Contributions in Fractional Diffusive Limit and Wave Turbulence in Kinetic Theory

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A mis padres (To my parents):

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Abstract

This thesis is split in two different topics. Firstly, we study anomalous transport from kinetic models. Secondly, we consider the equations coming from weak wave turbulence theory and we study them via mean-field limits of finite stochastic particle systems.

Anomalous transport from kinetic models.
The goal is to understand how fractional diffusion arises from kinetic equations. We explain how fractional diffusion corresponds to anomalous transport and its relation to the classical diffusion equation. In previous works it has been seen that particles systems undergoing free transport and scattering with the media can give rise to fractional phenomena in two cases: firstly, if in the dynamics of the particles there is a heavy-tail equilibrium distribution; and secondly, if the scattering rate is degenerate for small velocities.

We use these known results in the literature to study the emergence of fractional phenomena for some particular kinetic equations.

Firstly, we study BGK-type equations conserving not only mass (as in previous results), but also momentum and energy. In the hydrodynamic limit we obtain a fractional diffusion equation for the temperature and density making use of the Boussinesq relation and we also demonstrate that with the same rescaling fractional diffusion cannot be derived additionally for the momentum. But considering the case of conservation of mass and momentum only, we do obtain the incompressible Stokes equation with fractional diffusion in the hydrodynamic limit for heavy-tailed equilibria.

Secondly, we will study diffusion phenomena arising from transport of energy in an anharmonic chain. More precisely, we will consider the so-called FPU-β chain, which is a very simple model for a one-dimensional crystal in which atoms are coupled to their nearest neighbours by a harmonic potential, weakly perturbed by a nonlinear quartic potential. The starting point of our mathematical analysis is a kinetic equation; lattice vibrations, responsible for heat transport, are modelled by an interacting gas of phonons whose evolution is described by the Boltzmann Phonon Equation. Our main result is the derivation of an anomalous diffusion equation for the temperature.

Weak wave turbulence theory and mean-field limits for stochastic particle systems.

The isotropic 4-wave kinetic equation is considered in its weak formulation using model homogeneous kernels. Existence and uniqueness of solutions is proven in a particular setting. We also consider finite stochastic particle systems undergoing instantaneous coagulation-fragmentation phenomena and give conditions in which this system approximates the solution of the equation (mean-field limit).

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Statement of Originality

I hereby declare that my dissertation entitled 'Contributions in Fractional Diffusive Limit and Wave Turbulence in Kinetic Theory' is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is concurrently submitted for any such degree of diploma or other qualification. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text.

Chapter 1 gives an overview of the mathematical methods and techniques needed for Part I. The information provided there comes from multiple references which are given. However Section 1.5.5 was developed by the author with the help of Professor James Norris. Even though no new results are given, as far as we know no other reference does the type of computations performed there.

This literature review was done under the guidance, explanations and supervision of Professor Clément Mouhot.

Chapter 2 is original research work produced in collaboration with Doctor Sabine Hittmeir from the Johann Radon Institute for Computational and Applied Mathematics (RICAM), Linz, Austria.

The original research problem was suggested by Professor Clément Mouhot, who also established the collaboration and proofed-read the final work.

Chapter 3 is original research work produced in collaboration with Professor Antoine Mellet from the University of Maryland, US.

The original research problem was suggested by Professor Antoine Mellet and Professor Clément Mouhot. Professor Clément Mouhot was the one who suggested this collaboration.

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Contents

Part I

Fractional diffusion limit for some kinetic models

Chapter 1

Overview: anomalous diffusion in kinetic theory

In this chapter, we start by explaining the main idea of multiscale analysis, which is the derivation of macroscopic models from microscopic ones. We focus here on a particular type of multiscale analysis giving rise to the fractional diffusion equation. In Section 1.2 we define the diffusion and fractional diffusion equations followed by the linear Boltzmann equation which is a kinetic model suitable for the study of the diffusive limit. In Section 1.2.3 we review the main results on this direction and explain the methods used to obtain them in Section 1.5.

We will also spend some time explaining the type of phenomena that the fractional diffusion equation models (super-diffusions/anomalous transport) and relating it to the classical diffusion in Section 1.4.

Contents

1.1 Preliminaries: Multiscale analysis

In this first Section we give a general flavour of the idea of multiscale analysis. Much can be written about this subject. Here we present a very short introduction sketching some of the concepts in an informal way.

One of the greatest applications of mathematics is the description and **prediction** of physical phenomena, for example, using Newton's laws, we can predict with much precision the movements of the planets; Newton's laws are what is called a **model** in mathematics. It is through models of physical phenomena, like the movement of the planets, that mathematics are used to get predictions.

However, we can find two different models describing the same phenomenon. How is this be possible?

An example of this are the model called **Boltzmann equation** and the model called **diffusion equation**. Both can be used to describe a gas. However, these models are obtained by studying the gas from different *perspectives*. On one hand, the model of the Boltzmann equation is obtained by studying the particles of the gas and their **collisions**; that is why is called **microscopic model**. On the other hand, the diffusion equation describes what can be seen by the naked eye, i.e., how the **flow** of the gas behaves; this is a **macroscopic model**. The microscopic and macroscopic models are quantitatively and qualitatively different even if the physical phenomenon, the behaviour of a gas, is the same.

This example evokes an 'old' idea; all the matter of the universe is formed by atoms, so if we know how atoms behave, we would expect to know how the world (the one that can be observed by the naked eye) behaves. Mathematically, this means that macroscopic models should be derived from microscopic ones. However, in the models of the gas, the diffusion equation was not derived from the Boltzmann equation; both models were obtained independently. Nevertheless, since both are good models, one would expect to find an a posteriori relation between them.

In this document, it is shown how to **derive a posteriori** diffusion-type equations from Boltzmann-type equations. The set of methods for doing so are called **multiscale analysis** or **scaling process** or **limiting process**. This derivation gives insight on the relation between the two models.

The origins of multiscale analysis can be found in Hilbert's 6th problem. In the International Congress of Mathematics of 1900, Hilbert presented 23 main problems for the mathematics of the 20-th century. Problem number 6 is the axiomatization of physics. This means to find a set of axioms (describing how particles behave) from which to derive all the physical phenomena, in particular, the one that can be observed by the naked eye. Part of the problem states:

6-th Problem. Mathematical treatment of the axioms of physics[1]

> *"To treat [...], by means of axioms, those physical sciences in which mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.*
>
> *As to the axioms of the theory of probabilities, it seems to me desirable that their logical investigation should be accompanied by a rigorous and satisfactory development of the method of mean values in mathematical physics, and in particular in the* ***kinetic theory of gases****.*
>
> *Important investigations by physicists on the foundations of mechanics are at hand; I refer to the writings of Mach, Hertz,* ***Boltzmann*** *and Volkmann. It is therefore very desirable that the discussion of the foundations of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the* ***limiting processes****, there merely indicated, which* ***lead from the atomistic view to the laws of motion of continua****. [...] "*

The mathematical derivation of macroscopic models from microscopic ones seemed an impossible task. What Hilbert suggested is to use an intermediate step: kinetic theory. We comment next on the meaning of this quote.

Atomic systems are in general intractable at mathematical, computational and experimental level. Let us take for example Newton's laws applied to the modelling of a gas.

A gas can be described by giving the position and velocity of each particle at each instant of time. Therefore to each particle i, it corresponds the following dynamics:

$$\begin{cases} \dot{x}_i = v_i \\ \\ \dot{v}_i = \text{ sum of forces, which depends on all the particles .} \end{cases}$$

This atomic model presents three problems:

(i) mathematically, the problem of solving the system is related to the N-body problem, which is a very hard problem to study;

(ii) numerically, each particle has 6 degrees of freedom and the number of particles is too big to be computed (around 10^{20});

(iii) experimentally, the equations tell us how the gas behaves, but we need to have an initial description of the gas, i.e., we need to know all the positions and velocities

[1] From `http://aleph0.clarku.edu/ djoyce/hilbert/problems.html#prob6`, David E. Joyce.

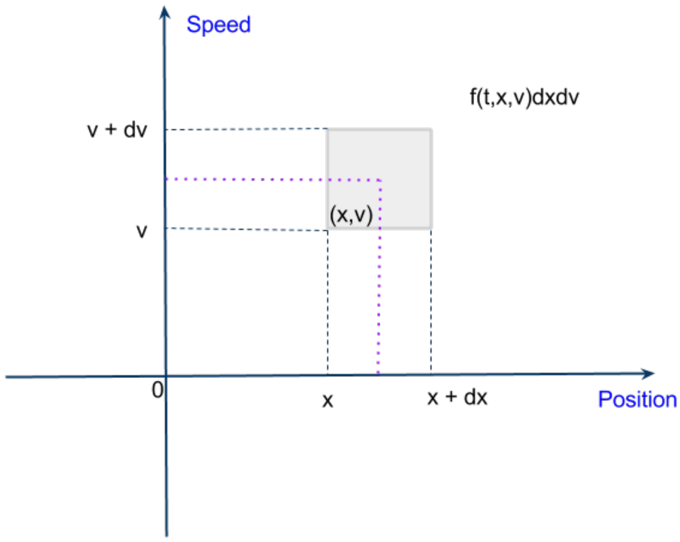

Figure 1.1: In the Boltzmann equation, $f(t,x,v)dxdv$ is the number of particles in an infinitesimal volume $dxdv$ centred at the point (x,v) at a time t.

of all the particles at a given time. However, technically, it is not known how to get those measurements.

Summarizing, the microscopic model is very hard to deal with.

Boltzmann proposed to model the same system but considering that the only known information is the distribution of the particles in the **phase space** (space of positions and velocities). Therefore, he assumes that we do not know exactly the position and velocity of each particle. The goal is then to study $f_t = f_t(x,v)$ probability distribution of the particles in space and velocity for every given time. $f(t,x,v)dxdv$ gives the number of particles in an infinitesimal volume $dxdv$ around (x,v) at a time t (see figure 1.1).

Originally Maxwell and afterwards Boltzmann avoided the intractability of the atomic model at the price of adding some 'uncertainty' into it: we do not have the whole information of the system, we will just look at how the positions and velocities of the particles evolve in average.

This idea gave birth to **Statistical Mechanics** and the Boltzmann equation for rarefied gases is the most fundamental one in what is called **kinetic theory**.

Kinetic theory studies mathematical models giving the evolution of a statistical or probability distribution for a given quantity. It started with the study of gas dynamics but it now extends to other areas like the study of plasmas.

Hilbert's suggestion is to derive macroscopic models from microscopic ones using an intermediate scale (mesoscale) which corresponds to kinetic models.

1.1.1 Multiscale analysis from kinetic models

The idea of multiscale analysis is to derive mathematically a particular physical model from another one that contains more information than the original one. The two models or equations are at different scales and deriving one model from the other requires, in the cases that will be treated here, averaging and a limiting process. This means that a model at atomistic scale explains how particular physical phenomena may arise at observable scale.

The kinetic equation has a solution that depends on space, time and velocity. The macroscopic equation depends only on space and time. The latter will be derived by averaging over the velocities the solution to the kinetic equation, and by performing a limiting process.

In physical terms, rescaling space and time means the following. The micro time scale is the typical time a particle takes to change its velocities. For observable changes to happen in the bulk of the particles, we need to speed up time and consider macro time scales. In the same manner, we also make a zoom out in space, to focus on the bulk of particles instead of on the individual particles.

This rescaling in time and space has to be done properly so that it stands out interesting phenomena: if we speed up time too much, the particles may escape to infinity and we will see nothing, i.e., in the limit we will get zero. If we do not speed up time fast enough, no changes will occur on the bulk of particles and no phenomena will arise.

Summarising, in the limiting process information is lost and at the same time, the dynamics of the bulk of particles, that were only implicit in the kinetic equation, stand out. Notice that due to the loss of information during the limiting process, it is possible that different kinetic models lead to the same macroscopic equation. Example of limiting process from kinetic equations can be found in the classical references [SR09], [CIP94] and [Vil02].

In this part of the dissertation we will deal with a particular type of multiscale analysis called **diffusive limit**. We will present the diffusive limit from kinetic equations giving rise to the fractional diffusion equation.

1.2 Mathematical models and previous results

1.2.1 Classical and fractional diffusion equations

In this section we will study the fractional diffusion equation and its classical counterpart, the diffusion equation. We give here the basic definitions and afterwards in Section 1.4 we will study their properties.

The classical diffusion equation is written as

$$\partial_t \rho(t,x) - \nabla_x \cdot (D\,\nabla_x \rho(t,x)) = 0 \quad \text{in } (0,\infty)\times\mathbb{R}^N \tag{1.1}$$
$$\rho(0,\cdot) = \rho^{in} \quad \text{in } \mathbb{R}^N \tag{1.2}$$

where D is a diffusion matrix (positive definite). The Cauchy theory and the properties of this equation are well known and can be found in the classical reference [Eva98].

The fractional heat equation is a generalisation of the classical one:

$$\partial_t \rho(t,x) + \kappa(-\Delta_x)^{(\alpha/2)}\rho(t,x) = 0 \qquad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{1.3}$$
$$\rho(0,\cdot) = \rho^{in} \qquad \text{in } \mathbb{R}^N \tag{1.4}$$

with $\kappa \in \mathbb{R}_+$. The fractional laplacian corresponds to the classical laplacian in the case $\alpha = 2$ and it is defined as

$$(-\Delta_x)^{(\alpha/2)}\rho := \mathcal{F}^{-1}(|k|^\alpha \mathcal{F}(\rho)(k)), \qquad \alpha \in (0,2) \tag{1.5}$$

where $\mathcal{F}$ stands for the Fourier transform in the space variable. This definition is the one considered in Laplace-Fourier methods (Section 1.5.3) but it is useful to have the following equivalent definition in terms of a principal value integral:

$$(-\Delta)^{\alpha/2} f(x) := c_{N,\alpha}\mathrm{PV}\int_{\mathbb{R}^N} \frac{f(x)-f(y)}{\|x-y\|^{N+\alpha}}\,dy \qquad \alpha \in (0,2) \tag{1.6}$$

for $c_{N,\alpha}$ a constant depending only on the dimension and the exponent α. This is the definition used in the moments methods (Section 1.5.4) and it already shows that the fractional laplacian is a non-local operator.

Fractional and classical diffusion equations model transport phenomena. The case of the fractional diffusion is a particular instance of **anomalous transport** called **super-diffusions**. All these ideas are explained in Section 1.4.

For the Cauchy problem on this equation the reader is referred to [dPQRV12, MLP01] and for regularity results to [VdPQR13] and the references therein.

For some applications of fractional diffusive phenomena see [HBW+00] and [MS12].

1.2.2 Linear Boltzmann equation

We will use the linear Boltzmann equation to illustrate how fractional diffusion phenomena arises from this kinetic model and the different methods that exist to obtain this derivation. Notice that the equation presented here has some variants that appear in other contexts under different names, like radiative transport equation.

The linear Boltzmann equation is a kinetic equation giving the distribution $f = f(t,x,v)$

of particles undergoing free transport and collisions with the background (scattering), see figure 1.1 in page 21. The equation conserves the total mass and the scattering process makes the distribution of the particles relax to an equilibrium. In its general form, the linear Boltzmann equation is

$$\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) = L(f)(t,x,v) \qquad \text{in } (0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N \tag{1.7}$$
$$f(0,\cdot) = f^{in}(x,v) \qquad \text{in } \mathbb{R}^N\times\mathbb{R}^N \tag{1.8}$$

where L is a linear Boltzmann operator

$$\begin{aligned} L(f) &:= \int_{\mathbb{R}^N}\left[\sigma(v,v')f(v') - \sigma(v',v)f(v)\right]\, dv' \\ &= K(f) - \nu f \end{aligned} \tag{1.9}$$

for

$$K(f) = \int_{\mathbb{R}^N}\sigma(v,v')f(v')\,dv', \qquad \nu(v) = \int_{\mathbb{R}^N}\sigma(v',v)\,dv'.$$

The **collision kernel** or cross-section $\sigma \equiv \sigma(v',v)$ is non-negative and ν is called **collision frequency**. The collision kernel indicates the proportion of particles whose velocity changes from v to v'. The operator L is linear, defined in $L^1(\nu)$ and conservative, i.e., it preserves the total mass of the distribution:

$$\int_{\mathbb{R}^N} L(f)\,dv = 0 \qquad \text{for } f\in L^1(\nu)\ (^2)\,.$$

We say that a function M is an **equilibria** for L if $L(M)=0$. We will consider in this document linear Boltzmann equations with two types of equilibria: either, **Maxwellian** distributions, whose normalise form is

$$M(v) = \frac{1}{(2\pi)^{N/2}}\exp(-\|v\|^2/2) \tag{1.10}$$

or, we will consider heavy-tail functions under the following shape

$$M(v) \le c_0|v|^{-N-\alpha} \text{ for all } v\in\mathbb{R}^N, \quad M(v) = c_0|v|^{-N-\alpha} \text{ if } |v|\ge 1, \quad M\in C^1(\mathbb{R}^N) \tag{1.11}$$

with $s_0 > 0$ and $\alpha > 0$.

We will not study here the properties of this equation (existence of solutions, positivity, dissipation of energy (entropy), maximum principle,...) but they can be found in [AG13] and [Mou13]. We will look for a priori estimates in Section 1.5.1.

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$$f\in L^1(\nu) \text{ if (definition) } \int |f|\nu\,dv < \infty$$

1.2.3 Diffusive limit in the literature

1.2.3.1 Scaling

Considering the conceptual meaning of the functions $\rho = \rho(t,x)$ and $f = f(t,x,v)$ as the distribution of particles in space and phase-space, respectively, we have:

$$\rho(t,x) = \int_{\mathbb{R}^N} f(t,x,v)\, dv\,.$$

Given this relation one may think that solutions of the linear Boltzmann equation will give, by integration with respect to v, solutions of the diffusion equation. However, this first idea proves to be wrong; integrating the linear Boltzmann equation over the velocities and using the mass conservation we have that

$$\partial_t \rho + \nabla_x \cdot \int_{\mathbb{R}^N} v f(t,x,v)\, dv = 0$$

which is not the diffusion equation. Therefore, solutions to the Boltzmann equation do not give *directly* solutions to the diffusion equation.

We see next that we need to consider f_ε solution of the **rescaled** linear Boltzmann equation (for a well-chosen rescaling) to obtain in the limit $\rho = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^N} f_\varepsilon\, dv$ a solution of the (fractional) diffusion equation.

To perform the diffusive limit for the linear Boltzmann equation space and time are rescaled introducing the macroscopic variables

$$x' = \varepsilon x, \quad t' = \theta(\varepsilon) t \tag{1.12}$$

and the rescaled distribution function

$$f_\varepsilon(t', x', v) = f(t,x,v). \tag{1.13}$$

The rescaling means that a small variation in the macroscopic variables implies a big variation in the microscopic ones. Consider a point x_0' expressed in macroscopic variables and define x_1' as a small perturbation Δ of x_0', $x_1' := x_0' + \Delta$. Dividing by ε, we get

$$\frac{x_1'}{\varepsilon} = \frac{x_0'}{\varepsilon} + \frac{\Delta}{\varepsilon}$$

where $x_1 := x_1'/\varepsilon$ and $x_0 := x_0'/\varepsilon$ are expressed in microscopic variables. Therefore a small perturbation Δ in the macroscopic variables corresponds to a big perturbation Δ/ε in the microscopic variables.

The rescaling corresponds to making the mean free path (distance between collisions) small and making the time scale very large. In other words, to observe macroscopic phenomena from the equation we make a zoom out in space and speed up time. In the limit the diffusion equation is obtained. The diffusion approximation to kinetic equations has been studied in various works, see for example [BSS84], [BLP79], [DGP00], [LK74], [PS00].

Now f_ε satisfies the following rescaled linear Boltzmann equation (we have skipped the primes):

$$\begin{aligned} \theta(\varepsilon)\partial_t f_\varepsilon(t,x,v) + \varepsilon\, v\cdot\nabla_x f_\varepsilon(t,x,v) &= K(f_\varepsilon)(t,x,v) - \nu f_\varepsilon(t,x,v) && (1.14)\\ f_\varepsilon(0,\cdot) &= f^{in}(x,v) && (1.15) \end{aligned}$$

for $(t,x,v)\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$. Notice that the initial condition f^{in} is independent of ε.

1.2.3.2 A formal computation

A formal computation will help us understand how the diffusive limit comes out. Let us consider the classical diffusive limit.

Formally, as $\varepsilon\to 0$ we have that $L(f_\varepsilon)\to 0$ so the limit $f^0\in\ker L = \mathrm{span}\{M\}$, therefore $f^\varepsilon\to\rho(t,x)M(v)$.

For the classical diffusion limit $\theta(\varepsilon)=\varepsilon^2$. Integrating (1.14) on the velocities we have:

$$\partial_t \underbrace{\int f^\varepsilon\,dv}_{\rho^\varepsilon} + \underbrace{\varepsilon^{-1}\nabla_x\cdot\int v f^\varepsilon\,dv}_{=:\,I} = 0$$

and using the self-adjointness of L we obtain formally

$$\begin{aligned} I &:= \varepsilon^{-1}\nabla_x\cdot\int_{\mathbb{R}^N} L^{-1}(v)L(f^\varepsilon)\,dv\\ &= \nabla_x\cdot\int_{\mathbb{R}^N} L^{-1}(v)v\cdot\nabla_x f^\varepsilon\,dv + \varepsilon\nabla_x\cdot\int_{\mathbb{R}^N} L^{-1}(v)\partial_t f^\varepsilon\,dv\\ &\quad\to -\nabla_x\cdot(\kappa\nabla_x\rho(t,x)) \end{aligned}$$

where

$$\kappa = -\int_{\mathbb{R}^N} L^{-1}(v)\otimes v\,M\,dv.$$

Hence, we have obtained formally that $f_\varepsilon\to\rho(t,x)M(v)$ with ρ satisfying:

$$\partial_t\rho = \nabla_x\cdot(\kappa\nabla_x\rho(t,x))$$

as long as $\kappa<\infty$.

Since L is formed by a compact operator K and a multiplicative operator ν, κ can be formally approximated as a constant matrix:

$$\kappa \approx \int_{\mathbb{R}^N} \frac{\|v\|^2}{\nu} M \, dv \tag{1.16}$$

Observe that we require $\kappa < \infty$ in order to have a finite diffusion coefficient. Fractional diffusion phenomena takes place when this integral is not finite. This can happen because there is a degeneracy in ν (Section 1.2.3.4) and/or not all the moments in M are finite, for example, because M is a heavy-tail function (1.11) (Section 1.2.3.3).

Heavy-tail equilibria (also called **power laws**) are important distributions. We repeat here the account given in reference [MMM11] on different contexts where heavy-tail functions appear: on the velocity distribution in astrophysical plasmas [ST91], [MR94]; on dissipative collision mechanisms in granular gases [EB02], see also the review [Vil06]; on elastic collision mechanisms in mixture of gases with Maxwellian collision kernel [BG06]; in economy, through the Pareto distribution, see for example [New05], [Wri05].

In the case of a divergent diffusion coefficient κ, the multiscale analysis gives a fractional diffusion equation in the limit and has a fundamental change compared with the classical diffusion limit: we need a different time rescaling $t' := \varepsilon^{\gamma} t$ for some $\gamma > 0$.

1.2.3.3 Fractional diffusion due to heavy-tail equilibria

Next, we put together the main results in [MMM11]. The fractional diffusion is obtained as the limit of the linear Boltzmann equation and it is due to having a heavy-tail equilibria. We start with the assumptions on the equation

Assumption (A1). The cross-section σ is locally integrable on $\mathbb{R}^{2N}$, non-negative, and the collision frequency ν is locally integrable on $\mathbb{R}^N$ and satisfies

$$\nu(-v) = \nu(v) > 0 \quad \text{for all } v \in \mathbb{R}^N.$$

Assumption (A2). There exists a function $0 \leq M \in L^1(\nu)$ such that $\|v\|^2 \nu(v)^{-1} M$ is locally integrable and

$$\nu(v) M(v) = K(F)(v) = \int_{\mathbb{R}^N} \sigma(v, v') M(v') dv',$$

and

$$M(-v) = M(v) > 0 \text{ for all } v \in \mathbb{R}^N \text{ and } \int_{\mathbb{R}^N} M(v) \, dv = 1.$$

The existence of this function is a consequence of Krein-Rutman's theorem (see [MMM11] and the references therein).

Assumption (B1). There exists $\alpha > 0$ and a slowly varying function l (explained below) such that

$$M(v) = M_0(v) l(\|v\|),$$

where M_0 is such that

$$\|v\|^{\alpha+N} M_0(v) \to \kappa_0 \in (0, \infty) \text{ as } \|v\| \to \infty.$$

A slowly varying function is a measurable function $l : \mathbb{R}_+ \to \mathbb{R}$ such that

$$l(\lambda s) \sim l(s) \text{ as } s \to \infty \text{ for all } \lambda > 0.$$

Assumption (B2). There exists $\beta \in \mathbb{R}$ and a positive constant ν_0 such that

$$\|v\|^{-\beta} \nu(v) \to \nu_0 \text{ as } \|v\| \to \infty.$$

Assumption (B3). There exists a constant C such that

$$\int_{\mathbb{R}^N} M' \frac{\nu}{b} dv' + \left(\int_{\mathbb{R}^N} \frac{M'}{\nu'} \frac{b^2}{\nu^2} dv' \right)^{1/2} \leq C \text{ for all } v \in \mathbb{R}^N,$$

where $b = b(v, v') := \sigma(v, v') M^{-1}(v)$.

Theorem 1.1 (Fractional diffusion limit, Theorem 3.2 in [MMM11])**.** *Assume that Assumptions (A1-A2) and (B1-B2-B3) hold with $\alpha > 0$ and $\beta < \min\{\alpha; 2 - \alpha\}$. Define*

$$\gamma := \frac{\alpha - \beta}{1 - \beta}, \quad \textit{and} \quad \theta(\varepsilon) := l(\varepsilon^{-\frac{1}{1-\beta}}) \varepsilon^{\gamma}.$$

Observe that this implies that $\beta < 1$ and $\gamma < 2$. Assume furthermore that $f^{in} \in L^2(M^{-1})$ and let f_ε be the solution to (1.14), with that choice of θ and initial data f^{in}.

Then, f_ε converges in $L^\infty(0, T; L^2(\mathbb{R}^N \times \mathbb{R}^N))$-weak to a function $\rho(t, x) F(v)$ where $\rho(t, x)$ is the unique solution of the fractional diffusion equation of order γ:

$$\begin{aligned} \partial_t \rho + \kappa (-\Delta_x)^{\gamma/2} \rho &= 0 \quad \textit{in } (0, \infty) \times \mathbb{R}^N \\ \rho(0, \cdot) &= \rho^{in} \quad \textit{in } \mathbb{R}^N, \end{aligned}$$

with κ given by

$$\kappa = \frac{\kappa_0 \nu_0}{1 - \beta} \int_{\mathbb{R}^N} \frac{w_1^2}{\nu_0^2 + w_1^2} \frac{1}{\|w\|^{N+\gamma}} dw.$$

Theorem 1.2 (Classical diffusion limit with anomalous time scale, Theorem 3.4 in [MMM11])**.** *Assume that Assumptions (A1-A2) and (B1-B2-B3) hold with*

$$\alpha > 1 \quad \textit{and} \quad \beta = 2 - \alpha \quad \textit{(i.e. } \gamma = 2),$$

and l such that

$$l(r)\ln(r) \to +\infty \quad \text{as } r \to +\infty$$

(in particular, the second moment of F is infinite).

Then define

$$\theta(\varepsilon) = \varepsilon^2 l(\varepsilon^{-\frac{1}{1-\beta}} \ln(\varepsilon^{-1}).$$

Assume furthermore that $f^{in} \in L^2(M^{-1})$ and let f_ε be the solution of (1.14), with $\theta(\varepsilon)$ defined as above and initial data f^{in}.

Then, f_ε converges in $L^\infty(0,T; L^2(\mathbb{R}^N \times \mathbb{R}^N))$-weak to ρF where $\rho = \rho(t,x)$ is the unique solution to the standard diffusion equation

$$\partial_t \rho - \kappa \Delta_x \rho = 0$$

with κ given by

$$\kappa = \frac{\kappa_0 \nu_0}{(1-\beta} \lim_{\lambda \to 0} \frac{1}{\ln(\lambda^{-1})} \int_{\|w\| \geq \lambda} \frac{w_1^2}{\nu_0^2 + w_1^2} \frac{1}{\|w\|^{N+2}} dw.$$

Theorem 1.3 (Classical diffusion limit with classical time scale, Theorem 3.6 in [MMM11])**.** *Assume that Assumptions (A1-A2) hold as well as the following bounds*

$$\int_{\mathbb{R}^N} \left(\frac{\nu(v)}{b(v,v')} + \frac{\|v'\|^2}{\nu(v')} \right) F'\, dv' \leq C \quad \text{for all } v \in \mathbb{R}^N.$$

Assume, furthermore, that $f^{in} \in L^2(M^{-1})$ and let f_ε be the solution of (1.14), with $\theta(\varepsilon) = \varepsilon^2$ and initial data f^{in}.

Then f_ε converges in $L^\infty(0,T; L^2(\mathbb{R}^N \times \mathbb{R}^N))$-weak and in $L^2((0,T) \times \mathbb{R}^N \times \mathbb{R}^N))$-strong to a function ρF where $\rho = \rho(t,x)$ is the unique solution of the standard diffusion equation with diffusion constant

$$D = \int_{\mathbb{R}^N} (v \otimes \chi)\, dv \quad \text{where } L(\chi) = -vM.$$

1.2.3.4 Fractional diffusion due to a degeneracy of the collision frequency

This result in presented in [BAMP11]. Consider now that the Boltzmann operator is defined as:

$$L(f) = \nu(v)[\rho_\nu M(v) - f(v)] \tag{1.17}$$

where $\rho_\nu = \rho_\nu(t,x)$ is defined such that there is conservation of the total mass:

$$\int_{\mathbb{R}^N} L(f)\, dv = 0$$

and M is a given probability density.

Theorem 1.4 ([BAMP11]). *Suppose that ν is bounded and that for $\delta > 0$, $\beta > 0$, $\nu_0 > 0$*

$$\begin{aligned} \nu(v) &= \nu_0 \|v\|^{N+2+\beta}, \quad \textit{for } \|v\| \le \delta \\ M(v) &= M_0 > 0 \quad \textit{for } \|v\| \le \delta. \end{aligned}$$

where M is a probability distribution with

$$\int_{\mathbb{R}^N} vM(v)\,dv = 0$$

and

$$\int_{\|v\|\ge\delta} \frac{\|v\|^2}{\nu(v)} M(v)\,dv < \infty, \qquad \int_{\mathbb{R}^N} \nu(v)^2 M(v)\,dv < \infty.$$

Define

$$\gamma := 2 - \frac{\beta}{\beta + N + 1}.$$

Then the solution f_ε of

$$\begin{aligned} \varepsilon^\gamma \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon &= \nu(v)[\rho_{\nu,\varepsilon} M(v) - f_\varepsilon] \\ f(0,\cdot) &= f^{in} \end{aligned}$$

converges weakly in $L^2_{\nu M^{-1}}(\mathbb{R}^N \times \mathbb{R}^N \times (0,T))$ for all $T > 0$ to a function $\rho(x,t)M(v)$ where ρ solves for some $\kappa > 0$

$$\begin{cases} \partial_t \rho + \kappa(-\Delta)^{\gamma/2}\rho = 0 \\ \rho(x,0) = \int_{\mathbb{R}^N} f^{in}\,dv\,. \end{cases}$$

See Section 1.6.1 for an interpretation on why a degeneracy in the collision frequency gives fractional phenomena.

1.2.3.5 Methods

In the literature, we have the following methods showing the fractional diffusion limit for the linear Boltzmann equation.

- In [MMM11], explained in Section 1.2.3.3, the authors use a **Laplace-Fourier transform** method (Section 1.5.3). This method may not work for collision kernels depending on space or non linear operators.

- In [JKO09] the limit is obtained using **probabilistic methods**. In Section 1.5.5 we will interpret the linear Boltzmann equation as giving the evolution of the probability distribution of a particle undergoing a Markov process. We will show the fractional diffusive limit by proving the convergence to a stable Lévy process.

- In [Mel10] the limit is obtained via a **moments method** (Section 1.5.4). This allows to consider collision frequencies ν depending on the x variable (which was not possible with the Laplace-Fourier Transform method).
- In [BAMP11] the fractional diffusion is obtained using also the moments method but for any given equilibria considering that the collision frequency ν has a singularity near $v = 0$, see the next section.
- In [AMP10] a **fractional Hilbert expansion** is used (Section 1.5.2), this gives stronger convergence results than in the previous methods.

1.2.4 From atomic models to diffusion phenomena

We conclude this section by mentioning that there exists results on the derivation of the linear Boltzmann equation with gaussian equilibrium starting from atomic (deterministic) models, [DP99], [vBLLS80].

Recently, in [BGSR13] the heat equation was derived directly from a deterministic particle system without using the kinetic scale as an intermediate step.

No results of this type are known (to the best of our knowledge) for the fractional diffusive case or for linear Boltzmann-type equations with heavy-tail equilibria. The closest result in the literature can be found in [MT14], in this reference, the authors prove a 'superdiffusive' central limit theorem for a periodic Lorentz gas. The limit in the end is classical but the rescaling in the central limit theorem is anomalous.

1.3 Contributions in this part of the dissertation

1.3.1 Kinetic derivation of fractional Stokes and Stokes-Fourier systems (joint work with Dr. Sabine Hittmeir)

In Chapter 2 we extend the results presented in Section 1.2.3 to kinetic models preserving not only the total mass (0th moment) but also the first and second moment. We will consider a kinetic equation that resembles a linearised BGK equation. The BGK equation is the equation for a density distribution $f = f(t, x, v)$. In dimension 3 it reads [SR09]:

$$\partial_t f + v \cdot \nabla_x f = \mathcal{M}_f - f$$

where

$$\mathcal{M}_f := \frac{\rho(t,x)}{(2\pi T(t,x))^{3/2}} \exp\left(-\frac{|v - U(t,x)|^2}{2T(t,x)}\right)$$

where ρ, U, T are the density, the momentum and the energy respectively, defined as

$$\rho(t,x) = \int f(t,x,v)\, dv;\ \ \rho U(t,x) = \int v f(t,x,v)\, dv;\ \ \rho(|U|^2 + 3T)(t,x) = \int |v|^2 f(t,x,v)\, dv.$$

Observe that $\mathcal{M}_f$ is a Maxwellian distribution. In our linearised equation sometimes we will consider that we have a Maxwellian equilibria and sometimes it will be substituted by a heavy-tail distribution. In the limit we obtain what we call the fractional Stokes or Fourier-Stokes equation.

Next we explain the classical Stokes limit, done in [GL02], as a background for our result.

1.3.1.1 Classical hydrodynamic limit for the Stokes equation

The classical Stokes limit starts from the Boltzmann equation. Here we will give a sketch of this result.

The Boltzmann equation has the following shape

$$\partial_t h(t,x,v) + v \cdot \nabla_x h(t,x,v) = Q(h,h) \qquad \text{in } (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N$$

for Q a particular bilinear operator. We will not describe here this equation, since it will not be necessary for the future but the reader is referred to [SR09], [CIP94] and [Vil02] for more information.

The linearisation of the collision operator Q around the equilibrium distribution (Maxwellian) M is written as:

$$h(t,x,v) = M + \delta g(t,x,v) \text{ for some } \delta > 0.$$

Then the linearised term g fulfils the equation

$$\partial_t g + v \cdot \nabla_x g = -L(g) + \delta \mathcal{Q}(g,g) \tag{1.18}$$

where L is the linearised Boltzmann operator and $\mathcal{Q}$ is a modified bilinear operator [SR09].

To perform the hydrodynamic limit, the kinetic equation (1.18) is rescaled in space and time (and in relation with the Knudsen number). Different rescaling produce in the limit different macroscopic equations like Euler, Navier-Stokes, Stokes or Acoustic equations (see for example [Gol98], [SR09], [DMEL89], [Vil01]). In [GL02] (1.18) is rescaled as

$$\varepsilon \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon = -\frac{1}{\varepsilon} L(g_\varepsilon) + \frac{\delta_\varepsilon}{\varepsilon} \mathcal{Q}(g_\varepsilon, g_\varepsilon) \tag{1.19}$$

assuming that $\delta_\varepsilon/\varepsilon \to 0$ as $\varepsilon \to 0$. In this manner, it is proven in [GL02] that the nonlinear term in (1.19) vanishes in the limit and hence, the limiting behaviour is given by the linearised equation

$$\varepsilon^2 \partial_t g_\varepsilon + \varepsilon v \cdot \nabla_x g_\varepsilon = -L(g_\varepsilon) \tag{1.20}$$

where we have abused notation writing again g_ε.

Theorem 1.5 (Hydrodynamic limit for the Stokes equation, [GL02]). *g_ε in (1.20) satisfies*

$$g_\varepsilon \rightharpoonup g = \left(\underbrace{\int_{\mathbb{R}^N} g dv}_{=:\rho} + \underbrace{\int_{\mathbb{R}^N} vg\, dv}_{=:\overrightarrow{m}} \cdot v + \underbrace{\int_{\mathbb{R}^N} \left(\frac{1}{2}\|v\|^2 - \frac{N}{2}\right) g\, dv}_{=:\theta} \left(\frac{1}{2}\|v\|^2 - \frac{N}{2}\right) \right) M$$

where the convergence is in some particular space that we do not make precise here (since it is not relevant for our work). The macroscopic quantities $\rho, \overrightarrow{m}, \theta$ satisfy the

(i) incompressibility condition: $\nabla_x \cdot \overrightarrow{m} = 0$;

(ii) Boussinesq relation: $\rho + \theta = 0$;

(iii) Stokes equation:

$$\partial_t \overrightarrow{m} = \omega \Delta_x \overrightarrow{m} + \nabla_x p \tag{1.21}$$

$$\frac{N+2}{2}\partial_t \theta = \kappa \Delta_x \theta \tag{1.22}$$

for $\omega, \kappa > 0$ and $p = p(t,x)$ is a pressure term. The equations have some particular initial data (that we omit here, see [GL02]) fulfilling the incompressibility and Boussinesq relations.

Remark 1.6. The following two properties of the linearised Boltzmann equation are fundamental to perform the limit in Theorem (1.5) presented in [GL02]:

(i) The conservation of the moments: for $\psi(v) = 1, v, \|v\|^2$, it holds

$$\int_{\mathbb{R}^N} \psi(v) L(g) dv = 0.$$

(ii) The Kernel(L)=span$\{M, vM, \|v\|^2 M\}$; g corresponds to the limit of the projection of g_ε onto the Kernel of L in the weighted $L^2(M^{-1}dv)$ space.

We will see all this in more detail in Section 2.1.1.

1.3.1.2 Our contribution

Our goal is to find a result similar to the one in the preceding section (1.21)-(1.22) but having fractional diffusion equations for the 0th, 1st and 2nd moment. The structure of the equations and formulation is similar to the one for the classical Stokes limit but the proof is completely different, based on the moments methods [Mel10] explained in Section 1.5.4. The main results obtained are Theorems 2.1 (page 88) and 2.2 (page 89).

1.3.2 Anomalous transport in FPU-β chains (joint work with Professor Antoine Mellet)

In Chapter 3, we investigate some aspects of the transport of energy in one dimensional chains of oscillators. The goal is to derive Fourier's law, which is at the core of the heat equation and states that the heat flux $\vec{j}$ behaves as

$$\vec{j} = -\kappa \nabla_x T \tag{1.23}$$

where T is the temperature and κ is a positive constant that may depend on the temperature itself. This law has been observed experimentally, but to this day there is not a complete and full mathematical justification describing how it arises from the atomic laws of the solid. Nevertheless, many mathematical works have been devoted in this direction and major progress has been achieved, see [BLRB00] for a review of this very challenging problem.

At the microscopic level, solids can be modeled as lattices, were each node represents an atom. For insulating crystals, where heat is transported by lattice vibrations (see [LLP03]), one possible approach to derive Fourier's law relies on the introduction of the Boltzmann phonon equation, a kinetic equation that can play the role of an intermediate step between the microscopic atomic level and the macroscopic scale. It is this approach, first suggested by Peierls [Pei29], that we try to make rigorous in this paper in a very particular setting.

The particular framework we are considering was made popular by a famous numerical experiment performed by Fermi, Pasta and Ulam in the 1950's at Los Alamos National Laboratories. The goal of their experiment was to investigate numerically the dynamic (and relaxation toward equilibrium) of the simplest model for a crystal: a chain of oscillators coupled to their nearest neighbors by non-linear forces described by an Hamiltonian of the form

$$H = \frac{1}{2} \sum_{i \in \mathbb{Z}} \left[p_i^2 + V(q_{i+1} - q_i) \right].$$

When V is purely harmonic, the system has quasi-periodic solutions and does not relax to an equilibrium (see [BI05]). Fermi, Pasta and Ulam thus considered the next two simplest cases by adding a cubic potential $V(r) = r^2 + \alpha r^3$ (this model is now referred to as the FPU-α chain) or a quartic potential $V(r) = r^2 + \beta r^4$ (the FPU-β chain).

These models have been widely studied since that original experiment (see Lepri, Livi, Politi [LLP05] for a recent review of the work devoted to these models). Our goal in this paper is to derive Fourier's law for the FPU-β chain (we will see later why we do not consider the FPU-α chain). To achieve this, we rely on an idea of Peierls [Pei29], who describes lattice vibrations, responsible for heat transport, as an interacting gas of phonons whose density distribution function (denoted W below) solves a Boltzmann phonon equation (also known as Peierls equation in this context). The mathematical derivation of this Boltzmann phonon equation starting from the microscopic equations for the motion of the atoms (Hamiltonian dynamic) has written formally by H. Spohn in [Spo06b]. We will thus not focus on this step, though we will spend some time in this paper discussing the results of [Spo06b]. Our focus instead will be on the rigorous derivation of Fourier's law from the Boltzmann phonon equation. The most remarkable aspect of our result is that we will not recover (1.23), but instead a non-local Fourier law corresponding to an anomalous diffusion equation (in place of the usual heat equation). This was not unexpected, since anomalous heat diffusion phenomena in the FPU-β framework have actually been observed numerically in dimension one and two (while normal diffusion is observed in the three dimensional case), see in particular [SMY$^+$00], [LLP03], [LLP05], and also [AK01] for a study at the level of the kinetic equation. In fact, by using Peierls equation, it has previously been proved that the energy current correlation has a slow decay in time as $t^{-3/5}$ indicating anomalous diffusive behavior (see [Per03, LS08]).

Let us now describe our main result. As mentioned above, the starting point of our analysis is the Boltzmann phonon equation given by:

$$\partial_t W + \omega'(k)\partial_x W = C(W)$$

where the unknown $W(t, x, k)$ is a function of the time $t \geq 0$, the position $x \in \mathbb{R}$ and the wave vector $k \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$. This function is introduced in [Spo06b] as the Wigner transform of the displacement field of the atoms, but it can be interpreted as a density distribution function for a gas of interacting phonons (describing the chain vibrations). The function $\omega(k)$ is the dispersion relation for the lattice and the operator C describes the interactions between the phonons.

We will discuss in Sections 3.1 and 3.2 the particular form of ω and C corresponding to our microscopic models. For the FPU-β chain, the operator C will be the so-called *four phonon collision operator*, which is an integral operator of Boltzmann type but cubic instead of quadratic (see (3.18)).

As explained above, our goal is to derive a macroscopic equation for the temperature. This is, at least in spirit, similar to the derivation of Navier-Stokes equations from the Boltzmann equation for diluted gas (see [BGL91] and references therein). We will consider a perturbation of a thermodynamical equilibrium $\overline{W}(k) = \frac{\overline{T}}{\omega(k)}$ (note that the temperature is classically defined by the relation $E = k_B T$ where $E = \int_{\mathbb{T}} \omega(k) W(k)\, dk$ and k_B denotes Boltzmann's constant - here, we choose temperature units so that $k_B = 1$):

$$W^\varepsilon(t, x, k) = \overline{W}(k)(1 + \varepsilon f^\varepsilon(t, x, k)).$$

The function f^ε then solves

$$\partial_t f^\varepsilon + \omega'(k)\partial_x f^\varepsilon = L(f^\varepsilon) + \mathcal{O}(\varepsilon)$$

where L is the linearized operator

$$L(f) = \frac{1}{\overline{W}} DC(\overline{W})(\overline{W} f).$$

As usual a macroscopic equation is derived after an appropriate rescaling of the time and space variable. More precisely, we will show (see Theorem 3.4) that the solution of

$$\varepsilon^{\frac{8}{5}}\partial_t f^\varepsilon + \varepsilon\omega'(k)\partial_x f^\varepsilon = L(f^\varepsilon)$$

converges to a function $T(t,x)$ solution of

$$\partial_t T + \frac{\kappa}{\overline{T}^{6/5}}(-\Delta)^{\frac{4}{5}} T = 0$$

thus giving the anomalous Fourier's law (of order $3/5$)

$$\vec{j} = -\kappa(\overline{T})\nabla(-\Delta)^{-\frac{1}{5}} T.$$

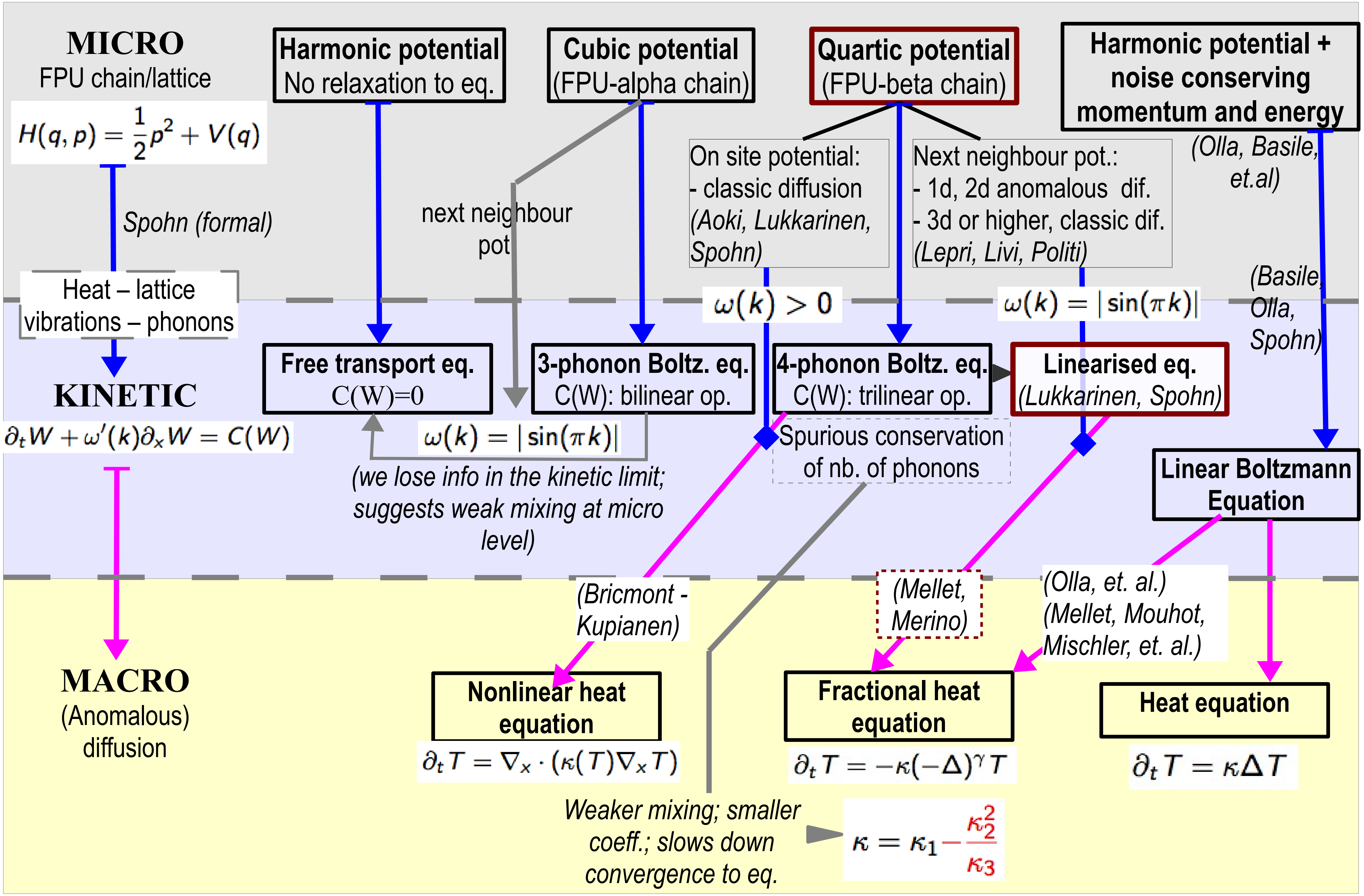
The derivation of such a fractional diffusion equation from a kinetic equation is now classical (see Section 1.2.3 and references in Section 1.2.3.5). As in previous results (see in particular [BAMP11]), the order of the limiting diffusion process is determined by the degeneracy of the collision frequency of the operator L. Our work is thus greatly indebted to the work of J. Lukkarinen and H. Spohn [LS08] who carefully study the properties of the operator L and show in particular that the collision frequency behaves as $|k|^{5/3}$ as $k \to 0$.

The main novelty here, compared with the results mentioned above, is the fact that the kernel of the collision operator L is 2 dimensional. The reason for that will be discussed in the next sections and it appears to be a mathematical artifact rather than being related to some physical phenomenon. It does, however, indicate some weakness in the mixing properties of the collision process (this will be even more obvious for the FPU-α chain, for which the collision operator vanishes altogether). And while the macroscopic behavior of f^ε is completely determined by the function $T(t,x)$, the other component of the projection of f^ε onto the kernel of L will play a role in reducing the value of the diffusion coefficient κ.

We point out that we will not attempt here to derive a nonlinear Fourier law by working with the nonlinear operator C (rather than the linearized operator L). Such a derivation is developed in [BK08] by Bricmont and Kupiainen, but under assumptions that ensure that regular diffusion, rather than anomalous diffusion, takes place (non degeneracy of the collision frequency).

To conclude this section, we mention that diffusive and superdiffusive heat transport has also been derived for FPU-type chains in a different mathematical setting using a probabilistic approach: in this setting the hamiltonian dynamics of the microscopic system are considered to have only an harmonic potential and the dynamics are perturbed by a stochastic noise conserving momentum and energy (see [BBO06], [BBO09], [BOS10] and the review paper [Oll09].)

For a summary on the previous works, their relations and the place were our result takes place inside this area of research see scheme in next page.



MICRO
FPU chain/lattice
$H(q,p) = \frac{1}{2}p^2 + V(q)$
Spohn (formal)
Heat – lattice vibrations – phonons
KINETIC
$\partial_t W + \omega'(k)\partial_x W = C(W)$
MACRO
(Anomalous) diffusion
Harmonic potential
No relaxation to eq.
Cubic potential
(FPU-alpha chain)
Quartic potential
(FPU-beta chain)
Harmonic potential + noise conserving momentum and energy
(Olla, Basile, et.al)
next neighbour pot
On site potential:
- classic diffusion
(Aoki, Lukkarinen, Spohn)
Next neighbour pot.:
- 1d, 2d anomalous dif.
- 3d or higher, classic dif.
(Lepri, Livi, Politi)
$\omega(k) > 0$
$\omega(k) = |\sin(\pi k)|$
(Basile, Olla, Spohn)
Free transport eq.
C(W)=0
3-phonon Boltz. eq.
C(W): bilinear op.
4-phonon Boltz. eq.
C(W): trilinear op.
Linearised eq.
(Lukkarinen, Spohn)
$\omega(k) = |\sin(\pi k)|$
Spurious conservation of nb. of phonons
(we lose info in the kinetic limit; suggests weak mixing at micro level)
Linear Boltzmann Equation
(Bricmont - Kupianen)
(Mellet, Merino)
(Olla, et. al.)
(Mellet, Mouhot, Mischler, et. al.)
Nonlinear heat equation
$\partial_t T = \nabla_x \cdot (\kappa(T)\nabla_x T)$
Fractional heat equation
$\partial_t T = -\kappa(-\Delta)^\gamma T$
Heat equation
$\partial_t T = \kappa \Delta T$
Weaker mixing; smaller coeff.; slows down convergence to eq.
$\kappa = \kappa_1 - \frac{\kappa_2^2}{\kappa_3}$

1.4 Anomalous transport: super-diffusions

Fractional diffusion equations model super-diffusion phenomena. In super-diffusions, particles distributed according to ρ are transported spreading 'faster than any diffusion'. Explaining the meaning of this will be the goal of this section as well as studying the properties of super-diffusions and compare them with the classical diffusion.

One can compare the 'speed of spreading' through the Mean Square Displacement (MSD) of the particles which we explain in the next section. In classical diffusion the Mean Square Displacement is linear in time, in fractional phenomena this is not the case, that is why it is called **anomalous transport**.

1.4.1 Rescaling invariance and self-similarity

The Mean Square Displacement (MSD) of a particle $X(t)$ whose position over time is distributed according to the density probability $\rho = \rho(t,x)$ is given by

$$\langle X(t)^2 \rangle := \int_0^t \int_{\mathbb{R}^N} \|X(s)\|^2 \rho(s,x) ds dx.$$

In the case of the diffusion equation it holds

$$\langle \|x\|^2 \rangle = 2Dt. \tag{1.24}$$

where $D > 0$ is the diffusive constant.

The relation between space and time established by the MSD is of paramount importance since it implies a **self-similarity** of the trajectories of the particles. We explain this next. Observe that if we multiply this relation by ε^2 we have that

$$\langle \|\varepsilon x\|^2 \rangle = 2D\varepsilon^2 t$$

and defining new variables $x' = \varepsilon x$ and $t' = \varepsilon^2 t$ we get again the relation

$$\langle \|x'\|^2 \rangle = 2Dt'.$$

Consequently, if space and time are rescaled by a factor of ε and ε^2 respectively, we *observe* the same trajectories, i.e., the trajectories are self-similar.

The MSD gives a good indication of the rescaling chosen in the diffusion limit since in most cases it is the one keeping the **scaling invariance** of the diffusion equation, i.e.,

$$(x,t) \mapsto (\varepsilon x, \varepsilon^2 t);$$

which corresponds to the self-similarity of the trajectories and keeping the speed of diffusion, $2D$, constant.

For the fractional diffusion equation something analogous happens, the Mean Square Displacement in this case is not well defined since the variance is not finite (the density behaves asymptotically as a power law), however it holds that (see [MK00])

$$\langle \|x\|^\delta \rangle \sim t^{\frac{\delta}{\alpha}}, \quad 0 < \delta < \alpha < 2.$$

Rescaling space and time as

$$(x,t) \mapsto (\varepsilon x, \varepsilon^\alpha t)$$

the previous expression stays invariant. This is the chosen rescaling in fractional diffusive limits.

As we said in the introduction one can compare the 'speed of spreading' through the Mean Square Displacement (MSD) of the particles. In classical diffusion it grows linearly in time, but in fractional diffusion phenomena it diverges because it scales faster, hence the spreading is faster.

1.4.2 Diffusion vs fractional diffusion equation

In the diffusion or heat equation

$$\partial_t \rho(t,x) = D\Delta_x \rho(t,x), \qquad D > 0$$

$\rho(t,x)$ is a probability density giving the distribution of the particles in space at each time; it describes *transport* of particles.

The diffusion equation is obtained by the combination of two rules:

(i) *Conservation of the total mass* (number of particles); expressed as

$$\partial_t \rho(t,x) = -\nabla_x \cdot \vec{j}(t,x).$$

where $\vec{j}$ is the flux of particles (rate at which particles cross an infinitesimal surface). Mathematically, the conservation of mass is a consequence of the divergence theorem.

(ii) *Fick's law* (or **Fourier law** for the temperature). This is a law observed experimentally:

$$\vec{j}(t,x) = -D\nabla_x \rho(t,x),$$

meaning that particles move linearly from places of high concentration to places of low concentration following the gradient. D is a positive constant called 'diffusivity constant' and it is proportional to the speed at which particles spread (diffuse); recall the Mean Square Displacement (MSD) in (1.24).

A first intuition is that, since D dictates how fast the diffusion is taking place, if we want particles to spread faster than any diffusion, we need D to increase to infinity; meaning that the mean square displacement goes to infinity; hence, the variance will not be finite. This corresponds to having the constant κ in (1.16) equal to infinity.

In super-diffusive phenomena for $\alpha \in (1,2)$ the total number of particles is conserved but Fourier law is violated; it will need to be replaced as we explain in Section 1.4.3.

1.4.2.1 Gaussian vs stable distributions

The fundamental solution of the diffusion equation is the gaussian (or Maxwellian) distribution [Eva98], i.e., given initial data

$$\rho(0,x) = \delta(x - x_0)$$

its solution is

$$\rho(t,x) = \frac{1}{\sqrt{(2\pi D t)^N}} \exp\left(-\frac{\|x - x_0\|^2}{2Dt}\right).$$

Stable distribution, and not Gaussians, are the fundamental solution of the fractional diffusion equation (1.3) [MLP01]. Their density behaves asymptotically as a power law [FN99]. Stable distributions are defined in Section 1.5.5.2.

In contrast with the gaussian distribution, stable distributions do not have all its moments finite. Intuitively, this is coherent with the idea of super-diffusions: suppose that the variance of the fundamental solution is not finite, then particles are more likely to be further from their starting point x_0 than they are to be with the Gaussian distribution. Consequently, particles spread faster than in a normal diffusion.

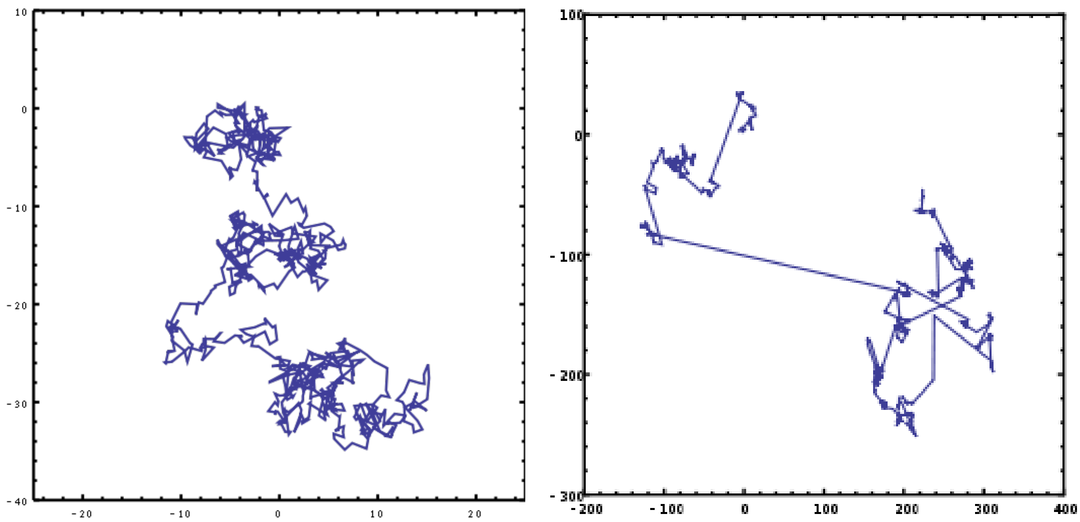
Stable Lévy processes. To the solutions of the diffusion and fractional diffusion equations one can associate stochastic processes called, stable Lévy processes (explained in Section 1.5.5.2). In the case of the classical diffusion, it corresponds to a 2-stable Lévy process, which is **Brownian motion**. This is a gaussian process and its law is determined by the density solution of the diffusion equation. Likewise, for a fractional diffusion equation of order α, there is associated an α-stable Lévy process, $\alpha \in (0,2)$.

This relation between diffusion equations and stable Lévy processes is explained in more detail in [RW00, Ber] for the classical case and in [MS12, Section 4.5] for the fractional case.

All these processes have in common the self-similarity of their trajectories, i.e., if $(L_t^{(\alpha)})_{t\geq 0}$ is an α-stable Lévy process then

$$(L_t^{(\alpha)})_{t\geq 0} \sim (\varepsilon L_{\varepsilon^{-\alpha} t}^{(\alpha)})_{t\geq 0}$$

in law [Sat99]. The main difference between $\alpha = 2$ and $\alpha \in (0, 2)$ is that Brownian motion is continuous almost everywhere, while the rest of stable processes are discontinuous; a particle makes a sequence of small jumps and from time to time it makes a large jump. This large jumps correspond to the idea that the particle is super-diffusing.



(a) Brownian motion.

(b) Lévy process (jump points joined with a line).

Figure 1.2: Source: by UserPAR, via Wikimedia Commons.

