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Endowing machines with the expert human ability to select representations: why and how

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18.1 Introduction

To achieve efficient human computer collaboration, computers need to be able to represent information in ways that humans can understand. Picking a good representation is critical for effective communication and human learning, especially on technical topics. To select representations appropriately, AI systems must have some understanding of how humans reason and comprehend the nature of representations. In this interdisciplinary research, we are developing the foundations for the analysis of representations for reasoning. Ultimately, our goal is to build AI systems that select representations intelligently, taking users' preferences and abilities into account.

Alternative representations commonly used in problem solving include formal mathematical notations (logics), graphs and other diagrams (data plots, networks), charts, tables and, of course, natural language. Particular knowledge domains typically have specialised variants of these representations. So, creating an AI system that is capable of selecting an effective representation for a given individual working on a particular class of problems is an ambitious goal. First, individuals vary greatly in their knowledge of the problem domain, and they will have differing degrees of familiarity in alternative representations. How can an automated system take these factors into account? Second, the process of picking a good representation, even for a *typical* problem solver, is a substantial challenge in itself. Problem solvers clearly have the meta-cognitive ability to recognise when they are struggling to find a solution and do things like trying an alternative strategy when they reach an impasse. Nevertheless, deliberately attempting to switch to an alternative representation is difficult for typical problem solvers; it seems this is an expert skill. Ordinary problem solvers often need instructors to tell them when to change representations and what representation to change to.

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Classic research in cognitive science highlights the challenge of changing representations. For example, in an experiment on analogy by Gick and Holyoak (Gick, 1989), participants were given a *source* story about how an army must storm a fortress but all the roads converging on the fortress are mined to stop the whole army passing along any one of them. To solve the problem, the general split up the army and sent each platoon along a separate road knowing that small groups would not trigger the mines. Shortly after the story, the participants in the experiment were given an isomorphic *target* problem about directing a strong x-ray beam through a patient to kill the tumour and asked to solve the problem of how to avoid damaging the healthy tissue around the tumour. The results of the experiment are normally interpreted as showing the power of analogy as 75% of participants given the source story successfully solved the target problem, but only 10% of the control group without the source story found the solution. However, in terms of representation switching, that is, the ability to extrapolate the isomorphic story to an analogous target problem, only 40% from the successful cohort (of 75%) were successful without additional hints, even though they had recently been told the story by the experimenter. The remaining 35% from the successful cohort (of 75%) were successful when told to try to apply the story (i.e., to switch representation). The implication of many such studies is that changing representation is hard.

The change of representation demanded in the analogy is relatively small, but for real problems finding an effective representation typically involves switching format, say from algebra to a diagram, or a table to a network. Such switches are far more demanding, both for problem solvers to adapt to the new representation, but most importantly, for our present goal to select the alternative representation in the first place. Cognitive problem solving theory (Newell and Simon, 1972) suggests that switching representation may itself be interpreted as a form of problem solving that involves changing the initial state, satisfying goal states, problem state expressions, finding operators to transform problem states, or a mixture of them all. This is why selecting alternative representations has, so far, been a task for instructors and domain experts.

In this chapter we explore how to give computers the ability to select effective representations for humans. One obvious benefit is that giving users representations that are better suited to their knowledge and levels of experience should enhance their ability to comprehend, solve problems and to learn about the target domain. There is another *human-like* computing benefit. Computer systems typically use some logical symbolic language, which may be efficient for the computer but is often inaccessible and thus a barrier for their interaction with humans. Thus, endowing a system with the ability to select a representation that is suited to its current human user may improve the communication between the system and the human. The system might translate its own inference steps and outputs from its logical symbolic language into the representation preferable for the user, and hence be able to provide explanations of its reasoning in a form that can be comprehended by the human.

The general hypothesis of our work is that *if we use foundational analysis of the user's expertise and the cognitive and formal properties of problems and representations in an AI system, then we can improve human interaction with it and their success at solving a task at* *hand.* Testing this hypothesis is ongoing work in the *rep2rep* project,[1] and here we describe some of its current contributions:[2]

- representation selection theory consisting of the language and measures to analyse representations;
- theory of cognitive properties for assessing the efficacy and suitability of a representation for particular users;
- computational algorithms and implementations for carrying out the analysis and assessing the suitability of a representation for a particular task and user.

In order to test that our theories of representations and computational models based on them are indeed more human-like and lead to humans being more successful in solving a task, we carried out preliminary empirical studies with encouraging results, but much more still needs to be done for evaluation. A practical application of this work in the domain of education and AI tutors is planned for future work.

The structure of this chapter is as follows. We start by exemplifying in Section 18.2 what changing a representation means. In Section 18.3 we discuss the benefits of changing representations that have been recognised in empirical studies. Then in Section 18.4 we explore more deeply the wide range of impacts that switching representations may have on human problem solving, and hence the complexity and challenge of selecting an effective representation. We explore in Section 18.5 how representations can be analysed computationally in terms of formal and cognitive properties, and how attributes of representations may relate across representations. Then, in Section 18.6 we demonstrate how this analysis can be automated in an intelligent system. Finally, in Section 18.7 we discuss how the automated representation choice based on the user model and the cognitive and formal properties could be applied in AI tutoring systems in order to personalise interaction and to improve users' abilities to solve problems.

18.2 Example of selecting a representation

To illustrate the approach to choosing representations in our framework, consider this *Birds* problem in probability:

> *One quarter of all animals are birds. Two thirds of all birds can fly. Half of all flying animals are birds. Birds have feathers. If X is an animal, what is the probability that it's not a bird and it cannot fly?*

[1] http://www.cl.cam.ac.uk/research/rep2rep

[2] Some of the work reported in this chapter has previously been published in (Raggi, Stapleton, Stockdill, Jamnik, Garcia Garcia and Cheng, 2020; Stockdill, Raggi, Jamnik, Garcia Garcia, Sutherland, Cheng and Sarkar, 2020; Raggi, Stockdill, Jamnik, Garcia Garcia, Sutherland and Cheng, 2020).

