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Orthogonal root numbers of tempered parameters

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Abstract
We show that an orthogonal root number of a tempered L-parameter φ decomposes as the product of two other numbers: the orthogonal root number of the principal parameter and the value on a central involution of Langlands's central character for φ. The formula resolves a conjecture of Gross and Reeder and computes root numbers of Weil–Deligne representations arising in a conjectural description of the Plancherel measure.

Contents

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1 Introduction

To every representation of the absolute Galois group Γ_k of a local field k, or more generally, of its Weil–Deligne group WD_k, we can attach various local constants, the γ-, L-, and ε-factors, each a function of a complex parameter s. Of these, the ε-factor has the simplest description as a function of s: it is an exponential function, $s \mapsto a \cdot b^{s-1/2}$. Understanding this local constant thus amounts to understanding the base b and constant term a.

The base of the ε-factor (or depending on one's choice of terminology, its logarithm) is known as the *Artin conductor*, taken as 1 for k archimedean. This quantity measures the ramification of the representation. Although the Artin conductor has its own subtleties, especially in the presence of wild ramification, there are good formulas to compute it [1, Chapter VI], formulas that make precise the sense in which the conductor measures ramification.

The constant term a is known as the *root number*. We denote it here by ω, an additive function of Weil–Deligne representations. The choice of exponent $s - 1/2$ ensures that the root number is a complex number of modulus one, so we can refer to it informally as the sign of the ε-factor. This sign is much more subtle than the Artin conductor.

Here is one example of its subtlety. The problem of computing a root number generalizes the problem of computing the sign of a Gauss sum. For quadratic Gauss sums, the sign is determined by a congruence condition on the modulus, a classical result of Gauss and the most difficult of the basic properties of these sums. For cubic Gauss sums, the situation is much more complicated: there is no congruence condition to describe the sign, and in fact, the sign is randomly and uniformly distributed on the unit circle [2].

In other words, for as simple a class of Galois representations as the cubic characters, the root number exhibits great complexity. Fortunately, there is a special class of Galois representations whose root numbers are more amenable to computation: the orthogonal representations. After all, the orthogonal characters are precisely the quadratic characters.

Using formal properties of root numbers, it is easy to see that the root number of an orthogonal representation is a fourth root of unity, and the square of the root number can be described in terms of the determinant of the representation. Ultimately, then, computing an orthogonal root number is a matter of distinguishing between two square roots. Deligne calculated the distinction by expressing the root number of an orthogonal representation in terms of the second Stiefel–Whitney class of the representation [3]. Stiefel–Whitney classes are a notion from algebraic topology, but they can be reinterpreted purely in terms of group cohomology, as the pullback of a certain universal cohomology class. In this way, Deligne's formula reduces computing the root number of an orthogonal representation to a problem in group cohomology.

In the Langlands program we generalize the study of Weil–Deligne representations to the study of L-parameters for a quasi-split reductive k-group G. To extend the definition of the local constants to this setting we reduce to the case of the general linear group by composing an L-parameter $\mathrm{WD}_k \to {}^L G$ with a complex representation $r\colon {}^L G \to \mathrm{GL}(V)$ and computing the local constants of the representation $r \circ \varphi$. Deligne's formula suggests the possibility of computing the root numbers $\omega(\varphi, r) \overset{\text{def}}{=} \omega(r \circ \varphi)$ for orthogonal r, which we call the *orthogonal root numbers* of φ. The hope is a formula for $\omega(\varphi, r)$ that incorporates information about the L-parameter φ and the complex orthogonal representation r.

In a 2010 paper, the direct inspiration for this article, Gross and Reeder gave a conjectural formula for a particular class of orthogonal root numbers and proved the formula when G is split. Motivated in part by a conjecture of Hiraga, Ichino, and Ikeda on the formal degree of discrete series [4, Conjecture 1.4], they took the adjoint representation $\mathrm{Ad}\colon {}^L G \to \mathrm{GL}(\hat{\mathfrak{g}})$, an orthogonal representation, and set about computing $\omega(\varphi, \mathrm{Ad})$. Their conjectural answer has three factors.

First, there is a recipe, due to Langlands, that constructs from the L-parameter φ a character χ_φ of the center of G. At the same time, we can conjecturally assign to φ a finite set Π_φ of smooth irreducible representations of $G(k)$. The set Π_φ is called the *L-packet* of φ and the assignment $\varphi \mapsto \Pi_\varphi$ is called the *local Langlands correspondence*. It is expected that all elements of Π_φ have the same central character, and further, that this character is χ_φ.

Second, Gross and Reeder evaluated this central character on a certain *canonical involution* z_{Ad} in the center of G, defined as the value of the sum of the positive coroots on -1.

Third, when k is nonarchimedean the L-group admits a particular distinguished parameter called the *principal parameter* $\varphi_{\mathrm{prin}}\colon \mathrm{WD}_k \to {}^L G$. This parameter is trivial on the Weil group, and its restriction to the Deligne SL_2 corresponds, via the Jacobson-Morozov theorem, to the sum of elements in a pinning of $\widehat{G}$, a nilpotent element of the Lie algebra. The L-packet of the principal parameter captures the Steinberg representation. When k is archimedean, we define the principal parameter to be the trivial crossed homomorphism $W_k \to \widehat{G}$.

Conjecture 1 [5, Conjecture 8.3] *If k is nonarchimdean of characteristic zero, the center of G is anisotropic, and the parameter $\varphi\colon \mathrm{WD}_k \to {}^L G$ is discrete then*

$$\frac{\omega(\varphi, Ad)}{\omega(\varphi_{prin}, Ad)} = \chi_\varphi(z_{Ad}).$$

The goal of this article is to verify Gross and Reeder's conjecture. Actually, we will prove a more general version that relaxes the base field to an arbitrary local field, the adjoint representation to an arbitrary orthogonal representation, and discreteness of the parameter to temperedness. We generalize Gross and Reeder's canonical involution z_{Ad} to an involution z_r (see Sect. 5.4) that depends only on the restriction of r to $\widehat{G}$.

The original motivation for our generalization was to compute the adjoint ε-factor that conjecturally describes the Plancherel measure [4, Conjecture 1.5], of which the formal degree conjecture that motivated Gross and Reeder is a special case. In the more general conjecture, the adjoint representation of ${}^LG$ is replaced by a certain "relative adjoint representation" of the L-group of a Levi. From this perspective it is natural, and in the end costs little, to relax the adjoint representation to an arbitrary orthogonal representation.

Theorem A *Let r be an orthogonal representation of ${}^LG$ and $\varphi\colon \mathrm{WD}_k \to {}^LG$ a tempered parameter. Then*

$$\frac{\omega(\varphi, r)}{\omega(\varphi_{prin}, r)} = \chi_\varphi(z_r).$$

The theorem does not fully compute the orthogonal root number $\omega(\varphi, r)$: instead, it disentangles the root number into an automorphic factor, $\chi_\varphi(z_r)$, and a Galois factor, $\omega(\varphi_{\mathrm{prin}}, r)$. If one wanted to use Theorem A to pin down an orthogonal root number precisely, it seems that the main challenge would be to compute the orthogonal root number of the principal parameter. In general I expect no better resolution to this problem than the rough answer provided by Clifford theory, though for special classes of representations, such as the adjoint representation [5, Equation (21)], it might be possible to say more.

Although our work is informed by the local Langlands correspondence, everything here takes place on the Galois side. A stronger version of the theorem would assert that $\chi_\varphi(z_r)$ is actually the value on z_r of the central character of the L-packet of φ; this is much more difficult because it requires some knowledge of L-packets. Lapid proved the stronger version of Theorem A for generic irreducible representations of certain classical groups [6].

Gross and Reeder proved Conjecture 1 for split G by an argument in group cohomology. This article is an outgrowth of an observation that their argument can be generalized in various directions.

To relax the split assumption in Gross and Reeder's proof, we generalize, in Lemma B of Sect. 2, the basic lemma from group cohomology that underlies their proof, taking into account the Galois action on $\widehat{G}$. Since the pin extension of a complex orthogonal group is not topologically split, we must use Borel cohomology here instead of continuous (group) cohomology. With this modification, Lemma B proves equal two particular Borel cohomology classes in $\mathrm{H}^2_{\mathrm{Borel}}({}^LG, \{\pm 1\})$. Theorem A then follows from the equation by pullback along the parameter φ, once we properly identify the factors of the pulled-back equation. Most of this article is devoted to identifying these factors.

Two of the factors are root numbers, and their recognition as such passes through Deligne's theorem. Gross and Reeder already reformulated the theorem in the language of group cohomology for determinant-one orthogonal representations of Galois type, and it is mostly a matter of collecting definitions from the algebraic topology literature, in Sect. 3, to extend their reformulation to tempered orthogonal representations.

The third factor of the pulled-back equation is the value of the central character χ_φ on the involution z_r. Identifying this factor requires several detours that are surely well known to experts in the field.

Any spin representation of a complex reductive group gives rise to a character of its topological fundamental group. The first detour, in Sect. 4, is to compute this character. Here we offer a small correction to an exercise of Bourbaki.