Link with the Central Limit Theorem. At the very basis, the hydrodynamic limit is a manifestation of the Central Limit Theorem; given a sequence $\{X_1, X_2, \dots\}$ of i.i.d random variables with expectation μ and variance $\sigma^2 < \infty$ then

$$\sqrt{n}\left(\frac{1}{n}\sum_{k=0}^{n} X_k - \mu\right) \xrightarrow{d} \mathcal{N}(0, \sigma^2) \quad \text{as } n \to \infty$$

where d indicates convergence in distribution

Note that:

- the result is universal; it does not depend on the particular distribution of the random variables; they always converge to a Gaussian distribution;
- it requires finite variance.

Classical diffusive limits extend the idea of the Central Limit Theorem (or Generalised Central Limit Theorems in the case of fractional diffusions [MS12, Section 4.2]). The analogous idea in the case of stochastic processes is that, Brownian motion and Lévy processes are obtained as limits of Random Walks (Donsker's Theorem, [RW00]) and Continuous Time Random Walks [MS12, Section 4.4-4.5], respectively.

1.4.3 Fractional derivatives, heavy-tailed functions and non-locality

The explanation given here comes from reference [MS12].

Let us focus in the 1-dimensional case. A definition of fractional derivative in dimension 1 is

$$\frac{d^\alpha}{dx^\alpha} f(x) = \lim_{h \to 0} \frac{\Delta^\alpha f(x)}{h^\alpha}, \qquad \alpha > 0$$

where

$$\Delta^\alpha f(x) := \sum_{k=0}^{\infty} \binom{\alpha}{k} (-1)^k f(x - kh) \tag{1.25}$$

and

$$\binom{\alpha}{k} := \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha - k + 1)}.$$

Note that if $\alpha = n$ this corresponds to the classical derivatives where

$$\Delta^n f(x) := \sum_{k=0}^{n} \binom{n}{k} (-1)^k f(x - kh).$$

The combinatorial number is generalised using the Gamma function $\Gamma(n+1) = n!$.

Non-locality. To compute expression (1.25), the information over the entire space ($x + \delta h$ for any $\delta \in \mathbb{N}$) is required. This implies that the fractional derivative ($\alpha \neq \mathbb{N}$) is **non-local**. Whereas the classical derivative is a local operator because the series defining $\Delta^n f$ is actually just a finite sum.

Heavy-tail functions. Expression (1.25) is a discrete convolution of f with the so called Grunwald weights, which have the following asymptotic property [MS12]:

$$w_j := (-1)^k \binom{\alpha}{k} \sim \frac{-\alpha}{\Gamma(1-\alpha)} k^{-\alpha-1} \qquad \text{as } k \to \infty.$$

This is related to the appearance of the heavy-tail function. Moreover, for $\alpha \in (1,2)$, $w_j > 0$ for all $j \geq 2$.

For more details on this and other alternative definitions of the fractional derivative, check reference [MS12].

1.4.3.1 Anomalous Fourier law

We focus on the case when $\alpha \in (1,2)$. Fourier law

$$\vec{j} = -D \nabla_x \rho(t,x)$$

is replaced by the fractional Fourier law

$$\vec{j} = -D\nabla^{\alpha-1}\rho, \quad \alpha \in (1,2) \text{ where } \nabla^{\alpha-1} = \left(\partial_1^{\alpha-1}, \ldots \partial_N^{\alpha-1}\right)$$

to give, combined with the conservation of the total mass ($\partial_t \rho = -\nabla \cdot \vec{j}$), the fractional diffusion equation

$$\partial_t \rho = -D(-\Delta_x)^{\alpha/2}\rho.$$

What does this equation mean? Considering the Grunwald weights w_j for $\alpha \in (1,2)$ ($w_j > 0$ for $j \geq 2$), the discrete convolution in expression (1.25) means that particles are transported *over the entire space* in a heavy-tailed way (see figure 1.3). In contrast, in the normal diffusion particles are transported into a neighbourhood. Therefore, the fact that particles spread faster in fractional diffusion than in a normal one is a consequence of the non-locality of the operator (though, of course, not all non-local operators have this effect).

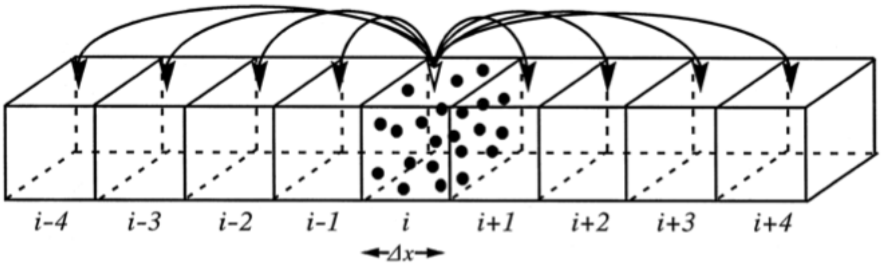


Figure 1.3: With the fractional Fourier law, particles spread over the entire space in a heavy tail way (convolution). The figure is from reference [MS12].

For a discussion on the qualitative difference between $\alpha < 1$ and $\alpha \in [1,2)$, the reader is referred to [UZ99, Section 12.3].

1.5 Methods in the diffusive limit

1.5.1 Toy example

To explain the existing methods for the (fractional) diffusive limit, we will consider a simple case of the linear Boltzmann equation (1.7) in which $\sigma(v,v') = M(v)$, where $M : \mathbb{R}^N \to \mathbb{R}$ has the following properties:

$$M = M(v) > 0 \text{ a.e. in } \mathbb{R}^N \tag{1.26}$$

$$\int_{\mathbb{R}^N} M\,dv = 1 \tag{1.27}$$

$$L(M) = 0, \tag{1.28}$$

$$M(v) = M(-v) \text{ a.e. in } \mathbb{R}^N. \tag{1.29}$$

M is the equilibrium distribution and we will consider either that it is Maxwellian (1.10) or a heavy-tail (1.11).

Under these assumptions the linear Boltzmann equation (1.7) simplifies into:

$$\partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + f(t,x,v) = M(v)\int_{\mathbb{R}^N} f(t,x,v')\,dv' \tag{1.30}$$

$$f(0,\cdot) = f^{in} \tag{1.31}$$

for $(t,x,v)\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$.

1.5.1.1 Diffusive limit

We give next an example of fractional diffusion limit for this simpler case.

Theorem 1.7 (Fractional diffusion limit for the linear Boltzmann equation, [MMM11])**.** *Let M be a function fulfilling (1.11) with $\alpha\in(0,2)$. Assume also that M satisfies (1.26), (1.27), (1.28), (1.29). Let $f^{in}\in L^2(M^{-1}dv)$ and let f_ε be the solution of the rescaled linear Boltzmann equation*

$$\varepsilon^\alpha\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon = M\int_{\mathbb{R}^N} f_\varepsilon\,dv - f_\varepsilon. \tag{1.32}$$

with initial data $f(0,\cdot)=f^{in}$ and for $(t,x,v)\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$. Then, when $\varepsilon\to0$, f_ε converges in $L^\infty(0,T;L^2(\mathbb{R}^N\times\mathbb{R}^N))$-weak to ρM with $\rho=\rho(t,x)$ the unique solution to the fractional diffusion equation

$$\begin{aligned} &\partial_t\rho + \kappa(-\Delta_x)^{\alpha/2}\rho = 0 && \text{in } (0,\infty)\times\mathbb{R}^N\\ &\rho(0,\cdot) = \int_{\mathbb{R}^N} f^{in}\,dv && \text{in } \mathbb{R}^N \end{aligned}$$

for some constant $\kappa>0$.

In the following section we will present these methods on this toy example. We will start first by studying its classical counterpart:

Theorem 1.8 (Classical diffusion limit)**.** *Let M be a function satisfying (1.26), (1.27), (1.28), (1.29) and also*

$$\int_{\mathbb{R}^N}\|v\|^2 M(v)\,dv<\infty.$$

Let $f^{in}\in L^2(M^{-1}dv)$ and let f_ε be the solution of the rescaled linear Boltzmann equation

$$\varepsilon^2\partial_t f_\varepsilon + \varepsilon v\cdot\nabla_x f_\varepsilon = M\int_{\mathbb{R}^N} f_\varepsilon\,dv - f_\varepsilon. \tag{1.33}$$

with initial data $f(0,\cdot)=f^{in}$ for $(t,x,v)\in(0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N$. Then, when $\varepsilon\to0$, f_ε converges in $L^\infty(0,T;L^2(\mathbb{R}^N\times\mathbb{R}^N))$-weak to ρM with $\rho=\rho(t,x)$ the unique solution to the diffusion *equation*

$$\begin{aligned} \partial_t \rho - \nabla_x \cdot (D \nabla_x \rho) &= 0 && \text{in } (0,\infty) \times \mathbb{R}^N \\ \rho(0,\cdot) &= \int_{\mathbb{R}^N} f(0,\cdot,v)\, dv && \text{in } \mathbb{R}^N \end{aligned}$$

for

$$D = \int_{\mathbb{R}^N} M(v)\, v \otimes v\, dv < \infty.$$

For some results on the classical diffusive limit for the linear Boltzmann equation see [AG13].

1.5.1.2 A priori estimates

To prove the diffusive limit we will need the following:

Proposition 1.9 (A priori estimates)**.** *Consider the rescaled linear Boltzmann equation:*

$$\begin{aligned} \varepsilon^\alpha \partial_t f_\varepsilon + \varepsilon v \cdot \nabla_x f_\varepsilon + f_\varepsilon - M \int_{\mathbb{R}^N} f_\varepsilon\, dv &= 0 && \text{in } (0,\infty) \times \mathbb{R}^N \times \mathbb{R}^N \\ f_\varepsilon(0,\cdot) &= f^{in} && \text{in } \mathbb{R}^N \times \mathbb{R}^N\,. \end{aligned}$$

Then, we have the two following estimates

$$\sup_{t\geq 0} \int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon(t,\cdot))^2}{M}\, dvdx \leq \int_{\mathbb{R}^{2N}} \frac{(f^{in})^2}{M}\, dvdx = \|f^{in}\|^2_{L^2(M^{-1})} \tag{1.34}$$

and

$$\int_0^\infty \int_{\mathbb{R}^{2N}} [f_\varepsilon - \rho_\varepsilon M]^2 M^{-1}\, dvdxdt \leq \frac{\varepsilon^\alpha}{2} \|f^{in}\|^2_{L^2(M^{-1})}\,. \tag{1.35}$$

Also, $\rho_\varepsilon(t,x)$, as well as $L(f_\varepsilon)$, are well defined a.e., and

$$\sup_{t\geq 0} \int_{\mathbb{R}^N} \rho_\varepsilon(t,\cdot)^2\, dx \leq \|f^{in}\|^2_{L^2(M^{-1})}\,. \tag{1.36}$$

Proof. We start proving first the following

Lemma 1.10.

$$\varepsilon^\alpha \frac{d}{dt} \int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon)^2}{2} M^{-1} dvdx = -\int_{\mathbb{R}^{2N}} [f_\varepsilon - \rho_\varepsilon M]^2 M^{-1} dvdx\,.$$

Proof of Lemma 1.10.

$$\begin{aligned}
\varepsilon^\alpha \frac{d}{dt}\int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon)^2}{2} M^{-1} dv dx &\overset{(A.1)}{=} \int_{\mathbb{R}^{2N}} (\rho_\varepsilon M - f_\varepsilon) f_\varepsilon M^{-1}\, dv dx \\
&\overset{(A.2)}{=} \int_{\mathbb{R}^{2N}} [(\rho_\varepsilon)^2 M - (f_\varepsilon)^2 M^{-1}]\, dv dx \\
&\overset{(A.3)}{=} -\int_{\mathbb{R}^{2N}} [f_\varepsilon - \rho_\varepsilon M]^2 M^{-1} dv dx\,.
\end{aligned}$$

Equality (A.1) is justified by the following:

$$\begin{aligned}
\varepsilon^\alpha \frac{d}{dt}\int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon)^2}{2} M^{-1} dv dx &= 2\varepsilon^\alpha \int_{\mathbb{R}^{2N}} f_\varepsilon\,(\partial_t f_\varepsilon)\, M^{-1}\, dv dx \\
&= -2\int_{\mathbb{R}^{2N}} \varepsilon(v\cdot\nabla_x f_\varepsilon) f_\varepsilon\, M^{-1}\, dv dx \\
&\quad + \int_{\mathbb{R}^{2N}} (\rho_\varepsilon M - f_\varepsilon) f_\varepsilon M^{-1}\, dv dx
\end{aligned}$$

and

$$2\int_{\mathbb{R}^{2N}} (v\cdot\nabla_x f_\varepsilon) f_\varepsilon\, M^{-1}\, dv dx = \int_{\mathbb{R}^{2N}} v\cdot\nabla_x f_\varepsilon^2\, M^{-1}\, dv dx = 0$$

where the last equality is due to the divergence theorem. Equality (A.2) is proven by rewriting $(\rho_\varepsilon M - f_\varepsilon) f_\varepsilon M^{-1} = \rho_\varepsilon f_\varepsilon - (f_\varepsilon)^2 M^{-1}$ and computing

$$\begin{aligned}
\int_{\mathbb{R}^{2N}} \rho_\varepsilon f_\varepsilon\, dv dx &= \int_{\mathbb{R}^N} \rho_\varepsilon \left(\int_{\mathbb{R}^N} f_\varepsilon\, dv\right) dx = \int_{\mathbb{R}^N} (\rho_\varepsilon)^2\, dx \\
&= \int_{\mathbb{R}^N} (\rho_\varepsilon)^2\, dx \int_{\mathbb{R}^N} M\, dv = \int_{\mathbb{R}^{2N}} (\rho_\varepsilon)^2 M\, dx dv\,.
\end{aligned}$$

Finally, we prove equality (A.3). Note $a := \rho_\varepsilon$, $b := M$, $c := f_\varepsilon$, then

$$\begin{aligned}
(\rho_\varepsilon)^2 M - (f_\varepsilon)^2 M^{-1} &= a^2 b - c^2 b^{-1} \\
(f_\varepsilon - \rho_\varepsilon M)^2 M^{-1} &= (c - ab)^2 b^{-1} = -c^2 b^{-1} + a^2 b - 2a^2 b + 2ac\,.
\end{aligned}$$

We just need to check that

$$\int_{\mathbb{R}^{2N}} \left(-2a^2 b + 2ac\right)\, dx dv = 0,$$

which is true:

$$\int_{\mathbb{R}^{2N}} (\rho_\varepsilon)^2 M\, dx dv - \int_{\mathbb{R}^{2N}} \rho_\varepsilon f_\varepsilon = \int_{\mathbb{R}^N} (\rho_\varepsilon)^2 \underbrace{\left(\int_{\mathbb{R}^N} M\, dv\right)}_{=1} dx - \int_{\mathbb{R}^N} \rho_\varepsilon \underbrace{\left(\int_{\mathbb{R}^N} f_\varepsilon\, dv\right)}_{=\rho_\varepsilon} dx = 0\,.$$

□

By Lemma 1.10 we know that

$$\int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon)^2}{2} M^{-1} dv dx$$

is a decreasing function on the time variable, hence we obtain the first estimate (1.34). The second estimate (1.35) is obtained by integrating in time the expression in Lemma 1.10:

$$\varepsilon^\alpha \int_0^\tau \frac{d}{dt} \int_{\mathbb{R}^{2N}} \frac{(f_\varepsilon)^2}{2} M^{-1} dv dx dt = \frac{\varepsilon^\alpha}{2} \left(\int_{\mathbb{R}^{2N}} f_\varepsilon(\tau,\cdot)^2 M^{-1} dv dx - \int_{\mathbb{R}^{2N}} f_\varepsilon(0,\cdot)^2 M^{-1} dv dx \right)$$

$$= \frac{\varepsilon^\alpha}{2} \left(\|f_\varepsilon(\tau,\cdot)\|^2_{L^2(M^{-1})} - \|f_\varepsilon^{in}\|^2_{L^2(M^{-1})} \right) \le \frac{\varepsilon^\alpha}{2} \|f_\varepsilon(\tau,\cdot)\|^2_{L^2(M^{-1})} \le \frac{\varepsilon^\alpha}{2} \|f_\varepsilon^{in}\|^2_{L^2(M^{-1})} \,.$$

Now, by the Cauchy-Schwarz inequality, we obtain estimate (1.36):

$$\rho_\varepsilon(t,x) = \int_{\mathbb{R}^N} \frac{f_\varepsilon}{M^{1/2}} M^{1/2}\, dv \le \left(\frac{(f_\varepsilon)^2}{M}\, dv \right)^{1/2} ,$$

so that $\rho_\varepsilon(t,x)$, as well as $L(f_\varepsilon)$, are well defined a.e., and

$$\sup_{t\ge 0} \int_{\mathbb{R}^N} \rho_\varepsilon(t,\cdot)^2\, dx \le \|f^{in}\|^2_{L^2(M^{-1})} \,.$$

□

1.5.2 Hilbert expansion (classical diffusion)

In this part we study the classical diffusion limit. This requires that the linear Boltzmann equation has an equilibrium distribution with finite second moment. We will use the Hilbert expansion, which is a common technique in multiscale analysis. This proof is an adaptation and combines the ones in [AG13] and [MMM11].

Examples of diffusion limits for some non linear collision operators can be found in [GM03], [MLT10], [Mel02].

1.5.2.1 Classical diffusion limit

In this section we will prove theorem 1.8.

The idea of the proof is based on an 'approximation' (ansatz) of the solution f_ε of the form

$$F_\varepsilon = f_0 + \varepsilon f_1 + \varepsilon^2 f_2 \text{ with } \int_{\mathbb{R}^N} F_\varepsilon\, dv = \rho \;\text{ for all } \varepsilon,$$

where ρ will satisfy the diffusion equation (1.1).

Specifically, we define the approximation to the solution f_ε as follows,

$$F_\varepsilon(t,x,v) = f_0(t,x,v) + \varepsilon f_1(t,x,v) + \varepsilon^2 f_2(t,x,v)$$

where

$$f_0 := M\,\rho \tag{1.37}$$

$$f_1 := -v\cdot\nabla_x f_0 = -M\ v\cdot\nabla_x\rho \tag{1.38}$$

$$f_2 := -\partial_t f_0 - v\cdot\nabla_x f_1 = -M\ \left(\partial_t\rho - v\cdot\nabla_x(v\cdot\rho)\right) \tag{1.39}$$

where ρ is the solution of the diffusion equation (1.1) with initial condition $\rho^{in} = \int_{\mathbb{R}^N} f^{in}\,dv$. The construction of F_ε will be explained in Section 1.5.2.2.
Observe the following properties

$$f_0 - M\int_{\mathbb{R}^N} f_0\,dv = 0 \tag{1.40}$$

$$\int_{\mathbb{R}^N} f_1\,dv = 0 \tag{1.41}$$

$$\int_{\mathbb{R}^N} f_2\,dv = 0 \tag{1.42}$$

$$\text{(1.41)-(1.42) imply that } \int_{\mathbb{R}^N} F_\varepsilon\,dv = \rho \quad \text{for any } \varepsilon>0\,. \tag{1.43}$$

The properties (1.40)-(1.41) are readily proven, property (1.42) requires the diffusion matrix

$$D := \int_{\mathbb{R}^N} M(v)\,v\otimes v\,dv$$

to be well defined. For that, it is necessary and sufficient that the second moment of M to be bounded (as we will see).

To prove Theorem 1.8 we are left to check the

Proposition 1.11. *It holds that*

$$\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} f_\varepsilon\,dv = \lim_{\varepsilon\to 0}\int_{\mathbb{R}^N} F_\varepsilon\,dv = \rho.$$

Proof of Proposition 1.11. We substitute the function F_ε in the rescaled Boltzmann equation (1.33) and denote

$$\mathcal{Q}\,F_\varepsilon := (\varepsilon^2\partial_t + \varepsilon v\cdot\nabla_x + \mathrm{Id} - \mathcal{K})F_\varepsilon \tag{1.44}$$

where Id is the identity operator and $\mathcal{K}F_\varepsilon = M\int_{\mathbb{R}^N} F_\varepsilon\,dv$. Observe that the operator $\mathcal{Q}$ is linear. We compute

$$
\begin{aligned}
\mathcal{Q}\, f_0 &= \varepsilon^2 \partial_t f_0 + \varepsilon v \cdot \nabla_x f_0 + f_0 - M \int_{\mathbb{R}^N} f_0 \, dv \overset{(1.40)\text{-}(1.38)}{=} \varepsilon^2 \partial_t f_0 - \varepsilon f_1 \,, \\
\mathcal{Q}\, \varepsilon f_1 &= \varepsilon^2 \partial_t \varepsilon f_1 + \varepsilon v \cdot \nabla_x \varepsilon f_1 + \varepsilon f_1 - F \int_{\mathbb{R}^N} \varepsilon f_1 \, dv \overset{(1.41)\text{-}(1.39)}{=} \varepsilon^3 \partial_t f_1 + \varepsilon^2 (-f_2 - \partial_t f_0) + \varepsilon f_1 \,, \\
\mathcal{Q}\, \varepsilon^2 f_2 &= \varepsilon^2 \partial_t \varepsilon^2 f_2 + \varepsilon v \cdot \varepsilon^2 \nabla_x f_2 + \varepsilon^2 f_2 - F \int_{\mathbb{R}^N} \varepsilon^2 f_2 \, dv \overset{(1.42)}{=} \varepsilon^4 \partial_t f_2 + \varepsilon^3 v \cdot \nabla_x f_2 + \varepsilon^2 f_2 \,.
\end{aligned}
$$

Since $\mathcal{Q}$ is a linear operator,

$$
\mathcal{Q}\, F_\varepsilon = \mathcal{Q}\,(f_0 + \varepsilon f_1 + \varepsilon^2 f_2) = \varepsilon^3 \partial_t f_1 + \varepsilon^4 \partial_t f_2 + \varepsilon^3 v \cdot \nabla_x f_2 \,.
$$

Therefore $R_\varepsilon := F_\varepsilon - f_\varepsilon$ satisfies

$$
\mathcal{Q}\, R_\varepsilon = \mathcal{Q}\,(F_\varepsilon - f_\varepsilon) = \varepsilon^3 \partial_t f_1 + \varepsilon^4 \partial_t f_2 + \varepsilon^3 v \cdot \nabla_x f_2 \,. \tag{1.45}
$$

At this stage, sometimes the convergence of solutions can be proven using estimate results on the Boltzmann equation. However, we do not use this technique here, see [AG13] for more details.

It holds that

$$
\int_{\mathbb{R}^N} R_\varepsilon \, dv = \int_{\mathbb{R}^N} F_\varepsilon - f_\varepsilon \, dv = \rho_\varepsilon - \int_{\mathbb{R}^N} f_\varepsilon(v) dv \,.
$$

Integrating the equation on R_ε (1.44) w.r.t v, we obtain

$$
\int_{\mathbb{R}^N} \mathcal{Q}\, R_\varepsilon \, dv = \varepsilon^2 \partial_t \int_{\mathbb{R}^N} R_\varepsilon \, dv + \varepsilon \int_{\mathbb{R}^N} v \cdot \nabla_x R_\varepsilon \, dv
$$

using that $\int_{\mathbb{R}^N} M(v') \int_{\mathbb{R}^N} R_\varepsilon \, dv \, dv' = \int_{\mathbb{R}^N} R_\varepsilon \, dv \int_{\mathbb{R}^N} M(v') \, dv' = \int_{\mathbb{R}^N} R_\varepsilon \, dv$.

The factor $\int_{\mathbb{R}^N} v \cdot \nabla_x R_\varepsilon \, dv$ prevents to go further in the study of the limit of $\int_{\mathbb{R}^N} R_\varepsilon \, dv$. In order to overcome this obstacle, we consider the Laplace-Fourier transform in t and x defined as follows,

$$
\widehat{g}(p,k,v) := \int_{\mathbb{R}^N} \int_0^\infty e^{-pt} e^{-ik\cdot x} g(t,x,v) dt dx, \quad g \in L^\infty(0,\infty) \times L^1(\mathbb{R}^N_x), \; p > 0, \; k \in \mathbb{R}^N \tag{1.46}
$$

and we apply this transformation to the equation on R_ε (1.45).

The function $\widehat{R_\varepsilon}$ satisfies

$$
\varepsilon^2 p \, \widehat{R_\varepsilon} - \varepsilon^2 \widehat{R_\varepsilon^{in}} + \varepsilon i (v\cdot k) \widehat{R_\varepsilon} + \widehat{R_\varepsilon} - M \int_{\mathbb{R}^N} \widehat{R_\varepsilon} \, dv = \underbrace{\varepsilon^3 p \widehat{f_1} - \varepsilon^3 \widehat{f_1^{in}} + \varepsilon^4 p \widehat{f_2} - \varepsilon^4 \widehat{f_2^{in}} + \varepsilon^3 i (v \cdot k) f_2}_{=: b^\varepsilon(p,k,v)}
$$

where $g^{in} = g(t=0,\cdot)$ and the $\widehat{\ }$ symbol means Fourier Transform (and not Laplace-

Fourier Transform) in the functions with the in label. Isolate $\widehat{R_\varepsilon}$

$$\widehat{R_\varepsilon} = \frac{M \int_{\mathbb{R}^N} \widehat{R_\varepsilon}\, dv + \varepsilon^2 \widehat{R_\varepsilon^{in}} + b^\varepsilon(t,x,v)}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1}\,.$$

Integrate w.r.t the variable v over $\mathbb{R}^N$, and using $\int_{\mathbb{R}^N} M\, dv = 1$

$$\underbrace{\left(\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N}\left(\frac{1}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1)} - 1\right) M(v)dv\right)}_{=:a^\varepsilon(p,k,v)} \widehat{G_\varepsilon} + I^\varepsilon(p,k,v) = 0$$

where

$$I^\varepsilon(p,k,v) := \int_{\mathbb{R}^N}\left(\frac{\widehat{R_\varepsilon^{in}} + \varepsilon p \widehat{f}_1 - \varepsilon \widehat{f_1^{in}} + \varepsilon^2 p \widehat{f}_2 - \varepsilon^2 \widehat{f_2^{in}} + \varepsilon i(v\cdot k)\widehat{f}_2}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1}\right) dv\,.$$

Finally, we compute the limit of $\widehat{G_\varepsilon} := \widehat{\rho} - \int_{\mathbb{R}^N} \widehat{f}_\varepsilon(v) dv$.
From the previous step, we have

$$a^\varepsilon(p,k,v)\widehat{G_\varepsilon} = -I^\varepsilon(p,k,v)\,.$$

Lemma 1.12 (Laplace symbol)**.** *With the previous notations,*

$$a^\varepsilon(p,k,v) \underset{\varepsilon\to 0}{\longrightarrow} -p - s|k|^2 \quad \textit{for some } s > 0. \tag{1.47}$$

(cf. proof in Section 1.5.3.1.)

Lemma 1.13. *With the previous notations,*

$$I^\varepsilon(p,k,v) \underset{\varepsilon\to 0}{\longrightarrow} \widehat{\rho^{in}} - \int_{\mathbb{R}^N} \widehat{f^{in}}\, dv\,.$$

Proof of Lemma 1.13. By definition, $R_\varepsilon^{in} = F_\varepsilon^{in} - f^{in} = f_0^{in} + \varepsilon f_1^{in} + \varepsilon^2 f_2^{in} - f^{in}$. Hence, we can write

$$I^\varepsilon(p,k,v) = \int_{\mathbb{R}^N} \frac{\widehat{f_0^{in}} - \widehat{f^{in}} + \varepsilon p \widehat{f}_1 + \varepsilon^2 p \widehat{f}_2 + \varepsilon i(v\cdot k)\widehat{f}_2 + \varepsilon f_1^{in} + \varepsilon^2 f_2^{in}}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1}\, dv\,. \tag{1.48}$$

By Lebesgue dominated convergence theorem:

$$\int_{\mathbb{R}^N} \frac{\widehat{f_0^{in}}}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1}\, dv = \widehat{\rho^{in}} \int_{\mathbb{R}^N} \frac{M(v)}{\varepsilon^2 p + \varepsilon i(v\cdot k) + 1}\, dv \longrightarrow \widehat{\rho^{in}} \int_{\mathbb{R}^N} M(v)\, dv = \widehat{\rho^{in}}\,.$$

Using that if a function $g \in L^2(\mathbb{R}^N)$, then its Laplace-Fourier Transform $\widehat{g} \in L^2(\mathbb{R}^N)$ by Parseval equality, we can apply again Lebesgue dominated convergence theorem (since $f^{in} \in L^2_x(\mathbb{R}^N) \times L^2(M^{-1}, \mathbb{R}^N)$ by hypothesis). Therefore, we obtain

$$\int_{\mathbb{R}^N} \frac{\widehat{f^{in}}}{\varepsilon^2 p + \varepsilon i(v \cdot k) + 1} \, dv \longrightarrow \int_{\mathbb{R}^N} \widehat{f^{in}} \, dv \, .$$

The rest of the terms in the integral (1.48) goes to zero as $\varepsilon \to 0$ applying also the Lebesgue dominated convergence theorem. We can apply it with the next observation; $f_1, f_2, (v \cdot k) f_2 \in L^2(\mathbb{R}^N)$, therefore $\widehat{f_1}, \widehat{f_2}, (v \cdot k)\widehat{f_2} \in L^2(\mathbb{R}^N)$. □

Considering Lemma (1.13) and that we chose the initial condition $\rho(0, \cdot)$ to be

$$\rho^{in} = \int_{\mathbb{R}^N} f^{in}(v) \, dv \, ,$$

we have that

$$I^{\varepsilon}(p, k, v) \underset{\varepsilon \to 0}{\longrightarrow} 0 \, .$$

Also, it will be proven that

Lemma 1.14. *The limit* $\widehat{G_0} := \lim_{\varepsilon \to 0} \widehat{G_\varepsilon}$ *is well defined.*

(cf. proof of Lemma 1.18).

Then, letting $\varepsilon \to 0$ in equation (1.47), we obtain

$$(-p - s|k|^2)\widehat{G_0} = 0 \, .$$

This equation holds for all $p > 0$ and $k \in \mathbb{R}^N$, therefore

$$0 \equiv \widehat{G_0} = \lim_{\varepsilon \to 0} \widehat{G_\varepsilon} = \widehat{\rho} - \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} \widehat{f_\varepsilon}(v) \, dv.$$

Since the Laplace-Fourier Transform is a one-to-one map, we deduce that $\rho = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} f_\varepsilon \, dv$ a.e..

□

1.5.2.2 Construction of the ansatz

In this section we explain how F_ε is built. The key concept is that of *Hilbert expansion*, which is used to construct an **ansatz**, i.e., we suppose *a priori* that the solution of the equation has a particular shape, called the ansatz. Afterwards, the ansatz is substituted in the equation to study its properties.

Consider a solution $f_\varepsilon(t,x,v)$ to the rescaled linear Boltzmann equation (1.33) written as a **Hilbert expansion** of powers of ε

$$f_\varepsilon(t,x,v) = \sum_{n\geq 0} \varepsilon^n f_n(t,x,v) \tag{1.49}$$

where the functions $f_n(t,x,v)$ are at least differentiable w.r.t t, twice differentiable w.r.t x and integrable w.r.t v. Observe that the Hilbert formal series do not converge in general for any value of $\varepsilon > 0$ and the functions f_n are not necessarily non-negative (remember that the solution of the linear Boltzmann equation, to have physical coherence, must be non-negative).

Formally, we substitute the Hilbert expansion (1.49) in the rescaled linear Boltzmann equation (1.33) and develop

$$\begin{aligned}
0 \quad = \quad & \varepsilon^2\partial_t f_0 + \varepsilon v\cdot\nabla_x f_0 + f_0 - M(v)\int_{\mathbb{R}^N} f_0\,dv \\
& + \varepsilon^3\partial_t f_1 + \varepsilon^2 v\cdot\nabla_x f_1 + \varepsilon f_1 - \varepsilon M(v)\int_{\mathbb{R}^N} f_1\,dv \\
& + \varepsilon^4\partial_t f_2 + \varepsilon^3 v\cdot\nabla_x f_2 + \varepsilon^2 f_2 - \varepsilon^2 M(v)\int_{\mathbb{R}^N} f_2\,dv \\
& + \varepsilon^5\partial_t f_3 + \varepsilon^4 v\cdot\nabla_x f_3 + \varepsilon^3 f_3 - \varepsilon^3 M(v)\int_{\mathbb{R}^N} f_3\,dv + \ldots \quad .
\end{aligned}$$

For each degree on ε an equation is obtained.

Degree 0

$$f_0 - M(v)\rho_0 = 0 \Longrightarrow (\mathrm{Id}-\mathcal{K})f_0 = 0\,.$$

Degree 1

$$v\cdot\nabla_x f_0 + f_1 - M(v)\rho_1 = 0 \Longrightarrow (\mathrm{Id}-\mathcal{K})f_1 = -v\cdot\nabla_x f_0\,.$$

Degree 2

$$\partial_t f_0 + v\cdot\nabla_x f_1 + f_2 - M(v)\rho_2 = 0 \Longrightarrow (\mathrm{Id}-\mathcal{K})f_2 = -\partial_t f_0 - v\cdot\nabla_x f_1\,.$$

Degree $n>1$ In general, we will have the equation

$$(\mathrm{Id}-\mathcal{K})f_n = -\partial_t f_{n-2} - v\cdot\nabla_x f_{n-1}\,.$$

We want to solve this system of equations:

Solving the degree 0.

$$(\mathrm{Id}-\mathcal{K})f_0 = 0 \Longrightarrow f_0 \in \mathrm{Ker}(\mathrm{Id}-\mathcal{K})\,.$$

Therefore

$$f_0(t,x,v) = M(v)\rho_0(t,x)\,.$$

Solving the degree 1.

$$(\mathrm{Id}-\mathcal{K})f_1 = -v\cdot\nabla_x f_0\,.$$

Now, we study the existence of solutions of this integral equation. Observe that the operator $\mathcal{K}$ is self-adjoint in $L^2(M^{-1})$: consider two functions $f,g\in L^2(M^{-1})$ and write

$$\int_{\mathbb{R}^N} g(v)\,\mathcal{K}(f)(v)\,M^{-1}(v)\,dv = \int_{\mathbb{R}^N} g(v)M(v)\left(\int_{\mathbb{R}^N} f(v')dv'\right)M^{-1}(v)\,dv = \int_{\mathbb{R}^N} f\,dv\int_{\mathbb{R}^N} g\,dv\,.$$

In the same manner

$$\int_{\mathbb{R}^N} f\mathcal{K}(g)\,M^{-1}\,dv = \int_{\mathbb{R}^N} g\,dv\int_{\mathbb{R}^N} f\,dv\,,$$

which shows the self-adjointness. $\mathcal{K}$ is a compact operator in $L^2(M^{-1})$; consider a sequence $(\psi_n)_{n\in\mathbb{N}}$ bounded in $L^2(M^{-1})$, in particular, $(\psi_n)_{n\in\mathbb{N}}$ is bounded in L^1; therefore, $(\int_{\mathbb{R}^N}\psi_n\,dv)_{n\in\mathbb{N}}$ is a bounded sequence that does not depend on v; therefore, we can find a convergent subsequence in $L^2(M^{-1})$. We deduce that $\mathcal{K}$ is a compact operator since $\mathcal{K}\psi_n = M(v)\int_{\mathbb{R}^N}\psi_n\,dv$. Since $\mathcal{K}$ is an operator self-adjoint and compact, we have that

$$\mathrm{Ker}(\mathrm{Id}-\mathcal{K})^\perp = \mathrm{Im}(\mathrm{Id}-\mathcal{K})$$

and

$$\mathrm{Ker}(\mathrm{Id}-\mathcal{K}) = \left\{\phi\in L^2(M^{-1})\quad \text{s.t}\quad \phi(t,x,v) = M(v)\int_{\mathbb{R}^N}\phi(v')dv' = M(v)\rho^\phi(t,x)\right\}\,.$$

Therefore,

$$\begin{aligned}
\mathrm{Ker}(\mathrm{Id}-\mathcal{K})^\perp &= \left\{\psi\in L^2(M^{-1})\quad \text{s.t}\quad \int_{\mathbb{R}^N}\psi(v)\phi(v)M^{-1}(v)\,dv = 0\quad \forall\phi\in\mathrm{Ker}(\mathrm{Id}-\mathcal{K})\right\}\\
&= \left\{\psi\in L^2(M^{-1})\quad \text{s.t}\quad \rho^\phi\int_{\mathbb{R}^N}\psi(v)\,dv = 0\quad \forall\phi\in\mathrm{Ker}(\mathrm{Id}-\mathcal{K})\right\}\\
&= \{\psi\in L^2(M^{-1})\quad \text{s.t}\quad \int_{\mathbb{R}^N}\psi\,dv = 0\}\,.
\end{aligned}$$

For the integral equation $(\mathrm{Id}-\mathcal{K})\phi = \psi$ to have solution, it is required that

$$\psi\in\mathrm{Im}(\mathrm{Id}-\mathcal{K}) = \mathrm{Ker}(\mathrm{Id}-\mathcal{K})^\perp,\ \text{i.e.,}\ \int_{\mathbb{R}^N}\psi\,dv = 0\,.$$

If this condition is fulfilled, then

$$\left\{\text{solutions } \phi \in L^2(M^{-1}) \text{ of } (\mathrm{Id}-\mathcal{K})\phi = \psi,\ \text{for } \psi \in L^2(M^{-1}),\ \int_{\mathbb{R}^N} \psi\, dv = 0\right\} = \psi + \mathrm{Ker}(\mathrm{Id}-\mathcal{K})$$

(since we have $(\mathrm{Id}-\mathcal{K})\psi = \psi$ when $\int_{\mathbb{R}^N} \psi\, dv$=0).

Let us consider again, the equation on f_1

$$(\mathrm{Id}-\mathcal{K})f_1 = -v\cdot\nabla_x f_0\,,$$

we have seen that, for this equation to have solution, we require

$$\int_{\mathbb{R}^N} -v\cdot\nabla_x f_0\, dv = 0$$

which is true because

$$\int_{\mathbb{R}^N} -v\cdot\nabla_x f_0\, dv = \int_{\mathbb{R}^N} -M(v)\, v\cdot\nabla_x \rho_0(t,x)\, dv$$

and M is even. Therefore, the set of possible solutions f_1 are

$$f_1 = -v\cdot\nabla_x f_0 + \mathrm{Ker}(\mathrm{Id}-\mathcal{K}) = -M(v)\, v\cdot\nabla_x\rho_0(t,x) + \mathrm{Ker}(\mathrm{Id}-\mathcal{K})\,.$$

We consider

$$f_1 = -v\cdot\nabla_x f_0 = -M(v)\, v\cdot\nabla_x\rho_0(t,x)\,,$$

note that we already take the element in $\mathrm{Ker}(\mathrm{Id}-\mathcal{K})$ to be zero, because this term will be killed in equation (1.50) by symmetry.

Solving the degree 2

$$(\mathrm{Id}-\mathcal{K})f_2 = -\partial_t f_0 - v\cdot\nabla_x f_1\,.$$

In the same way as before, for this integral equation to have solution, we require

$$\int_{\mathbb{R}^N} \left(-\partial_t f_0 - v\cdot\nabla_x f_1\right)\, dv = 0 \tag{1.50}$$

this is equivalent to the condition

$$\boxed{\partial_t\rho_0 - \int_{\mathbb{R}^N} M(v)\ \left(v\cdot\nabla_x(v\cdot\nabla_x\rho_0)\right)\, dv = 0}\,. \tag{1.51}$$

This condition is a **diffusion equation** as we see next. If this condition (1.51) is satisfied, then

$$f_2(t,x,v) = M(v)\left(-\partial_t\rho_0 + v\cdot\nabla_x(v\cdot\nabla_x\rho_0)\right) + \mathrm{Ker}(\mathrm{Id}-\mathcal{K})\,.$$

As in the previous case, to have $\int_{\mathbb{R}^N} f_2\, dv = 0$, we will need to choose the solution

$$f_2(t,x,v) = M(v)\left(-\partial_t \rho_0 + v\cdot\nabla_x(v\cdot\nabla_x\rho_0)\right).$$

Let us study the integral

$$\int_{\mathbb{R}^N} M(v)\ (v\cdot\nabla_x(v\cdot\nabla_x\rho))\ dv = \nabla_x\cdot\left[\left(\int_{\mathbb{R}^N} M(v)\, v\otimes v\, dv\right)\nabla_x\rho\right]. \tag{1.52}$$

Define the matrix

$$D := \int_{\mathbb{R}^N} M(v)\, v\otimes v\, dv\,,$$

then the condition

$$\int_{\mathbb{R}^N} f_2\, dv = 0 \quad\Longleftrightarrow\quad \partial_t\rho_0 - \nabla_x\cdot(D\,\nabla_x\rho_0)) = 0$$

is the diffusion equation. However, we need to check that the diffusion matrix D is well defined.

Lemma 1.15. *The diffusion matrix D is well defined if and only if the second moment order of M is finite.*

Proof. If the matrix D is well defined, then the second moment order of M is finite since it corresponds to the trace of D. If

$$\int_{\mathbb{R}^N} \|v\|^2 M(v)\, dv < \infty$$

then the matrix is well defined because, for each component of the matrix,

$$v_i v_j \le \frac{v_i^2}{2} + \frac{v_j^2}{2} \le \frac{\|v\|^2}{2} + \frac{\|v\|^2}{2} = \|v\|^2$$

(where the first inequality is Young's inequality). Therefore, the second moment order of M bounds from above all the components of D and therefore it is well defined. $\square$

1.5.2.3 The fractional Hilbert expansion

We conclude this section by mentioning that there exists a version of the Hilbert expansion method that allows to derive the fractional diffusion equation from the linear Boltzmann equation. This has been done in reference [AMP10] and the reader is referred to it for more information.

1.5.3 Laplace-Fourier Transform

Theorem 1.8 requires the second moment order of M to be finite. What happens when the second moment of M is not finite?

As we saw in Section 1.2.3.2, the multiscale analysis gives a fractional diffusion equation in the limit and has a fundamental change compared with the classical diffusion limit: we need a different time rescaling $t' := \varepsilon^\alpha t$, $\alpha \in (0, 2)$.

Here we will proof Theorem 1.7 for $\alpha < 2$ along with the classical diffusion (Theorem 1.8) for M having bounded second moment with the rescale in time $t' = \varepsilon^2 t$. We will assume in the classical diffusion case that M is rotationally invariant to simplify the proof.

By performing the Laplace-Fourier Transform in space and time (1.46) on the linear Boltzmann equation, we will find the equation satisfied by $\widehat{\rho_\varepsilon} = \int_{\mathbb{R}^N} \widehat{f_\varepsilon}\, dv$ (where the hat indicates the Laplace-Fourier Transform) and we will take the limit on this equation.

We apply the Laplace-Fourier Transform to equation (1.30)

$$\varepsilon^\alpha p \widehat{f_\varepsilon} - \varepsilon^\alpha \widehat{f^{in}} + \varepsilon i (v \cdot k) \widehat{f_\varepsilon} + \widehat{f_\varepsilon} - M \int_{\mathbb{R}^N} f_\varepsilon \, dv = 0$$

where $\widehat{f^{in}}$ denotes the Fourier Transform in space (and not the Laplace-Fourier Transform in time and space as for the other terms).
We isolate $\widehat{f_\varepsilon}$

$$\widehat{f_\varepsilon} = \frac{M}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} \widehat{\rho_\varepsilon} + \frac{\varepsilon^\alpha \widehat{f^{in}}}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k}$$

and we integrate w.r.t v:

$$\widehat{\rho_\varepsilon} = \left(\int_{\mathbb{R}^N} \frac{M}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} \, dv \right) \widehat{\rho_\varepsilon} + \left(\int_{\mathbb{R}^N} \frac{\varepsilon^\alpha \widehat{f^{in}}}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} \, dv \right) .$$

This last equation can be rewritten (because $\int_{\mathbb{R}^N} M \, dv = 1$) as

$$a^\varepsilon(p, k) \widehat{\rho_\varepsilon} + \int_{\mathbb{R}^N} \frac{\widehat{f^{in}}}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} \, dv = 0 \tag{1.53}$$

where

$$a^\varepsilon(p, k) := \frac{1}{\varepsilon^\alpha} \int_{\mathbb{R}^N} \left(\frac{1}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} - 1 \right) M(v) \, dv \, .$$

We now compute the limit of each term in the previous equation when $\varepsilon \to 0$.

Proposition 1.16. *It holds that*

$$\int_{\mathbb{R}^N} \frac{\widehat{f^{in}}}{1 + \varepsilon^\alpha p + \varepsilon i v \cdot k} \, dv \to \int_{\mathbb{R}^N} \widehat{f^{in}} \, dv$$

for $f^{in} \in L^2(M^{-1})$.

Proof. The assumption $f^{in} \in L^2(M^{-1})$ implies in particular that $f^{in} \in L^2_x(L^1_v)$. Hence, its Fourier Transform $\widehat{f^{in}}$ also belongs to $L^2_k(L^1_v)$ by Parseval equality, which means that $\widehat{f^{in}}$ is integrable in v for almost all k. This allows to apply the Lebesgue dominated convergence theorem, which yields, for almost every k,

$$\int_{\mathbb{R}^N} \frac{\widehat{f^{in}}}{1+\varepsilon^\alpha p + \varepsilon i v \cdot k}\, dv \longrightarrow \int_{\mathbb{R}^N} \widehat{f^{in}} = \widehat{\rho^{in}}\,.$$

□

Proposition 1.17 (The fractional symbol)**.** *If* $\alpha \in (0,2)$*, then*

$$a^\varepsilon(p,k) \to -p - s|k|^\alpha$$

with $s \in (0,\infty)$ *given by*

$$s = \int_{\mathbb{R}^N} \frac{w_1^2}{1+w_1^2} \frac{s_0}{|w|^{N+\alpha}}\, dw$$

(w_1 *indicates the first coordinate of the vector* w*). Furthermore,* $a^\varepsilon(p,k)$ *satisfies*

$$|a^\varepsilon(p,k)| \le |p| + s|k|^\alpha\,.$$

Proof. We postpone the proof of this crucial step to Section 1.5.3.1. □

Lemma 1.18. *The limit* $\widehat{\rho}_0 := \lim_{\varepsilon\to 0} \widehat{\rho}_\varepsilon$ *is well defined.*

Before going into the proof of Lemma 1.18, let us explain how it allows to conclude that letting $\varepsilon \to 0$, we obtain

$$\widehat{\rho^{in}} + (-p - s|k|^\alpha)\widehat{\rho}_0 = 0 \text{ for a.e } p > 0,\ k \in \mathbb{R}^N\,.$$

This equation is also satisfied by the unique solution of the fractional diffusion equation, so that $\widehat{\rho} = \widehat{\rho}_0$, and then $\rho = \rho_0$ because the Laplace-Fourier Transform is a one-to-one mapping (say in $\mathcal{S}'([0,\infty) \times \mathbb{R}^N)$, the dual of the Schwartz space). Moreover, we have that $f_\varepsilon \to \rho M$ weakly in $L^\infty(0,T; L^2(\mathbb{R}^N \times \mathbb{R}^N))$ by the a priori estimates 1.9.

We prove now the previous lemma:

Proof of Lemma 1.18. From the bound

$$\sup_{t\ge 0} \int_{\mathbb{R}^N} \rho_\varepsilon(t,\cdot)^2\, dx \le \|f^{in}\|^2_{L^2(M^{-1})} \tag{1.54}$$

established in 1.9, and up to the extraction of a subsequence, we know that there exists $\eta \in L^\infty(0,\infty; L^2(\mathbb{R}^N))$ such that $\rho_\varepsilon \rightharpoonup \eta$ weakly in $L^\infty(0,\infty; L^2(\mathbb{R}^N))$. On one hand, from the precedent estimate, we have $\widehat{\rho_\varepsilon}$ is bounded in $L^\infty(a,\infty;L^2(\mathbb{R}^N))$ for any $a>0$. On the other hand, consider the space of test functions $\mathcal{D}$ and the space of Schwartz $\mathcal{S}$. Then, for any $\phi=\phi(t)\in\mathcal{D}(0,\infty)$ its Laplace transform $\mathcal{L}\phi$ belongs to $L^1(0,\infty)$ and for any $\psi=\psi(x)\in\mathcal{S}(\mathbb{R}^N)$ its Fourier transform $\mathcal{F}\psi$ belongs to $\mathcal{S}(\mathbb{R}^N)$ so that

$$\int_{\mathbb{R}^N}\widehat{\rho_\varepsilon}\,(\phi\otimes\psi)\,dv=\int_{\mathbb{R}^N}\rho_\varepsilon\,(\mathcal{L}\phi\otimes\mathcal{F}\psi)\,dv\to\int_{\mathbb{R}^N}\eta\,(\mathcal{L}\phi\otimes\mathcal{F}\psi)\,dv=\int_{\mathbb{R}^N}\widehat{\eta}\,(\phi\otimes\psi)\,dv$$

as $\varepsilon\to 0$. We deduce that $\widehat{\rho_\varepsilon}\to\widehat{\eta}$ weakly in $L^\infty(a,\infty;L^2(\mathbb{R}^N))$ for any $a>0$.

□

1.5.3.1 The fractional symbol

In this proof we see how the fractional symbol appears:

Proof of Proposition 1.17. Remember that

$$a^\varepsilon(p,k):=\frac{1}{\varepsilon^\alpha}\int_{\mathbb{R}^N}\left(\frac{1}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k}-1\right)M(v)\,dv$$

where

$$M(-v)=M(v)>0,\qquad\int_{\mathbb{R}^N}M\,dv=1\qquad\alpha\in(0,2]\,.$$

We split a^ε into a sum of three terms:

$$\begin{aligned}a^\varepsilon(p,k)&=-\frac{1}{\varepsilon^\alpha}\int_{\mathbb{R}^N}\frac{\varepsilon^\alpha p+\varepsilon i v\cdot k}{1+\varepsilon^\alpha p+\varepsilon i v\cdot k}M(v)\,dv\\&\overset{(1)}{=}\underbrace{-p\int_{\mathbb{R}^N}\frac{1+\varepsilon^\alpha p}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2}M(v)\,dv}_{=:a_1^\varepsilon(p,k)}\\&\quad\underbrace{-\frac{1}{\varepsilon^\alpha}\int_{\mathbb{R}^N}\frac{\varepsilon i v\cdot k}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2}M(v)\,dv}_{=:a_2^\varepsilon(p,k)}\\&\quad\underbrace{-\frac{1}{\varepsilon^\alpha}\int_{\mathbb{R}^N}\frac{\varepsilon^2(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2+\varepsilon^2(v\cdot k)^2}M(v)\,dv}_{=:a_3^\varepsilon(p,k)}\,.\end{aligned}$$

In (1), we multiply numerator and denominator by the conjugate of the denominator. Now, we look at each term:

- the integrand of $a_1^\varepsilon(p,k)$ is bounded by pF uniformly on ε and the dominated convergence implies

$$a_1^\varepsilon(p,k)\underset{\varepsilon\to 0}{\longrightarrow}-p\int_{\mathbb{R}^N}M(v)\,dv\,.$$

- The integrand of $a_2^\varepsilon(p,k)$ is an odd function on v, therefore $a_2^\varepsilon(p,k)\equiv 0$.

- To study the limit of $a_3^\varepsilon(p,k)$ we will consider two cases:

  (i) $\alpha = 2$ and the second moment order of M is bounded.

$$a_3^\varepsilon(p,k) = \int_{\mathbb{R}^N} \frac{(v\cdot k)^2}{(1+\varepsilon^2 p)^2 + \varepsilon^2 (v\cdot k)^2} M(v)\, dv$$

  is uniformly bounded on ε by the maximum of $|k|^2 F$ and $|v|^2 M$. By dominated convergence,

$$a_3^\varepsilon(p,k) \underset{\varepsilon\to 0}{\longrightarrow} \int_{\mathbb{R}^N} (v\cdot k)^2 M(v)\, dv\,.$$

  In particular, if we also assume that M is rotational invariant, we have that

$$(v\cdot k)^2 = \sum_i^N v_i^2 k_i^2 + 2\sum_{i\neq j}^N v_i k_i v_j k_j$$

  and then

$$\int_{\mathbb{R}^N} (v\cdot k)^2 M(v)\, dv = \sum_i^N k_i^2 \int_{\mathbb{R}^N} v_1^2 M(v)\, dv + 2\sum_{i\neq j}^N k_i k_j \int_{\mathbb{R}^N} v_1 v_2 M(v)\, dv\,.$$

  The last integral is zero applying Fubini and that F is an even function. Therefore the limit of $a_3^\varepsilon(p,k)$ is $|k|^2 s$ with

$$s = \int_{\mathbb{R}^N} v_1^2 M(v)\, dv\,.$$

  (ii) $\alpha \in (0,2)$ and the second moment order of M is unbounded. We consider that M is of the form indicated in (1.11).

Lemma 1.19. *Suppose $\alpha \in (0,2)$, then $d^\varepsilon := a_3^\varepsilon$ fulfills, for any $p > 0$, $k \in \mathbb{R}^N$*

$$|d^\varepsilon(p,k)| \le s|k|^\alpha, \quad d^\varepsilon(p,k) \underset{\varepsilon\to 0}{\longrightarrow} s|k|^\alpha \tag{1.55}$$

with $s \in (0,\infty)$ given by

$$s = \int_{\mathbb{R}^N} \frac{w_1^2}{1+w_1^2} \frac{c_0}{|w|^{N+\alpha}} dw\,.$$

Proof of Lemma 1.19. The inequality (1.55) follows from estimate (1.34) in Proposition 1.9

$$0 \le d^\varepsilon(p,k) \le \int_{\mathbb{R}^N} \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{1+\varepsilon^2 (v\cdot k)^2} \frac{c_0}{|v|^{N+\alpha}} dv = s|k|^\alpha$$

where the last equality is obtained by making the change of variables $w := \varepsilon|k|v$. We split

d^ε between small and large velocities in the following way:

$$d^\varepsilon(p,k) = d_1^\varepsilon(p,k) + d_2^\varepsilon(p,k)$$

with

$$\begin{aligned} d_1^\varepsilon(p,k) &= \int_{|v|\leq 1} \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2 + \varepsilon^2 (v\cdot k)^2} M(v)dv \\ &\leq \varepsilon^{2-\alpha} \int_{|v|\leq 1} (v\cdot k)^2 M(v) dv \\ &\leq \varepsilon^{2-\alpha}|k|^2 \int_{|v|\leq 1} F(v)dv \longrightarrow 0 \end{aligned}$$

and

$$\begin{aligned} d_2^\varepsilon(p,k) &= \int_{|v|\geq 1} \frac{\varepsilon^{2-\alpha}(v\cdot k)^2}{(1+\varepsilon^\alpha p)^2 + \varepsilon^2 (v\cdot k)^2} \frac{c_0}{|v|^{N+\alpha}} dv \\ &= |k|^\alpha \int_{|w|\geq \varepsilon |k|} \frac{w_1^2}{(1+\varepsilon^\alpha p)^2 + w_1^2} \frac{c_0}{|w|^{N+\alpha}} dw \longrightarrow s|k|^\alpha, \end{aligned}$$

where we use again the change of variables $w := \varepsilon |k| v$ and the dominated convergence theorem.

□

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1.5.4 Mellet's moments methods

In this Section we will prove theorem 1.7 using Mellet's moments methods. We start by explaining how this method works.

1.5.4.1 The idea behind the method: weak formulation

Consider equation (1.30) in distribution sense, i.e., for all distribution functions $\chi \in \mathcal{D}((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N)$ we make the $L^2(M^{-1})$ product of the equation with χ:

$$\int_0^\infty \int_{\mathbb{R}^{2N}} \partial_t f_\varepsilon \chi M^{-1} dx dv dt \tag{1.56}$$

$$= \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} \left[(\rho_\varepsilon M - f_\varepsilon)\chi M^{-1} + f_\varepsilon\, \varepsilon v \cdot \nabla_x \chi M^{-1} \right] dx dv dt. \tag{1.57}$$

Now the a priori estimates (1.34) imply that

$$f_\varepsilon \rightharpoonup \rho(t,x)M(v) \quad \text{in } L^\infty((0,\infty); L^2(\mathbb{R}^N\times\mathbb{R}^N; M^{-1}dv)) - \text{weak}*$$

and strong in $L^1((0,T), L^2(\mathbb{R}^N \times \mathbb{R}^N, M^{-1}dv))$. We can write f_ε as

$$f_\varepsilon(t,x,v) = \rho_\varepsilon(t,x)M(v) + g_\varepsilon(t,x,v)$$

with $\rho_\varepsilon = \int_{\mathbb{R}^N} f_\varepsilon \, dv$ and where g_ε is the remainder. This remainder gives the behaviour of ρ_ε in the hydrodynamic limit (fluctuations of f_ε). However,

(i) the specific shape for g_ε is unknown as well as some properties like rotationally invariance; the velocities are not decoupled from time and space (like in the expression $\rho(t,x)M(v)$).

(ii) We do not have enough control on g_ε. We know that

$$\|g_\varepsilon\|_{L^2((0,\infty)\times\mathbb{R}^N\times\mathbb{R}^N, M^{-1}dv)} \le C\varepsilon^{\alpha/2}$$

by estimate (1.35) which is insufficient to balance the factor $\varepsilon^{-\alpha}$ in (1.57).

Since the information that we have on g_ε is not enough, the idea of the moments method is to avoid computing directly on g_ε. For that, we consider a test function $\bar{\chi}_\varepsilon$ of the shape

$$\bar{\chi}_\varepsilon(t,x,v) = \varphi(t,x)M(v) + \varepsilon\psi(t,x,v)$$

where the first term $\varphi \in \mathcal{D}((0,\infty)\times\mathbb{R}^N)$ is a test function. Observe that $\varphi(t,x)M(v)$ has the same shape as the limit $\rho(t,x)M(v)$. ψ will help to 'control' the remainder term g_ε.

Plugging-in $\bar{\chi}_\varepsilon$ in equation (1.56), we obtain

$$\begin{aligned}
& \int_0^\infty \int_{\mathbb{R}^{2N}} \partial_t f_\varepsilon \bar{\chi}_\varepsilon M^{-1} dx dv dt && (1.58)\\
= \; & \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} \left[(\rho_\varepsilon M - f_\varepsilon)(\varphi M + \varepsilon\psi) M^{-1} + f_\varepsilon \varepsilon v \cdot \nabla_x \bar{\chi}_\varepsilon M^{-1} \right] dx dv dt && (1.59)\\
= \; & \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} \left[(\rho_\varepsilon M - f_\varepsilon) \varepsilon\psi M^{-1} + f_\varepsilon \varepsilon v \cdot \nabla_x \bar{\chi}_\varepsilon M^{-1} \right] dx dv dt. && (1.60)
\end{aligned}$$

We want to get rid of the term

$$\int_0^\infty \int_{\mathbb{R}^{2N}} \left(-f_\varepsilon(\varepsilon\psi) + f_\varepsilon \varepsilon v \cdot \nabla_x \bar{\chi}_\varepsilon \right) M^{-1} dx dv dt \tag{1.61}$$

which can be achieved imposing

$$-\varepsilon\psi + \varepsilon v \cdot \nabla_x \bar{\chi}_\varepsilon = 0$$

which is equivalent to

$$\bar{\chi}_\varepsilon - \varepsilon v \cdot \nabla_x \bar{\chi}_\varepsilon = \varphi M(v)$$

for which we know an explicit solution

$$\bar{\chi_\varepsilon}(t,x,v) = M(v)\int_0^\infty e^{-z}\varphi(t,x+\varepsilon vz)dz.$$

Hence, the equation becomes

$$\begin{aligned}\int_0^\infty\int_{\mathbb{R}^{2N}} \partial_t f_\varepsilon \bar{\chi_\varepsilon} M^{-1} dxdvdt &= \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbb{R}^{2N}} (\rho_\varepsilon M)\,\varepsilon\psi M^{-1} dxdvdt && (1.62)\\ &= \varepsilon^{-\alpha}\int_0^\infty\int_{\mathbb{R}^{2N}} (\rho_\varepsilon M)\,(\bar{\chi_\varepsilon} M^{-1}-\varphi) dxdvdt && (1.63)\end{aligned}$$

This way we have gotten rid of g_ε. ρM is the limit of f_ε and, on the other hand, the balance between $\varepsilon^{-\alpha}$ exploding in the limit and $\bar{\chi_\varepsilon} - \varphi M$ going to zero in the limit will give the fractional symbol.