Here are three different ways one can go about solving this (see Figure 18.1):

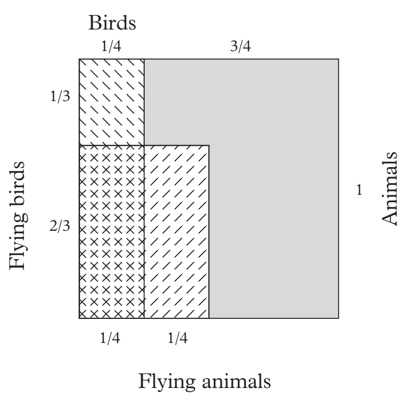
(a) You could divide areas of a rectangle to represent parts of the animal population that can fly and parts that are birds.

(b) You could use contingency tables to enumerate in its cells all possible divisions of animals with relation to being birds or being able to fly.

(c) You could use formal Bayesian notation about conditional probability.

Which of these are effective representations for the problem? It depends; the first is probably best for school children; the last for more advanced mathematicians. Can this choice of appropriate representation be mechanised, and how? In our work we lay the foundations for new cognitive theories that would allow us to understand the relative benefits of different representations of problems and their solutions, including taking into account individual differences. We automate an appropriate choice of problem representation for both humans and machines to improve human-machine communication (see Sections 18.5 and 18.6).[3] But first, let us examine what the benefits of switching representations are and why doing so is hard.

(a)
Geometric representation—the solution is the area of the solid shaded region $\frac{3}{4} - \left(\frac{2}{3}\right)\left(\frac{1}{4}\right) = \frac{7}{12}$:



(b)
Contingency table representation – the solution is in the shaded cell:

	birds	non-birds	total
flying	(2/3)(1/4)	(2/3)(1/4)	
non-flying		3/4−(2/3)(1/4)	
total	1/4	3/4	1

(c) Bayesian representation:

$$\begin{aligned}\Pr(\bar{b}\cap\bar{f}) &= \Pr(\bar{b}) - \Pr(\bar{b}\cap f)\\ &= \Pr(\bar{b}) - \Pr(b\cap f)\\ &= (1-\Pr(b)) - \Pr(f\,|\,b)\Pr(b)\\ &= \tfrac{3}{4} - \left(\tfrac{2}{3}\right)\left(\tfrac{1}{4}\right) = \tfrac{7}{12}\end{aligned}$$

Figure 18.1 *The* Birds *example.*

[3] It is worth pointing out that we are not devising new representations. Instead, we are introducing a new language and framework that enable us to describe and compare different representations with respect to the task and the user.

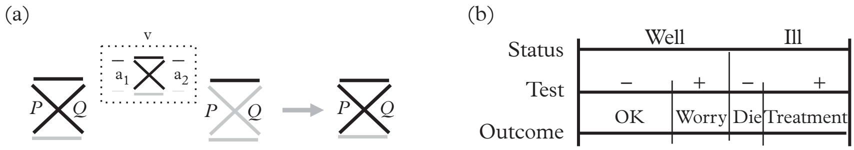
18.3 Benefits of switching representations

The three examples in the previous section illustrate the potential for alternative representation to provide varying benefits to different users. But do the benefits justify the effort needed for the development of automated systems to select representations for humans? We review the wide range of *epistemic* and *cognitive* factors from the cognitive and computational literature to show that switching from a poor representation to a good one does provide substantial benefits in reasoning and problem solving.

18.3.1 Epistemic benefits of switching representations

At the most fundamental level, alternative representations can differentially support problem solving through the particular information they encode and thus make available. This is not merely a matter of *what* information about the target domain is provided by a representation's expressions, but concerns *how* information is encoded. For example, here is a transitive inference problem: "A is to the left of B, A is to the left of C; what is the relation between B and C?". In capturing the premises in a sentential notation, such as "left-of(A,B)" and "left-of(A,C)", nothing is said about the relation between B and C. However, in this diagram "A C B" we easily see that C is to the left of B. The diagram naturally has greater specificity: we are forced to draw B somewhere relative to C. Using the diagram we can successfully answer the question, but whether this specificity is a benefit or not depends on whether the extra information in the diagram is appropriate for the task in the first place, and whether the extra information is uniquely determined or it needs to be split into cases. The correct answer could have been "not known". Nevertheless, the general point still holds that alternative representations may usefully encode more, or disadvantageously less, relevant domain information, or vice versa. The idea has been extensively discussed in the literature; for example, by Stenning and Oberlander (1995) and Shimojima (2015).

The information that can be directly encoded by a representation may improve problem solving performance of users. For example, for propositional calculus, diagrammatic representations are an alternative to the traditional formulae and truth table representations, which are commonly used for teaching. A review of Frege, Wittegnstein, Pierce and Gardner's representations by Cheng (2020) shows how their different formats substantially impact the accessibility of information: not just the content of propositional relations but also the form of inference rules and the strategic information needed to manage proof making. Cheng (2020) designed a novel diagrammatic representation for propositional calculus that makes directly accessible these multiple levels of information that are not easily available in the sentential and other representations (Figure 18.2a). Furthermore, Cheng (2011*a*) designed a novel representation for probability theory that simultaneously encodes information that would normally be distributed over algebraic expressions, set notations, contingency tables and tree diagrams (Figure 18.2b). Experimental and classroom studies have shown the beneficial impact on user performance of representations that directly express the full range of the relevant domain information,



A Truth Diagram for the inference (P ∨ Q) ∨ (P ∧ Q) → P ∨ Q.

A Probability Space Diagram for disease diagnosis.

Figure 18.2 *Novel accessible diagrammatic representations.*

in particular, that these novel representations substantially improve students' problem solving and learning (e.g., Cheng (2002), Cheng and Shipstone (2003)).

The key point here is that different representations can substantially impact how effectively users solve problems by providing them with the information that is needed in a readily accessible form. In this regard, a poor representation will force the problem solver to spawn sub-problems in order to obtain the missing information, either by pursuing extra chains of reasoning or by bearing the cost of switching among multiple representations, when domain knowledge is dispersed.

