline{\mathcal{M}}_{g,\ell}(\mathbb{P}^N, m)$ is an open, $\mathrm{PGL}(N+1, \mathbb{C})$-invariant smooth quasi-projective subscheme of the expected $\mathbb{C}$-dimension $(N-3)(1-g) + (N+1)m + \ell$. An explanation is given in Definition 2.2.8.

Definition 2.1.6. A *polarisation* on X taming J is a Hermitian line bundle $\mathcal{O}_X(1) \to X$ equipped with a Hermitian connection ∇ with curvature form $-2\pi \mathrm{i}\Omega$ where Ω is a symplectic form on X taming J.

Remark 2.1.7. Given any compact subset $K \subset \mathcal{J}_\tau(X, \omega)$, there exists a polarisation on X taming all the almost complex structures in K. This is shown in Lemma 2.2.1.

Definition 2.1.8 (Framed stable maps)**.** Fix a polarisation $\mathcal{O}_X(1) \to X$ taming J as in Definition 2.1.6. An Ω-*stable map* (of genus g and class A) is a smooth map $u : C \to X$ satisfying the following conditions.

(i) C is a prestable curve of genus g and $u_*[C] = A$.

(ii) For each irreducible (resp. unstable irreducible) component $C' \subset C$, we have $\int_{C'} u^*\Omega \geqslant 0$ (resp. > 0).

We will refer to Ω-stable maps of genus g and class A as just stable maps when Ω, g, A are clear from the context. Given any stable map $u : C \to X$, we define the holomorphic line bundle $\mathfrak{L}_u \to C$ to be

$$\mathfrak{L}_u := \omega_C \otimes (u^*\mathcal{O}_X(1))^{\otimes 3} \tag{2.1.0.2}$$

where ω_C denotes the dualizing line bundle of C and the holomorphic structure on the line bundle $u^*\mathcal{O}_X(1)$ is defined by $(u^*\nabla)^{0,1}$. We say the stable map $u : C \to X$ is *framed* if we additionally have the data of a degree m holomorphic embedding $C \subset \mathbb{P}^N$ corresponding to a point of $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ for some $N \geqslant 2$ and $m \geqslant 1$. The set of framed stable maps (for a fixed choice of N and m) is equipped with a natural topology described in Definition 2.2.11.

Remark 2.1.9. The line bundle $\mathfrak{L}_u$ defined in (2.1.0.2) is ample because of the stability condition imposed on $u : C \to X$. This means that we can promote $u : C \to X$ to a framed stable map by choosing a complex basis $(s_0, \dots, s_N)$ of the vector space $H^0(C, \mathfrak{L}_u^{\otimes p})$ for an integer $p \gg 1$ and taking the associated projective embedding $[s_0 : \dots : s_N] : C \to \mathbb{P}^N$. Refer to Lemma 2.2.2 for more details.

Remark 2.1.10. For a fixed choice of N and m, there is a natural $\mathrm{PGL}(N+1, \mathbb{C})$-action on the space of framed stable maps. The natural forgetful map to $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ is equivariant with respect to this action.

Before defining the auxiliary data needed for our construction of a global Kuranishi chart, we need a few preliminary definitions.

Definition 2.1.11 (Auxiliary data)**.** An *auxiliary datum* for the moduli space $\overline{\mathcal{M}}_{g,n}(X, A; J)$ is a tuple $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ where

(i) ∇^X is a $\mathbb{C}$-linear connection on the tangent bundle T_X (with $\mathbb{C}$-linear structure induced by J),

(ii) $\mathcal{O}_X(1) \to X$ is a polarisation taming J as in Definition 2.1.6. We set $d := \langle [\Omega], A \rangle$,

(iii) $p \geqslant 1$ is an integer. For later reference below, we introduce the following related notation.

(a) $m := p(2g - 2 + 3d) = p \deg(\mathfrak{L}_u)$ and $N := m - g$,

(b) $\mathcal{G} := \mathrm{PGL}(N+1, \mathbb{C})$ and $G := \mathrm{PU}(N+1) = U(N+1)/S^1$,

Note that $m = p \deg(\mathfrak{L}_u)$ while $N = \dim H^0(C, \mathfrak{L}_u^{\otimes p}) - 1$ by the Riemann-Roch formula.

(iv) $\mathcal{U}$ is a good covering in the sense of Definition 2.2.12.

(v) $k \geqslant 1$ is an integer.

Remark 2.1.12 (Motivation for the good covering)*.* In the construction described below, we need to pick out a class of *unitarily framed stable maps* from the space of all framed stable maps. Morally, this can be done by choosing local slices for the $\mathcal{G}$-action on the space of framed stable maps followed by a partition of unity argument. The datum of a good covering allows for the construction of a continuous G-equivariant map $\lambda_{\mathcal{U}} \colon \mathcal{T} \to \mathcal{G}/G$, which determines the class of unitarily framed stable maps, that is a desired slice. While more details can be found in §2.2.2, the reader is advised to take the existence of $\lambda_{\mathcal{U}}$ on good faith at first reading. See [AMS23, §4.3] for a different approach to this problem.

In order to ensure that our thickening defined below is a manifold (and $\mathcal{E}$ is a vector bundle), at least near $\mathfrak{s}^{-1}(0)$, we will need to restrict to a special subclass of auxiliary data.

Definition 2.1.13. We call an auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ *unobstructed* if the following properties hold for any stable J-holomorphic map $u : C \to X$ in $\overline{\mathcal{M}}_g(X, A; J)$.

(a) The line bundle $\mathfrak{L}_u^{\otimes p} \to C$ is very ample and $H^1(C, \mathfrak{L}_u^{\otimes p}) = 0$.

(b) For every complex linear basis $\mathcal{F} = (s_0, \cdots, s_N)$ of $H^0(C, \mathfrak{L}_u^{\otimes p})$, we obtain a framed stable map $(\iota_{\mathcal{F}} \colon C \hookrightarrow \mathbb{P}^N, u)$, in the sense of Definition 2.1.8, satisfying the following.

   (1) We have $H^1(C, T^{*0,1}_{\mathbb{P}^N}|_C \otimes u^*T_X \otimes \mathcal{O}_C(k)) = 0$.

   (2) If $\lambda_{\mathcal{U}}(\iota_{\mathcal{F}}, u) = [\mathrm{Ide}] \in \mathcal{G}/G$, then

$$D(\bar{\partial}_J)_u \oplus (\langle \cdot \rangle \circ d\iota_{\tilde{C},\mathcal{F}}) : \Omega^0(C, u^*T_X) \oplus E_{(\iota_{\mathcal{F}},u)} \to \Omega^{0,1}(\tilde{C}, \tilde{u}^*T_X) \qquad (2.1.0.3)$$

   is surjective. Here, $D(\bar{\partial}_J)_u$ is the linearization of the non-linear Cauchy–Riemann operator $\bar{\partial}_J$ at the map u and the map $\langle \cdot \rangle$ is as in (2.1.0.6).

We can now describe the global Kuranishi chart associated to an unobstruced auxiliary datum.

Construction 2.1.14. Having fixed an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$, we define

(i) (Thickening) $\mathcal{T}$ consists of all tuples $(u, \iota, C, \eta, \alpha)$ satisfying the following properties.

   (a) (u, ι, C) is a framed stable map lying in the domain of $\lambda_{\mathcal{U}}$.

   (b) η belongs to the finite dimensional $\mathbb{C}$-vector space

$$E_{(\iota,u)} := H^0(C, T^{*0,1}_{\mathbb{P}^N}|_C \otimes u^*T_X \otimes \mathcal{O}_C(k)) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))} \qquad (2.1.0.4)$$

   where we use the complex linear identification $T^{*0,1}_{\mathbb{P}^N} \simeq T_{\mathbb{P}^N}$ (given by the Fubini–Study metric) to endow the former with a holomorphic structure while we endow u^*T_X with the holomorphic structure given by $(u^*\nabla^X)^{0,1}$. On the normalization $\tilde{C} \to C$, the equation

$$\bar{\partial}_J \tilde{u} + \langle \eta \rangle \circ d\iota_{\tilde{C}} = 0 \in \Omega^{0,1}(\tilde{C}, \tilde{u}^*T_X) \qquad (2.1.0.5)$$

   is satisfied. Here, $\tilde{u}$ and $\iota_{\tilde{C}}$ denote the pullbacks to $\tilde{C}$ of the map u and the inclusion $C \subset \mathbb{P}^N$ respectively. The $\mathbb{C}$-linear contraction operator

$$\langle \cdot \rangle : E_{(\iota,u)} \to \Omega^0(C, T^{*0,1}_{\mathbb{P}^N}|_C \otimes u^*T_X) \qquad (2.1.0.6)$$

   is induced by the standard Hermitian metric on the line bundle $\mathcal{O}_{\mathbb{P}^N}(k)$.

(c) $\alpha \in H^1(C, \mathcal{O}_C)$ is such that we have the identity

$$[\mathcal{O}_C(1)] = p \cdot [\mathfrak{L}_u] + \alpha \in \mathrm{Pic}(C) \tag{2.1.0.7}$$

in the Picard group of C (the group of isomorphism classes of holomorphic line bundles) with group operation (given by tensor product of line bundles) written additively. Here, we are identifying $H^1(C, \mathcal{O}_C)$ with the (abelian) Lie algebra of $\mathrm{Pic}(C)$ and $+\alpha$ denotes translation by α using the exponential map of the group $\mathrm{Pic}(C)$.

(d) $H^1(C, T^{*0,1}_{\mathbb{P}^N}|_C \otimes u^*T_X \otimes \mathcal{O}_C(k)) = 0$,

(e) the linearised operator associated to Equation (2.1.0.5) is surjective when restricted to $C^\infty(C, u^*T_X) \oplus E_{\iota,u}$.

The natural $U(N+1)$-action on $\mathcal{T}$ descends to a G-action. Moreover, the natural forgetful morphism $\pi : \mathcal{T} \to \overline{\mathcal{M}}^*_g(\mathbb{P}^N, m)$ is G-equivariant.

(ii) (Obstruction bundle) $\mathcal{E} \to \mathcal{T}$ is the family of vector spaces over $\mathcal{T}$ whose fibre over a given point $(C \subset \mathbb{P}^N, u, \eta, \alpha) \in \mathcal{T}$ is given by

$$\mathfrak{su}(N+1) \oplus E_{(\iota,u)} \oplus H^1(C, \mathcal{O}_C) \tag{2.1.0.8}$$

and carries a natural (fibrewise linear) action of G which lifts the G-action on $\mathcal{T}$.

(iii) (Obstruction section) The section $\mathfrak{s} : \mathcal{T} \to \mathcal{E}$ is defined by the formula

$$\mathfrak{s}(C \subset \mathbb{P}^N, u, \eta, \alpha) = (\mathrm{i} \log \lambda_{\mathcal{U}}(C \subset \mathbb{P}^N, u), \eta, \alpha). \tag{2.1.0.9}$$

For the definition of the 'polar decomposition' map $\mathrm{i}\log : \mathcal{G}/G \to \mathfrak{su}(N+1)$, see Definition 2.2.5. The section $\mathfrak{s}$ is G-equivariant and there is a natural forgetful map $\mathfrak{s}^{-1}(0)/G \to \overline{\mathcal{M}}_g(X, A; J)$.

The associated global Kuranishi chart $\mathcal{K}_n = (G, \mathcal{T}_n, \mathcal{E}_n, \mathfrak{s}_n)$ for $\overline{\mathcal{M}}_{g,n}(X, A; J)$ is defined by pulling back $\mathcal{K}$ along the forgetful map $\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m) \to \overline{\mathcal{M}}^*_g(\mathbb{P}^N, m)$.

Remark 2.1.15. When $A = 0$, J-holomorphic stable maps are just stable curves mapping to a point in X, $\mathfrak{L}_u = \omega_C$, which does not have positive degree on each component. Hence once has to use $\mathfrak{L}'_u := \omega_C(x_1 + \cdots + x_n)$ in Construction 2.1.14 instead.

Remark 2.1.16. Note that Construction 2.1.14 defines only the points of the spaces $\mathcal{T}$ and $\mathcal{E}$ and does not describe any additional structure on these (e.g. a rel–C^∞ structure, a vector bundle structure). This additional structure is explained in §2.3.1 and §2.3.2.

Remark 2.1.17 (Comparison with the construction of [AMS21]). The construction of the global Kuranishi chart in [AMS21] for moduli spaces of genus 0 stable maps depends on a slightly different set of auxiliary data than the more general construction described above. More precisely, to define the global Kuranishi chart in [AMS21], one needs to make a choice of $\nabla^X, \mathcal{O}_X(1)$ and k (and a relatively ample line bundle $\mathcal{L}$ on the relevant universal curve; see the paragraph preceding [AMS21, Definition 6.11] for further details). It is easy to show, using a slight variant of the argument which proves Theorem 2.1.18(3a), that the construction of [AMS21] is equivalent to ours when $g = 0$. We did not check whether the construction in [AMS23] is equivalent to the one presented here.

We can now formulate our main result on global Kuranishi charts for Gromov–Witten moduli spaces.

Theorem 2.1.18 (Global Kuranishi charts for GW moduli spaces)**.** *Fix (X,ω) and A, g, n as before.*

(1) *Fix $J \in \mathcal{J}_\tau(X,\omega)$. Unobstructed auxiliary data, in the sense of Definition 2.1.13, exist. Moreover, any choices of connection ∇^X and polarisation $\mathcal{O}_X(1)$ taming J can be extended to an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$.*

(2) *Given $J \in \mathcal{J}_\tau(X,\omega)$ and an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$, the associated global Kuranishi chart $\mathcal{K}_n = (G, \mathcal{T}_n, \mathcal{E}_n, \mathfrak{s}_n)$ from Construction 2.1.14 has the following properties.*

   (a) *The projection $\mathcal{T}_n \to \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m)$ carries a natural rel–C^∞ structure of the expected dimension in a G-invariant neighborhood $\mathcal{T}_n^{\mathrm{reg}}$ of $\mathfrak{s}_n^{-1}(0)$.*

   (b) *$\mathcal{E}_n^{\mathrm{reg}} := \mathcal{E}_n|_{\mathcal{T}_n^{\mathrm{reg}}}$ naturally carries the structure of a rel–C^∞ vector bundle of the expected rank over $\mathcal{T}_n^{\mathrm{reg}} \to \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m)$ for which the section $\mathfrak{s}_n$ is of class rel–C^∞.*

   (c) *The G-action on $\mathcal{T}_n^{\mathrm{reg}}$ and $\mathcal{E}_n^{\mathrm{reg}}$ is rel–C^∞ and fibrewise locally linear in the sense of* [AMS21, Definition 4.20]*. The stabilizer of every point of $\mathcal{T}_n$ in a neighborhood of $\mathfrak{s}_n^{-1}(0)$ is finite.*

   (d) *The natural forgetful map $\mathfrak{s}_n^{-1}(0)/G \to \overline{\mathcal{M}}_{g,n}(X, A; J)$ is a homeomorphism.*

   (e) *The virtual vector bundle given by*

$$T_{\mathcal{T}_n^{\mathrm{reg}}/\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N,m)} - (\mathcal{E}_n^{\mathrm{reg}} \oplus \underline{\mathfrak{g}}) \tag{2.1.0.10}$$

   has a natural stably complex (virtual) vector bundle lift in a neighborhood of the zero locus $\mathfrak{s}_n^{-1}(0)$ where $T_{\mathcal{T}_n^{\mathrm{reg}}/\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N,m)}$ is the vertical tangent bundle and $\underline{\mathfrak{g}}$ is the trivial bundle with fibre $\mathfrak{g} = \mathrm{Lie}(G)$.

(3) *The global Kuranishi charts of Construction 2.1.14 have the following uniqueness properties.*

   (a) *Fix $J \in \mathcal{J}_\tau(X,\omega)$. Then, the global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X, A; J)$ associated to any two unobstructed auxiliary data are stably complex rel–C^∞ equivalent in the sense of Definition 2.4.1.*

   (b) *Given $J_0, J_1 \in \mathcal{J}_\tau(X,\omega)$, there exist unobstructed auxiliary data $(\nabla^{X,i}, \mathcal{O}_X(1), p, \mathcal{U}_i, k)$ for $i = 0, 1$ so that the associated global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X, A; J_0)$ and $\overline{\mathcal{M}}_{g,n}(X, A; J_1)$ are stably complex rel–C^∞ cobordant in the sense of Definition 2.4.3.*

We will prove Theorem 2.1.18(1), (2) and (3) in §2.2, §2.3 and §2.4 respectively.

Remark 2.1.19. Recall the equivalence moves relating global Kuranishi charts (namely *germ equivalence*, *stabilisation* and *group enlargement*) described in [AMS21, §4]. In §2.4 we formulate a slight refinement of these moves in order to keep track of the natural rel–C^∞ structures yielded by our construction. The notion of *rel–C^∞ cobordism* for global Kuranishi charts is also formulated in the same section (see Definition 2.4.3).

Remark 2.1.20 (Virtual fundamental classes). The virtual fundamental class of an oriented global Kuranishi chart for a compact space Z (lying in the dual of Čech cohomology of Z with $\mathbb{Q}$-coefficients) is invariant under equivalence and cobordism (see [AMS21, §5.1]). Theorem 2.1.18 therefore provides a construction of virtual fundamental classes for the Gromov–Witten moduli spaces of a closed symplectic manifold. The explicit definitions are given in §2.5.1 and §2.5.2.

2.2 Transversality for auxiliary data

In this section, we will prove Theorem 2.1.18(1) by showing how to choose the parameters $p, \mathcal{U}, k$ as in Definition 2.1.11 so that the auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ is unobstructed. We may assume that we are already given choices of ∇^X, $\mathcal{O}_X(1)$. Indeed, it is obvious that J-linear connections on T_X exist, while the existence of polarisations on X taming J is guaranteed by Lemma 2.2.1 below.

Lemma 2.2.1. *There exists a complex line bundle $\mathcal{O}_X(1) \to X$ with Hermitian metric $\langle\cdot,\cdot\rangle$ and a Hermitian connection ∇ with curvature given by $-2\pi i\Omega$, where Ω is a symplectic form taming J. In fact, given any compact subset $\mathcal{F} \subset \mathcal{J}_\tau(X,\omega)$ containing J, we can choose $\mathcal{O}_X(1)$ to be such that Ω tames each $J' \in \mathcal{F}$.*

Proof. By approximating $[\omega] \in H^2(X,\mathbb{R})$ by an element of $H^2(X,\mathbb{Q})$ and multiplying by a large positive integer to clear denominators, we first choose a symplectic form Ω taming (each almost complex structure in) $\mathcal{F}$ such that $[\Omega]$ has an integral lift $h \in H^2(X,\mathbb{Z})$. This is possible as being symplectic and taming $\mathcal{F}$ are both open properties of closed 2-forms. Now, let $\mathcal{L}$ be a complex line bundle on X with $c_1(\mathcal{L}) = h$. Choose any Hermitian metric $\langle\cdot,\cdot\rangle$ on $\mathcal{L}$ and a compatible Hermitian connection ∇' on $\mathcal{L}$ and write the curvature as $-2\pi i\Omega'$. Since h is a common integral lift of $[\Omega]$ and $[\Omega']$, we can find a smooth (real) 1-form β such that $\Omega' = \Omega + d\beta$. The connection $\nabla = \nabla' + 2\pi i\beta$ now is also Hermitian for $\langle\cdot,\cdot\rangle$ and has curvature given by $-2\pi i\Omega$. Thus, we may take $\mathcal{O}_X(1)$ to be the line bundle $\mathcal{L}$ equipped with the metric $\langle\cdot,\cdot\rangle$ and compatible Hermitian connection ∇. □

As in Definition 2.1.11, we set $d := \langle[\Omega], A\rangle$.

2.2.1 Choosing the integer p

In this subsection, we will show how to choose the integer p.

Lemma 2.2.2 (Positivity)**.** *There exists a positive integer p depending only on g, d with the following property. Following the notation of equation (2.1.0.2), for any stable map*

$[u, C]$ and any integer $q \geqslant p$, the line bundle $\mathfrak{L}_u^{\otimes q} \to C$ is very ample and we have $H^1(C, \mathfrak{L}_u^{\otimes q}) = 0$.

Proof. Given an irreducible component C' of C, we have that $\deg_{C'}(\omega_C) \geqslant -2$ and the degree is non-positive if and only if C' is unstable. In this case $u|_{C'}$ is nonconstant, so $\deg_{C'}(u^*\mathcal{O}_X(1)) \geqslant 1$. Thus $\deg_{C'}(\mathfrak{L}_u) > 0$ and $\mathfrak{L}_u$ has total degree $2g - 2 + 3d$ on C. It follows that C has $\leqslant 2g - 2 + 3d$ irreducible components, so the number of possibilities for the dual graph of C, decorated by genus labels on the vertices, can be bounded in terms of g and d. Hence it suffices to find a p for each possible decorated dual graph Γ of C.

For each vertex $v \in \Gamma$ of C, let C_v be the normalization of the irreducible component of C corresponding to v. Let g_v be the genus of C_v, $D_v \subset C_v$ be the subset consisting of the inverse images of the nodal points and L_v be the pullback of $\mathfrak{L}_u$ to C_v. For any two points $a, b \in C_v$, Serre duality [Har77, Theorem III.7.6] yields

$$H^1(C_v, L_v^{\otimes q}(-D_v - a - b)) = H^0(C_v, \omega_{C_v}(D_v + a + b) \otimes L_v^{*\otimes q})^* = 0 \qquad (2.2.1.1)$$

for $q \geqslant p_\Gamma := 1 + \max_{v \in \Gamma}(2g_v + |D_v|)$ once we recall that $\deg_{C_v} L_v \geqslant 1$. This cohomology vanishing statement (for each $v \in \Gamma$) implies that $\mathfrak{L}_u^{\otimes q}$ is very ample. To see that $\mathfrak{L}_u^{\otimes q}$ has vanishing H^1, denote by $x(v, e)$ the point on C_v corresponding to the node associated to the edge $e = \{v, v'\}$ of Γ. Twisting the normalisation sequence

$$0 \to \mathcal{O}_C \to \bigoplus_{v \in V(\Gamma)} \mathcal{O}_{C_v} \to \bigoplus_{e = \{v, v'\} \in E(\Gamma)} T_{x(v,e)} C_v \otimes T_{x(v',e)} C_{v'} \to 0$$

by $\mathfrak{L}_u^{\otimes q}$ and taking its long exact sequence in cohomology, we obtain that $H^1(C, \mathfrak{L}_u^{\otimes q}) = 0$. Since the lower bound p_Γ on q depends only on the decorated dual graph Γ, the proof is complete. $\square$

We fix $p \geqslant 1$ to be the smallest integer which satisfies the conclusion of Lemma 2.2.2 above. Having fixed the choice of p, we define the associated quantities $m, N, \mathcal{G}, G$ exactly as in Definition 2.1.11(iii). The following observation will be useful.

Lemma 2.2.3 (Unobstructed projective embedding)**.** *Let $[u, C]$ be a stable map as in Definition 2.1.8. Then, any complex linear basis $\mathcal{F} = (s_0, \cdots, s_N)$ of $H^0(C, \mathfrak{L}_u^{\otimes p})$ determines a point of $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ via the holomorphic projective embedding*

$$\iota_{C,\mathcal{F}} = [s_0 : \cdots : s_N] : C \to \mathbb{P}^N. \qquad (2.2.1.2)$$

Moreover, the map $\iota_{C,\mathcal{F}}$ is unobstructed, i.e., $H^1(C, \iota_{C,\mathcal{F}}^* T_{\mathbb{P}^N}) = 0$.

Proof. Since $\iota_{C,\mathcal{F}}$ is the map obtained from the complete linear system defined by the very ample line bundle $\mathfrak{L}_u^{\otimes p}$, we obtain a point of $\overline{\mathcal{M}}_{g,n}^*(\mathbb{P}^N, m)$ once we note the identification $\iota_{C,\mathcal{F}}^* \mathcal{O}_{\mathbb{P}^N}(1) \simeq \mathfrak{L}_u^{\otimes p}$. To prove unobstructedness, pull back the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N}(1)^{N+1} \to T_{\mathbb{P}^N} \to 0$ to C via $\iota_{C,\mathcal{F}}$ and use the long exact sequence in cohomology to obtain a surjective map

$$H^1(C, \iota_{C,\mathcal{F}}^* \mathcal{O}_{\mathbb{P}^N}(1))^{N+1} \to H^1(C, \iota_{C,\mathcal{F}}^* T_{\mathbb{P}^N}). \qquad (2.2.1.3)$$

This shows that $H^1(C, \iota_{C,\mathcal{F}}^* T_{\mathbb{P}^N}) = 0$ as desired since $H^1(C, \iota_{C,\mathcal{F}}^* \mathcal{O}_{\mathbb{P}^N}(1)) = 0$. □

Remark 2.2.4. Observe that our choice of p is such that condition (a) of Definition 2.1.13 is satisfied.

2.2.2 Constructing a good covering $\mathcal{U}$ and the map $\lambda_{\mathcal{U}}$

At the end of this subsection, we will show how to construct a good covering $\mathcal{U}$, defined in Definition 2.2.12, and the associated map $\lambda_{\mathcal{U}}$. We need a number of preparatory results for this purpose which we now turn to.

Definition 2.2.5 (Logarithm for Hermitian matrices)**.** Let $\mathcal{H}$ be the space of $(N+1) \times (N+1)$ Hermitian positive definite matrices modulo the action of positive real scalars. The group $\mathcal{G}$ has a left action on $\mathcal{H}$ as follows: an element $[T] \in \mathcal{G}$ maps an element $[A] \in \mathcal{H}$ to $[TAT^*] \in \mathcal{H}$. The Lie algebra $\mathfrak{su}(N+1)$ consisting of skew-Hermitian trace-free $(N+1)\times(N+1)$ matrices carries a natural adjoint G-action. With this, the exponential map

$$\mathfrak{su}(N+1) \to \mathcal{H} \tag{2.2.2.1}$$
$$\mathrm{i}M \mapsto [\exp M] \tag{2.2.2.2}$$

is a G-equivariant diffeomorphism. We let $\mathrm{i}\log : \mathcal{H} \to \mathfrak{su}(N+1)$ denote its inverse. We can identify this with a map $\mathcal{G}/G \to \mathfrak{su}(N+1)$, also denoted $\mathrm{i}\log$, via the isomorphism $P : \mathcal{G}/G \to \mathcal{H}$ provided by Lemma 2.2.6(iii) below.

Lemma 2.2.6 (Polar decomposition for $\mathcal{G}$)**.** *We have the following assertions.*

(i) *The multiplication map $\mathcal{H} \times G \to \mathcal{G}$ is a diffeomorphism.*

(ii) *If Λ is a finite set, then the linear combination map*

$$c_\Lambda : (\mathbb{R}^\Lambda_{\geqslant 0}\backslash\{0\}) \times \mathcal{H}^\Lambda \to \mathcal{H} \tag{2.2.2.3}$$
$$(\{t_i\}_{i\in\Lambda}, \{[A_i]_{i\in\Lambda}\}) \mapsto \textstyle\sum_{i\in\Lambda} t_i[A_i] \tag{2.2.2.4}$$

is $\mathcal{G}$-equivariant, where $\mathbb{R}^\Lambda_{\geqslant 0}\backslash\{0\}$ carries the trivial action.

(iii) *The map $\mathcal{G} \to \mathcal{H}$ given by $T \mapsto TT^*$ descends to a $\mathcal{G}$-equivariant diffeomorphism $P : \mathcal{G}/G \to \mathcal{H}$. The identity G-coset is mapped to the class $[Ide] \in \mathcal{H}$ of the identity matrix under P.*

Proof. The first assertion is an immediate consequence of the polar decomposition in $\mathrm{GL}(N+1, \mathbb{C})$. Note that $\mathcal{G}$-equivariance in the second and third assertions are true by definition. Using the first assertion, we can view the map P as the squaring map $[A] \mapsto [A^2]$ on $\mathcal{H}$, which shows that it is a diffeomorphism. □

Lemma 2.2.7 (Reduction of structure group)**.** *If M is a second countable smooth manifold with a proper action of $\mathcal{G}$ with finite stabilisers, then there exists a smooth $\mathcal{G}$-equivariant map $M \to \mathcal{G}/G$.*

Proof. We first construct such a map in a $\mathcal{G}$-invariant neighborhood of any point in M. Given $x \in M$, replace it by a point in its orbit to assume that its stabiliser Γ is contained in G. Using a Γ-invariant tubular neighbourhood of x, choose a locally closed Γ-invariant submanifold $S \subset M$ passing through x such that T_xS is a complement to the linearized action map $\mathfrak{g} \hookrightarrow T_xM$. Restricting the action map $\mathcal{G} \times M \to M$ gives a smooth $\mathcal{G}$-equivariant map

$$\Phi : (\mathcal{G} \times S)/\Gamma \to M \tag{2.2.2.5}$$

where Γ acts on $\mathcal{G} \times S$ by $\gamma \cdot (g, s) = (g\gamma^{-1}, \gamma s)$. As $d\Phi_{(1,x)} : \mathfrak{g} \oplus T_xS \to T_xM$ is an isomorphism, we may shrink S to assume that Φ is a $\mathcal{G}$-equivariant local diffeomorphism. Properness of the $\mathcal{G}$-action implies that Φ is injective after shrinking S further. Define a $\mathcal{G}$-equivariant map $\mathcal{N} := \Phi((\mathcal{G} \times S)/\Gamma) \to \mathcal{G}/G$ by applying Φ^{-1} followed by the obvious projection. Thus, we have solved the problem in the $\mathcal{G}$-invariant neighborhood $\mathcal{N}$ of x. Further, any smooth Γ-invariant compactly supported cutoff function χ on S admits a $\mathcal{G}$-invariant smooth extension $\tilde{\chi}$ to $\mathcal{N}$ and can be extended by zero to obtain a $\mathcal{G}$-invariant cutoff function $\tilde{\chi}$ on M (we are using the fact that $\mathcal{G} \cdot \operatorname{supp} \chi$ is closed in M which follows from properness of the action).

Therefore, the statement follows if we can cover M by a locally finite collection of $\mathcal{G}$-invariant open subsets, each admitting a smooth $\mathcal{G}$-equivariant map to $\mathcal{G}/G$ and then use a $\mathcal{G}$-invariant smooth partition of unity to patch them. Here we use (ii),(iii) of Lemma 2.2.6 to make sense of convex combinations in $\mathcal{G}/G$.

To obtain such a locally finite cover, it suffices to show that the quotient space $N = M/\mathcal{G}$ is metrizable (and therefore paracompact). By properness of the $\mathcal{G}$-action, we know that N is Hausdorff. Since M is second countable, so is N. By the Urysohn metrization theorem, it remains to show that N is a regular space (i.e., given $y \in N$ and a closed subset $C \subset N$ with $y \notin C$, there exist open neighborhoods in N separating them). Equivalently, given a closed $\mathcal{G}$-invariant subset $F \subset M$ and a point $x \notin F$, we need to find $\mathcal{G}$-invariant disjoint neighborhoods $\mathcal{U}$, $\mathcal{V}$ of x, F respectively. As before, let Γ be the stabiliser of x and S be a local Γ-invariant slice at x for the $\mathcal{G}$-action. Then, $x \in S\backslash F$ and thus, we can choose a Γ-invariant open neighborhood U of x in $S\backslash F$ such that $\overline{U}$ is compact and is contained in $S\backslash F$. Now, we can take $\mathcal{U} = \mathcal{G} \cdot U$ and $\mathcal{V} = M\backslash(\mathcal{G} \cdot \overline{U})$. Note that $\mathcal{V} \subset M$ is open because the action is proper. $\square$

Definition 2.2.8 (Algebraic base space and universal curve on it)**.** For any integer $\ell \geqslant 0$, define $\overline{\mathcal{M}}^*_{g,\ell}(\mathbb{P}^N, m)$ exactly as in Definition 2.1.4. This is an algebraic scheme of finite type over $\mathbb{C}$ by [FP97, §4.1]. By Lemma 2.2.3, all points of $\overline{\mathcal{M}}^*_{g,\ell}(\mathbb{P}^N, m)$ are unobstructed and have no non-trivial automorphisms. By [RRS08], it is therefore a smooth manifold of the expected complex dimension $(N-3)(1-g) + m(N+1) + \ell$. Denote by

$$\pi : \mathcal{C}_{g,\ell} \subset \overline{\mathcal{M}}^*_{g,\ell}(\mathbb{P}^N, m) \times \mathbb{P}^N \to \overline{\mathcal{M}}^*_{g,\ell}(\mathbb{P}^N, m) \tag{2.2.2.6}$$

the universal curve, which is also an algebraic scheme of finite type over $\mathbb{C}$.

For $\ell \geqslant N+1$ define $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,\ell}(\mathbb{P}^N, m) \subset \overline{\mathcal{M}}^*_{g,\ell}(\mathbb{P}^N, m)$ to be the open subset of maps

$[\iota, C, x_1, \ldots, x_n]$ where

- $(C, x_1, \ldots, x_n)$ is stable,
- ι is regular, has no automorphisms and is nondegenerate, i.e., does not lie in a hyperplane.

Denote by $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell]}(\mathbb{P}^N, m)$ the quotient of $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,\ell}(\mathbb{P}^N, m)$ under the (free) action of the permutation group S_ℓ on the marked points.

Lemma 2.2.9. *The $\mathcal{G}$-action on $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell]}(\mathbb{P}^N, m)$ is proper and almost free for any integer $\ell \geqslant N+1$.*

Proof. Since $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,\ell}(\mathbb{P}^N, m) \to \overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell]}(\mathbb{P}^N, m)$ is a finite unbranched covering map, it suffices to consider the $\mathcal{G}$-action on the former. Since the statement to prove concerns an algebraic action of an algebraic group on a variety, we may use the Noetherian valuative criterion [SPa22, Tag 0208] to test properness. To this end, let R be a discrete valuation ring and K its fraction field. Given any three morphisms

$$\alpha, \alpha' : \operatorname{Spec} R \to \overline{\mathcal{M}}^{*,\mathrm{st}}_{g,\ell}(\mathbb{P}^N, m) \tag{2.2.2.7}$$

$$\gamma : \operatorname{Spec} K \to \mathcal{G} \tag{2.2.2.8}$$

such that $\gamma \cdot \alpha_K = \alpha'_K$, we need to extend γ to a morphism $\operatorname{Spec} R \to \mathcal{G}$. Lift γ to an element $\delta \in \mathrm{GL}(N+1, K)$ which is unique up to an element of $K^\times$. We will lift δ to $GL(N+1, R)$ up to an element of $K^\times$.

The morphism α yields a projective flat family $\pi_R : \mathcal{C}_R \subset \mathbb{P}^N_R \to \operatorname{Spec} R$ of stable ℓ-pointed genus g curves with the marked points given by sections $\sigma_1, \ldots, \sigma_\ell : \operatorname{Spec} R \to \mathcal{C}_R$ of π_R. Moreover, the restriction

$$R^{N+1} = H^0(\mathbb{P}^N_R, \mathcal{O}_{\mathbb{P}^N_R}(1)) \to H^0(\mathcal{C}_R, \mathcal{O}_{\mathcal{C}_R}(1)) \tag{2.2.2.9}$$

gives an isomorphism of R-modules where we use [Har77, Theorem III.5.1(a)] to compute the H^0 group on the left explicitly. Similarly, we get $(\pi'_R : \mathcal{C}'_R \subset \mathbb{P}^N_R \to \operatorname{Spec} R, \sigma'_1, \ldots, \sigma'_\ell)$ associated to α'. The element δ now yields an isomorphism $\varphi : \mathcal{C}_K \to \mathcal{C}'_K$ over $\operatorname{Spec} K$ (mapping σ_i to σ'_i for $1 \leqslant i \leqslant \ell$) of the two families and an isomorphism $\Phi : \mathcal{O}_{\mathcal{C}_K}(1) \simeq \varphi^*\mathcal{O}_{\mathcal{C}'_K}(1)$. Taking global sections of Φ recovers δ.

By the uniqueness of stable reduction [SPa22, Tag 0E97], we obtain a unique extension $\widehat{\varphi} : \mathcal{C}_R \to \mathcal{C}'_R$ of φ to an isomorphism of families of stable ℓ-pointed genus g curves over $\operatorname{Spec} R$. Since $\operatorname{Spec} K \subset \operatorname{Spec} R$ is dense, we get $\mathcal{O}_{\mathcal{C}_R}(1) \simeq \widehat{\varphi}^*\mathcal{O}_{\mathcal{C}'_R}(1)$ from [FP97, Proposition 1]. Taking global sections of this isomorphism now yields an element of $\mathrm{GL}(N+1, R)$ whose restriction to K differs from δ by an element of $K^\times$.

For the second assertion, suppose $A \in \mathcal{G}$ fixes $y = [\iota, C, x_1, \ldots, x_n]$. Then the associated biholomorphism ϕ_A of $\mathbb{P}^N$ preserves $\iota(C) \cong C$ setwise and thus induces an automorphism $\psi_A : C \to C$. If A, B are two elements of the stabiliser such that $\psi_A = \psi_B$, then AB^{-1} fixes $\iota(C)$ pointwise. This contradicts the assumption that ι is nondegenerate. Thus we get an injection $\mathcal{G}_y \hookrightarrow \operatorname{Aut}(C, x_1, \ldots, x_n)$, the later of which is finite. $\square$

Corollary 2.2.10. *For any integer $\ell \geqslant N+1$, there exists a smooth $\mathcal{G}$-equivariant map $\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell]}(\mathbb{P}^N, m) \to \mathcal{G}/G$.*

Proof. This is an immediate consequence of Lemmas 2.2.7 and 2.2.9. $\square$

Definition 2.2.11 (Polyfolds of stable maps)**.** We define $Z = Z_{A,g}(X)$ to be the polyfold of Ω-stable genus g maps to X in class A which are not necessarily J-holomorphic. For the construction of the polyfold structure we refer to [HWZ17, Theorem 3.37]. Since the polyfold Z is locally modeled on (retracts of sc-)Hilbert spaces, it admits (sc-)smooth cutoff functions. (To see this, combine Proposition 5.5 and Theorem 12.6 in [HWZ21] with the example of sc-Hilbert spaces discussed in the paragraph following Definition 5.13 therein.)

Similarly, define $\tilde{Z} = Z_{\tilde{A},g}(\mathbb{P}^N \times X)$ to be the polyfold of Ω-stable genus g maps to $\mathbb{P}^N \times X$ in the class $\tilde{A} = [\mathrm{pt}] \times A + m[\mathbb{P}^1] \times [\mathrm{pt}]$. There is an obvious $\mathcal{G}$-action on $\tilde{Z}$. Observe that the set of framed stable maps, introduced in Definition 2.1.11(iii), acquires a natural topology via its natural inclusion into $\tilde{Z}$ as a subspace. Let π_Z be the natural projection from the space of framed stable maps to Z.

Definition 2.2.12. A *good covering* is a collection $\mathcal{U} = \{(U_i, \ell_i, D_i, \lambda_i, \chi_i)\}_{i\in\Lambda}$ of tuples indexed by a finite set Λ such that we have the following properties.

1. For each $i \in \Lambda$, $U_i \subset Z$ is an open subset, $\ell_i \geqslant N+1$ is an integer and $D_i \subset X$ is a codimension 2 submanifold-with-boundary satisfying the following properties for any $[u, C] \in U_i$.

   (i) The map u is transverse to D_i with $u(C) \cap \partial D_i = \varnothing$.

   (ii) $u^{-1}(D_i)$ consists of exactly ℓ_i distinct non-nodal points of C.

   (iii) The curve C equipped with the marked points $u^{-1}(D_i)$ is stable.

   As a result, there is a well-defined map $\mathrm{st}_{\ell_i,D_i} : \pi_Z^{-1}(U_i) \to \overline{\mathcal{M}}^{*}_{g,[\ell_i]}(\mathbb{P}^N, m)$ given by including the intersections of a stable map with D_i as ℓ_i unordered marked points. Set $\widetilde{U}_i := \mathrm{st}^{-1}_{\ell_i,D_i}(\overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell_i]}(\mathbb{P}^N, m))$.

2. For each $i \in \Lambda$, $\lambda_i : \overline{\mathcal{M}}^{*,\mathrm{st}}_{g,[\ell_i]}(\mathbb{P}^N, m) \to \mathcal{G}/G$ is a smooth $\mathcal{G}$-equivariant map.

3. For each $i \in \Lambda$, $\chi_i : Z \to [0,1]$ is a nonzero sc-smooth function supported in U_i. Moreover, for every point $[u, C]$ in $\overline{\mathcal{M}}_g(X, A; J)$, there exists an index $i \in \Lambda$ with $[u, C] \in \pi_Z(\widetilde{U}_i)$ and $\chi_i[u, C] > 0$.

Remark 2.2.13 (Existence of good coverings). By [Par16, Lemma 9.2.7], we can find U, ℓ, D with properties 1(i)–(iii) in Definition 2.2.12 near any point $[u, C] \in \overline{\mathcal{M}}_g(X, A : J)$. Now, using Corollary 2.2.10, the existence of sc-smooth cut-off functions on Z and the compactness of $\overline{\mathcal{M}}_g(X, A; J)$, we deduce that good coverings exist.

With these preparations in place, we fix a good covering $\mathcal{U}$ as above and finally construct the associated map $\lambda_{\mathcal{U}}$. Let $V^J_{A,g} \subset Z$ be the open subset where the sc-smooth function $\sum_{i\in\Lambda} \chi_i$ is strictly positive. Now, define the function

$$\lambda_{\mathcal{U}} := c_\Lambda(\{\chi_i \circ \pi_Z\}_{i\in\Lambda}, \{\lambda_i \circ \mathrm{st}_{\ell_i, D_i}\}_{i\in\Lambda}) : \pi_Z^{-1}(V_{A,g}^J) \to \mathcal{G}/G \tag{2.2.2.10}$$

where we are using the notation of Lemma 2.2.6. It is easy to deduce from Lemma 2.2.6 that $\lambda_{\mathcal{U}}$ is indeed a $\mathcal{G}$-equivariant map.

2.2.3 Choosing the integer k

At the end of this subsection, we will show how to choose the integer k. We begin by investigating the cohomology of restrictions of $\mathcal{O}_{\mathbb{P}^N}(k)$ to embedded curves in $\mathbb{P}^N$.

Lemma 2.2.14 (Extension of sections)**.** *There exists a positive integer k_0 with the following property. For any $C \subset \mathbb{P}^N$ corresponding to a point of $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and any integer $k' \geqslant k_0$, the restriction map*

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k')) \to H^0(C, \mathcal{O}_C(k')) \tag{2.2.3.1}$$

is surjective.

Proof. Using the short exact sequence

$$0 \to \mathcal{I}_{C/\mathbb{P}^N} \to \mathcal{O}_{\mathbb{P}^N} \to \mathcal{O}_C \to 0 \tag{2.2.3.2}$$

of coherent sheaves on $\mathbb{P}^N$, where $\mathcal{I}_{C/\mathbb{P}^N}$ is the ideal sheaf of $C \subset \mathbb{P}^N$, it suffices to show that we have $H^1(\mathbb{P}^N, \mathcal{I}_{C/\mathbb{P}^N}(k')) = 0$ for any $C \subset \mathbb{P}^N$ in $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and any $k' \geqslant k_0$ for some uniform constant k_0.

To this end, let us define the function $k' : \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \to \mathbb{Z}_{\geqslant m}$ which assigns to any $C \subset \mathbb{P}^N$ the smallest $k' \geqslant m$ for which $H^1(C, \mathcal{I}_{C/\mathbb{P}^N}(k')) = 0$. This function is well-defined by Serre's vanishing theorem [Har77, Theorem III.5.2]. By [Har77, Theorem III.12.8], it is upper semicontinuous with respect to the Zariski topology on $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$. Being an algebraic scheme of finite type, $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ is quasi-compact in the Zariski topology. Thus, the function k' achieves a maximum k_0 which has the desired property by Lemma 2.2.15 below. $\square$

Lemma 2.2.15. *Let $C \subset \mathbb{P}^N$ be an embedded prestable genus g curve of degree m with ideal sheaf $\mathcal{I}_{C/\mathbb{P}^N}$. Then, the function given by $k' \mapsto \dim H^1(\mathbb{P}^N, \mathcal{I}_{C/\mathbb{P}^N}(k'))$ is monotonically decreasing for $k' \geqslant m$.*

Proof. After performing a generic linear change of coordinates, we may assume that the linear hyperplane $Y = \mathbb{P}^{N-1} \subset \mathbb{P}^N$ (defined by the vanishing of the homogeneous coordinate x_0) meets C in a set F of m distinct non-singular points. Using local analytic equations of C and Y at their (transverse) intersection points, it is easy to verify that we obtain a short exact sequence

$$0 \to \mathcal{I}_{C/\mathbb{P}^N}(-Y) \xrightarrow{x_0} \mathcal{I}_{C/\mathbb{P}^N} \xrightarrow{\mathrm{res}} \mathcal{I}_{F/Y} \to 0 \tag{2.2.3.3}$$

of coherent sheaves on $\mathbb{P}^N$ (with res denoting the natural restriction map from the ideal sheaf of $C \subset \mathbb{P}^N$ to the ideal sheaf of $F \subset Y$). Tensor (2.2.3.3) with $\mathcal{O}_{\mathbb{P}^N}(Y) \otimes \mathcal{O}_{\mathbb{P}^N}(k') = \mathcal{O}_{\mathbb{P}^N}(k'+1)$ for some $k' \geqslant m$ and observe that the long exact sequence in cohomology contains

$$H^1(\mathbb{P}^N, \mathcal{I}_{C/\mathbb{P}^N}(k')) \to H^1(\mathbb{P}^N, \mathcal{I}_{C/\mathbb{P}^N}(k'+1)) \to H^1(Y, \mathcal{I}_{F/Y}(k'+1)). \tag{2.2.3.4}$$

To obtain the desired inequality, it thus suffices to show that $H^1(Y, \mathcal{I}_{F/Y}(k'+1)) = 0$. To this end, consider the tautological short exact sequence

$$0 \to \mathcal{I}_{F/Y} \to \mathcal{O}_Y \to \mathcal{O}_F \to 0 \tag{2.2.3.5}$$

on Y. Tensor this with $\mathcal{O}_Y(k'+1)$ and observe that the long exact sequence in cohomology contains

$$\begin{aligned} H^0(Y, \mathcal{O}_Y(k'+1)) \xrightarrow{\text{res}} H^0(F, \mathcal{O}_F(k'+1)) \to H^1(Y, \mathcal{I}_{F/Y}(k'+1)) \\ \to H^1(Y, \mathcal{O}_Y(k'+1)). \end{aligned} \tag{2.2.3.6}$$

Now, $H^1(Y, \mathcal{O}_Y(k'+1)) = 0$ by [Har77, Theorem III.5.1(b)] while the map res in (2.2.3.6) is surjective since $k' \geqslant m$. This allows us to conclude that $H^1(Y, \mathcal{I}_{F/Y}(k'+1)) = 0$ as desired. $\square$

Lemma 2.2.16. *Let $\hat{\sigma}_0, \ldots, \hat{\sigma}_N$ be an orthonormal basis of $\mathbb{C}^{N+1}$ with respect to the standard Hermitian inner product and let $\sigma_0, \ldots, \sigma_N$ be the corresponding elements of $H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$ under the obvious identification $\mathbb{C}^{N+1} = H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. Then, using the standard Hermitian metric on the line bundle $\mathcal{O}_{\mathbb{P}^N}(1)$, we have the identity*

$$\sum_{j=0}^{N} |\sigma_j(x)|^2 = 1$$

for all points $x \in \mathbb{P}^N$.

Proof. Let $x \in \mathbb{P}^N$ be given. Choose a unit vector $\hat{x} \in \mathbb{C}^{N+1}$ spanning the complex line corresponding to x. By the definition of the standard Hermitian metric on $\mathcal{O}_{\mathbb{P}^N}(1)$, we have $|\sigma_j(x)| = |\langle \hat{x}, \hat{\sigma}_j \rangle|$ for each $0 \leqslant j \leqslant N$ (where we are using the standard Hermitian inner product on $\mathbb{C}^{N+1}$ on the right side). It follows from the orthonormality of $\{\hat{\sigma}_j\}_{j=0}^N$ in $\mathbb{C}^{N+1}$, that we can write $\hat{x} = \sum_{j=0}^N \langle \hat{x}, \hat{\sigma}_j \rangle \hat{\sigma}_j$. Applying Pythagoras' theorem to this decomposition of the unit vector $\hat{x}$ now gives the desired result. $\square$

Lemma 2.2.17 (Cohomology vanishing I)**.** *There exists a positive integer k_1 with the following property. For any point $[u, C]$ in $\overline{\mathcal{M}}_g(X, A; J)$, any complex basis $\mathcal{F} = (s_0, \ldots, s_N)$ of $H^0(C, \mathfrak{L}_u^{\otimes p})$ and any integer $k' \geqslant k_1$, we have*

$$H^1(C, \iota_{C,\mathcal{F}}^*(T^{*0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k')) \otimes u^* T_X) = 0 \tag{2.2.3.7}$$

where $\iota_{C,\mathcal{F}} : C \hookrightarrow \mathbb{P}^N$ is the embedding appearing in Lemma 2.2.3.

Proof. Observe that we may replace ${T^*}^{0,1}_{\mathbb{P}^N}$ by $T_{\mathbb{P}^N}$ because of the definition of the holomorphic structure on the former bundle. This has the effect of making (2.2.3.7) manifestly independent of the particular choice of basis $\mathcal{F}$ and associated embedding $\iota_{C,\mathcal{F}}$. Thus, for each point $\hat{u} = [u, C]$, we can define the positive integer $k'(\hat{u})$ to be the smallest $k' \geqslant 1$ for which (2.2.3.7) holds. The existence of such a k', i.e. the finiteness of $k'(\hat{u})$ follows from Serre's vanishing theorem [Har77, Theorem III.5.2]. Since the vanishing of H^1 is an open condition, it follows that $k' : \overline{\mathcal{M}}_g(X, A; J) \to \mathbb{Z}_{\geqslant 1}$ is upper semicontinuous and thus, attains a maximum k_1 on the compact space $\overline{\mathcal{M}}_g(X, A; J)$.