Finally, we write $\chi_\varepsilon = \bar{\chi_\varepsilon} M^{-1}$, χ_ε satisfies the equation

$$\chi_\varepsilon - \varepsilon v\cdot\nabla_x\chi_\varepsilon = \varphi$$

whose solution is

$$\chi_\varepsilon(t,x,v) = \int_0^\infty e^{-z}\varphi(t,x+\varepsilon vz)\,dz. \tag{1.64}$$

By integration by parts twice we obtain

$$\begin{aligned}\chi_\varepsilon - \varphi &= \int_0^\infty e^{-z}\left(\varphi(t,x+\varepsilon vz)-\varphi(t,x)\right)\,dz\\ &= \varepsilon\int_0^\infty e^{-z} v\cdot\nabla_x\varphi(t,x+\varepsilon vz)\,dz\\ &= \varepsilon v\cdot\nabla_x\varphi(t,x) + \varepsilon^2\int_0^\infty e^{-z} v^T D^2\varphi(t,x+\varepsilon vz)\,v\,dz && (1.65)\end{aligned}$$

where T indicates transpose. We have that

$$|\chi_\varepsilon - \varphi| \le \|D\varphi\|_{L^\infty}\varepsilon\|v\|$$

but we will need stronger convergence results:

Lemma 1.20 (Convergence properties, [Mel10]). *It holds that for any $\varphi \in \mathcal{D}(\mathbb{R}^N\times[0,\infty))$ and χ_ε defined as in (1.64) that*

$$\int(\chi_\varepsilon - \varphi)\,dv \to 0 \quad \textit{strongly in } L^2((0,\infty)\times\mathbb{R}^N),$$

$$\int(\partial_t\chi_\varepsilon - \partial_t\varphi)\,dv \to 0 \quad \textit{strongly in } L^2((0,\infty)\times\mathbb{R}^N).$$

The proof for this lemma can be found in [Mel10] and analogous ones will be proven in Chapter 2.

1.5.4.2 The fractional symbol

We need to compute the limit in:

$$\int_0^\infty \int_{\mathbb{R}^{2N}} \partial_t f_\varepsilon \chi_\varepsilon \, dxdvdt \quad = \quad \varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} (\rho_\varepsilon M)\,(\chi_\varepsilon - \varphi)\, dxdvdt \tag{1.66}$$

for

$$\chi_\varepsilon - \varepsilon v \cdot \nabla_x \chi_\varepsilon = \varphi.$$

On one hand we have that

$$\int_0^\infty \int_{\mathbb{R}^{2N}} \partial_t f_\varepsilon \chi_\varepsilon dxdvdt \to \int_0^\infty \int_{\mathbb{R}} \partial_t \rho \varphi \, dxdt$$

using Lemma 1.20 and the convergence properties of f_ε. On the other hand we have the following:

Proposition 1.21 (Fractional symbol). *It holds that*

$$\varepsilon^{-\alpha} \int_0^\infty \int_{\mathbb{R}^{2N}} (\rho_\varepsilon M)\,(\chi_\varepsilon - \varphi) dxdvdt \to \int_0^\infty \int_{\mathbb{R}^N} \rho(-\Delta_x)^{\alpha/2} \varphi \, dxdt$$

as $\varepsilon \to 0$.

To prove this theorem we will use here the definition of the fractional Laplacian based on the singular integral (1.6).

Proof of Proposition 1.21. We will focus on the integral in the velocities and split it into large and small velocities. We will need to prove the two following Lemmas:

Lemma 1.22 (Large velocities). *It holds that*

$$\varepsilon^{-\alpha} \int_{\|v\|\geq 1} M(\chi_\varepsilon - \varphi) dv = c_0 \int_0^\infty e^{-z} z^\alpha \int_{\|w\|\geq \varepsilon z} \frac{\varphi(t, x + \varepsilon v z) - \varphi(t, x)}{\|w\|^{N+\alpha}} \, dzdw.$$

Lemma 1.23 (Small velocities). *The following estimate holds:*

$$\left| \varepsilon^{-\alpha} \int_{\|v\|\leq 1} M(\chi_\varepsilon - \varphi) dv \right| \leq C\varepsilon^{2-\alpha}$$

for some $C > 0$.

To conclude the proof we will need to prove:

Lemma 1.24 (Strong convergence of the fractional symbol)**.**

$$\int_0^\infty e^{-z} z^\alpha \int_{\|w\|\geq \varepsilon z} \frac{\varphi(t, x+\varepsilon vz) - \varphi(t,x)}{\|w\|^{N+\alpha}} \, dz dw \to -\kappa(-\Delta_x)^{\alpha/2}\varphi$$

strongly in $L^2((0,\infty)\times\mathbb{R}^N)$ for some $\kappa > 0$.

Proof of Lemma 1.23. The statement is consequence of the expansion (1.65) (using the parity of v for the term of order $\varepsilon^{1-\alpha}$). ☐

Proof of Lemma 1.22. We compute on the integral to obtain (remember the shape of M in (1.11))

$$\begin{aligned}
\varepsilon^{-\alpha}\int_{\|v\|\geq 1} M(\chi_\varepsilon - \varphi)\, dv &= \varepsilon^{-\alpha}\int_{\|v\|\geq 1} M \int_0^\infty e^{-z}\left(\varphi(t,x+\varepsilon vz) - \varphi(t,x)\right) dz dv \\
&= c_0 \varepsilon^{-\alpha}\int_{\|v\|\geq 1}\int_0^\infty e^{-z}\frac{\varphi(t,x+\varepsilon vz) - \varphi(t,x)}{\|v\|^{N+\alpha}} dz dv \\
&= c_0 \varepsilon^{-\alpha}\int_{\|w\|\geq \varepsilon z}\int_0^\infty e^{-z}|\varepsilon z|^{N+\alpha}\frac{\varphi(t,x+w) - \varphi(t,x)}{\|w\|^{N+\alpha}}\frac{dw}{|\varepsilon z|^N} dz \\
&= c_0 \int_0^\infty e^{-z} z^\alpha \int_{\|w\|\geq \varepsilon z}\frac{\varphi(t,x+w) - \varphi(t,x)}{\|w\|^{N+\alpha}} dw dz
\end{aligned}$$

doing the change of variables $w = \varepsilon v z$.

☐

Proof of Lemma 1.24. The proof can be found in [Mel10] and also in this document in Chapter 2, Lemma 2.13. ☐

The combination of these three lemmas allows to conclude the statement using the weak convergence of ρ_ε in $L^\infty((0,\infty); L^2(\mathbb{R}^N))$.

☐

1.5.5 Probabilistic approach

The solution to the linear Boltzmann equation is a probability density $f = f(t,x,v)$, hence the probabilistic formulation of the equation in terms of the dynamics of a stochastic particle will provide insight on the equation. In this section we explain the dynamics of a stochastic particle $(X_t, V_t)_{t\geq 0}$ under the law given by f. [AG13] gives a hint to what is explained here for a modified equation.

Our goal is to model the dynamics of a **single** particle given by its position and velocity $(X(t), V(t)) \in \mathbb{R}^N \times \mathbb{R}^N$ over time $t \in [0,\infty)$.

Suppose that we are given some random initial data $(X_0, V_0) \in \mathbb{R}^N \times \mathbb{R}^N$ whose distribution has density function $f(0,x,v)$

$$(X_0, V_0) \sim f(0,x,v)\,.$$

Consider now $U_1, U_2, \ldots$ i.i.d random variables with density function M and $T_1, T_2, \ldots$ also i.i.d with exponential distribution of parameter 1, $E(1)$. We assume T_i, U_i to be pairwise independent.

Modeling assumptions. In our model, the particle travels at a constant velocity and after an exponential time, the velocity 'jumps' to a new one with distribution given by the density function M (note that, therefore, we are assuming that the velocity after the jump is independent from the velocity before the jump). With these considerations in mind, the exponential random variables T_n give the lapse of time between two consecutive jumps in the velocity and U_n is the velocity of the particle after n jumps.

1.5.5.1 Derivation of the linear Boltzmann equation

Consider the single particle model described in the previous section, our goal is to find an equation satisfied by the law of (X_t, V_t).

Steps:

- One finds the equations for (X_t, V_t).
- One observes that $(X_t, V_t)_{t\geq 0}$ is a Markov process in $\mathbb{R}^N \times \mathbb{R}^N$ and finds its infinitesimal generator.
- One identifies a martingale from the Markov process and using the martingale, finds an equation for the law of the process $(X_t, V_t)_{t\geq 0}$.

Characterization of the Markov process. Firstly, we find the equations for (X_t, V_t). Define, $J_n = T_1 + \cdots + T_n$, i.e., the lapse of time between time zero and the n-th jump. Then,

$$\begin{aligned} X_t &= X_0 + \int_0^t V_s\, ds && (1.67)\\ V_t &= V_0 \quad \text{for } t < J_1 && (1.68)\\ V_t &= U_n \quad \text{for } J_n \leq t < J_{n+1},\ n \geq 1 && (1.69) \end{aligned}$$

Proposition 1.25. *$(V_t)_{t\geq 0}$ is a pure jump Markov process in $\mathbb{R}^N \times \mathbb{R}^N$ with jump kernel*

$$\pi(v, dv') = M(v')dv'\,.$$

(This measure is the probability of having post-jump values dv' if the pre-jump value is v.) The holding times are given by the i.i.d random variables $(T_n)_{n\in\mathbb{N}}$, hence the rate of jumping is constant equal to 1.

Before tackling the diffusive limit, our goal will be to prove the next theorem:

Theorem 1.26. *Consider the random process $(X_t, V_t)_{t\geq 0}$ defined in Section 1.5.5.1 with initial distribution $f(0,x,v) \in C^1_x \times C^0_v$. Then, the law ρ of the process has density function $f \in C^1_x \times C^0_v$ fulfilling the linear Boltzmann equation*

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = M(v) \int_{\mathbb{R}^N} f(t,x,v')dv' - f(t,x,v)\,.$$

Identification of a martingale.

General results. This section is extracted and adapted from [DN08]. Consider a continuous-time Markov chain on a measurable state space, E, with holding times $(S_n)_{n\in\mathbb{N}}$ and jump chain $(Y_n)_{n\in\mathbb{N}}$. Define $J_0 = 0$ and $J_n = S_0 + \ldots + S_n$. Then:

Definition 1.27. The **jump measure** μ and the **compensator** ν of the Markov chain are random measures on $(0,\infty) \times E$ given by

$$\mu = \sum_{t:X_t \neq X_{t-}} \delta(t, X_t) = \sum_{n=1}^{\infty} \delta_{(J_n, Y_n)}$$

and

$$\nu(dt, B) = q(X_{t-})\pi(X_{t-}, B)dt$$

for all $B \in \mathcal{E}$, the set of all sets of E. The term $q(X)$ is the jump rate at X and π is the jump kernel.

Definition 1.28. The previsible σ-algebra $\mathcal{P}$ on $\Omega\times(0,\infty)$ is the σ-algebra generated by all the left-continuous adapted processes. A function defined on $\Omega\times(0,\infty)\times E$ is **previsible** if it is $\mathcal{P}\otimes\mathcal{E}$-measurable.

Theorem 1.29 ([DN08], Appendix)**.** *Let H be previsible and assume that, for all $t \geq 0$,*

$$\mathbb{E} \int_0^t \int_E |H(s,y)|\nu(ds,dy) < \infty. \tag{1.70}$$

Then the following process is a well-defined martingale

$$\bar{M}_t = \int_{(0,t]\times E} H(s,y)(\mu - \nu)(ds,dy).$$

The martingale. In this section we apply theorem 1.29 to construct a martingale. Proposition 1.25 provides the values of the jump rate, $q(X) \equiv 1$, and jump kernel, $\pi(v, dv') = M(v')dv'$ of our Markov process; hence, the compensator defined in Definition 1.27 is in our case $\nu(ds, (X_s, dv)) = M(v)\, dvds$. Now, for a function $g \in C^1_x \times L^1_v$ we have that

$$g(X_t, V_t) - g(X_0, V_0) = \int_0^t V_s \cdot \nabla_x g(X_s, V_s) V_s\, ds + \int_{(0,t]\times\mathbb{R}^N} \{g(X_s, v) - g(X_s, V_{s-})\}\mu(ds, dv)$$

where μ is the jump measure.

Now, we have that

$$g(X_t, V_t) - g(X_0, V_0) = \int_0^t V_s \cdot \nabla_x g(X_s, V_s)\, ds + \bar{M}_t + \int_{(0,t]\times\mathbb{R}^N} \{g(X_s, v) - g(X_s, V_{s-})\} \underbrace{M(v)\, dvds}_{\nu(ds,(X_s,dv))}$$

where

$$\bar{M}_s = \int_{(0,t]\times\mathbb{R}^N} \{g(X_s, v) - g(X_s, V_{s-})\}(\mu - \nu)(ds, dv)$$

is a martingale by Theorem 1.29. Observe that condition (1.70) is fulfilled since we expect a finite number of jumps on a finite time. See the Appendix in reference [DN08] for more information on how to identify martingales out of Markov chains.

An equation for the distribution.

Lemma 1.30. *There exists a density function for ρ_t, i.e., the measure ρ_t is absolutely continuous w.r.t the Lebesgue measure.*

Proof. We can write

$$\begin{aligned} (X_t, V_t) &= (X_t - X_{J_1}, V_t) 1_{J_1 \le t} && (1.71)\\ &+ (X_{J_1}, 0) 1_{J_1 \le t} && (1.72)\\ &+ (X_0 + tV_0, V_0) 1_{J_1 > t}\,. && (1.73) \end{aligned}$$

Remember that $J_1 \sim E(1)$. Using the previous decomposition of (X_t, V_t), we have

$$\begin{aligned} \mathbb{E}(g(X_t, V_t)) &= e^{-t} \int_{\mathbb{R}^{2N}} g(x + tv, v) f(0, x, v)\, dxdv && (1.74)\\ &+ \int_{\mathbb{R}^{2N}\times\mathbb{R}^{2N}} \int_0^t e^{-s} g(x + sv + x', v') f(0, x, v) \rho^*_{t-s}(dx', dv')\, dxdvds, && (1.75) \end{aligned}$$

where ρ^*_t is the measure of the process starting from $\delta_0(dx)M(v)dv$ (ρ_t is the measure of the process starting from $f(0,x,v)dv$).

The first integral corresponds to the expected values taken by the function g when $J_1 > t$, i.e., there is no jump in the velocity, and the second integral corresponds to the expected values taken by g when $J_1 < t$. In integral (1.75) the parameter s represents the value of J_1, i.e., the time of the first jump in the velocity, and $x + sv$ is the position at time $J_1 = s$. The values of x and v are given through the density function $f(0,x,v)$ and the value of s is distributed as an exponential law of parameter 1, hence the presence of $\exp(-s)$. Now, the values of (x', v') correspond to the values of position and velocity taken by the particle after the first jump in velocity, i.e., for $t \geq J_1 = s$, and therefore their values are given by the distribution $\rho^*_{t-s}(dx', dv')$.

This expectation can be written as:

$$\mathbb{E}(g(X_t, V_t)) = \int_{\mathbb{R}^{2N}} g(x,v) f(t,x,v)\, dxdv$$

where

$$\boxed{f(t,x,v) = e^{-t} f(0, x - tv, v) + M(v) \int_0^t e^{-s} \int_{\mathbb{R}^{2N}} f(0, x - sv' - x', v') \rho^*_{t-s}(dx', v') dv' ds}\,. \tag{1.76}$$

To prove (1.76), we perform the change of variables $y = x + sv + x'$ and the distribution $\rho^*_{t-s}(dx', dv')$ is written as the product of the the distribution conditioning on $\{V_{t-s} = v'\}$ times the probability of that being so, i.e.:

$$\rho^*_{t-s}(dx', dv') = \rho^*_{t-s}(dx', v')\, M(v') dv' \tag{1.77}$$

where we have abused notation and written $\rho^*_{t-s}(dx', v')$ for the conditional distribution on $\{V_{t-s} = v'\}$ (observe that it can be proven that ρ^*_{t-s} can be written in this fashion). Then (1.75) gives:

$$\begin{aligned}
&\int_{\mathbb{R}^{2N} \times \mathbb{R}^{2N}} \int_0^t e^{-s} g(y, v') f(0, y - sv - x', v) M(v') dv' \rho^*_{t-s}(dx', v')\, dydvds \\
&= \int_{\mathbb{R}^{2N}} g(y, v') \left(\int_{\mathbb{R}^{2N}} \int_0^t e^{-s} f(0, y - sv - x', v) M(v') \rho^*_{t-s}(dx', v') ds \right) dydv'
\end{aligned}$$

and then by changing variables again $y = x$, $v' = v$, $v = v'$. □

Remark 1.31. As a corollary of the previous result, if the initial distribution is bounded in the following sense

$$f(0,x,v) \leq f_0(v) \quad \forall\, x, \quad f_0 \text{ integrable},$$

then

$$f(t,x,v) \leq e^{-t} f_0(v) + (1 - e^{-t}) \|f_0\|_{L^1} M(v)\,.$$

Write ρ_t for the distribution of (X_t, V_t), a probability measure on $\mathbb{R}^N \times \mathbb{R}^N$. From the previous result we have that

$$\rho_t(dx, dv) = f(t, x, v)dxdv \,. \tag{1.78}$$

Proof of Theorem 1.26. By Lemma 1.30 we know that ρ has a probability density f. We also know from the previous Section 1.5.5.1, we know that

$$\begin{aligned} \bar{M}(X_t, V_t) \quad &:= \quad g(X_t, V_t) - g(X_0, V_0) - \int_0^t V_t \cdot \nabla_x g(X_s, V_s)\, ds \\ &\quad - \int_{(0,t]\times\mathbb{R}^N} \{g(X_s, v) - g(X_s, V_{s-})\} M(v)\, dv ds \end{aligned}$$

is a martingale. Hence,

$$\mathbb{E}[\bar{M}(X_t, V_t)] = \mathbb{E}[\bar{M}(X_0, V_0)] = 0$$

or, expressed in integral form,

$$\int_{\mathbb{R}^N\times\mathbb{R}^N} g(x, v)\rho_t(dx, dv) \quad = \quad \int_{\mathbb{R}^N\times\mathbb{R}^N} f(0, x, v)g(x, v)dxdv \tag{1.79}$$

$$+ \quad \int_0^t \int_{\mathbb{R}^N\times\mathbb{R}^N} \mathcal{G}g(x, v)\rho_s(dx, dv)ds \tag{1.80}$$

where

$$\mathcal{G}g(x, v) := v \cdot \nabla_x g(x, v) + \int_{\mathbb{R}^N} \{g(x, v') - g(x, v)\} M(v')\, dv' \,.$$

The integral (1.80) is equal to

$$\int_0^t \int_{\mathbb{R}^N\times\mathbb{R}^N} g(x, v)\mathcal{G}^* \rho_s(dx, dv)$$

where $\mathcal{G}^*$ is the adjoint of $\mathcal{G}$.

Since the equation holds for arbitrary $g \in C^1_x \times L^1_v$ and using (1.78) we have that (x, v)-a.e.

$$f(t, x, v) = f(0, x, v) + \int_0^t \mathcal{G}^* f(s, x, v)ds \,. \tag{1.81}$$

where, using integration by parts and Fubini's theorem,

$$\mathcal{G}^* f(s, x, v) = -v \cdot \nabla_x f(s, x, v) + M(v) \int_{\mathbb{R}^N} f(s, x, v')\, dv' - f(s, x, v) \,.$$

This expression is well defined because, thanks to (1.76), and assuming that $f(0, x, v) \in C^1_x \times C^0_v$, f is continuous in the variables (x, v) for all t; it is defined as an integral on (x, v) plus a continuous function. Also, f is differentiable w.r.t x (using that expression (1.76) is differentiable). Moreover, we conclude that (1.81) is defined pointwise.

Finally, observe that in (1.81) f is defined in terms of an integral w.r.t t, therefore it is continuous w.r.t t, hence, f is defined as an integral of a continuous function, so it is differentiable. Now, deriving w.r.t t in (1.81), one obtains the linear Boltzmann equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = M(v) \int_{\mathbb{R}^N} f(t,x,v')dv' - f(t,x,v)\,,$$

which concludes the proof. □

1.5.5.2 Stable Lévy processes

Here we expand what it was explained in Section 1.4.2.1. This will be needed to prove the fractional diffusive limit. Let us start with some introductory definitions and properties.

Definition 1.32 (In [Ber96])**.** Let $Y = (Y_t, t \geq 0)$ be a stochastic process taking values in $\mathbb{R}^N$. We say that Y has the **scaling property** of index $\alpha > 0$ if, for every $k > 0$, the rescaled process

$$(k^{-1/\alpha} Y_{kt}, t \geq 0)$$

has the same finite-dimensional distributions as Y.

If Y is a Lévy process with scaling property for α, then we say that Y is an α-**stable Lévy process**.

Definition 1.33 (In [RY99], Section III.4)**.** A **random variable** Y **is stable** if, for every k, there are independent random variables $Y_1, \ldots, Y_k$ with the same law as Y and constants $a_k > 0, b_k$ such that

$$Y_1 + \ldots + Y_k \stackrel{(d)}{=} a_k Y + b_k$$

It can be proved that this equality forces $a_k = k^{1/\alpha}$ for some $\alpha \in [0,2)$.

As long as the

Theorem 1.34 (In [RY99], Section III.4.)**.** *If Y is stable with index $\alpha \in (0,2)$, then $\sigma = 0$ and the Lévy measure has density $(m_1 \mathbb{1}_{(x<0)} + m_2 \mathbb{1}_{(x>0)})|x|^{-(1+\alpha)}$ with m_1 and $m_2 \geq 0$.*

Proposition 1.35 (In [Sat99], Representation of a non-trivial α-stable distribution μ with $0 < \alpha < 2$.)**.** *If $0 < \alpha < 1$, then μ has drift $\gamma_0 \in \mathbb{R}$ and*

$$\widehat{\mu}(z) = \exp\left[\int_S \lambda(d\xi) \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1)\frac{dr}{r^{1+\alpha}} + i\langle \gamma_0, z\rangle\right]. \tag{1.82}$$

If $1 < \alpha < 2$, then μ has center $\gamma_1 \in \mathbb{R}$ and

$$\widehat{\mu}(z) = \exp\left[\int_S \lambda(d\xi) \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1 - i\langle z, r\xi\rangle)\frac{dr}{r^{1+\alpha}} + i\langle \gamma_1, z\rangle\right]. \tag{1.83}$$

If $\alpha = 1$, then for $\gamma \in \mathbb{R}$ we have

$$\widehat{\mu}(z) = \exp\left[\int_S \lambda(d\xi) \int_0^\infty (e^{i\langle z, r\xi\rangle} - 1 - i\langle z, r\xi\rangle 1_{(0,1]}(r))\frac{dr}{r^2} + i\langle\gamma, z\rangle\right]. \tag{1.84}$$

In all cases λ is a finite measure on S, the unit circle in $\mathbb{R}^N$.

(cf. the proof is in [Sat99], chapter 3, remark 14.6.)

1.5.5.3 Fractional diffusive limit

Recall the single particle model defined in Section 1.5.5. In this section, we study the evolution of the spatial component X_t after rescaling space and time.

Assume now that $X_0 = 0$ and $V_0 \sim M$. Define Y_t to be the random variable that describes the position of the particle at the jumps of velocity, i.e.,

$$\begin{cases} Y_0 &= 0 \\ Y_t &= X_{J_n} \quad \text{for } J_n \le t < J_{n+1} \end{cases}$$

Y jumps at rate 1 with increment $\Delta Y = Y_t - Y_{t-} = UT$ (example, $Y_1 = X_{J_1} = J_1 \times V_0 = T_1 U_0$). Hence, $(Y_t)_{t\ge 0}$ is a Lévy process.

Now, we know that a Lévy process is characterised by a Lévy triplet:

- drift = 0 (because we are considering only the jumps),
- diffusion = 0,
- the Lévy measure $K(dy)$ is the distribution of UT (which corresponds to the displacement in position between two jumps in velocity), hence, for suitable function g
$$\int g(y)\, K(dy) = \mathbb{E}(g(UT)) = \int_{\mathbb{R}^N} \int_0^\infty e^{-t} g(ut) M(u)\, du dt \tag{1.85}$$
since $U \sim M$ and $T \sim E(1)$ are independent by hypothesis.

We characterise the distribution of Y through its characteristic function. We know that for Lévy processes

$$\mathbb{E}(e^{i\theta^T Y_t}) = e^{t\psi(\theta)} \tag{1.86}$$

with characteristic exponent

$$\psi(\theta) = \int_{\mathbb{R}^N} \{e^{i\theta^T y} - 1 - i\theta^T y 1_{\|y\|\le 1}\} K(dy)\,. \tag{1.87}$$

Assume that $M \in C^1$, it is even and

$$\boxed{F(v) = \frac{c_0}{\|v\|^{N+\alpha}} \text{ for } \|v\| \geq 1}, \tag{1.88}$$

for $\alpha \in (0, 2]$.

Theorem 1.36. *The rescaled process $Z^\varepsilon = \varepsilon Y_{\varepsilon^{-\alpha}t}$ converges weakly to a random variable Z which is an α-stable process with characteristic exponent*

$$\psi^\alpha(\theta) = C(\alpha) \int_S \int_0^\infty \frac{e^{i\theta^T \xi r} - 1}{r^{1+\alpha}} \, dr d\xi$$

where

$$C(\alpha) := c_0 \int_0^\infty s^\alpha e^{-s} \, ds \,. \tag{1.89}$$

Proof. We prove the result by showing that the characteristic functions converge to the characteristic function of a particular α-stable process and then by applying Lévy's continuity theorem for characteristic functions.

So, firstly, we consider the characteristic function of the rescaled process

$$\begin{aligned}
\log \mathbb{E}[\exp(i\theta^T \varepsilon Y_{\varepsilon^{-\alpha}t})] &= \varepsilon^{-\alpha} t \psi(\varepsilon\theta) \quad \text{by (1.86)} \\
&= \varepsilon^{-\alpha} t \int_{\mathbb{R}^N} \int_0^\infty \{e^{i\theta^T \varepsilon y s} - 1 - i\varepsilon\theta^T y s 1_{\|ys\| \leq 1}\} e^{-s} M(y) \, dyds \\
&\qquad \text{by (1.85) and (1.87)} \\
&= \underbrace{\varepsilon^{-\alpha} t \int_{\|y\| \leq 1} \int_0^\infty \{e^{i\theta^T \varepsilon y s} - 1\} e^{-s} M(y) \, dyds}_{=:I} \\
&\quad \underbrace{+ \varepsilon^{-\alpha} t \int_{\|y\| \geq 1} \int_0^\infty \{e^{i\theta^T \varepsilon y s} - 1\} e^{-s} M(y) \, dyds}_{=:II}
\end{aligned}$$

where the term $i\varepsilon\theta^T y s 1_{\|ys\| \leq 1}$ disappears in the last equality because it produces an odd integrand.

Then, for the term I we have that

$$e^{i\theta^T \varepsilon y s} - 1 = \cos(\theta^T \varepsilon y s) + i \sin(\theta^T \varepsilon y s) - 1 \,.$$

On one hand, observe that when we integrate the sin term, the integral gives 0 because the integrand is an odd function of y. On the other hand, we can bound using $\|y\| \leq 1$

$$\cos(\theta^T \varepsilon ys) - 1 = \sum_{n=0}^{\infty} (-1)^n \frac{(\theta^T \varepsilon ys)^{2n}}{(2n)!} - 1 \leq \sum_{n>0}^{\infty} (-1)^n \varepsilon^{2n} \frac{(\|\theta\| s)^{2n}}{(2n)!} = \mathcal{O}(\varepsilon^2) \quad (\leq \cos(\varepsilon \|\theta\| s))$$

and we can exchange limit and integral with this bound using dominated convergence and the limit yields zero. Hence, $I \to 0,$ when $\varepsilon \to 0.$

For the second term, we perform the change of variables $\varepsilon ys = z$, ($\varepsilon^N s^N dy = dz$)

$$\begin{aligned} II &= t\varepsilon^{-\alpha-N} s^{-N} \int_{\|z\| \geq \varepsilon s} \int_0^\infty \{e^{i\theta^T z} - 1\} e^{-s} M(z/\varepsilon s)\, dz ds \\ &= t \int_{\|z\| \geq \varepsilon s} \int_0^\infty \{e^{i\theta^T z} - 1\} e^{-s} s^\alpha M(z)\, dz ds \quad \text{by (1.88)} \\ &= t \int_0^\infty s^\alpha e^{-s} \int_{\|z\| \geq \varepsilon s} \{e^{i\theta^T z} - 1\} M(z)\, dz ds\,. \end{aligned}$$

In the end

$$\lim_{\varepsilon \to 0} \log \mathbb{E}[\exp(i\theta^T \varepsilon Y_{\varepsilon^{-\alpha} t})] = t \int_0^\infty s^\alpha e^{-s}\, ds \int_{\mathbb{R}^N} \{e^{i\theta^T z} - 1\} \frac{c_0}{\|z\|^{N+\alpha}}\, dz\,.$$

Define

$$C(\alpha) := c_0 \int_0^\infty s^\alpha e^{-s}\, ds\,,$$

then we rewrite

$$\lim_{\varepsilon \to 0} \log \mathbb{E}[\exp(i\theta^T \varepsilon Y_{\varepsilon^{-\alpha} t})] = tC(\alpha) \int_{\mathbb{R}^N} \frac{e^{i\theta^T z} - 1}{\|z\|^{N+\alpha}}\, dz\,.$$

Observe that $C(\alpha)$ is a gamma function $C(\alpha) = c_0 \Gamma(\alpha + 1)$, where $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$.

In our case, we apply Proposition 1.35 with λ proportional to the identity, and therefore

$$\int_{\mathbb{R}^N} \xi \lambda(d\xi) = 0\,.$$

Now, performing the change of variables $z = \xi r$, (with $\xi = \|z\|$) $dz = r^{N-1} dr d\xi$, we have that

$$C(\alpha) \int_{\mathbb{R}^N} \frac{e^{i\theta^T z} - 1}{\|z\|^{N+\alpha}}\, dz = C(\alpha) \int_S \int_0^\infty \frac{e^{i\theta^T \xi r} - 1}{r^{1+\alpha}}\, dr d\xi$$

Hence, in the notation of proposition 1.35, we have that

$$\lambda = C(\alpha) \mathrm{Id}\,,$$

(observe that $C(\alpha) > 0$).

Summarizing, the limit of the characteristic functions of $\varepsilon Y_{\varepsilon^{-\alpha}t}$ corresponds to the characteristic function ψ^α of an α-stable process. Now, we can apply Lévy's continuity theorem and since the characteristic function ψ^α is continuous at zero, there exists a random variable Z that has ψ^α as characteristic function and is the weak limit of $\varepsilon Y_{\varepsilon^{-\alpha}t}$.

□

Using this probability approach we interpret how fractional phenomena arises in Section 1.6.1.

1.6 Summary and final remarks

Here we summarise some of the concepts presented in this chapter and interpret why fractional phenomena arises in some particular cases using the probabilistic picture.

- About the diffusion and fractional diffusion equations.
  - Both equations give the evolution of a probability density over time.
  - Both equations model transport phenomena, in the case of fractional diffusion it is called anomalous transport, in particular, it models superdiffusions. It is called anomalous diffusion because the Mean Square Displacement of the particles is not linear in time, as it happens with the classical diffusion equation. It is called super-diffusion because particles spread faster than in the classical diffusion.
  - The solution of the diffusion equation is associated to a gaussian process, who is linked to Brownian motion. In the same way, fractional diffusion equations are linked to stable Lévy processes, whose density behaves asymptotically as a power law. Brownian motion is just a particular case of stable Lévy process.
  - The trajectories of the stochastic particles following stable processes are self-similar, this is linked to the scaling invariance of the equations.
  - Fractional Laplacians are non-local operators, classical Laplacian is a local operator. Brownian motion is continuous and the other stable Lévy processes are jump processes and therefore discontinuous.
- About the diffusive limit from the linear Boltzmann equation.
  - The rescaling in time $t' = \varepsilon^\alpha t$ needed for the diffusive limit corresponds to the order of the fractional Laplacian obtained in the limit $(-\Delta_x)^{\alpha/2}$. This corresponds to the scaling invariance of the equation.
  - The existing methods in Partial Differential Equations are the Hilbert expansion method, Laplace-Fourier Transform method and Moments method. There exist also some Probabilistic approaches.

1.6.1 How does fractional phenomena arise

Remember that fractional phenomena occurs when in the formal computation presented in 1.2.3.2 we have that the diffusive constant obtained in the limit

$$\kappa \sim \int_{\mathbb{R}^N} \frac{\|v\|^2}{\nu(v)} M(v)\, dv = +\infty$$

diverges. This can happen if the collision frequency ν and/or the equilibrium M take a particular shapes (degeneracy at zero/ heavy tail). We cannot conclude, however, that these are the only scenarios giving rise to fractional diffusion phenomena.

Also, we have seen in Section 1.5.5 that the linear Boltzmann equation (1.30) gives the evolution of a single particle that undergoes free transport and scattering with the media. The scattering takes place at exponential rate of parameter ν, then the velocity of the particles changes and takes a new one with distribution M.

With this information in mind, we interpret the causes for the emergence of fractional phenomena.

Fractional phenomena due to a heavy-tail equilibria (large velocities) (Section1.2.3.3), [MMM11]. When the particle scatters and changes its velocity, in the presence of a heavy-tail equilibria, it is more probable to choose a large velocity since the variance is not finite. Hence, by choosing larger velocities, the particle makes larger displacements, therefore, it spreads faster than in the classical diffusive case. This gives rise to super-diffusion phenomena.

Fractional phenomena due to a degeneracy in the collision frequency (small velocities) (Section 1.2.3.4), [BAMP11]. In the probabilistic model we have considered a collision frequency equal to 1. This can be generalised considering non-trivial collision frequencies, the change is in the rate at which the jumps happen; instead of being exponentials of rate 1, they will we exponential of rate $\nu(V_{t-})$, i.e., the rate depends on the current velocity of the particle. Therefore, modifying this rate can contribute to having a faster spreading of the particles; making the rate of jumping very small for very small velocities. This is the case when the collision frequency presents a degeneracy at zero:

$$\nu(v) \sim \nu_0 \|v\|^{N+2+\beta}, \quad \text{as } \|v\| \to 0 \quad \text{for some } \beta > 0.$$

This means that the smaller the velocity is, the more unlikely is to change. Then, a particle that moves at small speed but (almost) in a straight line ends up further from its starting point than a particle who takes higher velocities but changes the direction frequently, since most of the displacement is averaged out. This is why this degeneracy at zero gives fractional phenomena at macroscopic level (Theorem 1.4).

Remember that we have also seen in Section 1.2.3.3 that ν can contribute to the fractional phenomena by favouring high velocities (therefore large displacements), i.e., for a particular choice of ν, for higher velocities the rate of change is smaller.

Chapter 2

Kinetic derivation of fractional Stokes and Stokes-Fourier systems

Joint work with Dr. Sabine Hittmeir

Contents

2.1 Introduction

In this section we aim to extend the fractional diffusion limit presented in Section 1.2.3 to a kinetic transport equation conserving not only mass, but also momentum and energy. Many works have been investigating the incompressible fluid dynamical limit of the Boltzmann equation, see e.g. [BU91], [GL02], [SR09] and references therein. We shall review the basic formal derivation of the linear equations of the corresponding hydrodynamic limit below. On the other hand Navier-Stokes type of equations with a fractional Laplacian have gained also a lot of interest, and have been e.g. related to a model with modified dissipativity arising in turbulence in [BPFS79]. For an existence and uniqueness result in Besov spaces we refer to [Wu06]. A derivation of fractional fluid dynamical equations from kinetic transport equations would therefore be desirable to obtain. As a first step towards this direction we here analyse the linear case, i.e. we start from a linear kinetic transport equation of the form

$$\partial_t f + v \cdot \nabla_x f = \mathcal{L} f \,, \tag{2.1}$$

where we assume the null space of $\mathcal{L}$ to be spanned by the equilibrium distribution $M(v)$ satisfying

$$M(v) = M(|v|) \geq 0, \quad M(v) < \infty, \quad \int_{\mathbb{R}^N} M(v) dv = 1 \,,$$

with the moment conditions

$$\int_{\mathbb{R}^N} M(v)\, dv = 1, \quad \int_{\mathbb{R}^N} |v|^2 M(v)\, dv = N, \quad \int_{\mathbb{R}^N} |v|^4 M(v)\, dv = N(N+2)\,. \tag{2.2}$$

We assume in the following $M(v)$ to be either the classical Gaussian

$$M^*(v) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{|v|^2}{2}} \,, \tag{2.3}$$

or a heavy tailed distribution satisfying

$$\tilde{M}(v) = \frac{c_0}{|v|^{\alpha+N}} \quad \text{for } |v| \geq 1 \tag{2.4}$$

for some $\alpha > 4$, that will be specified below and for some positive constant c_0. For the Gaussian $M^*(v)$ the moment conditions in (2.2) can be easily verified. For the second class of equilibrium distributions with heavy tails we only prescribe the behaviour for $|v| \geq 1$ and assume $\tilde{M}(v)$ to be smooth and bounded from above and below for small velocities. Hence for $\alpha > 4$ the particularly chosen constants in (2.2) mean no loss of generality. If in the following we keep the general notation $M(v)$, the statement holds for both $M(v) = M^*(v)$ and $M(v) = \tilde{M}(v)$.

The macroscopic moments for density, momentum and temperature (actually, temperature times density) of f are given by

$$U_f = \begin{pmatrix} \rho_f \\ m_f \\ \theta_f \end{pmatrix} = \int_{\mathbb{R}^N} \zeta(v) f dv, \qquad \text{where} \quad \zeta(v) = \begin{pmatrix} 1 \\ v \\ \frac{|v|^2-N}{N} \end{pmatrix}.$$

We consider a linear collision operator of the form

$$\mathcal{L}f = \nu(v)\left(\mathcal{K}f - f\right) \tag{2.5}$$

with the operator $\mathcal{K}$ being defined as

$$\mathcal{K}f = M(v)\,\phi(v) \cdot U_{\nu,f} = M(v)\left(\rho_{\nu,f} + v \cdot m_{\nu,f} + \frac{|v|^2 - N}{2}\theta_{\nu,f}\right), \tag{2.6}$$

where

$$\phi(v) = \begin{pmatrix} 1 \\ v \\ \frac{|v|^2-N}{2} \end{pmatrix} \tag{2.7}$$

differs from $\zeta(v)$ only due to a normalising constant in the last component. The collision frequency is assumed to be velocity dependent in the sense that

$$\nu(v) = \nu(|v|) \geq 0\,.$$

For the Gaussian equilibrium distribution the corresponding collision frequency $\nu(v) = \nu^*(v)$ is assumed to have a degeneracy as $|v| \to 0$ of the following form

$$\nu^*(v) = |v|^{\beta^*} \qquad \text{for} \quad |v| \leq 1\,, \tag{2.8}$$

for some $\beta^* > 0$ specified below. Moreover $\nu^*(v)$ is assumed to be smooth and bounded from above and below by a positive constant for $|v| \geq 1$. For the heavy-tailed equilibrium distribution the following far-field behaviour of the collision frequency $\nu(v) = \tilde{\nu}(v)$ is assumed

$$\tilde{\nu}(v) = |v|^{\tilde{\beta}} \qquad \text{for} \quad |v| \geq 1\,, \tag{2.9}$$

where $\tilde{\beta} < 1$ will be coupled to the parameter α determining the tail of $\tilde{M}(v)$. Here $\tilde{\nu}(v)$ is assumed to be smooth and bounded from above and below by a positive constant for small velocities. The macroscopic quantity $U_{\nu,f} = (\rho_{\nu,f}, m_{\nu,f}, \theta_{\nu,f})^T$ is defined via

$$\int_{\mathbb{R}^N} \nu \phi f dv = A U_{\nu,f} \tag{2.10}$$

in such a way that the collision operator satisfies the conservation laws

$$\int_{\mathbb{R}^N} \phi \mathcal{L} f dv = 0\,. \tag{2.11}$$

Using (2.6) this implies for the matrix A in (2.10)

$$A = \int_{\mathbb{R}^N} \nu\, \phi \otimes \phi M dv,$$

where invertibility of A can be checked by direct calculation. Observe that for f of the form $f = M\phi \cdot U$ we have $U_{\nu,f} = U_f = U$. We can then express the linear operator $\mathcal{K}$ as

$$\mathcal{K} f = M\,\phi \cdot U_{\nu,f} = M\,\phi \cdot A^{-1} \int_{\mathbb{R}^N} \nu \phi f dv\,. \tag{2.12}$$

Now the conservation properties can easily be checked

$$\begin{aligned} \int_{\mathbb{R}^N} \phi \mathcal{L} f\, dv &= \int_{\mathbb{R}^N} \nu \phi f dv - \int_{\mathbb{R}^N} \nu \phi M \phi \cdot A^{-1} dv \int_{\mathbb{R}^N} \nu \phi\, f\, dv \\ &= \left(I - \int_{\mathbb{R}^N} \nu \phi \otimes \phi M dv A^{-1} \right) \int_{\mathbb{R}^N} \nu \phi f dv = \left(I - A A^{-1} \right) \int_{\mathbb{R}^N} \nu \phi f dv = 0\,. \end{aligned}$$

Clearly the vector $\phi(v)$ in (2.11) can be replaced by the vector $\zeta(v)$, since their only difference is a normalising constant factor in the last component. Integrating the kinetic transport equation against $\zeta(v)$, the conservation laws in terms of the macroscopic moments read:

$$\begin{aligned} &\partial_t \rho_f + \nabla \cdot m_f = 0\,, \\ &\partial_t m_f + \nabla_x \cdot \int_{\mathbb{R}^N} v \otimes v\, f\, dv = 0\,, \\ &\partial_t \theta_f + \nabla_x \cdot \int_{\mathbb{R}^N} v \frac{|v|^2 - N}{N} f\, dv = 0\,. \end{aligned}$$

As mentioned above we will also investigate the limit to the fractional Stokes equation, hence in this case we shall only assume the conservation of mass and momentum. In this case we have

$$\bar{\phi}(v) = \begin{pmatrix} 1 \\ v \end{pmatrix}, \qquad \bar{U}_f = \begin{pmatrix} \rho_f \\ m_f \end{pmatrix} = \int_{\mathbb{R}^N} \bar{\phi}(v) f dv\,. \tag{2.13}$$

Observe in particular that the corresponding $\bar{A}$ is a diagonal matrix.

In the remainder of introduction we are going to motivate the choice of the linear BGK model and recall the formal classical Stokes-Fourier limit as well as point out the difference to the regime with fractional rescaling. We then summarise the assumptions on the equilibrium distributions and the parameters involved and state the main results. Section 2 contains well-posedness and a priori estimates. We then introduce in a similar fashion to [Mel10] and [BAMP11] an auxiliary function on which the moments method is based upon in Section 3 and prove the necessary convergence properties for the individual terms arising in the weak formulation. These are then unified for deriving the macroscopic dynamics in the fractional Stokes and Stokes-Fourier limit in Section 4.

Before demonstrating the classical formal Stokes-Fourier limit we shall give a brief motivation for the choice of our collision operator. Struchtrup [SR09] and e.g. also [CDL05] used a power law form of ν in terms of $|v - m/\rho|$ for large absolute values of the latter to obtain the correct Prandtl number out of a nonlinear BGK model of the following type:

$$\partial_t F + v \cdot \nabla_x F = \nu(|v - m_F/\rho_F|) \left(\mathcal{M}(\rho_{\nu,F}, m_{\nu,F}, \theta_{\nu,F}) - F\right) , \tag{2.14}$$

where $\mathcal{M}$ denotes the Maxwellian

$$\mathcal{M}(U) = \frac{\rho}{(2\pi\theta)^{N/2}} e^{-\frac{|v-m/\rho|^2}{2\theta}} .$$

The macroscopic quantities $U_{\nu,F}$ are again defined such that the conservation laws are guaranteed:

$$\int_{\mathbb{R}^N} \nu(|v - m_F/\rho_F|)\, \phi(v) \mathcal{M}(U_{\nu,F}) dv = \int_{\mathbb{R}^N} \nu(|v - m_F/\rho_F|)\, \phi(v) F \, dv .$$

We assume to be close to the global equilibrium $\mathcal{M}(1, 0, 1)$ (which corresponds to $M^*(v)$ from (2.3)). This means we can write for the remainder $F - \mathcal{M}(1, 0, 1) = \delta f$ for a small parameter δ. Then the linearised equation reads as follows

$$\partial_t f + v \cdot \nabla_x f = \nu(|v|) \left(\nabla_U \mathcal{M}(1, 0, 1) \cdot U_{\nu,f} - f\right) , \tag{2.15}$$

where $U_{\nu,f}$ is given by relation (2.10). Observing moreover $\nabla_U \mathcal{M}(1, 0, 1) = \phi(v) M^*(v)$ we arrive at (2.1) with the operator given by (2.6).

2.1.1 The (classical) Stokes-Fourier Limit

We shall briefly outline the formal derivation of the Stokes-Fourier system as the (classical) diffusion limit from the linear kinetic transport equation with the diffusion scaling

$\gamma = 2$:

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \nu(M\phi \cdot U^\varepsilon_\nu - f^\varepsilon)\,, \tag{2.16}$$

where here and in the following we denote the macroscopic moments of f^ε by $U^\varepsilon := U_{f^\varepsilon}$. For more details we refer e.g. to the work of [GL02], where the limit for the Boltzmann equation is carried out. Integration in v gives the macroscopic equation

$$\varepsilon \partial_t \rho^\varepsilon + \nabla_x \cdot m^\varepsilon = 0\,, \tag{2.17}$$

which is closed in terms of the macroscopic moments. This equation formally provides the incompressibility condition for m in the limit $\varepsilon \to 0$. Integrating (2.16) against v implies

$$\partial_t m^\varepsilon + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} v \otimes v f^\varepsilon dv = 0\,. \tag{2.18}$$

We shall split the second moment as follows

$$\partial_t m^\varepsilon + \frac{1}{\varepsilon}\nabla_x \int_{\mathbb{R}^N} \frac{|v|^2}{N} f^\varepsilon dv + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} \left(v \otimes v - \frac{|v|^2}{N} I\right) f^\varepsilon dv = 0\,. \tag{2.19}$$

The second term can be expressed in terms of the macroscopic moments as follows:

$$\int_{\mathbb{R}^N} \frac{|v|^2}{N} f^\varepsilon dv = \int_{\mathbb{R}^N} \left(\frac{|v|^2 - N}{N}\right) f^\varepsilon dv + \int_{\mathbb{R}^N} f^\varepsilon dv = \theta^\varepsilon + \rho^\varepsilon\,,$$

which provides the Boussinesq relation at leading order. The remaining terms of order 1 in the equation for m are of gradient type and therefore correspond to a pressure term, which vanishes when using divergence-free test functions. To analyse the behaviour of the third integral in (2.19) we employ the macro-micro decomposition

$$f^\varepsilon = M\phi \cdot U^\varepsilon_\nu + g^\varepsilon_\nu\,,$$

which inserted into the kinetic equation (2.16) formally gives

$$g^\varepsilon_\nu = -\varepsilon \frac{v}{\nu} M \cdot \nabla_x(\phi \cdot U^\varepsilon_\nu) + O(\varepsilon^2) = -\varepsilon \frac{v}{\nu} M \cdot (\phi \cdot \nabla_x U^\varepsilon) + O(\varepsilon^2)\,,$$

since knowing that g^ε_ν is $O(\varepsilon)$, implies that $U^\varepsilon_\nu = U^\varepsilon + O(\varepsilon)$. Now one can see that the macroscopic part of the antisymmetric integral term in (2.19) vanishes and we are left with

$$\partial_t m^\varepsilon + \frac{1}{\varepsilon}\nabla_x(\rho^\varepsilon + \theta^\varepsilon) + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} \left(v \otimes v - \frac{|v|^2}{N} I\right) g^\varepsilon_\nu dv = 0\,.$$

The leading order term of g_ν^ε implies

$$
\begin{aligned}
&-\frac{1}{\varepsilon}\nabla_x \int_{\mathbb{R}^N} \left(v\otimes v - \frac{|v|^2}{N} I \right) g_\nu^\varepsilon dv = \nabla_x \cdot \int_{\mathbb{R}^N} \left(v\otimes v - \frac{|v|^2}{N} I \right) \frac{M}{\nu} (v\otimes v \,:\, \nabla_x m^\varepsilon) dv + O(\varepsilon) \\
&\quad = \mu_0 \nabla_x \cdot (\nabla_x m^\varepsilon + (\nabla_x m^\varepsilon)^T) + O(\varepsilon) = \mu_0 \Delta_x m^\varepsilon + O(\varepsilon)\,,
\end{aligned}
$$

for $\mu_0 = \int_{\mathbb{R}^N} v_1^2 v_2^2 \frac{M}{\nu} dv$, where we have used the incompressiblity condition to leading order. Summarising we obtain from the equation for m^ε

$$
\nabla_x(\rho^\varepsilon + \theta^\varepsilon) = O(\varepsilon)\,, \tag{2.20}
$$

$$
\partial_t m^\varepsilon = \mu_0 \Delta_x m^\varepsilon + \nabla_x p^\varepsilon + O(\varepsilon)\,. \tag{2.21}
$$

We shall now turn to the equation for the temperature and therefore consider the following moment

$$
\partial_t \int_{\mathbb{R}^N} \frac{|v|^2 - (N+2)}{2} f^\varepsilon dv + \frac{1}{\varepsilon} \nabla_x \cdot \int_{\mathbb{R}^N} v\, \frac{|v|^2 - (N+2)}{2} f^\varepsilon dv = 0\,. \tag{2.22}
$$

Note that due to the Boussinesq equation we have

$$
\int_{\mathbb{R}^N} \frac{|v|^2 - (N+2)}{2} f^\varepsilon dv = \frac{N}{2}(\theta^\varepsilon - \rho^\varepsilon) = N\theta^\varepsilon + O(\varepsilon)\,.
$$

The choice of the moment is such that inserting the decomposition into the second integral, the leading term vanishes:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} v \frac{|v|^2 - (N+2)}{2} f^\varepsilon dv \\
&\quad = \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} v\otimes v \frac{|v|^2 - (N+2)}{2} M m^\varepsilon dv + \frac{1}{\varepsilon}\nabla_x \cdot \int_{\mathbb{R}^N} v \frac{|v|^2 - (N+2)}{2} g^\varepsilon dv \\
&\quad = -\nabla_x \cdot \int_{\mathbb{R}^N} v\otimes v\, \frac{|v|^2 - (N+2)}{2} \frac{M}{\nu} \nabla_x (\phi \cdot U^\varepsilon) dv + O(\varepsilon) \\
&\quad = -\nabla_x \cdot \int_{\mathbb{R}^N} v\otimes v\, \frac{|v|^2 - (N+2)}{2} \frac{M}{\nu} \nabla_x \left(\rho^\varepsilon + \frac{|v|^2 - N}{2}\theta^\varepsilon \right) dv + O(\varepsilon) \\
&\quad = -N\kappa_0 \Delta_x \theta^\varepsilon + O(\varepsilon)\,,
\end{aligned}
$$

for $\kappa_0 = \int_{\mathbb{R}^N} \frac{|v|^2(|v|^2-(N+2))^2}{4N} \frac{M}{\nu} dv > 0$, where we used the Boussinesq relation to leading order. Hence formally we arrive in the limit $\varepsilon \to 0$ at the incompressible Stokes-Fourier system:

$$
\begin{aligned}
\rho + \theta = 0\,, \qquad & \nabla_x \cdot m = 0 \\
\partial_t m \;&=\; \mu_0 \Delta_x m + \nabla_x p \\
\partial_t \theta \;&=\; \kappa_0 \Delta_x \theta
\end{aligned}
$$

Note that the momentum satisfies a heat equation up to a pressure gradient. This pressure term vanishes when using divergence-free testfunctions, which are typically used for incompressible fluid dynamical equations.

2.1.2 Rescaled equation for fractional Stokes-Fourier limit and function spaces

As already mentioned in the introduction above it is our aim to analyse the Cauchy problem for the kinetic equation with a rescaling in time of order $\gamma \in (1,2)$:

$$\begin{aligned}
&\varepsilon^\gamma \partial_t f^\varepsilon + \varepsilon v \cdot \nabla_x f^\varepsilon = \mathcal{L} f^\varepsilon \\
&f^\varepsilon(0,v,x) = f^{in}(v,x) \quad \in L^2_{x,v}(M^{-1}), \qquad \text{satisfying} \quad \nabla \cdot \int_{\mathbb{R}^N} v f^{in} dv = 0\,.
\end{aligned} \tag{2.23}$$

Note that the latter condition guarantees that the initial data verifies the incompressibility condition $\nabla_x \cdot m^{in} = 0$. Here and in the following we denote weighted L^2-spaces as:

$$\|h\|^2_{L^2_{t,x,v}(\omega)} = \int_0^\infty \int_{\mathbb{R}^{2N}} h^2\, \omega\, dv dx dt\,. \tag{2.24}$$

The weight functions we are considering will only depend on v. To be more precise we will need the weight functions $M^{-1}, \nu M^{-1}$ and M. The spaces $L^2_{x,v}(\omega)$ and $L^2_v(\omega)$ are defined in a similar way, where integration in (2.24) is performed over x, v or v respectively. Also we shall use the abbrevations $L^p_t = L^p(0,\infty)$, $L^p_{x,v} = L^p(\mathbb{R}^N \times \mathbb{R}^N)$ and $L^p_{t,x} = L^p((0,\infty)\times\mathbb{R}^N)$.

The conservation property of $\mathcal{L}$ implies for the zeroth moment of (2.23) after dividing by ε

$$\varepsilon^{\gamma-1}\partial_t \rho^\varepsilon + \nabla_x \cdot m^\varepsilon = 0\,, \tag{2.25}$$

which provides again the incompressibility condition to leading order. Using the same macro-micro decomposition as above, we obtain for the first and second moment similar to before

$$\begin{aligned}
\partial_t m^\varepsilon + \varepsilon^{1-\gamma}\nabla_x(\rho^\varepsilon + \theta^\varepsilon) &= \varepsilon^{2-\gamma}\nabla_x \cdot \int_{\mathbb{R}^N} \left(v\otimes v - \frac{|v|^2}{N} I\right) \frac{v}{\nu} M \cdot \nabla_x (v \cdot m^\varepsilon) dv + O(\varepsilon)\,, \\
\partial_t \theta^\varepsilon &= \varepsilon^{2-\gamma}\nabla_x \cdot \int_{\mathbb{R}^N} \frac{|v|^2(|v|^2-(N+2))^2}{4N}\frac{M}{\nu} dv \nabla_x \theta^\varepsilon + O(\varepsilon)\,.
\end{aligned}$$

If we consider the fractional Stokes limit, then either the 2nd or the 6th moment of M/ν will be unbounded, but in such a way that it is balanced by the order $\varepsilon^{2-\gamma}$ in the limit $\varepsilon \to 0$. Considering the fractional Stokes limit (i.e. there is no equation for θ) requires the 4th moment to be unbounded. This also explains why we cannot derive a fractional Stokes-Fourier system with a fractional Laplacian appearing in both equations for m and θ.

We shall also note that the scaling $\gamma = 1$ corresponds to the scaling for the acoustic limit.

2.1.3 Summary of the assumptions and results

Assumption 1. [Assumptions on the parameters for the fractional Fourier-Stokes limit]

(i) For the case of heavy-tailed equilibrium distributions $\tilde{M}$ we shall make the following assumptions on the parameters $\alpha, \tilde{\beta}$ determining the behaviour of $\tilde{M}$ and the corresponding collision frequency $\tilde{\nu}$ for large $|v|$ (see (2.4) and (2.9)):

Let $\alpha > 5$ and $\tilde{\beta} < 1$ satisfy

$$5 < \alpha + \tilde{\beta} < 6\,, \qquad \tilde{\beta} < \frac{\alpha - 4}{2}\,. \tag{2.26}$$

The parameter $\tilde{\gamma}$ used for the rescaling in time then satisfies

$$\tilde{\gamma} = \frac{\alpha - \tilde{\beta} - 4}{1 - \tilde{\beta}} \;\in\; (1, 2)\,.$$

Observe that this also includes a velocity independent collision frequency $\tilde{\nu}(v) \equiv 1$. In this case the requirements on the parameters are

$$\tilde{\beta} = 0\,, \qquad \alpha = 5 + \delta \quad \text{for} \quad \delta \in (0, 1)\,, \qquad \tilde{\gamma} = 1 + \delta\,.$$

(ii) For the Gaussian equilibrium distributions M^* the collision frequency ν^* is degenerate as $|v| \to 0$ with exponent $\beta^* > 1$, see (2.8). For this exponent β^* and the corresponding parameter γ^* for the rescaling in time we assume

$$N + 2 < \beta^* < N + 3\,, \qquad \gamma^* = \frac{\beta^* + N}{\beta^* - 1} \;\in\; (1, 2)\,.$$

These conditions stated in Assumption 1 imply for the heavy-tailed equilibrium distribution the following integrability properties

$$\int_{\mathbb{R}^N} \frac{|v|^k}{\tilde{\nu}} \tilde{M} dv \leq C \;\; (k \leq 5), \qquad \int_{\mathbb{R}^N} \frac{|v|^6}{\tilde{\nu}} \tilde{M} dv = \infty\,, \tag{2.27}$$

whereas for the Gaussian equlibrium distribution the unboundedness occurs at the lowest order

$$\int_{\mathbb{R}^N} \frac{|v|^2}{\nu^*} M^* dv = \infty, \qquad \int_{\mathbb{R}^N} \frac{|v|^j}{\nu^*} M^* dv \leq C \;\; (j \geq 3)\,. \tag{2.28}$$

If in the following the statements do hold for both cases of equilibrium distributions in Assumption 1 we write (M, γ), which can be either $(\tilde{M}, \tilde{\gamma})$ or (M^*, γ^*).

Theorem 2.1. *Let Assumption 1 hold. Then the solution f^ε to (2.23) converges as $\varepsilon \to 0$ to*

$$f^\varepsilon(t,x,v) \rightharpoonup^* f(t,x,v) = M\left(v\cdot m(x) + \frac{|v|^2-(N+2)}{2}\theta(t,x)\right) \quad \text{in} \quad L^\infty(0,T;L^2_{x,v}(\nu M^{-1}))\,, \tag{2.29}$$

where the macroscopic quantities are determined by

$$\begin{aligned} &m(x) = m^{in}(x),\\ &\partial_t\theta = -\kappa(-\Delta)^{\gamma/2}\theta, \qquad \theta(0,x) = \theta^{in}(x)\,, \end{aligned}$$

for a positive constant $\kappa > 0$, where the equations are understood in the weak sense. In particular $\partial_t m = 0$ holds in distribution sense restricted to divergence-free testfunctions. The initial data

$$U^{in} = \int_{\mathbb{R}^N} \zeta(v) f^{in}(x,v)dv$$

is hereby assumed to satisfy

$$\nabla_x\cdot m^{in}(x) = 0, \quad \rho^{in}(x) + \theta^{in}(x) = 0\,.$$

The derivation of this theorem shows that one cannot obtain a fractional derivative in all moments at the same time, since the chosen time scale is not the right one for the diffusive terms in the momentum equation. For the sake of completeness we shall recall here that the fractional Laplacian can be defined using the Fourier Transform

$$\mathcal{F}((-\Delta_x)^{\gamma/2}h)(k) = |k|^\gamma \mathcal{F}(h)(k)\,.$$

We will rather use the following alternative representation as a singular integral

$$(-\Delta_x)^{\gamma/2}h = C_{N,\gamma} PV\int_{\mathbb{R}^N}\frac{h(x)-h(y)}{|x-y|^{N+\gamma}}dy\,,$$

see e.g. also [DNPV12].

Assumption 2. [Assumptions on the parameters for the fractional Stokes system without temperature] We shall here only consider the case of heavy-tailed equilibrium distributions $\tilde{M}$ with corresponding collision frequency $\tilde{\nu}$. For the parameters α and $\tilde{\beta}$ (see (2.4) and (2.9)) we make the following assumptions:

Let $\alpha > 3$ and $\tilde{\beta} < 1$ satisfy

$$3 < \alpha + \tilde{\beta} < 4\,, \qquad \tilde{\beta} < \frac{\alpha - 2}{2}\,. \tag{2.30}$$

The parameter used for the rescaling in time then satisfies

$$\tilde{\gamma} = \frac{\alpha - \tilde{\beta} - 2}{1 - \tilde{\beta}} \in (1, 2) .$$

Again this includes the case $\tilde{\nu} \equiv 1$ with the choice of parameters

$$\tilde{\beta} = 0 , \qquad \alpha = 3 + \delta \quad \text{for} \quad \delta \in (0, 1) , \qquad \tilde{\gamma} = 1 + \delta .$$

The corresponding conditions to (2.27) for these heavy-tailed equilibrium distribution read

$$\int_{\mathbb{R}^N} \frac{|v|^k}{\tilde{\nu}} \tilde{M} dv \leq C \;\; (k \leq 3), \qquad \int_{\mathbb{R}^N} \frac{|v|^4}{\tilde{\nu}} \tilde{M} dv = \infty . \tag{2.31}$$

Theorem 2.2. *Let Assumption 2 hold. Then the solution f^ε to (2.23) converges as $\varepsilon \to 0$ to*

$$f^\varepsilon(t, x, v) \rightharpoonup^* f(t, x, v) = M(\rho(x) + v \cdot m(t, x)) \quad \text{in} \quad L^\infty(0, T; L^2_{x,v}(\nu M^{-1})) , \tag{2.32}$$

where the macroscopic quantities solve

$$\begin{aligned} \rho(x) \;&=\; \rho^{in}(x) , \\ \nabla \cdot m &= 0 , \\ \partial_t m &= -\kappa(-\Delta)^{\tilde{\gamma}/2} m + \nabla_x p , \qquad m(0, x) = m^{in}(x) \end{aligned}$$

where the equation for the evolution of m holds in the weak sense. The pressure term $p \in L^2_{t,x}$ vanishes when using divergence-free testfunctions. The initial data $\bar{U}^{in} = \int_{\mathbb{R}^N} \bar{\phi} f^{in} dv$ is assumed to satisfy $\nabla \cdot m^{in} = 0$.

