Homogenization of Random Media: Random Walks, Diffusions and Stochastic Interface Models

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared here or specified in the text. Chapters 2 and 3 are based on the joint work [AT21] with Sebastian Andres, while Chapter 4 is based on [Tay21]. In the collaboration, the contribution of each collaborator was equal. This dissertation is not substantially the same as any that I have submitted, or am concurrently submitting, for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or is being concurrently submitted, for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution. It does not exceed the prescribed word limit for the relevant Degree Committee.

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Abstract

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This thesis concerns homogenization results, in particular scaling limits and heat kernel estimates, for random processes moving in random environments and for stochastic interface models. The first chapter will survey recent research and introduce three models of interest: the random conductance model, the Ginzburg-Landau $\nabla\phi$ model, and the symmetric diffusion process in a random medium.

In the second chapter we present some novel research on the random conductance model; a random walk on an infinite lattice, usually taken to be $\mathbb{Z}^d$ with nearest neighbour edges, whose law is determined by random weights on the edges. In the setting of degenerate, ergodic weights and general speed measure, we present a quenched local limit theorem for this model. This states that for almost every instance of the random environment, the heat kernel, once suitably rescaled, converges to that of Brownian motion with a deterministic, non-degenerate covariance matrix. The quenched local limit theorem is proven under ergodicity and moment conditions on the environment. Under stronger, non-optimal moment conditions, we also prove annealed local limit theorems for the static RCM with general speed measure and for the dynamic RCM. The dynamic model allows for the random weights, or conductances, to vary with time.

Our focus turns to the Ginzburg-Landau gradient model in the subsequent chapter. This is a model for a stochastic interface separating two distinct thermodynamic phases, using an infinite system of coupled stochastic differential equations (SDE). Our main assumption is that the potential in the SDE system is strictly convex with second derivative uniformly bounded below. The aforementioned annealed local limit theorem for the dynamic RCM is applied via a coupling relation to prove a scaling limit result for the space-time covariances in the Ginzburg-Landau model. We also show that the associated Gibbs distribution scales to a Gaussian free field.

In the final chapter, we study a symmetric diffusion process in divergence form in a stationary and ergodic random environment. This is a continuum analogue of the random conductance model and similar analytical techniques are applicable here. The coefficients are assumed to be degenerate and unbounded but satisfy a moment condition. We derive upper off-diagonal estimates on the heat kernel of this process for general speed measure. Lower off-diagonal estimates are also proven for a natural choice of speed measure under an additional decorrelation assumption on the environment. Finally, using these estimates, a scaling limit for the Green's function is derived.

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Contents

Chapter 1

Introduction

The overarching theme of this thesis is the homogenization of random processes moving in random media. Homogenization is a rather broad term; the essential aim of the field is to describe the macroscopic behaviour of some physical system, where on a microscopic level there are heterogeneous fluctuations in the medium. From our perspective as probabilists, we model these fluctuations as random variables. This problem is inspired by questions in physics or material sciences, where one may be interested in modelling the flow of heat or electrical conductivity for instance. The heterogeneous fluctuations then represent variation in the thermal conductivity or electrical resistivity of the medium. Homogenization theory can be approached using many different techniques from physics, mathematical analysis or probability theory; for a detailed exposition see [ZKO94] or [BLP11]. As a motivating example, we give here a probabilistic formulation of such a problem.

Suppose we wish to describe the propagation of heat through some material, taking the microscopic structure to be the lattice $\mathbb{Z}^d$ with heterogeneous conductances $\{\omega(x,y) \in (0,\infty) : x, y \in \mathbb{Z}^d, |x-y| = 1\}$. Then given an initial temperature profile $f : \mathbb{Z}^d \to \mathbb{R}$, the evolution of the heat profile is described by the Cauchy problem

$$\partial_t u(t,x) = \mathcal{L}^\omega u(t,x), \qquad t \geq 0,\, x \in \mathbb{Z}^d, \tag{1.1}$$

with initial condition $u(0,\cdot) = f(\cdot)$. Here, $\mathcal{L}^\omega$ is an elliptic operator acting on functions $g : \mathbb{Z}^d \to \mathbb{R}$, given by

$$\mathcal{L}^\omega g(x) = \sum_{y\,:|x-y|=1} \omega(x,y)\big(g(y) - g(x)\big), \qquad x \in \mathbb{Z}^d. \tag{1.2}$$

Due to the well-studied connections between partial differential equations (PDE) and Markov processes, if we define $(X_t)_{t\geq 0}$ to be the Markov process with infinitesimal generator given by (1.2), then the unique solution to (1.1) is precisely

$$u(t,x) := \mathrm{E}_x^{\omega}\big[f(X_t)\big], \tag{1.3}$$

where E_x^{ω} denotes the expectation under the law of $(X_t)_{t\geq 0}$ started from x. When the configuration of conductances ω is sampled from some probability measure $\mathbb{P}$, the process $\big(X_t\big)_{t\geq 0}$ is a random walk in a random environment, known as the *random conductance model*. Furthermore, we can describe the macroscopic evolution of heat using the diffusive scaling limit of the random conductance model. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a suitably regular initial distribution of heat for the macroscopic model, and for $n \in \mathbb{N}$ write $u_n(t,x)$ for the solution to (1.3) with initial condition $f_n(\cdot) := f(\frac{\cdot}{n})$. This corresponds to the microscopic structure being a lattice with mesh size $\frac{1}{n}$. Then the functional central limit theorem for the random conductance model (see Section 1.1 for a detailed discussion) implies that $\mathbb{P}$-a.s. for all $x \in \mathbb{R}^d$ and $t > 0$,

$$u_n(n^2 t, \lfloor nx \rfloor) = \mathrm{E}_{\lfloor nx \rfloor}^{\omega}\big[f(X_{n^2 t}/n)\big] \longrightarrow \mathrm{E}_x^{\mathrm{BM}}\big[f(B_t)\big], \qquad \text{as } n \to \infty, \tag{1.4}$$

where $(B_t)_{t\geq 0}$ is a Brownian motion with some deterministic, non-degenerate covariance matrix Σ, and $\mathrm{E}_x^{\mathrm{BM}}$ denotes the expectation under its law started from x. If we define the right-hand side of the above equation to be

$$\bar{u}(t,x) := \mathrm{E}_x^{\mathrm{BM}}\big[f(B_t)\big], \qquad x \in \mathbb{R}^d,\, t > 0.$$

Then, again by the connection between PDEs and Markov processes, this function governs the distribution of heat in the macroscopic medium, with initial distribution $\bar{u}(0,x) = f(x)$. More specifically, it solves the Cauchy equation

$$\partial_t \bar{u}(t,x) = Q\bar{u}(t,x), \qquad t \geq 0,\, x \in \mathbb{R}^d, \tag{1.5}$$

with initial configuration $\bar{u}(0,\cdot) = f(\cdot)$. Here, Q is the generator of the aforementioned Brownian motion, explicitly

$$Qf := \sum_{i,j=1}^{d} q_{ij} \frac{\partial^2 f}{\partial x_i\, \partial x_j},$$

where $q = \Sigma^2$. Observe that while the heterogeneous conductances ω are sampled as random variables and may be rather irregular, the macroscopic distribution of heat corresponds to a PDE with deterministic coefficients (1.5). This is the essence of homogenization theory.

We have seen an example of how the homogenization of random media can be understood in terms of scaling limits of random processes. In the remainder of this chapter, we introduce and discuss three specific models of interest.

1.1 Random Conductance Model

In this section we introduce and survey some of the existing literature on the *random conductance model* (RCM), an example of a random walk in a random environment. This model is the subject of the research presented in Chapter 2, where the focus is on local limit theorems. We refer the reader to [Zei06, Bis11, Kum14] for reviews of this model and generalised versions of it.

Consider a countable set of vertices V and suppose we are given a collection of non-negative real numbers $\omega = \{\omega(x,y) \in [0,\infty) \,:\, x,y \in V\}$ which we refer to as weights or conductances. For $x \in V$ let $\mu^\omega(x) := \sum_{y\in V} \omega(x,y)$ and assume this is positive for each $x \in V$. In general, the random walk in environment ω is a Markov process with state space V that if currently at $x \in V$, chooses its next position to be $y \in V$ with probability

$$p^\omega(x,y) = \frac{\omega(x,y)}{\mu^\omega(x)}. \tag{1.6}$$

Herein, we specify to the case of the hypercubic lattice $V = \mathbb{Z}^d$ with undirected nearest-neighbour edges $E_d := \{\{x,y\} \,:\, x,y \in \mathbb{Z}^d, |x-y| = 1\}$, meaning we enforce $\omega(x,y) = 0$ whenever $\{x,y\} \notin E_d$. Also, since the edges are taken as undirected we have $\omega(x,y) = \omega(y,x)$ for all $x,\, y \in \mathbb{Z}^d$. In this thesis we focus solely on the continuous-time model which can be constructed by Poissonizing the discrete-time Markov chain with transition probabilities given by (1.6), namely, consider the transition kernel

$$Q_t^\omega(x,y) := \sum_{n\in\mathbb{N}} \frac{t^n}{n!}\mathrm{e}^{-t}(p^\omega)^n(x,y), \qquad x,y \in \mathbb{Z}^d.$$

This gives a continuous-time process where the waiting time at each vertex is distributed as $\mathrm{Exp}(1)$. Since this distribution is independent of the vertex in question, this model is known as the *constant speed random walk* (CSRW). More generally, we can consider the case where the jump rate is determined by some function $\theta^\omega : \mathbb{Z}^d \to \mathbb{R}$ of the current position, which may also depend on the environment. This is constructed by considering the process with generator

$$\mathcal{L}_\theta^\omega f(x) := \frac{1}{\theta^\omega(x)} \sum_{y\sim x} \omega(x,y) \big(f(y) - f(x)\big), \tag{1.7}$$

acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$. The waiting time at vertex $x \in \mathbb{Z}^d$ then has distribution $\mathrm{Exp}\big(\mu^\omega(x)/\theta^\omega(x)\big)$.

We can glean insight from a simple example. Consider $\omega(e) = 1$ for all $e \in E_d$, then the discrete-time Markov chain, $\big(X_n\big)_{n\geq 1}$, with transition probabilities given by (1.6) becomes the simple symmetric random walk on $\mathbb{Z}^d$. The increments of this walk are independent and identically distributed so simply by Donsker's invariance principle [Don51] we have that the diffusively rescaled random walk converges in distribution to a Brownian motion.

$$\Big(\frac{1}{n}X_{\lfloor n^2 t\rfloor}\Big)_{t\in[0,1]} \xrightarrow{d} \big(B_t\big)_{t\in[0,1]} \qquad \text{as } n\to\infty, \tag{1.8}$$

where $(B_t)_{t\geq 0}$ denotes a standard Brownian motion on $\mathbb{R}^d$. In the case of varying weights however, the increments are no longer stationary nor independent. This difficulty can be somewhat overcome by sampling ω from a stationary probability space. Henceforth we introduce the *random environment*. Let $(\Omega,\mathcal{F}) = \big(\mathbb{R}_{\geq 0}^{E_d}, \mathcal{B}(\mathbb{R}_{\geq 0})^{\otimes E_d}\big)$ be the measurable space of all possible environments. Now let $\mathbb{P}$ be an arbitrary probability measure on $(\Omega,\mathcal{F})$ and let $\mathbb{E}$ denote the respective expectation. We equip Ω with a group of spatial translations $(\tau_x)_{x\in\mathbb{Z}^d}$ acting on the environment as

$$\tau_x\,\omega(y,z) = \omega(y+x, z+x) \qquad \forall\, x\in\mathbb{Z}^d,\ \{y,z\}\in E_d.$$

Generally, one assumes that the law of the environment is invariant under such translations, i.e. $\mathbb{P}\circ\tau_x^{-1} = \mathbb{P}$ for all $x\in\mathbb{Z}^d$. This is clear whenever the conductances are independent of each other in which case $\mathbb{P}$ becomes the product measure. However, recent results, including ours in Chapter 2, focus on a general ergodic environment, that is $\mathbb{P}(A)\in\{0,1\}$ for any $A\in\mathcal{F}$ such that $\tau_x(A)=A$ for all $x\in\mathbb{Z}^d$. The conductances $\omega(e)$, $e\in E_d$, are now random variables and we refer to the process with generator given by (1.7) as the random conductance model with *speed measure* θ^ω. For $x\in\mathbb{Z}^d$, denote P_x^ω the law of this process $(X_t)_{t\geq 0}$ subject to the initial condition $P_x^\omega(X_0=x)=1$ and write E_x^ω for the expectation under this measure. In Chapter 2 we will prove results for general speed measure, however, in the present discussion we mainly concern ourselves with the CSRW given by $\theta^\omega\equiv\mu^\omega$ and another canonical choice of speed measure, $\theta^\omega\equiv 1$. The latter gives rise to the *variable speed random walk* (VSRW), so-called because its waiting time at $x\in\mathbb{Z}^d$ now depends on the conductances around this vertex. More precisely, the jump time is exponentially distributed with rate $\mu^\omega(x)$. The process $(X_t)_{t\geq 0}$ with general speed measure θ^ω can alternatively be defined via a time-change of the VSRW $(Y_t)_{t\geq 0}$. Consider the functional

$$A_t := \int_0^t \theta^\omega(Y_s)\,ds, \qquad t>0,$$

and denote its right-continuous inverse $a_t := \inf\{s>0 \,:\, A_s>t\}$. Then $X_t = Y_{a_t}$ for all $t\geq 0$.

There are two natural laws on the path space of the above process:

1. The quenched law, concerning $\mathbb{P}$-a.s. phenomena; fix an instance of the environment ω and study the process $(X_t)_{t\geq 0}$ under P_x^ω. Under the quenched law the process is always Markovian, but the irregularities of the heterogeneous environment may need to be controlled.

2. The annealed law, $\mathbb{E}P_x^\omega$, describing the evolution of the process averaged over instances of the environment ω. Under this law the process is not necessarily Markovian, however, the environment is homogenized.

The random conductance model has been the subject of extensive research that has intensified in the last decade. One of the key questions of interest is the functional central limit theorem, akin to (1.8): when does the RCM in the diffusive scaling limit converge to Brownian motion? This question can be approached under either the quenched or annealed law, and we give the precise quenched statement below. In general, for this result, the quenched statement implies the annealed one. The ultimate aim of this field of research is to describe the minimal conditions on the probability measure $\mathbb{P}$ such that homogenization results hold. As such, there has been a concerted effort in recent years to establish the *quenched functional central limit theorem* (QFCLT) under very general conditions on $\mathbb{P}$.

Quenched Functional Central Limit Theorem. Let $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$ for $t \geq 0$. For all $T > 0$ and every bounded continuous function F on the Skorokhod space $\mathcal{D}([0,T], \mathbb{R}^d)$, we have that for $\mathbb{P}$-a.e. ω, $E_0^\omega\big[F(X^{(n)})\big] \to E_0^{\mathrm{BM}}\big[F(B)\big]$ as $n \to \infty$. Here, E_0^{BM} denotes the expectation under the law of a Brownian motion $B = (B_t)_{t\geq 0}$ on $\mathbb{R}^d$ started at 0, with some deterministic, non-degenerate covariance matrix.

The above is also known as the *quenched invariance principle*. A weak version of this result, where the convergence is in probability rather than almost surely in $\mathbb{P}$, was first established for the VSRW in [DMFGW89], cf. also [KV86]. The authors prove this whenever $\mathbb{P}$ is ergodic and the first moment of the conductances is finite, $\mathbb{E}[\omega(e)] < \infty$. Once the QFCLT is obtained for the VSRW then it holds for the RCM with speed measure θ^ω provided this is stationary, i.e. $\theta^\omega(x) = \theta^{\tau_x \omega}(0)$, and $0 < \mathbb{E}[\theta^\omega(0)] < \infty$ [ABDH13, Remark 1.5]. That the process is reversible with respect to the invariant measure, $\theta^\omega \equiv 1$ in the case of the VSRW, is of key importance. For the asymmetric random walk in a random environment i.e. when $\omega(x,y) \neq \omega(y,x)$, the analysis is hard and the theory less complete, cf. [Zei06]. However, there are many results in the case of balanced environments, see [Law83, GZ12, BD14, DGR18] for invariance principles.

Some time later, the FCLT was extended to a quenched, $\mathbb{P}$-a.s. statement as above. This was first proven in [SS04] in the case where the conductances $\omega(e)$ are independent and identically distributed under $\mathbb{P}$, and uniformly elliptic: there exists $c \in (0, 1)$ such that,

$$c \le \omega(e) \le c^{-1}, \qquad \forall\, e \in E_d. \tag{1.9}$$

The strategy in [SS04] is to approximate X by a martingale term and a *corrector* term. The former converges in the diffusive limit by a martingale central limit theorem and the latter is shown to be sublinear using Gaussian bounds on the heat kernel of the process.

The *heat kernel* is the transition density of X with respect to the invariant measure θ^ω, defined for $t \ge 0$ and $x, y \in \mathbb{Z}^d$ as

$$p^\omega_\theta(t, x, y) \;:=\; \frac{P^\omega_x\,(X_t = y)}{\theta^\omega(y)}. \tag{1.10}$$

The function $u_t(y) := p^\omega_\theta(t, x, y)$ solves the formal parabolic equation with random coefficients

$$\partial_t u_t(y) = \mathcal{L}^\omega_\theta u_t(y), \qquad \forall\, t > 0,\; y \in \mathbb{Z}^d. \tag{1.11}$$

Gaussian, or off-diagonal, estimates are another homogenization result of interest and the focus of Chapter 4 where the topic is diffusions in random environments. Typically this means bounds like the following, where d_θ denotes a metric on $\mathbb{Z}^d$ (for the CSRW it is the graph metric but there are complexities for general speed measure).

Off-Diagonal Heat Kernel Estimates. For $\mathbb{P}$-a.e. ω, there exist constants $c_i > 0$ such that for $t > 0$ and $x,\, y \in \mathbb{Z}^d$,

$$c_1\, t^{-d/2} \mathrm{e}^{-\frac{c_2\, d_\theta(x,y)^2}{t}} \le p^\omega_\theta(t, x, y) \le c_3\, t^{-d/2} \mathrm{e}^{-\frac{c_4\, d_\theta(x,y)^2}{t}}. \tag{1.12}$$

For a uniformly elliptic environment, said Gaussian estimates were obtained in [Del99] in the general context of weighted graphs. Such bounds are known to be intimately connected to the existence of a parabolic Harnack inequality for solutions to (1.11), cf. [GT02, BB04]. The Harnack inequality gives regularity of the solution to a parabolic or elliptic equation such as (1.11) by bounding its maximum value on a space-time cylinder by a constant times its minimum value on a larger but comparable cylinder. It is derived in this context by the celebrated method of Moser iteration [Mos61, Mos63, Mos71]. We draw on this method in Chapter 4 and on the related De Giorgi iteration [DG57] in Chapter 2. The strategy of proof in [SS04] is inspired by the arguments in [Osa83] for diffusion processes with random coefficients - this model is our focus in Chapter 4 where we derive Gaussian heat kernel estimates in a degenerate, ergodic environment.

One interesting model that can be recovered via a specific choice of law $\mathbb{P}$ is the simple random walk on a supercritical percolation cluster. Independently for each $e \in E_d$ set $\omega(e) \sim \text{Ber}(p)$ for $p > p_c(d)$, where $p_c(d)$ is the bond percolation threshold in dimension d. For this model in dimension $d \geq 2$ the quenched invariance principle was proven in [BB07] and independently in [MP07]. Gaussian bounds are established in [Bar04].

For i.i.d. conductances with $d \geq 2$, the RCM is by now quite well understood. Building on a series of papers [BP07, Mat08, BD10], one significant development was the QFCLT in [ABDH13] for unbounded conductances. No ellipticity condition is required here, only the assumption that $\mathbb{P}(\omega(e) > 0) > p_c(d)$. The authors prove this for the VSRW and also for the CSRW under the additional condition of a finite first moment, $\mathbb{E}[\omega(e)] < \infty$. One might wonder why the case of unbounded conductances is more difficult than the uniformly elliptic case. This is due to the 'trapping' phenomenon; consider an edge $e_0 \in E_d$ with high conductance $\omega(e_0) = \mathcal{O}(K) >> 1$ surrounded by weights $\omega(e) = \mathcal{O}(1)$, as illustrated in Figure 1.1. Then the CSRW is expected to spend $\mathcal{O}(K)$ time in this 'trap' before escaping and this can lead to sub-diffusive behaviour. For example, if $\mathbb{E}[\omega(e)] = \infty$ and the conductances have suitably long polynomial tails, the trapping effect means the QFCLT may fail for the CSRW (although under a different scaling the process converges to the fractional kinetics process) [BZ10,BČ11,Čer11]. So the work [ABDH13] is essentially optimal for an i.i.d. environment.

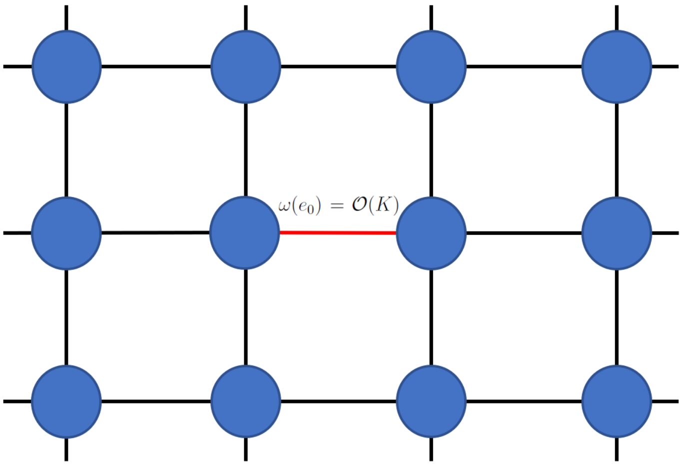

Figure 1.1: The red edge e_0 has high conductance $\mathcal{O}(K) >> 1$ and is surrounded by black edges of conductance $\mathcal{O}(1)$. The red bond acts as a 'trap'.

The next challenge after relaxing the uniform ellipticity condition was to transfer the arguments from an i.i.d. environment to general ergodic $\mathbb{P}$. In the i.i.d. setting just discussed, many methods rely on the independence of conductances together with probabilistic bounds to control the environment, see the definitions of *good* and *very good* balls in [Bar04] for instance. With an ergodic environment this is no longer possible, however, ergodic theory does allow for the control of averages of the conductances on suitably large balls. The QFCLT is obtained for a general ergodic environment in dimension $d = 2$ in [Bis11] under the optimal moment assumptions $\mathbb{E}[\omega(e)] < \infty$ and $\mathbb{E}[\omega(e)^{-1}] < \infty$, where a thorough survey of the model is also presented. The arguments therein are inspired by the percolation setting and it is not clear how to extend these to higher dimensions. However, under the moment condition: there exist $p,\, q \in (1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ such that

$$\mathbb{E}[\omega(e)^p] + \mathbb{E}[\omega(e)^{-q}] < \infty, \tag{1.13}$$

the QFCLT is established for dimensions $d \geq 2$ in [ADS15]. The proof, again motivated by ideas from diffusions in random environments [FK97, FK99], uses the Moser iteration technique to derive a maximal inequality for the corrector. The inequality for the exponents p and q was recently relaxed in [BS20b]. In both Chapter 2 on the RCM and Chapter 4 concerning diffusions in random environments, we will work with ergodic environments and moment conditions of this form. Moment conditions are indeed necessary for the QFCLT and aren't an overly restrictive assumption, as evidenced by the work in [BBT16] where ergodic environments are constructed for $d \geq 2$ satisfying $\mathbb{E}[\omega(e)^p] + \mathbb{E}[\omega(e)^{-p}] < \infty$ for any $p \in (0, 1)$ such that the weak functional central limit theorem holds but the quenched statement fails. As has been proven in the two-dimensional case, the condition $\mathbb{E}[\omega(e)] + \mathbb{E}[\omega(e)^{-1}]$ is conjectured to be the optimal moment condition in dimension $d > 2$ for the QFCLT to hold in a general ergodic environment [Bis11] and to prove it under these conditions is still an open problem.

The RCM has also been studied extensively in dimension $d = 1$. The QFCLT has recently been proven under optimal moment conditions for a time-dependent ergodic environment in [Bis19], cf. also [DS16]. Nevertheless, in Chapter 2 we focus solely on dimensions $d \geq 2$, since the techniques we utilize there are not easily transferable to the one-dimensional setting.

Another homogenization result is the *local limit theorem* (LLT). This states that a rescaled version of the heat kernel (1.10) converges to the transition density of a Brownian motion. Again, we are interested in proving this result under very general assumptions on the law of the conductances $\mathbb{P}$. Our main results in Chapter 2 are local limit theorems under both the quenched and annealed law for an ergodic environment satisfying moment conditions. We give an example of the basic quenched statement below.

Quenched Local Limit Theorem. Let $t > 0$, $x \in \mathbb{Z}^d$ and $a := \mathbb{E}[\theta^\omega(0)]^{-1}$. For $\mathbb{P}$-a.e. ω, $n^d p^\omega_\theta(n^2 t, 0, nx) \to a\, p_{\mathrm{BM}}(t, 0, x)$ as $n \to \infty$, where $p_{\mathrm{BM}}(t, 0, x)$ denotes the transition density of a Brownian motion on $\mathbb{R}^d$ with some deterministic, non-degenerate covariance matrix.

In general, the above is a stronger statement than the QFCLT. The QFCLT is a statement about the whole path of the random walk whereas the local limit theorem concerns pointwise transition probabilities, hence it is possible that the latter, along with Gaussian bounds, may fail while the former holds. This is due to the aforementioned trapping phenomenon and examples of sub-diffusive heat kernel decay for i.i.d. environments are constructed in [BBHK08, Bou10, BB12]. For sharp conditions on the polynomial lower tails of i.i.d. conductances near zero for the parabolic Harnack inequality and local limit theorem to hold we refer to [BKM15]. This is one reason why the random conductance model is an interesting model to study, and whilst remarkable progress has been made in the last decade, many questions are still to be fully resolved. For the FCLT, the quenched statement implies the annealed one, yet this is not the case for the LLT. Indeed there are regimes where the LLT fails under the annealed measure; in [FM06] an environment of unbounded and independent conductances is constructed such that the annealed return probability decays arbitrarily slowly. In addition to quenched results, in Chapter 2 we will present local limit theorems for the RCM and the dynamic RCM under the annealed law. The context of these results and the techniques applied to prove them are discussed in greater detail therein.

The dynamic RCM, introduced precisely in Chapter 2, is a generalisation where the environment ω evolves in time. Variations of the RCM include random walks on point processes in $\mathbb{R}^d$ [CFP13]; the simple random walk on Penrose tilings [BT14]; random walks on percolation clusters with long-range correlations [DRS14, Sap17]; and random conductance models with long-range jumps [CKW20, BCKW21].

The random conductance model has found applications in models for electrical networks, see for instance [DS84]. There are also connections with stochastic interface models - this is the topic of the following subsection.

1.2 Ginzburg Landau $\nabla\phi$ Model

A somewhat unexpected context in which one encounters random conductance models is that of stochastic interface models. Such models are proposed to study the boundary or interface separating two distinct phases in a statistical physics system when phase coexistence occurs. For example, at zero degrees celsius, the distinct macroscopic states of water and ice can coexist and interface models aim to describe the sharp hypersurfaces that separate these two phases. Similar phenomena arise in alloys consisting of various metals and in crystal formation. One well-established model for a stochastic interface is the *Ginzburg-Landau gradient model* (or $\nabla\phi$ model), which is the central object studied in Chapter 3. Here we give a brief introduction to this fascinating model which, as we will see, is intimately connected to the RCM of the previous section.

The principal goal of statistical mechanics is to understand the macroscopic behaviour of materials from an atomic or molecular, microscopic level. The scaling parameter $n \in \mathbb{N}$ connects these two scales and can be interpreted as the ratio between a typical unit on the macroscopic scale, for example a metre or centimetre, and distance on the microscopic scale such as a nanometre. Since this ratio is rather large, the macroscopic phenomena can then be understood by taking a scaling limit $n \to \infty$. The model for the microscopic interactions involves randomness and typically has ergodic or mixing properties, so scaling limits can often be characterised using homogenization results from probability theory. Chapter 3 consists of specific results in this direction.

In the $\nabla\phi$ model, the interface is described by a field of height variables evolving in time $\{\phi_t(x) : x \in \mathbb{Z}^d,\, t \geq 0\}$. One may take a different base graph to $\mathbb{Z}^d$ which we refer to as the infinite volume model, for example we will subsequently work with increasing finite graphs $L_n \uparrow \mathbb{Z}^d$ to construct the infinite volume process. The dynamics are governed by the following infinite system of stochastic differential equations involving nearest neighbour interaction:

$$\phi_t(x) = \phi_0(x) - \int_0^t \sum_{y:|x-y|=1} V'(\phi_s(x) - \phi_s(y))\, ds + \sqrt{2}\, w_t(x), \qquad x \in \mathbb{Z}^d,\, t > 0. \quad (1.14)$$

Here $\{w(x) : x \in \mathbb{Z}^d\}$ is a collection of independent Brownian motions and $V \in C^2(\mathbb{R}, \mathbb{R})$ is an even potential, usually taken to be convex. The Hamiltonian assigns an energy to a configuration by penalising the gradients between neighbouring heights. It is defined as

$$H(\phi) = \frac{1}{2} \sum_{\substack{x,y \in \mathbb{Z}^d \\ |x-y|=1}} V(\phi(x) - \phi(y)).$$

Note that the drift term in (1.14) can also be written as

$$-\sum_{y:|x-y|=1} V'(\phi_t(x) - \phi_t(y)) = -\frac{\partial H(\phi)}{\partial \phi_x}.$$

So, intuitively, the drift term encourages the surface to remain broadly level and one might expect that there exists an equilibrium measure for the dynamic. This is known as the Gibbs measure. In the infinite volume case it is given by the formal expression

$$\mu(d\phi) \propto \exp(-H(\phi)) \prod_x d\phi(x),$$

acting on $\phi \in \mathbb{R}^{\mathbb{Z}^d}$. It is not a priori clear how to define this measure rigorously.

Investigating the fluctuations of the macroscopic interface has been quite an active field of research, see [Fun05] for a survey. Many results have been established under the assumption that there exist constants c_-, $c_+ > 0$ such that

$$c_- \leq V''(x) \leq c_+, \qquad \forall\, x \in \mathbb{R}. \tag{1.15}$$

Under the above assumption, infinite volume Gibbs measures for the heights can be constructed by taking a limit of finite domains for dimensions $d \geq 3$. In dimension $d = 2$, the heights diverge as the size of the domain approaches infinity, however, one can construct Gibbs measures on the gradients $\nabla\phi$. Existence and uniqueness of such gradient Gibbs measures are proven in [FS97], along with a homogenisation result relating the scaling limit of the heights to a deterministic partial differential equation. Large deviation principles for the rescaled heights on a finite domain are established in [DGI00] for zero boundary conditions, and in [FS04] for non-zero boundary conditions when the potential has an additional self-interaction term. Another problem of interest is studying the extremal values of the surface; various bounds are established on the maximum of static heights under a Gibbs measure in [DG00], also for ϕ_t in the limit $t \to \infty$ in [DN07].

Central limit theorems (CLT), stating that the fluctuations of the interface scale to a continuum Gaussian free field, have also been extensively studied. For results on the gradient field $\nabla\phi$ see [NS97] for static heights, [GOS01] for the time-varying case and [Mil11] for the case of a finite domain. A central limit theorem for the field ϕ is given in [BS11] for a certain class of non-convex potentials.

One special case is when the potential is quadratic, $V(x) = \frac{1}{2}x^2$. This model, depicted in Figure 1.2, is known as the *discrete Gaussian free field* (DGFF), also as the *harmonic crystal*. Large deviation principles and the CLT for this Gaussian model are derived in [BAD96]. For bounds on the extremal values of the surface in dimension $d = 2$ see [BDG01, Dav06]. Deeper results have also been established such as if one takes a DGFF on a grid with mesh size converging to zero, then the zero contour line converges to a Schramm-Loewner evolution SLE_4 curve [SS11].

One tool that is imperative in analysing the $\nabla\phi$ model and proving the results discussed above is the *Helffer-Sjöstrand representation* [HS94]. This elegant coupling relation allows us to express correlation functions of the heights sampled under a Gibbs measure in terms of a random walk amongst dynamic random conductances. More concretely, consider the dynamic RCM (defined precisely in Section 2.1.3) with conductances given by

$$\omega_t(x,y) = V''(\phi_t(y) - \phi_t(x)), \qquad \{x,y\} \in E_d,\ t > 0, \tag{1.16}$$

and denote the transition probabilities of this process by $p^\omega(s,t,x,y)$ for $s \leq t$, $x, y \in \mathbb{Z}^d$.

Then if $F, G \in C^1(\mathbb{R}^{\mathbb{Z}^d})$ are functions with bounded derivatives depending only on finitely many coordinates, we have the following expression,

$$\mathbb{C}\mathrm{ov}_\mu(F(\phi_0), G(\phi_t)) = \int_0^\infty \sum_{x,\, y \in \mathbb{Z}^d} \mathbb{E}_\mu\left[\frac{\partial F(\phi_0)}{\partial \phi(x)} \frac{\partial G(\phi_{t+s})}{\partial \phi(y)}\, p^\omega(0, t+s, x, y)\right] ds, \tag{1.17}$$

where μ is a stationary, ergodic Gibbs measure and $\mathbb{E}_\mu$, $\mathbb{C}\mathrm{ov}_\mu$ denote expectation and covariance respectively, under the law of $(\phi_t)_{t\geq 0}$ started from the distribution μ.

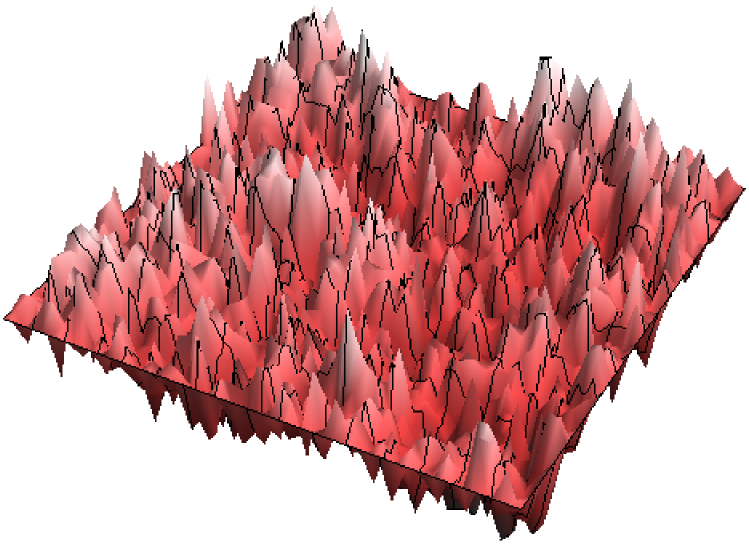

Figure 1.2: A discrete Gaussian free field sampled from the equilibrium ϕ-Gibbs measure on a 60 x 60 grid with Dirichlet boundary conditions (Source: en.wikipedia.org/wiki/Gaussian_free_field).

In the case of the DGFF, we see that the dynamic RCM with conductances given by (1.16) is merely the simple random walk on $\mathbb{Z}^d$. This simplifies the analysis of the model as many quantities such as the Green's function can be estimated rather precisely. A second observation is that for more general potentials, the condition (1.15) directly corresponds to the conductances being uniformly bounded. Many questions on the $\nabla\phi$ model can then be rephrased in terms of homogenization results for the RCM in a uniformly elliptic environment, such as those discussed in the previous section. For instance, the celebrated CLT in [GOS01] is proven by directly applying the FCLT for the RCM [KV86] together with Gaussian heat kernel estimates.

So relaxing the assumption (1.15) on the potential amounts to dealing with unbounded conductances in the RCM. In Chapter 3, we exploit this beautiful coupling relation together with recent advances for degenerate random environments to prove homogenization results when the potential may have unbounded second derivative. In particular, we prove a scaling limit for the pointwise covariance functions of the interface and a CLT for the fluctuations under a static Gibbs measure by applying respectively the annealed local limit theorem and the annealed functional central limit theorem for the dynamic RCM. The requisite local limit theorem is proven in Chapter 2 and the FCLT is found in [ACDS18].

1.3 Symmetric Diffusions in Random Media

In Chapter 4, we turn our attention to a third model, the symmetric diffusion process in a random environment. The state space is $\mathbb{R}^d$ and the process corresponds to the following generator in divergence form,

$$\mathcal{L}^\omega u(x) = \nabla \cdot (a^\omega(x) \nabla u(x)), \quad x \in \mathbb{R}^d, \tag{1.18}$$

where $a^\omega(x)$ is a symmetric d-dimensional matrix depending on a random parameter ω. As for the RCM, we will assume that ω is drawn from some probability space $\mathbb{P}$ upon which we have a measurable group of translations, $\{\tau_x\}_{x\in\mathbb{R}^d}$. This space will typically be assumed to be ergodic under such translations. In the case when the field of coefficients $\{a^\omega(x)\}_{x\in\mathbb{R}^d}$ is sufficiently smooth, we have by the seminal theory of Itô, see for instance [SV79, RY99], that the diffusion $X_t = (X_t^1, ..., X_t^d)$ with generator given by (1.18) corresponds to the stochastic differential equations,

$$dX_t^i = \sqrt{2} \sum_{j=1}^d \sigma_{ij}(X_t)\, dW_t^j + \sum_{j=1}^d \partial_j a_{ij}^\omega(X_t)\, dt, \quad t > 0, \tag{1.19}$$

for $i = 1, \dots, d$, where $W_t = (W_t^1, ..., W_t^d)$ denotes a standard d-dimensional Brownian motion and σ is given by $a^\omega(x) = \sigma\sigma^T(x)$ for all $x \in \mathbb{R}^d$. The operator (1.18) can be interpreted as a continuum analogue of (1.7). As such, there is a significant interplay between this model and the random conductance model, and the homogenization results we discussed in Section 1.1 are also of particular interest herein.

In this context, the quenched functional central limit theorem, or quenched invariance principle, is formulated in the same way as in Section 1.1: for almost every ω, the rescaled process $(X_t^{(n)})_{t\geq 0} := (\frac{1}{n} X_{n^2 t})_{t\geq 0}$ converges to a Brownian motion $(B_t)_{t\geq 0}$ with deterministic, non-degenerate covariance matrix. In [PV81], this problem is considered for the case of differentiable, periodic coefficients. The QFCLT is established in [PV81] under the conditions that the environment is ergodic, the coefficient matrix $a^\omega(x)$ is symmetric and differentiable with bounded derivatives, and the coefficients are uniformly elliptic, i.e. there exist constants $\lambda,\, \Lambda \in (0,\infty)$ such that

$$\lambda \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,\,j=1}^{d} a_{ij}^{\omega}(x)\xi_i\xi_j \le \Lambda \sum_{i=1}^{d} \xi_i^2, \tag{1.20}$$

for all $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^d$ and $\mathbb{P}$-a.e. ω. The case without drift is studied in [PV82], however, this is an exception to the divergence form of (1.18). These results were extended in [Osa83] to more general forms of operator, provided the coefficients are smooth, periodic and uniformly elliptic.

However, many random drifts of interest, Gaussian fields for example, do not satisfy the assumption of bounded coefficients. So, as for the RCM, there has recently been a concerted effort to establish results outside of the uniformly elliptic regime. A non-symmetric operator is studied in [FK97] with uniformly elliptic symmetric part and unbounded anti-symmetric part with finite p^{th} moment under $\mathbb{P}$ for $p > d$. In one recent development in the degenerate setting [BM15], the QFCLT is proven for operators of the specific type $\mathcal{L}u := \mathrm{e}^{V}\nabla\cdot(\mathrm{e}^{-V}\nabla u)$, where $V : \mathbb{R}^d \to \mathbb{R}$ is periodic, measurable and satisfies a local integrability condition. Results in this vein are often stated either for periodic/almost periodic coefficients or for coefficients which are realisations of stationary random fields. The former being a special case of the latter.

Inspired by recent results on the RCM, in [CD16] a general ergodic environment is considered with measurable, unbounded coefficients. The environment satisfies a weak version of ellipticity: there exist random functions $\lambda^{\omega},\, \Lambda^{\omega} : \mathbb{R}^d \to (0,\infty)$ such that

$$\lambda^{\omega}(x) \sum_{i=1}^{d} \xi_i^2 \le \sum_{i,\,j=1}^{d} a_{ij}^{\omega}(x)\xi_i\xi_j \le \Lambda^{\omega}(x) \sum_{i=1}^{d} \xi_i^2, \qquad \forall\, x \in \mathbb{R}^d,\, \forall\, \xi \in \mathbb{R}^d. \tag{1.21}$$

Under a moment assumption on the degenerate, unbounded functions λ^{ω} and Λ^{ω}, the construction of the diffusion associated to (1.18) and the QFCLT are the main results of [CD16]. Clearly, if the coefficients are merely measurable then the generator is only defined in a formal sense and one cannot directly use Itô calculus to define the diffusion. Some care must therefore be taken to construct it. This problem is overcome in [CD16] using Dirichlet forms, drawing on the theory developed in [FOT94]. The stochastic calculus of Dirichlet forms had previously been applied to construct diffusions associated to measurable coefficients in [BM15] and for the uniformly elliptic, periodic case in [Lej01]. The QFCLT is proven in a similar way to how it is proven for the RCM: first, the diffusion process is decomposed into a martingale and a corrector term. Then, in what constitutes the main step, a maximal inequality is derived using Moser iteration. This in turn gives sublinearity of the corrector which is what is required for the invariance principle.

