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Abstract

In this paper we propose to use Polya urn processes to model the emergence of conformity in an environment where people interact with each other sequentially and indirectly, through a common physical facility. Examples include rewinding video tapes, erasing blackboards, and switching headlights, etc. We find that a minimum amount of imitation is able to generate a maximum level of conformity. We then reinterpret the result in a group imitation setup, and show that as long as groups imitate each other with positive probabilities, they will end up with the same population composition, irrespective of the initial conditions, and the imitating probabilities.

JEL Classification Codes: C6, D7.

Keywords: Polya urn, conformity, emergence, social norms.

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1 Introduction

Everyday we give and receive numerous small courtesies. When we leave a building we hold the door for the person closely behind us. When we check out a video tape from a public library, we find that the tape is already rewound for us by the last borrower. When we arrive at a classroom to give a lecture, the blackboard is cleaned by the last instructor, and after we finish our lecture, we clean the board for whoever is the next instructor. In each of these examples, our everyday experience seems to suggest that there exists a prevalent pattern of behavior, or a norm, in which most people do the same thing most of the times. Such experience is so commonplace that we often take it for granted. But it is not obvious how a norm in these contexts can ever be established, for the following reasons. 1. There is no policing from a central authority. 2. People are anonymous. We do not know who the last borrower of a video tape is, or who is going to be the next instructor. We often interact with strangers, whom we do not even see. Hence a punishment/reward scheme that targets specific individuals is not feasible. 3. People behave very naively. The payoffs at stake are rather small, and people do not spend their scarce computational resources in petty things like whether I should erase the board. Instead, they simply respond to their past experience in some very mechanical way. For example, the more they observe other people do something in the past, the more likely they will do the same thing. 4. People are different in many ways. They come from different backgrounds, which endow them with different propensities to do one thing or another. They observe different things. They respond to their observations in different ways, some people may be more insulating from other people's influences, while other people might be more conforming. In one word, we have a population of simple-minded, heterogeneous, and anonymous agents who live in the absence of a regulating authority. How could order emerge in such an environment?

In this paper we propose to look at such problems through the lens of Polya urn processes. The simplest possible example of a Polya urn process is the following. Imagine an urn that contains a red ball and a blue ball initially. Then randomly draw a ball from the urn, look at its color, put it back together with another ball of the same color. Then repeat this procedure. The question is, does the fraction of, say, red balls, converge in the long run? If so, where does it converge to? It turns out that the fraction of red balls converges almost surely, but it could converge to any point in the unit interval with equal probability (Johnson and Kotz, 1977; Chung et al., 2003). Put it another way, if we run the process on 1,000 computers independently, then after a while we will find that all the processes will settle down, but the 1,000 processes will settle down around 1,000 different points in the unit interval.

The above example, simple as it is, possesses two interesting properties. 1. A definite pattern is able to emerge spontaneously. 2. Ex ante it is hard to

predict which pattern will emerge. We believe that these properties are also shared by the formation of a social norm. This is why we propose to use Polya urn processes to study the formation of norms.

In the economics literature, the Polya process has been applied to the problem of industry location (Arthur 1987). The idea is that there is an initial distribution of firms across different regions. Each firm is then equally likely to "spin off" new firms. New firms stay in their parent region. The limiting regional pattern of an industry is thus determined by initial conditions and historical random events. Recently, Skyrms and Pemantle (2000) model the dynamics of network formation as a Polya urn process. In their benchmark model a finite number of agents decide with whom to interact in each period. The more agent A has visited agent B in the past, the more likely A will visit B in the future. Formally, such a reinforcement scheme is reduced to a Polya urn process.

The general theme of all these applications is about the spontaneous emergence of order. This paper puts the theme in yet another concrete context. Consider a finite number of people interacting with each other sequentially and indirectly, through a common physical facility, a blackboard, for example. Each agent observes what the immediate predecessor does, and then chooses her own action according to her experience in the past, which consists of her observation of other people's choices, and her own choices. We assume that agents behave very naively. They simply respond to their past experience in a monotonic and probabilistic way. Our concern is whether a pattern will emerge in such a system, and if so, what kind of pattern will emerge?

The rest of the paper is organized as follows. Section 2 introduces the model. Section 3 presents the main result of conformity, which is reinterpreted in a group imitation setup in Section 4. Section 5 concludes.

2 The Model

There are I players. Each player has an urn, with unlimited capacity. Initially player i has $R_i > 0$ red balls and $B_i > 0$ blue balls. Let $N_i = R_i + B_i$ be the total number of balls in player i 's urn initially. The I players make decisions sequentially and repeatedly. The order of moves is fixed to be $1, 2, \dots, I, 1, 2, \dots, I$, and so on. Each player observes the decision made by his immediate predecessor. There are two actions to choose, the red action and the blue action. When it is player i 's turn to move, player i chooses the red action with probability identical to the fraction of red balls in his urn, and the blue action with the remaining probability. If player i chooses the red action, then he adds $\gamma_i \geq 0$ red balls to his urn, and player $i + 1$ adds $\gamma'_{i+1} > 0$ red balls to her urn. The rules are the same if player i chooses the blue action.

The model has a simple interpretation. We think of each player's urn as his "memory". The red balls and blue balls correspond to his experiences with the red action and the blue action. The initial configuration (R_i, B_i) represents player i 's "prior", or background. Without any experience of other people's choices or his own choices, player i will play the red action with probability $\frac{R_i}{N_i}$. The parameters γ_i and γ'_i try to capture the idea of habit formation and imitation, respectively. Thus if player i chooses the red action, he reinforces this experience by adding γ_i red balls to his urn. At the same time, player i 's choice is observed by player $i + 1$, hence player $i + 1$ reinforces this experience by adding γ'_{i+1} red balls to her urn. We allow γ_i to be zero, but require γ'_i to be strictly positive, so that the society is minimally "connected" in some sense, nobody is completely autistic.

Our model is rather special in several ways. First, the order of moves is fixed. This may be appropriate for the erasing blackboard example, but not appropriate for the video tape example, where borrowers arrive randomly, and some borrowers may arrive more frequently than others. Second, there is no "forgetting" in our model. What happens a long time ago has the same impact on one's behavior as what happens most recently. In reality, however, people tend to discount what they do or what they observe in the distant past relative to their recent experience. Third, the two actions are symmetric. There are no preferences and no intrinsic benefits or costs attached to them. This is again, oversimplifying. It is certainly costly to erase the board after a long lecture, and some people certainly derive some satisfaction from doing this anyway. Realism is compromised in exchange for analytical tractability. With the above simplifications, we show in the next section that in the long run, the players' behavior in terms of the probability with which to choose a certain action must be the same, even if they come from different backgrounds, and the ways they form habit or imitate others are different, so long as the society is minimally "connected". Thus the weakest "connectivity" suffices to generate the strongest "conformity".