To show that k_1 defined this way satisfies the desired property, it will suffice to show that if (2.2.3.7) holds at $[u, C]$ for some k', then it does so even when k' is replaced by a larger integer. Suppose to the contrary that we have $H^1(C, \iota^*_{C,\mathcal{F}}(T_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k' + a)) \otimes u^*T_X) \neq 0$ for some $a \geqslant 1$. By Serre duality [Har77, Theorem III.7.6] on C, we get a nonzero holomorphic section σ of $(\iota^*_{C,\mathcal{F}}(T_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k' + a)) \otimes u^*T_X)^* \otimes \omega_C$, where ω_C denotes the dualizing line bundle of the curve C. Taking a suitable holomorphic section s of $\mathcal{O}_{\mathbb{P}^N}(1)$, we deduce that $\sigma \otimes s^{\otimes a}$ is a nonzero holomorphic section of $(\iota^*_{C,\mathcal{F}}(T_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k')) \otimes u^*T_X)^* \otimes \omega_C$. Applying Serre duality again, we see that this contradicts (2.2.3.7). ☐

Lemma 2.2.18 (Cohomology vanishing II)**.** *There exists a positive integer k_2 with the following property. For any point $[u, C]$ in $\overline{\mathcal{M}}_g(X, A; J)$, any integer $k' \geqslant k_2$ and any complex basis $\mathcal{F}$ of $H^0(C, \mathfrak{L}_u^{\otimes p})$ such that $\lambda_{\mathcal{U}}$ maps the associated framed stable map $(C \subset \mathbb{P}^N, u)$ to the identity coset in $\mathcal{G}/G$, the map*

$$\langle \cdot \rangle \circ d\iota_{\tilde{C},\mathcal{F}} : H^0(C, \iota^*_{C,\mathcal{F}}{T^*}^{0,1}_{\mathbb{P}^N} \otimes u^*T_X \otimes \iota^*_{C,\mathcal{F}}\mathcal{O}_{\mathbb{P}^N}(k')) \otimes \overline{H^0(C, \iota^*_{C,\mathcal{F}}\mathcal{O}_{\mathbb{P}^N}(k'))} \to \Omega^{0,1}(\tilde{C}, \tilde{u}^*T_X) \quad (2.2.3.8)$$

has image spanning the cokernel of the linearized Cauchy–Riemann operator $D(\bar{\partial}_J)_u$.

Proof. Let $\mathcal{M} \subset \tilde{Z}$ (see Definition 2.2.11) be the subset consisting of framed J-holomorphic stable maps $(C \subset \mathbb{P}^N, u)$ which are mapped by $\lambda_{\mathcal{U}}$ to the identity coset in $\mathcal{G}/G$ and satisfy $\mathcal{O}_C(1) \simeq \mathfrak{L}_u^{\otimes p}$. Since G is compact, $\lambda_{\mathcal{U}}$ is $\mathcal{G}$-equivariant and $\overline{\mathcal{M}}_g(X, A; J)$ is compact, it follows that $\mathcal{M}$ is compact. For each point $\hat{u} = (C \subset \mathbb{P}^N, u)$ in $\mathcal{M}$, define $k'(\hat{u})$ to be the smallest positive integer k' for which the map $D(\bar{\partial}_J)_u \oplus (\langle \cdot \rangle \circ d\iota_{\tilde{C},\mathcal{F}})$ is surjective (the existence of such a k', i.e., the finiteness of $k'(\hat{u})$, follows from [AMS21, Lemma 6.24 and Proposition 6.26]). Since surjectivity is an open condition, it follows that the function $k' : \mathcal{M} \to \mathbb{Z}_{\geqslant 1}$ is upper semicontinuous and thus, it attains a maximum on the compact space $\mathcal{M}$.

Define $k_2 = \sup_{\hat{u} \in \mathcal{M}} k'(\hat{u})$. To see that k_2 defined this way satisfies the desired property, it will suffice to show that if (2.2.3.8) maps onto the cokernel of $D(\bar{\partial}_J)_u$ for some k', then it does so even when k' is replaced by a larger integer. But this is an easy consequence of Lemma 2.2.16. Indeed, for any element $f \otimes \overline{g}$ in the domain of (2.2.3.8), we may put $f_j := f \otimes \iota^*_{C,\mathcal{F}}\sigma_j$ and $g_j := g \otimes \iota^*_{C,\mathcal{F}}\sigma_j$ (with $\sigma_0, \ldots, \sigma_N$ as in Lemma 2.2.16) to get $\langle \sum_{j=0}^N f_j \otimes \overline{g_j} \rangle = \sum_{j=0}^N |\iota^*_{C,\mathcal{F}}\sigma_j|^2 \langle f \otimes \overline{g} \rangle = \langle f \otimes \overline{g} \rangle$. But now $\sum_{i=0}^N f_j \otimes \overline{g_j}$ lies in the domain of (2.2.3.8) (but with k' replaced by $k' + 1$) and has the same image as $f \otimes \overline{g}$. ☐

We now choose k to be the maximum of k_0, k_1, k_2.

Remark 2.2.19. Observe that our choice of k is such that condition (b) of Definition 2.1.13 is satisfied. More precisely, we get condition (b1) from Lemma 2.2.17 while condition (b2) follows from Lemmas 2.2.14 and 2.2.18.

2.2.4 Completing the proof of Theorem 2.1.18(1)

The preceding subsections show how to choose an unobstructed auxiliary datum. More precisely, it is always possible to find a complex linear connection ∇^X on T_X and a polarisation $\mathcal{O}_X(1) \to X$, the latter following from Lemma 2.2.1. Given $\nabla^X, \mathcal{O}_X(1)$, §2.2.1, §2.2.2 and §2.2.3 respectively show how to choose $p, \mathcal{U}$ and k while Remarks 2.2.4 and 2.2.19 respectively show that conditions (a) and (b) of Definition 2.1.13 are satisfied for these choices. This completes the proof of Theorem 2.1.18(1).

2.3 Global Kuranishi chart

In this section, we will prove Theorem 2.1.18(2). Let $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ be an unobstructed auxiliary datum. The case with n marked points is a formal consequence of the case without marked points and so, we shall focus on the latter. We prove the (relative) smoothness of the thickening, obstruction bundle, group action and obstruction section in the subsections below. Except for the (relative) smoothness of the obstruction section, treated in Lemma 2.3.16, these are all direct consequences of unobstructedness and some basic results from [Swa21].

Remark 2.3.1 (Regular loci). To keep the notation readable, we use the same notation for the thickening $\mathcal{T}$ and the open locus $\mathcal{T}^{\text{reg}} \subset \mathcal{T}$ where it is cut-out transversely (and similarly for $\mathcal{E}$). Thus, all the statements made in §2.3 are to be interpreted as being valid only over a sufficiently small G-invariant open neighborhood of the zero locus $\mathfrak{s}^{-1}(0) \subset \mathcal{T}$.

2.3.1 Thickening

In this subsection, we consider the space $\mathcal{T}'$ of tuples $(C \subset \mathbb{P}^N, u : C \to X, \eta)$ satisfying conditions (ia) and (ib) of Construction 2.1.14. Specifically, we endow $\mathcal{T}'$ with a natural structure of a rel–C^∞ manifold over the algebraic base space $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ from Definition 2.1.4. Observing that the projection $\mathcal{T} \to \mathcal{T}'$ is a local homeomorphism, this will define a natural rel–C^∞ structure on $\mathcal{T}$ as well. To this end, let us first introduce rel–C^∞ manifolds and their morphisms. A detailed exposition can be found in [Swa21].

Definition 2.3.2. Given a topological space S, an *S-space* $Y/S = (Y, p)$ is a topological space Y equipped with a map $p\colon Y \to S$. An *S-chart* (φ, U) *of dimension* n consists of an open subset $U \subset Y$ so that $p(U)$ is open and an open embedding $\varphi\colon U \to p(U) \times \mathbb{R}^n$ with $\mathrm{pr}_1\varphi = p|_U$.

Definition 2.3.3. Let S and S' be topological spaces and $n, k, m \geqslant 0$. Suppose $U \subset S$ and $A \subset \mathbb{R}^n_k$ are open subsets. A continuous map $\varphi\colon U \times A \to S' \times \mathbb{R}^m$ is *of class rel–C^ℓ* for $\ell \in \mathbb{N} \cup \{\infty\}$ if

- $\varphi_1 := \mathrm{pr}_2\varphi\colon U \times A \to S'$ is continuous and only depends on the first variable,
- $\varphi_2(s, \cdot) \in C^\ell(A, \mathbb{R}^m)$ for each $s \in U$,
- the induced map $U \to C^\ell(A, \mathbb{R}^m)$ is continuous with respect to the C^ℓ-topology.

If $V \subset S \times \mathbb{R}^n_k$ is an open subset, we say $\varphi\colon V \to S \times \mathbb{R}^m$ is *of class rel–C^ℓ* if it is locally of class rel–C^ℓ. The *relative derivative* of φ is given by $d\varphi(s,x)v = (\varphi_1(s), d\varphi_2(s,x)v)$.

We call two charts for X/S compatible if the transition function is of class rel–C^∞.

Definition 2.3.4. A *rel–C^∞ manifold with corners over S* is an S-space X/S equipped with a maximal atlas of S-charts with corners which are pairwise rel–C^∞ compatible.

Definition 2.3.5. A *morphism of rel–C^∞-manifolds* is a pair $(F, f)\colon X/S \to X'/S'$ where $f\colon S \to S'$ is continuous, $p'F = fp$, and F is of class rel–C^∞ in local coordinates.

The compposition of two such morphisms is the obvious one.

Remark 2.3.6. While a rel–C^∞ manifold X/S (of dimension n) might not have a tangent bundle, it always has a *relative tangent bundle* $T_{X/S}$ (of rank n) which is defined by letting it be $\varphi^*\mathbb{R}^n$ over a local chart (U, φ). A *relative submersion* is a morphism (F, f) whose relative derivative $T_{X/S} \to F^*T_{X'/S'}$ is surjective.

Now, to realize $\mathcal{T}'$ over $\overline{\mathcal{M}}^*_g(\mathbb{P}^N, m)$ as a rel–C^∞ manifold, we will use the existence result from [Swa21] after recasting $\mathcal{T}'$ as a holomorphic curve moduli space using Gromov's shearing trick as in [AMS21].

Define a complex vector bundle over $X \times \mathbb{P}^N$ by

$$E = T_X \boxtimes (T^{*0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))}). \tag{2.3.1.1}$$

Using the evident connections on all of the bundles involved, we obtain a splitting of the tangent bundle of E into the vertical E-direction and the horizontal X- and $\mathbb{P}^N$-directions. At any point $(x, y, \eta) \in E$, with $x \in X$, $y \in \mathbb{P}^N$ and $\eta \in E_{(x,y)}$, use this splitting to define the endomorphism

$$\tilde{J}_{(x,y,\eta)}(v, w, \zeta) := (J_x v + \langle \eta \rangle w, J^{\mathrm{std}}_y w, J^{E_{(x,y)}}\zeta), \tag{2.3.1.2}$$

where J_x is the almost complex structure on X at x, J^{std}_y is the standard complex structure of $\mathbb{P}^N$ at y, $J^{E_{(x,y)}}$ denotes multiplication by i on the fibres of E and $\langle \cdot \rangle$ denotes the evident inner product pairing $\mathcal{O}_{\mathbb{P}^N}(k) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))} \to \mathbb{C}$. This defines an almost complex structure $\tilde{J}$ on E.

It is easy to verify that the projection $E \to X \times \mathbb{P}^N$ is pseudo-holomorphic, with the latter carrying the product almost complex structure. Clearly, a smooth map $C \to E$ is $\tilde{J}$-holomorphic if and only if the corresponding map $\iota : C \to \mathbb{P}^N$ is holomorphic and the corresponding map $u : C \to X$ and element

$$\eta \in H^0(C, u^*T_X \otimes \iota^*(T^{*0,1}_{\mathbb{P}^N} \otimes \mathcal{O}_{\mathbb{P}^N}(k))) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))} \tag{2.3.1.3}$$

satisfy $\bar{\partial}_J u + \langle \eta \rangle|_C = 0$. Thus, $\mathcal{T}'$ is realized a space of pseudo-holomorphic embedded curves in E, with the map to the base space $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ induced by the natural projection $p_{\mathbb{P}^N} : E \to X \times \mathbb{P}^N \to \mathbb{P}^N$. These curves are of genus g and lie in the class $\tilde{A} := A \times [\mathrm{pt}] + [\mathrm{pt}] \times m[\mathbb{P}^1] \in H_2(X \times \mathbb{P}^N, \mathbb{Z}) = H_2(E, \mathbb{Z})$.

To apply [Swa21, Corollary 3.7], we will realize $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ as the object representing the following sheaf of sets $\mathfrak{F}$ on the category $(C^\infty/\cdot)$ of rel–C^∞ manifolds.

Definition 2.3.7. Given a rel–C^∞ manifold Y/S, we define the set $\mathfrak{F}(Y/S)$ to consist of all commutative diagrams of the form

$$\begin{tikzcd} Y \arrow[d] & \mathcal{C}_Y \arrow[l] \arrow[r, "H"] \arrow[d] & E \arrow[d, "p_{\mathbb{P}^N}"] \\ S & \mathcal{C} \arrow[l, "\pi"'] \arrow[r, "h"] & \mathbb{P}^N \end{tikzcd}$$

where

- $\mathcal{C} \xrightarrow{(h,\pi)} \mathbb{P}^N \times S$ is a family of curves over S arising by pullback along a continuous map $S \to \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and the square on the left is cartesian (and so, $\mathcal{C}_Y/\mathcal{C}$ acquires a natural rel–C^∞ structure),
- $(H, h) : \mathcal{C}_Y/\mathcal{C} \to E/\mathbb{P}^N$ is of class rel–C^∞,
- for each $y \in Y$ with image $s \in S$, the restriction $H|_y : \pi^{-1}(s) \to E$ of the map H to the fibre of $\mathcal{C}_Y/Y$ over y is a transversely cut-out pseudo-holomorphic stable embedding into E of genus g, class $\tilde{A}$, with the property that the projection map from the kernel of the linearized operator of $H|_y : \pi^{-1}(s) \to E$ to the kernel of the linearized operator of $p_{\mathbb{P}^N} \circ (H|_y) = h|_s : \pi^{-1}(s) \to \mathbb{P}^N$ is surjective.

For rel–C^∞ morphisms $Y'/S' \to Y/S$, the associated functorial maps $\mathfrak{F}(Y/S) \to \mathfrak{F}(Y'/S')$ are given by pullbacks of such diagrams.

To show that the sheaf $\mathfrak{F}$ is representable, we need the following simple observation which was not explicitly stated in [Swa21].

Lemma 2.3.8 (Rel–C^∞ submersions)**.** *Suppose $Y'/S \to Y/S$ is a rel–C^∞ submersion of rel–C^∞ manifolds. Then, Y'/Y is naturally a rel–C^∞ manifold and is given by the categorical fibre product*

$$Y'/Y = (Y'/S) \times_{(Y/S)} (Y/Y) \tag{2.3.1.4}$$

in the category $(C^\infty/\cdot)$ of rel–C^∞ manifolds.

Proof. Straightforward. □

Proposition 2.3.9. *The sheaf $\mathfrak{F}$ is representable by a rel–C^∞ structure on an open subset of transversely cut-out points in $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$. This open subset contains the zero locus of the obstruction section $\mathfrak{s}$.*

Proof. By [Swa21, Proposition 2.16], the representability of $\mathfrak{F}$ is a local question on both $\mathcal{T}'$ and $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$. Near a given $\tilde{p} \in \mathcal{T}'$ (and the corresponding point $p \in \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$), defining an element of $\mathfrak{F}(\mathrm{pt}/\mathrm{pt})$, use the construction of [Swa21, §6.3] to find a family $\mathfrak{C}/\mathfrak{M}$ of stable pointed genus g curves, such that we can realize $\mathcal{T}'$ (resp. $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$) near $\tilde{p}$ (resp. p) as a space of pseudo-holomorphic stable maps from the fibres of $\mathfrak{C}/\mathfrak{M}$ to E (resp. $\mathbb{P}^N$) with suitable divisor constraints imposed on the added marked points. For more details, see [Swa21, Proposition 6.8].

Using the functorial description of the space of pseudo-holomorphic maps from fibres of $\mathfrak{C}/\mathfrak{M}$ given in [Swa21, Definitions 3.1, 3.2, 3.3] and applying [Swa21, Corollary 3.7] to this functor now gives canonical rel–C^∞ structures on $\mathcal{T}'/\mathfrak{M}$ near $\tilde{p}$ (resp. $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)/\mathfrak{M}$ near p) and inspection of (vertical) tangent spaces shows that the natural projection $\mathcal{T}'/\mathfrak{M} \to \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)/\mathfrak{M}$ is a rel–C^∞ submersion near $\tilde{p}$ (resp. p). Lemma 2.3.8 now implies that we get a canonical rel–C^∞ structure on $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ near $(\tilde{p}, p)$. Using the explicit categorical fibre product description of this rel–C^∞ structure (given in Lemma 2.3.8), we check that this rel–C^∞ structure indeed represents the functor $\mathfrak{F}$ of Definition 2.3.7.

The final assertion about the zero locus of $\mathfrak{s}$ follows from the fact that the auxiliary datum fixed in the beginning of this section was unobstructed. □

To complete the discussion of the thickening, we note that the natural projection $\mathcal{T} \to \mathcal{T}'$ has discrete fibres and is a local homeomorphism since, for any prestable curve C of genus g, the exponential map $H^1(C, \mathcal{O}_C) \to \mathrm{Pic}^0(C)$ has this property. We use this to endow $\mathcal{T}/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ with a natural rel–C^∞ structure.

Remark 2.3.10. Proposition 2.3.9 and the preceding paragraph complete the proof of Theorem 2.1.18(2a).

2.3.2 Obstruction bundle

Recall from Construction 2.1.14 that the obstruction bundle $\mathcal{E}$ has three summands. The first summand is a trivial bundle with fibre $\mathfrak{su}(N+1)$ and therefore has a natural rel–C^∞ vector bundle structure. The third summand acquires a natural rel–C^∞ structure as it is the pullback to $\mathcal{T}$ of an algebraic vector bundle on $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ by the next lemma.

Lemma 2.3.11 ($R^1\pi_*\mathcal{O}$ has rank g)**.** *$\mathbb{E}^* := R^1\pi_*\mathcal{O}_{\mathcal{C}_g}$ is a rank g algebraic vector bundle on $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ dual to the Hodge bundle $\mathbb{E} := \pi_*\omega_\pi$ where ω_π is the relative dualizing line bundle of $\pi : \mathcal{C}_g \to \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$. The fibre of the bundle $\mathbb{E}^*$ at a point $C \subset \mathbb{P}^N$ is naturally identified with $H^1(C, \mathcal{O}_C)$.*

Proof. The fact that $\mathbb{E}^*$ defines a locally free sheaf follows from the theorem on cohomology and base change [Har77, Theorem III.12.11]. The assertion that it is dual to the Hodge bundle is a consequence of Serre duality [Har77, Theorem III.7.6] for the case of curves. □

It remains to consider the second summand of the obstruction bundle. This summand is actually pulled back under the projection map $\mathcal{T} \to \mathcal{T}'$ (recall the definition of $\mathcal{T}'$ from the previous subsection) and its fibre over $(C \subset \mathbb{P}^N, u : C \to X, \eta)$ is given by the vector space $E_{(C\subset\mathbb{P}^N,u)}$ of (2.1.0.4). As the auxiliary datum fixed in the beginning of §2.3 is unobstructed, we may apply [Swa21, Theorem 5.18] to endow this with the structure of a rel–C^∞ vector bundle.

Remark 2.3.12. We have now completed the proof of the first half of Theorem 2.1.18(2b).

Remark 2.3.13 (Stable almost complex structure). To establish the assertion of Theorem 2.1.18(2e), we begin by observing that the bundle $\mathcal{E} \oplus \underline{\mathfrak{g}}$ is already a complex vector bundle. Indeed, the second and third summand in (2.1.0.8) are complex vector spaces and we have an obvious identification $\mathfrak{g} \oplus \mathfrak{su}(N+1) = \mathfrak{sl}(N+1,\mathbb{C})$. It remains to find a natural stable complex structure on the vertical tangent bundle $T_{\mathcal{T}/\overline{\mathcal{M}}_g^*(\mathbb{P}^N,m)}$. This comes from the identification of the vertical tangent bundle with bundle of kernels of the associated (surjective) linearized real Cauchy–Riemann type operators. This is stably equivalent to the index (virtual) bundle of the corresponding complex Cauchy–Riemann type operators and the latter has an obvious stable complex structure. For more details, see the orientation argument in the proof of [MS12, Theorem 3.1.6(i)].

2.3.3 Group action

Since the rel–C^∞ structure on $\mathcal{T}/\overline{\mathcal{M}}_g^*(\mathbb{P}^N,m)$ is pulled back from $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N,m)$, the G-action defines a rel–C^∞ map

$$(G\times\mathcal{T})/(G\times\overline{\mathcal{M}}_g^*(\mathbb{P}^N,m)) \to \mathcal{T}/\overline{\mathcal{M}}_g^*(\mathbb{P}^N,m) \tag{2.3.3.1}$$

if the corresponding map with $\mathcal{T}$ replaced by $\mathcal{T}'$ is of class rel–C^∞. But this is clear once we observe that the corresponding natural transformation at the level of the functor $\mathfrak{F}$ (from Definition 2.3.7) is well-defined and is therefore represented by a rel–C^∞ map by the Yoneda lemma. The next lemma now establishes the fibrewise local linearity (in the sense of [AMS21, Definition 4.20]) of the G-action.

Lemma 2.3.14. *Let V be a finite dimensional vector space, $\pi : M \to V$ a rel–C^∞ manifold and Γ a finite group. Assume that we are given a rel–C^∞ action of Γ on M/V which covers a linear Γ-representation θ on V. Let $x \in \pi^{-1}(0)$ be fixed by Γ. Then, M/V has a rel–C^∞ chart at x in which the Γ-action is linear.*

Proof. By shrinking to a Γ-invariant coordinate neighborhood of $x \in M$, we may assume that M is an open subset of a product $V\times W$, with W another finite dimensional vector space with $x = (0,0)$. The action of any $g \in \Gamma$ is given, in these coordinates, by a rel–C^∞ map $(v,w) \mapsto (\theta(g)v, \varphi_g(v,w))$ with

$$\varphi_g(\theta(h)v, \varphi_h(v,w)) \equiv \varphi_{gh}(v,w). \tag{2.3.3.2}$$

Define the representation $\rho : \Gamma \to GL(W)$ by $\rho(g) = \frac{\partial\varphi_g}{\partial w}(0,0)$. Define the map $(v,w) \mapsto (v, T(v,w))$ by

$$T(v,w) = \frac{1}{|\Gamma|}\sum_{g\in\Gamma}\rho(g)^{-1}\varphi_g(v,w) \tag{2.3.3.3}$$

and observe that $\frac{\partial T}{\partial w}(0,0)$ is the identity map. Using the implicit function theorem with parameters, [Swa21, Lemma 5.10], it follows that $(v,w) \mapsto (v, T(v,w))$ is a rel–C^∞ coordinate change on M/V near x. A direct check shows that in these new coordinates the Γ-action is given by $\theta \oplus \rho$. □

Remark 2.3.15. We have now completed the proof of the first half of Theorem 2.1.18(2c).

2.3.4 Obstruction section

Recall from Construction 2.1.14 that the obstruction section $\mathfrak{s}$ has three components corresponding to the three summands of the obstruction bundle $\mathcal{E}$. It is immediate from the construction of the rel–C^∞ vector bundle structure on $\mathcal{E}$ and the Yoneda lemma that the second component of $\mathfrak{s}$ is rel–C^∞.

The first component of $\mathfrak{s}$, denoted $\lambda_{\mathcal{U}}(C \subset \mathbb{P}^N, u)$, is an $\mathfrak{su}(N+1)$-valued function which is defined using the good covering $\mathcal{U}$ (recall Definition 2.2.12). To see that this is a rel–C^∞ function on $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ (and therefore on $\mathcal{T}/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$), we need the two following observations. The first is that the formula (2.2.2.10) defining $\lambda_{\mathcal{U}}$ is sc-smooth on the polyfold $\pi_Z^{-1}(V_{A,g}^J)$ since the cutoff functions $\chi_i : Z \to [0,1]$ were chosen to be sc-smooth. The second is that the rel–C^∞ structure on $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$, defined via Proposition 2.3.9, has local rel–C^∞ charts given by (finite dimensional) submanifolds of the polyfold $\tilde{Z}$ of Definition 2.2.11 (for more details on the local charts see [Swa21, §4.3 and §4.4], specifically Lemma 4.12–Theorem 4.16 therein). Putting these two observations together, we find that $\lambda_{\mathcal{U}} : \mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \to \mathfrak{su}(N+1)/\mathrm{pt}$ is rel–C^∞.

We finally turn to the relative smoothness, near $\mathfrak{s}^{-1}(0) \subset \mathcal{T}$, of the third component of $\mathfrak{s}$ which we recall takes values in the (pullback of the) vector bundle $\mathbb{E}^*$ specified by Lemma 2.3.11. The desired assertion will follow from the next lemma once we recall that the projection $\mathcal{T} \to \mathcal{T}'$ is a local homeomorphism.

Lemma 2.3.16. *Consider a transversely cut-out $(C \subset \mathbb{P}^N, u, \eta) \in \mathcal{T}'$ satisfying $\mathcal{O}_C(1) \simeq \mathfrak{L}_u^{\otimes p}$. Then there is a rel–$C^\infty$ section σ, defined on $\mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ near $(C \subset \mathbb{P}^N, u, \eta)$, of the pullback of the bundle $\mathbb{E}^*$ such that we have $\sigma(C \subset \mathbb{P}^N, u, \eta) = 0$ and the identity*

$$[\mathcal{O}_{\hat{C}}(1)] = p \cdot [\mathfrak{L}_{\hat{u}}] + \sigma(\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta}) \tag{2.3.4.1}$$

in $\mathrm{Pic}(\hat{C})$ *holds for all points $(\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta}) \in \mathcal{T}'$ at which σ is defined.*

Proof. Let $\mathcal{C}_g^\circ \subset \mathcal{C}_g$ be the set of smooth points in the fibres of the universal curve $\mathcal{C}_g \to \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$. We claim that it suffices to find an integer $r \geqslant 1$ and rel–C^∞ maps

$$\tau_1, \ldots, \tau_r, \tau_1', \ldots, \tau_r' : \mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \to \mathcal{C}_g^\circ/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m), \tag{2.3.4.2}$$

defined on a neighborhood of $q := (C \subset \mathbb{P}^N, u, \eta)$, such we have $\tau_i(q) = \tau_i'(q)$ for $1 \leqslant i \leqslant r$ and, for all points $\hat{q} = (\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta})$ in this neighborhood, a holomorphic line bundle isomorphism

$$\mathcal{O}_{\hat{C}}(1) \simeq \mathfrak{L}_u^{\otimes p} \otimes \mathcal{O}_{\hat{C}}(D_{\hat{q}}) \tag{2.3.4.3}$$

where $D_{\hat{q}} := \sum_{i=1}^{r} \tau_i(\hat{q}) - \sum_{i=1}^{r} \tau_i'(\hat{q})$. Indeed, in this case, we can define the desired section σ by the explicit formula $\sigma(\hat{q}) := \sum_{i=1}^{r} \rho(\tau_i(\hat{q}), \tau_i'(\hat{q}))$ where ρ denotes the (holomorphic) Abel–Jacobi map

$$\rho : \mathcal{C}_g^\circ \times_{\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)} \mathcal{C}_g^\circ \to \mathbb{E}^* \tag{2.3.4.4}$$

which is defined near the diagonal $\Delta_{\mathcal{C}_g^\circ}$ and has the property that for any $\hat{C} \subset \mathbb{P}^N$ in $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and any two points $x, y \in \hat{C}$, the element $\rho(x, y) \in H^1(\hat{C}, \mathcal{O}_{\hat{C}})$ satisfies

$$[\mathcal{O}_{\hat{C}}(x)] = [\mathcal{O}_{\hat{C}}(y)] + \rho(x, y) \in \operatorname{Pic}(\hat{C}) \tag{2.3.4.5}$$

and $\rho(x, y) = 0$ if $x = y$.

To complete the proof, we will now show how to construct an integer r and maps τ_i, τ_i' as above. We first choose an integer $\ell \gg 1$ such that the (isomorphic) holomorphic line bundles on C

$$L_1 := \mathcal{O}_C(\ell + 1) \otimes (\omega_C^*)^{\otimes p} \tag{2.3.4.6}$$

$$L_2 := \mathcal{O}_C(\ell) \otimes (u^*\mathcal{O}_X(1))^{\otimes 3p} \tag{2.3.4.7}$$

are very ample and have vanishing first cohomology. Let $r := \deg(L_1) = \deg(L_2)$ and s_1 (resp. s_2) be a holomorphic section of L_1 (resp. L_2) such that the sections s_1, s_2 have the same vanishing locus on C consisting of r distinct non-singular points $z_1, \ldots, z_r \in C$.

Let $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \xleftarrow{\pi} \mathcal{C}_g \xrightarrow{F} \mathbb{P}^N$ be the universal map on $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and let ω_π be the relative dualizing line bundle of π. Define the coherent sheaf

$$\mathcal{L}_1 := \pi_*(F^*\mathcal{O}_{\mathbb{P}^N}(\ell + 1) \otimes \omega_\pi^{-\otimes p}) \tag{2.3.4.8}$$

on $\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ and observe, using $H^1(C, L_1) = 0$, that $\mathcal{L}_1$ is locally free near the point $C \subset \mathbb{P}^N$ by the theorem on cohomology and base change [Har77, Theorem III.12.11]. Thus, we can find a local holomorphic section σ_1 of $\mathcal{L}_1$ with $\sigma_1(C \subset \mathbb{P}^N) = s_1 \in H^0(C, \mathcal{O}_C(\ell + 1) \otimes \omega_C^{-\otimes p})$. Using this, for $\hat{q} = (\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta})$ close to $q = (C \subset \mathbb{P}^N, u, \eta)$, we define $\tau_1(\hat{q}), \ldots, \tau_r(\hat{q}) \in \hat{C}$ to be the unique zeros of $\sigma_1(\hat{C} \subset \mathbb{P}^N)$ close to $z_1, \ldots, z_r \in C$. The functions $\tau_1, \ldots, \tau_r$ are obviously rel–C^∞, since they are pullbacks to $\mathcal{T}'$ of continuous (even holomorphic) sections of π.

Finally, we need to construct $\tau_1', \ldots, \tau_r'$. For this, we consider the space $\mathcal{S}'$ parametrizing all tuples $(\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta}, \hat{s}_2)$, where $(\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta}) \in \mathcal{T}_1$ and $\hat{s}_2 \in H^0(\hat{C}, \mathcal{O}_{\hat{C}}(\ell) \otimes (\hat{u}^*\mathcal{O}_X(1))^{\otimes 3p})$. By regarding such tuples as pseudo-holomorphic maps (with respect to a suitable almost complex structure as in §2.3.1) into the total space of the line bundle $\mathcal{O}_X(1)^{\otimes 3p} \boxtimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^N}(\ell) \to X \times \mathbb{P}^N$ pulled back to the space E from (2.3.1.1), we argue exactly as in Proposition 2.3.9 to conclude that, near the point $(C \subset \mathbb{P}^N, u, \eta, s_2)$, the space $\mathcal{S}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ carries a natural rel–C^∞ structure representing a functor analogous to $\mathfrak{F}$. Further, $H^1(C, L_2) = 0$ implies that the natural forgetful map $\mathcal{S}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \to \mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ is a rel–C^∞ submersion near $(C \subset \mathbb{P}^N, u, \eta, s_2)$. This last assertion allows us to produce a rel–C^∞ local section $\sigma_2 : \mathcal{T}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m) \to \mathcal{S}'/\overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ of the forgetful map which is defined near $q = (C \subset \mathbb{P}^N, u, \eta)$ and maps it to $(C \subset \mathbb{P}^N, u, \eta, s_2)$. This local section σ_2 yields, for $\hat{q} = (\hat{C} \subset \mathbb{P}^N, \hat{u}, \hat{\eta}) \in \mathcal{T}'$ sufficiently close to q, a section

$$\sigma_2(\hat{q}) \in H^0(\hat{C}, \mathcal{O}_{\hat{C}}(\ell) \otimes (\hat{u}^*\mathcal{O}_X(1))^{\otimes 3p}) \tag{2.3.4.9}$$

with unique zeros $\tau_1'(\hat{q}), \ldots, \tau_r'(\hat{q}) \in \hat{C}$ close to $z_1, \ldots, z_r \in C$. Using the fact that σ_2 is rel–C^∞ and the holomorphicity of each $\sigma_2(\hat{q})$, it follows that the functions $\tau_1', \ldots, \tau_r'$ are also rel–C^∞. This completes the construction of r, τ_i, τ_i' and concludes the proof. □

Remark 2.3.17. We have now completed the proof of the second half of Theorem 2.1.18(2b).

Remark 2.3.18 (Zeros of the obstruction section). To see that the natural continuous map

$$\mathfrak{s}^{-1}(0)/G \to \overline{\mathcal{M}}_g(X, A; J) \tag{2.3.4.10}$$

is a homeomorphism, and therefore establish Theorem 2.1.18(2d), we argue as follows. Bijectivity follows by noting that $\mathfrak{s}^{-1}(0)$ exactly parametrizes J-holomorphic stable maps (C, u) in $\overline{\mathcal{M}}_g(X, A; J)$ together with a choice of projective embedding $C \subset \mathbb{P}^N$ given by the holomorphic sections of $\mathfrak{L}_u^{\otimes p}$ such that $\lambda_{\mathcal{U}}(C \subset \mathbb{P}^N, u)$ is the identity coset in $\mathcal{G}/G$. Using the $\mathcal{G}$-equivariance of $\lambda_{\mathcal{U}}$ and the fact that the projective embeddings in question correspond to a choice of basis for $H^0(C, \mathfrak{L}_u^{\otimes p})$, it follows that the preimage of each point under the projection $\mathfrak{s}^{-1}(0) \to \overline{\mathcal{M}}_g(X, A; J)$ is a G-orbit. It now follows that $\mathfrak{s}^{-1}(0)$ is compact and thus, the continuous bijection between $\mathfrak{s}^{-1}(0)/G$ and $\overline{\mathcal{M}}_g(X, A; J)$ is a homeomorphism since the latter space is known to be Hausdorff.

Remark 2.3.19 (Finite stabilizers). The stabilizer $\Gamma \subset G$ of any point $(C \subset \mathbb{P}^N, u)$ of $\mathfrak{s}^{-1}(0)$ is naturally identified with the automorphism group of the stable map (C, u) and is therefore finite. This implies that the action of G on $\mathcal{T}$ also has finite stabilizers in a neighborhood of the compact set $\mathfrak{s}^{-1}(0)$. This establishes the second half of Theorem 2.1.18(2c).

2.3.5 Completing the proof of Theorem 2.1.18(2)

The preceding subsections show that if we fix an unobstructed auxiliary datum, then the output of Construction 2.1.14 has all the properties claimed in the statement of Theorem 2.1.18(2) and, in particular, is of class rel–C^∞. Refer to Remarks 2.3.10, 2.3.12, 2.3.13, 2.3.15, 2.3.17, 2.3.18 and 2.3.19. This completes the proof of Theorem 2.1.18(2).

2.4 Uniqueness up to equivalence and cobordism

In this section, we will prove Theorem 2.1.18(3). We begin by formulating the notions of equivalence and cobordism for rel–C^∞ global Kuranishi charts. These are slight variations on the definitions presented in [AMS21, §5.1] that allow us to keep track of the rel–C^∞ structures involved.

Definition 2.4.1 (Rel–C^∞ equivalence). Let $\mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$ be a rel–C^∞ global Kuranishi chart as in Definition 2.1.2. Consider the following moves applied to $\mathcal{K}$.

(i) (Germ equivalence) Given a G-invariant open neighborhood $U \subset \mathcal{T}$ of $\mathfrak{s}^{-1}(0)$, replace $\mathcal{K}$ by $(G, U/\mathcal{M}, \mathcal{E}|_U, \mathfrak{s}|_U)$.

(ii) (Stabilization) Given a rel–C^∞ vector bundle $p : W \to \mathcal{T}/\mathcal{M}$ carrying a compatible rel–C^∞ action of G, replace $\mathcal{K}$ by $(G, W/\mathcal{M}, p^*\mathcal{E} \oplus p^*W, p^*\mathfrak{s} \oplus \Delta_W)$. Here, Δ_W denotes the tautological diagonal section of $p^*W \to W$.

(iii) (Group enlargement) Given a compact Lie group G' and a rel–C^∞ principal G'-bundle $q : P \to \mathcal{T}/\mathcal{M}$ carrying a compatible rel–C^∞ action of G, replace $\mathcal{K}$ by $(G \times G', P/\mathcal{M}, q^*\mathcal{E}, q^*\mathfrak{s})$.

(iv) (Base modification) Given a smooth manifold $\mathcal{M}'$ (equipped with a smooth submersion to $\mathcal{M}$) and a rel–C^∞ submersion $\mathcal{T}/\mathcal{M} \to \mathcal{M}'/\mathcal{M}$ (covering the identity map of $\mathcal{M}$), replace $\mathcal{K}$ by $(G, \mathcal{T}/\mathcal{M}', \mathcal{E}, \mathfrak{s})$.

We say that two rel–C^∞ global Kuranishi charts $\mathcal{K}, \mathcal{K}'$ are *rel–C^∞ equivalent* if there exists a finite sequence of rel–C^∞ global Kuranishi charts $\mathcal{K} = \mathcal{K}_0, \dots, \mathcal{K}_N = \mathcal{K}'$ such that for each $0 \leqslant i < N$, the chart $\mathcal{K}_i$ is obtained from $\mathcal{K}_{i+1}$ (or $\mathcal{K}_{i+1}$ is obtained from $\mathcal{K}_i$) by applying one of the moves (i)–(iv) above. There is an obvious refinement of this notion of equivalence when $\mathcal{K}$, $\mathcal{K}'$ are stably complex (resp. oriented) by allowing only G-equivariant stably complex (resp. oriented) W in (Stabilization) and pseudo-holomorphic (resp. oriented) submersions $\mathcal{M}' \to \mathcal{M}$ in (Base modification).

Remark 2.4.2. The move (Base modification) is not present in [AMS21] since the definition of global Kuranishi charts therein does not make explicit reference to the base space of the thickening.

Definition 2.4.3. Let $\mathcal{K}_0 = (G, \mathcal{T}_0/\mathcal{M}, \mathcal{E}_0, \mathfrak{s}_0)$ and $\mathcal{K}_1 = (G, \mathcal{T}_1/\mathcal{M}, \mathcal{E}_1, \mathfrak{s}_1)$ be rel–C^∞ global Kuranishi charts having the same symmetry group G and base space $\mathcal{M}$. We say that $\mathcal{K}_0$ and $\mathcal{K}_1$ are *rel–C^∞ cobordant* if there exists $\mathcal{K}_{01} = (G, \mathcal{T}_{01}/\mathcal{M}, \mathcal{E}_{01}, \mathfrak{s}_{01})$ with the following properties.

(i) $\mathcal{T}_{01} \to \mathcal{M}$ is a rel–C^∞ manifold-with-boundary with $\partial(\mathcal{T}_{01}/\mathcal{M}) = (\mathcal{T}_0/\mathcal{M}) \sqcup (\mathcal{T}_1/\mathcal{M})$.

(ii) $\mathcal{E}_{01} \to \mathcal{T}_{01}/\mathcal{M}$ is a rel–C^∞ vector bundle which restricts on the boundary to $\mathcal{E}_0 \sqcup \mathcal{E}_1$.

(iii) $\mathfrak{s}_{01}$ is a rel–C^∞ section of $\mathcal{E}_{01}$ with compact zero locus and it restricts on the boundary to $\mathfrak{s}_0 \sqcup \mathfrak{s}_1$.

(iv) There is a rel–C^∞ G-action on $\mathcal{E}_{01} \to \mathcal{T}_{01}/\mathcal{M}$ which makes $\mathfrak{s}_{01}$ a G-equivariant section and is compatible with the given actions on the boundary.

There is an obvious refinement of this notion of cobordism when $\mathcal{K}_0, \mathcal{K}_1$ are stably complex (resp. oriented) by requiring the cobordism to carry compatible stable complex structures (resp. orientations).

We will prove the uniqueness of the global Kuranishi charts of Construction 2.1.14 up to stably complex rel–C^∞ equivalence and cobordism in the subsections below. The case with n marked points is a formal consequence of the case with no marked points and so, we shall focus on the latter.

2.4.1 Equivalence

Fix $J \in \mathcal{J}_\tau(X,\omega)$ and let $(\nabla^{X,i}, \mathcal{O}_{X,i}(1), p_i, \mathcal{U}_i, k_i)$ for $i = 0,1$ be any two choices of unobstructed auxiliary data for $\overline{\mathcal{M}}_g(X,A;J)$ and let $\mathcal{K}_i = (G_i, \mathcal{T}_i/\mathcal{M}_i, \mathcal{E}_i, \mathfrak{s}_i)$ for $i = 0,1$ be the associated rel–C^∞ global Kuranishi charts from Construction 2.1.14. Consider the doubly thickened rel–C^∞ global Kuranishi chart

$$\mathcal{K} = (G_0 \times G_1, \mathcal{T}_{01}/\mathcal{M}_{01}, \mathcal{E}_{01}, \mathfrak{s}_{01}) \tag{2.4.1.1}$$

which is defined as follows.

(i) $\mathcal{M}_{01}$ is the space of embedded algebraic prestable genus g curves $C \subset \mathbb{P}^{N_0} \times \mathbb{P}^{N_1}$ such that applying the coordinate projections $\mathbb{P}^{N_0} \times \mathbb{P}^{N_1} \to \mathbb{P}^{N_i}$ yields a point $\iota_i : C \hookrightarrow \mathbb{P}^{N_i}$ of $\mathcal{M}_i$ for $i = 0,1$. As in Definition 2.2.8, $\mathcal{M}_{01}$ is a smooth quasi-projective variety of the expected dimension and the natural forgetful maps $\mathcal{M}_{01} \to \mathcal{M}_i$ are algebraic submersions.

(ii) $\mathcal{T}_{01}$ consists of tuples $(C \subset \mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, u : C \to X, \eta_0, \alpha_0, \eta_1, \alpha_1)$ satisfying the following properties.

   (a) $C \subset \mathbb{P}^{N_0} \times \mathbb{P}^{N_1}$ lies in $\mathcal{M}_{01}$.

   (b) $(C \hookrightarrow \mathbb{P}^{N_i}, u : C \to X)$ is a framed stable map lying in the domain of $\lambda_{\mathcal{U}_i}$ for $i = 0,1$.

   (c) η_i belongs to the finite dimensional vector space $E^i_{(C \hookrightarrow \mathbb{P}^{N_i}, u)}$ from (2.1.0.4) for $i = 0,1$ and on the normalization $\tilde{C} \to C$, we have the equation

$$\bar{\partial}_J \tilde{u} + \langle \eta_0 \rangle \circ d\tilde{\iota}_0 + \langle \eta_1 \rangle \circ d\tilde{\iota}_1 = 0 \in \Omega^{0,1}(\tilde{C}, \tilde{u}^* T_X) \tag{2.4.1.2}$$

   with $\tilde{\iota}_i$ is the pullback of ι_i along $\tilde{C} \to C$ for $i = 0,1$.

   (d) $\alpha_i \in H^1(C, \mathcal{O}_C)$ satisfies the analogue of (2.1.0.7) for $i = 0,1$.

(iii) The fibre of $\mathcal{E}_{01}$ over a point $(C \subset \mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, u : C \to X, \eta_0, \alpha_0, \eta_1, \alpha_1)$ is given by

$$\bigoplus_{i=0,1} \Big(\mathfrak{su}(N_i + 1) \oplus E^i_{(C \hookrightarrow \mathbb{P}^{N_i}, u)} \oplus H^1(C, \mathcal{O}_C) \Big) \tag{2.4.1.3}$$

   and $\mathfrak{s}_{01} = \mathfrak{s}_0 \oplus \mathfrak{s}_1$ at this point is given by $\mathfrak{s}_i = (\mathrm{i} \log \lambda_{\mathcal{U}_i}(C \hookrightarrow \mathbb{P}^{N_i}, u), \eta_i, \alpha_i)$ for $i = 0,1$. By abuse of notation, we denote the vector bundle summands for $i = 0,1$ from (2.4.1.3) by $\mathcal{E}_i$.

(iv) The group $G_0 \times G_1$ acts on $\mathcal{T}_{01} \to \mathcal{M}_{01}$ and $\mathcal{E}_{01}$ in the evident way.

Arguing as in §2.3, we conclude that $\mathcal{K}$ is a stably complex rel–C^∞ global Kuranishi chart for $\overline{\mathcal{M}}_g(X,A;J)$ and that $\mathcal{T}_{01}, \mathcal{E}_{01}$ and $\mathfrak{s}_{01}$ are actually rel–C^∞ with base $\mathcal{M}_0$ or $\mathcal{M}_1$ (and therefore also with base $\mathcal{M}_{01}$). By symmetry, it suffices to show that $\mathcal{K}_0$ and $\mathcal{K}$ are stably complex rel–C^∞ equivalent. To see this, we first apply (Base modification) $\mathcal{K}$ and the submersion $\mathcal{M}_{01} \to \mathcal{M}_0$ to obtain

$$\mathcal{K}_0' := (G_0 \times G_1, \mathcal{T}_{01}/\mathcal{M}_0, \mathcal{E}_{01}, \mathfrak{s}_{01}). \tag{2.4.1.4}$$

The next observation is the key to showing that $\mathcal{K}_0'$ and $\mathcal{K}_0$ can be related by the moves (Germ equivalence), (Stabilization) and (Group enlargement). The explicit use of (Germ equivalence) will be hidden as we will always work in a sufficiently small $(G_0 \times G_1)$-invariant neighborhood of $\mathfrak{s}_{01}^{-1}(0)$.

Lemma 2.4.4. *At any* $\hat{x} = (C \subset \mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, u : C \to X, 0, 0, 0, 0) \in \mathfrak{s}_{01}^{-1}(0)$, *the vertical linearization*

$$d\mathfrak{s}_1|_{\hat{x}} : T_{\mathcal{T}_{01}/\mathcal{M}_0}|_{\hat{x}} \to \mathcal{E}_1|_{\hat{x}} \tag{2.4.1.5}$$

is surjective.

Proof. Write $d\mathfrak{s}_1|_{\hat{x}} = L_1 \oplus L_2 \oplus L_3$ using the direct sum decomposition of $\mathcal{E}_1|_{\hat{x}}$ from (2.4.1.3). Since the auxiliary datum $(\nabla^{X,0}, \mathcal{O}_{X,0}(1), p_0, \mathcal{U}_0, k_0)$ is unobstructed, it follows that the restriction

$$L_2 : T_{\mathcal{T}_{01}/\mathcal{M}_{01}}|_{\hat{x}} \to E^1_{(C \hookrightarrow \mathbb{P}^{N_1}, u)} \tag{2.4.1.6}$$

is surjective. It therefore suffices to argue that $(L_1 \oplus L_3)|_{\ker L_2}$ is surjective. The projection

$$\ker L_2 \to T_{\mathcal{M}_{01}/\mathcal{M}_0}|_{\hat{x}} = H^0(C, \iota_1^* T_{\mathbb{P}^{N_1}}) \tag{2.4.1.7}$$

has a natural splitting σ corresponding to taking $\eta_1 = 0$, keeping

$$(\iota_0 : C \hookrightarrow \mathbb{P}^{N_0}, u : C \to X, \eta_0, \alpha_0) \tag{2.4.1.8}$$

constant and infinitesimally deforming the embedding $\iota_1 : C \hookrightarrow \mathbb{P}^{N_1}$ (observe that the infinitesimal deformation of α_1 is determined by that of ι_1 and the fact that u is fixed). The description of σ makes it clear that the operator $L_3|_{\ker L_2}$ actually factors through the projection (2.4.1.7). The resulting map

$$\hat{L}_3 : T_{\mathcal{M}_{01}/\mathcal{M}_0}|_{\hat{x}} = H^0(C, \iota_1^* T_{\mathbb{P}^{N_1}}) \to H^1(C, \mathcal{O}_C) \tag{2.4.1.9}$$

is identified with the connecting map of the long exact sequence in cohomology obtained by pulling back the Euler exact sequence $0 \to \mathcal{O}_{\mathbb{P}^{N_1}} \to \mathcal{O}_{\mathbb{P}^{N_1}}(1)^{N_1+1} \to T_{\mathbb{P}^{N_1}} \to 0$ along ι_1. This shows that (2.4.1.9) is surjective and that $\ker \hat{L}_3$ is identified with $\mathfrak{sl}(N_1 + 1, \mathbb{C})$, corresponding to the infinitesimal action of $PGL(N_1 + 1, \mathbb{C}) = PSL(N_1 + 1, \mathbb{C})$ on $T_{\mathcal{M}_{01}/\mathcal{M}_0}|_{\hat{x}}$. It remains to show that $L_1|_{\ker(L_2 \oplus L_3)}$ is surjective. For this, we use the splitting σ, restricted to $\ker \hat{L}_3$ and observe that $PSL(N_1 + 1, \mathbb{C})$-equivariance of the map $\lambda_{\mathcal{U}_1}$ gives us the desired surjectivity statement. □

From Lemma 2.4.4, it is immediate that $\mathcal{K}_0'' := (G_0 \times G_1, \mathfrak{s}_1^{-1}(0)/\mathcal{M}_0, \mathcal{E}_0, \mathfrak{s}_0)$ is a rel–C^∞ global Kuranishi chart related to $\mathcal{K}_0$ by (Group enlargement). To conclude, we need to show that $\mathcal{K}_0''$ and $\mathcal{K}_0'$ are related by (Stabilization) with the role of W in Definition 2.4.1 played by $\mathcal{E}_1$. This readily follows from Lemma 2.4.4 and a rel–C^∞ tubular neighborhood argument.

Remark 2.4.5. We have now established Theorem 2.1.18(3a).

2.4.2 Cobordism

Fix $J_0, J_1 \in \mathcal{J}_\tau(X, \omega)$. We first connect them by a smooth path $\gamma : [0,1] \to \mathcal{J}_\tau(X,\omega)$ and write $J_t := \gamma(t)$. Choose a smooth family of J_t-linear connections $\nabla^{X,t}$ on T_X. By Lemma 2.2.1, we can find a polarization $\mathcal{O}_X(1)$ with associated symplectic form Ω taming the image of γ. Choose p as in §2.2.1, depending only on g and $d = \langle [\Omega], A\rangle$. Using the compactness of the parametrized moduli space $\overline{\mathcal{M}}_g(X, A; \gamma)$, we can now choose $\mathcal{U}$ and k as in §2.2.2 and §2.2.3 such that, for each $t \in [0,1]$, the auxiliary datum $(\nabla^{X,t}, \mathcal{O}_X(1), p, \mathcal{U}, k)$ is unobstructed for J_t. This yields a stably complex rel–C^∞ global Kuranishi chart $\mathcal{K}_t$ for $\overline{\mathcal{M}}_g(X, A; J_t)$ for each $t \in [0,1]$. Repeating the arguments of §2.3 with the parameter t, we see that the family $\{\mathcal{K}_t\}_{t\in[0,1]}$ fits together to exhibit a stably complex rel–C^∞ cobordism between $\mathcal{K}_0$ and $\mathcal{K}_1$.