Regarding the local limit theorem, under the same assumptions as in [CD16], the authors extend this work in [CD15] and prove that the process has a transition density, or heat kernel, $p^\omega(t,x,y)$. Further, they prove that when this heat kernel is suitably rescaled, it converges to the heat kernel of a Brownian motion [CD15]. Of course, in the continuum, the heat kernel doesn't have such an explicit definition as (1.10) for the RCM. Nevertheless, one can make use of the fact that the transition density formally solves a parabolic partial differential equation with random coefficients. Let $u_t(y) := p^\omega(t,x,y)$ for $t > 0$ and $x, y \in \mathbb{R}^d$, then

$$\partial_t u_t(y) - \mathcal{L}^\omega u_t(y) = 0, \qquad t > 0,\, y \in \mathbb{R}^d. \tag{1.22}$$

In truth, it solves a weak version of this equation given in terms of the Dirichlet form (for the proper formulation see Definition 4.2.1). The proof of the local limit theorem in [CD15] follows a similar method to [ADS16a] for the RCM, first establishing a parabolic Harnack inequality for solutions to (1.22), then applying it to derive Hölder continuity of the heat kernel. As discussed in Section 1.1, heat kernel estimates are connected to the QFCLT and LLT. However, for a degenerate environment, whilst the Harnack inequality leads to near-diagonal heat kernel estimates such as those in Proposition 4.3.1, it does not directly imply Gaussian-type estimates. In Chapter 4, our main contribution is extending the theory of symmetric diffusions in a degenerate, ergodic environment by establishing upper and lower off-diagonal estimates on the heat kernel. Our upper estimate also covers the case of general speed measures θ^ω, which is the process with generator given by

$$\mathcal{L}^\omega u(x) = \frac{1}{\theta^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(x)), \quad x \in \mathbb{R}^d. \tag{1.23}$$

Alternatively, this process may be obtained via a time-change of the original one. It has a transition density, $p^\omega_\theta(t,x,y)$, with respect to the measure $\theta^\omega(x)\,dx$ rather than w.r.t. the Lebesgue measure. For the process with speed measure $\theta^\omega \equiv \Lambda^\omega$, these heat kernel estimates are given in terms of the Euclidean metric (see Corollary 4.4.1). This choice can be interpreted as the 'constant speed' diffusion corresponding to the canonical speed measure μ^ω in the context of the RCM. However, in the setting of general speed, the upper Gaussian estimate is given in terms of the *intrinsic metric*, a metric on $\mathbb{R}^d$ that depends on the random field a^ω and the speed measure θ^ω. In this case, an additional assumption that for $\mathbb{P}$-a.e. ω, the functions $a^\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\theta^\omega : \mathbb{R}^d \to \mathbb{R}$ are continuous is required for sufficient regularity of this environment-dependent metric, in order to derive the upper estimate. Because the intrinsic metric is not comparable to the Euclidean metric in general, for the lower heat kernel estimate we focus solely on the case $\theta^\omega \equiv \Lambda^\omega$. This is due to the chaining argument applied for the derivation which requires balls in the intrinsic and Euclidean metrics to be comparable. Finally, as an application of the off-diagonal heat kernel estimates, we will also prove a scaling limit for the Green kernel of the process, given by

$$g^\omega(x,y) := \int_0^\infty p_\theta^\omega(t,x,y)\,dt, \qquad x,\, y \in \mathbb{R}^d.$$

All of the concepts discussed above will be introduced precisely in Chapter 4, where the methods used to prove these results are also discussed in detail. Regarding applications of this model, we are not aware of any connections to stochastic interface models. But homogenization of the PDE with random coefficients (1.22) corresponds to a heat conduction problem; the setup is similar to the discussion at the beginning of this chapter regarding the RCM.

Chapter 2

Local Limit Theorems for the RCM

Abstract. In this chapter we study a continuous-time random walk on $\mathbb{Z}^d$ in an environment of random conductances taking values in $(0, \infty)$. For a static environment, we extend the quenched local limit theorem to the case of a general speed measure, given suitable ergodicity and moment conditions on the conductances and on the speed measure. Under stronger moment conditions, an annealed local limit theorem is also derived. Furthermore, an annealed local limit theorem is exhibited in the case of time-dependent conductances, under analogous moment and ergodicity assumptions.

2.1 Introduction

2.1.1 The Model

We consider the graph $G = (\mathbb{Z}^d, E_d)$ of the hypercubic lattice with the set of nearest-neighbour edges $E_d := \{\{x, y\} : x, y \in \mathbb{Z}^d, |x - y| = 1\}$ in dimension $d \geq 2$. We place upon G positive weights $\omega = \{\omega(e) \in (0, \infty) : e \in E_d\}$, and define two measures on $\mathbb{Z}^d$,

$$\mu^\omega(x) := \sum_{y \sim x} \omega(x, y), \qquad \nu^\omega(x) := \sum_{y \sim x} \frac{1}{\omega(x, y)}.$$

Let $(\Omega, \mathcal{F}) := (\mathbb{R}_+^{E_d}, \mathcal{B}(\mathbb{R}_+)^{\otimes E_d})$ be the measurable space of all possible environments. We denote by $\mathbb{P}$ an arbitrary probability measure on $(\Omega, \mathcal{F})$ and $\mathbb{E}$ the respective expectation. The measure space $(\Omega, \mathcal{F})$ is naturally equipped with a group of space shifts $\left\{\tau_z : z \in \mathbb{Z}^d\right\}$, which act on Ω as

$$(\tau_z \omega)(x, y) \ := \ \omega(x + z, y + z), \qquad \forall \{x, y\} \in E_d. \tag{2.1}$$

Let $\theta^\omega : \mathbb{Z}^d \to (0, \infty)$ be a positive function which may depend upon the *environment*

$\omega \in \Omega$. The random walk $(X_t)_{t\geq 0}$ defined by the following generator,

$$\mathcal{L}_\theta^\omega f(x) := \frac{1}{\theta^\omega(x)} \sum_{y\sim x} \omega(x,y) \big(f(y) - f(x)\big) ,$$

acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$, is reversible with respect to θ^ω, and we call this process the *random conductance model (RCM)* with *speed measure* θ^ω. We denote P_x^ω the law of this process started at $x \in \mathbb{Z}^d$ and E_x^ω the corresponding expectation. There are two natural laws on the path space, which is the Skorokhod space $\mathcal{D}\big([0,\infty), \mathbb{Z}^d\big)$ of all càdlàg functions from $[0,\infty)$ to $\mathbb{Z}^d$, that are considered in the literature - the quenched law $P_x^\omega(\cdot)$ which concerns $\mathbb{P}$-almost sure phenomena, and the annealed law $\mathbb{E}P_x^\omega(\cdot)$.

If the random walk X is currently at x, it will next move to y with probability $\omega(x,y)/\mu^\omega(x)$, after waiting an exponential time with mean $\theta^\omega(x)/\mu^\omega(x)$ at the vertex x. The main results of this chapter are statements about the heat kernel of X,

$$p_\theta^\omega(t,x,y) \;:=\; \frac{P_x^\omega\left(X_t = y\right)}{\theta^\omega(y)}, \qquad t \geq 0,\; x,y \in \mathbb{Z}^d.$$

Perhaps the most natural choice for the speed measure is $\theta^\omega \equiv \mu^\omega$, for which we obtain the constant speed random walk (CSRW) that spends i.i.d. Exp(1)-distributed waiting times at all vertices it visits. Another well-studied process, the variable speed random walk (VSRW), is recovered by setting $\theta^\omega \equiv 1$, so called because as opposed to the CSRW, the waiting time at a vertex x does indeed depend on the location; it is an Exp$(\mu^\omega(x))$-distributed random variable.

2.1.2 Main Results on the Static RCM

As our first main results we obtain quenched and annealed local limit theorems for the static random conductance model. A general assumption required is stationarity and ergodicity of the environment.

Assumption 2.1.1. *(i)* $\mathbb{P}[0 < \omega(e) < \infty] = 1$ *and* $\mathbb{E}[\omega(e)] < \infty$ *for all* $e \in E_d$.

(ii) $\mathbb{P}$ *is ergodic with respect to spatial translations of* $\mathbb{Z}^d$*, i.e.* $\mathbb{P} \circ \tau_x^{-1} = \mathbb{P}$ *for all* $x \in \mathbb{Z}^d$ *and* $\mathbb{P}(A) \in \{0,1\}$ *for any* $A \in \mathcal{F}$ *such that* $\tau_x(A) = A$ *for all* $x \in \mathbb{Z}^d$.

(iii) θ *is stationary, i.e.* $\theta^\omega(x+y) = \theta^{\tau_y\omega}(x)$ *for all* $x,y \in \mathbb{Z}^d$ *and* $\mathbb{P}$*-a.e.* $\omega \in \Omega$*. Further,* $\mathbb{E}[\theta^\omega(0)] < \infty$ *and* $\mathbb{E}[\theta^\omega(0)/\mu^\omega(0)] \in (0,\infty)$.

In particular, the last condition in Assumption 2.1.1(iii) ensures that the process X is non-explosive. As discussed in Chapter 1, considerable effort has been invested in the last decade into the derivation of quenched invariance principles or quenched functional central limit theorems (QFCLT). The following QFCLT for random walks under ergodic conductances is the main result of [ADS15].

Theorem 2.1.2 (QFCLT). *Suppose Assumption 2.1.1 holds. Further assume that there exist* $p,\, q \in (1, \infty]$ *satisfying* $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ *such that* $\mathbb{E}\left[\omega(e)^p\right] < \infty$ *and* $\mathbb{E}\left[\omega(e)^{-q}\right] < \infty$ *for any* $e \in E_d$*. For* $n \in \mathbb{N}$*, define* $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$*,* $t \geq 0$*. Then, for* $\mathbb{P}$*-a.e.* ω*,* $X^{(n)}$ *converges (under* P_0^ω*) in law towards a Brownian motion on* $\mathbb{R}^d$ *with a deterministic non-degenerate covariance matrix* Σ^2*.*

Proof. For the VSRW, this is [ADS15, Theorem 1.3]. As noted in [ADS15, Remark 1.5] the QFCLT extends to the random walk with general speed measure θ^ω provided that $\mathbb{E}[\theta^\omega(0)] \in (0, \infty)$. See [ABDH13, Section 6.2] for a proof of this extension in the case of the CSRW. □

Recently the moment condition in Theorem 2.1.2 has been improved in [BS20b].

Remark 2.1.3. If we let $\bar{\Sigma}^2$ denote the covariance matrix of the above Theorem in the case of the VSRW, the corresponding covariance matrix of the random walk X with speed measure θ^ω is given by $\Sigma^2 = \mathbb{E}\left[\theta^\omega(0)\right]^{-1} \bar{\Sigma}^2$ – see [ADS15, Remark 1.5].

Assumption 2.1.4. *There exist* $p, q, r \in (1, \infty]$ *satisfying*

$$\frac{1}{r} + \frac{1}{p}\frac{r-1}{r} + \frac{1}{q} < \frac{2}{d} \tag{2.2}$$

such that

$$\mathbb{E}\left[\left(\frac{\mu^\omega(0)}{\theta^\omega(0)}\right)^p \theta^\omega(0)\right] + \mathbb{E}\left[\nu^\omega(0)^q\right] + \mathbb{E}\left[\theta^\omega(0)^{-1}\right] + \mathbb{E}\left[\theta^\omega(0)^r\right] \;<\; \infty. \tag{2.3}$$

While under Assumptions 2.1.1 and 2.1.4 Gaussian-type upper bounds on the heat kernel p_θ have been obtained in [ADS19], in the present work our focus is on local limit theorems. A local limit theorem constitutes a scaling limit of the heat kernel towards the normalized Gaussian transition density of the Brownian motion with covariance matrix Σ^2, which appears as the limit process in the QFCLT in Theorem 2.1.2. The Gaussian heat kernel associated with that process will be denoted

$$k_t(x) \equiv k_t^\Sigma(x) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma^2}} \exp\Big(- x \cdot (\Sigma^2)^{-1} x / (2t)\Big). \tag{2.4}$$

Our first main result is the following local limit theorem for the RCM under general speed measure. For $x \in \mathbb{R}^d$ write $\lfloor x \rfloor = (\lfloor x_1 \rfloor, ..., \lfloor x_d \rfloor) \in \mathbb{Z}^d$.

Theorem 2.1.5 (Quenched local limit theorem)**.** *Let $T_2 > T_1 > 0$, $K > 0$ and suppose that Assumptions 2.1.1 and 2.1.4 hold. Then,*

$$\lim_{n\to\infty} \sup_{|x|\le K} \sup_{t\in[T_1,T_2]} \Big| n^d p_\theta^\omega(n^2 t, 0, \lfloor nx \rfloor) - a\, k_t(x) \Big| = 0, \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega,$$

with $a := \mathbb{E}\big[\theta^\omega(0)\big]^{-1}$.

Remark 2.1.6. (i) In the case of the CSRW or VSRW Assumption 2.1.4 coincides with the moment condition in Theorem 2.1.2. Indeed, for the CSRW, $\theta^\omega \equiv \mu^\omega$, choose $p = \infty$ and relabel r by p; for the VSRW, $\theta^\omega \equiv 1$, choose $r = \infty$.

(ii) For the sake of a simpler presentation, Theorem 2.1.5 is stated for the RCM on $\mathbb{Z}^d$ only. However, its proof extends to RCMs with ergodic conductances satisfying a slightly modified moment condition on a general class of random graphs including supercritical i.i.d. percolation clusters and clusters in percolation models with long range correlations, see e.g. [DRS14, Sap17]. The corresponding QFCLT has been shown in [DNS18] and a local limit theorem for the VSRW in [ACS21, Section 5].

(iii) The quenched local limit theorem has also been established for symmetric diffusions in a stationary and ergodic environment, under analogous assumptions to the above theorem. This is the main result of [CD15], see Appendix A therein for a discussion of the general speed case.

Theorem 2.1.5 extends the local limit theorem in [ADS16a, Theorem 1.11] for the CSRW to the case of a general speed measure. In general, a local limit theorem is a stronger statement than an FCLT. In fact, even in the i.i.d. case, where the QFCLT does hold [ABDH13], we see the surprising effect that due to a trapping phenomenon the heat kernel may behave subdiffusively (see [BBHK08]), in particular a local limit theorem may fail. Nevertheless it does hold, for instance, in the case of uniformly elliptic conductances, where $\mathbb{P}(c^{-1} \le \omega(e) \le c) = 1$ for some $c \ge 1$, or for random walks on supercritical percolation clusters (see [BH09]). For sharp conditions on the tails of i.i.d. conductances at zero for Harnack inequalities and a local limit theorem to hold we refer to [BKM15]. Hence, it is clear that some moment condition is necessary. In the case of the CSRW under general ergodic conductances the moment condition in Assumption 2.1.4 is known to be optimal, see [ADS16a, Theorem 5.4]. Furthermore, for the VSRW a quenched local limit theorem has very recently been shown in [BS20a] under a weaker moment condition with $1/p + 1/q < 2/(d-1)$. Local limit theorems have also been obtained in slightly different settings, see [CH08], where some general criteria for local limit theorems are provided based on the arguments in [BH09]. Finally, stronger quantitative homogenization results for heat kernels and Green functions can be established by using techniques from quantitative stochastic homogenization, see [AKM19, Chapters 8–9] for details in the uniformly elliptic case. This technique has been adapted to the VSRW on percolation clusters in [DG21], and it is expected that it also applies to other degenerate models.

The proof of the local limit theorem has two main ingredients, the QFCLT in Theorem 2.1.2 and a Hölder regularity estimate for the heat kernel. For the latter it is common to use a purely analytic approach and to interpret the heat kernel as a fundamental solution of the heat equation $(\partial_t - \mathcal{L}_\theta^\omega)u = 0$. Here we will follow the arguments in [ACS21] based on De Giorgi's iteration technique. This approach to show Hölder regularity directly circumvents the need for a parabolic Harnack inequality, in contrast to the proofs in [ADS16a, BH09], which makes it significantly simpler. As a by-product to our argument we do obtain a weak parabolic Harnack inequality (Proposition 2.2.15) and a lower near-diagonal heat kernel estimate (Corollary 2.2.16). In [DG21, Theorem 3], following again the approach in [AKM19], stronger Lipschitz continuity of the heat kernel on i.i.d. percolation clusters has been shown, which matches the gradient of the Gaussian heat kernel.

Applications of homogenisation results such as FCLTs and local limit theorems in statistical mechanics often require convergence under the annealed measure. While a QFCLT does imply an annealed FCLT in general, the same does not apply to the local limit theorem. Next we provide an annealed local limit theorem under a stronger moment condition, which we do not expect to be optimal.

Theorem 2.1.7 (Annealed local limit theorem). *Suppose Assumption 2.1.1 holds. There exist exponents $p, q, r_1, r_2 \in (1, \infty)$ (only depending on d) such that if*

$$\mathbb{E}\big[\mu^\omega(0)^p\big] + \mathbb{E}\big[\nu^\omega(0)^q\big] + \mathbb{E}\big[\theta^\omega(0)^{-r_1}\big] + \mathbb{E}\big[\theta^\omega(0)^{r_2}\big] \;<\; \infty$$

then the following holds. For all $K > 0$ and $0 < T_1 \le T_2$,

$$\lim_{n\to\infty} \mathbb{E}\bigg[\sup_{|x|\le K}\sup_{t\in[T_1,T_2]}\Big|n^d p_\theta^\omega(n^2 t, 0, \lfloor nx \rfloor) - a k_t(x)\Big|\bigg] = 0. \tag{2.5}$$

Remark 2.1.8. In the case of the VSRW, i.e. $\theta^\omega \equiv 1$, the moment condition required in Theorem 2.1.7 is more explicitly given by $\mathbb{E}\big[\omega(e)^{2(\kappa'\vee p)}\big] + \mathbb{E}\big[\omega(e)^{-2(\kappa'\vee q)}\big] < \infty$, $e \in E_d$, for some $p,\, q \in (1,\infty)$ such that $1/p + 1/q < 2/d$ and $\kappa' = \kappa'(d,p,q,\infty)$ defined in Proposition 2.3.1 below. Similarly, in the case of the CSRW, $\theta^\omega \equiv \mu^\omega$, the condition reduces to $\mathbb{E}\big[\omega(e)^{4\kappa'\vee 2p}\big] + \mathbb{E}\big[\omega(e)^{-(4\kappa'+2)\vee 2q}\big] \;<\; \infty$, $e \in E_d$, again for some $p,\, q \in (1,\infty)$ such that $1/p + 1/q < 2/d$ and $\kappa' = \kappa'(d,\infty,q,p)$ defined as in Proposition 2.3.1.

As mentioned above, the proofs of the quenched local limit theorems in [ADS16a] and Theorem 2.1.5 rely on Hölder regularity estimates on the heat kernel, which involve random constants depending on the exponential of the conductances. Those constants can be controlled almost surely, but naively taking expectations would require exponential moment conditions stronger than the polynomial moment conditions in Assumption 2.1.4. To derive the annealed local limit theorem given the corresponding quenched result, one might hope to employ the dominated convergence theorem, which requires that the integrand above can be dominated uniformly in n by an integrable function. We achieve this using a maximal inequality from [ADS19]. Then it is the form of the random constants in this inequality that allows us to anneal the result using only polynomial moments, together with a simple probabilistic bound.

2.1.3 Main Results on the Dynamic RCM

Next we introduce the dynamic random conductance model. We endow $G = (\mathbb{Z}^d, E_d)$, $d \geq 2$, with a family $\omega = \{\omega_t(e) \in (0, \infty) : e \in E_d,\, t \in \mathbb{R}\}$ of positive, time-dependent weights. For $t \in \mathbb{R}$, $x \in \mathbb{Z}^d$, let

$$\mu_t^\omega(x) := \sum_{y \sim x} \omega_t(x, y), \qquad \nu_t^\omega(x) := \sum_{y \sim x} \frac{1}{\omega_t(x, y)}.$$

We define the *dynamic variable speed random walk* starting in $x \in \mathbb{Z}^d$ at $s \in \mathbb{R}$ to be the continuous-time Markov chain $(X_t : t \geq s)$ with time-dependent generator

$$\left(\mathcal{L}_t^\omega f\right)(x) := \sum_{y \sim x} \omega_t(x, y) \left(f(y) - f(x)\right),$$

acting on bounded functions $f : \mathbb{Z}^d \to \mathbb{R}$. Note that the counting measure, which is time-independent, is an invariant measure for X. In contrast to Section 2.1.2, the results in this subsection, like many results on the dynamic RCM, are restricted to this specific speed measure. We denote by $P_{s,x}^\omega$ the law of this process started at $x \in \mathbb{Z}^d$ at time s, and $E_{s,x}^\omega$ the corresponding expectation. For $x, y \in \mathbb{Z}^d$ and $t \geq s$, we denote by $p^\omega(s, t, x, y)$ the heat kernel of $(X_t)_{t \geq s}$, that is

$$p^\omega(s, t, x, y) := P_{s,x}^\omega \left[X_t = y\right].$$

Let Ω be the set of measurable functions from $\mathbb{R}$ to $(0, \infty)^{E_d}$ equipped with a σ-algebra $\mathcal{F}$ and let $\mathbb{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Upon it we consider the $d+1$-parameter group of translations $(\tau_{t,x})_{(t,x) \in \mathbb{R} \times \mathbb{Z}^d}$ given by

$$\tau_{t,x} : \Omega \to \Omega, \quad \big(\omega_s(e)\big)_{s \in \mathbb{R},\, e \in E_d} \mapsto \big(\omega_{t+s}(x + e)\big)_{s \in \mathbb{R},\, e \in E_d}. \tag{2.6}$$

The required ergodicity and stationarity assumptions on the time-dependent random environment are as follows.

Assumption 2.1.9. *(i)* $\mathbb{P}$ *is ergodic with respect to time-space translations, i.e. for all* $x \in \mathbb{Z}^d$ *and* $t \in \mathbb{R}$*,* $\mathbb{P} \circ \tau_{t,x}^{-1} = \mathbb{P}$*. Further,* $\mathbb{P}(A) \in \{0,1\}$ *for any* $A \in \mathcal{F}$ *such that* $\tau_{t,x}(A) = A$ *for all* $x \in \mathbb{Z}^d$*,* $t \in \mathbb{R}$*.*

(ii) For every $A \in \mathcal{F}$*, the mapping* $(\omega, t, x) \mapsto \mathbb{1}_A(\tau_{t,x}\omega)$ *is jointly measurable with respect to the* σ*-algebra* $\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) \otimes 2^{\mathbb{Z}^d}$*.*

Theorem 2.1.10 (Quenched FCLT and local limit theorem)**.** *Suppose Assumption 2.1.9 holds and there exist* $p, q \in (1, \infty]$ *satisfying*

$$\frac{1}{p-1} + \frac{1}{(p-1)q} + \frac{1}{q} < \frac{2}{d}$$

such that $\mathbb{E}\big[\omega_0(e)^p\big] < \infty$ *and* $\mathbb{E}\big[\omega_0(e)^{-q}\big] < \infty$ *for any* $e \in E_d$*.*

(i) The QFCLT holds with a deterministic non-degenerate covariance matrix Σ^2*.*

(ii) For any $T_2 > T_1 > 0$ *and* $K > 0$*,*

$$\lim_{n\to\infty} \sup_{|x|\le K} \sup_{t\in[T_1,T_2]} \big| n^d p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) - k_t(x) \big| \;=\; 0, \quad \textit{for } \mathbb{P}\textit{-a.e. } \omega,$$

where k_t *still denotes the heat kernel of a Brownian motion on* $\mathbb{R}^d$ *with covariance* Σ^2*.*

Proof. The QFLCT in (i) has been proven in [ACDS18], see [BR18] for a similar result. For the quenched local limit theorem in (ii) we refer to [ACS21]. □

Similarly as in the static case we establish an annealed local limit theorem for the dynamic RCM under a stronger, but still polynomial moment condition.

Theorem 2.1.11 (Annealed local limit theorem)**.** *Suppose Assumption 2.1.9 holds. There exist exponents* $p, q \in (1, \infty)$ *(specified more explicitly in Assumption 2.4.2 below) such that if* $\mathbb{E}\big[\omega_0(e)^p\big] < \infty$ *and* $\mathbb{E}\big[\omega_0(e)^{-q}\big] < \infty$ *for any* $e \in E_d$*, then the following holds. For all* $K > 0$ *and* $0 < T_1 \le T_2$*,*

$$\lim_{n\to\infty} \mathbb{E}\bigg[\sup_{|x|\le K} \sup_{t\in[T_1,T_2]} \big| n^d p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) - k_t(x) \big| \bigg] = 0. \tag{2.7}$$

An annealed local limit theorem has been stated in the uniformly elliptic case in [And14]. We do not expect the moment conditions in Theorem 2.1.11 to be optimal and that they can be relaxed. In an upcoming paper [DKS] an annealed local limit theorem is obtained for ergodic conductances uniformly bounded from below but only having a finite first moment by using an entropy argument from [BDCKY15].

Relevant examples of dynamic RCMs include random walks in an environment generated by some interacting particle systems like zero-range or exclusion processes (cf.

[MO16]). Some on-diagonal heat kernel upper bounds for a degenerate time-dependent conductance model are obtained in [MO16]. Full two-sided Gaussian bounds have been shown in the uniformly elliptic case for the VSRW [DD05] or for the CSRW under effectively non-decreasing conductances [DHZ19]. However, unlike for the static environments, two-sided Gaussian heat kernel bounds are much less regular and some pathologies may arise as they are not stable under perturbations, see [HK16]. Moreover, in the degenerate case such bounds are expected to be governed by the intrinsic distance. Even in the static case, in contrast to the CSRW, the intrinsic distance of the VSRW is not comparable to the Euclidean distance in general, cf. [ADS19], and the exact form of a time-dynamic version of the distance is still unknown. These facts make the derivation of Gaussian bounds for the dynamic RCM with unbounded conductances a subtle open challenge.

2.1.4 Notation

We finally introduce some further notation used in this chapter. We write c to denote a positive, finite constant which may change on each appearance. Constants denoted by c_i will remain the same. We endow the graph $G = (\mathbb{Z}^d, E_d)$ with the natural graph distance d, i.e. $d(x,y)$ is the minimal length of a path between x and y. Denote $B(x,r) := \{y \in \mathbb{Z}^d : d(x,y) \leq r\}$ the closed ball with centre x and radius r. For a non-empty, finite, connected set $A \subseteq \mathbb{Z}^d$, we denote by $\partial A := \{x \in A : d(x,y) = 1 \text{ for some } y \in A^c\}$ the inner boundary and by $\partial^+ A := \{x \in A^c : d(x,y) = 1 \text{ for some } y \in A\}$ the outer boundary of A. We write $\overline{A} = A \cup \partial^+ A$ for the closure of A. The graph is given the counting measure, i.e. the measure of $A \subseteq \mathbb{Z}^d$ is the number $|A|$ of elements in A. For $f\colon \mathbb{Z}^d \to \mathbb{R}$ we define the operator ∇ by

$$\nabla f\colon E_d \to \mathbb{R}, \qquad E_d \ni e \;\longmapsto\; \nabla f(e) \;:=\; f(e^+) - f(e^-),$$

where for each non-oriented edge $e \in E_d$ we specify one of its two endpoints as its initial vertex e^+ and the other one as its terminal vertex e^-. Further, the corresponding adjoint operator $\nabla^* F\colon \mathbb{Z}^d \to \mathbb{R}$ acting on functions $F\colon E_d \to \mathbb{R}$ is defined in such a way that $\langle \nabla f, F\rangle_{\ell^2(E_d)} = \langle f, \nabla^* F\rangle_{\ell^2(\mathbb{Z}^d)}$ for all $f \in \ell^2(\mathbb{Z}^d)$ and $F \in \ell^2(E_d)$. Notice that in the discrete setting the product rule reads

$$\nabla(fg) \;=\; \mathrm{av}(f)\nabla g + \mathrm{av}(g)\nabla f, \tag{2.8}$$

where $\mathrm{av}(f)(e) := \frac{1}{2}(f(e^+) + f(e^-))$. We denote inner products as follows; for $f,\, g \in \ell^2(\mathbb{Z}^d)$ and a weighting function $\phi : \mathbb{Z}^d \to (0,\infty)$, $\langle f, g\rangle_{\ell^2(\mathbb{Z}^d,\phi)} := \sum_{x\in\mathbb{Z}^d} f(x)g(x)\phi(x)$ and if $f,\, g \in \ell^2(\mathbb{E}_d)$, $\langle f, g\rangle_{\ell^2(E_d)} := \sum_{e\in E_d} f(e)g(e)$. The corresponding weighted norm is denoted $\left\|f\right\|_{\ell^2(\mathbb{Z}^d,\phi)}$. The Dirichlet form associated with the operator $\mathcal{L}^\omega_\theta$ is defined for $f,\, g : \mathbb{Z}^d \to \mathbb{R}$ in its domain $\mathcal{D}(\mathcal{E}^\omega) \subseteq L^2(\mathbb{Z}^d, \theta^\omega\, dx)$ by

$$\mathcal{E}^\omega(f,g) := \langle f, -\mathcal{L}^\omega_\theta g\rangle_{\ell^2(\mathbb{Z}^d,\theta)} \equiv \langle \nabla f, \omega \nabla g\rangle_{\ell^2(E_d)}.$$

We will use the shorthand $\mathcal{E}^\omega(f) := \mathcal{E}^\omega(f,f)$. For non-empty, finite $B \subseteq \mathbb{Z}^d$ and $p \in (0,\infty)$, space-averaged ℓ^p-norms on functions $f : B \to \mathbb{R}$ will be used,

$$\|f\|_{p,B} := \left(\frac{1}{|B|}\sum_{x\in B}|f(x)|^p\right)^{1/p} \quad \text{and} \quad \|f\|_{\infty,B} := \max_{x\in B}|f(x)|.$$

Now let $Q = I \times B$ where $I \subseteq \mathbb{R}$ is compact. Let $u : Q \to \mathbb{R}$ and denote $u_t : B \to \mathbb{R}$, $u_t(\cdot) := u(t,\cdot)$ for $t \in I$. For $p' \in (0,\infty)$, we define the space-time averaged norms

$$\left\|u\right\|_{p,p',Q} := \left(\frac{1}{|I|}\int_I \left\|u_t\right\|^{p'}_{p,B} dt\right)^{1/p} \quad \text{and} \quad \left\|u\right\|_{p,\infty,Q} := \operatorname{ess\,sup}_{t\in I}\left\|u_t\right\|_{p,B}.$$

Furthermore, we will work with two varieties of weighted norms

$$\begin{aligned}
\left\|f\right\|_{p,B,\phi} &:= \left(\frac{1}{\phi(B)}\sum_{x\in B}\left|f(x)\right|^p\,\phi(x)\right)^{1/p}, \quad \left|f\right|_{p,B,\phi} := \left(\frac{1}{|B|}\sum_{x\in B}\left|f(x)\right|^p\,\phi(x)\right)^{1/p},\\
\left\|u\right\|_{p,p',Q,\phi} &:= \left(\frac{1}{|I|}\int_I \left\|u_t\right\|^{p'}_{p,B,\phi}\,dt\right)^{1/p'}, \quad \left\|u\right\|_{p,\infty,Q,\phi} := \operatorname{ess\,sup}_{t\in I}\left\|u_t\right\|_{p,B,\phi},\\
\left|u\right|_{p,p',Q,\phi} &:= \left(\frac{1}{|I|}\int_I \left|u_t\right|^{p'}_{p,B,\phi}\,dt\right)^{1/p'},
\end{aligned}$$

for a weighting function $\phi : B \to (0,\infty)$, where $\phi(B) := \sum_{x\in B}\phi(x)$.

2.1.5 Structure of the Chapter

Section 2.2 is devoted to the proof of the quenched local limit theorem for general speed measures - Theorem 2.1.5. The annealed local limit theorems for the static and dynamic RCM, Theorem 2.1.7 and Theorem 2.1.11, are shown in Sections 2.3 and 2.4, respectively.

2.2 Local Limit Theorem for the Static RCM under General Speed Measure

For the proof of Theorem 2.1.5 we shall follow a method first developed in [CH08] and [BH09], for which the main ingredients are the QFCLT in Theorem 2.1.2 and a

Hölder regularity estimate for the heat kernel. To derive the latter we adapt the techniques employed in [ACS21] to the general speed measure case. The key result in Theorem 2.2.5 is an oscillation inequality for solutions of $\partial_t u - \mathcal{L}_\theta^\omega u = 0$, such as the heat kernel, which implies the required Hölder regularity by a simple iteration argument (see Proposition 2.2.13 below). For the proof of the oscillation inequality, we first derive a maximal inequality (see Theorem 2.2.3) using a De Giorgi iteration scheme in Section 2.2.2. Then we bound the measure of the level sets of a solution u in terms of $(-\ln u)_+$ (see Lemmas 2.2.8 and 2.2.9 below). These two steps are sufficient to prove the oscillation inequality following an idea in [WYW06], see Section 2.2.3. To begin with, we collect the required functional inequalities in Section 2.2.1.

For the rest of Section 2.2 we assume $d \geq 2$ and we fix $p, q, r \in (1, \infty]$ such that

$$\frac{1}{r} + \frac{1}{p}\frac{r-1}{r} + \frac{1}{q} \;<\; \frac{2}{d}. \tag{2.9}$$

2.2.1 Sobolev and Weighted Local Poincaré Inequalities

One auxiliary result which will prove useful is a modification of the Sobolev inequality derived in [ADS15].

Proposition 2.2.1. *Let $B \subset \mathbb{Z}^d$ be finite and connected. There exists $c_1 = c_1(d,q)$ such that for any $v : \mathbb{Z}^d \to \mathbb{R}$ with $v \equiv 0$ on ∂B,*

$$\left\| v^2 \right\|_{\rho,B} \;\leq\; c_1 \, |B|^{2/d} \left\| \nu^\omega \right\|_{q,B} \left\| \theta^\omega \right\|_{1,B} \frac{\mathcal{E}^\omega(v)}{\theta^\omega(B)},$$

where $\rho := qd/(q(d-2)+d)$.

Proof. By [ADS15, equation (28)],

$$\left\| v^2 \right\|_{\rho,B} \;\leq\; c_1 \, |B|^{2/d} \left\| \nu^\omega \right\|_{q,B} \, \frac{\mathcal{E}^\omega(v)}{|B|},$$

and since $\left\| \theta^\omega \right\|_{1,B} = \theta^\omega(B)/|B|$ this gives the claim. □

Another input is a weighted Poincaré inequality which will be applied in deriving the aforementioned oscillations bound. We denote the weighted average of any $u : \mathbb{Z}^d \to \mathbb{R}$ over a finite subset $B \subset \mathbb{Z}^d$ with respect to some $\phi : \mathbb{Z}^d \to (0, \infty)$,

$$(u)_{B,\phi} \;:=\; \frac{1}{\phi(B)} \sum_{x \in B} u(x)\, \phi(x).$$

We shall also write $(u)_B := (u)_{B,1}$ when $\phi \equiv 1$.

Proposition 2.2.2. *There exists $c_2 = c_2(d) < \infty$ such that for any ball $B(n) := B(x_0, n)$ with $x_0 \in \mathbb{Z}^d$ and $n \geq 1$, any non-empty $\mathcal{N} \subseteq B$ and $u : \mathbb{Z}^d \to \mathbb{R}$,*

$$\left\| u - (u)_{B(n),\theta} \right\|_{1,B(n),\theta}^2 \;\leq\; c_2\, \mathcal{A}_1^\omega(n)\, \frac{n^2}{|B(n)|} \sum_{\substack{x,y \in B(n)\\ x \sim y}} \omega(x,y)\, \big(u(x) - u(y)\big)^2, \tag{2.10}$$

and

$$\begin{aligned} &\left\| u - (u)_{\mathcal{N},\theta} \right\|_{1,B(n),\theta}^2 \\ &\qquad \leq\; c_2\, \mathcal{A}_1^\omega(n) \left(1 + \frac{\theta^\omega(B(n))}{\theta^\omega(\mathcal{N})}\right)^2 \frac{n^2}{|B(n)|} \sum_{\substack{x,y \in B(n)\\ x \sim y}} \omega(x,y)\, \big(u(x) - u(y)\big)^2 \end{aligned} \tag{2.11}$$

with $\mathcal{A}_1^\omega(n) := \left\| 1/\theta^\omega \right\|_{1,B(n)}^2 \left\| \theta^\omega \right\|_{r,B(n)}^2 \left\| \nu^\omega \right\|_{q,B(n)}$.

Proof. By a discrete version of the co-area formula the classical local ℓ^1-Poincaré inequality on $\mathbb{Z}^d$ can be easily established, see e.g. [SC97, Lemma 3.3.3], which also implies an ℓ^α-Poincaré inequality for any $\alpha \in [1, d)$. Note that, by [Cou96, Théorème 4.1], the volume regularity of balls and the local ℓ^α-Poincaré inequality on $\mathbb{Z}^d$ implies that for $d \geq 2$ and any $u \colon \mathbb{Z}^d \to \mathbb{R}$,

$$\inf_{a \in \mathbb{R}} \left\| u - a \right\|_{\frac{d\alpha}{d-\alpha}, B(n)} \;\leq\; c\, n \left(\frac{1}{|B(n)|} \sum_{\substack{x,y \in B(n)\\ x \sim y}} \left| u(x) - u(y) \right|^\alpha \right)^{1/\alpha}. \tag{2.12}$$

Further, for any $\alpha \in [1, 2)$, Hölder's inequality yields

$$\left(\frac{1}{|B(n)|} \sum_{\substack{x,y \in B(n)\\ x \sim y}} \left| u(x) - u(y) \right|^\alpha \right)^{\frac{1}{\alpha}} \leq \left\| \nu^\omega \right\|_{\frac{\alpha}{2-\alpha}}^{1/2} \left(\frac{1}{|B(n)|} \sum_{\substack{x,y \in B(n)\\ x \sim y}} \omega(x,y) \big(u(x) - u(y)\big)^2 \right)^{\frac{1}{2}}. \tag{2.13}$$

Note that by [DK13, Lemma 2], we have for any $a \in \mathbb{R}$,

$$\left\| u - (u)_{B(n),\theta} \right\|_{1,B(n),\theta} \leq c \left\| u - a \right\|_{1,B(n),\theta}.$$

Now we prove (2.10) by distinguishing two cases. In the case $r \geq 2$ we have by Cauchy-Schwarz,

$$\left\| u - a \right\|_{1,B(n),\theta} \;\leq\; \left\| \theta^\omega \right\|_{1,B(n)}^{-1} \left\| \theta^\omega \right\|_{2,B(n)} \left\| u - a \right\|_{2,B(n)}.$$

Hence we obtain the assertion (2.10) by using (2.12) and (2.13) with the choice $\alpha = 2d/(d+2)$ and Jensen's inequality. Note that $\alpha/(2-\alpha) = d/2 < q$.

Similarly, in the case $r \in (1,2)$, denoting its Hölder conjugate r_* we have by Hölder's inequality

$$\left\|u - a\right\|_{1,B(n),\theta} \;\le\; \left\|\theta^\omega\right\|_{1,B(n)}^{-1} \left\|\theta^\omega\right\|_{r,B(n)} \left\|u - a\right\|_{r_*,B(n)},$$

and we may use (2.12) and (2.13) with the choice $\alpha = dr_*/(d + r_*)$.

Notice that $d\alpha/(d-\alpha) = r_*$, $\alpha/(2-\alpha) \le q$ and $\alpha \in [1,2)$ since $r \in (1,d]$ and satisfies (2.9). This finishes the proof of (2.10).

To see (2.11), note that by the triangle inequality

$$\begin{aligned}
\left\|u - (u)_{\mathcal{N},\theta}\right\|_{1,B(n),\theta} \;&\le\; \left\|u - (u)_{B(n),\theta}\right\|_{1,B(n),\theta} \;+\; \left|(u)_{\mathcal{N},\theta} - (u)_{B(n),\theta}\right| \\
&\le\; \left\|u - (u)_{B(n),\theta}\right\|_{1,B(n),\theta} \;+\; \frac{1}{\theta^\omega(\mathcal{N})} \sum_{y \in \mathcal{N}} \left|u(y) - (u)_{B(n),\theta}\right| \theta^\omega(y) \\
&\le\; \left(1 + \frac{\theta^\omega(B(n))}{\theta^\omega(\mathcal{N})}\right) \left\|u - (u)_{B(n),\theta}\right\|_{1,B(n),\theta},
\end{aligned}$$

so (2.11) follows from (2.10). □

2.2.2 Maximal Inequality

For the analysis, we work with space-time cylinders defined as follows. For any $x_0 \in \mathbb{Z}^d$ and $t_0 \in \mathbb{R}$ let $I_\tau := [t_0 - \tau n^2, t_0]$ and $B_\sigma := B(x_0, \sigma n)$ for $\sigma \in (0,1]$, $\tau \in (0,1]$. We write $Q(n) := [t_0 - n^2, t_0] \times B(x_0, n)$ and

$$Q_{\tau,\sigma}(n) := I_\tau \times B_\sigma \quad \text{and} \quad Q_\sigma := Q_\sigma(n) := Q_{\sigma,\sigma}(n).$$

The main result in this subsection is the following maximal inequality.