3 Conformity

For ease of exposition, let one round be such that everybody has moved once, and the balls have been added to the urns according to the rules of the model. We illustrate this by an example in which $I = 3$. In the first round, player 1 moves first. Suppose player 1 chooses the red action, then he adds γ_1 red balls to his own urn, and γ'_2 red balls to player 2's urn. Now player 2 moves, suppose player 2 chooses the blue action, then he adds γ_2 blue balls to his own urn, and γ'_3 blue balls to player 3's urn. Then player 3 moves and chooses, say, the blue action, as a result player 3 adds γ_3 blue balls to his own urn, and γ'_1 blue balls to player 1's urn. This completes the first round. Hence after the first round,

player i has $N_i + \gamma_i + \gamma'_i$ balls in his urn. In general, after the n th round, player i has $N_i + n(\gamma_i + \gamma'_i)$ balls in his urn.

For $i = 1, 2, \dots, I$, let x_n^i denote the fraction of red balls in i 's urn at the beginning of the n th round. Let $x_n = (x_n^i)_{i=1}^I$. We are interested in the limiting behavior of x_n . Does it converge? If so, where does it converge to? The following proposition answers these questions.

Proposition 1 *There exists a random vector x , such that (a) x_n converges to x almost surely, and (b) The support of x is contained in the diagonal of $[0, 1]^I$.*

Proof: See the Appendix. ■

The proposition says two things. First, the process always settles down. The sample paths along which the process diverges can be neglected. Second, the probabilities with which people choose a certain action are the same in the limit, even if different people have different background and different γ and γ' 's. We are not able to show that the support of the limiting distribution is actually *identical* to the diagonal, not just contained in it. But computer simulations suggest that this seems to be true. We report a simulation result below to illustrate Proposition 1.

PUT FIGURE 1 AND FIGURE 2 HERE.

Both simulations begin with the same parameters: $I = 3$, $\gamma_1 = 6$, $\gamma_2 = 5$, $\gamma_3 = 10$, $\gamma'_1 = 4$, $\gamma'_2 = 3$, $\gamma'_3 = 5$, $R_1 = 5$, $R_2 = 2$, $R_3 = 1$, $B_1 = 3$, $B_2 = 8$, and $B_3 = 7$. In both cases the system converge, but it converges to two different points.

The proof of Proposition 1 is based on a theorem in Arthur, Ermoliev, and Kaniovski (1984), hereafter referred to as the AEK theorem. Arthur, Ermoliev, and Kaniovski (1984) studies the following problem.

Consider an urn containing balls of I colors. Initially there are γ balls in the urn. At the beginning of round n , a ball of color i is added to the urn with probability $q_n^i(x_n)$, where $x_n = (x_n^i)_{i=1}^I$ is the vector that summarizes the fraction of each color at the beginning of round n . After some manipulation, the law of motion of the process $(x_n)_n$ can be written as follows.

$$x_{n+1}^i = x_n^i + \frac{1}{n + \gamma} (q_n^i(x_n) - x_n^i) + \frac{1}{n + \gamma} (\beta_n^i(x_n) - q_n^i(x_n)), \quad (1)$$

where

$$\beta_n^i(x_n) = \begin{cases} 1 & \text{with prob. } q_n^i(x_n) \\ 0 & \text{with prob. } 1 - q_n^i(x_n) \end{cases} .$$

Let S denote the simplex contained in $[0, 1]^I$, Arthur, Ermoliev, and Kaniovski (1984) prove the following theorem.

Theorem (Arthur, Ermoliev, and Kaniovski (1984)) Let $\{q_n\}$ be continuous functions. If there exists a continuous function $q : S \rightarrow S$, a sequence of constants $\{a_n\}$, and a function $v : S \rightarrow \mathbb{R}$, such that

- (a) $\sup_{x \in S} \|q_n(x) - q(x)\| \leq a_n$, and $\sum_{n=1}^{\infty} a_n/n < \infty$.
- (b) $B = \{x \in S | q(x) = x\}$ contains a finite number of components.
- (c) (i) v is twice differentiable.
(ii) $v(x) \geq 0, \forall x \in S$.
(iii) $\langle q(x) - x, v_x(x) \rangle < 0, \forall x \in S \setminus B$.

Then x_n converges to a point in B or to the border of a connected component.

It turns out that the problem of I colors with one urn is not that different from the problem of two colors with I urns. In the proof of Proposition 1, We first write the problem in the form as in the AEK theorem, then we prove that the conditions required by the theorem are met in our problem. In particular, we construct a Lyapunov function v required by the AEK theorem.

Proposition 1 suggests that an arbitrary level of imitation suffices to generate the highest level of conformity, but it says nothing about where the system conforms to. However, if one of the players, say player 1, is a confederate seeded by some outside planner (e.g., the school headmaster who would like all her teachers to erase the board), and player 1 always plays the red action, then applying the same techniques as in Proposition 1, one can show that eventually everybody conforms to the red action.

4 Group Imitation

In this section we explore a different interpretation of the conformity result in the previous section. Instead of having individuals interacting with each other, and each individual ending up with the same probabilities to choose certain actions, we now consider groups interacting with each other, and ask whether all the groups end up with the same population composition, i.e. whether the fractions of the population who choose certain actions are the same across all groups.

We then examine the limiting distribution of the population composition and compare it with that of the isolated groups.

Imagine there are I groups, located on a circle. Initially, group i has R_i people who choose the red action, and B_i people who choose the blue action. In each period, a new person arrives at each and every group. Consider the person who arrives at group i . She samples a group at random, then turns into the same color as the person she samples and becomes a member of group i . Suppose she samples group i with probability γ_i , group $i - 1$ with probability γ'_i , and group $i + 1$ with probability γ''_i , where $\gamma_i + \gamma'_i + \gamma''_i = 1$. Assume for simplicity that the new arrivals do not interfere with each other, i.e. new arrivals are never sampled by other new arrivals before they become members of a group, and two new arrivals could sample the same person.

The first question we ask is, is the population composition the same across all groups in the long run? We first provide the answer when $I = 2$, then we partially generalize the result to any number of groups.