The second detour, in Sect. 5, is to compute the second cohomology group of the Weil group with coefficients in the character lattice of a k-torus, generalizing a standard computation for the absolute Galois group. It turns out that this cohomology group is the character group of the Harish–Chandra subgroup of the torus. In Sect. 1 we extend these results, for k nonarchimedean of characteristic zero, to any finite type k-group of multiplicative type, using a generalization of Tate duality due to Karpuk.

Finally, in the brief Sect. 6 we weave together these disparate threads to prove Theorem A, and then explain the connection with the conjectures on the Plancherel measure.

1.1 Notational conventions

Let k be a local field. If k is nonarchimedean, let p denote its residue characteristic. We assume the characteristic of k is odd because Theorem A is trivially satisfied when $+1 = -1$. Let Γ_k be the absolute Galois group of k, let W_k be the Weil group of k, and let

$$\mathrm{WD}_k \stackrel{\text{def}}{=} \begin{cases} W_k & \text{if } k \text{ is archimedean} \\ W_k \times \mathrm{SL}_2(\mathbb{C}) & \text{if not} \end{cases}$$

be the Weil–Deligne group of k. In practice we can immediately forget WD_k and work with W_k because root numbers of orthogonal Weil–Deligne representations are unaffected by restriction to the Weil group [5, Section 2.3]. For k nonarchimedean, let $I_k \subset W_k$ be the inertia subgroup.

We reserve the letter G for a group, of two kinds: either a reductive group or a topological group. When G is a topological group, we assume it to be Hausdorff. When G is a reductive group, we assume it to be connected and quasi-split.

To best align with the statement of the key Lemma B we work with the Weil form of the L-group, ${}^L G \stackrel{\text{def}}{=} \widehat{G} \rtimes W_k$. The choice of the Weil form over the Galois form is not essential because in practice, we can replace the Weil-group factor of this semidirect product with the Galois group of any extension of k that is large enough to contain the splitting field of G and to trivialize the L-parameter relevant to the problem at hand.

For us, a *representation* of the L-group ${}^L G$ is a representation in the sense of Borel's Corvallis article [7, (2.6)], that is, a finite-dimensional complex vector space V together with a continuous homomorphism ${}^L G \to \mathrm{GL}(V)$ whose restriction to $\widehat{G}$ is a morphism of complex algebraic groups. Similarly, a *representation* of a complex reductive group is assumed algebraic.

In general, the ε-factor, and hence the root number, depends not only on a Weil–Deligne representation but also on a nontrivial additive character of k and a Haar measure on k. Since there are formulas that explain the dependence of these factors on the character and the measure, nothing is lost in fixing them. We follow Gross and Reeder in computing ε-factors with respect to the Haar measure that assigns $\mathcal{O}_k$

measure one and an additive character that is trivial on $\mathcal{O}_k$ and nontrivial on the inverse of some uniformizer.

In this article we use three kinds of cohomology: the singular cohomology of a topological space, denoted by $\mathrm{H}^\bullet_{\mathrm{sing}}$; the continuous cohomology of a topological group, denoted by $\mathrm{H}^\bullet$; and the Borel cohomology of a topological group, denoted by $\mathrm{H}^\bullet_{\mathrm{Borel}}$, whose definition we review in Sect. 2.2.

Given any group A of order two, let $\mathrm{sgn}\colon A \to \{\pm 1\}$ denote the canonical isomorphism to the group $\{\pm 1\}$ of order two.

2 Group cohomology

This section collects general facts on group cohomology that inform the rest of this article. In Sect. 2.1, we review the relationship between extensions and cohomology for discrete groups. In Sect. 2.2 we explain how the relationship behaves in the presence of topology. And in Sect. 2.3 we state and prove the main cohomological lemma, Lemma B, underlying the proof of Theorem A. The remaining sections of the article will flesh out the connection between Theorem A and Lemma B.

2.1 Group extensions

In this subsection, we work in the category of discrete groups. An *extension* of the group G by the group A is an exact sequence

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1,$$

permitting one to identify A with a subgroup of E and G with the quotient of G by this subgroup. We call A, E, and G, the first, second, and third terms of the extension, respectively. We always assume that the extension is *abelian*, in other words, that A is an abelian group. In fact, in our application all extensions are *central*, that is, having A as a central subgroup of G. Since A is abelian, the conjugation action of G on E descends to an action of G on A: that is, A is a G-module. Conversely, starting from a G-module A, an extension of G by A is an extension of G by the abelian group A with the property that the G-action on A induced by the extension agrees with the given action.

A *morphism* of extensions is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & A' & \longrightarrow & E' & \longrightarrow & G' & \longrightarrow & 1. \end{array}$$

As for extensions, we can speak of the first, second, and third terms of a morphism, namely, the maps $A \to A'$, $E \to E'$, and $G \to G'$, respectively. The isomorphisms in this category are precisely the morphisms whose terms are all isomorphisms. An easy diagram chase shows that a morphism is an isomorphism as soon as its first and third terms are isomorphisms. For this reason, we often restrict attention to the isomorphisms whose first and third terms are equalities; we call such an isomorphism an *equivalence*. Besides the isomorphisms, there are two kinds of morphisms of interest.

First, given two G-modules A and A' and a G-equivariant homomorphism $\alpha\colon A \to A'$, we can construct the morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \| & & \\ 1 & \longrightarrow & A' & \longrightarrow & \alpha_*(E) & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

in which $\alpha_*(E)$ is the cokernel of the map $A \to A' \rtimes E$ sending a to $\alpha(a)a$. We call the bottom extension (or sometimes, its second term) the *pushout* of the top extension.[1] The pushout morphism is universal in the sense that any morphism with first term α from the top extension to an extension of G by A' factors through the pushout morphism.

Second, given a homomorphism $\gamma\colon G' \to G$, we can construct the morphism of extensions

$$\begin{array}{ccccccccc} 1 & \longrightarrow & A & \longrightarrow & \gamma^*(E) & \longrightarrow & G' & \longrightarrow & 1 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle\gamma} & & \\ 1 & \longrightarrow & A & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 1 \end{array}$$

in which $\gamma^*(E)$ is the pullback (of sets, hence of groups) of the right square. We call the top extension (or sometimes, its second term) the *pullback* of the bottom extension. The pullback morphism is universal in the sense that any morphism with third term γ from an extension of A by G' to the bottom extension factors through the pullback morphism.

The theory of group extensions is relevant to us because of its connection to group cohomology, which relates to root numbers, in turn, by Deligne's theorem.

Given an extension

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1,$$

let $s\colon G \to E$ be a set-theoretic section of $E \to G$. The formula

$$z_s(g, g') = s(g)s(g')s(gg')^{-1}$$

defines a 2-cocyle $z_s \in \mathrm{Z}^2(G, A)$. The cohomology class of the resulting cocycle is independent of the choice of section: any two such sections differ by a function $G \to A$, and the coboundary of this function exhibits a cohomology between the cocycles.

Conversely, given a 2-cocycle $z \in \mathrm{Z}^2(G, A)$, define the group extension

$$1 \longrightarrow A \longrightarrow A \boxtimes_z G \longrightarrow G \longrightarrow 1$$

in which $A \boxtimes_z G = A \times G$ with multiplication

$$(a, g) \cdot (a', g') = (a \cdot {}^{g}a' \cdot z(g, g'), gg').$$

[1] Our pushout is usually not isomorphic to the pushout in the category of groups. The category-theoretic pushout is the amalgamated free product.

The equivalence class of the resulting extension depends only on the cohomology class of the cocycle: any function $G \to A$ whose coboundary exhibits a cohomology between two different cocycles gives rise to an equivalence of the corresponding extensions.

In summary, given a group G and a G-module A, we have constructed a canonical bijection between equivalence classes of extensions of G by A and the cohomology group $\mathrm{H}^2(G, A)$. The bijection is compatible with pushforward and pullback of cohomology classes and extensions.

The dictionary between extensions and cohomology classes nicely answers, or rather, reformulates, a natural question in the theory of group extensions: when does the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] & G \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & A' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

extend to a morphism of extensions? The answer to the question is a special case of our key lemma, Lemma B, and is also used in the proof of the lemma.

To answer the question, use the universal properties of pullback and pushforward to extend the candidate morphism to the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A' \arrow[r] \arrow[d, equal] & \alpha_*(E) \arrow[r] & G \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & A' \arrow[r] \arrow[d, equal] & \gamma^*(E') \arrow[r] \arrow[d] & G \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & A' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

It follows that the original diagram can be extended to a morphism if and only if the middle extensions in the diagram are isomorphic. In other words, letting $c \in \mathrm{H}^2(G, A)$ and $c' \in \mathrm{H}^2(G', A')$ denote the cohomology classes classifying the top and bottom extensions, the original diagram can be extended to a morphism if and only if

$$\alpha_*(c) = \gamma^*(c') \qquad \text{in } \mathrm{H}^2(G, A').$$

2.2 Borel cohomology

A sequence of topological groups

$$1 \longrightarrow A \longrightarrow E \longrightarrow G \longrightarrow 1$$

is a *topological extension* of G by A if $A \to E$ is a closed subgroup and the induced map from the cokernel of $A \to E$ to G, where the source has the quotient topology, is an isomorphism. The pullback and pushout constructions from Sect. 2.1 work just as well in this setting, provided all maps in question are continuous.