In this regime the fractional diffusion only appears in the equation for the momentum, whereas the density does not change with time. This resembles well the Navier-Stokes equations, where the density (and temperature) are assumed to be constant and the continuity equation reduces to the incompressibility condition.

Remark 2.3. The reason why the fractional Stokes limit cannot be carried out for the Gaussian equilibrium distribution is that in this case the fractional derivative arises from the unbounded second moment of M/ν and therefore appears for the density term. In the case of the Stokes-Fourier system the Boussinesq equation then relates the density to the temperature. In the Stokes limit however no such relation is available.

2.2 A priori estimates and the Cauchy problem

2.2.1 Integrability conditions on M

The above Assumptions 1 and 2 on the parameters determining the behaviour of M and ν guarantee the boundedness of the moments required for carrying out the macroscopic limit. We summarise these integrability conditions in the following Lemma:

Lemma 2.4. *Let (M,ν) be either given by $(\tilde{M},\tilde{\nu})$ or (M^*,ν^*). In both cases we assume that the corresponding conditions on the parameters stated in Assumption 1 are satisfied. Then the following integrability conditions hold*

$$\int_{|v|\geq\delta}\frac{|v|^2M(v)}{\nu(v)}dv\leq C\,,\qquad\int_{\mathbb{R}^N}\frac{|v|^{j+3}M(v)}{\nu(v)}dv\leq C\qquad\text{for } 0\leq j\leq 2\,,\tag{2.33}$$

$$\int_{\mathbb{R}^N}|v|^k\nu^2(v)M(v)dv\leq C\,,\qquad\int_{\mathbb{R}^N}|v|^k\nu(v)M(v)dv\leq C\qquad\text{for } 0\leq k\leq 4\,,\tag{2.34}$$

where $\delta=0$ in the case of heavy-tailed equilibrium distributions, and $0<\delta=1$ (w.l.o.g.) in the case of the Gaussian equilibrium distributions.

If only the conservation of mass and momentum hold, the order of integrable moments reduces as follows:

Lemma 2.5. *For the heavy-tailed equilibrium distributions satisfying Assumption 2 the integrability conditions (2.33) hold for $j=0$ and (2.34) is satisfied for $0\leq k\leq 2$.*

2.2.2 A priori estimates and well-posedness

Lemma 2.6. *Let the equilibrium distribution M satisfy Assumption 1 or 2, then*

$$\|\nu\mathcal{K}f\|_{L^2_v(M^{-1})}\leq C\|f\|_{L^2_v(M^{-1})}\,.$$

Proof. The proof can be easily seen by first observing that

$$\|\nu\mathcal{K}f\|_{L^2_v(M^{-1})}=\int_{\mathbb{R}^N}\nu^2M(\phi\cdot U_\nu)^2dv\leq C|U_\nu|^2\,,\tag{2.35}$$

where we have used the boundedness of M in (2.34), which can now be employed again together with the Cauchy-Schwarz inequality to conclude

$$|U_\nu|^2=\left|A^{-1}\int_{\mathbb{R}^N}\nu\phi f dv\right|^2\leq C\int_{\mathbb{R}^N}\frac{f^2}{M}dv\int_{\mathbb{R}^N}\nu^2|\phi|^2M\,dv\leq C\|f\|^2_{L^2_v(M^{-1})}\,.\tag{2.36}$$

□

This continuity property of the linear collision operator allows to deduce well-posedness of the Cauchy-problem (2.1) with initial data $f^{in} \in L^2_{x,v}(M^{-1})$. The mild formulation reads

$$f(t,x,v) = f^{in}(x-vt,v)e^{-\nu t} + \int_0^t e^{-\nu(t-s)}\nu\mathcal{K}f(s,x-(t-s)v,v)ds\,.$$

If the assumptions guaranteeing continuity of $\mathcal{K}$ as in Lemma 2.6 hold, then a standard contraction argument yields local well-posedness, which can be extended to a global result using the a priori estimate (2.41) below for $\varepsilon = 1$. Clearly also the Cauchy problem for the rescaled kinetic equation is well posed for any $\varepsilon > 0$:

Corollary 2.7. *Let Assumption 1 or Assumption 2 hold and let $f^{in} \in L^2_{x,v}(M^{-1})$. Then there exists a unique solution $f^\varepsilon \in L^\infty_t(L^2_{x,v}(M^{-1}))$ to (2.23).*

Since we want to determine the convergence of f^ε as $\varepsilon \to 0$ we shall now investigate the a priori estimates for the rescaled problem. The basic L^2-estimate for kinetic transport equations is obtained by integrating the equation against f^ε/M. Similar to the formal derivation of the Fourier-Stokes limit in the introduction we shall introduce the micro-macro decompositions

$$f^\varepsilon = M\,\phi\cdot U^\varepsilon + g^\varepsilon\,, \tag{2.37}$$

$$f^\varepsilon = M\,\phi\cdot U^\varepsilon_\nu + g^\varepsilon_\nu = \mathcal{K}f^\varepsilon + g^\varepsilon_\nu\,, \tag{2.38}$$

whose remainder terms fulfill

$$\int_{\mathbb{R}^N} \phi g^\varepsilon dv = 0\,, \qquad \int_{\mathbb{R}^N} \nu\phi g^\varepsilon_\nu dv = 0\,, \tag{2.39}$$

due to the definition of the macroscopic moments and the conservation properties respectively. In a similar fashion to [MMM11] and [BAMP11] we obtain the following lemma:

Lemma 2.8. *Let Assumption 1 or Assumption 2 hold. Then the operator $\frac{1}{\nu}\mathcal{L}$ is bounded in $L^2_v(\nu M^{-1})$ and satisfies*

$$\int_{\mathbb{R}^N} \mathcal{L}f\frac{f}{M}dv = -\int_{\mathbb{R}^N}\frac{\nu}{M}|f-\mathcal{K}f|^2dv \tag{2.40}$$

for a positive constant C and for all $f \in L^2_v(\nu M^{-1})$.

Proof. To prove the boundedness of $\frac{1}{\nu}\mathcal{L}$ it remains to check the boundedness of $\mathcal{K}$. In a similar fashion to (2.35) one can show that $\|\mathcal{K}f\|_{L^2_v(\nu M^{-1})} \leq C|U_\nu|^2$, and we conclude the boundedness with a slight modification of (2.36):

$$|U_\nu|^2 = \left|A^{-1}\int_{\mathbb{R}^N}\nu\phi f dv\right|^2 \leq C\int_{\mathbb{R}^N}\frac{\nu}{M}f^2dv\int_{\mathbb{R}^N}\nu|\phi|^2Mdv \leq C\|f\|^2_{L_v(\nu M^{-1})}\,.$$

To show (2.40) we first observe that due to the conservation properties of $\mathcal{L}$ (2.11) we have

$$\int_{\mathbb{R}^N} \mathcal{L}f \frac{\mathcal{K}f}{M} dv = \int_{\mathbb{R}^N} \phi \, \mathcal{L}f \, dv \cdot U_\nu = 0 \, .$$

Using this we can rewrite

$$\int_{\mathbb{R}^N} \mathcal{L}f \frac{f}{M} dv = \int_{\mathbb{R}^N} \mathcal{L}f \frac{f - \mathcal{K}f}{M} dv = - \int_{\mathbb{R}^N} \frac{\nu}{M} |f - \mathcal{K}f|^2 dv \, .$$

$\square$

This lemma now yields the basic ingredient for deriving the following a priori estimates:

Proposition 2.9. *Let Assumption 1 be satisfied. Then the solution f^ε of (2.23) is bounded in $L^\infty_t(L^2_{x,v}(M^{-1}))$ uniformly with respect to ε. Moreover it satisfies the decomposition (2.38), where U^ε_ν and g^ε_ν are bounded by the initial data f^{in} in the sense that*

$$\begin{aligned} \sup_{t>0} \|f^\varepsilon\|_{L^2_{x,v}(M^{-1})} &\leq \|f^{in}\|_{L^2_{x,v}(M^{-1})} \, , && (2.41) \\ \|g^\varepsilon_\nu\|_{L^2_{t,x,v}(\nu M^{-1})} &\leq \varepsilon^{\gamma/2} \|f^{in}\|_{L^2_{x,v}(M^{-1})} \, , && (2.42) \\ \sup_{t>0} \|U^\varepsilon_\nu(t,.)\|_{L^2_x} &\leq C \|f^{in}\|_{L^2_{x,v}(M^{-1})} \, . && (2.43) \end{aligned}$$

Proof. Using (2.40), the basic L^2-estimate for the solution is obtained as follows

$$\frac{\varepsilon^\gamma}{2} \frac{d}{dt} \|f^\varepsilon\|^2_{L^2_{x,v}(M^{-1})} = \int_{\mathbb{R}^{2N}} \mathcal{L}f^\varepsilon \frac{f^\varepsilon}{M} dv dx = - \int_{\mathbb{R}^{2N}} \frac{\nu}{M} |f^\varepsilon - \mathcal{K}f^\varepsilon|^2 dv dx = - \int_{\mathbb{R}^{2N}} \frac{\nu}{M} (g^\varepsilon_\nu)^2 dv dx \, .$$

Integration in time implies (2.41) and (2.42). For the boundedness of the macroscopic moments U^ε_ν in (2.43) it only remains to integrate (2.36) over x and taking the supremum in time. $\square$

Lemma 2.10. *Let the assumptions of Proposition 2.9 hold. Then there exists a $U \in L^\infty_t(L^2_x)$, such that $f^\varepsilon \rightharpoonup^* M\phi \cdot U$ in $L^\infty((0,T); L^2_{x,v}(\nu M^{-1}))$ for any $T > 0$. In particular we have the convergence of the macroscopic moments U^ε_ν, $U^\varepsilon \rightharpoonup^* U$ in $L^\infty((0,T); L^2_x)$. In the case of heavy tailed equilibrium distributions $\tilde{M}$ moreover strong convergence of $U^\varepsilon_\nu - U^\varepsilon \to 0$ in $L^\infty((0,T); L^2_x)$ holds. Under Assumption 2 the same statements are valid for $\bar{U}^\varepsilon_\nu$ and $\bar{U}^\varepsilon$ respectively.*

Proof. To see the weak*-convergence we first observe that the uniform bound of U^ε_ν in $L^\infty_t(L^2_x)$ given in (2.43) implies the existence of a $U \in L^\infty_t(L^2_x)$ such that $U^\varepsilon_\nu \rightharpoonup^* U$ in $L^\infty_t(L^2_x)$. Moreover the bound (2.42) implies that $f^\varepsilon - M\phi \cdot U^\varepsilon_\nu \to 0$ in $L^2_{t,x,v}(\nu M^{-1})$, which allows to deduce $f^\varepsilon \rightharpoonup^* M\phi \cdot U$ in $L^\infty((0,T); L^2_{x,v}(\nu M^{-1}))$ for any $T > 0$, implying also for the macroscopic moment $U^\varepsilon \rightharpoonup^* U$ in $L^\infty((0,T); L^2_x)$.

To show the strong convergence of $U^\varepsilon_\nu - U^\varepsilon$ in the case of heavy-tailed equilibria we first note that integrating the difference of the decompositions (2.37)-(2.38) against ϕ gives

$$A(U^\varepsilon_\nu - U^\varepsilon) = \int_{\mathbb{R}^N} \phi(g^\varepsilon - g^\varepsilon_\nu)dv = -\int_{\mathbb{R}^N} \phi g^\varepsilon_\nu dv\,.$$

In the case of $M(v) = \tilde{M}(v)$ the integrability of M in (2.33) holds for $\delta = 0$ and we can thus employ the Cauchy-Schwarz inequality as follows

$$\begin{aligned} \|U^\varepsilon_\nu(t,.) - U^\varepsilon(t,.)\|^2_{L^2(\mathbb{R}^N)} &\leq C\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\phi g^\varepsilon_\nu dv\right)^2 dx \\ &\leq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{\nu(g^\varepsilon_\nu)^2}{M}dvdx\int_{\mathbb{R}^N}\frac{|\phi|^2 M}{\nu}dv \leq C\varepsilon^\gamma\,, \end{aligned}$$

where for the last inequality we applied (2.42). □

2.3 Weak formulation and auxiliary equation

2.3.1 An auxiliary equation

Analogously to Mellet [Mel10] and Ben-Abdallah et al. [BAMP11] we introduce an auxiliary function $\chi^\varepsilon(t,v,x)$ defined as the solution of

$$\nu(v)\chi^\varepsilon - \varepsilon v\cdot\nabla_x\chi^\varepsilon = \nu(v)\varphi(t,x)\,, \tag{2.44}$$

where $\varphi(t,x)$ is a test function in $\mathcal{D}([0,\infty)\times\mathbb{R}^N)$ and hence $\chi^\varepsilon \in L^\infty_{t,v}((0,\infty)\times\mathbb{R}^N; L^2_x(\mathbb{R}^N))$. It is easy to verify that

$$\chi^\varepsilon = \int_0^\infty e^{-\nu(v)z}\nu(v)\varphi(t,x+\varepsilon vz)dz\,.$$

Considering

$$\chi^\varepsilon - \varphi = \int_0^\infty \nu e^{-\nu z}(\varphi(t,x+\varepsilon vz) - \varphi(t,x))dz\,, \tag{2.45}$$

it can easily be deduced that $|\chi^\varepsilon - \varphi| \leq \|D\varphi\|_\infty\varepsilon|v|$, which implies uniform convergence in space and time, but not with respect to v. The proof of Lemma 2.5 in [BAMP11] can easily be extended to give the following convergence results:

$$\phi\,\chi^\varepsilon \to \phi\,\varphi \quad \text{strongly in } L^\infty_t(L^2_{x,v}(M))\,, \tag{2.46}$$

$$\phi\,\partial_t\chi^\varepsilon \to \phi\,\partial_t\varphi \quad \text{strongly in } L^\infty_t(L^2_{x,v}(M))\,, \tag{2.47}$$

where the extension from $\phi \equiv 1$ in [BAMP11] to ϕ given as in (2.7) is straightforward due to the weight M. The proof relies on a estimate of the form

$$
\begin{aligned}
\|\phi(\chi^\varepsilon - \varphi)\|^2_{L^2_{x,v}(M)} &= \int_{\mathbb{R}^{2N}} M \left| \int_0^\infty e^{-\nu z} \nu \phi(\varphi(x + \varepsilon v z) - \varphi(x)) dz \right|^2 dx dv \\
&\leq \int_{\mathbb{R}^N} \int_0^\infty M e^{-\nu z} \nu |\phi|^2 \|\varphi(\cdot + \varepsilon v z) - \varphi\|^2_{L^2_x} dz dv
\end{aligned}
$$

The fact that $\|\varphi(\cdot + \varepsilon v z) - \varphi\|_{L^2_x} \to 0$ as $\varepsilon \to 0$ for all v and z, together with the integrability condition (2.34), allow to apply the Lebesgue dominated convergence theorem. A similar proof holds for the time derivative.

2.3.2 The weak formulation

Since the macroscopic equation for ρ^ε is closed in terms of the macroscopic moments U^ε (see (2.25)), it is sufficient to consider test functions $\varphi(t,x) \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ independent of v. Note that this corresponds to building the inner product in $L^2_{t,x,v}(M^{-1})$ of the kinetic equation with $\varphi(t,x)M(v)$.

$$
-\int_0^\infty \int_{\mathbb{R}^N} \rho^\varepsilon \partial_t \varphi dx dt - \int_{\mathbb{R}^N} \rho^{in} \varphi(t=0) dx = \varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^N} \nabla_x \varphi \cdot m^\varepsilon dx dt \quad (2.48)
$$

This equation will in the limit provide the incompressibility condition.

In order to derive equations for the macroscopic momentum and temperature we consider the weak formulation of the rescaled kinetic equation (2.23) using testfunctions as introduced in the previous subsection. As for the classical Stokes-Fourier equations we shall consider the following moments corresponding to

$$
\psi(v) = \begin{pmatrix} v \\ \frac{|v|^2 - (N+2)}{2} \end{pmatrix}.
$$

We shall for each moment ψ_i consider a separated testfunction $\phi_i \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ with its corresponding auxiliary function χ_i^ε. Integrating the kinetic equation against $\psi_i \chi_i^\varepsilon$ gives

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i f^\varepsilon \partial_t \chi_i^\varepsilon dv dx dt - \int_{\mathbb{R}^{2N}} \psi_i f^{in} \chi_i^\varepsilon(t=0) dv dx \\
&= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i \mathcal{L} f^\varepsilon \, \chi_i^\varepsilon dv dx dt + \varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i v \, f^\varepsilon \cdot \nabla_x \chi_i^\varepsilon dv dx dt \\
&= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i M \, \phi \cdot U_\nu^\varepsilon \, \chi_i^\varepsilon dv dx dt + \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i f^\varepsilon (-\nu \chi_i^\varepsilon + \varepsilon v \cdot \nabla_x \chi_i^\varepsilon) dv dx dt \\
&= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \nu \psi_i M \, \phi \cdot U_\nu^\varepsilon \, \chi_i^\varepsilon dv dx dt - \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \nu \psi_i f^\varepsilon \, dv \, \varphi_i \, dx dt \,,
\end{aligned}
$$

where we have used the auxiliary equation (2.44). Taking into account the conservation property of the collision operator (2.11) in the latter integral we finally obtain the weak formulation

$$
\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i f^\varepsilon \partial_t \chi_i^\varepsilon dv dx dt - \int_{\mathbb{R}^{2N}} \psi_i f^{in} \chi^\varepsilon(t=0) dx dv \\
&= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i M \, \phi \cdot U_\nu^\varepsilon \, \nu (\chi_i^\varepsilon - \varphi_i) dv dx dt \,. \qquad (2.49)
\end{aligned}
$$

In the following we will analyse the convergence properties of this weak form, in particular the right hand side. In the next subsection we will analyse the limiting behaviours of the separate terms. These Lemmas will then be used in Section 2.4 to conclude the proofs of the Theorems 2.1 and 2.2.

2.3.3 Convergence properties

We first derive the convergence results required for the macroscopic limit to the fractional Stokes-Fourier system. At the end of the subsection we will derive the corresponding convergence properties for the fractional Stokes limit for conservation of density and momentum only.

In the following we will several times have to bound integrals of the form

$$
I(t,x) = \int_{\mathbb{R}^N} f(v) g(t, x + \tau v) dv
$$

in $L^2_{t,x}$ for some $\tau \in \mathbb{R}$. This can be done by first applying the Cauchy-Schwarz inequality and then interchanging the order of integration:

$$
\begin{aligned}
\|I\|^2_{L^2_{t,x}} &= \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} f(v) g(t, x + \tau v) dv \right)^2 dx dt \\
&\le \int_0^\infty \int_{\mathbb{R}^N} |f(v)| dv \int_{\mathbb{R}^N} |f(v)| g^2(t, x + \tau v) dv dx dt \\
&= \int_0^\infty \int_{\mathbb{R}^N} g^2(t,x) dx dt \left(\int_{\mathbb{R}^N} |f(v)| dv \right)^2 = \|g\|_{L^2_{t,x}} \|f\|^2_{L^1_v} \,. \qquad (2.50)
\end{aligned}
$$

We shall first consider the terms arising from the time derivative on the left hand side of the weak formulation in (2.49):

Lemma 2.11. *Let Assumption 1 hold and let χ_i^ε be auxiliary functions satisfying (2.44) for $\varphi_i \in \mathcal{D}([0,\infty) \times \mathbb{R}^N)$ $(i \in \{1, \dots, N\})$. Let moreover f_ε be the weak solution as in Proposition 2.9. Then, as $\varepsilon \to 0$, the weak form of the time derivatives in (2.49) converges in the sense that*

$$
\int_0^\infty \int_{\mathbb{R}^{2N}} \psi_i f^\varepsilon \partial_t \chi_i^\varepsilon dv dx dt + \int_{\mathbb{R}^{2N}} \psi_i f^{in} \chi_i^\varepsilon(t=0) dv dx
$$

$$\to \int_{\mathbb{R}^N} \psi_i \phi \, M \, dv \cdot \left(\int_0^\infty \int_{\mathbb{R}^N} U \partial_t \varphi_i dx dt + \int_{\mathbb{R}^N} U^{in} \varphi_i (t=0) dx \right)$$

Proof. Due to the strong convergence of $\psi \partial_t \chi_i^\varepsilon \to \psi \partial_t \varphi_i$ in $L^\infty((0,\infty); L^2_{x,v}(M))$ in (2.47) the weak convergence of $f^\varepsilon \rightharpoonup M\phi \cdot U$ in $L^\infty((0,T); L^2_{x,v}(M^{-1}))$ and the fact that φ_i is a test function, the stated convergence can be deduced. □

For passing to the limit in the right hand side of the weak formulation in (2.49) we will make use of the following expansions of the auxiliary function obtained by integration by parts:

$$\begin{aligned} \nu(v)(\chi^\varepsilon(t,x,v) - \varphi(t,x)) &= \varepsilon v \cdot \nabla_x \varphi(t,x) \\ &\quad + \varepsilon^2 \int_0^\infty e^{-\nu z} v^T \cdot D_x^2 \varphi(t, x + \varepsilon v z) \cdot v dz \end{aligned} \tag{2.51}$$

$$\nu(v)(\chi^\varepsilon(t,x,v) - \varphi(t,x)) = \varepsilon \int_0^\infty \nu e^{-\nu z} v \cdot \nabla_x \varphi(t, x + \varepsilon v z) dz \tag{2.52}$$

We start with deriving the behaviour of the right hand side of (2.49) for $\psi_i = v_i$ ($i \in \{1, \ldots, N\}$):

Lemma 2.12. *Let the assumptions of Lemma 2.11 hold, then*

$$\begin{aligned} &\varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \, \phi \cdot U_\nu^\varepsilon \, \nu (\chi_i^\varepsilon - \varphi_i) dv dx dt \\ &\quad = \varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^N} (\rho_\nu^\varepsilon + \theta_\nu^\varepsilon) \, \partial_{x_i} \varphi_i dx dt + R^\varepsilon \qquad i \in \{1, \ldots, N\}, \end{aligned}$$

where $R^\varepsilon \to 0$ *as* $\varepsilon \to 0$.

Proof. We shall employ the expansion of $\nu(\chi_i^\varepsilon - \varphi_i)$ according to (2.51):

$$\begin{aligned} &\varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \, \phi \cdot U_\nu^\varepsilon \, \nu(\chi_i^\varepsilon - \varphi_i) dv dx dt \\ &= \varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} v_i M \, \phi \cdot U_\nu^\varepsilon \, v dv \right) \cdot \nabla_x \varphi_i \, dx dt \\ &\quad + \varepsilon^{2-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \int_0^\infty v_i e^{-\nu z} v^T D_x^2 \varphi_i (t, x + \varepsilon v z) v dz M \, \phi \cdot U_\nu^\varepsilon \, dv dx dt \\ &=: I_1^\varepsilon + I_2^\varepsilon \, . \end{aligned}$$

We start with showing that $I_2^\varepsilon \to 0$ performing an estimation of the type (2.50):

$$|I_2^\varepsilon| \le C \varepsilon^{2-\gamma} \| D_x^2 \varphi_i \|_{L^2_{t,x}} \| U_\nu^\varepsilon \|_{L^2_{t,x}} \int_{\mathbb{R}^N} \frac{|v|^3 + |v|^5}{\nu} M dv \le C \varepsilon^{2-\gamma} \to 0 \, .$$

The integral I_1^ε gives rise to the Boussinesq equation. The integrand of I_1^ε containing the macroscopic momentum is odd and hence vanishes, such that

$$I_1^\varepsilon \quad = \quad \varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^N} (\rho_\nu^\varepsilon + \theta_\nu^\varepsilon) \partial_{x_i} \varphi_i dx dt \, ,$$

which concludes the proof. □

Lemma 2.13. *Let the assumptions of Lemma 2.11 hold. Then the fractional derivative arises from the following integrals as $\varepsilon \to 0$:*

(i) For the case of heavy-tailed equilibrium distributions, i.e. $M = \tilde{M}$ and $\nu = \tilde{\nu}$, we have

$$\varepsilon^{-\tilde{\gamma}} \int_{\mathbb{R}^N} \tilde{\nu} \tilde{M} \frac{|v|^4}{2N} (\chi^\varepsilon - \varphi) dv \to -\tilde{\kappa} (-\Delta_x)^{\tilde{\gamma}/2} \varphi$$

strongly in $L^2_{t,x}$.

(ii) For the case of Gaussian equilibrium distributions, i.e. $M = M^*$ and $\nu = \nu^*$, we have

$$\varepsilon^{-\gamma^*} \int_{\mathbb{R}^N} \nu^* M^* (\chi^\varepsilon - \varphi) dv \to -\kappa^* (-\Delta_x)^{\gamma^*/2} \varphi$$

strongly in $L^2_{t,x}$.

Proof. We shall first demonstrate the convergence for the heavy-tailed equilibrium distributions stated in (i). We therefore split the domain of integration as follows:

$$\tilde{J}_1^\varepsilon = \varepsilon^{-\tilde{\gamma}} \int_{|v| \le 1} |v|^4 \tilde{\nu} \tilde{M} (\chi^\varepsilon - \varphi) dv, \quad \tilde{J}_2^\varepsilon = \varepsilon^{-\tilde{\gamma}} \int_{|v| \ge 1} |v|^4 \tilde{\nu} \tilde{M} (\chi^\varepsilon - \varphi) dv \, .$$

We expand the first integral using (2.51):

$$\tilde{J}_1^\varepsilon \quad = \quad \varepsilon^{1-\tilde{\gamma}} \int_{|v| \le 1} |v|^4 v \tilde{M} dv \cdot \nabla_x \varphi + \varepsilon^{2-\tilde{\gamma}} \int_{|v| \le 1} \int_0^\infty |v|^4 e^{-\tilde{\nu} z} v^T D_x^2 \varphi(t, x + \varepsilon v z) v \tilde{M} dz dv \, .$$

The first integrand is odd, therefore the integral vanishes. The second integrand is uniformly bounded in $|v| \le 1$, hence $\tilde{J}_1^\varepsilon \to 0$ as $\varepsilon \to 0$ uniformly in t, x and also $L^2_{t,x}$. For the integral $\tilde{J}_2^\varepsilon$ we use the behaviours of $\tilde{M}$ and $\tilde{v}$, as well as (2.45):

$$\begin{aligned} \tilde{J}_2^\varepsilon \quad &= \quad \varepsilon^{-\tilde{\gamma}} c_0 \int_{|v| \ge 1} |v|^{4-N-\alpha+\tilde{\beta}} \int_0^\infty \tilde{\nu} \varepsilon^{-\tilde{\nu} z} (\varphi(t, x + \varepsilon v z) - \varphi(t, x)) dz dv \\ &= \quad \varepsilon^{-\tilde{\gamma}} c_0 \int_{|v| \ge 1} |v|^{4-N-\alpha+\tilde{\beta}} \int_0^\infty e^{-s} \left(\varphi \left(t, x + \varepsilon \frac{v}{\tilde{\nu}} s \right) - \varphi(t, x) \right) ds dv \end{aligned}$$

where we substituted $s = \tilde{\nu} z$. We recall that $\tilde{\beta} < 1$ and perform the change of variables

$$w = \varepsilon \frac{v}{|v|^{\tilde{\beta}}}, \qquad dv = \frac{1}{1-\tilde{\beta}} \left(\frac{|w|^{\tilde{\beta}}}{\varepsilon} \right)^{\frac{N}{1-\tilde{\beta}}} dw\,, \tag{2.53}$$

where for the calculation of the determinant of the Jacobian-matrix Silvester's theorem can be applied. Recalling $\tilde{\gamma} = (\alpha - \tilde{\beta} - 4)/(1-\tilde{\beta})$, we obtain

$$\begin{aligned}
\tilde{J}_2^{\varepsilon} &= \frac{c_0}{1-\tilde{\beta}} \int_{|w|\geq \varepsilon} \int_0^\infty e^{-s} \frac{\varphi(t,x+ws) - \varphi(t,x)}{|w|^{N+\tilde{\gamma}}} ds dw \\
&= \frac{c_0}{1-\tilde{\beta}} \int_0^\infty \int_{|y|>\varepsilon s} \frac{\varphi(t,x+y) - \varphi(t,x)}{|y|^{N+\tilde{\gamma}}} dy\, e^{-s} s^{\tilde{\gamma}} ds
\end{aligned}$$

where substituted $y = ws$. Due to the definition of the principle value we have the pointwise convergence in t, x of

$$\tilde{J}_2^{\varepsilon} \to \tilde{J}^0$$

with J^0 being defined as

$$\begin{aligned}
\tilde{J}^0 &= \frac{c_0}{1-\tilde{\beta}} PV \int_{\mathbb{R}^N} \int_0^\infty e^{-s} \frac{\varphi(t,x+sw) - \varphi(t,x)}{|w|^{N+\tilde{\gamma}}} ds dw \\
&= \frac{c_0}{1-\tilde{\beta}} PV \int_{\mathbb{R}^N} \frac{\varphi(t,x+y) - \varphi(t,x)}{|y|^{N+\tilde{\gamma}}} dy \int_0^\infty e^{-s} s^{\tilde{\gamma}} ds \\
&= \Gamma(1+\tilde{\gamma})\, \tilde{\kappa}\, PV \int_{\mathbb{R}^N} \frac{\varphi(t,x+y) - \varphi(t,x)}{|y|^{N+\tilde{\gamma}}} dy \qquad (2.54) \\
&= -\tilde{\kappa}(-\Delta)^{\tilde{\gamma}/2} \varphi
\end{aligned}$$

with $\tilde{\kappa} = \frac{c_0 \Gamma(\tilde{\gamma}+1)}{1-\tilde{\beta}}$. For proving convergence in $L^2_{t,x}$ we proceed as in [BAMP11] and split $\tilde{J}^0$ into

$$\begin{aligned}
\frac{1}{\tilde{\kappa}} \tilde{J}^0 &= \int_{|w|\geq 1} \int_0^\infty e^{-s} \frac{\varphi(t,x+sw) - \varphi(t,x)}{|w|^{N+\tilde{\gamma}}} ds dw \\
&\quad + \int_{|w|\leq 1} \int_0^\infty e^{-s} \frac{\varphi(t,x+sw) - \varphi(t,x) - sw \cdot \nabla_x \varphi(t,x)}{|w|^{N+\tilde{\gamma}}} ds dw\,. \qquad (2.55)
\end{aligned}$$

These integrals are defined in the classical sense. Splitting $\tilde{J}_2^{\varepsilon}$ into the integral over the domain $\{|w| \geq 1\}$ and $\{\varepsilon < |w| < 1\}$ respectively, we obtain

$$\begin{aligned}
\frac{1}{\tilde{\kappa}}(\tilde{J}_2^{\varepsilon} - \tilde{J}^0) &= -\int_{|w|\leq \varepsilon} \int_0^\infty e^{-s} \frac{\varphi(t,x+sw) - \varphi(t,x) - sw \cdot \nabla_x \varphi(t,x)}{|w|^{N+\tilde{\gamma}}} ds dw \\
&= -\int_{|w|\leq \varepsilon} \int_0^\infty e^{-s} \frac{w^T D_x^2 \varphi(t,x+sw) \cdot w}{|w|^{N+\tilde{\gamma}}} ds dw\,, \qquad (2.56)
\end{aligned}$$

where we have performed integration by parts twice. Due to the fact that

$$\int_0^\infty e^{-s} ds \int_{|w|\le\varepsilon} \frac{1}{|w|^{N+\tilde{\gamma}-2}} dw \le C\varepsilon^{2-\tilde{\gamma}} \to 0$$

we deduce the (strong) $L^2_{t,x}$-convergence of $\tilde{J}_2^\varepsilon - \tilde{J}^0 \to 0$, which concludes the proof for the heavy-tailed equilibrium distributions.

We shall now derive the fractional Laplacian for the Gaussian equilibrium distributions $M^*(v) = \frac{1}{(2\pi)^{N/2}} e^{-\frac{|v|^2}{2}}$ as stated in (ii). We proceed in a similar fashion to [BAMP11] and split the integral in (ii) as follows:

$$J_1^{\varepsilon *} = \varepsilon^{-\gamma^*} \int_{|v|\le 1} \nu^* M^* (\chi^\varepsilon - \varphi) dv, \quad J_2^{\varepsilon *} = \varepsilon^{-\gamma^*} \int_{|v|\ge 1} \nu^* M^* (\chi^\varepsilon - \varphi) dv\,.$$

As we shall see below the degeneracy occurs in the first integral, whereas the second integral vanishes in the limit. Expanding $J_2^{\varepsilon *}$ according to (2.51) we obtain

$$J_2^{\varepsilon *} = \varepsilon^{1-\gamma^*} \int_{|v|\ge 1} M^* v dv \cdot \nabla_x \varphi + \varepsilon^{2-\gamma^*} \int_{|v|\ge 1} \int_0^\infty e^{-\tilde{\nu}^* z} v^T D_x^2 \varphi(t, x + \varepsilon v z) v M^* dz dv\,.$$

The first integral vanishes, since the integrand is odd. The second integrand is uniformly bounded in $\{|v| \ge 1\}$, hence the second integral also converges to 0 uniformly and in $L^2_{t,x}$. We shall now turn to the integral $J_1^{\varepsilon *}$ over the domain of small velocities. Observe that we cannot expand $\nu^*(\chi^\varepsilon - \varphi)$ according to (2.51) as above, since $\int_{|v|\le 1} \frac{|v|^2 M^*}{\nu^*} dv$ is unbounded. Hence we expand $\nu^*(\chi^\varepsilon - \varphi)$ only up to first order as given in (2.52) and proceed as in [BAMP11]:

$$\begin{aligned} J_1^{\varepsilon *} &= \varepsilon^{1-\gamma^*} \int_{|v|\le 1} \int_0^\infty e^{-\nu^* z} \nu^* v \cdot \nabla_x \varphi(t, x + \varepsilon v z) dz M^* dv \\ &= \varepsilon^{1-\gamma^*} \int_{|v|\le 1} \int_0^\infty e^{-s} v \cdot \nabla_x \varphi\left(t, x + \varepsilon \frac{v}{\nu^*} s\right) ds M^* dv\,. \end{aligned}$$

We again perform a change of variables similar to (2.53), noting that here $\beta^* > 1$, such that the domain of integration is inverted:

$$w = \varepsilon \frac{v}{|v|^{\beta^*}}\,, \qquad dv = \frac{1}{\beta^* - 1} \left(\frac{\varepsilon}{|w|^{\beta^*}} \right)^{\frac{N}{\beta^* - 1}} dw\,.$$

Recalling $\gamma^* = (\beta^* + d)/(\beta^* - 1)$ we obtain

$$\begin{aligned} J_1^{\varepsilon *} &= \frac{1}{\beta^* - 1} \int_{|w|\ge\varepsilon} \int_0^\infty e^{-s} w \cdot \nabla_x \varphi(t, x + sw) ds |w|^{-\frac{\beta^* + N}{\beta^* - 1}} M^* \left((\varepsilon/|w|)^{\frac{1}{\beta^* - 1}} \right) dw \\ &= \frac{1}{(2\pi)^{N/2} (\beta^* - 1)} \int_{|w|\ge\varepsilon} \int_0^\infty e^{-s} \frac{\varphi(t, x + sw) - \varphi(t, x)}{|w|^{N+\gamma^*}} ds\, e^{-\frac{1}{2} \left(\frac{\varepsilon}{|w|} \right)^{\frac{1}{\beta^* - 1}}} dw\,. \end{aligned}$$

As above we introduce the integral

$$J^{0*} \quad = \quad \frac{1}{(2\pi)^{N/2}(\beta^*-1)} PV \int_{\mathbb{R}^N} \int_0^\infty e^{-s} \frac{\varphi(t,x+sw)-\varphi(t,x)}{|w|^{N+\gamma^*}} dsdw\,,$$

satisfying the analogous relations given in (2.56). Moreover J^{0*} can be split into two integrals according to (2.55), from which we can deduce the $L^2_{t,x}$ convergence of $J^{\varepsilon*} \to J^{0*}$. From the Gaussian equilibrium distributions being non-constant for small velocities two more terms arise here compared to (2.56) and [BAMP11]:

$$\begin{aligned}
(\beta^*-1)(2\pi)^{N/2}(J_1^{\varepsilon*}-J^{0*}) &= \int_{|w|\geq 1}\int_0^\infty e^{-s}\frac{\varphi(t,x+sw)-\varphi(t,x)}{|w|^{N+\gamma^*}}ds\left(e^{-\frac{1}{2}\left(\frac{\varepsilon}{|w|}\right)^{\frac{2}{\beta^*-1}}}-1\right)dw\\
&+\int_{\varepsilon\leq|w|\leq 1}\int_0^\infty e^{-s}\frac{\varphi(t,x+sw)-\varphi(t,x)-sw\cdot\nabla_x\varphi(t,x)}{|w|^{N+\gamma^*}}ds\left(e^{-\frac{1}{2}\left(\frac{\varepsilon}{|w|}\right)^{\frac{2}{\beta^*-1}}}-1\right)dw\\
&+\int_{|w|\leq\varepsilon}\int_0^\infty e^{-s}\frac{\varphi(t,x+sw)-\varphi(t,x)-sw\cdot\nabla_x\varphi(t,x)}{|w|^{N+\gamma^*}}dsdw\\
&=: L_1^{\varepsilon*}+L_2^{\varepsilon*}+L_3^{\varepsilon*}\,.
\end{aligned}$$

For the third integral $L_3^{\varepsilon*}$ the convergence to 0 in $L^2_{t,x}$ is obtained in the same fashion to (2.56) above. For $L_1^{\varepsilon*}$ we employ an estimation as in (2.50):

$$\|L_1^{\varepsilon*}\|_{L^2_{t,x}} \leq 2\|\varphi\|_{L^2_{t,x}}\int_0^s e^{-s}ds\int_{|w|\geq 1}|w|^{-(N+\gamma^*)}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{|w|}\right)^{\frac{2}{\beta^*-1}}}\right)dw \leq C\left(1-e^{-\frac{1}{2}\varepsilon^{\frac{2}{\beta^*-1}}}\right)\to 0$$

To see the convergence of the remaining term $L_2^{\varepsilon*}$ we perform integration by parts twice and bound

$$\|L_2^{\varepsilon*}\|_{L^2_{t,x}} \leq \|D_x^2\varphi\|_{L^2_{t,x}}\int_{\varepsilon\leq|w|\leq 1}\int_0^s e^{-s}|w|^{-(N+\gamma^*-2)}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{|w|}\right)^{\frac{2}{\beta^*-1}}}\right)dsdw$$

We now split the domain of integration in the latter integral once more. For any $a\in(0,1)$

$$\begin{aligned}
\int_{\varepsilon\leq|w|\leq 1}\int_0^\infty \frac{e^{-s}}{|w|^{N+\gamma^*-2}}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{|w|}\right)^{\frac{2}{\beta^*-1}}}\right)dsdw &\leq C\int_\varepsilon^1 r^{1-\gamma^*}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{r}\right)^{\frac{2}{\beta^*-1}}}\right)dr\\
= C\int_\varepsilon^{\varepsilon^a} r^{1-\gamma^*}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{r}\right)^{\frac{2}{\beta^*-1}}}\right)dr &+ \int_{\varepsilon^a}^1 r^{1-\gamma^*}\left(1-e^{-\frac{1}{2}\left(\frac{\varepsilon}{r}\right)^{\frac{2}{\beta^*-1}}}\right)dr\\
\leq Cr^{2-\gamma^*}\Big|_\varepsilon^{\varepsilon^a} + C\left(1-e^{-\frac{\varepsilon^{\frac{2(1-a)}{\beta^*-1}}}{2}}\right) &\to 0\,.
\end{aligned}$$

By dominated convergence, this implies the strong convergence of $J_2^{\varepsilon*}$ to J^{0*} in $L^2_{t,x}$, which concludes the proof of the Lemma. □

Lemma 2.14. *Let the assumptions of Lemma 2.11 hold and recall that* $\psi_{N+1} = \frac{|v|^2-(N+2)}{2}$. *Then, as* $\varepsilon \to 0$, *we have*

$$\varepsilon^{-\gamma}\int_0^\infty \int_{\mathbb{R}^{2N}} \psi_{N+1} M\,\phi\cdot U_\nu^\varepsilon\,\nu(\chi^\varepsilon-\varphi)\,dvdxdt \;\to\; -\kappa\int_0^\infty\int_{\mathbb{R}^N}\theta(-\Delta)^{\gamma/2}\varphi\,dxdt\,.$$

Proof. We shall again employ the expansion of $\nu(\chi^\varepsilon-\varphi)$ according to (2.51):

$$\begin{aligned}
&\varepsilon^{-\gamma}\int_0^\infty\int_{\mathbb{R}^{2N}}\psi_{N+1}M\,\phi\cdot U_\nu^\varepsilon\,\nu(\chi^\varepsilon-\varphi)\,dvdxdt\\
&\quad=\varepsilon^{1-\gamma}\int_0^\infty\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\psi_{N+1}M\,\phi\cdot U_\nu^\varepsilon v\;dv\cdot\nabla_x\varphi\,dxdt\\
&\quad\quad+\varepsilon^{2-\gamma}\int_0^\infty\int_{\mathbb{R}^{2N}}\int_0^\infty\psi_{N+1}e^{-\nu z}v^TD_x^2\varphi(x+\varepsilon vz,t)v\,dzM\,\phi\cdot U_\nu^\varepsilon\,dvdxdt\\
&\quad=: I_1^\varepsilon+I_2^\varepsilon\,.
\end{aligned}$$

The part in the integrand of I_1^ε containing the macroscopic density and temperature is odd and hence vanishes, therefore we are left with computing only the part containing the momentum:

$$2I_1^\varepsilon \;=\; \varepsilon^{1-\gamma}\int_0^\infty\int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N}\left(|v|^2-(N+2)\right)v\otimes vM\,dv\cdot m_\nu^\varepsilon\right)\cdot\nabla_x\varphi\,dxdt\;=\;0\,,$$

which holds due to the moment conditions in (2.2). We now turn to the second integral term I_2^ε, which gives rise to the fractional Laplacian. We first order the moments accordingly

$$\begin{aligned}
2\,I_2^\varepsilon&=\varepsilon^{2-\gamma}\int_0^\infty\int_{\mathbb{R}^{2N}}\int_0^\infty(|v|^2-(N+2))e^{-\nu z}v^TD_x^2\varphi(t,x+\varepsilon vz)\,vdzM\,\phi\cdot U_\nu^\varepsilon\,dvdxdt\\
&=\varepsilon^{2-\gamma}(N+2)\int_0^\infty\int_{\mathbb{R}^{2N}}\int_0^\infty e^{-\nu z}v^TD_x^2\varphi(x+\varepsilon vz,t)vdzMdv\left(-\rho_\nu^\varepsilon+\frac{N}{2}\theta_\nu^\varepsilon\right)dxdt\\
&\quad+\varepsilon^{2-\gamma}\int_0^\infty\int_{\mathbb{R}^{2N}}\int_0^\infty\left((|v|^2-(N+2))v\cdot m_\nu^\varepsilon+|v|^2\left(\rho_\nu^\varepsilon-(N+1)\theta_\nu^\varepsilon\right)\right)\cdot\\
&\quad\cdot e^{-\nu z}v^TD_x^2\varphi(x+\varepsilon vz,t)vdzM\,dvdxdt\\
&\quad+\frac{\varepsilon^{2-\gamma}}{2}\int_0^\infty\int_{\mathbb{R}^{2N}}\int_0^\infty|v|^4\theta_\nu^\varepsilon e^{-\nu z}v^TD_x^2\varphi(x+\varepsilon vz,t)vdzM\,dvdxdt\\
&=: L_1^\varepsilon+L_2^\varepsilon+L_3^\varepsilon\,.
\end{aligned}$$

We start with showing that $L_2^\varepsilon\to 0$ for both cases of equilibrium distributions due to (2.27) and (2.28)

$$|L_2^\varepsilon|\le C\varepsilon^{2-\gamma}\|D_x^2\varphi\|_{L^2_{t,x}}\|U_\nu^\varepsilon\|_{L^2_{t,x}}\int\frac{|v|^3+|v|^5}{\nu}M\,dv\le C\varepsilon^{2-\gamma}\to 0\,.$$

Moreover for the heavy-tailed equilibrium distributions the integral term L_1^ε also vanishes in the limit due to (2.27) using the same argumentation. The third integral term L_3^ε corresponds, after integration by parts twice and inserting the definition of $\nu(\chi^\varepsilon - \varphi)$, to the integral in Lemma 2.13 (i) and hence converges towards the fractional Laplacian. For the case of Gaussian equilibrium the roles of the integrals L_1^ε and L_3^ε are interchanged, namely L_3^ε vanishes and from L_1^ε we obtain the fractional Laplacian according to Lemma 2.13 (ii). □

We shall now state the corresponding convergence properties for the fractional Stokes limit without temperature. In fact, in the weak form (2.49) we only need to consider the moment $\bar{\psi}(v) = v$. Since in this case we only treat the case of heavy-tailed equilibrium distributions as stated in Assumption 2, no distinction between the types of equilibrium distributions has to made here. Hence for the fractional Stokes limit we skip the tildes for M and ν in the following.

Lemma 2.15. *Let Assumption 2 hold and let χ_i^ε be the auxiliary functions as defined above (2.44) for corresponding $\varphi_i \in \mathcal{D}((0,\infty) \times \mathbb{R}^N)$ and let f^ε be the weak solution as in Proposition 2.9.*

(i) The weak form of the time derivatives in (2.49) for $\bar{\psi} = v$ converges in the sense of Lemma 2.11 with the macroscopic moments U being replaced by $\bar{U}$ as $\varepsilon \to 0$.

(ii) For $\bar{\psi}_i = v_i$ we have for the right hand side in the weak formulation of (2.49):

$$\varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \,\bar{\phi} \cdot \bar{U}_\nu^\varepsilon \,\nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt$$
$$= -\varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^N} \varphi_i \partial_{x_i} \rho_\nu^\varepsilon dxdt - \kappa \int_0^\infty \int_{\mathbb{R}^N} m_i (-\Delta)^{\gamma/2} \varphi_i \, dxdt + \bar{R}_i^\varepsilon \,,$$

where $\bar{R}_i^\varepsilon \to 0$ for all $i \in \{1, \dots, N\}$.

Proof. The convergence of the terms involving time derivatives in (i) is similar to the proof of Lemma 2.11. To derive the integral identity in (ii) we first split the integral into the terms containing ρ_ν^ε and m_ν^ε respectively:

$$\varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \,\bar{\phi} \cdot \bar{U}_\nu^\varepsilon \,\nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt$$
$$= \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \rho_\nu^\varepsilon \,\nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt + \varepsilon^{-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} v_i M \, v \cdot m_\nu^\varepsilon \nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt$$
$$=: \bar{I}_1^\varepsilon + \bar{I}_2^\varepsilon \,.$$

We expand $\nu(\chi_i^\varepsilon - \varphi_i)$ according to (2.51) in $\bar{I}_1^\varepsilon$:

$$\begin{aligned}
\bar{I}_1^\varepsilon &= \varepsilon^{1-\gamma}\int_0^\infty \int_{\mathbb{R}^{2N}} v_i v M dv \cdot \nabla_x \varphi_i\, \rho_\nu^\varepsilon dxdt \\
&\quad + \varepsilon^{2-\gamma}\int_0^\infty \int_{\mathbb{R}^{2N}} \int_0^\infty e^{-\nu z} v_i v D_x^2 \varphi(t, x+\varepsilon v z) v M\, dzdv \rho_\nu^\varepsilon dxdt \\
&= \varepsilon^{1-\gamma}\int_0^\infty \int_{\mathbb{R}^N} \rho_\nu^\varepsilon \partial_{x_i}\varphi_i dxdt + \hat{R}_i^\varepsilon
\end{aligned}$$

where the latter integral vanishes in the limit $\varepsilon \to 0$:

$$|\hat{R}_i^\varepsilon| \le C\varepsilon^{2-\gamma}\|D_x^2\varphi_i\|_{L^2_{t,x}}\|\rho_\nu^\varepsilon\|_{L^2_{t,x}}\int_{\mathbb{R}^N}\frac{|v|^3}{\nu}M\, dv \le C\varepsilon^{2-\gamma} \to 0\,.$$

We shall now derive the fractional Laplacian from the integral $\bar{I}_2^\varepsilon$ and therefore, similar to above, split the integral into

$$\begin{aligned}
\bar{I}_2^\varepsilon &= \varepsilon^{-\gamma}\int_0^\infty \int_{\mathbb{R}^N}\int_{|v|\le 1} v_i M\, v \cdot m_\nu^\varepsilon \nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt \\
&\quad + \varepsilon^{-\gamma}\int_0^\infty \int_{\mathbb{R}^N}\int_{|v|\ge 1} v_i M \rho_\nu^\varepsilon\, v \cdot m_\nu^\varepsilon \nu(\chi_i^\varepsilon - \varphi_i)\, dvdxdt \\
&=: \bar{J}_1^\varepsilon + \bar{J}_2^\varepsilon\,.
\end{aligned}$$

Inserting (2.51) it is easy to see that $\bar{J}_1^\varepsilon$ vanishes in the limit $\varepsilon \to 0$. We insert (2.45) in the integrand of $\bar{J}_2^\varepsilon$ to obtain after substituting $s = \nu z$

$$\bar{J}_2^\varepsilon = \varepsilon^{-\gamma}\int_0^\infty \int_{\mathbb{R}^N}\int_{|v|\ge 1}\int_0^\infty \nu e^{-s} v_i v \cdot m_\nu^\varepsilon M \left(\varphi_i\left(t, x+\varepsilon\frac{v}{\nu}s\right) - \varphi_i(t,x)\right) dsdvdxdt$$

Recalling the definition of $\gamma = (\alpha-\beta-2)/(1-\beta)$ and using the same change of variables as in (2.53) we obtain

$$\begin{aligned}
&(1-\beta)(\gamma+N)\bar{I}_2^\varepsilon \\
&= (\gamma+N)\int_0^\infty \int_{\mathbb{R}^N}\left(\int_{|w|\ge\varepsilon}\int_0^\infty e^{-s}\frac{w_i w}{|w|^2}\frac{1}{|w|^{\gamma+N}}(\varphi_i(t,x+sw)-\varphi_i(t,x))dsdw\right)\cdot m_\nu^\varepsilon dxdt \\
&= -\int_0^\infty \int_{\mathbb{R}^N}\left(\int_{|w|\ge\varepsilon}\int_0^\infty e^{-s}\nabla_w\left(\frac{1}{|w|^{\gamma+N}}\right) w_i(\varphi_i(t,x+sw)-\varphi_i(t,x))dsdw\right)\cdot m_\nu^\varepsilon dxdt \\
&= \int_0^\infty \int_{\mathbb{R}^N}\left(\int_{|w|\ge\varepsilon}\int_0^\infty e^{-s}\frac{1}{|w|^{\gamma+N}}(\varphi_i(t,x+sw)-\varphi_i(t,x))dsdw\right) m_{\nu i}^\varepsilon dxdt \\
&\quad + \int_0^\infty \int_{\mathbb{R}^N}\left(\int_{|w|\ge\varepsilon}\int_0^\infty e^{-s}\frac{w_i}{|w|^{\gamma+N}} s\nabla_x\varphi_i(t,x+sw)dsdw\right)\cdot m_\nu^\varepsilon dxdt
\end{aligned}$$

$$+\int_0^\infty \int_{\mathbb{R}^N} \left(\int_{|w|=\varepsilon} \int_0^\infty e^{-s} \frac{w_i}{|w|^{\gamma+N}} s\varphi_i(t,x+sw) \frac{w}{|w|} ds d\sigma \right) \cdot m_\nu^\varepsilon dxdt$$
$$=: \bar{L}_1^\varepsilon + \bar{L}_2^\varepsilon + \bar{b}^\varepsilon \,,$$

where we performed integration by parts and used the fact that the outer unit normal on the sphere is $w/|w|$. The convergence of $\bar{L}_1^\varepsilon$ towards the integral involving the fractional Laplacian

$$\bar{L}_1^\varepsilon \to \kappa \int_0^\infty \int_{\mathbb{R}^N} m_i (-\Delta)^{\frac{\gamma}{2}} \varphi_i \, dxdt$$

is deduced as in the proof of Lemma 2.14. Hence to conclude the proof it remains to show that $\bar{L}_2^\varepsilon$ and $\bar{b}^\varepsilon$ vanish in the limit. Therefore we first observe

$$\begin{aligned}
(1-\beta)(\gamma+N)\bar{L}_2^\varepsilon &= \int_0^\infty \int_{\mathbb{R}^N} \left(\int_{|w|\geq\varepsilon} \int_0^\infty e^{-s} \frac{w_i}{|w|^{\gamma+N}} s\nabla_x \varphi_i(t,x+sw) ds dw \right) \cdot m_\nu^\varepsilon dxdt \\
&= -\int_0^\infty \int_{\mathbb{R}^N} \int_{|w|\geq\varepsilon} \int_0^\infty e^{-s} \frac{w_i}{|w|^{\gamma+N}} s (\nabla_x \cdot m_\nu^\varepsilon) \varphi_i(t,x+sw) ds dw dx dt \\
&= -\int_0^\infty \int_{\mathbb{R}^{2N}} \int_0^\infty e^{-s} \frac{w_i}{|w|^{\gamma+N}} s (\nabla_x \cdot m_\nu^\varepsilon) \varphi_i(t,x+sw) ds dw dx dt \\
&\quad + \int_0^\infty \int_{\mathbb{R}^N} \int_{|w|\leq\varepsilon} \int_0^\infty e^{-s} \frac{w_i}{|w|^{\gamma+N}} s (\nabla_x \cdot m_\nu^\varepsilon) \varphi_i(t,x+sw) ds dw dx dt \\
&=: \bar{K}_1^\varepsilon + \bar{K}_2^\varepsilon \,.
\end{aligned}$$

For the first integral $\bar{K}_1^\varepsilon$ we shall use the fact that $\nabla \cdot m_\nu^\varepsilon \rightharpoonup 0$ in $L^2_{t,x}$. Hence, if

$$\int_{\mathbb{R}^N} \frac{w_i}{|w|^{\gamma+N}} \int_0^\infty s e^{-s} \varphi_i(t,x+sw) ds dw \tag{2.57}$$

is bounded in $L^2_{t,x}$, then $\bar{K}_1^\varepsilon \to 0$. Proceeding as in (2.50) we can bound the $L^2_{t,x}$-norm of the integral (2.57) over the domain $\{|w| \geq 1\}$ directly by

$$C\|\varphi_i\|_{L^2_{t,x}} \int_1^\infty |w|^{-\gamma-N+1} dw \leq C \,.$$

For the integral (2.57) over the domain $\{|w| \leq 1\}$ we observe that $se^{-s} = \partial_s((s+1)e^{-s})$. Integrating by parts in s we can then bound the $L^2_{t,x}$-norm using an estimation of the type (2.50) by

$$C\|\nabla_x \varphi\|_{L^2_{t,x}} \int_{|w|\leq 1} |w|^{-\gamma-N+2} dw \leq C$$

from which we can now deduce $\bar{K}_1^\varepsilon \to 0$ (note that the boundary term is odd in w and hence vanishes). To see $\bar{K}_2^\varepsilon \to 0$ we integrate by parts additionally in x

$$|K_2^\varepsilon| \le C\|m_\nu^\varepsilon\|_{L^2_{t,x}}\|D_x^2\varphi\|_{L^2_{t,x}}\int_{|w|\le\varepsilon}|w|^{-\gamma-N+2}dw \le C\varepsilon^{2-\gamma}\to 0\,.$$

It now remains to show that the boundary terms vanish. We employ integration in parts twice

$$\begin{aligned}|\bar{b}^\varepsilon| &= \left|\frac{1}{\gamma+N}\int_0^\infty\int_{\mathbb{R}^N}\int_{|w|=\varepsilon}\int_0^\infty e^{-s}wD_x^2\varphi(t,x+sw)w\frac{w_i}{|w|^{\gamma+N}}m_\nu^\varepsilon\cdot\frac{w}{|w|}d\sigma dwdxdt\right|\\ &\le C\|D_x^2\varphi\|_{L^2_{t,x}}\|m_\nu^\varepsilon\|_{L^2_{t,x}}\int_{|w|=\varepsilon}\frac{|w|^4}{|w|^{\gamma+N+1}}d\sigma \le C\varepsilon^{2-\gamma}\to 0\,.\end{aligned}$$

□

2.4 Derivation of the macroscopic dynamics

2.4.1 Derivation of the fractional Stokes-Fourier system

The convergence of the solution f^ε of the Cauchy problem in (2.23) was already shown. We will now derive the macroscopic equations determining the limiting solution stated in Theorem 2.1.

Proof of Theorem 2.1. We start by deriving the incompressibility condition from equation (2.48). Since $\partial_t\varphi$ and ρ^ε are both uniformly bounded in $L^2_{t,x}$, multiplying (2.48) with $\varepsilon^{\gamma-1}$ and using the fact that $m^\varepsilon \rightharpoonup m$ in $L^2_{t,x}$ we obtain the incompressibility condition in the limit $\varepsilon\to 0$.

We shall now turn to the weak form of the first moments. Due to Lemma 2.12 we know that

$$\begin{aligned}&-\int_0^\infty\int_{\mathbb{R}^{2N}}v_if^\varepsilon\partial_t\chi_i^\varepsilon dvdxdt-\int_{\mathbb{R}^{2N}}v_if^{in}\chi_i^\varepsilon(t=0)dxdv\\ &=\frac{\varepsilon^{1-\gamma}}{N}\int_0^\infty\int_{\mathbb{R}^N}(\rho_\nu^\varepsilon+\theta_\nu^\varepsilon)\,\partial_{x_i}\varphi_idxdt+R_i^\varepsilon\,,\qquad i\in\{1,\dots,N\}\,.\end{aligned}\tag{2.58}$$

Again, due to the boundedness of the terms on the left hand side and the remainder R^ε, which vanishes in the limit $\varepsilon\to 0$, we obtain after multiplying by $\varepsilon^{\gamma-1}$:

$$\left|\int_0^\infty\int_{\mathbb{R}^N}(\rho_\nu^\varepsilon+\theta_\nu^\varepsilon)\,\partial_{x_i}\varphi_idxdt\right|\le C\varepsilon^{\gamma-1}\qquad\text{for all } i\in\{1,\dots,N\}\,.\tag{2.59}$$

Hence using the fact that $U_\nu^\varepsilon \rightharpoonup U$ in $L^2_{t,x}$, we obtain the Boussinesq relation. Moreover, carrying out the limit in the equation for m^ε we obtain

$$\partial_t m = \nabla_x p$$

in the weak sense, where $p(t,x)$ is the remainder of the Boussinesq relation:

$$p(t,x) = \lim_{\varepsilon\to 0} \varepsilon^{1-\gamma} \left(\rho_\nu^\varepsilon + \theta_\nu^\varepsilon\right) = \lim_{\varepsilon\to 0} \varepsilon^{1-\gamma} \left(\rho_\nu^\varepsilon - \rho + (\theta_\nu^\varepsilon - \theta)\sqrt{2/N}\right)$$

which is bounded in $L^2_{t,x}$ due to (2.59). Using divergence-free testfunctions, i.e. $\sum_i \partial_{xi}\varphi_i = 0$, we obtain $\partial_t m = 0$.

We shall now turn to the equation for θ. Herefore we use the weak form of the moment corresponding to $\psi_{N+1} = \frac{|v|^2-(N+2)}{2}$. Lemma 2.11 and the Boussinesq relation imply

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^{2N}} \psi_{N+1} f^\varepsilon \partial_t \chi^\varepsilon dv dx dt - \int_{\mathbb{R}^{2N}} \psi_{N+1} f^{in} \chi^\varepsilon(t=0) dx dv \\
&\to \left(1+\frac{N}{2}\right) \int_0^\infty \int_{\mathbb{R}^{2N}} \theta \partial_t \varphi dx dt - \left(1+\frac{N}{2}\right) \int_{\mathbb{R}^{2N}} \theta^{in} \varphi(t=0) dx dv
\end{aligned}$$

where we have used the Boussinesq equation for the limiting solution and the assumption on the initial data $\rho^{in}+\theta^{in} = 0$. Lemma 2.14 completes the derivation of the dynamics for the limiting function $f = M\phi \cdot U$. □

2.4.2 Derivation of the dynamics for fractional Stokes limit

We finally give the proof for the limiting solution stated in Theorem 2.2.