18.3.2 Cognitive benefits of switching representations

The impact of alternative representations is attributable to cognitive processing (rather than informational differences) when the representations being compared are informationally equivalent: specifically, when all the information held by one representation can also be inferred from the information in the other representation (Larkin and Simon, 1987). Even when two representations are informationally equivalent, the user's ease of problem solving can dramatically vary. For example, in isomorphic, informationally equivalent, versions of the Tower of Hanoi puzzle, the difficulty of reaching the goal can vary by a factor of 18 (Kotovsky, Hayes and Simon, 1985) when operators that move disks between pegs are replaced by operators that transform the size of objects. The seminal work of Larkin and Simon (1987) showed (using computational models of a pulley system problem) that diagrams are (sometimes) superior to sentential representations. Namely, they permit the simple perceptual matching of information to inference rules and the efficient use of locational indexing of information that substantially reduces the need to search deliberately for relevant information. In a study using the same pulley system representations, Cheng (2004) found that solutions using diagrams were arrived at six times faster than with sentential representations.

Another aspect differentiating good from poor representations concerns how coherently elements of the representations encode domain concepts. Many have recognised the importance of isomorphic mappings between domain concepts and the tokens standing for them (Gurr, 1998; Moody, 2009; Barwise and Etchemendy, 1995). One-to-one mappings are easier for users than one-to-many mappings. Cheng (2011*b*) goes further contending that effective representations for rich domains should maintain such isomorphic mappings at successively higher conceptual and symbolic levels.

Cognitive differences may not only impact immediate problem solving but might determine the long term survival and propagation of representations. Numeration systems are informationally equivalent, but Zhang and Norman (1995) demonstrate that different ways of encoding number information can influence how simply numbers are expressed and also how much they can cause unnecessary work in computations. Arguably, the Hindu-Arabic number system's particular properties, its medium size base and separation of number and power information into different perceptual dimensions, are responsible for its world wide adoption.

At the opposite end of the spectrum, significant cognitive differences can occur with relatively small variants in representations. For instance, Peebles and Cheng (2003) investigated how reading Cartesian graphs is affected when data for two dependent measures varying with an independent measure is plotted in different ways (e.g., oil and coal prices over time): either as a conventional "function" graph with the two dependent variables on the y-axis, or as an unfamiliar "parametric" graph with both dependent variables plotted on each of the two primary axes and the independent variable plotted as data points along the curve in the graph. Despite participants' lack of familiarity with the parametric graphs, they responded significantly more quickly than with the conventional graph, without sacrificing accuracy. Thus, it is quite feasible to improve task performance not just by changing the format of the representations, but by changing the particular way in which different types of variables are plotted.

This is but a small sampling of a diverse literature (for more see the review by Hegarty (2011)). The key point here is that different representations can and do substantially impact on how effectively users solve problems in cognitive terms, potentially by over an order of magnitude. Myriad of factors contribute to explanations of why alternative representations may enhance or hinder problem solving, which presents a major challenge to our goal of building an automated system for representation selection. We next explore the reasons why selecting a good representation is challenging.

18.4 Why selecting a good representation is hard

The examples in Section 18.2 showed that alternative representations require different knowledge about the formats (structures) of the representations, and thus support distinct problem solution strategies. In Section 18.3 we saw the large array of benefits that can flow from switching representations. But selecting representations is intrinsically challenging: simply, the myriad of ways and the extent to which representations directly impact problem solving means that there is no small set of core factors that determine the relative efficacy of representations. However, we contend that it may nevertheless be feasible to systematically define a space that organises the wealth of cognitive factors to serve as a cognitive analysis framework.

18.4.1 Representational and cognitive complexity

Why is using a cognitive theory to selected an effective representation so challenging? In general terms, the number of different aspects of cognition that conceivably have an impact are manifold and each of these aspects are themselves complex (Markman, 1999; Anderson, 2000; Stillings, Weisler, Chase, Feinstein, Garfield and Rissland, 1995). Let us consider two examples.

Despite the growing renaissance in cognitive neuroscience, how the brain physically implements mental representations remains largely unknown. So, cognitive scientists still find it essential to formulate accounts in terms of information structures and processes such as: declarative propositions stored as semantic networks with spreading activation (Markman, 1999; Anderson, 2000); mental imagery that exploits much of the functionality of the visual perceptual system but in the mind's eye (Finke, 1989); procedural information encoded as condition-action rules (Klahr, Langley and Neches, 1987; Anderson, 2007); hierarchies of concepts stored in discrimination networks to aggregate related concepts and separate dissimilar ones (Gobet, Lane, Croker, Cheng, Jones, Oliver and Pine, 2001); and, even heterogenous structures that tightly coordinate declarative, procedural and diagrammatic information (Koedinger and Anderson, 1990). Accounts of how problem solvers work with a representation must invoke several of these internal mental representations; for example, the table in Figure 18.1b includes declarative propositions, rules for computing cell values, and a hierarchical conceptual scheme to coordinate the rows and columns. Each representation imposes different cognitive demands, which interact, so it is by no means simple to assess the overall cognitive cost of using each representation.

The idea that the demands placed on working memory (WM) could be a key to assess task difficulty is wide spread and it is tempting to apply it to evaluating alternative representations. For instance, how much does the use of a certain representation tend to breach the user's WM capacity? However, the notation of WM capacity is tricky. Although the famous magic number of 7 ± 2 items (Miller, 1956) is widely known, the modern estimate of approximately four chunks (Cowan, 2001) is a more appropriate general capacity limit for typical tasks. Nevertheless, finer grained models of the sub-processes of the cognitive architecture suggest that WM capacity should not be considered as one fixed-limit general store, but as a collection of lower level mechanisms each possessing their own small capacity buffer (Anderson, 2007, Anderson, 2000). Thus, the prospects of developing a generic account based on WM loading is remote.