Remark 2.4.6. We have now established Theorem 2.1.18(3b).

2.4.3 Completing the proof of Theorem 2.1.18(3)

The preceding subsections show that, for fixed J, different choices of unobstructed auxiliary data lead to global Kuranishi charts related by stably complex rel–C^∞ equivalence (see Remark 2.4.5) and that for different choices of J, it is possible to find auxiliary data for which the resulting global Kuranishi charts are related by stably complex rel–C^∞ cobordism (see Remark 2.4.6). This completes the proof of Theorem 2.1.18(3).

2.5 Product formula for GW invariants

In this section, we prove the product formula for Gromov–Witten invariants (Theorem 2.5.9) as an application of the global Kuranishi chart construction. We begin by briefly recalling the construction of the virtual fundamental class associated to an equivalence class of global Kuranishi charts and the definition of Gromov–Witten invariants coming from Theorem 2.1.18. We then discuss stable map moduli spaces of a product of symplectic manifolds and derive the product formula as a consequence.

2.5.1 Virtual fundamental classes

Let $\mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$ be an oriented rel–C^∞ global Kuranishi chart as in Definition 2.1.2 for a compact Hausdorff space Z. The *virtual dimension* of Z with respect to $\mathcal{K}$ is defined to be

$$\operatorname{vdim}_{\mathcal{K}} Z = \dim \mathcal{T} - \dim G - \operatorname{rank} \mathcal{E} \qquad (2.5.1.1)$$

where all dimensions are understood to be over $\mathbb{R}$. Abbreviating $d = \mathrm{vdim}_{\mathcal{K}} Z$ and $r = \dim \mathcal{T} - \dim G$, the *virtual fundamental class* $[Z]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^d(Z, ; \mathbb{Q})^\vee$ of Z with respect to $\mathcal{K}$ is defined to be the composite

$$\check{H}^d(Z;\mathbb{Q}) \xrightarrow{\mathfrak{s}^*\tau(\mathcal{E}/G)} H^r_c(\mathcal{T}/G;\mathbb{Q}) \xrightarrow{\int_{\mathcal{T}/G}} \mathbb{Q}$$

where the first map is given by

$$\begin{aligned}\check{H}^*(Z;\mathbb{Q}) \cong \varinjlim_{W \supseteq \mathfrak{s}^{-1}(0)/G} \check{H}^*(W;\mathbb{Q}) &\xrightarrow{\mathfrak{s}^*\tau(\mathcal{E}/G)} \varinjlim_{W \supseteq \mathfrak{s}^{-1}(0)/G} \check{H}^*(W, W\backslash \mathfrak{s}^{-1}(0)/G;\mathbb{Q}) \\ &\cong \check{H}^*(\mathcal{T}/G, (\mathcal{T}\backslash \mathfrak{s}^{-1}(0))/G;\mathbb{Q}) \to \check{H}^*_c(\mathcal{T};\mathbb{Q}). \qquad (2.5.1.2)\end{aligned}$$

We take the direct limit over neighbourhoods of $\mathfrak{s}^{-1}(0)/G$ in $\mathcal{T}/G$, while $\tau(\mathcal{E}/G)$ is the Thom class of the orbibundle. The second map is integration over the homology $\mathbb{Q}$-manifold $\mathcal{T}/G$. The number $\mathrm{vdim}_{\mathcal{K}} Z$ and the class $[Z]^{\mathrm{vir}}_{\mathcal{K}}$ are unchanged if we replace $\mathcal{K}$ by an equivalent oriented global Kuranishi chart (see [AMS21, §5] for more details).

2.5.2 Gromov–Witten classes

We now specialize to the case of Gromov–Witten theory. As in §2.1, let (X, ω) be a closed symplectic manifold, $A \in H_2(X, \mathbb{Z})$ and $g, n \geqslant 0$ be integers. For any $J \in \mathcal{J}_\tau(X, \omega)$, we have a natural map

$$\overline{\mathcal{M}}_{g,n}(X, A; J) \xrightarrow{\mathrm{ev}\times\mathrm{st}} X^n \times \overline{\mathcal{M}}_{g,n} \qquad (2.5.2.1)$$

$$(C, x_1, \ldots, x_n, u) \mapsto (u(x_1), \ldots, u(x_n), (C, x_1, \ldots, x_n)^{\mathrm{st}}) \qquad (2.5.2.2)$$

defined on the moduli space of stable J-holomorphic maps given by evaluation at the marked points and stabilization of the domain. We adopt the convention of taking $\overline{\mathcal{M}}_{g,n} = \mathrm{pt}$ when $2g - 2 + n \leqslant 0$. Using §2.5.1, we may define the *virtual fundamental class* of the moduli space

$$[\overline{\mathcal{M}}_{g,n}(X, A; J)]^{\mathrm{vir}} := [\overline{\mathcal{M}}_{g,n}(X, A; J)]^{\mathrm{vir}}_{\mathcal{K}} \in \check{H}^{\mathrm{vdim}}(\overline{\mathcal{M}}_{g,n}(X, A; J); \mathbb{Q})^\vee \qquad (2.5.2.3)$$

by choosing any $\mathcal{K}$ from the set of equivalent global Kuranishi charts provided by Theorem 2.1.18. Define the associated *Gromov–Witten class* to be

$$\mathrm{GW}^{(X,\omega)}_{A,g,n} := (\mathrm{ev} \times \mathrm{st})_*[\overline{\mathcal{M}}_{g,n}(X, A; J)]^{\mathrm{vir}} \in H_{\mathrm{vdim}}(X^n \times \overline{\mathcal{M}}_{g,n}, ; \mathbb{Q}) \qquad (2.5.2.4)$$

where $\mathcal{K}$ is any global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X, A; J)$ provided by Theorem 2.1.18. We are justified in omitting J from the notation for the Gromov–Witten class since the global Kuranishi charts of Theorem 2.1.18 are unique up to equivalence and cobordism. Pairing the Gromov–Witten class with cohomology classes $\alpha_1, \ldots, \alpha_n \in H^*(X, ; \mathbb{Q})$ and $\beta \in H^*(\overline{\mathcal{M}}_{g,n}, ; \mathbb{Q})$ yields *Gromov–Witten invariants*. Specializing to $g = 0$ and $n = 3$, we obtain the *small quantum cohomology ring* $QH^*(X, \omega)$ over the universal Novikov ring Λ_0 as in [MS12, Chapter 11].

2.5.3 Gromov–Witten invariants of a product

Let (X_i, ω_i) be closed symplectic manifolds for $i = 0, 1$ and set $(X, \omega) := (X_0, \omega_0) \times (X_1, \omega_1)$. Given homology classes $A_i \in H_2(X_i, \mathbb{Z})$ for $i = 0, 1$, let $\mathfrak{A} \subset H_2(X, \mathbb{Z})$ denote the set of classes A which project to A_i under the coordinate projection $\mathrm{pr}_i : X \to X_i$ for $i = 0, 1$. Fix $g, n \geqslant 0$ and $J_i \in \mathcal{J}_\tau(X_i, \omega_i)$ and set $J := J_0 \times J_1$. Define the moduli space

$$\overline{\mathcal{M}}_{g,n}(X, \mathfrak{A}; J) = \coprod_{A \in \mathfrak{A}} \overline{\mathcal{M}}_{g,n}(X, A; J). \tag{2.5.3.1}$$

All the non-empty components of this finite disjoint union have the same virtual dimension. Using Theorem 2.1.18 and polarizations $\mathcal{O}_{X_i}(1)$ on X_i taming J_i for $i = 0, 1$, choose a single unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ with $\mathcal{O}_X(1) = \mathcal{O}_{X_0}(1) \boxtimes \mathcal{O}_{X_1}(1)$. We obtain a global Kuranishi chart

$$\mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s}) \tag{2.5.3.2}$$

for the whole of $\overline{\mathcal{M}}_{g,n}(X, \mathfrak{A}; J)$ by taking the disjoint union over $A \in \mathfrak{A}$ of the resulting charts $\mathcal{K}_A$ for $\overline{\mathcal{M}}_{g,n}(X, A; J)$. There is a natural map

$$\Phi : \overline{\mathcal{M}}_{g,n}(X, \mathfrak{A}; J) \to \overline{\mathcal{M}}_{g,n}(X_0, A_0; J_0) \times \overline{\mathcal{M}}_{g,n}(X_1, A_1; J_1). \tag{2.5.3.3}$$

Theorem 2.1.18(1) yields unobstructed auxiliary data $(\nabla^{X_i}, \mathcal{O}_{X_i}(1), p_i, \mathcal{U}_i, k_i)$ for the moduli spaces $\overline{\mathcal{M}}_{g,n}(X_i, A_i; J_i)$. Let $\mathcal{K}_i = (G_i, \mathcal{T}_i/\mathcal{M}_i, \mathcal{E}_i, \mathfrak{s}_i)$ be the associated global Kuranishi charts provided by Theorem 2.1.18(2) for $i = 0, 1$. Recall that $\mathcal{M}_i = \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_i}, m_i)$. Define $\mathcal{N}$ to be the inverse image of $\mathcal{M}_0 \times \mathcal{M}_1$ under the natural morphism

$$\overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, (m_0, m_1)) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_0}, m_0) \times \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_1}, m_1). \tag{2.5.3.4}$$

We let

$$\Psi : \mathcal{N} \to \mathcal{M}_0 \times \mathcal{M}_1 \tag{2.5.3.5}$$

denote the morphism induced by the restriction. It naturally factors through $\mathcal{M}_0 \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{M}_1$.

Lemma 2.5.1. *$\mathcal{N}$ is a smooth quasi-projective scheme of the expected dimension. Moreover, when we have $2g - 2 + n > 0$, the natural morphism $\mathcal{N} \to \mathcal{M}_0 \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{M}_1$ is proper and birational.*

Proof. Let $(C, x_1, \ldots, x_n, u : C \to \mathbb{P}^{N_0} \times \mathbb{P}^{N_1})$ be a point of $\mathcal{N}$ and, for $i = 0, 1$, let $(C_i, x_1^i, \ldots, x_n^i, u_i : C_i \to \mathbb{P}^{N_i})$ be the corresponding point of $\mathcal{M}_i$. Let $\kappa_i : C \to C_i$ be the natural morphism. Any component of C which is contracted by both κ_0 and κ_1 must be a sphere with $\geqslant 3$ special points. Since $\mathrm{Aut}(C_i, x_1^i, \ldots, x_n^i, u_i)$ is trivial for $i = 0, 1$, it follows that $\mathrm{Aut}(C, x_1, \ldots, x_n, u)$ is also trivial. Finally, we observe that $H^1(C, u^* T_{\mathbb{P}^{N_i}}) = H^1(C_i, u_i^* T_{\mathbb{P}^{N_i}}) = 0$ for $i = 0, 1$ where the first equality comes from the fact that C_i is obtained from C by sequentially contracting spheres with $\leqslant 2$ special points on which the map u is constant. Thus, $\mathcal{N}$ is a smooth quasi-projective scheme of the expected dimension.

Let us now assume that $2g-2+n>0$ and show that $\mathcal{N}\to\mathcal{M}_0\times_{\overline{\mathcal{M}}_{g,n}}\mathcal{M}_1$ is proper and birational. Properness is clear since the source and the target of the map are proper over $\mathcal{M}_0\times\mathcal{M}_1$. To argue that the map is birational (i.e. of degree 1), it suffices, thanks to the fact that all spaces are unobstructed, to show that the induced map

$$\mathcal{N}^{\mathrm{sm}}\to\mathcal{M}_0^{\mathrm{sm}}\times_{\mathcal{M}_{g,n}}\mathcal{M}_1^{\mathrm{sn}}$$

is an isomorphism. To this end, consider a point of the target given by $(\iota_i,C_i,x_1^i,\dots,x_n^i)$ in $\mathcal{M}_i$ for $i=0,1$ and an isomorphism $\varphi:(C_0,x_1^0,\dots,x_n^0)^{\mathrm{st}}\to(C_1,x_1^1,\dots,x_n^1)^{\mathrm{st}}$. As C_i is smooth, $\varphi:C_0\to C_1$ is an isomorphism. But in this case it is obvious that the lift to $\mathcal{N}^{\mathrm{sm}}$ of this point of $\mathcal{M}_0\times_{\mathcal{M}_{g,n}}\mathcal{M}_1$ is unique and is given by $(C_0,x_1^0,\dots,x_n^0,u=(u_0,u_1\circ\varphi))$. $\square$

Remark 2.5.2. It is crucial in Lemma 2.5.1 that the fibre product is taken in the sense of orbifolds (or stacks) and not over the underlying coarse moduli space $\overline{M}_{g,n}$. Indeed, when (g,n) is $(1,1)$ or $(2,0)$, the corresponding map $\mathcal{N}\to\mathcal{M}_0\times_{\overline{M}_{g,n}}\mathcal{M}_1$ is still proper but of degree 2.

Lemma 2.5.3. *The natural map*

$$(\mathfrak{s}_0^{-1}(0)\times\mathfrak{s}_1^{-1}(0))\times_{\mathcal{M}_0\times\mathcal{M}_1}\mathcal{N}\to\overline{\mathcal{M}}_{g,n}(X,\mathfrak{A};J)\tag{2.5.3.6}$$

descends to a homeomorphism on the $(G_0\times G_1)$-quotient.

Proof. Continuity of the map is evident. Since the source is compact and the target is Hausdorff, it suffices to argue that we get a bijection after passing to the $(G_0\times G_1)$-quotient. Suppose we are given a point $(C,x_1,\dots,x_n,u:C\to X_0\times X_1)$ of $\overline{\mathcal{M}}_{g,n}(X,\mathfrak{A};J)$. We get points $(C_i,x_1^i,\dots,x_n^i,u_i:C\to X_i)$ of $\overline{\mathcal{M}}_{g,n}(X_i,A_i;J_i)$ and associated contraction maps $\kappa_i:C\to C_i$ for $i=0,1$ by applying the map Φ from (2.5.3.3). Since $\mathcal{K}_i$ is a global Kuranishi chart for $i=0,1$, we can lift $(C_i,x_1^i,\dots,x_n^i,u_i:C\to X_i)$ to a point $(C_i,x_1^i,\dots,x_n^i,u_i,\iota_i:C_i\to\mathbb{P}^{N_i})\in\mathfrak{s}_i^{-1}(0)$ which is unique up to the action of G_i. The contraction maps $C\to C_i$ and the maps $\iota_i:C_i\to\mathbb{P}^{N_i}$ now uniquely determine a map $C\to\mathbb{P}^{N_0}\times\mathbb{P}^{N_1}$ whose stability follows from that of u. Thus, each point in $\overline{\mathcal{M}}_{g,n}(X,\mathfrak{A};J)$ has an inverse image in $(\mathfrak{s}_0^{-1}(0)\times\mathfrak{s}_1^{-1}(0))\times_{\mathcal{M}_0\times\mathcal{M}_1}\mathcal{N}$ which is unique up to the action of $G_0\times G_1$. $\square$

Remark 2.5.4. Lemma 2.5.3 shows that

$$\mathcal{K}_\Psi:=\Psi^*(\mathcal{K}_0\times\mathcal{K}_1)\tag{2.5.3.7}$$

defines a rel–C^∞ global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,\mathfrak{A};J)$.

Lemma 2.5.5. *$\mathcal{K}_\Psi$ is stably complex rel–C^∞ equivalent to $\mathcal{K}$ from* (2.5.3.2).

Proof. Straightforward adaptation of the argument used in §2.4.1 to prove Theorem 2.1.18(3a). $\square$

Remark 2.5.6. Lemma 2.5.5 corresponds to the comparison of obstructions theories in [Beh99, Proposition 6].

The following fact will be crucial for our proof of the product formula.

Lemma 2.5.7. *Assume $2g-2+n>0$ and (g,n) is neither $(1,1)$ nor $(2,0)$. Then, we have*

$$\Psi_*[\mathcal{N}] = (\mathrm{st}\times\mathrm{st})^*\mathrm{PD}(\Delta)\cap[\mathcal{M}_0\times\mathcal{M}_1] \tag{2.5.3.8}$$

in the Borel–Moore homology of $\mathcal{M}_0\times\mathcal{M}_1$ over $\mathbb{Q}$, where $\Delta:\overline{\mathcal{M}}_{g,n}\to\overline{\mathcal{M}}_{g,n}\times\overline{\mathcal{M}}_{g,n}$ is the diagonal map.

Proof. Since (g,n) is neither $(1,1)$ nor $(2,0)$, the forgetful map $\overline{\mathcal{M}}_{g,n}\to\overline{M}_{g,n}$ is an isomorphism over a Zariski open subset $\overline{M}^*_{g,n}\subset\overline{M}_{g,n}$. From this and Lemma 2.5.1, it follows that Ψ maps $\mathcal{N}$ onto the closed subscheme $\mathcal{M}_{01}:=\mathcal{M}_0\times_{\overline{M}_{g,n}}\mathcal{M}_1\subset\mathcal{M}_0\times\mathcal{M}_1$ birationally.

Let $\mathcal{N}'\subset\mathcal{M}_{01}$ be the maximal Zariski open subset for which $\mathcal{N}'\to\overline{M}_{g,n}$ has image contained in $\overline{M}^*_{g,n}$, $\Psi^{-1}(\mathcal{N}')\to\mathcal{N}'$ is an isomorphism and the $\mathcal{M}_0\times\mathcal{M}_1\to\overline{M}_{g,n}\times\overline{M}_{g,n}$ is a submersion at the points of $\mathcal{N}'$. The set $F:=\mathcal{M}_{01}\backslash\mathcal{N}'$ is then Zariski closed in $\mathcal{M}_0\times\mathcal{M}_1$ and $\dim_{\mathbb{C}}F\leqslant\dim_{\mathbb{C}}\mathcal{N}'-1$. By construction, it follows that (2.5.3.8) holds over the complement of F in $\mathcal{M}_0\times\mathcal{M}_1$. To conclude that (2.5.3.8) holds over all of $\mathcal{M}_0\times\mathcal{M}_1$, use the excision exact sequence in Borel–Moore homology and the fact that the Borel–Moore homology of F is supported in degrees $\leqslant\dim_{\mathbb{R}}\mathcal{N}-2$. □

Remark 2.5.8. By interpreting the Poincaré dual class of the diagonal map Δ in the sense of orbifolds, it is possible to extend Lemma 2.5.7 to cover the cases when (g,n) is $(1,1)$ or $(2,0)$. We do not pursue this generalisation here.

Theorem 2.5.9 (Product formula for GW classes)**.** *Using the notation of this subsection, we have*

$$\begin{aligned}&\Phi_*[\overline{\mathcal{M}}_{g,n}(X,\mathfrak{A};J)]^{\mathrm{vir}}\\ &\quad=(\mathrm{st}\times\mathrm{st})^*\mathrm{PD}(\Delta)\cap([\overline{\mathcal{M}}_{g,n}(X_0,A_0;J_0)]^{\mathrm{vir}}\times[\overline{\mathcal{M}}_{g,n}(X_1,A_1;J_1)]^{\mathrm{vir}})\end{aligned} \tag{2.5.3.9}$$

whenever $2g-2+n>0$ and $(g,n)\notin\{(1,1),(2,0)\}$.

Proof. The left side is *a priori* defined using the global Kuranishi chart $\mathcal{K}$, but by virtue of Lemma 2.5.5 we can replace $\mathcal{K}$ by $\mathcal{K}_\Psi$. Let $\tilde{\mathcal{T}}/\mathcal{N}$ be the thickening of $\mathcal{K}_\Psi$ and let $\tilde{\Psi}:\tilde{\mathcal{T}}\to\mathcal{T}_0\times\mathcal{T}_1$ be the natural map. Denote by $W\subset\overline{\mathcal{M}}_{g,n}$ the complement of the automorphism free smooth locus $\mathcal{M}^{\mathrm{free}}_{g,n}$. This has real codimension $\geqslant 2$, so the map

$$H^{3g-3+n}(\overline{\mathcal{M}}_{g,n}\times\overline{\mathcal{M}}_{g,n}\mid\Delta;\mathbb{Q})\to H^{3g-3+n}(\mathcal{M}^{\mathrm{free}}_{g,n}\times\mathcal{M}^{\mathrm{free}}_{g,n}\mid\Delta_{\mathcal{M}^{\mathrm{free}}_{g,n}};\mathbb{Q})$$

is an isomorphism mapping $\mathrm{PD}(\Delta)$ to $\mathrm{PD}(\Delta_{\mathcal{M}^{\mathrm{free}}_{g,n}})$. Since the restriction of the stabilisation map to $\mathrm{st}^{-1}(\mathcal{M}^{\mathrm{free}}_{g,n})$ is a submersion, this implies

$$\tilde{\Psi}_*[\tilde{\mathcal{T}}/(G_0\times G_1)]=(\mathrm{st}\times\mathrm{st})^*\mathrm{PD}(\Delta)\cap[(\mathcal{T}_0/G_0)\times(\mathcal{T}_1/G_1)] \tag{2.5.3.10}$$

in the Borel–Moore homology of $\mathcal{T}_0/G_0 \times \mathcal{T}_1/G_1$ over $\mathbb{Q}$. As the obstruction bundle of $\mathcal{K}_\Psi$ is given by the pullback of the obstruction bundle of $\mathcal{K}_0 \times \mathcal{K}_1$, the definition of the virtual fundamental class implies (2.5.3.9). $\square$

Specialising to the case $g = 0$ and $n = 3$, we obtain the following consequence.

Corollary 2.5.10 (Künneth formula for quantum cohomology)**.** *The canonical Künneth map*

$$\mathrm{QH}(X_0, \omega_0) \otimes_{\Lambda_0} \mathrm{QH}(X_1, \omega_1) \to \mathrm{QH}(X, \omega) \tag{2.5.3.11}$$

is an isomorphism of Λ_0-algebras.

Corollary 2.5.11. *Suppose ω and ω' are symplectic forms on a closed manifold X such that*

$$\mathrm{GW}^{(X,\omega)}_{A,0,n} \neq \mathrm{GW}^{(X,\omega')}_{A,0,n} \tag{2.5.3.12}$$

for some $A \in H_2(X, \mathbb{Z})$ and $n \geqslant 3$. Then, for any $k \geqslant 1$ and any $\varphi \in \mathrm{Diff}(X \times (S^2)^k)$ isotopic to the identity, it is impossible to connect $\varphi^*(\omega \oplus \sigma^{\oplus k})$ and $\omega' \oplus \sigma^{\oplus k}$ by a path of symplectic forms on $X \times (S^2)^k$.

Chapter 3

A fibre product formula

3.1 Virtual fundamental classes of cut-down moduli spaces

This section is the technical backbone of §4.1 and §3.2. Readers more interested in applications are advised to skip this section and refer to it as needed.

3.1.1 Embeddings of global Kuranishi charts

We investigate how geometric relations between global Kuranishi charts translate into relations between their virtual fundamental classes.

Definition 3.1.1. A *morphism of global Kuranishi charts* $\mathfrak{f}\colon \mathcal{K}' = (G', \mathcal{T}', \mathcal{E}', \mathfrak{s}') \to \mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ consists of a group morphism $\alpha\colon G \to G'$, an α-equivariant map $f\colon \mathcal{T} \to \mathcal{T}'$ and an α-equivariant vector bundle morphism $\tilde{f}\colon \mathcal{E} \to f^*\mathcal{E}'$ so that $tilde f\mathfrak{s} = \mathfrak{s}'f$. We call $\mathfrak{f}$ an *embedding* if $\alpha = \mathrm{id}$, f is an embedding of manifolds and $tilde f$ is an injection of vector bundles.
If $\mathcal{K}$ and $\mathcal{K}'$ are rel–C^∞ over base spaces $\mathcal{M}$ respectively $\mathcal{M}'$, we say the morphism is of *class rel–C^∞* if α is smooth and f and $\tilde{f}$ are rel–C^∞ covering a smooth morphism $\mathcal{M} \to \mathcal{M}'$.

Remark 3.1.2. If $\mathfrak{f} = (\alpha, f, \tilde{f})$ is such that α, f and F are embeddings in the respective category, we can replace $(G', \mathcal{T}', \mathcal{E}', \mathfrak{s}')$ with $(G, G \times_{G'} \mathcal{T}', G \times_{G'} \mathcal{E}', \mathrm{id} \times \mathfrak{s}')$.

If $\mathcal{T}/\mathcal{M}$ is a rel–C^∞ manifold with smooth base, its tangent microbundle has a canonical (equivariant) vector bundle lift given by $T_\mathcal{T} := q^*T_\mathcal{M} \oplus T_{\mathcal{T}/\mathcal{M}}$, where $q\colon \mathcal{T} \to \mathcal{M}$ is the structural map. Given a rel–C^∞ embedding $j\colon \mathcal{T}'/\mathcal{M}' \hookrightarrow \mathcal{T}/\mathcal{M}$, where $\mathcal{M}'$ is a smooth submanifold of $\mathcal{M}$, we define the *normal bundle* of $\mathcal{T}'/\mathcal{M}'$ inside $\mathcal{T}/\mathcal{M}$ to be

$$N_{\mathcal{T}'/\mathcal{T}} := q^*N_{\mathcal{M}'/\mathcal{M}} \oplus N^v_{\mathcal{T}'/\mathcal{T}_{\mathcal{M}'}},$$

where $N^v_{\mathcal{T}'/\mathcal{T}_{\mathcal{S}'}} := j^*T_{\mathcal{T}_{\mathcal{M}'}/\mathcal{M}'}/T_{\mathcal{T}'/\mathcal{M}'}$ is the *vertical normal bundle* with $\mathcal{T}_{\mathcal{M}'} := \mathcal{T} \times_\mathcal{M} \mathcal{M}'$.

Definition 3.1.3. Suppose $j\colon \mathcal{K}' \hookrightarrow \mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$ is a rel–C^∞ embedding. We call $N_{\mathcal{K}'/\mathcal{K}} := N_{\mathcal{T}'/\mathcal{T}} - \mathcal{D}$ its *virtual normal bundle*, where $\mathcal{D} = \mathrm{coker}(\tilde{j})$.

Throughout, we assume our global Kuranishi charts to be oriented in the following sense, equivalent to [AMS21, §5.4]. Clearly, if both $\mathcal{K}$ and $\mathcal{K}'$ are oriented, then so is $N_{\mathcal{K}'/\mathcal{K}}$.

Definition 3.1.4. A *(Borel equivariant) orientation* of a Kuranishi chart $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ consists of a $\mathbb{Q}$-orientation of the virtual vector bundle $(T\mathcal{T})_G - \underline{\mathfrak{g}}$ and $\mathcal{E}_G$ over $\mathcal{T}_G$.

We need orientations of both $\mathcal{T}/G$ as well as $\mathcal{E}$ in order to define the virtual fundamental class. By [AMS21, Lemma 5.11], this is equivalent to a $\mathbb{Q}$-orientation of $(T\mathcal{T})_G - \underline{\mathfrak{g}} - \mathcal{E}$.

Notation. Given $A \subset B$, we write $H_*(B \mid A; \mathbb{Q}) := H_*(B, B\backslash A; \mathbb{Q})$ and similarly for cohomology.

Example 3.1.5. If $\mathcal{T}$ and the action on it are smooth, there exists an embedding $\mathcal{T} \times \mathfrak{g} \hookrightarrow T\mathcal{T}$, where $\mathfrak{g} = \mathrm{Lie}(G)$. Taking a G-invariant complement $\mathcal{D}$ of this distribution, any choice of equivariant Thom class $\tau \in H_G^{\dim(\mathcal{T}/G)}(\mathcal{D} \mid \mathcal{T}, \mathbb{Q})$ defines a $\mathbb{Q}$-orientation of $\mathcal{T} - \underline{\mathfrak{g}}$.

Given an oriented orbifold $\overline{\mathcal{T}}$ and an oriented suborbifold $\overline{\mathcal{T}'} \hookrightarrow \overline{\mathcal{T}}$ of codimension k, we have the Poincaré duality isomorphisms

$$H_k(\overline{\mathcal{T}} \mid \overline{\mathcal{T}}'; \mathbb{Q}) \cong H_c^{\dim(\overline{\mathcal{T}'})}(\overline{\mathcal{T}} \mid \overline{\mathcal{T}}'; \mathbb{Q}) \cong H_0(\overline{\mathcal{T}}'; \mathbb{Q})$$

Thus $H^k(\overline{\mathcal{T}} \mid \overline{\mathcal{T}}'; \mathbb{Q}) \cong \mathbb{Q}^{|\pi_0(\overline{\mathcal{T}}')|}$ and taking the sum of all generators, we obtain the *Poincaré dual* $\mathrm{PD}(\overline{\mathcal{T}}')$ of $\overline{\mathcal{T}}'$ in $\overline{\mathcal{T}}$. The composite

$$j_! j^* \colon H^*(\overline{\mathcal{T}}; \mathbb{Q}) \to H^{*+k}(\overline{\mathcal{T}}; \mathbb{Q})$$

is given by multiplication with the image of $\mathrm{PD}(\overline{\mathcal{T}}')$ in $H^k(\overline{\mathcal{T}}; \mathbb{Q})$.

Remark 3.1.6. In the case of thickenings as above, we can give an explicit description of the Poincaré dual in terms of the normal bundle. Factor the inclusion $j \colon \mathcal{T}' \to \mathcal{T}$ as

$$\mathcal{T}' \xrightarrow{i} \hat{\mathcal{T}} := \mathcal{T}' \times_{\mathcal{M}} \mathcal{M}' \to \mathcal{T},$$

where we equip $\hat{\mathcal{T}}$ with the canonical G-orientation. A relative version of the equivariant tubular neighbourhood theorem shows that $\mathrm{PD}_{\hat{\mathcal{T}}}(\mathcal{T}'/G)$ corresponds to the equivariant Thom class of $N^v_{\mathcal{T}'/\mathcal{T}}$ under the canonical isomorphism induced by the tubular neighbourhood. Meanwhile, $\mathrm{PD}_{\mathcal{T}/G}(\hat{\mathcal{T}}/G) = q^*\mathrm{PD}_{\mathcal{M}/G}(\mathcal{M}'/G)$, which corresponds to the equivariant Thom class of $q^* N_{\mathcal{M}'/\mathcal{M}}$. Thus,

$$\mathrm{PD}_{\mathcal{T}/G}(\mathcal{T}'/G) = \mathrm{PD}_{\mathcal{T}/G}(\hat{\mathcal{T}}/G) \cdot \mathrm{PD}_{\hat{\mathcal{T}}}(\mathcal{T}'/G). \tag{3.1.1.1}$$

Proposition 3.1.7. *Let $j \colon \mathcal{K}' \hookrightarrow \mathcal{K}$ be a rel–C^∞ embedding of oriented rel–C^∞ global Kuranishi charts, covering a smooth embedding $\mathcal{M}' \hookrightarrow \mathcal{M}$ of oriented base spaces. If $N_{\mathcal{K}'/\mathcal{K}} \oplus \mathcal{D} = N_{\mathcal{T}'/\mathcal{T}}$, then*

$$j_*(e_G(\mathcal{D}) \cap [\mathfrak{M}']^{\mathrm{vir}}) = \mathrm{PD}(\mathcal{T}'/G) \cap [\mathfrak{M}]^{\mathrm{vir}}. \tag{3.1.1.2}$$

Here we identify $e_G(\mathcal{D}) \in H^*_G(\mathcal{T}';\mathbb{Q})$ with the corresponding element in $H^*(\mathcal{T}'/G;\mathbb{Q})$. Equip $\mathcal{D}$ with the unique orientation satisfying $e_G(\mathcal{E}'\oplus D) = e_G(\mathcal{E})|_{\mathcal{T}'}$.

Proof. Suppose first $j^*\mathcal{E} = \mathcal{E}'$. The equality

$$j_*[\mathfrak{M}']^{\mathrm{vir}}_{\mathcal{K}'} = \mathrm{PD}(\mathcal{T}'/G) \cap [\mathfrak{M}]^{\mathrm{vir}}_{\mathcal{K}} \tag{3.1.1.3}$$

follows from the commutativity of

$$\begin{tikzcd} \check{H}^{\mathrm{vdim}(\mathfrak{M}')+*}(\mathcal{M}';\mathbb{Q}) \arrow[rr, "\mathfrak{s}'^*\tau(\mathcal{E}'/G)"] & & H_c^{\dim(\mathcal{T}'/G)+*}(\mathcal{T}'/G \mid \mathcal{M}';\mathbb{Q}) \arrow[d, "j_!"] \\ \check{H}^{\mathrm{vdim}(\mathfrak{M}')+*}(\mathcal{M};\mathbb{Q}) \arrow[u, "j^*"] \arrow[r, "\cdot\mathrm{PD}(\mathcal{T}'/G)"] & \check{H}^{\mathrm{vdim}(\mathfrak{M})+*}(\mathcal{M};\mathbb{Q}) \arrow[r, "\mathfrak{s}^*\tau(\mathcal{E}/G)"] & H_c^{\dim(\mathcal{T}/G)+*}(\mathcal{T}/G;\mathbb{Q}) \end{tikzcd}$$

where we need the convention

$$\langle \alpha \cap [\mathfrak{M}]^{\mathrm{vir}}, \beta\rangle = \langle [\mathfrak{M}]^{\mathrm{vir}}, \beta\cdot\alpha\rangle.$$

Now assume $\mathrm{rank}(\mathcal{D}) > 0$. Let $\widetilde{K} := (G, \mathcal{T}', j^*\mathcal{E}, \mathfrak{s}')$. This is also a global Kuranishi chart for $\mathcal{M}'$, albeit with a larger obstruction bundle. By the definition of the virtual fundamental class,

$$e_G(\mathcal{D}) \cap [\mathfrak{M}']^{\mathrm{vir}}_{\mathcal{K}'} = [\mathfrak{M}']^{\mathrm{vir}}_{\widetilde{\mathcal{K}}}.$$

This completes the proof. □

Remark 3.1.8. For Proposition 3.1.7 it suffices to assume that the map $\mathcal{T}'/G \to \mathcal{T}/G$ since the Poincaré duality statement we use is on the level of coarse moduli spaces.

Proposition 3.1.7 is not quite ideal, since one might have $\mathcal{E}' = j^*\mathcal{E} \oplus N_{\mathcal{T}'/\mathcal{T}}$, in which case we would expect that the virtual fundamental classes agree, at least under certain assumptions.

Lemma 3.1.9. *Let $\mathcal{K}'$ and $\mathcal{K}$ be rel–C^∞ smooth global Kuranishi charts over $\mathcal{M}$ for M. Suppose there exists a rel–C^∞ embedding $j\colon \mathcal{K}' \hookrightarrow \mathcal{K}$ over $\mathcal{M}$, inducing a quasi-isomorphism*

$$[T_{\mathcal{T}'/\mathcal{M}}|_{\mathfrak{s}'^{-1}(0)} \xrightarrow{D\mathfrak{s}'} \mathcal{E}'|_{\mathfrak{s}'^{-1}(0)}] \to [T_{\mathcal{T}/\mathcal{M}}|_{\mathfrak{s}^{-1}(0)} \xrightarrow{D\mathfrak{s}} \mathcal{E}|_{\mathfrak{s}^{-1}(0)}] \tag{3.1.1.4}$$

of complexes of vector bundles. Then $[M]^{\mathrm{vir}}_{\mathcal{K}'} = [M]^{\mathrm{vir}}_{\mathcal{K}}$.

Proof. Using a relative tubular neighbourhood, we may assume that $\mathcal{T}$ admits a vector bundle structure $p\colon \mathcal{T} \to \mathcal{T}'$. Fix a splitting $\mathcal{E}|_{\mathcal{T}'} = \mathcal{E}' \oplus \mathcal{D}$. Using [tD08, Theorem], we may assume without loss of generality that $\mathcal{E} = p^*\mathcal{E}' \oplus p^*\mathcal{D}$, where $p\colon \mathcal{T} \to \mathcal{T}'$ is the bundle map. Write $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$.

As $\mathcal{K}' \hookrightarrow \mathcal{K}$, we have $\mathcal{T}' \subset \mathfrak{s}_2^{-1}(0)$. Given $x \in Z := \mathfrak{s}'^{-1}(0) = \mathfrak{s}^{-1}(0)$ we have an associated commutative diagram of vertical derivatives

$$\begin{tikzcd} T_{\mathcal{T}'/\mathcal{M},x} \arrow[r, "d^v\mathfrak{s}'(x)"] \arrow[d, hook] & \mathcal{E}'_x \arrow[d, hook] \\ T_{\mathcal{T}'/\mathcal{M},x} \oplus \mathcal{T}_x \arrow[r, "d^v\mathfrak{s}_1(x) \oplus d^v\mathfrak{s}_2(x)"] & \mathcal{E}'_x \oplus \mathcal{D}_x \end{tikzcd}$$

Since $\operatorname{coker}(d^v\mathfrak{s}(x)) = \operatorname{coker}(d^v\mathfrak{s}'(x))$ is a quotient of $\mathcal{E}'_x$, it follows that $\operatorname{coker}(d^v\mathfrak{s}_2(x)) = 0$. Replacing $\mathcal{T}$ by a neighbourhood of Z, we may assume $\mathfrak{s}_2 \pitchfork 0$. Set $S := \mathfrak{s}_2^{-1}(0)$. As $\dim(S) = \dim(\mathcal{T}')$, the two global Kuranishi charts $\mathcal{K}_2 := (G, S, p^*\mathcal{E}'|_S, \mathfrak{s}_1|_S)$ and $\mathcal{K}'$ are related by (Germ equivalence).

Finally, $[M]^{\text{vir}}_{\mathcal{K}_2} = [M]^{\text{vir}}_{\mathcal{K}}$, since the Poincaré dual of S in $\mathcal{T}$ is $\mathfrak{s}_2^*\tau_{p^*\mathcal{D}}$. $\square$

In other words, the virtual fundamental class only depends on the global Kuranishi chart up to quasi-isomorphism, similar to [BF97, Proposition 5.3].

Example 3.1.10. Both the embedding condition and (3.1.1.4) are necessary for this to hold. To see this, consider $\mathcal{T} = \mathbb{R} = \mathcal{T}'$ and $\mathcal{E} = \underline{\mathbb{R}^2} = \mathcal{E}'$ with $\mathfrak{s}(t) = t^2$ and $\mathfrak{s}(t) = t^3$. Or $\mathcal{T}' = \mathbb{R} \times \{0\} \subset \mathcal{T} = \mathbb{R}^2$ with $\mathcal{E}' = \underline{\mathbb{R}}$ and $\mathcal{E} = \underline{\mathbb{R}^2}$ and $\mathfrak{s}(t,r) = (t^2, r^2)$ and $\mathfrak{s}'(t) = t^2$.

3.1.2 Fibre products of global Kuranishi charts

In this section we construct a fibre product of global Kuranishi charts over another global Kuranishi chart. In the previous section we considered embeddings of global Kuranishi charts with the same covering group. Here we require a more general notion of morphism.

Suppose we are given a morphism

$$(p_1, \pi_i, \Pi_i) \colon \mathcal{K}_i = (G \times G_i, \mathcal{T}_i/\mathcal{M}_i, \mathcal{E}_i, \mathfrak{s}_i) \to \mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$$

of oriented global Kuranishi charts for $i \in \{0, 1\}$. Set $\widetilde{G}_i := G \times G_i$ and $\widetilde{G} := \widetilde{G}_0 \times \widetilde{G}_1$.

Assumption 3.1.11. *We assume that*

a) π_0 is a relative submersion covering a smooth submersion $p_0 \colon \mathcal{M}_0 \to \mathcal{M}$ and Π_0 is fibrewise surjective.

b) We have $G_x = G_{\pi_j(x)}$ for any $x \in \mathcal{T}_j$ and $j \in \{0, 1\}$. Moreover, $(G \times G_j)_x = G_x \times (G_j)_x$.

Definition 3.1.12. The *fibre product chart* is $\mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_1 := (\widetilde{G}_0 \times \widetilde{G}_1, \widetilde{\mathcal{T}}, \widetilde{\mathcal{E}}, \widetilde{\mathfrak{s}})$, where

$$\widetilde{\mathcal{T}} := \{(x_0, g, x_1) \in \mathcal{T}_0 \times G \times \mathcal{T}_1 \mid g \cdot \pi_0(x_0) = \pi_1(x_1)\}$$

while

$$\widetilde{\mathcal{E}} := \{(e_0, g, e_1) \in \mathcal{E}_0 \times G \times \mathcal{E}_1 \mid g \cdot \Pi_0(e_0) = \Pi_1(e_1)\}$$

on which $\widetilde{G}$ acts by

$$((g_0, h_0), (g_1, h_1)) \cdot (y_0, g, y_1) = (((g_0, h_0) \cdot y_0, g_1 g g_0^{-1}, (g_1, h_1) \cdot y_1)$$

for $y_i \in \mathcal{T}_i$, respectively $\mathcal{E}_i$. The obstruction section is given by $\widetilde{\mathfrak{s}} = \mathfrak{s}_0 \times \mathrm{id}_G \times \mathfrak{s}_1$.

In this fibre product of global Kuranishi charts we use the orbifold fibre product applied instead of the naive fibre product of thickenings and obstruction bundles. It is a rel–C^∞ manifold over $\widetilde{\mathcal{M}} := \{(w_0, g, w_1) \in \mathcal{M}_0 \times G \times \mathcal{M}_1 \mid g \cdot p_0(w_0) = p_1(w_1)\}$. Assumption 3.1.11(b) implies that the canonical map $\widetilde{\mathcal{T}} \to \mathcal{T}_0 \times \mathcal{T}_1$ descends to an embedding on the level of orbit spaces.

Remark 3.1.13. Assumption 3.1.11(a) is more than enough to ensure that this is a well-defined global Kuranishi chart. Explicitly, it would suffice to require that $\pi_0 \pitchfork \pi_1$ and that $\Pi_0 \oplus \Pi_1 \colon \mathcal{E}_0 \oplus \mathcal{E}_1 \to \mathcal{E}$ is a surjective vector bundle morphisms. This is essentially the notion of *d-transversality* in [Joy12, §4.6]. For our main application we can arrange the stronger assumptions above and thus work with them.

Moreover, Assumption 3.1.11(b) ensures that it defines a global Kuranishi chart *for* $\mathfrak{M}_0 \times_{\mathfrak{M}} \mathfrak{M}_1$. It admits a canonical orientation.

Lemma 3.1.14. *We have* $\mathrm{PD}(\widetilde{\mathcal{T}}/\widetilde{G}) = (\bar\pi_0 \times \bar\pi_1)^*\mathrm{PD}(\Delta_{\mathcal{T}/G})$ *where* $\bar\pi_i \colon \mathcal{T}_i/\widetilde{G}_i \to \mathcal{T}/G$ *is the map induced by* π_i*.*

Proof. Note that $\Delta_{\mathcal{T}/G}$ is the coarse moduli space of the orbifold $[\hat{\mathcal{T}}/G \times G]$, where

$$\hat{\mathcal{T}} = \{(x, g, x') \in \mathcal{T} \times G \times \mathcal{T} \mid g \cdot x = x'\}.$$

Then $\widetilde{\mathcal{T}} = \hat{\mathcal{T}} \times_{\mathcal{T}\times\mathcal{T}} (\mathcal{T}_0 \times \mathcal{T}_1)$, so the claim follows from Corollary A.1.5. □

By Proposition 3.1.7 and Remark 3.1.6, we therefore obtain the following fibre-product formula.

Theorem 3.1.15. *Given global Kuranishi charts* $\mathcal{K}, \mathcal{K}_0, \mathcal{K}_1$ *as above, we have*

$$j_*(e_{\widetilde{G}}(\pi^*\mathcal{E}) \cap [\mathfrak{M}_0 \times_{\mathfrak{M}} \mathfrak{M}_1]^{\mathrm{vir}}) = (\bar\pi_0 \times \bar\pi_1)^*\mathrm{PD}(\Delta_{\mathcal{T}/G}) \cap [\mathfrak{M}_0 \times \mathfrak{M}_1]^{\mathrm{vir}}.$$

Remark 3.1.16 (Fibre products along embeddings). In the case where $G_1 = G$ and $\pi_1 \colon \mathcal{T}_1 \to \mathcal{T}$ is an embedding and Π_1 is fibrewise injective, the induced map $\bar\pi_1 \colon \mathfrak{M}_1 \to \mathfrak{M}$ is an embedding as well. Thus $\mathfrak{M}' := \mathfrak{M}_0 \times_{\mathfrak{M}} \mathfrak{M}'$ embeds into $\mathfrak{M}_0$ and the canonical morphism $\mathcal{K}_0 \times_{\mathcal{K}} \mathcal{K}_1 \to \mathcal{K}_0$ is an embedding of global Kuranishi charts (with the same covering group). By Proposition 3.1.7,

$$p_{0*}(e_{\widetilde{G}_1}(p_1^*\mathcal{D}) \cap [\mathfrak{M}_0 \times_{\mathfrak{M}} \mathfrak{M}_1]^{\mathrm{vir}}) = \pi_0^*\mathrm{PD}(\mathcal{T}_1/G) \cap [\mathfrak{M}_1]^{\mathrm{vir}}. \tag{3.1.2.1}$$

where $p_i \colon \mathcal{T}_0 \times_{\mathcal{T}} \mathcal{T}_1 \to \mathcal{T}_i$ is the projection and $\mathcal{D} = \pi_1^*\mathcal{E}/\mathcal{E}_1$.

Example 3.1.17. Suppose we have a global Kuranishi chart $\mathcal{K} = (G, \mathcal{T}/\mathcal{M}, \mathcal{E}, \mathfrak{s})$ for $\mathfrak{M}$ and $f \colon \mathcal{M}' \to \mathcal{M}$ is a G-equivariant submersion. Then

$$[f^*\mathfrak{M}]^{\mathrm{vir}} = (\bar f \times \bar\pi)^*\mathrm{PD}(\Delta_{\mathcal{M}/G}) \times ([\mathcal{M}'/G] \otimes [\mathfrak{M}]^{\mathrm{vir}}),$$

where $\pi \colon \mathcal{T} \to \mathcal{M}$ is the structural map.

3.2 GW invariants of a fibre product

In this section we establish a formula for the GW invariants of a fibre product of closed symplectic manifolds, stated in Theorem 3.2.1. The first step is phrased in terms of abstract global Kuranishi charts and was carried out in the previous section.The second step consists of lifting for a strongly Hamiltonian fibre bundle $\pi\colon X \to B$, the map π to a morphism between global Kuranishi charts, shown in Proposition 3.2.5.

Here, a *strongly Hamiltonian fibre bundle* $\pi\colon (X, \omega_X) \to (B, \omega_B)$ is a smooth submersion, so that $\ker(d\pi)$ is a symplectic subbundle of TX. Suppose $\pi_X\colon (X, \omega_X) \to (B, \omega_B)$ and $\pi_Y\colon (Y, \omega_Y) \to (B, \omega_B)$ are two strongly Hamiltonian fibre bundles over a closed symplectic manifold with compact fibres. We fix symplectically orthogonal splittings

$$T_X = H_X \oplus \ker(d\pi_X) \qquad\qquad T_Y = H_Y \oplus \ker(d\pi_Y)$$

identifying H_X with $\pi_X^* T_B$ and similarly for Y. Let

$$Z := X \times_B Y$$

be the fibre product with bundle map $\pi_Z\colon Z \to B$ and inclusion $j\colon Z \hookrightarrow X \times Y$. Fix $J_B \in \mathcal{J}_\tau(B, \omega_B)$ and extend it via the above splitting to *fibred* almost complex structures $J_X \in \mathcal{J}_\tau(X, \omega_X)$ and $J_Y \in \mathcal{J}_\tau(Y, \omega_Y)$. In particular, π_X and π_Y are pseudoholomorphic, so they induce maps

$$\pi_X\colon \overline{\mathcal{M}}_{g,n}(X, A_X, J_X) \to \overline{\mathcal{M}}_{g,n}(B, \pi_{X*}A_X, J_B)$$

and

$$\pi_Y\colon \overline{\mathcal{M}}_{g,n}(Y, A_Y, J_Y) \to \overline{\mathcal{M}}_{g,n}(B, \pi_{Y*}A_Y, J_B).$$

The almost complex structure $J_X \oplus J_Y$ restricts to an ω_Z-tame almost complex structure J_Z on Z. Given $A_X \in H_2(X, \mathbb{Z})$ and $A_Y \in H_2(Y, \mathbb{Z})$ with $\pi_{X*}A_X = A_B = \pi_{Y*}A_Y$, define

$$\mathfrak{B} := \{A \in H_2(Z, \mathbb{Z}) \mid p_{X*}A = A_X,\, p_{Y*}A = A_Y\}.$$

We have the following generalisation of Theorem 2.5.9.

Theorem 3.2.1. *For any $g, n \geqslant 0$ we have*

$$\sum_{A \in \mathfrak{B}} j_*(e_G(\pi_Z^*\mathcal{E}_B) \cap [\overline{\mathcal{M}}_{g,n}(Z, A; J_Z)]^{\mathrm{vir}})$$
$$= (\pi_X \times \pi_Y)^* \mathrm{PD}(\Delta_{\mathcal{T}_B/G}) \cap [\overline{\mathcal{M}}_{g,n}(X, A_X; J_X) \times \overline{\mathcal{M}}_{g,n}(Y, A_Y; J_Y)]^{\mathrm{vir}}.$$

Proof. The key ingredient of the proof is Proposition 3.2.5. It constructs oriented global Kuranishi charts $\mathcal{K}_X, \mathcal{K}_Y, \mathcal{K}_B$ for the moduli spaces of stable maps, which satisfy Assumption 3.1.11. Moreover, they are equivalent to global Kuranishi charts given by the construction of §2.