Classifying topological extensions is more subtle than classifying extensions of discrete groups, however, because not every continuous surjection of topological spaces admits a continuous section. If we were to carry out the work of Sect. 2.1 in the category of topological groups, where all maps are required to be continuous, we would find that the continuous cohomology $\mathrm{H}^2(G, A)$ classifies extensions of G by A that are *topologically split*, that is, whose second term is a direct product $G \times A$ as a topological space. The collection of topologically split extensions is much too small for most purposes. For instance, the universal-cover group extension of a Lie group is never topologically split, but we need a theory that can see the spin extension of the special orthogonal group because of its great relevance to the study of Stiefel–Whitney classes.

To capture the topological extensions that are not topologically split one must enlarge the continuous cohomology group. The correct enlargement is known as *Borel cohomology*,[2] a variant of discrete group cohomology in which one requires that cochains be Borel measurable. I refer the reader to Stasheff's survey article [8] as well as Moore's papers on the subject [9–12] for more information on Borel cohomology.

Theorem 2 *Let G and A be separable locally compact groups with A abelian. There is a canonical natural isomorphism between $H^2_{Borel}(G, A)$ and the set of equivalence classes of topological extensions of G by A.*

Proof Mackey proved this result in his thesis [13]. The same construction as in Sect. 2.1 yields the bijection, with the following additional argument. Given an extension of G by A, one must find a section whose associated cocycle is Borel measurable. This is Mackey's Théorème 3, and its proof shows that the section need only be Borel-measurable. Conversely, given a Borel-measurable cocycle $z \colon G \times G \to A$, one must show endow the extension $A \boxtimes_z G$ with a locally compact topology making it an extension of topological groups. This is Mackey's Théorème 2. A uniqueness statement in that theorem ensures that the topology on the extension can be recovered from any measurable cocycle classifying it. $\square$

Example 3 Using the fact that $\mathbb{C}^\times$ is connected and $\{\pm 1\}$ is discrete, it is easy to show by direct computation that the continuous cohomology group $\mathrm{H}^2(\mathbb{C}^\times, \{\pm 1\})$ is trivial. On the other hand, since $\mathbb{C}^\times = W_{\mathbb{C}}$, we know [3, (4.2)] that $\mathrm{H}^2_{\mathrm{Borel}}(\mathbb{C}^\times, \{\pm 1\}) \simeq \{\pm 1\}$. The nontrivial cohomology class represents the topologically non-split squaring extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \mathbb{C}^\times \xrightarrow{z \mapsto z^2} \mathbb{C}^\times \longrightarrow 1.$$

Since Borel cohomology is not widely used in the Langlands program, we point out several relevant properties.

First, in the nonarchimedean case, the pullback of a Borel cocycle along an L-parameter is automatically continuous.

Lemma 4 *Let W and G be topological groups, let A be a continuous G-module, and let $\varphi : W \to G$ be a continuous homomorphism. If φ factors through a discrete*

[2] Borel cohomology is named after Émile, not Armand. It is sometimes also called *Moore cohomology*.

quotient of W then pullback by φ induces a map

$$\varphi^* : H^i_{Borel}(G, A) \to H^i(W, A).$$

Proof Pullback along φ takes an arbitrary cocycle $G^n \to A$ to a cocycle $W^n \to A$ that is constant on cosets of an open subgroup, hence continuous. $\square$

Second, in the archimedean case, Borel H^1 works just as well as continuous H^1 in the local Langlands correspondence.

Lemma 5 *Let G and A be separable completely metrizable topological groups, in other words, Polish groups, and let A carry a continuous G-module structure. The natural map*

$$H^1(G, A) \to H^1_{Borel}(G, A)$$

is an isomorphism.

Results of this kind are known as *automatic continuity*.

Proof A theorem of Banach and Pettis [14], nicely explained in Rosendal's overview of automatic continuity [15, Theorem 2.2], proves the result when the action of G on A is trivial, in which case $\mathrm{H}^1(G, A) = \mathrm{Hom}_{\mathrm{cts}}(G, A)$. In general, use their theorem together with the fact that a crossed homomorphism $G \to A$ is the same as a homomorphism $A \rtimes G \to G$ that restricts to the identity on G. $\square$

Remark 6 It would be interesting to see if Clausen and Scholze's condensed mathematics [16] can replace Borel cohomology. We will not pursue this idea here.

2.3 The key lemma

The following lemma is the main tool underpinning the proof of Theorem A. There is nothing essential in its use of Borel cohomology.

Lemma B *Let A, A', E, G, G', and W be topological groups with A and A' abelian. Assume that either all groups are discrete or all groups are separable and locally compact. Let the group W act continuously on the groups G and E by group automorphisms, let $E \to G$ be a continuous W-equivariant homomorphism, and let*

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & E \arrow[r] \arrow[d, "\varepsilon"] & G \arrow[r] \arrow[d, "\gamma"] & 1 \\ 1 \arrow[r] & A' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 \end{tikzcd}$$

be a morphism of topological extensions. Let $f : W \to G'$ be a continuous homomorphism such that the map $\gamma f : G \rtimes W \to G'$ is a homomorphism. Consider the diagram

$$\begin{tikzcd} 1 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & E \rtimes W \arrow[r] & G \rtimes W \arrow[r] \arrow[d, "\gamma f"] & 1 & (c) \\ 1 \arrow[r] & A' \arrow[r] & E' \arrow[r] & G' \arrow[r] & 1 & (c'). \end{tikzcd}$$

with top and bottom extensions classified by the cohomology classes c and c', respectively, and let $p\colon G \rtimes W \to W$ denote the canonical projection. Then

$$(\gamma f)^*(c') = \alpha_*(c) \cdot p^* f^*(c').$$

The assumption on the topologies of the groups involved implies, by Theorem 2 in the locally compact case and the discussion of Sect. 2.1 in the discrete case, that the relevant extensions are classified by Borel cohomology classes.

Proof The proof rests on an understanding of $\mathrm{H}^2_{\mathrm{Borel}}(G \rtimes W, A)$. The semidirect product fits into a split short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & G \arrow[r, "i"] & G \rtimes W \arrow[r, "p"] & W \arrow[l, bend left, dashed, "s"] \arrow[r] & 1 \end{tikzcd}$$

in which $i\colon G \to G \rtimes W$ is the canonical inclusion, $p\colon G \rtimes W \to W$ is the canonical projection, and $s\colon W \to G \rtimes W$ is the canonical inclusion. This sequence dualizes to a split exact sequence of abelian groups

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{H}^2_{\mathrm{Borel}}(W, A) \arrow[r, "p^*"] & \mathrm{H}^2_{\mathrm{Borel}}(G \rtimes W, A) \arrow[l, bend left, dashed, "s^*"] \arrow[r, "i^*"] & \mathrm{H}^2_{\mathrm{Borel}}(G, A). \end{tikzcd}$$

The map p^* realizes $\mathrm{H}^2_{\mathrm{Borel}}(W, A)$ as a direct summand of $\mathrm{H}^2_{\mathrm{Borel}}(G \rtimes W, A)$ with a canonical complement, the kernel of s^*. The map p^*s^* is projection onto the summand, and the map i^* identifies its complement, the kernel of s^*, with a subgroup of $\mathrm{H}^2_{\mathrm{Borel}}(G, A)$. To prove the lemma, it therefore suffices to show that $s^*(\gamma f)^*(c') = f^*(c')$ and that $i^*(\gamma f)^*(c') = i^*\alpha_*(c)$. The first equation follows from the identity $f = (\gamma f) \circ s$. The second equation amounts to showing that $\gamma^*(c') = \alpha_* i^*(c)$, and this is a consequence of the existence of ε. $\square$

Later, in Sect. 5.4, we need the following compatibility condition.

Lemma 7 *In the setting of Lemma B, a Borel-measurable crossed homomorphism $\varphi\colon W \to G$ maps under the coboundary $H^1_{Borel}(W, G) \to H^2_{Borel}(W, A)$ to the pullback of c along the homomorphism $\varphi \cdot id\colon W \to G \rtimes W$.*

By Lemma 5, we could have written "continuous" instead of "Borel-measurable".