For other aspects, a similar tale of complex interactions between representations and cognitive structures and processes can be told. Novices and experts mentally represent information in very different ways (Koedinger and Anderson, 1990). As real world cognition involves continuing cycles of perception, internal cognition, and motor output, we can use the external environment to off-load information from WM or to replace laborious mental reasoning with perceptual inferences (Larkin and Simon, 1987). We can use different strategies on the same problem, which may be knowledge-rich or knowledge-lean (Newell and Simon, 1972). Hierarchies of goals are used to solve problems but the organisation of goals can vary from person to person and from representation to representation (Stillings, Weisler, Chase, Feinstein, Garfield and Rissland, 1995). In sum, there is no doubt that developing a method to predict the efficacy of alternative representation is a substantial endeavour.

18.4.2 Cognitive framework

Although the difficulty of selecting effective representations is revealed by studies in cognitive science, the discipline has matured to the point where it now provides a reasonable map covering the terrain for comparing representations that is to be explored (albeit incomplete and with connections between areas still sketchy). This map combines two fundamental dimensions considered in cognition.

The first is a dimension ranging across the size of cognitive objects that encode meaning, or in other words, a granularity scale of representations. One can think of it as a decision tree, where the scale spans a hierarchy of levels from symbols at the leaves to whole systems of representations at the trunk. The branches in between are constituted by compound cognitive forms such as expressions, chunks and schemas (Stillings, Weisler, Chase, Feinstein, Garfield and Rissland, 1995; Gobet, Lane, Croker, Cheng, Jones, Oliver and Pine, 2001; Schank, 1982).

The second dimension is time. Newell (1990) and Anderson (2002) both identify multiple temporal levels at which cognitive processes operate, ranging from 100 milliseconds to years, for instance, from the time to retrieve a fact from memory to the time required to acquire expertise. Both authors recognise the relatively strong interactions between processes at a particular characteristic time scale, and relatively weak interactions between different time scales. Thus, cognitive processes with durations differing by an order of magnitude may be treated as nearly independent for the sake of analysis, although short processes will cumulatively impact long processes.

Our novel framework, composed of the representational granularity dimension and a temporal dimension, will be elaborated in the following section.

18.5 Describing representations: *rep2rep*

To endow a machine with the ability to select representations, it must first be able to describe them to then analyse them for ranking and selection. The analysis explores if a representation is informationally adequate to express a particular problem, before we even consider whether it will be effective for human users.

We devised a framework, within which different representations can be described and analysed for their suitability.[4] This analysis is based on two main measures: *informational suitability* and *cognitive costs* (Raggi, Stapleton, Stockdill, Jamnik, Garcia Garcia and Cheng, 2020). Informational suitability is described in terms of *formal properties* of a representation, whereas cognitive cost is described in terms of *cognitive properties* (based on our two dimensional—spatial and temporal—cognitive map) that may be

[4] The *rep2rep* framework is a developing project, so formalisations, mechanisms and implementations are evolving. Current *rep2rep* implementation can be found here: https://github.com/rep2rep/robin.

related to formal properties and are crucially dependent also on the user profile. We formalise and describe the implementation of informational suitability and cognitive cost measures in Section 18.6. But first, we introduce the framework components for describing representations.

We distinguish between *cognitive* and *formal* properties of a representation, in an approach that radically, but systematically, reconfigures previously descriptive accounts of the nature of representations and notations (Moody, 2009; Hegarty, 2011; Engelhardt and Richards, 2018). We use this to devise methods for measuring competency in alternative representation use, and also to engineer a system to automatically select representations. *Cognitive properties* characterise cognitive processes demanded of a particular representation (e.g., problem state space characteristics; applicable state space search methods; attention demands of recognition; inference operator complexity (Cheng, Lowe and Scaife, 2001; Cheng, 2016)). *Formal properties* characterise the nature of the content of the representation domain (e.g., operation types like associative or commutative, symmetries, coordinate systems, quantity or measurement scales) (Raggi, Stapleton, Stockdill, Jamnik, Garcia Garcia and Cheng, 2020).

18.5.1 A description language for representations

The language for describing representations in terms of their properties is general as it must be able to deal with very diverse objects of representations. For example, in the above *Birds* problem, the candidate representations include elements like natural language, formal notation, a geometric figure, and a table. Each has its own symbols, grammar and manipulation rules. Their differences yield a different cognitive cost, that is, the effort that is demanded of the user when working with a particular representation. For example, the simpler and fewer inferences in the Geometric representation will result in a less costly solution than in the say Bayesian representation.

Our language describes representations in terms of tokens, expressions, types, laws, tactics and patterns. Each of these can have attributes which specify records associated with them. The attributes encode the informational content of the problem and a representation. Tokens are the atomic symbols from which expressions are built. Types classify expressions and tokens. Tactics specify how to manipulate the representation, and laws determine rules around how tactics can work. Patterns specify the structure of expressions using types, tokens, and attribute holes. For example, if a representation uses a token + four times, we can describe this as:

$$\begin{aligned}\text{token} + : \{&\text{type} := \texttt{real} \times \texttt{real} \to \texttt{real},\\ &\text{occurrences} := 4\}\end{aligned}$$

Moreover, the pattern for expressing conditional probability $\Pr(\_ \mid \_) = \_$ can be described as:

$$\begin{aligned}\text{pattern } \texttt{CP} : \{&\text{type} := \texttt{formula},\\ &\text{holes} := [\texttt{event}^2, \texttt{real}],\\ &\text{tokens} := [\Pr, |, =, (,)]\}.\end{aligned}$$

18.5.2 Importance

Some properties of a representation are more important than others, and some may even be irrelevant (noise) and need to be ignored in the analysis of suitability of representation for solving a problem.[5] We use the notion of *importance* to express this. Clearly, the importance is strictly relevant to the task, so we express it only when describing a problem (like the *Birds* problem above) in a particular representation (e.g., in Bayesian representation). Importance is defined as a function from the properties to the interval ranging between 0 and 1, where 0 is noise and 1 denotes a maximally informationally relevant property. For example, the token Pr for the Bayesian representation of the *Birds* problem is important. Assigning importance is like finding good heuristics – in our framework, the *domain expert* who is setting up our framework for deployment assigns these values. In the future, we will explore if there is a principled approach to assigning these values, and if these importance parameters can be generated automatically by analysing a sufficiently large set of problems. We propose to use the theory of observational advantages of representations that we developed elsewhere (Stapleton, Jamnik and Shimojima, 2017; Stapleton, Shimojima and Jamnik, 2018) as a starting point.