Abbreviate $\overline{\mathcal{M}}(W) := \overline{\mathcal{M}}_{g,n}(W, A_W; J_W)$ for $W \in \{X, Y, B, Z\}$. By Theorem 3.1.15, we thus see that $\overline{\mathcal{M}}(X) \times_{\overline{\mathcal{M}}(B)} \overline{\mathcal{M}}(Y)$ admits a global Kuranishi chart $\widetilde{\mathcal{K}}$ which is rel–C^∞ over $\widetilde{\mathcal{M}} := \mathcal{M}'_X \times_{\mathcal{M}_B} \mathcal{M}'_Y$ and whose virtual fundamental class satisfies

$$j_*(e_G(\tilde{\pi}^*\mathcal{E}) \cap [\overline{\mathcal{M}}(X) \times_{\overline{\mathcal{M}}(B)} \overline{\mathcal{M}}(Y)]^{\mathrm{vir}}) = (\pi_X \times \pi_Y)^*\mathrm{PD}(\Delta_{\mathcal{T}_B/G}) \cap [\overline{\mathcal{M}}(X) \times \overline{\mathcal{M}}(Y)]^{\mathrm{vir}}. \tag{3.2.0.1}$$

To relate $\overline{\mathcal{M}}(X) \times_{\overline{\mathcal{M}}(B)} \overline{\mathcal{M}}(Y)$ with $\overline{\mathcal{M}}(Z)$, we use a similar argument as in §2.5. Let $\mathcal{M}'_Z$ be the preimage of $\widetilde{\mathcal{M}}$ under

$$\begin{array}{c} \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_X} \times \mathbb{P}^N \times \mathbb{P}^{N_Y}, (m_X, m, m_Y)) \\ \big\downarrow \\ \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_X} \times \mathbb{P}^N, (m_X, m)) \times \overline{\mathcal{M}}_{g,n}(\mathbb{P}^{N_Y} \times \mathbb{P}^N, (m_Y, m)) \end{array}$$

and let $\Psi\colon \mathcal{M}'_Z \to \widetilde{\mathcal{M}}$ be the induced map. By the same reasoning as in the proof of Lemma 2.5.1, we show that $\mathcal{M}'_Z$ is a smooth quasi-projective variety of the expected dimension and Ψ is a proper birational equivalence.

Set $\mathcal{K}'_Z := \Psi^*\widetilde{\mathcal{K}}$. As in §2.5.3, we see that $\mathcal{K}'_Z$ defines a rel–C^∞ global Kuranishi chart for $\bigsqcup_{A\in\mathfrak{B}} \overline{\mathcal{M}}_{g,n}(Z, A; J_Z)$, which is equivalent to the disjoint union of global Kuranishi charts given by Construction 2.1.14. Thus

$$\Psi_* \sum_{A\in\mathfrak{B}} [\overline{\mathcal{M}}_{g,n}(Z, A; J_Z)]^{\mathrm{vir}} = [\overline{\mathcal{M}}(X) \times_{\overline{\mathcal{M}}(B)} \overline{\mathcal{M}}(Y)]^{\mathrm{vir}},$$

so Theorem 3.2.1 follows from (3.2.0.1).

□

Corollary 3.2.2. *If $\overline{\mathcal{M}}_{g,n}(B, A_B; J_B)$ is unobstructed, then*

$$\begin{aligned} &\sum_{A\in\mathfrak{B}} j_*[\overline{\mathcal{M}}_{g,n}(Z, A; J_Z)]^{\mathrm{vir}} \\ &\quad = (\pi_X \times \pi_Y)^*\mathrm{PD}(\Delta_{\overline{\mathcal{M}}_{g,n}(B,A_B;J_B)}) \cap [\overline{\mathcal{M}}_{g,n}(X, A_X; J_X) \times \overline{\mathcal{M}}_{g,n}(Y, A_Y; J_Y)]^{\mathrm{vir}}. \end{aligned}$$

Example 3.2.3. By [MS12, Proposition 7.4.3], $\overline{\mathcal{M}}_{0,n}(B, A_B; J_B)$ is unobstructed if (B, ω_B, J_B) is Kähler and there exists a transitive compact Lie group action by biholomorphisms on B.

Remark 3.2.4. Taking $B = *$ in Corollary 3.2.2 we can extend Theorem 2.5.9 to the two cases $(g, n) \in \{(1, 1), (2, 0)\}$.

Proposition 3.2.5. *We can choose an unobstructed auxiliary datum $(\nabla^B, \mathcal{O}_B(1), p, \mathcal{U}, k)$ (resulting in a global Kuranishi chart $\mathcal{K}_B$) for $\overline{\mathcal{M}}_{g,n}(B, A_B; J_B)$ such that there exists a global Kuranishi chart $\mathcal{K}'_X = (G \times G'_X, \mathcal{T}'_X, \mathcal{E}'_X, \mathfrak{s}'_X)$ for $\overline{\mathcal{M}}_{g,n}(X, A_X; J_X)$ with the following properties.*

a) *$\mathcal{K}'_X$ is equivalent to the global Kuranishi chart given by Construction 2.1.14.*

b) *We have for any $x \in \mathcal{T}'_X$ that $G_x = G_{\tilde{\pi}_X(x)}$ and $(G \times G'_X)_x = G_x \times (G'_X)_x$.*

c) *There exists an isomorphism* $\mathcal{E}'_X \cong \tilde{\pi}_X^*\mathcal{E}_B \oplus \hat{\mathcal{E}}_X$ *of rel–*C^∞ $(G \times G')$*-vector bundles, so that the projection* $\mathcal{E}'_X \to \mathcal{E}_B$ *intertwines* $\mathfrak{s}'_X$ *with* $\mathfrak{s}_B$*.*

There exists a topological submersion $\mathcal{T}'_X \to \mathcal{M}'_X := \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N \times \mathbb{P}^{N_X}, (m, m_X))$ *and* $\mathcal{K}'$ *is a rel–*C^∞ *global Kuranishi chart over* $\mathcal{M}'_X$*. If* $\mathcal{M}_B$ *denotes the base space of* $\mathcal{K}_B$*, then there exists a relatively smooth equivariant submersion* $\tilde{\pi}_X \colon \mathcal{T}'_{X'}/\mathcal{M}'_X \to \mathcal{T}_B/\mathcal{M}_B$*.*

Proof. We assume that either $g \geqslant 2$ or $A_B \neq 0$. In this case we can assume $n = 0$. Otherwise one has to adapt the following construction as in Remark 2.1.15. Given a stable map $u \colon C \to X$ we denote by $u_B \colon C_B \to B$ the map induced by $\pi_X u$ with stabilised domain. If ι is a function from C to some manifold M so that ι descends to C_B, we denote the induced map $C_B \to M$ by ι_B as well.

Fix ∇^B and a polarisation $\mathcal{O}_B(1)$. Let $\tilde{\nabla}$ be any complex linear connection on $\ker(d\pi_X)$ and set $\nabla^X := \pi_X^*\nabla^B \oplus \tilde{\nabla}$. Let $\mathcal{O}_X(1) \to X$ be any polarisation as in Definition 2.1.6. Given a smooth stable map $u \colon C \to X$ define

$$\mathfrak{L}_u := \omega_C \otimes u^*\mathcal{O}_X(1)^{\otimes 3} \qquad\qquad \hat{\mathfrak{L}}_u := \omega_C \otimes (\pi_X u)^*\mathcal{O}_B(1)^{\otimes 3} = \kappa_u^*\mathfrak{L}_{u_B}$$

where $\kappa_u \colon C \to C_B$ is the contraction map. Choose $p \geqslant 1$ so that the conclusion of Lemma 2.2.2 holds for any $\mathfrak{L}_u$ with $u \in \overline{\mathcal{M}}_{g,n}(X, A; J_X)$ or $u \in \overline{\mathcal{M}}_{g,n}(B, A_B; J_B)$. Set $m_X := p \otimes \deg(\mathfrak{L}_u)$ and $m = p \cdot \deg(\hat{\mathfrak{L}}_u) = p \cdot \deg(\mathfrak{L}_{u_B})$ and let $N_X = m_X - g$ and $N = m - g$.

A choice of basis of $H^0(C, \mathfrak{L}_u)$ respectively $H^0(C, \hat{\mathfrak{L}}_u)$ induces maps $\iota_X \colon C \hookrightarrow \mathbb{P}^{N_X}$ and $\hat{\iota} \colon C \to \mathbb{P}^N$ where ι_X is an unobstructed nondegenerate embedding and $\hat{\iota}$ descends to an unosbtructed nondegenerate embedding $\iota_B \colon C_B \hookrightarrow \mathbb{P}^N$, induced by a choice of basis of $H^0(C_B, \mathfrak{L}_{u_B})$. Denote by $\mathcal{M}_B \subset \overline{\mathcal{M}}_g(\mathbb{P}^N, m)$ the space of nondegenerate regular embeddings. Let $\mathcal{M}'_X \subset \overline{\mathcal{M}}_g(\mathbb{P}^N \times \mathbb{P}^{N_X}, (m, m_X))$ be given by those curves which p_{1*} maps to $\overline{\mathcal{M}}^*_g(\mathbb{P}^N, m)$ and p_{2*} maps to $\overline{\mathcal{M}}^*_g(\mathbb{P}^{N_X}, m_X)$, and where p_{2*} does not contract irreducible components of the domain. By Lemma 2.5.1, $\mathcal{M}'_X$ is unobstructed and has no isotropy. Denote $G := \mathrm{PU}(N+1)$. The map $p_{1*} \colon \mathcal{M}'_X \to \mathcal{M}_B$ is a G-equviariant submersion and invariant with respect to the action by $G_X := \mathrm{PU}(N_X + 1)$, and (b) is satisfied.

Complete $(\nabla^B, \mathcal{O}_B(1), p)$ to an unobstructed auxiliary datum $(\nabla^B, \mathcal{O}_B(1), p, \mathcal{U}, k)$ where we might have to increase k depending on the following construction. Define $\mathcal{T}'_X$ to be the set of tuples $(u, C, \iota, \alpha, \alpha_X, \eta, \eta_X)$ (modulo reparametrisations of the domain) where

- $u \colon C \to X$ is a smooth stable map of type $(g, 0)$ with $u_*[C] = A_X$,
- $\iota = (\iota_1, \iota_2) \colon C \to \mathbb{P}^N \times \mathbb{P}^{N_X}$ defines an element of $\mathcal{M}'_X$,
- $\alpha, \alpha_X \in H^1(C, \mathcal{O}_C)$ are such that

$$[\iota_1^*\mathcal{O}_{\mathbb{P}^N}(1)] = p \cdot [\hat{\mathfrak{L}}_u] + \alpha \qquad\qquad [\iota_2^*\mathcal{O}_{\mathbb{P}^{N_X}}(1)] = p \cdot [\mathfrak{L}_u] + \alpha_X$$

  in $\mathrm{Pic}(C)$,

- $\eta \in E'^B_{(u,\iota)} := H^0(C, \iota_1^* T^{0,1*}_{\mathbb{P}^N} \otimes (\pi_X u)^* TB \otimes \iota_1^* \mathcal{O}_{\mathbb{P}^N}(k)) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(k))}$ and $\eta_X \in E'^X_{(u,\iota)} := H^0(C, \iota_1^* T^{0,1*}{}_{\mathbb{P}^{N_X}} \otimes u^* \ker(d\pi_X) \otimes \iota_2^* \mathcal{O}_{\mathbb{P}^{N_X}}(k)) \otimes \overline{H^0(\mathbb{P}^{N_X}, \mathcal{O}_{\mathbb{P}^{N_X}}(k))}$ satisfy
$$\overline{\partial}_J \tilde{u} + \langle \eta \rangle \circ d\tilde{\iota}_1 + \langle \eta_X \rangle \circ d\tilde{\iota}_2 = 0$$
on the normalisation $\tilde{C}$ of C.

Choose a good covering $\mathcal{V}$ of the polyfold of stable maps to $X \times \mathbb{P}^{N_X}$ in the sense of Definition 2.2.12 and let $\lambda_{\mathcal{V}} \colon \mathcal{T}'_X \to \mathrm{PGL}(N_X+1)/G_X$ be the induced $(G\times G_X)$-equivariant map. Define the vector bundle $\mathcal{E}'_X \to \mathcal{T}'_X$ by letting its fibre over $(u, C, \iota, \alpha, \alpha_X, \eta, \eta_X)$ be given by

$$\mathfrak{su}(N+1) \oplus \mathfrak{su}(N_X+1) \oplus H^1(C, \mathcal{O}_C) \oplus H^1(C, \mathcal{O}_C) \oplus E'^B_{(u,\iota)} \oplus E'^X_{(u,\iota)}$$

and define the obstruction section $\mathfrak{s} \colon \mathcal{T}'_X \to \mathcal{E}'_X$ by

$$\mathfrak{s}'_X(u, C, \iota, \alpha, \alpha_X, \eta, \eta_X) = (i\log(\lambda_{\mathcal{U}}(u_B, \iota_{1,B})), i\log(\lambda_{\mathcal{V}}(u, \iota_2)), \alpha, \alpha_X, \eta, \eta_X).$$

This map is clearly equivariant under the canonical action of $G \times G_X$ on $\mathcal{T}'_X$ and $\mathcal{E}'_X$. By the arguments of §2.2, we can choose k sufficiently large (and shrink $\mathcal{T}'_X$) such that $\mathcal{K}'_X := (G \times G'_X, \mathcal{T}'_X/\mathcal{M}'_X, \mathcal{E}'_X, \mathfrak{s}'_X)$ is a global Kuranishi chart for $\overline{\mathcal{M}}_g(X, A_X; J_X)$ with the properties listed in Theorem 2.1.18. By the same reasoning as in §2.4.1 $\mathcal{K}'_X$ satisfies (a).

Define $\tilde{\pi}_X \colon \mathcal{T}'_X \to \mathcal{T}_B$ by

$$\tilde{\pi}_X(u, \iota, C, \alpha, \alpha_X, \eta, \eta_X) = (u_B, \iota_{1,B}, C_B, \alpha, \eta_B)$$

where we identify $H^1(C, \mathcal{O}_C)$ with $H^1(C_B, \mathcal{O}_{C_B})$ via κ_u^*. This is well-defined, since for any irreducible component Z which is contracted by κ_u we have $g(Z) = 0$ and $\eta|_Z$ has to be constant.

Thus it defines a rel–C^∞ map $\mathcal{T}'_X/\mathcal{M}'_X \to \mathcal{T}_B/\mathcal{M}_B$, which is a rel–$C^\infty$ submersion as can be seen by considering the induced map between the relative tangent bundles. Clearly, there exists an isomorphism $\Pi \colon \mathcal{E}'_X \cong \tilde{\pi}_X^* \mathcal{E}_B \oplus \hat{\mathcal{E}}_X$ where

$$(\hat{\mathcal{E}}_X)_y = \mathfrak{su}(N_X+1) \oplus H^1(C, \mathcal{O}_C) \oplus E'^X_{(u,\iota)}$$

for $y = (u, C, \iota, \alpha, \alpha_X, \eta, \eta_X) \in \mathcal{T}'_X$. If $\Pi \colon \mathcal{E}'_X \to \mathcal{E}_B$ is the induced $(G \times G_X)$-equivariant surjective map covering $\tilde{\pi}_X$, then $\Pi \mathfrak{s}'_X = \mathfrak{s}_B \tilde{\pi}_X$ as claimed. □

Corollary 3.2.6. *If $A_B = 0$, then*

$$\sum_{A \in \mathfrak{B}} j_* \mathrm{GW}^{(Z,\omega_Z)}_{A,0,n} = (\pi^* \mathrm{PD}(\Delta_B) \times \mathrm{PD}(\Delta_{\overline{\mathcal{M}}_{0,n}})) \cap (\mathrm{GW}^{(X,\omega_X)}_{A_X,0,n} \otimes \mathrm{GW}^{(Y,\omega_Y)}_{A_Y,0,n}).$$

Restriction of fibre bundles

If $j\colon Y \to B$ is the inclusion of a symplectic submanifold instead, let J be an almost complex structure on B which preserves TY. Abbreviate

$$\overline{\mathcal{M}}_{g,n}(Y, A_B; J_Y) := \bigsqcup_{\pi_{Y*}A_Y = A_B} \overline{\mathcal{M}}_{g,n}(Y, A_Y; J_Y).$$

Lemma 3.2.7. *We can choose $\mathcal{K}_B$ in Proposition 3.2.5 so that there exists a global Kuranishi chart $(G, \mathcal{T}_Y/\mathcal{M}_B, \mathcal{E}_Y, \mathfrak{s}_Y)$ for $\overline{\mathcal{M}}_{g,n}(Y, A_B; J_Y)$ which embeds relatively smoothly into $\mathcal{K}_B$.*

Proof. Choose a J_B-linear connection ∇^B that restricts to a connection ∇^Y on TY and pick any polarisation $\mathcal{O}_B(1)$. Extend to an unobstructed auxiliary datum $(\nabla^B, \mathcal{O}_B(1), p, \mathcal{U} \cap \mathcal{Z}(Y, A), k)$, where the choice of $\mathcal{U}$ has to be done compatible with Y and k has to be chosen sufficiently large. Then $(\nabla^Y, \mathcal{O}_B(1)|_Y, p, \mathcal{U} \cap \mathcal{Z}(Y, A), k)$ is an unobstructed auxiliary datum for $\overline{\mathcal{M}}_{g,n}(Y, A_B; J_Y)$. To obtain the embedding $\mathcal{T}_Y \hookrightarrow \mathcal{T}_B$, we increase k (on both sides) so that the operator

$$D\overline{\partial}_J(u) + \langle\cdot\rangle \circ d\iota\colon C^\infty(C, u^*N_{Y/X}) \oplus N_E \to \Omega^{0,1}(\tilde{C}, u^*N_{Y/X})$$

is surjective for (u, C, x_*, ι) with $\lambda(u, \iota) = [\mathrm{Id}]$.

Here $N_E = \overline{\mathrm{Hom}}_{\mathbb{C}}(\iota^*T_{\mathbb{P}^N}, u^*N_{Y/X}) \otimes \iota^*\mathcal{O}(k) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}(k))}$. $\square$

Combining Proposition 3.2.5 with (3.1.2.1) we deduce that

$$\sum_{p_{X*}A = A_X} j_*(e_{G\times G'_X}(\pi^*\mathcal{E}_B) \cap [\overline{\mathcal{M}}_{g,n}(X|_Y, A; J_X)]^{\mathrm{vir}}) = \pi_X^*\mathrm{PD}_G(\mathcal{T}_Y) \cap [\overline{\mathcal{M}}_{g,n}(X, A_X; J_X)]^{\mathrm{vir}}. \tag{3.2.0.2}$$

Example 3.2.8. Suppose (B, ω_B, J_B) is a convex Kähler manifold and Y the zero locus of a transverse holomorphic section of a convex holomorphic vector bundle $p\colon V \to B$. Denote $\mathcal{V} = \overline{\mathcal{M}}_{0,n}(V, p_*^{-1}A_B; J_L)$. Then $\tilde{Y} := \pi_X^{-1}(Y) \subset X$ satisfies

$$\sum_{p_{X*}A = A_X} j_*[\overline{\mathcal{M}}_{0,n}(\tilde{Y}, A; J_X)]^{\mathrm{vir}} = \pi_X^* e(\mathcal{V}) \cap [\overline{\mathcal{M}}_{0,n}(X, A_X; J_X)]^{\mathrm{vir}} \tag{3.2.0.3}$$

Proof. The reasoning of the first step was used in [Kon95] and [KKP03] to outline a proof of, respectively prove the Quantum Lefschetz Hyperplane Theorem. By the conditions on B and V, $\overline{\mathcal{M}}_{g,n}(B, A_B; J_B)$ is unobstructed and $\mathcal{V}$ is a smooth vector bundle over it. The section ρ defines a section $\tilde{\rho}$ of $\mathcal{V}$ with zero locus given by $\bigsqcup_{j_*A_Y = A_B} \overline{\mathcal{M}}_{0,n}(Y, A_Y; J_Y)$. As V is convex, a long exact sequence arguemnt shows that $\tilde{\rho}$ intersects the zero section transversely. Hence,

$$\sum_{j_*A_Y = A_B} j_*[\overline{\mathcal{M}}_{0,n}(Y, A_Y; J_Y)]^{\mathrm{vir}} = e(\mathcal{V}) \cap [\overline{\mathcal{M}}_{0,n}(B, A_B; J_B)]^{\mathrm{vir}}. \tag{3.2.0.4}$$

Thus (3.2.0.3) follows from Lemma 3.1.9 and (3.2.0.2). $\square$

Chapter 4

Relations between GW invariants

4.1 Kontsevich–Manin axioms

The Kontsevich-Manin axioms are a catchphrase used to describe the properties in [KM94] that GW invariants are expected to satisfy. In contrast to the introduction, we will consider here the *Gromov–Witten classes*

$$\mathrm{GW}^{X,\omega}_{g,n,A} := (\mathrm{ev} \times \mathrm{st})_*[\overline{\mathcal{M}}_{g,n}(X, A; J)]^{\mathrm{vir}}$$

where $J \in \mathcal{J}_\tau(X, \omega)$ is arbitrary. While the axioms are less elegant in terms of the GW classes, their proof is slightly more transparent.

The Effective, Homology, and Grading axioms follow directly from the construction.

Lemma 4.1.1 (Symmetry)**.** *The GW invariants* $\mathrm{GW}^{X,A}_{g,n}$ *satisfy*

$$\langle \alpha_{\sigma(1)} \times \cdots \times \alpha_{\sigma(n)} \times \mathrm{PD}(\sigma_*\beta), \mathrm{GW}^{X,A}_{g,n}\rangle = (-1)^{\epsilon(\sigma,\alpha)}\langle \alpha_1 \times \cdots \times \alpha_n \times \mathrm{PD}(\beta), \mathrm{GW}^{X,A}_{g,n}\rangle$$

for any permutation $\sigma \in S_n$ and classes $\alpha_i \in H^*(X;\mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$, where

$$\epsilon(\sigma, \alpha) = |\{i > j \mid \sigma(i) < \sigma(j),\ |\alpha_i|, |\alpha_j| \in 2\mathbb{Z} + 1\}|.$$

Proof. Given a global Kuranishi charts $\mathcal{K}_n$ as in Construction 2.1.14, the holomorphic S_n-action on the base space $\mathcal{M}_n \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)$ lifts to a continuous action by rel–C^∞ diffeomorphisms on the thickening and the obstruction bundle. As the equivariant Thom class of the obstruction bundle is S_n-invariant, so is the virtual fundamental class. □

Lemma 4.1.2 (Mapping to a point)**.** *We have*

$$[\overline{\mathcal{M}}_{g,n}(X, 0; J)]^{\mathrm{vir}} = c_{g\dim_{\mathbb{C}}(X)}(T_X \boxtimes \mathbb{E}^\vee) \cap [X \times \overline{\mathcal{M}}_{g,n}]$$

in $H_*(X \times \overline{\mathcal{M}}_{g,n};\mathbb{Q})$, where $\mathbb{E}$ denotes the Hodge bundle over $\overline{\mathcal{M}}_{g,n}$.

Proof. Given a constant stable map $u : C \to X$ with image x, we have

$$H^1(C, u^*T_X) = H^1(C, \mathcal{O}_C) \otimes T_xX.$$

Thus the cokernel of $D\bar{\partial}_J(u)$ has rank $2g\dim(X)$ and the obstruction bundles of $\overline{\mathcal{M}}_{g,n}(X,0;J) = X \times \overline{\mathcal{M}}_{g,n}$ is given by $\mathrm{Ob} := T_X \boxtimes \mathbb{E}^*$. Let s_0 denote its zero section. Fix a global Kuranishi chart $\mathcal{K}_n = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ as given by Construction 2.1.14 with base space $\mathcal{M} \subset \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m)$. Denote by $\mathcal{C} \to \mathcal{M}$ the universal curve and set $\mathcal{L} := R^1\pi_*\mathcal{O}_{\mathcal{C}}$. Then $\mathcal{K}_{\mathrm{ob}} := (G, \mathcal{T}_{\mathrm{ob}}, \widetilde{\mathrm{Ob}}, \tilde{\mathfrak{s}})$ with

$$\mathcal{T}_{\mathrm{ob}} := X \times \{([\iota, C, x_1, \ldots, x_n], \alpha) \in \mathcal{L} \mid [\iota^*\mathcal{O}(1)] = p \cdot [\omega_C(x_1 + \cdots + x_n)] + \alpha\}$$

is a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,0;J)$ of the expected virtual dimension. The obstruction bundle is

$$\widetilde{\mathrm{Ob}} = \mathrm{Ob} \boxplus \mathcal{L} \oplus \mathfrak{su}(N+1)$$

and $\tilde{\mathfrak{s}}$ is given by the zero section in the first summand, the obvious map in the second one and $i\log(\lambda)$ in the last. Let $j\colon \mathcal{T}_{\mathrm{ob}} \hookrightarrow \mathcal{T}$ be the inclusion. There exists a natural equivariant morphism $\Phi\colon j^*\mathcal{E} \to \widetilde{\mathrm{Ob}}$ of complex vector bundles; it is given by the identity on $\mathfrak{su}(N+1)$ and $\mathcal{L}$ and maps the perturbation term η to the image of $\langle \eta \rangle \circ d\iota$ under the quotient map $\Omega^{0,1}_J(\tilde{C}, \tilde{u}^*T_X) \to H^1(C, u^*T_X)$. By the construction of $\mathcal{K}_n$, the map Φ is surjective. Moreover, its kernel agrees with the normal bundle $N_{X \times \mathcal{M}/\mathcal{T}}$ of $X \times \mathcal{M}$ in $\mathcal{T}$ (as rel–C^∞ manifolds over $\mathcal{M}$). Fixing a splitting $L\colon \widetilde{\mathrm{Ob}} \to \mathcal{E}|_{\mathcal{T}'}$ of Φ we obtain that the two-term complexes associated to $\mathcal{K}_{\mathrm{ob}}$ and $\mathcal{K}_n$ respectively are quasi-isomorphic in the sense of Lemma 3.1.9 whence the claim follows. □

Remark 4.1.3. This argument can be applied in any situation where $\overline{\mathcal{M}}_{g,n}(X,A;J)$ is smooth with obstruction bundle Ob to see that $[\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}} = e(\mathrm{Ob}) \cap [\overline{\mathcal{M}}_{g,n}(X,A;J)]$ under the identification of the dual of Čech cohomology with singular homology.

We observe the following vanishing statement, alluded to in [KM94] and corresponding to [RT97, Proposition 2.14(3)].

Lemma 4.1.4. *If* $(1-g)(\dim_{\mathbb{C}}(X) - 3) + 2\langle c_1(T_X), A\rangle < 0$, *then* $\mathrm{GW}^{X,\omega}_{g,n,A} = 0$ *for any* $n \geqslant 0$.

Proof. Let $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ be a global Kuranishi chart for $\overline{\mathcal{M}}_g(X,A;J)$ and let $(G, \mathcal{T}_n, \mathcal{E}_n, \mathfrak{s}_n)$ be the induced global Kuranishi chart for $\overline{\mathcal{M}}_g(X,A;J)$. By construction, there exist equivariant maps $\pi_n\colon \mathcal{T}_n \to \mathcal{T}$ and $\tilde{\pi}_n\colon \mathcal{E}_n = \pi_n^*\mathcal{E} \to \mathcal{E}$ satisfying $\tilde{\pi}_n\mathfrak{s}_n = \mathfrak{s}_n\pi_n$. As $|\mathfrak{s}^*\tau_{\mathcal{E}/G}| > \dim(\mathcal{T}/G)$, it follows that $\mathfrak{s}_n^*\tau_{\mathcal{E}_n/G} = \pi_n^*\mathfrak{s}^*\tau_{\mathcal{E}/G} = 0$. □

4.1.1 Fundamental class axiom

Suppose $n \geqslant 1$ and $A \neq 0$. Let $\mathcal{K}$ be a global Kuranishi chart for $\overline{\mathcal{M}}_g(X,A;J)$ as given by Construction 2.1.14. Let $\mathcal{K}_n$ and $\mathcal{K}_{n-1}$ be the induced global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X,A;J)$, respectively $\overline{\mathcal{M}}_{g,n-1}(X,A;J)$. Denote by $\mathcal{M}_n$ and $\mathcal{M}_{n-1}$ their bases spaces.

Proposition 4.1.5 (Fundamental class)**.** *For* $\alpha_1, \ldots, \alpha_{n-1} \in H^*(X^n;\mathbb{Q})$ *and* $\beta \in H_*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ *we have*

$$\langle \alpha_1 \times \cdots \times \alpha_{n-1} \times 1_X \times \mathrm{PD}(\beta), \mathrm{GW}^{X,A}_{g,n}\rangle = \langle \alpha_1 \times \cdots \times \alpha_{n-1} \times \mathrm{PD}(\pi_{n*}\beta), \mathrm{GW}^{X,A}_{g,n-1}\rangle,$$

where $1_X \in H^0(X;\mathbb{Q})$ *is the unit.*

Proof. Let $\pi_n\colon \mathcal{T}_n \to \mathcal{T}_{n-1}$ be the forgetful map. By construction, $\mathcal{E}_n = \pi_n^*\mathcal{E}_{n-1}$ and $\mathfrak{s}_n = \pi_n^*\mathfrak{s}_{n-1}$. We will construct a principal H-bundle $\widetilde{\mathcal{T}}_{n-1} \to \mathcal{T}_{n-1}$ for some compact Lie group H so that the pullback $\pi_n'\colon \widetilde{\mathcal{T}}_n := \mathcal{T}_n \times_{\mathcal{M}_{n-1}} \widetilde{\mathcal{T}}_{n-1} \to \widetilde{\mathcal{T}}_{n-1}$ satisfies

$$(\widetilde{\pi}_n)_!\,\mathrm{st}^* = \mathrm{st}^*\,\pi_n!.$$

This proves the claim since the global Kuranishi chart $\widetilde{\mathcal{K}}_\ell$ determined by $\widetilde{\mathcal{T}}_\ell$ is equivalent to $\mathcal{K}_\ell$ and the obstruction bundle and section of $\widetilde{\mathcal{K}}_n$ are obtained by pullback from $\widetilde{\mathcal{K}}_{n-1}$.

Let thus P_{n-1}/H be a presentation of $\overline{\mathcal{M}}_{g,n-1}$ as a global quotient. Let $P_n = P_{n-1}$ be the pullback of P_{n-1} along the representable morphism π_n. Then P_n/H is a presentation of $\overline{\mathcal{M}}_{g,n}$. The canonical map $\pi_n''\colon P_n \to P_{n-1}$ is the pullback of π_n and satisfies $\pi''_{n!}\mathrm{st}^* = \mathrm{st}^*\pi_{n!}$ by Corollary A.1.3. By Lemma A.1.7, $\mathcal{N}_\ell := \mathcal{M}_\ell \times_{\overline{\mathcal{M}}_{g,\ell}} P_\ell$ is a principal H-bundle over $\mathcal{M}_\ell$ for $\ell = n-1, n$ and $\mathcal{N}_n = \mathcal{N}_{n-1} \times_{\mathcal{M}_{n-1}} \mathcal{M}_n$. Then $\widetilde{\mathcal{T}}_\ell := \mathcal{T}_\ell \times_{\mathcal{M}_\ell} \widetilde{\mathcal{N}}_\ell$ is a principal H-bundle over $\mathcal{T}_\ell$ and $(G\times H, \widetilde{\mathcal{T}}_\ell, p_\ell^*\mathcal{E}_\ell, p_\ell^*\mathfrak{s}_\ell)$ is a global Kuranishi chart equivalent to $\mathcal{K}_\ell$ for $\ell \in \{n, n-1\}$. The induced map $\widetilde{\pi}_n\colon \widetilde{\mathcal{T}}_n \to \widetilde{\mathcal{T}}_{n-1}$ is the pullback of π_n'' along $p_{n-1}\colon \widetilde{\mathcal{T}}_{n-1} \to P_{n-1}$. The map p_{n-1}, factoring through $\mathcal{N}_{n-1}$, is G-invariant and a submersion away from a subset of real codimension at least 2. By Lemma A.1.6 combined with the functoriality of the exceptional pushforward, it follows that $(\widetilde{\pi}_n)_!\,\mathrm{st}^* = \mathrm{st}^*\pi_{n!}$. The case of $A = 0$ meanwhile follows from Lemma 4.1.2. $\square$

Remark 4.1.6. If $2g-2+n-1 \leqslant 0$, we can repeat the argument by replacing $\overline{\mathcal{M}}_{g,n-1}$ (and possibly $\overline{\mathcal{M}}_{g,n}$) by a point. This shows that

$$\langle \alpha_1 \times \cdots \times \alpha_{n-1} \times 1_X, \mathrm{ev}_*[\overline{\mathcal{M}}_{g,n}(X,A;J]^{\mathrm{vir}}\rangle = 0$$

for any $g, n \geqslant 0$.

4.1.2 Divisor axiom

Suppose $n \geqslant 1$ and that $A \neq 0$. The Divisor axiom is the second recursion relation satisfied by the GW invariants of X.

Proposition 4.1.7 (Divisor)**.** *For* $\alpha_1, \ldots, \alpha_n \in H^*(X;\mathbb{Q})$ *with* $|\alpha_n| = 2$ *and* $\beta \in H_*(\overline{\mathcal{M}}_{g,n-1};\mathbb{Q})$ *we have*

$$\langle \alpha_1 \times \cdots \times \alpha_n \times \pi_n^*\mathrm{PD}(\beta), \mathrm{GW}^{X,A}_{g,n}\rangle = \langle \alpha_n, A\rangle\langle \alpha_1 \times \cdots \times \alpha_{n-1} \times \mathrm{PD}(\beta), \mathrm{GW}^{X,A}_{g,n-1}\rangle.$$

The first, crucial observation is that we can construct our global Kuranishi chart in such a way that the evaluation maps become relative submersions.

Lemma 4.1.8. *We can choose the auxiliary datum in the construction of* $\mathcal{K}_{g,n,A}$ *in Theorem 2.1.18 so that the evaluation map* $\mathrm{ev}\colon \mathcal{T}_n \to X^n$ *is a relative submersion.*

Proof. Given an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$, fix a point $[u, C, x_1, \ldots, x_n]$ in $\overline{\mathcal{M}}_{g,n}(X,A;J)$, and let $\mathcal{F}$ be a basis of $H^0(C, \mathfrak{L}_u^{\otimes p})$ with $\lambda_{\mathcal{U}}(u, \iota_{C,\mathcal{F}}) = 0$. By [AMS21, Proposition 6.26], using [AMS21, Lemma 6.24] with the divisor consisting of the nodal *and* marked points, there exists an integer $k'_u \geqslant k$ so that the linearisation of $\overline{\partial}_J + \langle\cdot\rangle$ at (u,ι) restricted to

$$H^0(C, \overline{\mathrm{Hom}}_{\mathbb{C}}(\iota^*_{C,\mathcal{F}}T_{\mathbb{P}^N}, u^*T_X) \otimes \iota^*_{C,\mathcal{F}}\mathcal{O}_{\mathbb{P}^N}(k') \otimes \mathcal{O}_C(-x)) \otimes \overline{H^0(C, \iota^*_{C,\mathcal{F}}\mathcal{O}_{\mathbb{P}^N}(k'))}$$

surjects onto the cokernel of $D(\overline{\partial}_J)u$ restricted to

$$V_{\{x_j\}} := \{\xi \in C^\infty(C, u^*T_X) \mid \forall j : \xi(x_j) = 0\}.$$

As surjectivity is an open condition, the claim follows from the compactness of $\mathfrak{s}^{-1}(0)$. $\square$

Let now $Y \subset X$ be a smooth hypersurface Poincaré dual to a class $\gamma \in H^2(X,\mathbb{Z})$. Let $\mathcal{K}_{g,n,A}$ be a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A;J)$ satisfying the conclusion of Lemma 4.1.8. Let $\mathcal{K}_Y$ be the global Kuranishi chart with thickening $\mathcal{T}_Y := \mathrm{ev}_n^{-1}(Y)$ and all other data given by restriction. Denote $\overline{\mathcal{M}}_Y := \mathfrak{s}_Y^{-1}(0)/G$ and let $j\colon \mathcal{T}_Y/G \hookrightarrow \mathcal{T}_n/G$ be the inclusion. By Proposition 3.1.7,

$$j_*[\overline{\mathcal{M}}_Y]^{\mathrm{vir}} = \mathrm{ev}_n^*\gamma \cap [\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}.$$

The forgetful map π_n restricts to a proper map $\mathcal{T}_Y \to \mathcal{T}_{n-1}$ of manifolds of the same dimension.

Lemma 4.1.9. $\pi_{n*}[\overline{\mathcal{M}}_Y]^{\mathrm{vir}} = \langle\gamma, A\rangle\,[\overline{\mathcal{M}}_{g,n-1}(X,A;J)]^{\mathrm{vir}}$.

Proof. We show that $\mathcal{T}_Y \to \mathcal{T}_{n-1}$ has degree $\langle\gamma, A\rangle$. It suffices to check the claim for a generic point in each connected component of $\mathcal{T}_{n-1}$. Since $\mathcal{T}_{n-1}$ is a covering of a space of regular stable maps, we may choose each such point to have smooth domain. Fix thus $y = [u,\iota,C,x_1,\dots,x_{n-1},\alpha,\eta] \in \mathcal{T}_{n-1}$ with $u \pitchfork Y$. By the definition of $\mathcal{T}_n$ and $\mathcal{T}_{n-1}$ we can find a neighbourhood U_{n-1} so that $U_{n-1} \cong V_{n-1} \times B_y$ where $B_y \subset T_{\mathcal{T}_{n-1}/\mathcal{M}_{n-1},y}$ is an open neighbourhood of the origin and $V_{n-1} \subset \mathcal{M}_{n-1}$. Then $U_n := \pi_n^{-1}(U_{n-1})$ is canonically isomorphic to $\pi_{n-1}^{-1}(V_{n-1}) \times B_y \cong V_{n-1} \times C \times B_y$; the latter is guaranteed by shrinking V_{n-1}. Shrinking V_{n-1} further if necessary, $U_n \cap \mathcal{T}_Y \cong V_{n-1} \times W$, where W is smooth with tangent space

$$T_{(x,0)}W = \{(v,\xi) \in T_xC \times B_y \mid du(x)v - \xi(x) \in T_{u(y)}Y\}.$$

for $x \in u^{-1}(Y)$. The forgetful map $U_n \cap \mathcal{T}_Y \to U_{n-1}$ is identified with $\mathrm{id} \times \psi$, where $\psi(z,\xi) = \xi$. As

$$\begin{tikzcd}
T_{(x,0)}W \arrow[r, "\mathrm{pr}_1"] \arrow[d, "d\psi(x,0)"] & T_xC \arrow[d, "du(x)"] \\
T_{\mathcal{T}_{n-1}/\mathcal{M}_{n-1},y} \arrow[r, "\mathrm{ev}_x"] & T_{u(x)}X/T_{u(x)}Y
\end{tikzcd}$$

is cartesian for any $x \in u^{-1}(Y)$, we have $\deg(\psi,(x,0)) = \mathrm{ind}(u,Y,x)$. This completes the proof. $\square$

The Divisor axiom is an immediate consequence.

4.1.3 Splitting axiom

Fix $g = g_0 + g_1$ and $n = n_0 + n_1$ with $2g_i - 2 + n_i + 1 > 0$. Let $S \subset \{1, \ldots, n\}$ be a subset with $|S| = n_0$. The clutching map

$$\varphi_S : \overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_0,n_0+1} \to \overline{\mathcal{M}}_{g,n}$$

given by gluing two curves together at the $(n_0+1)^{\text{th}}$ and first marked point and renumbering according to the partition induced by S is a closed local embedding. This map lifts to maps

$$\varphi_{S,X} : \overline{\mathcal{M}}_{g_0,n_0+1}(X, A_0; J) \times_X \overline{\mathcal{M}}_{g_1,n_1+1}(X, A_1; J) \to \overline{\mathcal{M}}_{g,n}(X, A_0 + A_1; J).$$

Together with the clutching maps described in the next subsection, the images of the maps φ_S form the boundary divisor of $\overline{\mathcal{M}}_{g,n}$. The same is true for moduli spaces of stable maps where one has an additional choice of how to 'split' the homology class. The Splitting axiom is an algebraic reflection thereof.

Proposition 4.1.10 (Splitting)**.** *Write* $\mathrm{PD}(X) = \sum_{i\in I} \gamma_i \times \gamma_i'$ *for* $\gamma_i, \gamma_i' \in H^*(X;\mathbb{Q})$. *We have for* $\alpha_1, \ldots, \alpha_n \in H^*(X;\mathbb{Q})$ *and* $\beta_i \in H_*(\overline{\mathcal{M}}_{g_i,n_i+1};\mathbb{Q})$ *that*

$$\begin{aligned}
&\langle \alpha_1 \times \cdots \times \alpha_n \times \mathrm{PD}(\varphi_{S*}(\beta_0 \otimes \beta_1)), \mathrm{GW}^{X,A}_{g,n}\rangle \\
&\quad = (-1)^{\epsilon(\alpha,S)} \sum_{A_0+A_1=A} \sum_i \langle \alpha_{f_0(1)} \times \cdots \times \alpha_{f_0(n_0)} \times \gamma_i \times \mathrm{PD}(\beta_0), \mathrm{GW}^{X,A_0}_{g_0,n_0+1}\rangle \\
&\qquad\qquad\qquad \langle \gamma_i' \times \alpha_{f_1(2)} \times \cdots \times \alpha_{f_1(n_1+1)} \times \mathrm{PD}(\beta_1), \mathrm{GW}^{X,A_1}_{g_0,n_0+1}\rangle
\end{aligned}$$

where $\epsilon(\alpha, S) := |\{i < j \mid j \in S,\ i \notin S,\ |\alpha_i|, |\alpha_j| \in 2\mathbb{Z}+1\}|$.

By Lemma 4.1.1, we may assume $S = \{1, \ldots, n_0\}$ and omit it from the notation. The domain of φ_X admits a global Kuranishi chart $\mathcal{K}_{g_0,n_0+1,A_0} \times_X \mathcal{K}_{g_1,n_1+1,A_1}$ of the expected virtual dimension by Lemma 4.1.8, which embeds into $\mathcal{K}_{g_0,n_0+1,A_0} \times \mathcal{K}_{g_1,n_1+1,A_1}$. For the sake of brevity, we denote the base space of $\mathcal{K}_{g,n,A}$ by $\mathcal{M}$. For $0 \leqslant m_0 \leqslant m$, let $\widehat{\mathcal{M}}_{g_0,n_0,m_0}$ be the preimage of $\mathcal{M}$ under

$$\varphi_{\mathbb{P}^N} : \overline{\mathcal{M}}_{g_0,n_0+1}(\mathbb{P}^N, m_0) \times_{\mathbb{P}^N} \overline{\mathcal{M}}_{g_1,n_1+1}(\mathbb{P}^N, m_1) \to \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)$$

and set

$$\widehat{\mathcal{M}}_{g_0,n_0} := \bigsqcup_{0\leqslant m_0\leqslant m} \widehat{\mathcal{M}}_{g_0,n_0,m_0}.$$

Lemma 4.1.11. $\widehat{\mathcal{M}}_{g_0,n_0,m_0}$ *is a complex manifold of the expected dimension.*

Proof. Suppose $\varphi_{\mathbb{P}^N}([\iota, C, x_*], [\iota', C', x_*']) = [u, \Sigma, y_*] \in \mathcal{M}$. As the normalisation of Σ is $\tilde{\Sigma} = \tilde{C} \sqcup \tilde{C}'$ and u is unobstructed, so are ι and ι'. If ρ is an automorphism of (ι, C, x_*), it can be extended by the identity to an automorphism of (u, Σ, y_*). Hence $\rho = \mathrm{id}_C$. Similarly, we see that (ι', C', x'_*) has no isotropy. Thus we may conclude by [RRS08]. $\square$

Denote by $\hat{\varphi}\colon \widehat{\mathcal{M}}_{g_0,n_0} \to \mathcal{M}$ the map induced by $\varphi_{\mathbb{P}^N}$. It is a $\mathrm{PGL}_{\mathbb{C}}(N+1)$-equivariant immersion whose image has normal crossing singularities.

Let $[u, C, x_*] \in \overline{\mathcal{M}}_{g,n}(X, A; J)$ lie in the image of φ_X. Any splitting of the domain into two curves of the prescribed genus and prescribed set of points defines (via the restriction of a framing coming from $\mathcal{L}_u$) a unique element in the image of $\hat{\varphi}$. Conversely, any decomposition of the domain of the framing leads to a corresponding splitting of $[u, C, x_*]$, where the degree of the restrictions of u may vary. This shows that $\mathcal{K}_{g_0,n_0} := \hat{\varphi}^*\mathcal{K}_{g,n,A}$ is a global Kuranishi chart for

$$\overline{\mathcal{M}}_{g_0,n_0}(X) := \bigsqcup_{A_0+A_1=A} \overline{\mathcal{M}}_{g_0,n_0+1}(X, A_0; J) \times_X \overline{\mathcal{M}}_{g_1,n_1+1}(X, A_1; J).$$

as is $\mathcal{K}'_{g,n,A} := \bigsqcup_{A_0+A_1=A} \mathcal{K}_{g_0,n_0+1,A_0} \times_X \mathcal{K}_{g_1,n_1+1,A_1}$.

Lemma 4.1.12. *$\mathcal{K}_{g_0,n_0}$ and $\mathcal{K}'_{g,n,A}$ are equivalent.*

Proof. This follows from a double-sum construction as in §2.4.1. $\square$

We can now prove Proposition 4.1.10. The strategy of proof is the same as in Proposition 4.1.5 with the additional complication that $\tilde{\varphi}$ is not the pullback of φ.

Proof. We first sketch the proof in the case of genus zero. In higher genus, the fact that we have to take fibre product of orbifolds adds a layer of complexity, however the strategy is the same.

We factor $\varphi\colon \widehat{\mathcal{M}}_{g_0,n_0} \to \mathcal{M}$ as

$$\widehat{\mathcal{M}}_{0,n_0} \xrightarrow{\vartheta} \mathcal{M} \times_{\overline{\mathcal{M}}_{0,n}} (\overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1}) \xrightarrow{\varphi'} \mathcal{M}.$$

Then $\mathcal{M} \times_{\overline{\mathcal{M}}_{0,n}} (\overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1})$ is a quai-projective variety over $\mathbb{C}$ but not necessarily smooth. However, it is a homology $\mathbb{Q}$-manifold. As φ' is the pullback of the clutching map on the level of moduli space of stable curves, it is a closed immersion of schemes. We will show below that ϑ has degree 1. Lifting this factorisation to the level of thickenings, we obtain that $\hat{\varphi}_!\,\mathrm{st}^* = \mathrm{st}^*\,\varphi_!$ as maps $H^*(\overline{\mathcal{M}}_{0,n_0+1} \times \overline{\mathcal{M}}_{0,n_1+1}; \mathbb{Q}) \to H^*(\mathcal{T}/G; \mathbb{Q})$ by the results of §A.

For the general case, let P/H be a presentation of $\overline{\mathcal{M}}_{g,n}$ as a global quotient. By [ACG11], we may assume that $P \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^\ell, \ell + g)$ for some $\ell \geqslant 1$. As φ is representable by [ACG11, Proposition 12.10.11], $P' := P \times_{\overline{\mathcal{M}}_{g,n}} (\overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1})$ is a G-smooth manifold with P'/H representing $\overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1}$. Moreover, $\mathcal{N} := P \times_{\overline{\mathcal{M}}_{g,n}} \mathcal{M}$ is a principal G'-bundle over $\mathcal{M}$, while

$$\mathcal{N}_{g_0,n_0} := \mathcal{N} \times_{\mathcal{M}} \widehat{\mathcal{M}}_{g_0,n_0} = P' \times_{\overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1}} \widehat{\mathcal{M}}_{g_0,n_0}$$

is one over $\widehat{\mathcal{M}}_{g_0,n_0}$. Denote by $\mathrm{st}\colon \mathcal{N} \to \overline{\mathcal{M}}_{g,n}$ and $\mathrm{st}\colon \mathcal{N}_{g_0,n_0} \to \overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1}$ the induced stabilisation maps. Let $\tilde{\varphi}\colon \mathcal{N}_{g_0,n_0} \to \mathcal{N}$ be the morphism of principal bundles covering $\hat{\varphi}$. The diagram

$$\begin{tikzcd}
\mathcal{N}_{g_0,n_0} \arrow[r, "\vartheta"] \arrow[dr, "\mathrm{st}"'] & \mathcal{N}' \arrow[r, "\varphi'"] \arrow[d, "\mathrm{st}"] & \mathcal{N} \arrow[d, "\mathrm{st}"] \\
& \overline{\mathcal{M}}_{g,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1} \arrow[r, "\varphi"] & \overline{\mathcal{M}}_{g,n}
\end{tikzcd}$$

commutes, where $\tilde{\varphi} = \varphi'\vartheta$ and $\mathcal{N}' = P' \times_P \mathcal{N}$, so the square is cartesian. To see that ϑ is a birational equivalence, let $\mathcal{N}'^{\mathrm{sm}} \subset \mathcal{N}'$ be preimage (under $\mathcal{N}' \to \mathcal{N} \to \mathcal{M}$) of the subset of $\mathcal{M}$ consisting of curves with one node. As all elements of $\mathcal{M}$ are unobstructed, the complement of $\mathcal{N}'^{\mathrm{sm}}$ is of codimension at least 2. It follows from a straightforward consideration of the fibre product $\mathcal{N}'$ that the induced map $\mathcal{N}^{\mathrm{sm}}_{g_0,n_0} \to \mathcal{N}'^{\mathrm{sm}}$ is an isomorphism. Thus ϑ is a birational equivalence. In particular, it has degree 1.