Theorem 2.2.3. *Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$ and $u > 0$ be such that $\partial_t u - \mathcal{L}_\theta^\omega u \le 0$ on $Q(n)$ for any $n \ge 1$. Then, for any $0 \le \Delta < 2/(d+2)$ there exists $N_1 = N_1(\Delta) \in \mathbb{N}$ and $c_3 = c_3(d,p,q,r)$ such that for all $n \ge N_1$, $h \ge 0$ and $1/2 \le \sigma' < \sigma \le 1$ with $\sigma - \sigma' > n^{-\Delta}$,*

$$\max_{(t,x) \in Q_{\sigma'}(n)} u(t,x) \;\le\; h \;+\; c_3 \left(\frac{\mathcal{A}_2^\omega(n)}{(\sigma - \sigma')^2}\right)^{\kappa} \left\|(u-h)_+\right\|_{2p_*,2,Q_\sigma(n),\theta}.$$

Here $p_* := p/(p-1)$, $\kappa := 1 + p_*\rho/2(\rho - p_* r_*)$ with ρ as in Proposition 2.2.1, and

$$\mathcal{A}_2^\omega(n) \;:=\; \left\|1 \vee (\mu^\omega/\theta^\omega)\right\|_{p,B(n),\theta} \left\|1 \vee \nu^\omega\right\|_{q,B(n)} \left\|1 \vee \theta^\omega\right\|_{r,B(n)}^2 \left\|1 \vee (1/\theta^\omega)\right\|_{1,B(n)}. \tag{2.14}$$

An energy estimate is required in proving the above, cf. [ADS19, Lemma 3.7].

Lemma 2.2.4. *Suppose $Q = I \times B$ where $I = [s_1, s_2] \subseteq \mathbb{R}$ is an interval and $B \subset \mathbb{Z}^d$ is finite and connected. Let u be a non-negative solution of $\partial_t u - \mathcal{L}_\theta^\omega u \le 0$ on Q. Let*

$\eta : \mathbb{Z}^d \to [0,1]$ and $\xi : \mathbb{R} \to [0,1]$ be cutoff functions such that $\operatorname{supp} \eta \subseteq B$, $\operatorname{supp} \xi \subseteq I$ and $\eta \equiv 0$ on ∂B, $\xi(s_1) = 0$. Then there exists c_4 such that for any $k \geq 0$ and $p, p_* \in (1, \infty)$ with $\frac{1}{p} + \frac{1}{p_*} = 1$,

$$\frac{1}{|I|} \big\| \xi \eta^2 (u-k)_+^2 \big\|_{1,\infty,Q,\theta} + \frac{1}{|I|} \int_I \xi(t) \, \frac{\mathcal{E}^\omega(\eta v)}{\theta^\omega(B)} \, dt \\ \leq c_4 \Big(\big\| \mu^\omega / \theta^\omega \big\|_{p,B,\theta} \big\| \nabla \eta \big\|_{l^\infty(E_d)}^2 + \big\| \xi' \big\|_{L^\infty(I)} \Big) \, \big\| (u-k)_+^2 \big\|_{p_*,1,Q,\theta}. \tag{2.15}$$

Proof. Set $v = (u-k)_+$ for abbreviation. Then, on Q, by applying the chain rule we obtain $\partial_t v^2 = 2v \partial_t u \leq 2v \mathcal{L}_\theta^\omega u$. Furthermore note that $\nabla(\eta^2 v_t) \nabla v_t \leq \nabla(\eta^2 v_t) \nabla u_t$, which can be verified by distinguishing several cases. Thus, a summation by parts gives for any $t \in [s_1, s_2]$,

$$\frac{1}{2} \, \partial_t \big\| \eta v_t \big\|_{\ell^2(\mathbb{Z}^d, \theta)}^2 \; \leq \; - \big\langle \nabla(\eta^2 v_t), \omega \nabla v_t \big\rangle_{\ell^2(E_d)}.$$

Now by the product rule (2.8),

$$\big\langle \nabla(\eta v_t), \omega \, \nabla(\eta v_t) \big\rangle_{\ell^2(E_d)} \; \leq \; \big\langle \nabla(\eta^2 v_t), \omega \, \nabla v_t \big\rangle_{\ell^2(E_d)} + \big\langle \operatorname{av}(v_t)^2, \omega \, (\nabla \eta)^2 \big\rangle_{\ell^2(E_d)},$$

where we used that $\operatorname{av}(\eta)^2 \leq \operatorname{av}(\eta^2)$ by Jensen's inequality. By combining the last two inequalities we get

$$\frac{1}{2} \, \partial_t \big\| \eta v_t \big\|_{\ell^2(\mathbb{Z}^d, \theta)} + \mathcal{E}^\omega \big(\eta v_t \big) \; \leq \; \big\langle \operatorname{av}(v_t)^2, \omega \, (\nabla \eta)^2 \big\rangle_{\ell^2(E_d)},$$

therefore by Hölder's inequality

$$\frac{1}{2} \, \partial_t \big\| \eta^2 v_t^2 \big\|_{1,B,\theta} + \frac{\mathcal{E}^\omega(\eta v_t)}{\theta^\omega(B)} \; \leq \; \big\| \mu^\omega / \theta^\omega \big\|_{p,B,\theta} \big\| \nabla \eta \big\|_{\ell^\infty(E_d)}^2 \big\| v_t^2 \big\|_{p_*,B,\theta}. \tag{2.16}$$

Finally, since $\xi(s_1) = 0$, applying integration by parts and Jensen's inequality

$$\int_{s_1}^{s} \xi(t) \, \partial_t \big\| (\eta v_t)^2 \big\|_{1,B,\theta} \, \mathrm{d}t \; = \; \int_{s_1}^{s} \Big(\partial_t \big(\xi(t) \big\| (\eta v_t)^2 \big\|_{1,B,\theta} \big) - \xi'(t) \big\| (\eta v_t)^2 \big\|_{1,B,\theta} \Big) \, \mathrm{d}t \\ \geq \; \xi(s) \big\| (\eta v_s)^2 \big\|_{1,B,\theta} \; - \; \|\xi'\|_{L^\infty(I)} \, |I| \, \big\| v^2 \big\|_{p_*,1,Q,\theta}$$

for any $s \in (s_1, s_2]$. Thus, by multiplying both sides of (2.16) with $\xi(t)$ and integrating the resulting inequality over $[s_1, s]$ for any $s \in I$, the assertion (2.15) follows.

□

Proof of Theorem 2.2.3. The proof is based on an iteration argument and will be divided into two steps. First we will derive the estimate needed for a single iteration step, then the actual iteration will be carried out. Set $\alpha := 1 + \frac{1}{p_*} - \frac{r_*}{\rho}$ with $r_* := r/(r-1)$. Notice that for any $p, q, r \in (1, \infty]$ satisfying (2.9), $\alpha > 1$ and therefore $1/\alpha_* := 1 - 1/\alpha > 0$.

Step 1: Let $1/2 \leq \sigma' < \sigma \leq 1$ and $0 \leq k < l$ be fixed. Note that, due to the discrete structure of the underlying space $\mathbb{Z}^d$, the balls B_σ and $B_{\sigma'}$ may coincide. To ensure that $B_{\sigma'} \subsetneq B_\sigma$ we assume in this step that $(\sigma - \sigma')n \geq 1$. Then, it is possible to define a spatial cut-off function $\eta : \mathbb{Z}^d \to [0, 1]$ such that $\operatorname{supp} \eta \subseteq B_\sigma$, $\eta \equiv 1$ on $B_{\sigma'}$, $\eta \equiv 0$ on ∂B_σ and $\|\nabla\eta\|_{l^\infty(E)} \leq 1/((\sigma - \sigma')n)$. Further, let $\xi \in C^\infty(\mathbb{R})$ be a cut-off in time satisfying $\operatorname{supp} \xi \subseteq I_\sigma$, $\xi \equiv 1$ on $I_{\sigma'}$, $\xi(t_0 - \sigma n^2) = 0$ and $\|\xi'\|_{L^\infty([0,\infty))} \leq 1/((\sigma - \sigma')n^2)$. By Hölder's inequality, followed by applications of Hölder's and Young's inequalities,

$$\begin{aligned}\big\|(u-l)_+^2\big\|_{p_*,1,Q_{\sigma'},\theta} \;&\leq\; \big\|(u-k)_+^2\big\|_{\alpha p_*,\alpha,Q_{\sigma'},\theta}\,\big\|\mathbb{1}_{\{u\geq l\}}\big\|_{\alpha_* p_*,\alpha_*,Q_{\sigma'},\theta}\\ &\leq\; \Big(\big\|(u-k)_+^2\big\|_{1,\infty,Q_{\sigma'},\theta} + \big\|(u-k)_+^2\big\|_{\rho/r_*,1,Q_{\sigma'},\theta}\Big)\,\big\|\mathbb{1}_{\{u\geq l\}}\big\|_{p_*,1,Q_{\sigma'},\theta}^{1/\alpha_*}. \qquad (2.17)\end{aligned}$$

Note that by Jensen's inequality

$$\frac{\theta^\omega(B_\sigma)}{\theta^\omega(B_{\sigma'})} \;\leq\; c\,\|\theta^\omega\|_{1,B_\sigma}\,\|1/\theta^\omega\|_{1,B_{\sigma'}}. \qquad (2.18)$$

We use Hölder's inequality, the Sobolev inequality in Proposition 2.2.1, the fact that $r_*/\rho < 1$ and Lemma 2.2.4 to obtain

$$\begin{aligned}\big\|(u-k)_+^2\big\|_{\rho/r_*,1,Q_{\sigma'},\theta} \;&\leq\; c\Big(\|\theta^\omega\|_{1,B_\sigma}\,\|1/\theta^\omega\|_{1,B_{\sigma'}}\Big)^{\frac{r_*}{\rho}}\,\big\|\xi\eta^2(u-k)_+^2\big\|_{\rho/r_*,1,Q_\sigma,\theta}\\ &\leq\; c\,n^2\,\|\nu^\omega\|_{q,B_\sigma}\Big(\|\theta^\omega\|_{r,B_\sigma}^2\,\|1/\theta^\omega\|_{1,B_\sigma}\Big)^{\frac{r_*}{\rho}}\,\frac{1}{|I_\sigma|}\int_{I_\sigma}\xi(t)\,\frac{\mathcal{E}^\omega\big(\eta\,(u_t-k)_+\big)}{\theta^\omega(B_\sigma)}\,dt\\ &\leq\; c\,\frac{\tilde{\mathcal{A}}_2^\omega(n)}{(\sigma-\sigma')^2}\,\big\|(u-k)_+^2\big\|_{p_*,1,Q_\sigma,\theta}, \qquad (2.19)\end{aligned}$$

with $\tilde{\mathcal{A}}_2^\omega(n) := \mathcal{A}_2^\omega(n) \,/\, \|1 \vee (1/\theta^\omega)\|_{1,B_\sigma}$. Further, again by (2.18) and Lemma 2.2.4,

$$\begin{aligned}\big\|(u-k)_+^2\big\|_{1,\infty,Q_{\sigma'},\theta} \;&\leq\; c\,\|\theta^\omega\|_{1,B_\sigma}\,\|1/\theta^\omega\|_{1,B_{\sigma'}}\big\|\xi\eta^2(u-k)_+^2\big\|_{1,\infty,Q_\sigma,\theta}\\ &\leq\; c\,\frac{\|1\vee(\mu^\omega/\theta^\omega)\|_{p,B_\sigma,\theta}\,\|\theta^\omega\|_{1,B_\sigma}\,\|1/\theta^\omega\|_{1,B_\sigma}}{(\sigma-\sigma')^2}\big\|(u-k)_+^2\big\|_{p_*,1,Q_\sigma,\theta}\\ &\leq\; c\,\frac{\tilde{\mathcal{A}}_2^\omega(n)}{(\sigma-\sigma')^2}\,\big\|(u-k)_+^2\big\|_{p_*,1,Q_\sigma,\theta}. \qquad (2.20)\end{aligned}$$

Moreover, note that

$$\begin{aligned}\big\|\mathbb{1}_{\{u\geq l\}}\big\|_{p_*,1,Q_{\sigma'},\theta} \;&\leq\; c\,\|\theta^\omega\|_{1,B_\sigma}\,\|1/\theta^\omega\|_{1,B_{\sigma'}}\big\|\mathbb{1}_{\{u-k\geq l-k\}}\big\|_{p_*,1,Q_\sigma,\theta}\\ &\leq\; \frac{\tilde{\mathcal{A}}_2^\omega(n)}{(l-k)^2}\,\big\|(u-k)_+^2\big\|_{p_*,1,Q_\sigma,\theta}. \qquad (2.21)\end{aligned}$$

Therefore, combining (2.17) with (2.19), (2.20) and (2.21) yields

$$\big\|(u-l)_+^2\big\|_{p_*,1,Q_{\sigma'},\theta} \;\le\; \frac{c\,\tilde{\mathcal{A}}_2^\omega(n)^{1+\frac{1}{\alpha_*}}}{(l-k)^{2/\alpha_*}(\sigma-\sigma')^2}\,\big\|(u-k)_+^2\big\|_{p_*,1,Q_\sigma,\theta}^{1+\frac{1}{\alpha_*}}.$$

Introducing $\varphi(l,\sigma') := \big\|(u-l)_+^2\big\|_{p_*,1,Q_{\sigma'},\theta}$ and setting $M := c\,\tilde{\mathcal{A}}_2^\omega(n)^{1+\frac{1}{\alpha_*}}$ the above inequality reads

$$\varphi(l,\sigma') \;\le\; \frac{M}{(l-k)^{2/\alpha_*}(\sigma-\sigma')^2}\,\varphi(k,\sigma)^{1+\frac{1}{\alpha_*}} \tag{2.22}$$

and holds for any $0 \le k < l$ and $1/2 \le \sigma' < \sigma \le 1$.

Step 2: For any $\Delta \in [0, 2/(d+2))$ let $n \ge N_2(\Delta)$ where $N_2(\Delta) < \infty$ is such that $n^{2/(d+2)-\Delta} \ge 2$ for all $n \ge N_2$. Let $h \ge 0$ be arbitrary and $1/2 \le \sigma' < \sigma \le 1$ be chosen in such a way that $\sigma - \sigma' > n^{-\Delta}$. Further, for $j \in \mathbb{N}$ we set

$$\sigma_j \;:=\; 2^{-j}(\sigma-\sigma'), \qquad k_j \;:=\; h \,+\, K\big(1-2^{-j}\big),$$

where $K := 2^{2(1+\alpha_*)^2}\big(M/(\sigma-\sigma')^2\big)^{\alpha_*/2}\varphi(h,\sigma)^{1/2}$, and $J := \lfloor d\ln n/2\alpha_*\ln 2\rfloor$. Since $\alpha_* \ge (d+2)/2$, we have

$$(\sigma_{j-1}-\sigma_j)n \;=\; 2^{-j}(\sigma-\sigma')n \;>\; 1, \qquad \forall j = 1,\dots,J.$$

Next we claim that, by induction,

$$\varphi(k_j,\sigma_j) \;\le\; \frac{\varphi(h,\sigma)}{r^j}, \qquad \forall j = 1,\dots,J, \tag{2.23}$$

where $r = 2^{4(1+\alpha_*)}$. Indeed for $j = 0$ the bound (2.23) is trivial. Now assuming that (2.23) holds for any $j-1 \in \{0,\dots,J-1\}$, we obtain by (2.22) that

$$\begin{aligned}
\varphi(k_j,\sigma_j) \;&\le\; M\left(\frac{2^j}{K}\right)^{2/\alpha_*}\left(\frac{2^j}{(\sigma-\sigma')}\right)^2\varphi(k_{j-1},\sigma_{j-1})^{1+\frac{1}{\alpha_*}}\\
&\le\; M\left(\frac{2^j}{K}\right)^{2/\alpha_*}\left(\frac{2^j}{(\sigma-\sigma')}\right)^2\left(\frac{\varphi(h,\sigma)}{r^{j-1}}\right)^{1+\frac{1}{\alpha_*}} \;\le\; \frac{\varphi(h,\sigma)}{r^j},
\end{aligned}$$

which completes the proof of (2.23). Note that by the choice of J, $(n^{2d}2^{2J})/r^J \le 1$ and $(\sigma_J - \sigma_{J+1})n \ge 1$.

By using the Cauchy-Schwarz inequality, (2.20) and (2.23), we have that

$$\begin{aligned}\max_{(t,x)\in Q_{\sigma_{J+1}}} \big(u(t,x) - k_{J+1}\big)_+ \;&\le\; c\, n^d \,\big\|1/\theta^\omega\big\|_{1,B_\sigma}^{1/2} \,\big\|\big(u - k_{J+1}\big)_+^2\big\|_{1,\infty,Q_{\sigma_{J+1}},\theta}^{1/2} \\ &\le\; c\,\big\|1/\theta^\omega\big\|_{1,B_\sigma}^{1/2} \bigg(n^{2d}\, 2^{2J}\, \frac{\tilde{\mathcal{A}}_2^\omega(n)}{(\sigma-\sigma')^2}\,\varphi(k_J,\sigma_J)\bigg)^{1/2} \;\le\; c\,\bigg(\frac{\mathcal{A}_2^\omega(n)}{(\sigma-\sigma')^2}\,\varphi(h,\sigma)\bigg)^{1/2} \\ &=\; c\,\bigg(\frac{\mathcal{A}_2^\omega(n)}{(\sigma-\sigma')^2}\bigg)^{1/2}\, \big\|(u-h)_+\big\|_{2p_*,2,Q_\sigma(n),\theta}.\end{aligned}$$

Hence,

$$\max_{(t,x)\in Q_{\sigma'}} u(t,x) \;\le\; h \,+\, K \,+\, c\,\bigg(\frac{\mathcal{A}_2^\omega(n)}{(\sigma-\sigma')^2}\bigg)^{1/2}\, \big\|(u-h)_+\big\|_{2p_*,2,Q_\sigma(n),\theta},$$

and the claim follows with $\kappa = (1+\alpha_*)/2$ as in the statement. □

2.2.3 Oscillation Inequality

The next significant result allows us to control the oscillations of a space-time harmonic function. We denote the oscillation of a function u on a cylinder $Q \subseteq \mathbb{R}\times\mathbb{Z}^d$, $\operatorname{osc}_Q u := \max_{(t,x)\in Q} u(t,x) - \min_{(t,x)\in Q} u(t,x)$. Recall the definition of $\mathcal{A}_1^\omega(n)$ and $\mathcal{A}_2^\omega(n)$ in Proposition 2.2.2 and (2.14), respectively. For $n \ge 4$ we also set $\mathcal{A}_3^\omega(n) := \big\|1/\theta^\omega\big\|_{1,B(\frac{n}{4})}\big\|\theta^\omega\big\|_{1,B(\frac{n}{2})}$.

Theorem 2.2.5 (Oscillation inequality)**.** *Fix $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$. Let $u : \mathbb{Z}^d \to \mathbb{R}$ be such that $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ on $Q(n)$ for $n \ge 1$. There exists $N_3 = N_3(d)$ (independent of x_0) such that for all $n \ge N_3$ the following holds. There exists*

$$\gamma^\omega(x_0,n) = \gamma\big(\mathcal{A}_1^\omega(n), \mathcal{A}_2^\omega(n), \mathcal{A}_3^\omega(n), \big\|\mu^\omega\big\|_{1,B(n)}, \big\|\theta^\omega\big\|_{1,B(n)}, \big\|1/\theta^\omega\big\|_{1,B(n)}\big) \in (0,1),$$

which is continuous and increasing in all components, such that

$$\operatorname{osc}_{Q(n/4)} u \;\le\; \gamma^\omega(x_0,n)\, \operatorname{osc}_{Q(n)} u.$$

Before we prove Theorem 2.2.5 we briefly record the following continuity statement for space-time harmonic functions as one of its consequences.

Corollary 2.2.6. *Suppose that Assumptions 2.1.1 and 2.1.4 hold. Let $\delta > 0$, $x_0 \in \mathbb{Z}^d$ and $\sqrt{t_0}/2 > \delta$ be fixed. Suppose $\partial_t u - \mathcal{L}_t^\omega u = 0$ on $[0,t_0]\times B(x_0,n)$. For $\mathbb{P}$-a.e. ω, there exist $N_4 = N_4(x_0,\omega)$ and $\bar\gamma \in (0,1)$ (only depending on the law of ω and θ^ω) such that if*

$\delta n \geq N_4$, *then for any* $t \in n^2[t_0 - \delta^2, t_0]$ *and* $x_1, x_2 \in B(x_0, \delta n)$,

$$\big|u(t,x_1) - u(t,x_2)\big| \leq c_5 \left(\frac{\delta}{\sqrt{t_0}}\right)^{\varrho} \max_{[3t_0/4, t_0] \times B(x_0, \sqrt{t_0}/2)} u,$$

where $\varrho := \ln \bar{\gamma} / \ln(1/4)$ *and* c_5 *depends only on* $\bar{\gamma}$.

Proof. This follows from Theorem 2.2.5 as in [ACS21, Corollary 2.6], see also Proposition 2.2.13 below for a similar proof. □

In the remainder of this subsection we will prove Theorem 2.2.5 by following the method in [ACS21], originally used in [WYW06] for parabolic equations in continuous spaces. Consider the function $g : (0, \infty) \to [0, \infty)$, which may be regarded as a continuously differentiable version of the function $x \mapsto (-\ln x)_+$, defined by

$$g(z) := \begin{cases} -\ln z & \text{if } z \in (0, \bar{c}], \\ \frac{(z-1)^2}{2\bar{c}(1-\bar{c})} & \text{if } z \in (\bar{c}, 1], \\ 0 & \text{if } z \in (1, \infty), \end{cases}$$

where $\bar{c} \in [\frac{1}{4}, \frac{1}{3}]$ is the smallest solution of the equation $2c \ln(1/c) = 1 - c$. Note that $g \in C^1(0, \infty)$ is convex and non-increasing. Although $g(u)$ is not space-time harmonic, we can still bound its Dirichlet energy as follows.

Lemma 2.2.7. *Suppose* $u > 0$ *satisfies* $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ *on* $Q = I \times B$ *with* I *and* B *as in Lemma 2.2.4. Let* $\eta : \mathbb{Z}^d \to [0,1]$ *be a cut-off function with* $\operatorname{supp} \eta \subseteq B$ *and* $\eta \equiv 0$ *on* ∂B. *Then,*

$$\partial_t \big\|\eta^2 g(u_t)\big\|_{1,B,\theta} + \frac{\mathcal{E}^{\omega,\eta^2}(g(u_t))}{6\,\theta^\omega(B)} \;\leq\; 6\,\frac{\big\|1 \vee \mu^\omega\big\|_{1,B}}{\big\|\theta^\omega\big\|_{1,B}}\; \operatorname{osr}(\eta)^2\, \big\|\nabla \eta\big\|^2_{l^\infty(E_d)}, \tag{2.24}$$

where $\operatorname{osr}(\eta) := \max\{(\eta(y)/\eta(x)) \vee 1 \,|\, \{x,y\} \in E_d,\ \eta(x) \neq 0\}$ *and*

$$\mathcal{E}^{\omega,\eta^2}(f) \;:=\; \sum_{e \in E_d} \big(\eta^2(e^+) \wedge \eta^2(e^-)\big)\, \omega(e)\, (\nabla f)^2(e).$$

Proof. Since $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ on $Q = I \times B$,

$$\begin{aligned} \partial_t \big\langle \eta^2, g(u_t)\big\rangle_{\ell^2(\mathbb{Z}^d,\theta)} \;&=\; \big\langle \eta^2 g'(u_t), \partial_t u_t\big\rangle_{\ell^2(\mathbb{Z}^d,\theta)} \\ &=\; \big\langle \eta^2 g'(u_t), \mathcal{L}_\theta^\omega u_t\big\rangle_{\ell^2(\mathbb{Z}^d,\theta)} = -\big\langle \nabla(\eta^2 g'(u_t)), \omega \nabla u_t\big\rangle_{\ell^2(E_d)}. \end{aligned}$$

Now, g' is piecewise differentiable and $1/3 g'(z)^2 \leq g''(z)$ for a.e. $z \in (0, \infty)$. In particular,

$-zg'(z) \leq 4/3$ for any $z \in (0,\infty)$. So by [ACS21, Lemma A.1],

$$-\big\langle \nabla(\eta^2 g'(u_t)), \omega \nabla u_t \big\rangle_{\ell^2(E_d)} \;\leq\; -\frac{1}{6}\, \mathcal{E}^{\omega,\eta^2}\big(g(u_t)\big) \;+\; 6\, \mathrm{osr}(\eta)^2 \,\big\langle \nabla \eta, \omega \nabla \eta \big\rangle_{\ell^2(E_d)}.$$

The result follows by combining the above two estimates. $\square$

Now, define

$$M_n \;:=\; \sup_{(t,x)\in Q(n)} u(t,x) \quad \text{and} \quad m_n \;:=\; \inf_{(t,x)\in Q(n)} u(t,x). \tag{2.25}$$

For the purposes of the next lemma, given $k_0 \in \mathbb{R}$, we denote

$$k_j \;:=\; M_n - 2^{-j}(M_n - k_0), \qquad j \in \mathbb{N}. \tag{2.26}$$

Also recall the definition of $\mathcal{A}_3^\omega(n)$ right before Theorem 2.2.5.

Lemma 2.2.8. *Let $t_0 \in \mathbb{R}, x_0 \in \mathbb{Z}^d$, and u be such that $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ on $Q(n)$ for $n \geq 4$. Let $\eta : \mathbb{Z}^d \to [0,1]$ be the spatial cut-off function $\eta(x) := [1 - 2d(x_0,x)/n]_+$. Suppose, for some $k_0 \in \mathbb{R}$,*

$$\frac{1}{n^2}\int_{t_0-n^2}^{t_0} \big\|\mathbb{1}_{\{u_t \leq k_0\}}\big\|_{1,B(n),\eta^2\theta}\, dt \;\geq\; \frac{1}{2}. \tag{2.27}$$

Then there exist $c_6, c_7 > 0$ such that for any $\delta \in (0, 1/4c_7\mathcal{A}_3^\omega(n))$ and any

$$j \geq 1 + \frac{c_6 \,\big\|1 \vee \mu^\omega\big\|_{1,B(n)}\, \big\|1 \vee (1/\theta^\omega)\big\|_{1,B(n)}}{\frac{1}{4} - c_7 \delta \mathcal{A}_3^\omega(n)}$$

we have that

$$\big\|\mathbb{1}_{\{u_t \leq k_j\}}\big\|_{1,B(n/2),\theta} \;\geq\; \delta, \qquad \forall t \in \big[t_0 - \tfrac{1}{4}n^2, t_0\big].$$

Proof. Set

$$v_t(x) := \frac{M_n - u_t(x)}{M_n - k_0}, \qquad h_j = \epsilon_j := 2^{-j}, \quad j \in \mathbb{N}.$$

Then $\partial_t(v+\epsilon_j) - \mathcal{L}_\theta^\omega(v+\epsilon_j) = 0$ on $Q(n)$ for all $j \in \mathbb{N}$ and, for any $x \in \mathbb{Z}^d$, $u_t(x) > k_j$ if and only if $v_t(x) < h_j$. By (2.27) there exists $s_0 \in [t_0 - n^2, t_0 - \frac{1}{3}n^2]$ such that

$$\big\|\mathbb{1}_{\{v_{s_0} < 1\}}\big\|_{1,B(n),\eta^2\theta} \;\leq\; \frac{3}{4}. \tag{2.28}$$

To see this, assume the contrary is true, that is $\big\|\mathbb{1}_{\{v_s<1\}}\big\|_{1,B(n),\eta^2\theta} > \frac{3}{4}$ for all $s \in [t_0 -$

$n^2, t_0 - \frac{1}{3}n^2]$. Then

$$\begin{aligned}\frac{1}{2} \;&\geq\; \frac{1}{n^2}\int_{t_0-n^2}^{t_0} \left\|\mathbb{1}_{\{u_t>k_0\}}\right\|_{1,B,\eta^2\theta} dt \;=\; \frac{1}{n^2}\int_{t_0-n^2}^{t_0} \left\|\mathbb{1}_{\{v_t<1\}}\right\|_{1,B,\eta^2\theta} dt \\ &>\; \frac{1}{n^2}\int_{t_0-n^2}^{t_0-\frac{1}{3}n^2} \frac{3}{4}\, dt \;=\; \frac{1}{2},\end{aligned}$$

which is a contradiction. Let $t \in [t_0 - \frac{1}{4}n^2, t_0]$. By integrating the estimate (2.24) over the interval $[s_0, t]$, noting that $\left\|\nabla\eta\right\|_{l^\infty(E)} \leq 2/n$, $\operatorname{osr}(\eta) \leq 2$ and $t - s_0 \leq n^2$,

$$\left\|g(v_t+\epsilon_j)\right\|_{1,B(n),\eta^2\theta} \;\leq\; \left\|g(v_{s_0}+\epsilon_j)\right\|_{1,B(n),\eta^2\theta} + c\left\|1\vee\mu^\omega\right\|_{1,B(n)} \left\|\theta^\omega\right\|_{1,B(n)}^{-1}.$$

Since g is non-increasing and identically zero on $[1,\infty)$, using (2.28) we have

$$\left\|g(v_{s_0}+\epsilon_j)\right\|_{1,B(n),\eta^2\theta} \;\leq\; g(\epsilon_j)\left\|\mathbb{1}_{\{v_{s_0}<1\}}\right\|_{1,B(n),\eta^2\theta} \;\leq\; \frac{3}{4}\,g(\epsilon_j),$$

and

$$\left\|g(v_t+\epsilon_j)\right\|_{1,B(n),\eta^2\theta} \;\geq\; g(h_j+\epsilon_j)\left\|\mathbb{1}_{\{v_t<h_j\}}\right\|_{1,B(n),\eta^2\theta}.$$

So, combining the above, for $j \geq 2$

$$\begin{aligned}\left\|\mathbb{1}_{\{v_t<h_j\}}\right\|_{1,B(n),\eta^2\theta} \;&\leq\; \frac{3}{4}\,\frac{g(\epsilon_j)}{g(h_j+\epsilon_j)} + \frac{c}{g(h_j+\epsilon_j)}\left\|1\vee\mu^\omega\right\|_{1,B(n)}\left\|\theta^\omega\right\|_{1,B(n)}^{-1} \\ &\leq\; \frac{3}{4}\left(1+\frac{1}{j-1}\right) + \frac{c}{j-1}\left\|1\vee\mu^\omega\right\|_{1,B(n)}\left\|1\vee(1/\theta^\omega)\right\|_{1,B(n)}.\end{aligned}$$

Then, since $\eta \equiv 0$ on $B(n/2)^c$,

$$\begin{aligned}\left\|\mathbb{1}_{\{u_t\leq k_j\}}\right\|_{1,B(n/2),\theta} \;&=\; \frac{\langle\eta^2\theta^\omega,1\rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))}\left(1-\left\|\mathbb{1}_{\{v_t<h_j\}}\right\|_{1,B(n),\eta^2\theta}\right) \\ &\geq\; \frac{\langle\eta^2\theta^\omega,1\rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))}\left(\frac{1}{4}-\frac{c}{j-1}\left\|1\vee\mu^\omega\right\|_{1,B(n)}\left\|1\vee(1/\theta^\omega)\right\|_{1,B(n)}\right).\end{aligned} \tag{2.29}$$

Note that $\langle\eta^2\theta^\omega,1\rangle_{\ell^2(\mathbb{Z}^d)}/\theta^\omega(B(n/2)) \in (0,1)$ and since $\eta \geq 1/2$ on $B(n/4)$,

$$\frac{\langle\eta^2\theta^\omega,1\rangle_{\ell^2(\mathbb{Z}^d)}}{\theta^\omega(B(n/2))} \;\geq\; c\left\|\theta^\omega\right\|_{1,B(n/4)}\left\|\theta^\omega\right\|_{1,B(n/2)}^{-1}.$$

By combining this inequality above with (2.29) and using that

$$j-1 \;\geq\; \frac{c_6\left\|1\vee\mu^\omega\right\|_{1,B(n)}\left\|1\vee(1/\theta^\omega)\right\|_{1,B(n)}}{\frac{1}{4}-c_7\delta\left\|\theta^\omega\right\|_{1,B(n/4)}^{-1}\left\|\theta^\omega\right\|_{1,B(n/2)}}$$

by Jensen's inequality, we get the claim. $\square$

Lemma 2.2.9. *Set $\tau := 1/4$ and $\sigma := 1/2$. Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$, $n \geq 4$, and suppose u satisfies $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ on $Q(n)$. Assume there exist $\delta > 0$ and $i_0 \in \mathbb{N}$ such that*

$$\left\| \mathbb{1}_{\{u_t \leq k_{i_0}\}} \right\|_{1,B(x_0,\sigma n),\theta} \;\geq\; \delta, \qquad \forall t \in I_\tau = \left[t_0 - \tfrac{1}{4} n^2, t_0\right]. \tag{2.30}$$

Let $\epsilon \in (0,1)$ be arbitrary. Then there exists

$$j_0 = j_0\big(\epsilon, \delta, i_0, \mathcal{A}_1^\omega(n), \left\|\mu^\omega\right\|_{1,B(n)}, \left\|\theta^\omega\right\|_{1,B(n)}\big) \in \mathbb{N} \quad \textit{with } j_0 \geq i_0,$$

which is continuous and decreasing in the first two components and continuous and increasing in the other components, such that

$$\left\| \mathbb{1}_{\{u > k_j\}} \right\|_{1,1,Q_{\tau,\sigma}(n),\theta} \;\leq\; \epsilon, \qquad \forall\, j \geq j_0.$$

Proof. Let $\eta : \mathbb{Z}^d \to [0,1]$ be a cut-off function such that $\operatorname{supp} \eta \subseteq B(n)$, $\eta \equiv 1$ on B_σ and $\eta \equiv 0$ on $\partial B(n)$ with linear decay on $B(n) \setminus B_\sigma$. So $\left\|\nabla \eta\right\|_{\ell^\infty(E_d)} \leq 2/n$ and $\operatorname{osr}(\eta) \leq 2$. Now, let

$$w_t(x) \;:=\; \frac{M_n - u_t(x)}{M_n - k_{i_0}} \quad \text{and} \quad h_j = \epsilon_j := 2^{-j}.$$

Then $w \geq 0$ and $\partial_t(w + \epsilon_j) - \mathcal{L}_\theta^\omega(w + \epsilon_j) = 0$ on $Q(n)$ for $j \in \mathbb{N}$. For any $t \in I_\tau$, let $\mathcal{N}_t := \{x \in B_\sigma : g(w_t(x) + \epsilon_j) = 0\}$. Since $g \equiv 0$ on $(1, \infty)$ by its definition,

$$\frac{\theta^\omega(\mathcal{N}_t)}{\theta^\omega(B_\sigma)} \;=\; \left\| \mathbb{1}_{\{g(w_t+\epsilon_j)=0\}} \right\|_{1,B_\sigma,\theta} \;\geq\; \left\| \mathbb{1}_{\{w_t \geq 1\}} \right\|_{1,B_\sigma,\theta} \;=\; \left\| \mathbb{1}_{\{u_t \leq k_{i_0}\}} \right\|_{1,B_\sigma,\theta} \;\geq\; \delta,$$

where we used (2.30) in the last step. By Proposition 2.2.2 we have

$$\left\| g(w_t + \epsilon_j) \right\|_{1,B_\sigma,\theta}^2 \;\leq\; c_7\, n^2\, \mathcal{A}_1^\omega(\sigma n) \left(1 + \frac{\theta^\omega(B_\sigma)}{\theta^\omega(\mathcal{N}_t)}\right)^2 \frac{\mathcal{E}^{\omega,\eta^2}\big(g(w_t + e_j)\big)}{|B_\sigma|},$$

so that by Jensen's inequality and by integrating (2.24) over I_τ,

$$\begin{aligned}
\left\| g(w + \epsilon_j) \right\|_{1,1,Q_{\tau,\sigma},\theta}^2 \;&\leq\; \frac{1}{\tau n^2} \int_{I_\tau} \left\| g(w_t + \epsilon_j) \right\|_{1,B_\sigma,\theta}^2 \, dt \\
&\leq\; \frac{c}{\delta^2}\, \mathcal{A}_1^\omega(\sigma n) \left\|\theta^\omega\right\|_{1,B(n)} \int_{I_\tau} \frac{\mathcal{E}^{\omega,\eta^2}\big(g(w_t + \epsilon_j)\big)}{\theta^\omega(B(n))} \, dt \\
&\leq\; \frac{c}{\delta^2}\, \mathcal{A}_1^\omega(n) \Big(\left\|\theta^\omega\right\|_{1,B(n)} \left\| \eta^2 g(w_{t_0 - \tau n^2} + \epsilon_j) \right\|_{1,B(n),\theta} + \left\| 1 \vee \mu^\omega \right\|_{1,B(n)} \Big).
\end{aligned}$$

Since g is non-increasing and $w_t > 0$ for all $t \in I_\tau$,

$$\begin{aligned}
\big\| \mathbb{1}_{w<h_j} \big\|_{1,1,Q_{\tau,\sigma},\theta}^2 \;&\le\; \frac{\big\| g(w+\epsilon_j) \big\|_{1,1,Q_{\tau,\sigma},\theta}^2}{g(h_j+\epsilon_j)^2} \\
&\le\; \frac{c}{\delta^2}\, A_1^\omega(n) \left(\big\|\theta\big\|_{1,B(n)} \frac{g(\epsilon_j)}{g(h_j+\epsilon_j)^2} \;+\; \big\|1 \vee \mu^\omega\big\|_{1,B(n)} \frac{1}{g(h_j+\epsilon_j)^2} \right) \\
&\le\; \frac{c}{\delta^2}\, A_1^\omega(n)\, \big\|1 \vee \theta^\omega\big\|_{1,B(n)}\, \big\|1 \vee \mu^\omega\big\|_{1,B(n)} \left(\frac{j}{(j-1)^2} \;+\; \frac{1}{(j-1)^2} \right).
\end{aligned} \tag{2.31}$$

Therefore, for any $\epsilon > 0$, there exists some $j_0 \ge i_0$ as in the statement such that $\big\|\mathbb{1}_{\{u>k_j\}}\big\|_{1,1,Q_{\tau,\sigma},\theta} = \big\|\mathbb{1}_{\{w<h_{j-i_0}\}}\big\|_{1,1,Q_{\tau,\sigma},\theta} \le \epsilon$ for all $j \ge j_0$. □

Proof of Theorem 2.2.5. We may assume without loss of generality that $u > 0$, otherwise consider $u - \inf_{Q(n)} u$. Set $\tau = 1/4$, $\sigma = 1/2$ as before in Lemma 2.2.9. Define $k_0 := (M_n + m_n)/2$ with M_n and m_n as in (2.25) and let k_j be defined by (2.26). Further, let η be the cut-off function $\eta(x) := \big[1 - d(x_0,x)/\sigma n\big]_+$. We may assume

$$\frac{1}{n^2} \int_{I_1} \big\|\mathbb{1}_{\{u_t \le k_0\}}\big\|_{1,B(n),\eta^2\theta}\, dt \;\ge\; \frac{1}{2}.$$

Otherwise, consider $M_n + m_n - u$ in place of u. Set $\epsilon := \big(2c_3\,(4\mathcal{A}_2^\omega(\sigma n))^\kappa\big)^{-2p_*}$ with $\mathcal{A}_2^\omega(n)$ as in Theorem 2.2.3. Fix any $\Delta \in (0, \frac{2}{d+2})$ and $N_3 \ge 2N_1(\Delta)$ such that $\frac{1}{2} > (\sigma N_3)^{-\Delta}$. Now for all $n \ge N_3$, applying consecutively Lemma 2.2.8 and Lemma 2.2.9, there exists

$$l = l^\omega(x_0,n) = l\big(\mathcal{A}_1^\omega(n), \mathcal{A}_2^\omega(n), \mathcal{A}_3^\omega(n), \big\|\mu^\omega\big\|_{1,B(n)}, \big\|\theta^\omega\big\|_{1,B(n)}, \big\|1/\theta^\omega\big\|_{1,B(n)}\big),$$

which is continuous and increasing in all components, such that

$$\big\|\mathbb{1}_{\{u>k_j\}}\big\|_{1,1,Q_{\tau,\sigma}(n),\theta} \;\le\; \epsilon, \qquad \forall j \ge l.$$

By an application of Jensen's inequality,

$$\begin{aligned}
\big\|(u-k_l)_+\big\|_{2p_*,2,Q_1(\sigma n),\theta} \;&\le\; \big(M_n - k_l\big)\, \big\|\mathbb{1}_{\{u>k_l\}}\big\|_{2p_*,2,Q(\sigma n),\theta} \\
&\le\; \big(M_n - k_l\big)\, \big\|\mathbb{1}_{\{u>k_l\}}\big\|_{1,1,Q_1(\sigma n),\theta}^{1/2p_*} \;\le\; \big(M_n - k_l\big)\, \epsilon^{1/2p_*}.
\end{aligned}$$

Now, let $\vartheta = \frac{\sigma}{2} = \frac{1}{4}$. Then Theorem 2.2.3 implies that

$$\begin{aligned}
M_{\vartheta n} \;&\le\; \max_{Q_{1/2}(\sigma n)} u(t,x) \;\le\; k_l + c_3\big(4\mathcal{A}_2^\omega(\sigma n)\big)^\kappa\, \big\|(u-k_l)_+\big\|_{2p_*,2,Q_1(\sigma n),\theta} \\
&\le\; k_l \;+\; \frac{1}{2}\big(M_n - k_l\big) \;=\; M_n \;-\; 2^{-(l+2)}\big(M_n - m_n\big).
\end{aligned}$$

Hence

$$M_{\vartheta n} - m_{\vartheta n} \;\leq\; M_n \;-\; 2^{-(l+2)}\big(M_n - m_n\big) - m_{\vartheta n} \;\leq\; \big(1 - 2^{-(l+2)}\big)\big(M_n - m_n\big),$$

and the theorem is proven. □

2.2.4 Proof of the Local Limit Theorem

As mentioned at the beginning of this section, we will derive the required Hölder regularity estimate from the oscillation inequality in Theorem 2.2.5. The following version of the ergodic theorem will help us to control ergodic averages on scaled balls with varying centre-points.