18.5.3 Correspondences

We use the notion of *correspondences* to encode informational links between different representations. For example, in the *Birds* problem above, there is a correspondence between the areas used in the Geometric representation and the probability used in the Bayesian representation. These links can be, for example, analogies between representations or structures that are preserved through transformations. We formalise correspondences probabilistically which also allows us to inherit some provable consequences such as reversability of a property (if property a is related to property b, then b is related to a), or composability (if property a is related to property b, and b is related to c, then a is related to c). Similarly to the importance parameter of a property, some correspondences are stronger than others, for example, a token "intersection" in the Natural language representation strongly corresponds to $\cap$ in the Bayesian representation. We use the notion of *strength* to capture this, so a correspondence is a triple $\langle a, b, s \rangle$ where a and b are properties and s is the strength of their correspondence. Similarly to importance, we leave it with the domain expert to assign its value. In addition, we have devised algorithms that automatically generate some correspondences and compute their strengths probabilistically, which eases the load on the domain expert (Stockdill, Raggi, Jamnik, Garcia Garcia, Sutherland, Cheng and Sarkar, 2020).

[5] One could imagine that a property of a representation may be detrimental if the representation is used for a particular problem solving task (e.g., loss of accuracy, correctness), so its importance may need to be negatively accounted for in the suitability analysis. Our framework does not currently provide this feature—it is left for future work.

To illustrate, let us formalise the correspondence between areas in the Geometric representation and events in the Bayesian representation. Specifically, there is a strong correspondence between the type ‘region’ and the type ‘event’:

$$\langle \text{type region, type event, } 1.0 \rangle.$$

Immediately we know the reverse is also a correspondence, although less strong:[6]

$$\langle \text{type event, type region, } 0.8 \rangle.$$

Similarly, we might consider how intersection is represented in a Geometric representation – we intersect areas by *overlapping* them:

$$\langle \text{token } \cap \text{, pattern overlap, } 0.8 \rangle.$$

Consider again the statement of the *Birds* problem in the Natural language. We see tokens such as ‘all’, and ‘and’. These have clear analogues in the Bayesian representation with which we can make correspondences (where Ω is the probability space, i.e., the universe).

$$\langle \text{token all, token } \Omega, 0.9 \rangle$$
$$\langle \text{token and, token } \cap, 1.0 \rangle$$

The more strong correspondences occur between representations, the better a potential representation is as a candidate for re-representation.

Note that we have a correspondence between the Natural language and the Geometric representation by composition: we know that ‘and’ corresponds to $\cap$, and that $\cap$ corresponds to overlapping, and thus we can derive the correspondence from ‘and’ to overlapping:

$$\frac{\langle \text{token and, token } \cap, 1.0 \rangle \quad \langle \text{token } \cap \text{, pattern overlap, } 0.8 \rangle}{\langle \text{token and, pattern overlap, } 0.8 \rangle}$$

The value for the strength of the resulting correspondence can be computed, here by multiplication of the originating correspondences (or probabilistically from a dataset of co-occurring properties, if such a dataset is available).

Further details of the formalisation and implementation of correspondences, their strength, automatic generation and use in the analysis of the suitability of a representation can be found in (Stockdill, Raggi, Jamnik, Garcia Garcia, Sutherland, Cheng and Sarkar, 2020).

6 The strength is reduced because there are many ways to encode events in the Geometric representation. An event might become a line segment, or a point in space.

18.5.4 Formal properties for assessing informational suitability

We can now give an example of how some of the representations from our *Birds* problem can be described. Notice that we need to be able to describe *representational systems* in general. In addition, we must be able to express the problem (i.e., the *question*), for example, our *Birds* problem, in these representational systems. For the problem, the notion of importance is used too.

We catalogue formal properties using templates of attributes that (currently) the domain expert who wants to use our framework assigns values to. Table 18.1 gives snippets from a formal property catalogue for the *Birds* problem stated in the Natural language representation. The colours code the importance of the property relative to the information content (top to bottom in decreasing importance). Table 18.2 gives snippets of the catalogue of formal properties for the Bayesian representational system (used in the solution in Figure 18.1c).

It is important to note that our representation language does not provide a complete formal description of a representation as, for example, formal logics do. For many representations, this is not possible. Instead, our language describes representations with sufficient specificity to be able to analyse them, draw analogies between them, and to assess their informational suitability and cognitive cost (see Section 18.6).

Table 18.1 *Formal properties of the* Birds *problem in the Natural language representation (note colour for* importance *parameter).*

Kind	Value
error allowed	0
answer type	ratio
tokens	probability : {occurrences := 1}, and :{occurrences := 1}, not : {occurrences := 1}
types	ratio, class
patterns	Class-ratio : { holes := [class ⇒ 2, ratio ⇒ 1], tokens := [of, are], occurrences := 3, token-registration := 1}, Class-probability : { holes := [class ⇒ 2], tokens := [probability, is], occurrences := 1, token-registration := 1}
laws	Bayes' theorem, law of total probability, unit measure, additive inverse, . . .
tactics	re-represent : {occurrences := 1,inference-type := transformation}
tokens	one : {occurrences := 1}, quarter :{occurrences := 1}, all : {occurrences := 3}, animals :{occurrences := 2}, birds : {occurrences := 4}, . . .
mode	sentential
tokens	feathers : {occurrences := 1}

Table 18.2 *A section of formal properties for Bayesian representational system.*