Pull back $\widehat{\mathcal{K}}_{g_0,n_0}$ along $\mathcal{N}_{g_0,n_0} \to \widehat{\mathcal{M}}_{g_0,n_0}$ and change the covering group from G to $G \times G'$ to obtain an equivalent global Kuranishi chart $\widehat{\mathcal{K}}^{\mathcal{N}}_{g_0,n_0}$. Define $\mathcal{K}^{\mathcal{N}}_{g,n}$ analogously to obtain a global Kuranishi chart which is rel–C^∞ over $\mathcal{N}$ and equivalent to $\mathcal{K}_{g,n}$. The pullback of $\hat{\varphi}$ to a map between thickenings descends to φ_X when restricted to the zero locus of the obstruction section. It factors as $\widetilde{\varphi} = \widetilde{\varphi}'\widetilde{\vartheta}$, which are lifts of φ' respectively ϑ. Then $\widetilde{\vartheta}$ has degree 1, so $\widetilde{\varphi}$ satisfies

$$\widetilde{\varphi}_! \,\mathrm{st}^* = \mathrm{st}^* \,\varphi_!$$

as maps $H^*(\overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1};\mathbb{Q}) \to H^*(\mathcal{T}^{\mathcal{N}}/G \times H;\mathbb{Q})$ by Lemma A.1.3 and Lemma A.1.6. As the obstruction bundle of $\widehat{\mathcal{K}}^{\mathcal{N}}_{g_0,n_0}$ is the pullback of the obstruction bundle of $\mathcal{K}^{\mathcal{N}}_{g,n}$, this shows that

$$\sum_{A_0+A_1=A} \varphi_{X*}(\mathrm{st}^*\gamma \cap [\overline{\mathcal{M}}_{g_0,n_0+1}(X,A_0;J) \times_X \overline{\mathcal{M}}_{g_1,n_1+1}(X,A_1;J)]^{\mathrm{vir}})$$
$$= \mathrm{st}^*(\varphi_!\gamma) \cap [\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}. \quad (4.1.3.1)$$

for any $\gamma \in H^*(\overline{\mathcal{M}}_{g_0,n_0+1} \times \overline{\mathcal{M}}_{g_1,n_1+1};\mathbb{Q})$. Applying Proposition 3.1.7 to $\mathcal{K}'_{g,n,A}$, we obtain an expression for the left hand side that exact gives the Splitting axiom. ☐

4.1.4 Genus reduction axiom

Fix g and $n \geqslant 2$ and let $\mathrm{ev}_{n-1,n}$ be the evaluation at the $(n-1)^{\text{th}}$ and n^{th} marked point. Given an almost complex manifold (Y, J_Y) and $B \in H_2(Y,\mathbb{Z})$, define

$$\overline{\mathcal{M}}_{g,n}(Y,B;J_Y)_{n-1,n} := \mathrm{ev}^{-1}_{n-1,n}(\Delta_Y) \subset \overline{\mathcal{M}}_{g,n}(Y,B;J_Y).$$

Let

$$\psi_Y : \overline{\mathcal{M}}_{g,n}(Y,B;J_Y)_{n-1,n} \to \overline{\mathcal{M}}_{g+1,n-2}(Y,B;J_Y)$$

be given by gluing the $(n-1)^{\text{nth}}$ and the n^{nth} marked point of the domain together. It covers the corresponding smooth map $\psi\colon \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g+1,n-2}$.

Proposition 4.1.13 (Genus reduction)**.** *We have for any $\alpha_1, \ldots, \alpha_{n-2} \in H^*(X;\mathbb{Q})$ and $\beta \in H_*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$ that*

$$\begin{aligned}\langle \alpha_1 \times \cdots \times \alpha_{n-2} \times \mathrm{PD}(\Delta_X) \times \mathrm{PD}(\beta), \mathrm{GW}^{X,A}_{g,n}\rangle \\ = \langle \alpha_1 \times \cdots \times \alpha_{n-2} \times \mathrm{PD}(\psi_*\beta), \mathrm{GW}^{X,A}_{g+1,n-2}\rangle.\end{aligned}$$

Lemma 4.1.14. *The preimage*

$$\widetilde{\mathcal{M}} := \psi_{\mathbb{P}^N}^{-1}(\overline{\mathcal{M}}^*_{g+1,n-2}(\mathbb{P}^N, m)) \subset \overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)$$

is a smooth manifold of the expected dimension. The induced map

$$\psi_{\mathbb{P}^N} : \widetilde{\mathcal{M}} \to \overline{\mathcal{M}}^*_{g+1,n-2}(\mathbb{P}^N, m)$$

is smooth and G-equivariant. It factors as the composition of a double cover with a map which is generically an embedding.

Proof. Suppose $\psi([\iota, C, x_*]) = [\iota', C', x'_*]$ and let $\rho \in \mathrm{Aut}(\iota, C, x_*)$. Then ρ descends to an automorphism of (ι', C', x'_*), which has to be the identity. As the gluing map $\kappa\colon C \to C'$ is injective on a dense subset, $\rho = \mathrm{id}_C$. We have a short exact sequence $0 \to \kappa^*\iota'^*\mathcal{O}_{\mathbb{P}^N}(1) \to \iota^*\mathcal{O}_{\mathbb{P}^N}(1) \to \mathbb{C}_x \to 0$ where $\mathbb{C}_x$ denotes the skyscraper sheaf over x and the last map is given by $s \mapsto s(x_{n-1}) - s(x_n)$ (using a trivialisation of $\mathcal{O}_{\mathbb{P}^N}(1)$ near $\iota(x_n)$). Part of its long exact sequence is

$$0 = H^1(C', \iota'^*\mathcal{O}_{\mathbb{P}^N}(1)) \to H^1(C, \iota^*\mathcal{O}_{\mathbb{P}^N}(1)) \to H^1(\{x\}, \mathbb{C}) = 0.$$

Hence ι is unobstructed.

$\mathbb{Z}/2$ acts freely and smoothly on $\widetilde{\mathcal{M}}$ by permuting the last two points and $\psi_{\mathbb{P}^N}$ factors through $\widetilde{\mathcal{M}}/(\mathbb{Z}/2)$. As $\psi_{\mathbb{P}^N}$ is an immersion and $\widetilde{\mathcal{M}}/(\mathbb{Z}/2) \to \overline{\mathcal{M}}^*_{g+1,n-2}(\mathbb{P}^N, m)$ is injective over the locus of curves with smooth domain, the claim follows. □

Let $\mathcal{K}_{g,n,A}$ be a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X, A; J)$ satisfying the conclusion of Lemma 4.1.8. Let $\mathcal{K}'$ be the global Kuranishi chart with thickening $\mathcal{T}' = \mathrm{ev}^{-1}_{n-1,n}(\Delta_X)$ and whose other data are given by restriction from $\mathcal{K}_{g,n,A}$. Similar reasoning as in §2.4.1 can be used to show that $\mathcal{K}'$ and $\psi^*_{\mathbb{P}^N}\mathcal{K}_{g+1,n-2}$ are equivalent global Kuranishi charts for $\overline{\mathcal{M}}_{g,n}(X, A; J)_{n-1,n}$.

Pulling back $\psi^*_{\mathbb{P}^N}\mathcal{K}_{g+1,n-2}$ along $\mathcal{N} \to \overline{\mathcal{M}}^*_{g+1,n-2}(\mathbb{P}^N, m)$ and taking the product of the covering group with G', we obtain an equivalent global Kuranishi chart $\tilde{\mathcal{K}}_{g+1,n-2}$ for $\overline{\mathcal{M}}_{g,n}(X, A; J)_{n-1,n}$. The proof of Proposition 4.1.10 now carries over in a straightforward manner.

4.2 Gravitational descendants

The theory of GW invariants can be enriched by additionally integrating natural classes on the moduli space of stable maps itself over the virtual fundamental class. This extension is motivated by theoretical physics, [Wit91, EHX97]. In the symplectic setting, these generalised GW invariants were first defined in [RT97, §6] and in [KM98] for projective varieties. In the general symplectic setting the definition is not as immediate. This can already seen from the case where X is a point. While there are natural vector bundles on the moduli *stack* of stable curves, they do not descend to the coarse moduli space $\overline{M}_{g,n}$. Refer [ACG11, Chapter 13] for a discussion and the definition of a vector bundle on a stack. We will simply define the necessary objects as we need them rather than in full generality.

Definition 4.2.1. Given an n-pointed family $(\pi\colon \mathcal{C} \to \mathcal{V}, \sigma_1, \dots, \sigma_n)$ of stable curves and $i \leqslant n$, we define the i^{th} *tautological line bundle* of $\mathcal{V}$ to be $\mathbb{L}_i^{\mathcal{V}} := \sigma_i^*(\ker(d\pi)^*)$. We define the i^{th} ψ*-class* to be

$$\psi_i^{\mathcal{V}} := c_1(\mathbb{L}_i^{\mathcal{V}}) \in H^2(\mathcal{V};\mathbb{Z}).$$

The *Hodge bundle* of $\mathcal{C} \to \mathcal{V}$ is the complex rank-g vector bundle $\mathbb{E}^{\mathcal{V}} := \pi_*\omega_{\mathcal{C}/\mathcal{V}}$. The λ*-classes* are

$$\lambda_j^{\mathcal{V}} := c_j(\mathbb{E}^{\mathcal{V}}).$$

These vector bundles patch together to form the i^{th} *tautological line bundle*, respectively the *Hodge bundle*, on the Deligne-Mumford stack $\overline{\mathcal{M}}_{g,n}$. In this case, or if the family $\mathcal{V}$ is clear from the context, we omit the superscripts. See [HKK+03, Chapter 25] and [FP00] for context and relations satisfied by the integrals of these classes of $\overline{\mathcal{M}}_{g,n}$ and moduli stacks of stable maps.

Fix a closed symplectic manifold (X,ω) with $J \in \mathcal{J}_\tau(X,\omega)$ and $A \in H_2(X;\mathbb{Z})$. Let $\mathcal{K}_n := (G, \mathcal{T}_n/\mathcal{M}_n, \mathcal{E}_n, \mathfrak{s}_n)$ be a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A;J)$ as given by Construction 2.1.14. Recall that $\mathcal{M}_n$ is a G-invariant open subset of the automorphism free locus of regular maps $\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^N, m)$ in $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^N, m)$, admitting a quasi-projective smooth universal family $\mathcal{U}_n \subset \mathcal{M}_{n+1}$ on which G acts almost freely. Let $\Pi_n\colon \mathcal{T}_n \to \mathcal{M}_n$ be the structure map and define

$$\mathbb{L}_i := \mathbb{L}_{n,i} := \Pi_n^*(\mathbb{L}_i^{\mathcal{M}_n}) \qquad\qquad \mathbb{E} := \Pi_n^*(\mathbb{E}^{\mathcal{M}_n}).$$

They are, by definition, relatively smooth vector bundles over $\mathcal{T}_n$. The G-action on $\mathcal{T}_n$ lifts to a fibrewise linear G-action on $\mathbb{L}_i$ and $\mathbb{E}$. Define

$$\psi_i := \psi_{n,i} := c_1(\mathbb{L}_i)_G \qquad\qquad \lambda_j := c_j(\mathbb{E})_G$$

in $H^*_G(\mathcal{T};\mathbb{Q}) \cong H^*(\mathcal{T}/G;\mathbb{Q})$. They restrict to classes on $\overline{\mathcal{M}}_{g,n}(X,A;J)$, also denoted ψ_i and λ.

Lemma 4.2.2. *The ψ- and λ-classes on $\overline{\mathcal{M}}_{g,n}(X,A;J)$ are independent of the choice of unobstructed auxiliary datum.*

Proof. As $\mathbb{E} \cong \mathrm{st}^*\mathbb{E}$ is isomorphic to the pullback of the Hodge bundle on $\overline{\mathcal{M}}_{g,n}$, this is clear for the Hodge classes. Given two unobstructed auxiliary data, we obtain global Kuranishi charts $\mathcal{K}_{n,j}$ with base space $\mathcal{N}_0 \subset \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_0}, m_0)$. By §2.4.1 there exists a double-sum global Kuranishi chart of $\overline{\mathcal{M}}_{g,n}(X,A;J)$ with covering group $G_0 \times G_1$ and base $\mathcal{N}_{01}$. Here $\mathcal{N}_{01} \subset \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, (m_1, m_1))$ is the preimage of $\mathcal{N}_0 \times \mathcal{N}_1$ under the product map Φ. The induced map $\Phi_j \colon \mathcal{N}_{01} \to \mathcal{N}_j$ is a principal $G_{j'}$-bundle (for $\{j,j'\} = \{0,1\}$), so

$$\mathbb{L}_i^{\mathcal{N}_{01}} = \Phi_j^*\mathbb{L}_i^{\mathcal{N}_j} \qquad \mathbb{E}^{\mathcal{N}_{01}} = \Phi_j^*\mathbb{E}^{\mathcal{N}_j}.$$

In particular, $\mathbb{L}_i^{\mathcal{N}_{01}}$ is a principal $G_{j'}$-bundle over $\mathbb{L}_i^{\mathcal{N}_j}$, so $c_1(\mathbb{L}_i^{\mathcal{N}_{01}})_{G_0\times G_1} = \Phi_j^* c_1(\mathbb{L}_i^{\mathcal{N}_j})_{G_j}$. Similarly for the Hodge bundle. Pulling these relations back to the thickening of the double-sum Kuranishi chart, we obtain the claim. □

In particular, if $\overline{\mathcal{M}}_{g,n}(X,A;J)$ is unobstructed, we recover the standard definition of ψ- and λ-classes.

Definition 4.2.3. The *gravitational descendant* (or *descendent Gromov–Witten invariant*) of (X,ω) associated to (A,g,n) is

$$\langle \tau_{k_1}\alpha_1, \ldots, \tau_{k_n}\alpha_n; \sigma\rangle^{X,\omega}_{A,g,n} := \langle \psi_1^{k_1}\mathrm{ev}_1^*\alpha_1 \cdots \psi_n^{k_n}\mathrm{ev}_n^*\alpha_n \cdot \mathrm{st}^*\mathrm{PD}(\sigma), [\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}\rangle$$

for $k_1, \ldots, k_n \geqslant 0$, $\alpha_1, \ldots, \alpha_n \in H^*(X;\mathbb{Q})$ and $\sigma \in H_*(\overline{\mathcal{M}}_{g,n};\mathbb{Q})$. The *Hodge integrals* of X are the numbers

$$\langle \lambda_1^{b_1} \cdots \lambda_g^{b_g} \cdot \psi_1^{k_1}\mathrm{ev}_1^*\alpha_1 \cdots \psi_n^{k_n}\mathrm{ev}_n^*\alpha_n \cdot \mathrm{st}^*\mathrm{PD}(\sigma), [\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}\rangle,$$

where $b_1, \ldots, b_g \geqslant 0$ are integers.

Remark 4.2.4. The construction in §2.4.2 shows that the gravitational descendants of X do not depend on the choice of ω-tame almost complex structure.

Remark 4.2.5. The ψ-classes in [RT97] are defined to be $\psi_i := \mathrm{st}^*\psi_i$. These (respectively a slightly modified definition thereof) are called *gravitational descendents* in [Giv01a] and differ from our definition of ψ-classes, called *gravitational ancestors* by Givental. In [KM98] (whose definition agrees with ours) the exact relationship between the two definitions is elucidated. In [KKP03], *modified ψ-classes* $\bar{\psi}_i$ are defined, which satisfy $\pi_{n+1}^*\bar{\psi}_i = \bar{\psi}_i$.

Remark 4.2.6. The fact that the ψ-classes and λ-classes on $\overline{\mathcal{M}}_{g,n}(X,A;J)$ can be defined as pullbacks of the respective classes on moduli spaces of stable maps to projective space allows us to lift relations shown for the the gravitational descendants of projective space to those of arbitrary symplectic manifolds.

Fix now a global Kuranishi chart $\mathcal{K}_n$ with base space $\mathcal{M}_n$ and structure map $\Pi_n \colon \mathcal{T}_n \to \mathcal{M}_n$. By construction, the forgetful map restricts to a proper smooth maps $\mathcal{M}_{n+1} \to \mathcal{M}_n$

with canonical sections $\sigma_i := \sigma_i^{(n)}$. We abbreviate

$$D_{n+1,i} := \operatorname{im}(\sigma_i^{(n)}) \qquad\qquad \overline{D}_{n+1,i} := D_{n+1,i}/G$$

and

$$D_{n+1,i}(A) := \Pi_n^{-1}(D_{n+1,i}) \qquad\qquad \overline{D}_{n+1,i}(A) := D_{n+1,i}(A)/G).$$

$\overline{D}_{n+1,i}(A)$ is an oriented suborbifold of $\mathcal{T}_{n+1}/G$, Poincaré dual to $\delta_{n+1,i,A} = \Pi_n^*(\mathrm{PD}(\overline{D}_{n+1,i}))$.

Lemma 4.2.7. *Let $\pi_{n+1}\colon \overline{\mathcal{M}}_{g,n+1}(X,A;J) \to \overline{\mathcal{M}}_{g,n}(X,A;J)$ be the forgetful map. Then*

$$\psi_{n+1,i} = \pi_{n+1}^*\psi_{n,i} + \delta_{n+1,i,A} \tag{4.2.0.1}$$

in $H_G^*(\mathcal{M}_{n+1},\mathbb{Q})$. Moreover, $\psi_{n+1,i}\cdot\delta_{n+1,i,A} = 0$.

Proof. This is an immediate corollary of the previous remark and classical arguments. Let $U := \mathcal{M}_{n+1}\backslash D_{n+1,i}$. Given $[\iota, C, x_1,\dots,x_{n+1}] \in U$, the forgetful map π_{n+1} does not change the irreducible component of C containing x_i. Thus $L_i := \mathbb{L}_i \otimes \pi_{n+1}^*\mathbb{L}_i^{-1}$ is trivial over U. Set $\rho := \sigma_i^{(n+1)}\sigma_i^{(n)}$ and $V := \operatorname{im}(\rho)$. Then

$$\left(\sigma_i^{(n)}\right)^*(\mathbb{L}_i \otimes \pi_{n+1}^*\mathbb{L}_i^{-1}) = \rho^*\omega_{\pi_{n+2}} \otimes \left(\sigma_i^{(n+1)}\right)^*\omega_{\pi_{n+1}}.$$

As the normal bundle of $D_{n+1,i}$ is $(\sigma_i^{(n+1)})^*\omega_{\pi_{n+1}}$, it remains to see that $\rho^*\omega_{\pi_{n+2}}$ is equivariantly trivial. Let $\tilde{\pi}\colon \mathcal{M}_{n+1} \to \mathcal{M}_4$ be the map which forgets all marked points except the first, the i^{th} and the last two. Then $\tilde{\pi}$ is G-equivariant and maps $D_{i,n+2}$ to

$$\mathcal{M}_{n,2} \times_{\mathbb{P}^N} (\mathbb{P}^N \times \overline{\mathcal{M}}_{0,4}) \cong \mathcal{M}_2 \times \overline{\mathcal{M}}_{0,4}$$

and V to $\mathcal{M}_2 \times \{*\}$ under this identificaton. Hence $\rho^*\omega_{\pi_{n+2}}^{-1} \cong \tilde{\pi}^*T_{\overline{\mathcal{M}}_{0,4},*}|_V$ equivariantly. If $\iota\colon \overline{D}_{n+1,i} \hookrightarrow \mathcal{M}_{n+1}/G$ is the inclusion, then

$$\psi_{n+1,i,G}\cdot \mathrm{PD}(\overline{D}_{n+1,i}) = \iota_! c_1(\rho^*\omega_{\pi_{n+2}})_G = 0,$$

implying the last claim by Lemma A.1.4. $\square$

There are three further relations for the generalised GW invariants, the first two specialising to the Fundamental class axiom and the last to the Divisor axiom if we have no ψ-insertions. Denote by 1_X the unit of $H^*(X;\mathbb{Q})$.

Proposition 4.2.8 (String equation)**.** *We have*

$$\langle \psi^{k_1}\alpha_1,\dots,\psi^{k_n}\alpha_n, 1_X;\sigma\rangle_{A,g,n+1}^{X,\omega} = \sum_{i=1}^n \langle \psi^{k_1}\alpha_1,\dots,\psi^{k_i-1}\alpha_i,\dots,\psi^{k_n}\alpha_n;\pi_{n+1*}\sigma\rangle_{A,g,n}^{X,\omega} \tag{4.2.0.2}$$

for any $\alpha_1,\dots,\alpha_n \in H^*(X;\mathbb{Q})$ and $\sigma \in H_*(\overline{\mathcal{M}}_{g,n+1};\mathbb{Q})$.

Proof. Write $\alpha := \alpha_1 \times \cdots \times \alpha_n$ and $\beta := \mathrm{PD}(\sigma)$. By (4.2.0.1),

$$\begin{aligned}&\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n, 1_X; \sigma\rangle^{X,\omega}_{A,g,n+1} \\ &\qquad = \langle \prod_{i=1}^{n} (\pi^*_{n+1}\psi_i + \delta_{n+1,i,A})^{k_i} \cdot \pi^*_{n+1}(\mathrm{ev}^*\alpha \times \mathrm{st}^*(\pi_{n+1!}\beta)), [\overline{\mathcal{M}}_{g,n+1}(X, A; J)]^{\mathrm{vir}}\rangle\end{aligned}$$

so the claim follows from the Fundamental class axiom (Proposition 4.1.5), the last assertion of Lemma 4.2.7, the fact that $D_{n+1,i} \cap D_{n+1,j} = \varnothing$ for $i \neq j$ and the equality

$$\pi_{n+1*}(\delta_{n+1,i,A} \cap [\overline{\mathcal{M}}_{g,n+1}(X, A; J)]^{\mathrm{vir}}) = [\overline{\mathcal{M}}_{g,n+1}(X, A; J)]^{\mathrm{vir}}.$$

□

Proposition 4.2.9 (Dilaton equation)**.** *We have*

$$\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n, \psi^1 1_X; \sigma\rangle^{X,\omega}_{A,g,n+1} = (2g - 2 + n)\, \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n; \pi_{n+1*}\sigma\rangle^{X,\omega}_{A,g,n}$$

for any $\alpha_1, \ldots, \alpha_n \in H^*(X;\mathbb{Q})$ *and* $\sigma \in H_*(\overline{\mathcal{M}}_{g,n+1};\mathbb{Q})$.

Proof. By the proof of [Man99, Lemma VI.3.7.2],

$$\sigma^*_{n+1}\omega_{\pi_{n+2}} \cong \omega_{\pi_{n+1}}\Big(\sum_{i=1}^{n} \sigma_i^{(n)}\Big)$$

as G*-equivariant* line bundles, so we can express ψ_{n+1} in terms of $c_1(\omega_{n+1})$ and the canonical sections of π_{n+1}. As $\deg(\omega_{\pi_{n+1}}|_C) = 2g - 2$ for any fibre C of π_{n+1},

$$\pi_{n+1*}(\psi_{n+1} \cap [\mathcal{M}_{n+1}]) = (2g - 2 + n)\, [\mathcal{M}_n].$$

By Corollary A.1.3, the same equality holds for the corresponding classes on the quotient. All other terms vanish because

$$\psi_{n+1,G} \cdot \delta_{n+1,j,G} = c_1(\omega_{\pi_{n+1}}(D_{n+1,j}))_G \cdot \delta_{n+1,j,G} + \sum_{i \neq j} \delta_{n+1,i,G} \cdot \delta_{n+1,j,G} = 0.$$

□

Proposition 4.2.10 (Divisor equation)**.** *If* $(A, g, n) \notin \{(0, 0, 2), (0, 1, 0)\}$ *and* $\gamma \in H^2(X;\mathbb{Q})$, *then*

$$\begin{aligned}\langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n, \gamma; \sigma\rangle^{X,\omega}_{A,g,n+1} &= \langle \gamma, A\rangle \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n; \pi_{n+1*}\sigma\rangle^{X,\omega}_{A,g,n} \\ &\quad + \sum_{i=1}^{n} \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_i - 1}(\alpha_i \cdot \gamma), \ldots, \psi^{k_n}\alpha_n; \pi_{n+1*}\sigma\rangle^{X,\omega}_{A,g,n}\end{aligned}$$

for any $\alpha_1, \ldots, \alpha_n \in H^*(X;\mathbb{Q})$ *and* $\sigma \in H_*(\overline{\mathcal{M}}_{g,n+1};\mathbb{Q})$.

Proof. Set $\beta := \mathrm{PD}(\sigma)$ and $\alpha := \alpha_1 \times \cdots \times \alpha_n$. By Lemma 4.2.7, the right hand side splits into a sum of terms where one is of the form

$$\langle \pi_{n+1}^*\Big(\mathrm{ev}^*\alpha \times \prod_{i=1}^n \psi_{n,i}^{k_i}\Big), (\mathrm{st}^*(\beta)\cdot \mathrm{ev}_{n+1}^*\gamma) \cap [\overline{\mathcal{M}}_{g,n+1}(X,A;J)]^{\mathrm{vir}}\rangle$$
$$= \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_n}\alpha_n; \pi_{n+1*}\sigma\rangle_{A,g,n}^{X,\omega}$$

by Proposition 4.1.7. The other terms are given by

$$\begin{aligned}
&\langle \pi_{n+1}^*(\mathrm{ev}^*\alpha \times \prod_{i=1}^n \psi_{n,i}^{k_i-\delta_{ij}}) \cdot \delta_{n+1,j,A} \cdot \mathrm{ev}_{n+1}^*\gamma, \mathrm{st}^*(\beta) \cap [\overline{\mathcal{M}}_{g,n+1}(X,A;J)]^{\mathrm{vir}}\rangle \\
&\qquad = \langle \pi_{n+1}^*(\mathrm{ev}^*\alpha \times \prod_{i=1}^n \psi_{n,i}^{k_i-\delta_{ij}}) \cdot \mathrm{ev}_j^*\gamma \cdot \delta_{n+1,j,A}, \mathrm{st}^*(\beta) \cap [\overline{\mathcal{M}}_{g,n+1}(X,A;J)]^{\mathrm{vir}}\rangle \\
&\qquad = \langle \psi^{k_1}\alpha_1, \ldots, \psi^{k_i-1}(\alpha_i\cdot\gamma), \ldots, \psi^{k_n}\alpha_n; \pi_{n+1*}\sigma\rangle_{A,g,n}^{X,\omega}
\end{aligned}$$

for $1 \leqslant j \leqslant n$, where the second equality holds because $\mathrm{ev}_{n+1}|_{\overline{D}_{n+1,j}(A)} = \mathrm{ev}_j|_{\overline{D}_{n+1,j}(A)}$ and the last because $\overline{D}_{n+1,j}(A)$ is the image of a section of π_{n+1}. □

Chapter 5

A comparison and equivariant extensions

5.1 A comparision to the GW invariants of Ruan–Tian

5.1.1 Definition of GW invariants via pseudocycles

The first construction of Gromov–Witten invariants in the symplectic setting was given by Ruan and Tian in [RT95, RT97], where they restrict to semipositive symplectic manifolds. For this class of manifolds, the moduli space of stable maps which satisfy a perturbed Cauchy-Riemann equation admits a stratification by smooth manifold, where the top stratum is orientable and of the expected dimension, while all other strata are of codimension at least 2. Hence one can define an intersection theory, respectively define a pseudocycle. In genus 0, one can even avoid perturbing the Cauchy-Riemann equation by allowing the almost complex structure to be domain-dependent, as was done in [MS12, Chapter 6]. Another advantage is the fact that in genus 0 the GW invariants are $\mathbb{Z}$-valued by construction.

To our knowledge, save for the relative virtual fundamental class defined in [IP19a], it is not known whether the GW invariants obtained via a virtual framework agree with the invariants of Ruan-Tian for a semipositive symplectic manifold.

A symplectic manifold (X, ω) is *semipositive* if for any $A \in \pi_2(X)$

$$\omega(A) > 0,\ c_1(A) \geqslant 3 - n \quad \Rightarrow \quad c_1(A) \geqslant 0. \tag{5.1.1.1}$$

In particular, any symplectic manifold of complex dimension at most 3 is semipositive. Let us recall the definition of GW invariants in [RT97] for (X, ω) satisfying (5.1.1.1).

A *good cover* $p_\mu \colon \overline{\mathcal{M}}^\mu_{g,n} \to \overline{M}_{g,n}$ of the (coarse) moduli space of stable curves is a finite cover such that $\overline{\mathcal{M}}^\mu_{g,n}$ admits a universal family that is a projective normal variety. Such good covers can be constructed using level-m structures; refer to [ACG11, Chapter XVI] or [Mum83] for the details. Let $\mathfrak{f}_\mu \colon \overline{\mathcal{U}}^\mu_{g,n} \to \overline{\mathcal{M}}^\mu_{g,n}$ be the universal curve. Fix a closed embedding $\phi \colon \overline{\mathcal{U}}^\mu_{g,n} \hookrightarrow \mathbb{P}^k$.

Given $J \in \mathcal{J}_\tau(X, \omega)$ we will consider perturbations $\nu \in C^\infty(\mathbb{P}^k \times X, \overline{\mathrm{Hom}}_{\mathbb{C}}(p_1^* T_{\mathbb{P}^k}, p_2^* T_X))$.

Let $\overline{\mathcal{M}}^{\mu}_{g,n}(A;J,\nu)$ be the space of equivalence classes of *stable (J,ν)-maps of type* (g,n) (u,j,C,j,x_*) where

1. (C,j,x_*) is of type (g,n),
2. $j\colon C\to\overline{\mathcal{U}}^{\mu}_{g,n}$ is a holomorphic map onto a fibre F of $\overline{\mathcal{U}}^{\mu}_{g,n}$, inducing a contraction $C\to F$ of nodal surfaces,
3. $u\colon C\to X$ is a stable smooth map, representing A and satisfying

$$\overline{\partial}_J u = \nu(\phi j, u)\circ d(\phi j).$$

We say that (u,j,C,x_*) is *equivalent* to (u',j',C',x'_*) if there exists a biholomorphism $\psi\colon (C,x_*)\to(C',x'_*)$ with $\psi^*u'=u$ and $\psi^*j'=j$. A stable (J,ν)-map is *simple* if

1. for each irreducible component $Z\subset C$ on which u is nonconstant, $u|_Z$ is a simple map, i.e., does not factor through a branched holomorphic covering,
2. $u(Z)\neq u(Z')$ for any two irreducible components $Z\neq Z'$ of C on which u is nonconstant.

The space of simple (J,ν)-maps is denoted by $\overline{\mathcal{M}}^{\mu,*}_{g,n}(A;J,\nu)$. It admits a canonical forgetful map $\overline{\mathcal{M}}^{\mu}_{g,n}(A;J,\nu)\to\overline{\mathcal{M}}^{\mu}_{g,n}$ through which the stabilisation map $\overline{\mathcal{M}}^{\mu}_{g,n}(A;J,\nu)\to\overline{\mathcal{M}}_{g,n}$ factors.

$\overline{\mathcal{M}}^{\mu}_{g,n}(A;J,\nu)$ can be stratified by the topological type of the domains together with the distribution of the homology class: To each stable (J,ν)-map we can associate a *marked graph* γ consisting of a n-marked graph G together with a maps $\mathfrak{d}\colon V(G)\to H_2(X,\mathbb{Z})$ and $g\colon V(G)\to\mathbb{N}_0$ so that

$$\dim(H^1(G))+\sum_{v\in V(G)} g(v)=g \qquad\qquad \sum_{v\in V(G)}\mathfrak{d}(v)=A$$

and for any $v\in V(G)$ the stability condition $2g(v)+|\{f\in \mathrm{Fl}(G): s(f)=v\}|\geqslant 3$ holds, where $\mathrm{Fl}(G)$ is the set of flags of G. We denote by $\overline{\mathcal{M}}^{\mu}_{\gamma}(A;J,\nu)$ the stratum of stable maps whose dual graph is given by γ and by $\overline{\mathcal{M}}^{\mu,*}_{\gamma}(A;J,\nu)$ its intersection with the space of simple maps. We denote by $\mathcal{M}^{\mu,*}_{g,n}(A;J,\nu)$ the locus of simple maps with smooth domain.

By [RT97, Proposition 2.3, Theorem 3.1], respectively [Zin17, Theorem 3.3] (whose arguments simplify to our setting), the following holds for generic (J,ν).

1. $\overline{\mathcal{M}}^{\mu,*}_{\gamma}(A;J,\nu)$ is a smooth oriented manifold of dimension

$$2(1-g)\dim_{\mathbb{C}}(X)+2\langle c_1(T_X),A\rangle+\dim_{\mathbb{R}}(\overline{\mathcal{M}}^{\mu}_{\gamma}).$$

2. The maps ev and st_μ define a pseudocycle $\mathrm{ev}\times\mathrm{st}_\mu\colon\mathcal{M}^{\mu,*}_{g,n}(A;J,\nu)\to X\times\overline{\mathcal{M}}^{\mu}_{g,n}$.

We use the following definition of a pseudocycle from [IP19a]; see also [MS12, Chapter 6.1] or [Zin08].

Definition 5.1.1. A d-dimensional pseudocycle $f\colon M \to N$ is a continuous map from a d-dimensional oriented manifold to a locally compact space so that $f(M)$ has compact closure and

$$\Omega_f := \bigcap_{\substack{K\subset M\\ \text{compact}}} \overline{f(M\backslash K)}$$

has Lebesgue covering dimension $\leqslant d-2$.

Given a d-dimensional pseudocycle $f\colon M \to N$, we can define a class $[f] \in H_d^{\mathrm{BM}}(N,\mathbb{Z})$ as follows. Let $M^\circ := f^{-1}(N\backslash\Omega_f)$. Then M° is an oriented d-dimensional manifold and $f^\circ\colon M^\circ \to N\backslash\Omega_f$ is proper. We define $[f] \in H_d^{\mathrm{BM}}(N,\mathbb{Z})$ to be the image of $[M^\circ]$ under

$$H_d^{\mathrm{BM}}(M^\circ,\mathbb{Z}) \xrightarrow{f^\circ_*} H_d^{\mathrm{BM}}(N\backslash\Omega_f,\mathbb{Z}) \cong H_d^{\mathrm{BM}}(N,\mathbb{Z}).$$

Here the isomorphism is an immediate consequence of $\dim(\Omega_f) < d-1$ and the long exact sequence in Borel–Moore homology. Refer to [IP19a, §3, §A.3] for more details.

Definition 5.1.2. [RT97] The *pseudocycle Gromov–Witten class* of (X,ω) associated to (g,n,A) is

$$\sigma^A_{g,n} = \frac{1}{d_\mu}(\mathrm{id}\times p_\mu)_*[\mathrm{ev}\times \mathrm{st}_\mu] \in H_*(X^n \times \overline{\mathcal{M}}_{g,n};\mathbb{Q}).$$

Theorem 5.1.3. *Suppose (X,ω) is semipositive. Then the pseudocycle GW classes agree with the GW classes defined in Chapter 2. Explicitly,*

$$\sigma^A_{g,n} = (\mathrm{ev}\times\mathrm{st})_*[\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}} \tag{5.1.1.2}$$

for any $g,n \geqslant 0$ with $2g-2+n>0$.

5.1.2 Proof of Theorem 5.1.3

Fix (g,n) with $2g-2+n>0$ and let $p_\mu\colon \overline{\mathcal{M}}^\mu_{g,n} \to \overline{\mathcal{M}}_{g,n}$ be a good finite cover. Set $d_\mu := \deg(p_\mu)$. Define

$$\mathcal{W}^\mu_{g,n,A} := \left\{(t,u,C,x_*,j)\ \middle|\ \begin{array}{c} u\colon (C,x_*)\to X \text{ smooth stable of type } (g,n),\\ t\in[0,1],\ j\colon (C,x_*)\to\overline{\mathcal{U}}^\mu_{g,n},\ u_*[C]=A,\\ \bar\partial_J u = t\,(\phi j\times u)^*\nu \end{array}\right\},$$

where j is a contraction onto a fibre of $\overline{\mathcal{U}}^\mu_{g,n}$.

Lemma 5.1.4. *$\mathcal{W}^\mu_{g,n,A}$ is compact and Hausdorff when endowed with the topology induced by Gromov convergence.*

Proof. Compactness follows from [RT95, Proposition 3.1], while the uniqueness of the limit follows by the arguments of the proof of [MS12, Theorem 5.5.3]. □

Remark 5.1.5. Denote by $\overline{\mathcal{U}}^{\mu,\mathrm{sing}}_{g,n}$ the nodes and marked points of the universal curve. In order to apply [Swa21] later on, we restrict to perturbations ν that are supported away from $\phi(\overline{\mathcal{U}}^{\mu,\mathrm{sing}}_{g,n})$. By elliptic regularity, such perturbations ν suffice to achieve transversality.

We will construct a global Kuranishi chart with boundary for $\mathcal{W}^{\mu}_{g,n,A}$, which restricts to a cover of the previously constructed global Kuranishi chart for $\overline{\mathcal{M}}_{g,n}(X,A;J)$ over one fibre and to a global Kuranishi chart for $\overline{\mathcal{M}}^{\mu}_{g,n}(X,A;J,\nu)$ on the other boundary fibre. By Lemma 5.1.9 and Lemma A.2.7, this will imply that

$$d_\mu\,(\mathrm{ev}\times\mathrm{st})_*[\overline{\mathcal{M}}_{g,n}(X,A;J)]^{\mathrm{vir}}=(\mathrm{ev}\times\mathrm{st})_*[\overline{\mathcal{M}}^{\mu}_{g,n}(X,A;J,\nu)]^{\mathrm{vir}}$$

in $H_*(X^n\times\overline{\mathcal{M}}_{g,n};\mathbb{Q})$. Finally we compare $(\mathrm{ev}\times\mathrm{st})_*[\overline{\mathcal{M}}^{\mu}_{g,n}(X,A;J,\nu)]^{\mathrm{vir}}$ with $\sigma^{A}_{g,n}$.

Fix an unobstructed auxiliary datum $(\nabla^X,\mathcal{O}_X(1),p,\mathcal{U},k)$ where

1. $p\gg 0$ is sufficiently large that $\mathfrak{L}_u^{\otimes p}$ is very ample for any $(t,u)\in\mathcal{W}^{\mu}_{g,n,A}$;
2. $\mathcal{U}$ is a good covering in the sense of Definition 2.2.12 where we take the image of $\mathcal{W}^{\mu}_{g,n,A}$ in the polyfold of smooth maps to X instead of $\overline{\mathcal{M}}_{g,n}(X,A;J)$ in the third condition;
3. $k\in\mathbb{N}$ will be determined later.

Define $\widetilde{\mathcal{T}}$ to be the set $\{(t,u,C,x_*,j,\iota,\alpha,\eta)\}/\sim$ so that

- $u\colon(C,x_*)\to X$ is a smooth stable map of genus g representing A.
- $j\colon(C,x_*)\to\overline{\mathcal{U}}_{g,n}$ is a contraction of nodal surfaces onto a fibre of the universal curve,
- $\iota\colon(C,x_*)\to\mathbb{P}^N$ is an element of $\overline{\mathcal{M}}^{*}_{g,n}(\mathbb{P}^N,m)$,
- $\alpha\in H^1(C,\mathcal{O}_C)$ satisfies $[\iota^*\mathcal{O}_{\mathbb{P}^N}(1)]=p\cdot[\mathfrak{L}_u]+\alpha$ in $\mathrm{Pic}(C)$,
- $\eta\in E_{(\iota,u)}:=H^0(C,\iota^*T^{*}_{\mathbb{P}^N}{}^{0,1}\otimes u^*T_X\otimes\iota^*\mathcal{O}(k))\otimes\overline{H^0(\mathbb{P}^N,\mathcal{O}(k))}$ is such that
$$\bar\partial_J\tilde u+\langle\eta\rangle\circ d\tilde\iota-t\,\nu(\phi\tilde j,\tilde u)=0\tag{5.1.2.1}$$
on the normalisation $\tilde C$ of C.

We quotient by reparametrisations of the domain. Let $P\colon\widetilde{\mathcal{T}}\to[0,1]$ be the obvious projection and $\widetilde{\mathcal{T}}_t:=P^{-1}(\{t\})$.

Set $\widetilde{\mathcal{M}}:=\overline{\mathcal{M}}^{\mu}_{g,n}\times_{\overline{M}_{g,n}}\overline{\mathcal{M}}^{*}_{g,n}(\mathbb{P}^N,m)$ and let $\pi\colon\widetilde{\mathcal{T}}\to\widetilde{\mathcal{M}}$ be the forgetful map. Define $\widetilde{\mathcal{E}}\to\widetilde{\mathcal{T}}$ by letting its fibre over $y=(t,u,C,x_*,j,\iota,\alpha,\eta)$ be

$$\widetilde{\mathcal{E}}_y=\mathfrak{su}(N+1)\oplus H^1(C,\mathcal{O}_C)\oplus E_{(\iota,u)},$$

while the obstruction section $\tilde{\mathfrak{s}}$ is given by $\tilde{\mathfrak{s}}(y)=(i\log(\lambda(u,\iota)),\alpha,\eta)$. Let $G:=\mathrm{PU}(N+1)$ acting via post-composition on the framings and the perturbation terms η. For $i\in\{0,1\}$, denote

$$\widetilde{\mathcal{K}}_i:=(G,\widetilde{\mathcal{T}}_i,\widetilde{\mathcal{E}}_i|_{\widetilde{\mathcal{T}}_i},\tilde{\mathfrak{s}}|_{\widetilde{\mathcal{T}}_i}).$$

Lemma 5.1.6. *We can choose k sufficiently large so that the linearisation of (5.1.2.1) restricted to $C^\infty(C,u^*T_X)\oplus E_{(\iota,u)}$ is surjective for any element in $\tilde{\mathfrak{s}}^{-1}(0)$.*

Proof. This follows from the proof of Lemma 2.2.18. □

Proposition 5.1.7. *For $k \gg 0$, $\widetilde{\mathcal{T}}^{\mathrm{reg}}$ is naturally a rel–C^∞ manifold over $[0,1] \times \widetilde{\mathcal{M}}$ and the structure map is a topological submersion. The restriction $\widetilde{\mathcal{E}}^{\mathrm{reg}} := \widetilde{\mathcal{E}}|_{\widetilde{\mathcal{T}}^{\mathrm{reg}}}$ is a rel–C^∞ vector bundle and the restriction of $\widetilde{\mathfrak{s}}$ is of class rel–C^∞. Moreover, $\mathrm{ev}\colon \widetilde{\mathcal{T}} \to X^n$ is a rel–C^∞ submersion.*

Proof. Forgetting the α-parameter and using Gromov's shearing trick, we can consider $\widetilde{\mathcal{T}}$ as a subset of the moduli space of embedded regular perturbed holomorphic maps to the total space of a vector bundle $E \to \mathbb{P}^N \times X$. Fixing a splitting $T_E = \pi^* T_{\mathbb{P}^N \times X} \oplus \pi^* E$, we define the family of almost complex structures on E by

$$\tilde{J}^t_e(\hat{x}, v, e') = (J_0 \hat{x}, Jv + \langle e\rangle(\hat{x}), J^E e')$$

for $(\hat{x}, v) \in T_{\mathbb{P}^N \times X, \pi_E(e)}$ and $e' \in E_e$.

By Remark 5.1.5, we can use [Swa21] as in Proposition 2.3.9 to deduce the relative smoothness of $\widetilde{\mathcal{T}}$ over $[0,1] \times \widetilde{\mathcal{M}}$. The structural map is a submersion since we obtain transversality without variation of the domain or the t-parameter. The other claims follow from the same reasoning as in §2.3. □

As the arguments in §2.4 carry over word by word, we obtain the first step of our proof.

Corollary 5.1.8. *$\mathcal{W}^{\mu}_{g,n,A}$ admits an oriented global Kuranishi chart $\widetilde{\mathcal{K}}_n$ with boundary of the expected dimension.*

In particular, $\widetilde{\mathcal{K}}_{n,0}$ is an oriented global Kuranishi chart for $\overline{\mathcal{M}}^{\mu}_X := \overline{\mathcal{M}}_{g,n}(X, A; J) \times_{\overline{M}_{g,n}} \overline{\mathcal{M}}^{\mu}_{g,n}$ with

$$(\mathrm{ev} \times \mathrm{st})_* [\overline{\mathcal{M}}^{\mu}_X]^{\mathrm{vir}}_{\widetilde{\mathcal{K}}_0} = d_\mu \, (\mathrm{ev} \times \mathrm{st})_* [\overline{\mathcal{M}}_{g,n}(X, A; J)]^{\mathrm{vir}}. \tag{5.1.2.2}$$

due to

Lemma 5.1.9. *Suppose $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ is an oriented global Kuranishi chart of a space M and $\mathcal{T}$ admits a degree-d cover $p\colon \mathcal{T}' \to \mathcal{T}$. If p is G-equivariant with respect to some G-action on $\mathcal{T}'$, then $\mathcal{K}' := (G, \mathcal{T}', p^*\mathcal{E}, p^*\mathfrak{s})$ is a global Kuranishi chart for $M' := (p^*\mathfrak{s})^{-1}(0)/G$. The canonical map $\overline{p}\colon M' \to M$ is a degree-d cover and*

$$\overline{p}_* [M']^{\mathrm{vir}}_{\mathcal{K}'} = d \, [M]^{\mathrm{vir}}_{\mathcal{K}}.$$

Proof. The first part is straightforward. The relation between the virtual fundamental classes follows from the functoriality of Thom classes and because the map $\mathcal{T}'_G \to \mathcal{T}_G$ of homotopy quotients has degree d. □

It remains to show that

$$d_\mu \sigma^A_{g,n} = (\mathrm{ev} \times \mathrm{st}_\mu)_* [\overline{\mathcal{M}}^{\mu}_{g,n}(A; J, \nu)]^{\mathrm{vir}}_{\widetilde{\mathcal{K}}_1}.$$

This is a consequence of the following general result. It is the analogue of [IP19a, Theorem 5.2] in our setting. Compare also with Lemma 3.6 op. cit..

Lemma 5.1.10. *Let M be an oriented manifold of dimension d inside a compact space $\overline{M}$ that admits a global Kuranishi chart $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ of dimension d. Suppose $\mathfrak{s}$ intersects the zero section transversely over the preimage of M and G acts freely on that locus. Let $f\colon \overline{M} \to N$ be a continuous map to (the orbit space of) a smooth compact oriented orbifold, so that $f|_M$ is a pseudocycle. Then $f_*[\overline{M}]^{\mathrm{vir}} = [f|_M]$ in $H_*(N;\mathbb{Q})$.*

Proof. Set $P := f(\overline{M}\backslash M)$ and $M^\circ := f^{-1}(N\backslash P)$. Then M° is an open submanifold of M and $f\colon M^\circ \to N\backslash P$ is proper. In particular, $[f] = f_*[M^\circ] \in H_d^{\mathrm{BM}}(N\backslash P;\mathbb{Q}) \cong H_d^{\mathrm{BM}}(N;\mathbb{Q})$. Let $j\colon M \hookrightarrow \overline{M}$ be the inclusion inducing $j_!\colon \check{H}^*_c(M^\circ;\mathbb{Q}) \to \check{H}^*(\overline{M};\mathbb{Q})$. By assumption on the Kuranishi section $\mathfrak{s}$, the class $(j_!)^*[\overline{M}]^{\mathrm{vir}}$ in $\check{H}^d_c(M^\circ;\mathbb{Q})^\vee$ corresponds to evaluation at the fundamental class $[M^\circ]$. This implies that the diagram

$$\begin{tikzcd}
\check{H}^d_c(N\backslash P;\mathbb{Q}) \arrow[r, "(f|_M)^*"] \arrow[d, "i_!"] & \check{H}^d_c(M;\mathbb{Q}) \arrow[r, "\mathrm{PD}"] \arrow[d, "j_!"] & H_0(M;\mathbb{Q}) \arrow[d] \\
\check{H}^d(N;\mathbb{Q}) \arrow[r, "f^*"] & \check{H}^d(\overline{M};\mathbb{Q}) \arrow[r, "{[\bar{M}]^{\mathrm{vir}}}"] & \mathbb{Q}
\end{tikzcd}$$

commutes. Hence, $f_*[\overline{M}]^{\mathrm{vir}}$ agrees with the evaluation at $[f|_M]$ and thus the two define the same class in homology. $\square$

5.2 Virtual localisation and equivariant GW theory

In this section, we define global Kuranishi charts endowed with a compatible group action and construct an equivariant virtual fundamental class. We prove a localisation formula, analogous to [AB84], in the setting of global Kuranishi charts, see Theorem 5.2.10 and show that it applies to the equivariant GW invariants of Hamiltonian symplectic manifolds constructed in §5.2.3.

5.2.1 Equivariant virtual fundamental classes

We define what it means for a global Kuranishi chart to carry a compatible group action and construct the associated equivariant virtual fundamental class. The technical background for this can be found in §A.2. The construction can be considered a special case of parameterised virtual fundamental classes.

Definition 5.2.1. Suppose K is a compact Lie group acting on a space $\mathfrak{M}$. A global Kuranishi chart $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ for $\mathfrak{M}$ is K-*compatible* if

- $\mathcal{T}$ admits a K-action that commutes with the G-action,
- $\mathcal{E}$ admits a K-linearisation that commutes with the given G-action, so that
- $\mathfrak{s}$ is K-equivariant.

We say it is rel–C^∞ K-compatible if it is rel–C^∞ over a base space S so that $\mathcal{T} \to S$ is K-invariants and K acts relatively smoothly.

Definition 5.2.2. Let $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ be an oriented global Kuranishi chart for $\mathfrak{M}$ with a locally linear compatible $\mathbb{T}$-action. The *equivariant virtual fundamental class* $[\mathfrak{M}]^{\mathrm{vir}}_K$ is the element of $\mathrm{Hom}_{H^*_K}(\check{H}^{*+\mathrm{vdim}}_K(\mathfrak{M};\mathbb{Q}), H^*_K)$ given by the composite

$$\check{H}^{*+\mathrm{vdim}}_K(\mathfrak{M};\mathbb{Q}) \xrightarrow{\mathfrak{s}^*\tau^K_{\mathcal{E}/G}\cdot} H^{*+\dim(\mathcal{T}/G)}_{K,fc}(\mathcal{T}/G;\mathbb{Q}) \xrightarrow{\int^K_{\mathcal{T}/G}} H^*_K \tag{5.2.1.1}$$

where the subscript fc denotes cohomology with fibrewise compact support (of the fibration $(\mathcal{T}/G)_K \to BK$) and $H^*_K := H^*_K(\mathrm{pt};\mathbb{Q})$.

Here we use that

$$\check{H}^*_{\mathbb{T}}(\mathfrak{M};\mathbb{Q}) = \varinjlim_{\substack{n\to\infty \\ W\supseteq\mathfrak{M}}} \check{H}^*(B_nW;\mathbb{Q})$$

where we take the direct limit over n and open neighbourhoods of $\mathfrak{M}$ in $\mathcal{T}/G$. The first map in (5.2.1.1) is induced by the composition

$$\begin{aligned}\check{H}^*(W_K;\mathbb{Q}) \xrightarrow{\mathfrak{s}^*\tau^K_{\mathcal{E}/G}|_W\cdot} \check{H}^{*+\mathrm{rank}(\mathcal{E})}_K(W \mid \mathfrak{M};\mathbb{Q}) \xrightarrow{\simeq} \check{H}^{*+\mathrm{rank}(\mathcal{E})}_K(\mathcal{T}/G \mid \mathfrak{M};\mathbb{Q}) \\ \to \check{H}^{*+\mathrm{rank}(\mathcal{E})}_{K,c}(\mathcal{T}/G;\mathbb{Q})\end{aligned}$$

while the second map is the trace map of $(\mathcal{T}/G)_K \to BK$ defined in §A.2.