Let x_n^i denote the fraction of red people in group i at the beginning of period n . Let $x_n = (x_n^i)_i$.

Proposition 2 *For all $B_i > 0$, $R_i > 0$, $\gamma_i > 0$, $\gamma'_i > 0$, $i = 1, 2$, x_n converges almost surely. In the limit, the two groups have the same population composition.*

Proof: See the Appendix. ■

When there are more than two groups, constructing a Lyapunov function is not so easy. We have the following partial extension of the two group result.

Proposition 3 *Fix any $(B_i, R_i)_{i=1}^I$. If for all i , $\gamma''_i = \gamma'_{i+1}$, then all groups will have the same population composition in the limit with probability one.*

Proof: See the Appendix. ■

If we interpret red people as, for example, religious people, and blue people as atheists, then the above propositions predict that geographically neighboring areas should have similar proportion of religious people relative to atheists. But how is the common proportion distributed? We do not have a closed form solution, but we report some simulation findings here. For simplicity, assume that there are only two groups and $B_i = R_i = 1$, for $i = 1, 2$. If the two groups are isolated, namely $\gamma_1 = \gamma_2 = 1$, then we come back to the standard Polya urn processes, and we know that the composition in the two groups are *i.i.d.* uniform over $[0, 1]$. Now imagine the two isolated groups merge into one large group, with 2 red people and 2 blue people initially. We know that the limiting distribution of the population composition in the large group is the beta

distribution with $\alpha = \beta = 2$ (Johnson and Kotz (1977)). The question is, what is the limiting distribution when two small groups interact with each other? If $\gamma_1 = \gamma_2 = \frac{1}{2}$, then the two small group case is similar to the one large group case, with only one difference: in the two group case, two people arrive at a time and they do not interfere with each other; in the one group case, one person arrives at a time. Intuitively, the distribution in the two group case should be less scattered than $\text{beta}(2, 2)$, because after any history, extreme outcomes realize with smaller probability in the two group case. Simulation suggests that there is nothing special about γ_1 and γ_2 being $\frac{1}{2}$. As long as they are *equal*, the limiting distribution is the same. However, we do not have a proof. We report the simulation result below, and state the conjecture.

In Figure 3, we simulate two interacting groups, imitating each other with equal probabilities, ranging from 1 to 0.25, and compare the frequency distributions with the beta distribution $\beta(2, 2)$. All the four distributions seem to be more centered than the beta distribution, but it is hard to distinguish among themselves.

Let $\lambda(\gamma)$ denote the limiting distribution of of the common composition of the two groups, which imitate each other with probabilities $\gamma_1 = \gamma_2 = \gamma$.

Conjecture: $\lambda(\gamma)$ *does not depend on* γ .

PUT FIGURE 3 HERE.

5 Conclusion

In this paper we first propose to use Polya urn models to study the conformity of individual behavior. We show that conformity is able to emerge spontaneously at the aggregate level from very simple behavioral rules at the individual level. The emerging behavioral pattern is however, impossible to predict *ex ante*. Random events that happened early on play an important role in shaping future outcomes. We then reinterpret the conformity result in a group imitation setup. We show that as long as groups imitate each other with positive probabilities, they will end up with the same population composition, irrespective of the initial conditions, and how small the imitating probabilities are.

The model developed here can potentially serve as a building block to consider more interesting problems. There are no payoffs in the current model, and no intrinsic differences between the red action and the blue action. Suppose the red action is intrinsically better than the blue action, and an agent is not reinforced by how many times he chooses one action, but how many times he chooses one action with a satisfactory payoff. Does imitation prevent the agents

from learning which action is better? Another element that is missing in the model is forgetting. In our model, things happening in the distant past have the same influence over decision making as things happening recently. A more plausible assumption is that people discount their past experience by a discount factor δ . For example, a ball added ten periods ago is counted as δ^{10} balls, compared to a ball added today. If people forget, and if they forget at different rates, do we still have the conformity result? If so, what does the support of the limiting distribution look like? In the one person case, once discounting is added, the person will always settle down with either action deterministically. Only extreme points remain in the support (Skyrms and Pemantle (2000)). Is this also true in our model?

6 Appendix

Proof of Proposition 1: Fix n and x_n . Fix $i \in \{1, 2, \dots, I\}$. Let $\Delta_i(n, x_n)$ denote the number of red balls added to i 's urn in round n , given x_n being the fractions of red balls at the beginning of round n . Then,

$$\begin{aligned} x_{n+1}^i &= \left(\frac{1}{N_i + n(\gamma_i + \gamma'_i)} \right) (x_n^i (N_i + (n-1)(\gamma_i + \gamma'_i)) + \Delta_i(n, x_n)) \\ &= x_n^i + \left(\frac{1}{N_i + n(\gamma_i + \gamma'_i)} \right) (-x_n^i (\gamma_i + \gamma'_i) + \Delta_i(n, x_n)). \end{aligned}$$

By definition,

$$\Delta_i(n, x_n) = \begin{cases} 0 & \text{with prob. } P((i-1)B \cap iB | n, x_n) \\ \gamma_i & \text{with prob. } P((i-1)B \cap iR | n, x_n) \\ \gamma'_i & \text{with prob. } P((i-1)R \cap iB | n, x_n) \\ \gamma_i + \gamma'_i & \text{with prob. } P((i-1)R \cap iR | n, x_n). \end{cases}$$

where $P((i-1)B \cap iB | n, x_n)$ is the probability that player $i-1$ chooses the blue action and player i also chooses the blue action, conditional on the round being the n th round and the fractions of red balls at the beginning of round n being x_n . The other probabilities have similar interpretations.

Let

$$\beta_n^i(x_n) := \frac{1}{\gamma_i + \gamma'_i} \Delta_i(n, x_n).$$

Then

$$x_{n+1}^i = x_n^i + \left(\frac{\gamma_i + \gamma'_i}{N_i + n(\gamma_i + \gamma'_i)} \right) (\beta_n^i(x_n) - x_n^i).$$

Let

$$q_n^i(x_n) := \frac{1}{\gamma_i + \gamma'_i} E(\Delta_i(n, x_n)),$$

where $E(\Delta_i(n, x_n))$ denote the expectation of $\Delta_i(n, x_n)$, conditional on fixed n and x_n .