Kind	Value
mode	sentential
rigorous	TRUE
types	real, event
tokens	Ω : {type := event }, $\emptyset$: {type := event }, 0 : {type := real }, 1 : {type := real }, = : {type := $\alpha \times \alpha \rightarrow$ formula }, + : {type := real × real → real}, − : {type := real × real → real}, × : {type := real × real → real}, ÷ : {type := real × real → real}, $\cup$: {type := event × event → event}, $\cap$: {type := event × event → event}, \ : {type := event × event → event}, $^{-}$: {type := event → event}, Pr : {type := event × event → real}, | : {type := delimiter }
patterns	Conditional-probability : { holes := [event $\Rightarrow$ 2, real $\Rightarrow$ 1], tokens := [Pr, |, =] }, Simple-probability : { holes := [event $\Rightarrow$ 1, real $\Rightarrow$ 1], tokens := [Pr, =] }, Joint-probability : { holes := [event $\Rightarrow$ 2, real $\Rightarrow$ 1], tokens := [$\cap$, Pr, =] }, Equality-chain : { holes := [real $\Rightarrow$ $O(n)$], tokens := [=] }
laws	Bayes' theorem, law of total probability, non-negative probability, unit-measure, sigma-additivity, commutativity, ...
tactics	rewrite, apply lemma, arithmetic calculation

18.5.5 Cognitive properties for assessing cognitive cost

Every human user of a system has a different background, expertise and preferences in terms of their ability to solve a problem. In our approach we account for these cognitive aspects through assessing representations in terms of their cognitive properties. Since these are not general characteristics of a representational system but are specific to the problem that the user wants to solve, we attribute cognitive properties to specific representations of specific problems. We then assess the cost of these cognitive properties with respect to particular users. This enables personalisation in terms of adapting representations to users.

Picking up the cognitive map sketched in Section 18.4, we propose 9 key cognitive properties to populate the space defined by the dimensions of *notation granularity* and *temporal scale* (see Table 18.3).

We now give a brief explanation of these 9 key cognitive properties. We computationally modelled these properties and implemented algorithms for calculating the *cognitive cost* that they entail (see Section 18.6).

Registration: refers to the process of identifying some object in a representation as a token (or expression), and acknowledging their existence and location. The registration of tokens depends on users ability to observe their role in the pattern. Analogously, the registration of patterns depends on the *mode* (which describes a higher level of notational granularity) of the representation. Patterns are given an attribute *token registration*,

Table 18.3 *Cognitive properties organised according to spatial (columns) and temporal (rows) aspects of cognitive processing.*

	token	expression	whole
registration	registration		subRS variety
semantic encoding	number of types concept mapping		
inference	quantity scale	expression complexity inference type	branching factor
solution			solution depth

which assigns *icon*, *notation index* or *search* to the tokens used by the pattern, with their corresponding individual costs in the increasing order as implied by (Larkin and Simon, 1987).

Number of types: refers to processing semantics, that is, identifying the types of tokens and expressions in a representation. A larger variety in the *types* of tokens and expressions means a higher semantic processing cost for the user.

Concept-mapping: refers to the mapping of tokens and expressions to their corresponding concepts in the user's internal representation (Zhang, 1997). Its cost is associated with the accumulated effort of processing various defects of representations: specifically, *excess* (symbol that does not match to an important concept), *redundancy* (two symbols for the same concept), *deficit* (a concept with no symbol to represent it), or *overload* (one symbol for multiple concepts). These incur cognitive cost increasing in the order implied by (Gurr, 1998; Moody, 2009). The total cost is a weighted sum of these individual defect costs.

Quantity scale: refers to a well documented scale hierarchy, specifically, *nominal*, *ordinal*, *interval* or *ratio*, all of which effect cognitive costs (Zhang and Norman, 1995). These are associated with arithmetic operations, so we use the correspondences to the Arithmetic representation to estimate the cost.

Expression complexity: encodes the assumption in cognitive science that more complex expressions demand greater processing resources, and that complexity rises with increasing the breadth and depth of expressions (all else being equal). Our algorithm takes each pattern and instantiates its holes recursively with other patterns or tokens of the appropriate type until no holes remain uninstantiated. This results in an encoding of *parse trees* for expressions. Thus, we can generate, for every pattern, a sample of possible expression trees that satisfy it. The average number of nodes in each tree gives us a measure of the complexity of such a pattern.

Inference type: refers to the difficulty intrinsic to applying tactics. We assume an attribute *inference type* for each tactic, valued as *assign*, *match*, *substitute*, *calculate*, or *transform*. These classes are associated with cost that is increasing in the order listed here. Note that we assume that we have a specific solution for a problem in the relevant representation.

subRS variety: considers heterogeneity of a representation, that is, if it consists in part from a sub-representation that could be considered an independent representation in its own right (e.g., a table where the cell values are arithmetic formulae has two subRSs). This heterogeneity incurs a heavy cognitive cost (Van Someren, Reimann, Boshuizen *et al.*, 1998): we estimate it from the number of modes.

Branching factor: refers to the breadth of possible manipulations (estimated from tactics and their attributes, like branching factor).

Solution depth: is simply the total number of tactic uses (from the tactic attribute *uses*). Note that we assume that we have a specific solution for a problem in the relevant representation.

The challenge of assessing the cognitive cost of a representation is greater than just taking cognitive properties into account, because individuals vary in their degree of familiarity and hence proficiency in using particular representations. To adjust cognitive costs from a typical user to individual's abilities, we are devising a small but diverse set of user profiling tests—this is currently under development. The measures extracted from these profiles should enable us to scale the level of contributions of each cognitive property to the overall cost of a representational system for an individual. This can give us a basis for automating the representation selection that is sensitive to individual's cognitive differences, which we address next.

18.6 Automated analysis and ranking of representations

Within the *rep2rep* framework we describe the representations and problems with the language of formal and cognitive properties outlined in Section 18.5. In addition, we built algorithms that automatically analyse these encodings for a given problem (like the one in Table 18.1) with respect to candidate representational systems (like the one in Table 18.2) in order to rank the representations, and ultimately suggest the most appropriate one. This analysis is based on the evaluation of the informational suitability and cognitive cost.