Proposition 2.2.10. *Let $\mathcal{B} := \big\{B : B$ closed Euclidean ball in $\mathbb{R}^d\big\}$. Suppose that Assumption 2.1.1 holds. Then, for any $f \in L^1(\Omega)$,*

$$\lim_{n\to\infty} \sup_{B\in\mathcal{B}} \left| \frac{1}{n^d} \sum_{x\in(nB)\cap\mathbb{Z}^d} f \circ \tau_x \;-\; |B| \cdot \mathbb{E}\big[f\big] \right| \;=\; 0, \qquad \mathbb{P}\text{-a.s.},$$

where $|B|$ denotes the Lebesgue measure of B.

Proof. See, for instance, [KP87, Theorem 1]. □

Lemma 2.2.11. *Suppose Assumptions 2.1.1 and 2.1.4 hold. Let γ^ω be as in Theorem 2.2.5. Then, $\mathbb{P}$-a.s., for any $x \in \mathbb{R}^d$ and $\delta \in (0,1)$,*

$$\limsup_{n\to\infty} \gamma^\omega(\lfloor nx \rfloor, \delta n) \;\leq\; \bar{\gamma} \in (0,1),$$

with $\bar{\gamma}$ only depending on the law of ω and θ^ω.

Proof. Recall that γ^ω is continuous and increasing in all components. Now, by Proposition 2.2.10 we have, for instance, for any $x \in \mathbb{R}^d$ and $\delta \in (0,1)$,

$$\limsup_{n\to\infty} \big\|\mu^\omega\big\|_{1,B(\lfloor nx\rfloor,\delta n)} \;\leq\; \mathbb{E}\big[\mu^\omega(0)\big] =: \bar{\mu}, \qquad \mathbb{P}\text{-a.s.}$$

Analogous statements hold for the other components of γ^ω, that is $\mathcal{A}_1^\omega$, $\mathcal{A}_2^\omega$ etc. Since γ^ω is continuous and increasing in all components we get the claim for some $\bar{\gamma} \in (0,1)$ depending only on the respective moments of $\mu^\omega(0)$, $\nu^\omega(0)$ and $\theta^\omega(0)$. □

Lemma 2.2.11 facilitates applying the oscillations inequality iteratively with a common, deterministic constant. Together with the upper heat kernel bound cited below, this will produce a Hölder continuity statement for the rescaled heat kernel in Proposition 2.2.13 below.

Lemma 2.2.12. *Suppose Assumptions 2.1.1 and 2.1.4 hold. For $\mathbb{P}$-a.e. ω, any $\lambda > 0$ and $x \in \mathbb{Z}^d$ there exist $c_8 = c_8(d, p, q, r, \lambda)$ and $N_5 = N_5(x, \omega)$ such that for any t with $\sqrt{t} \geq N_5$ and all $y \in B(x, \lambda\sqrt{t})$,*

$$p_\theta^\omega(t, x, y) \;\leq\; c_8\, t^{-d/2}.$$

Proof. This can be directly read off [ADS19, Theorem 3.2] or derived from Theorem 2.2.3 by the method in [ACS21, Corollary 2.10]. □

Proposition 2.2.13. *Let $\delta > 0$, $\sqrt{t}/2 \geq \delta$ and $x \in \mathbb{R}^d$ be fixed. Then, there exists $c_9 > 0$ such that for $\mathbb{P}$-a.e. ω,*

$$\limsup_{n\to\infty} \sup_{\substack{y_1,y_2 \in B(\lfloor nx \rfloor, \delta n) \\ s_1,s_2 \in [t-\delta^2, t]}} n^d \left| p_\theta^\omega\big(n^2 s_1, 0, y_1\big) - p_\theta^\omega\big(n^2 s_2, 0, y_2\big) \right| \;\leq\; c_9 \left(\frac{\delta}{\sqrt{t}}\right)^{\varrho} t^{-d/2},$$

where $\varrho = \ln(\bar{\gamma})/\ln(1/4)$.

Proof. Set $\delta_k := 4^{-k}\sqrt{t}/2$ and with a slight abuse of notation let

$$Q_k \;:=\; n^2[t - \delta_k^2, t] \times B(\lfloor nx \rfloor, \delta_k n), \qquad k \geq 0.$$

Choose $k_0 \in \mathbb{N}$ such that $\delta_{k_0} \geq \delta > \delta_{k_0+1}$. In particular, for every $k \leq k_0$ we have $\delta_k \in [\delta, \sqrt{t}]$. Now apply Theorem 2.2.5 and Lemma 2.2.11, which give that there exists $N_6 = N_6(\omega, x, \delta)$ such that for $\mathbb{P}$-a.e. ω and all $n \geq N_6$,

$$\operatorname{osc}_{Q_k} p_\theta^\omega\big(\cdot, 0, \cdot\big) \;\leq\; \bar{\gamma}\, \operatorname{osc}_{Q_{k-1}} p_\theta^\omega\big(\cdot, 0, \cdot\big), \qquad \forall k = 1, ..., k_0.$$

We iterate the above inequality on the chain $Q_0 \supset Q_1 \supset \cdots \supset Q_{k_0}$ to obtain

$$\operatorname{osc}_{Q_{k_0}} p_\theta^\omega\big(\cdot, 0, \cdot\big) \;\leq\; \bar{\gamma}^{k_0} \max_{Q_0} p_\theta^\omega\big(\cdot, 0, \cdot\big). \tag{2.32}$$

Note that

$$Q_{k_0} \;=\; n^2[t - \delta_{k_0}^2, t] \times B(\lfloor nx \rfloor, \delta_{k_0} n) \;\supset\; n^2[t - \delta^2, t] \times B(\lfloor nx \rfloor, \delta n).$$

Hence, since $\overline{\gamma}^{k_0} \leq c(\delta/\sqrt{t})^{\varrho}$, the claim follows from (2.32) and Lemma 2.2.12. □

We shall now apply the above Hölder regularity to prove a pointwise version of the local limit theorem.

Proposition 2.2.14. *Suppose Assumptions 2.1.1 and 2.1.4 hold. For any $x \in \mathbb{Z}^d$ and*

$t > 0$,

$$\lim_{n\to\infty} \Big| n^d \, p_\theta^\omega\big(n^2 t, 0, \lfloor nx \rfloor\big) - a\, k_t(x) \Big| \;=\; 0, \qquad \mathbb{P}\text{-}a.s.$$

with k_t *as defined in* (2.4) *and* $a := \mathbb{E}\big[\theta^\omega(0)\big]^{-1}$.

Proof. For any $x \in \mathbb{R}^d$ and $\delta > 0$ let $C(x,\delta) := x + [-\delta,\delta]^d$ and $C^n(x,\delta) := n\, C(x,\delta) \cap \mathbb{Z}^d$, i.e. $C(x,\delta)$ is a ball in $\mathbb{R}^d$ with respect to the supremum norm. Note that the cubes $C^n(x,\delta)$ are comparable with $B(\lfloor nx \rfloor, \delta n)$ and we may apply Proposition 2.2.13 with $B(\lfloor nx \rfloor, \delta n)$ replaced by $C^n(x,\delta)$. Let

$$J := \Big(p_\theta^\omega\big(n^2 t, 0, \lfloor nx \rfloor\big) - n^{-d}\, a\, k_t(x) \Big)\, \theta^\omega\big(C^n(x,\delta)\big).$$

We can rewrite this, for any $\delta > 0$, as $J = \sum_{i=1}^4 J_i$ where

$$\begin{aligned}
J_1 &:= \sum_{z \in C^n(x,\delta)} \Big(p_\theta^\omega\big(n^2 t, 0, \lfloor nx \rfloor\big) - p_\theta^\omega\big(n^2 t, 0, z\big) \Big)\, \theta^\omega(z),\\
J_2 &:= P_0^\omega\Big[X_t^{(n)} \in C(x,\delta) \Big] - \int_{C(x,\delta)} k_t(y)\, dy,\\
J_3 &:= k_t(x) \Big((2\delta)^d - \theta^\omega\big(C^n(x,\delta)\big)\, n^{-d}\, a \Big),\\
J_4 &:= \int_{C(x,\delta)} \big(k_t(y) - k_t(x) \big)\, dy,
\end{aligned}$$

with $X_t^{(n)} := \frac{1}{n} X_{n^2 t}$, $t \geq 0$, being the rescaled random walk. It suffices to prove that, for each $i = 1, \ldots, 4$, as $n \to \infty$, $|J_i| / n^{-d} \theta^\omega\big(C^n(x,\delta)\big)$ converges $\mathbb{P}$-a.s. to a limit which is small with respect to δ.

First note that $J_2 \to 0$ by Theorem 2.1.2 and $n^{-d} \theta^\omega\big(C^n(x,\delta)\big) \to (2\delta)^d / a$ by the arguments of Lemma 2.2.11. Thus, $\lim_{n\to\infty} |J_i| / n^{-d} \theta^\omega\big(C^n(x,\delta)\big) = 0$ for $i = 2, 3$. Further, by the Lipschitz continuity of the heat kernel k_t in its space variable it follows that $\lim_{n\to\infty} |J_4| / n^{-d}\, \theta^\omega\big(C^n(x,\delta)\big) = O(\delta)$. To deal with the remaining term, we apply Proposition 2.2.13, which yields

$$\limsup_{n\to\infty} \max_{z \in C^n(x,\delta)} n^d \left| p_\theta^\omega\big(n^2 t, 0, z\big) \;-\; p_\theta^\omega\big(n^2 t, 0, \lfloor nx \rfloor\big) \right| \;\leq\; c\, \delta^\varrho\, t^{-\frac{d}{2} - \frac{\varrho}{2}}.$$

Hence, $\limsup_{n\to\infty} |J_1| / n^{-d}\, \theta^\omega\big(C^n(x,\delta)\big) \;=\; O(\delta^\varrho)$, $\mathbb{P}$-a.s. Finally, the claim follows by letting $\delta \to 0$. $\square$

Proof of Theorem 2.1.5. Having proven the pointwise result Proposition 2.2.14, the full local limit theorem follows by extending over compact sets in x and t. This is done using a covering argument, exactly as in Step 2 in the proof of [ACS21, Theorem 3.1], which in turn is a slight modification of the proofs in [CH08] and [BH09]. $\square$

2.2.5 Weak Parabolic Harnack Inequality and Near-Diagonal Heat Kernel Bounds

The above method of proving the local limit theorem is simpler than the derivations of [ADS16a, BH09], in part because it does not require a full parabolic Harnack inequality. However, the above analysis still provides a weak parabolic Harnack inequality.

Proposition 2.2.15. *Suppose Assumptions 2.1.1 and 2.1.4 hold. For any $x_0 \in \mathbb{Z}^d$, $t_0 \in \mathbb{R}$ and $\mathbb{P}$-a.e. ω, there exists $N_7 = N_7(\omega, x_0)$ such that for all $n \geq N_7$ the following holds. Let $u > 0$ be such that $\partial_t u - \mathcal{L}_\theta^\omega u = 0$ on $Q(n) := [t_0 - n^2, t_0] \times B(x_0, n)$. Assume there exists $\epsilon > 0$ such that*

$$\frac{1}{n^2} \int_{t_0 - n^2}^{t_0} \left\| \mathbb{1}_{\{u_t \geq \epsilon\}} \right\|_{1, B(n), \eta^2 \theta} dt \;\geq\; \frac{1}{2} \tag{2.33}$$

with η as in Lemma 2.2.8. Then there exists $\gamma = \gamma(\epsilon, p, q, d)$ (also depending on the law of ω and θ^ω) such that

$$u(t, x) \geq \gamma \qquad \forall\, (t, x) \in Q_{\frac{1}{2}}(n/2) = [t_0 - n^2/8, t_0] \times B(x_0, n/4).$$

Proof. This follows by the same method as [ACS21, Theorem 2.14]. Theorem 2.2.3, Lemma 2.2.8 and Lemma 2.2.9 are all necessary ingredients. □

Finally, we can also derive from Theorem 2.1.5 a near-diagonal lower heat kernel estimate, which complements the upper bounds obtained in [ADS19].

Corollary 2.2.16. *Suppose Assumptions 2.1.1 and 2.1.4 hold. For $\mathbb{P}$-a.e. ω, there exists $N_8(\omega) > 0$ and $c_{10} = c_{10}(d) > 0$ such that for all $t \geq N_8(\omega)$ and $x \in B(0, \sqrt{t})$,*

$$p_\theta^\omega(t, 0, x) \geq c_{10}\, t^{-d/2}.$$

Proof. This follows from the local limit theorem exactly as for the constant speed case in [ADS16a, Lemma 5.3]. □

2.3 Annealed Local Limit Theorem under General Speed Measure

2.3.1 Maximal Inequality for the Heat Kernel

The first step to show the annealed local limit theorem in Theorem 2.1.7 is to establish an L^1 form of the maximal inequality in [ADS19], which involves space-time cylinders of a more convenient form for this section. So for $\epsilon \in (0,1/4),\ x_0 \in \mathbb{Z}^d$, we redefine

$$Q_\sigma(n) := \big[(1-\sigma)\epsilon n^2, n^2 - (1-\sigma)\epsilon n^2\big] \times B(x_0, \sigma n)$$

where $n \in \mathbb{N}$ and $\sigma \in [\frac{1}{2}, 1]$.

Proposition 2.3.1. *Fix $\epsilon \in (0,1/4)$, $x_0 \in \mathbb{Z}^d$ and let $p,q,r \in (1,\infty]$ be such that* (2.2) *holds. There exists $c_{11} = c_{11}(d,p,q,r)$ such that for all $n \geq 1$ and $1/2 \leq \sigma' < \sigma \leq 1$,*

$$\max_{(t,x)\in Q_{\sigma'}(n)} p_\theta^\omega(t,0,x) \leq c_{11} \big\| 1 \vee (1/\theta^\omega) \big\|_{1,B(n)} \left(\frac{\mathcal{A}_4^\omega(n)}{\epsilon(\sigma-\sigma')^2} \right)^{\kappa'} \big| p_\theta^\omega(\cdot,0,\cdot) \big|_{1,1/p_*,Q_\sigma(n),\theta},$$

where $\kappa' = \kappa'(d,p,q,r) := p_* + p_*^2 \rho/(\rho - r_* p_*)$ with ρ as in Proposition 2.2.1 and

$$\mathcal{A}_4^\omega(n) \ := \ \big| 1 \vee (\mu^\omega/\theta^\omega) \big|_{p,B(n),\theta} \big\| 1 \vee \nu^\omega \big\|_{q,B(n)} \big\| 1 \vee \theta^\omega \big\|_{r,B(n)} \big\| 1 \vee (1/\theta^\omega) \big\|_{q,B(n)}. \tag{2.34}$$

Proof. For abbreviation we set $u = p_\theta^\omega(\cdot,0,\cdot)$ and $\sigma_k := \sigma - (\sigma - \sigma')2^{-k}$. Further, write $B_k := B(x_0, \sigma_k n)$ and $Q_k := Q_{\sigma_k}(n)$. Note that $|B_k|/|B_{k+1}| \leq c\, 2^d$. Let $\gamma = 1/(2p_*)$. Then by Hölder's inequality

$$|u|_{2p_*,2,Q_k,\theta} \ \leq \ |u|_{1,2\gamma,Q_k,\theta}^\gamma \, \|u\|_{\infty,\infty,Q_k}^{1-\gamma},$$

and by the proof of [ADS19, Proposition 3.8] (cf. last line on page 14), setting $\phi = 1$ and $\delta = 1$ there, we have

$$\|u\|_{\infty,\infty,Q_{k-1}} \ \leq \ c \left(\frac{\mathcal{A}_4^\omega(n)}{\epsilon(\sigma_k - \sigma_{k-1})^2} \right)^{\kappa/p_*} |u|_{2p_*,2,Q_k,\theta}$$

with $\kappa = \kappa(d,p,q,r)$ as throughout [ADS19]. Combining the above equations yields

$$\|u\|_{\infty,\infty,Q_{k-1}} \ \leq \ 2^{2\kappa k/p_*} \, J \, |u|_{1,2\gamma,Q_\sigma,\theta}^\gamma \, \|u\|_{\infty,\infty,Q_k}^{1-\gamma},$$

where we have introduced $J := c\big(\mathcal{A}_4^\omega(n)/\epsilon(\sigma-\sigma')^2\big)^{\kappa/p_*} \geq 1$ for brevity. By iteration, we have for any $K \in \mathbb{Z}_+$,

$$\|u\|_{\infty,\infty,Q_{\sigma'}} \ \leq \ 2^{2\kappa/p_* \sum_{k=0}^{K-1}(k+1)(1-\gamma)^k} \Big(J \, |u|_{1,2\gamma,Q_\sigma,\theta}^\gamma \Big)^{\sum_{k=0}^{K-1}(1-\gamma)^k} \|u\|_{\infty,\infty,Q_K}^{(1-\gamma)^K}. \tag{2.35}$$

Note that $p_\theta^\omega(t,0,x)\,\theta^\omega(x) \leq 1$ for all $t > 0$ and $x \in \mathbb{Z}^d$. Therefore,

$$\|u\|_{\infty,\infty,Q_K} \ \leq \ \max_{x\in B_K} \theta^\omega(x)^{-1} \max_{(t,x)\in Q(n)} u(t,x)\,\theta^\omega(x) \ \leq \ |B_K| \ \|1/\theta^\omega\|_{1,B_K}.$$

Since $\limsup_{K\to\infty} |B_K|^{(1-\gamma)^K} \leq c$ with c independent of n and $\|1/\theta^\omega\|_{1,B_K}^{(1-\gamma)^K} \leq c\,\|1 \vee$

$(1/\theta^\omega)\big\|_{1,B(n)}$, we obtain by letting $K \to \infty$ in (2.35),

$$\big\|u\big\|_{\infty,\infty,Q_{\sigma'}} \;\le\; 2^{\frac{2\kappa}{p_* \gamma^2}} \,\big\|1 \vee (1/\theta^\omega)\big\|_{1,B(n)}\, J^{1/\gamma}\,\big|u\big|_{1,2\gamma,Q_\sigma,\theta},$$

which completes the proof, with $\kappa' := 2\kappa$. □

2.3.2 Proof of Theorem 2.1.7

Here we anneal the results of Section 2.2 to derive the annealed local limit theorem for the static RCM under a general speed measure stated in Theorem 2.1.7. This will require a stronger moment condition. For any $p, q, r_1, r_2 \in [1, \infty]$ set

$$M(p,q,r_1,r_2) \;:=\; \mathbb{E}\left[\mu^\omega(0)^p\right] + \mathbb{E}\left[\nu^\omega(0)^q\right] + \mathbb{E}\left[\theta^\omega(0)^{-r_1}\right] + \mathbb{E}\left[\theta^\omega(0)^{r_2}\right] \in (0,\infty].$$

Proposition 2.3.2. *Suppose Assumption 2.1.1 holds. Then there exist $p, q, r_1, r_2 \in (0,\infty)$ (only depending on d) such that, under the moment condition $M(p,q,r_1,r_2) < \infty$, for all $K > 0$ and $0 < T_1 \le T_2$,*

$$\mathbb{E}\Bigg[\sup_{n\ge 1, |x|\le K,\, t\in[T_1,T_2]} n^d\, p^\omega_\theta(n^2 t, 0, \lfloor nx \rfloor)\Bigg] \;<\; \infty.$$

Before we prove Proposition 2.3.2 we remark that it immediately implies the annealed local limit theorem.

Proof of Theorem 2.1.7. Given the quenched result in Theorem 2.1.5, the statement follows from Proposition 2.3.2 by the dominated convergence theorem. □

As a by-product we obtain an annealed on-diagonal estimate on the heat kernel.

Corollary 2.3.3. *Under the assumptions of Proposition 2.3.2, for any $\lambda > 0$ there exists $c_{12} = c_{12}(\lambda, d, p, q, r)$ such that for all $t \ge 1$,*

$$\mathbb{E}\Bigg[\sup_{x\in B(0,\lambda\sqrt{t})} p^\omega_\theta(t,0,x)\Bigg] \;\le\; c_{12}\, t^{-d/2}. \tag{2.36}$$

Proof. Choosing $K = 2\lambda$, $n = \lceil \sqrt{t} \rceil$, $T_2 = 1$ and any $T_1 \in (0,1)$, this follows directly from Proposition 2.3.2. □

The rest of this section is devoted to the proof of Proposition 2.3.2. We start with a consequence of the maximal inequality in Proposition 2.3.1.

Lemma 2.3.4. *Let p, q, $r \in (1,\infty]$ be such that* (2.2) *holds. For all $K > 0$, $0 < T_1 \le T_2$, there exist $c_{13} = c_{13}(d,p,q,r,K,T_1,T_2)$ and $c_{14} = c_{14}(K,T_2)$ such that*

$$\sup_{|x|\le K,\, t\in[T_1,T_2]} n^d\, p_\theta^\omega(n^2t, 0, \lfloor nx \rfloor) \;\le\; c_{13} \left\| 1 \vee (1/\theta^\omega) \right\|_{1,B(n)} \mathcal{A}_4^\omega(c_{14}n)^{\kappa'}, \qquad \forall n \ge 1,$$

with $\mathcal{A}_4^\omega$ as in (2.34).

Proof. First note that by definition of the heat kernel p_θ^ω,

$$\begin{aligned} \left| p_\theta^\omega(\cdot,0,\cdot) \right|_{1,1/p_*,Q_1(n),\theta} \;&=\; \left(\frac{1}{|I_1|} \int_{I_1} \left(\frac{1}{|B(n)|} \sum_{y\in B(n)} p_\theta^\omega(t,0,y)\, \theta^\omega(y) \right)^{1/p_*} dt \right)^{p_*} \\ &\le\; c\, n^{-d} \left(\frac{1}{|I_1|} \int_{I_1} P_0^\omega\big[X_t \in B(n) \big]^{1/p_*} dt \right)^{p_*} \;\le\; c\, n^{-d}, \end{aligned} \tag{2.37}$$

for all $n \in \mathbb{N}$. Choose $x_0 = 0$ and set $N = c_{14}n$ for any $c_{14} > 2\lceil K \vee \sqrt{T_2} \rceil$. Then we can find $\epsilon \in (0, 1/4)$ such that

$$\left\{ (n^2t, \lfloor nx \rfloor) :\, t \in [T_1,T_2], |x| \le K \right\} \;\subseteq\; Q_{1/2}(N) = \left[\tfrac{\epsilon}{2}N^2, (1-\tfrac{\epsilon}{2})N^2 \right] \times B(0, N/2).$$

The claim follows now from Proposition 2.3.1 with the choice $\sigma = 1$ and $\sigma' = 1/2$ together with (2.37). $\square$

Another ingredient in the proof of Proposition 2.3.2 will be the following version of the maximal ergodic theorem, which we recall for the reader's convenience.

Lemma 2.3.5. *Suppose Assumption 2.1.1 holds. Then for all $p \in (1,\infty)$, $x_0 \in \mathbb{Z}^d$ and $f \in L^p(\Omega)$,*

$$\mathbb{E}\left[\sup_{n\ge1} \left(\frac{1}{|B(n)|} \sum_{x\in B(x_0,n)} f \circ \tau_x \right)^p \right] \le \left(\frac{p}{p-1} \right)^p \mathbb{E}\left[f^p \right]. \tag{2.38}$$

Proof. See [Kre85, Chapter 6, Theorem 1.2]. $\square$

Proof of Proposition 2.3.2. By Lemma 2.3.4 it suffices to show that, under a suitable moment condition, $\mathbb{E}\left[\sup_{n\ge1} \left\| 1 \vee (1/\theta^\omega) \right\|_{1,B(n)} \mathcal{A}_4^\omega(c_{14}n)^{\kappa'} \right] < \infty$. Recall that

$$\mathcal{A}_4^\omega(n) \;=\; \left| 1 \vee (\mu^\omega/\theta^\omega) \right|_{p,B(n),\theta} \left\| 1 \vee \nu^\omega \right\|_{q,B(n)} \left\| 1 \vee \theta^\omega \right\|_{r,B(n)} \left\| 1 \vee (1/\theta^\omega) \right\|_{q,B(n)},$$

for any $p, q, r \in (1,\infty]$ satisfying (2.2). After an application of Hölder's inequality it suffices to show

$$\mathbb{E}\left[\sup_{n\ge1} \left\| \nu^\omega \right\|_{q,B(n)}^{4\kappa'} \right] \;<\; \infty,$$

and similar moment bounds on the other terms. Now suppose that $\mathbb{E}\big[\nu^\omega(0)^{4\kappa'\vee q'}\big] < \infty$ for any $q' > q$. Then, if $4\kappa' > q$, given Assumption 2.1.1, we can apply Lemma 2.3.5 to deduce

$$\mathbb{E}\left[\sup_{n\geq 1} \left\|\nu^\omega\right\|_{q,B(n)}^{4\kappa'}\right] \;\leq\; c\,\mathbb{E}\left[\nu^\omega(0)^{4\kappa'}\right] \;<\; \infty.$$

In the case $4\kappa' \leq q < q'$, we have by Jensen's inequality followed by Lemma 2.3.5,

$$\begin{aligned}\mathbb{E}\left[\sup_{n\geq 1} \left\|\nu^\omega\right\|_{q,B(n)}^{4\kappa'}\right] \;&\leq\; \mathbb{E}\left[\sup_{n\geq 1}\left(\frac{1}{|B(n)|}\sum_{x\in B(n)} \nu^\omega(x)^q\right)^{\frac{q'}{q}}\right]^{\frac{4\kappa'}{q'}} \\ &\leq\; c\,\mathbb{E}\left[v^\omega(0)^{q'}\right]^{\frac{4\kappa'}{q'}} \;<\; \infty.\end{aligned}$$

The other terms involving $\left\|\theta^\omega\right\|_{r,B(n)}$ etc. can be treated similarly.

□

2.4 Annealed Local Limit Theorem for the Dynamic RCM

Similarly as in the static case our starting point is establishing an L^1 maximal inequality for space-time harmonic functions. Once again, we redefine our space-time cylinders. For $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d, n \in \mathbb{N}$, and $\sigma \in (0,1]$, let

$$Q_\sigma(n) := [t_0, t_0 + \sigma n^2] \times B(x_0, \sigma n).$$

Throughout this section we fix p, $q \in (1,\infty]$ satisfying

$$\frac{1}{p-1}\frac{q+1}{q} + \frac{1}{q} < \frac{2}{d}. \tag{2.39}$$

Proposition 2.4.1. *Let $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{Z}^d$ and $\Delta \in (0,1)$. There exist $N_9 = N_9(\Delta) \in \mathbb{N}$ and $c_{15} = c_{15}(d,p,q)$ such that for all $n \geq N_9$ and $\frac{1}{2} \leq \sigma' < \sigma \leq 1$ with $\sigma - \sigma' > n^{-\Delta}$,*

$$\max_{(t,x)\in Q_{\sigma'}(n)} p^\omega(0,t,0,x) \;\leq\; c_{15}\left(\frac{\mathcal{A}_5^\omega(n)}{(\sigma-\sigma')^2}\right)^{\kappa'} \left\|p^\omega(0,\cdot,0,\cdot)\right\|_{1,1,Q_\sigma(n)}^{\beta_n},$$

where $\kappa' := \alpha^2 p_*/(\alpha - 1)$ with $\alpha := \frac{1}{p_*} + \frac{1}{p_*}(1-\frac{1}{\rho})\frac{q}{q+1}$, ρ as in Proposition 2.2.1, and

$$\mathcal{A}_5^\omega(n) \;:=\; \left\|1\vee\mu^\omega\right\|_{p,p,Q(n)}\left\|1\vee\nu^\omega\right\|_{q,q,Q(n)}, \quad \beta_n \;:=\; \vartheta\sum_{k=0}^{K_n-1}(1-\vartheta)^k,$$

with $\vartheta := 1/(2\alpha p_*) \in (0,1)$ and $K_n := \left\lfloor\frac{\Delta\ln n - \ln(\sigma-\sigma')}{\ln 2}\right\rfloor$.

Proof. Write $u(\cdot,\cdot) = p^\omega(0,\cdot,0,\cdot)$ and $\sigma_k := \sigma - (\sigma-\sigma')2^{-k}$ for $k \in \mathbb{N}$. Then,

$$\|u\|_{2\alpha p_*,2\alpha p_*,Q_{\sigma_k}} \leq \|u\|_{1,1,Q_{\sigma_k}}^{\vartheta} \|u\|_{\infty,\infty,Q_{\sigma_k}}^{1-\vartheta},$$

by Hölder's inequality. Note that by the definition of K_n we have $\sigma_k - \sigma_{k-1} > n^{-\Delta}$ for all $k \in \{1,\ldots,K_n\}$. By [ACDS18, Theorem 5.5] (notice that $f=0$ in the present setting which leads to $\gamma = 1$ therein), there exist $c = c(d) \in (1,\infty)$, $N_9(\Delta) \in \mathbb{N}$ such that for $n \geq N_9(\Delta)$ and $k \in \{1,...,K_n\}$,

$$\|u\|_{\infty,\infty,Q_{\sigma_{k-1}}} \leq c\left(\frac{\mathcal{A}_5^\omega(n)}{(\sigma_k-\sigma_{k-1})^2}\right)^{\kappa} \|u\|_{2\alpha p_*,2\alpha p_*,Q_{\sigma_k}} \leq 2^{2\kappa k} J\, \|u\|_{1,1,Q_{\sigma_k}}^{\vartheta} \|u\|_{\infty,\infty,Q_{\sigma_k}}^{1-\vartheta},$$

with $\kappa := \frac{\alpha}{2(\alpha-1)}$ and $J := c\left(\frac{\mathcal{A}_5^\omega(n)}{(\sigma-\sigma')^2}\right)^{\kappa} \geq 1$. Then by iteration,

$$\begin{aligned} \|u\|_{\infty,\infty,Q_{\sigma'}} &\leq 2^{2\kappa\sum_{k=0}^{K_n-1}(k+1)(1-\vartheta)^k}\left(J\|u\|_{1,1,Q_\sigma}^{\vartheta}\right)^{\sum_{k=0}^{K_n-1}(1-\vartheta)^k} \|u\|_{\infty,\infty,Q_{\sigma_{K_n}}}^{(1-\vartheta)^{K_n}} \\ &\leq 2^{2\kappa/\vartheta^2} J^{1/\vartheta}\, \|u\|_{1,1,Q_\sigma}^{\beta_n}, \end{aligned}$$

where we used that $u \leq 1$. □

Assumption 2.4.2. *Suppose that* $\mathbb{E}\big[\omega_0(e)^{2(\kappa'\vee p)}\big] < \infty$ *and* $\mathbb{E}\big[\omega_0(e)^{-2(\kappa'\vee q)}\big] < \infty$ *for any* $e \in E_d$ *with* $p, q \in (1,\infty]$ *satisfying* (2.39) *and* κ' *as in Proposition 2.4.1.*

Proposition 2.4.3. *Suppose Assumption 2.1.9 and Assumption 2.4.2 hold. Then for all* $K > 0$ *and* $0 < T_1 \leq T_2$*, there exists* $c_{16} = c_{16}(d,p,q,K,T_1,T_2)$ *such that*

$$\mathbb{E}\left[\sup_{n\in\mathbb{N},|x|\leq K,\, t\in[T_1,T_2]} n^d p^\omega(0,n^2t,0,\lfloor nx\rfloor)\right] \leq c_{16}.$$

We postpone the proof of the above to the end of this section. First, we deduce the annealed local limit theorem from it.

Proof of Theorem 2.1.11. The statement follows from the corresponding quenched result, see Theorem 2.1.10-(ii) above, together with Proposition 2.4.3 by an application of the dominated convergence theorem. Note that the moment condition in Assumption 2.4.2 is stronger than the one required in Theorem 2.1.10. □

As in the static case Proposition 2.4.3 also directly implies an annealed on-diagonal heat kernel estimate (cf. Corollary 2.3.3 above).

Corollary 2.4.4. *Suppose Assumptions 2.1.9 and 2.4.2 hold. Then for any* $\lambda > 0$ *there* *exists* $c_{17} = c_{17}(d,p,q)$ *such that for all* $t \geq 1$*,*

$$\mathbb{E}\Bigg[\sup_{x \in B(0,\lambda\sqrt{t})} p^\omega(0,t,0,x)\Bigg] \;\leq\; c_{17}\, t^{-d/2}.$$

The proof of Proposition 2.4.3 begins with a consequence of Proposition 2.4.1.

Lemma 2.4.5. *For all* $K > 0$*,* $0 < T_1 \leq T_2$*, there exist* $N_{10} = N_{10}(T_2, K) \in \mathbb{N}$ *and constants* $c_{18} = c_{18}(d,p,q,K,T_1,T_2)$*,* $c_{19} = c_{19}(K,T_2)$ *such that for all* $n \geq N_{10}$*,*

$$\sup_{|x| \leq K,\, t \in [T_1,T_2]} n^d\, p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) \;\leq\; c_{18}\, \mathcal{A}_5^\omega (c_{19} n)^{\kappa'}$$

with $\mathcal{A}_5^\omega$ *and* κ' *as in Proposition 2.4.1.*

Proof. First note that by definition of the heat kernel p^ω,

$$\begin{aligned} \|p^\omega(0,\cdot,0,\cdot)\|_{1,1,Q(n)} \;&=\; \frac{1}{|I_1|} \int_{I_1} \frac{1}{|B(n)|} \sum_{y \in B(n)} p^\omega(0,t,0,y)\, dt \\ &=\; c\, n^{-d} \frac{1}{|I_1|} \int_{I_1} P_{0,0}^\omega\big[X_t \in B(n)\big]\, dt \;\leq\; c\, n^{-d}, \end{aligned} \tag{2.40}$$

for all $n \in \mathbb{N}$. Set $x_0 = 0$, $t_0 = T_1$ and let $N = c_{19}\, n$ with c_{19} chosen such that

$$\big\{(n^2 t, \lfloor nx \rfloor) : t \in [T_1,T_2], |x| \leq K\big\} \;\subseteq\; Q_{1/2}(N) = \big[t_0, t_0 + \tfrac{1}{2}N^2\big] \times B(x_0, N/2).$$

Then by applying Proposition 2.4.1 with the choice $\Delta = 1/2$, $\sigma = 1$ and $\sigma' = 1/2$ we get that for all $n \geq \lceil \frac{N_9}{c_{19}} \rceil \vee 4$,

$$\sup_{|x| \leq K,\, t \in [T_1,T_2]} n^d\, p^\omega(0, n^2 t, 0, \lfloor nx \rfloor) \;\leq\; c\, \mathcal{A}_5^\omega (c_{19} n)^{\kappa'}\, n^{d(1-\beta_n)}.$$

Since $n^{d(1-\beta_n)} \to 1$ as $n \to \infty$ the claim follows. □

For the proof of Proposition 2.4.3 we also require a maximal ergodic theorem for space-time ergodic environments.

Lemma 2.4.6. *Suppose Assumption 2.1.9 holds. Let* $x_0 \in \mathbb{Z}^d$*,* $t_0 \geq 0$ *and* $p \geq 1$*. Then there exists* $c_{20} = c_{20}(p) > 0$ *such that for all* $f \in L^p(\Omega)$*,*

$$\mathbb{E}\left[\left(\sup_{n \geq 1} \frac{1}{n^2} \int_{t_0}^{t_0+n^2} \frac{1}{|B(x_0,n)|} \sum_{x \in B(x_0,n)} f \circ \tau_{t,x}\, dt\right)^p\right] \;\leq\; c_{20}\, \mathbb{E}\left[f^p\right]. \tag{2.41}$$

Proof. See the discussion following [Kre85, Chapter 6, Theorem 4.4, p.224]. □

Proof of Proposition 2.4.3. By Lemma 2.4.5, it suffices to bound $\mathbb{E}\big[\sup_{n\geq 1}\mathcal{A}_5^\omega(c_{19}n)^{\kappa'}\big]$ under the moment condition of Assumption 2.4.2. This follows by using the maximal ergodic theorem of Lemma 2.4.6, similarly to the proof of Proposition 2.3.2. $\square$

2.5 Future Directions

In this chapter we have presented local central limit theorems for the static and dynamic random conductance model, under both the quenched and the annealed law. Some natural follow-up questions include the following:

(i) What are the optimal moment conditions for the quenched local limit theorem to hold in the setting of a degenerate, ergodic environment and general speed measure? The LLT in [ADS16a] is proven therein to be optimal for the CSRW, and our moment condition, Assumption 2.1.4, agrees with the moment condition in [ADS16a] for this choice of speed measure. On the other hand, for the VSRW, the inequality that the exponents p and q must satisfy has recently been improved in [BS20a]. So it's possible that, using a stronger technique, our moment assumption for general speed measure could also be relaxed.

(ii) In the same setting as above, what are the optimal moment conditions for the annealed local limit theorem to hold? We do not expect the moment assumption in Theorem 2.1.7 to be optimal, due to its dependence on the exponent from the L^1 maximal inequality (Proposition 2.3.1). During the course of this project, some avenues were explored to relax the requisite moment condition, such as attempting to establish a parabolic Harnack inequality similar to the one in [ADS16a], but for annealed conductances.

(iii) Regarding the dynamic RCM of Section 2.4, one open problem is to establish off-diagonal heat kernel estimates for a degenerate, ergodic environment. In the case of uniformly elliptic conductances, these have been established in [DD05] (see also [GOS01, Appendix B]). For the VSRW, such bounds are known to be governed by the intrinsic metric or *chemical distance* - see [ADS19] for this result in a static environment. The subtle challenge remains of how to define such an intrinsic metric when the environment is time-dependent. If this result were to be established, it would have an immediate application to the Ginzburg-Landau $\nabla\phi$ model of the following chapter. In particular, one could extend Theorem 3.1.2 to a time-dynamic situation - see the discussion in Remark 3.1.3(ii).

Chapter 3

Scaling Limit for the Ginzburg-Landau $\nabla\phi$ Model

Abstract. In this chapter we apply the annealed local limit theorem for the dynamic random conductance model of the previous chapter to prove a scaling limit result for the space-time covariances in the Ginzburg-Landau $\nabla\phi$ model. We also show that the associated Gibbs distribution scales to a Gaussian free field. These results apply to convex potentials for which the second derivative may be unbounded.

3.1 Introduction and Main Results

As introduced in Section 1.2, the Ginzburg-Landau $\nabla\phi$ model describes a hypersurface (interface) embedded in $d+1$-dimensional space, $\mathbb{R}^{d+1}$, which separates two pure thermodynamical phases. The interface is represented by a field of height variables $\phi = \{\phi(x) \in \mathbb{R} : x \in \Gamma\}$, which measure the vertical distances between the interface and $\Gamma \subseteq \mathbb{Z}^d$, a fixed d-dimensional reference hyperplane. The Hamiltonian H represents the energy associated with the field of height variables ϕ. In general, for $\Gamma = \mathbb{Z}^d$ or a finite subset $\Gamma \Subset \mathbb{Z}^d$,

$$H(\phi) \equiv H_\Gamma^\psi(\phi) = \frac{1}{2} \sum_{\{x,y\} \in \Gamma^*} V(\phi(x) - \phi(y)), \tag{3.1}$$

where Γ^* denotes the set of undirected, nearest-neighbour edges with at least one vertex in Γ. Note that boundary conditions $\psi = \{\psi(x) : x \in \partial^+\Gamma\}$, where $\partial^+\Gamma := \{x \in \mathbb{Z}^d \setminus \Gamma : |x-y| = 1 \text{ for some } y \in \Gamma\}$ denotes the outer boundary, are required to define the sum in the case $\Gamma \Subset \mathbb{Z}^d$, i.e. we set $\phi(x) = \psi(x)$ for $x \in \partial^+\Gamma$. The sum in (3.1) is merely formal when $\Gamma = \mathbb{Z}^d$. The dynamics of the $\nabla\phi$ model are governed by the following infinite system of SDEs for $\phi_t = \{\phi_t(x) : x \in \Gamma\} \in \mathbb{R}^\Gamma$,

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t)\, dt + \sqrt{2}\, dw_t(x), \quad x \in \Gamma,\ t > 0,$$

where $w_t = \{w_t(x) : x \in \mathbb{Z}^d\}$ is a collection of independent one-dimensional standard Brownian motions. Due to the form of the Hamiltonian, only nearest neighbour interactions are involved. Equivalent to the above in the case $\Gamma = \mathbb{Z}^d$ is

$$\phi_t(x) = \phi_0(x) - \int_0^t \sum_{y:|x-y|=1} V'(\phi_t(x) - \phi_t(y))\, dt + \sqrt{2}\, w_t(x), \qquad x \in \mathbb{Z}^d,\ t \geq 0. \tag{3.2}$$

We are interested in the decay of the space-time covariances of height variables under an equilibrium Gibbs measure. By the Helffer-Sjöstrand representation [HS94] (cf. also [DD05, GOS01]) such covariances can be written in terms of the *annealed* heat kernel of a random walk amongst dynamic random conductances. More precisely,

$$\mathbb{C}\mathrm{ov}_\mu\big(\phi_0(0), \phi_t(y)\big) = \int_0^\infty \mathbb{E}_\mu\left[p^\omega(0, t+s, 0, y)\right] ds.$$

Here $\mathbb{E}_\mu$ and $\mathbb{C}\mathrm{ov}_\mu$ denote expectation and covariance with respect to the law of the process $(\phi_t)_{t\geq 0}$ started from an ergodic Gibbs measure μ, and p^ω denotes the heat kernel of the dynamic RCM of Section 2.4 with time-dependent conductances given by

$$\omega_t(x, y) := V''\big(\phi_t(y) - \phi_t(x)\big), \qquad \{x, y\} \in E_d,\ t \geq 0. \tag{3.3}$$

Thus far, applications of the aforementioned Helffer-Sjöstrand relation have mostly been restricted to gradient models with a strictly convex potential function that has second derivative bounded above. This corresponds to uniformly elliptic conductances in the random walk picture. However, recent developments in the setting of degenerate conductances will also allow some potentials that are strictly convex but may have faster than quadratic growth at infinity. As our first main result in this direction, we use the annealed local limit theorem of Theorem 2.1.11 to derive a scaling limit for the space-time covariances of the ϕ-field for a wider class of potentials.