Lemma 5.2.3. *If K acts freely on $\mathfrak{M}$ and $\mathcal{K}$ is a K-compatible global Kuranishi chart, then $[\mathfrak{M}]^{\mathrm{vir}}_K = 0$.*

Proof. Write $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$. As K acts freely in a neighbourhood of $\mathfrak{M}$ in $\mathcal{T}/G$, we may shrink to assume it acts freely on all of $\mathcal{T}/G$. Then $H^{\dim(\mathcal{T}/G)+*}_{K,c}(\mathcal{T}/G;\mathbb{Q}) \to H^*_K(\mathrm{pt};\mathbb{Q})$ vanishes and thus so does $[\mathfrak{M}]^{\mathrm{vir}}_K$. □

Remark 5.2.4. By Corollary A.2.3 we have a commutative square

$$\begin{array}{ccc} \check{H}^{*+\mathrm{vdim}}_K(\mathfrak{M};\mathbb{Q}) & \xrightarrow{[\mathfrak{M}]^{\mathrm{vir}}_K} & H^*_K(\mathrm{pt};\mathbb{Q}) \\ \downarrow & & \downarrow \\ \check{H}^{*+\mathrm{vdim}}(\mathfrak{M};\mathbb{Q}) & \xrightarrow{[\mathfrak{M}]^{\mathrm{vir}}} & H^0(\mathrm{pt};\mathbb{Q}) \end{array} \tag{5.2.1.2}$$

allowing us to recover $[\mathfrak{M}]^{\mathrm{vir}}$ (partially) from $[\mathfrak{M}]^{\mathrm{vir}}_K$.

The arguments of §3.1.1 can be carried over to the equivariant setting, using Lemma A.2.5. We obtain the analogous statement for equivariant virtual fundamental classes.

Proposition 5.2.5. *Suppose $j\colon \mathcal{K}' \hookrightarrow \mathcal{K}$ is a rel–C^∞ embedding of oriented global Kuranishi charts as in Proposition 3.1.7 and that K acts relatively smoothly on each global Kuranishi chart. If the embedding is K-equivariant, then*

$$j_*(e_K(j^*(\mathcal{E}/\mathcal{E}')/G) \cap [\mathfrak{M}']^{\mathrm{vir}}_K) = \mathrm{PD}_K(\mathcal{T}'/G) \cap [\mathfrak{M}]^{\mathrm{vir}}_K.$$

Remark 5.2.6. Suppose $\mathfrak{M}$ is a moduli space parameterised by a topological space B. Using the obvious definition of a parameterised global Kuranishi chart, the results in §A.2

allow for the definition of a parameterised virtual fundamental class of the form

$$[\mathfrak{M}]^{\mathrm{vir}}_B \colon \check{H}^{d+*}_{fc}(\mathfrak{M};\mathbb{Q}) \to H^*(B;\mathbb{Q}).$$

In particular, by Lemma A.2.2 and Lemma A.2.4, the results of §3.1.1 carry over. The equivariant virtual fundamental class defined here is just a special case of this construction. This is also discussed (for smooth parameter spaces) in [AMS23, §4.7].

5.2.2 Virtual localisation

Let $\mathfrak{M}$ be a moduli space with an oriented global Kuranishi chart $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$. Suppose $\mathfrak{M}$ admits a continuous $\mathbb{T}$-action which lifts to a compatible $\mathbb{T}$-action on $\mathcal{K}$. Here $\mathbb{T} = (S^1)^k$ for some $k \geqslant 1$. The aim of the localisation statement is to reduce the computation of the virtual fundamental class of $\mathfrak{M}$ (and any invariants arising from it) to a computation on the fixed point locus $\mathfrak{M}^{\mathbb{T}}$. To see that $\mathfrak{M}^{\mathbb{T}}$ admits a global Kuranishi chart, we need the following preliminary lemma.

Lemma 5.2.7. *Suppose H is a compact connected Lie group and $\mathbb{T}\times H$ acts locally linearly on an oriented topological manifold Y. If the action of H is additionally almost free, then $(Y/H)^{\mathbb{T}}$ is an oriented homology $\mathbb{Q}$-manifold.*

Proof. Let $q_Y \colon Y \to \overline{Y} := Y/H$ be the quotient map and set $Y^{qf} := q_Y^{-1}((Y/H)^{\mathbb{T}})$.Then $x \in Y^{qf}$ if and only if for any $t \in \mathbb{T}$, we have $t\cdot x = h \cdot x$ for some $h \in H$. As H acts almost freely,

$$Y^{qf} = \{y \in Y \mid \dim((\mathbb{T}\times H)_y) = \dim(\mathbb{T})\}\,.$$

Each path component of Y^{qf} is a topological manifold of Y and consists of elements whose stabilisers lie in the same conjugacy class. Given $\Gamma \leqslant \mathbb{T}\times H$, let $\Lambda_{[\Gamma]}$ be the set of path-components of Y^{qf} with stabiliser group in the conjugacy class of Γ. Then the path-components of $\overline{Y}^{\mathbb{T}}$ are indexed by $\Lambda = \bigsqcup \Lambda_{[\Gamma]}$ and $q_Y^{-1}(\overline{Y}_\lambda)$ is path-connected, since H is connected. As the right hand side is an oriented manifold on which H acts almost freely and locally linearly, the claim follows. $\square$

By Lemma 5.2.7, the preimage of $(\mathcal{T}/G)^{\mathbb{T}}$ in $\mathcal{T}$ is given by $\mathcal{T}^{qf} = \bigsqcup\limits_{\lambda} \mathcal{T}_\lambda$, where each $\mathcal{T}_\lambda$ is a G-invariant submanifold. Note that the dimension of $\mathcal{T}_\lambda$ might depend on λ. For each λ we have a splitting $\mathcal{E}|_{\mathcal{T}_\lambda} = \mathcal{E}^f_\lambda \oplus \mathcal{E}^m_\lambda$, where

$$(\mathcal{E}^f_\lambda)_x := \{v \in \mathcal{E}_x \mid \forall h \in (\mathbb{T}\times G)_{x,0} : h\cdot v = v\} \quad (\mathcal{E}^m_\lambda)_x := \left\{ v - \int_{(\mathbb{T}\times G)_{x,0}} h\cdot v \, dm(h) \;\middle|\; v \in \mathcal{E}_x \right\}$$

for $x \in \mathcal{T}_\lambda$, where $(\mathbb{T}\times G)_{x,0}$ is the identity component of the stabiliser group and m the normalised Haar measure. Since $\mathfrak{s}$ is $(\mathbb{T}\times G)$-equivariant, $\mathfrak{s}(\mathcal{T}_\lambda) \subset \mathcal{E}^f_\lambda$. Moreover, the rank of $\mathcal{E}^f_\lambda$ is constant along $\mathcal{T}_\lambda$. This shows that

Lemma 5.2.8. *$\mathcal{K}_\lambda := (G, \mathcal{T}_\lambda, \mathcal{E}^f_\lambda, \mathfrak{s}|_{\mathcal{T}_\lambda})$ is a global Kuranishi chart for $\mathfrak{M}_\lambda := \mathfrak{s}^{-1}(0)\cap\mathcal{T}_\lambda/G$*

and

$$\mathfrak{M}^{\mathbb{T}} = \bigsqcup_{\lambda} \mathfrak{M}_\lambda.$$

The proof of [AB84, Theorem (3.5)] relies on the existence of local equivariant maps $U \to \mathbb{T}/\Gamma$ where $\Gamma < \mathbb{T}$ is a closed subgroup and U is an open $\mathbb{T}$-invariant subset of $M\backslash M^{\mathbb{T}}$. One can construct such sets U using equivariant tubular neighbourhoods of $\mathbb{T}$-orbits. Given a global Kuranishi chart $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$ with a compatible $\mathbb{T}$-action, we want to emulate this argument for $\mathcal{T}/G$. We do require relative smoothness of $\mathcal{T}$ over a base space; note that we do not show relative smoothness for the submanifolds $\mathcal{T}_\lambda$.

Lemma 5.2.9. *Let X/S be a rel–C^∞ manifold with S a smooth manifold. Suppose G is a compact Lie group acting continuously on X and smoothly on S so that the map $G \times X \to X$ is relatively smooth over $G \times S \to S$. Then for each $x \in X$ there exists an equivariant retraction $r\colon W \to G \cdot x$ defined on a G-invariant neighbourhood of x.*

Proof. Let π be the structural map $X \to S$. By Lemma 2.3.14 and [AMS21, Proposition 4.25], there is a G-equivariant fibre submersion $\phi\colon U \subset X \times X \to X$ in the sense of [AMS21, Definition 4.22].[1] Fix $x \in X$ and set $s := \pi(x)$. Let Z be a G-slice through x so that $\{x\} \times Z \subset U$. Let $V := G \cdot Z \cong G \times_{G_x} Z$. Then we can define $r_0\colon Z \to X_s$ by $r_0(z) = \phi(x, z)$. This map is G_x-equivariant, so we can extend it to $r_1\colon V \to X' := \pi^{-1}(G \cdot s)$ by $r_1(g \cdot z) = g \cdot r_0(z)$. As $r_0(x) = x$ by assumption on ϕ, r_1 fixes $G \cdot x$ pointwise. The induced action by $H := G_s$ on X_s is smooth. Using an H-invariant Riemannian metric, we can define an H-equivariant retraction $r_2\colon W' \to H \cdot x$. As before we can extend r_2 to a G-equivariant retraction $r_3\colon W := G \cdot W' \to G \cdot x$ by setting $r_3(g \cdot y) = g \cdot r_2(y)$. By the compactness of G, W is open. Shrinking Z we may assume $\mathrm{im}(r_1) \subset W$. Then $r := r_3 \circ r_1$ is the desired map. $\square$

Theorem 5.2.10 (Virtual localisation)**.** *Let $\mathcal{K} = (G, \mathcal{T}/S, \mathcal{E}, \mathfrak{s})$ be a rel–C^∞ global Kuranishi chart for $\mathfrak{M}$. Assume $\mathcal{K}$ is endowed with a compatible rel–C^∞ action by a torus $\mathbb{T}$. Suppose each path-component of the fixed point locus $\mathcal{T}^{\mathbb{T}}$ is a rel–C^∞ manifold over a submanifold of S.*[2] *Then,*

$$[\mathfrak{M}]^{\mathrm{vir}}_{\mathbb{T}} = \sum_{\lambda} j_{\lambda *} \frac{e_{\mathbb{T}}(\mathcal{E}^m_\lambda/G) \cap [\mathfrak{M}_\lambda]^{\mathrm{vir}}_{\mathbb{T}}}{e_{\mathbb{T}}(N_{\mathcal{T}_\lambda/\mathcal{T}}/G)} \tag{5.2.2.1}$$

in $(\check{H}^*_{\mathbb{T}}(\mathfrak{M}; \mathbb{Q}) \otimes_{H^*_{\mathbb{T}}} \mathrm{Frac}(H^*_{\mathbb{T}}))^\vee$.

Proof. Using the compactness of $\mathfrak{s}^{-1}(0)$, we may assume $\mathcal{T}^{qf}$ has finitely many path components. By Lemma 5.2.9 we can find for each $x \in \mathcal{T}$ a $(\mathbb{T} \times G)$-invariant open neighbourhood W_x and a $(\mathbb{T} \times G)$-equivariant retraction $W_x \to (\mathbb{T} \times G) \cdot x$. The pullback $H^*_{\mathbb{T}} \to H^*_{\mathbb{T}}(W_x/G; \mathbb{Q})$ factors through $H^*_{\mathbb{T}/\mathbb{T}_{\bar{x}}} = H^*_{(\mathbb{T}/\mathbb{T}_{\bar{x}})_0}$, where $\bar{x}$ is the image of x in W_x/G. As $H^*_{(\mathbb{T}/\mathbb{T}_{\bar{x}})_0}$ is torsion over $H^*_{\mathbb{T}}$ for $x \notin \mathcal{T}^{qf}$, the same is true for $H^*_{\mathbb{T}}(W_x/G; \mathbb{Q})$. Covering $\mathfrak{s}^{-1}(0)\backslash \mathcal{T}^{qf}$ by finitely many such neighbourhoods, it follows that $H^*_{\mathbb{T}}(\overline{\mathcal{T}} \backslash \overline{\mathcal{T}}^{\mathbb{T}} \mid \mathfrak{M}\backslash \mathfrak{M}^{\mathbb{T}}; \mathbb{Q})$

[1] That is, U is a neighbourhood of the diagonal that is invariant under the diagonal G-action, ϕ is open and equivariant so that $\phi|_{U \cap \{x\} \times X_{\pi(x)}}$ is the inclusion, and $\pi\phi = \pi \mathrm{pr}_1$.

[2] We impose this condition so each path-component has a well-defined normal bundle.

is a torsion $H^*_{\mathbb{T}}$-module by the Mayer-Vietoris sequence, where $\overline{\mathcal{T}} := \mathcal{T}/G$. In contrast to the case where $\mathcal{T}$ is a smooth manifold and $\mathbb{T} \times G$ acts smoothly, we do not have global tubular neighbourhoods. However, we have them locally, so we can use again induction and the Mayer-Vietoris sequence to see that

$$j^*\colon H^*_{\mathbb{T}}(\overline{\mathcal{T}} \mid \mathfrak{M}; \mathbb{Q}) \to H^*_{\mathbb{T}}(\overline{\mathcal{T}}^{\mathbb{T}} \mid \mathfrak{M}^{\mathbb{T}}; \mathbb{Q})$$

admits an inverse up to torsion given by

$$j_!\colon H^*_{\mathbb{T}}(\overline{\mathcal{T}}^{\mathbb{T}} \mid \mathfrak{M}^{\mathbb{T}}; \mathbb{Q}) \to H^*_{\mathbb{T}}(\overline{\mathcal{T}} \mid \mathfrak{M}; \mathbb{Q}).$$

As $j_{\lambda !}j^*_\lambda = \mathrm{PD}_{\mathbb{T}\times G}(\mathcal{T}_\lambda)\cdot$ by (A.1.0.1), the sum $\sigma := \sum_\lambda \mathrm{PD}_{\mathbb{T}\times G}(\mathcal{T}_\lambda)$ is invertible in the localised module $H^*_{\mathbb{T}}(\overline{\mathcal{T}}; \mathbb{Q}) \otimes_{H^*_{\mathbb{T}}} \mathrm{Frac}(H^*_{\mathbb{T}})$. By Proposition 5.2.5

$$\mathrm{PD}_{\mathbb{T}}(\mathcal{T}_\lambda/G) \cap [\mathfrak{M}]^{\mathrm{vir}}_{\mathbb{T}} = j_{\lambda *}(e_{\mathbb{T}}(\mathcal{E}^m_\lambda/G) \cap [\mathfrak{M}_\lambda]^{\mathrm{vir}}_{\mathbb{T}}).$$

Taking the sum over the path-components of $\mathfrak{M}^{\mathbb{T}}$ and inverting σ, we obtain (5.2.2.1). □

Corollary 5.2.11. *Suppose $\mathcal{K}$ and $\mathcal{M}$ are as in Theorem 5.2.10. If $\mathcal{E} = \mathcal{E}' \oplus V$, where V is a trivial bundle on which $\mathbb{T}$ acts trivially, then we only have to consider path-components $\mathcal{T}_\lambda$ of $\mathcal{T}^{qf}$ with stabiliser group $\mathbb{T}$.*

Proof. Indeed, if $H \leqslant \mathbb{T} \times G$ is the stabiliser group of some $y \in \mathcal{T}_\lambda$ with $H \neq \mathbb{T}$, then $V^m_\lambda \neq 0$. Thus $e(\mathcal{E}^m/G) = 0$ and so there is no contribution of $[\mathfrak{M}_\lambda]^{\mathrm{vir}}$ □

5.2.3 Equivariant GW invariants

Suppose (X, ω) is a closed symplectic manifold equipped a Hamiltonian action by a compact connected Lie group K with moment map μ. Let $A \in H_2(X; \mathbb{Z})$. We first note the following compatibility with our construction of a global Kuranishi chart.

Lemma 5.2.12. *If $J \in \mathcal{J}_\tau(X, \omega)$ is K-invariant, the following holds.*

1. *There exists an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ such that ∇^X is K-invariant, $\mathcal{O}_X(1)$ admits a unitary K-linearisation, and $\lambda_{\mathcal{U}}$ is K-invariant.*

2. *The resulting global Kuranishi chart obtained by Construction 2.1.14 using this auxiliary datum admits a relatively smooth compatible K-action.*

3. *If $(\nabla^{X'}, \mathcal{O}_{X'}(1), p', \mathcal{U}', k')$ is another unobstructed auxiliary datum satisfying the conditions of (1), then the associated global Kuranishi charts are equivalent via K-compatible charts such that the moves respect the K-actions. If J' is another ω-compatible almost complex structure such that the K-action is J'-holomorphic, then the cobordism constructed in §2.4.2 can be chosen to be K-compatible.*

Proof. By [Rie01, Corollary 1.4] we can find a polarisation $\mathcal{O}_X(1)$ as in Definition 2.1.6 such that the K-action on X lifts to a fibrewise linear unitary K-action on $\mathcal{O}_X(1)$. If

$\tilde{\nabla}^X$ is a J-linear connection on T_X, then so is the connection ∇^X obtained by averaging $\tilde{\nabla}^X$ over K. Given these two data, Theorem 2.1.18 asserts that we can complete them to an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \tilde{\mathcal{U}}, \tilde{k})$. Averaging the cut-off functions in $\tilde{\mathcal{U}}$ over K and possibly increasing $\tilde{k}$ to k, we obtain an unobstructed auxiliary datum $(\nabla^X, \mathcal{O}_X(1), p, \mathcal{U}, k)$ as claimed. By Construction 2.1.14, the associated global Kuranishi chart is K-compatible. The statements about relative smoothness follow from the description of its universal property in §2.3.1. Finally, (3) can be seen by noting that the proofs of §2.4.1 carry over verbatim to the equivariant setting. □

Definition 5.2.13. The *equivariant Gromov–Witten invariants* of (X, ω, μ) are the maps

$$\mathrm{GW}^{X,\omega,\mu}_{g,n,A} := ((\mathrm{ev} \times \mathrm{st})_K)_*[\overline{\mathcal{M}}_{g,n}(X, A;)J]^{\mathrm{vir}}_K : H^{*+\mathrm{vdim}}_K(X^n \times \overline{\mathcal{M}}_{g,n}; \mathbb{Q}) \to H^*_K \quad (5.2.3.1)$$

where J is any K-invariant ω-tame almost complex structure on X.

Remark 5.2.14. By [Kir84, Proposition 5.8], (X, μ) is equivariantly formal. This allows us to recover the non-equivariant GW invariants from the equivariant ones by (5.2.1.2) as follows. Given $\alpha = \alpha_1 \times \cdots \times \alpha_n \in H^*(X^n; \mathbb{Q})$, we can find $\widetilde{\alpha} \in H^*_K(X^n; \mathbb{Q})$ so that $\iota^*\widetilde{\alpha} = \alpha$, where $\iota\colon X \to X_{\mathbb{T}}$ is the canonical inclusion. Then

$$\langle \alpha_1, \ldots, \alpha_n \rangle^{X,\omega}_{g,n,A} = c_* \mathrm{GW}^{X,\omega,\mu}_{g,n,A}(\widetilde{\alpha}),$$

where $c\colon BK \to \mathrm{pt}$ is the constant map. In particular, if the equivariant GW invariants of X vanish, then so do the non-equivariant GW invariants.

Proposition 5.2.15. *The invariants* $\mathrm{GW}^{X,\omega,\mu}_{g,n,A}$ *satisfy the equivariant analogue of the Kontsevich–Manin axioms.*

Proof. By the equivariant Kontsevich-Manin axioms we mean the generalisation of the relations listed in the introduction to equivariant cohomology. The arguments of §4.1 carry over, using Proposition 5.2.5 instead of Proposition 3.1.7. As an example, we discuss the Fundamental class axiom; it says

$$\langle \alpha_1 \times \cdots \times \alpha_n \times 1_X \times \beta; \mathrm{GW}^{X,\omega,\mu}_{g,n+1,A}\rangle = \langle \alpha_1 \times \cdots \times \alpha_n \times \pi_{n+1!}\beta; \mathrm{GW}^{X,\omega,\mu}_{g,n,A}\rangle \quad (5.2.3.2)$$

as elements of H^*_K. Let $\mathcal{K}_n$ be a global Kuranishi chart for $\overline{\mathcal{M}}_{g,n-1}(X, A; J)$ equipped with a compatible K-action and let $\mathcal{K}_n n + 1$ be its pullback along the forgetful map π_{n+1}. By the proof of Proposition 4.1.5, we may replace them with global Kuranishi charts $\widetilde{\mathcal{K}}_n$ and $\widetilde{\mathcal{K}}_n$ which are still compatible with the group action and where the forgetful map $\widetilde{\pi}_{n+1}\colon \widetilde{\mathcal{T}}_{n+1}/\widetilde{G} \to \widetilde{\mathcal{T}}_n/\widetilde{G}$ satisfies

$$(\widetilde{\pi}_{n+1})_! \,\mathrm{st}^* = \mathrm{st}^* \,\pi_{n+1!}$$

in ordinary cohomology. Now we may conclude by using the straightforward generalisation of Lemma A.1.6 to equivariant cohomology. The other axioms are left to the interested reader. □

Set $QH^*_K(X,\omega) := H^*_K(X;\mathbb{Q}) \otimes_{\mathbb{Q}} \Lambda$ and endow it with the product

$$\alpha * \beta = \sum_{A \in H_2(X;\mathbb{Z})} (\alpha * \beta)_A t^{\omega(A)},$$

where

$$\int_X^K (\alpha * \beta)_A \cdot \gamma = \mathrm{GW}^{X,\omega,\mu}_{0,3,A}(\alpha,\beta,\gamma) \tag{5.2.3.3}$$

for any $\gamma \in H^*_K(X;\mathbb{Q})$. By the equivariant Symmetry and Splitting axiom, this is graded-commutative and associate. Note that (5.2.3.3) determines $(\alpha * \beta)_A$ uniquely since (X,μ) is equivariantly formal.

Chapter 6

Exotic tori in higher projective spaces

In this chapter we prove Theorem 1.2.1. It is an immediate consequence of the following result about the Newton polytope of the disk potential of a lifted Vianna tori, defined in §6.1.2. More precisely, we prove the following refinement.

Theorem 6.0.1. *The Newton polytope of the disk potential of $\overline{T}_{(a,b,c)}$ is a nondegenerate simplex in $\mathbb{R}^n$. One 2-dimensional face is a triangle with affine edge lengths a, b and c, while the affine length of any other edge is 1. In particular, if $\{a, b, c\} \neq \{a', b', c'\}$, then there is no symplectomorphism of $\mathbb{P}^n$ that maps the lifted Vianna torus $\overline{T}_{(a,b,c)}$ to $\overline{T}_{(a',b',c')}$.*

6.1 Geometric preliminaries

In this section we define the necessary geometric constructions in order to apply [PT20, Theorem 1.1] in §6.2. We introduce the notion of a solid mutation configuration and solid mutations, generalising mutations of a 2-dimensional Lagrangian torus along a disk to higher dimensions. Subsequently, we define the lifts of the Vianna tori and show that they are related by solid mutations.

6.1.1 Solid mutation configurations

We generalise the results of [PT20, §4.4, §4.5] to higher dimensions. Compare with [PT20, §5.3], where the ambient manifolds are required to be toric. In particular, the definition of solid mutation matches the definition of higher mutation in [PT20] with mutation configuration (F, w) where F is an $(n-1)$-dimensional face of a moment polytope.

Definition 6.1.1. Let (M^{2n}, ω) be a symplectic manifold. A pair $(L, \mathfrak{T})$ is a *solid mutation configuration (SMC)* in M if

- L is a Lagrangian torus;
- $\mathfrak{T}$ is a Lagrangian solid torus, i.e. $\mathfrak{T}$ is diffeomorphic to $\mathbb{D} \times \mathbb{T}^{n-2}$;
- L and $\mathfrak{T}$ intersect cleanly along the boundary of $\mathfrak{T}$;

- the pair $(L, \mathfrak{T} \cap L)$ is diffeomorphic to the standard pair $(\mathbb{T}^n, \mathbb{T}^{n-1})$ for some n.

Here two submanifolds N_0 and N_1 of M *intersect cleanly* if $K = N_0 \cap N_1$ is a smooth submanifold of M and $T_xK = T_xN_0 \cap T_xN_1$ for any $x \in K$.

Let us make the following observation, which is the higher dimensional analogue of [PT20, Corollary 3.4].

Lemma 6.1.2. *Suppose $(L, \mathfrak{T})$ is a SMC in (M, ω) and L is monotone. Then there exists a divisor $D \subset M \backslash (L \cup \mathfrak{T})$ Poincaré dual to $dc_1(M)$ for some $d \gg 1$, so that L is exact in $M \backslash D$.*

Proof. The assertion follows by the same argument as in [PT20, Corollary 3.4] from [PT20, Theorem 3.3]. We sketch the argument. Since L is monotone, we can replace ω by $\omega' = \tau\omega$ and thus assume that $\omega(\pi_2(M, L)) \subset \mathbb{Z}$. Hence we can find a Hermitian line bundle $\mathcal{L} \to M$ with Hermitian curvature $F^\nabla = -2\pi i \omega$. Then $\mathcal{L}|_L$ is flat, as is the restriction of $\mathcal{L}^{\otimes k}$ for any $k \geqslant 1$. Now construct for each k sections s_{k1}, s_{k2}, s_{k3} of $\mathcal{L}^{\otimes k}$ which are bounded away from 0 on L, $\mathfrak{T}$ and on $L \cap \mathfrak{T}$ respectively and show that their sum is bounded away from zero. Also, choose the sections s_{ki} so that they are covariantly constant over L. □

As in [PT20] we will construct a model neighbourhood for SMCs, which will allows us to define solid mutations. The key ingredient is a Weinstein neighbourhood theorem for SMCs (compare to [PT20, Lemma 4.11]), for which we need the following technical result.

Lemma 6.1.3. *Suppose $\psi \colon [\frac{1}{2}, 1] \times \mathbb{T}^n \to [\frac{1}{2}, 1] \times \mathbb{T}^n$ is a diffeomorphism with $\psi(1, x) = (1, x)$ for $x \in \mathbb{T}^n$. Then there exists $\epsilon > 0$ and a diffeomorphism $\Psi \colon \mathbb{D} \times \mathbb{T}^{n-1} \to \mathbb{D} \times \mathbb{T}^{n-1}$ which agrees with ψ on $[1-\epsilon, 1] \times \mathbb{T}^n$. If ψ is equivariant with respect to a torus action on $\mathbb{T}^n$, then we can choose Ψ to be equivariant as well.*

Here we identify $[\frac{1}{2}, 1] \times S^1$ with the corresponding annulus inside the closed unit disk $\mathbb{D} \subset \mathbb{C}$.

Proof. Write $\psi = (\psi', \psi'')$ and define $\psi_r(x) := \psi''(r, x)$ for $x \in \mathbb{T}^n$. Then there is $0 < \epsilon < \frac{1}{9}$ so that ψ_r is a diffeomorphism for $|1 - r| < 3\epsilon$. In particular, ψ_r defines an (equivariant) isotopy from $\psi_{1-2\epsilon}$ to the identity. Let $\rho \colon [0,1] \to [0,1]$ be a smooth cutoff function with $\rho(t) = t$ for $t \geqslant 1 - \epsilon$, $\rho \equiv 1 - 2\epsilon$ on $[0, 1-2\epsilon]$ and $\rho'(t) > 0$ on $(1 - 2\epsilon, 1]$. Similarly, let $\beta \colon [\frac{1}{2}, 1] \to [0,1]$ be a smooth cutoff function so that $\beta(t) = \frac{1}{2}t$ near $\frac{1}{2}$, $\beta < \frac{1}{2}$ on $[\frac{1}{2}, 1 - 2\epsilon]$, $\beta \equiv 1$ on $[1-\epsilon, 1]$ and β is strictly increasing on $[1 - 2\epsilon, 1 - \epsilon]$. Define $\psi^{(1)} \colon [\frac{1}{2}, 1] \times \mathbb{T}^n \to (0, 1] \times \mathbb{T}^n$ by

$$\psi^{(1)}(t, x) = (\beta(t)\psi'(\rho(t), x), \psi''(\rho(t), x)).$$

This is a diffeomorphism by the choice of ρ and β and agrees with ψ near $\{1\} \times \mathbb{T}^n$.

It suffices thus to extend the closed embedding

$$\psi^{(1)} \colon [\frac{1}{2}, r] \times \mathbb{T}^n \to [0,1] \times \mathbb{T}^n \backslash \psi^{(1)}\Big((r, 1]) \times \mathbb{T}^n\Big)$$

for some $r < 1 - 2\epsilon$. We can write it as

$$\psi^{(1)}(t,x) = (th(x), \varphi(x))$$

for $t \in [\frac{1}{2}, r]$, where $\varphi \in \mathrm{Diff}(\mathbb{T}^n)$ is (equivariantly) isotopic to the identity and $h\colon \mathbb{T}^n \to (0,1]$ is smooth. Note that $\mathrm{im}(4h) \times \mathbb{T}^n$ is the inner boundary of $\psi([1-2\epsilon, 1] \times \mathbb{T}^n)$. Given an (equivariant) isotopy $\{\varphi_s\}_{s\in[0,1]}$ from the identity to φ, let $\chi\colon [0,1/3] \to [0,1]$ be a smooth cutoff function, so that $\chi \equiv 0$ near 0 and $\chi \equiv 1$ near $\frac{1}{3}$. Fix also a smooth cutoff function $\eta\colon [0,r] \to [0,1]$ so that $\eta \equiv 0$ near $\frac{1}{3}$, $\eta \equiv 1$ on $[\frac{1}{2}, r]$ and $\eta'(t) \geqslant 0$. Set $a := \min h$. Then define $\tilde{\Psi}\colon [0,1] \times \mathbb{T}^n \to [0,1] \times \mathbb{T}^n$ by

$$\tilde{\Psi}(t,x) = \begin{cases} \psi^{(1)}(t,x) & r \leqslant t \leqslant 1 \\ \left(at\,(\frac{h(x)}{a})^{\eta(t)}, \varphi(x)\right) & \frac{1}{3} \leqslant t \leqslant r \\ (ta, \varphi_{\chi(t)}(x)) & 0 \leqslant t \leqslant \frac{1}{3} \end{cases}$$

This descends to the desired diffeomorphism of the solid torus. If we start with an equivariant ψ, then $\tilde{\Psi}$ is equivariant by construction, and thus so is Ψ. □

Lemma 6.1.4 (Weinstein neighbourhood theorem for solid mutation configurations)**.** *Suppose $(L_i, \mathfrak{T}_i) \subset (M_i, \omega_i)$ is a solid mutation configuration for $i \in \{0,1\}$ with $\dim(M_0) = \dim(M_1)$. Then there exist neighbourhoods $U_i \subset M_i$ of $L_i \cup \mathfrak{T}_i$ and a symplectomorphism $\psi\colon U_0 \to U_1$ mapping $(L_0, \mathfrak{T}_0)$ to $(L_1, \mathfrak{T}_1)$.*

Proof. By definition there exists a diffeomorphism $\phi\colon L_0 \to L_1$ which maps $L_0 \cap \mathfrak{T}_0$ to $L_1 \cap \mathfrak{T}_1$. Extend ϕ to a symplectic bundle isomorphism $\Phi\colon TM_0|_{L_0} \to TM_1|_{L_1}$, which maps $T\mathfrak{T}_0|_{\partial\mathfrak{T}_0}$ to $T\mathfrak{T}_1|_{\partial\mathfrak{T}_1}$. Now the proof proceeds along the lines of [PT20, Lemma 4.11]. By the Weinstein neighbourhood theorem, Φ defines a symplectomorphism $\psi'\colon U_0' \to U_1'$ for neighbourhoods U_0' and U_1' of L_0, respectively L_1. Then ψ' maps $U_0' \cap \mathfrak{T}_0$ to a manifold tangent to $U_1' \cap \mathfrak{T}_1$ with the same boundary. As cleanly intersecting submanifolds admits a normal form near their intersection, we may post-compose ψ' with a Hamiltonian isotopy to assume ψ' maps $U_0' \cap \mathfrak{T}_0$ to $U_1' \cap \mathfrak{T}_1$. By Lemma 6.1.3 we can extend its restriction to $U_0' \cap \mathfrak{T}_0$ to a diffeomorphism $\varphi\colon \mathfrak{T}_0 \to \mathfrak{T}_1$, possibly shrinking U_i'. As φ is induced by Φ near $\partial\mathfrak{T}_0$, the standard lift of φ to a symplectic vector bundle isomorphism $TM_0|_{\mathfrak{T}_0} \to TM_1|_{\mathfrak{T}_1}$ extends Φ. Using the Weinstein neighbourhood theorem again, we obtain the desired symplectomorphism ψ. □

Suppose a torus $\mathbb{T}$ embeds as subtorus of $\partial\mathfrak{T}_0$ and $\partial\mathfrak{T}_1$. Multiplication by elements of $\mathbb{T}$ induces a torus action on L_i and $\mathfrak{T}_i$ which extends to a Hamiltonian $\mathbb{T}$-action on a neighbourhood of $L_i \cup \mathfrak{T}_i$. Using the equivariant version of the Weinstein neighbourhood theorem and Lemma 6.1.3, we can choose ψ in the previous statement to be equivariant.

We now construct our model neighbourhood, see also [PT20, §4.5]. Set $X_1 := \mathbb{C}^2 \backslash \{x_1 x_2 = 1\}$ and endow it with the Lefschetz fibration $\pi\colon X_1 \to \mathbb{C}\backslash\{1\}$ defined by $\pi(x) = x_1 x_2$. Given a simple loop (or more generally an embedded path) γ in $\mathbb{C}\backslash\{0,1\}$ define the Lagrangian

$$T_\gamma := \left\{(x,y) \in \mathbb{C}^2 \;\middle|\; |x| = |y|,\ \pi(x,y) \in \operatorname{im}(\gamma)\right\}. \tag{6.1.1.1}$$

By [PT20, Lemma 4.13], there exists a primitive θ of $\omega_{\mathrm{std}}|_{X_1}$ and $A > 0$ so that T_γ is exact with respect to θ if and only if γ encloses 1 and a disk of area A. Given such a loop γ we say T_γ (or just γ) is of *Clifford type* if γ encloses 0 and of *Chekanov type* otherwise. If ℓ is a line segment starting at 0, then T_ℓ is a vanishing cycle, in particular, a Lagrangian disk.

Given $n \geqslant 3$, set $X := X_1 \times \mathbb{C}^{n-2}$ and endow it with the restriction of the standard symplectic form on $\mathbb{C}^n$. Denote $\overline{T}_\gamma := T_\gamma \times \mathbb{T}^{n-2}$. This is exact, with respect to $\tilde{\theta} := \theta \oplus \theta_{n-2}$ for the standard primitive θ_{n-2} of ω_{std} on $\mathbb{C}^{n-2}$, if T_γ is exact. The following is a straightfoward exercise.

Lemma 6.1.5. *Suppose γ is a loop enclosing both 0 and r and let ℓ be the line segment from 0 to $\min(\{\operatorname{im}(\gamma) \cap i\mathbb{R}\})$. Then $(\overline{T}_\gamma, \overline{T}_\ell)$ is an SMC.*

We obtain the following corresponding generalisation of [PT20, Lemma 4.17] by applying Lemma 6.1.4 and the discussion afterwards.

Corollary 6.1.6. *If $(L, \mathfrak{T})$ is an SMC in (M^{2n}, ω), there exists a neighbourhood $U \subset M$ of $L \cup \mathfrak{T}$ and an equivariant symplectic embedding $\psi\colon U \hookrightarrow X_1 \times \mathbb{C}^{n-2}$ so that $\psi(L, \mathfrak{T}) = (\overline{T}_\gamma, \overline{T}_\ell)$ for γ of Clifford type and ℓ a line segment as above.*

By [PT20, Lemma 4.14], $\overline{T}_\gamma$ and $\overline{T}_{\gamma'}$ are isotopic through a compactly supported Hamiltonian if and only if γ and γ' (as above) are smoothly isotopic in $\mathbb{C}\backslash\{0,1\}$ and enclose disks of the same area. Thus the following definition is well-defined up to Hamiltonian isotopy.

Definition 6.1.7. Let $(L, \mathfrak{T})$ be an SMC in (M^{2n}, ω) and let ψ be a symplectomorphism as in Corollary 6.1.6. The *solid mutation of L along $\mathfrak{T}$* is $L_{\mathfrak{T}} := \psi^{-1}(\overline{T}_{\gamma'})$ for any simple loop γ' in $\psi(U)$ of Chekanov type.

Lemma 6.1.8. *If $(L, \mathfrak{T})$ is an SMC in (M, ω) and L is monotone, then so is $L_{\mathfrak{T}}$.*

Proof. Using Corollary 6.1.6, the proof is analogous to the proof of [Cha23, Lemma 2.5]. □

6.1.2 Lifting Vianna tori

A *Markov triple* (a, b, c) is a triple of positive integers satisfying the Diophantine equation $a^2 + b^2 + c^2 = 3abc$. The set of these triples forms the vertices of the *Markov tree*, which is connected and infinite by [Aig13, Chapter 3]. A triple (a, b, c) is connected to (a', b', c') by an edge if and only if (a', b', c') is a *Markov mutation* of (a, b, c), i.e., of the form $(3bc - a, b, c)$, $(a, 3ac - b, c)$ or $(a, b, 3ab - c)$.

For any Markov triple (a, b, c), [Via16] constructs a monotone Lagrangian torus $T_{(a,b,c)}$ in $\mathbb{P}^2$, whose Hamiltonian isotopy class is uniquely determined by (a, b, c). To make this precise, let $\mathbb{P}(a^2, b^2, c^2)$ be the weighted projective space associated to a Markov triple

(a,b,c) with associated degenerations from $\mathbb{P}^2$. By [Via16], $\mathbb{P}^2$ can be obtained from $\mathbb{P}(a^2,b^2,c^2)$ by performing at most three rational blow-downs.

Definition 6.1.9 (Vianna tori)**.** The *Vianna torus* $T_{(a,b,c)}$ associated to a Markov triple (a,b,c) is the central fiber of the almost toric fibration of $\mathbb{P}^2$ obtained from the rational blow-down of $\mathbb{P}(a^2,b^2,c^2)$.[1]

In [PT20], the authors relate Vianna's construction to local mutations, which are a special case of our solid mutations. In particular, they show that each $T_{(a,b,c)}$ admits canonical mutation configurations $(T_{(a,b,c)}, D_{(a',b',c')})$, indexed by the Markov triples (a',b',c') obtained from (a,b,c) by a Markov mutation. The mutation along $D_{(a',b',c')}$ results in a torus that is Hamiltonian isotopic to $T_{(a',b',c')}$.

We will lift these tori to monotone Lagrangian tori in $\mathbb{P}^n$ for $n \geqslant 3$ using symplectic reduction. In Lemma 6.1.12, we show that a mutation of $T_{(a,b,c)}$ along $D_{(a',b',c')}$ corresponds to a solid mutation of its lift to $\mathbb{P}^n$ along the lift of $D_{(a',b',c')}$.

Fix $n \geqslant 3$ and let

$$\mu_n \colon \mathbb{P}^n \to \mathbb{R}^{n-2} : [z] \mapsto \frac{1}{|z|^2}(|z_3|^2, \dots, |z_n|^2)$$

be the moment map of the standard Hamiltonian $\mathbb{T}^{n-2}$-action acting on the last $n-2$ homogeneous coordinates. In particular, the action is free on $F_n := \mu_n^{-1}\left(\left\{\frac{1}{n+1}\sum_{i=1}^{n-2} e_i\right\}\right)$, where $e_1, \dots, e_{n-2}$ is the standard basis of $\mathbb{R}^{n-2}$. A computation in local coordinates shows that $F_n/\mathbb{T}^{n-2}$ equipped with the reduced symplectic form is symplectomorphic to $\mathbb{P}^2$. Let $q \colon F_n \to \mathbb{P}^2$ be induced by the quotient map. A straightforward computation shows that the preimage of the Clifford torus in $\mathbb{P}^2$ under q is the Clifford torus in $\mathbb{P}^n$.

Definition 6.1.10. Given a Markov triple (a,b,c), we define the *lifted Vianna torus* to be

$$T^{(n)}_{(a,b,c)} := q^{-1}(T_{(a,b,c)}).$$

We see that $T^{(n)}_{(a,b,c)}$ is a Lagrangian torus by the definition of the symplectic structure on the symplectic reduction. If n is clear from the context, we write $\overline{T}_{(a,b,c)}$ instead of $T^{(n)}_{(a,b,c)}$.

Remark 6.1.11. As we can also do the reduction inductively, at each step reducing by an S^1-action, we see that for each (a,b,c), we obtain a tower $T^{(n)}_{(a,b,c)} =: T_n \to T_{n-1} \to \dots \to T_2 := T_{(a,b,c)}$, where $T_{j+1} \to T_j$ is the restriction of a principal S^1-bundle on $\mathbb{P}^j$. As the Euler class of this S^1-bundle is a multiple of $[\omega_{\mathrm{FS}}]$, its restriction to T_j vanishes. Thus $T^{(n)}_{(a,b,c)} \to T_{(a,b,c)}$ is a trivial $\mathbb{T}^{n-2}$-bundle.

Lemma 6.1.12. *If (a',b',c') is a Markov mutation of (a,b,c), then $T^{(n)}_{(a,b,c)}$ and $T^{(n)}_{(a',b',c')}$ are solid mutations of each other. In particular, each $T^{(n)}_{(a,b,c)}$ is monotone.*

Proof. By Vianna's construction and [PT20, Lemma 4.21], there exists a mutation configuration $(T_{(a,b,c)}, D)$ in $\mathbb{P}^2$, so that the associated mutation of $T_{(a,b,c)}$ is $T_{(a',b',c')}$. Let

[1] Note that Vianna denotes $T_{(a,b,c)}$ by $T(a^2,b^2,c^2)$ instead.

$\mathfrak{T} = q^{-1}(D)$. As q is the quotient map of a free $\mathbb{T}^{n-2}$-action, $(T^{(n)}_{(a,b,c)}, \mathfrak{T})$ is a solid mutation configuration in $\mathbb{P}^n$. It follows from the local model in [PT20], respectively §6.1.1 that the solid mutation of $T^{(n)}_{(a,b,c)}$ along $\mathfrak{T}$ is the lift of the mutation of $T_{(a,b,c)}$ along D, and thus Hamiltonian isotopic to $T^{(n)}_{(a',b',c')}$. The last assertion follows from Lemma 6.1.8. $\square$

6.2 A wall-crossing formula for the lifted Vianna tori

In this section, we explain how to obtain the wall-crossing formula for the disk potential under a solid mutation. This is a variation of [PT20, Theorem 5.7], where we do not require the toric assumption due to our Weinstein neighbourhood theorem for general solid mutation configurations.

Let $X_1 := \mathbb{C}^2 \backslash \{x_1 x_2 = 1\}$ be as in the previous section and let γ be a loop of Clifford type in X_1. Suppose ℓ is a straight line segment in $\mathbb{C}^*$ joining the origin to a point $p \in \gamma$ and only intersecting γ at p. Set $L_0 := T_\gamma$ and $D_0 := T_\ell$ as defined in (6.1.1.1). By [PT20, Lemma 4.15], there is a small neighbourhood U_0 of $L_0 \cup D_0$ such that U_0 is Liouville and the Liouville completion of $\overline{U_0}$ (which we can take to be a Liouville domain) agrees with X_1. Let L_1 denote a Chekanov type torus in U_0.

Set $\overline{L}_i := L_i \times \mathbb{T}^{n-2}$. This is an exact Lagrangian in $(U_0 \times (A_\epsilon)^{n-2}, \tilde{\theta}))$ for $i \in \{0,1\}$, where $A_\epsilon := \{z \in \mathbb{C} \,|\, |z-1| < \epsilon\}$ for $\epsilon > 0$ and $\widetilde{\theta}$ was defined in §6.1.1. Let U_s be a Liouville domain obtained by smoothing the corners in $\overline{U_0} \times (\overline{A_\epsilon})^{n-2}$. Since the completion of A_ϵ is $\mathbb{C}^*$, the completion of U_s is $X_1 \times (\mathbb{C}^*)^{n-2}$, see [Oan06, §3.d.].

Remark 6.2.1. Suppose L is a monotone Lagrangian torus in a symplectic manifold (M^{2n}, ω). Its disk potential W_L can be considered as a Laurent polynomial as follows. Let $\rho : H_1(L;\mathbb{Z}) \to \mathbb{C}^*$ be a local system on L. Fix a basis $v_1, \ldots, v_n$ of $H_1(L;\mathbb{Z}) \cong \mathbb{Z}^n$ and let $(x_1, \ldots, x_n)$ be the image of $(v_1, \ldots, v_n)$ under the holonomy map ρ. This tuple specifies the holonomy of a flat $\mathbb{C}^*$ line bundle over L uniquely. Thus, one can identify $(\mathbb{C}^*)^n$ with the space of flat line bundles over L. We can then write the disk potential W_L, defined in (1.2.0.1), as

$$W_L : (\mathbb{C}^*)^n \to \mathbb{C}$$

$$W_L(x_1, \ldots, x_n) = \sum_{\substack{\beta \in \pi_2(M,L) \\ \mu(\beta)=2}} |\mathcal{M}(L,\beta)|\, x^{\partial \beta},$$

where $\mathcal{M}(L,\beta)$ is the moduli space of $J-$holomorphic discs in the class β such that the boundary of the disk lies on L and passes through a generic (but throughout fixed) $p \in L$. We identify $\partial\beta$ with a point in $\mathbb{Z}^n$ via the chosen basis and use multi-index notation.

We now extract a local wall-crossing formula from [Sei13]. This requires the choice of a certain basis of $H_1(\overline{L}_i;\mathbb{Z})$. Write $\overline{L}_0 = T_\gamma \times \mathbb{T}^{n-2}$ with $\pi \colon T_\gamma \to \operatorname{im}(\gamma)$ the restriction of the Lefschetz fibration. Note that L_0 and L_1 intersect cleanly in two circles, so $\overline{L}_0$ and $\overline{L}_1$ intersect cleanly in two tori $\mathbb{T}^{n-1}$.

Definition 6.2.2 (Admissible pair of bases)**.** We call any two bases of $H_1(\overline{L}_0;\mathbb{Z})$ and $H_1(\overline{L}_0;\mathbb{Z})$ *admissible* if they are obtained via the following construction. We have $L_0 \cap L_1 =$

$C_+ \sqcup C_-$, where $C_\pm$ are circles lying in a fibre of $\pi\colon X_1 \to \mathbb{C}$. Let $\alpha_0 = -[C_+]$. Choose β_0 to be the circle in T_γ given by a lift of γ to X_1 and let $\gamma_{0,i}$ be the class of the i^{th} factor of $\mathbb{T}^{n-2}$ for $1 \leqslant i \leqslant n-2$. This forms a basis of $H_1(\overline{L}_0;\mathbb{Z})$ with respect to which we denote the coordinates of the disk potential by $x_0, y_0, z_{0,1}, \ldots, z_{0,n-2}$. Given any smooth isotopy from $\overline{L}_0$ to $\overline{L}_1$ which preserves α_0 and which is the identity on the last $(n-1)$-factors, we denote by $\alpha_1, \beta_1, \gamma_{1,1}, \ldots, \gamma_{1,n-2}$ the induced basis of $H_1(\overline{L}_1;\mathbb{Z})$.

Given any basis of $H_1(\overline{L}_i,\mathbb{Z})$, we can and will identify a local system ρ_i on $\overline{L}_i$ with a point $(x_i, y_i, z_{ij}) \in (\mathbb{C}^*)^n$ using these coordinates. Abbreviate $\mathbf{L}_i := (\overline{L}_i, \rho_i)$.

Lemma 6.2.3 (Local Wall-Crossing)**.** *Given an admissible pair of bases, we have*

$$HF_{X_1\times(\mathbb{C}^*)^{n-2}}(\mathbf{L}_1,\mathbf{L}_2) \neq 0$$

if and only if $(x_1, y_1, z_{1,1}, z_{1,2} \ldots, z_{1,n-2}) = (x_0, y_0(1+x_0), z_{0,1}, \ldots, z_{0,n-2})$.

Proof. This is a generalisation of [Sei13, Proposition 11.8] which treats the case of $n = 2$. Instead of using Morse-Blott Floer homology as in op. cit., one can also count strips by hand. To this end, perturb L_1 slightly in the fibre direction so that it intersects L_0 transversely in two fibres of $(x,y) \mapsto xy$. Perturb the i^{th} S^1, denoted S_{1i}, in the $\mathbb{T}^{n-2}$-factor of L_1 so that it intersects the i^{th} circle S_{0i} of the $\mathbb{T}^{n-2}$-factor of $\overline{L_0}$ transversely. Let p_i be the composition $X_1 \times (\mathbb{C}^\times)^{n-2} \to (\mathbb{C}^\times)^{n-2} \xrightarrow{\mathrm{pr}_i} \mathbb{C}^\times$.