Then

$$x_{n+1}^i = x_n^i + \left(\frac{\gamma_i + \gamma'_i}{N_i + n(\gamma_i + \gamma'_i)} \right) (q_n^i(x_n) - x_n^i) + \left(\frac{\gamma_i + \gamma'_i}{N_i + n(\gamma_i + \gamma'_i)} \right) (\beta_n^i(x_n) - q_n^i(x_n)). \quad (2)$$

Definition: Let $\{f_n\}_n$ be a sequence of functions, each mapping $[0, 1]^I$ into $[0, 1]^I$. Let $\{f_n\}_n$ have a pointwise limit f . We say that f_n converges to f *reasonably rapidly* if there exists a sequence of positive constants $\{a_n\}$, such that $\sup_{x \in [0, 1]^I} \|f_n(x) - f(x)\| \leq a_n$, and $\sum_{n=1}^{\infty} a_n/n < \infty$.

Lemma 1 (1) For any i , for any $x \in [0, 1]^I$,

$$q_n^i(x) \longrightarrow q^i(x) = \frac{\gamma_i}{\gamma_i + \gamma'_i} x_i + \frac{\gamma'_i}{\gamma_i + \gamma'_i} x_{i-1},$$

and (2) q_n converges to q reasonably rapidly, where $q_n(\cdot) := (q_n^i(\cdot))_i$, and $q(\cdot) := (q^i(\cdot))_i$.

Proof of Lemma 1:

(1) By definition of $q_n^i(x)$, it suffices to show that

$$E\Delta_i(n, x) \longrightarrow \gamma_i x_i + \gamma'_i x_{i-1}.$$

Consider the conditional probability $P((i-1)B \cap iB | n, x)$. If $i = 1$, then we show that

$$P(IB \cap 1B | n, x) \longrightarrow (1 - x_1)(1 - x_I).$$

Let \underline{IB} denote the event that "out of $N_I + (n-1)(\gamma_I + \gamma'_I) + \gamma'_I$ balls, in which $x_I(N_I + (n-1)(\gamma_I + \gamma'_I)) + \gamma'_I$ balls are red, a blue ball is chosen in I 's urn".

Let \overline{IB} denote the event that "out of $N_I + (n-1)(\gamma_I + \gamma'_I) + \gamma'_I$ balls, in which $x_I(N_I + (n-1)(\gamma_I + \gamma'_I))$ balls are red, a blue ball is chosen in I 's urn". Then,

$$P(\underline{IB})P(1B|n, x) \leq P(IB \cap 1B|n, x) \leq P(\overline{IB})P(1B|n, x).$$

But

$$P(1B|n, x) = 1 - x_1,$$

$$P(\underline{IB}) = 1 - \frac{x_I(N_I + (n-1)(\gamma_I + \gamma'_I)) + \gamma'_I}{N_I + (n-1)(\gamma_I + \gamma'_I) + \gamma'_I},$$

and

$$P(\overline{IB}) = 1 - \frac{x_I(N_I + (n-1)(\gamma_I + \gamma'_I))}{N_I + (n-1)(\gamma_I + \gamma'_I) + \gamma'_I}.$$

Hence

$$P(\underline{IB}) \longrightarrow 1 - x_I,$$

and

$$P(\overline{IB}) \longrightarrow 1 - x_I.$$

By the sandwich theorem,

$$P(IB \cap 1B|n, x) \longrightarrow (1 - x_1)(1 - x_I).$$

Similarly,

$$P(IB \cap 1R|n, x) \longrightarrow x_1(1 - x_I),$$

$$P(IR \cap 1B|n, x) \longrightarrow (1 - x_1)x_I,$$

and

$$P(IR \cap 1R|n, x) \longrightarrow x_1x_I.$$

Therefore,

$$E\Delta_1(n, x) \longrightarrow \gamma_1x_1 + \gamma'_1x_I.$$

If $i > 1$, then we show that

$$P((i-1)B \cap iB|n, x) \longrightarrow (1 - x_{i-1})(1 - x_i).$$

Let $(i-1)B$ denote the event that "out of $N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}) + \gamma'_{i-1}$ balls, in which $x_{i-1}(N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1})) + \gamma'_{i-1}$ balls are red, a blue ball is chosen in $i-1$'s urn".

Let $\overline{(i-1)B}$ denote the event that "out of $N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}) + \gamma'_{i-1}$ balls, in which $x_{i-1}(N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}))$ balls are red, a blue ball is chosen in $i-1$'s urn".

Similarly, Let \underline{iB} denote the event that "out of $N_i + (n-1)(\gamma_i + \gamma'_i) + \gamma'_i$ balls, in which $x_i(N_i + (n-1)(\gamma_i + \gamma'_i)) + \gamma'_i$ balls are red, a blue ball is chosen in i 's urn".

Let \overline{iB} denote the event that "out of $N_i + (n-1)(\gamma_i + \gamma'_i) + \gamma'_i$ balls, in which $x_i(N_i + (n-1)(\gamma_i + \gamma'_i))$ balls are red, a blue ball is chosen in i 's urn".

Then

$$P(\underline{(i-1)B}) P(\underline{iB}) \leq P((i-1)B \cap iB | n, x) \leq P(\overline{(i-1)B}) P(\overline{iB}).$$

But

$$P(\underline{(i-1)B}) = 1 - \frac{x_{i-1}(N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1})) + \gamma'_{i-1}}{N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}) + \gamma'_{i-1}},$$

$$P(\overline{(i-1)B}) = 1 - \frac{x_{i-1}(N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}))}{N_{i-1} + (n-1)(\gamma_{i-1} + \gamma'_{i-1}) + \gamma'_{i-1}},$$

$$P(\underline{iB}) = 1 - \frac{x_i(N_i + (n-1)(\gamma_i + \gamma'_i)) + \gamma'_i}{N_i + (n-1)(\gamma_i + \gamma'_i) + \gamma'_i},$$

and

$$P(\overline{iB}) = 1 - \frac{x_i(N_i + (n-1)(\gamma_i + \gamma'_i))}{N_i + (n-1)(\gamma_i + \gamma'_i) + \gamma'_i}.$$

Hence

$$\begin{aligned} P(\underline{(i-1)B}) &\longrightarrow 1 - x_{i-1}, \\ P(\overline{(i-1)B}) &\longrightarrow 1 - x_{i-1}, \\ P(\underline{iB}) &\longrightarrow 1 - x_i, \\ P(\overline{iB}) &\longrightarrow 1 - x_i. \end{aligned}$$