Theorem 3.1.1. *Suppose $d \geq 3$ and let $V \in C^2(\mathbb{R})$ be even with $V'' \geq c_- > 0$. Then for all $h \in \mathbb{R}$ there exists a stationary, shift-invariant, ergodic ϕ-Gibbs measure μ of mean h, i.e. $\mathbb{E}_\mu[\phi(x)] = h$ for all $x \in \mathbb{Z}^d$. Further, assume that*

$$\mathbb{E}_\mu\Big[V''\big(\phi(y) - \phi(x)\big)^{\bar{p}}\Big] < \infty, \qquad \textit{for any } \{x, y\} \in E_d, \tag{3.4}$$

with $\overline{p} := (2+d)(1+2/d+\sqrt{1+1/d^2})$*. Then for all* $t > 0$ *and* $x \in \mathbb{R}^d$*,*

$$\lim_{n\to\infty} n^{d-2}\,\mathbb{C}\mathrm{ov}_\mu\big(\phi_0(0),\phi_{n^2t}(\lfloor nx\rfloor)\big) = \int_0^\infty k_{t+s}(x)\,ds,$$

where k_t *is the heat kernel from Theorem 2.1.10 with conductances as given in* (3.3).

Theorem 3.1.1 extends the scaling limit result of [And14, Theorem 5.2] to hold for potentials V for which V'' may be unbounded above. Note that Theorem 3.1.1 also contains an existence result for stationary, shift-invariant, ergodic ϕ-Gibbs measures whose derivation in the present setting requires some extra consideration. We obtain the existence from the Brascamp-Lieb inequality together with an existence and uniqueness result for the system of SDEs (3.2), see Proposition 3.2.3, which we derive by adapting arguments from [Roy07, Chapter 4].

Our second main result is a scaling limit for the time-static height variables under the ϕ-Gibbs measure towards a Gaussian free field (GFF). We refer to [NS97, Theorem A], [GOS01, Corollary 2.2], [BS11, Theorem 2.4] and [NW18, Theorem 9] for similar results. For $f \in C_0^\infty(\mathbb{R}^d)$, we denote a rescaled version of this $f_n(x) := n^{-(1+d/2)} f(x/n)$ for $n \in \mathbb{Z}_+$. We will consider the field of heights acting as a linear functional on such a test function,

$$\phi(f_n) := n^{-(1+d/2)} \int_{\mathbb{R}^d} f(x)\,\phi(\lfloor nx\rfloor)\,dx. \tag{3.5}$$

Theorem 3.1.2 (Scaling to GFF)**.** *Suppose* $d \geq 3$ *and let* $V \in C^2(\mathbb{R})$ *be even with* $V'' \geq c_- > 0$*. Let* μ *be a stationary, ergodic* ϕ*-Gibbs measure of mean* 0*. Assume*

$$\mathbb{E}_\mu\Big[V''\big(\phi(y)-\phi(x)\big)^p\Big] \;<\; \infty, \qquad \text{for any } \{x,y\} \in E_d,$$

for some $p > 1 + \frac{d}{2}$*. Then for any* $\lambda \in \mathbb{R}$ *and* $f \in C_0^\infty(\mathbb{R}^d)$*,*

$$\lim_{n\to\infty} \mathbb{E}_\mu\Big[\exp(\lambda\phi(f_n))\Big] = \exp\Big(\frac{\lambda^2}{2}\int_{\mathbb{R}^d} f(x)(-Q^{-1}f)(x)\,dx\Big),$$

where Q^{-1} *is the inverse of* $Qf := \sum_{i,j=1}^d q_{ij}\frac{\partial^2 f}{\partial x_i\,\partial x_j}$ *and* $q = \Sigma^2$ *the covariance matrix from Theorem 2.1.10 with conductances given by* (3.3).

Remark 3.1.3. (i) Note that in (3.5) the height variables are scaled by $n^{-(1+d/2)}$ while the conventional scaling for a central limit theorem is $n^{-d/2}$. This stronger scaling is required due to strong correlations of the height variables (cf. [NS97,BS11]), in contrast to the scaling limit of the gradient field, which has weaker correlations and only requires the standard scaling $n^{-d/2}$ (cf. [GOS01,NW18]).

(ii) Having established Theorem 3.1.1, a natural next goal is to study the equilibrium space-time fluctuation of the interface and to derive a stronger, time-dynamic version of Theorem 3.1.2. See [GOS01, Theorem 2.1] for the case where the potential additionally has second derivative uniformly bounded above. However, this requires extending Theorem 3.1.1 from a pointwise result to a scaling limit for the covariances of the ϕ-field integrated against test functions, cf. [GOS01, Proposition 5.1]. In order to control the arising remainder term, we believe such an extension would require upper off-diagonal heat kernel estimates for the dynamic RCM in a degenerate, ergodic environment, which are not available at present, as discussed at the end of Section 2.1.3.

(iii) We state the above for a Gibbs measure of mean 0 for ease of presentation but the result holds for any stationary, ergodic ϕ-Gibbs measure.

Finally, we provide a verification of the moment assumptions in Theorems 3.1.1 and 3.1.2 for a class of potentials V with V'' having polynomial growth.

Proposition 3.1.4. *Suppose $d \geq 3$ and let the potential $V \in C^2(\mathbb{R})$ be even, satisfying $V'' \geq c_- > 0$. Let μ be any ergodic, shift-invariant, stationary ϕ-Gibbs measure. Then for all $p > 0$, $\mathbb{E}_\mu\big[|\phi_t(x)|^p\big] < \infty$ for any $x \in \mathbb{Z}^d$ and $t \geq 0$.*

Example 3.1.5. The above proposition shows that Theorems 3.1.1 and 3.1.2 apply to polynomial potentials of interest, for example the anharmonic crystal potential $V(x) = x^2 + \lambda x^4$ ($\lambda > 0$), for which the spatial correlation decay is discussed in [BFLS81].

Notation

We write c to denote a positive, finite constant which may change on each appearance. Constants denoted by c_i will remain the same. As already introduced, we will write $\Gamma \Subset \mathbb{Z}^d$ for Γ a finite subset of $\mathbb{Z}^d$. $\partial^+\Gamma := \{x \in \Gamma^c : |x-y| = 1 \text{ for some } y \in \Gamma\}$ denotes the outer boundary of a set $\Gamma \subseteq \mathbb{Z}^d$. We write $\overline{\Gamma} = \Gamma \cup \partial^+\Gamma$ for the closure of Γ. E_d denotes the set of undirected, nearest-neighbour edges in $\mathbb{Z}^d$ and Γ^* denotes the set of all undirected edges in Γ, i.e. $\Gamma^* = \{\{x,y\} \in E_d : x, y \in \overline{\Gamma}\}$. We write $\mathcal{P}(S)$ for the family of Borel probability measures on some topological space S. We also write $E = \mathbb{R}^{\mathbb{Z}^d}$ and $E_\Gamma = \mathbb{R}^\Gamma$ for $\Gamma \Subset \mathbb{Z}^d$. If $\Gamma \subset \mathbb{Z}^d$ and $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ then ϕ_Γ denotes the canonical projection onto Γ, i.e. $\phi_\Gamma = \phi \mathbb{1}_\Gamma$. For $\phi \in \mathbb{R}^{\mathbb{Z}^d}$ and $x \in \mathbb{Z}^d$, we use the shorthand

$$\partial_x H(\phi) = \frac{\partial H}{\partial \phi(x)}(\phi).$$

For $f\colon \mathbb{Z}^d \to \mathbb{R}$ we define the operator ∇ by

$$\nabla f\colon E_d \to \mathbb{R}, \qquad E_d \ni e \;\longmapsto\; \nabla f(e) \;:=\; f(e^+) - f(e^-),$$

where for each non-oriented edge $e \in E_d$ we specify one of its two endpoints as its initial vertex e^+ and the other one as its terminal vertex e^-. Given another function $g : \mathbb{Z}^d \to \mathbb{R}$, we denote the discrete convolution

$$f * g(x) := \sum_{y \in \mathbb{Z}^d} f(y) g(x-y), \qquad x \in \mathbb{Z}^d.$$

For $\Gamma \subseteq \mathbb{Z}^d$ and some weight $\alpha : \mathbb{Z}^d \to (0, \infty)$, we will work with inner products as follows. For $f,\, g \in \ell^2(\Gamma)$ and $f,\, g \in \ell^2(\Gamma, \alpha)$ respectively,

$$\langle f, g \rangle_{\ell^2(\Gamma)} := \sum_{x \in \Gamma} f(x) g(x), \qquad \langle f, g \rangle_{\ell^2(\Gamma,\alpha)} := \sum_{x \in \Gamma} f(x) g(x) \alpha(x).$$

We denote the associated norms by $\| \cdot \|_{\ell^2(\Gamma)}$ and $\| \cdot \|_{\ell^2(\Gamma,\alpha)}$ respectively. Finally, given $h : E_d \to \mathbb{R}$, two further norms we employ are

$$\|f\|_{l^1(\Gamma)} := \sum_{x \in \Gamma} |f(x)|, \qquad \|h\|_{l^2(E_d)} := \Big(\sum_{e \in E_d} h(e)^2 \Big)^{1/2}.$$

3.2 Setup and Existence of ϕ-Gibbs Measures

If $\Gamma \Subset \mathbb{Z}^d$, the finite volume process is governed for $x \in \Gamma$ and $t \geq 0$ by

$$\phi_t^{\Gamma,\psi}(x) = \phi_0^{\Gamma,\psi}(x) - \int_0^t \sum_{y \in \overline{\Gamma} : |x-y|=1} V'\big(\phi_s^{\Gamma,\psi}(x) - \phi_s^{\Gamma,\psi}(y)\big)\, ds + \sqrt{2}\, w_t(x). \tag{3.6}$$

We complete this definition by setting the boundary condition $\phi_t^{\Gamma,\psi}(y) = \psi(y)$, for all $y \in \mathbb{Z}^d \setminus \Gamma$ and $t \geq 0$. Note that for a convex potential V, the above system of SDEs has a unique strong solution, continuous in t, by [Roy07, Theorem 2.2.19] for instance (the hypothesis (2.2.6) therein is satisfied with $a = b = 0$). Furthermore, when we need to work with the initial configuration as a variable we will write, for $x \in \Gamma$ and $t \geq 0$,

$$\phi_t^{\Gamma,\psi,\chi}(x) = \chi(x) - \int_0^t \sum_{y \in \overline{\Gamma} : |x-y|=1} V'\big(\phi_s^{\Gamma,\psi,\chi}(x) - \phi_s^{\Gamma,\psi,\chi}(y)\big)\, ds + \sqrt{2}\, w_t(x), \tag{3.7}$$

where again $\phi_t^{\Gamma,\psi,\chi}(y) = \psi(y)$, for all $y \in \mathbb{Z}^d \setminus \Gamma$ and $t > 0$. Generally, we take $\psi,\, \chi \in \mathbb{R}^{\mathbb{Z}^d}$ but of course only the values of χ in Γ and of ψ in $\mathbb{Z}^d \setminus \Gamma$ enter the definition.

The evolution of ϕ_t is designed such that it is stationary and reversible under the equilibrium ϕ-Gibbs measure μ_Γ^ψ or μ (see (3.9) below). We denote by $\mathbb{P}_\mu$ the law of the process ϕ_t started under the distribution μ (and $\mathbb{E}_\mu$ the corresponding expectation). By a slight abuse of notation we will also write $\mathbb{E}_\mu$, $\mathbb{V}\mathrm{ar}_\mu$ and $\mathbb{C}\mathrm{ov}_\mu$ for the expectation, variance and covariance under μ. When the initial configuration ϕ_0 is fixed, we write E for expectation under the law of the collection of Brownian motions w.

Most of the mathematical literature on the $\nabla\phi$ model treats the case of a suitably smooth, even and strictly convex interaction potential V such that V'' is bounded above. However, we will relax these conditions; throughout the rest of this chapter we work with V as in the following assumption.

Assumption 3.2.1. *The potential $V \in C^2(\mathbb{R})$ is even and there exists $c_- > 0$ such that*

$$c_- \leq V''(x), \quad \textit{for all } x \in \mathbb{R}. \tag{3.8}$$

Note that under Assumption 3.2.1, the coefficients of the SDE (3.2) are not necessarily globally Lipschitz continuous. However, it is still possible to construct an almost surely continuous solution ϕ_t, see Proposition 3.2.3. The assumption that the potential has second derivative bounded away from zero is helpful for the existence of an equilibrium ϕ-Gibbs measure. For $\Gamma \Subset \mathbb{Z}^d$, the finite volume ϕ-Gibbs measure for the field of heights $\phi \in \mathbb{R}^\Gamma$ is defined as

$$\mu(d\phi) \equiv \mu_\Gamma^\psi(d\phi) = \frac{1}{Z_\Gamma^\psi} \exp\Big(-H_\Gamma^\psi(\phi)\Big)\, d\phi_\Gamma, \tag{3.9}$$

with boundary condition $\psi \in \mathbb{R}^{\partial^+\Gamma}$, where $d\phi_\Gamma$ is the Lebesgue measure on $\mathbb{R}^\Gamma$ and Z_Γ^ψ is a normalisation constant. Then (3.8) implies $Z_\Gamma^\psi < \infty$ for every $\Gamma \Subset \mathbb{Z}^d$ and hence $\mu_\Gamma^\psi \in \mathcal{P}(\mathbb{R}^\Gamma)$ is a probability measure. In the infinite volume case $\Gamma = \mathbb{Z}^d$, (3.9) has no rigorous meaning but one can still define Gibbs measures as follows.

Definition 3.2.2. A probability measure $\mu \in \mathcal{P}\big(\mathbb{R}^{\mathbb{Z}^d}\big)$ is a ϕ-Gibbs measure if its conditional probability on $\mathcal{F}_{\Gamma^c} = \sigma\{\phi(x) : x \in \mathbb{Z}^d \setminus \Gamma\}$ satisfies the DLR (Dobrushin-Lanford-Ruelle) equation

$$\mu(\cdot|\mathcal{F}_{\Gamma^c})(\psi) = \mu_\Gamma^\psi(\cdot), \quad \text{for } \mu\text{-a.e. } \psi, \tag{3.10}$$

for all $\Gamma \Subset \mathbb{Z}^d$.

In order to prove the existence of a solution to the infinite system of SDEs we will adapt the arguments in [Roy07, Chapter 4] to our setting, cf. also [Zit08, Section 1.1]. We will also show reversibility of the Gibbs measure. To study the properties of solutions to the system of SDEs (3.2), it is necessary to restrict to a suitable class of initial configurations. Let $\mathcal{S} := \{(\phi(x))_{x\in\mathbb{Z}^d} : \big|\phi(x)\big| \leq a + |x|^n, \text{ for some } a \in \mathbb{R},\, n \in \mathbb{N}\}$ denote the configurations of heights with at most polynomial growth. The process will be constructed in the weighted space $\ell^2(\mathbb{Z}^d, \alpha)$ where $\alpha = (\alpha(x))_{x\in\mathbb{Z}^d}$ is a positive, symmetric sequence. To define α, first let $p(x) := c_- \mathbb{1}_{|x|=1}$ for $x \in \mathbb{Z}^d$, let $\sigma := \sum_{x\in\mathbb{Z}^d} p(x) = 2dc_-$

and fix $\sigma' > \sigma$. Then,

$$\alpha := \sum_{k=0}^{\infty} (\sigma')^{-k} (*p)^k. \tag{3.11}$$

Note that α and $\alpha * \alpha$ are summable sequences that decay exponentially on $\mathbb{Z}^d$, also α is superharmonic with respect to the convolution kernel $(\sigma')^{-1}p$, i.e. $\alpha * p \leq \sigma' \alpha$.

Proposition 3.2.3. *Given any initial configuration $\phi_0 \in \mathcal{S}$, there exists a unique solution to the system of SDEs* (3.2) *such that for any $x \in \mathbb{Z}^d$ the process $\phi_t(x)$ is almost surely continuous and for all $t > 0$ the configuration $\phi_t \in \mathcal{S}$ almost surely. Any Gibbs measure concentrated on $\mathcal{S}$ is stationary and reversible with respect to the process ϕ_t.*

In order to prove the above proposition we will establish comparison inequalities for finite volume processes on different boxes with varying boundary and initial conditions. We first require the following auxiliary lemma.

Lemma 3.2.4. *Let $t > 0$ and $\Gamma \Subset \mathbb{Z}^d$, there exist $c_i(d, c_-) > 0$ such that for any $\psi, \chi \in \mathcal{S}$ and $z \in \mathbb{Z}^d$, we have*

$$\mathrm{E}\bigg[\sum_{x \in \mathbb{Z}^d} \alpha(x) \sup_{0 \leq s \leq t} \big(\phi_s^{\Gamma,\psi,\chi}(x)\big)^2\bigg] \leq \mathrm{e}^{c_1 t}\bigg(\sum_{x \in \Gamma} \alpha(x)(\chi(x)^2 + c_2) + \sum_{x \in \Gamma^c} \alpha(x)\psi(x)^2\bigg), \tag{3.12}$$

$$\alpha(0)\mathrm{E}\bigg[\sup_{0 \leq s \leq t} \big(\phi_s^{\Gamma,\psi,\chi}(z)\big)^2\bigg] \leq \mathrm{e}^{c_3 t}\big((\alpha * \chi^2)(z) + c_4|\alpha| + (\alpha * \psi_{\Gamma^c}^2)(z)\big). \tag{3.13}$$

Proof. Let $Y_t := \phi_t^{\Gamma,\psi,\chi}$. By Itô's formula, for $x \in \Gamma$,

$$Y_t(x)^2 = \chi(x)^2 + 2\,M_t(x) - 2\int_0^t Y_s(x)\partial_x H_\Gamma^\psi(Y_s)\,ds + t, \tag{3.14}$$

where we have defined the martingale $M_t(x) := \int_0^t Y_s(x)\,dw_s(x)$. By Assumption 3.2.1, we have as in [Roy07, Lemma 4.2.9] that for all $x \in \Gamma$ and $s > 0$,

$$-Y_s(x)\partial_x H_\Gamma^\psi(Y_s) \leq \frac{\sigma}{2} Y_s(x)^2 + \frac{1}{2}\sum_{y \in \mathbb{Z}^d} p(y-x) Y_s(y)^2. \tag{3.15}$$

For a real-valued process $(Z_t)_{t \geq 0}$ we write $\widehat{Z}_t := \sup_{0 \leq s \leq t} Z_s$. Combining the above equations gives

$$\widehat{Y}_t(x)^2 \leq \chi(x)^2 + 2\,\widehat{M}_t(x) + t + \sigma\int_0^t \widehat{Y}_s(x)^2\,ds + \int_0^t \sum_{y \in \mathbb{Z}^d} p(y-x)\widehat{Y}_s(y)^2\,ds. \tag{3.16}$$

Now, Doob's submartingale inequality implies

$$\mathrm{E}\Big[\widehat{M}_t(x)^2\Big] \leq 4\,\mathrm{E}\Big[M_t(x)^2\Big] = 4\,\mathrm{E}\bigg[\int_0^t Y_s(x)^2\,ds\bigg].$$

Therefore, together with Jensen's inequality and a simple bound,

$$\mathrm{E}\Big[\widehat{M}_t(x)\Big] \leq 4\,\mathrm{E}\Big[\int_0^t \widehat{Y}_s(x)^2\,ds\Big]^{1/2} \leq 4 + \frac{1}{2}\mathrm{E}\Big[\int_0^t \widehat{Y}_s(x)^2\,ds\Big].$$

Denote $u_t(x) := \mathrm{E}\big[\sup_{0\leq s\leq t} Y_s(x)^2\big]$. By substituting the above into (3.16) and taking expectation,

$$u_t(x) \leq \chi(x)^2 + 8 + t + (1+\sigma)\int_0^t u_s(x)\,ds + \int_0^t \sum_{y\in\mathbb{Z}^d} p(y-x)u_s(y)\,ds. \tag{3.17}$$

On the other hand if $x \in \mathbb{Z}^d \setminus \Gamma$, we simply have $u_t(x) = \psi(x)^2$. Therefore, if we let $f(t) := \sum_{x\in\mathbb{Z}^d} \alpha(x)u_t(x)$, we have

$$f(t) \leq \sum_{x\in\mathbb{Z}^d} \alpha(x)\big((\chi(x)^2 + 8 + t)\mathbb{1}_{x\in\Gamma} + \psi(x)^2\mathbb{1}_{x\in\Gamma^c}\big) + \int_0^t (1+\sigma+\sigma')f(s)\,ds, \tag{3.18}$$

where we used that $\alpha * p \leq \sigma'\alpha$. Then by Gronwall's lemma,

$$f(t) \leq \exp\big((1+\sigma+\sigma')t\big)\sum_{x\in\mathbb{Z}^d} \alpha(x)\big((\chi(x)^2 + 8 + t)\mathbb{1}_{x\in\Gamma} + \psi(x)^2\mathbb{1}_{x\in\Gamma^c}\big), \tag{3.19}$$

which proves the first inequality.

We now prove the second statement. Since α is symmetric, (3.12) is equivalent to

$$\mathrm{E}\Big[\alpha * \sup_{0\leq s\leq t}\big(\phi_s^{\Gamma,\psi,\chi}\big)^2(0)\Big] \leq \mathrm{e}^{ct}\big(\alpha * \chi^2(0) + c\,|\alpha| + \alpha * \psi_{\Gamma^c}^2(0)\big).$$

The above derivation of (3.12) still holds if α is replaced by the translation $\tilde{\alpha} := \alpha(z+\cdot)$ throughout, since $\tilde{\alpha} * p \leq \sigma'\tilde{\alpha}$. Therefore,

$$\mathrm{E}\Big[\tilde{\alpha} * \sup_{0\leq s\leq t}\big(\phi_s^{\Gamma,\psi,\chi}\big)^2(0)\Big] \leq \mathrm{e}^{ct}\big(\tilde{\alpha} * \chi^2(0) + c\,|\tilde{\alpha}| + \tilde{\alpha} * \psi_{\Gamma^c}^2(0)\big).$$

Note that $|\tilde{\alpha}| = |\alpha|$ and for any $k \in \ell^2(\mathbb{Z}^d)$, $\tilde{\alpha} * k(0) = \alpha * k(z)$. Hence,

$$\begin{aligned}\alpha(0)\mathrm{E}\Big[\sup_{0\leq s\leq t}\big(\phi_s^{\Gamma,\psi,\chi}(z)\big)^2\Big] = \mathrm{E}\Big[\tilde{\alpha}(-z)\sup_{0\leq s\leq t}\big(\phi_s^{\Gamma,\psi,\chi}(z)\big)^2\Big] &\leq \mathrm{E}\Big[\tilde{\alpha} * \sup_{0\leq s\leq t}\big(\phi_s^{\Gamma,\psi,\chi}\big)^2(0)\Big] \\ &\leq \mathrm{e}^{ct}\big(\alpha * \chi^2(z) + c\,|\alpha| + \alpha * \psi_{\Gamma^c}^2(z)\big).\end{aligned}$$

□

Proposition 3.2.5. *There exist constants $c_i(d, c_-) > 0$ such that for all $\chi \in \mathcal{S}$ and $\Gamma \Subset \mathbb{Z}^d$:*

(i) For any $t > 0$ and $L \subset \Gamma$,

$$\alpha(0)\mathrm{E}\left[\sum_{x\in\mathbb{Z}^d}\alpha(x)\sup_{0\le s\le t}\big(\phi_s^{\Gamma,0,\chi}(x)-\phi_s^{L,0,\chi}(x)\big)^2\right]\le \mathrm{e}^{c_5 t}\sum_{y\in\Gamma\setminus L}\alpha*\alpha(y)\chi(y)^2+c_6|\alpha|\,\alpha(y). \tag{3.20}$$

(ii) For any $t > 0$ and $\psi, \psi' \in \mathcal{S}$,

$$\sum_{x\in\mathbb{Z}^d}\alpha(x)\big(\phi_t^{\Gamma,\psi,\chi}(x)-\phi_t^{\Gamma,\psi',\chi}(x)\big)^2\le \mathrm{e}^{c_7 t}\sum_{y\in\Gamma^c}\alpha(y)\big(\psi(y)-\psi'(y)\big)^2. \tag{3.21}$$

(iii) For any $t > 0$ and $\psi, \chi' \in \mathcal{S}$,

$$\sum_{x\in\mathbb{Z}^d}\alpha(x)\big(\phi_t^{\Gamma,\psi,\chi}(x)-\phi_t^{\Gamma,\psi,\chi'}(x)\big)^2\le \mathrm{e}^{c_8 t}\sum_{y\in\Gamma}\alpha(y)\big(\chi(y)-\chi'(y)\big)^2. \tag{3.22}$$

Proof. As in [Roy07, Lemma 4.2.10], for any $\phi_1, \phi_2 \in \mathcal{S}$ and $x \in \mathbb{Z}^d$, we have the following inequality,

$$\begin{aligned}\big(\phi_1(x)-\phi_2(x)\big)&\Big(\partial_x H(\phi_1)-\partial_x H(\phi_2)\Big)\\ &\ge -\frac{1}{2}\sum_{y\in\mathbb{Z}^d}p(y-x)\big((\phi_1(x)-\phi_2(x))^2+(\phi_1(y)-\phi_2(y))^2\big).\end{aligned} \tag{3.23}$$

Therefore by the SDE (3.7), for any $x \in L$ we have

$$\begin{aligned}\big(\phi_t^{\Gamma,0,\chi}(x)-\phi_t^{L,0,\chi}(x)\big)^2 &\le \sigma\int_0^t\big(\phi_s^{\Gamma,0,\chi}(x)-\phi_s^{L,0,\chi}(x)\big)^2\,ds\\ &\quad+\int_0^t\sum_{y\in\mathbb{Z}^d}p(y-x)\big(\phi_s^{\Gamma,0,\chi}(y)-\phi_s^{L,0,\chi}(y)\big)^2\,ds.\end{aligned} \tag{3.24}$$

On the other hand if $x \in \Gamma \setminus L$, we have a bound on $\mathrm{E}\big[\sup_{s\le t}\phi_s^{\Gamma,0,\chi}(x)^2\big]$ by Lemma 3.2.4. So defining

$$D_t(x) := \mathrm{E}\Big[\sup_{s\le t}(\phi_s^{\Gamma,0,\chi}(x)-\phi_s^{L,0,\chi}(x))^2\Big],$$

we have in all cases that

$$D_t(x)\le \mathrm{e}^{ct}\alpha^{-1}(0)\big((\alpha*\chi^2)(x)+c|\alpha|\,\big)\mathbb{1}_{x\in\Gamma\setminus L}+\int_0^t\sigma D_s(x)+\sum_{y\in\mathbb{Z}^d}p(y-x)D_s(y)\,ds.$$

Now let $g(t) := \sum_{x \in \mathbb{Z}^d} \alpha(x) D_t(x)$. Then using $\alpha * p \leq \sigma' \alpha$ gives

$$g(t) \leq \mathrm{e}^{ct} \alpha^{-1}(0) \sum_{x \in \Gamma \setminus L} \big((\alpha * \alpha)(x) \chi(x)^2 + c|\alpha| \, \alpha(x) \big) + (\sigma + \sigma') \int_0^t g(s) \, ds. \tag{3.25}$$

Finally, an application of Gronwall's lemma gives the first statement.

To prove the second statement, arguing as for (3.24) we have that for all $x \in \Gamma$,

$$\begin{aligned} \big(\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi',\chi}(x)\big)^2 \leq \sigma \int_0^t \big(\phi_s^{\Gamma,\psi,\chi}(x) - \phi_s^{\Gamma,\psi',\chi}(x)\big)^2 \, ds \\ + \int_0^t \sum_{y \in \mathbb{Z}^d} p(y-x) \big(\phi_s^{\Gamma,\psi,\chi}(y) - \phi_s^{\Gamma,\psi',\chi}(y)\big)^2 \, ds. \end{aligned} \tag{3.26}$$

Alternatively, if $x \in \Gamma^c$, we simply have

$$\big(\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi',\chi}(x)\big)^2 = \big(\psi(x) - \psi'(x)\big)^2. \tag{3.27}$$

Then, letting

$$D_t(x) := \big(\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi',\chi}(x)\big)^2, \tag{3.28}$$

we have for all $x \in \mathbb{Z}^d$,

$$D_t(x) \leq \big(\psi(x) - \psi'(x)\big)^2 \mathbb{1}_{x \in \Gamma^c} + \int_0^t \sigma D_s(x) + \sum_{y \in \mathbb{Z}^d} p(y-x) D_s(y) \, ds.$$

Now, let $g(t) := \sum_{x \in \mathbb{Z}^d} \alpha(x) D_t(x)$. Then, since $\alpha * p \leq \sigma' \alpha$,

$$g(t) \leq \sum_{y \in \Gamma^c} \alpha(y) \big(\psi(y) - \psi'(y)\big)^2 + (\sigma + \sigma') \int_0^t g(s) \, ds. \tag{3.29}$$

This implies the second statement by Gronwall's lemma.

For the third statement, combining the SDE (3.7) with the inequality (3.23) gives, for all $x \in \Gamma$,

$$\begin{aligned} \big(\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi,\chi'}(x)\big)^2 \leq \big(\chi(x) - \chi'(x)\big)^2 + \sigma \int_0^t \big(\phi_s^{\Gamma,\psi,\chi}(x) - \phi_s^{\Gamma,\psi,\chi'}(x)\big)^2 \, ds \\ + \int_0^t \sum_{y \in \mathbb{Z}^d} p(y-x) \big(\phi_s^{\Gamma,\psi,\chi}(y) - \phi_s^{\Gamma,\psi,\chi'}(y)\big)^2 \, ds. \end{aligned} \tag{3.30}$$

On the other hand, if $x \in \Gamma^c$,

$$\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi,\chi'}(x) = \psi(x) - \psi(x) = 0.$$

Let

$$D_t(x) := \big(\phi_t^{\Gamma,\psi,\chi}(x) - \phi_t^{\Gamma,\psi,\chi'}(x)\big)^2,$$

then for all $x \in \mathbb{Z}^d$,

$$D_t(x) \le \mathbb{1}_{x\in\Gamma}\Big(\big(\chi(x) - \chi'(x)\big)^2 + \int_0^t \sigma D_s(x) + \sum_{y\in\mathbb{Z}^d} p(y-x) D_s(y)\, ds\Big). \tag{3.31}$$

Then proceeding, as before, to multiply (3.31) by $\alpha(x)$ and sum over $x \in \mathbb{Z}^d$, with $g(t) := \sum_{x\in\mathbb{Z}^d} \alpha(x) D_t(x)$,

$$g(t) \le \sum_{y\in\Gamma} \alpha(y)\big(\chi(y) - \chi'(y)\big)^2 + (\sigma + \sigma') \int_0^t g(s)\, ds. \tag{3.32}$$

Once more, we conclude by employing Gronwall's lemma.

□

Proof of Proposition 3.2.3. Let $T > 0$, we will construct a solution in the Banach space H_T of continuous processes on $[0, T]$ taking values in E, with norm given by

$$\|\phi\|_{H_T} := \Big(\sum_{x\in\mathbb{Z}^d} \alpha(x) \mathrm{E}\Big[\sup_{t\le T} \phi_t(x)^2\Big]\Big)^{1/2}.$$

Let $\chi = \phi_0$ and define boxes $L_n := [-n, n]^d \cap \mathbb{Z}^d$ for $n \in \mathbb{N}$. The bound in Proposition 3.2.5(i) gives that the sequence $(\phi^{L_n,0,\chi})_{n\ge1}$ is Cauchy in H_T, therefore there exists a limit $\phi \in H_T$ (which we will also denote ϕ^χ) and we may extract a subsequence such that for all $x \in \mathbb{Z}^d$,

$$\lim_{k\to\infty} \sup_{t\le T} \Big|\phi_t^{L_{n_k},0,\chi}(x) - \phi_t(x)\Big|^2 = 0,$$

almost surely. Note that for any $n \ge 1$, $\phi^{L_n,0,\chi}$ solves the SDE (3.2) for all $x \in L_{n-1}$. Therefore taking the limit, ϕ satisfies (3.2) for all $x \in \mathbb{Z}^d$. Continuity and $\phi_t \in \mathcal{S}$ for all $t > 0$ follow from the fact that $\phi \in H_T$.

Regarding uniqueness, we can show a stronger statement governing continuity with respect to the initial condition. Let ϕ^χ and $\phi^{\chi'}$ be solutions corresponding to the initial distributions $\chi, \chi' \in \mathcal{S}$ respectively. Then it follows from Proposition 3.2.5(iii) that

$$\|\phi_t^\chi - \phi_t^{\chi'}\|_{\ell^2(\mathbb{Z}^d,\alpha)} \le \mathrm{e}^{ct}\|\chi - \chi'\|_{\ell^2(\mathbb{Z}^d,\alpha)}, \tag{3.33}$$

for all $t > 0$. Uniqueness follows by setting $\chi' = \chi$.

Next we prove the reversibility of Gibbs measures with respect to $(\phi_t)_{t\ge0}$. Let $\mathcal{D} \subset \mathcal{S}$ be a countable subset, dense in $\ell^2(\mathbb{Z}^d, \alpha)$ and fix $t > 0$. By a diagonalisation procedure, we can construct a subsequence of boxes $(L_{n_k})_{k\ge1}$ such that for any $\chi \in \mathcal{D}$, $\phi_t^{L_{n_k},0,\chi} \to \phi_t^\chi$

in $\ell^2(\mathbb{Z}^d, \alpha)$ as $k \to \infty$ almost surely. Now by Proposition 3.2.5(iii), the mapping $\chi \mapsto \phi_t^{L_{n_k},0,\chi}$ is uniformly continuous on $\ell^2(\mathbb{Z}^d, \alpha)$, so we also have the above convergence for any $\chi \in \mathcal{S}$.

Let $f,\, g : E \to \mathbb{R}$ be bounded and continuous, depending only on a finite number of coordinates within some set $\Gamma \subset L \Subset \mathbb{Z}^d$ and fix $\psi \in \mathcal{S}$. Since $\phi^{L,\psi,\chi}$ is a Kolmogorov process on E_L, its restriction to L is reversible with respect to the finite volume Gibbs measure μ_L^ψ. So $\phi^{L,\psi,\chi}$ is reversible on E with respect to $\pi_L(d\xi, \psi) := \mu_L^\psi(d\xi) \otimes \delta_{\psi_{L^c}}$. Hence,

$$\int_E \mathrm{E}\big[f(\phi_0^{L,\psi,\chi})g(\phi_t^{L,\psi,\chi})\big]\pi_L(d\chi, \psi) = \int_E \mathrm{E}\big[f(\phi_t^{L,\psi,\chi})g(\phi_0^{L,\psi,\chi})\big]\pi_L(d\chi, \psi).$$

Now let μ be an arbitrary Gibbs measure on E. Integrating over $\psi \in \mathcal{S}$ with respect to μ and applying the D.L.R. equations (3.10) gives

$$\int_E \mathrm{E}\big[f(\phi_0^{L,\chi,\chi})g(\phi_t^{L,\chi,\chi})\big]\mu(d\chi) = \int_E \mathrm{E}\big[f(\phi_t^{L,\chi,\chi})g(\phi_0^{L,\chi,\chi})\big]\mu(d\chi). \tag{3.34}$$

Now, for all $x \in \mathbb{Z}^d$, $t > 0$ and $\chi \in \mathcal{S}$, $\phi_t^{L_{n_k},0,\chi}(x) \to \phi_t^\chi(x)$ almost surely. Together with Proposition 3.2.5(ii) this gives $\phi_t^{L_{n_k},\chi,\chi} \to \phi_t^\chi$ in E_Γ as $k \to \infty$. Setting $L = L_{n_k}$ in (3.34), since f and g are bounded we may use the dominated convergence theorem to pass to the limit as $k \to \infty$, giving

$$\int_E \mathrm{E}\big[f(\phi_0^\chi)g(\phi_t^\chi)\big]\mu(d\chi) = \int_E \mathrm{E}\big[f(\phi_t^\chi)g(\phi_0^\chi)\big]\mu(d\chi). \tag{3.35}$$

The monotone class theorem then allows us to extend to any measurable, bounded f, g on E. Therefore ϕ is reversible with respect to μ. $\square$

Brascamp-Lieb inequalities state that for $\Gamma \Subset \mathbb{Z}^d$, covariances under the aforementioned ϕ-Gibbs measure μ_Γ^ψ are bounded by those under $\mu_\Gamma^{\psi,G}$, the Gaussian finite volume ϕ-Gibbs measure determined by the quadratic potential $V^*(x) = \frac{c_-}{2}x^2$.

Proposition 3.2.6 (Brascamp-Lieb inequality for exponential moments). *Let* $\Gamma \Subset \mathbb{Z}^d$. *For every* $\nu \in \mathbb{R}^\Gamma$*,*

$$\mathbb{E}_{\mu_\Gamma^\psi}\left[\exp\Big(\big|\langle \nu, \phi - \mathbb{E}_{\mu_\Gamma^\psi}[\phi]\rangle_{\ell^2(\Gamma)}\big|\Big)\right] \le 2\exp\Big(\frac{1}{2}\,\mathbb{V}\mathrm{ar}_{\mu_\Gamma^{\psi,G}}\big(\langle \nu, \phi\rangle_{\ell^2(\Gamma)}\big)\Big). \tag{3.36}$$

Proof. This is [Fun05, Theorem 4.9]. Note that the condition $V''(x) \le c_+,\ \forall\, x \in \mathbb{R}$, for some $c_+ > 0$, is not needed for the proof. $\square$

This inequality is pivotal in proving the following existence result, which constitutes the first part of Theorem 3.1.1. We shall also employ the massive Hamiltonian

$$H_m(\phi) \equiv H^{\psi}_{m,\Gamma}(\phi) \;:=\; H^{\psi}_{\Gamma}(\phi) + \frac{m^2}{2}\sum_{x\in\Gamma}\phi(x)^2, \qquad m>0. \tag{3.37}$$

Remark 3.2.7. Note that Proposition 3.2.6 also holds for the massive Hamiltonian $H^{\psi}_{m,\Gamma}$ and in that case the Gaussian potential can be taken to be $V^*(x) = \frac{c_-+2dm^2}{2}x^2$.

Theorem 3.2.8 (Existence of ϕ-Gibbs measures)**.** *If $d \geq 3$ then for all $h \in \mathbb{R}$ there exists a stationary, shift-invariant, ergodic ϕ-Gibbs measure $\mu \in \mathcal{P}\big(\mathbb{R}^{\mathbb{Z}^d}\big)$ of mean h, i.e. $\mathbb{E}_{\mu}[\phi(x)] = h$ for all $x \in \mathbb{Z}^d$.*

Proof. Let $m > 0$ and first take a sequential limit as $n \to \infty$ of finite volume ϕ-Gibbs measures μ^0_{m,Γ_n} with periodic boundary conditions, corresponding to the massive Hamiltonian H^0_{m,Γ_n} on the torus $\Gamma_n := \big(\mathbb{Z}/n\mathbb{Z}\big)^d$. Precisely, this is defined by setting

$$\mu^0_{m,\Gamma_n}(d\phi) = \frac{1}{Z_{\Lambda_n}}\exp\Big(-H^{\psi}_{m,\Lambda_n}(\phi)\Big)\,d\phi_{\Lambda_n},$$

for $\Lambda_n = \big([0,n)\cap\mathbb{Z}\big)^d$ with boundary condition $\psi(x) = \phi(x \bmod n)$ for all $x \in \partial^+\Lambda_n$, where $d\phi_{\Lambda_n}$ is the Lebesgue measure on $\mathbb{R}^{\Lambda_n}$ and Z_{Λ_n} is a normalisation constant.