Proof Let $s\colon G \to E$ be a Borel-measurable section of $E \to G$ and let $\widetilde{\varphi} = s \circ \varphi$ be a lift of φ to E. The coboundary of φ is given by the formula [17, Chapter I.5.6]

$$(w, w') \mapsto \widetilde{\varphi}(w) \cdot {}^{w}\widetilde{\varphi}(w') \cdot \widetilde{\varphi}(ww')^{-1}.$$

On the other hand, the class c is represented by the 2-cocycle

$$(g, w; g', w') \mapsto s(g)w \cdot s(g')w' \cdot (s(g \cdot {}^{w}g')ww')^{-1} = s(g) \cdot {}^{w}s(g') \cdot s(g \cdot {}^{w}g')^{-1}$$

corresponding to the section $(s, \mathrm{id})\colon G \rtimes W \to E \rtimes W$. Pulling back this function along φ replaces g by $\varphi(w)$ and recovers the coboundary of φ. $\square$

For our application of Lemma B to the proof of Theorem A, the group W is the Weil group, the top extension is the universal cover of the dual group, the bottom extension is the universal cover of a complex orthogonal group, and the morphism between them arises from the given orthogonal representation $r\colon {}^L G \to \mathrm{O}(V)$:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\widehat{G}) & \longrightarrow & \widehat{G}_{\mathrm{univ}} & \longrightarrow & \widehat{G} & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle e_r} & & \big\downarrow & & \big\downarrow{\scriptstyle r|_{\widehat{G}}} & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \qquad (c_{\mathrm{pin}}). \end{array} \tag{1}$$

Here e_r is the “spin character”, which we study in Sect. 4, and the class c_{pin} classifies the bottom extension. Let $c_G \in \mathrm{H}^2_{\mathrm{Borel}}({}^L G, \pi_1(\widehat{G}))$ classify the extension $\widehat{G}_{\mathrm{univ}} \rtimes W_k$ of ${}^L G$ by $\pi_1(\widehat{G})$, as in the Theorem B.

With this setup, we prove the theorem by pulling back the conclusion of Lemma B along the given L-parameter. After pullback, the quotient of the cohomology classes $r^*(c_{\mathrm{pin}})$ and $p^* r|^*_{W_k}(c_{\mathrm{pin}})$ becomes a quotient of root numbers and the cohomology class $e_{r,*}(c_G)$ becomes the value of a central character on an involution. The goal of the remainder of the article is to explain these identifications, thus proving Theorem A.

3 Stiefel–Whitney classes

Deligne’s formula for orthogonal root numbers is a key technical tool supporting the main results of this article. To use his formula effectively, we need a workable definition of the second Stiefel–Whitney class. The goal of this largely expository section is to explain how to interpret in terms of group cohomology the second Stiefel–Whitney class of a bounded complex orthogonal representation of a countable discrete group. In the end, the class is the pullback of a certain pin-group extension.

This interpretation is surely well known to the experts, and special cases have already appeared in the literature, for instance, in a paper of Gunarwardena, Kahn, and Thomas on real orthogonal representations of finite groups [18]. We generalize their results by working with countable discrete groups instead of finite groups and complex representations instead of real representations.

3.1 Classifying spaces

One way to construct characteristic classes is to pull them back from universal cohomology classes of a certain classifying space. In this subsection, we review the theory of classifying spaces, loosely following Mitchell’s notes on classifying spaces [19] and Section 6 of Stasheff’s survey article [8].

Let G be a Lie group. We assume G to be second countable but we do not assume G to be connected. So G could be a complex reductive group or a countable discrete group.

Let $P \to B$ be a principal G-bundle. Although G is a manifold we do not impose any smoothness or differentiability assumption on bundles: they are simply continuous. If P is weakly contractible, that is, having trivial homotopy groups, then we call B a *classifying space* for G and P a *universal G-bundle*. The bundle is universal in the following sense: for every CW-complex X, the canonical map from the set of homotopy classes of maps $X \to B$ to the set of equivalence classes of principal G-bundles over X, defined by pulling back the universal G-bundle, is a bijection. As the universal property makes reference only to homotopy classes of maps, a classifying space for G is defined uniquely only up to homotopy equivalence.

It turns out, though this is not clear from the definition, that classifying spaces exist for every G. Often we can construct the classifying space by hand. For example, the classifying space of a compact orthogonal group of rank n is the Grassmannian of n-planes in $\mathbb{R}^{\oplus\mathbb{N}}$. But for a general group such a geometric construction is difficult, and we can instead construct the classifying space by simplicial methods [20, Chapter 16, Section 5]. The simplicial construction of classifying spaces makes clear their functoriality: a homomorphism $r\colon G \to H$ of topological groups gives rise to a map $Br\colon BG \to BH$. Functoriality also follows from Yoneda's Lemma, without needing to choose a specific model for the classifying space: the map Br represents the balanced product functor $P \mapsto H \times_G P$ from principal G-bundles to principal H-bundles.

The homotopy class of the classifying space BG depends only on homotopy type of the group G in the following sense: any group homomorphism $G \to H$ that is a homotopy equivalence induces a homotopy equivalence of classifying spaces. At the same time, a theorem of Iwasawa and Malcev [21, Section 7] implies that the inclusion into G of a maximal compact subgroup K is a homotopy equivalence. Hence the map $BK \to BG$ is a homotopy equivalence.

Universal characteristic classes live in the singular cohomology of classifying spaces. To translate characteristic classes into the language of group cohomology, therefore, we should strive to interpret the singular cohomology of a classifying space in terms of group cohomology. In his thesis [22], Wigner gave such an interpretation using Borel cohomology.

Theorem 8 *Let A be a discrete abelian group. There is a canonical natural isomorphism*

$$H^i_{sing}(B(\cdot), A) \simeq H^i_{Borel}(\cdot, A)$$

of contravariant functors from the category of Lie groups to the category of abelian groups.

Proof Let G be a Lie group. Wigner defined cohomology groups $\mathrm{H}^\bullet_{\mathrm{Wig}}(G, A)$ which he showed to agree with the Borel cohomology groups $\mathrm{H}^\bullet_{\mathrm{Borel}}(G, A)$. Wigner's Theorem 4 shows that $\mathrm{H}^\bullet(G, A)$ agrees with the cohomology of the constant sheaf $\underline{A}$ on the classifying space BG. Since G is a Lie group, BG is sufficiently nice (homotopy equivalent to a CW complex, say) that this sheaf cohomology agrees with the singular cohomology $\mathrm{H}^i_{\mathrm{sing}}(BG, A)$ [23]. $\square$

3.2 Definition for vector bundles

Let X be a CW complex. The theory of Stiefel–Whitney classes assigns to a rank-n real vector bundle V over X a family of cohomology classes

$$w_i(V) \in \mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{F}_2), \qquad i = 0, 1, \ldots, n.$$

These characteristic classes, along with others like the Chern classes, provide a powerful algebraic framework for computations with vector bundles. The standard source for the subject is Milnor and Stasheff's book on characteristic classes [24]; for our application, Chapters 4 to 9 and 14 are especially relevant.

Stiefel–Whitney classes are characterized abstractly, via a series of axioms: unitality, naturality, multiplicativity of the total class, and nontriviality. To show that these axioms do indeed define a collection of cohomology classes, and that this collection is unique, one defines the classes as the pullback of certain universal cohomology classes on a classifying space. The naturality axiom forces such a description.

The classifying space $B\mathrm{GL}_n(\mathbb{R})$ is the Grassmannian of n-planes in $\mathbb{R}^{\oplus\mathbb{N}}$, topologized as a direct limit, and the universal bundle over $B\mathrm{GL}_n(\mathbb{R})$ is just the tautological bundle over the Grassmannian, whose fiber over a point is the n-plane the point represents. The $\mathbb{F}_2$ singular cohomology ring of this infinite Grassmannian is a graded polynomial ring with one generator for each $i = 1, 2, \ldots, n$. We call the generator in degree i the ith *universal singular Stiefel–Whitney class*. Given a rank-n real vector bundle V on X classified by the map $f_V : X \to B\mathrm{GL}_n(\mathbb{R})$, we define the ith Stiefel–Whitney class of V as the image under the pullback

$$f_V^* : \mathrm{H}^i_{\mathrm{sing}}(B\mathrm{GL}_n(\mathbb{R}), \mathbb{F}_2) \to \mathrm{H}^i_{\mathrm{sing}}(X, \mathbb{F}_2)$$

of the ith universal singular Stiefel–Whitney class.

Since the compact orthogonal group O_n of rank n is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{R})$, the classifying spaces of the two groups are homotopy equivalent to each other and real vector bundles are classified by maps to $B\mathrm{O}_n$. In the literature one often works with the classifying spaces of this maximal compact subgroup instead of the ambient general linear group, as we do here.

3.3 Definition for representations

It is now clear how to define the Stiefel–Whitney classes of a real representation. Let $w_{i,\mathrm{univ}} \in \mathrm{H}^i_{\mathrm{Borel}}(\mathrm{GL}_n(\mathbb{R}), \{\pm 1\})$ denote the Borel cohomology class corresponding, via Theorem 8, to the ith universal singular Stiefel–Whitney class. We call $w_{i,\mathrm{univ}}$ the *ith universal Stiefel–Whitney class*. Set $w_{i,\mathrm{univ}} \stackrel{\mathrm{def}}{=} 0$ if $i < 0$ or $i > n$.

Definition 9 Let G be a Lie group. The *ith Stiefel–Whitney class* of a real representation $r : G \to \mathrm{GL}_n(\mathbb{R})$ is the cohomology class

$$w_i(r) \stackrel{\mathrm{def}}{=} r^*(w_{i,\mathrm{univ}}) \in \mathrm{H}^i_{\mathrm{Borel}}(G, \{\pm 1\}).$$

This definition is not enough for us, however. Since the relevant representations of L-groups are complex, not real, we need to define the Stiefel–Whitney classes of a complex representation, the goal of this subsection. In outline, to make the definition we simply require that base change from $\mathbb{R}$ to $\mathbb{C}$ leave Stiefel–Whitney classes unchanged. This stipulation defines Stiefel–Whitney classes for all complex representations that descend to $\mathbb{R}$, in particular, the bounded orthogonal representations.