Again, by the sandwich theorem,

$$P((i-1)B \cap iB | n, x) \longrightarrow (1 - x_{i-1})(1 - x_i).$$

Similarly,

$$P((i-1)B \cap iR | n, x) \longrightarrow (1 - x_{i-1})x_i,$$

$$P((i-1)R \cap iB | n, x) \longrightarrow x_{i-1}(1-x_i),$$

and

$$P((i-1)R \cap iR | n, x) \longrightarrow x_{i-1}x_i.$$

Therefore

$$E\Delta_i(n, x) \longrightarrow \gamma_i x_i + \gamma'_i x_{i-1}.$$

(2) In order to show that the conditional expectations converge reasonably rapidly, it suffices to show that the conditional probabilities converge reasonably rapidly. But by the proof of part (1), it is easy to see that, for example, $P(\underline{IB})$ converges to $1 - x_I$ reasonably rapidly, hence it is easy to show, again by a sandwich argument, that $P(IB \cap 1B | n, x)$ converges to $(1 - x_1)(1 - x_I)$ reasonably rapidly. This completes the proof of Lemma 1.

The proof of the AEK theorem is based on Theorem 7.3 in Nevelson and Hasminskii (1976). Nowhere in the proof of the AEK theorem requires that the q_n functions be probability functions. Hence the AEK theorem still holds if we replace S by $[0, 1]^I$ everywhere in the statement of the theorem. By Lemma 1, condition (a) of the AEK theorem is satisfied. Also by Lemma 1,

$$q(x) - x = \begin{pmatrix} l_1(x_I - x_1) \\ \vdots \\ l_I(x_{I-1} - x_I) \end{pmatrix},$$

where $l_i = \frac{\gamma'_i}{\gamma_i + \gamma'_i}$.

Hence $B = \text{diagonal of } [0, 1]^I$, and B contains only one component, which is B itself. Moreover, the boundary of B is also B itself, since every neighborhood of every point along the diagonal contains both points in B and points not in B . Therefore, to establish the proposition it suffices to construct a Lyapunov function v .

Before we give the general construction, we illustrate the idea in a simple example, where $I = 3$, and $\gamma_i = \gamma'_i = 1$, $i = 1, 2, 3$.

In this case

$$q(x) - x = \begin{pmatrix} \frac{1}{2}(x_3 - x_1) \\ \frac{1}{2}(x_1 - x_2) \\ \frac{1}{2}(x_2 - x_3) \end{pmatrix}.$$

We need to find a function $v(x)$ which is twice differentiable and non-negative, and the inner product between $v_x(x)$ and $(q(x) - x)$ is strictly negative at every non fixed point of q , and zero at every fixed point of q .

Such v is easy to find. Let

$$v(x) = \frac{1}{4}(x_1^2 + x_2^2 + x_3^2) - \frac{1}{2}(x_1x_2 + x_1x_3 + x_2x_3) + A,$$

where A is some positive number to make sure that v is non-negative. Then

$$\begin{aligned} v_x(x) &= \begin{pmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_3 - \frac{1}{2}x_2 \\ \frac{1}{2}x_2 - \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ \frac{1}{2}x_3 - \frac{1}{2}x_2 - \frac{1}{2}x_1 \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{2}x_1 - \frac{1}{2}x_3 \\ \frac{1}{2}x_2 - \frac{1}{2}x_1 \\ \frac{1}{2}x_3 - \frac{1}{2}x_2 \end{pmatrix} - \begin{pmatrix} \frac{1}{2}x_2 \\ \frac{1}{2}x_3 \\ \frac{1}{2}x_1 \end{pmatrix} \\ &: = (a) - (b) \end{aligned}$$

This v function works because $\langle q(x) - x, (a) \rangle \leq 0$, with equality only at fixed points of q , and $\langle q(x) - x, (b) \rangle = 0$ for all x . This is going to be the general approach as well: we construct v as a quadratic function so that $v_x(x)$ can be written in two parts, (a) and (b) , such that $\langle q(x) - x, (a) \rangle \leq 0$, with equality only at fixed points of q , and $\langle q(x) - x, (b) \rangle = 0$ for all x .

In general, let

$$v(x) = \sum_{i=1}^I a_i \frac{x_i^2}{2} - \sum_{1 \leq i < j \leq I} b_{ij} x_i x_j + A.$$

Since $x \in [0, 1]^I$, we can always choose A large enough to make sure that v is nonnegative. Hence conditions $c(i)$ and $c(ii)$ are trivially satisfied. It remains to be shown that there exist a_i and b_{ij} such that condition $c(iii)$ is also satisfied.

Let $b_{12} = a_2$, $b_{23} = a_3$, $b_{34} = a_4$, ..., $b_{I-1I} = a_I$, and $b_{1I} = a_1$, then

$$\begin{aligned} &v_x(x) \\ &= \begin{pmatrix} a_1(x_1 - x_I) \\ a_2(x_2 - x_1) \\ a_3(x_3 - x_2) \\ \vdots \\ a_I(x_I - x_{I-1}) \end{pmatrix} \\ &\quad - \begin{pmatrix} a_2x_2 + b_{13}x_3 + b_{14}x_4 + b_{15}x_5 + \dots + b_{1I-3}x_{I-3} + b_{1I-2}x_{I-2} + b_{1I-1}x_{I-1} \\ a_3x_3 + b_{24}x_4 + b_{25}x_5 + b_{26}x_6 + \dots + b_{2I-2}x_{I-2} + b_{2I-1}x_{I-1} + b_{2I}x_I \\ a_4x_4 + b_{35}x_5 + b_{36}x_6 + b_{13}x_1 + \dots + b_{3I-1}x_{I-1} + b_{3I}x_I + b_{13}x_1 \\ \vdots \\ a_1x_1 + b_{2I}x_2 + b_{3I}x_3 + b_{4I}x_4 + \dots + b_{I-4I}x_{I-4} + b_{I-3I}x_{I-3} + b_{I-2I}x_{I-2} \end{pmatrix} \\ &: = (a) - (b). \end{aligned}$$

On the other hand,

$$q(x) - x = \begin{pmatrix} l_1(x_I - x_1) \\ l_2(x_1 - x_2) \\ l_3(x_2 - x_3) \\ \vdots \\ l_I(x_{I-1} - x_I) \end{pmatrix} := (c),$$

where $l_i = \frac{\gamma'_i}{\gamma_i + \gamma'_i}$.