Proof of Theorem 2.2. The incompressibility condition from equation (2.48) can be deduced as in the proof of Theorem 2.1 above. Lemma (2.15) implies

$$\begin{aligned}
&-\int_0^\infty \int_{\mathbb{R}^{2N}} v_i f^\varepsilon \partial_t \chi_i^\varepsilon dv dx dt - \int_{\mathbb{R}^{2N}} v_i f^{in} \chi_i^\varepsilon(t=0) dv dx \\
&= -\varepsilon^{1-\gamma} \int_0^\infty \int_{\mathbb{R}^{2N}} \varphi_i \partial_{x_i} \rho_\nu^\varepsilon dx dt - \kappa \int_0^\infty \int_{\mathbb{R}^N} m_i (-\Delta)^{\gamma/2} \varphi_i \, dx dt + \bar{R}_i^\varepsilon \,,
\end{aligned}$$

where $\bar{R}_i^\varepsilon \to 0$ as $\varepsilon \to 0$. Using divergence-free testfunctions, i.e. considering $\Phi = (\varphi_1, \ldots, \varphi_N)^T$ with $\nabla \cdot \Phi = 0$, we obtain in the limit

$$-\int_0^\infty \int_{\mathbb{R}^N} m \cdot \partial_t \Phi dx dt - \int_{\mathbb{R}^N} m^{in} \cdot \Phi(t=0) dx = \kappa \int_0^\infty \int_{\mathbb{R}^N} m \cdot (-\Delta)^{\gamma/2} \Phi \, dx dt \,,$$

which gives

$$\partial_t m = -\kappa(-\Delta_x)^{-\frac{\gamma}{2}} m$$
$$m(0,x) = m^{in}(x)$$

in the distribution sense for divergence-free testfunctions. □

Chapter 3

Anomalous energy transport in FPU-β chain

Joint work with Dr. Antoine Mellet

We recall the reader that we have motivated this chapter in Section 1.3.2 where an introduction and previous results are given.

This chapter is organized as follows: In the next section, we describe the original problem (chains of coupled harmonic oscillators) and its relation to the Boltzmann phonon equation. We then introduce the collision operators C that appears in the context of FPU chains. In that section, we will see in particular that this kinetic description cannot be used to study the FPU-α chain because the collision operator C vanishes in that case. This section is mostly based on the paper of H. Spohn [Spo06b].

In Section 3.2, we investigate the properties of the four phonon collision operators, appearing in the context of the FPU-β chain as well as its linearization around an equilibrium (this section is largely based on the work of J. Lukkarinen and H. Spohn [LS08]). The main result of our paper is finally stated in Section 3.3 and its proof is divided between Sections 3.4 and 3.5.

Contents

3.1 Crystal vibrations: A kinetic description

In this section, we recall the results from the paper of H. Spohn [Spo06b] that are relevant to our present study. Our goal is to detail the relation between the Boltzmann phonon equation that we are considering in this paper and the microscopic models. At the microscopic level, we consider an infinite lattice $\mathbb{Z}^n$ describing the equilibrium positions of the atoms of a crystal (we briefly introduce the model in general dimension, though starting in the next section, we will focus solely on the one-dimensional case). The deviation of the atom $i \in \mathbb{Z}^n$ from its equilibrium position is denoted by q_i, and the conjugate momentum variable is denoted by p_i. We consider the dynamics associated to the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + V_h(q) + \sqrt{\lambda}V(q)$$

where V_h is a harmonic potential (quadratic) and $\sqrt{\lambda}V$ is a small anharmonic perturbation (the kinetic equation is obtained in the limit $\lambda \to 0$). The general form of the harmonic potential is

$$V_h(q) = \frac{1}{2}\sum_{i,j\in\mathbb{Z}^n} \overline{\alpha}(i-j)q_iq_j + \frac{\omega_0^2}{2}\sum_{i\in\mathbb{Z}^n} q_i^2, \tag{3.1}$$

while V is typically a cubic or quartic potential of the form

$$V(q) = \sum_{i\in\mathbb{Z}^n} \gamma(q_i) \quad \text{or} \quad V(q) = \sum_{\substack{i,j\,\in\,\mathbb{Z}^n \\ |i-j|=1}} \gamma(q_j - q_i).$$

In order to understand how energy is being transported by the vibration of the atoms in the lattice, we will replace this very large system of ODE by a kinetic equation (the so-called Botzmann phonon equation) whose unknown $W(x,k,t)$ will be interpreted as a density distribution function for a gas of interacting phonons. The idea of describing the lattice vibrations by interacting phonons, whose evolution would be described by a Boltzmann type equation first appeared in a paper of Peierls [Pei29]. This derivation was made more rigorous by H. Spohn [Spo06b] using Wigner transforms and asymptotic analysis.

We will not give any details concerning this derivation (we refer the interested reader to the work of H. Spohn [Spo06b]). We just claim that (formally at least) an appropriately rescaled Wigner transform of the displacement field q converges when $\lambda \to 0$ to a function $W(t,x,k)$ solution of the Boltzmann phonon equation

$$\partial_t W + \nabla_k \omega(k) \cdot \nabla_x W = C(W). \tag{3.2}$$

The function W depends on the time $t \geq 0$, the position $x \in \mathbb{R}^n$ and a wave vector k which lies in the Torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$. The function $\omega(k)$ is the dispersion relation of the lattice. It is determined by the harmonic part of the potential. For general potential given by (3.1), we have:

$$\omega(k) = (\omega_0^2 + \widehat{\overline{\alpha}}(k))^{1/2} \tag{3.3}$$

where $\widehat{\overline{\alpha}}(k)$ is the Fourier transform of $\overline{\alpha}$, defined by

$$\widehat{\overline{\alpha}}(k) = \sum_{j\in\mathbb{Z}^n} e^{-i2\pi k\cdot j}\alpha(j).$$

The operator C in the right hand side of (3.2) is an integral collision operator which depends on the anharmonic potential $V(q)$. Of course this operator C is crucial in determining the long time behavior of the solutions of this equation, so we will spend a bit of time discussing its properties in this introduction.

Note that while the relation between $W(t,x,k)$ and the microscopic variable q_i and p_i is rather complicated, the total energy of the system is given by

$$\begin{aligned}\int_{\mathbb{R}^n}\int_{\mathbb{T}^n} \omega(k)W(t,x,k)\,dk\,dx &= \frac{1}{2}\int |\widehat{p}(k)|^2 + \omega(k)^2|\widehat{q}(k)|^2\,dk \\ &= \sum_{i\in\mathbb{Z}^n} \frac{1}{2}p_i^2 + V_h(q). \end{aligned} \tag{3.4}$$

3.1.1 The FPU framework

As explained in the introduction, we now focus on the FPU chain model. For this model, we have $N = 1$ (we denote by $\mathbb{T}$ the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$) and the potential describes only nearest neighbors interactions. The harmonic potential is thus given by:

$$V_h(q) = \frac{1}{8} \sum_{i\in\mathbb{Z}} (q_{i+1} - q_i)^2,$$

and the anharmonic potential V is either cubic (FPU-α chain) or quartic (FPU-β chain):

$$V(q) = \sum_{i\in\mathbb{Z}^n} \gamma(q_{i+1} - q_i), \qquad \gamma(q) = \frac{1}{3} q^3 \text{ or } \gamma(q) = \frac{1}{4} q^4.$$

The corresponding microscopic dynamics is given by

$$\begin{aligned} \frac{d}{dt} q_i(t) &= p_i(t) \\ \frac{d}{dt} p_i(t) &= \frac{1}{4} q_{i+1}(t) - \frac{1}{2} q_i(t) + \frac{1}{4} q_{i-1}(t) - \sqrt{\lambda}[\gamma'(q_i - q_{i-1}) - \gamma'(q_{i+1} - q_i)]. \end{aligned} \tag{3.5}$$

3.1.2 The dispersion relation

When V_h is given by

$$V_h(q) = \frac{1}{2}\omega_0^2 \sum_{i\in\mathbb{Z}} q_i^2 + \frac{1}{8} \sum_{i\in\mathbb{Z}} (q_{i+1} - q_i)^2, \tag{3.6}$$

equation (3.3) gives the following formula for the dispersion relation:

$$\omega(k)^2 = \omega_0^2 + \frac{1}{2} - \frac{1}{4}\left(e^{i2\pi k} + e^{-i2\pi k}\right)$$

and so

$$\omega(k) = \left(\omega_0^2 + \frac{1}{2}(1 - \cos(2\pi k))\right)^{1/2}, \quad k \in \mathbb{T}. \tag{3.7}$$

For the FPU model, we have $\omega_0 = 0$, and so the dispersion relation is given by

$$\omega(k) = \sqrt{\frac{1}{2}(1 - \cos(2\pi k))} = |\sin(\pi k)|.$$

3.1.3 The interaction operator C

The operator C in the right hand side of (3.2) is determined by the non-harmonic perturbation of the potential V.

Cubic potentials: Three phonons operator When the anharmonic potential is cubic, that is

$$V = \frac{1}{3} \sum_{i\in\mathbb{Z}} q_i^3, \tag{3.8}$$

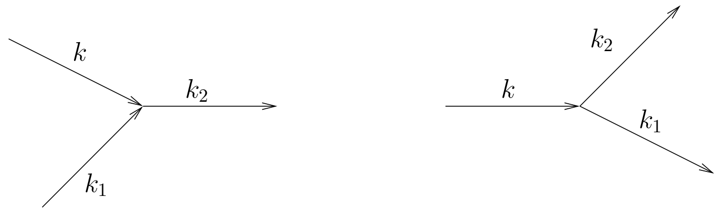

Figure 3.1: Three phonons interactions

or

$$V = \frac{1}{3}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^3 \tag{3.9}$$

(the latter one corresponds to the FPU-α chain), the collision operator is given by

$$\begin{aligned} C(W) = 4\pi \int\int & F(k,k_1,k_2)^2 \\ & \times \Big[2\delta(k+k_1-k_2)\delta(\omega+\omega_1-\omega_2)(W_1W_2+WW_2-WW_1) \\ & + \delta(k-k_1-k_2)\delta(\omega-\omega_1-\omega_2)(W_1W_2-WW_1-WW_2)\Big] dk_1 dk_2 \end{aligned} \tag{3.10}$$

where we used the notation $\omega_i = \omega(k_i)$ and $W_i = W(k_i)$.

The formula for the collision rate $F(k,k_1,k_2)$ can be found in [Spo06b]. In particular, when V is given by (3.8) (on-site potential) then

$$F(k,k_1,k_2)^2 = (8\omega\omega_1\omega_2)^{-1}$$

When V is the nearest neighbor interaction potential (3.9) and $\omega_0 = 0$ (that is for the FPU-α chain), the collision rate becomes

$$F(k,k_1,k_2)^2 = (8\omega\omega_1\omega_2)^{-1}|[\exp(i2\pi k)-1][\exp(i2\pi k_1)-1][\exp(i2\pi k_2)-1]|^2.$$

Using the fact that

$$|\exp(i2\pi k)-1|^2 = 4\sin^2(\pi k),$$

we find

$$F(k,k_1,k_2)^2 = 8\frac{\sin^2(\pi k)\sin^2(\pi k_1)\sin^2(\pi k_2)}{\omega\omega_1\omega_2}.$$

Going back to (3.10), we note that the first term can be interpreted as describing a wave vector k merging with a wave vector k_1 and leading to a new wave vector k_2 ($k + k_1 \to k_2$), while the second term describes the splitting of wave vector k into k_1 and k_2 ($k \to k_1 + k_2$). See Figure 3.1. These interactions conserve the energy ($\omega + \omega_1 = \omega_2$), but the momentum is conserved only modulo integers: the δ-function in the first term yields the constraint $k + k_1 = k_2 + n$, $n \in \mathbb{Z}$, $k, k_1, k_2 \in \mathbb{T}$ (one talks of normal process when $n = 0$, and *umklapp* process when $n \neq 0$).

This quadratic operator is reminiscent of the Boltzmann operator for the theory of dilute gas. There is however an essential difference: The kinetic energy $\frac{1}{2}v^2$ is replaced here by the dispersion relation $\omega(k)$. In order to further study this integral operator, it is thus essential to characterize the set of (k, k_1, k_2) such that the δ-functions are not zero, that is:

$$\begin{cases} k + k_1 = k_2 \\ \omega(k) + \omega(k_1) = \omega(k_2) \end{cases}$$

or

$$\omega(k) + \omega(k_1) = \omega(k + k_1), \qquad (k, k_1) \in \mathbb{T}, \tag{3.11}$$

This is much more delicate than for the usual Bolzmann operator and for general dispersion relation ω, it is not obvious that (3.11) has any solutions.

In our framework, that is when ω is given by (3.7) (nearest neighbor harmonic coupling) we actually can prove that

$$\omega(k) + \omega(k_1) - \omega(k + k_1) \geq \frac{\omega_0}{2}$$

so (3.11) has no solutions when $\omega_0 > 0$ and only the trivial solution $k_1 = 0$ when $\omega_0 = 0$.

It follows [Spo06b]:

Theorem 3.1. *When ω is given by* (3.7) *with $\omega_0 \geq 0$, then the three phonon collision operator* (3.10) *satisfies $C(W) = 0$ for all W.*

In particular, this implies that for the FPU-α chain, the collision operator vanishes, and the corresponding Boltzmann phonon equation reduces to pure transport. This suggests poor relaxation to equilibrium for the microscopic model, and it means that this kinetic approach is of no use in studying the long time behavior of the hamiltonian system. This is of course the reason why we focus in this paper on the FPU-β chain.

Remark 3.2. As noted in [Spo06b], equation (3.11) might have non trivial solutions for other dispersion relations (for instance $\omega(k) = \omega_0 + 2(1 - \cos(2\pi k))$), so this three phonon operator is of interest in other framework (different harmonic potential V_h).

Quartic potentials: Four phonons operator. We now consider the quartic potential given by

$$V(q) = \frac{1}{4}\sum_{i\in\mathbb{Z}} q_i^4 \tag{3.12}$$

Figure 3.2: Four phonons interactions

or

$$V = \frac{1}{4}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^4. \tag{3.13}$$

The corresponding collision operator then reads

$$\begin{aligned} C(W) = 12\pi \sum_{\sigma_1,\sigma_2,\sigma_3=\pm 1} \int\int\int F(k,k_1,k_2,k_3)^2 \\ \times\, \delta(k+\sigma_1 k_1+\sigma_2 k_2+\sigma_3 k_3)\delta(\omega+\sigma_1\omega_1+\sigma_2\omega_2+\sigma_3\omega_3) \\ \times\, (W_1W_2W_3 + W(\sigma_1 W_2W_3 + W_1\sigma_2 W_3 + W_1W_2\sigma_3))\, dk_1\, dk_2\, dk_3 \end{aligned} \tag{3.14}$$

with

$$F(k,k_1,k_2,k_3)^2 = (16\omega\omega_1\omega_2\omega_3)^{-1}$$

for on-site potential (3.12) and

$$F(k,k_1,k_2,k_3)^2 = \prod_{i=0}^{3}\frac{2\sin^2(\pi k_i)}{\omega(k_i)}. \tag{3.15}$$

for nearest neighbor coupling (3.13).

The term proportional to W is the loss term, while the gain term is $W_1W_2W_3$ (which is always positive). Again, we can interpret the different terms as pair collisions or merging/splitting of phonons (see Figure 3.2). In order to understand the collision rule, we note that for pair collisions $(k,k_1)\to(k_2,k_3)$ (which correspond to the terms such that $\sum_{j=1}^{3}\sigma_j = -1$ in the integral), we need to solve

$$\omega(k)+\omega(k_1) = \omega(k_2)+\omega(k+k_1-k_2) \tag{3.16}$$

while for three phonons mergers (or splitting) $(k,k_1,k_2)\to k_3$ we have

$$\omega(k)+\omega(k_1)+\omega(k_2) = \omega(k+k_1+k_2). \tag{3.17}$$

In general, it is not possible to solve these equations explicitly, and it is not obvious that either of these equations should be satisfied on a set of positive measure. In fact, when ω is given by (3.7) (nearest neighbor couplings), it can be shown (see [Spo08]) that (3.17) has no solution (so collision processes in which three phonons are merged into one, or one phonon splits into three are impossible). As a consequence, the only interactions that are allowed are pair collisions, which, in particular, preserve the total number of phonons. This preservation of the number of phonons, reminiscent of the preservation of the number of particles in gas dynamics, does not follow here from a fundamental physical principle, but is instead a mathematical artifact. This property is however stable under small perturbation of ω, and it also holds for the nonlinear wave equation for which $\omega(k) = |k|$ ($k \in \mathbb{R}^3$).

As a consequence, the operator C can be rewritten as

$$C(W) = 36\pi \int\int\int F(k, k_1, k_2, k_3)^2 \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3)$$
$$[W_1 W_2 W_3 + W W_2 W_3 - W W_1 W_3 - W W_1 W_2]\, dk_1\, dk_2\, dk_3. \tag{3.18}$$

When ω is given by (3.7), we will see later on that (3.16) has non trivial solutions on a set of full measure, that is

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \delta(\omega(k) + \omega(k_1) - \omega(k_2) - \omega(k + k_1 - k_2))\, dk_1\, dk_2 \neq 0.$$

In particular this operator C is non trivial.

3.2 FPU-β chain: The four phonon collision operator

In this section, we briefly summarize the properties of the four phonon collision operator (3.18) which arises in the modeling of the FPU-β chain.

3.2.1 Conserved quantities

All the collision operators C mentioned above conserve the energy. This can be expressed by the following condition:

$$\int_{\mathbb{T}} \omega(k) C(W)(k)\, dk = 0$$

for all functions W.

The four phonon collision operator (3.18), corresponding to the quartic potential, also satisfies

$$\int_{\mathbb{T}} C(W)(k)\, dk = 0$$

which can be interpreted as the conservation of the number of phonons $\int_{\mathbb{T}} W\, dk$. However, this quantity has no microscopic equivalent, and does not correspond to any physical principle. Rather it is a consequence of the symmetry of the operator, which follows from the fact that 3 phonon merger cannot take place ((3.17) has no solutions). In particular, this equality does not hold for the three phonon operator.

Note that the first moment k is preserved in the wave kinetic equation case (where $k \in \mathbb{R}^N$). However, this conservation is broken here by umklapp processes.

3.2.2 Entropy

The Boltzmann phonon operators satisfy an entropy inequality, similar to Boltzmann H-Theorem in gas dynamic. In particular, for the four phonon operator we can rewrite (3.18) as follows:

$$\begin{aligned} C(W) &= 36\pi \int\int\int F(k)^2 \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ & \quad WW_1W_2W_3[W^{-1}+W_1^{-1}-W_2^{-1}-W_3^{-1}]dk_1\,dk_2\,dk_3 \end{aligned}$$

and we then see that (assuming all integrals are well defined):

$$\begin{aligned} &\int_{\mathbb{T}^1} W^{-1}(k)C(W)(k)\,dk \\ &\quad = 9\pi \int\int\int\int F(k)^2\delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ &\qquad \cdot WW_1W_2W_3[W^{-1}+W_1^{-1}-W_2^{-1}-W_3^{-1}]^2 dk_1\,dk_2\,dk_3 \\ &\quad \geq 0. \end{aligned} \tag{3.19}$$

3.2.3 Stationary solutions

It is easy to check that the distributions

$$W_b(k) = \frac{1}{b\omega(k)}$$

for any $b > 0$ satisfy $C(W_b) = 0$ for all the operators C considered above. This fact is in accordance with equilibrium statistical mechanics (see [Spo06b]). It is more delicate to check that these are the only solutions. In fact it is not always true.

For the four phonon collision operator (3.18), we can check that

$$W_{a,b}(k) = \frac{1}{a + b\omega(k)} \tag{3.20}$$

is an equilibrium for all $a, b > 0$.

Conversely, the entropy inequality (3.19) implies that if $C(W) = 0$ then $\psi(k) = W(k)^{-1}$ is a collision invariant, that is

$$\psi(k) + \psi(k_1) = \psi(k_2) + \psi(k_3)$$

for all k, k_1, k_2, k_3 such that

$$k + k_1 = k_2 + k_3, \quad \text{and} \quad \omega(k) + \omega(k_1) = \omega(k_2) + \omega(k_3).$$

An obvious candidate is $\psi(k) = a + b\omega(k)$. Under general conditions on ω, Spohn proved that these are indeed the only collision invariants in dimension $N \geq 2$ [Spo06a]. The same result is proved by Lukkarinen and Spohn [LS08] in our framework (dimension 1).

As a conclusion, (3.20) are the only solutions of $C(W) = 0$ for the four phonon collision operator (3.18). Note that the fact that we can take $a \neq 0$ is a consequence of the conservation of the number of phonons for the four phonon collision operator (which, as explained above, follows from the fact that equation (3.17) describing merging and splitting of phonons has no solutions).

3.2.4 The linearized operator

As mentioned in the introduction, we will be interested in the behavior of the solutions of the Boltzmann phonon equation in the neighborhood of a thermodynamical equilibrium. Given $\overline{W}(k) = \frac{T}{\omega(k)}$, we thus introduce the linearized operator

$$L(f) = \frac{1}{\overline{W}} DC(\overline{W})(\overline{W} f)$$

where DC denotes the derivative of the operator C.

By differentiating the equation $C(W_{a,b}) = 0$ with respect to a and b, we get:

$$L(1) = 0 \quad \text{and } L(\omega^{-1}) = 0,$$

which suggests (as will be proved later) that the kernel of L is two dimensional and spanned by 1 and ω^{-1}. In our framework, the later mode, ω^{-1} is singular (not integrable) for $k = 0$. Because of natural a priori bounds on the solutions of the Boltzmann Phonon equation, it will be easy to see that this mode is not present in the macroscopic limit. It will however play an important role in the derivation of a macroscopic model. Note that it comes from the derivation with respect to the spurious coefficient a.

Similarly, differentiating the conservation equations

$$\int \omega C(\overline{W} + t\overline{W} f)\, dk = 0 \text{ and } \int C(\overline{W} + t\overline{W} f)\, dk = 0$$

with respect to t, we deduce that

$$\int L(f)\, dk = 0, \text{ and } \int \omega^{-1} L(f)\, dk = 0.$$

The properties of L will be further investigated in Section 3.4. For now, we just state the following proposition without proof, since it is all we need to formally derive a macroscopic equation.

Proposition 3.3. *The operator* $L : L^2(\mathbb{T}^1, V(k)\,dk) \longrightarrow L^2(\mathbb{T}^1, V(k)^{-1}\,dk)$ *(where* V *is defined by* (3.31)*) is a bounded self-adjoint operator which satisfies*

(i) $\ker(L) = \mathit{Span}\,\{1, \omega(k)^{-1}\}$

(ii) $R(L) = \{h \in L^2(\mathbb{T}^1, V(k)^{-1}\,dk)\,;\, \int_{\mathbb{T}} h(k)\,dk = \int_{\mathbb{T}} \omega^{-1}(k)h(k)\,dk = 0\,\}$

We end this section by deriving the explicit formula for the operator L: A direct computation gives (when $W(k) = \frac{\overline{T}}{\omega(k)}$):

$$
\begin{aligned}
&DC(W)(Wf)\\
&= 36\pi \int\!\!\int\!\!\int F(k,k_1,k_2,k_3)^2 \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3)\\
&\qquad \times WW_1W_2W_3\Big[f_3W_3^{-1} + f_2W_2^{-1} - f_1W_1^{-1} - fW^{-1}\Big]\,dk_1\,dk_2\,dk_3\\
&= 36\pi\overline{T}^3 \int\!\!\int\!\!\int \frac{F(k,k_1,k_2,k_3)^2}{\omega\omega_1\omega_2\omega_3}\delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3)\\
&\qquad \times \Big[\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f\Big]\,dk_1\,dk_2\,dk_3
\end{aligned}
$$

Using (3.15), we see that

$$
\frac{F(k,k_1,k_2,k_3)^2}{\omega\omega_1\omega_2\omega_3} = 16
$$

and we deduce:

$$
\begin{aligned}
L(f) = 576\pi\overline{T}^2\omega \int\!\!\int\!\!\int &\delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3)\\
&\times \Big[\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f\Big]\,dk_1\,dk_2\,dk_3. \qquad (3.21)
\end{aligned}
$$

3.2.5 Formal asymptotic limit

We now have all the ingredient to perform the usual asymptotic analysis and attempt to derive (formally) a diffusion equation from the Boltzmann phonon equation (we will see however that it fails in our framework). The starting point is the following rescaled equation in the FPU-β chain framework detailed above:

$$
\varepsilon^2\partial_t W + \varepsilon\omega'(k)\partial_x W = C(W), \qquad (3.22)
$$

where C is the four phonon collision operator (3.18) with collision frequency given by (3.15), and we consider a solution which is a perturbation of a thermodynamical equilibrium:

$$
W^\varepsilon(t,x,k) = \overline{W}(k)(1 + \varepsilon f^\varepsilon(t,x,k))
$$

where $\overline{W} = \frac{\overline{T}}{\omega(k)}$ for some constant $\overline{T} > 0$.

We introduce the operators

$$Q(f,f) = \frac{1}{\overline{W}} D^2 C(\overline{W})(\overline{W}f, \overline{W}f),$$

and

$$R(f,f,f) = \frac{1}{\overline{W}} D^3 C(\overline{W})(\overline{W}f, \overline{W}f, \overline{W}f)$$

so that (we recall that C is a cubic operator):

$$\frac{1}{\overline{W}} C(W^\varepsilon) = \varepsilon L(f) + \varepsilon^2 \frac{1}{2} Q(f,f) + \varepsilon^3 \frac{1}{6} R(f,f,f)$$

where L is given by (3.21).

The function f^ε solves

$$\varepsilon^2 \partial_t f^\varepsilon + \varepsilon w'(k) \partial_x f^\varepsilon = L(f^\varepsilon) + \varepsilon \frac{1}{2} Q(f^\varepsilon, f^\varepsilon) + \varepsilon^2 \frac{1}{6} R(f^\varepsilon, f^\varepsilon, f^\varepsilon). \tag{3.23}$$

Taking the limit $\varepsilon \to 0$ in (3.23), we get

$$L(f^0) = 0$$

and so Proposition 3.3 (i) implies

$$f^0(t,x,k) = T(t,x) + S(t,x)\omega(k)^{-1}.$$

Since equation (3.23) preserves the L^1 norm, it is natural to assume that $f^0(t,x,k) \in L^1(\mathbb{R} \times \mathbb{T})$. We note however that $\omega(k) \sim |k|$ as $|k| \to 0$, and so we must have

$$S(t,x) = 0.$$

Next, integrating (3.23) with respect to k yields

$$\partial_t T^\varepsilon + \partial_x J^\varepsilon = 0$$

with

$$T^\varepsilon = \langle f^\varepsilon \rangle, \quad J^\varepsilon(t,x) = \frac{1}{\varepsilon} \langle \omega' f^\varepsilon \rangle$$

where we use the notation $\langle \cdot \rangle = \int_{\mathbb{T}} \cdot \, dk$.

We now need to compute $J = \lim_{\varepsilon \to 0} J^\varepsilon$. Recalling that L is a self adjoint operator, we write

$$\varepsilon^{-1} \langle \omega' f^\varepsilon \rangle = \langle L^{-1}(\omega') L(f^\varepsilon) \rangle$$

and using (3.23), we replace $L(f^\varepsilon)$ in the right hand side:

$$\varepsilon^{-1}\langle \omega' f^\varepsilon\rangle = \langle L^{-1}(\omega')\omega'\partial_x f^\varepsilon\rangle - \langle L^{-1}(\omega')Q(f^\varepsilon, f^\varepsilon)\rangle + \mathcal{O}(\varepsilon).$$

Formally, we thus get

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}\langle \omega' f^\varepsilon\rangle = \langle L^{-1}(\omega')\omega'\rangle\partial_x T - \langle L^{-1}(\omega')Q(T,T)\rangle.$$

Finally, a direct computation gives

$$Q(f,f) = 576\pi\overline{T}^2\omega \int\int\int \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3)$$
$$\Big[2(\omega-\omega_3)[f_1f_2 - ff_3] + (\omega+\omega_1)[f_2f_3 - ff_1]\Big]\, dk_1\, dk_2\, dk_3,$$

and it is readily seen that $Q(T,T) = 0$. We thus get the following relation

$$J = \langle L^{-1}(\omega')\omega'\rangle\partial_x T$$

which is Fourier's law with diffusion coefficient

$$\kappa = -\langle L^{-1}(\omega')\omega'\rangle > 0.$$

We conclude this section with the following remarks:

(i) The non linear term $Q(T,T) = 0$ does not contribute to the limiting equation. In the next section, we will drop this term altogether.

(ii) The fact that $S = 0$ will need to be addressed very carefully in the rigorous proof. In particular, we will see that while we do indeed have $f^0 = T$, the term S plays a significant role in the rigorous derivation of the diffusion equation (see next section).

(iii) Perhaps the most important remark is that one need to check that κ is well defined. In fact, it can be proved that the integrand in the definition of the diffusion coefficient behaves like $|k|^{-5/3}$ for small k. It follows that

$$\kappa = +\infty$$

so the limit presented above does not give any equation for the evolution of T. Such a phenomenon is not uncommon, and based on previous work (see [MMM11]), we expect that by taking a different time scale in (3.23) we can derive an anomalous diffusion equation for the evolution of the temperature T. This is of course the goal of this paper as explained in the next section.

3.3 Main result

In view of the formal asymptotic limit detailed in the previous section, we now consider the following linear equation:

$$\varepsilon^{\alpha}\partial_t f^{\varepsilon} + \varepsilon\omega'(k)\partial_x f^{\varepsilon} = \overline{T}^2 L(f^{\varepsilon}), \qquad x \in \mathbb{R},\ k \in \mathbb{T} \tag{3.24}$$

where

$$\omega(k) = |\sin(\pi k)|$$

and L is defined by

$$L(f) = \omega \int\int\int \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \left[\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f\right] dk_1\, dk_2\, dk_3. \tag{3.25}$$

Note also that we have made L independent of the equilibrium temperature $\overline{T}$ and set all other constant in L equal to 1 for the sake of clarity.

The existence of a solution to this equation is fairly classical. We recall it for the sake of completeness in Proposition 3.15.

Our main result is then the following:

Theorem 3.4 (Fractional diffusion limit for the linearised equation)**.** *Let f^{ε} be a solution of equation (3.24) and take $\alpha = \frac{8}{5}$, with initial data $f_0 \in L^2(\mathbb{R} \times \mathbb{T})$. Then*

$$f^{\varepsilon}(t, x, k) \rightharpoonup T(t, x) \qquad L^{\infty}((0, \infty); L^2(\mathbb{R} \times \mathbb{T}))\text{-weak} *$$

where T solves the fractional diffusion equation

$$\partial_t T + \frac{\kappa}{\overline{T}^{6/5}}(-\Delta_x)^{4/5} T = 0 \qquad \text{in } (0, \infty) \times \mathbb{R} \tag{3.26}$$

with initial condition

$$T(0, x) = T_0(x) := \int_0^1 f_0(x, k)\, dk. \tag{3.27}$$

The diffusion coefficient $\kappa \in (0, \infty)$ is given by

$$\kappa = \kappa_1 - \frac{\kappa_2^2}{\kappa_3} \in (0, \infty)$$

where $\kappa_1, \kappa_2, \kappa_3$ are defined in Proposition 3.19.

First, we note that it is enough to consider the case

$$\overline{T} = 1$$

since we can recover the general case by a simple rescaling $t \mapsto \overline{T}^2 t$, $x \mapsto \overline{T}^2 x$.

The main difficulty here, compared with previous work devoted to fractional diffusion limit of kinetic equations, is the fact that the kernel of L is spanned by 1 and $\omega(k)^{-1}$. This last mode should not appear in the limit since it is not square integrable, but it will nevertheless play an important role.

In fact, we will prove that f^ε can be expanded as follows:

$$f^\varepsilon(t,x,k) = T^\varepsilon(t,x) + \varepsilon^{\frac{3}{5}} S^\varepsilon(t,x)\omega(k)^{-1} + \varepsilon^{\frac{4}{5}} h^\varepsilon(t,x,k)$$

where T^ε is bounded in $L^\infty(0,\infty; L^2(\mathbb{R}))$, h^ε is bounded in $L^2_V(\mathbb{T}\times\mathbb{R})$ and S^ε converges in some weak sense to a non trivial function. More precisely we will prove in Section 3.6:

Proposition 3.5. *The function $S^\varepsilon(t,x)$ converges in distribution sense to*

$$S(t,x) = -\frac{\kappa_2}{\kappa_3}(-\Delta)^{3/10} T(t,x).$$

In particular, as mentioned above, this means that the mode $\omega(k)^{-1}$ vanishes in the limit and the macroscopic behavior of the phonon distribution is completely described by $T = \lim_{\varepsilon\to 0} T^\varepsilon$. However, projecting equation (3.24) onto the constant mode of the kernel of L, we will find the following equation of the evolution of T:

$$\partial_t T + \kappa_1(-\Delta)^{4/5} T + \kappa_2(-\Delta)^{1/2} S = 0. \tag{3.28}$$

We see that $S = \lim_{\varepsilon\to 0} S^\varepsilon$ plays a role in the evolution of T. To understand this, we note (anticipating a bit on the result of the next section) that the reason we are observing anomalous diffusion phenomena here (as opposed to standard diffusion as described in the previous section), is the fact that phonons with frequency k close to zero encounter very few collisions (degenerate collision frequency). And the term $\varepsilon^{\frac{3}{5}} S^\varepsilon(t,x)\omega(k)^{-1}$, while small, is heavily concentrated around $k=0$ (non integrable singularity at $k=0$). The competition between the smallness and the singularity gives rise to a term of order 1 in the equation.

In order to describe the evolution of T, we now need to obtain an equation for S. By projecting equation (3.24) onto the $\omega(k)^{-1}$ mode of the kernel of L, we will prove that:

$$\kappa_2(-\Delta)^{1/2} T + \kappa_3(-\Delta)^{1/5} S = 0. \tag{3.29}$$

We note that there is no $\partial_t S$ in (3.29) (unlike the corresponding equation for T). The reason is that due to the singularity of $\omega(k)^{-1}$ for $k=0$, the quantity S diffuses faster than T (so we would have to take a smaller α in (3.24) in order to observe the diffusion of S). At our time scale (given by $\alpha = \frac{8}{5}$), S has thus already reached equilibrium, and can be expressed (in view of (3.29)) as

$$S = -\frac{\kappa_2}{\kappa_3}(-\Delta)^{3/10}T.$$

Inserting this expression into (3.28), we find

$$\partial_t T + \kappa(-\Delta)^{4/5}T = 0$$

where $\kappa = \kappa_1 - \frac{\kappa_2^2}{\kappa_3}$. Of course, we will show that $\kappa > 0$ (once the explicit expressions for the κ_i are given, it will be a very simple consequence of Cauchy-Schwarz inequality - see Lemma 3.21). It is interesting to note that the effect of the mode ω^{-1} on the macroscopic equation is to reduce the diffusion coefficient (and thus to slow down the diffusion). This can be understood by noting that the fact that the kernel of L does not contain only the natural constant mode, is due to the lack of merging $k + k_1 + k_2 \to k_3$ and splitting $k \to k_1 + k_2 + k_3$ interactions for phonons in the non linear collision operator C (fewer interactions $\Rightarrow$ slower relaxation).

3.4 Properties of the operator L

The asymptotic behavior of the solution of (3.24) depends very strongly on the properties of the operator L. This operator is studied in great detail in [LS08], and we will recall their main results in this section.

The operator L can be written as

$$L(f) = \int K(k,k')f(k')\,dk' - V(k)f(k)$$

where

$$K(k,k') = \omega(k)\omega(k')\int_{\mathbb{T}} 2\,\delta(\omega(k)+\omega(k_1)-\omega(k')-\omega(k+k_1-k')) \\ - \delta(\omega(k)+\omega(k')-\omega(k_1)-\omega(k+k'-k_1)\,dk_1 \qquad (3.30)$$

and

$$V(k) = \omega(k)^2\int_{\mathbb{T}} \delta(\omega(k)+\omega(k_1)-\omega(k')-\omega(k+k_1-k'))\,dk_1dk'. \qquad (3.31)$$

The fact that $\int_{\mathbb{T}} L(f)\,dk = 0$ for all f implies

$$V(k) = \int_{\mathbb{T}} K(k',k)\,dk'$$

(this equality can be checked also from the formula for K and V, but it is much easier this way) and a short computation shows that

$$K(k,k') = K(k',k).$$

In particular, L is a self adjoint operator in $L^2(\mathbb{T})$. It is also positive since we have

$$\begin{aligned} -\int_{\mathbb{T}} L(f)f\,dk &= \frac{1}{4}\int\int\int\int \delta(k+k_1-k_2-k_3)\delta(\omega+\omega_1-\omega_2-\omega_3) \\ &\quad [\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f]^2\,dk\,dk_1\,dk_2\,dk_3 \\ &\geq 0 \end{aligned} \tag{3.32}$$

for all f. One of our goals will be to improve this inequality and show that L has a spectral gap property in the appropriate functional spaces. For that, we will need to show that the integral operator

$$K(f) = \int K(k,k')f(k')\,dk' \tag{3.33}$$

is a compact operator (in an appropriate functional spaces)

The first step, in view of (3.30) is to study the solution set of the equation of conservation of energy:

$$\omega(k) + \omega(k_1) = \omega(k') + \omega(k+k_1-k'). \tag{3.34}$$

We recall the following result:

Proposition 3.6 ([LS08]). *The equation* (3.34) *has the trivial solutions* $k'=k$ *and* $k'=k_1$*, and the (non trivial) solution*

$$k_1 = h(k,k')$$

where

$$h(k,k') = \frac{k'-k}{2} + 2\arcsin\left(\tan\frac{|k'-k|}{4}\cos\frac{k+k'}{4}\right)$$

(and there are no other solutions of (3.34)*).*

With this proposition in hand, one can now compute the kernel $K(k,k')$ and the multiplicative function $V(k)$. We recall here the main result of [LS08]. The first one states that the function $V(k)$ is degenerate for $k\to 0$ (note that W in [LS08] corresponds to our V):

Proposition 3.7 ([LS08, Lemma 4.1]). *The function* $V:\mathbb{R}\to\mathbb{R}_+$ *is symmetric* ($V(1-k) = V(k)$)*, continuous and satisfies*

$$c_1|\sin(\pi k)|^{5/3} \leq V(k) \leq c_2|\sin(\pi k)|^{5/3} \tag{3.35}$$

for all $k\in\mathbb{R}$*, for some* $c_1, c_2 > 0$*. Moreover,*

$$\lim_{k\to 0}\left(|\sin\pi k|^{-5/3}V(k)\right) = v_0 > 0.$$

Because of the degeneracy of V for $k = 0$, we do not expect the operator L to have a spectral gap in L^2. We thus introduce the operator

$$L_0(f) := V^{-1/2} L(V^{-1/2} f)$$

We note that this operator has the form

$$L_0(f) = K_0(f) - f$$

with

$$K_0(f) = V^{-1/2} K(V^{-1/2} f).$$

To prove that L_0 has good properties in $L^2(\mathbb{T})$, we need to study the operator K_0. Again, it is proved in [LS08] that $K_0 : L^2(\mathbb{T}^1) \to L^2(\mathbb{T}^1)$ is a compact, self-adjoint operator, which implies that $K : L^2(\mathbb{T}^1, V\,dk) \to L^2(\mathbb{T}^1, V^{-1}dk)$ is a compact, self-adjoint operator.

To be more precise, in [LS08], the kernel K is first written as

$$K(k,k') = 2\omega(k)K_2(k,k')\omega(k') - \omega(k)K_1(k,k')\omega(k')$$

where

$$K_1(k,k') := 4\frac{\mathbb{1}\left(F_-(k,k') > 0\right)}{\sqrt{F_-(k,k')}} \quad \text{and} \quad K_2(k,k') := \frac{2}{\sqrt{F_+(k,k')}} \tag{3.36}$$

for $k, k' \in [0,1]$ and

$$F_\pm(k,k') = \left(\cos(\pi k) + \cos(\pi k')\right)^2 \pm 4\sin(\pi k)\sin(\pi k').$$

and the main result of [LS08] is the following:

Proposition 3.8 ([LS08, Propositions 4.3 and 4.4.]). *Let $\psi : [0,1] \to \mathbb{R}$ be given, and assume that there are $C, p > 0$ such that*

$$|\psi(k)| \le C\,(\sin \pi k)^p$$

for all $k \in [0,1]$. Then the kernels

$$\psi(k)^* K_2(k,k')\psi(k') \qquad \textit{and} \qquad \psi(k)^* K_1(k,k')\psi(k')$$

define compact, self-adjoint integral operators in $L^2(\mathbb{T})$.

We immediately conclude:

Corollary 3.9. *The kernel*

$$K_0(k,k') = V^{-1/2}(k)\omega(k)\left(2K_2(k,k') - K_1(k,k')\right)\omega(k')V^{-1/2}(k') \tag{3.37}$$

defines a compact self-adjoint operator in $L^2((0,1))$*. As a consequence, the kernel*

$$K(k,k') = V^{1/2}(k)K_0(k,k')V^{1/2}(k')$$

defines a compact self-adjoint operator from $L^2(\mathbb{T}^1, V(k)\,dk)$ *onto* $L^2(\mathbb{T}^1, V(k)^{-1}\,dk)$*. In particular,*

$$\int_{\mathbb{T}} |K(f)(k)|^2 V(k)^{-1}\,dk \leq C\int_{\mathbb{T}} |f(k)|^2 V(k)\,dk. \tag{3.38}$$

Proof. Indeed, by Proposition 3.7 we have that

$$V^{-1/2}(k)\omega(k) \leq c_2\,(\sin\pi k)^{1/6}$$

and the claim follows from Proposition 3.8. □

Furthermore, we note that we have not used the full potential of Proposition (3.8). We can thus improve (3.38) as follows:

Corollary 3.10. *The kernel*

$$\widetilde{K_0}(k,k') := (\sin(\pi k))^{-1/6+\eta}\,K_0(x,k')\,\left(\sin(\pi k')\right)^{-1/6+\eta} \quad \eta > 0$$

defines a compact self-adjoint operator in $L^2((0,1))$*. In particular, for all* $\eta > 0$*, there exists* $C(\eta)$ *such that*

$$\int_{\mathbb{T}} |K(f)(k)|^2 (\sin(\pi k))^{-\frac{1}{3}+\eta} V(k)^{-1}\,dk \leq C\int_{\mathbb{T}} |f(k)|^2 (\sin(\pi k))^{\frac{1}{3}-\eta} V(k)\,dk \tag{3.39}$$

Proof. Using Proposition 3.7 we have that

$$\begin{aligned} V^{-1/2}(k)\omega(k)\,(\sin\pi k)^{-1/6+\eta} &\leq c_2\,(\sin\pi k)^{1/6}\,(\sin\pi k)^{-1/6+\eta} \\ &= c_2\,(\sin\pi k)^{\eta} \end{aligned}$$

the claim follows from Proposition 3.8. □

We have thus showed that $L : L^2(\mathbb{T}^1, V(k)\,dk) \longrightarrow L^2(\mathbb{T}^1, V(k)^{-1}\,dk)$ was a bounded operator. Next, we characterize the kernel of L: First, we note that given $f \in L^2(\mathbb{T}^1, V(k)\,dk)$, inequality (3.32) implies that if $L(f) = 0$ then

$$\begin{aligned} \iiiint \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ \times [\omega_3 f_3 + \omega_2 f_2 - \omega_1 f_1 - \omega f]^2\,dk\,dk_1\,dk_2\,dk_3 = 0. \end{aligned}$$

So f must satisfy

$$\omega(k)f(k) + \omega(k_1)f(k_1) = \omega(k_2)f(k_2) + \omega(k + k_1 - k_2)f(k + k_1 - k_2)$$

whenever

$$\omega(k) + \omega(k_1) = \omega(k_2) + \omega(k + k_1 - k_2).$$

We also say that $\omega(k) f(k)$ must be a collision invariant. Such invariants have been characterized in [LS08]:

Theorem 3.11 ([LS08]). *A function $\psi \in L^1(\mathbb{T})$ is a collisional invariant if and only if there exists c_1 and c_2 such that*

$$\psi(k) = c_1 + c_2 \omega(k).$$

As a consequence, we deduce:

Corollary 3.12. *The kernel of L is the two dimensional subspace of $L^2(\mathbb{T}^1, V(k)\,dk)$ spanned by the functions 1 and $\omega(k)^{-1}$ (note that both of those functions belongs to $L^2(\mathbb{T}^1, V(k)\,dk)$ thanks to (3.35))*

Finally, the compactness of K and inequality (3.32) implies

Lemma 3.13. *There exists $c_0 > 0$ such that*

$$- \int_{\mathbb{T}^1} L(f) f \, dk \geq c_0 \int V(k) |f - \Pi(f)|^2 \, dk$$

for all $f \in L^2(\mathbb{T}^1, V(k)\,dk)$, where $\Pi(f)$ denotes the orthogonal projection of f onto $\ker(L)$.

To summarize, we have thus showed:

Proposition 3.14. *The operator $L : L^2(\mathbb{T}^1, V(k)\,dk) \longrightarrow L^2(\mathbb{T}^1, V(k)^{-1}\,dk)$ is bounded and satisfies:*

(i) The kernel of L has dimension 2 and is spanned by 1 and $\frac{1}{\omega(k)}$.

(ii) For all $f \in L^2(\mathbb{T}^1, V(k)\,dk)$, we have

$$\int_{\mathbb{T}^1} L(f) \, dk = 0 \quad \textit{and} \quad \int_{\mathbb{T}^1} \frac{1}{\omega(k)} L(f) \, dk = 0. \tag{3.40}$$

(iii) There exists $c_0 > 0$ such that

$$- \int_{\mathbb{T}^1} L(f) f \, dk \geq c_0 \int V(k) |f - \Pi(f)|^2 \, dk$$

for all $f \in L^2(\mathbb{T}^1, V(k)\,dk)$, where $\Pi(f)$ denotes the orthogonal projection of f onto $\ker(L)$.

Note that the projection of f onto $\ker(L)$ can be written as

$$\Pi(f) = T + S \left[\langle V \rangle \omega(k)^{-1} - \langle V \omega^{-1} \rangle \right]$$

with

$$T = \frac{1}{\langle V\rangle}\int V(k)f(k)\,dk \text{ and } S = \frac{1}{m_0}\int\left[\langle V\rangle\frac{V(k)}{\omega(k)} - \langle V\omega^{-1}\rangle V(k)\right] f(k)\,dk$$

where $m_0 = \langle V\rangle^2\langle V\omega^{-2}\rangle - \langle V\omega^{-1}\rangle^2\langle V\rangle$ is a normalization constant. The operator Π is a continuous operator in $L^2(V(k)\,dk)$.

We finish this section commenting on the existence of solutions for the equation for the sake of completeness:

Proposition 3.15 (Cauchy Problem)**.** *There exists a unique solution in* $L^\infty((0,\infty); L^2(\mathbb{R}\times\mathbb{T}))$ *for equation* (3.24) *with initial data* $f_0 \in L^2(\mathbb{R}\times\mathbb{T})$.

Proof. A traditional method for solving the Cauchy problem for this type of equations uses an iterative scheme based on the mild formulation:

$$f(t,x,k) = f_0(x-\omega'(k)t,k) + \int_0^t Lf(x-(t-s)\omega'(k),s)ds$$

together with the estimate

$$\|L(f)\|_{L^2(\mathbb{R}\times\mathbb{T})} \le C\|f\|_{L^2(\mathbb{R}\times\mathbb{T})}.$$

This last estimate is consequence of (3.38) and the boundedness of the function V. We refer to [AG13] and [Mou13] for further details on this method. □

3.5 Proof of Theorem 3.4

3.5.1 A priori estimates

As a first step in the proof of Theorem 3.4, we establish some a priori estimates. The coercivity property of L (Lemma 3.13) gives the following proposition:

Proposition 3.16. *Assume that* $f_0 \in L^2(\mathbb{R}\times\mathbb{T})$. *Then, the function* $f^\varepsilon(t,x,k)$, *solution of* (3.24) *satisfies*

$$||f^\varepsilon(t)||_{L^2(\mathbb{R}\times\mathbb{T})} \le ||f_0||_{L^2(\mathbb{R}\times\mathbb{T})} \qquad \textit{for all } t\ge 0. \tag{3.41}$$

Furthermore, f^ε *can be expanded as follows:*

$$f^\varepsilon = \Pi(f^\varepsilon) + \varepsilon^{4/5}h^\varepsilon, \tag{3.42}$$

where

$$\|h^\varepsilon\|_{L^2_V((0,\infty)\times\mathbb{R}\times\mathbb{T})} \le C||f_0||_{L^2(\mathbb{R}\times\mathbb{T})} \tag{3.43}$$

and $\Pi(f^\varepsilon)$ *is the projection of* f^ε *onto* $\ker(L)$*, given by*

$$\Pi(f^\varepsilon)(t,x,k) = \tilde{T}^\varepsilon(t,x) + \tilde{S}^\varepsilon(t,x)\left[\langle V\rangle\omega(k)^{-1} - \langle V\omega^{-1}\rangle\right]$$

with

$$\begin{aligned} \tilde{T}^\varepsilon(t,x) &= \frac{1}{\langle V\rangle}\int V(k) f^\varepsilon(t,x,k)\,dk\,, \\ \tilde{S}^\varepsilon(t,x) &= \frac{1}{m_0}\int \left[\langle V\rangle\frac{V(k)}{\omega(k)} - \langle V\omega^{-1}\rangle V(k)\right] f^\varepsilon(t,x,k)\,dk \end{aligned} \tag{3.44}$$

where $\tilde{T}^\varepsilon, \tilde{S}^\varepsilon$ *are bounded in* $L^\infty((0,\infty);L^2(\mathbb{R}))$.

Proof. Multiplying (3.24) by f^ε and integrating with respect to x and k, we get

$$\frac{1}{2}\frac{d}{dt}\|f^\varepsilon(t)\|^2_{L^2(\mathbb{R}\times\mathbb{T}^1)} - \frac{1}{\varepsilon^\alpha}\int_{\mathbb{R}}\int_{\mathbb{T}^1} L(f^\varepsilon)f^\varepsilon\,dk\,dx = 0.$$

Integrating with respect to t and using (Lemma 3.13), we deduce

$$\frac{1}{2}\|f^\varepsilon(t)\|^2_{L^2(\mathbb{R}\times\mathbb{T}^1)} + \frac{c_0}{\varepsilon^\alpha}\int_0^t\int_{\mathbb{R}}\int_{\mathbb{T}^1} V(k)|f^\varepsilon - \Pi(f^\varepsilon)|^2\,dk\,dx\,ds \le \frac{1}{2}\|f_0^\varepsilon\|^2_{L^2(\mathbb{R}\times\mathbb{T}^1)}.$$

which implies the proposition. The fact that $\tilde{T}^\varepsilon, \tilde{S}^\varepsilon \in L^\infty((0,\infty);L^2(\mathbb{R}))$ is a direct consequence of this estimate and Cauchy-Schwartz. □

Because the singular terms in $\Pi(f^\varepsilon)$ (those involving $\omega(k)^{-1}$) play a particular role in the sequel, we will prefer to write $\Pi(f^\varepsilon)$ as follows:

$$\Pi f^\varepsilon = T^\varepsilon + \frac{\langle V\rangle}{\omega}\tilde{S}^\varepsilon(x,t)$$

with

$$T^\varepsilon(t,x) = \tilde{T}^\varepsilon(t,x) - \tilde{S}^\varepsilon(t,x)\langle V\omega^{-1}\rangle$$

Finally, we set

$$S^\varepsilon(t,x) = \varepsilon^{-3/5}\langle V\rangle\tilde{S}^\varepsilon(t,x), \tag{3.45}$$

leading to the following expansion of f^ε:

$$f^\varepsilon(t,x,k) = T^\varepsilon(t,x) + \varepsilon^{\frac{3}{5}}S^\varepsilon(t,x)\omega(k)^{-1} + \varepsilon^{\frac{4}{5}}h^\varepsilon(t,x,k). \tag{3.46}$$

Note that while T^ε and h^ε are clearly bounded (in appropriate functional spaces) in view of Proposition 3.16, the scaling of S^ε may seem arbitrary at this point. However, we will see later on that S^ε defined as in (3.45) indeed converges to a non trivial function (in some weak sense).

3.5.2 Laplace Fourier Transform

As in [MMM11], the main tool in deriving the macroscopic equation for T is the use of the Laplace-Fourier transform. More precisely, we define

$$\widehat{f^\varepsilon}(p,\xi,k) = \int_{\mathbb{R}}\int_0^\infty e^{-pt}e^{-i\xi x}f^\varepsilon(t,x,k)\,dt\,dx.$$

We also denote by $\widehat{f_0}(\xi,k)$ the Fourier transform of $f_0(x,k)$.

Remark 3.17. We recall that the Fourier transform preserves the $L^2(\mathbb{R})$ norm (Parseval's theorem). It is also easy to see that the Laplace transform of an L^1 function is in L^∞. However our functions are not L^1 with respect to t. Instead, we will make use of the simple fact that for a given function $g(t)$, its Laplace transform $\widehat{g}(p)$ satisfies

$$|\widehat{g}(p)| \le \frac{1}{p}\|g\|_{L^\infty(0,\infty)} \quad \text{and} \quad |\widehat{g}(p)| \le \frac{1}{\sqrt{2p}}\|g\|_{L^2(0,\infty)} \tag{3.47}$$

for all $p > 0$.

Taking the Laplace Fourier transform of the equation, we obtain:

$$\varepsilon^\alpha p\widehat{f^\varepsilon} - \varepsilon^\alpha\widehat{f_0} + i\varepsilon\omega'(k)\xi\widehat{f^\varepsilon} = K(\widehat{f^\varepsilon}) - V\widehat{f^\varepsilon}$$

which easily yields

$$\widehat{f^\varepsilon}(p,\xi,k) = \frac{\varepsilon^\alpha}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}\widehat{f_0} + \frac{1}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}K(\widehat{f^\varepsilon}). \tag{3.48}$$

We recall that $L(f) = K(f) - Vf$ with $K(f) = \int K(k,k')f(k')\,dk'$. The fact that $\int L(f)\,dk = 0$ and $\int \frac{1}{\omega(k)}L(f)\,dk = 0$ for all f implies

$$V(k) = \int K(k',k)dk', \qquad \frac{V(k)}{\omega(k)} = \int K(k',k)\frac{1}{\omega(k')}\,dk'$$

Multiplying (3.48) by $K(k',k)$ and integrating with respect to k and k', we get

$$\begin{aligned}
\int_{\mathbb{T}} K(\widehat{f^\varepsilon})(k')dk' &= \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\varepsilon^\alpha K(k',k)}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}\widehat{f_0}(\xi,k)\,dk\,dk' \\
&\quad + \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{K(k',k)}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}K(\widehat{f^\varepsilon})(k)\,dk\,dk' \\
&= \int_{\mathbb{T}} \frac{\varepsilon^\alpha V(k)}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}\widehat{f_0}(\xi,k)\,dk \\
&\quad + \int_{\mathbb{T}} \frac{V(k)}{\varepsilon^\alpha p + V(k) + i\varepsilon\omega'(k)\xi}K(\widehat{f^\varepsilon})(k)\,dk.
\end{aligned}$$

We deduce

$$\begin{aligned}0 = &\int_{\mathbb{T}} \frac{V(k)}{\varepsilon^{\alpha} p + V(k) + i\varepsilon\omega'(k)\xi} \widehat{f}_0(\xi,k)\,dk \\ &+ \varepsilon^{-\alpha} \int_{\mathbb{T}} \left(\frac{V(k)}{\varepsilon^{\alpha} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right) K(\widehat{f^{\varepsilon}})(k)\,dk. \end{aligned} \tag{3.49}$$

Similarly, multiplying (3.48) by $K(k',k)\frac{\varepsilon^{\frac{3}{5}}}{\omega(k')}$, and we get:

$$\begin{aligned}0 = \varepsilon^{\frac{3}{5}} &\int_{\mathbb{T}} \frac{V(k)}{\varepsilon^{\alpha} p + V(k) + i\varepsilon\omega'(k)\xi} \frac{\widehat{f}_0(\xi,k)}{\omega(k)}\,dk \\ &+ \varepsilon^{-\alpha}\varepsilon^{\frac{3}{5}} \int_{\mathbb{T}} \left(\frac{V(k)}{\varepsilon^{\alpha} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right) \frac{K(\widehat{f^{\varepsilon}})(k)}{\omega(k)}\,dk. \end{aligned} \tag{3.50}$$

Next, we write

$$K(\widehat{f^{\varepsilon}}) = K(\Pi(\widehat{f^{\varepsilon}})) + K(\widehat{f^{\varepsilon}} - \Pi(\widehat{f^{\varepsilon}})) = V\Pi(\widehat{f^{\varepsilon}}) + K(\widehat{f^{\varepsilon}} - \Pi(\widehat{f^{\varepsilon}}))$$

where we rewrite

$$\Pi(\widehat{f^{\varepsilon}}) = \widehat{T}^{\varepsilon} + \varepsilon^{3/5}\frac{1}{\omega(k)}\widehat{S}^{\varepsilon}.$$

We can thus rewrite (3.49) as follows:

$$\mathcal{F}_1^{\varepsilon}(\widehat{f}^0) + a_1^{\varepsilon}(p,\xi)\widehat{T}^{\varepsilon}(p,\xi) + a_2^{\varepsilon}(p,\xi)\widehat{S}^{\varepsilon}(p,\xi) + R_1^{\varepsilon}(p,\xi) = 0 \tag{3.51}$$

and (3.50) as follows:

$$\mathcal{F}_2^{\varepsilon}(\widehat{f}^0) + a_2^{\varepsilon}(p,\xi)\widehat{T}^{\varepsilon}(p,\xi) + a_3^{\varepsilon}(p,\xi)\widehat{S}^{\varepsilon}(p,\xi) + R_2^{\varepsilon}(p,\xi) = 0, \tag{3.52}$$

where for $\alpha = 8/5$, we have:

$$\begin{aligned}\mathcal{F}_1^{\varepsilon}(\widehat{f}^0) &= \int_{\mathbb{T}} \frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} \widehat{f}_0(\xi,k)\,dk \\ \mathcal{F}_2^{\varepsilon}(\widehat{f}^0) &= \varepsilon^{\frac{3}{5}} \int_{\mathbb{T}} \frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} \frac{\widehat{f}_0(\xi,k)}{\omega(k)}\,dk, \end{aligned}$$

$$\begin{aligned}a_1^{\varepsilon}(p,\xi) &:= \varepsilon^{-\frac{8}{5}} \int_{\mathbb{T}} \left(\frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right) V(k)\,dk \\ a_2^{\varepsilon}(p,\xi) &:= \varepsilon^{-\frac{8}{5}} \int_{\mathbb{T}} \left(\frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right) \frac{\varepsilon^{\frac{3}{5}} V(k)}{\omega(k)}\,dk \\ &= \varepsilon^{-1} \int_{\mathbb{T}} \left(\frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right) \frac{V(k)}{\omega(k)}\,dk \end{aligned}$$

$$a_3^\varepsilon(p,\xi) := \varepsilon^{-1}\int_{\mathbb{T}}\left(\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right)\frac{\varepsilon^{\frac{3}{5}}V(k)}{\omega(k)^2}\,dk$$

and

$$R_1^\varepsilon(\xi,p) := \varepsilon^{-\frac{8}{5}}\int_{\mathbb{T}}\left(\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right)K(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))(k)\,dk$$

$$R_2^\varepsilon(\xi,p) := \varepsilon^{-1}\int_{\mathbb{T}}\left(\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right)\frac{1}{\omega(k)}K(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))(k)\,dk$$

In order to prove the main theorem, we now need to pass to the limit in (3.51) and (3.52). The following three propositions give the necessary results for that.

First, we have the following limits for the terms involving the initial data:

Proposition 3.18. *The following limits hold for all $P \geq 0$:*

$$\mathcal{F}_1^\varepsilon(\widehat{f^0})(\xi,p) \longrightarrow \int_{\mathbb{T}}\widehat{f^0}(\xi,k)\,dk = \widehat{T}_0(\xi) \qquad \text{in } L^2((0,P)\times\mathbb{R})$$

$$\mathcal{F}_2^\varepsilon(\widehat{f^0})(\xi,p) \longrightarrow 0 \qquad \text{in } L^1((0,P)\times\mathbb{R})$$

when $\varepsilon \to 0$.