An $\mathbb{R}$-*structure* on a complex vector space V is a real subspace $V_0 \subseteq V$ such that the map $V_0 \otimes_{\mathbb{R}} \mathbb{C} \to V$ is an isomorphism. An $\mathbb{R}$-*structure* on a complex representation (r, V) is a real representation (r_0, V_0) such that V_0 is an $\mathbb{R}$-structure on V and r factors through r_0. An *isomorphism* of $\mathbb{R}$-structures on a complex representation is an isomorphism of the corresponding real representations.

It is not the case that every complex orthogonal representation admits an $\mathbb{R}$-structure: for instance, the two-dimensional representation of $\mathbb{Z}$ that sends 1 to the diagonal matrix with entries $(2i, -i/2)$ is orthogonal but does not admit an $\mathbb{R}$-structure because its character takes imaginary values. However, if the representation is in addition *bounded*, meaning the closure of its image is compact, then it does admit an $\mathbb{R}$-structure.

Lemma 10 *All bounded complex orthogonal representations admit an* $\mathbb{R}$-*structure.*

Proof An exercise in point-set topology shows that the closure of the image of the representation is again a group, and by hypothesis the closure is compact. Hence the representation factors through some compact orthogonal group O_n, where n is the complex dimension of the representation, because O_n is a maximal compact subgroup of the complex orthogonal group. □

The base change problem appears in many other settings than representation theory. For example, we can study the base change of varieties from $\mathbb{R}$ to $\mathbb{C}$. In that setting, it can happen that $\mathbb{R}$-varieties are nonisomorphic but become isomorphic upon base change to $\mathbb{C}$. In our setting the story is simpler: any two real forms of a (semisimple) representation must be isomorphic.

Lemma 11 *Let G be a group and $r : G \to \mathrm{GL}(V)$ a finite-dimensional complex representation. If r is semisimple then any two of its* $\mathbb{R}$-*structures are isomorphic.*

Proof Since r is semisimple, the group $\mathrm{Aut}_G(r) \subseteq \mathrm{GL}(V)$ of linear G-equivariant automorphisms of V is a product of complex general linear groups. At the same time, the isomorphism classes of $\mathbb{R}$-structures on r are classified by the Galois cohomology set $\mathrm{H}^1(\Gamma_{\mathbb{R}}, \mathrm{Aut}_G(r))$. By Hilbert's Theorem 90, this cohomology set is trivial. □

By the lemma, the following definition does not depend on the choice of $\mathbb{R}$-structure.

Definition 12 Let G be a Lie group and let (r, V) be a semisimple complex representation of G that admits an $\mathbb{R}$-structure (r_0, V_0). The *ith Stiefel–Whitney class* of (r, V) is

$$w_i(r) \stackrel{\text{def}}{=} w_i(r_0) \in \mathrm{H}^i_{\mathrm{Borel}}(G, \{\pm 1\}).$$

We are most interested in the second Stiefel–Whitney class because of its appearance in Deligne's theorem on root numbers.

Lemma 13 *Let V_0 be a real anisotropic quadratic space and let $V \stackrel{def}{=} V_0 \otimes_{\mathbb{R}} \mathbb{C}$, a complex quadratic space. There exists an extension* $\mathrm{Pin}(V)$ *of* $O(V)$ *by* $\{\pm 1\}$*, called the (complex)* pin group*, with the following property: the class $c_{pin} \in H^2_{Borel}(O(V), \{\pm 1\})$ that classifies* $\mathrm{Pin}(V)$ *is the pullback of the second universal Stiefel–Whitney class*

$$w_{2,univ} \in H^2_{Borel}(O(V_0), \{\pm 1\}) \simeq H^2_{Borel}(\mathrm{GL}(V_0), \{\pm 1\}).$$

We call the class $c_{\mathrm{pin}} \in \mathrm{H}^2_{\mathrm{Borel}}(\mathrm{O}(V), \{\pm 1\})$ the *pin class*.

Proof Let $\mathrm{Pin}(V_0)$ be the extension of $\mathrm{O}(V_0)$ by $\{\pm 1\}$ classified by $w_{2,\mathrm{univ}}$. It is well-known that $\mathrm{Pin}(V_0)$ is (the rational points of) the standard (real) algebraic pin group defined using the Clifford algebra $\mathrm{Cl}(V_0)$; see, for instance, Appendix I of [25] or the introduction to [18].

To be specific, the group $\mathrm{Pin}(V_0)$ is the stabilizer of the subspace $V_0 \subseteq \mathrm{Cl}(V_0)$ under the conjugation action of the units group $\mathrm{Cl}(V_0)^\times$ on $\mathrm{Cl}(V_0)$. The main anti-involution of $\mathrm{Cl}(V_0)$ is the automorphism α induced by the order-reversing automorphism $v_1 \otimes \cdots \otimes v_d \mapsto v_d \otimes \cdots \otimes v_1$ of the tensor algebra on V_0. The spinor norm is the homomorphism $\mathrm{Cl}(V_0) \to \mathbb{C}^\times$ sending $x \in \mathrm{Cl}(V_0)$ to $x\alpha(x)$. Finally, $\mathrm{Pin}(V_0)$ is the kernel of the spinor norm.

It is now clear that we may take as $\mathrm{Pin}(V)$ the complex algebraic group obtained from (the algebraic group underlying) $\mathrm{Pin}(V_0)$ by base change from $\mathbb{R}$ to $\mathbb{C}$. ◻

It follows from the lemma that the second Stiefel–Whitney class of a bounded complex orthogonal representation $r : G \to \mathrm{O}(V)$ is classified by the pullback of group extensions $r^* \mathrm{Pin}(V)$. This conclusion is our final reformulation of the second Stiefel–Whitney class in the language of group cohomology.

Remark 14 There are two variant and non-isomorphic definitions of the pin group, stemming from the fact that the elements $w_{2,\mathrm{univ}}$ and $w_{1,\mathrm{univ}}^2$ of $\mathrm{H}^2_{\mathrm{Borel}}(\mathrm{O}(V), \{\pm 1\})$ are distinct. Conrad's SGA 3 article [26], whose notation for the Pin group agrees with ours, nicely explains the difference; Remarks C.4.9 and C.5.1 are especially relevant. The other variant of the pin group, which we do not use here, is due to Atiyah, Bott, and Shapiro [27], and is denoted by Pin^- in Conrad's article. The definition is similar to ours but one modifies the spinor norm by a sign twist.

3.4 Deligne's theorem

We start by defining the Stiefel–Whitney classes of a complex orthogonal representation of the Weil group. When the field k is archimedean, this task is already complete because $W_{\mathbb{R}}$ and $W_{\mathbb{C}}$ are Lie groups. When k is nonarchimedean, we use the fact that every complex representation of W_k factors through a discrete quotient.

Definition 15 Let k be nonarchimedean, let V be a complex quadratic space, and let $r\colon W_k \to \mathrm{O}(V)$ be a complex orthogonal representation of W_k. The ith *Stiefel–Whitney class* of (r, V) is the image of $w_i(r)$ under the inflation map

$$\mathrm{H}^i(W_k/\ker r, \{\pm 1\}) \to \mathrm{H}^i(W_k, \{\pm 1\}).$$

The evident compatibility of Stiefel–Whitney classes with inflation shows that we are free to replace $\ker r$ by any open subgroup of W_k on which r is trivial. We can now state our reformulation of Deligne's theorem on root numbers.

Theorem 16 *Let $r\colon W_k \to O(V)$ be a bounded complex orthogonal representation and let $c_{pin} \in H^2_{Borel}(O(V), \{\pm 1\})$ be the pin class of Lemma 13. Then*

$$\frac{\omega(r)}{\omega(\det r)} = \operatorname{sgn} r^*(c_{pin}).$$

The statement of the theorem uses that the group $\mathrm{H}^2(W_k, \{\pm 1\})$ is cyclic of order two. The function sgn was defined in Sect. 1.1; it uniquely identifies this group with the group $\{\pm 1\}$.

Proof Applying Deligne's theorem [3, Proposition 5.2] to the virtual representation

$$r - \det r - (\dim r - 1) \cdot \mathrm{triv}$$

of dimension zero and determinant one shows that the lefthand side of the equation equals $\operatorname{sgn}(w_2(r))$. And by Lemma 13, $w_2(r) = r^*(c_{\mathrm{pin}})$. □

4 Spin lifting

Let G be a complex reductive group, identified with its set of $\mathbb{C}$-points. In our application G will be the Langlands dual of a reductive k-group, but in this section we restrict attention to complex groups so there is no need to decorate G with a hat.

Consider an (algebraic, complex) orthogonal representation $r\colon G \to \mathrm{O}(V)$. As G is (by assumption) connected, the representation r factors through the identity component $\mathrm{SO}(V)$ of $\mathrm{O}(V)$, and we may without loss of generality replace $\mathrm{O}(V)$ by $\mathrm{SO}(V)$ as the target of r.