We will show that there exist $(a_i)_{i=1}^I$ and $(b_{ij})_{i < j}$, all strictly positive, such that $(b) \cdot (c) = 0$, for all $x \in [0, 1]^I$.

When we take the inner product of (b) and (c) , we obtain $C_I^2 = \frac{I(I-1)}{2!}$ terms of cross products between x_i and x_j . Letting the coefficient of each term be zero, we obtain C_I^2 equations. Notice that we also have C_I^2 unknowns, I a 's and $(C_I^2 - I)$ b 's. We write these equations out as follows.

$$\begin{aligned} x_I x_1 : \quad & l_I a_1 = l_2 b_{2I} \\ x_1 x_2 : \quad & l_1 a_2 = l_3 b_{13} \\ x_2 x_3 : \quad & l_2 a_3 = l_4 b_{24} \\ x_3 x_4 : \quad & l_3 a_4 = l_5 b_{35} \\ x_4 x_5 : \quad & l_4 a_5 = l_6 b_{46} \\ & \cdot \\ & \cdot \\ x_{I-1} x_I : \quad & l_{I-1} a_I = l_1 b_{1I-1} \\ x_1 x_3 : \quad & (l_1 + l_3) b_{13} = l_2 a_3 + l_4 b_{14} \\ x_1 x_4 : \quad & (l_1 + l_4) b_{14} = l_2 b_{24} + l_5 b_{15} \\ x_1 x_5 : \quad & (l_1 + l_5) b_{15} = l_2 b_{25} + l_6 b_{16} \\ & \cdot \\ & \cdot \\ x_1 x_{I-1} : \quad & (l_1 + l_{I-1}) b_{1I-1} = l_2 b_{2I-1} + l_I a_1 \\ x_2 x_4 : \quad & (l_2 + l_4) b_{24} = l_3 a_4 + l_5 b_{25} \\ x_2 x_5 : \quad & (l_2 + l_5) b_{25} = l_3 b_{35} + l_6 b_{26} \\ & \cdot \end{aligned} \tag{3}$$

$$\begin{array}{c}
\cdot \\
x_2x_I : \quad (l_2 + l_I) b_{2I} = l_3b_{3I} + l_1a_2 \\
\cdot \\
\cdot \\
x_{I-2}x_I : \quad (l_{I-2} + l_I) b_{I-2I} = l_{I-1}a_I + l_1b_{1I-2}
\end{array}$$

System (3) is a system of C_I^2 equations with C_I^2 unknowns. Notice that all the C_I^2 unknowns are displayed on the *l.h.s.* of the system. We claim that every term on the *l.h.s.* of system (3) shows up exactly once on the *r.h.s.*, and since the two sides have the same number of terms, this implies that if we add up all the C_I^2 equations, we obtain an identity, which in turn, implies that the coefficient matrix of system (3) is singular.

To see that every term on the *l.h.s.* of system (3) shows up exactly once on the *r.h.s.*, consider a_i first. When we take the inner product between (b) and (c), a_i shows up twice, once in the coefficient of $x_i x_{i-1}$, and once in the coefficient of $x_i x_{i-2}$. The coefficient of both terms is $l_{i-1} a_i$, but the signs are opposite. Hence $l_{i-1} a_i$ appears once on both sides of system (3). Now consider b_{ij} . When we take the inner product between (b) and (c), b_{ij} shows up four times, twice in the coefficients of $x_i x_j$, once in the coefficient of $x_i x_{j-1}$, and once in the coefficient of $x_{i-1} x_j$. The coefficient of $x_i x_j$ is $(l_i + l_j) b_{ij}$, the coefficient of $x_i x_{j-1}$ is $l_j b_{ij}$, and the coefficient of $x_{i-1} x_j$ is $l_i b_{ij}$. Again, $(l_i + l_j) b_{ij} x_i x_j$ shows up negative in the inner product, while $l_j b_{ij} x_i x_{j-1}$ and $l_i b_{ij} x_{i-1} x_j$ show up positive in the inner product. Hence $l_i b_{ij}$ shows up once on both sides of (3), and so does $l_j b_{ij}$.

Since system (3) is a homogenous system with singular coefficient matrix, it has a non-zero solution. Next we show that the system has a non-zero *and* non-negative solution. We prove a more general result in Lemma 2. The a 's and b 's in Lemma 2 are just for notational convenience. They are different from the a 's and b 's outside Lemma 2.

Lemma 2 Consider the n -variate linear homogeneous system of equations,

$$\begin{array}{rcl}
a_1x_1 & = & b_{12}x_2 + \dots + b_{1n}x_n \\
a_2x_2 & = & b_{21}x_1 + \dots + b_{2n}x_n \\
& & \cdot \\
& & \cdot \\
a_nx_n & = & b_{n1}x_1 + \dots + b_{nn-1}x_{n-1}
\end{array}$$

If $\forall j, a_j \geq 0$; $\forall i, j, b_{ij} \geq 0$; and $\forall j, \sum_{i \neq j} b_{ij} = a_j$, then $\exists x \neq 0$, and $\forall i, x_i \geq 0$, such that x solves the system.

Proof of Lemma 2: Proof by induction. Lemma 2 holds trivially for $n = 2$. Suppose it is true for $n > 2$, we need to show that it is true for $n + 1$. With $n + 1$ variables, the system becomes

$$\begin{aligned}
a_1 x_1 &= b_{12} x_2 + \dots + b_{1n+1} x_{n+1} \\
a_2 x_2 &= b_{21} x_1 + \dots + b_{2n+1} x_{n+1} \\
&\vdots \\
&\vdots \\
a_{n+1} x_{n+1} &= b_{n+11} x_1 + \dots + b_{n+1n} x_n
\end{aligned}$$