Next, we pass to the limit in the symbol $a_i^\varepsilon(p,\xi)$:

Proposition 3.19. *The following limits hold pointwise $(p,\xi) \in (0,\infty)\times\mathbb{R}$ and strongly in $L^p_{loc}((0,\infty)\times\mathbb{R})$ for all $p \in (1,\infty)$:*

$$a_1^\varepsilon(p,\xi) \longrightarrow -p-\kappa_1|\xi|^{\frac{8}{5}} \quad \textit{with} \quad \kappa_1 = \frac{6}{5}\left(\frac{\pi}{v_0}\right)^{3/5}\int_0^\infty \frac{z^{3/5}}{z^2+1}\,dz \tag{3.53}$$

$$a_2^\varepsilon(p,\xi) \longrightarrow -\kappa_2|\xi| \quad \textit{with } \kappa_2 = \frac{6}{5}\int_0^\infty\frac{1}{z^2+1}\,dz \tag{3.54}$$

$$a_3^\varepsilon(p,\xi) \longrightarrow -\kappa_3|\xi|^{\frac{2}{5}} \quad \textit{with } \kappa_3 = \frac{6}{5}\left(\frac{v_0}{\pi}\right)^{3/5}\int_0^\infty\frac{z^{-3/5}}{z^2+1}\,dz \tag{3.55}$$

Furthermore, $a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon \in L^\infty_{loc}((0,\infty)\times\mathbb{R})$ uniformly with respect to ε.

Finally, we need to show that the remainder terms, involving $f^\varepsilon - \Pi(f^\varepsilon)$, go to zero:

Proposition 3.20. *For all $0 < a < P$ and $K > 0$, we have*

$$R_i^\varepsilon \to 0 \quad \textit{in } L^2((a,P)\times(-K,K))$$

as $\varepsilon \to 0$ for $i = 1,2$.

The proofs of these three propositions are given in Section 3.5.3.

Proof of Theorem 3.4. We are now ready to prove Theorem 3.4. First, using Proposition 3.16, we see that up to a subsequence, $T^\varepsilon(t,x)$ converges weakly to $T(t,x)$ in $L^2((0,\tau)\times\mathbb{R})$ for all τ (the uniqueness of the limit will give the convergence of the whole sequence).

Next, for a given test function $\varphi(p,\xi)$ in $\mathcal{D}((0,\infty)\times\mathbb{R})$, we then have

$$\int_0^\infty \int_{\mathbb{R}} \widehat{T}^\varepsilon(p,\xi)\varphi(p,\xi)\,d\xi\,dp = \int_0^\infty \int_{\mathbb{R}} T^\varepsilon(t,x)\widehat{\varphi}(t,x)\,dx\,dt \tag{3.56}$$

where $\widehat{\varphi}\in L^2((0,\infty)\times\mathbb{R})$. This last fact is the classical Parseval inequality for the Fourier transform, while for the Laplace transform, it follows from Minkowski's integral inequality:

$$\begin{aligned}\left(\int_0^\infty \left(\int_0^\infty e^{-pt}\varphi(p)\,dp\right)^2 dt\right)^{1/2} &\le \int_0^\infty \left(\int_0^\infty e^{-2pt}\,dt\right)^{1/2}\varphi(p)\,dp\\ &\le \int_0^\infty \frac{1}{\sqrt{2p}}\varphi(p)\,dp<\infty.\end{aligned}$$

Thus $\widehat{T}^\varepsilon$ converges to $\widehat{T}$ in $\mathcal{D}'((0,\infty)\times\mathbb{R})$. Since $\widehat{T}^\varepsilon$ is also bounded in $L^2_{loc}((0,\infty)\times\mathbb{R})$ (using (3.47)), we deduce that (up to another subsequence) it converges weakly in $L^2_{loc}((0,\infty)\times\mathbb{R})$ to $\widehat{T}$.

In order to derive the equation satisfied by $\widehat{T}$, we need to pass to the limit in (3.51) and (3.52). However, we do not know that S^ε (defined in (3.45)) is bounded in some functional space. So we multiply equation (3.51) by a_3^ξ and (3.52) by a_2^ξ and consider their difference, in order to get rid of the terms in $\widehat{S}^\varepsilon$:

$$\begin{aligned}0 \;=\;& a_3^\xi(p,\xi)\mathcal{F}_1^\varepsilon(\widehat{f}^0)+\left(a_3^\xi(p,\xi)a_1^\xi(p,\xi)-(a_2^\xi(p,\xi))^2\right)\widehat{T}^\varepsilon(p,\xi)\\ &+a_3^\xi(p,\xi)R_1^\varepsilon(p,\xi)-a_2^\xi(p,\xi)\mathcal{F}_2^\varepsilon(\widehat{f}^0)-a_2^\xi(p,\xi)R_2^\varepsilon(p,\xi).\end{aligned}$$

Using Proposition 3.19, Proposition 3.20 and Proposition 3.18, we can now pass to the limit in this equation in $\mathcal{D}'((0,\infty)\times\mathbb{R})$ and deduce:

$$-\kappa_3|\xi|^{2/5}\widehat{T}_0+\left(-\kappa_3|\xi|^{2/5}(-p-\kappa_1|\xi|^{8/5})-\kappa_2^2|\xi|^2\right)\widehat{T}=0 \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}).$$

Furthermore, factorizing $-\kappa_3|\xi|^{2/5}$ in this last equation we get

$$-\kappa_3|\xi|^{2/5}\left(\widehat{T}_0-p\widehat{T}-(\kappa_1+\frac{\kappa_2^2}{\kappa_3})|\xi|^{8/5}\widehat{T}\right)=0 \quad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}).$$

This implies that the function

$$g(p,\xi):=\widehat{T}_0-p\widehat{T}-\left(\kappa_1-\frac{\kappa_2^2}{\kappa_3}\right)|\xi|^{8/5}\widehat{T}, \tag{3.57}$$

which belongs to $L^2_{loc}((0,\infty)\times\mathbb{R})$, satisfies

$$g(p,\xi)=0 \text{ a.e. in } (0,\infty)\times\mathbb{R}$$

which gives (3.26)-(3.27).

To complete the proof of Theorem 3.4, it remains to show that f^ε converges to $T(t,x)$ (weakly in $L^\infty((0,\infty),L^2(\mathbb{R}\times\mathbb{T}))$). Since f^ε is bounded in $L^\infty(0,\infty;L^2(\mathbb{R}\times\mathbb{T})$, and in view of the expansion (3.46), it is enough to show that $\varepsilon^{3/5}S^\varepsilon$ converges to zero in some weak sense.

This follows from Proposition 3.5, the proof of which uses equation (3.52) and some bounds from below on $a_3^\varepsilon(p,\xi)$ and will be detailed in Section 3.6. □

We end this section by proving that the diffusion coefficient κ is indeed positive:

Lemma 3.21. *The coefficients κ_1, κ_2 and κ_3 are such that*

$$\kappa_1-\frac{\kappa_2^2}{\kappa_3}>0.$$

Proof. Indeed, this is equivalent to

$$\kappa_2^2<\kappa_1\kappa_3$$

and using the explicit formula for κ_1, κ_2 and κ_3, we see that this is equivalent to

$$\left(\int_0^\infty\frac{1}{1+z^2}\,dz\right)^2<\int_0^\infty\frac{z^{3/5}}{1+z^2}\,dz\int_0^\infty\frac{z^{-3/5}}{1+z^2}\,dz$$

which is an immediate consequence of Hölder inequality. □

3.5.3 Proofs of the asymptotic results

We recall here that $\mathbb{T}$ denotes the torus $\mathbb{R}/\mathbb{Z}$ and that $\omega(k)=|\sin(\pi k)|$. Since the dispersion relation ω is degenerate at $k=0\pm n$, it will be easier in the computation below to work with k in the symmetric interval $(-\frac{1}{2},\frac{1}{2})$ (when working with the interval $(0,1)$, we have to deal with both endpoints 0 and 1). Note that the function ω is even in that interval and that

$$\omega'(k)=\operatorname{sgn}(k)\pi\cos(\pi k).$$

Finally, Proposition 3.7 implies:

Proposition 3.22. *The function $k\mapsto V(k)$ is even and non-negative on the interval $(-\frac{1}{2},\frac{1}{2})$. Furthermore the function $W(k):=V(k)|k|^{-5/3}$ for $k\in(-\frac{1}{2},\frac{1}{2})$ satisfies*

$$\lim_{k\to 0}W(k)=w_0:=v_0\pi^{5/3}$$

and

$$C_0^{-1} \le W(k) \le C_0$$

for some $C_0 > 0$.

Proof of Proposition 3.18. The first part of the proposition follows immediately from Lebesgue dominated convergence theorem, since

$$\left| \frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k) + i\varepsilon\omega'(k)\xi} \right| = \frac{V(k)}{\sqrt{\left(\varepsilon^{\frac{8}{5}}p + V(k)\right)^2 + (\varepsilon\omega'(k)\xi)^2}} \le 1$$

and

$$\frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k) + i\varepsilon\omega'(k)\xi} \longrightarrow 1 \quad \text{as } \varepsilon \to 0.$$

For the second part, we note that

$$\left| \frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k) + i\varepsilon\omega'(k)\xi} \right| \le \frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k)}$$

and so

$$\begin{aligned}
|\mathcal{F}_2^{\varepsilon}(\widehat{f}^0)|(\xi,p) &\le \varepsilon^{\frac{3}{5}} \int_{\mathbb{T}} \frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k)} \frac{\widehat{f}_0(\xi,k)}{\omega(k)}\, dk \\
&\le C\varepsilon^{\frac{3}{5}} ||\widehat{f}_0(\xi,\cdot)||_{L^\infty(\mathbb{T})} \int_0^{1/2} \frac{|k|^{2/3}}{\varepsilon^{\frac{8}{5}}p + |k|^{5/3}}\, dk \\
&\le C\varepsilon^{\frac{3}{5}} ||\widehat{f}_0(\xi,\cdot)||_{L^\infty(\mathbb{T})} (1 + |\ln(\varepsilon^{\frac{8}{5}}p)|)
\end{aligned} \tag{3.58}$$

hence the result, since this last inequality implies (integrating with respect to ξ and p)

$$||\mathcal{F}_2^{\varepsilon}(\widehat{f}^0)||_{L^1((0,P)\times\mathbb{R})} \le C\varepsilon^{\frac{3}{5}} ||f_0||_{L^\infty(\mathbb{R}\times\mathbb{T})} P(1 + |\ln(\varepsilon^{\frac{8}{5}}P)|).$$

□

Proof of Proposition 3.19. First, we write

$$\begin{aligned}
1 - \frac{V(k)}{\varepsilon^{\frac{8}{5}}p + V(k) + i\varepsilon\omega'(k)\xi} =& \frac{\varepsilon^{\frac{8}{5}}p + i\varepsilon\omega'(k)\xi}{\varepsilon^{\frac{8}{5}}p + V(k) + i\varepsilon\omega'(k)\xi} \\
=& \frac{\varepsilon^{\frac{8}{5}}p + V(k)}{(\varepsilon^{\frac{8}{5}}p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \varepsilon^{\frac{8}{5}}p \\
&+ \frac{Vi\varepsilon\omega'(k)\xi}{(\varepsilon^{\frac{8}{5}}p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2}
\end{aligned}$$

$$+\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2} \tag{3.59}$$

Using the fact that $V(k)=V(-k)$, $\omega'(-k)=-\omega'(k)$, we deduce that

$$\begin{aligned}
a_1^\varepsilon(p,\xi) := & -p\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{\varepsilon^{\frac{8}{5}}p+V(k)}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,dk \\
& -\varepsilon^{-\frac{8}{5}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,dk.
\end{aligned}$$

Dominated convergence immediately implies that the first term converges to $-p$, so we only have to consider the term

$$d^\varepsilon(p,\xi)=\varepsilon^{-\frac{8}{5}}\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,dk.$$

For some $\delta\in(0,\frac{1}{2})$, we write

$$d^\varepsilon(p,\xi)=d_1^\varepsilon(p,\xi)+d_2^\varepsilon(p,\xi)$$

where

$$\begin{aligned}
d_1^\varepsilon(p,\xi) &= \varepsilon^{-\frac{8}{5}}\int_{\substack{k\,\in\,(-\frac{1}{2},\,\frac{1}{2})\\ |k|\,\geq\,\delta}}\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,dk \\
&\leq C\varepsilon^{-\frac{8}{5}}\int_{\substack{k\,\in\,(-\frac{1}{2},\,\frac{1}{2})\\ |k|\,\geq\,\delta}}\frac{(\varepsilon\xi)^2}{V(k)}\,dk \\
&\leq C(\delta)|\xi|^2\varepsilon^{\frac{2}{5}}
\end{aligned}$$

and

$$\begin{aligned}
d_2^\varepsilon(p,\xi) &= \varepsilon^{-\frac{8}{5}}\int_{|k|\leq\delta}\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,dk \\
&= 2\varepsilon^{-\frac{8}{5}}\int_0^\delta\frac{(\varepsilon\pi\cos(\pi k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+W(k)|k|^{5/3})^2+(\varepsilon\pi\cos(\pi k)\xi)^2}W(k)|k|^{5/3}\,dk \\
&= 2\varepsilon^{-\frac{8}{5}}\int_0^\delta\frac{(\pi\cos(\pi k))^2}{(\varepsilon^{\frac{3}{5}}\frac{p}{|\xi|}+W(k)\frac{|k|^{5/3}}{\varepsilon|\xi|})^2+(\pi\cos(\pi k))^2}W(k)|k|^{5/3}\,dk.
\end{aligned}$$

We now do the change of variable

$$w=\frac{|k|^{5/3}}{\varepsilon|\xi|},\quad dk=\frac{3}{5}(\varepsilon|\xi|)^{3/5}w^{-2/5}dw, \tag{3.60}$$

which yields

$$
\begin{aligned}
d_2^\varepsilon(p,\xi) &= 2\varepsilon^{-\frac{8}{5}}\int_0^{\frac{\delta^{5/3}}{\varepsilon|\xi|}} \frac{z^\varepsilon(w)}{\left(\varepsilon^{\frac{3}{5}}\frac{p}{|\xi|}+W^\varepsilon(w)w\right)^2+z^\varepsilon(w)} W^\varepsilon(w)\varepsilon|\xi|w\frac{3}{5}(\varepsilon|\xi|)^{3/5}w^{-2/5}\,dw \\
&= |\xi|^{8/5}\frac{6}{5}\int_0^{\frac{\delta^{5/3}}{\varepsilon|\xi|}} \frac{z^\varepsilon(w)}{\left(\varepsilon^{\frac{3}{5}}\frac{p}{|\xi|}+W^\varepsilon(w)w\right)^2+z^\varepsilon(w)} W^\varepsilon(w)w^{3/5}\,dw
\end{aligned}
$$

where

$$
\begin{aligned}
z^\varepsilon(w) &= (\pi\cos(\pi(\varepsilon|\xi|w)^{3/5}))^2 \\
W^\varepsilon(w) &= W((\varepsilon|\xi|w)^{3/5}).
\end{aligned}
$$

In particular, the integrand converges pointwise (for all w and ξ), as ε goes to zero, to

$$
\frac{\pi^2}{(w_0 w)^2+\pi^2}w_0 w^{3/5}
$$

and it is bounded by

$$
\frac{\pi^2}{(C_0^{-1}w)^2+(\pi\cos(\pi\delta))^2}C_0 w^{3/5}.
$$

We deduce that

$$
|d_2^\varepsilon(p,\xi)|\leq C|\xi|^{8/5}
$$

for some constant C and that

$$
d_2^\varepsilon(p,\xi)\longrightarrow |\xi|^{8/5}\frac{6}{5}\int_0^\infty \frac{\pi^2}{(w_0 w)^2+\pi^2}w_0 w^{3/5}\,dw = \kappa_1|\xi|^{\frac{8}{5}}
$$

(recall that $w_0 = v_0\pi^{5/3}$) which concludes the proof of the first part. Note that we have also proved that

$$
|a_1^\varepsilon(p,\xi)|\leq p + C\varepsilon^{\frac{2}{5}}|\xi|^2 + C|\xi|^{8/5}.
$$

In particular, $a_1^\varepsilon(p,\xi)$ is bounded in $L^\infty_{loc}((0,\infty)\times\mathbb{R})$. Since it converges pointwise, a classical argument shows that it also converges strongly in $L^p_{loc}((0,\infty)\times\mathbb{R})$ for all $0<p<\infty$.

The convergence of a_2^ε is proved similarly: Using (3.59), we find

$$
\begin{aligned}
a_2^\varepsilon(p,\xi) := &-\varepsilon^{\frac{3}{5}}p\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\varepsilon^{\frac{8}{5}}p+V(k)}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}\frac{V(k)}{\omega(k)}\,dk \\
&-\varepsilon^{-1}\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}\frac{V(k)}{\omega(k)}\,dk.
\end{aligned}
$$

The first term is bounded by

$$
\begin{aligned}
\varepsilon^{\frac{3}{5}} p \int_{-\frac{1}{2}}^{\frac{1}{2}} & \frac{1}{\varepsilon^{\frac{8}{5}} p + V(k)} \frac{V(k)}{\omega(k)}\, dk \\
& \leq C \varepsilon^{\frac{3}{5}} p \int_0^{\frac{1}{2}} \frac{1}{\varepsilon^{\frac{8}{5}} p + C_0^{-1} k^{\frac{5}{3}}} k^{\frac{2}{3}}\, dk \\
& \leq C \varepsilon^{-1} \int_0^{\frac{1}{2}} \frac{1}{1 + \varepsilon^{-\frac{8}{5}} p^{-1} k^{\frac{5}{3}}} k^{\frac{2}{3}}\, dk \\
& \leq C \varepsilon^{-1} p \varepsilon^{\frac{8}{5}} \int_0^{C\varepsilon^{-\frac{8}{5}} p^{-1}} \frac{1}{1+w} dw \\
& \leq C p \varepsilon^{\frac{3}{5}} \ln(1 + C\varepsilon^{-\frac{8}{5}} p^{-1})
\end{aligned}
$$

and thus converges to zero as $\varepsilon \to 0$ (here we used the change of variable $w = \varepsilon^{-\frac{8}{5}} p^{-1} k^{\frac{5}{3}}$). For the second term the same decomposition of the integral in the interval $|k| \leq \delta$ and $|k| \geq \delta$. The integral in $|k| \geq \delta$ is bounded by $C(\delta)\varepsilon|\xi|^2$. For the integral in $|k| \leq \delta$, the change of variable (3.60) gives that it is bounded by $C|\xi|$ and converges to

$$
|\xi| \frac{6}{5} \int_0^{\infty} \frac{\pi^2}{(w_0 w)^2 + \pi^2} \frac{w_0}{\pi}\, dw = \kappa_2 |\xi|.
$$

Note that

$$
|a_2^{\varepsilon}(p,\xi)| \leq C p \varepsilon^{\frac{3}{5}} \ln(1 + C\varepsilon^{-\frac{8}{5}} p^{-1}) + C(\delta)\varepsilon|\xi|^2 + C|\xi| \tag{3.61}
$$

so $a_2^{\varepsilon} \in L^{\infty}_{loc}((0,\infty) \times \mathbb{R})$ implying, next to the pointwise convergence, the $L^p_{loc}((0,\infty)\times\mathbb{R})$ strong convergence for $0 < p < \infty$.

Finally, using (3.59), we find

$$
\begin{aligned}
a_3^{\varepsilon}(p,\xi) := & - \varepsilon^{\frac{6}{5}} p \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\varepsilon^{\frac{8}{5}} p + V(k)}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon \omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2}\, dk \\
& - \varepsilon^{-\frac{2}{5}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon \omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2}\, dk.
\end{aligned} \tag{3.62}
$$

The first term is bounded by

$$
\begin{aligned}
\varepsilon^{\frac{6}{5}} p \int_{-\frac{1}{2}}^{\frac{1}{2}} & \frac{1}{\varepsilon^{\frac{8}{5}} p + V(k)} \frac{V(k)}{(\omega(k))^2}\, dk \\
& \leq C \varepsilon^{\frac{6}{5}} p \int_0^{\frac{1}{2}} \frac{1}{\varepsilon^{\frac{8}{5}} p + C_0^{-1} k^{\frac{5}{3}}} k^{-\frac{1}{3}}\, dk \\
& \leq C \varepsilon^{-\frac{2}{5}} \int_0^{\frac{1}{2}} \frac{1}{1 + \varepsilon^{-\frac{8}{5}} p^{-1} k^{\frac{5}{3}}} k^{-\frac{1}{3}}\, dk
\end{aligned}
$$

$$\leq C\varepsilon^{\frac{24}{25}}p^{\frac{4}{5}}\int_0^{C\varepsilon^{-\frac{8}{5}}p^{-1}}\frac{w^{-\frac{3}{5}}}{1+w}dw$$
$$\leq C\varepsilon^{\frac{24}{25}}p^{\frac{4}{5}}\int_0^{\infty}\frac{w^{-\frac{3}{5}}}{1+w}dw$$

and thus converges to zero as $\varepsilon\to 0$. For the second term, the same decomposition of the integral in the interval $|k|\leq\delta$ and $|k|\geq\delta$. The integral in $|k|\geq\delta$ is bounded by $C(\delta)\varepsilon^{8/5}|\xi|$. The integral in $|k|\leq\delta$, the change of variable (3.60) gives that it is bounded by $C|\xi|^{2/5}$ and converges to

$$|\xi|^{\frac{2}{5}}\frac{6}{5}\int_0^{\infty}\frac{\pi^2}{(w_0w)^2+\pi^2}\frac{w_0}{\pi^2}w^{-\frac{3}{5}}\,dw=\kappa_3|\xi|^{\frac{2}{5}}.$$

Analogously as in the previous cases, we have that

$$|a_3^{\varepsilon}(p,\xi)|\leq C\varepsilon^{\frac{24}{25}}p^{\frac{4}{5}}+C(\delta)\varepsilon^{8/5}|\xi|+C|\xi|^{2/5}$$

so $a_3^{\varepsilon}\in L^{\infty}_{loc}((0,\infty)\times\mathbb{R})$ which, next to the pointwise convergence, implies the $L^{p}_{loc}((0,\infty)\times\mathbb{R})$ strong convergence $p\in(0,\infty)$.

$\square$

It only remain to prove Proposition 3.20. For that we will require the following lemma:

Lemma 3.23. *For all $\eta\in(0,\frac{1}{3}]$, we have*

$$\int_{\mathbb{T}}\left|\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right|^2V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk \leq C\left[(\varepsilon^{\frac{8}{5}}p)^{\frac{8}{5}}+(\varepsilon|\xi|)^{\frac{9}{5}-\frac{3\eta}{5}}+(\varepsilon|\xi|)^2\right] \tag{3.63}$$

and

$$\int_{\mathbb{T}}\left|\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right|^2\frac{V(k)}{\omega(k)^2}(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk \leq C\left[(\varepsilon^{\frac{8}{5}}p)^{\frac{3}{5}(1-\eta)}+(\varepsilon\xi)^{\frac{3}{5}(1-\eta)}+(\varepsilon|\xi|)^2\right] \tag{3.64}$$

We note that when $\eta=\frac{1}{3}$ (that is when we do not have the term $(\sin(\pi k))^{\frac{1}{3}-\eta}$ in the integral), then the integral behaves like $\varepsilon^{\frac{8}{5}}$. As we will see below, this would be just enough to show that the remainder term R_1^{ε} is bounded, but not to show that it converges to zero. The improvement of the norm of K given by (3.39) is thus essential here.

We first prove Proposition 3.20 (using Lemma 3.23), before giving the proof of Lemma 3.23:

Proof of Proposition 3.20. Using (3.39), we get:

$$
\begin{aligned}
|R_1^\varepsilon(p,\xi)| &= \varepsilon^{-\frac{8}{5}}\left|\int_{\mathbb{T}}\left(\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right)K(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))(k)\,dk\right| \\
&\le \varepsilon^{-\frac{8}{5}}\left(\int_{\mathbb{T}}\left|\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right|^2 V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{T}}K(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))^2(\sin(\pi k))^{-\frac{1}{3}+\eta}V^{-1}(k)\,dk\right)^{1/2} \\
&\le C\varepsilon^{-\frac{8}{5}}\left(\int_{\mathbb{T}}\left|\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right|^2 V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{T}}(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))^2(\sin(\pi k))^{\frac{1}{3}-\eta}V(k)\,dk\right)^{1/2}
\end{aligned}
$$

and using (3.63), we deduce that for $p<P$ and $|\xi|\le K$, we have

$$
|R_1^\varepsilon(p,\xi)| \le C(P,K)\varepsilon^{-\frac{8}{5}}\varepsilon^{\frac{9-3\eta}{10}}\left(\int_{\mathbb{T}}(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))^2V(k)\,dk\right)^{1/2}.
$$

For all $0<a<P$ and $K>0$, we deduce

$$
\begin{aligned}
\int_a^P\int_{-K}^K |R_1^\varepsilon(p,\xi)|^2\,d\xi\,dp &\le C(P,K)\varepsilon^{-\frac{16}{5}}\varepsilon^{\frac{9-3\eta}{5}}\|\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon})\|^2_{L^\infty((a,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))} \\
&\le C(P,K)\frac{1}{2a}\varepsilon^{\frac{-7-3\eta}{5}}\|f^\varepsilon-\Pi(f^\varepsilon)\|^2_{L^2((0,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))} \\
&\le C(a,P,K)\varepsilon^{\frac{-7-3\eta}{5}}\varepsilon^{\frac{8}{5}} \\
&\le C(a,P,K)\varepsilon^{\frac{1-3\eta}{5}}
\end{aligned}
$$

where we have used (3.47).

Clearly, this implies Proposition 3.20 for $i=1$.

Proceeding similarly, we have that

$$
\begin{aligned}
|R_2^\varepsilon(p,\xi)| \le C\varepsilon^{-1}&\left((\varepsilon^{\frac{8}{5}}p)^{\frac{3}{5}(1-\eta)}+(\varepsilon\xi)^{\frac{3}{5}(1-\eta)}+(\varepsilon|\xi|)^2\right)^{1/2} \\
&\times\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon-\Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2}
\end{aligned}
\tag{3.65}
$$

and therefore

$$
\int_a^P\int_{-K}^K |R_2^\varepsilon(p,\xi)|^2\,dp\,d\xi \le C(P,K)\varepsilon^{-2}\varepsilon^{\frac{3(1-\eta)}{5}}\|(\widehat{f^\varepsilon}-\Pi(\widehat{f^\varepsilon}))\|^2_{L^\infty((a,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))}
$$

$$
\begin{aligned}
&\leq C(a,P,K)\varepsilon^{\frac{-7-3\eta}{5}}\|f^\varepsilon-\Pi(f^\varepsilon)\|^2_{L^2((0,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))}\\
&\leq C(a,P,K)\varepsilon^{\frac{1-3\eta}{5}} \qquad (3.66)
\end{aligned}
$$

which converges to zero for any $\eta\in(0,\frac{1}{3})$. □

Proof of Lemma 3.23. We write:

$$
\begin{aligned}
&\int_{\mathbb{T}}\left|\frac{V(k)}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}-1\right|^2 V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\quad=\int_{\mathbb{T}}\left|\frac{\varepsilon^{\frac{8}{5}}p+i\varepsilon\omega'(k)\xi}{\varepsilon^{\frac{8}{5}}p+V(k)+i\varepsilon\omega'(k)\xi}\right|^2 V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\quad=\int_{\mathbb{T}}\frac{(\varepsilon^{\frac{8}{5}}p)^2+(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\quad=I_1+I_2
\end{aligned}
$$

where

$$
\begin{aligned}
I_1&:=\int_{\mathbb{T}}\frac{(\varepsilon^{\frac{8}{5}}p)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\leq 2\int_0^{1/2}\frac{(\varepsilon^{\frac{8}{5}}p)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2}V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\leq 2\int_0^{1/2}\frac{(\varepsilon^{\frac{8}{5}}p)^2}{(\varepsilon^{\frac{8}{5}}p+k^{5/3})^2}k^{5/3}\,dk
\end{aligned}
$$

(note we do not need to use the $(\sin(\pi k))^{\frac{1}{3}-\eta}$ to control this term) and

$$
\begin{aligned}
I_2&:=\int_{\mathbb{T}}\frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}}p+V(k))^2+(\varepsilon\omega'(k)\xi)^2}V(k)(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk\\
&\leq 2\int_0^{1/4}\frac{(\varepsilon\omega'(k)\xi)^2}{V(k)^2+(\varepsilon\omega'(k)\xi)^2}V(k)\,(\sin(\pi k))^{\frac{1}{3}-\eta}\,dk+2\int_{1/4}^{1/2}\frac{(\varepsilon\pi\xi)^2}{V(k)}\,dk\\
&\leq C\int_0^{1/4}\frac{(\varepsilon\xi)^2}{k^{10/3}+(\varepsilon\xi)^2}k^{2-\eta}\,dk+C\varepsilon^2|\xi|^2
\end{aligned}
$$

here the $(\sin(\pi k))^{\frac{1}{3}-\eta}$ is essential.

Using the change of variable $w=\frac{k^{5/3}}{\varepsilon^{8/5}p}$ in I_1 and $w=\frac{k^{5/3}}{\varepsilon\xi}$ in I_2, we find

$$
\begin{aligned}
I_1&\leq C(\varepsilon^{8/5}p)^{8/5}\int_0^\infty\frac{w^{3/5}}{(1+w)^2}\,dw\\
I_2&\leq C(\varepsilon\xi)^{9/5-3\eta/5}\int_0^\infty\frac{w^{4/5-3\eta/5}}{1+w^2}\,dw+C\varepsilon^2|\xi|^2
\end{aligned}
$$

where the integral in the right hand side are clearly finite (recall that $\eta \in (0, \frac{1}{3})$. Inequality (3.63) follows.

We now proceed similarly to prove (3.64): First, we write

$$\begin{aligned}
&\int_{\mathbb{T}} \left| \frac{V(k)}{\varepsilon^{\frac{8}{5}} p + V(k) + i\varepsilon\omega'(k)\xi} - 1 \right|^2 \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta}\, dk \\
&\quad = \int_{\mathbb{T}} \frac{(\varepsilon^{\frac{8}{5}} p)^2 + (\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta}\, dk \\
&\quad = \tilde{I}_1 + \tilde{I}_2
\end{aligned}$$

where

$$\begin{aligned}
\tilde{I}_1 &:= \int_{\mathbb{T}} \frac{(\varepsilon^{\frac{8}{5}} p)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta}\, dk \\
&\leq 2 \int_0^{1/2} \frac{(\varepsilon^{\frac{8}{5}} p)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2} \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta} dk \\
&\leq 2 \int_0^{1/2} \frac{(\varepsilon^{\frac{8}{5}} p)^2}{(\varepsilon^{\frac{8}{5}} p + k^{5/3})^2} k^{-\eta}\, dk
\end{aligned}$$

and

$$\begin{aligned}
\tilde{I}_2 &:= \int_{\mathbb{T}} \frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta}\, dk \\
&\leq 2 \int_0^{1/2} \frac{(\varepsilon\omega'(k)\xi)^2}{V(k)^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2} (\sin(\pi k))^{\frac{1}{3}-\eta}\, dk \\
&\leq 2 \int_0^{1/4} \frac{(\varepsilon\xi)^2}{k^{10/3} + (\varepsilon\xi)^2} k^{-\eta}\, dk + C(\varepsilon|\xi|)^2
\end{aligned}$$

Using the change of variable $w = \frac{k^{5/3}}{\varepsilon^{8/5} p}$ in $\tilde{I}_1$ and $w = \frac{k^{5/3}}{\varepsilon\xi}$ in $\tilde{I}_2$, we find

$$\begin{aligned}
\tilde{I}_1 &\leq C(\varepsilon^{\frac{8}{5}} p)^{\frac{3}{5}(1-\eta)} \int_0^\infty \frac{w^{-3/5\eta}}{(1+w)^2}\, dw \\
\tilde{I}_2 &\leq C(\varepsilon\xi)^{\frac{3}{5}(1-\eta)} \int_0^\infty \frac{w^{-2/5-3\eta/5}}{1+w^2}\, dw + C(\varepsilon|\xi|)^2
\end{aligned}$$

which yields (3.64). □

3.6 Proof of Proposition 3.5

The proof of Proposition 3.5 relies on the following crucial bound:

Lemma 3.24. *There exists a constant c such that for all K and for all ε such that $\varepsilon K \leq 1$, the following lower bound holds*

$$|a_3^\varepsilon(p,\xi)| \geq c\,\varepsilon^{\frac{6}{25}} p^{\frac{2}{5}} + c|\xi|^{\frac{2}{5}} \qquad \textit{for } 0 \leq p \leq K, \quad |\xi| \leq K. \tag{3.67}$$

Proof of Lemma 3.24. We recall that $a_3^\varepsilon(p,\xi)$ is given by (3.62). In particular, we note that for all $(p,\xi) \neq (0,0)$, we have $a_3^\varepsilon(p,\xi) < 0$. Furthermore, we can write (using the fact that all the terms in (3.62) have the same sign):

$$\begin{aligned}
-a_3^\varepsilon(p,k) \geq \varepsilon^{\frac{6}{5}} p \int_0^{\frac{1}{4}} & \frac{\varepsilon^{\frac{8}{5}} p + V(k)}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2}\, dk \\
+ \varepsilon^{-\frac{2}{5}} \int_0^{\frac{1}{4}} & \frac{(\varepsilon\omega'(k)\xi)^2}{(\varepsilon^{\frac{8}{5}} p + V(k))^2 + (\varepsilon\omega'(k)\xi)^2} \frac{V(k)}{\omega(k)^2}\, dk.
\end{aligned}$$

Using the fact that for $k \in (0, 1/4)$ we have $C_0^{-1}|k|^{5/3} \leq V(k) \leq C_0|k|^{5/3}$, $\frac{\pi}{\sqrt{2}} \leq \omega'(k) \leq \pi$ and $\frac{\pi}{2}k \leq \omega(k) \leq \pi k$, we obtain the following lower bound (for some constant $c > 0$):

$$\begin{aligned}
-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{\frac{6}{5}} p \int_0^{\frac{1}{4}} & \frac{|k|^{4/3}}{(\varepsilon^{\frac{8}{5}} p + C_0|k|^{5/3})^2 + (\varepsilon\pi\xi)^2}\, dk \\
+ c\,\varepsilon^{-\frac{2}{5}} \int_0^{\frac{1}{4}} & \frac{(\varepsilon\pi\xi)^2 k^{-1/3}}{(\varepsilon^{\frac{8}{5}} p + C_0|k|^{5/3})^2 + (\varepsilon\pi\xi)^2}\, dk.
\end{aligned} \tag{3.68}$$

From now on, we fix K and assume that $0 < p \leq K$ and that $|\xi| \leq K$. We also assume that ε is such that $\varepsilon K \leq 1$. In order to establish (3.67), we consider two cases, and in each case we use only one of the integrals in (3.68):

(i) First, assume that p and ξ are such that

$$|\xi| \leq \varepsilon^{\frac{3}{5}} p. \tag{3.69}$$

Then, using only the first integral in (3.68), we get (using (3.69)):

$$-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{\frac{6}{5}} p \int_0^{\frac{1}{4}} \frac{|k|^{4/3}}{(\varepsilon^{\frac{8}{5}} p + C_0|k|^{5/3})^2 + (\pi\varepsilon^{\frac{8}{5}} p)^2}\, dk$$

and the change of variable $w = (\varepsilon^{\frac{8}{5}} p)^{-\frac{3}{5}} k$ yields

$$-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{\frac{6}{5}} p \frac{(\varepsilon^{\frac{8}{5}} p)^{\frac{7}{5}}}{(\varepsilon^{\frac{8}{5}} p)^2} \int_0^{\frac{1}{4(\varepsilon^{\frac{8}{5}} p)^{\frac{3}{5}}}} \frac{|w|^{4/3}}{(1 + C_0|w|^{5/3})^2 + \pi^2}\, dw$$

and using the fact that $\varepsilon^{\frac{8}{5}}p \leq 1$, we deduce (for a different constant c):

$$-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{\frac{6}{25}}p^{\frac{2}{5}}.$$

Finally, using (3.69), we also get

$$-a_3^\varepsilon(p,k) \geq c|\xi|^{\frac{2}{5}}$$

and so (3.67) holds in this case.

(ii) Next, we assume that p and ξ are such that

$$\varepsilon^{\frac{3}{5}}p \leq |\xi| \tag{3.70}$$

and using only the second integral in (3.68), we get (using (3.70)):

$$-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{-\frac{2}{5}} \int_0^{\frac{1}{4}} \frac{(\varepsilon\pi\xi)^2 k^{-1/3}}{(\varepsilon|\xi| + C_0|k|^{5/3})^2 + (\varepsilon\pi\xi)^2}\,dk$$

and the change of variable $w = (\varepsilon|\xi|)^{-\frac{3}{5}}k$, yields:

$$\begin{aligned}-a_3^\varepsilon(p,k) &\geq c\,\varepsilon^{-\frac{2}{5}}\pi^2 \int_0^{\frac{1}{4(\varepsilon\xi)^{\frac{3}{5}}}} \frac{(\varepsilon\xi)^{\frac{2}{5}} w^{-1/3}}{(1 + C_0|w|^{5/3})^2 + \pi^2}\,dw \\ &\geq c|\xi|^{\frac{2}{5}}\end{aligned}$$

(using the fact that $\varepsilon|\xi| \leq 1$). Finally, using (3.70), we also get

$$-a_3^\varepsilon(p,k) \geq c\,\varepsilon^{\frac{6}{25}}p^{\frac{2}{5}}$$

and so (3.67) holds also in this case.

□

Proof of Proposition 3.5. We use equation (3.52) to determine $\widehat{S}^\varepsilon$:

$$\widehat{S}^\varepsilon = \frac{1}{a_3^\varepsilon}\left(-\mathcal{F}_2^\varepsilon(\widehat{f}_0) - R_2^\varepsilon - a_2^\varepsilon \widehat{T}^\varepsilon\right). \tag{3.71}$$

Note that we can do this since $a_3^\varepsilon(p,\xi) < 0$ as long as p and ξ are not simultaneously zero.

We now need to show that we can pass to the limit in all the terms in the right hand side. First, using Lemma 3.24 and the estimate (3.61), we deduce that for a given K and for all $\varepsilon \leq K^{-1}$, we have

$$
\begin{aligned}
\frac{a_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)} = \left|\frac{a_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)}\right| &\leq \frac{Cp\varepsilon^{3/5}\ln(1+C\varepsilon^{-8/5}p^{-1})}{c\varepsilon^{\frac{6}{25}}p^{\frac{2}{5}}} + \frac{C\varepsilon|\xi|^2}{c|\xi|^{\frac{2}{5}}} + \frac{C|\xi|}{c|\xi|^{\frac{2}{5}}} \\
&\leq Cp^{\frac{3}{5}}\varepsilon^{\frac{9}{25}}\ln(1+C\varepsilon^{-8/5}p^{-1}) + C\varepsilon|\xi|^{\frac{8}{5}} + C|\xi|^{\frac{3}{5}} \\
&\leq C(K)
\end{aligned}
$$

for all $0 \leq p \leq K$ and $|\xi| \leq K$. Furthermore, this uniform bound, together with Proposition 3.19 implies that

$$
\frac{a_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)} \longrightarrow \frac{\kappa_2}{\kappa_3}|\xi|^{3/5}
$$

pointwise and in $L^p_{loc}((0,\infty)\times\mathbb{R})$ strong.

Next, for ε sufficiently small we can use Lemma 3.24 along with the estimates on $\mathcal{F}_2^\varepsilon(\hat{f}_0)$ in (3.58) to conclude that

$$
\frac{1}{a_3^\varepsilon}\mathcal{F}_2^\varepsilon(\hat{f}_0) \to 0 \qquad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R}).
$$

Finally, we need to bound the quantity

$$
\left|\frac{R_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)}\right|.
$$

For that, we fix $0 < a < P$ and for $p \in (a,P)$ and $|\xi| \leq K$, estimate (3.65) then implies

$$
\begin{aligned}
|R_2^\varepsilon(p,\xi)| &\leq C\varepsilon^{-1}\left(\varepsilon^{\frac{24}{25}(1-\eta)} + (\varepsilon\xi)^{\frac{3}{5}(1-\eta)}\right)^{1/2}\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon - \Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2} \\
&\leq \begin{cases} C\varepsilon^{-1}\varepsilon^{\frac{12}{25}(1-\eta)}\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon - \Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2} & \text{if } |\xi| \leq \varepsilon^{3/5} \\ C\varepsilon^{-1}(\varepsilon\xi)^{\frac{3}{10}(1-\eta)}\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon - \Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2} & \text{if } |\xi| \geq \varepsilon^{3/5} \end{cases}
\end{aligned}
$$

and we are going to use the following consequence of Lemma 3.24:

$$
-a_3^\varepsilon(p,\xi) \geq \begin{cases} c(a)\varepsilon^{6/25} & \text{if } |\xi| \leq \varepsilon^{3/5} \\ c|\xi|^{2/5} & \text{if } |\xi| \geq \varepsilon^{3/5} \end{cases}
$$

We deduce

$$
\left|\frac{R_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)}\right| \leq \begin{cases} C\varepsilon^{\frac{-19-12\eta}{25}}\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon - \Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2} & \text{if } |\xi| \leq \varepsilon^{3/5} \\ C\varepsilon^{-1}\varepsilon^{\frac{3}{10}(1-\eta)}|\xi|^{\frac{-1-3\eta}{10}}\left(\int_{\mathbb{T}}(\widehat{f}^\varepsilon - \Pi(\widehat{f}^\varepsilon))^2V(k)dk\right)^{1/2} & \text{if } |\xi| \geq \varepsilon^{3/5} \end{cases}.
$$

Finally, using the condition $|\xi| \geq \varepsilon^{3/5}$ in the second case, we deduce that

$$\left|\frac{R_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)}\right| \leq C(a,p,K)\varepsilon^{\frac{-19-12\eta}{25}} \left(\int_{\mathbb{T}} (\widehat{f^\varepsilon} - \Pi(\widehat{f^\varepsilon}))^2 V(k)dk\right)^{1/2}$$

for all $p \in (a,P)$ and $|\xi| \leq K$.

We deduce

$$\begin{aligned}\left(\int_a^P \int_{-K}^K \left|\frac{R_2^\varepsilon(p,\xi)}{a_3^\varepsilon(p,\xi)}\right|^2 d\xi\, dp\right)^{1/2} &\leq C(a,P,K)\varepsilon^{\frac{-19-12\eta}{25}} \|(\widehat{f^\varepsilon} - \Pi(\widehat{f^\varepsilon}))\|_{L^\infty((a,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))} \\ &\leq C(a,P,K)\varepsilon^{\frac{-19-12\eta}{25}} \|f^\varepsilon - \Pi(f^\varepsilon)\|_{L^2((0,\infty);L^2_V(\mathbb{T}\times\mathbb{R}))} \\ &\leq C(a,P,K)\varepsilon^{\frac{1-12\eta}{25}}\end{aligned}$$

which goes to zero as $\varepsilon \to 0$.

We can now pass to the limit in (3.71) to conclude that

$$\widehat{S}^\varepsilon \longrightarrow \widehat{S} = \frac{\kappa_2}{\kappa_3}|\xi|^{3/5}\widehat{T} \qquad \text{in } \mathcal{D}'((0,\infty)\times\mathbb{R})$$

which completes the proof of Proposition 3.5. □

3.7 Appendix: Origin of the collision frequency

Microscopic dynamics. Consider the microscopic level with the Hamiltonian

$$H(q,p) = \frac{1}{2}\sum_{i\in\mathbb{Z}} p_i^2 + \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2 + \frac{1}{4}\beta\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^4$$

which gives the following dynamics

$$\begin{aligned}\frac{d}{dt}q_i(t) &= p_i(t) \\ \frac{d}{dt}p_i(t) &= \frac{1}{4}q_{i+1} - \frac{1}{2}q_i + \frac{1}{4}q_{i-1} \\ &\quad + \beta(q_{i+1}-q_i)^3 - \beta(q_i - q_{i-1})^3.\end{aligned} \tag{3.72}$$

Dispersion relation. The dispersion relation is defined in [Spo06b] expression (2.14). The dispersion relation comes only from the harmonic part of the potential. In our case, from

$$V_{harm} = \frac{1}{8}\sum_{i\in\mathbb{Z}}(q_{i+1}-q_i)^2.$$

Consider the discrete Fourier Transform defined as

$$\widehat{f}(k) = \sum_{x \in \mathbb{Z}} e^{-i2\pi kx} f_x.$$

The wave vector k lies in the torus $\mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$. Alternatively, we can take $k \in I = [0,1]$ and assume that all functions are 1-periodic with respect to k.

We compute the discrete Fourier Transform of

$$\frac{\partial}{\partial q_i} V_{harm} = -\frac{1}{4} q_{i+1} + \frac{1}{2} q_i - \frac{1}{4} q_{i-1}$$

to obtain

$$\widehat{\frac{\partial}{\partial q_i} V_{harm}}(k) = \frac{1}{2} - \frac{1}{4}\left(e^{i2\pi k} + e^{-i2\pi k}\right) \widehat{q}(k) := \sqrt{\omega(k)}\widehat{q}(k).$$

The dispersion relation is given by

$$\omega(k) = \sqrt{\frac{1}{2}(1 - \cos(2\pi k))} = \sin(\pi k).$$

Note that $\omega(k) \geq 0$ for $k \in I$, and one can indeed look at ω as a 1-periodic function defined on $\mathbb{R}$ by $\omega(k) = |\sin(\pi k)|$ rather than a function defined on $\mathbb{T}^1$. However, ω is not differentiable at $k = 0$.

3.7.1 Four phonons collision operator.

The kinetic limit of the previous hamiltonian will give the 4-phonon Boltzmann equation

$$\partial_t W + \omega'(k)\partial_x W = C(W) \tag{3.73}$$

as we already saw in Section 3.1, with

$$\begin{aligned} C(W) \quad = \quad & 36\pi \int\int\int F(k,k_1,k_2,k_3)^2 \delta(k + k_1 - k_2 - k_3)\delta(\omega + \omega_1 - \omega_2 - \omega_3) \\ & [W_1W_2W_3 + WW_2W_3 - WW_1W_3 - WW_1W_2]\, dk_1\, dk_2\, dk_3. \end{aligned} \tag{3.74}$$

The dispersion relation ω and the collision frequency F depend on the shape of the original Hamiltonian. In most frameworks an on-site potential for the Hamiltonian is considered. This gives rise to a collision kernel F of the form

$$F(k,k_1,k_2,k_3)^2 = (\omega(k)\omega(k_1)\omega(k_2)\omega(k_3))^{-1}.$$

However, for FPU-β case the potential depends on the nearest neighbours (3.72)), giving

$$F(k,k_1,k_2,k_3)^2 = \prod_{i=0}^{3} \frac{2\sin^2(\pi k_i)}{\omega(k_i)}.$$

We explain here how this expression is obtained.

Origin of the collision frequency. From the Hamiltonian dynamics (3.72), we define as in [Spo06b]

$$a(k) = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\hat{q}(k) + i\frac{1}{\sqrt{\omega}}\hat{p}(k)\right).$$

The collision frequency appears in the dynamics of a:

$$\begin{aligned}\frac{d}{dt}a(k,\sigma) &= -i\sigma\omega(k)a(k,\sigma)\\ &\quad - i\sigma\beta\sum_{\sigma'\in\{\pm1\}^3}\int_{[0,1]^3} d^3k'\,\delta\left(k-\sum_{j=1}^{3}k'_j\right)F(k')\prod_{j=1}^{3}a(k'_j,\sigma'_j)\end{aligned}$$

Therefore we compute

$$\begin{aligned}\frac{d}{dt}a(k) &= \frac{1}{\sqrt{2}}\sqrt{\omega}\frac{d}{dt}\hat{q}(k) + i\frac{1}{\sqrt{2}\sqrt{\omega}}\hat{p}(k)\\ &= \frac{1}{\sqrt{2}}\sqrt{\omega}\hat{p}(k) + i\frac{1}{\sqrt{2}\sqrt{\omega}}\frac{1}{4}\left(e^{2\pi ik}\hat{q}(k) - 2\hat{q}(k) + e^{-2\pi ik}\hat{q}(k)\right)\\ &\quad + i\beta\underbrace{\frac{1}{\sqrt{2}\sqrt{\omega}}\left(\widehat{(q_{i+1}-q_i)^3} - \widehat{(q_i-q_{i-1})^3}\right)}_{=:\hat{Q}}\\ &= \frac{1}{\sqrt{2}}\sqrt{\omega}\hat{p}(k) + i\frac{1}{\sqrt{2}\sqrt{\omega}}\omega^2\hat{q(k)} + i\beta\hat{Q}\\ &= -i\omega a(k,t) + i\beta\hat{Q}.\end{aligned}$$

The first term will give the transport part in equation (3.73) and the β part gives the collision term. Now we study $\hat{Q}$. We have that

$$\begin{aligned}\widehat{(q_{i+1}-q_i)^3} &= \left((e^{2\pi ik}-1)\hat{q}(k)\right)^{*3}\\ &= \int\int\int(e_1-1)(e_2-1)(e_3-1)\hat{q}_1\hat{q}_2\hat{q}_3\delta(k-k_1-k_2-k_3)dk_1dk_2dk_3\end{aligned}$$

denoting

$$e_i = e^{2\pi ik_i} \quad \text{and} \quad \hat{q}_i = \hat{q}(k_i).$$

Therefore

$$\widehat{(q_{i+1}-q_i)^3} - \widehat{(q_i-q_{i-1})^3} = \int\left((e_1-1)(e_2-1)(e_3-1) - (\overline{1-e_1})(\overline{1-e_2})(\overline{1-e_3})\right)$$

$$\times\, \hat{q}_1\hat{q}_2\hat{q}_3\delta(k-k_1-k_2-k_3)dk_1dk_2dk_3$$

Now we use that

$$\hat{q}_i = \frac{1}{\sqrt{2}\sqrt{\omega(k_i)}}\underbrace{\left(a(k_i)+\overline{a(-k_i)}\right)}_{=:A_i}$$

so that

$$\hat{Q} = \int \underbrace{\frac{((e_1-1)(e_2-1)(e_3-1)-(\overline{1-e_1})(\overline{1-e_2})(\overline{1-e_3}))}{(\sqrt{2})^4\sqrt{\omega(k)\omega(k_1)\omega(k_2)\omega(k_3)}}}_{=:\Phi} A_1A_2A_3\delta(k-k_1-k_2-k_3)dk_1dk_2dk_3 \tag{3.75}$$

Φ is what will give the collision frequency which is the square of Φ.

Now by observing that

$$e^{2\pi i k_i}-1 = e^{2\pi i k_i}\left(1-e^{-2\pi i k_i}\right) \quad \text{i.e.} \quad (e_i-1) = e_i\overline{(1-e_i)}$$

we can rewrite the numerator of Φ as

$$N(k,k_1,k_2,k_3) = e_1e_2e_3\overline{(1-e_1)(1-e_2)(1-e_3)} - \overline{(1-e_1)(1-e_2)(1-e_3)}$$

since $k-k_1-k_2-k_3=0$ in (3.75) we have that

$$e_1e_2e_3 = e^{2\pi i(k_1+k_2+k_3)} = e^{2\pi i k}$$

so

$$N(k,k_1,k_2,k_3) = (e_k-1)\overline{(1-e_1)(1-e_2)(1-e_3)}.$$

Using analogously that $\overline{1-e_i} = \overline{e_i}(e_i-1)$ we have that also

$$N(k,k_1,k_2,k_3) = \overline{(1-e_k)}(e_1-1)(e_2-1)(e_3-1).$$

So summing the two previous expressions we check that N is a real number since one is the conjugate of the other, i.e.,

$$\begin{aligned} N(k,k_1,k_2,k_3) &= \frac{1}{2}\left((e_k-1)\overline{(1-e_1)(1-e_2)(1-e_3)} + \overline{(1-e_k)}(e_1-1)(e_2-1)(e_3-1)\right) \\ &= Re\left((e_k-1)\overline{(1-e_1)(1-e_2)(1-e_3)}\right). \end{aligned}$$

To compute the collision frequency, we square expression Φ. However since expression N is a real number, squaring it corresponds to taking the modulus square that we will denote by $|\cdot|$. We obtain

$$
\begin{aligned}
F(k, k_1, k_2, k_3) &= |\Phi|^2 = \frac{|(e_k - 1)\overline{(1 - e_1)(1 - e_2)(1 - e_3)}|^2}{\left((\sqrt{2})^4 \sqrt{\omega(k)\omega(k_1)\omega(k_2)\omega(k_3)}\right)^2} \\
&= \prod_{j=0}^{3} \frac{|e_i - 1|^2}{2\omega(k_i)} \\
&= \prod_{j=0}^{3} \frac{4\sin^2(\pi k_i)}{2\omega(k_i)}
\end{aligned}
$$

where we have used that

$$
\begin{aligned}
|e_i - 1|^2 &= (\cos(2\pi k_i) - 1)^2 + \sin^2(2\pi k_i) \\
&= \cos^2(2\pi k_i) + 1 - 2\cos(2\pi k_i) + \sin^2(2\pi k_i) \\
&= 2 - 2\cos(2\pi k_i) \\
&= 4\sin^2(\pi k_i).
\end{aligned}
$$

Part II

Wave turbulence theory and mean-field limits for stochastic particle systems

Chapter 4

Isotropic Wave Turbulence with simplified kernels: existence, uniqueness and mean-field limit for a class of instantaneous coagulation-fragmentation processes

This work has been done under the supervision of Professor James Norris.

The isotropic 4-wave kinetic equation is considered in its weak formulation using model homogeneous kernels. Existence and uniqueness of solutions is proven in a particular setting. We also consider finite stochastic particle systems undergoing instantaneous coagulation-fragmentation phenomena and give conditions in which this system approximates the solution of the equation (mean-field limit).

Contents

Notation

$$
\begin{aligned}
\mathbb{R}_+ &= [0, \infty); \\
\mathcal{B} &= \text{space of bounded measurable functions with bounded support;} \\
D &= \{(\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3_+ \,|\, \omega_1 + \omega_2 \geq \omega_3\}; \\
\mathbf{k} & \quad \text{wavevector, it belongs to } \mathbb{R}^N; \\
\omega(\mathbf{k}) & \quad \text{dispersion relation;} \\
\overline{T} &= \overline{T}(\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3, \mathbf{k}) \text{ interaction coefficient;} \\
\mathcal{P}(\mathbb{R}_+) & \quad \text{space of probability measures in } \mathbb{R}_+ \\
\mathcal{M}(\mathbb{R}_+) & \quad \text{set of finite measures on } \mathbb{R}_+.
\end{aligned}
$$

4.1 Introduction

Wave turbulence ([ZDP04, ZLF92, Naz11], [S$^+$06, Entry turbulence]) describes weakly non-linear systems of dispersive waves. The present work focuses in the case of 4 interacting waves.

We start with a brief presentation of the general 4-wave kinetic equation and move quickly to consider the isotropic case with simplified kernels, which is the object of study of the present work, and present the main results.

We give a brief account on the theory of wave turbulence in Section 4.1.3. The rest of the text consists on the proofs of the main theorems.

4.1.1 The 4-wave kinetic equation.

Using in shorthand $n_i = n(\mathbf{k}_i, t)$, $n_k = n(\mathbf{k}, t)$, $\omega_i = \omega(\mathbf{k}_i)$ and $\omega = \omega(\mathbf{k})$, the **4-wave kinetic equation** is given by

$$\begin{aligned} \frac{d}{dt}n(\mathbf{k},t) &= 4\pi \int_{\mathbb{R}^{3N}} \overline{T}^2(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})(n_1n_2n_3 + n_1n_2n_k - n_1n_3n_k - n_2n_3n_k) \\ &\quad \times \delta(\omega_1+\omega_2-\omega_3-\omega)\delta(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k})d\mathbf{k}_1 d\mathbf{k}_2 d\mathbf{k}_3. \end{aligned} \tag{4.1}$$

where $\mathbf{k} \in \mathbb{R}^N$ is called **wavevector**; the function $n = n(\mathbf{k}, t)$ can be interpreted as the spectral density (in $\mathbf{k}$-space) of a wave field and it is called **energy spectrum**; $\omega(\mathbf{k})$ is the **dispersion relation**; and

$$\overline{T}_{123k} := \overline{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})$$

is the **interaction coefficient**.

$$E = \int_{\mathbb{R}^N} \omega(\mathbf{k})\, n(\mathbf{k})d\mathbf{k}, \quad W = \int_{\mathbb{R}^N} n(\mathbf{k})d\mathbf{k}$$

correspond to the total energy and the waveaction (total number of waves), respectively. These two quantities are conserved formally.

Properties of the dispersion relation and the interaction coefficient. $\omega(\mathbf{k})$ and T_{123k} are homogeneous, i.e., for some $\alpha > 0$ and $\beta \in \mathbb{R}$

$$\omega(\xi\mathbf{k}) = \xi^\alpha \omega(\mathbf{k}), \qquad \overline{T}(\xi\mathbf{k}_1,\xi\mathbf{k}_2,\xi\mathbf{k}_3,\xi\mathbf{k}) = \xi^\beta \overline{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}) \qquad \xi > 0.$$

Moreover the interaction coefficient possesses the following symmetries

$$\overline{T}_{123k} = \overline{T}_{213k} = \overline{T}_{12k3} = \overline{T}_{3k12}.$$

Example: shallow water. In the case of shallow water we deal with weakly-nonlinear waves on the surface of an ideal fluid in an infinite basin of constant depth h small. In this case ([Zak99]) we have that $\alpha = 1$, $\beta = 2$, dimension is 2 and

$$T(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}) = -\frac{1}{16\pi^2 h}\frac{1}{(k_1k_2k_3k)^{1/2}}\left[(\mathbf{k}_1\cdot\mathbf{k}_2)(\mathbf{k}_3\cdot\mathbf{k}) + (\mathbf{k}_1\cdot\mathbf{k}_3)(\mathbf{k}_2\cdot\mathbf{k}) + (\mathbf{k}_1\cdot\mathbf{k})(\mathbf{k}_2\cdot\mathbf{k}_3)\right]. \tag{4.2}$$

In general T will be given by very complex expressions, see for example [ZLF92].

Resonant conditions and the δ distributions. The delta distributions appearing in equation (4.1) correspond to the so-called resonant conditions:

$$\mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k}$$

$$\omega(\mathbf{k}_1) + \omega(\mathbf{k}_2) \quad = \quad \omega(\mathbf{k}_3) + \omega(\mathbf{k}).$$

This imposes the conservation of energy and momentum in the wave interactions.

4.1.2 The *simplified* weak isotropic 4-wave kinetic equation.

We focus our study on the weak formulation of the isotropic 4-wave kinetic equation defined against functions in $\mathcal{B}(\mathbb{R}^N)$; the set of bounded measurable functions with bounded support in $\mathbb{R}^N$.

More specifically, we assume that $n(\mathbf{k}) = n(k)$ is a radial function (isotropic). Then, using the relation $\omega(\mathbf{k}) = k^\alpha$, we study the evolution of the **angle-averaged frequency spectrum** $\mu = \mu(d\omega)$ which corresponds to

$$\mu(d\omega) := \frac{|S^{N-1}|}{\alpha} \omega^{\frac{N-\alpha}{\alpha}} n(\omega^{1/\alpha}) d\omega,$$

where S^{N-1} is the N dimensional sphere. The total number of waves (waveaction) and the total energy are now expressed respectively as

$$W \quad = \quad \int_0^\infty \mu(d\omega) \tag{4.3}$$

$$E \quad = \quad \int_0^\infty \omega\mu(d\omega). \tag{4.4}$$

The weak form of the isotropic equation is given formally by

$$\mu_t = \mu_0 + \int_0^t Q(\mu_s, \mu_s, \mu_s)\, ds \tag{4.5}$$

where Q is defined against functions $f \in \mathcal{B}(\mathbb{R}_+)$ as

$$\begin{aligned}\langle f, Q(\mu,\mu,\mu)\rangle \quad &= \quad \frac{1}{2}\int_D \mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3) K(\omega_1,\omega_2,\omega_3) \\ &\qquad \times[f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_2) - f(\omega_1)]\end{aligned}$$

where $D := \{\mathbb{R}^3_+ \cap (\omega_1 + \omega_2 \geq \omega_3)\}$. See appendix 4.6 for the formal derivation of this equation.

Formally $K = K(\omega_1, \omega_2, \omega_3)$ is written as

$$\begin{aligned}K(\omega_1,\omega_2,\omega_3) \quad &= \quad \frac{8\pi}{\alpha|S^{N-1}|^4}(\omega_1+\omega_2-\omega_3)^{\frac{N-\alpha}{\alpha}} \\ &\int_{(S^{N-1})^4} d\mathbf{s}_1 d\mathbf{s}_2 d\mathbf{s}_3 d\mathbf{s}\, \overline{T}^2(\omega_1^{1/\alpha}\mathbf{s}_1, \omega_2^{1/\alpha}\mathbf{s}_2, \omega_3^{1/\alpha}\mathbf{s}_3, (\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}) \\ &\qquad \times\delta(\omega_1^{1/\alpha}\mathbf{s}_1 + \omega_2^{1/\alpha}\mathbf{s}_2 - \omega_3^{1/\alpha}\mathbf{s}_3 - (\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}).\end{aligned} \tag{4.6}$$

Notice that formally K is homogeneous of degree

$$\lambda := \frac{2\beta - \alpha}{\alpha}. \tag{4.7}$$

Our starting point is equation (4.5) considering *simplified kernels* K. In this work we do not study the relation between the interaction coefficient $\overline{T}$ and K. Specifically, we will consider the following type of kernels:

Definition 4.1. We say that K is a **model kernel** if

- $K : \mathbb{R}_+^3 \to \mathbb{R}_+$;
- K is continuous in $\mathbb{R}_+^3 = [0,\infty)^3$;
- K is homogeneous of degree λ;
- $K(\omega_1, \omega_2, \omega_3) = K(\omega_2, \omega_1, \omega_3)$ for all $(\omega_1, \omega_2, \omega_3) \in \mathbb{R}_+^3$.