The special orthogonal group $\mathrm{SO}(V)$ is not simply connected. Supposing that $\dim V > 2$, its universal cover $p\colon \mathrm{Spin}(V) \to \mathrm{SO}(V)$, a double cover, is an algebraic group called the *spin group*. We can construct the spin group as a subgroup of the units group of a Clifford algebra, though for our purposes, we can understand the group using the combinatorics of root systems alone. When $\dim V = 2$, so that $\mathrm{SO}(V) = \mathbb{G}_{\mathrm{m}}$ is a one-dimensional torus, we define $\mathrm{Spin}(V) = \mathbb{G}_{\mathrm{m}}$ with double cover $p\colon \mathrm{Spin}(V) \to \mathrm{SO}(V)$ the squaring map.

The existence of the spin group creates a dichotomy in the orthogonal representations $r\colon G \to \mathrm{SO}(V)$ of G: either the representation lifts to the spin group or it does not. We can study and refine this lifting question by passing to the universal cover G_{univ} of G:

$$G_{\mathrm{univ}} = \mathfrak{z} \times G_{\mathrm{sc}}$$

where $\mathfrak{z}$ is the Lie algebra of the center Z of G and G_{sc} is the simply-connected cover of the derived subgroup of G. Standard facts about covering spaces imply that r lifts to a homomorphism $r_{\mathrm{univ}} \colon G_{\mathrm{univ}} \to \mathrm{Spin}(V)$. Restricting r_{univ} to the kernels of the projections yields the following commutative diagram, which is essentially equivalent to (1):

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(G) \arrow[r] \arrow[d, "e_r"] & G_{\mathrm{univ}} \arrow[r] \arrow[d, "r_{\mathrm{univ}}"] & G \arrow[r] \arrow[d, "r"] & 1 \\
1 \arrow[r] & \{\pm 1\} \arrow[r] & \mathrm{Spin}(V) \arrow[r] & \mathrm{SO}(V) \arrow[r] & 1.
\end{tikzcd}$$

Then r lifts to the spin group if and only if the character $e_r \colon \pi_1(G) \to \{\pm 1\}$, which we call the *spin character* of r, is trivial.

Our goal in this section is to give a formula for the spin character of a representation in terms of its weights. We start in Sect. 4.1 with a criterion for r to lift to the spin group. From this criterion we then deduce, in Sect. 4.3, a formula for the spin character.

All that is new here is our exposition: this calculation forms part of the canon of the representation theory of compact Lie groups. With that said, there is a small discrepancy between our answer and Bourbaki's, which I believe to be an error on Bourbaki's part. This discrepancy is discussed in Sect. 4.2.

Given a representation (r, V) of G and a maximal torus T, let $\Phi(V) \subseteq X^*(T)$ denote the set of weights of T on V.

4.1 Lifting criterion

The goal of this subsection is to describe which orthogonal representations of G lift to the spin group. We start with the essential case where $G = T$ is a torus; the general case reduces easily to this one.

Let S be a maximal torus of $\mathrm{SO}(V)$ and let $\widetilde{S} \subseteq \mathrm{Spin}(V)$ be its preimage in the spin group, again a maximal torus. Any homomorphism $T \to \mathrm{SO}(V)$ can be conjugated to take values in S, after which point the lift to the spin group, if it exists, factors through $\widetilde{S}$. Passing to character lattices, the lifting question reduces to the algebra question of whether the homomorphism $X^*(S) \to X^*(T)$ can be extended to the group $X^*(\widetilde{S})$, which contains $X^*(S)$ as an index-two subgroup because $\widetilde{S} \to S$ is a double-cover.

$$\begin{tikzcd}
 & \widetilde{S} \arrow[d, "p"] \\
T \arrow[ur, dashed, "\widetilde{r}"] \arrow[r, "r"] & S
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
 & X^*(\widetilde{S}) \arrow[dl, dashed, "\widetilde{r}^*"'] \\
X^*(T) & X^*(S) \arrow[l, "r^*"] \arrow[u, "p^*"']
\end{tikzcd}$$

To solve this extension problem, we need to understand the double cover $\widetilde{S} \to S$ as well as the relationship between the homomorphism $T \to S$ and the weights of the original orthogonal representation. Let n be the rank of $\mathrm{Spin}(V)$, so that $\dim V = 2n$ or $2n+1$. Fix a set I of cardinality $\dim V$ equipped with an involution $i \mapsto -i$ that fixes no element of I when $\dim V$ is even and exactly one element of I, which we denote by 0, when $\dim V$ is odd.

First, the double cover. Choose an I-indexed basis $e = (e_i)_{i \in I}$ of V for which $\langle e_i, e_j \rangle = [i = -j]$; [3] we call such a basis a *Witt basis*. Let S_e denote the group of $s = (s_i)_{i \in I} \in \mathbb{G}_{\mathrm{m}}^I$ such that $s_i \cdot s_{-i} = 1$ and such that, when $\dim V$ is odd, $s_0 = 1$. The rank-n torus S_e acts on V by

$$s \cdot e_i \stackrel{\text{def}}{=} s_i e_i,$$

realizing S_e as a subgroup of $\mathrm{SO}(V)$. Evidently S_e is a maximal torus, and every other maximal torus of $\mathrm{SO}(V)$ arises from a Witt basis e by this construction. Passing to the character lattice, describing a basis of $X^*(S_e)$ requires a choice of gauge $p\colon I \setminus \{0\} \to \{\pm 1\}$, that is, a negation-equivariant function. Given p, say $i > 0$ if $p(i) = +1$ and $i < 0$ if $p(i) = -1$, for $i \in I$. The characters $f_i\colon s \mapsto s_i$ for $i > 0$ give a basis for $X^*(S)$.

We can use the Clifford-algebra description of $\mathrm{Spin}(V)$, or even easier, the Bourbaki root-system tables [29, Planches], to work out the character lattice of $X^*(\widetilde{S})$. Taking $f_p = (f_i)_{i>0}$ as a basis for $X^*(S)_{\mathbb{Q}}$, the character lattice of $X^*(\widetilde{S})$ is the set of elements of $\frac{1}{2}X^*(S)$ whose coordinates are all integers or all half-integers. In particular, $X^*(\widetilde{S})$ is generated by $X^*(S)$ and the vector $\tilde{f}_p = \frac{1}{2}\sum_{i>0} f_i$. It follows that for A an abelian group, a homomorphism $X^*(S) \to A$ extends to a homomorphism $X^*(\widetilde{S}) \to A$ if and only if, letting a_i denote the image of f_i, the sum $\sum_{i>0} a_i$ lies in $2A$. If this property is satisfied then $\tilde{f}_p$ can map to any element whose double is $\sum_{i>0} a_i$. When A is 2-torsionfree there is at most one such element, hence at most one extension.

Consequently, $\widetilde{S}$ can be described as the quotient S/B where B is the group of $(\varepsilon_i)_{i \in I} \in \{\pm 1\}^I$ with $\varepsilon_i = \varepsilon_{-i}$, $\varepsilon_0 = 1$, and $\prod_{i>0} \varepsilon_i = 1$. In this description the twofold cover $\widetilde{S} \to S$ is induced by the squaring map on S.

Next, the weights. Since V is an orthogonal representation of T, its set of weights is negation-invariant. Let $p\colon \Phi(V) \setminus \{0\} \to \{\pm 1\}$ be a gauge. Using the gauge we can write down an orthogonal decomposition

$$V = V_0 \oplus \bigoplus_{\alpha > 0} (V_\alpha \oplus V_{-\alpha})$$

in which V_α is the orthogonal complement of $V_{-\alpha}$ when $\alpha \neq 0$. At this point the gauge is only a notational convenience since the summands in the decomposition do not depend on it. Choose a Witt basis $(e_i)_{i \in I}$ for V consisting of weight vectors of S and let α_i denote the weight of e_i. If $\dim V$ is odd then e_0 is of weight 0. The map

[3] Here $[\mathcal{P}]$ is the Iverson bracket popularized by Knuth [28]: it equals 0 if the property $\mathcal{P}$ is false and 1 if $\mathcal{P}$ is true.

$X^*(S) \to X^*(T)$ dual to the homomorphism $T \to S$ sends the basis vector f_i to the character α_i.

These two analyses combine to a criterion for the map $T \to S$ to lift to $\widetilde{S}$. Choose a gauge p on $X^*(T)$ and define the element

$$\rho_r \stackrel{\text{def}}{=} \frac{1}{2} \sum_{\alpha>0} (\dim V_\alpha)\alpha \in \frac{1}{2} X^*(T). \tag{2}$$

Although ρ_r depends on the choice of gauge, we will only ever use it in a way that is independent of the choice of gauge. Now our criterion is this: the representation r lifts to the spin group if and only if $\rho_r \in X^*(T)$.

When G is no longer abelian, we can reduce the spin-lifting problem to the abelian case using the observation that all the obstructions to lifting lie in a maximal torus.

Lemma 17 *Let $f : G \to H$ be a homomorphism of complex reductive groups, $T \subseteq G$ a split maximal torus of G, and $\widetilde{H} \to H$ an isogeny. Then f lifts to $\widetilde{H}$ if and only if $f|_T$ lifts to $\widetilde{H}$.*

The lemma is true over much more general bases than the complex numbers. Counter to the spirit of SGA 3, we prove it using the analytic topology on G.