Informational suitability. We define *informational suitability* in terms of correspondences between the formal properties of the problem q in the given representation (e.g., the *Birds* problem in the Natural language representation) and the formal properties of a candidate alternative representation r (e.g., the Bayesian representation). This is modulated by the *importance* score of the property and the *strength* s of the correspondence. If there is no correspondence between the properties of the original representation of q and the candidate representation r, this means that the candidate representation r cannot convey the information carried in the original representation of the problem q. Thus, the algorithm for computing informational suitability first takes the original representation of the problem q, identifies the corresponding properties of the alternative candidate representation r, and then sums the strengths of all corresponding properties multiplied by their importance.

The set C of identified correspondences is computed based on minimally redundant and maximally covering properties. Minimally redundant properties means that for any two pairs of corresponding properties, they must be independent. This ensures that redundancy of informationally similar properties is avoided. Maximally covering properties in the set C are those important properties that are the most information carrying to express the problem q.

More formally, given a minimally redundant and maximally covering set of corresponding properties C for a problem q and a candidate representation r, the informational suitability IS can be computed as:

$$IS(q,r) = \sum_{\langle p_1,p_2,s\rangle \in C} s \cdot \text{importance}_q(p_1) \tag{18.1}$$

where p_1 is a property of q, p_2 is a corresponding property of r, s is a strength of that correspondence $\langle p_1, p_2, s\rangle$, and $\text{importance}_q(p_1)$ is the importance of property p_1 for q.

In the *Birds* example given in the Natural language representation, we computed the IS for Bayesian representation, Contingency tables representation, Geometric representation and Natural language representation. We compared these automatically produced rankings with the rankings that human experts would give for a very similar probability example (the *Medical* problem) in an online survey, and the results are comparable (higher value is better) – see Table 18.4.

Cognitive cost. Informational suitability only takes into account how well a representation can express a problem in terms of information theory. It is the *cognitive cost* that takes a particular user into account in terms of cognitive properties described in Section 18.5. The total cognitive cost Cost is defined as:

$$\text{Cost}(q,u) = \sum_{p} c_p(u) \cdot \text{norm}_p(\text{cost}_p(q,u)) \tag{18.2}$$

where:

- q is the problem;
- u is the user expertise parameter where we assign each user a value $0 < u < 1$ where $0 < u < 1/3$ represents a *novice*, $1/3 < u < 2/3$ an *average* user, and $2/3 < u < 1$ an *expert*;
- p is a particular cognitive property listed in Section 18.5;
- $\text{cost}_p(q,u)$ is a cost of an individual cognitive property p; it encodes values for the attributes that determine the cost of that property ordered as described above; for example, for registration, according to the literature and as explained above, we need to give an increasing cost to icons, notation index and search, correspondingly,

Table 18.4 *Informational suitability was computed for the* Medical *and* Birds *example (including Probability trees and Set algebra representations). The original statement of both problems was given in Bayesian representation (bay) and for the* Birds *problem also in Natural language (nl) representation. Human experts were surveyed about the use of representations for the Medical problem. Here is one-tailed Pearson's correlation* $r = 0.89$ $(p = 0.053)$. *The r-value indicates that there is a strong positive correlation between the scores the algorithm assigns, and the scores the experts assign. The p-value gives a good indication that this is of statistical significance.*

Informational Suitability				
	Medical		Birds	
	survey	computed (bay)	computed (nl)	computed (bay)
Bayesian	6	17.4	12.6	18.9
Geometric	4.8	11.4	12.8	12.2
Contingency	4.9	8.38	8.5	9.4
NatLang	3.5	6.9	11.9	9.0
Pr-trees	–	9.04	5.7	9.5
Set Algebra	–	10.3	4.4	12.8

but what the exact value should be is unclear; in the future this value could be empirically informed; for now we set them based on the cognitive science literature;

- $c_p(u)$ is a moderating factor for cost of a property according to the expertise of the user: higher-granularity property costs are inflated for novices and deflated for experts;
- norm_p is a function that normalises and makes the scales of each property comparable; this too should be empirically informed, but for now, we pick these provisional values.

Notice that, in principle, to personalise recommendations to individuals, the user can be profiled, and all of the parameter values above can be adjusted using this profile. Based on the literature and expertise, we used provisional values for our prototype *rep2rep* framework and based them for a typical average user ($u = 0.5$). Currently, we devised a profiling test specifically for the quantity scale property. In future work, we plan to design profiling methods for the other properties to inform the values of these parameters. Table 18.5 gives the results of computing the total cognitive cost of a representation, and similarly to the IS score, these are in line with what the experts reported in our survey (lower is better).

Table 18.5 *Computed cognitive costs for* $u = 0.5$ *(average user). For this u, we make* $c_p(u) = 1$ *for all properties: tr = token registration, er = expression registration, tt = number of token types, et = number of expression types, cm = concept-mapping, qs = quantity scale, ec = expression complexity, it = inference type, sr = subRS variety, bf = branching factor, sd = solution depth. Also,* η_p *normalises the value to a number between 0 and 100 (*$\eta_p(x) = 100(x - \min_p)/(\max_p - \min_p)$*), while a constant scales it according to the p's proposed total effect on cognitive cost. The rest of the columns are representational systems.*