Some examples of model kernels are:

$$\begin{aligned} K(\omega_1,\omega_2,\omega_3) &= \frac{1}{2}\left(\omega_1^p\omega_2^q\omega_3^r + \omega_1^q\omega_2^p\omega_3^r\right) \quad \text{with } p+q+r=\lambda, \\ K(\omega_1,\omega_2,\omega_3) &= (\omega_1\omega_2\omega_3)^{\lambda/3}, \\ K(\omega_1,\omega_2,\omega_3) &= \frac{1}{3}(\omega_1^\lambda + \omega_2^\lambda + \omega_3^\lambda). \end{aligned} \tag{4.8}$$

The main question we want to address is:

> FOR WHICH TYPES OF KERNELS K THERE IS EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR EQUATION (4.5) AND, MOREOVER, CAN THIS SOLUTION(S) BE TAKEN AS THE MEAN-FIELD LIMIT OF A SPECIFIC STOCHASTIC PARTICLE SYSTEM?

The present work gives a positive answer for a particular class of kernels as explained in the next section, but first, for the motivation of the problem, we need to answer the two following questions:

a) Why is it relevant to study the weak isotropic 4-wave kinetic equation with simplified kernels?

The present work is inspired on the article [Con09] from the physics literature on wave turbulence. In [Con09] the author works with the 3-wave kinetic equation and considers its isotropic version also assuming simplified kernels. The idea is that the 3-wave kinetic equation can be interpreted as a process where particles coagulate and fragment. This interpretation allows to use numerical methods coming from the theory of coagulation-fragmentation processes, which can be applied to this type of simplified kernels.

As in [Con09], ignoring the specific shape of the interaction coefficient $\overline{T}$ is not uncommon in the wave turbulence literature; in general the shape of $\overline{T}$ is too complex, too messy to extract information. Moreover, the most important feature in wave turbulence, the steady states called KZ-spectrum, depend only on the parameters α, β and N. That is why in the physics literature $\overline{T}$ plays a secondary role, sometimes no role at all.

It is believed that only the asymptotic scaling properties of the kernel will affect the asymptotic behaviour of the solution. This is similar to what happens in the case of the Smoluchowski's coagulation equation, where homogeneous kernels give rise to self-similar solutions (scaling solutions) in some cases. The hypothesis that solutions become self-similar in the long run under the presence of an homogeneous kernel is called **dynamical scaling hypothesis**, see [MC11] for more on this. In the case of wave turbulence we expect this self-similar solutions to correspond to the steady states given by the KZ-spectrum.

Proving the dynamical scaling hypothesis for the simplified isotropic 4-wave kinetic equation under the assumptions of Theorem 4.6 (existence of solutions) will imply proving the validity of the KZ-spectrum for this simplified kernels (if there is correspondence between the two). This would provide a great indication of the mathematical validity of the theory of wave turbulence.

b) Why consider the isotropic case? There are examples in the physics literature where the phenomena are considered to behave isotropically (like in Langmuir waves for isotropic plasmas and shallow water with flat bottom).

The main reason though to consider the isotropic case is that it makes easier to get a mean-field limit from discrete stochastic particle systems. Suppose that we want to find a discrete particle system that approximates the dynamics of (4.1). For given waves with wavenumbers $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$, we want to see if they interact. On one hand, due to the resonance conditions $\mathbf{k}$ defined as

$$\mathbf{k} = \mathbf{k}_1 + \mathbf{k}_2 - \mathbf{k}_3$$

is uniquely determined. On the other hand, on top we must add the constraint

$$\omega = \omega_1 + \omega_2 - \omega_3$$

and this in general will not be satisfied. Therefore, if we consider systems with a finite number of particles, in general, interactions will not occur and the dynamics will be constant.

We go around this problem by considering the isotropic case. By assuming that $n =$

$n(k)$ is a rotationally invariant function, we add the degree of freedom that we need.

4.1.2.1 Summary of results

Next we summarise the main results in the present work. These results are the analogous ones presented in the papers [Nor99, Nor00] for the Smoluchowski equation (coagulation model).

Remark 4.2 (Strategy). We will adapt the proofs by Norris in [Nor99] and [Nor00] for coagulation phenomena. In the proof by Norris in [Nor99] sublinear functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ are used, i.e.,

$$\begin{aligned} \varphi(\lambda x) &\leq \lambda\varphi(x), \quad \lambda \geq 1 \\ \varphi(x+y) &\leq \varphi(x)+\varphi(y). \end{aligned}$$

These functions are the key to get bounds because of the following property: let $(\mu_t^n)_{t\geq 0}$ be a stochastic coagulation process with n particles, if initially

$$\langle \varphi, \mu_0^n \rangle \leq \Lambda$$

for some $\Lambda < \infty$, for all $n \in \mathbb{N}$, then

$$\langle \varphi, \mu_t^n \rangle \leq \Lambda \text{ for all } n, t.$$

Actually, what we obtain is that

$$\langle \varphi, \mu_t^n \rangle \leq \langle \varphi, \mu_0^n \rangle$$

thanks to the sublinearity of φ; say that two particles of masses $x, y \in \mathbb{R}_+$ coagulate creating a particle of mass $x+y$, then

$$\varphi(x+y) \leq \varphi(x)+\varphi(y) \tag{4.9}$$

by sublinearity.

In general, this idea to get bounds cannot be applied to the type of stochastic particle processes that we are going to consider because they also include fragmentation phenomena; we will have that in an interaction two particles of masses $\omega_1, \omega_2 \in \mathbb{R}_+$ disappear and two particles of masses $\omega_1+\omega_2-\omega_3, \omega_3 \in \mathbb{R}_+$ are created.

To get bounds on this stochastic process using the method above we need an expression analogous to (4.9), i.e.,

$$\varphi(\omega_1+\omega_2-\omega_3)+\varphi(\omega_3) \leq \varphi(\omega_1)+\varphi(\omega_2).$$

Therefore we can use Norris method with the appropriate adaptations for the particular case where $\varphi(\omega) = \omega + c$ for a constant c, which we will take to be one.

Notice that this works as a consequence of the conservation of the energy (given by the ω's, see (4.4)) and the conservation of the total number of particles at each interaction.

Definition 4.3. Consider $\varphi(\omega) = \omega + 1$. We say that a kernel K is **sub-multiplicative** if

$$K(\omega_1, \omega_2, \omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3). \tag{4.10}$$

A. Existence and uniqueness of solutions.

Definition 4.4 (Solution and types of solutions)**.** We will say that $(\mu_t)_{t<T}$ is a *local solution* if it satisfies (4.5) for all bounded measurable functions f of bounded support and such that $\langle \omega, \mu_t \rangle \leq \langle \omega, \mu_0 \rangle$ for all $t < T$. If $T = +\infty$ then we have a *solution*. If moreover,

$$\int_0^\infty \omega \mu_t(d\omega)$$

is finite and constant, then we say that $(\mu_t)_{t<T}$ is *conservative*.

We call any local solution $(\mu_t)_{t<T}$ such that

$$\int_0^t \langle \varphi^2, \mu_s \rangle \, ds < \infty \quad \text{for all } t < T$$

a *strong solution*.

Remark 4.5. Observe that we consider the possibility of having not conservative solutions, implying loss of mass. This will correspond to gelation in coagulation and the concept of finite capacity cascades in Wave Turbulence (see [Con09]).

Theorem 4.6 (Existence and uniqueness of solutions)**.** *Consider equation* (4.5) *and a given* μ_0 *measure in* $\mathbb{R}_+$*. Define* $\varphi(\omega) = \omega + 1$ *and assume that* K *is submultiplicative model kernel. Assume further that* $\langle \varphi, \mu_0 \rangle < \infty$ *(i.e., initially the total number of waves* (4.3) *and the total energy* (4.4) *are finite). Then, if* $(\mu_t)_{t<T}$ *and* $(\nu_t)_{t<T}$ *are local solutions, starting from* μ_0*, and if* $(\nu_t)_{t<T}$ *is strong, then* $\mu_t = \nu_t$ *for all* $t < T$*. Moreover, any strong solution is conservative.*

Also, if $\langle \varphi^2, \mu_0 \rangle < \infty$*, then there exists a unique maximal strong solution* $(\mu_t)_{t<\zeta(\mu_0)}$ *with* $\zeta(\mu_0) = \langle \varphi^2, \mu_0 \rangle^{-1} \langle \varphi, \mu_0 \rangle^{-1}$.

The proof of this theorem will be an adaptation of [Nor99, Theorem 2.1].

B. Mean-field limit (coagulation-fragmentation phenomena). We will consider a system of stochastic particles undergoing coagulation-fragmentation phenomena. The basic idea is that three particles $(\omega_1, \omega_2, \omega_3)$ with $\omega_1 + \omega_2 \geq \omega_3$ will interact at a given rate

$K(\omega_1, \omega_2, \omega_3)$. In the interaction, first ω_1 and ω_2 coagulate to form $\omega_1+\omega_2$ and then, under the presence of ω_3 the coagulant splits into two other components which are $\omega_1+\omega_2-\omega_3$ and a new ω_3 (fragmentation). So interactions are

$$[\omega_1, \omega_2, \omega_3] \mapsto [\omega_1+\omega_2-\omega_3, \omega_3, \omega_3].$$

Note that we assume that K is symmetric in the first two variables because in the interactions the role of ω_1 and ω_2 is symmetric.

We will define and build for each $n \geq 1$, $(X_t^n)_{t\geq 0}$ a instantaneous coagulation-fragmentation stochastic particle system of n particles (Section 4.3.1) following the previous ideas.

We will approximate the solutions to the isotropic 4-wave kinetic equation using this coagulation-fragmentation phenomena. We present here two mean-field limits each of them requiring a different set of assumptions:

Theorem 4.7 (First mean-field limit)**.** *Assume that for $\tilde{\varphi}(\omega) = \omega^{1-\gamma}$, $\gamma \in (0,1)$ it holds that K is a model kernel with*

$$K(\omega_1, \omega_2, \omega_3) \leq \tilde{\varphi}(\omega_1)\tilde{\varphi}(\omega_2)\tilde{\varphi}(\omega_3).$$

Assume also that $\langle \omega, X_0^n \rangle$ is bounded uniformly in n by $\langle \omega, \mu_0 \rangle < \infty$, and

$$X_0^n \to \mu_0 \quad \textit{weakly.}$$

Then the sequence of laws $(X_t^n)_{n\in\mathbb{N}}$ is tight in the Skorokhod topology. Moreover, under any weak limit law, $(\mu_t)_{t\geq 0}$ is almost surely a solution of equation (4.5). In particular, this equation has at least one solution.

The proof of this theorem will be an adaptation of [Nor99, Theorem 4.1].

Denote by d some metric on $\mathcal{M}$, the set of finite measures on $\mathbb{R}_+$, which is compatible with the topology of weak convergence, i.e.,

$$d(\mu_n, \mu) \to 0 \quad \text{if and only if} \quad \langle f, \mu_n \rangle \to \langle f, \mu \rangle \tag{4.11}$$

for all bounded continuous functions $f : \mathbb{R}_+ \to \mathbb{R}$. We choose d so that $d(\mu, \mu') \leq \|\mu - \mu'\|$ for al $\mu, \mu' \in \mathcal{M}$.

Theorem 4.8 (Second mean-field limit)**.** *Let K be a model kernel and let μ_0 be a measure on $\mathbb{R}_+$. Assume that for $\varphi(\omega) = \omega + 1$ it holds*

$$K(\omega_1, \omega_2, \omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)$$

and that $\langle \varphi, \mu_0 \rangle < \infty$ and $\langle \varphi^2, \mu_0 \rangle < \infty$. Denote by $(\mu_t)_{t<T}$ the maximal strong solution to (4.5) provided by Theorem 4.6. Let $(X_t^n)_{n\in\mathbb{N}}$ be a sequence of instantaneous coagulation-fragmentation

particle system, with jump kernel K. Suppose that

$$d(\varphi X_0^n, \varphi\mu_0) \to 0$$

as $n \to \infty$. Then, for all $t < T$,

$$\sup_{s\leq t} d(\varphi X_s^n, \varphi\mu_s) \to 0$$

in probability, as $n \to \infty$.

The proof of this theorem will be an adaptation of [Nor99, Theorem 4.4].

Many mathematical works have been devoted to the study of the coagulation-fragmentation equation. We base our work on [Nor99] and [Nor00] but the reader is also referred to [EW00], [EMRR05], [LM02], [W$^+$05], as an example.

C. Applications For the physical applications we consider K given by expression (4.8), i.e. $K(\omega_1, \omega_2, \omega_3) = (\omega_1\omega_2\omega_3)^{\lambda/3}$, which is submultiplicative (since $\omega^\lambda \leq \omega + 1,\ \lambda \in [0, 1]$).

If $\lambda \in [0, 3)$ then we can apply all the previous theorems. For the case $\lambda = 3$ the theorems also apply with the exception of the first mean-field limit, Theorem 4.7.

Here are some examples:

- *Langmuir waves in isotropic plasmas and spin waves*: $\beta = 2$, $\alpha = 2$, so $\lambda = 1$ (the dimension is $N = 3$).
- *Shallow water* (isotropic in a flat bottom, [Zak99]): $\beta = 2$, $\alpha = 1$, so $\lambda = 3$ (dimension $N = 2$).
- *Waves on elastic plates*: $\beta = 3$, $\alpha = 2$, so $\lambda = 2$ (dimension $N = 2$).

However, these results cannot be applied to other systems like gravity waves on deep water, nonlinear optics and Bose-Einstein condensates.

4.1.3 Some notes on the physical theory of Wave Turbulence

The theory of Wave Turbulence is a relatively recent field where most of the results are due to physicists. Next, we present some concepts of the theory extracted from [ZDP04, ZLF92, Naz11], [S$^+$06, Entry turbulence]. All the results are formal and require a rigorous mathematical counterpart.

Wave turbulence is formed by the so-called weak wave turbulence (whose central object is the kinetic wave equation) and the so-called 'coherent structures'.

Wave turbulence takes place on the onset of weakly non-linear dispersive waves. The assumption on weak non-linearity allows the derivation of the kinetic wave equation of which (4.1) is an example for the case of 4 interacting waves. In the general case, N waves interact in resonant sets transferring energy.

Differences between physical systems are given by the dimension of the system, the number of interacting waves and the medium itself (which is described by the dispersion relation and the interaction coefficient).

A. Derivation of the wave kinetic equation and the Cauchy theory. There is not a rigorous mathematical derivation and Cauchy theory for the kinetic wave equation. In this work we prove existence and uniqueness of solutions for the isotropic weak 4-wave kinetic equation in some restricted setting.

- *General procedure:* in [Naz11, Section 6.1.1] it is given a scheme of the general procedure to derive the kinetic wave equation. We do not reproduce here the explanation there but point at some of the key steps:
  - the starting point is a nonlinear wave equation (mostly written in Hamiltonian form);
  - then the equation is written in Fourier space in $\mathbf{k}$ using the interaction representation between waves;
  - using the weakness of the nonlinearity hypothesis, a perturbation analysis is done expanding around a small nonlinearity parameter;
  - perform statistical averaging.
- *Example: shallow water*. In the case of shallow (or deep water) the vertical coordinate is considered to be
$$-h < z < \eta(\mathbf{r}), \qquad \mathbf{r} = (x, y)$$
and the velocity field V is incompressible and a potential field,
$$\text{div } V = 0, \qquad V = \nabla\Phi$$
where the potential satisfies the Laplace equation
$$\Delta\Phi = 0$$
with boundary conditions
$$\Phi|_{z=\eta} = \Psi(\mathbf{r}, t), \qquad \Phi|_{z=-h} = 0.$$
The Hamiltonian is consider to by the sum $H = T + U$ of kinetic and potential

energies defined as follows:

$$
\begin{aligned}
T &= \frac{1}{2}\int dr \int_{-h}^{\eta} (\nabla\Phi)^2 dz, \\
U &= \frac{1}{2} g \int \eta^2 dr + \sigma \int \left(\sqrt{1+(\nabla\eta)^2} - 1\right) dr
\end{aligned}
$$

where g is the acceleration of gravity and σ is the surface tension coefficent. Zakharov [Zak98] derived the equations of motion for η and Ψ as

$$
\frac{\partial\eta}{\partial t} = \frac{\delta H}{\delta\Psi}, \qquad \frac{\partial\Psi}{\partial t} = -\frac{\delta H}{\delta\eta}.
$$

In [Zak99], Zakharov derives the kinetic wave equation for shallow and deep water starting from these equations.

- *The delta distribution*. One of the main issues to study the validity of the kinetic wave equation is the presence of the two delta distributions that make sure that the energy and the total momentum are conserved.
- *N-waves*. At the beginning of this work the 4-wave equation was presented. In the general case, the kinetic equation will correspond to N interacting waves, where N is the minimal number such that the interaction operator is non-zero, i.e., such that

  (i) the N-wave resonant conditions are satisfied for a non-trivial set of wave vectors (here 'non-trivial set' must be made precise):

  $$
  \begin{aligned}
  \omega(\mathbf{k}_1) \pm \omega(\mathbf{k}_2) \pm \ldots \pm \omega(\mathbf{k}_N) &= 0; \\
  \mathbf{k}_1 \pm \mathbf{k}_2 \pm \ldots \pm \mathbf{k}_N;
  \end{aligned}
  $$

  (ii) the N-wave interaction coefficient $\overline{T}$ must be non-zero over this set.

B. The Kolmogorov-Zakharov (KZ) spectra. The Kolmogorov-Zakharov spectrum corresponds to steady states of the system.

- *Derivation, validity (locality) and stability.* The derivation of the KZ spectrum is explained in [ZLF92, Chapter 3]. For the derivation, only the homogeneity index of the interaction coefficient $\overline{T}$ is needed. However, the validity of the KZ spectrum depends on the condition of 'locality', i.e., that only waves with similar wavelength interact. This condition is translated in the finiteness of the interaction integral (see [ZLF92] for more details) and it does depend on the particular shape of $\overline{T}$. On the other hand, one should check the stability of the KZ to small perturbations.

- *Case of shallow water:* for this case, the corresponding Kolmogorov-Zakharov solutions are ([Zak99]):

$$n_k^{(1)} \sim k^{-10/3}$$
$$n_k^{(2)} \sim k^{-3}.$$

Observe that there are two solutions; the first one corresponds to the energy flux and the second to the flux of action (corresponding to the waveaction).

Historical note. The kinetic wave equation was first derived by Nordheim in 1928 [Nor28] in the context of a Bose gas and by Peierls [Pei29] in 1929 in the context of thermal conduction in crystals.

C. Some examples. We have already seen the case of shallow water, but there are many more examples.

The Majda-McLaughlin-Tabak model is explained in [ZDP04] in dimension 1 where the dispersion relation is given by

$$\omega(\mathbf{k}) = k^\alpha, \quad \alpha > 0$$

where $k = \|\mathbf{k}\|$ and

$$T_{123k} = (k_1 k_2 k_3 k)^{\beta/4} \tag{4.12}$$

for some $\beta \in \mathbb{R}$. The particular case $\alpha = \frac{1}{2}$ corresponds to the Majda-McLaughlin-Tabak (MMT) model.

We have a four-wave interaction process with **resonant conditions**:

$$\begin{cases} \mathbf{k}_1 + \mathbf{k}_2 = \mathbf{k}_3 + \mathbf{k} \\ |k_1|^{1/2} + |k_2|^{1/2} = |k_3|^{1/2} + |k|^{1/2}. \end{cases}$$

In this case wave numbers that are non-trivial solutions to these conditions cannot have all the same sign. Moreover, non-trivial solutions can be parametrized by a two parameter family A and $\xi > 0$:

$$k_1 = -A^2\xi^2, \quad k_2 = A^2(1+\xi+\xi^2)^2, \quad k_3 = A^2(1+\xi)^2, \quad k = A^2\xi^2(1+\xi)^2. \tag{4.13}$$

When $\beta = 0$ the collision rate is bounded. In [ZDP04] the authors obtain the following Kolmogorov-type solutions for $\alpha = 1/2$ and $\beta = 0$:

$$n \sim |k|^{-5/6} \tag{4.14}$$
$$n \sim |k|^{-1}. \tag{4.15}$$

The derivation of the kinetic wave equation is done from the equation

$$i\frac{\partial\psi}{\partial t} = \underbrace{\left|\frac{\partial}{\partial x}\right|^{\alpha}\psi}_{\text{dispersive}} \underbrace{+\lambda\left|\frac{\partial}{\partial x}\right|^{\beta/4}\left(\left|\left|\frac{\partial}{\partial x}\right|^{\beta/4}\psi\right|^{2}\left|\frac{\partial}{\partial x}\right|^{\beta/4}\psi\right)}_{\text{non-linearity}} \qquad \lambda=\pm 1$$

where $\psi(x,t)$ denotes a complex wave field.

Other examples in wave turbulence are (taken from [Naz11]):

- 4-wave examples
  - surface gravity waves; $N=2$, $\alpha=1/2$, $\beta=3$;
  - langmuir waves in isotropic plasmas, spin waves; $N=3$, $\alpha=2$, $\beta=2$;
  - waves on elastic plates: $N=2$, $\alpha=2$, $\beta=3$;
  - Bose-Einstein condensates and non-linear optics: $\alpha=2$, $\beta=0$;
  - Gravity waves on deep water: $N=2$, $\alpha=1/2$, $\beta=3$.
- 3-wave examples
  - capillary waves: $N=2$, $\alpha=3/2$;
  - acoustic turbulence, waves in isotropic elastic media; $N=3$, $\alpha=1$;
  - interval waves in stratified fluids: $N=1$, $\alpha=-1$;
- other examples
  - Kelvin waves on vortex filaments: $N=1$, 6-wave interaction, $\alpha=2$.

4.2 Existence of solutions for unbounded kernel

In this section we will follow the steps in [Nor99, Theorem 2.1] (see Remark 4.2).

Remark 4.9. We make some comments about Theorem 4.6:

(i) The statement is correct even if

$$K(\omega_1,\omega_2,\omega_3)\le C\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)$$

for some positive constant $C<\infty$. This only changes the $\zeta(\mu_0)$ into

$$\zeta(\mu_0)=\langle\varphi^2,\mu_0\rangle^{-1}C^{-1}\langle\varphi,\mu_0\rangle^{-1}.$$

Also notice that by scaling time, we can eliminate the multiplicative constant.

(ii) Notice that in the coagulation case, existence of strong solutions is assured for times $T' = \langle \varphi^2, \mu_0 \rangle^{-1}$. We expect that in the 4-wave equations we can assure existence of strong solutions for larger times. The reason that we do not get that is because when bounding (4.30), we ignore some negative factors.

(iii) We will need to use that for $\varphi(\omega) = \omega + 1$, it holds that for any local solution $(\mu_t)_{t<T}$

$$\langle \varphi, \mu_t \rangle \leq \langle \varphi, \mu_0 \rangle \quad \text{for all } t < T. \tag{4.16}$$

This is a condition for μ_t being a solution (see Definition 4.4). Notice that in particular strong solutions fulfilled this condition automatically as they are conservative (this is explained in expression (4.29)).

(iv) We could have defined our set of test functions as including also measurable functions with linear growth (and in an unbounded interval). This way the theorem works the same and we would have that $\langle \varphi, \mu_t \rangle = \langle \varphi, \mu_0 \rangle$ for all t where the solution exists, i.e., for that particular set of test functions we would only consider conservative solutions.

(v) A main difference with the result obtained in [Nor99] and [Nor00] is that we do not allow K to blow up at zero.

4.2.1 Proof of Theorem 4.6

The rest of this section will consist on the proof of this theorem, which we will split in different propositions. We will follow the idea and structure as in [Nor99, Theorem 2.1]. We want to apply an iterative scheme on the equation to prove existence of solutions and for that we need estimates on $\|Q(\mu)\|$ and $\|Q(\mu) - Q(\mu')\|$, which, unfortunately, are unavailable in our present case for unbounded kernels. To sort this problem, we will consider an auxiliary process that approximates our looked for solution and that operates on bounded sets.

This auxiliary process will take the form $(X_t^B, \Lambda_t^B)_{t\geq 0}$ for some bounded set B. Λ_t^B gives an upper estimate of the effect on X_t^B of the particles outside B and X_t^B will be a lower bound for our process in B.

Let $B \subset [0,\infty)$ be bounded. Denote by $\mathcal{M}_B$ the space of finite signed measures supported on B. We define $L^B : \mathcal{M}_B \times \mathbb{R} \to \mathcal{M}_B \times \mathbb{R}$ by the requirement:

$$\begin{aligned}
\langle (f,a), L^B(\mu,\lambda) \rangle \;&=\; \frac{1}{2}\int_D \big(f(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + a\varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} \\
&\qquad + f(\omega_3) - f(\omega_1) - f(\omega_2)\big) K(\omega_1,\omega_2,\omega_3)\mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3) \\
&+ \; (\lambda^2 + 2\lambda\langle \varphi, \mu\rangle) \int_0^\infty (a\varphi(\omega) - f(\omega))\,\varphi(\omega)\mu(d\omega)
\end{aligned}$$

for all bounded measurable functions f on $(0,\infty)$ and all $a \in \mathbb{R}$ where $D = \{\mathbb{R}^3_+ \cap \omega_1 + \omega_2 - \omega_3 \geq 0\}$. We used the notation $\langle (f,a),(\mu,\lambda)\rangle = \langle f,\mu\rangle + a\lambda$.

Consider the equation

$$(\mu_t, \lambda_t) = (\mu_0, \lambda_0) + \int_0^t L^B(\mu_s, \lambda_s)\, ds. \tag{4.17}$$

We admit as a *local solution* any continuous map

$$t \mapsto (\mu_t, \lambda_t) : [0,T] \to \mathcal{M}_B \times \mathbb{R}$$

where $T \in (0,\infty)$, which satisfies equation (4.17) for all $t \in [0,T]$.

Proposition 4.10 (Existence for the auxiliary process)**.** *Suppose $\mu_0 \in \mathcal{M}_B$ with $\mu_0 \geq 0$ and that $\lambda_0 \in [0,\infty)$. The equation (4.17) has a unique solution $(\mu_t,\lambda_t)_{t\geq 0}$ starting from (μ_0,λ_0). Moreover, $\mu_t \geq 0$ and $\lambda_t \geq 0$ for all t.*

The proof is obtained by adapting the one in [Nor99, Proposition 2.2].

Proof. By assumption (4.10) it holds that for $\varphi(\omega) = \omega + 1$

$$K(\omega_1,\omega_2,\omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3).$$

Observe that $\varphi \geq 1$. By a scaling argument we may assume, without loss, that

$$\langle \varphi, \mu_0\rangle + \lambda_0 \leq 1,$$

which implies that

$$\|\mu_0\| + |\lambda_0| \leq 1.$$

We will show next by a standard iterative scheme, that there is a constant $T > 0$ depending only on φ and B, and a unique local solution $(\mu_t,\lambda_t)_{t\leq T}$ starting from (μ_0,λ_0). Then we will see that $\mu_t \geq 0$ for all $t \in [0,T]$.

This will be enough to prove the proposition: if we put $f = 0$ and $a = 1$ in (4.17) we get

$$\begin{aligned}\frac{d}{dt}\lambda_t &= \frac{1}{2}\int_D \varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} K(\omega_1,\omega_2,\omega_3)\mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)\\ &\quad +(\lambda^2 + 2\lambda\langle\varphi,\mu\rangle)\int_0^\infty \varphi(\omega)^2\mu(d\omega).\end{aligned}$$

So, since $\mu_t \geq 0$, we deduce that $\lambda_t \geq 0$ for all $t \in [0,T]$. Next, we put $f = \varphi$ and $a = 1$

to see that

$$\begin{aligned}\frac{d}{dt}\langle\varphi,\mu_t\rangle+\lambda_t &= \frac{1}{2}\int_D(\varphi(\omega_1+\omega_2-\omega_3)+\varphi(\omega_3)-\varphi(\omega_1)-\varphi(\omega_2)) \\ &\qquad\times K(\omega_1,\omega_2,\omega_3)\mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)=0\end{aligned} \tag{4.18}$$

which is zero given that $\varphi(\omega)=\omega+1$. Therefore,

$$\|\mu_T\|+|\lambda_T|\leq\langle\varphi,\mu_T\rangle+\lambda_T=\langle\varphi,\mu_0\rangle+\lambda_0\leq 1.$$

We can now start again from (μ_T,λ_T) at time T to extend the solution to $[0,2T]$, and so on, to prove the proposition.

We use the following norm on $\mathcal{M}_B\times\mathbb{R}$:

$$\|(\mu,\lambda)\|=\|\mu\|+|\lambda|.$$

Note the following estimates: there is a constant $C=C(\varphi,B)<\infty$ such that for all $\mu,\mu'\in\mathcal{M}_B$ and all $\lambda,\lambda'\in\mathbb{R}$

$$\|L^B(\mu,\lambda)\| \leq C\|(\mu,\lambda)\|^3 \tag{4.19}$$

$$\begin{aligned}\|L^B(\mu,\lambda)-L^B(\mu',\lambda')\| &\leq C\Big(\|\mu-\mu'\|\left(\|\mu\|^2+\|\mu\|\|\mu'\|+\|\mu'\|^2\right) \\ &\quad+(|\lambda|+|\lambda'|)|\lambda-\lambda'|\|\mu\|+|\lambda'|^2\|\mu-\mu'\| \\ &\qquad+|\lambda-\lambda'|\|\mu\|^2+|\lambda'|\left(\|\mu\|\|\mu-\mu'\|+\|\mu'\|\|\mu'-\mu\|\right)\Big)\end{aligned} \tag{4.20}$$

Observe that we get these estimates because we are working on a bounded set B.

We turn to the iterative scheme. Set $(\mu_t^0,\lambda_t^0)=(\mu_0,\lambda_0)$ for all t and define inductively a sequence of continuous maps

$$t\mapsto(\mu_t^n,\lambda_t^n):\ [0,\infty)\to\mathcal{M}_B\times\mathbb{R}$$

by

$$(\mu_t^{n+1},\lambda_t^{n+1})=(\mu_0,\lambda_0)+\int_0^t L^B(\mu_s^n,\lambda_s^n)\,ds.$$

Set

$$f_n(t)=\|(\mu_t^n,\lambda_t^n)\|$$

then $f_0(t)=f_n(0)=\|(\mu_0,\lambda_0)\|\leq 1$ and by the estimate (4.19) we have that

$$f_{n+1}(t)\leq 1+C\int_0^t f_n(s)^3\,ds.$$

Hence

$$f_n(t) \le (1-2Ct)^{-1/2} \quad \text{for } t < (2C)^{-1}.$$

This last assertion is checked by induction. Suppose that it holds for n then

$$f_{n+1}(t) \le 1 + C\int_0^t (1-2Cs)^{-3/2}\, ds = 1 + (1-2Cs)^{-1/2}|_{s=0}^{s=t}.$$

Therefore, for all n setting $T = (4C)^{-1}$, we have

$$\|(\mu_t^n, \lambda_t^n)\| \le \sqrt{2} \qquad t \le T. \tag{4.21}$$

Next set $g_0(t) = f_0(t)$ and for $n \ge 1$

$$g_n(t) = \|(\mu_t^n, \lambda_t^n) - (\mu_t^{n-1}, \lambda_t^{n-1})\|.$$

By estimates (4.20) and (4.21), there is a constant $C = C(B, \varphi) < \infty$ such that

$$g_{n+1}(t) \le C\int_0^t g_n(s)\, ds \quad t \le T.$$

Hence by the usual arguments (Gronwall, Cauchy sequence), (μ_t^n, λ_t^n) converges in $\mathcal{M}_B \times \mathbb{R}$ uniformly in $t \le T$, to the desired local solution, which is also unique. Moreover, for some constant $C < \infty$ depending only on φ and B we have

$$\|(\mu_t, \lambda_t)\| \le C \qquad t \le T.$$

Finally, we are left to check that $\mu_t \ge 0$. For this, we need the following result:

Proposition 4.11. *Let*

$$(t, \omega) \mapsto f_t(\omega) : [0, T] \times B \to \mathbb{R}$$

be a bounded measurable function, having a bounded partial derivative $\partial f/\partial t$. Then, for all $t \le T$,

$$\frac{d}{dt}\langle f_t, \mu_t\rangle = \langle \partial f/\partial t, \mu_t\rangle + \langle (f_t, 0), L^B(\mu_t, \lambda_t)\rangle.$$

The proof is a straightforward adaptation of the same Proposition (with different L^B) in [Nor99, Proposition 2.3].

For $t \le T$, set

$$\theta_t(\omega_1) = \exp \int_0^t \left(\int_{\mathbb{R}_+^2 \cap (\omega_1+\omega_2 \ge \omega_3)} K(\omega_1, \omega_2, \omega_3)\mu_s(d\omega_2)\mu_s(d\omega_3) + \left(\lambda_s^2 + 2\lambda_s\langle \varphi, \mu_s\rangle\right)\varphi(\omega_1) \right) ds$$

and define $G_t : \mathcal{M}_B \to \mathcal{M}_B$ by

$$\begin{aligned}\langle f, G_t(\mu)\rangle &= \frac{1}{2}\int_D ((f\theta_t)(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + (f\theta_t)(\omega_3)) \\ &\quad \times K(\omega_1,\omega_2,\omega_3)\theta_t(\omega_1)^{-1}\theta_t(\omega_2)^{-1}\theta_t(\omega_3)^{-1} \\ &\quad \times \mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)\end{aligned}$$

Note that $G_t(\mu) \geq 0$ whenever $\mu \geq 0$ and for some $C = C(\varphi, B) < \infty$ we have

$$\|G_t(\mu)\| \leq C\|\mu\|^3 \tag{4.22}$$

$$\|G_t(\mu) - G_t(\mu')\| \leq C\|\mu-\mu'\|\left(\|\mu\|^2 + \|\mu'\|\|\mu\| + \|\mu'\|^2\right). \tag{4.23}$$

Set $\tilde{\mu}_t = \theta_t\mu_t$. By Proposition 4.11, for all bounded measurable function f we have

$$\frac{d}{dt}\langle f, \tilde{\mu}_t\rangle = \langle f\frac{\partial\theta}{\partial t}, \mu_t\rangle + \langle (f\theta_t, 0), L^B(\mu_t, \lambda_t)\rangle$$

so, using the symmetry of ω_1 and ω_2 in L^B we get

$$\frac{d}{dt}\langle f, \tilde{\mu}_t\rangle = \langle f, G_t(\tilde{\mu}_t)\rangle. \tag{4.24}$$

Thus, the function θ_t is simply designed as an integrating factor, which removes the negative terms appearing in L^B.

Define inductively a new sequence of measures $\tilde{\mu}_t^n$ by setting $\tilde{\mu}_t^0 = \mu_0$ and for $n \geq 0$

$$\tilde{\mu}^{n+1} = \mu_0 + \int_0^t G_s(\tilde{\mu}_s^n)\, ds.$$

By an argument similar to that used for the original iterative scheme, the proof is completed: we can show, first, and possibly for a smaller value of $T > 0$, but with the same dependence, that $\|\tilde{\mu}_t^n\|$ is bounded, uniformly in n, for $t \leq T$, and then that $\|\tilde{\mu}_t^n - \tilde{\mu}_t\| \to 0$ as $n \to \infty$. Since $\tilde{\mu}_t^n \geq 0$ for all n, we deduce $\tilde{\mu}_t \geq 0$ and hence $\mu_t \geq 0$ for all $t \leq T$. □

We fix now $\mu_0 \in \mathcal{M}$ with $\mu_0 \geq 0$ and $\langle \varphi, \mu_0\rangle < \infty$. For each bounded set $B \subset [0,\infty)$, let

$$\mu_0^B = \mathbb{1}_B\mu_0, \qquad \lambda_0^B = \int_{[0,\infty)\setminus B} \varphi(\omega)\mu_0(d\omega) \tag{4.25}$$

and denote by $(\mu_t^B, \lambda_t^B)_{t\geq 0}$ the unique solution to (4.17), starting from (μ_0^B, λ_0^B), provided by Proposition 4.10. We have that for $B \subset B'$,

$$\mu_t^B \leq \mu_t^{B'}, \qquad \langle \varphi, \mu_t^B\rangle + \lambda_t^B = \langle \varphi, \mu_t^{B'}\rangle + \lambda_t^{B'}.$$

The inequality will be proven in Proposition 4.12 and the equality is consequence of expression (4.18) and the fact that

$$\langle \varphi, \mu_0^B \rangle + \lambda_0^B = \langle \varphi, \mu_0^{B'} \rangle + \lambda_0^{B'}$$

by expression (4.25).

Moreover, it holds that for any local solution $(\nu_t)_{t<T}$ of the 4-wave kinetic equation (4.5), for all $t < T$,

$$\mu_t^B \le \nu_t, \qquad \langle \varphi, \mu_t^B \rangle + \lambda_t^B \ge \langle \varphi, \nu_t \rangle. \tag{4.26}$$

We prove the first inequality in Proposition 4.13. The second inequality is consequence of

$$\langle \varphi, \nu_t \rangle \le \langle \varphi, \mu_0 \rangle \le \langle \varphi, \mu_0 \rangle + \lambda_0^B = \langle \varphi, \mu_t^B \rangle + \lambda_t^B. \tag{4.27}$$

We now show how these facts lead to the proof of Theorem 4.6. Set $\mu_t = \lim_{B\uparrow[0,\infty)} \mu_t^B$ and $\lambda_t = \lim_{B\uparrow[0,\infty)} \lambda_t^B$. Note that

$$\langle \varphi, \mu_t \rangle = \lim_{B\uparrow[0,\infty)} \langle \varphi, \mu_t^B \rangle \le \langle \varphi, \mu_0 \rangle < \infty.$$

So, by dominated convergence, using that K is submultiplicative, for all bounded measurable functions f,

$$\int_D f(\omega_1+\omega_2-\omega_3) \mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B} K(\omega_1,\omega_2,\omega_3) \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3) \to 0,$$

and we can pass to the limit in (4.17) to obtain

$$\begin{aligned} \frac{d}{dt}\langle f, \mu_t \rangle &= \frac{1}{2}\int_D (f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)) \\ &\quad \times K(\omega_1,\omega_2,\omega_3)\mu_t(d\omega_1)\mu_t(d\omega_2)\mu_t(d\omega_3) \\ &\quad -(\lambda_t^2 + 2\lambda_t \langle \varphi, \mu_t \rangle)\langle f\varphi, \mu_t \rangle. \end{aligned}$$

For any local solution $(\nu_t)_{t<T}$, for all $t < T$,

$$\mu_t \le \nu_t, \qquad \langle \varphi, \mu_t \rangle + \lambda_t \ge \langle \varphi, \nu_t \rangle.$$

Hence, if $\lambda_t = 0$ for all $t < T$, then $(\mu_t)_{t<T}$ is a local solution and, moreover, is the only local solution on $[0, T)$. If $(\nu_t)_{t<T}$ is a strong local solution, then

$$\int_0^t \langle \varphi^2, \mu_s \rangle \, ds \le \int_0^t \langle \varphi^2, \nu_s \rangle \, ds < \infty$$

for all $t < T$; this allows us to pass to the limit in (4.17) to obtain

$$\frac{d}{dt}\lambda_t = (\lambda_t^2 + 2\lambda_t\langle\varphi,\mu_t\rangle)\langle\varphi^2,\mu_t\rangle \tag{4.28}$$

and to deduce from this equation that $\lambda_t = 0$ for all $t < T$. It follows that $(\nu_t)_{t<T}$ is the only local solution on $[0,T)$. For any local solution $(\nu_t)_{t<T}$,

$$\begin{aligned}\int_0^\infty \omega 1\!\!1_{\omega\le n}\nu_t(d\omega) &= \int_0^\infty \omega 1\!\!1_{\omega\le n}\nu_0(d\omega) \\ &+ \frac{1}{2}\int_0^t\int_D \{(\omega_1+\omega_2-\omega_3)1\!\!1_{\omega_1+\omega_2-\omega_3\le n} + \omega_3 1\!\!1_{\omega_3\le n} - \omega_1 1\!\!1_{\omega_1\le n} - \omega_2 1\!\!1_{\omega_2\le n}\} \\ &\quad\times K(\omega_1,\omega_2,\omega_3)\nu_s(d\omega_1)\nu_s(d\omega_2)\nu_s(d\omega_3).\end{aligned} \tag{4.29}$$

Hence, if $(\nu_t)_{t<T}$ is strong we have that

$$\int_0^t \langle\omega^2,\nu_s\rangle\, ds \le \int_0^t \langle\varphi^2,\nu_s\rangle\, ds < \infty.$$

Then, by dominated convergence, the second term on the right tends to 0 as $n \to \infty$, showing that $(\nu_t)_{t<T}$ is conservative.

Suppose now that $\langle\varphi^2,\mu_0\rangle < \infty$ and set $T = \langle\varphi^2,\mu_0\rangle^{-1}\langle\varphi,\mu_0\rangle^{-1}$. For any bounded set $B \subset [0,\infty)$, we have

$$\begin{aligned}\frac{d}{dt}\langle\varphi^2,\mu_t^B\rangle &\le \frac{1}{2}\int_D \{\varphi(\omega_1+\omega_2-\omega_3)^2 + \varphi(\omega_3)^2 - \varphi(\omega_1)^2 - \varphi(\omega_2)^2\} \\ &\quad\times K(\omega_1,\omega_2,\omega_3)\mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3) \\ &\le \int_D \varphi(\omega_1)^2\varphi(\omega_2)^2\varphi(\omega_3)\mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3) \\ &\le \langle\varphi,\mu_t^B\rangle\langle\varphi^2,\mu_t^B\rangle^2 \\ &\le \langle\varphi,\mu_0\rangle\langle\varphi^2,\mu_t^B\rangle^2\end{aligned} \tag{4.30}$$

where we used that

$$\begin{aligned}(\omega_1+\omega_2-\omega_3+1)^2 + (\omega_3+1)^2 - (\omega_1+1)^2 - (\omega_2+1)^2 &= (\tilde\omega_1+\tilde\omega_2-\tilde\omega_3)^2 + \tilde\omega_3{}^2 - \tilde\omega_1{}^2 - \tilde\omega_2{}^2 \\ &= 2\tilde\omega_1\tilde\omega_2 + 2\tilde\omega_3\left(\tilde\omega_3 - \tilde\omega_1 - \tilde\omega_2\right) \\ &\le 2\tilde\omega_1\tilde\omega_2\end{aligned}$$

with $\tilde\omega_i = \omega_i + 1$, and using that in our domain $\omega_1+\omega_2-\omega_3 \ge 0$, so for $t < T$

$$\langle\varphi^2,\mu_t\rangle \le (S - \langle\varphi,\mu_0\rangle t)^{-1}$$

where $S = \langle\varphi^2,\mu_0\rangle^{-1}$. Hence (4.28) holds and forces $\lambda_t = 0$ for $t < T$ as above, so $(\mu_t)_{t<T}$

is a strong local solution.

Proposition 4.12. *Suppose $B \subset B'$ and that $(\mu_t^B, \lambda_t^B)_{t\geq 0}$, $(\mu_t^{B'}, \lambda_t^{B'})_{t\geq 0}$ are the solutions of (4.17) for each one of these sets corresponding to the initial data given by (4.25). Then for all $t \geq 0$, $\mu_t^B \leq \mu_t^{B'}$.*

The proof is obtained by adapting the one in [Nor99, Proposition 2.4].

Proof. Set

$$\theta_t(\omega_1) = \exp \int_0^t \left(\int_{\mathbb{R}_+^2 \cap (\omega_1+\omega_2 \geq \omega_3)} K(\omega_1,\omega_2,\omega_3)\mu_s^B(d\omega_2)\mu_s^B(d\omega_3) + ((\lambda_s^B)^2 + 2\lambda_s^B \langle \varphi, \mu_s^B\rangle)\varphi(\omega_1) \right) ds.$$

Denote by $\pi_t = \theta_t(\mu_t^{B'} - \mu_t^B)$. Note that $\pi_0 \geq 0$. By Proposition 4.11, for any bounded measurable function f,

$$\begin{aligned}
\frac{d}{dt}\langle f, \pi_t\rangle &= \langle f\frac{\partial \theta_t}{\partial t}, \mu_t^{B'} - \mu_t^B\rangle \\
&\quad + \langle (f\theta_t, 0), L^{B'}(\mu_t^{B'}, \lambda_t^{B'}) - L^B(\mu_t^B, \lambda_t^B)\rangle \\
&= \int_D (f\theta_t)(\omega_1) K(\omega_1,\omega_2,\omega_3)\left(\mu_t^{B'}(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&+ \left((\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi, \mu_t^B\rangle\right) \int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)(\mu_t^{B'}(d\omega_1) - \mu_t^B(d\omega_1)) \\
&+ \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times \left(\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B'}\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&+ \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times \left(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&- \int_D (f\theta_t)(\omega_1)K(\omega_1,\omega_2,\omega_3)\left(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&- \left((\lambda_t^{B'})^2 + 2\lambda_t^{B'}\langle\varphi, \mu_t^{B'}\rangle\right)\int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)\mu_t^{B'}(d\omega_1) \\
&+ \left((\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi, \mu_t^B\rangle\right)\int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)\mu_t^B(d\omega_1) \\
&= I \\
&+ \int_D (f\theta_t)(\omega_1)K(\omega_1,\omega_2,\omega_3)\left(\mu_t^{B'}(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3) - \mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3)\right) \\
&+ \left(\left((\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi, \mu_t^B\rangle\right) - \left((\lambda_t^{B'})^2 + 2\lambda_t^{B'}\langle\varphi, \mu_t^{B'}\rangle\right)\right)\langle f\theta_t\varphi, \mu_t^{B'}\rangle
\end{aligned}$$

where

$$\begin{aligned} I \;:=\; & \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3) \\ & \times\Big(\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B'}\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B}(d\omega_3)\Big) \\ + \; & \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3) \\ & \times\Big(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B}(d\omega_3)\Big). \end{aligned}$$

Now, squaring the equality

$$\langle\varphi,\mu_t^B\rangle + \lambda_t^B = \langle\varphi,\mu_t^{B'}\rangle + \lambda_t^{B'}$$

we have that

$$\big((\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi,\mu_t^B\rangle\big) - (\lambda_t^{B'})^2 - 2\lambda_t^{B'}\langle\varphi,\mu_t^{B'}\rangle = \langle\varphi,\mu_t^{B'}\rangle^2 - \langle\varphi,\mu_t^B\rangle^2$$

and therefore

$$\begin{aligned} \frac{d}{dt}\langle f,\pi_t\rangle \;=\; & I \\ + \; & \int_{\mathbb{R}^3_+\setminus D} (f\theta_t)(\omega_1)\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3) \\ & \Big(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3) - \mu_t^{B'}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B}(d\omega_3)\Big) \\ + \; & \int_D (f\theta_t)(\omega_1)(\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3) - K(\omega_1,\omega_2,\omega_3)) \\ & \Big(\mu_t^{B'}(d\omega_1)\mu_t^{B'}(d\omega_3)\mu_t^{B'}(d\omega_3) - \mu_t^{B'}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B}(d\omega_3)\Big). \end{aligned}$$

Therefore, π_t satisfies an equation of the form

$$\frac{d}{dt}\pi_t = H_t(\pi_t)$$

where $H_t : \mathcal{M}_{B'} \to \mathcal{M}_{B'}$ and it holds $H_t(\pi) \geq 0$ whenever $\pi \geq 0$ and where we have estimates, for $t \leq 1$,

$$\|H_t(\pi)\| \leq C\|\pi\|$$

for some constant $C < \infty$ depending only on φ and B'. Therefore, we can apply the same sort of argument that we used for nonnegativity to see that $\pi_t \geq 0$ for all $t \leq 1$, and then for all $t < \infty$.

Explicitly, H_t is

$$H_t \;=\; \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3)$$

$$
\begin{aligned}
&\times\Big(\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B'}\theta_t^{-1}(\omega_1)\pi(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3)\\
&\quad+\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B'}\mu_t^{B}(d\omega_1)\theta_t^{-1}(\omega_2)\pi(d\omega_2)\mu_t^{B'}(d\omega_3)\\
&\quad+\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\theta_t^{-1}(\omega_3)\pi(d\omega_3)\Big)\\
+&\ \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3)\\
&\times\Big(\theta_t^{-1}(\omega_1)\pi(d\omega_1)\mu_t^{B'}(d\omega_2)\mu_t^{B'}(d\omega_3)+\theta_t^{-1}(\omega_2)\pi(d\omega_2)\mu_t^{B}(d\omega_1)\mu_t^{B'}(d\omega_2)\\
&\quad+\theta_t^{-1}(\omega_3)\pi(d\omega_3)\mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\Big)\\
+&\ \int_{\mathbb{R}^3_+\setminus D}(f\theta_t)(\omega_1)\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)\\
&\times\Big(\mu_t^{B'}(d\omega_1)\theta_t^{-1}(\omega_2)\pi(d\omega_2)\mu_t^{B'}(d\omega_3)+\mu_t^{B'}(d\omega_1)\mu_t^{B}(d\omega_2)\theta_t^{-1}(\omega_3)\pi(d\omega_3)\Big)\\
+&\ \int_D (f\theta_t)(\omega_1)(\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)-K(\omega_1,\omega_2,\omega_3))\\
&\times\Big(\mu_t^{B'}(d\omega_1)\theta_t^{-1}(\omega_2)\pi(d\omega_2)\mu_t^{B'}(d\omega_3)+\mu_t^{B'}(d\omega_1)\mu_t^{B}(d\omega_2)\theta_t^{-1}(\omega_3)\pi(d\omega_3)\Big).
\end{aligned}
$$

where we have used that

$$\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B'}\mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B'}(d\omega_3)=\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\mu_t^{B}(d\omega_1)\mu_t^{B}(d\omega_2)\mu_t^{B'}(d\omega_3),$$

given that if $\omega_1,\omega_2<b$ and $\omega_3>b$ then it must hold that $\omega_1+\omega_2-\omega_3<b$.

□

Proposition 4.13. *Suppose that $(\nu_t)_{t<T}$ is a local solution of the 4-wave kinetic equation (4.5), starting from μ_0. Then, for all bounded sets $B\subset[0,\infty)$ and all $t<T$, $\mu_t^B\le\nu_t$.*

The proof is obtained by adapting the one in [Nor99, Proposition 2.5].

Proof. Set θ_t as in the previous Proposition and denote $\nu_t^B=\mathbb{1}_B\nu_t$ and $\pi_t=\theta_t(\nu_t^B-\mu_t^B)$. By a modification of Proposition 4.11, we have, for all bounded measurable functions f,

$$\frac{d}{dt}\langle f,\pi_t\rangle=\langle f\partial\theta/\partial t,\nu_t^B-\mu_t^B\rangle+\langle f\theta_t\mathbb{1}_B,Q(\nu_t)\rangle-\langle(f\theta_t,0),L^B(\mu_t^B,\lambda_t^B)\rangle.$$

Now, proceeding as before we have that

$$
\begin{aligned}
\frac{d}{dt}\langle f,\pi_t\rangle &= \int_D (f\theta_t)(\omega_1)K(\omega_1,\omega_2,\omega_3)(\nu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)-\mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3))\\
&+\ \left((\lambda_t^B)^2+2\lambda_t^B\langle\varphi,\mu_t^B\rangle\right)\int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)\nu_t^B(d\omega_1)\\
&+\ \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3)\\
&\qquad\times\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3)-\mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times \left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t^B(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&- \quad \int_D (f\theta_t)(\omega_1)K(\omega_1,\omega_2,\omega_3)\left(\nu_t^B(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&= \quad \chi_t \int_0^\infty (f\theta_t)(\omega_1)\varphi(\omega_1)\nu_t^B(d\omega_1) \\
&+ \quad \frac{1}{2}\int_D (f\theta_t)(\omega_1+\omega_2-\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times \mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&+ \quad \frac{1}{2}\int_D (f\theta_t)(\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times \left(\nu_t(d\omega_1)\nu_t(d\omega_2)\nu_t^B(d\omega_3) - \mu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&+ \quad \int_{\mathbb{R}^3_+\setminus D} (f\theta_t)(\omega_1)\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3)\left(\nu_t^B(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \nu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right) \\
&+ \quad \int_D (f\theta_t)(\omega_1)(\varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3) - K(\omega_1,\omega_2,\omega_3)) \\
&\quad \times \qquad \left(\nu_t^B(d\omega_1)\nu_t(d\omega_2)\nu_t(d\omega_3) - \nu_t^B(d\omega_1)\mu_t^B(d\omega_2)\mu_t^B(d\omega_3)\right)
\end{aligned}
$$

where $\chi_t = (\lambda_t^B)^2 + 2\lambda_t^B\langle\varphi,\mu_t^B\rangle + \langle\varphi,\mu_t^B\rangle^2 - \langle\varphi,\nu_t\rangle^2 \geq 0$.

Therefore, analogously as in the previous Proposition 4.12, we have that

$$\frac{d}{dt}(\pi_t) = \tilde{H}_t(\pi_t)$$

where $\tilde{H}_t : \mathcal{M}_B \to \mathcal{M}_B$ is linear and $\tilde{H}_t(\pi) \geq 0$ whenever $\pi \geq 0$. Moreover for $t \leq 1$

$$\|\tilde{H}_t(\pi)\| \leq C\|\pi\|$$

for some constant $C < \infty$ depending only on φ and B. □

4.3 Mean-field limit

4.3.1 The instantaneous coagulation-fragmentation stochastic process

Define

$$D = \{(\omega_1,\omega_2,\omega_3) \in \mathbb{R}^3_+ \,|\, \omega_1+\omega_2 \geq \omega_3\}.$$

We consider X_0^n a probability measure on $\mathbb{R}_+$ written as a sum of unit masses

$$X_0^n = \frac{1}{n}\sum_{i=1}^n \delta_{\omega_i}$$

for $\omega_1,\ldots,\omega_n \in \mathbb{R}_+$. X_0^n represents a system of n waves labelled by their dispersion

$\omega_1, \ldots, \omega_n$.

We define a Markov process $(X_t^n)_{t\geq 0}$ of probability measures on $\mathbb{R}_+$. For each triple $(\omega_i, \omega_j, \omega_l) \in D$ of distinct particles, take an independent exponential random time T_{ijl}, $i < j$, with parameter

$$\frac{1}{n^2} K(\omega_i, \omega_j, \omega_l). \tag{4.31}$$

Set $T_{ijk} = T_{jik}$ and set $T = \min_{ijl} T_{ijl}$. Then set

$$X_t^n = X_0^n \qquad \text{for } t < T$$

and

$$X_T^n = X_0^n + \frac{1}{n}(\delta_\omega + \delta_{\omega_l} - \delta_{\omega_i} - \delta_{\omega_j})$$

with $\omega = \omega_i + \omega_j - \omega_l$. Then begin the construction afresh from X_T^n.

We call the process $(X_t^n)_{t\geq 0}$ an instantaneous n-coagulation-fragmentation stochastic process.

Remark 4.14. Note that we should be careful not to pick the same particle twice as one particle cannot interact with itself. Suppose that $\omega_i = \omega_j = \omega_l$ then, the Markov Chain does not make a jump. The same happens with $\omega_i = \omega_l$ or $\omega_j = \omega_l$. Finally the case $\omega_i = \omega_j$ needs to be considered. For that, we define

$$\mu^{(1)}(A \times B \times C) = \mu(A)\mu(B)\mu(C) - \mu(A \cap B)\mu(C)$$

as the counting measure of triples of particles with different particles in the first and second position. Also, define

$$\mu^{(n)}(A \times B \times C) = \mu(A)\mu(B)\mu(C) - n^{-1}\mu(A \cap B)\mu(C). \tag{4.32}$$

Note that

$$n^3 \mu^{(n)} = (n\mu)^{(1)}. \tag{4.33}$$

Generator of the Markov Chain: For all $F \in C_b$:

$$GF(X) = \frac{n}{2} \int_D \left[F(X^{\omega_1,\omega_2,\omega_3}) - F(X)\right] K(\omega_1, \omega_2, \omega_3) X^{(n)}(d\omega_1, d\omega_2, d\omega_3)$$

where

$$X^{\omega_1,\omega_2,\omega_3} = X + \frac{1}{n}\left(\delta_{\omega_3} + \delta_{\omega_1+\omega_2-\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right).$$

Interpretation of the stochastic process. Three different particles, say $\omega_1, \omega_2, \omega_3$ interact at a random time given by the rate (4.31).

The outcome of the interaction is that ω_1 and ω_2 merge and then, under the presence of ω_3, they split, creating a new particle ω_3 and another one with the rest $\omega = \omega_1 + \omega_2 - \omega_3$. (Coagulation-fragmentation phenomena, which takes place instantaneously).

The martingale formulation. Now, for each function $f \in C_b(\mathbb{R}_+)$ the Markov chain can be expressed as

$$\langle f, X_t^n \rangle = \langle f, X_0^n \rangle + M_t^{n,f} + \int_0^t \langle f, Q^{(n)}(X_s^n) \rangle \, ds \tag{4.34}$$

where $(M_t^{n,f})_{t \geq 0}$ is a martingale. Note that using (4.33) we have that

$$\begin{aligned} & \langle f, Q^{(n)}(\mu) \rangle \\ = \; & \frac{1}{2} \int_D \frac{1}{n} (f(\omega_1 + \omega_2 - \omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)) \frac{1}{n^2} K(\omega_1, \omega_2, \omega_3)(n\mu)^{(1)}(d\omega_1, d\omega_2, d\omega_3) \\ = \; & \frac{1}{2} \int_D (f(\omega_1 + \omega_2 - \omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)) K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \end{aligned}$$

from this expression it is clear why we needed to rescaled the collision frequency by n^2.

4.3.2 First result on mean-field limit

We will start working in the simpler case where K is bounded and see that the unbounded case will come as a 'modification' of the bounded one.

4.3.2.1 Mean-field limit for bounded jump kernel

Uniqueness of solutions for bounded kernel

Lemma 4.15. *It holds that Q given in (4.5) is linear in each one of its terms and the following symmetry*

$$\langle f, Q(\mu, \nu, \tau) \rangle = \langle f, Q(\nu, \mu, \tau) \rangle$$

but

$$\begin{aligned} \langle f, Q(\mu, \nu, \tau) \rangle & \neq \langle f, Q(\mu, \tau, \nu) \rangle \\ \langle f, Q(\mu, \nu, \tau) \rangle & \neq \langle f, Q(\tau, \nu, \mu) \rangle. \end{aligned}$$

Moreover,

$$Q(\mu, \mu, \mu) - Q(\nu, \nu, \nu) = Q(\mu + \nu, \mu - \nu, \mu) + Q(\mu + \nu, \nu, \mu - \nu) + Q(\mu, \nu, \nu - \mu) \tag{4.35}$$

Proof. The first part of the statement is immediate from the definition. The second part will make use of this symmetry property along with the linearity in each component:

$$\begin{aligned}
Q(\mu,\mu,\mu) - Q(\nu,\nu,\nu) &= Q(\mu,\mu,\mu) + Q(\nu,\mu,\mu) - Q(\mu,\nu,\mu) + Q(\mu,\nu,\nu) \\
&\quad -Q(\mu,\nu,\nu) - Q(\nu,\nu,\nu) \\
&= Q(\mu+\nu,\mu,\mu) + Q(\mu,\nu,\nu-\mu) - Q(\mu+\nu,\nu,\nu) \\
&= Q(\mu+\nu,\mu,\mu) - Q(\mu+\nu,\nu,\mu) + Q(\mu+\nu,\nu,\mu) - Q(\mu+\nu,\nu,\nu) \\
&\quad + Q(\mu,\nu,\nu-\mu) \\
&= Q(\mu+\nu,\mu-\nu,\mu) + Q(\mu+\nu,\nu,\mu-\nu) + Q(\mu,\nu,\nu-\mu)
\end{aligned}$$

□

Proposition 4.16 (Uniqueness of solutions)**.** *Suppose that the jump kernel in* (4.5) *is bounded by* Λ. *Then for any given initial data, if there exists a solution for* (4.5)*, then the solution is unique.*

Proof. Suppose that we have $\mu_t, \nu_t \in \mathcal{P}(\mathbb{R}_+)$ solutions to (4.5) with the same initial data. We will compare these solutions in the total variation norm:

$$\|\mu_t - \nu_t\|_{TV} = \sup_{\|f\|_\infty = 1} \langle f, \mu_t - \nu_t \rangle = \sup_{\|f\|_\infty=1} \int_0^t \langle f, \dot{\mu}_t - \dot{\nu}_t \rangle.$$

Then by expression (4.35) we have that

$$\dot{\mu}_s - \dot{\nu}_s = Q(\mu_s+\nu_s, \mu_s-\nu_s, \mu_s) + Q(\mu_s+\nu_s,\nu_s,\mu_s-\nu_s) + Q(\mu_s,\nu_s,\nu_s-\mu_s).$$

Therefore, for any $f \in C_b(\mathbb{R}_+)$ such that $\|f\|_\infty = 1$ it holds

$$|\langle f, \dot{\mu}_s - \dot{\nu}_s\rangle| \leq 24\Lambda \|\mu_s - \nu_s\|_{TV}.$$

Finally applying Gronwall on

$$\|\mu_t - \nu_t\|_{TV} \leq 24\Lambda \int_0^t \|\mu_s - \nu_s\|_{TV}\, ds$$

we have that the two solutions must coincide. □

Remark 4.17. Existence of solutions for this case can be proven directly using a classical argument of iterative scheme (as done previously for the unbounded case).