Proof Isogenies are covering spaces. This claim follows from the lifting criterion for covering spaces [30, Proposition 1.33] together with the surjectivity of the map $\pi_1(T) \to \pi_1(G)$ [31, Section 4.6]. □

Combining Lemma 17 with our analysis of the abelian case completely solves the problem of lifting an orthogonal representation to the spin group.

Theorem 18 *Let G be a reductive group, $T \subseteq G$ a maximal torus, and (r, V) an orthogonal representation. Then r lifts to the spin group if and only if $\rho_r \in X^*(T)$.*

4.2 Comparison with Bourbaki

Our Theorem 18 differs from at least one work in the canon of Lie groups, Chapter 9 of Bourbaki's *Groupes et algèbres de Lie* [31]. I believe there is a small error in Bourbaki's account of the lifting criterion. In light of the famous scrupulousness with which the Bourbaki group prepared their treatises, a few words are in order to explain the discrepancy.

Chapter 9 of Bourbaki's book studies compact connected Lie groups. Although this setting is different from ours, the algebraic setting, there is a standard dictionary [32, Sections VIII.6–7] between the compact and algebraic settings, and this dictionary gives a comparison between our work and Bourbaki's. Exercise 7a of Section 7 of [31] concerns the spin lifting question. The difference between Bourbaki's answer and ours, the quantity ρ_r defined in Eq. (2), is that we take into account the multiplicity of the weights, in other words, the dimensions of the weight spaces, while Bourbaki's analogue of ρ_r,

$$\frac{1}{2}\sum_{0<\alpha\in\Phi(V)}\alpha,$$

does not weight the sum by multiplicity. In this subsection, we give an example explaining why Bourbaki's criterion is incorrect.

Our theory predicts that the double of an orthogonal representation lifts to the spin group. Here is an independent proof of this prediction. It shows that Bourbaki's exercise cannot be correct, as we explain after the proof.

Lemma 19 *Let $r\colon G \to \mathrm{SO}(V)$ be an orthogonal representation of a reductive group G. Then $r \oplus r$ lifts to the spin group.*

Proof It suffices to prove the claim in the case where r is the tautological (identity) orthogonal representation of $G = \mathrm{SO}(V)$. Consider the following commutative diagram, in which the left horizontal arrows are diagonal inclusions, the right horizontal arrows are multiplication maps, and the vertical arrows are the canonical projection from the spin group to the special orthogonal group.

$$\begin{array}{ccccc}
\mathrm{Spin}(V) & \xrightarrow{\Delta} & \mathrm{Spin}(V\oplus 0)\times\mathrm{Spin}(0\oplus V) & \xrightarrow{\bullet} & \mathrm{Spin}(V\oplus V)\\
\downarrow & & \downarrow & & \downarrow\\
\mathrm{SO}(V) & \xrightarrow{\Delta} & \mathrm{SO}(V\oplus 0)\times\mathrm{SO}(0\oplus V) & \xrightarrow{\bullet} & \mathrm{SO}(V\oplus V).
\end{array}$$

The claim amounts to showing that the bottom horizontal composite arrow lifts to $\mathrm{Spin}(V\oplus V)$. Recall that the kernel of $\mathrm{Spin}(V)\to\mathrm{SO}(V)$ is negation in the Clifford algebra, which we denote by -1. The claim is equivalent to the statement that -1 lies in the kernel of the horizontal top composite arrow. And this statement follows from the fact that for any isometric embedding $W \to V$ of quadratic spaces, the induced map $\mathrm{Spin}(W)\to\mathrm{Spin}(V)$ sends the center to the center. This fact is a consequence of the Clifford-algebra definition of the spin group: the induced spin-group map is the restriction of the induced Clifford algebra map $\mathrm{Cl}(W)\to\mathrm{Cl}(V)$, and this map is the identity on the copy of $\mathbb{C}$ inside both algebras. □

It is clear that the tautological orthogonal representation of $\mathrm{SO}(V)$ does not lift to the spin group: otherwise, the spin cover of $\mathrm{SO}(V)$ would split and $\mathrm{Spin}(V)$ would be disconnected. On the other hand, since $\Phi(V)=\Phi(V\oplus V)$ and Bourbaki's criterion is sensitive only to the set of weights, that criterion, along with Lemma 19, would imply that the tautological representation does lift.

4.3 Spin character

In this subsection, we build on our work from Sect. 4.1 to give a formula for the spin character of an orthogonal representation $r\colon G\to\mathrm{SO}(V)$.

First, let's review the construction of the universal covering projection of a complex torus T. The universal cover of T can be identified with the Lie algebra $\mathfrak{t}$ and the universal covering map $\mathfrak{t}\to T$ is then the exponential map from the theory of Lie groups.

The universal cover is functorial because formation of Lie algebras is functorial: the morphism on universal covers induced by a morphism of tori is the differential of the morphism on the Lie algebras.

For our purposes it is more useful, however, to describe the universal cover $\mathfrak{t}$ using cocharacter lattices. The evaluation map $X_*(T) \otimes \mathbb{C}^\times \to T$ is an isomorphism, forming the tensor product over $\mathbb{Z}$. It is conventional to use exponential notation for these tensors, writing a^λ for $\lambda \otimes a$. Similarly, evaluation of the derivative at 1 gives a canonical isomorphism $X_*(T) \otimes \mathbb{C} \to \mathfrak{t}$. In these coordinates, the exponential cover $\mathfrak{t} \to T$ is simply the map $X_*(T) \otimes \mathbb{C} \to X_*(T) \otimes \mathbb{C}^\times$ induced by the exponential function

$$\lambda \otimes a \mapsto \exp(2\pi i a)^\lambda.$$

The universal cover map identifies its kernel, $X_*(T)$, with the fundamental group of T. More concretely, the identification $X_*(T) \simeq \pi_1(T)$ is restriction of cocharacters to the unit circle of $\mathbb{C}^\times$. In this tensor product model, functoriality of the universal cover follows from functoriality of the cocharacter lattice: a homomorphism $f : T \to S$ of tori induces a map $f_* : X_*(T) \to X_*(S)$, and the induced map $f_{\text{univ}} : T_{\text{univ}} \to S_{\text{univ}}$ is simply

$$f_{\text{univ}} : \lambda \otimes a \mapsto f_*(\lambda) \otimes a.$$

Of particular importance are the isogenies $f : T \to S$, for which f_{univ} relates two different descriptions of the same universal cover.

We first work out the character e_r in the case where $G = T$ is a torus; the general case follows immediately from this special case. Choose a gauge on $X^*(T) \setminus \{0\}$ so that we may speak of positive and negative characters. Retain the notation from Sect. 4.1, so that I indexes a Witt basis $(e_i)_{i \in I}$ of V of weight vectors, S is the maximal torus of $\mathrm{SO}(V)$ corresponding to the basis, and $\widetilde{S}$ is its double cover in $\mathrm{Spin}(V)$. Here $X_*(S) = \ker(\mathbb{Z}^I \to \mathbb{Z}^{I/\pm})$, a basis for $X_*(S)$ is $(f_i - f_{-i})_{i>0}$ where $(f_i)_{\in I}$ is the standard basis of $\mathbb{Z}^I$, and $X_*(\widetilde{S})$ is the set of elements of $\frac{1}{2}X_*(S)$ whose f_i-coefficients are either all integers or all half-integers. Now consider the following diagram, where $S' = S$ and the dashed arrow is squaring.

$$\begin{tikzcd} T_{\text{univ}} \arrow[rrr] \arrow[d, "r_{\text{univ}}"] & & & T \arrow[d, "r"] \\ S_{\text{univ}} \arrow[r] & S' \arrow[r] \arrow[rr, bend left, dashed, "2"] & \widetilde{S} \arrow[r] & S \end{tikzcd}$$

The map $X_*(S') = X_*(S) \to X_*(S)$ induced by squaring is multiplication by 2. Therefore, under the identification $S_{\text{univ}} = X_*(S) \otimes \mathbb{C}$, the map $S_{\text{univ}} \to X_*(S') \otimes \mathbb{C}^\times = S'$ is

$$\lambda \otimes a \mapsto \exp(2\pi i a/2)^\lambda \qquad \lambda \in X_*(S).$$

At the same time, our work in Sect. 4.1 shows that the map $r_*\colon X_*(T) \to X_*(S)$ is

$$r_*\colon \lambda \mapsto \sum_{i>0} \langle \lambda, \alpha_i \rangle (f_i - f_{-i}).$$

As $\alpha_{-i} = \alpha_i^{-1}$, it follows that the map $X_*(T) \to \widetilde{S}$ is given by the formula

$$\lambda \mapsto \big(\exp(\pi i \langle \lambda, \alpha_i \rangle)\big)_{i \in I},$$

where the target element is interpreted as a coset in S following the discussion in Sect. 4.1. Belying the notational complexity of this formula, every component of the tuple is ± 1. We know that the image in $\widetilde{S}$ of this tuple lies in the center of $\mathrm{Spin}(V)$, an order-two subgroup of $\widetilde{S}$. To identify the image as $+1$ or -1, we take the product of the elements of the tuple with positive index i. The final formula, therefore, is

$$e_r(\lambda) = \prod_{i>0} \exp\big(\pi i \langle \lambda, \alpha_i \rangle\big) = \exp\big(\pi i \langle \lambda, 2\rho_r \rangle\big) = (-1)^{\langle \lambda, 2\rho_r \rangle}$$

where $\lambda \in X_*(T) \simeq \pi_1(T)$

When G is not a torus, we choose a split maximal torus T in G and use the fact that the inclusion $i\colon T \to G$ induces a surjection $\pi_1(T) \to \pi_1(G)$. A diagram chase shows that the spin character $e_{r \circ i}$ for the restriction of r to T factors through e_r. In this way we reduce to the case where G is a torus.