Since V is even, $\mathbb{E}_{\mu^0_{m,\Gamma_n}}\big[\phi(x)\big] = 0$. When $d \geq 3$, the variance of the Gaussian system corresponding to the potential $V^*(x) = \frac{c_-}{2}x^2$ is uniformly bounded in n, so by the Brascamp-Lieb inequality,

$$\sup_{x\in\mathbb{Z}^d,\, n\in\mathbb{N}} \mathbb{E}_{\mu^0_{m,\Gamma_n}}\Big[\exp(\lambda\big|\phi(x)\big|)\Big] < \infty, \quad \forall\,\lambda > 0.$$

In order to show tightness of $(\mu^0_{m,\Gamma_n})_{n\in\mathbb{N}}$ we introduce another weighted $\ell^2(\mathbb{Z}^d)$ norm

$$\big\|\phi\big\|_r^2 := \sum_{x\in\mathbb{Z}^d}\phi(x)^2\mathrm{e}^{-2r|x|}, \qquad \text{for } r>0.$$

Then sets of the form $K_M = \{\phi \in \mathbb{R}^{\mathbb{Z}^d} : \big\|\phi\big\|_r \leq M\}$ are compact, cf. e.g. [Fun05, proof of Proposition 3.3]. For any $n \in \mathbb{N}$,

$$\mathbb{P}_{\mu^0_{m,\Gamma_n}}\big(K^c_M\big) \leq \mathbb{E}_{\mu^0_{m,\Gamma_n}}\Big[\big\|\phi\big\|_r^2/M^2\Big] = \frac{1}{M^2}\sum_{x\in\mathbb{Z}^d}\mathbb{E}_{\mu^0_{m,\Gamma_n}}\Big[\phi(0)^2\Big]\mathrm{e}^{-2r|x|},$$

where we have used shift-invariance of μ^0_{m,Γ_n}. Then the above Brascamp-Lieb inequality implies

$$\sup_{n\in\mathbb{N}}\mathbb{P}_{\mu^0_{m,\Gamma_n}}\big(K^c_M\big) \leq c\sum_{x\in\mathbb{Z}^d}\frac{\mathrm{e}^{-2r|x|}}{M^2} \leq \frac{c}{M^2}.$$

This tends to zero as $M \to \infty$. Therefore $(\mu^0_{m,\Gamma_n})_{n\in\mathbb{N}}$ is tight and along some proper subsequence there exists a limit $\mu^0_m := \lim_{k\to\infty} \mu^0_{m,\Gamma_{n_k}}$, a shift-invariant Gibbs measure on $\mathbb{Z}^d$ of mean 0. Now for all $m > 0$ and $x \in \mathbb{Z}^d$, $\frac{\partial^2 H_m(\phi)}{\partial \phi(x)^2} \geq c_-$ so by the Brascamp-Lieb inequality again (taking an infinite volume limit)

$$\sup_{x\in\mathbb{Z}^d,\, 0<m\leq 1} \mathbb{E}_{\mu^0_m}\Big[\exp(\lambda\big|\phi(x)\big|)\Big] < \infty, \quad \forall\, \lambda > 0.$$

With this bound we can then show that the limit $\mu^0 = \lim_{m\downarrow 0} \mu^0_m$ exists, analogously to the above argument. The distribution of $\phi + h$, where ϕ is μ^0 distributed, is a shift-invariant ϕ-Gibbs measure on $\mathbb{Z}^d$ under which $\phi(x)$ has mean h for all $x \in \mathbb{Z}^d$. Having shown that the convex set of shift-invariant ϕ-Gibbs measures of mean h is non-empty, there exists an extremal element of this set which is ergodic, see [Geo11, Theorem 14.15]. Finally, by Proposition 3.2.3 this Gibbs measure is reversible and hence stationary for the process ϕ_t. $\square$

Remark 3.2.9. The ϕ-Gibbs measures exist when $d \geq 3$ but not for $d = 1, 2$. An infinite volume (thermodynamic) limit for μ^0_Γ as $\Gamma \uparrow \mathbb{Z}^d$ exists only when $d \geq 3$.

3.3 Proofs of Theorems 3.1.1 and 3.1.2 via the Helffer-Sjöstrand Representation

Our first aim is to investigate the decay of the space-time correlation functions under the equilibrium Gibbs measures. The idea, originally from Helffer and Sjöstrand [HS94], is to describe the correlation functions in terms of a certain random walk in a dynamic random environment (cf. also [DD05, GOS01]). Let $(X_t)_{t\geq 0}$ be the random walk on $\mathbb{Z}^d$ with jump rates given by the random dynamic conductances

$$\omega_t(e) := V''(\nabla_e \phi_t) = V''(\phi_t(y) - \phi_t(x)), \qquad e = \{x, y\} \in E_d. \tag{3.38}$$

Note that the conductances are positive by Assumption 3.2.1 and, since V is even, the jump rates are symmetric, i.e. $\omega_t(\{x,y\}) = \omega_t(\{y,x\})$. Further, let $p^\omega(s,t,x,y)$, $x, y \in \mathbb{Z}^d$, $s \leq t$, denote the transition densities of the random walk X. Then the Helffer-Sjöstrand representation (see [Fun05, Theorem 4.2] or [DD05, Equation (6.10)]) states that if F, $G \in C^1_b(\mathcal{S})$ are differentiable functions with bounded derivatives depending only on finitely many coordinates then for all $t > 0$,

$$\mathrm{Cov}_\mu(F(\phi_0), G(\phi_t)) = \int_0^\infty \sum_{x,\,y\in\mathbb{Z}^d} \mathbb{E}_\mu\left[\frac{\partial F(\phi_0)}{\partial\phi(x)}\,\frac{\partial G(\phi_{t+s})}{\partial\phi(y)}\, p^\omega(0, t+s, x, y)\right] ds, \tag{3.39}$$

where μ is a stationary, ergodic, shift-invariant ϕ-Gibbs measure. Note that for $d \geq 3$ the integral in (3.39) is finite due to the following on-diagonal heat kernel estimate.

Lemma 3.3.1. *There exists deterministic $c_9 = c_9(d, c_-) < \infty$ such that*

$$p^\omega(0, t, x, y) \leq c_9 \, t^{-d/2}, \qquad \forall \, t \geq 1, \; x, y \in \mathbb{Z}^d. \tag{3.40}$$

Proof. Note that by Assumption 3.2.1, $\omega_t(e) \geq c_-$ for all $t \in \mathbb{R}$ and $e \in E_d$. By the Nash inequality, cf. [MO16, Theorem 2.1] for instance, there exists $c(d, c_-) > 0$ such that for any $t \in \mathbb{R}$ and $f : \mathbb{Z}^d \to \mathbb{R}$,

$$\left\| f \right\|_{\ell^2(\mathbb{Z}^d)} \leq c \left\| \omega_t^{1/2} \nabla f \right\|_{\ell^2(E_d)}^{\frac{d}{d+2}} \left\| f \right\|_{\ell^1(\mathbb{Z}^d)}^{\frac{2}{d+2}}.$$

We adapt standard arguments, cf. [CKS87, Theorem 2.1] and [MO16, Theorem 5.1], to prove the on-diagonal estimate from the Nash inequality above. Write $u_t(\cdot) = p^\omega(0, t, 0, \cdot)$ and

$$\mathcal{E}_t := \left\| u_t \right\|_{l^2(\mathbb{Z}^d)}^2, \qquad \mathcal{D}_t := \left\| \omega_t^{1/2} \nabla u_t \right\|_{l^2(E_d)}^2. \tag{3.41}$$

Recall that the generator of the dynamic RCM is given by

$$\left(\mathcal{L}_t^\omega f \right)(x) := \sum_{y \sim x} \omega_t(x, y) \big(f(y) - f(x) \big), \tag{3.42}$$

acting on bounded $f : \mathbb{Z}^d \to \mathbb{R}$. By [ACS21, Proposition B.3], u_t solves the caloric equation, for $\mathbb{P}$-a.e. ω,

$$\partial_t u_t(x) = \big(\mathcal{L}_t^\omega u_t \big)(x), \qquad \forall \, x \in \mathbb{Z}^d \text{ and a.e. } t \in (0, \infty). \tag{3.43}$$

This implies that for a.e. $t \in (0, \infty)$,

$$-\frac{d\mathcal{E}_t}{dt} = 2 \, \mathcal{D}_t. \tag{3.44}$$

Note that $\left\| u_t \right\|_{l^1(\mathbb{Z}^d)} = 1$ for all $t > 0$, so by the Nash inequality, with $\alpha = d/(d+2)$,

$$\mathcal{E}_t \leq c \, \mathcal{D}_t^\alpha. \tag{3.45}$$

Combining (3.44) and (3.45) gives

$$\frac{d}{dt} \big(\mathcal{E}_t^{1 - \frac{1}{\alpha}} \big) = \frac{\alpha - 1}{\alpha} \mathcal{E}_t^{-\frac{1}{\alpha}} \frac{d\mathcal{E}_t}{dt} \geq c > 0, \tag{3.46}$$

where we have used that $\alpha \in (0, 1)$. Also $\mathcal{E}_0 = 1$, therefore by integrating the above relation, for a.e. $t \in [1/2, \infty)$,

$$\mathcal{E}_t^{1-\frac{1}{\alpha}} \geq c\,t. \tag{3.47}$$

So for a.e. $t \in [1/2, \infty)$,

$$\mathcal{E}_t = \sum_{z \in \mathbb{Z}^d} p^\omega(0, t, 0, z)^2 \leq c\,t^{-\frac{d}{2}}. \tag{3.48}$$

For any $s \in \mathbb{R}$, $z \in \mathbb{Z}^d$, with the time-space translation $\tau_{s,z}$ defined as in (2.6), the above also holds for the environment $\tau_{s,z}\omega$. Furthermore, by [ACS21, Equation (B.2)],

$$p^{\tau_{s,z}\omega}(t_1, t_2, x, y) = p^\omega(t_1 + s, t_2 + s, x + z, y + z), \qquad \forall\, t_1,\, t_2 \in \mathbb{R},\ \forall\, x,\, y \in \mathbb{Z}^d. \tag{3.49}$$

So for any $s \in \mathbb{R}$, $x \in \mathbb{Z}^d$,

$$\sum_{z \in \mathbb{Z}^d} p^\omega(s, s+t, x, z)^2 \leq c\,t^{-\frac{d}{2}}, \quad \text{for a.e. } t \in [1/2, \infty). \tag{3.50}$$

Now, define the time-reversed environment by $\tilde{\omega}_t(e) := \omega_{-t}(e)$ for all $e \in E_d$, $t \in \mathbb{R}$. Note that (3.50) also holds with $\tilde{\omega}$ in place of ω. We also have, by [ACS21, Lemma B.2], that for $t > 0,\ y,\ z \in \mathbb{Z}^d$,

$$p^\omega\big(t/2, t, z, y\big) = p^{\tilde{\omega}}\big(-t, -t/2, y, z\big). \tag{3.51}$$

Then, by the Markov property,

$$\begin{aligned}
p^\omega(0, t, x, y) &= \sum_{z \in \mathbb{Z}^d} p^\omega(0, t/2, x, z)\, p^\omega(t/2, t, z, y) \\
&= \sum_{z \in \mathbb{Z}^d} p^\omega(0, t/2, x, z)\, p^{\tilde{\omega}}(-t, -t/2, y, z) \quad \text{by (3.51)} \\
&\leq \Big(\sum_{z \in \mathbb{Z}^d} p^\omega(0, t/2, x, z)^2\Big)^{\frac{1}{2}} \Big(\sum_{z \in \mathbb{Z}^d} p^{\tilde{\omega}}(-t, -t/2, y, z)^2\Big)^{\frac{1}{2}} \\
&\leq c\,t^{-\frac{d}{2}},
\end{aligned}$$

for a.e. $t \in [1, \infty)$, where we applied the Cauchy-Schwartz inequality in the penultimate line and (3.50) for the final line. Finally, since $t \mapsto p^\omega(0, t, x, y)$ is right-continuous, the above estimate holds for all $t \geq 1$. $\square$

A consequence of the above is the following variance estimate, an example of algebraic decay to equilibrium, in contrast to the exponential decay to equilibrium which would follow from a spectral gap estimate or Poincaré inequality. For this model, these inequalities hold on finite boxes but fail on the whole lattice.

Corollary 3.3.2. *Suppose $d \geq 3$ and let $\mu \in \mathcal{P}\big(\mathbb{R}^{\mathbb{Z}^d}\big)$ be any ergodic, shift-invariant,* *stationary ϕ-Gibbs measure.*

(i) There exists $c_{10} = c_{10}(d, c_-) > 0$ such that for all $F,\, G \in C_b^1(\mathcal{S})$ and $t > 0$,

$$\Big|\mathbb{C}\mathrm{ov}_\mu\big(F(\phi_0), G(\phi_t)\big)\Big| \leq \frac{c_{10}}{(t\vee 1)^{\frac{d}{2}-1}} \sum_{x,\, y\in\mathbb{Z}^d} \mathbb{E}_\mu\Big[\Big(\frac{\partial F}{\partial\phi(x)}(\phi_0)\Big)^2\Big]^{\frac{1}{2}} \mathbb{E}_\mu\Big[\Big(\frac{\partial G}{\partial\phi(y)}(\phi_0)\Big)^2\Big]^{\frac{1}{2}}.$$

(ii) $\mathbb{E}_\mu\big[\phi(0)^2\big] < \infty$.

Proof. (i) follows by the Cauchy-Schwarz inequality applied to (3.39) together with Lemma 3.3.1 and stationarity of μ. Further, by taking $t \downarrow 0$ we deduce from (i) and the dominated convergence theorem that $\sup_M \mathbb{E}_\mu\big[(\phi_0(0)\wedge M)^2\big] < \infty$, which in turn implies (ii) by the monotone convergence theorem. □

Proof of Theorem 3.1.1. In Theorem 3.2.8 above, the existence of a stationary, shift-invariant, ergodic ϕ-Gibbs measure μ has been shown. Further, the environment ω defined in (3.38) satisfies Assumption 2.1.9 by the ergodicity of μ. Note that $\omega_t(e) \geq c_-$ for any $e \in E_d$ and $t > 0$ by Assumption 3.2.1, so we may set $q = \infty$ in Assumption 2.4.2, which then reduces to (3.4). The Helffer-Sjöstrand relation (3.39) gives

$$\mathbb{C}\mathrm{ov}_\mu(\phi_0(0), \phi_t(x)) = \int_0^\infty \mathbb{E}_\mu\big[p^\omega(0, t+s, 0, x)\big]\, ds.$$

Now, applying Theorem 2.1.11,

$$\begin{aligned} n^{d-2}\,\mathbb{C}\mathrm{ov}_\mu(\phi_0(0), \phi_{n^2t}\left(\lfloor nx\rfloor\right)) &= \int_0^\infty \mathbb{E}_\mu\Big[n^d\, p^\omega\big(0, n^2(t+s), 0, \lfloor nx\rfloor\big)\Big]\, ds \\ &\xrightarrow[n\to\infty]{} \int_0^\infty k_{t+s}(x)\, ds, \end{aligned} \tag{3.52}$$

which is the claim. Note that Theorem 2.1.11 gives uniform convergence of the integrand on any compact interval $[0, T]$ and Lemma 3.3.1 tells us that the integrand is dominated by $c\, s^{-\frac{d}{2}}$, integrable on $[T, \infty)$ since $d \geq 3$. Therefore, by the dominated convergence theorem we are justified in interchanging the limit and the integral. □

Having applied the annealed local limit theorem to prove the above space-time covariance scaling limit, we now present an application of the invariance principle in Theorem 2.1.10-(i). We use this to characterise the scaling limit of the equilibrium fluctuations as a Gaussian free field. Recall that for $f \in C_0^\infty(\mathbb{R}^d)$ and $n \in \mathbb{Z}_+$,

$$f_n(x) := n^{-(1+d/2)} f(x/n) \quad \text{and} \quad \phi(f_n) := n^{-(1+d/2)} \int_{\mathbb{R}^d} f(x)\,\phi(\lfloor nx\rfloor)\, dx.$$

The key step in proving this scaling limit is to establish the following lemma. Recall that $(X_t)_{t\geq 0}$ denotes the random walk on $\mathbb{Z}^d$ under the conductances given by (3.38). For simplicity, we write P_x for the law of the walk started from $x \in \mathbb{Z}^d$ at time $s = 0$ and E_x for the corresponding expectation.

Lemma 3.3.3. *Under the hypotheses of Theorem 3.1.2, the following holds.*

$$\lim_{n\to\infty} n^{-d} \int_0^\infty \sum_{x\in\mathbb{Z}^d} f(x/n)\,\mathrm{E}_x\big[f(X_{n^2t}/n)\big]\,dt = \int_{\mathbb{R}^d} f(x)(-Q^{-1}f)(x)\,dx \quad \textit{in } L^1(\mu). \tag{3.53}$$

Proof. The annealed functional central limit theorem for the random walk X is an immediate consequence of the QFCLT in Theorem 2.1.10-(i) with $q = \infty$. This implies that for any $t > 0$, the annealed law of $\frac{1}{n}X_{n^2t}$ started from 0 tends weakly to that of B_t, where $(B_t)_{t\geq 0}$ is a Brownian motion started from 0, with covariance matrix $q = \Sigma^2$. We denote the expectation under the law of this Brownian motion E^{BM}. In particular, for any family $\mathcal{G} \subset C_0^\infty(\mathbb{R}^d)$ of uniformly equicontinuous and uniformly bounded functions,

$$\mathbb{E}_\mu\Big[\sup_{g\in\mathcal{G}}\big(\mathrm{E}_0\big[g(X_{n^2t}/n)\big] - \mathrm{E}^{\mathrm{BM}}\big[g(B_t)\big]\big)^2\Big] \to 0, \quad \text{as } n\to\infty. \tag{3.54}$$

For $n \in \mathbb{N}$ and $t > 0$, define

$$\begin{aligned} I_t^{(n)} &:= n^{-d}\sum_{x\in\mathbb{Z}^d} f(x/n)\,\mathrm{E}_x\big[f(X_{n^2t}/n)\big], \\ H_t &:= \int_{\mathbb{R}^d} f(x)\,\mathrm{E}^{\mathrm{BM}}\big[f(x+B_t)\big]\,dx. \end{aligned}$$

Let $R > 0$ be such that $\operatorname{supp} f \subset \big\{x\in\mathbb{R}^d \ : \ |x| < R\big\}$ and define $\mathcal{R}_n := [-nR, nR]^d \cap \mathbb{Z}^d$. Note that, by a change of variables, we can write

$$H_t = \int_{[0,1]^d} n^{-d}\sum_{x\in\mathcal{R}_n} f\big((x+z)/n\big)\,\mathrm{E}^{\mathrm{BM}}\big[f\big((x+z)/n + B_t\big)\big]\,dz. \tag{3.55}$$

Then (3.53) is equivalent to

$$\lim_{n\to\infty}\int_0^\infty I_t^{(n)}\,dt = \int_0^\infty H_t\,dt \quad \text{in } L^1(\mu). \tag{3.56}$$

We first prove $L^2(\mu)$ convergence pointwise in t. Since $f \in C_0^\infty(\mathbb{R}^d)$, the family

$$\mathcal{G} := \Big\{f\Big(\frac{x}{n} + \cdot\Big) \ : \ x\in\mathbb{Z}^d, |x| \leq nR\Big\}$$

is uniformly equicontinuous and uniformly bounded. This allows us to apply (3.54); it also gives the existence of a sequence $\epsilon_n \downarrow 0$ such that

$$\left| f\Big(\frac{x+z}{n}\Big) - f\Big(\frac{x}{n}\Big) \right| < \epsilon_n, \quad \forall\, x \in \mathbb{Z}^d,\, \forall\, z \in [0,1]^d. \tag{3.57}$$

This will effectively allow us to replace $I_t^{(n)}$, H_t by

$$\tilde{I}_t^{(n)} := \int_{[0,1]^d} n^{-d} \sum_{x \in \mathbb{Z}^d} f\big((x+z)/n\big)\, \mathrm{E}_0\big[f(x/n + X_{n^2 t}/n)\big]\, dz, \tag{3.58}$$

$$\tilde{H}_t := \int_{[0,1]^d} n^{-d} \sum_{x \in \mathcal{R}_n} f\big((x+z)/n\big)\, \mathrm{E}^{\mathrm{BM}}\big[f\big(x/n + B_t\big)\big]\, dz, \tag{3.59}$$

respectively, since

$$\left| I_t^{(n)} - \tilde{I}_t^{(n)} \right| < c\, \epsilon_n R^d \sup_{x \in \mathbb{R}^d} \big| f(x) \big|\,, \qquad \left| H_t^{(n)} - \tilde{H}_t^{(n)} \right| < c\, \epsilon_n R^d \sup_{x \in \mathbb{R}^d} \big| f(x) \big|\,. \tag{3.60}$$

Then, by the Cauchy-Schwartz inequality and (3.54),

$$\begin{aligned} \mathbb{E}_\mu\big[(\tilde{I}_t^{(n)} - \tilde{H}_t)^2\big] &\le c\, R^{2d}\, \big\|f\big\|_{L^2(\mathbb{R}^d)}^2 \mathbb{E}_\mu\Big[\sup_{g \in \mathcal{G}} \big(\mathrm{E}_0\big[g(X_{n^2 t}/n)\big] - \mathrm{E}^{\mathrm{BM}}\big[g(B_t)\big]\big)^2\Big] \\ &\to 0, \quad \text{as } n \to \infty. \end{aligned} \tag{3.61}$$

Together, the above and (3.60) imply that

$$\mathbb{E}_\mu\big[(I_t^{(n)} - H_t)^2\big] \to 0, \quad \text{as } n \to \infty. \tag{3.62}$$

In particular, the above implies $L^1(\mu)$ convergence, pointwise in t. Now, by Lemma 3.3.1, which implies an on-diagonal estimate for the semigroup P_t associated with X_t,

$$\big|I_t^{(n)}\big| = \Big| n^{-d} \sum_{x \in \mathbb{Z}^d} f(x/n) P_{n^2 t}\, f(x/n) \Big| \le c\, \big(n^2 (t \vee 1)\big)^{-\frac{d}{2}} \big\|f\big\|_{L^2(\mathbb{R}^d)}^2, \tag{3.63}$$

for all $t \ge 0$, where we again used that $\mathcal{G}$ is uniformly equicontinuous to bound

$$n^{-d} \sum_{x \in \mathcal{R}_n} f(x/n)^2 \le c\, \big\|f\big\|_{L^2(\mathbb{R}^d)}^2.$$

Similarly, by the on-diagonal decay of the Gaussian heat kernel, we have

$$|H_t| \le c\, (1 \vee t)^{-\frac{d}{2}}\, \big\|f\big\|_{L^2(\mathbb{R}^d)}^2, \tag{3.64}$$

for all $t \geq 0$. In summary, for all $t > 0$,

$$\mathbb{E}_\mu\big[|I_t^{(n)} - H_t|\big] \to 0, \quad \text{as } n \to \infty,$$

and there exists $c > 0$ such that for all $n \in \mathbb{Z}_+$,

$$|I_t^{(n)} - H_t| \leq c\,(t \vee 1)^{-\frac{d}{2}}.$$

Therefore, we conclude the $L^1(\mu)$ convergence in (3.56) by consecutive applications of the dominated convergence theorem; first in $L^1([0,\infty))$, then in $L^1(\mu)$. □

Proof of Theorem 3.1.2. Let $n \in \mathbb{Z}_+$ and set $G_n(\lambda) := \mathbb{E}_\mu[\exp(\lambda\phi(f_n))]$, $\lambda \in \mathbb{R}$. By the Brascamp-Lieb inequality, Proposition 3.2.6, $\{G_n(\lambda)\}_{n\geq 1}$ is uniformly bounded for λ in a compact interval. So differentiating gives

$$\frac{dG_n(\lambda)}{d\lambda} = \mathbb{E}_\mu\big[\phi(f_n)\exp(\lambda\phi(f_n))\big].$$

Then by the Helffer-Sjöstrand relation (3.39), the above equals

$$\begin{aligned} &\lambda\, n^{-(d+2)} \int_0^\infty \sum_{x\in\mathbb{Z}^d} \mathbb{E}_\mu\Big[f(x/n)\,\mathrm{E}_x\big[f(X_t/n)\big]\exp(\lambda\phi_t(f_n))\Big]\,dt \\ =\, &\lambda\, n^{-d} \int_0^\infty \sum_{x\in\mathbb{Z}^d} f(x/n)\,\mathbb{E}_\mu\Big[\mathrm{E}_x\big[f(X_{n^2t}/n)\big]\exp(\lambda\phi_{n^2t}(f_n))\Big]\,dt, \end{aligned}$$

where we have employed a simple change of variable. Now, using Lemma 3.3.3, we deduce that

$$\frac{dG_n(\lambda)}{d\lambda} = \lambda\, G_n(\lambda) \int_{\mathbb{R}^d} f(x)(-Q^{-1}f)(x)\,dx + \lambda\, o(1),$$

as $n \to \infty$. This relies on the fact that by the stationarity property of the Gibbs measure, $\mathbb{E}_\mu[\exp(\lambda\phi_{n^2t}(f_n))] = \mathbb{E}_\mu[\exp(\lambda\phi(f_n))]$ for any λ and $t \geq 0$, and this term is uniformly bounded in n. Therefore, denoting $G(\lambda) = \lim_{n\to\infty} G_n(\lambda)$, we have by letting $n \to \infty$,

$$\frac{dG(\lambda)}{d\lambda} = \lambda\, G(\lambda) \int_{\mathbb{R}^d} f(x)(-Q^{-1}f)(x)\,dx$$

which together with $G(0) = 1$ gives the claim. □

3.4 Moments of ϕ-Gibbs Measures

Finally, we shall derive Proposition 3.1.4 giving polynomial moment bounds on the heights ϕ under any ergodic, shift-invariant, stationary ϕ-Gibbs measure. Hence we can verify the conditions in Theorem 3.1.1 and Theorem 3.1.2 for any polynomial potential satisfying Assumption 3.2.1. The proof will require the following comparison estimate for ϕ_t and $\phi_t^{L_n}$.

Lemma 3.4.1. *Let μ be a shift-invariant Gibbs measure. There exists a positive, symmetric, summable sequence $\alpha = \big(\alpha(x)\big)_{x\in\mathbb{Z}^d}$ such that the following holds. There exist constants $c_{11}, c_{12}, c_{13} > 0$ such that for any $n \in \mathbb{N}$, $t > 0$ and for any bounded Lipschitz function $f : \ell^2(\mathbb{Z}^d, \alpha) \to \mathbb{R}$,*

$$\mathbb{E}_\mu\Big[\big(f\big(\phi_t\big) - f\big(\phi_t^{L_n,0}\big)\big)^2\Big] \le c_{11}\, \big\|f\big\|_{\mathrm{lip},\alpha}^2\, \mathrm{e}^{c_{12}t - c_{13}n}\Big(1 + \mathbb{E}_\mu\big[\phi_0(0)^2\big]\Big), \tag{3.65}$$

where $\big\|f\big\|_{\mathrm{lip},\alpha} := \sup_{\phi\neq\phi'}\big|f(\phi) - f(\phi')\big|\,\big\|\phi - \phi'\big\|_{\ell^2(\mathbb{Z}^d,\alpha)}^{-1}$.

Proof. It follows from Proposition 3.2.5(i) that

$$\alpha(0)\,\mathbb{E}_\mu\Big[\big(f\big(\phi_t\big) - f\big(\phi_t^{L_n,0}\big)\big)^2\Big] \le \mathrm{e}^{c\,t}\,\big\|f\big\|_{\mathrm{lip},\alpha}^2 \mathbb{E}_\mu\Bigg[\sum_{x\in L_n^c} \alpha * \alpha(x)\,\phi_0(x)^2 + c|\alpha|\,\alpha(x)\Bigg].$$

Then by shift-invariance of μ and exponential decay of α and $\alpha * \alpha$ on $\mathbb{Z}^d$, we get (3.65). □

Proof of Proposition 3.1.4. Since μ is stationary and shift-invariant it suffices to show that $\mathbb{E}_\mu\big[\big|\phi_0(0)\big|^p\big] < \infty$ for all $p > 0$. By Jensen's inequality it is enough to consider $p > 2$. For any $M > 1$ let $f_M(\phi) := \big(\big|\phi(0)\big| \wedge M\big)^{p/2}$ which is Lipschitz continuous on $\ell^2(\mathbb{Z}^d, \alpha)$ with $\big\|f_M\big\|_{\mathrm{lip},\alpha}^2 \le c\, M^{p-2}$. For arbitrary $t > 0$ and $n \in \mathbb{Z}_+$,

$$\mathbb{E}_\mu\Big[f_M(\phi_t)^2\Big] \le 2\,\mathbb{E}_\mu\Big[\big(f_M\big(\phi_t\big) - f_M\big(\phi_t^{L_n,0}\big)\big)^2\Big] + 2\,\mathbb{E}_\mu\Big[f_M\big(\phi_t^{L_n,0}\big)^2\Big]. \tag{3.66}$$

To control the first term on the right hand side of (3.66), we fix $\epsilon > 0$. As argued in [Roy07, Theorem 5.1.3], for arbitrary $\lambda > 0$ we introduce an increasing sequence of boxes $L_{n(t)}$ such that $c_{14}t \le n(t) \le c_{14}(t+1)$ where $c_{14} > 0$ is chosen such that $c_{12}t - c_{13}n(t) < -\lambda t$, with c_{12}, c_{13} as in Lemma 3.4.1. Therefore, by (3.65) and Corollary 3.3.2-(ii), there exists $T(\epsilon, M) > 0$ such that

$$\mathbb{E}_\mu\Big[\Big(f_M\big(\phi_t\big) - f_M\big(\phi_t^{L_{n(t)},0}\big)\Big)^2\Big] \le c\,M^{p-2}\Big(1 + \mathbb{E}_\mu\big[\phi_0(0)^2\big]\Big)\mathrm{e}^{-\lambda t} \le \epsilon \tag{3.67}$$

for all $t > T(\epsilon, M)$. For the latter term in (3.66), the constant zero boundary condition allows us, via the DLR equation (3.10), to reduce the expectation to that over a finite

Gibbs measure as defined in (3.9),

$$\mathbb{E}_{\mu}\Big[f_M\big(\phi_t^{L_n,0}\big)^2\Big] \;=\; \mathbb{E}_{\mu^0_{L_n}}\Big[f_M\big(\phi_t^{L_n,0}\big)^2\Big]. \tag{3.68}$$

Now, the finite volume process $\phi^{L_n,0}$ is stationary with respect to $\mu^0_{L_n}$ so by the Brascamp-Lieb inequality, as argued in Theorem 3.2.8,

$$\sup_{n\in\mathbb{N},\, M>0,\, t>0} \mathbb{E}_{\mu^0_{L_n}}\Big[f_M\big(\phi_t^{L_n,0}\big)^2\Big] = \sup_{n\in\mathbb{N},\, M>0} \mathbb{E}_{\mu^0_{L_n}}\Big[f_M\big(\phi_0^{L_n,0}\big)^2\Big] < \infty. \tag{3.69}$$

Substituting (3.69) and (3.67) into (3.66) gives

$$\mathbb{E}_{\mu}\Big[\big(\big|\phi_t(0)\big| \wedge M\big)^p\Big] \;<\; \epsilon + c, \tag{3.70}$$

for all $t > T(\epsilon, M)$, with the constant c independent of M. However, ϕ_t is stationary with respect to μ so (3.70) in fact holds for all $t \geq 0$. We conclude by the monotone convergence theorem, letting $M \uparrow \infty$, that $\mathbb{E}_{\mu}\big[\big|\phi_0(0)\big|^p\big] < \infty$. □

3.5 Future Directions

In this chapter, we utilised the elegant Helffer-Sjoštrand coupling relation to prove a number of statements on the Ginzburg-Landau $\nabla\phi$ model, using recent developments in the homogenization of random walks in random environments. Following this work, some natural future aims are:

(i) Extending the scaling limit in Theorem 3.1.2 to a time-dynamic result. As discussed in Remark 3.1.3, we believe this would require off-diagonal heat kernel estimates for the dynamic RCM in a degenerate, ergodic environment.

(ii) Relaxing the condition on the potential, $V'' \geq c_- > 0$, in Theorem 3.1.1 and Theorem 3.1.2. We are still likely to require that $V'' \geq 0$ for applications of RCM results, so that the conductances remain non-negative. However, our main reason for working with a uniformly convex potential in this chapter is for the construction of ϕ-Gibbs measures, and there are a number of existence results in this direction for certain non-convex potentials, see for instance [BK07, BS11, Ye19]. Furthermore, Biskup and Rodriguez recently established the QFCLT for a dynamic RCM where the conductances may take the value zero for finite intervals of time [BR18].

(iii) Establishing a covariance scaling limit for the gradient field $\{\nabla\phi(e)\}_{e\in E_d}$, akin to Theorem 3.1.1 for the field of heights. To prove this result via the Helffer-Sjöstrand relation would require an annealed local limit theorem for the gradient of the heat kernel, which is not available at present. Some bounds on the gradient of the heat kernel are established and applied to the gradient field in [DD05], for a uniformly elliptic environment.

Chapter 4

Heat Kernel Estimates for Symmetric Diffusions

Abstract. In this chapter we study a symmetric diffusion process on $\mathbb{R}^d$, $d \geq 2$, in divergence form in a stationary and ergodic random environment. The coefficients are assumed to be degenerate and unbounded but satisfy a moment condition. We derive upper off-diagonal estimates on the heat kernel of this process for general speed measure. Lower off-diagonal estimates are also shown for a natural choice of speed measure, under an additional decorrelation assumption on the environment. Using these heat kernel estimates, a scaling limit for the Green's function is proven.

4.1 Introduction

Our focus is the diffusion process on $\mathbb{R}^d$ formally associated with the following generator

$$\mathcal{L}^\omega u(x) = \frac{1}{\theta^\omega(x)} \nabla \cdot (a^\omega(x) \nabla u(x)), \quad x \in \mathbb{R}^d, \tag{4.1}$$

where the random field $\{a^\omega(x)\}_{x\in\mathbb{R}^d}$ is a symmetric d-dimensional matrix for each $x \in \mathbb{R}^d$, and θ^ω is a positive speed measure which may also depend on the random environment ω. Firstly, we set out the precise assumptions on the random environment. Let $\left(\Omega, \mathcal{G}, \mathbb{P}, \{\tau_x\}_{x\in\mathbb{R}^d}\right)$ be a probability space together with a measurable group of translations. $\mathbb{E}$ will denote the expectation under this probability measure. To construct the random field let $a : \Omega \to \mathbb{R}^{d\times d}$ be a $\mathcal{G}$-measurable random variable and define $a^\omega(x) := a(\tau_x\omega)$. The speed measure is defined similarly, take a $\mathcal{G}$-measurable random variable $\theta : \Omega \to (0, \infty)$ and let $\theta^\omega(x) := \theta(\tau_x\omega)$. We refer to this function as the speed measure because the process with general θ^ω can be obtained from the process with $\theta^\omega \equiv 1$ via a time-change. As made precise in the following, we assume throughout that the random environment is stationary, ergodic and satisfies a non-uniform ellipticity condition.

Assumption 4.1.1. *The probability space satisfies:*

(i) $\mathbb{P}(\tau_x A) = \mathbb{P}(A)$ *for all* $A \in \mathcal{G}$ *and any* $x \in \mathbb{R}^d$.

(ii) If $\tau_x A = A$ *for all* $x \in \mathbb{R}^d$ *then* $\mathbb{P}(A) \in \{0, 1\}$.

(iii) The mapping $(x, \omega) \mapsto \tau_x \omega$ *is* $\mathcal{B}(\mathbb{R}^d) \otimes \mathcal{G}$ *measurable.*

Furthermore for each $x \in \mathbb{R}^d$, $a^\omega(x)$ *is symmetric and there exist positive,* $\mathcal{G}$*-measurable* $\lambda,\ \Lambda : \Omega \to (0, \infty)$ *such that for* $\mathbb{P}$*-a.e.* $\omega \in \Omega$ *and all* $\xi \in \mathbb{R}^d$, $x \in \mathbb{R}^d$,

$$\lambda(\tau_x \omega)|\xi|^2 \leq \xi \cdot (a^\omega(x)\xi) \leq \Lambda(\tau_x \omega)|\xi|^2 . \tag{4.2}$$

Also, defining $\Lambda^\omega(x) := \Lambda(\tau_x \omega)$ *and* $\lambda^\omega(x) := \lambda(\tau_x \omega)$ *for* $x \in \mathbb{R}^d$*, assume that* $\mathbb{P}$*-a.s.*

$$\Lambda^\omega,\ (\lambda^\omega)^{-1},\ \theta^\omega,\ (\theta^\omega)^{-1} \in L^\infty_{loc}(\mathbb{R}^d). \tag{4.3}$$

The final assumption of local boundedness will allow us to pass from estimates on the semigroup of the diffusion process to pointwise bounds on the heat kernel. Rather than assuming these functions are uniformly bounded, we work with moment conditions given in terms of the following, for $p, q, r \in (0, \infty]$ define

$$\begin{aligned} M_1(p, q, r) &:= \mathbb{E}[\theta^\omega(0)^r] + \mathbb{E}[\lambda^\omega(0)^{-q}] + \mathbb{E}[\Lambda^\omega(0)^p \theta^\omega(0)^{1-p}], \\ M_2(p, q) &:= \mathbb{E}[\lambda^\omega(0)^{-q}] + \mathbb{E}[\Lambda^\omega(0)^p]. \end{aligned} \tag{4.4}$$

By the ergodic theorem, these conditions together with Assumption 4.1.1 allow us to control average values of the functions on large balls. For instance, denoting by $B(x, r)$ the closed Euclidean ball of radius r centred at x, $\bar{\Lambda}_p := \mathbb{E}[\Lambda^\omega(0)^p]$ and $\bar{\lambda}_q := \mathbb{E}[\lambda^\omega(0)^q]$, then $M_2(p, q) < \infty$ implies that for $\mathbb{P}$-a.e. ω, there exists $N_1^\omega(x) > 0$ such that for all $r \geq N_1^\omega(x)$,

$$\frac{1}{|B(x, r)|} \int_{B(x,r)} \Lambda^\omega(u)^p \, du < 2\,\bar{\Lambda}_p, \qquad \frac{1}{|B(x, r)|} \int_{B(x,r)} \lambda^\omega(u)^q \, du < 2\,\bar{\lambda}_q. \tag{4.5}$$

As discussed in Chapter 1, when the coefficients are smooth and uniformly elliptic, i.e. when $\Lambda^\omega(x)$ and $\lambda^\omega(x)$ are bounded above and below respectively, uniformly in ω, the model we are considering is fairly well understood. However, in the present setting we do not assume differentiability of the random field $\{a^\omega(x)\}_{x \in \mathbb{R}^d}$ so some work is required to construct the process associated with (4.1) in a general ergodic environment. The diffusion is constructed using the theory of Dirichlet forms, with the corresponding form being

$$\mathcal{E}^\omega(u,v) := \sum_{i,j=1}^{d} \int_{\mathbb{R}^d} a_{ij}^\omega(x)\partial_i u(x)\partial_j v(x)\, dx, \tag{4.6}$$

for $f,\ g$ in a proper class of functions $\mathcal{F}^\theta \subset L^2(\mathbb{R}^d, \theta^\omega\, dx)$, defined precisely in Section 4.2. The construction of a diffusion process $(X_t^\theta)_{t\geq 0}$ associated to (4.1) is a recent result of [CD16]. This is done under Assumption 4.1.1 together with the moment condition $M_2(p,q) < \infty$ for some $p,\ q \in (1,\infty]$ satisfying $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$. The main result in [CD16, Theorem 1.1] is the quenched invariance principle, that is for $\mathbb{P}$-a.e. ω the law of the process $\big(\frac{1}{n}X_{n^2t}\big)_{t\geq 0}$ on $C([0,\infty), \mathbb{R}^d)$ converges weakly as $n \to \infty$ to the law of Brownian motion. This is first proven for $\theta^\omega \equiv 1$ and then for general speed measure satisfying $\mathbb{E}[\theta^\omega(0)] < \infty$ and $\mathbb{E}[\theta^\omega(0)^{-1}] < \infty$, after showing that the general speed process can be obtained via a time change.

Regarding the heat kernel of the operator $\mathcal{L}^\omega$, it is also shown therein that the semigroup P_t of the above diffusion process has a transition kernel $p_\theta^\omega(t,x,y)$ with respect to $\theta^\omega(x)\, dx$, furthermore this is jointly continuous in x and y. Explicitly, for continuous, bounded $f : \mathbb{R}^d \to \mathbb{R}$,

$$P_t f(x) = \int_{y\in\mathbb{R}^d} f(y) p_\theta^\omega(t,x,y)\theta^\omega(y)\, dy, \qquad \forall\, x \in \mathbb{R}^d,\ t > 0. \tag{4.7}$$

A second, stronger result that has recently been established under Assumption 4.1.1 and moment condition $M_1(p,q,r) < \infty$ for some $p,q,r \in (1,\infty]$ satisfying $\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}$ is the quenched local central limit theorem [CD15, Theorem 1.1]. This states that the rescaled transition kernel $p_\theta^\omega(n^2t, 0, nx)$ converges as $n \to \infty$ to the heat kernel of a Brownian motion $k_t^\Sigma(0,x)$ with some deterministic, positive definite covariance matrix Σ implicitly depending on the law $\mathbb{P}$. Namely, for $t > 0$ and $x,\ y \in \mathbb{R}^d$,

$$k_t^\Sigma(x,y) := \frac{1}{\sqrt{(2\pi t)^d \det \Sigma}} \exp\Big(- \frac{(y-x)\cdot \Sigma^{-1}(y-x)}{2t} \Big). \tag{4.8}$$

The convergence is uniform on compact sets in t and x and the key step is to apply a parabolic Harnack inequality to obtain Hölder regularity of the heat kernel; this is achieved via Moser iteration which will also play an important role in our analysis. Many of the techniques herein take inspiration from the random conductance model which was the focus of Chapter 2, where we established local limit theorems in a degenerate, ergodic setting. The diffusion studied in this chapter is a continuum analogue of that model. Importantly, the RCM literature indicates that moment conditions are indeed necessary for a general ergodic environment. For instance, [ADS16a] proves a local limit theorem under a moment assumption equivalent to $M_2(p,q) < \infty$ for $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ and shows that this condition is optimal for the canonical choice of speed measure (known as the constant speed random walk). Another recent result under moment conditions is a Liouville theorem for the elliptic equation associated to (4.1) in [BFO18], cf. also [BCF19] for a related result on the parabolic equation associated to a time-dynamic, uniformly elliptic version of (4.1). Local boundedness and a Harnack inequality for solutions to the elliptic equation were recently proven in [BS21] under moment conditions.