	$\mathrm{norm}_p(x)$	nl	bay	geo	cont	tree	s-alg	eul
tr	$0.5 \cdot \eta_{\mathrm{tr}}(x)$	6.6	0	20.4	29.8	50	14.2	14.1
er	$0.5 \cdot \eta_{\mathrm{er}}(x)$	0.8	0	0	2.2	50	1.3	0.2
tt	$1 \cdot \eta_{\mathrm{tt}}(x)$	27.8	88.9	27.8	0	88.9	11.1	100
et	$1 \cdot \eta_{\mathrm{et}}(x)$	4.2	12.5	45.8	58.3	100	0	75
qs	$1 \cdot \eta_{\mathrm{qs}}(x)$	10.7	48.9	100	10.3	16.1	0	4
cm	$2 \cdot \eta_{\mathrm{cm}}(x)$	108	0	200	198.4	181.7	124.7	143.8
ec	$2 \cdot \eta_{\mathrm{ec}}(x)$	22.3	0	46.8	71.1	200	9.6	20.1
it	$2 \cdot \eta_{\mathrm{it}}(x)$	200	25.5	17	0	45.7	25.5	20
sr	$4 \cdot \eta_{\mathrm{sr}}(x)$	0	0	0	400	0	0	400
bf	$4 \cdot \eta_{\mathrm{bf}}(x)$	260	233.4	216.5	400	0	253.3	267.4
sd	$4 \cdot \eta_{\mathrm{sd}}(x)$	260	369.9	0	177	223	369.9	400
	total	81.8	70.8	61.3	122.5	86.9	73.6	131.3
	rank	4	2	1	6	5	3	7

18.7 Applications and future directions

In this work we are laying the foundations for a new class of adaptive technology that aims to automatically select problem solving representations that are suited both to individuals and the particular class of problems that they wish to solve. The answer to the "why" of this endeavour is that representations are fundamental to human cognition and that good representations are intrinsically hard for humans to pick for themselves without expert instructor assistance. The answer to the "how" of the endeavour is to decompose the problem at multiple levels. The highest level distinguishes informational or epistemic requirements of effective representations and the cognitive requirements concerning how humans use representations. On the next level, we are addressing the informational component in terms of formal properties of representations and have proposed algorithms to assess the sufficiency of competing representations. We address the human user component in terms of cognitive properties, at different spatial and temporal scales. Our pilot work has produced encouraging results, where the suitability values computed by our algorithm rank representations in line with how the human experts would rank them. We are currently conducting a more extensive empirical study with teachers as experts. To take into account competence differences, we are developing profiling tests of individuals' abilities relating to specific cognitive properties of representations, targeting quantity scales first given their fundamental role in representations. It remains future work to test how human users' ability to solve tasks is affected by using representations recommended by our system.

Some representations are more effective for humans and some for machines. Indeed, these could come into conflict, for example, when a machine-centric representation is opaque for humans, or when a human-centric representation is computationally inefficient for machines. Should a representation that our system recommends for a user be the one that the tool uses? Our approach is deliberately focused on benefits for the human user and on developing a system to select representations that are suited to users' individual knowledge and abilities. We envisage that such a system could be used as an interface for any AI tool to enable adaptability and personalisation. If a tool uses a machine-centric representation, then our human-centric recommended representation could serve as a target representation that the interface of the tool could translate its output into. But these are questions for future work.

There are many areas of applications of this work. One area is education, with the potential for AI tutoring systems to be adaptive and personalised for individuals in terms of their level of experience with different representations. Switching representation is an instructional strategy that has received little attention in AI and Education (with one exception (Cox and Brna, 1995)), even though effective representation choice has been acknowledged in the field for decades (Kaput, 1992). Intelligent tutoring systems can achieve learning gains of about 0.5 SDs over conventional instruction using techniques like student modelling to drive tutoring actions (Ma, Adesope, Nesbit and Liu, 2014), but representation switching has far greater potential (e.g., (Cheng, 1999; Cheng, 2002; Cheng, 2011*a*) showed a factor of two in learning gains). For instruction specifically about subject matter content, the system might recommend a familiar representation to the student. For (meta–)instruction about representations themselves, the system could pick representations that are stretching but not beyond the potential capability of the learner. Our tutoring system will host a library of alternative representations and its main intervention will be to recommend representations to the student. Another application area is to tailor the explanations given by knowledge-based or decision-support systems by selecting a representation to meet the user's level of sophistication. For example, suppose the probability problem in Figure 18.1 concerned the interpretation of a test outcome knowing that the base proportion of non flying birds differs for a particular sub-population. We envisage a system would administer a few key representation profiling tests to pick which of the three formats in Figure 18.1, or others, to show to the problem solver, and to trade-off what information to provide that has the best likelihood of being correctly comprehended.

Currently, most mechanised problem solving systems have a single fixed representation available to them, typically logics (Kovács and Voronkov, 2013; Harrison, 2009) and diagrams (Jamnik, Bundy and Green, 1999; Barwise and Etchemendy, 1994). There are a few exceptions like Openbox (Barker-Plummer, Etchemendy, Liu, Murray and Swoboda, 2008) and MixR (Urbas and Jamnik, 2014) that implement multiple representations, but these are either deprecated or not targeted at tutoring but formal reasoning. In machine learning, representation learning has become a rich area of research (Bengio, Courville and Vincent, 2013). However, the gap between the computer processing and the user understanding seems to be increasing.

To what extent has our hypothesis been confirmed? Our framework introduces a language and a number of measures that enable the analysis of diverse formal and informal representations. The user's level of expertise in using and the cognitive load of employing a particular representation are captured by our cognitive properties and the formalisation of cognitive costs. We implemented these concepts in a system that can automatically carry out the suitability analysis. We carried out pilot studies that give us confidence that our system produces results in line with those of human experts. There is much to be further developed including the theoretical characteristics of our framework, further empirical evaluations of the effect of using our system's recommendation on human task solving ability, and the application of this work in an AI tutoring environment.

Why is this work important? AI engines that can choose representations of problems in a similar way to humans will be an essential component of human-like computers. They will give machines a powerful ability to adapt representations so that they are better suited to the particular preferences or abilities of the human user. This is especially important when the machine must give users intelligible explanations about its reasoning. Human choices of representations give us clues of their problem solving approaches. This will aid the construction of the world model that reflects that of a human. Consequently, the machines will be able to not only better adapt to the individual user, but also better interpret instructions or information provided by humans. Ultimately, this will lead to machines collaborating with humans in more intuitive ways, as well as tutoring humans to develop their creative problem-solving skills.

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