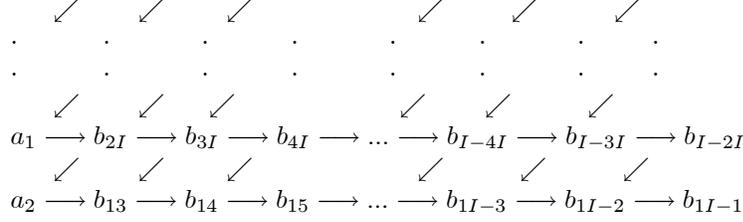
If $a_j = 0, \forall j$, then all the coefficients are zero, Lemma 2 is trivially true. Without loss of generality, assume $a_1 \neq 0$. Then $x_1 = \frac{1}{a_1} (b_{12} x_2 + \dots + b_{1n+1} x_{n+1})$. Substituting it into the rest of the system, then multiplying both sides of the rest of the system by a_1 , and rearranging, the rest of the system becomes,

$$\begin{aligned}
(a_1 a_2 - b_{21} b_{12}) x_2 &= (b_{21} b_{13} + a_1 b_{23}) x_3 + \dots + (b_{21} b_{1n+1} + a_1 b_{2n+1}) x_{n+1} \\
(a_1 a_3 - b_{31} b_{13}) x_3 &= (b_{31} b_{12} + a_1 b_{32}) x_2 + \dots + (b_{31} b_{1n+1} + a_1 b_{3n+1}) x_{n+1} \\
&\vdots \\
&\vdots \\
(a_1 a_{n+1} - b_{n+11} b_{1n+1}) x_{n+1} &= (b_{n+11} b_{12} + a_1 b_{n+12}) x_2 + \dots + (b_{n+11} b_{1n+1} + a_1 b_{n+1n}) x_n.
\end{aligned}$$

By the assumptions on the a 's and the b 's, we are allowed to use the induction hypothesis. Therefore, there exists a non-zero and non-negative vector $(x_2, x_3, \dots, x_{n+1})$ that solves the above system, hence there exists a non-zero and non-negative x that solves the original system. This completes the proof of Lemma 2.

Now we go back to system (3). Notice that each unknown is a positive linear combination (i.e., linear combination with positive coefficients) of one or two other unknowns. If we think of each unknown as a node, and think of the unknowns in the linear combination as each receiving a link from the original unknown, we obtain the following directed graph from system (3).

$$\begin{array}{cccccccccccccccc}
a_2 & \longrightarrow & b_{13} & \longrightarrow & b_{14} & \longrightarrow & b_{15} & \longrightarrow & \dots & \longrightarrow & b_{1I-3} & \longrightarrow & b_{1I-2} & \longrightarrow & b_{1I-1} \\
& \swarrow & \\
a_3 & \longrightarrow & b_{24} & \longrightarrow & b_{25} & \longrightarrow & b_{26} & \longrightarrow & \dots & \longrightarrow & b_{2I-2} & \longrightarrow & b_{2I-1} & \longrightarrow & b_{2I} \\
& \swarrow & \\
a_4 & \longrightarrow & b_{35} & \longrightarrow & b_{36} & \longrightarrow & b_{37} & \longrightarrow & \dots & \longrightarrow & b_{3I-1} & \longrightarrow & b_{3I} & \longrightarrow & b_{13} \\
& \swarrow & \\
a_5 & \longrightarrow & b_{46} & \longrightarrow & b_{47} & \longrightarrow & b_{48} & \longrightarrow & \dots & \longrightarrow & b_{4I} & \longrightarrow & b_{14} & \longrightarrow & b_{24}
\end{array}$$



The pattern of the graph is, except for the leftmost and the rightmost columns, each node is pointing to the right next node, and the node in the next row left to it. For example, b_{14} is pointing to b_{15} and b_{24} , because when we take the inner product between (b) and (c) , b_{14} , b_{15} , and b_{24} are the only unknowns associated with the term x_1x_4 . In general, for every b not in the first, second and last column, b_{ij} is pointing to b_{ij+1} and b_{i+1j} (remember that $I+1=1$ and $b_{ij} = b_{ji} = \partial^2 v / \partial x_i \partial x_j$), because when we compute the inner product between (b) and (c) , the only unknowns associated with the term $x_i x_j$ are b_{ij} , b_{ij+1} and b_{i+1j} . For every b in the second column, b_{ij} is pointing to b_{ij+1} and a_j , for the same reason. For the leftmost column, each a points to the right next node, following directly from system (3). Each node in the rightmost column should also point to two nodes, but we consider only one of them, which suffices for our purposes. Finally, we copy the first row after the last row to indicate the cyclic property of the graph. We can imagine that it is drawn on a cylinder.

By Lemma 2, there exists a non-negative, non-zero solution to system (3). We fix this solution. It is easy to see from the graph that if any unknown is 0, then to keep non-negativity, all the other unknowns must also be 0, which is impossible since we begin with a non-zero solution. Therefore all the unknowns in this non-zero and non-negative solution must be strictly positive, which is what we need for the Lyapunov function. This completes the proof of Proposition 1. ■

Proof of Proposition 2: We follow the same steps as in the proof of Proposition 1. For $i = 1, 2$, let $N_i = B_i + R_i$. Let $\Delta_i(n, x_n)$ denote the number of red people added to group i in period n , given the composition profile x_n at the beginning of period n . Then

$$\Delta_i(n, x_n) = \begin{cases} 0 & \text{with prob. } \gamma_i(1 - x_n^i) + \gamma'_i(1 - x_n^j) \\ 1 & \text{with prob. } \gamma_i x_n^i + \gamma'_i x_n^j. \end{cases}$$

Hence

$$E\Delta_i(n, x_n) = \gamma_i x_n^i + \gamma'_i x_n^j,$$

where the expectation is conditional on n and x_n .

Let

$$q_n^i(x) := E\Delta_i(n, x_n),$$

then

$$q_n^i(x) = q^i(x) = \gamma_i x^i + \gamma'_i x^j.$$

Hence

$$q(x) = \begin{pmatrix} q^1(x) \\ q^2(x) \end{pmatrix} = \begin{pmatrix} \gamma_1 x_1 + \gamma'_1 x_2 \\ \gamma_2 x_2 + \gamma'_2 x_1 \end{pmatrix}.$$

Hence

$$q(x) - x = \begin{pmatrix} \gamma'_1(x_2 - x_1) \\ \gamma'_2(x_1 - x_2) \end{pmatrix}.$$

Let

$$v(x) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 - x_1x_2,$$

and the rest of the proof follows from the AEK theorem. \blacksquare

Proof of Proposition 3: For notational simplicity, consider $I = 3$. Following the same definitions and steps as in the proof of Proposition 2, we have

$$q(x) - x = \begin{pmatrix} \gamma'_1(x_3 - x_1) + \gamma''_1(x_2 - x_1) \\ \gamma'_2(x_1 - x_2) + \gamma''_2(x_3 - x_2) \\ \gamma'_3(x_2 - x_3) + \gamma''_3(x_1 - x_3) \end{pmatrix}.$$

First we show that (x_1, x_2, x_3) solves $q(x) - x = 0$ if and only if $x_1 = x_2 = x_3$.