A local limit theorem quantifies the limiting behaviour of the heat kernel and is known to provide near-diagonal estimates on the kernel prior to rescaling – see Proposition 4.3.1. In this chapter, our aim is to derive full Gaussian estimates on the heat kernel, $p_\theta^\omega(t,x,y)$, for all x and y, also known as off-diagonal estimates. For general speed measure, it is known that these bounds should be governed by the intrinsic metric, cf. [CKS87, Dav87, Str88]. In the random environment setting, this is a metric on $\mathbb{R}^d$ dependent on a^ω and θ^ω, defined as

$$d_\theta^\omega(x,y) := \sup\Big\{\phi(y)-\phi(x) \,:\, \phi\in C^1(\mathbb{R}^d),\, h^\omega(\phi)^2 = \sup_{z\in\mathbb{R}^d}\frac{(\nabla\phi\cdot a^\omega\nabla\phi)(z)}{\theta^\omega(z)} \le 1\Big\}.$$

Outside of the uniformly elliptic case it is clear that the above is not in general comparable to the Euclidean metric, which we denote $d(\cdot\,,\cdot)$. A natural follow-up question to this work would be to find the minimal conditions on an ergodic environment for which these two metrics are comparable. However here we require some regularity of the intrinsic metric in order to derive off-diagonal heat kernel estimates in terms of it. Specifically we must show it is strictly local, meaning it generates the Euclidean topology on $\mathbb{R}^d$. We therefore make the following additional assumption.

Assumption 4.1.2 (Continuity of the Environment)**.** *For $\mathbb{P}$-a.e. ω, the functions $a^\omega : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ and $\theta^\omega : \mathbb{R}^d \to (0,\infty)$ are continuous.*

Our first main result is an upper off-diagonal heat kernel estimate for the symmetric diffusion process with general speed measure in an ergodic, degenerate environment, and is proven in Section 4.2.

Theorem 4.1.3. *Suppose Assumption 4.1.1 and Assumption 4.1.2 hold. Let $d\ge 2$ and assume $M_1(p,q,r)<\infty$ for some $p,q,r\in(1,\infty]$ satisfying $\frac{1}{r}+\frac{1}{q}+\frac{1}{p-1}\frac{r-1}{r}<\frac{2}{d}$. Then for $\mathbb{P}$-a.e. $\omega\in\Omega$ and every $x\in\mathbb{R}^d$, there exist $N_2^\omega(x)>0$, $c_1(d,p,q,r)>0$ and $\gamma(d,p,q,r)>0$ such that the following holds for all $y\in\mathbb{R}^d$ and $\sqrt{t}>N_2^\omega(x)$,*

$$p_\theta^\omega(t,x,y) \le c_1\, t^{-\frac{d}{2}}\Big(1+\frac{d(x,y)}{\sqrt{t}}\Big)^\gamma \exp\Big(-\frac{d_\theta^\omega(x,y)^2}{8t}\Big). \tag{4.9}$$

Remark 4.1.4. (i) In the 'constant speed' setting of $\theta^\omega\equiv\Lambda^\omega$ (and more generally whenever $\theta^\omega\ge c\,\Lambda^\omega$), we obtain off-diagonal heat kernel estimates in terms of the

Euclidean metric d, without the need for Assumption 4.1.2. We get a full Gaussian upper estimate here because the polynomial prefactor in (4.9) can be absorbed into the exponential when the two metrics are comparable. See Corollary 4.4.1 for the precise statement.

(ii) We restrict to $d \geq 2$ because the derivation of the maximal inequality in Section 4.2.1 changes in dimension one, due to the Sobolev inequality (Lemma 4.2.4) taking a different form. Similarly, the lower estimate (Theorem 4.1.6) relies on the parabolic Harnack inequality in [CD15] which is derived therein using Moser iteration for $d \geq 2$ only.

To prove the above estimate we use Davies' perturbation method, a technique for deriving upper off-diagonal estimates, well-established in the elliptic and parabolic equations literature for uniformly elliptic operators cf. [Dav87, Dav89, CKS87, Str88, GT12]. The idea also translates to heat kernels on graphs [Dav93, Del99] and recently the RCM in a degenerate, ergodic environment [ADS16b, ADS19]. The first step of Davies' method is to consider the Cauchy problem associated to the perturbed operator $\mathcal{L}^\omega_\psi := \mathrm{e}^{\psi}\mathcal{L}^\omega \mathrm{e}^{-\psi}$ where ψ is an arbitrary test function, and use a maximal inequality to bound the fundamental solution. In [Zhi11], off-diagonal estimates are derived for solutions of a parabolic equation in a uniformly elliptic setting with degenerate, locally integrable weight; this was a useful inspiration for the Cauchy problem we consider in Section 4.2.1. In particular, [Zhi11] considers a parabolic equation $\rho\, \partial_t u = \nabla \cdot (\rho\, a \nabla u)$, where ρ is a non-negative function, $\rho,\ \rho^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ and a is a uniformly elliptic, symmetric matrix of measurable coefficients. To derive the maximal inequality we use a Moser iteration scheme adapted to the perturbed operator, similar to the method used to derive the parabolic Harnack inequality for the original operator $\mathcal{L}^\omega$ in [CD15]. Ergodic theory plays a key role here in controlling constants which depend on the random environment. Moser iteration has previously been applied to prove the corresponding RCM results – the quenched invariance principle in [ADS15], the Harnack inequality in [ADS16a] and off-diagonal estimates in [ADS16b, ADS19].

The second part of the argument is to optimise over the test function ψ. In the uniformly elliptic case this is straightforward as one can work with the Euclidean metric, however in our general setting of degenerate coefficient matrix and speed measure the off-diagonal estimate is governed by the intrinsic metric defined above. Utilising a test function related to this metric requires certain regularity properties, for instance that it generates the Euclidean topology on $\mathbb{R}^d$. In Section 4.2.3 we first relate the intrinsic metric to a Riemannian metric, then apply a recent result from geometric analysis [Bur15] to prove the necessary regularity properties under Assumption 4.1.2.

As a counterpart to the preceding upper estimate, we also present a lower off-diagonal estimate for the heat kernel, in the 'constant speed' case of $\theta^\omega \equiv \Lambda^\omega$. Whilst

Assumption 4.1.2 for regularity of the intrinsic metric is no longer required, we need a stronger control on the environment than given in Assumption 4.1.1. In particular, a decorrelation assumption suffices for the proof and we assume finite-range dependence of the environment.

Assumption 4.1.5. *Suppose there exists a positive constant $\mathcal{R} > 0$ such that for all $x \in \mathbb{R}^d$ and $\mathbb{P}$-a.e. ω, $\tau_x\omega$ is independent of $\{\tau_y\omega : y \in B(x, \mathcal{R})^c\}$.*

To prove the lower estimate we adapt the established chaining argument to the diffusion in a degenerate random environment, the method originated in [FS86] for solutions to parabolic equations, building on the ideas of Nash [Nas58]. It was adapted to the weighted graph setting in [Del99], to random walks on percolation clusters in [Bar04], and it was recently applied to the RCM [AH21]. The strategy is to repeatedly apply near-diagonal lower estimates, derived from the parabolic Harnack inequality established in [CD16], along a sequence of balls. The form of the constant in the Harnack inequality means that averages of the functions λ^ω and Λ^ω on balls with varying centre-points must be controlled simultaneously to derive the lower off-diagonal estimate. Something stronger than the classical ergodic theorem is required to do this, so given Assumption 4.1.5 we establish a specific form of a concentration inequality in Proposition 4.3.3 for this purpose. By an argument similar to [AH21] this inequality is then used to control the environment-dependent terms arising from the Harnack inequality, see Proposition 4.3.4. The statement is given below and proven in Section 4.3.

Theorem 4.1.6. *Suppose $d \geq 2$ and Assumptions 4.1.1 and 4.1.5 hold. There exist $p_0, q_0 \in (1, \infty)$ such that if $M_2(p_0, q_0) < \infty$ then for $\mathbb{P}$-a.e. ω and every $x \in \mathbb{R}^d$, there exist $c_i(d) > 0$ and a random constant $N_3^\omega(x) > 0$ satisfying*

$$\mathbb{P}(N_3^\omega(x) > n) \leq c_2\, n^{-\alpha} \qquad \forall\, n > 0, \tag{4.10}$$

for some $\alpha > d(d-1) - 2$, such that the following holds. For all $y \in \mathbb{R}^d$ and $t \geq N_3^\omega(x)\big(1 \vee d(x, y)\big)$,

$$p_\Lambda^\omega(t, x, y) \geq c_3\, t^{-d/2} \exp\Big(- c_4 \frac{d(x, y)^2}{t}\Big). \tag{4.11}$$

Remark 4.1.7. (i) In [AH21], three other assumptions such as an FKG inequality or a spectral gap inequality are offered as alternatives to finite-range dependence. Some of these are specific to the discrete setting and Assumption 4.1.5 is the most natural for our context, but it may be possible to replace it with other similar conditions.

(ii) We state the above only for $\theta^\omega \equiv \Lambda^\omega$ because for general speed measure the intrinsic metric is not necessarily comparable to the Euclidean metric which is used for the chaining argument. It may be possible to adapt the argument to general speed however and it is unclear whether this would require further assumptions in order to compare the two metrics.

Our final result is a scaling limit for the Green's function of the diffusion process, defined for dimension $d \geq 3$ as

$$g^\omega(x, y) := \int_0^\infty p_\Lambda^\omega(t, x, y)\, dt.$$

As already noted, the diffusion with general speed measure may be obtained from the process with speed measure $\theta^\omega \equiv \Lambda^\omega$ via a time change [CD15, Theorem 2.4]. Therefore the Green's function is independent of the speed measure θ^ω. Also, observe that the Green's function exists in dimension $d \geq 3$ due to the upper off-diagonal heat kernel estimate above, in fact a near-diagonal estimate is sufficient for this. Applying Theorem 4.1.3 together with a long-range bound obtained in Section 4.4, we obtain sufficient bounds to apply the local limit theorem [CD15, Theorem 1.1] and show that an appropriately rescaled version of the Green's function converges to that of a Brownian motion,

$$g_{\mathrm{BM}}(x, y) := \int_0^\infty k_t^\Sigma(x, y)\, dt. \tag{4.12}$$

Theorem 4.1.8. *Let $d \geq 3$ and suppose Assumption 4.1.1 holds. Also assume there exist p, $q \in (1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$ such that $M_2(p, q) < \infty$. Then for $x_0 \in \mathbb{R}^d$, $0 < r_1 < r_2$ and the annulus $A := \{x \in \mathbb{R}^d \,:\, 0 < r_1 \leq d(x_0, x) \leq r_2\}$,*

$$\lim_{n\to\infty} \sup_{x \in A} |n^{d-2} g^\omega(x_0, nx) - a\, g_{BM}(x_0, x)| = 0 \quad \text{for } \mathbb{P}\text{-a.e. } \omega. \tag{4.13}$$

In the above we have $a := \mathbb{E}[\Lambda^\omega(0)]^{-1}$.

Remark 4.1.9. Analogous results have been proven for the RCM in [ADS16a, Theorem 1.14] and [Ger20, Theorem 5.3]. See also [AH21, Theorem 1.6] for further estimates on the Green's function which can be derived from off-diagonal heat kernel estimates.

Notation and Structure of the Chapter. For $x \in \mathbb{R}^d$, $|x|$ denotes the standard Euclidean norm. For vectors u, $v \in \mathbb{R}^d$, the canonical scalar product is given by $u \cdot v$ and gradient ∇u. We write c to denote a positive, finite constant which may change on each appearance. Constants denoted by c_i will remain the same. For α, $\beta \in \mathbb{R}$, we write $\alpha \simeq \beta$ to mean there exist constants c, $\tilde{c} > 0$ such that $c\,\alpha \leq \beta \leq \tilde{c}\,\alpha$. For a countable set A, its cardinality is denoted $|A|$. Otherwise if $A \subset \mathbb{R}^d$, $|A|$ is the Lebesgue measure. For any $p \in (1, \infty)$, the Hölder conjugate is written $p_* := \frac{p}{p-1}$. We will work with inner products as follows. For functions $f,\ g \in L^2(\mathbb{R}^d)$,

$$(f,g) := \int_{\mathbb{R}^d} f(x)g(x)\,dx.$$

Given a positive weight $\nu : \mathbb{R}^d \to (0,\infty)$ and $f,\ g \in L^2(\mathbb{R}^d, \nu\,dx)$,

$$(f,g)_\nu := \int_{\mathbb{R}^d} f(x)g(x)\nu(x)\,dx. \tag{4.14}$$

Furthermore, for $p \in (0,\infty)$ and bounded $B \subset \mathbb{R}^d$, we define norms

$$\begin{aligned}
\|f\|_p &:= \Big(\int_{\mathbb{R}^d} |f(x)|^p\ dx \Big)^{1/p}, & \|f\|_{p,\nu} &:= \Big(\int_{\mathbb{R}^d} |f(x)|^p\,\nu(x)\,dx \Big)^{1/p}, \\
\|f\|_{p,B} &:= \Big(\frac{1}{|B|} \int_B |f(x)|^p\ dx \Big)^{1/p}, & \|f\|_{p,B,\nu} &:= \Big(\frac{1}{|B|} \int_B |f(x)|^p\,\nu(x)\,dx \Big)^{1/p}.
\end{aligned}$$

For $q \in (0,\infty)$, $I \subset \mathbb{R}$, $B \subset \mathbb{R}^d$, $Q = I \times B$ and $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, let

$$\begin{aligned}
\|u\|_{p,q,Q} &:= \Big(\frac{1}{|I|} \int_I \|u_t\|_{p,B}^q\,dt \Big)^{1/q}, & \|u\|_{p,q,Q,\nu} &:= \Big(\frac{1}{|I|} \int_I \|u_t\|_{p,B,\nu}^q\,dt \Big)^{1/q}, \\
\|u\|_{p,\infty,Q} &:= \operatorname{ess\,sup}_{t\in I} \|u_t\|_{p,B}, & \|u\|_{p,\infty,Q,\nu} &:= \operatorname{ess\,sup}_{t\in I} \|u_t\|_{p,B,\nu}.
\end{aligned}$$

Finally for $f : \mathbb{R}^d \to \mathbb{R}^m$, let

$$\|f\|_\infty := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} |f(x)|\,.$$

All of the results herein will be quenched, in that they hold for $\mathbb{P}$-a.e. instance of the environment ω unless stated otherwise. Regarding the structure of the chapter, Section 4.2 is devoted to the proof of the upper off-diagonal heat kernel estimate Theorem 4.1.3. The lower estimate, Theorem 4.1.6, is then proven in Section 4.3. Finally, Section 4.4 concerns the proof of the Green's function scaling limit.

4.2 Davies' Method

Throughout this section assume $d \geq 2$, Assumption 4.1.1 holds and let $p,\ q,\ r \in (1,\infty]$ satisfy $\frac{1}{r} + \frac{1}{q} + \frac{1}{p-1}\frac{r-1}{r} < \frac{2}{d}$. One important space we will work with is $\mathcal{F}_G^\theta$ which, for open $G \subseteq \mathbb{R}^d$, is the closure of $C_0^\infty(G)$ in $L^2(G, \theta^\omega dx)$ with respect to $\mathcal{E}^\omega + (\cdot\,,\cdot)_\theta$. We write $\mathcal{F}^\theta$ in the case $G = \mathbb{R}^d$ and if $\theta^\omega \equiv 1$ also we simply write $\mathcal{F}$. Define $\mathcal{F}_{\text{loc}}^\theta$ by $u \in \mathcal{F}_{\text{loc}}^\theta$ if for all balls $B \subset \mathbb{R}^d$ there exists $u_B \in \mathcal{F}_B^\theta$ such that $u = u_B$ $\mathbb{P}$-a.s. In the case $\theta^\omega \equiv 1$ this space is denoted $\mathcal{F}_{\text{loc}}$. Following [CD15], we define the parabolic equation that the heat kernel formally satisfies.

Definition 4.2.1 (Caloric function)**.** Let $I \subseteq \mathbb{R}$ and $G \subseteq \mathbb{R}^d$ be open sets. A function $u : I \to \mathcal{F}_G^\theta$ is caloric if the map $t \mapsto (u(t,\cdot),\phi)_\theta$ is differentiable for any $\phi \in L^2(G,\theta^\omega dx)$ and

$$\frac{d}{dt}(u_t,\phi)_\theta + \mathcal{E}^\omega(u_t,\phi) = 0, \tag{4.15}$$

for all $\phi \in \mathcal{F}_G^\theta$.

4.2.1 Maximal Inequality for the Perturbed Cauchy Equation

The first step in applying Davies' method is to establish a bound on solutions to the following Cauchy problem.

Lemma 4.2.2 (Cauchy Problem)**.** *Let u be caloric on $\mathbb{R} \times \mathbb{R}^d$ and $u(0,\cdot) = f(\cdot)$ for some $f \in L^2(\mathbb{R}^d,\theta^\omega dx)$. Let $\psi \in W_{loc}^{1,\infty}(\mathbb{R}^d)$ satisfy $\|\psi\|_\infty < \infty$ and*

$$h^\omega(\psi)^2 := \operatorname{ess\,sup}_{x\in\mathbb{R}^d} \frac{(\nabla\psi \cdot a^\omega \nabla\psi)(x)}{\theta^\omega(x)} < \infty.$$

Then writing $v(t,x) := \mathrm{e}^{\psi(x)}u(t,x)$, we have for all $t > 0$,

$$\|v_t\|_{2,\theta}^2 \le \mathrm{e}^{h^\omega(\psi)^2 t}\|\mathrm{e}^\psi f\|_{2,\theta}^2.$$

Proof. Formally, $v_t = v(t,\cdot)$ solves the caloric equation

$$\frac{d}{dt}(v_t,\phi)_\theta + J^\omega(v_t,\phi) = 0, \tag{4.16}$$

for suitable test functions ϕ, where the operator

$$J^\omega(v,\phi) = \int_{\mathbb{R}^d} (a^\omega\nabla v)\cdot\nabla\phi + \phi(a^\omega\nabla v)\cdot\nabla\psi - v(a^\omega\nabla\psi)\cdot\nabla\phi - v\phi(a^\omega\nabla\psi)\cdot\nabla\psi\,dx.$$

More precisely, for $t > 0$, $u_t = u(t,\cdot) \in \mathcal{F}^\theta$ by Definition 4.2.1 and the supposed properties of ψ guarantee that $\mathrm{e}^{2\psi}u_t \in \mathcal{F}^\theta$ also. Therefore, setting $\phi = \mathrm{e}^{2\psi}u_t$ in (4.15) and rearranging, we have

$$\frac{d}{dt}(v_t,v_t)_\theta + J^\omega(v_t,v_t) = 0.$$

Since a^ω is symmetric,

$$\begin{aligned} J^\omega(v_t,v_t) &= \int_{\mathbb{R}^d} (a^\omega\nabla v_t)\cdot\nabla v_t - v_t^2(a^\omega\nabla\psi)\cdot\nabla\psi\,dx \\ &\ge -\int_{\mathbb{R}^d} v_t^2(a^\omega\nabla\psi)\cdot\nabla\psi\,dx \\ &\ge -h^\omega(\psi)^2\,\|v_t\|_{2,\theta}^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt}\left\|v_t\right\|_{2,\theta}^2 \leq h^\omega(\psi)^2\left\|v_t\right\|_{2,\theta}^2,$$

from which the result follows. $\square$

We now establish an energy estimate which we will go on to apply iteratively in order to derive a maximal inequality for v.

Lemma 4.2.3. *Let $I = (t_1, t_2) \subseteq \mathbb{R}_+$ and $B \subseteq \mathbb{R}^d$ be any Euclidean ball, $Q := I \times B$. Let u be a locally bounded positive caloric function on $I \times B$, and $v(t,x) := \mathrm{e}^{\psi(x)}u(t,x)$ for $\psi \in W^{1,\infty}_{loc}(\mathbb{R}^d)$ with $\left\|\psi\right\|_\infty < \infty$ and $h^\omega(\psi)^2 < \infty$. Define cut-off functions $\eta \in C_0^\infty(B)$ such that $0 \leq \eta \leq 1$ and $\xi : \mathbb{R} \to [0,1]$ with $\xi \equiv 0$ on $(-\infty, t_1]$. Then, there exists $c_5 > 1$ such that for any $\alpha \geq 1$,*

$$\begin{aligned}\frac{1}{|I|}\left\|\xi(\eta v^\alpha)^2\right\|_{1,\infty,Q,\theta} &+ \frac{1}{|I|}\int_I \xi(t)\frac{\mathcal{E}^\omega(\eta v_t^\alpha, \eta v_t^\alpha)}{|B|}\,dt \\ &\leq c_5\left(\left\|\nabla\eta\right\|_\infty^2\left\|\Lambda^\omega/\theta^\omega\right\|_{p,B,\theta}\left\|v^{2\alpha}\right\|_{p_*,1,Q,\theta} + \left(\alpha^2\, h^\omega(\psi)^2 + \left\|\xi'\right\|_\infty\right)\left\|(\eta v^\alpha)^2\right\|_{1,1,Q,\theta}\right).\end{aligned}$$

Proof. One can show using (4.16) and the same argument as [CD15, Lemma B.3] that

$$\frac{d}{dt}(v_t^{2\alpha}, \eta^2)_\theta + 2\alpha\, J^\omega(v_t, \eta^2 v_t^{2\alpha-1}) \leq 0, \quad \forall\, t \geq 0. \tag{4.17}$$

Then,

$$\begin{aligned}\alpha\, J^\omega(v_t, \eta^2 v_t^{2\alpha-1}) = \int_{\mathbb{R}^d} &\alpha\,(a^\omega\nabla v_t)\cdot\nabla(\eta^2 v_t^{2\alpha-1}) + \alpha\,\eta^2 v_t^{2\alpha-1}(a^\omega\nabla v_t)\cdot\nabla\psi \\ &- \alpha\, v_t(a^\omega\nabla\psi)\cdot\nabla(\eta^2 v_t^{2\alpha-1}) - \alpha\,\eta^2 v_t^{2\alpha}(a^\omega\nabla\psi)\cdot\nabla\psi\, dx.\end{aligned}$$

We label these integrands $J_1, \dots, J_4$ in order.

$$J_1 = \alpha\,(a^\omega\nabla v_t)\cdot\nabla(\eta^2 v_t^{2\alpha-1}).$$

By algebraic manipulation,

$$\begin{aligned}J_1 &= \frac{2\alpha-1}{\alpha}(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - \frac{2(\alpha-1)}{\alpha}v_t^\alpha(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla\eta - \frac{1}{\alpha}v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta \\ &= 2\big((a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - v_t^\alpha(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla\eta\big) - \frac{1}{\alpha}K.\end{aligned}$$

Where

$$K := (a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla(\eta v_t^\alpha) - 2\, v_t^\alpha(a^\omega\nabla(\eta v_t^\alpha))\cdot\nabla\eta + v_t^{2\alpha}(a^\omega\nabla\eta)\cdot\nabla\eta \geq 0.$$

Then since $\alpha \geq 1$,

$$J_1 \geq 2\big((a^\omega \nabla(\eta v_t^\alpha)) \cdot \nabla(\eta v_t^\alpha) - v_t^\alpha (a^\omega \nabla(\eta v_t^\alpha)) \cdot \nabla \eta\big) - K$$
$$= (a^\omega \nabla(\eta v_t^\alpha)) \cdot \nabla(\eta v_t^\alpha) - v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta. \tag{4.18}$$

Similarly, by rearranging and applying Young's inequality,

$$|J_2| \leq \frac{1}{8}(a^\omega \nabla(\eta v_t^\alpha)) \cdot \nabla(\eta v_t^\alpha) + 3\eta v_t^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi + v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta. \tag{4.19}$$

$$|J_3| \leq \frac{1}{8}(a^\omega \nabla(\eta v_t^\alpha)) \cdot \nabla(\eta v_t^\alpha) + 8\alpha^2 \eta^2 v^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi$$
$$+ \eta^2 v_t^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi + v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta. \tag{4.20}$$

Substituting the above estimates into (4.17),

$$\frac{d}{dt}\big\|\eta^2 v_t^{2\alpha}\big\|_{1,\theta} \leq \int_{\mathbb{R}^d} 2\, v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta - \frac{5}{4}(a^\omega \nabla(\eta\, v_t^\alpha)) \cdot \nabla(\eta\, v_t^\alpha)\, dx$$
$$+ (9\alpha^2 + 4) \int_{\mathbb{R}^d} \eta^2 v_t^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi \, dx. \tag{4.21}$$

Therefore,

$$\frac{d}{dt}\big\|\eta^2 v_t^{2\alpha}\big\|_{1,B,\theta} + \frac{\mathcal{E}^\omega(\eta v_t^\alpha, \eta v_t^\alpha)}{|B|}$$
$$\leq \frac{1}{|B|}\Big(2\int_{\mathbb{R}^d} v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta \, dx + (9\alpha^2 + 4)\int_{\mathbb{R}^d} \eta^2 v_t^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi \, dx\Big).$$

We can then bound these terms as follows

$$\int_{\mathbb{R}^d} v_t^{2\alpha}(a^\omega \nabla \eta) \cdot \nabla \eta \, dx \leq \big\|\nabla \eta\big\|_\infty^2 \int_B v_t^{2\alpha} \Lambda^\omega \, dx \leq \big\|\nabla \eta\big\|_\infty^2 |B| \, \big\|v_t^{2\alpha}\big\|_{1,B,\Lambda}$$
$$\leq \big\|\nabla \eta\big\|_\infty^2 |B| \, \big\|\Lambda^\omega / \theta^\omega\big\|_{p,B,\theta} \big\|v_t^{2\alpha}\big\|_{p_*,B,\theta}.$$
$$\int_{\mathbb{R}^d} \eta^2 v_t^{2\alpha}(a^\omega \nabla \psi) \cdot \nabla \psi \, dx \leq h^\omega(\psi)^2 \int_{\mathbb{R}^d} \eta^2 v_t^{2\alpha} \theta^\omega \, dx$$
$$= h^\omega(\psi)^2 |B| \, \big\|\eta^2 v_t^{2\alpha}\big\|_{1,B,\theta},$$

where we used Hölder's inequality on the first term. So,

$$\frac{d}{dt}\big\|\eta^2 v_t^{2\alpha}\big\|_{1,B,\theta} + \frac{\mathcal{E}^\omega(\eta v_t^\alpha, \eta v_t^\alpha)}{|B|}$$
$$\leq 2\, \big\|\nabla \eta\big\|_\infty^2 \big\|\Lambda^\omega / \theta^\omega\big\|_{p,B,\theta} \big\|v_t^{2\alpha}\big\|_{p_*,B,\theta} + (9\alpha^2 + 4) h^\omega(\psi)^2 \big\|\eta^2 v_t^{2\alpha}\big\|_{1,B,\theta}. \tag{4.22}$$

Now let $t \in (t_1, t_2)$, multiply the above by $\xi(s)$ and integrate from $s = t_1$ to $s = t$,

$$
\begin{aligned}
\frac{1}{|I|}\Big(\xi(t)\big\|\eta^2 v_t^{2\alpha}\big\|_{1,B,\theta} + \int_{t_1}^{t} \frac{\mathcal{E}^\omega(\eta v_s^\alpha, \eta v_s^\alpha)}{|B|}\, ds\Big) \\
\leq 2\big\|\nabla\eta\big\|_\infty^2 \big\|\Lambda^\omega/\theta^\omega\big\|_{p,B,\theta}\big\|v^{2\alpha}\big\|_{p_*,1,I\times B,\theta} + (9\alpha^2+4)h^\omega(\psi)^2\big\|\eta^2 v^{2\alpha}\big\|_{1,1,I\times B,\theta} \\
+ \sup_{s\in I}\big|\xi'(s)\big|\,\big\|\eta^2 v^{2\alpha}\big\|_{1,1,I\times B,\theta}.
\end{aligned} \tag{4.23}
$$

Note that the final term on the right-hand side appears by integration by parts with the first term on the left-hand side. Finally, take supremum over $t \in I$ on the left-hand side. $\square$

The following Sobolev inequality is another component in deriving the maximal inequality in Proposition 4.2.5.

Lemma 4.2.4 (Sobolev Inequality). *Let $B \subseteq \mathbb{R}^d$ be a Euclidean ball and $\eta \in C_0^\infty(B)$ a cut-off function. Then there exists $c_6(d,q) > 0$ such that for all $u \in \mathcal{F}^\theta_{loc} \cup \mathcal{F}_{loc}$,*

$$
\big\|\eta^2 u^2\big\|_{\rho/r_*,B,\theta} \leq c_6\,|B|^{\frac{2}{d}}\,\big\|(\lambda^\omega)^{-1}\big\|_{q,B}\big\|\theta^\omega\big\|_{r,B}^{r_*/\rho}\frac{\mathcal{E}^\omega(\eta u, \eta u)}{|B|}, \tag{4.24}
$$

where $\rho := qd/(q(d-2)+d)$.

Proof. Firstly, by Hölder's inequality,

$$
\big\|\eta^2 u^2\big\|_{\rho/r_*,B,\theta} \leq \big\|\theta^\omega\big\|_{r,B}^{r_*/\rho}\big\|\eta^2 u^2\big\|_{\rho,B}. \tag{4.25}
$$

Also by [CD15, Proposition 2.3],

$$
\big\|\eta^2 u^2\big\|_{\rho} \leq \big\|\mathbb{1}_B(\lambda^\omega)^{-1}\big\|_q \mathcal{E}^\omega(\eta u, \eta u).
$$

After averaging over B this yields

$$
\big\|\eta^2 u^2\big\|_{\rho,B} \leq c\,|B|^{2/d}\,\big\|(\lambda^\omega)^{-1}\big\|_{q,B}\frac{\mathcal{E}^\omega(\eta u, \eta u)}{|B|}, \tag{4.26}
$$

for some $c = c(d,q) > 0$. The result then follows from (4.25) and (4.26). $\square$

We now derive the maximal inequality for v using Moser iteration. For $x_0 \in \mathbb{R}^d$, $\delta \in (0,1]$ and $n \in \mathbb{R}_+$ we denote a space-time cylinder $Q_\delta(n) := [0, \delta n^2] \times B(x_0, n)$. Furthermore, for $\sigma \in (0,1]$ and $\epsilon \in (0,1]$, let $s' = \epsilon\delta n^2$, $s'' = (1-\epsilon)\delta n^2$ and define

$$
Q_{\delta,\sigma}(n) := [(1-\sigma)s', (1-\sigma)s'' + \sigma\delta n^2] \times B(x_0, \sigma n). \tag{4.27}
$$

Proposition 4.2.5. *Let $x_0 \in \mathbb{R}^d$, $\delta \in (0,1]$, $\epsilon \in (0,1/4)$, $1/2 \le \sigma' < \sigma \le 1$ and $n \in [1,\infty)$. Let v be as in Lemma 4.2.3. Then there exist constants $c_7(d,p,q,r)$ and $\kappa(d,p,q,r)$ such that*

$$\max_{(t,x)\in Q_{\delta,1/2}(n)} v(t,x) \le c_7 \left((1+\delta n^2 h^\omega(\psi)^2) \frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2} \right)^{\frac{\kappa}{p_*}} \|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}. \tag{4.28}$$

In the above, $\mathcal{A}^\omega(n) := \left\|1 \vee (\Lambda^\omega/\theta^\omega)\right\|_{p,B(x_0,n),\theta} \left\|1 \vee (\lambda^\omega)^{-1}\right\|_{q,B(x_0,n)} \left\|1 \vee \theta^\omega\right\|_{r,B(x_0,n)}$.

Proof. Define $\alpha := 1 + \frac{1}{p_*} - \frac{r_*}{\rho} > 1$ and write $\alpha_k := \alpha^k$ for $k \in \mathbb{N}$. Let $\sigma_k := \sigma' + 2^{-k}(\sigma - \sigma')$ and $\tau_k := 2^{-k-1}(\sigma-\sigma')$. Also introduce shorthand $I_k = [(1-\sigma_k)s', (1-\sigma_k)s'' + \sigma_k \delta n^2]$, $B_k := B(x_0, \sigma_k n)$ and $Q_k = I_k \times B_k = Q_{\delta,\sigma_k}(n)$. Note that $|I_k|/|I_{k+1}| \le 2$ and $|B_k|/|B_{k+1}| \le c\,2^d$. We begin by applying Hölder's and Young's inequalities,

$$\left\|v^{2\alpha_k}\right\|_{\alpha p_*,\alpha,Q_{k+1},\theta} \le \left\|v^{2\alpha_k}\right\|_{1,\infty,Q_{k+1},\theta} + \left\|v^{2\alpha_k}\right\|_{\rho/r_*,1,Q_{k+1},\theta}, \tag{4.29}$$

with ρ as in Lemma 4.2.4. Now let $k \in \mathbb{N}$ and define a sequence of cut-off functions in space, $\eta_k : \mathbb{R}^d \to [0,1]$ such that $\operatorname{supp} \eta_k \subseteq B_k$, $\eta_k \equiv 1$ on B_{k+1} and $\left\|\nabla \eta_k\right\|_\infty \le \frac{2}{\tau_k n}$. Similarly, let $\xi_k : \mathbb{R} \to [0,1]$ be time cut-offs such that $\xi_k \equiv 1$ on I_{k+1}, $\xi_k \equiv 0$ on $(-\infty, (1-\sigma_k)s']$ and $\left\|\xi'\right\|_\infty \le \frac{2}{\tau_k \delta n^2}$. Then by (4.29),

$$\left\|v^{2\alpha_k}\right\|_{\alpha p_*,\alpha,Q_{k+1},\theta} \le c \left(\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta} + \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta} \right). \tag{4.30}$$

We will bound both terms on the right-hand side. By the Sobolev inequality (4.24),

$$\begin{aligned} &\left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta} \\ &\qquad \le c\,n^2 \left\|(\lambda^\omega)^{-1}\right\|_{q,B_k} \left\|\theta^\omega\right\|_{r,B_k}^{r_*/\rho} \frac{1}{|I_k|} \int_{I_k} \xi_k(t) \frac{\mathcal{E}^\omega(\eta_k v_t^{\alpha_k}, \eta_k v_t^{\alpha_k})}{|B_k|}\,dt. \end{aligned} \tag{4.31}$$

So,

$$\begin{aligned} \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta} + \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{\rho/r_*,1,Q_k,\theta} &\le \frac{c\,n^2}{|I_k|} \left\|\xi_k(\eta_k v^{\alpha_k})^2\right\|_{1,\infty,Q_k,\theta} \\ &\quad + \frac{c\,n^2}{|I_k|} \left\|(\lambda^\omega)^{-1}\right\|_{q,B_k} \left\|\theta^\omega\right\|_{r,B_k}^{r_*/\rho} \int_{I_k} \xi_k(t) \frac{\mathcal{E}^\omega(\eta_k v_t^{\alpha_k}, \eta_k v_t^{\alpha_k})}{|B_k|}\,dt. \end{aligned} \tag{4.32}$$

By Lemma 4.2.3 and Hölder's inequality,

$$(4.32) \le c\,\alpha_k^2\, \mathcal{A}^\omega(n) \Big(\frac{1}{\delta \tau_k^2} + n^2 h^\omega(\psi)^2 \Big) \left\|v^{2\alpha_k}\right\|_{p_*,1,Q_k,\theta}.$$

Returning to (4.29),

$$\begin{aligned}\|v\|_{2\alpha_{k+1}p_*,2\alpha_{k+1},Q_{k+1},\theta} &= \|v^{2\alpha_k}\|^{1/(2\alpha_k)}_{\alpha p_*,\alpha,Q_{k+1},\theta} \\ &\leq \Big(c\,\alpha_k^2\,2^{2k}\frac{(1+\delta n^2 h^\omega(\psi)^2)}{\delta(\sigma-\sigma')^2}\mathcal{A}^\omega(n)\Big)^{\frac{1}{2\alpha_k}}\|v\|_{2\alpha_k p_*,2\alpha_k,Q_k,\theta}.\end{aligned}$$

Iterating the above, for any $K \in \mathbb{Z}_+$,

$$\|v\|_{2\alpha_K p_*,2\alpha_K,Q_K,\theta} \leq c\prod_{k=0}^{K-1}\Big(\alpha_k^2\,2^{2k}\frac{(1+\delta n^2 h^\omega(\psi)^2)}{\delta(\sigma-\sigma')^2}\mathcal{A}^\omega(n)\Big)^{\frac{1}{2\alpha_k}}\|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}.$$

Letting $K\to\infty$, observe that $Q_K \downarrow Q_{\delta,\frac{1}{2}}(n)$ and $\prod_{k=0}^{K-1}(\alpha_k^2\,2^{2k})^{\frac{1}{2\alpha_k}}$ is uniformly bounded in K, so we have that

$$\max_{(t,x)\in Q_{\delta,\frac{1}{2}}(n)} v(t,x) \leq c\Big((1+\delta n^2 h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2}\Big)^{\frac{\kappa}{p_*}}\|v\|_{2p_*,2,Q_{\delta,\sigma}(n),\theta}, \tag{4.33}$$

where $\kappa := \frac{p_*}{2}\sum_{k=0}^{\infty}\frac{1}{\alpha_k} < \infty$. □

Corollary 4.2.6. *In the same setting as Proposition 4.2.5, there exists $c_8(d,p,q,r) > 0$ such that*

$$\max_{(t,x)\in Q_{\delta,\frac{1}{2}}(n)} v(t,x) \leq c_8\Big((1+\delta n^2 h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon(\sigma-\sigma')^2}\Big)^{\kappa}\|v\|_{2,\infty,Q_\delta(n),\theta}. \tag{4.34}$$

Proof. This is derived from Proposition 4.2.5, in a similar fashion to [DSC96, Theorem 2.2.3]. □

4.2.2 Heat Kernel Bound

We first conglomerate the two results of the preceding section – the Cauchy problem estimate and the maximal inequality.