Suppose $v\colon \mathbb{R} + i[0,1] \to X_1 \times (\mathbb{C}^\times)^{n-2}$ is a Floer strip of Maslov index one. Then, by the open mapping theorem and the maximum principle πv as well as each $v_i := p_i v$ have to map onto the two lunes enclosed by γ and γ', respectively between S_{0i} and S_{1i}. Since all factors are monotone Lagrangians, it follows that at most one of $\pi v, v_1, \ldots, v_{n-2}$ can be nonconstant. The later $n-2$ cases imply that we must have $z_{1,j} = z_{0,j}$, while the study of πv was done in [Sei13], respectively [Sei97, Chapter 17]. $\square$

Theorem 6.2.4. *Given a pair of adimissible bases of* $H_1(\overline{T}_{(a,b,c)};\mathbb{Z})$ *and* $H_1(\overline{T}_{(a',b',c')};\mathbb{Z})$*, the disk potentials are related by*

$$W_{\overline{T}_{(a,b,c)}}(x, y, z_1, \ldots, z_{n-2}) = W_{\overline{T}_{(a',b',c')}}(x, y(1+x), z_1, \ldots, z_{n-2}). \tag{6.2.0.1}$$

Proof. We first reduce this to a computation in the local model described in §6.1.1. By [PT20], we can find a disk $\mathbb{D} \subset \mathbb{P}^2$so that $(T_{(a,b,c)}, \mathbb{D})$ is a Lagrangian seed whose mutation is Hamiltonian isotopic to $T_{(a',b',c')}$. Denote by $\mathfrak{T}$ the lift of $\mathbb{D}$ to a solid torus in $\mathbb{P}^n$. Fix an equivariant symplectic embedding $\phi : U_0 \times (A_\epsilon)^{n-2} \to \mathbb{P}^n$ as in Corollary 6.1.6 and Lagrangians $L_0, D_0, L_1 \subset U_0$ as above so that $\phi(\overline{L}_0) = \overline{T}_{(a,b,c)}$ and $\phi(D_0 \times \mathbb{T}^{n-2}) = \mathfrak{T}$. Then,

$$\phi(\overline{L}_1) = \psi(\overline{T}_{(a',b',c')})$$

for some Hamiltonian diffeomorphism ψ on $\mathbb{P}^n$. Let $D \subset \mathbb{P}^n$ be a Donaldson divisor for $(\overline{T}_{(a,b,c)}, \mathfrak{T})$ as in Lemma 6.1.2. Choosing U_0 and ϵ sufficiently small, we may assume $\phi(U_0 \times (A_\epsilon)^{n-2}) \subset \mathbb{P}^n \backslash D$. We claim that $\phi(U_0 \times (A_\epsilon)^{n-2})$ is a Liouville subdomain of $\mathbb{P}^n \backslash D$, i.e., that $\lambda := \tilde{\theta} - \phi^*\theta_{\mathbb{P}^n\backslash D}$ is exact, where $\theta_{\mathbb{P}^n\backslash D}$ is a primitive of $\omega_{\mathrm{FS}}|_{\mathbb{P}^n\backslash D}$. Let α

be any loop in $U_0 \times (A_\epsilon)^{n-2}$. As $U_0 \times (A_\epsilon)^{n-2}$ deformation retracts onto $\overline{L}_0 \cup D_0 \times \mathbb{T}^{n-2}$, we may assume $\alpha = \iota_* \alpha'$ for some $\alpha' \in \pi_1(\overline{L}_0)$ and $\iota\colon \overline{L}_0 \hookrightarrow U_0 \times (A_\epsilon)^{n-2}$ the inclusion. Thus

$$\int_\alpha \lambda = \int_{\alpha'} \iota^* \lambda = 0,$$

since $\overline{L}_0$ is exact with respect to both primitive 1-forms. Thus λ is exact, so by [PT20, Theorem 3.1],

$$HF_{\mathbb{P}^n \setminus D}((\overline{T}_{(a,b,c)}, \hat{\rho}_0), (\overline{T}_{(a',b',c')}, \hat{\rho}_1)) \cong HF_{X_1 \times (\mathbb{C}^*)^{n-2}}((\overline{L}_0, \rho_0), (\overline{L}_1, \rho_1)). \tag{6.2.0.2}$$

for any local system ρ_i on $\overline{L}_i$ and the corresponding one on $\overline{T}_{(a,b,c)}$ respectively $\overline{T}_{(a',b',c')}$. By Lemma 6.2.3 and our choice of bases,

$$HF_{X_1 \times (\mathbb{C}^*)^{n-2}}((\overline{L}_0, \rho_0), (\overline{L}_1, \rho_1)) \neq 0$$

if and only if $\rho_0 = (x, y, z_1, \ldots, z_{n-2})$ and $\rho_1 = (x, y(1+x), z_1, \ldots, z_{n-2})$. As the disk potential is invariant under Hamiltonian isotopy, we see that $W_{\overline{T}_{(a',b',c')}} = W_{\phi(\overline{L}_1)}$ as disk potentials of Lagrangians in $\mathbb{P}^n$. Therefore [PT20, Theorem 1.1] implies (6.2.0.1). □

Remark 6.2.5. The theorem holds for general solid mutation configurations since it only uses the local models of §6.1.1.

Remark 6.2.6 (Algebraic mutations). Theorem 6.2.4 asserts that, given admissible bases of the first homology, the Laurent polynomials $W_{\overline{T}_{(a,b,c)}}$ and $W_{\overline{T}_{(a',b',c')}}$ are related by the algebraic mutation $(x, y, z_1, \ldots, z_{n-2}) \mapsto (x, y(1+x), z_1, \ldots, z_{n-2})$. Refer to [PT20, §4] and [ACG$^+$12, Definition 2] for the general definition of an algebraic mutation of a Laurent polynomial.

6.3 Distinguishing the tori in $\mathbb{P}^n$

In order to distinguish the tori $T_{(a,b,c)}$, Vianna shows that the Newton polytope associated to the disk potential of $T_{(a,b,c)}$ is a triangle with edges of affine length a, b and c. As the Newton polytope of $W_{T_{(a,b,c)}}$ is an invariant of the symplectomorphism class of $T_{(a,b,c)}$, the symplectomorphism class of $T_{(a,b,c)}$ in $\mathbb{P}^2$ is therefore uniquely determined by (a,b,c).

We will use this result from [Via16] to prove a similar result for the lifted Vianna tori. The argument uses induction on the Markov tree and properties of the Vianna tori and their Newton polytopes.

6.3.1 Newton polytopes

Given $n \geqslant 3$, let $R := \mathbb{R}[x_1^\pm, \ldots, x_n^\pm]$ be the ring of Laurent polynomials in n variables. We identify the set of monomials in R with $\mathbb{Z}^n$ in the obvious way. The *Newton polytope* of a Laurent polynomial $f = \sum_{k \in \mathbb{Z}^n} a_k x_1^{k_1} \cdots x_n^{k_n} \in R$ is the the closed convex hull

$$\mathrm{Newt}(f) := \mathrm{Conv}(\{k \in \mathbb{Z}^n : a_k \neq 0\}).$$

This association is equivariant with respect to the $\mathrm{GL}(n,\mathbb{Z})$-action on R, defined in [PT20, Remark 4.2], and the standard action on $\mathbb{R}^n$.

Example 6.3.1. Using the coordinates on $H_1(\overline{T}_{(1,1,1)};\mathbb{Z})$ coming from the the standard moment polytope of $\mathbb{P}^n$ scaled appropriately, the disk potential of the Clifford torus in $\mathbb{P}^n$ is

$$W_{\overline{T}_{(1,1,1)}}(x) = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}. \tag{6.3.1.1}$$

by [Aur07, Proposition 4.3]. Thus $\mathrm{Newt}(W_{\overline{T}_{(1,1,1)}}) = \mathrm{Conv}(e_1, \dots, e_n, -\sum_{i=1}^n e_i)$ is an n-simplex, where $e_1, \dots, e_n$ denotes the standard basis of $\mathbb{R}^n$.

In particular, the Newton polytope of the Clifford torus is a *Fano polytope* as defined in [ACG+12]; i.e.,

1. the polytope is convex;

2. it contains 0 in its interior;

3. its vertices are primitive in $\mathbb{Z}^n$.

In §6.2 we show that a solid mutation of a suitable Lagrangian torus corresponds to a specific algebraic mutation of its disk potential. By [ACG+12], an algebraic mutation of $f \in R$ corresponds to a *combinatorial mutation* of $\mathrm{Newt}(f)$. Refer to [ACG+12, Definition 5] for a precise description and to Remark 5 op. cit. for a discussion of the relationship between algebraic and combinatorial mutations. Note that we are only interested in algebraic mutations of the form $(x_1, \dots, x_n) \mapsto (x_1, x_2(1 + x_1), x_3, \dots, x_n)$. See [PT20, §5] for more general algebraic mutations that occur if one uses different geometric mutations of Lagrangians.

Example 6.3.2. We obtain $\overline{T}_{(1,1,2)}$ by solid-mutating $\overline{T}_{(1,1,1)}$ in $\mathbb{P}^2$ along $\overline{D}_{(1,1,2)} = D_{(1,1,2)} \times \mathbb{T}^{n-2}$. In the standard coordinates of $\overline{T}_{(1,1,1)}$, the class $[\partial D_{(1,1,2)}]$ is given by $(1, 1, 0, \dots, 0)$ by [PT20, Example 4.12]. Since the boundary of the disk in the local model corresponds to the vector $(1, 0, \dots, 0)$ in an admissible basis, it follows that we have to make the base change

$$(v_1, \dots, v_n) \mapsto (v_1, v_2 - v_1, v_3, \dots, v_n).$$

If a local system is given by $(x_1, \dots, x_n)$ in the previous basis, it corresponds to $x' = (\frac{x_1}{x_2}, x_2, \dots, x_n)$ in the new one. Then $W_{\overline{T}_{(1,1,1)}}(x') = \frac{x_1'}{x_2'} + x_2' + \cdots + x_n' + \frac{1}{x_1' x_3' \cdots x_n'}$ and hence

$$W_{\overline{T}_{(1,1,2)}}(x') = x_2' + x_3' + \cdots + x_n' + \frac{(1 + x_1')^2}{x_1' {x_2'}^2 x_3' \cdots x_n'}.$$

By [ACG+12, Proposition 2], the combinatorial mutation of a Fano polytope is again a Fano polytope. From Example 6.3.1 it follows that $\mathrm{Newt}(W_{\overline{T}_{(a,b,c)}})$ is a Fano polytope for any Markov triple (a, b, c). The proof of Theorem 6.0.1 relies heavily on going back and forth between Laurent polynomials and their associated Newton polytopes.

From now on we will use $x, y, z_1, \dots, z_{n-2}$ instead of $x_1, \dots, x_n$ to distinguish between the variables which are involved in the algebraic mutation and those which are not. We denote the associated coordinates of $\mathbb{R}^n$ by $\mathbf{x}, \mathbf{y}, \mathbf{z}_1, \dots, \mathbf{z}_{n-2}$.

Now we prove some lemmas in preparation for the proof of the main theorem in §6.3.2.

Lemma 6.3.3. a) *Suppose the Newton polytope of*

$$f(x,y) = \sum_{i=m}^{M} y^i f_i(x)$$

is Fano with $f_m \neq 0$ and $f_M \neq 0$. Then $m < 0 < M$. Moreover, if $\mathrm{Newt}(f)$ *is a triangle, then either f_M or f_m is a monomial, and if $f(x, y(1+x)^{-1})$ is a Laurent polynomial, then f_m is a monomial.*

b) *Suppose the Newton polytope of*

$$g(x,y,z) = \sum_{i=m}^{M} y^i z_1^{j_{1i}} \cdots z_{n-2}^{j_{(n-2)i}} g_i(x)$$

is contained in the affine plane $H = w + \langle v, e_1 \rangle$ for some $v = (v_1, \ldots, v_n) \in \mathbb{Z}^n$ such that $v_2 \neq 0$ and $w \in \mathbb{Z}^n$. Then j_{ri} depends linearly on i for any $r \in \{1, \ldots, n-2\}$.

Proof. a) By (2), $\mathrm{Newt}(f)$ has nonempty interior, so the first property is immediate. To see the second, suppose f_M is not a monomial. Then $\mathrm{Newt}(f) \cap \{\mathbf{y} = M\}$ is an edge of $\mathrm{Newt}(f)$; in particular it contains two vertices of $\mathrm{Newt}(f)$. As $\mathrm{Newt}(f) \cap \{\mathbf{y} = m\}$ is nonempty and only touches the boundary of $\mathrm{Newt}(f)$ it must therefore be a single point and thus f_m a monomial. If $f(x, y(1+x)^{-1})$ is still a Laurent polynomial, then $(1+x)^M$ has to divide f_M, so f_M cannot be a monomial.

b) Any monomial in g is of the form $x^k y^i z_1^{j_{1i}} \cdots z_{n-2}^{j_{(n-2)i}}$ where

$$(k, i, j_{1i}, \ldots, j_{(n-2)i}) = w + av + be_1 = (w_1 + av_1 + b, av_2 + w_2, av'_1 + w'_1, \ldots, av'_{n-2} + w'_{n-2})$$

for some $a, b \in \mathbb{R}$. As $v_2 \neq 0$, we obtain $j_{ri} = (i - w_2)\frac{v'_r}{v_2} + w'_r$.

□

We will use the following properties of the Newton polytopes associated to the Vianna tori. The first result is a summary of [Via16, Lemma 4.11] and the discussion loc. cit.

Lemma 6.3.4. *The Newton polytope* $\mathrm{Newt}(W_{T_{(a,b,c)}})$ *is a triangle whose edges have affine lengths a, b, and c. Moreover, the coefficients of the monomials in $W_{T_{(a,b,c)}}$ corresponding to the vertices of* $\mathrm{Newt}(W_{T_{(a,b,c)}})$ *are ± 1.*

The following statement is probably known to experts, but we did not find a proof in the literature. We provide a proof for completeness.

Lemma 6.3.5. *Suppose (a,b,c) and (a',b',c') are Markov triples related by a Markov mutation and*

$$W_{T_{(a,b,c)}}(x,y) = \sum_{i=m}^{M} y^i C_i(x)$$

with respect to a pair of admissible basis for $H_1(T_{(a,b,c)}; \mathbb{Z})$ and $H_1(T_{(a',b',c')}; \mathbb{Z})$. Then $C_M(x) = \pm x^k (1+x)^M$ for some $k \geqslant 0$. In particular, the monomials corresponding to the vertices on $\mathrm{Newt}(W_{T_{(a,b,c)}}) \cap \{\mathbf{y} = M\}$ *have the same coefficient.*

Proof. As $W_{T_{(a,b,c)}}$ is mutable, we must have $C_m(x) = ax^{k_m}$ for some $k_m \in \mathbb{Z}$ by Lemma 6.3.3(a). Applying (a) to $W_{T_{(a',b',c')}}$, we deduce that $\frac{C_M(x)}{(1+x)^M}$ is a monomial with the same coefficient as $y^M C_M(x)$ in $W_{T_{(a,b,c)}}$. By Lemma 6.3.4, the coefficient of $\frac{y^M C_M(x)}{(1+x)^M}$ in $W_{T_{(a',b',c')}}$ is ± 1, so $C_M(x) = \pm x^{k_M}(1+x)^M$. This shows that the coefficients along the edge $e = \mathrm{Conv}(\{(i, M) : x^i \in C_M(x)\})$ corresponding to $y^M C_M(x)$ are binomial. In particular, the monomials corresponding to the vertices of e have exactly the same coefficients, i.e., both are either 1 or -1. $\square$

6.3.2 Distinguishing tori by induction

To prove Theorem 6.0.1 , we use an induction based on the Markov tree. Explicitly, we verify that $(1, 1, 1)$ has the desired property $\mathcal{P}$ as the base step. Then, assuming $\mathcal{P}$ is satisfied by any Markov triple (a, b, c) of (graph) distance d away from $(1, 1, 1)$, we prove that $\mathcal{P}$ holds for an elementary mutation of (a, b, c).

The following result is a more precise formulation of Theorem 6.0.1. Set $\underline{z} := \sum_{i=1}^{n-2} z_i$.

Proposition 6.3.6. *Let (a, b, c) be a Markov triple. Given a basis of $H_1(\overline{T}_{(a,b,c)}; \mathbb{Z})$ as constructed in Definition 6.2.2, the following holds.*

i) $\mathrm{Newt}(W_{\overline{T}_{(a,b,c)}} - \underline{z})$ *is a triangle.*

ii) $W_{\overline{T}_{(a,b,c)}}(x, y, 1, \dots, 1) = W_{T_{(a,b,c)}}(x, y) + n - 2.$

iii) $\mathrm{Newt}(W_{\overline{T}_{(a,b,c)}})$ *is a simplex with one 2-dimensional face given by* $\mathrm{Newt}(\overline{W}_{T_{(a,b,c)}} - \underline{z})$ *and the other vertices being $e_3, \dots, e_n$. Moreover, they are at affine unit length from all other vertices.*

iv) The affine lengths of the edges of the triangle $\mathrm{Newt}(W_{\overline{T}_{(a,b,c)}} - \underline{z})$ *are a, b and c.*

Proof. Abbreviate $W_{(a,b,c)} := W_{T_{(a,b,c)}}$ and $\mathrm{Newt}((a, b, c)) := \mathrm{Newt}(W_{T_{(a,b,c)}})$ and similarly for $\overline{T}_{(a,b,c)}$. In Example 6.3.1 and Example 6.3.2, the disk potentials of the respect torus have been explicitly computed. By a direct verification, one can show that the claims hold for $(1, 1, 1)$ and $(1, 1, 2)$. Let (a', b', c') be a Markov triple at a distance $d + 1$ from $(1, 1, 1)$ for $d \geqslant 1$. Assume it is connected to a triple (a, b, c) at distance d from $(1, 1, 1)$. Fix an admissible pair of bases for $H_1(\overline{T}_{(a,b,c)}; \mathbb{Z})$ and $HH_1(\overline{T}_{(a',b',c')}; \mathbb{Z})$. By the induction hypothesis, (i)-(iv) hold for (a, b, c). Write

$$W_{\overline{(a,b,c)}} = \sum_{i=m}^{M} y^i \widetilde{C}_i(x, z_1, \dots, z_{n-2}).$$

By (i) and (ii), there exists for each monomial $x^i y^j$ in $W_{(a,b,c)}$ a unique $x^i y^j z_1^{k_{ij1}} \cdots z_{n-2}^{k_{ij(n-2)}}$ in $W_{\overline{(a,b,c)}}$ whose coefficient is given by the coefficient of $x^i y^j$ by (ii). Thus, by Lemma 6.3.5,

$$\widetilde{C}_M(x, z_1, \dots, z_{n-2}) = \pm x^k \sum_{j=0}^{M} \binom{M}{j} x^j z_1^{k_{j1}} \cdots z_{n-2}^{k_{j(n-2)}}$$

for some $k_{jr} := k_{Mjr}$. By (ii) and Lemma 6.3.3(a), $\mathrm{Newt}(W_{\overline{(a,b,c)}} - \underline{z}) \cap \{\mathbf{y} = M\}$ is a line. In particular, k_{jr} depends linearly on j, i.e. $k_{jr} = \ell_r j + c_r$ for some $\ell_r, c_r \in \mathbb{Q}$. Hence

$$\widetilde{C}_M(x, z_1, \ldots, z_{n-2}) = \pm x^k z_1^{c_1} \cdots z_{n-2}^{c_{n-2}} (1 + x z_1^{\ell_1} \cdots z_{n-2}^{\ell_{n-2}})^M. \tag{6.3.2.1}$$

By our choice of basis, $W_{\overline{(a,b,c)}}$ is mutable, so $(1+x)^M$ divides $\widetilde{C}_M$. This implies that $\ell = 0$ and $c_r \in \mathbb{Z}$. Thus $\mathrm{Newt}(W_{\overline{(a,b,c)}} - \underline{z})$ is contained in the affine plane $(k, M, c_1, \ldots, c_{n-2}) + \langle v, e_1 \rangle$ for some $v \in \mathbb{Z}^n \backslash \{0, e_1\}$. Lemma 6.3.3(b) implies that the $\mathbf{z}_r$-coordinate of points in $\mathrm{Newt}(W_{\overline{(a,b,c)}} - \underline{z})$ depends linearly on the $\mathbf{y}$-coordinate. Hence

$$W_{\overline{(a,b,c)}}(x, y, z_1, \ldots, z_{n-2}) = \underline{z} + \sum_{i=m}^{M} y^i z_1^{f_1(i)} \cdots z_{n-2}^{f_{n-2}(i)} C_i(x) \tag{6.3.2.2}$$

where $W_{(a,b,c)}(x, y) = \sum_{i=m}^{M} y^i C_i(x)$, and each f_r is a linear function. By Theorem 6.2.4,

$$W_{\overline{(a',b',c')}}(x, y, z_1, \ldots, z_{n-2}) = \underline{z} + \sum_{i=m}^{M} y^i z_1^{f_1(i)} \cdots z_{n-2}^{f_{n-2}(i)} \frac{C_i(x)}{(1+x)^i} \tag{6.3.2.3}$$

which immediately implies that (ii) holds for (a', b', c'). Since $f_1, \ldots, f_{n-2}$ are linear functions, (i) also holds for (a', b', c'). Since

$$\mathrm{Newt}(W_{\overline{(a',b',c')}}) = \mathrm{Conv}(\{e_3, \ldots, e_n, \mathrm{Conv}(\mathrm{Newt}(W_{\overline{(a',b',c')}} - \underline{z}))$$

is Fano, the points $e_3, \ldots, e_n$ are not contained in $\mathrm{Conv}(\mathrm{Newt}(W_{\overline{(a',b',c')}} - \underline{z})$. This implies the first claim of (iii). To see the second claim, note that the vertices of $\mathrm{Newt}(W_{(a',b',c')})$ are primitive in $\mathbb{Z}^2$ as $\mathrm{Newt}(W_{(a',b',c')})$ is Fano. By (ii), this shows that any edge e between e_r and a vertex of $\mathrm{Newt}(W_{\overline{(a',b',c')}} - \underline{z})$ is of the form $e = \{e_r + t(v, v') : t \in [0, 1]\}$, where v is primitive in $\mathbb{Z}^2$ and $v' \in \mathbb{Z}^{n-2}$. Thus the affine length of e is 1, as is the affine length of the edge between e_r and $e_{r'}$.

Finally, by Lemma 6.3.5 and (ii), any lattice point on an edge of $\mathrm{Newt}(W_{(a',b',c')})$ lifts to a unique lattice point on an edge of $\mathrm{Newt}(W_{\overline{(a',b',c')}} - \underline{z})$. Therefore, the affine lengths of the edges of the two triangles agree. By Lemma 6.3.4, this proves (iv) and concludes the induction. □

6.3.3 Proof of Theorem 6.0.1

The first two assertions of Theorem 6.0.1 are immediate from Proposition 6.3.6. We now show the last assertion. The *boundary Maslov-2 convex hull* $\mho_{\overline{T}_{(a,b,c)}}$ of $\overline{T}_{(a,b,c)}$ is the convex hull of $\{\partial[u] \,|\, u \colon (\mathbb{D}, S^1) \to (\mathbb{P}^n, \overline{T}_{(a,b,c)}) \text{ holomorphic with } \mu(u) = 2\} \subset \pi_1(\overline{T}_{(a,b,c)})$. By [Via16, Remark 4.5] and the simply-connectedness of $\mathbb{P}^n$, we can identify the Newton polytope of $W_{\overline{T}_{(a,b,c)}}$ with $\mho_{\overline{T}_{(a,b,c)}}$. As the latter is an invariant of the Lagrangian up to symplectomorphism by [Via16, Corollary 4.3], we obtain that $\overline{T}_{(a,b,c)}$ is not symplectomorphic to $\overline{T}_{(a',b',c')}$ for $(a, b, c) \neq (a', b', c')$.

Chapter 7

Cuplength and Lagrangian intersections

Let (X, ω) be a symplectic manifold and let L and L' be two Hamiltonian isotopic Lagrangian submanifolds in X. We will not assume that L and L' are transverse, but instead make the following assumption in order to exclude disk and sphere bubbling.

Assumption 7.0.1. *Throughout this paper, we will assume that*

1. *X is either closed or a Liouville manifold.*
2. *L is connected, closed and* relatively exact*, i.e.,*

$$\omega \cdot \pi_2(X, L) = 0.$$

We are interested in the degenerate Arnol'd conjecture:

Conjecture 7.0.2. *Given Assumption 7.0.1, there is a lower bound*

$$\#L \cap L' \geqslant \mathrm{Crit}(L)$$

where $\mathrm{Crit}(L)$ *is the minimal number of critical points of any smooth map* $L \to \mathbb{R}$.

It was recognised early on that Ljusternik-Schnirelmann theory, which was developped to study critical points of non-Morse functions, might be useful to obtain (partial) answers to this question. Its main invariant, the *Ljusternik-Schnirelmann category* provides a lower bound for the critical number of a manifold. Refer to [CLOT03] for the definition and to §7.3 for a related invariant. However, the LS category is hard to compute and the cuplength in any generalised cohomology theory bounds it from below. Thus

$$\mathrm{Crit}(L) \geqslant c_R(L)$$

for any ring spectrum R, where the *R-cuplength* of L is the natural number (or ∞)

$$c_R(L) := \inf\{k \in \mathbb{N} : \forall \alpha_1, \ldots, \alpha_k \in \tilde{R}^*(L) : \alpha_1 \cdot \cdots \cdot \alpha_k = 0\}. \tag{7.0.0.1}$$

when L is connected and the sum of the cuplengths of its connected components otherwise.

Theorem 7.0.3 ([Hof88, Theorem 3], [Flo89, Theorem 1], [Hof85, Theorem 1])**.** *Under Assumption 7.0.1, there is a lower bound*

$$\#L \cap L' \geqslant c_{\mathbb{Z}/2}(L). \tag{7.0.0.2}$$

If $X = T^*L$ is a contangent bundle with L the zero section, there is a lower bound

$$\#L \cap L' \geqslant c_{\mathbb{Z}}(L). \tag{7.0.0.3}$$

Hofer's proof uses Ljusternik-Schnirelmann theory, while Floer's proof proceeds via Conley indices. In general, (7.0.0.2) and (7.0.0.3) are weaker bounds than the one given by Conjecture 7.0.2. There are examples in which it is a strictly weaker bound, such as [Rud99, Example 3.7].

We replace cohomology with $\mathbb{Z}/2$-coefficients in Theorem 7.0.3 with certain generalised cohomology theories. In §7.4, we provide two examples where this leads to stronger lower bounds. The proof uses that real K-theory captures more information than singular cohomology.

Fix $J \in \mathcal{J}(X, \omega)$. If X is Liouville, assume J is convex at infinity. Denote by

$$\mathcal{M}_{L,L'} := \left\{u \in C^\infty(\mathbb{R} + i[0,1], X) : \overline{\partial}_J u = 0,\ E(u) < \infty,\ u(\mathbb{R}) \subset L,\ u(\mathbb{R}+i) \subset L'\right\}$$

the moduli space of pseudoholomorphic strips with Lagrangian boundary conditions. Let $\pi \colon \mathcal{M}_{L,L'} \to L$ be the evaluation at 0. The key technical point is the injectivity of

$$\pi^* \colon R^*(L) \to R^*(\mathcal{M}_{L,L'})$$

for certain ring spectra R. The proof could go via a cobordism argument if our moduli spaces were cut out transversely. As they are not, we use an approximation argument and a very simple version of a Kuranishi chart. We crucially use a certain virtual vector bundle, the index bundle associated to a Cauchy–Riemann operator.

We impose the following assumption on (X, L), respectively the chosen ring spectrum R, throughout the paper. We require it in order to apply the Thom isomorphism later on.

Assumption 7.0.4. *This index bundle of the moduli space of finite-energy pseudoholomorphic maps from a compact convex domain with smooth boundary in $\mathbb{C}$ to X, mapping the boundary to L, is R-orientable.*

[Por22] provides the following criteria for Assumption 7.0.4 to be satisfied.

Proposition 7.0.5 ([Por22, Proposition 1.13])**.** *Assumption 7.0.4 holds when*

1. *R is the Eilenberg-MacLane spectrum $H\mathbb{Z}/2$.*

2. *R is the Eilenberg-MacLane spectrum $H\mathbb{Z}$, and L is (relatively) spin.*

3. R^* is complex K-theory, and L is spin.

4. R^* is real K-theory, and TL admits a stable trivialisation over a 3-skeleton of L which extends (after complexification) to a stable trivialisation of TX over a 2-skeleton of X (as a complex vector bundle).

As in [Hof88], we combine the injectivity of π^* with standard Ljusternik-Schnirelmann theory to obtain the following lower bound.

Theorem 7.0.6. *Suppose L satisfies Assumption 7.0.1 and Assumption 7.0.4 for a ring spectrum R. Then the number of intersection points between L and L' satisfies*

$$\#L \cap L' \geqslant c_R(L).$$

Remark 7.0.7. If L is not connected, we may apply Theorem 7.0.6 to each path component of L and the corresponding component of L' to obtain the same inequality.

Suppose X is closed and symplectically aspherical. If ψ is a (possibly degenerate) Hamiltonian diffeomorphism of X, we can apply Theorem 7.0.6 to the graph of ψ in $X \times X$ to deduce the Hamiltonian version of this inequality.

Corollary 7.0.8. *The number of fixed points of ψ satisfies*

$$\#\mathrm{Fix}(\psi) \geqslant c_R(X).$$

In this setting, Conjecture 7.0.2 (which implies Corollary 7.0.8) has already been verified; see [Rud99, Theorem A], [OR99, Corollary 4.2] and [CLOT03, Theorem 8.28].

7.1 Homotopy and Fredholm theory

7.1.1 Generalised cohomology and Thom spectra

For our purposes, it suffices to work with classical spectra as defined in [Rud98] or [Ada95]. However, our definitions require some care, as we will take the generalised (co)homology of spaces which are not necessarily homotopy equivalent to a CW complex. A generalised cohomology theory defined on CW complexes can always be extended to all spaces, but this extension may not be unique. We need an extension that satisfies a certain continuity property, namely the statement of [AMS21, Lemma 5.2].

Unless otherwise specified, we work with spectra whose level spaces are homotopy equivalent to CW complexes. We denote by $\mathbb{S}$ the sphere spectrum. All our spaces are assumed to be compactly generated and Hausdorff.

Given an Ω-spectrum R, we define, for a pointed space X, the n^{th} *R-homology and R-cohomology groups* to be

$$R_n(X) := \pi_n(X \wedge R_n) \qquad\qquad R^n(X) := [X, R_n]_*$$

respectively, for n in $\mathbb{Z}$, where $[\cdot,\cdot]_*$ denotes the set of pointed homotopy classes of maps. As $R_n \simeq \Omega^2 R_{n+2}$, the sets $R_n(X)$ and $R^n(X)$ carry a natural abelian group structure for all n.

We recall the definition of relative R-(co)homology.
Given an inclusion $j\colon A \hookrightarrow X$ of pointed spaces, we denote by

$$C_X A := C_A \cup_j X$$

the *(reduced) mapping cone* of j, where C_A is the cone over A. We view this as a pointed space, with basepoint the vertex of C_A (or equivalently the basepoint of X).

The inclusion $X \hookrightarrow C_X A$ is a cofibration and collapsing X induces a natural map $C_X A \to \Sigma A$. By [tD08, Theorem 4.6.4] these maps fit into an h-coexact sequence

$$A \to X \to C_X A \to \Sigma A \to \Sigma X \to \dots$$

In particular, there exists for any pointed space W a long exact sequence of pointed sets

$$\dots \to [\Sigma X, W]_* \to [\Sigma A, W]_* \to [C_X A, W]_* \to [X, W]_* \to [A, W]_* \tag{7.1.1.1}$$

The *relative R-(co)homology* of (X, A) is defined by

$$R_*(X,A) := R_*(C_X A) \qquad\qquad R^*(X,A) := R^*(C_X A).$$

We recover the usual long exact sequence of a pair in R-cohomology due to (7.1.1.1). Similarly, there is a long exact sequence of a pair in R-homology.

In the case of a pair of unpointed spaces (X, A), one simply considers the pair (X_+, A_+), where $\cdot_+$ denotes the addition of a disjoint basepoint.

A *ring spectrum* consists of an Ω-spectrum R endowed with both a multiplication map $\mu\colon R \wedge R \to R$ and a unit $\iota\colon \mathbb{S} \to R$ such that the usual associativity and unit diagrams commute up to homotopy. In this case we can define a cup product

$$\cdot : R^*(X) \otimes R^*(X) \to R^*(X)$$

by letting $\alpha \cdot \beta$ be the composite

$$X \xrightarrow{\Delta} X \wedge X \xrightarrow{\alpha\wedge\beta} R_n \wedge R_k \to (R \wedge R)_{n+k} \xrightarrow{\mu} R_{n+k} \tag{7.1.1.2}$$

for $\alpha \in R^n(X)$ and $\beta \in R^k(X)$, where the map $R_n \wedge R_k \to (R \wedge R)_{n+k}$ comes from the construction of the smash product.

When one works outside the setting of CW complexes, the cup product does not necessarily descend to a map

$$R^*(X, A) \otimes R^*(X, B) \to R^*(X, A \cup B).$$

However, we can make the following two observations, which will be useful in §7.3.

Remark 7.1.1. Suppose $\alpha \in R^n(X)$ and $\beta \in R^m(X)$ admit representatives $\tilde{\alpha} : X \to R_n$ and $\tilde{\beta} : X \to R_m$ which send subspaces $A, B \subseteq X$ to the basepoints $*$ respectively. Then, by construction, $\alpha \cdot \beta$ admits a representative $X \to R_{n+m}$ sending $A \cup B$ to $*$. Induction shows the same result for classes $\alpha_1, \ldots, \alpha_k \in R^*(X)$ that admit representatives $\tilde{\alpha}_i$ mapping subspaces $A_i \subseteq X$ to $*$ respectively. In particular,

$$\alpha_1 \cdot \ldots \cdot \alpha_k = 0 \quad \text{in} \quad R^*(A_1 \cup \ldots \cup A_k).$$

Lemma 7.1.2. *Let (X, d) be a compact metric space, with an open cover $U_1, \ldots, U_k$. Let $\alpha_i \in R^{n_i}(X)$ be cohomology classes such that $\alpha_i|_{U_i} = 0$ in $R^{n_i}(U_i)$ for all i. Then the product $\alpha_1 \cdot \ldots \cdot \alpha_k$ vanishes in $R^*(X)$.*

Proof. Let $V_1, \ldots, V_k$ be an open cover of X so that $\overline{V_i} \subset U_i$ for each i. Set $d_i := d(\cdot, \overline{V_i})$. As $A_i := X \backslash U_i$ is disjoint from $\overline{V_i}$ and compact, there exists $\epsilon_i > 0$ so that $d_i^{-1}([0, \varepsilon_i]) \subset U_i$.

Pick maps $\tilde{\alpha}_i : X \to R_{n_i}$ representing each $\alpha_i \in R^{n_i}(X)$. Since $\alpha_i|_{U_i} = 0$, we can choose nullhomotopies

$$H_i : U_i \times [0,1] \to R_{n_i}$$

such that $H_i(\cdot, 0) \equiv *$ and $H_i(\cdot, 1) = \tilde{\alpha}_i|_{U_i}$.

Define maps $\beta_i : X \to R_{n_i}$ by

$$\beta_i(x) := \begin{cases} \tilde{\alpha}_i(x) & \text{if } \varepsilon_i \leqslant d_i(x) \\ H_i(x, \varepsilon_i^{-1} d_i(x)) & \text{if } 0 \leqslant d_i(x) \leqslant \varepsilon_i. \end{cases}$$

Then $\beta_i \simeq \tilde{\alpha}_i$, via the homotopy $G_i : X \times [0,1] \to R_{n_i}$ given by

$$G_i(x, s) := \begin{cases} \tilde{\alpha}_i(x) & \text{if } \varepsilon_i \leqslant d_i(x) \\ H_i(x, s + (1-s)\varepsilon_i^{-1} d_i(x)) & \text{if } 0 \leqslant d_i(x) \leqslant \varepsilon_i. \end{cases}$$

Hence β_i is a representative of $\alpha_i \in R^{n_i}(X)$. β_i sends V_i to the basepoint $*$ so the product $\alpha_1 \cdot \ldots \cdot \alpha_k$ is zero in $R^*(X)$ by Remark 7.1.1. □

Suppose now that X is a compact space and that $\xi \colon F \to X$ is a vector bundle of rank k. Let F_∞ be its fibrewise one-point compactification. This is a sphere bundle over X with a canonical section s_∞ given by the point at infinity in every fibre. The *Thom space* of ξ is defined to be the pointed space

$$\mathrm{Th}(\xi) := C_{F_\infty} \mathrm{im}(s_\infty)$$

and its *Thom spectrum*, written X^ξ or X^F, to be the spectrum $\Sigma^\infty \mathrm{Th}(\xi)$. In particular, if ξ is the trivial bundle of rank k, then $\mathrm{Th}(\xi) = \Sigma^k X_+$ and $X^\xi = \Sigma^{\infty + k} X_+$. Note that if X is not a CW complex, $\mathrm{Th}(\xi)$ might not be either. These are the only spectra whose level spaces are not CW complexes that we will encounter.

For a virtual vector bundle of the form $\eta - \mathbb{R}^N$, we let $\operatorname{rank}(\eta) - N$ be its *rank* and define its *Thom spectrum* to be

$$X^{\eta-\mathbb{R}^N} := \Sigma^{-N} X^{\eta}.$$

All our virtual bundles will be of this form, so this definition is sufficient for our purposes.[1]

Let R be a ring spectrum and ξ a virtual vector bundle over X of rank k. An *R-orientation* of ξ is an element $u \in R^k(X^\xi)$ such that for any map $j : \Sigma^k \mathbb{S} \to X^\xi$ which is a (stable) homotopy equivalence to a fibre, we have

$$j^* u = \pm[\iota] \in R^k(\Sigma^k \mathbb{S}) \cong R^0(\mathbb{S}).$$

where $[\iota]$ is the homotopy class of the unit map. Any trivial bundle is R-orientable, and if two out of ξ, η and $\xi \oplus \eta$ are R-oriented, then so is the third.

By the Thom isomorphism theorem, [Rud98, Theorem V.1.3] any R-orientation on a virtual vector bundle ξ over a compact CW complex X induces a natural isomorphism

$$R^{*+k}(X^\xi) \cong R^*(X).$$

By [AMS21, Lemma 5.2], this also holds when X is a compact subset of a manifold M, and both ξ and its R-orientation are pulled back from M.

We will need the following form of Atiyah duality, which can be viewed as a form of Poincaré duality for generalised cohomology theories.

Theorem 7.1.3 ([AMS21, Theorem 5.2])**.** *Let M be a smooth (not necessarily compact) manifold, possibly with boundary, and suppose $Z \subset M$ is any compact subset. Then there is a canonical isomorphism*

$$R_{-*}(M, M\backslash Z) \;\cong\; R^*\left(Z^{-TM|_Z}\right)$$

compatible with restriction to smaller closed subsets $Z' \subseteq Z$.

If M is R-oriented on a neighbourhood of Z and of dimension n, the Thom isomorphism theorem then gives an isomorphism

$$R^{*+n}(Z) \;\cong\; R_{-*}(M, M\backslash Z).$$

for compact subsets $Z \subset M$. In this case, we define the *fundamental class of M along Z* $[M]_Z \in R_n(M, M\backslash Z)$ to be the image of the unit in $R^0(Z)$ under this isomorphism. The class $[M]_Z$ depends only on the choice of orientation of which there can be more than two.

We will need the following version of the fact that, given an (oriented) compact manifold with boundary M, the fundamental class of ∂M has vanishing image in the homology

[1] The homotopy type of the Thom spectrum only depends on the stable isomorphism class of the virtual vector bundle. Due to compactness, any virtual bundle on X is stably isomorphic to one of the considered form, so it suffices for our applications.

of M.

Lemma 7.1.4. *Let W^{n+1} be an R-oriented smooth manifold with boundary and $K \subset W$ a compact subset. Then the image of $[\partial W]_{K\cap\partial W}$ in $R_n(W, W\backslash K)$ is* 0.

Proof. Suppose first that W is compact. The map ∂ in the exact sequence of a pair

$$R_{n+1}(W, \partial W) \xrightarrow{\partial} R_n(\partial W) \to R_n(W)$$

sends $[W]$ to $[\partial W]$ - see [Rud98, Remark V.2.14.a)]. The claim with $K = W$ then follows from the exactness of this sequence.

Now assume W is non-compact and set $C = K \cap \partial W$. By excision, we may modify W away from K, and replace W with a compact smooth neighbourhood of K. Then $[\partial W]_C$ is the restriction of the fundamental class $[\partial W] \in R_n(\partial W)$. We may deduce the claim now from the first step and the commutativity of the following diagram.

$$\begin{tikzcd}
R_{n+1}(W, \partial W) \arrow[r, "\partial"] & R_n(\partial W) \arrow[r] \arrow[d] & R_n(W) \arrow[d] \\
& R_n(\partial W, \partial W\backslash C) \arrow[r] & R_n(W, W\backslash K)
\end{tikzcd}$$

□

7.1.2 Index bundles

Let $s\colon \mathcal{B} \to \mathcal{E}$ be a smooth Fredholm section of a Banach bundle over a Banach manifold intersecting the zero section of $\mathcal{E}$ transversely. By the infinite-dimensional implicit function theorem [MS12, Theorem A.3.3], the zero locus $s^{-1}(0)$ is a smooth manifold of dimension $\mathrm{ind}(Ds) = \dim(\ker(Ds))$ with tangent bundle $\ker(Ds) \to s^{-1}(0)$. If Ds is not fibrewise surjective, the zero locus is not necessarily smooth or may have excess dimension. The natural replacement of $\ker(Ds)$ in this case is the *index bundle*, a virtual vector bundle constructed below. It relies on the notion of the stabilisation of a Fredholm operator.

Definition 7.1.5. Let $D\colon X \to Y$ be a Fredholm operator between two Banach spaces. We call an operator $T\colon \mathbb{R}^N \to Y$ a *stabilisation of* D if $D + T\colon X \oplus \mathbb{R}^N \to Y$ is surjective.

As T is compact, $D + T$ is still a Fredholm operator and

$$\mathrm{ind}(D + T) = \mathrm{ind}(D) + N$$

by [MS12, Theorem A.1.5(i)]. Given a smoothly varying family of Fredholm operators we will show the existence of a smoothly varying family of stabilisations near a compact subset in Lemma 7.1.6.

Let us fix our setting for the rest of this subsection. Let Y be a separable Hilbert manifold, $\mathcal{H}$ a separable Hilbert space and Λ a compact finite-dimensional manifold with boundary. We assume that $\psi\colon V \to \mathcal{H}$ is a smooth Fredholm map with $V \subset Y \times \Lambda$ an open subset. Define the open subset

$$V^{\text{reg}} := \{(x, \lambda) \in V : d_1\psi(x, \lambda) \text{ is surjective}\}$$

where d_1 is the derivative with respect to the first argument.

Lemma 7.1.6. *For any closed subset $A \subset V^{\text{reg}}$ and any compact subset $K \subset V$, there exists a neighbourhood $U \subset V$ of K and a smooth map $T\colon V \times \mathbb{R}^k \to \mathcal{H}$ which is linear in the second variable, vanishes on $(A \cup V\backslash U) \times \mathbb{R}^k$ and satisfies that*

$$d_1\psi(x, \lambda) + T(x, \lambda, \cdot)\colon T_{(x,\lambda)}V \oplus \mathbb{R}^k \to \mathcal{H}$$

is surjective for $(x, \lambda) \in U$.

Proof. For each $z \in K$ there exists an open neighbourhood $U_z \subset V$ of z, an integer $k_z \geqslant 0$, and an operator $T_z\colon \mathbb{R}^{k_z} \to \mathcal{H}$ such that

$$d_1\psi(y, \mu) + T_z\colon T_yY \oplus \mathbb{R}^{k_z} \to \mathcal{H}$$

is surjective for $(y, \mu) \in U_z$. Let $Z \subset K$ be a finite subset such that $U := \bigcup_{z \in Z} U_z$ contains K and set $k := \sum_{z \in Z} k_z$. Using a smooth partition of unity subordinate to $\{U_z\}_{z \in Z} \cup \{V\backslash K\}$, we obtain an operator $T'\colon V \times \mathbb{R}^k \to \mathcal{H}$ satisfying all conditions save for the vanishing on $A \times \mathbb{R}^k$. Multiplying T' with a smooth bump function which is identically one on $V\backslash V^{\text{reg}}$ and vanishes on A, we obtain the desired map. ☐

Definition 7.1.7. A family of operators T as in Lemma 7.1.6 is said to be a *stabilisation of ψ along K relative to A, of rank k.* We call

$$\text{Ind}_K(\psi; T) := \ker(d_1\psi + T)|_U - \mathbb{R}^k_U$$

the *index bundle of ψ along K (with respect to T)*, defined over a neighbourhood U of K.

Lemma 7.1.8. *Any two index bundles of ψ along K are stably equivalent as germs near K.*

Proof. Suppose T and S are two stabilisations along K. We may assume without loss of generality that $d_1\psi + T$ and $d_1\psi + S$ are surjective over the same subset. As we may add factors of $\mathbb{R}$ to the domain of T, respectively S, without changing the index bundle, we may assume that T and S are smooth maps $V \times \mathbb{R}^{k+\ell} \to \mathcal{H}$ with T vanishing on $V \times \mathbb{R}^k \times \{0\}$ and S vanishing on $V \times \{0\} \times \mathbb{R}^\ell$. Now we can linearly interpolate between them and apply [tD08, Theorem 14.3.2]. ☐

Definition 7.1.9. We let $\text{Ind}_K(\psi)$ denote the stable equivalence class of any $\text{Ind}_K(\psi; T)$ and call it the *index bundle of ψ along K.*

Definition 7.1.10. Given a ring spectrum R, the map ψ is *R-orientable along K* if $\text{Ind}_K(\psi)$ is R-orientable on a neighbourhood of K. We say that ψ is *R-orientable* if it is R-orientable along any compact subset.

7.1.3 Proof of Theorem 7.1.12

The following result generalises [Hof88, Theorem 5] to multiplicative generalised cohomology theories.

Proposition 7.1.11. *Let $\mathcal{Y}$ be a smooth separable Hilbert manifold and $\mathcal{H}$ be a separable Hilbert space. Let $\psi : \mathcal{Y} \times [0,1] \to \mathcal{H}$ be a smooth Fredholm map of index $n+1$, and write ψ_t for its restriction to $\mathcal{Y} \times \{t\}$. Given a ring spectrum R, assume*

1. *ψ is proper with respect to a neighbourhood of 0 in $\mathcal{H}$ and R-orientable along $\psi^{-1}(0)$,*
2. *ψ_0 is submersive near $\psi_0^{-1}(0)$,*
3. *there exists a smooth map $\pi\colon \mathcal{Y} \to N$ to a connected closed manifold N such that*
$$\pi|_{\psi_0^{-1}(0)}\colon \psi_0^{-1}(0) \to N$$
is a diffeomorphism.

If N is R-oriented, then $\pi^*\colon R^*(N) \to R^*(\psi_1^{-1}(0))$ is injective.

Proof. Set $K := \psi^{-1}(0)$ and $I := [0,1]$. By (1), K is compact. Given any subset $W \subset \mathcal{Y}\times I$ we denote by W_t its fibre over $t \in I$. Let T be a stabilisation of ψ along K relative to $K_0 \times \{0\}$ of rank k. Set

$$S := (\psi + T)^{-1}(0) \subset \mathcal{Y} \times I \times \mathbb{R}^k.$$

Then S is a smooth (non-compact) cobordism from S_0 to S_1 with $TS = \ker(d\psi + T)$. Assumption (1) on ψ implies that S is R-oriented on a neighbourhood of K. By the compactness of $T(v,t,\cdot)$ for $(v,t) \in \mathcal{Y} \times I$ and [MS12, Theorem A.1.5.i],

$$\dim(S) = n + k + 1.$$

Note that $K = \{(x,t,z) \in S : z = 0\}$ and $S_0 = K_0 \times \mathbb{R}^k$. Set

$$\tilde{\pi} := \pi \times \mathrm{id}_I \times \mathrm{id}_{\mathbb{R}^k} : S \to N \times I \times \mathbb{R}^k$$

and let $\tilde{\pi}_t$ be the restriction to $\mathcal{Y} \times \{t\} \times \mathbb{R}^k$. This fits into a commutative diagram of pairs

$$\begin{tikzcd}
(S_0, S_0\backslash K_0) \arrow[r, "\tilde{\pi}_0"] \arrow[d, hook, "i_0"] & (N \times \mathbb{R}^k, N \times (\mathbb{R}^k\backslash 0)) \arrow[d, hook, "\iota_0"] \\
(S, S\backslash K) \arrow[r, "\tilde{\pi}"] & (N \times I \times \mathbb{R}^k, N \times I \times (\mathbb{R}^k\backslash 0)) \\
(S_1, S_1\backslash K_1) \arrow[r, "\tilde{\pi}_1"] \arrow[u, hook, "i_1"] & (N \times \mathbb{R}^k, N \times (\mathbb{R}^k\backslash 0)) \arrow[u, hook, "\iota_1"]
\end{tikzcd}$$

Consider the composition

$$R^*(N) \xrightarrow{\pi_1^*} R^*(K_1) \xrightarrow[\cong]{\mathrm{AD}} R_{n+k-*}(S_1, S_1\backslash K_1) \xrightarrow{(\pi_1)_*} R_{n+k-*}(N\times\mathbb{R}^k, N\times(\mathbb{R}^k\backslash 0)) \xrightarrow[\cong]{\mathrm{AD}} R^*(N) \tag{7.1.3.1}$$

where AD denotes the Atiyah duality isomorphism. Note that in the second map we use the R-orientability assumption in (1).

By construction, this map is given by multiplication by $\mathrm{AD}((\pi_1)_*[S_1]_{K_1}) \in R^0(N)$, which is equal to $\mathrm{AD}((\pi_0)_*[S_0]_{K_0})$ by Lemma 7.1.4. As π_0 is a diffeomorphism, this cohomology class is a unit. Hence (7.1.3.1) is an isomorphism. Because it factors through the pullback $\pi_1^* : R^*(N) \to R^*(K_1)$, the latter must be injective. $\square$

We apply this to our situation. Let $G \subset \mathbb{C}$ be a convex bounded domain smooth boundary, and suppose $\{L_z\}_{z \in \partial G}$ is a *Hamiltonian family* of Lagrangians in X. That is, there exists a (relatively exact) Lagrangian $L \subset X$ and a smooth family $\{\phi_z^t\}_{z\in\partial G, t\in[0,1]}$ of Hamiltonian isotopies (which we can assume to be compactly supported) of X such that $L_z = \phi_z^1(L)$ for all z. We can assume that $L = L_{z_0}$ for some $z_0 \in \partial G$.

Consider the following moduli space of pseudoholomorphic discs with moving Lagrangian boundary conditions:

$$\mathcal{P} := \left\{ u \in C^\infty(G, X) : \overline{\partial}_J u = 0,\ E(u) < \infty,\ \forall z \in \partial G : u(z) \in L_z \right\} \tag{7.1.3.2}$$

where $\overline{\partial}_J$ is the Cauchy-Riemann operator associated to J and E is the symplectic energy. Let $\pi : \mathcal{P} \to L$ be evaluation at z_0.

Theorem 7.1.12. *The pullback $\pi^* : R^*(L) \to R^*(\mathcal{P})$ is injective.*

Remark 7.1.13. If the moduli space $\mathcal{P}$ were cut out transversely, this could be proved using a cobordism argument as in [Por22]. On the other hand, following [Por22, Remark 4.6], Theorem 7.1.12 can be used to give a slightly different proof of [Por22, Corollary 1.9], without using any transversality results.

Remark 7.1.14. Hofer [Hof88] proves Theorem 7.1.12 as well as Theorem 7.2.1, 7.0.6 and Corollary 7.0.8 in the case where R^* is Čech cohomology with coefficients in $\mathbb{Z}/2$.

Using an extension of an associated family of Hamiltonians we may extend the family of Hamiltonian isotopies $\{\phi_z\}_{z\in\partial G}$ to a smooth family $\{\phi_z\}_{z\in G}$ of Hamiltonian isotopies, parametrised by G.