The following theorem is an adaptation of part of [Nor99, Theorem 4.1]. Much more detail is provided here than in the original reference. To give the details, the author was much guided by an unpublished report [CGM+12] that studied the homogoneous Boltzmann equation with bounded kernels.

Theorem 4.18 (Mean-field limit for bounded jump kernel)**.** *Suppose that for a given measure μ_0 it holds that*

$$\langle \omega, X_0^n \rangle \leq \langle \omega, \mu_0 \rangle$$

and that as $n \to \infty$

$$X_0^n \to \mu_0 \qquad \textit{weakly}$$

Assume that the kernel is uniformly bounded

$$K \leq \Lambda < \infty.$$

Then the sequence $(X^n)_{t\geq 0}$ converges as $n \to \infty$ in probability in $D([0,\infty) \times \mathcal{P}(\mathbb{R}_+))$. Its limit, $(\mu_t)_{t\geq 0}$ is continuous and it satisfies the isotropic 4-wave kinetic equation (4.5)*. In particular, for all $f \in C_b(\mathbb{R}_+)$*

$$\begin{aligned}
\sup_{s\leq t} \langle f, X_t^n \rangle &\to \langle f, \mu_t \rangle, \\
\sup_{s\leq t} |M_s^{f,n}| &\to 0, \\
\sup_{s\leq t} \int_0^t \langle f, Q^{(n)}(X_s^n) \rangle \, ds &\to \int_0^t \langle f, Q(\mu_s) \rangle \, ds
\end{aligned}$$

all in probability. As a consequence, equation (4.5) *is obtained as the limit in probability of* (4.34) *as $n \to \infty$.*

Corollary 4.19 (Existence of solutions for the weak wave kinetic equation)**.** *There exists a solution for* (4.5) *(expressed as the limit of the X_t^n).*

Proof. We have that the limit $(\mu_t)_{t\geq 0}$ satisfies $\langle \omega, \mu_t \rangle \leq \langle \omega, \mu_0 \rangle$ by the following

$$\langle \omega \mathbb{1}_{\omega \leq k}, \mu \rangle = \lim_{n\to\infty} \langle \omega \mathbb{1}_{\omega \leq k}, X_t^n \rangle$$

and we have that

$$\langle \omega \mathbb{1}_{\omega \leq k}, X_t^n \rangle \leq \langle \omega, X_t^n \rangle \leq \langle \omega, \mu_0 \rangle.$$

So by making $k \to \infty$ we get the bound. □

4.3.2.2 Proof of Theorem 4.18

We want to take the limit in the martingale formulation (4.34). For that we will follow the following steps in [Nor99]:

(i) The martingale $(M^{n,f})_{n\in\mathbb{N}}$ converges uniformly in time for bounded sets to zero

$$\sup_{0\leq s\leq t} |M_s^{n,f}| \to 0 \qquad \text{in probability}$$

(Proposition 4.20).

(ii) Up to a subsequence $(X_t^n)_{n\in\mathbb{N}}$ converges weakly as $n\to\infty$ in $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+)$ (Proposition 4.21). This will be split in three steps:

(1) We will prove that the laws of the sequence $(\langle f, X_t^n\rangle)_{n\in\mathbb{N}}$ are tight in $D([0,\infty),\mathbb{R})$ (Lemma 4.22).

(2) From this deduce that actually the laws of the sequence $(X_t^n)_{n\in\mathbb{N}}$ itself is tight in $\mathcal{P}(D([0,\infty)\times\mathcal{P}(\mathbb{R}_+)))$ (Lemma 4.23).

(3) Finally use Prokhorov theorem to prove the statement.

(iii) Compute the limit of the trilinear term (Proposition 4.24). For this we will need to prove that:

(1) The limit of $(X_t^n)_{t\geq 0}$ as $n\to\infty$ is uniformly in compact sets of the t variable (Lemma 4.26). This will be a consequence of proving that the limit itself is continuous (Lemma 4.25).

(2) Prove that actually in the limit we can forget about the counting measure $X^{(n)}$ and consider just the product of the three measures $X(d\omega_1)X(d\omega_2)X(d\omega_3)$ (Lemma 4.27).

(iv) Using the uniqueness of the wave kinetic equation, we have that all the convergent subsequences converge to the same limit. Hence the whole sequence converges; if a tight sequence has every weakly convergent subsequence converging to the same limit, then the whole sequence converges weakly to that limit ([Bil13]).

(v) We have that the weak limit of $(X_t^n)_{n\in\mathbb{N}}$ satisfies the kinetic wave equation (4.5) so it is deterministic. Therefore, we actually have convergence in probability.

(vi) Finally, as an application of the functional monotone class theorem we can extend this result to functions $f\in\mathcal{B}(\mathbb{R}_+)$.

Step 1: control on the martingale

Proposition 4.20 (Martingale convergence)**.** *For any $f\in C_b(\mathbb{R}_+)$, $t\geq 0$*

$$\sup_{0\leq s\leq t}|M_s^{n,f}|\to 0 \qquad \textit{in } L^2(\mathbb{R})$$

in particular, it also converges in probability.

Proof of Proposition 4.20. We use Proposition 8.7 in [DN08] that ensures that

$$\mathbb{E}\left[\sup_{s\leq T}|M_s^{n,f}|^2\right] \leq 4\mathbb{E}\int_0^T \alpha^{n,f}(\mu_s)ds$$

as long as the right hand side is finite, where

$$\begin{aligned}\alpha^{n,f}(\mu_s) &= \frac{1}{2}\int_D \left(\frac{1}{n}(f(\omega_1+\omega_2-\omega_3)+f(\omega_3)-f(\omega_1)-f(\omega_2))\right)^2 \\ &\quad \times\frac{1}{n^2}K(\omega_1,\omega_2,\omega_3)(n\mu_s)^{(1)}(d\omega_1,d\omega_2,d\omega_3)\end{aligned} \tag{4.36}$$

(this statement is consequence of Doob's L^2 inequality). Therefore, using (4.33) we have that

$$\mathbb{E}\left[\sup_{s\leq t}|M_s^{n,f}|^2\right] \leq \frac{1}{n}32\|f\|_\infty^2\Lambda^2 t. \tag{4.37}$$

This implies the convergence of the supremum towards 0 in L^2 which implies also the convergence in probability. □

Step 2: convergence for the measures

Proposition 4.21 (Weak convergence for the measures)**.** *There exists a weakly convergent subsequence $(X_t^{n_k})_{k\in\mathbb{N}}$ in $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$ as $k\to\infty$.*

Lemma 4.22. *The sequence of laws of $(\langle f, X_t^n\rangle)_{n\in\mathbb{N}}$ on $D([0,\infty),\mathbb{R})$ is tight.*

Lemma 4.23. *The laws of the sequence $(X_t^n)_{n\in\mathbb{N}}$ on $D([0,\infty)\times\mathcal{P}(\mathbb{R}_+))$ is tight.*

Proof of Proposition 4.21. By Lemma 4.23 we know that the laws of the sequence $(X_t^n)_{n\in\mathbb{N}}$ are tight. This implies relative compactness for the sequence by Prokhorov's theorem. □

Proof of Lemma 4.22. We use Theorem 4.40. To prove the first part (i) of the Theorem we use that

$$|\langle f, X_t^n\rangle| = \left|\frac{1}{n}\sum_{i=1}^n f(\omega_t^{i,n})\right| \leq \frac{1}{n}\sum_{i=1}^n |f(\omega_t^{i,n})| \leq \|f\|_\infty$$

so for all $t\geq 0$, $\langle f, X_t^n\rangle \in [-\|f\|_\infty, \|f\|_\infty]$.

The second condition (ii) of the theorem will be consequence of the following inequalities:

$$\mathbb{E}\left[\sup_{r\in[s,t)}|M_r^{n,f}-M_s^{n,f}|^2\right] \leq \frac{1}{n}32\|f\|_\infty^2\Lambda^2(t-s) \tag{4.38}$$

and

$$\mathbb{E}\left[\sup_{r\in[s,t)}\left(\int_s^r \langle f, Q^{(n)}(X_p^n)\rangle\, dp\right)^2\right] \leq 16\|f\|_\infty^2\Lambda^2(t-s)^2. \tag{4.39}$$

which imply that

$$\mathbb{E}\left[\sup_{r\in[s,t)}|\langle f, X_r^n - X_s^n\rangle|^2\right] \leq A\left((t-s)^2 + \frac{(t-s)}{n}\right) \tag{4.40}$$

for some $A > 0$ depending only on $\|f\|_\infty$ and Λ.

First we use Markov's and Jensen's inequalities to get

$$\mathbb{P}(w'(\langle f, X^n\rangle, \delta, T) \geq \eta) \leq \frac{\mathbb{E}[w'(\langle f, X^n\rangle, \delta, T)]}{\eta} \leq \frac{\left(\mathbb{E}[w'(\langle f, X^n\rangle, \delta, T)^2]\right)^{1/2}}{\eta}.$$

(w' is defined in Theorem 4.40). Now, for a given partition $\{t_i\}_{i=1}^n$,

$$\sup_{r_1,r_2\in[t_{i-1},t_i)}|\langle f, X_{r_1}^n - X_{r_2}^n\rangle| \leq 2\sup_{r\in[t_{i-1},t_i)}|\langle f, X_r^n - X_{t_{i-1}}^n\rangle|.$$

Denote by i^* the point where the maximum on the right hand side is attained (the number of points in each partition is always finite). Now we want to consider a partition such that $\max_i|t_i - t_{i-1}| = \delta + \varepsilon$ for some $\varepsilon > 0$ so

$$w'(\langle f, X^n\rangle, \delta, T) \leq 2\sup_{r\in[t_{i^*-1},t_{i^*-1}+\delta+\varepsilon)}|\langle f, X_r^n - X_{t_{i^*-1}}^n\rangle| \quad a.s..$$

Therefore we are just left to check that

$$\mathbb{E}\left[\sup_{r\in[s,s+\delta+\varepsilon)}|\langle f, X_r^n - X_s^n\rangle|^2\right] \leq \frac{\eta^4}{2}$$

which is fulfilled thanks to the bound (4.40) by taking, for example,

$$\delta = \sqrt{1+\frac{\eta^4}{2A}} - 1 - \varepsilon$$

for ε small enough. □

Proof of Lemma 4.23. We will use Theorem 4.39 to prove this. To check condition (i), we consider the compact set $W \in \mathcal{P}(\mathbb{R}_+)$ (compact with respect to the topology induced by the weak convergence of measures) defined as

$$W := \left\{\tau \in \mathcal{P}(\mathbb{R}_+) \,:\, \int_{\mathbb{R}_+}\omega\,\tau(d\omega) \leq C\right\}.$$

Consider $(\mathcal{L}^n)_{n\in\mathbb{N}}$ the family of probability measures in $\mathcal{P}(D([0,\infty);W))$ which are the laws of $(X^n)_{n\in\mathbb{N}}$. We have that

$$\mathcal{L}^n(D([0,\infty);W) = 1 \quad \text{for all } n \in \mathbb{N}$$

by the conservation of the total energy and its boundedness (assumption (B1)):

$$\int_{\mathbb{R}_+} \omega X_t^n(d\omega) = \frac{1}{n}\sum_{i=0}^{n} \omega_t^{n,i} = \frac{1}{n}\sum_{i=0}^{n} \omega_0^{n,i} = \int_{\mathbb{R}_+} \omega \mu_0^n(d\omega) \leq C \quad \text{a.s..}$$

Now, to check condition (ii) we will use the family of continuous functions in $\mathcal{P}(\mathbb{R}_+)$ defined as

$$\mathbb{F} = \{F : \mathcal{P}(\mathbb{R}_+) \to \mathbb{R} : F(\tau) = \langle f, \tau\rangle \text{ for some } f \in C_b(\mathbb{R}_+)\}.$$

This family is closed under addition since $C_b(\mathbb{R}_+)$ is, it is continuous in $\mathcal{P}(\mathbb{R}_+)$, and separates points in $\mathcal{P}(\mathbb{R}_+)$: if $F(\tau) = F(\bar{\tau})$ for all $F \in \mathbb{F}$ then

$$\int_{\mathbb{R}_+} f(k) d(\tau - \bar{\tau})(k) = 0 \quad \forall f \in C_b(\mathbb{R}_+)$$

hence $\tau \equiv \bar{\tau}$.

So we are left with proving that for every $f \in C_b(\mathbb{R}_+)$ the sequence $\{\langle f, X^n\rangle\}_{n\in\mathbb{N}}$ is tight. This was proven in Lemma 4.22. □

Step 3: convergence for the trilinear term

Proposition 4.24 (Convergence for the trilinear term)**.** *It holds that*

$$\int_0^t \langle f, Q^{(n)}(X_s^{n_k})\rangle\, ds \to \int_0^t \langle f, Q(\mu_s, \mu_s, \mu_s)\rangle\, ds \quad \textit{weakly.}$$

Lemma 4.25 (Continuity of the limit)**.** *The weak limit of $(X_t^{n_k})_{t\geq 0}$ as $k \to \infty$ is continuous in time a.e..*

Lemma 4.26 (Uniform convergence)**.** *For all $f \in C_b(\mathbb{R}_+)$, it holds*

$$\sup_{s\leq t} |\langle f, X_s^{n_k} - \mu_s\rangle| \to 0 \quad \textit{weakly}$$

as $k \to \infty$.

Lemma 4.27. *It holds that*

$$\sup_{s\leq t} |\langle f, Q^{(n)}(X_s^{n_k}) - Q(\mu_s)\rangle| \to 0 \quad \textit{weakly}$$

as $k \to \infty$.

Proof of Proposition 4.24. By Lemma 4.27 we can pass the limit inside the integral in time. □

Proof of Lemma 4.25. We have that for any $f \in C_b(\mathbb{R}_+)$

$$|\langle f, X_t^{n_k}\rangle - \langle f, X_{t-}^{n_k}\rangle| \leq \frac{4}{n_k}\|f\|_\infty$$

applying Theorem 4.41 we get that $\langle f, \mu_t\rangle$ is continuous for any $f \in C_b(\mathbb{R}_+)$ and this implies the continuity of $(\mu_t)_{t\geq 0}$. □

Proof of Lemma 4.26. We know by Lemma 4.25 that the limit of $(X^{n_k})_{k\in\mathbb{N}}$ is continuous. The statement is consequence of the continuity mapping theorem in the Skorokhod space (proven using the Skorokhod representation theorem 4.38) and the fact that $g(X)(t) = \sup_{s\leq t}|X|$ is a continuous function in this space. □

Proof of Lemma 4.27. We abuse notation and denote by $(X_t^n)_{n\in\mathbb{N}}$ the convergent subsequence. We split the proof in two parts, we will prove for all $f \in C_b(\mathbb{R}_+)$:

(i) $\sup_{s\leq t}|\langle f, \left(Q - Q^{(n)}\right)(X_s^n)\rangle| \to 0$ as $n \to \infty$,

(ii) $\sup_{s\leq t}|\langle f, Q\left(X_s^n\right) - Q\left(\mu_s\right)\rangle| \to 0$ as $n \to \infty$.

(i) is consequence of

$$\begin{aligned} |\langle f, \left(Q - Q^{(n)}\right)(X_s^n)\rangle| &= \frac{1}{2}\frac{1}{n}\int_{2\omega_2\geq\omega_3}(f(2\omega_2-\omega_3)+f(\omega_3)-2f(\omega_2)) \\ &\qquad \times K(\omega_2,\omega_2,\omega_3)X_s^n(d\omega_2)X_s^n(d\omega_3) \\ &\leq \frac{2}{n}\|f\|_\infty\Lambda. \end{aligned} \tag{4.41}$$

Now, for (ii) we compute we have that

$$\begin{aligned} \sup_{s\leq t}|\langle f, Q(X_s^n) - Q(\mu_s)\rangle| \leq{}& \frac{1}{2}\int_D K(\omega_1,\omega_2,\omega_3)\,|f(\omega_1+\omega_2-\omega_3)+f(\omega_3)-f(\omega_1)-f(\omega_2)| \\ & \times \sup_{s\leq t}|X_s^n(d\omega_1)X_s^n(d\omega_2)X_s^n(d\omega_3) - \mu_s(d\omega_1)\mu_s(d\omega_2)\mu_s(d\omega_3)| \end{aligned} \tag{4.42}$$

We conclude (ii) with an argument analogous to Lemma 4.26 and the fact that

$$X_t^n \otimes X_t^n \otimes X_t^n \to \mu_t \otimes \mu_t \otimes \mu_t$$

weakly (consequence of Lévy's continuity theorem). □

4.3.2.3 Proof of Theorem 4.7 (unbounded kernel)

Remark 4.28. The proof that we already wrote in the case of bounded kernels works here in most parts substituting Λ by M where

$$\int_{\mathbb{R}_+} \omega X^n(d\omega) \leq M = \langle \omega, \mu_0 \rangle.$$

The only places where we need to be careful are Lemmas 4.26 and 4.27.

Lemma 4.29 (Convergence of a subsequence). *There exists a subsequence $(X_t^{n_k})_{k\in\mathbb{N}}$ that converges weakly in $D([0,\infty) \times \mathcal{P}(\mathbb{R}_+))$ as $k \to \infty$.*

Proof. The proof is exactly the one as in Section 4.3.2.2 and Proposition 4.21 using the bound on the jump kernel K, for example in the proof of Lemma 4.22, in the bounds of expressions (4.38) and (4.39), the value of Λ will be substituted by M^3. $\square$

Lemma 4.30. *For any $f \in C_b(\mathbb{R}_+)$, $t \geq 0$ it holds that*

$$\mathbb{E}\left[\sup_{s\leq t} |M_s^{n,f}|^2\right] \leq \frac{1}{n} 32 \|f\|_\infty^2 M^6 t.$$

Proof. The proof is the same one as in Proposition 4.20 using the bound on the jump kernel K. $\square$

Lemma 4.31. *It holds that for any $t \geq 0$*

$$\sup_{s\leq t} |\langle f, Q^{(n)}(X_s^n) - Q(\mu_s)\rangle| \to 0 \quad \textit{weakly}$$

for f continuous and of compact support.

Proof. Here everything works as in Section 4.3.2.2, but we need to find the bounds (4.41) and (4.42). We use a similar approach as in [Nor99].

Firstly, we will prove an analogous bound to (4.42).

Fix $\varepsilon > 0$ and define $p(\varepsilon) = \varepsilon^{-1/\gamma}$. Then for $\omega \geq p(\varepsilon)$ it holds

$$\frac{\tilde{\varphi}(\omega)}{\omega} \leq \varepsilon.$$

Now choose $\kappa \in (0, \gamma/[2(1-\gamma)])$. We split the domain into $F_1^p := \{(\omega_1, \omega_2, \omega_3) : \omega_1 \leq p^\kappa(\varepsilon), \omega_2 \leq p^\kappa(\varepsilon), \omega_3 \leq p^\kappa(\varepsilon)\}$ and F_2^p its complementary. In F_1^p the kernel is bounded and we have, with obvious notations,

$$\sup_{s\leq t} |\langle f, Q_1(X_s^n) - Q_1(\mu_s)\rangle| \to 0 \quad \text{weakly.}$$

On the other hand, in F_2^p, at least one of the components is greater than $p^\kappa(\varepsilon)$. Assume, without loss of generality that $\omega_3 \geq p^\kappa(\varepsilon)$. Then

$$\begin{aligned}
|\langle f, Q_2(X_t^n)\rangle| &= \left| \int_D \{f(\omega_1+\omega_2-\omega_3)+f(\omega_3)-f(\omega_1)-f(\omega_2)\} K(\omega_1,\omega_2,\omega_3) \right. \\
&\quad \left. \times X_t^n(d\omega_1)X_t^n(d\omega_2)X_t^n(d\omega_3) \right| \\
&\leq 4\|f\|_\infty \int_D \tilde{\varphi}(\omega_1)\tilde{\varphi}(\omega_2)\tilde{\varphi}(\omega_3)X_t^n(d\omega_1)X_t^n(d\omega_2)X_t^n(d\omega_3) \\
&\leq 4\|f\|_\infty \max\left\{ (p^\kappa(\varepsilon))^{2(1-\gamma)}\,\varepsilon\langle\omega,\mu_0\rangle, (p^\kappa(\varepsilon))^{1-\gamma}\,\varepsilon^2\langle\omega,\mu_0\rangle^2, \varepsilon^3\langle\omega,\mu_0\rangle^3 \right\} \\
&\leq c\varepsilon^\eta \qquad \text{for } \eta = 1-2\kappa(1-\gamma)/\gamma > 0.
\end{aligned}$$

and analogously

$$|\langle f, Q_2(\mu_t)\rangle| \leq c\varepsilon^\eta.$$

This implies that

$$\limsup_{n\to\infty} \sup_{s\leq t} |\langle f, Q_2(X_s^n) - Q_2(\mu_s)\rangle| \leq 2c\varepsilon^\eta$$

but ε is arbitrary so the limit is proved.

We are left with proving an analogous estimate to (4.41), which is obtained straightforwardly since we restrict ourselves to continuous functions of compact support. □

Proof of Theorem 4.7. Thanks to the previous Lemmas we know that there exists convergent subsequence $X_t^{n_k} \to \mu_t$ weakly as $k \to \infty$ such that

$$\langle f, \mu_t\rangle = \langle f, \mu_0\rangle + \int_0^t \langle f, Q(\mu_s)\rangle ds$$

for any f is continuous of compact support. Now using the bounds on the jump kernel and that $\langle\omega,\mu_t\rangle \leq \langle\omega,\mu_0\rangle$ and a limit argument, we can extend this equation to all bounded measurable functions f. □

4.3.3 Second result on mean-field limit

4.3.3.1 A coupling auxiliary process

Write

$$X_0^n = \frac{1}{n}\sum_{i=1}^n \delta_{\omega_i},$$

for $\omega_i \in \mathbb{R}_+$. Define for $B \subset \mathbb{R}_+$ bounded

$$X_0^{B,n} = \frac{1}{n}\sum_{i\,:\,\omega_i\in B}^n \delta_{\omega_i}.$$

Consider $\Lambda_0^{B,n}$ such that for each $B' \subset \mathbb{R}_+$ bounded such that $B \subset B'$ it holds

$$X_0^{B,n} \leq X_0^{B',n}, \quad \langle \varphi, X_0^{B,n} \rangle + \Lambda_0^{B,n} = \langle \varphi, X_0^{B',n} \rangle + \Lambda_0^{B',n}. \tag{4.43}$$

Set

$$\nu^B = (\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n} \langle \varphi, X_0^{B,n} \rangle - \frac{1}{n^2} \sum_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} \varphi(\omega_j)\varphi(\omega_k).$$

Note that ν^B decreases as B increases and $\nu^{[0,\infty)} = (\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n} \langle \varphi, X_0^{B,n} \rangle \geq 0$.

For $i < j$ take independent exponential random variables T_{ijk} of parameter $K(\omega_i, \omega_j, \omega_k)/n^2$. Set $T_{ijk} = T_{jik}$. Also, for $i \neq j$, take independent exponential random variables S_{ijk} of parameter $(\varphi(\omega_i)\varphi(\omega_j)\varphi(\omega_k) - K(\omega_i, \omega_j, \omega_k))/n^2$ (in all these cases we assume that $\omega_i + \omega_j \geq \omega_k$). We can construct, independently for each i, a family of independent exponential random variables S_i^B, increasing in B, with S_i^B having parameter $\varphi(\omega_i)\nu^B$.

Set

$$T_i^B = \min_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} (T_{ijk} \wedge S_{ijk}) \wedge S_i^B,$$

T_i^B is an exponential random variable of parameter

$$\frac{1}{n^2} \sum_{k,j\,:\,\omega_j \notin B \text{ or } \omega_k \notin B} \varphi(\omega_i)\varphi(\omega_j)\varphi(\omega_k) + \varphi(\omega_i)\nu^B = \varphi(\omega_i)\left((\Lambda_0^{B,n})^2 + 2\Lambda_0^{B,n} \langle \varphi, X_0^{B,n} \rangle \right).$$

For each B, the random variables

$$(T_{ijk}, T_i^B \,:\, i, j, k \text{ such that } \omega_i, \omega_j, \omega_k \in B,\ i < j)$$

form an independent family. Suppose that i is such that $\omega_i \in B$ and that j is such that $\omega_j \notin B$ or k is such that $\omega_k \notin B$, then we have

$$T_i^B \leq T_{ijk}$$

and for $B \subset B'$ and all i, we have (as a consequence of (4.43))

$$T_i^B \leq T_i^{B'}.$$

Now set

$$T = \left(\min_{i<j,k} T_{ijk} \right) \wedge \left(\min_i T_i^B \right).$$

We set $(X_t^{B,n}, \Lambda_t^{B,n}) = (X_0^{B,n}, \Lambda_0^{B,n})$ for $t < T$ and set

$$(X_T^{B,n}, \Lambda_T^{B,n}) = \begin{cases} (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i} - \frac{1}{n}\delta_{\omega_j} + \frac{1}{n}\delta_{\omega_k} + \frac{1}{n}\delta_{\omega_i+\omega_j-\omega_k}, \Lambda_0^{B,n}) \\ \quad \text{if } T = T_{ijk},\, \omega_i, \omega_j, \omega_k, \omega_i + \omega_j - \omega_k \in B, \\ \\ (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i} - \frac{1}{n}\delta_{\omega_j} + \frac{1}{n}\delta_{\omega_k}, \Lambda_0^{B,n} + \frac{1}{n}\varphi(\omega_i + \omega_j - \omega_k)) \\ \quad \text{if } T = T_{ijk},\, \omega_i, \omega_j, \omega_k \in B,\ \omega_i + \omega_j - \omega_k \notin B, \\ \\ (X_0^{B,n} - \frac{1}{n}\delta_{\omega_i}, \Lambda_0^{B,n} + \frac{1}{n}\varphi(\omega_i)), \quad \text{if } T = T_i^B,\, \omega_i \in B, \\ \\ (X_0^{B,n}, \Lambda_0^{B,n}), \quad \text{otherwise} \end{cases}$$

One can check that $X_T^{B,n}$ is supported on B and for $B \subset B'$

$$X_T^{B,n} \leq X_T^{B',n}, \qquad \langle \varphi, X_T^{B,n} \rangle + \Lambda_T^B = \langle \varphi, X_T^{B',n} \rangle + \Lambda_T^{B'}. \tag{4.44}$$

We repeat the above construction independently from time T, again and again to obtain a family of Markov processes $(X_t^{B,n}, \Lambda_t^{B,n})_{t \geq 0}$ such that (4.44) holds for all time.

Remark 4.32. Notice that $\Lambda_0^{B,n}$ and $X_0^{B,n}$ in the definition of ν^B must be updated to $\Lambda_T^{B,n}$ and $X_T^{B,n}$ in the new step.

For a bounded set $B \subset [0, \infty)$, we will consider

$$X_0^{B,n} = \mathbb{1}_B X_0^n, \quad \Lambda_0^{B,n} = \langle \varphi \mathbb{1}_{B^c}, X_0^n \rangle.$$

Markov Chain generator For all $F \in C_b(\mathcal{M}^B)$, $\mu \in \mathcal{M}^B$ we have

$$\begin{aligned} \mathcal{G}F(\mu, \lambda) &= \frac{n}{2} \int_D \{F(\mu^{\omega_1,\omega_2,\omega_3}, \lambda) - F(\mu, \lambda)\} \mathbb{1}_{\omega_1+\omega_2-\omega_3 \in B} K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \\ &+ \frac{n}{2} \int_D \left\{F\left(\hat{\mu}^{\omega_1,\omega_2,\omega_3}, \lambda^{\omega_1+\omega_2-\omega_3}\right) - F(\mu, \lambda)\right\} \mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B} K(\omega_1, \omega_2, \omega_3) \mu^{(n)}(d\omega_1, d\omega_2, d\omega_3) \\ &+ n \int_{\mathbb{R}_+} \{F(\mu^\omega, \lambda^\omega) - F(\mu, \lambda)\} \left(\lambda^2 + 2\lambda \langle \varphi, \mu \rangle\right) \varphi(\omega) \mu(d\omega) \end{aligned}$$

where

$$\mu^{\omega_1,\omega_2,\omega_3} = \mu + \frac{1}{n} \left(\delta_{\omega_3} + \delta_{\omega_1+\omega_2-\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right);$$

$$
\begin{aligned}
\hat{\mu}^{\omega_1,\omega_2,\omega_3} &= \mu + \frac{1}{n}\left(\delta_{\omega_3} - \delta_{\omega_1} - \delta_{\omega_2}\right); \\
\lambda^{\omega_1+\omega_2-\omega_3} &= \lambda + \frac{1}{n}\varphi(\omega_1+\omega_2-\omega_3); \\
\lambda^{\omega} &= \lambda + \frac{1}{n}\varphi(\omega); \\
\mu^{\omega} &= \mu - \frac{1}{n}\delta_\omega
\end{aligned}
$$

Associated martingale. Remember the definition

$$\mu^{(n)}(A\times B\times C) = \mu(A)\mu(B)\mu(C) - n^{-1}\mu(A\cap B)\mu(C)$$

which has the property $n^3\mu^{(n)} = (n\mu)^{(1)}$. Define for any bounded measurable function f on $\mathbb{R}_+$ and $a\in\mathbb{R}$:

$$
\begin{aligned}
L^{B,(n)}(\mu,\lambda)(f,a) &= \langle (f,a), L^{B,(n)}(\mu,\lambda)\rangle \\
&= \frac{1}{2}\int_{\mathbb{R}_+^3} \big(f(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + a\varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} \\
&\qquad +f(\omega_3) - f(\omega_1) - f(\omega_2)\big) K(\omega_1,\omega_2,\omega_3)\mu^{(n)}(d\omega_1,d\omega_2,d\omega_3) \\
&\quad + \left(\lambda^2 + 2\lambda\langle\varphi,\mu\rangle\right)\int_{\mathbb{R}_+} \left(a\varphi(\omega) - f(\omega)\right)\varphi(\omega)\mu(d\omega)
\end{aligned}
$$

and

$$
\begin{aligned}
P^{B,(n)}(\mu,\lambda)(f,a) &= \frac{1}{2n}\int_D \Big(f(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} + a\varphi(\omega_1+\omega_2-\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} \\
&\qquad +f(\omega_3) - f(\omega_1) - f(\omega_2)\Big)^2 K(\omega_1,\omega_2,\omega_3)\mu^{(n)}(d\omega_1,d\omega_2,d\omega_3) \\
&\quad + \left(\lambda^2 + 2\lambda\langle\varphi,\mu\rangle\right)\int_{\mathbb{R}_+} \left(a\varphi(\omega) - f(\omega)\right)^2\varphi(\omega)\mu(d\omega).
\end{aligned}
$$

Then, for all f and a

$$M_t^n = \langle f, X_t^{B,n}\rangle + a\Lambda_t^{B,n} - \langle f, X_0^{B,n}\rangle - a\Lambda_0^{B,n} - \int_0^t L^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds$$

is a martingale with previsible increasing process

$$\langle M\rangle_t = \int_0^t P^{B,(n)}(X_s^{B,n},\Lambda_s^{B,n})(f,a)\,ds.$$

4.3.3.2 Proof of Theorem 4.8

Remember the metric d in $\mathcal{M}^f$ defined around expression (4.11).

Proposition 4.33. *Let $B \subset [0,\infty)$ be bounded and μ_0 be measure on $\mathbb{R}_+$ such that $\langle \varphi, \mu_0\rangle < \infty$ and that*

$$\mu_0^{*n}(\partial B) = 0 \quad \textit{for all } n \geq 1.$$

Assume that for $\varphi(\omega) = \omega + 1$ it holds

$$K(\omega_1, \omega_2, \omega_3) \leq \varphi(\omega_1)\varphi(\omega_2)\varphi(\omega_3).$$

Consider $(\mu_t^B, \lambda_t^B)_{t\geq 0}$ the solution to (4.17) given by Proposition 4.10. Suppose that

$$d(X_0^{B,n}, \mu_0^B) \to 0, \quad |\Lambda_0^{B,n} - \lambda_0^B| \to 0$$

as $n \to \infty$. Then for all $t \geq 0$,

$$\sup_{s\leq t} d(X_s^{B,n}, \mu_s^B) \to 0, \quad \sup_{s\leq t} |\Lambda_s^{B,n} - \lambda_s^B| \to 0$$

in probability.

Proof of Proposition 4.33. Set $M = \sup_n \langle \varphi, X_0^{B,n}\rangle < \infty$. For all B and all continuous bounded functions f and all $a \in \mathbb{R}$

$$\begin{aligned} M_t^n &= \langle f, X_t^{B,n}\rangle + a\Lambda_t^{B,n} - \langle f, X_0^{B,n}\rangle - a\Lambda_0^{B,n} \\ &\quad - \int_0^t L^{B,(n)}(X_s^{B,n}, \Lambda_s^{B,n})(f,a)\, ds \end{aligned} \tag{4.45}$$

is a martingale with previsible increasing process

$$\langle M^n\rangle_t = \int_0^t P^{B,(n)}(X_s^{B,n}, \Lambda_s^{B,n})(f,a)\, ds,$$

(which is the analogous expression to (4.36)).

There is a constant $C < \infty$, depending only on B, Λ, φ, such that

$$|L^B(X_t^{B,n}, \Lambda_t^{B,n})(f,a)| \leq C(\|f\|_\infty + |a|) \tag{4.46}$$

$$|(L^B - L^{B,(n)})(X_t^{B,n}, \Lambda_t^{B,n})(f,a)| \leq Cn^{-1}(\|f\|_\infty + |a|), \tag{4.47}$$

$$|P^{B,(n)}(X_t^{B,n}, \Lambda_t^{B,n})(f,a)| \leq Cn^{-1}(\|f\|_\infty + |a|)^2, \tag{4.48}$$

where L^B is defined in expression (4.17).

Hence by the same argument as in Theorem 4.18, the laws of the sequence $(X^{B,n}, \Lambda^{B,n})$ are tight in $D([0,\infty), \mathcal{M}_B \times \mathbb{R})$ (inequality (4.48) is the analogous to (4.36); the inequality (4.46) is analogous to (4.39)).

Similarly, the laws of the sequence $(X^{B,n}, \Lambda^{B,n}, I^n, J^n)$ are tight in $D([0,\infty), \mathcal{M}_B \times$

$\mathbb{R} \times \mathcal{M}_{B\times B\times B} \times \mathcal{M}_{B\times B\times B})$, where

$$\begin{aligned}
I_t^n(d\omega_1, d\omega_2, d\omega_3) &= K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B} X_t^{B,n}(d\omega_1)X_t^{B,n}(d\omega_2)X_t^{B,n}(d\omega_3),\\
J_t^n(d\omega_1, d\omega_2, d\omega_3) &= K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B} X_t^{B,n}(d\omega_1)X_t^{B,n}(d\omega_2)X_t^{B,n}(d\omega_3).
\end{aligned}$$

Let (X, Λ, I, J) some weak limit point of the sequence. Passing to a subsequence and using the Skorokhod representation theorem 4.38, we can consider that the sequence converges almost surely, i.e., as a pointwise limit in $D([0,\infty), \mathcal{M}_B \times \mathbb{R} \times \mathcal{M}_{B\times B\times B} \times \mathcal{M}_{B\times B\times B})$. Therefore, there exist bounded measurable functions

$$I, J : [0,\infty) \times B \times B \times B \to [0,\infty)$$

symmetric in the first two components, such that

$$\begin{aligned}
I_t(d\omega_1, d\omega_2, d\omega_3) &= I(t,\omega_1,\omega_2,\omega_3)X_t(d\omega_1)X_t(d\omega_2)X_t(d\omega_3)\\
J_t(d\omega_1, d\omega_2, d\omega_3) &= J(t,\omega_1,\omega_2,\omega_3)X_t(d\omega_1)X_t(d\omega_2)X_t(d\omega_3)
\end{aligned}$$

in $\mathcal{M}_{B\times B\times B}$ and such that

$$\begin{aligned}
I(t,\omega_1,\omega_2,\omega_3) &= K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\in B}\\
J(t,\omega_1,\omega_2,\omega_3) &= K(\omega_1,\omega_2,\omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3\notin B}
\end{aligned}$$

whenever $\omega_1+\omega_2-\omega_3 \notin \partial B$ (notice that we assumed K to be continuous).

Now, passing to the limit in (4.45) we obtain, for all continuous functions f and all $a \in \mathbb{R}$, for all $t \geq 0$, almost surely

$$\begin{aligned}
\langle (f,a), (X_t, \Lambda_t)\rangle &= \langle (f,a), (X_0, \Lambda_0)\rangle\\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+^3} \big(f(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2)\big)\\
&\qquad \times I(s,\omega_1,\omega_2,\omega_3)X_s(d\omega_1)X_s(d\omega_2)X_s(d\omega_3)\, ds\\
&\quad + \frac{1}{2}\int_0^t \int_{\mathbb{R}_+^3} (a\varphi(\omega_1+\omega_2-\omega_3) + f(\omega_3) - f(\omega_1) - f(\omega_2))\\
&\qquad \times J(s,\omega_1,\omega_2,\omega_3)X_s(d\omega_1)X_s(d\omega_2)X_s(d\omega_3)\, ds\\
&\quad + \int_0^t \big(\Lambda_s^2 + 2\Lambda_s\langle\varphi, X_s\rangle\big)\int_{\mathbb{R}_+} (a\varphi(\omega) - f(\omega))\,\varphi(\omega)X_s(d\omega)\, ds.
\end{aligned}$$

Consider now an analogous iterative scheme to the one done in Proposition 4.10 for this equation. Denote by $(\nu_t^n)_{n\in\mathbb{N}}$ the sequence approximating $(X_t)_{t\geq 0}$. We deduce that

$$\nu_t^0 = \mu_0, \quad \nu_t^{n+1} \ll \mu_0 + \int_0^t (\nu_s^n + \nu_s^n * \nu_s^n * \hat{\nu}_s^n)\, ds$$

for $\hat{\nu}(A) = \nu(-A)$ and for all $n \geq 0$, (notice that we have extended the measures in the previous expression to the whole $\mathbb{R}$ by taking value 0 in subsets of $(-\infty, 0)$)[1].

By induction we have that

$$\nu_t^n \ll \gamma_0 = \sum_{k=1}^{\infty} \sum_{l=0}^{\infty} \nu_0^{*k} * \hat{\nu}_0^{*l}.$$

This implies in our case (taking $n \to \infty$) that $X_t \otimes X_t \otimes X_t$ is absolutely continuous with respect to $\gamma_0^{\otimes 3}$ for all $t \geq 0$, almost surely. For $G = \{(\omega_1, \omega_2, \omega_3) \,|\, \omega_1 + \omega_2 - \omega_3 \in \partial B\}$, we have that $\gamma_0^{\otimes 3}(G) = 0$ because of the assumptions on μ_0 and that $\gamma_0^{\otimes 3}(G) = (\gamma_0 * \gamma_0 * \hat{\gamma}_0)(G)$.

Therefore we can replace $I(t, \omega_1, \omega_2, \omega_3)$ by $K(\omega_1, \omega_2, \omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3 \in B}$ and $J(t, \omega_1, \omega_2, \omega_3)$ by $K(\omega_1, \omega_2, \omega_3)\mathbb{1}_{\omega_1+\omega_2-\omega_3 \notin B}$. Since the equation obtained after this substitution is the same as (4.17) and (μ_t^B, λ_t^B) is its unique solution, we conclude that the unique weak limit point of $(X^{B,n}, \Lambda^{B,n})$ in $D([0,\infty), \mathcal{M}_B \times \mathbb{R})$ is precisely $(\mu_t^B, \lambda_t^B)_{t \geq 0}$.

□

The proof of Theorem 4.8 is exactly the same one as in [Nor99, Theorem 4.4] and we copy it here just for the sake of completeness.

Proof of Theorem 4.8, from [Nor99]. Fix $\delta > 0$ and $t < T$. Since $(\mu_t)_{t<T}$ is strong, we can find a compact set B satisfying $\mu_0^{*n}(\partial B) = 0$([2]) for all $n \geq 1$ and such that $\lambda_t^B < \delta/2$. Now

$$d(\varphi X_0^n, \varphi \mu_0) \to 0,$$

so

$$d(X_0^{B,n}, \mu_0^B) \to 0, \quad |\Lambda_0^{B,n} - \lambda_0^B| \to 0.$$

Hence, by Proposition 4.33,

$$\sup_{s \leq t} d(X_s^{B,n}, \mu_s^B) \to 0, \quad \sup_{s \leq t} |\Lambda_s^{B,n} - \lambda_s^B| \to 0,$$

in probability as $n \to \infty$. Since $\{\mu_s^B : s \leq t\}$ is compact (the support of μ_s is contained in

[1]
$$\langle f, \nu * \nu * \hat{\nu} \rangle = \int_{\mathbb{R}^3} f(\omega_1 + \omega_2 - \omega_3)\nu(d\omega_1)\nu(d\omega_2)\nu(d\omega_3)$$

[2] The reason for this being true is that, for any given μ_0, $\mu_0^{*n}(\partial B) = 0$ for all $n \geq 1$ holds for all but countably many closed intervals in $\mathbb{R}_+$.

B)([3]), we also have

$$\sup_{s\leq t} d(\varphi X_s^{B,n}, \varphi\mu_s^B) \to 0$$

in probability as $n \to \infty$. By (4.26) and by the bounds on the instantaneous coagulation-fragmentation particle system (4.44), we have that for $s \leq t$

$$\begin{array}{rcl} \|\varphi(\mu_s - \mu_s^B)\| & = & \langle \varphi, \mu_s - \mu_s^B\rangle \leq \lambda_s^B \leq \lambda_t^B < \delta/2 \\ \|\varphi(X_s^n - X_s^{B,n})\| & = & \langle \varphi, X_s^n - X_s^{B,n}\rangle \leq \Lambda_s^{B,n} \leq \Lambda_t^{B,n} \\ & \leq & \lambda_t^B + |\Lambda_t^{B,n} - \lambda_t^B| \\ & \leq & \delta/2 + |\Lambda_t^{B,n} - \lambda_t^B|. \end{array}$$

Now (remember the properties of the metric d defined in (4.11))

$$\begin{array}{rcl} d(\varphi X_s^n, \varphi\mu_s) & \leq & \|\varphi(X_s^n - X_s^{B,n})\| + d(\varphi X_s^{B,n}, \varphi\mu_s^B) + \|\varphi(\mu_s - \mu_s^B)\| \\ & \leq & \delta + d(\varphi X_s^{B,n}, \varphi\mu_s^B) + |\Lambda_t^{B,n} - \lambda_t^B|, \end{array}$$

so

$$\mathbb{P}\left(\sup_{s\leq t} d(\varphi X_s^n, \varphi\mu_s) > \delta\right) \to 0$$

as $n \to \infty$, as required. □

4.4 Conclusions

In this work we have dealt with the weak isotropic 4-wave kinetic equation with simplified kernels. When the kernels are at most linear we have given conditions for the local existence and uniqueness of solutions. We have also derived the equation as a mean-field limit of interacting particle system given by a simultaneous coagulation-fragmentation: three particles interact with a coagulation-fragmentation phenomenon where one of the particles seem to act as a catalyst.

As we saw in the introduction, this theory can be applied to physical scenarios that include Langmuir waves, shallow water and waves on elastic plates. Moreover, using the interacting particle system, numerical methods can be devised to simulate the solution of the equation (as done by [Con09] for the 3-wave kinetic equation), by adapting the methods in [EW00].

Finally, these numerical simulations would allow the study of steady state solutions

[3] Remember the definition of the metric d given in (4.11). Since $d(X_s^{B,n}, \mu_s^B) \to 0$, we have that for all f bounded continuous function on $\mathbb{R}_+$

$$\int f\varphi(X_s^{B,n} - \mu_s^B) = \int f\varphi\mathbb{1}_B(X_s^{B,n} - \mu_s^B) \to 0$$

since φ restricted to B is also bounded and continuous.

and to check if they match the Kolmogorov-Zakharov spectra.

4.5 Appendix: Some properties of the Skorokhod space

Theorem 4.34 (Prohorov's theorem ([EK09]), Chapter 3)**.** *Let (S, d) be complete and separable, and let $\mathcal{M} \in \mathcal{P}(S)$. Then the following are equivalent:*

(i) *$\mathcal{M}$ is tight.*

(ii) *For each $\varepsilon > 0$, there exists a compact $K \in S$ such that*

$$\inf_{P \in \mathcal{M}} P(K^\varepsilon) \geq 1 - \varepsilon$$

where $K^\varepsilon := \{x \in S : \inf_{y \in K} d(x, y) < \varepsilon\}$.

(iii) *$\mathcal{M}$ is relatively compact.*

Let (E, r) be a metric space. The space $D([0, \infty); E)$ of cadlag functions taking values in E is widely used in stochastic processes. In general we would like to study the convergence of measures on this space, however, most of the tools known for convergence of measures are for measures in $\mathcal{P}(S)$ for S a complete separable metric space. Therefore, it would be very useful to find a topology in $D([0, \infty) \times E)$ such that it is a complete and separable metric space. This can be done when E is also complete and separable; and the metric considered is the Skorokhod one. This is why in this case the space of càdlàg functions is called Skorohod space.

Some important properties of this space are the following:

Proposition 4.35 ([EK09], Chapter 3)**.** *If $x \in D([0, \infty); E)$, then x has at most countably many points of discontinuity.*

Theorem 4.36 ([EK09], Chapter 3)**.** *If E is separable, then $D([0, \infty); E)$ is separable. If (E, r) is complete, then $(D([0, \infty); E), d)$ is complete, where d is the Skorokhod metric.*

Theorem 4.37. *The Skorokhod space is a complete separable metric space.*

Theorem 4.38 (The almost sure Skorokhod representation theorem, [EK09], Theorem 1.8, Chapter 3)**.** *Let (S, d) be a separable metric space. Suppose P_n, $n = 1, 2, \ldots$ and P in $\mathcal{P}(S)$ satisfy $\lim_{n \to \infty} \rho(P_n, P) = 0$ where ρ is the metric in $\mathcal{P}(S)$. Then there exists a probability space $(\Omega, \mathcal{F}, \nu)$ on which are defined S- valued random variable X_n, $n = 1, 2, \ldots$ and X with distributions P_n, $n = 1, 2, \ldots$ and P, respectively such that $\lim_{n \to \infty} X_n = X$ almost surely.*

Theorem 4.39 (Tightness criteria for measures on the Skorokhod space, [Jak86] Theorem 3.1)**.** *Let $(S, \mathcal{T})$ be a completely regular topological space with metrisable compact sets. Let $\mathbb{G}$ be a family of continuous functions on S. Suppose that $\mathbb{G}$ separates points in S and that it is closed* *under addition. Then a family $\{\mathcal{L}^n\}_{n\in\mathbb{N}}$ of probability measures in $\mathcal{P}(D([0,\infty);S)$ is tight iff the two following conditions hold:*

(i) For each $\varepsilon > 0$ there is a compact set $K_\varepsilon \subset S$ such that

$$\mathcal{L}^n(D([0,\infty);K_\varepsilon)) > 1-\varepsilon, \quad n \in \mathbb{N}.$$

(ii) The family $\{\mathcal{L}^n\}_{n\in\mathbb{N}}$ is $\mathbb{G}$-weakly tight.

Theorem 4.40 (Criteria for tightness in Skorokhod spaces ([EK09], Corollary 7.4, Chapter 3))**.** *Let (E,r) be a complete and separable metric space, and let $\{X_n\}$ be a family of processes with sample paths in $D([0,\infty);E)$. Then $\{X_n\}$ is relatively compact iff the two following conditions hold:*

(i) For every $\eta > 0$ and rational $t \geq 0$, there exists a compact set $\Lambda_{\eta,t} \subset E$ such that

$$\liminf_{n\to\infty} \mathbb{P}\{X_n(t) \in \Lambda^\eta_{\eta,t}\} \geq 1-\eta.$$

(ii) For every $\eta > 0$ and $T > 0$, there exits $\delta > 0$ such that

$$\limsup_{n\to\infty} \mathbb{P}\{w'(X_n,\delta,T) \geq \eta\} \leq \eta.$$

*where we have used the **modulus of continuity** w' defined as follows: for $x \in D([0,\infty)\times E)$, $\delta > 0$, and $T > 0$:*

$$w'(x,\delta,T) = \inf_{\{t_i\}} \max_i \sup_{s,t\in[t_{i-1},t_i)} r(x(s),x(t)),$$

where $\{t_i\}$ ranges over all partitions of the form $0 = t_0 < t_1 < \ldots < t_{n-1} < T \leq t_n$ with $\min_{1\leq i\leq n}(t_i - t_{i-1}) > \delta$ and $n \geq 1$

Theorem 4.41 (Continuity criteria for the limit in Skorokhod spaces ([EK09], Theorem 10.2, Chapter 3))**.** *Let (E,r) be a metric space. Let X_n, $n = 1,2,\ldots,$ and X be processes with sample paths in $D([0,\infty);E)$ and suppose that X_n converges in distribution to X. Then X is a.s. continuous if and only if $J(X_n)$ converges to zero in distribution, where*

$$J(x) = \int_0^\infty e^{-u}[J(x,u)\wedge 1]\,du$$

for

$$J(x,u) = \sup_{0\leq t\leq u} r(x(t),x(t-)).$$

4.6 Appendix: Formal derivation of the weak isotropic 4-wave kinetic equation

Suppose that $n(\mathbf{k}) = n(k)$ is a radial function (isotropic).

The waveaction in the isotropic case can be written as

$$W = \int_{\mathbb{R}^N} n(\mathbf{k})d\mathbf{k} = \int_{\mathbb{R}_+\times S^{N-1}} n(k)k^{N-1}dkd\mathbf{s} = \frac{|S^{N-1}|}{\alpha}\int_0^\infty n(\omega)\omega^{\frac{N-\alpha}{\alpha}}\,d\omega,$$

where S^{N-1} is the $N-1$ dimensional sphere. From this expression, one can denote the angle-averaged frequency spectrum $\mu = \mu(d\omega)$ as

$$\mu(d\omega) := \frac{|S^{N-1}|}{\alpha}\omega^{\frac{N-\alpha}{\alpha}}n(\omega)d\omega.$$

The total number of waves (waveaction) and the total energy are respectively

$$\begin{aligned} W &= \int_0^\infty \mu(d\omega) \\ E &= \int_0^\infty \omega\mu(d\omega). \end{aligned}$$

The isotropic version of the weak 4-wave kinetic equation can be written as

$$\mu_t = \mu_0 + \int_0^t Q(\mu_s,\mu_s,\mu_s)\,ds \tag{4.49}$$

where Q is defined against test functions $g \in \mathcal{S}(\mathbb{R}_+)$ as

$$\begin{aligned} \langle g, Q(\mu,\mu,\mu)\rangle &= \frac{1}{2}\int_D \mu(d\omega_1)\mu(d\omega_2)\mu(d\omega_3)K(\omega_1,\omega_2,\omega_3) \\ &\quad\times[g(\omega_1+\omega_2-\omega_3)+g(\omega_3)-g(\omega_2)-g(\omega_1)] \end{aligned} \tag{4.50}$$

where $D := \{\mathbb{R}_+^3 \cap (\omega_1+\omega_2 \geq \omega_3)\}$ and

$$\begin{aligned} K(\omega_1,\omega_2,\omega_3) &= \frac{8\pi}{\alpha|S^{N-1}|^4}(\omega_1+\omega_2-\omega_3)^{\frac{N-\alpha}{\alpha}} \\ &\quad \int_{(S^{N-1})^4} d\mathbf{s}_1 d\mathbf{s}_2 d\mathbf{s}_3 d\mathbf{s}\,\overline{T}^2(\omega_1^{1/\alpha}\mathbf{s}_1,\omega_2^{1/\alpha}\mathbf{s}_2,\omega_3^{1/\alpha}\mathbf{s}_3,(\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}) \\ &\quad \times\delta(\omega_1^{1/\alpha}\mathbf{s}_1+\omega_2^{1/\alpha}\mathbf{s}_2-\omega_3^{1/\alpha}\mathbf{s}_3-(\omega_1+\omega_2-\omega_3)^{1/\alpha}\mathbf{s}) \end{aligned} \tag{4.51}$$

Next we explain the formal derivation of the weak isotropic 4-wave kinetic equation

(4.5). We have that

$$\begin{aligned}
\int_{(0,\infty)} \partial_t \mu(\omega) d\omega &= \int_{\mathbb{R}^N} \partial_t n(\mathbf{k}) d\mathbf{k} \\
&= 4\pi \int_{\Omega^4 \times S^4} \overline{T}^2(k_1 s_1, k_2 s_2, k_3 s_3, ks) \\
&\qquad \times \delta(k_1 s_1 + k_2 s_2 - k_3 s_3 - ks)\delta(\omega_1 + \omega_2 - \omega_3 - \omega) \\
&\qquad \times (n_1 n_2 n_3 + n_1 n_2 n - n_1 n_3 n - n_2 n_3 n)(k k_1 k_2 k_3)^{N-1} dk ds \\
&= \frac{4\pi}{\alpha |S^{N-1}|^4} \int_{\mathbb{R}_+^4 \times S^4} d\omega_{0123} ds_{0123} T^2(\omega_1^{1/\alpha} s_1, \omega_2^{1/\alpha} s_2, \omega_3^{1/\alpha} s_3, \omega^{1/\alpha} s) \\
&\qquad \times \delta(\omega_1^{1/\alpha} s_1 + \omega_2^{1/\alpha} s_2 - \omega_3^{1/\alpha} s_3 - \omega^{1/\alpha} s)\delta(\omega_1 + \omega_2 - \omega_3 - \omega) \\
&\qquad \times (\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} + \mu(\omega_1)\mu(\omega_2)\mu(\omega)\omega_3^{\frac{N-\alpha}{\alpha}} \\
&\qquad - \mu(\omega_1)\mu(\omega_3)\mu(\omega)\omega_2^{\frac{N-\alpha}{\alpha}} - \mu(\omega_2)\mu(\omega_3)\mu(\omega)\omega_1^{\frac{N-\alpha}{\alpha}}) \\
&= \int_{\mathbb{R}_+^4} d\omega_{0123} F(\omega_1, \omega_2, \omega_3, \omega)\delta(\omega_1 + \omega_2 - \omega_3 - \omega) \\
&\qquad \times (\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} + \mu(\omega_1)\mu(\omega_2)\mu(\omega)\omega_3^{\frac{N-\alpha}{\alpha}} \\
&\qquad - \mu(\omega_1)\mu(\omega_3)\mu(\omega)\omega_2^{\frac{N-\alpha}{\alpha}} - \mu(\omega_2)\mu(\omega_3)\mu(\omega)\omega_1^{\frac{N-\alpha}{\alpha}})
\end{aligned}$$

for $S^i = (S^{N-1})^i$, $d\omega_{0123} = d\omega d\omega_1 d\omega_2 d\omega_3$, $ds_{0123} = ds_1 ds_2 ds_3 ds$, and

$$\begin{aligned}
F(\omega_1, \omega_2, \omega_3, \omega) &= \frac{4\pi}{\alpha |S^{N-1}|^4} \int_{S^4} ds_{0123} \overline{T}^2(\omega_1^{1/\alpha} s_1, \omega_2^{1/\alpha} s_2, \omega_3^{1/\alpha} s_3, \omega^{1/\alpha} s) \\
&\qquad \times \delta(\omega_1^{1/\alpha} s_1 + \omega_2^{1/\alpha} s_2 - \omega_3^{1/\alpha} s_3 - \omega^{1/\alpha} s).
\end{aligned}$$

Hence, μ_ω satisfies

$$\begin{aligned}
\partial_t \mu(\omega) &= \int_{\mathbb{R}_+^3} d\omega_{123} F(\omega_1, \omega_2, \omega_3, \omega)\delta(\omega_1 + \omega_2 - \omega_3 - \omega) && (4.52) \\
&\qquad \times (\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} + \mu(\omega_1)\mu(\omega_2)\mu(\omega)\omega_3^{\frac{N-\alpha}{\alpha}} && (4.53) \\
&\qquad - \mu(\omega_1)\mu(\omega_3)\mu(\omega)\omega_2^{\frac{N-\alpha}{\alpha}} - \mu(\omega_2)\mu(\omega_3)\mu(\omega)\omega_1^{\frac{N-\alpha}{\alpha}})
\end{aligned}$$

Its weak formulation

$$\mu_t = \mu^{in} + \int_{\Omega^3} Q(\mu_s, \mu_s, \mu_s)\, ds$$

is defined against functions $g \in \mathcal{S}(\mathbb{R}_+)$ as

$$\begin{aligned}
\langle g, Q(\mu, \mu, \mu) \rangle &= \int_{\mathbb{R}_+^4} d\omega_{0123} \mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} \\
&\qquad \times [F_{1230}\delta(\omega_{30}^{12}) g(\omega) + F_{1203}\delta(\omega_{03}^{12}) g(\omega_3)
\end{aligned}$$

$$
\begin{aligned}
&\qquad -F_{1032}\delta(\omega_{32}^{10})g(\omega_2) - F_{0231}\delta(\omega_{31}^{02})g(\omega_1)] \\
&= \int_{\mathbb{R}_+^4} d\omega_{0123}\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)\omega^{\frac{N-\alpha}{\alpha}} F_{1230}\delta(\omega_{30}^{12}) \\
&\qquad \times[g(\omega) + g(\omega_3) - g(\omega_2) - g(\omega_1)] \\
&= \frac{1}{2}\int_D d\omega_{123}\mu(\omega_1)\mu(\omega_2)\mu(\omega_3)K(\omega_1,\omega_2,\omega_3) \\
&\qquad \times[g(\omega_1+\omega_2-\omega_3) + g(\omega_3) - g(\omega_2) - g(\omega_1)]
\end{aligned} \tag{4.54}
$$

To conclude we assumed that $\overline{T}$ is symmetric in all its variables. We used that changing labels we get that

$$
\begin{aligned}
&d\omega_{123}F_{1230}\delta(\omega_{30}^{12})g(\omega) + F_{1203}\delta(\omega_{03}^{12})g(\omega_3) - F_{1032}\delta(\omega_{32}^{10})g(\omega_2) - F_{0231}\delta(\omega_{31}^{02})g(\omega_1) \\
&\quad = d\omega_{123}F_{1230}\delta(\omega_{30}^{12})g(\omega) + F_{1203}\delta(\omega_{03}^{12})g(\omega_3) - F_{3012}\delta(\omega_{12}^{30})g(\omega_2) - F_{0321}\delta(\omega_{21}^{03})g(\omega_1)
\end{aligned}
$$

and the properties of the function F to factorise it. We used the notation $\delta(\omega_{lp}^{ij}) = \delta(\omega_i + \omega_j - \omega_l - \omega_p)$ and

$$
K(\omega_1,\omega_2,\omega_3) := 2(\omega_1+\omega_2-\omega_3)^{\frac{N-\alpha}{\alpha}} F(\omega_1,\omega_2,\omega_3,\omega_1+\omega_2-\omega_3).
$$

For the last line we used the *sifting property* of the delta distribution i.e.

$$
\int_a^b f(t)\delta(t-d)\,dt = \begin{cases} f(d) & \text{for } d\in[a,b] \\ 0 & \text{otherwise} \end{cases}. \tag{4.55}
$$

Remark 4.42. In reference [ZLF92, Section 3.1.3], the authors state that even in isotropic medium, the interaction coefficient $\overline{T}$ in the 4-wave case cannot be considered to be isotropic too. In the 3-wave case it is possible, but not for the 4-wave. We can rewrite

$$
|\overline{T}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k})|^2 = \overline{T}_0^2 k^{2\beta} f_2\left(\frac{\mathbf{k}_1}{k},\frac{\mathbf{k}_2}{k},\frac{\mathbf{k}_3}{k}\right) \tag{4.56}
$$

for some dimensionless constant $\overline{T}_0$ and some dimensionless function f_2.

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