Proposition 4.2.7. *In the same setting as Proposition 4.2.5, there exists $c_9(d,p,q,r,\epsilon) > 0$ such that*

$$\max_{(t,x)\in Q_{\delta,\frac{1}{2}}(n)} v(t,x) \leq \frac{c_9}{n^{d/2}}\Big(\frac{\mathcal{A}^\omega(n)}{\epsilon\delta}\Big)^{\kappa}\mathrm{e}^{2(1-\epsilon)h^\omega(\psi)^2\delta n^2}\|\mathrm{e}^\psi f\|_{2,\theta}. \tag{4.35}$$

Proof. By combining Corollary 4.2.6 with Lemma 4.2.2, we obtain

$$\max_{(t,x)\in Q_{\delta,\frac{1}{2}}(n)} v(t,x) \leq \frac{c}{n^{d/2}}\Big((1+\delta n^2 h^\omega(\psi)^2)\frac{\mathcal{A}^\omega(n)}{\epsilon\delta}\Big)^{\kappa}\mathrm{e}^{h^\omega(\psi)^2\delta n^2/2}\|\mathrm{e}^\psi f\|_{2,\theta}.$$

The result follows since for any $\epsilon \in (0, 1/2)$ there exists $c(\epsilon) < \infty$ such that

$$(1 + \delta n^2 h^\omega(\psi)^2)^\kappa \leq c(\epsilon)\, \mathrm{e}^{(1-2\epsilon)h^\omega(\psi)^2 \delta n^2},$$

for all $n \geq 1$, $\delta \in (0, 1]$. $\square$

Proposition 4.2.8 (Heat Kernel Bound). *Suppose $M_1(p, q, r) < \infty$ and let $x_0 \in \mathbb{R}^d$. Then $\mathbb{P}$-a.s. there exist $c_{10}(d, p, q, r)$, $\gamma(d, p, q, r) > 0$ such that for all $\sqrt{t} \geq N_2^\omega(x_0)$ and x, $y \in \mathbb{R}^d$,*

$$p_\theta^\omega(t, x, y) \leq c_{10}\, t^{-\frac{d}{2}} \Big(1 + \frac{d(x_0, x)}{\sqrt{t}}\Big)^\gamma \Big(1 + \frac{d(x_0, y)}{\sqrt{t}}\Big)^\gamma \mathrm{e}^{2h^\omega(\psi)^2 t - \psi(x) + \psi(y)}. \tag{4.36}$$

Proof. Fix $\epsilon = \frac{1}{8}$. By the ergodic theorem there exists $N_2^\omega(x_0) > 0$ such that

$$\mathcal{A}^\omega(n) \leq c\, \big(1 + \mathbb{E}[\Lambda^\omega(0)^p \theta^\omega(0)^{1-p}]\big)\big(1 + \mathbb{E}[\lambda^\omega(0)^{-q}]\big)\big(1 + \mathbb{E}[\theta^\omega(0)^r]\big) =: \bar{A} < \infty,$$

for all $n \geq N_2^\omega(x_0)$. For given $x \in \mathbb{R}^d$ and $\sqrt{t} > N_2^\omega(x_0)$, we choose δ, n such that $(t, x) \in Q_{\delta, \frac{1}{2}}(n)$, for example by setting $n = 2d(x_0, x) + \sqrt{8t/7}$ and $\delta := 8t/(7n^2)$. Then considering the caloric function $u(t, x) := P_t f(x)$ for $f \in \mathcal{F}^\theta$, by Proposition 4.2.7,

$$\begin{aligned} \mathrm{e}^{\psi(x)} u(t, x) &\leq c\, n^{-d/2} (n^2/t)^\kappa \mathrm{e}^{2h^\omega(\psi)^2 t} \big\|\mathrm{e}^\psi f\big\|_{2,\theta} \\ &\leq c\, n^\gamma t^{-\kappa} \mathrm{e}^{2h^\omega(\psi)^2 t} \big\|\mathrm{e}^\psi f\big\|_{2,\theta}, \end{aligned} \tag{4.37}$$

for some $c = c(\epsilon, d, p, q, r, \bar{A})$, where $\gamma := 2\kappa - \frac{d}{2}$. Write $r(t) := c\, t^{-\kappa} \mathrm{e}^{2h^\omega(\psi)^2 t}$ and $b_t(x) := \big(2d(x_0, x) + \sqrt{8t/7}\big)^\gamma$. Since the above holds for all $x \in \mathbb{R}^d$ and $\sqrt{t} > N_2^\omega(x_0)$, we have

$$\mathrm{e}^{\psi(x)} P_t f(x) \leq b_t(x) r(t) \big\|\mathrm{e}^\psi f\big\|_{2,\theta}.$$

That is,

$$\big\|b_t^{-1} \mathrm{e}^\psi P_t f\big\|_\infty \leq r(t) \big\|\mathrm{e}^\psi f\big\|_{2,\theta}. \tag{4.38}$$

Now define an operator $P_t^\psi(g) := \mathrm{e}^\psi P_t(\mathrm{e}^{-\psi} g)$ for $\mathrm{e}^{-\psi} g \in \mathcal{F}^\theta$. Then we can bound the operator norm

$$\big\|b_t^{-1} P_t^\psi(\mathrm{e}^\psi f)\big\|_{L^2(\mathbb{R}^d, \theta^\omega dx) \to L^\infty} \leq r(t).$$

The above also holds with ψ replaced by $-\psi$. Since the dual of P_t^ψ is $P_t^{-\psi}$, the dual of $b_t^{-1} P_t^{-\psi}(\cdot)$ is $P_t^\psi(b_t^{-1} \cdot)$. So by duality,

$$\big\|P_t^\psi(b_t^{-1} g)\big\|_{2,\theta} \leq r(t) \big\|g\big\|_{1,\theta}. \tag{4.39}$$

Since $b_{\frac{t}{2}}(x) \le b_t(x)$, we have

$$\begin{aligned} \big\|b_t^{-1}\mathrm{e}^{\psi} P_t f\big\|_\infty &\le \big\|b_{\frac{t}{2}}^{-1}\mathrm{e}^{\psi} P_{\frac{t}{2}} P_{\frac{t}{2}} f\big\|_\infty \\ &\le r(t/2)\, \big\|\mathrm{e}^{\psi} P_{\frac{t}{2}} f\big\|_{2,\theta} \quad \text{by (4.38)}, \\ &\le r(t/2)^2\, \big\|\mathrm{e}^{\psi} b_{\frac{t}{2}} f\big\|_{1,\theta} \quad \text{by (4.39)}. \end{aligned} \tag{4.40}$$

That is, for all $x \in \mathbb{R}^d$ and $\sqrt{t} \ge N_2^\omega(x_0)$, we have

$$P_t f(x) \le \frac{c}{t^{2\kappa}} \mathrm{e}^{2h^\omega(\psi)^2 t - \psi(x)} (d(x_0,x) + \sqrt{t})^\gamma \int_{\mathbb{R}^d} (d(x_0,y) + \sqrt{t})^\gamma \mathrm{e}^{\psi(y)} \big|f(y)\big|\, \theta^\omega(y)\, dy.$$

It is standard that the above implies the heat kernel estimate (4.36) for almost all $x,\, y \in \mathbb{R}^d$. Furthermore, local boundedness in Assumption 4.1.1 allows us to pass to all $x,\, y \in \mathbb{R}^d$. □

4.2.3 Properties of the Intrinsic Metric

In order to prove the off-diagonal estimate in Theorem 4.1.3 from Proposition 4.2.8, we aim to set the function $\psi(\cdot) = \beta\, d_\theta^\omega(x,\cdot)$ in (4.36), then optimise over the constant β. This requires checking that this function ψ satisfies the necessary regularity assumptions for the proofs in Section 4.2.1. Recall that the intrinsic metric is defined as follows,

$$d_\theta^\omega(x,y) := \sup\Big\{\phi(y) - \phi(x) \,:\, \phi \in C^1(\mathbb{R}^d),\, h^\omega(\phi)^2 = \sup_{z\in\mathbb{R}^d} \frac{(\nabla\phi \cdot a^\omega \nabla\phi)(z)}{\theta^\omega(z)} \le 1\Big\}.$$

In deriving the required regularity of d_θ^ω, we first show that it is equal to D_θ^ω, the Riemannian distance computed with respect to $(\frac{a^\omega}{\theta^\omega})^{-1}$. This Riemannian metric is defined via the following path relation. Consider the following Hilbert space

$$\mathcal{H} := \big\{f \in C\big([0,\infty),\mathbb{R}^d\big) \,:\, f(0) = 0,\, \dot{f} \in L^2\big([0,\infty),\mathbb{R}^d\big)\big\},$$

where $\dot{f}$ denotes the weak derivative of f, together with the following norm

$$\big\|f\big\|_{\mathcal{H}} := \big\|\dot{f}\big\|_{L^2([0,\infty),\mathbb{R}^d)}.$$

Given $f \in \mathcal{H}$, define $\Phi(t,x;f) : [0,\infty)\times\mathbb{R}^d \to \mathbb{R}^d$ via

$$\frac{d}{dt}\Phi(t,x;f) = \Big(\frac{a^\omega(\Phi(t,x;f))}{\theta^\omega(\Phi(t,x;f))}\Big)^{1/2} \dot{f}(t),$$

with initial condition $\Phi(0, x; f) = x$. The Riemannian distance is then given by

$$D_\theta^\omega(x, y) := t^{1/2} \inf \Big\{ \|f\|_{\mathcal{H}} \,:\, f \in \mathcal{H},\, \Phi(t, x; f) = y \Big\},$$

for any $t > 0$.

Lemma 4.2.9 (Riemannian Distance Representation). *For all $x, y \in \mathbb{R}^d$, $d_\theta^\omega(x, y) = D_\theta^\omega(x, y)$.*

Proof. This follows by the proof of [Str88, Lemma I.1.24]. □

Next we will apply the additional Assumption 4.1.2 on the environment to derive the regularity we require of d_θ^ω. Our objective is to pass a function resembling $\rho_x(\cdot) := d_\theta^\omega(x, \cdot)$ into (4.36). In order to do this we must show some conditions such as $\rho_x \in W^{1,\infty}_{\text{loc}}(\mathbb{R}^d)$ and $h^\omega(\rho_x)^2 \le 1$. The requisite property is that the metric d_θ^ω is strictly local i.e. that d_θ^ω induces the original topology on $\mathbb{R}^d$. For further discussion of the properties of such intrinsic metrics and the distance function ρ_x see [Sto10], [BdMLS09, Appendix A] and [Stu95]. In the following proposition, we invoke a recent result from geometric analysis to directly deduce strict locality of the intrinsic metric d_θ^ω under Assumption 4.1.2.

Proposition 4.2.10. *If Assumption 4.1.2 holds then the intrinsic metric d_θ^ω is strictly local for $\mathbb{P}$-a.e. ω.*

Proof. Given Assumption 4.1.2 and Lemma 4.2.9, this follows directly from Proposition 4.1ii) or Theorem 4.5 in [Bur15], noting that the Euclidean metric corresponds to the Riemannian metric given by the identity matrix [Sto10, Proposition 3.3]. □

4.2.4 Upper Off-Diagonal Estimate

Having proven the necessary regularity of the intrinsic metric in the preceding subsection, we are now in a position to optimise over the test function in Proposition 4.2.8 and derive the upper off-diagonal estimate.

Proof of Theorem 4.1.3. As a corollary to Proposition 4.2.10, we have for example by [Stu95, Lemma 1] that for any $x \in \mathbb{R}^d$, $\rho_x \in C(\mathbb{R}^d) \cap L^2_{\text{loc}}(\mathbb{R}^d, \theta)$ and $h^\omega(\rho_x)^2 \le 1$ almost surely. Furthermore ρ_x has a weak derivative and [Sto10, Theorem 5.1] implies that $\operatorname{ess\,sup}_{z \in \mathbb{R}^d} |\nabla \rho_x(z)| < \infty$. The final property to check is that our test function is essentially bounded, whilst ρ_x may be unbounded we can take a bounded version with the desired properties. In accordance with [BdMLS09, Eqn. (2)], consider $\eta_x = \xi \circ \rho_x$ for a continuously differentiable cut-off function ξ to construct a function such that $\eta_x(x) = d_\theta^\omega(x, x) = 0$, $\eta_x(y) = d_\theta^\omega(x, y)$, η_x is essentially bounded and η_x satisfies the aforementioned properties, including $h^\omega(\eta_x)^2 \leq 1$. This is another consequence of Proposition 4.2.10. Therefore we are justified in setting $\psi(\cdot) = -\beta\,\eta_x(\cdot)$ in (4.36) for $\beta \in \mathbb{R}$, and $h^\omega(\psi)^2 \leq \beta^2$. Then by choosing the constant $\beta = d_\theta^\omega(x,y)/(4t)$ and setting $x_0 = x$ in (4.36) we have for $\mathbb{P}$-a.e. ω, all $x,\, y \in \mathbb{R}^d$ and $\sqrt{t} \geq N_2^\omega(x)$,

$$p_\theta^\omega(t,x,y) \leq c\, t^{-\frac{d}{2}} \Big(1 + \frac{d(x,y)}{\sqrt{t}}\Big)^\gamma \exp\Big(- \frac{d_\theta^\omega(x,y)^2}{8t}\Big), \tag{4.41}$$

which completes the proof. □

4.3 Lower Off-Diagonal Estimate

The starting point for proving the lower off-diagonal estimate of Theorem 4.1.6 is the following near-diagonal estimate. Throughout this section suppose Assumptions 4.1.1 and 4.1.5 hold. Also let $p,\, q \in (1,\infty)$ satisfy $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$.

Proposition 4.3.1. *Let $t > 0$ and $x \in \mathbb{R}^d$, then for all $y \in B\big(x, \frac{\sqrt{t}}{2}\big)$ we have*

$$p_\Lambda^\omega(t,x,y) \geq \frac{t^{-d/2}}{C_{PH}\big(\left\|\Lambda^\omega\right\|_{p,B(x,\sqrt{t})}, \left\|\lambda^\omega\right\|_{q,B(x,\sqrt{t})}\big)}. \tag{4.42}$$

The constant C_{PH} is given explicitly by

$$C_{PH} = c_{11} \exp\Big(c_{12} \Big(\big(1 \vee \left\|\Lambda^\omega\right\|_{p,B(x,\sqrt{t})}\big) \big(1 \vee \left\|\lambda^\omega\right\|_{q,B(x,\sqrt{t})}\big) \Big)^\kappa \Big), \tag{4.43}$$

for $c_i(d,p,q),\ \kappa(d,p,q) > 0$.

Proof. A parabolic Harnack inequality with constant C_{PH} is established in [CD15, Theorem 3.9] and this is a standard consequence of it, see for instance [ADS16a, Proposition 4.7] or [Del99, Proposition 3.1]. □

The chaining method is to apply Proposition 4.3.1 along a sequence of balls. Let $x \in \mathbb{R}^d$, a radius $0 < r \leq 4\,d(0,x)$ and $k \in \mathbb{N}$ satisfying $\frac{12 d(0,x)}{r} \leq k \leq \frac{16 d(0,x)}{r}$. Consider the sequence of points $x_j = \frac{j}{k} x$ for $j = 0,\dots,k$ that interpolates between 0 and x. Let $B_{x_j} = B(x_j, \frac{r}{48})$ and $s := \frac{r\, d(0,x)}{k}$, noting $\frac{r^2}{16} \leq s \leq \frac{r^2}{12}$.

To apply estimate (4.42) along a sequence we will need to control the ergodic average terms in (4.43) simultaneously for balls with different centre-points. To this end we establish a moment bound in Proposition 4.3.3 which employs finite range dependence to get better control than in the general ergodic setting. First, a prerequisite lemma.

Lemma 4.3.2. *For any $k > 2$ and independent random variables $Y_1, ..., Y_n \in L^k(\mathbb{P})$ with $\mathbb{E}[Y_i] = 0$ for all i, there exists $c_{13}(k) > 0$ such that*

$$\mathbb{E}\Big[\big|\sum_{i=1}^{n} Y_i\big|^k\Big] \leq c_{13}\,\max\Big\{\sum_{i=1}^{n}\mathbb{E}\big[|Y_i|^k\big],\ \Big(\sum_{i=1}^{n}\mathbb{E}\big[|Y_i|^2\big]\Big)^{\frac{k}{2}}\Big\}. \tag{4.44}$$

Proof. This follows from [Ros70, Theorem 3]. □

For $u \in \mathbb{R}^d$, $p,\ q > 0$ we write $\Delta\Lambda_p^\omega(u) := \Lambda^\omega(u)^p - \mathbb{E}[\Lambda^\omega(0)^p]$ and $\Delta\lambda_q^\omega(u) := \lambda^\omega(u)^q - \mathbb{E}[\lambda^\omega(0)^q]$ for the deviation of these moments from their respective means.

Proposition 4.3.3. *Let $\xi > 1$ and assume $M_2(2\xi p, 2\xi q) < \infty$. Let $R \subset \mathbb{R}^d$ be any region which can be covered by a disjoint partition of K balls of radius $\mathcal{R}$ in the maximum norm, i.e. $R \subset \bigcup_{i=1}^K \{z_i + [0,\mathcal{R}]^d\}$ for some $z_1, \ldots, z_K \in \mathbb{R}^d$. There exists $c_{14}(d,\mathcal{R},\xi) > 0$ such that*

$$\mathbb{E}\Big[\big|\int_R \Lambda^\omega(u)^p - \mathbb{E}[\Lambda^\omega(0)^p]\,du\big|^{2\xi}\Big] \leq c_{14}\,K^\xi, \tag{4.45}$$

$$\mathbb{E}\Big[\big|\int_R \lambda^\omega(u)^q - \mathbb{E}[\lambda^\omega(0)^q]\,du\big|^{2\xi}\Big] \leq c_{14}\,K^\xi. \tag{4.46}$$

Proof. We prove the statement only for Λ^ω, since the one for λ^ω is analogous. Denote $f(u) := \Delta\Lambda_p^\omega(u)\mathbb{1}_{u\in R}$. Then by Jensen's inequality and Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\Big[\big|\int_R \Delta\Lambda_p^\omega(u)\,du\big|^{2\xi}\Big] &= \mathbb{E}\Big[\big|\int_{[0,\mathcal{R}]^d}\sum_{i=1}^K f(z_i+u)\,du\big|^{2\xi}\Big] \\
&\leq \mathcal{R}^{d(2\xi-1)}\mathbb{E}\Big[\int_{[0,\mathcal{R}]^d}\big|\sum_{i=1}^K f(z_i+u)\big|^{2\xi}\,du\Big] \\
&= c\int_{[0,\mathcal{R}]^d}\mathbb{E}\Big[\big|\sum_{i=1}^K f(z_i+u)\big|^{2\xi}\Big]\,du.
\end{aligned} \tag{4.47}$$

For fixed $u \in [0,\mathcal{R}]^d$ the sequence $\big(f(z_i+u)\big)_{i=1}^K$ has mean zero and is independent by Assumption 4.1.5. So we have by Lemma 4.3.2 and shift-invariance of the environment,

$$\begin{aligned}
\mathbb{E}\Big[\big|\sum_{i=1}^K f(z_i+u)\big|^{2\xi}\Big] &\leq c_{13}\,\max\Big\{\sum_{i=1}^K\mathbb{E}\big[|f(z_i+u)|^{2\xi}\big],\ \big(\sum_{i=1}^K\mathbb{E}\big[|f(z_i+u)|^2\big]\big)^\xi\Big\} \\
&\leq c\,K^\xi.
\end{aligned} \tag{4.48}$$

Combining (4.47) and (4.48) gives the result. □

Proposition 4.3.4. *Let $\xi > d$ and assume $M_2(2\xi p, 2\xi q) < \infty$. For $\mathbb{P}$-a.e. ω, there exists $N_4(\omega) \in \mathbb{N}$ such that for all $r > 0$ and $x \in \mathbb{R}^d$ with $N_4(\omega) < r \leq 4\,d(0,x)$, for any sequence $y_0, \ldots, y_k$ where $y_0 = 0$, $y_k = x$ and $y_j \in B_{x_j}$ for $1 \leq j \leq k-1$, we have $c_{15}(d,\mathcal{R},\xi) > 0$*

such that

$$\sum_{j=0}^{k-1}\Big(1\vee\big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}\Big)\Big(1\vee\big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}\Big)\le c_{15}\,k. \tag{4.49}$$

Furthermore, we have the following estimate on $N_4(\omega)$, there exists $c_{16}(d,\mathcal{R},\xi)>0$ such that

$$\mathbb{P}(N_4(\omega)>n)\le c_{16}\,n^{2-d(\xi-1)}\qquad \forall\, n\in\mathbb{N}. \tag{4.50}$$

Proof. Let x and r be as in the statement and denote $z=\lfloor x\rfloor\in\mathbb{Z}^d$, $r_0=\lceil r\rceil\in\mathbb{Z}$. We will work with these discrete approximations of the variables x and r in order to apply countable union bounds and the Borel-Cantelli lemma. Note that $x\in C_z:=z+[0,1]^d$ and $r\in I_{r_0}:=[r_0-1,r_0]$. Assuming w.l.o.g. that $r>1$ and $d(0,x)>d$ we have $r\simeq r_0$ and $|x|\simeq|z|$. We define a region that covers the union of balls of interest

$$\bigcup_{j=0}^{k}B_{y_j}\subset R_{z,r_0}:=\Big\{\tau z+\big[-2r_0,2r_0\big]^d\,:\,\tau\in\big[0,2\big]\Big\}.$$

This region has volume $|R_{z,r_0}|\le c\,r_0^{d-1}|z|\le c\,r^dk$ and can be covered by at most $K\le c\,r_0^{d-1}|z|\,/\mathcal{R}^d$ non-intersecting balls of radius $\mathcal{R}$ in the maximal norm. Also there exists $c_{17}(d)$ such that for all $w\in\mathbb{R}^d$, $|\{j\in\{0,\dots,k\}\,:\,w\in B_{y_j}\}|\le c_{17}$, therefore

$$\begin{aligned}\sum_{j=0}^{k-1}\big\|\Lambda^\omega\big\|^p_{p,B(y_j,\sqrt{s})}&\le c_{17}\,r^{-d}\int_{\bigcup_{j=0}^k B_{y_j}}\Lambda^\omega(u)^p\,du\le c\,r^{-d}\int_{R_{z,r_0}}\Lambda^\omega(u)^p\,du\\&\le c\,r^{-d}\big|R_{z,r_0}\big|\,\mathbb{E}\big[\Lambda^\omega(0)^p\big]+c\,r^{-d}\int_{R_{z,r_0}}\Delta\Lambda_p^\omega(u)\,du\\&\le c\,k+c\,r^{-d}\int_{R_{z,r_0}}\Delta\Lambda_p^\omega(u)\,du.\end{aligned}\tag{4.51}$$

By Markov's inequality and Proposition 4.3.3 we have

$$\begin{aligned}\mathbb{P}\Big(\int_{R_{z,r_0}}\Delta\Lambda_p^\omega(u)\,du>kr^d\Big)&\le\mathbb{P}\Big(\big|\int_{R_{z,r_0}}\Delta\Lambda_p^\omega(u)\,du\big|>c|z|\,r_0^{d-1}\Big)\\&\le c\,\mathbb{E}\Big[\big|\int_{R_{z,r_0}}\Delta\Lambda_p^\omega(u)\,du\big|^{2\xi}\Big]/\big(|z|\,r_0^{d-1}\big)^{2\xi}\\&\le c\,\big(|z|\,r_0^{d-1}\big)^{-\xi}.\end{aligned}\tag{4.52}$$

Now let ρ, $l\in\mathbb{N}$ with $\rho\le l$. By (4.52) and a union bound, summing over $\{z\in\mathbb{Z}^d\,:$

$|z| = l\}$ and $r_0 \geq \rho$,

$$\mathbb{P}\Big(\exists\, z \in \mathbb{Z}^d,\ r_0 \in \mathbb{N} : |z| = l, r_0 \in [\rho, 4|z|], \int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\, du > kr^d\Big) \\ \leq c\, l^{d-1-\xi} \rho^{-\xi(d-1)+1}. \quad (4.53)$$

Now consider the event

$$E_\rho := \Big\{\exists\, z \in \mathbb{Z}^d,\ r_0 \in \mathbb{N} : |z| \geq \rho, r_0 \in [\rho, 4|z|], \int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\, du > kr^d\Big\}.$$

Since $\xi > d$, we can take a countable union bound over l in (4.53) to obtain

$$\mathbb{P}\big(E_\rho\big) \leq c\, \rho^{d(1-\xi)+1}. \quad (4.54)$$

Also $d(1-\xi)+1 < -1$ so by the Borel-Cantelli lemma there exists $\tilde{N}(\omega) \in \mathbb{N}$ such that for all $z \in \mathbb{Z}^d$, $r_0 \in \mathbb{N}$ with $\tilde{N}(\omega) < r_0 < 4|z|$ we have

$$\int_{R_{z,r_0}} \Delta\Lambda_p^\omega(u)\, du \leq kr^d.$$

Together with (4.51), this implies the existence of $N_4(\omega) \in \mathbb{N}$ such that for all $x \in \mathbb{R}^d$ and $r > 1$ with $N_4(\omega) < r \leq 4\, d(0, x)$ we have for $y_0, \dots, y_k$ defined as in the statement,

$$\sum_{j=0}^{k-1} \big\|\Lambda^\omega\big\|_{p, B(y_j, \sqrt{s})}^p \leq c\, k. \quad (4.55)$$

By the exact same reasoning, one can show the corresponding inequality for λ^ω. Moreover by Hölder's inequality,

$$\sum_{j=0}^{k-1} \Big(1 \vee \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}\Big)\Big(1 \vee \big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}\Big) \\ \leq k^{1-\frac{1}{p}-\frac{1}{q}} \Big(\sum_{j=0}^{k-1} \big(1 \vee \big\|\Lambda^\omega\big\|_{p,B(y_j,\sqrt{s})}^p\big)\Big)^{\frac{1}{p}} \Big(\sum_{j=0}^{k-1} \big(1 \vee \big\|\lambda^\omega\big\|_{q,B(y_j,\sqrt{s})}^q\big)\Big)^{\frac{1}{q}}. \quad (4.56)$$

This together with (4.55) and the equivalent bound for λ^ω gives the result. The stated decay of $N_4(\omega)$ follows by taking a union bound over ρ in (4.54). □

Corollary 4.3.5. *Let $\xi > d$ and assume $M_2(2\xi\kappa p, 2\xi\kappa q) < \infty$. In the same setting as Proposition 4.3.4 there exists $N_5(\omega) \in \mathbb{N}$ with decay as in* (4.50) *and* $c_{18}(d, p, q, \mathcal{R}, \xi) > 0$

such that $\mathbb{P}$*-a.s. for all* $r > 0$*,* $x \in \mathbb{R}^d$ *with* $N_5(\omega) < r \le 4\, d(0,x)$ *we have*

$$\sum_{j=0}^{k-1} \Big(1 \vee \|\Lambda^\omega\|_{p,B(y_j,\sqrt{s})}\Big)^\kappa \Big(1 \vee \|\lambda^\omega\|_{q,B(y_j,\sqrt{s})}\Big)^\kappa \le c_{18}\, k. \tag{4.57}$$

Proof. By Jensen's inequality $\|\Lambda^\omega\|^\kappa_{p,B(y_j,\sqrt{s})} \le \|(\Lambda^\omega)^\kappa\|_{p,B(y_j,\sqrt{s})}$ and similarly for the λ^ω terms. Then proceed as for Proposition 4.3.4 to prove the result, with Λ^ω replaced by $(\Lambda^\omega)^\kappa$. □

Proof of Theorem 4.1.6. By shift-invariance of the environment it suffices to prove the estimate for $p^\omega_\Lambda(t,0,x)$. Fix $\xi > d$ and for the moment assumption $M_2(p_0,q_0) < \infty$ choose $p_0 = 2\xi\kappa p$, $q_0 = 2\xi\kappa q$, in order to apply Corollary 4.3.5. Let $N_3^\omega(0) := N_1^\omega(0)^2 \vee N_4^\omega \vee N_5^\omega$ and assume as in the statement that $t \ge N_3^\omega(0)\big(1 \vee d(0,x)\big)$. We split the proof into two cases.

Firstly in the case $|x|^2/t < 1/4$ we have $x \in B\big(0,\sqrt{t}/2\big)$ so we may apply the near-diagonal lower estimate of Proposition 4.3.1,

$$p^\omega_\Lambda(t,0,x) \ge \frac{t^{-d/2}}{C_{\mathrm{PH}}\big(\|\Lambda^\omega\|_{p,B(0,\sqrt{t})}, \|\lambda^\omega\|_{q,B(0,\sqrt{t})}\big)}.$$

Since $\sqrt{t} \ge N_1^\omega(0)$, recalling the form of C_{PH} we apply the ergodic theorem to bound

$$C_{\mathrm{PH}}\big(\|\Lambda^\omega\|_{p,B(0,\sqrt{t})}, \|\lambda^\omega\|_{q,B(0,\sqrt{t})}\big) \le c_{11} \exp\Big(c\,\big((1 \vee \bar{\Lambda}_p)(1 \vee \bar{\lambda}_q)\big)^\kappa\Big). \tag{4.58}$$

Therefore,

$$p^\omega_\Lambda(t,0,x) \ge c\, t^{-d/2}.$$

Secondly, consider the case $|x|^2/t \ge 1/4$. Since Λ^ω and λ^ω are locally bounded, it follows from the semigroup property that for any $0 < \tau < t$,

$$p^\omega_\Lambda(t,0,x) = \int_{\mathbb{R}^d} p^\omega_\Lambda(\tau,0,u)\, p^\omega_\Lambda(t-\tau,u,x)\,\Lambda^\omega(u)\, du. \tag{4.59}$$

We will employ the chaining argument over the sequence of balls introduced below Proposition 4.3.1, set $r = t/|x| \ge N_3^\omega(0)$ which gives $s = t/k$. Iterating the above relation $k-1$ times gives

$$p^\omega_\Lambda(t,0,x) \ge \int_{B_{x_1}} \cdots \int_{B_{x_{k-1}}} p^\omega_\Lambda(s,0,y_1) \dots p^\omega_\Lambda(s,y_{k-1},x)\,\Lambda^\omega(y_1)\dots\Lambda^\omega(y_{k-1})\, dy_1 \dots dy_{k-1}.$$

We have by Proposition 4.3.1, for all $y_j \in B_{x_j}$,

$$\prod_{j=0}^{k-1} p^\omega_\Lambda(s, y_j, y_{j+1}) \geq \frac{c\, s^{-dk/2}}{\exp\Big(c \sum_{j=0}^{k-1} \Big(\big(1 \vee \left\|\Lambda^\omega\right\|_{p,B(y_j,\sqrt{s})}\big)\big(1 \vee \left\|\lambda^\omega\right\|_{q,B(y_j,\sqrt{s})}\big)\Big)^\kappa\Big)}$$
$$\geq \frac{c\, s^{-dk/2}}{\exp(c\, k)}, \tag{4.60}$$

where the second step is due to Corollary 4.3.5. Therefore,

$$p^\omega_\Lambda(t, 0, x) \geq \frac{c\, s^{-dk/2} \prod_{j=1}^{k-1} |B_{x_j}| \left\|\Lambda^\omega\right\|_{1,B_{x_j}}}{\exp(c\, k)}$$
$$\geq \frac{c\, r^{-dk} r^{d(k-1)} \prod_{j=1}^{k-1} \left\|\Lambda^\omega\right\|_{1,B_{x_j}}}{c^k}. \tag{4.61}$$

To bound the remaining stochastic term in the numerator we apply the harmonic-geometric mean inequality,

$$\Big(\prod_{j=1}^{k-1} \left\|\Lambda^\omega\right\|_{1,B_{x_j}}\Big)^{\frac{1}{k-1}} \geq \frac{k-1}{\sum_{j=1}^{k-1} \left\|\Lambda^\omega\right\|^{-1}_{1,B_{x_j}}} \geq \frac{c(k-1)}{\sum_{j=1}^{k-1} \left\|\lambda^\omega\right\|_{1,B_{x_j}}}. \tag{4.62}$$

Since $r > N^\omega_4$, it follows from Proposition 4.3.4 with the choice $y_j = x_j$, that $\sum_{j=1}^{k-1} \left\|\lambda^\omega\right\|_{1,B_{x_j}} \leq c\, k$. Therefore,

$$\prod_{j=1}^{k-1} \left\|\Lambda^\omega\right\|_{1,B_{x_j}} \geq c^k. \tag{4.63}$$

Combining (4.61) and (4.63) gives for some $c_{19} > 0$, $c_{20} \in (0, 1)$,

$$p^\omega_\Lambda(t, 0, x) \geq c_{19}\, r^{-d}\, c_{20}^k. \tag{4.64}$$

Finally, since $\frac{|x|^2}{t} \geq \frac{1}{4}$ we have $r \leq 2\, t^{1/2}$. Also $k \simeq \frac{|x|}{r} = \frac{|x|^2}{t}$ so we arrive at

$$p^\omega_\Lambda(t, 0, x) \geq c_2\, t^{-\frac{d}{2}} \exp\Big(-\frac{c_3\, d(0, x)^2}{t}\Big), \tag{4.65}$$

which completes the proof. □

4.4 Green's Function Scaling Limit

We shall now prove the Green's function scaling limit in Theorem 4.1.8. The strategy is to apply the local limit theorem [CD15, Theorem 1.1], then control remainder terms using the off-diagonal estimate of Theorem 4.1.3 and the long range bound established below in Proposition 4.4.2. Throughout this section suppose Assumption 4.1.1 holds and let $d \geq 3$ so that the Green's function exists. Also, let $p,\, q \in (1, \infty]$ satisfying $\frac{1}{p} + \frac{1}{q} < \frac{2}{d}$.

Herein, since the Green's function is independent of the choice of speed measure, we specify the case $\theta^\omega \equiv \Lambda^\omega$. This choice is analogous to the constant speed random walk in the random conductance model setting, for which the jump rate is constant and independent of its position. The benefit of this choice of speed measure is that the intrinsic metric may be bounded in terms of the Euclidean metric, leading to particularly amenable off-diagonal bounds. By the definition of Λ^ω, the intrinsic metric satisfies

$$\begin{aligned} d^\omega_\Lambda(x,y) &:= \sup\Big\{\phi(y) - \phi(x) \,:\, \phi \in C^1(\mathbb{R}^d),\, h^\omega(\phi)^2 = \sup_{z\in\mathbb{R}^d} \frac{(\nabla\phi \cdot a^\omega \nabla \phi)(z)}{\Lambda^\omega(z)} \leq 1\Big\} \\ &\geq \sup\Big\{\phi(y) - \phi(x) \,:\, \phi \in C^1(\mathbb{R}^d),\, \big\|\nabla\phi\big\|_\infty \leq 1\Big\} = d(x,y). \end{aligned} \tag{4.66}$$

The final equivalence here is due to the fact that the Euclidean metric is the Riemannian metric corresponding to the identity matrix.

Corollary 4.4.1. *Suppose $M_2(p,q) < \infty$. For $\mathbb{P}$-a.e. ω, there exist $N^\omega_6(x) > 0$ and $c_{21}(d,p,q)$, $c_{22}(d,p,q) > 0$ such that for all x, $y \in \mathbb{R}^d$, $\sqrt{t} > N^\omega_6(x)$,*

$$p^\omega_\Lambda(t,x,y) \leq c_{21}\, t^{-\frac{d}{2}} \exp\Big(-c_{22}\frac{d(x,y)^2}{t}\Big). \tag{4.67}$$

Proof. This follows by the exact reasoning of Theorem 4.1.3, noting that $h^\omega(\phi)^2 \leq \big\|\nabla\phi\big\|^2_\infty$. Since the Euclidean metric is trivially strictly local, the justification involving Assumption 4.1.2 is no longer required. □

Whilst the above off-diagonal estimate provides optimal bounds on the heat kernel for large enough time t, it is clear that to control the convergence in (4.13) we also require a bound on the rescaled heat kernel that holds for small $t > 0$. We obtain this from the following long range bound, derived in a similar fashion to results in the graph setting such as [Dav93, Theorem 10]. Interestingly, we obtain stronger decay in the present diffusions context than for the aforementioned random walks on graphs [Dav93], [Pan93], where a logarithm appears in the exponent. See also [ADS16b, Theorem 1.6(ii)] for the degenerate environment.

Proposition 4.4.2. *Suppose $M_2(p,q) < \infty$. For $\mathbb{P}$-a.e. ω, there exist $c_{23} > 0$ and $N_7(\omega) > 0$ such that for all $n \geq N_7(\omega)$, $t \geq 0$ and $x \in \mathbb{R}^d$ with $|x| \leq 2$ we have*

$$p^\omega_\Lambda(t,0,nx) \leq c_{23}\, n^d \exp\Big(-\frac{n^2|x|^2}{2\,t}\Big). \tag{4.68}$$

Proof. Firstly note that by Lemma 4.2.2, for any $f \in L^2(\mathbb{R}^d, \theta^\omega dx)$ and suitable ψ,

$$\big\|\mathrm{e}^{\psi} P_t f\big\|_{2,\Lambda}^2 \leq \mathrm{e}^{h^\omega(\psi)^2 t} \big\|\mathrm{e}^{\psi} f\big\|_{2,\Lambda}^2. \tag{4.69}$$

By the local boundedness in Assumption 4.1.1, this implies the pointwise estimate

$$\mathrm{e}^{2\,\psi(x)} p_\Lambda^\omega(t,x,y)^2 \Lambda^\omega(y)^2 \Lambda^\omega(x) \leq \mathrm{e}^{h^\omega(\psi)^2 t + 2\,\psi(y)} \Lambda^\omega(y), \tag{4.70}$$

for all $t \geq 0$ and $x,\, y \in \mathbb{R}^d$. Rearranging,

$$\begin{aligned} p_\Lambda^\omega(t,x,y) &\leq \Lambda^\omega(y)^{-1/2} \Lambda^\omega(x)^{-1/2} \exp\big(h^\omega(\psi)^2 t/2 + \psi(y) - \psi(x)\big) \\ &\leq \Lambda^\omega(y)^{-1/2} \Lambda^\omega(x)^{-1/2} \exp\big(\big\|\nabla\psi\big\|_\infty^2 t/2 + \psi(y) - \psi(x)\big). \end{aligned}$$

Arguing as in Section 4.2.4 but with the Euclidean metric gives

$$p_\Lambda^\omega(t,x,y) \leq \Lambda^\omega(y)^{-1/2} \Lambda^\omega(x)^{-1/2} \exp\Big(-\frac{d(x,y)^2}{2\,t}\Big).$$

Now set $x = 0$ and re-label $y = nx$ with $|x| \leq 2$,

$$p_\Lambda^\omega(t,0,nx) \leq \Lambda^\omega(0)^{-1/2} \Lambda^\omega(nx)^{-1/2} \exp\Big(-\frac{n^2 |x|^2}{2\,t}\Big). \tag{4.71}$$

Appealing to ergodicity of the environment, by Assumption 4.1.1 and the moment condition, $\mathbb{P}$-a.s. there exists $N_7(\omega) > 0$ such that for all $n \geq N_7(\omega)$,

$$\begin{aligned} \Lambda^\omega(nx)^{-1} \leq \lambda^\omega(nx)^{-1} &\leq \int_{B(0,2n)} \lambda^\omega(u)^{-1}\, du \\ &\leq c\, n^d \big\|1/\lambda^\omega\big\|_{1,B(0,2n)} \leq c\, n^d \big\|1/\lambda^\omega\big\|_{q,B(0,2n)} \quad \text{(Jensen's inequality)} \\ &\leq c\, n^d\, \mathbb{E}\big[\lambda^\omega(0)^{-q}\big]^{1/q}. \end{aligned} \tag{4.72}$$

Similarly,

$$\Lambda^\omega(0)^{-1} \leq c\, n^d\, \mathbb{E}\big[\lambda^\omega(0)^{-q}\big]^{1/q} \quad \text{for all } n \geq N_7(\omega). \tag{4.73}$$

Substituting (4.72) and (4.73) into (4.71) gives the result. □

Finally, we prove the Green's function scaling limit.

Proof of Theorem 4.1.8. By shift-invariance of the environment it suffices to prove the result for $x_0 = 0$. For simplicity we set $r_1 = 1$, $r_2 = 2$, and in a slight abuse of notation we write $k_t^\Sigma(x) = k_t^\Sigma(0,x)$. For $1 \le |x| \le 2$, $T_1,\ T_2 > 0$ and $n > 0$ we have

$$\begin{aligned}
\Big|n^{d-2} g^\omega(0,nx) - a\, g_{BM}(0,x)\Big| &= \Big|n^d \int_0^\infty p_\Lambda^\omega(n^2t,0,nx)\,dt - a\int_0^\infty k_t^\Sigma(x)\,dt\Big| \\
\le n^d \int_0^{T_1} p_\Lambda^\omega(n^2t,0,nx)\,dt &+ a\int_0^{T_1} k_t^\Sigma(x)\,dt + \int_{T_1}^{T_2} \big|n^d p_\Lambda^\omega(n^2t,0,nx) - a\,k_t^\Sigma(x)\big|\,dt \\
&+ n^d \int_{T_2}^\infty p_\Lambda^\omega(n^2t,0,nx)\,dt + a\int_{T_2}^\infty k_t^\Sigma(x)\,dt. \qquad (4.74)
\end{aligned}$$

In controlling these terms we first employ the main result of this chapter; the off-diagonal estimate in Corollary 4.4.1 gives

$$n^d \int_{T_2}^\infty p_\Lambda^\omega(n^2t,0,nx)\,dt \le c_{21}\int_{T_2}^\infty t^{-d/2}\mathrm{e}^{-c_{22}/t}\,dt, \qquad (4.75)$$

provided $n > \sqrt{N_6^\omega(0)/T_2}$. Similarly, for the Gaussian heat kernel there exists $c > 0$ such that for all $t \ge 0$ and $1 \le |x| \le 2$,

$$k_t^\Sigma(x) \le c\,t^{-d/2}\mathrm{e}^{-c/t}. \qquad (4.76)$$

For the first term in (4.74) we apply both the off-diagonal estimate and the long range bound of Proposition 4.4.2. Provided $n > N_7(\omega) \vee \big(N_6^\omega(0)/\sqrt{T_1}\big)$, we have

$$\begin{aligned}
n^d \int_0^{T_1} p_\Lambda^\omega(n^2t,0,nx)\,dt &\le n^d \int_0^{\frac{N_6^\omega(0)^2}{n^2}} p_\Lambda^\omega(n^2t,0,nx)\,dt + n^d \int_{\frac{N_6^\omega(0)^2}{n^2}}^{T_1} p_\Lambda^\omega(n^2t,0,nx)\,dt \\
&\le n^d \int_0^{\frac{N_6^\omega(0)^2}{n^2}} c\,n^d \mathrm{e}^{-\frac{1}{2t}}\,dt + n^d \int_{\frac{N_6^\omega(0)^2}{n^2}}^{T_1} c\,n^{-d} t^{-\frac{d}{2}}\mathrm{e}^{-\frac{c}{t}}\,dt \quad \text{(by (4.68) and (4.67) resp.)} \\
&\le c\,N_6^\omega(0)^2 n^{2d-2}\exp\big(-n^2/\big(2N_6^\omega(0)^2\big)\big) + c\int_0^{T_1} t^{-\frac{d}{2}}\mathrm{e}^{-\frac{c}{t}}\,dt. \qquad (4.77)
\end{aligned}$$

Let $\epsilon > 0$. Combining the above we have that for suitably large n,

$$\begin{aligned}
\Big|n^{d-2} g^\omega(0,nx) - a\, g_{BM}(0,x)\Big| &\le c\,(1+a)\Big(\int_0^{T_1} t^{-d/2}\mathrm{e}^{-c/t}\,dt + \int_{T_2}^\infty t^{-d/2}\mathrm{e}^{-c/t}\,dt \\
&+ N_6^\omega(0)^2 n^{2d-2}\exp\big(-n^2/\big(2N_6^\omega(0)^2\big)\big) + \int_{T_1}^{T_2} \big|n^d p_\Lambda^\omega(n^2t,0,nx) - a\,k_t^\Sigma(x)\big|\,dt\Big).
\end{aligned}$$

Now, $t^{-d/2}\mathrm{e}^{-c/t}$ is integrable on $(0,\infty)$ so we may fix T_1, T_2 such that

$$\int_0^{T_1} t^{-d/2}\mathrm{e}^{-c_2/t}\,dt + \int_{T_2}^\infty t^{-d/2}\mathrm{e}^{-c_2/t}\,dt < \epsilon.$$

For large enough n,

$$N_6^\omega(0)^2 n^{2d-2} \exp\big(-n^2/\big(2N_6^\omega(0)^2\big)\big) < \epsilon.$$

Furthermore, by the local limit theorem [CD15, Theorem 1.1],

$$\int_{T_1}^{T_2} \big|n^d p_\Lambda^\omega(n^2 t, 0, nx) - a\, k_t^\Sigma(x)\big|\, dt < \epsilon,$$

for large enough n, uniformly over $x \in A$. This gives the claim. □

4.5 Future Directions

Some ideas for extensions of the developments in this chapter are the following:

(i) Proving off-diagonal heat kernel estimates for operators of a more general type than solely symmetric and divergence-form. For example, one could aim to extend to the class of operators in [Osa83], whilst relaxing the uniform ellipticity and smoothness assumptions therein; or a degenerate version of the non-symmetric operator considered in [FK97]. One would perhaps first have to extend the work of [CD16] to construct the diffusion process, since this relies on the theory of symmetric Dirichlet forms.

(ii) Extending to time-dependent environments. There are some results covering time-dependent, bounded coefficients in [LOY98, Rho07, Rho08]. Furthermore, a theory of stochastic calculus for time-dependent Dirichlet forms has been established by Oshima, [Ō04]. However, in the degenerate case, one would likely face the same obstacles as for the dynamic RCM, namely how to define the intrinsic metric for a time-dependent environment.

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