The *if* part is trivial. To prove the only if part, suppose by way of contradiction that $x_1 \neq x_2$. Then it must be that $x_1 \neq x_3$, and $x_2 \neq x_3$. Now let $x_1 - x_2 = a$, $x_2 - x_3 = b$, $x_3 - x_1 = c$, we have

$$\begin{aligned} \gamma'_1 c &= \gamma''_1 a \\ \gamma'_2 a &= \gamma''_2 b \\ \gamma'_3 b &= \gamma''_3 c. \end{aligned}$$

If $a > 0$, then $c > 0$, and $b > 0$, but this means that $x_1 > x_2 > x_3 > x_1$, contradiction.

Now let

$$v(x) = \frac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

We have

$$\begin{aligned} & (q(x) - x) \cdot v_x(x) \\ &= \begin{pmatrix} \gamma'_1 x_3 + \gamma''_1 x_2 - (\gamma'_1 + \gamma''_1) x_1 \\ \gamma'_2 x_1 + \gamma''_2 x_3 - (\gamma'_2 + \gamma''_2) x_2 \\ \gamma'_3 x_2 + \gamma''_3 x_1 - (\gamma'_3 + \gamma''_3) x_3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ &= -[(\gamma''_1 x_1 - \gamma'_2 x_2)(x_1 - x_2) + (\gamma''_2 x_2 - \gamma'_3 x_3)(x_2 - x_3) + (\gamma''_3 x_3 - \gamma'_1 x_1)(x_3 - x_1)]. \end{aligned}$$

Hence if

$$\gamma''_1 = \gamma'_2, \text{ and } \gamma''_2 = \gamma'_3, \text{ and } \gamma''_3 = \gamma'_1,$$

then the rest of the proof follows from the AEK theorem. \blacksquare

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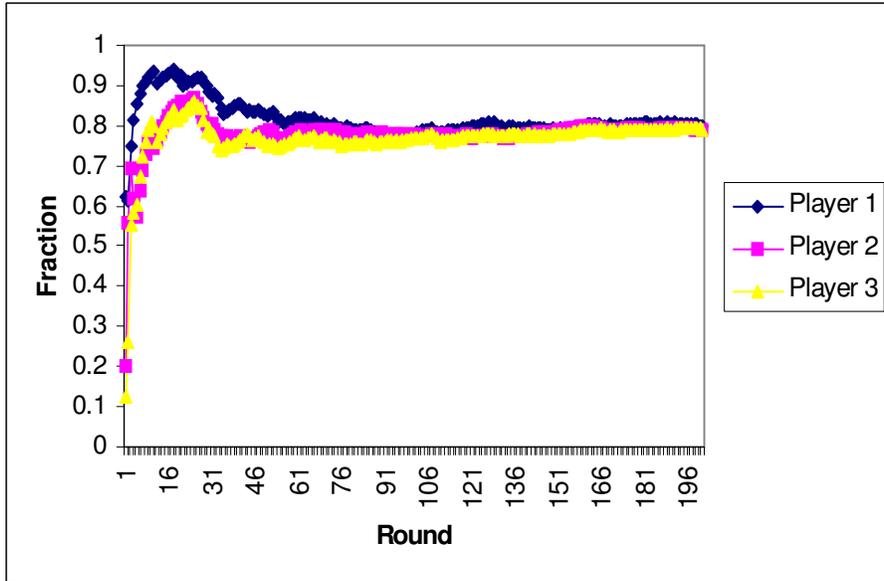


Figure 1: Proposition 1

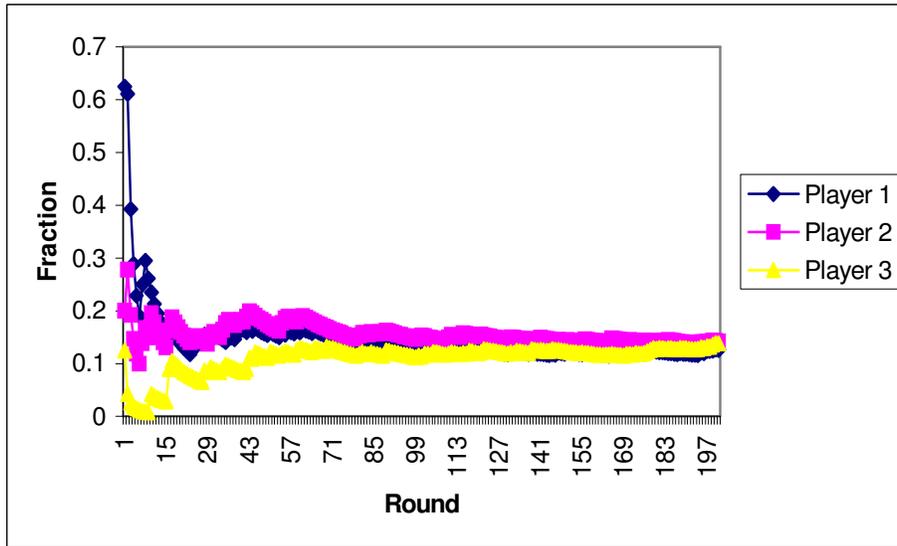


Figure 2: Proposition 1

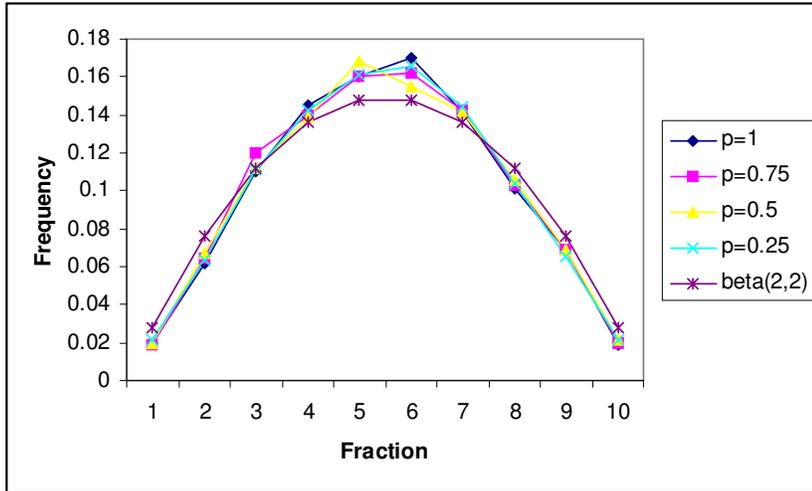


Figure 3: Two interacting groups