Fix $k \geqslant 3$. Given $t \in [0,1]$, we define $\psi_t \colon W^{k,2}(G, X) \to W^{k,2}(G, X)$ by setting

$$\psi_t(u)(z) := \phi_z^t(u(z))$$

for $z \in G$. By assumption, ψ_0 is the identity. Let

$$\mathcal{A} := \left\{ u \in W^{k,2}(G, X) : u(\partial G) \subset L \right\}.$$

The smooth Banach bundle $\mathcal{E} \to \mathcal{A}$ with fibre

$$\mathcal{E}_u := W^{k-1,2}(G, u^*TX).$$

admits a smooth Fredholm section $\overline{\partial}_J : \mathcal{A} \to \mathcal{E}$ given by

$$\overline{\partial}_J u = \partial_s u + J(u)\partial_t u.$$

The canonical evaluation map defines a map of pairs $\mathrm{ev}\colon \mathcal{A} \times (G, \partial G) \to (X, L)$. By pulling back, this defines a bundle pair

$$(F, F') := \mathrm{ev}^*(TX, TL) \to \mathcal{A} \times (G, \partial G).$$

Using a connection on TX, the linearisation of $\overline{\partial}_J$ defines a real Cauchy-Riemann operator on (F, F') by [MS12, Proposition 3.1.1]. Then Assumption 7.0.4 states exactly that its index bundle is R-oriented. For $u \in \overline{\partial}_J^{-1}(0)$ we have, by the Riemann-Roch theorem, [MS12, Theorem C.1.10], that

$$\mathrm{ind}(D_u\overline{\partial}_J) = \dim(L) + \mu(F|_u, F'|_u), \tag{7.1.3.3}$$

where $\mu(F|_u, F'|_u)$ is the boundary Maslov index of the pullback of (F, F') to $\{u\} \times G$.

Remark 7.1.15. If $u \in \mathcal{A}$ is pseudoholomorphic, then $\mu(F|_u, F'|_u) = 0$ as u must be constant due to relative exactness. However, if u instead satisfies that $\psi_t(u)$ is pseudoholomorphic for some $t > 0$, u need not be constant and may lie in a non-trivial relative homotopy class of discs. In this case, $\mu(F|_u, F'|_u)$ might not vanish.

By [Kui65, Theorem (3)] we can fix a smooth isometric trivialisation $\Psi\colon \mathcal{E} \to \mathcal{A} \times \mathcal{H}$, where $\mathcal{H}$ is some separable Hilbert space. Define $\mathcal{F}\colon \mathcal{A} \times [0, 1] \to \mathcal{H}$ by

$$\mathcal{F}_t(u) := \mathrm{pr}_2\Psi(\overline{\partial}_J\psi_t(u)) \tag{7.1.3.4}$$

letting pr_2 denote the projection to the second factor. Note that $\mathcal{F}_1^{-1}(0)$ is diffeomorphic via ψ_1 to the space $\mathcal{P}$ of pseudoholomorphic maps from G to M which have finite energy and map $z \in \partial G$ to L_z.

Proof of Theorem 7.1.12. Let

$$\mathcal{W} := \left\{(u, t) \in \mathcal{A} \times [0, 1] : \mu(F|_{\psi_t(u)}, F'|_{\psi_t(u)}) = 0\right\}.$$

This is an open subset of $\mathcal{A} \times [0, 1]$. We restrict to the subset where the Maslov index is 0 in order to have control over the index of $\mathcal{F}$, due to Remark 7.1.15. However, with a little more care the entire argument could also be applied without this restriction. Let $\pi : \mathcal{W} \to L$ be the evaluation map at z_0.

By [Hof88, Proposition 6] there exists a neighbourhood $U \subset \mathcal{W}$ of $\mathcal{F}^{-1}(0)$ such that $\mathcal{F}|_U : U \to \mathcal{H}$ is a Fredholm map of index $\dim(L) + 1$ and such that $\mathcal{F}|_U$ is proper with respect to a neighbourhood of $0 \in \mathcal{H}$. We note that U and $\mathcal{H}$ are separable Hilbert manifolds. Since pseudoholomorphic discs with boundary on L are constant, π defines a diffeomorphism $\mathcal{F}_0^{-1}(0) \to L$. Moreover, $\mathcal{F}_0$ is submersive by the proof of [Hof88, Lemma 5], and $\mathcal{F}$ is R-orientable by Assumption 7.0.4. Thus the claim follows from Proposition 7.1.11. □

7.2 Approximating pseudoholomorphic strips

We can now show the injectivity of the evaluation map from the moduli space of pseudoholomorphic strips.

Proposition 7.2.1. *The map* $\pi^*\colon R^*(L) \to R^*(\mathcal{M}_{L,L'})$ *is injective.*

The idea of the proof is to study a one-parameter family of moduli spaces of pseudoholomorphic discs $\mathcal{P}_\ell$ with moving boundary conditions. They approximate the moduli space of pseudoholomorphic strips $\mathcal{M}_{L,L'}$ as the parameter ℓ tends to ∞. Together with [AMS21, Lemma 5.2], this allows us to infer Proposition 7.2.1 from Theorem 7.1.12.

Throughout this section, we fix a convex domain G in $\mathbb{C}$ with smooth boundary, such that both $(-\eta, \eta)$ and $(-\eta, \eta) + i$ are contained in ∂G for some $\eta > 0$. For $\ell > 0$, define $Z_\ell := [-\ell, \ell] + [0,1]i$, and let

$$G_\ell := Z_\ell \cup (G + \ell) \cup (G - \ell)$$

be a smoothing of the truncated strip. Note that G_ℓ is diffeomorphic to a disk.

Definition 7.2.2. For a domain W in $\mathbb{C}$ and a smooth map $u : W \to X$, we define the *symplectic energy* to be

$$E(u) := \frac{1}{2}\int_W u^*\omega$$

whenever this integral is defined.

When u is pseudoholomorphic, the symplectic energy of u is defined and non-negative, although not necessarily finite.

We consider the following moduli spaces. Recall from the introduction that L is a closed, relatively exact Lagrangian in X and L' is Hamiltonian isotopic to L. We denote by $Z := \mathbb{R} + [0,1]i$ the infinite strip.

Definition 7.2.3. We define

$$\mathcal{D}_{L,L'} := \left\{u \in C^\infty(Z, X) : |E(u)| < \infty,\ u(\mathbb{R}) \subset L,\ u(\mathbb{R} + i) \subset L'\right\}.$$

It contains $\mathcal{M}_{L,L'} := \{u \in \mathcal{D}_{L,L'} : \overline{\partial}_J u = 0\}$ as the subspace of pseudoholomorphic maps.

Given $\ell > 0$ and $A \geqslant 0$, we set

$$\mathcal{F}_{\ell,A} := \left\{u \in C^\infty(Z_\ell, X) : E(u) \leqslant A,\ \overline{\partial}_J u = 0,\ u([-\ell,\ell]) \subset L,\ u([-\ell,\ell] + i) \subset L'\right\}.$$

Given $\ell \geqslant 0$ and a Hamiltonian family $\{L_t\}_{t\in[0,1]}$ of Lagrangians in X with $L_0 = L$ and $L_1 = L'$, we define the moduli space

$$\mathcal{P}_\ell := \left\{u \in C^\infty(G_\ell, X) : \overline{\partial}_J u = 0,\ u(s + it) \in L_t \text{ for } s + it \in \partial G_\ell\right\}.$$

All of these spaces are endowed with the weak C^∞ Whitney topology. By the Nash Embedding Theorem applied to the metric $g_J = \omega(\cdot, J\cdot)$, this topology is metrisable. Hofer showed in [Hof88, Theorems 1 and 2] that the moduli spaces $\mathcal{P}_\ell$ and $\mathcal{M}_{L,L'}$ are compact.

Evaluation at $0 \in \mathbb{C}$ defines a continuous map, denoted by π, from each of these spaces to L.

Remark 7.2.4. Pick some Hamiltonian isotopy $\{\psi^t\}_{t\in[0,1]}$ such that $\psi^t(L) = L_t$ for all t. Setting

$$L_{x+iy} := L_y \qquad \text{and} \qquad \psi^t_{x+iy} := \psi^{ty}$$

for $x + iy$ in ∂G shows that $\mathcal{P}_\ell$ is the space of pseudoholomorphic maps $G_\ell \to M$ of finite energy which map $z \in \partial G_\ell$ to L_z, i.e, of the form (7.1.3.2). This allows us to apply Theorem 7.1.12 with $\mathcal{P} = \mathcal{P}_\ell$.

We require the following uniform energy bound.

Lemma 7.2.5 ([Hof88, Lemma 2])**.** *The symplectic energy is uniformly bounded on all $\mathcal{P}_\ell$. More precisely, there exists a constant $C \geqslant 0$ such that for all $\ell > 0$ and all u in $\mathcal{P}_\ell$, we have $E(u) \leqslant C$.*

Fix a smooth cutoff function $\rho : \mathbb{R} \to [0, 1]$ with

$$\rho(t) = \begin{cases} 1 & t \leqslant \frac{1}{2} \\ 0 & t \geqslant \frac{3}{2} \end{cases}$$

and define for $\ell > 0$ the function $r_\ell : \mathcal{F}_{\ell,A} \to \mathcal{D}_{L,L'}$ by

$$r_\ell(u)(x + iy) := u(\rho(\ell^{-1}|x|)x + iy).$$

By construction, $r_\ell(u)$ agrees with u on $Z_{\frac{\ell}{2}}$.

Proposition 7.2.6 ([Hof88, Proposition 3])**.** *For any neighbourhood U of $\mathcal{M}_{L,L'}$ in $\mathcal{D}_{L,L'}$ and any $A \geqslant 0$, there exists $\ell_0 > 0$ such that $r_\ell(\mathcal{F}_{\ell,A}) \subseteq U$ for all $\ell \geqslant \ell_0$.*

Proof of Proposition 7.2.1. Let C be the uniform energy bound from Lemma 7.2.5. Any u in any $\mathcal{P}_\ell$ clearly satisfies $E(u|_{Z_\ell}) \leqslant C$. Picking U an open neighbourhood of $\mathcal{M}_{L,L'}$ in $\mathcal{D}_{L,L'}$, and taking ℓ_0 as in Proposition 7.2.6 with $A = C$, we obtain a commutative diagram

$$\begin{tikzcd} \mathcal{P}_{\ell_0} \arrow[r, "\cdot|_{Z_{\ell_0}}"] \arrow[drr, "\pi"] & \mathcal{F}_{\ell_0,C} \arrow[r, "r_{\ell_0}"] & U \arrow[d, "\pi"] \\ & & L \end{tikzcd}$$

By Theorem 7.1.12, the pullback $\pi^* : R^*(L) \to R^*(U)$ must be injective. Thanks to the isomorphism

$$R^*(\mathcal{M}_{L,L'}) \cong \varinjlim R^*(U),$$

taking the direct limit over open neighbourhoods of $\mathcal{M}_{L,L'}$ in $\mathcal{D}_{L,L'}$, given by [AMS21, Lemma 5.2] and the exactness of the direct limit functor, we may conclude. $\square$

7.3 Ljusternik-Schnirelmann theory

In this section we finish the proof of Theorem 7.0.6. Observe that there is a natural $\mathbb{R}$-action on $\mathcal{M}_{L,L'}$, by setting $t \cdot u := u(\cdot - t)$. The fixed points of this action are exactly the constant maps to points in $L \cap L'$. Hence there is a bijection between these fixed points and $L \cap L'$.

Lemma 7.3.1 ([Hof88, Lemma 4])**.** *There exists a continuous map $\sigma : \mathcal{M}_{L,L'} \to \mathbb{R}$ such that for any u which is not a fixed point of the $\mathbb{R}$-action, the function $t \mapsto \sigma(t \cdot u)$ is strictly decreasing.*

Sketch of the construction. One should think of σ as something akin to the Floer action functional. Indeed, if X is Liouville and L is an exact Lagrangian, we can take σ to be the usual Floer action functional.

If not, for each path component Q in $\mathcal{M}_{L,L'}$, we fix a basepoint u_0 in Q, and define $\sigma(u_0) := 0$. Then for some other u_1 in Q, we pick a path $\{u_t\}_{t \in [0,1]}$ from u_0 to u_1, and define

$$\sigma(u_1) = \int_{[0,1]^2} v^* \omega$$

where $v : [0,1]^2 \to X$ is a smoothing (rel endpoints) of the map sending (s,t) to $u_s(ti)$. This is well-defined due to relative exactness. □

Fix some basepoint $x_0 \in L$. For any subset S of $\mathcal{M}_{L,L'}$ or $\mathcal{D}_{L,L'}$, we consider the map of pairs

$$\pi_S : (S, \varnothing) \to (L, x_0) : u \mapsto u(0),$$

as well as the pullback

$$\pi_S^* : R^*(L, x_0) \to R^*(S).$$

Definition 7.3.2. To each subset S of $\mathcal{M}_{L,L'}$, we assign the non-negative integer

$$I(S) := \min \left\{ k \geqslant 1 : \exists\, U_1, \ldots, U_k \subset \mathcal{M}_{L,L'} \text{ open } : S \subset U_1 \cup \cdots \cup U_k \text{ and } \pi_{U_i}^* = 0 \right\}.$$

Note I has a uniform upper bound. Indeed, let N be the minimal number of contractible open subsets of L required to cover L. Then $I(S) \leqslant N$ for any $S \subset \mathcal{M}_{L,L'}$.

Lemma 7.3.3. *Fix subsets S and T of $\mathcal{M}_{L,L'}$.*

1. *If $S \subseteq T$, then $I(S) \leqslant I(T)$.*
2. *There is some open neighbourhood U of S such that $I(S) = I(U)$.*
3. *$I(S \cup T) \leqslant I(S) + I(T)$.*
4. *If $\{\varphi_t\}_{t \in \mathbb{R}}$ is a flow on $\mathcal{M}_{L,L'}$, then $I(S) = I(\varphi_t(S))$ for all $t \in \mathbb{R}$.*
5. *$I(\{u_1, \ldots, u_n\}) = 1$ for any $u_1, \ldots, u_n \in \mathcal{M}_{L,L'}$.*

Thus I is an *index function* in the sense of [Rud99, Definition 4.2].

Proof. If $S \leqslant T$, we take the minimum over a larger set, so the inequality is immediate. If $U_1, \ldots, U_{I(S)}$ are open subsets of $\mathcal{M}_{L,L'}$ covering S with $\pi^*_{U_i} = 0$ for all i, set

$$U = U_1 \cup \ldots \cup U_{I(S)}$$

Then $I(U) \leqslant I(S)$, so equality holds by (1). The union of two suitable open covers for S, respectively T defines a suitable open cover for $S \cup T$ which must have cardinality at least $I(S \cup T)$. This shows 3. As φ_t is homotopic to the identity, it takes a suitable cover for S to a suitable cover for $\varphi_t(S)$. Thus $I(\varphi_t(S)) \leqslant I(S)$, and equality follows from applying the same argument to $\varphi_{-t}(\varphi_t(S))$. To see the last claim, denote $\{p_1, \ldots, p_k\} = \{u_1(0), \ldots, u_n(0)\}$. For each $j \leqslant k$ choose a contractible open neighbourhood V_j of p_j in L, such that $\overline{V_i} \cap \overline{V_j} = \varnothing$ for $i \neq j$. Then the preimage $U = \pi^{-1}(V_1 \cup \cdots \cup V_k)$ defines a suitable cover for $\{u_1, \ldots, u_n\}$. □

Lemma 7.3.4. *$I(\mathcal{M}_{L,L'}) \geqslant c_R(L)$.*

Proof. Fix an open cover $U_1, \ldots, U_k$ of $\mathcal{M}_{L,L'}$ such that $\pi^*_{U_i} = 0$ for $i \leqslant k$ and let $\alpha_1, \ldots, \alpha_k \in R^*(L, x_0)$ be arbitrary. By Lemma 7.1.2 the product $\pi^*_{\mathcal{M}_{L,L'}}\alpha_1 \cdot \ldots \cdot \pi^*_{\mathcal{M}_{L,L'}}\alpha_k$ vanishes in $R^*(\mathcal{M}_{L,L'})$. By Theorem 7.1.12, $\pi^*_{\mathcal{M}_{L,L'}}$ is injective, so $\alpha_1 \cdot \ldots \cdot \alpha_k = 0$ in $R^*(L, x_0)$ and $c_R(L) \leqslant k$. Taking the infimum over all such open covers completes the proof. □

Given Lemma 7.3.4, the proof of Theorem 7.0.6 reduces to showing that

$$\# L \cap L' \geqslant I(\mathcal{M}_{L,L'}) \tag{7.3.0.1}$$

Since I is an index function, this follows from [Rud99, Theorem 4.2]. For the sake of exposition, we give a proof here, using standard Ljusternik-Schnirelman theory as in [Hof88, Section V].

Definition 7.3.5. For $1 \leqslant i \leqslant I(\mathcal{M}_{L,L'})$, we define

$$d_i := \inf_{I(S) \geqslant i} \sup \sigma(S)$$

where the infimum is taken over subsets of $\mathcal{M}_{L,L'}$.

For any $d \in \mathbb{R}$, we denote

$$Cr(d) := \left\{ u \in \mathcal{M}_{L,L'} : \sigma(u) = d, \; \mathbb{R} \cdot u = u \right\}.$$

It follows that

$$\sum_d \# Cr(d) = \# L \cap L'.$$

Lemma 7.3.6.

$$-\infty < d_1 \leqslant \ldots \leqslant d_{I(\mathcal{M}_{L,L'})} < \infty.$$

Proof. First observe that $d_j \leqslant d_{j+1}$ for all j since we take the infimum over a smaller set. The compactness of $\mathcal{M}_{L,L'}$ implies that $-\infty < d_1$ and $d_{I(\mathcal{M}_{L,L'})} < \infty$. $\square$

Lemma 7.3.7. *For any neighbourhood U of $Cr(d)$, there exists some $\varepsilon > 0$ such that*

$$u \in \sigma^{-1}((-\infty, d+\varepsilon])\backslash U \quad \Rightarrow \quad 1 \cdot u \in \sigma^{-1}((-\infty, d-\varepsilon]).$$

Proof. This follows from the compactness of $\sigma^{-1}((-\infty, d])\backslash U$ along with the continuity of the $\mathbb{R}$-action. $\square$

Lemma 7.3.8. *$Cr(d_j)$ is non-empty for all j.*

Proof. Suppose $Cr(d_j)$ is empty. Applying Lemma 7.3.7 to $U = \varnothing$ we obtain some $\varepsilon > 0$ such that

$$1 \cdot \sigma^{-1}((-\infty, d_j + \varepsilon]) \subseteq \sigma^{-1}((-\infty, d_j - \varepsilon]).$$

By definition of d_j, there exists $S \subseteq \mathcal{M}_{L,L'}$ such that $I(S) \geqslant j$ and

$$d_j \leqslant \sup \sigma(S) \leqslant d_j + \varepsilon.$$

But then $I(1 \cdot S) \geqslant j$ and $\sup \sigma(1 \cdot S) < d_j$, which is a contradiction. $\square$

Lemma 7.3.9. *If $d_j = d_{j+1}$ for any j, then $Cr(d_j)$ is infinite.*

Proof. If $Cr(d_j)$ is finite, then $I(Cr(d_j)) = 1$ by Lemma 7.3.3.(5). So it suffices to show that $I(Cr(d_j)) \geqslant 2$. Suppose by contradiction $I(Cr(d_j)) \leqslant 1$. Since $Cr(d_j)$ is non-empty, we must have equality. Then there is some open neighbourhood U of $Cr(d_j)$ such that $\pi_U^* = 0$. Given this U, fix $\varepsilon > 0$ as in the statement of Lemma 7.3.7.

Choose $S \subseteq \mathcal{M}_{L,L'}$ such that $I(S) \geqslant j+1$ and

$$d_j \leqslant \sup \sigma(S) \leqslant d_j + \varepsilon.$$

Then $I(1 \cdot (S\backslash U)) \geqslant j$ but $\sigma(1 \cdot (S\backslash U)) \leqslant d_j - \varepsilon$, a contradiction. $\square$

The inequality in (7.3.0.1), and hence Theorem 7.0.6, follows from Lemmas 7.3.8 and 7.3.9.

7.4 Two examples

Let

$$\mathrm{Sp}(n) := \mathrm{Sp}(2n; \mathbb{C}) \cap U(2n)$$

be the *compact symplectic group*. It is a compact simply-connected Lie group of dimension $n(2n+1)$. The zero section defines a Lagrangian embedding $\mathrm{Sp}(n) \hookrightarrow T^*\mathrm{Sp}(n)$, where we endow $T^*\mathrm{Sp}(n)$ with the canonical symplectic structure. As this embedding is a homotopy equivalence, $\mathrm{Sp}(n)$ is relatively exact. We will consider $\mathrm{Sp}(2)$ and $\mathrm{Sp}(3)$ since their cuplength with respect to a certain generalised cohomology theory was computed in [IM04] (see also [Kis07]) and is strictly greater than their cuplength with respect to integral cohomology.

Proposition 7.4.1 ([IMN03, IM04])**.** *The mod-2 and integral cuplengths of* $\mathrm{Sp}(2)$ *are*

$$c_{\mathbb{Z}/2}(\mathrm{Sp}(2)) = c_{\mathbb{Z}}(\mathrm{Sp}(2)) = 3,$$

while

$$c_{h^*}(\mathrm{Sp}(2)) = 4$$

where h^* *is the cohomology theory associated to the truncated sphere spectrum* $\mathbb{S}[0, 2]$*. In particular, [IM04] shows that its cuplength in real* K*-theory is*

$$c_{KO}(\mathrm{Sp}(2)) = 4.$$

Similarly the cuplengths of $\mathrm{Sp}(3)$ *in the same cohomology theories are given by*

$$c_{\mathbb{Z}/2}(\mathrm{Sp}(3)) = c_{\mathbb{Z}}(\mathrm{Sp}(3)) = 4$$

and

$$c_{h^*}(\mathrm{Sp}(3)) = c_{KO}(\mathrm{Sp}(3)) = 5.$$

Remark 7.4.2. We use Hofer's convention in [Hof88] for cuplengths which differs by one compared to that of [IM04].

Since $\mathrm{Sp}(2)$ and $\mathrm{Sp}(3)$ are Lie groups, they are parallelisable. By Proposition 7.0.5 we can therefore apply Theorem 7.0.6 with real K-theory to either one as the zero section lying inside its cotangent bundle. This gives a (strictly) stronger bound on the Arnol'd number than Hofer's cup length estimate, though this estimate was already known due to work of Laudenbach and Sikorav [LS85], using finite-dimensional approximations.

Corollary 7.4.3. *The minimum number of intersection points between a relatively exact Lagrangian embedding of* $\mathrm{Sp}(2)$ *(satisfying Proposition 7.0.5.4) and its image under any Hamiltonian diffeomorphism is at least* 4*. The same is true for* $\mathrm{Sp}(3)$ *with* 5 *instead of* 4*.*

Proposition 7.4.4. *The critical number of* $\mathrm{Sp}(2)$ *is 4.*

Proof. The critical number of $\mathrm{Sp}(2)$ is bounded below by 4 by Proposition 7.4.1. On the other hand, [Sma62, Theorem 6.1] combined with the computation of its integral homology in Lemma 7.4.5 implies that the Morse number of $\mathrm{Sp}(2)$ is 4, which is an upper bound for the critical number. □

The bound for $\mathrm{Sp}(2)$ (and a stronger bound for $\mathrm{Sp}(3)$) were shown by [LS85] for the respective zero section in the cotangent bundle. However, their methods are specifically geared towards cotangent bundles while our bounds persist under Weinstein handle attachments. As an example, one can plumb a copy of $T^*\mathrm{Sp}(2)$ containing $\mathrm{Sp}(2)$ as the zero section, with the cotangent bundle of any other 2-connected manifold of the same dimension to obtain a new Weinstein manifold X. Since $\mathrm{Sp}(2)$ is 2-connected, the resulting manifold admits the the same 2-skeleton (up to homotopy) as $T^*\mathrm{Sp}(2)$, which is trivial. Therefore Proposition 7.0.54 still holds for the embedding $\mathrm{Sp}(2) \hookrightarrow X$. This gives a stronger bound than Hofer's estimate, in a case for which the estimate of [LS85] does not apply.

We recap the computation of the integral cohomology rings of Sp(2) and Sp(3) here.

Lemma 7.4.5 ([IM04])**.** *The integral cuplengths of* Sp(2) *and* Sp(3) *are given by*

$$c_{\mathbb{Z}}(\mathrm{Sp}(2)) = 3 \quad \text{and} \quad c_{\mathbb{Z}}(\mathrm{Sp}(3)) = 4.$$

Furthermore, $H^*(\mathrm{Sp}(2))$ *and* $H^*(\mathrm{Sp}(3))$ *are free of rank 4 and 8 respectively, over both* $\mathbb{Z}$ *and* $\mathbb{Z}/2$*.*

Proof. Given n we can identify $\mathbb{H}^n$ with $\mathbb{R}^{4n}$ to see the existence of a principal $\mathrm{Sp}(n-1)$-bundle $\mathrm{Sp}(n) \to S^{4n-1}$ induced by the canonical action on the unit quaternions. Thus we can apply the Leray-Serre spectral sequence to compute the cohomology of Sp(2) with coefficients in $A = \mathbb{Z}$ or $A = \mathbb{Z}/2$. The E_2-page is given by

$$E_2^{p,q} = H^p(S^7, H^q(\mathrm{Sp}(1); A))$$

which vanishes for $p \notin \{0, 7\}$ and $q \notin \{0, 3\}$. Hence the spectral sequence collapses for degree reasons at the second page. As $\mathrm{Sp}(1) \cong S^3$, we obtain

$$H^*(\mathrm{Sp}(2); A) \cong H^*(S^7; A) \otimes_A H^*(S^3; A). \tag{7.4.0.1}$$

By the multiplicativity of the spectral sequence, (7.4.0.1) is an isomorphism of graded rings. Therefore,

$$H^n(\mathrm{Sp}(2); A) = \begin{cases} A & n \in \{0, 3, 7, 10\}, \\ 0 & \text{otherwise} \end{cases}$$

and we can deduce the values of $c_{\mathbb{Z}/2}(\mathrm{Sp}(2))$ and $c_{\mathbb{Z}}(\mathrm{Sp}(2))$.

The same argument gives an isomorphism of graded rings

$$H^*(\mathrm{Sp}(3); A) \cong H^*(S^{11}; A) \otimes_A H^*(\mathrm{Sp}(2); A).$$

Therefore,

$$H^n(\mathrm{Sp}(2); A) = \begin{cases} A & n \in \{0, 3, 7, 10, 11, 14, 18, 21\}, \\ 0 & \text{otherwise} \end{cases}$$

from which we deduce the corresponding statements for Sp(3). □

Appendix A

Intersection theory on orbifolds

In this appendix, we prove several results we needed in the previous chapters. They mainly consist of projection formulas and the definition of a trace map in a quite general setting. We found neither in the literature in the form, respectively, generality necessary for our purposes.

For us, an *orbifold* is a tuple $\mathfrak{X} = (\overline{\mathfrak{X}}, [X])$, where $\overline{\mathfrak{X}}$ is a topological space, called the *coarse moduli space of* $\mathfrak{X}$, X is a (topological or smooth) proper étale Lie groupoid and $[\,]$ denotes its Morita equivalence class. X is called a *presentation* of $\mathfrak{X}$. Refer to [Beh04] for more details. We will only consider smooth orbifolds, which can be represented by a *global quotient*, that is, a Lie groupoid of the form $[G \times M \rightrightarrows M]$ where G is a compact Lie group acting almost freely and smoothly on the manifold M. By [ALR07, Theorem 1.23] any effective orbifold is of this form, but we do not require our orbifolds to be effective. We denote by $[M/G]$ the stack presented by such a global quotient.

A Lie groupoid $X = [X_1 \rightrightarrows X_0]$ is *orientable* if X_0 and X_1 are orientable and if the source and target map from X_1 to X_0 are orientation-preserving. We call an orbifold *orientable* if each representing Lie groupoid is orientable. Thus, $[M/G]$ is orientable if and only if M is orientable and and G acts by orientation-preserving homeomorphisms.

Remark A.0.1. This notion of orientability is strictly stronger than orientability of the coarse moduli space as the example of the Klein bottle (whose quotient by an S^1-action is S^1 itself) shows.

By [Beh04, p.27], there is a canonical isomorphism

$$H^*([M/G], \mathbb{Q}) \cong H^*_G(M, \mathbb{Q}).$$

Let $q\colon M_G \to M/G$ denote the canonical map to the quotient; it defines by [Beh04, Proposition 36] an isomorphism

$$H^*(M/G, \mathbb{Q}) \to H^*_G(M, \mathbb{Q}).$$

As the coarse moduli space of an oriented orbifold is an oriented homology $\mathbb{Q}$-manifold, it satisfies rational Poincaré duality

$$\check{H}^*_c(Z, \mathbb{Q}) \cong H_{\dim(\mathfrak{X})-*}(\overline{\mathfrak{X}}, \overline{\mathfrak{X}} \backslash Z, \mathbb{Q})$$

by [Bre12].[1] Replacing singular homology with Borel–More homology, the same isomorphism holds with ordinary Čech cohomology on the left-hand side.

Remark A.0.2 (Orientation)*.* Since G acts smoothly and almost freely, we have an inclusion $\underline{\mathfrak{g}} \hookrightarrow TM$ of vector bundles, where $\mathfrak{g} = \mathrm{Lie}(G)$. Let $\mathcal{D} \subset TM$ be a G-invariant complement. For the sake of concreteness, we think of an *orientation* of $\mathfrak{X} = [M/G]$ as a Thom class of the vector bundle $\mathcal{D}_G \to M_G$.

A.1 Exceptional pushfoward

Let $f\colon M^m \to N^n$ be a G-equivariant map between smooth manifolds on which G acts smoothly and almost freely. Suppose $[M/G]$ and $[N/G]$ are oriented. Then f induces a morphism $[M/G] \to [N/G]$ and the *exceptional pushforward*

$$f_!\colon H^*(M/G, \mathbb{Q}) \to H^{*+n-m}(N/G, \mathbb{Q})$$

is defined by $f_! := \mathrm{PD}\, f_*\, \mathrm{PD}$. We clearly have $g_! f_! = (gf)_!$.

By [Bre72, Theorem 4.1], we can factor f as a composite $M \xrightarrow{j} N \times S^V \xrightarrow{\mathrm{pr}_1} N$, where S^V is the one-point compactification of a finite-dimensional orthogonal G-representation V and j is an equivariant embedding. We can describe $j_!$ and $\mathrm{pr}_{1!}$ explicitly.

Example A.1.1 (Embedding)*.* Suppose f is an embedding with Poincaré dual $\mathrm{PD}(M/G)$. Then

$$f_! f^*(\alpha) = \alpha \cdot \mathrm{PD}(M/G) \tag{A.1.0.1}$$

for $\alpha \in H^*(N/G; \mathbb{Q})$. If there exists an equivariant retraction $r\colon W \to M$, then $f_!(\alpha) = r^*\alpha \cdot \mathrm{PD}(M/G)$.

Example A.1.2 (Projection)*.* Suppose $M = N \times S^V$, where V is a finite-dimensional G-representation S^V is its one-point compactification (to which the G-action extends trivially) and f is the projection. Then $f_G\colon (N \times S^V)_G \to N_G$ is the sphere bundle of $(N \times (V \oplus \mathbb{R}))_G \to N_G$ and

$$\begin{tikzcd}
H^*_G(N \times S^V, \mathbb{Q}) \arrow[r, hook] \arrow[dr, "f_!"'] & H^{*+1}_G(N \times (V \oplus \mathbb{R}), N \times ((V \oplus \mathbb{R})\backslash 0), \mathbb{Q}) \arrow[d, "\cong"] \\
& H^{*-k}_G(N, \mathbb{Q})
\end{tikzcd}$$

commutes by [Dua03, §3], where the vertical map comes from the Thom isomorphism.

In particular, we have the following observation.

Corollary A.1.3. *Suppose we have a cartesian square*

[1] The proof of [Par16, Lemma A.6.4] also generalises easily to this setting.

$$\begin{tikzcd} P \arrow[r, "f"] \arrow[d, "q"] & P' \arrow[d, "q'"] \\ B \arrow[r, "\bar{f}"] & B' \end{tikzcd}$$

where q and q' are principal G-bundles for a compact Lie group G, B, B' are oriented smooth manifolds (and P, P' are equipped with the corresponding G-orientation), $\bar{f}$ is smooth and proper, a compact Lie group G' acts on the whole square almost freely. Then

$$f_! q^* = q'^* \bar{f}_!. \tag{A.1.0.2}$$

Lemma A.1.4 (Projection formula)**.** *Suppose we have a cartesian square*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "q"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

of oriented G-manifolds, where G acts almost freely. If f is proper and p is a submersion, then

$$f'_! q^* = p^* f_!.$$

If the G-action on X' and Y' extends to an almost $G \times G'$-action with respect to which f', q and p are equivariant, then (7.1.3) holds as maps $H^*(X/G;\mathbb{Q}) \to H^{k+*}(Y'/G \times G';\mathbb{Q})$.

Proof. If f is an embedding, so is f' and the claim follows from (A.1.0.1). If f is a projection $Y \times S^V \to Y$ for some finite-dimensional G-representation V, we can assume f' is the projection $Y' \times S^V \to Y'$. Denoting the induced map $Y'_{G\times G'} \to Y_G$ by p as well, we have $(Y' \times (V \oplus \mathbb{R}))_{G\times G'} = p^*(Y \times (V \oplus \mathbb{R}))_G$. Thus the claim follows from Example (A.1.2) and the functoriality of the Thom class. ☐

Corollary A.1.5. *Suppsoe we have a cartesian square as in Lemma A.1.4, where f is an embedding. Then $p^*\mathrm{PD}(X/G) = \mathrm{PD}(X'/G \times G')$.*

Proof. Let $\overline{Y'}$ and $\overline{X'}$ denote the quotients and suppose $k = \mathrm{codim}(X')$. Since both $p^*\mathrm{PD}(X/G)$ and $\mathrm{PD}(X'/G \times G')$ live in $H^k(\overline{Y'} \mid \overline{X'};\mathbb{Q}) \cong \mathbb{Q}^{|\pi_0(\overline{X'})|}$, they differ by multiplication with a locally constant function b. Thus $p^* f_! f^* \alpha = p^*\alpha \cdot p^*\mathrm{PD}(\overline{X})$ on one hand, while

$$p^* f_! f^* \alpha = f'_! q^* f^* \alpha = f'_! f'^* p^* \alpha == p^*\alpha \cdot \mathrm{PD}(\overline{X'}) = b p^*\alpha \cdot p^*\mathrm{PD}(\overline{X}).$$

As the same equality holds locally and p is admits local sections, it follows that $b \equiv 1$. ☐

Lemma A.1.6. *Let*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "q"] & Y' \arrow[d, "p"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

be a cartesian square of smooth manifolds, where f is proper. Suppose there exists an open subset $V \subset Y'$ so that $Y' \backslash V$ has codimension at least 2 and $p|_V : V \to Y$ is a (not necessarily surjective) submersion. Then

$$f'_! q^* = p^* f_!.$$

The same is true in equivariant cohomology if the square is cartesian in the category of almost free G-manifolds.

Proof. Set $U := X' \times_{Y'} V$ and let $i: U \to X'$ and $j: V \to Y'$ be the inclusions. Let $\hat{f}: U \to V$ be the induced map. The fundamental classes appearing below are elements of Borel–Moore homolgoy. By [Bre12, §V.10(57),Corollary V.10.2] and Lemma A.1.4, the claim holds in the nonequivariant case.

Suppose now G and G' are compact Lie groups so that $G \times G'$ acts almost freely on X' and Y' and G acts almost freely on Y with f and f' being equivariant and p, q being invariant under the G'-action and restricting to principal bundles over V, U. Since we can check the equality $f'_! q^* = p^* f!$ as maps $H^*_G(X, \mathbb{Q}) \to H^{*+k}_{G \times G'}(Y', \mathbb{Q})$ degree for degree, we can use finite-dimensional approximations of the classifying space of G. To these, apply Corollary A.1.3 to see that the projection formula holds for the pullbacks to U. Then use the same argument as in the first step to conclude. □

Lemma A.1.7. *Suppose $f: X \to Y$ is a smooth map of étale proper Lie groupoids and $[M/G]$ is a global resolution of Y. If X is a manifold, then the orbifold fibre product $M \times_Y X$ is a principal G-bundle over X.*

Proof. Since M and X are manifolds, so is $Z := M \times_Y X$. Let $q_0: M \to Y_0$ be the canonical map. Then $Z = \{(p, \alpha, x) \in M \times Y_1 \times X : \alpha: q(p) \to f(x)\}$ and $\pi: Z \to X$ is given by $\pi(p, \alpha, x) = x$. Define a G-action on Z by setting

$$g \cdot (p, \alpha, x) := (g \cdot p, \alpha \circ q_1(g, p)^{-1}, x).$$

Clearly, π is G-invariant and $g \cdot (p, \alpha, x) = (p, \alpha, x)$ implies that $g \cdot p = p$ and $q_1(g, p) = \mathrm{id}$. Since $q: [M/G] \to Y$ is étale, we must have $g = e$. Thus G acts freely on Z. To see that π is locally trivial, it suffices to consider the case where $Y = [V/\Gamma]$ for some finite group Γ and $M = [S/G_p]$ for some slice S through p. In this case π is the pullback of a covering map and thus a local diffeomorphism. This completes the proof. □

A.2 Trace maps

In the definition of the equivariant virtual fundamental class, we make use of a *trace map* $H^{*+m}_{K,fc}(\mathcal{T}; \mathbb{Q}) \to H^*_K(\mathrm{pt}; \mathbb{Q})$, which is a special case of integration along the fibre. While this is classical for fibre bundles of closed smooth manifolds,[BT82], and has been generalised in algebraic geometry, [Ive86, KS94], we found no results for the specific situation needed in this paper. Thus we give a brief definition and show the required properties. To avoid any subtleties with families of supports, we will assume that all spaces are locally compact, Hausdorff and paracompact.

Definition A.2.1 (Integration along the fibre)**.** Suppose $\pi\colon P \to B$ is an oriented fibre bundle over a paracompact base B with fibre an oriented topological orbifold $X = [M/G]$. Denote by $\mathcal{H}_c^*(X)$ the locally constant sheaf on B with stalks given by $\mathcal{H}_c^*(X)_b = H_c^*(P_b; \mathbb{Q})$ for $b \in B$. By [Bre12, Theorem 6.1] there exists a spectral sequence $\{E_r^{p,q}\}$ converging to $H_{fc}^*(P; \mathbb{Q})$ with

$$E_2^{p,q} = H^p(B; \mathcal{H}_c^q(X)).$$

In particular, $E_2^{p,q} = 0$ for $q > n := \dim(X)$, so $E_r^{p,n} \subset E_{r-1}^{p,n}$ for any $r > 2$. We have a canonical map $\int_X\colon \mathcal{H}_c^n(X) \to \underline{\mathbb{Q}}$ of locally constant sheaves on B; it is given at the stalk over $b \in B$ by

$$\mathcal{H}_c^n(X)_b = H_c^n(X_b; \mathbb{Q}) \xrightarrow{\mathrm{pt}_!} \mathbb{Q}.$$

We define the *integration along the fibre* $\pi_*\colon H_{fc}^{n+*}(P; \mathbb{Q}) \to H^*(B; \mathbb{Q})$ to be the composite

$$H_{fc}^{n+*}(P; \mathbb{Q}) \to E_\infty^{*,n} \hookrightarrow E_2^{*,n} = H^*(B; \mathcal{H}_c^n(X)) \xrightarrow{(\int_X)_\#} H^*(B; \mathbb{Q}).$$

By [Aue73], this agrees with the standard definition of integration along the fibre for smooth fibre bundles.

Lemma A.2.2 (Base change)**.** *Suppose $\pi\colon P \to B$ is an orientable fibre bundle over a paracompact base with fibre $\mathcal{T}$ an oriented orbifold and $f\colon B' \to B$ is a proper continuous map from another paracompact space. Then*

$$\begin{tikzcd}
H_c^{m+*}(P; \mathbb{Q}) \arrow[r] \arrow[d, "\tilde{f}^*"] & H^*(B; \mathbb{Q}) \arrow[d, "f^*"] \\
H_c^{m+*}(f^*P; \mathbb{Q}) \arrow[r] & H^*(B'; \mathbb{Q})
\end{tikzcd}$$

commutes.

Proof. This follows from the functoriality of the Leray-Serre spectral sequence associated to a fibration, see [Bre12, §6.2]. □

Corollary A.2.3. *Suppose $\mathcal{T}$ is an oriented topological orbifold of dimension m with a continuous action by a compact connected Lie group G. Then*

$$\begin{tikzcd}
H_{G,c}^{m+*}(\mathcal{T}; \mathbb{Q}) \arrow[r] \arrow[d, "i^*"] & H_G^*(\mathrm{pt}; \mathbb{Q}) \arrow[d] \\
H_c^{m+*}(\mathcal{T}; \mathbb{Q}) \arrow[r] & H^*(\mathrm{pt}; \mathbb{Q})
\end{tikzcd}$$

commutes, where $H_{G,c}^*(\mathcal{T}; \mathbb{Q}) = H_{fc}^*(\mathcal{T}_G; \mathbb{Q})$.

Lemma A.2.4 (Functoriality)**.** *Suppose $\pi\colon P \to B$ and $\rho\colon E \to P$ are two oriented fibre bundles with fibres X and Y the coarse moduli spaces of oriented orbifolds. Then*

$$(\pi\rho)_* = \pi_* \rho_* \iota,$$

where ι is the canonical map from cohomology with $(\pi\rho)$-fibrewise compact support to cohomology with π-fibrewise compact support.

Proof. We will assume both X and Y are compact, of dimension k, respectively ℓ, in order to simplify the notation. The general case can be obtained by restricting to cohomology with suitable support. Let $\theta := \pi\rho$ and set $Z_b := \theta^{-1}(\{b\})$ for $b \in B$. The maps $\rho_{b*}\colon H^{k+\ell}(Z_b;\mathbb{Q}) \to H^k(P_b;\mathbb{Q})$ induce a morphism $\rho_*\colon \mathcal{H}^{k+\ell}(Z) \to \mathcal{H}^k(X)$ and θ_* factors as

$$H^{k+\ell+*}(E;\mathbb{Q}) \to H^*(B,\mathcal{H}^{k+\ell}(Z)) \xrightarrow{(\rho_*)_\#} H^*(B,\mathcal{H}^k(X)) \xrightarrow{\pi_*} H^*(B;\mathbb{Q}).$$

By [Bre12, Corollary IV.7.3], the Leray sheaf $\mathscr{H}(\pi;\mathcal{H}^\ell(Y))$ of π with coefficients in $\mathcal{H}^\ell(Y)$, defined in [Bre12, §IV.4], is locally constant with stalks of the form $H^*(P_b;\mathcal{H}^\ell(Y))$.[2] Due to the functoriality of the Serre spectral sequence, there exists a canonical morphism $\mathcal{H}^{*+\ell}(Z) \to \mathscr{H}^*(\pi;\mathcal{H}^\ell(Y))$ of locally constant sheaves on B, given stalkwise by

$$\mathcal{H}^{*+\ell}(Z)_b = H^{*+\ell}(\theta^{-1}(\{b\};\mathbb{Q}) \twoheadrightarrow E(b)_\infty^{*,\ell} \hookrightarrow H^*(P_b;\mathcal{H}^\ell(Y)) = \mathscr{H}^*(\pi;\mathcal{H}^\ell(Y))_b.$$

where $\{E(b)_r^{p,q}\}$ is the Leray-Serre spectral sequence of $\theta^{-1}(\{b\}) \to P_b$. This stalkwise description shows that $\rho_*\colon \mathcal{H}^{*+\ell}(Z) \to \mathcal{H}^k(X)$ factors through $\mathscr{H}^*(\pi;\mathcal{H}^\ell(Y))$. Thus

$$\begin{tikzcd}
H^{k+\ell+*}(E;\mathbb{Q}) \arrow[r] \arrow[d] & H^*(B;\mathcal{H}^{k+\ell}(Z)) \arrow[r, "(\rho_*)_\#"] \arrow[d] & H^*(B;\mathcal{H}^k(X)) \\
H_c^{k+*}(P;\mathcal{H}^\ell(Y)) \arrow[r] & H^*(P;\mathscr{H}^*(\pi;\mathcal{H}^\ell(Y))) \arrow[ur] &
\end{tikzcd}$$

commutes. By [Bre12, §6.2],

$$\begin{tikzcd}
H_c^{k+*}(P;\mathcal{H}^\ell(Y)) \arrow[r] \arrow[d, "(\int_Y)_\#"] & H^*(P;\mathscr{H}^*(\pi;\mathcal{H}^\ell(Y))) \arrow[d] \\
H^{k+*}(P;\mathbb{Q}) \arrow[r] & H^*(B;\mathcal{H}^k(X))
\end{tikzcd}$$

commutes as well. The claim now follows by composing with π_*. ☐

Lemma A.2.5. *Let $\pi\colon P \to B$ be an oriented locally trivial fibration over a locally contractible space with fibre the orbit space of $[M/G]$. Suppose $P' \subset P$ is a subspace so that the induced map $\pi'\colon P' \to B$ is an oriented fibre bundle with fibre given by the orbit space of $[M'/G]$, for a G-invariant submanifold $M' \subset M$. Assume the inclusion $P' \hookrightarrow P$ admits a normal bundle and a tubular neighbourhood. Then*

$$\begin{tikzcd}
H_{fc}^{m'+*}(P';\mathbb{Q}) \arrow[r, "\pi'_*"] \arrow[d, "j_!"] & H^*(B;\mathbb{Q}) \\
H_{fc}^{m+*}(P;\mathbb{Q}) \arrow[ur, "\pi_*"] &
\end{tikzcd}$$

commutes, where $m = \dim([M/G])$ and $m' = \dim([M'/G])$. Moreover, $j_!j^* = \sigma\cdot$ for a class $\sigma \in H^k(P \mid P';\mathbb{Q})$ restricting to the Poincaré dual of $\mathfrak{X}'$ over a fibre.

[2] We use here that the monodromy of $\mathcal{H}^*(\ell)$ is trivial in degree ℓ since we work with oriented fibre bundles.

Proof. Let $\rho\colon \mathcal{N} \to P'$ be the normal bundle of the embedding and $\psi\colon V \to W \subset \mathcal{N}$ be a tubular neighbourhood. Then $\sigma = \psi^*\tau_{\mathcal{N}}$ and $j_!$ is given by the composite

$$H^*_{fc}(P';\mathbb{Q}) \xrightarrow{\simeq} H^{k+*}_{fc}(\mathcal{N} \mid P';\mathbb{Q}) \xrightarrow{\psi^*} H^{k+*}_{fc}(W \mid P';\mathbb{Q}) \to H^{k+*}_{fc}(P;\mathbb{Q}).$$

It suffices thus to show that the above triangle commutes with P replaced by $\mathcal{N}$. In this case $j_!$ is an isomorphism with inverse given by ρ_*. Thus the claim follows from Lemma A.2.4. $\square$

Lemma A.2.6. *Suppose X is the orbit space of an oriented global quotient orbifold of dimension n with boundary and $\pi\colon P \to B$ is a fibre bundle with fibre X. If $j\colon P' \hookrightarrow P$ denotes the subbundle with fibre ∂X, then the composition*

$$H^{*+n-1}_c(P;\mathbb{Q}) \xrightarrow{j^*} H^{*+n-1}_c(P';\mathbb{Q}) \xrightarrow{\pi_*} H^*(B;\mathbb{Q})$$

vanishes.

Proof. The homology $\mathbb{Q}$-manifold $\tilde{X} := X \cup_{\partial X} \partial X \times [0,1)$ admits a proper deformation retraction onto X, as does $\tilde{P} := X \cup_{P'} P' \times [0,1)$ onto P (where it is fibrewise proper). The deformation retraction fixes P' pointwise and is a map of fibre bundles over B. By the long exact sequence in compactly supported cohomology, it suffices to show that $H^{*+n-1}_c(P';\mathbb{Q}) \to H^*(B;\mathbb{Q})$ factors through $H^{*+n-1}_c(P';\mathbb{Q}) \to H^{*+n}_c(\tilde{P};\mathbb{Q})$.

The results of [Bro62] generalise directly to the setting of a topological manifolds with boundary, on which a compact group G acts almost freely and locally linearly, and to fibrations thereof. Hence ∂X admits a collar inside X and P' admits one inside P. Thus we can find a neighbourhood $U \subset \tilde{P}$ with $U \cong P' \times (-1,1)$ and the claim reduces to showing the commutativity of

$$\begin{tikzcd} H^{*+n-1}_c(P';\mathbb{Q}) \arrow[r] \arrow[d, "\pi_!"] & H^{*+n}_c(P' \times (-1,1);\mathbb{Q}) \arrow[dl, "\pi_!"] \\ H^*(B;\mathbb{Q}) & \end{tikzcd}$$

which is an immediate consequence of the Künneth theorem. $\square$

Lemma A.2.7. *Suppose two oriented global Kuranishi charts $\mathcal{K}_i = (G, \mathcal{T}_i, \mathcal{E}_i, \mathfrak{s}_i)$ for $\mathfrak{M}_i$ are cobordant via $\mathcal{K} = (G, \mathcal{T}, \mathcal{E}, \mathfrak{s})$. If $f_i\colon \mathfrak{M}_i \to N$ is a continuous map so that $f_0 \sqcup f_1$ extends over $\mathfrak{s}^{-1}(0)/G$, then $f_{0*}[\mathfrak{M}_0]^{\mathrm{vir}} = f_{1*}[\mathfrak{M}_1]^{\mathrm{vir}}$ in $\check{H}^*(N;\mathbb{Q})^\vee$. The same is true in the equivariant setting.*

Proof. Set $\mathfrak{W} := \mathfrak{s}^{-1}(0)/G$. The claim follows from Lemma A.2.6 and the commutativity of

$$\begin{tikzcd} \check{H}^{d+n}(N;\mathbb{Q}) \arrow[r, "f^*"] \arrow[dr, "f_0^* \sqcup -f_1^*"'] & \check{H}^{d+*}(\mathfrak{W};\mathbb{Q}) \arrow[r, "\mathfrak{s}^*\tau_{\mathcal{E}}"] \arrow[d] & H^{m+*}_c(\mathcal{T}_0/G \sqcup \mathcal{T}_1/G;\mathbb{Q}) \arrow[d] & \\ & \check{H}^{d+*}(\mathfrak{M}_0 \sqcup \mathfrak{M}_1;\mathbb{Q}) \arrow[r, "\mathfrak{s}^*\tau_{\mathcal{E}}"] & H^{m+*}_c(\mathcal{T}_0/G \sqcup \mathcal{T}_1/G;\mathbb{Q}) \arrow[r, "\mathrm{pt}_!"] & \mathbb{Q} \end{tikzcd}$$

where d is the virtual dimension of $\mathfrak{M}_0$ and $\mathfrak{M}_1$. The extension to the equivariant setting is straightforward. $\square$

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