Theorem 20 *Let G be a complex reductive group, let $T \subseteq G$ be a maximal torus, and let $r\colon G \to \mathrm{SO}(V)$ be an orthogonal representation. The spin character $e_r\colon \pi_1(G) \to \{\pm 1\}$ induced by r is given by the formula*

$$e_r(\lambda) = (-1)^{\langle \lambda, 2\rho_r \rangle},$$

where $\lambda \in X_*(T) \simeq \pi_1(G)$ and ρ_r is defined in (2).

5 Central characters and Weil cohomology

To apply Lemma B to prove Theorem A, we need to interpret the image of the class $\varphi^*(c_G)$ under the map

$$e_{r,*} : \mathrm{H}^2(W_k, \pi_1(\widehat{G})) \to \mathrm{H}^2(W_k, \{\pm 1\}).$$

It turns out that $\varphi^*(c_G)$ corresponds to Langlands's central character χ_φ – conjecturally, the central character of the L-packet of φ – and that the map $e_{r,*}$ corresponds to evaluation of the character on a certain involution z_r. The goal of this section is to justify and explain these claims, and ultimately, in Theorem 28, to show that $e_{r,*}\varphi^*(c_G) = \chi_\varphi(z_r)$.

Let Z be the center of the quasi-split reductive group G. The main difficulty is to interpret the cohomology group

$$\mathrm{H}^2(W_k, X^*(Z)).$$

After reviewing the general setting in which the cohomology of the Weil group is computed, in Sect. 5.1, we show in Sect. 5.2 that when Z is connected, this group is the character group of the Harish–Chandra subgroup $Z(k)^1$ of $Z(k)$. When char $k = 0$ and k is nonarchimedean this identification exists even for Z disconnected, as we explain in Sect. 1. Although the center of G need not be connected in general, Langlands's definition of χ_φ permits us to reduce, in Sect. 5.3, to the connected-center case. We conclude in Sect. 5.4 by defining the involution z_r and proving Theorem 28.

In what follows, we use continuous cohomology for the Weil group of a nonarchimedean field and Borel cohomology for the Weil group of an archimedean field. The difference is sometimes elided in the notation to avoid overburdening the reader.

5.1 Cohomology of the Weil group

What kind of group cohomology should we use to study the Weil group? Over an archimedean field the right answer is Borel cohomology, as Example 3 shows. Over a nonarchimedean field, one [4] right answer is continuous cohomology. Some care is required, however, because the Weil group is not profinite, only locally profinite. For profinite groups we can easily reduce most foundational problems to the setting of finite groups, where topology is irrelevant, but this is no longer the case for locally profinite groups.

Fortunately, Flach has written a nice article [34] that addresses technical concerns in the continuous cohomology of the Weil group. Flach first situates this cohomology in a general topos-theoretic setting, using theory from SGA 4, and then shows that this general theory recovers the usual definition of continuous cohomology by continuous cochains. One particularly useful consequence of his work is the existence of the usual long exact sequence for any short exact sequence of topological modules whose quotient map locally admits continuous sections [34, Lemma 6]. Another useful consequence is the following lemma.

Lemma 21 *Let V be a discrete W_k-module. Suppose the underlying abelian group of V is uniquely divisible, in other words, a $\mathbb{Q}$-vector space.*

1. *If k is nonarchimedean then $H^i(W_k, V) = 0$ for $i \geq 2$.*
2. *If k is archimedean then $H^i_{Borel}(W_k, V) = 0$ for i odd.*

Proof First, assume k is nonarchimedean. There is a Hochschild–Serre spectral sequence

$$\mathrm{H}^i(\mathbb{Z}, \mathrm{H}^j(I_k, V)) \implies \mathrm{H}^{i+j}(W_k, V)$$

[4] Lichtenbaum remarked that Borel and continuous cohomology agree in many cases, in particular, when the coefficient group is discrete and countable [33, Remark 2.2]. We will not use this result, though it would slightly simplify the exposition.

coming from the short exact sequence $1 \to I_k \to W_k \to \mathbb{Z} \to 1$. This spectral sequence is an instance of a more general spectral sequence that Flach constructed [34, Corollary 6]. In the more general sequence, the group $\mathrm{H}^j(I_k, V)$ is replaced by a sheaf which may or may not be representable. But since V is discrete and W_k is locally profinite, this sheaf is representable [34, Proposition 9.2] and there are no technical problems.

Since V is uniquely divisible and I_k is profinite, $\mathrm{H}^j(I_k, V) = 0$ for $j \geq 1$ [35, Proposition 1.6.2]. Since $\mathbb{Z}$ has cohomological dimension one, $\mathrm{H}^i(\mathbb{Z}, -)$ vanishes for $i \geq 2$. So all entries on the starting page of the spectral sequence vanish except possibly those in positions $(0, 0)$ and $(1, 0)$.

Next, suppose k is archimedean. If $k = \mathbb{C}$ then $W_{\mathbb{C}} = \mathbb{C}^\times = S^1 \times \mathbb{R}_{>0}$ has classifying space the infinite-dimensional complex projective space. Its integral, hence rational, cohomology is known to be concentrated in even degrees. If $k = \mathbb{R}$ then use the Hochschild–Serre spectral sequence in Borel cohomology [11, Theorem 9] for the short exact sequence $1 \to \mathbb{C}^\times \to W_{\mathbb{R}} \to \Gamma_{\mathbb{R}} \to 1$ together with the vanishing of $\mathrm{H}^{\geq 1}(\Gamma_{\mathbb{R}}, -)$ on uniquely divisible groups to show that the map $\mathrm{H}^i_{\mathrm{Borel}}(W_{\mathbb{R}}, V) \to \mathrm{H}^i_{\mathrm{Borel}}(\mathbb{C}^\times, V)^{\Gamma_{\mathbb{R}}}$ is an isomorphism for $i \geq 2$. □

5.2 Harish–Chandra subgroup

Recall that G is a quasi-split reductive k-group. Define the pairing

$$G(k) \otimes \mathrm{Hom}_{k-\mathrm{gp}}(G, \mathbb{G}_\mathrm{m}) \to \mathbb{R}$$

by $\langle g, \alpha \rangle = \mathrm{ord}_k(\alpha(g))$, where $\mathrm{ord}_k : k^\times \to \mathbb{R}$ is the absolute value for k archimedean and the discrete valuation for k nonarchimedean. Currying yields a map

$$\theta_G : G(k) \to \mathrm{Hom}(\mathrm{Hom}_{k\text{-gp}}(G, \mathbb{G}_\mathrm{m}), \mathbb{R}).$$

In particular, if $G = T$ is a torus then the target of θ_G is $X_*(T)_{\mathbb{R}}^{\Gamma_k}$. A character of $G(k)$ is *unramified* if it is inflated along θ_G.

Definition 22 Let G be a k-group. The *Harish–Chandra subgroup* $G(k)^1$ of $G(k)$ is the kernel of the homomorphism θ_G.

For example, if G is finite then $G(k)^1 = G(k)$ because roots of unity have valuation zero. If $G = T$ is a torus then the group $T(k)^1$ is compact, hence profinite. The Harish–Chandra subgroup is relevant because its character group is exactly the cohomology group of interest to us, as we now explain.

Lemma 23 *Let T be a k-torus. The image of the composite map (taken in Borel cohomology if k is archimedean)*

$$H^1(W_k, X^*(T)_{\mathbb{C}}) \longrightarrow H^1(W_k, \widehat{T}) \xrightarrow{LLC} \mathrm{Hom}_{cts}(T(k), \mathbb{C}^\times),$$

in which the second map is the local Langlands correspondence for tori, is the group of unramified characters of $T(k)$.

Proof When $T = \mathbb{G}_m$ the claim is clear, and Shapiro's lemma then proves the claim for any induced torus. In general, embed T in an induced torus S. Using the diagram

$$\begin{tikzcd}
\mathrm{H}^1(W_k, X^*(S)_{\mathbb{C}}) \arrow[r] \arrow[d, two heads] & \mathrm{H}^1(W_k, \widehat{S}) \arrow[r, "\mathrm{LLC}", "\sim"'] \arrow[d] & \mathrm{Hom}_{\mathrm{cts}}(S(k), \mathbb{C}^\times) \arrow[d] \\
\mathrm{H}^1(W_k, X^*(T)_{\mathbb{C}}) \arrow[r] & \mathrm{H}^1(W_k, \widehat{T}) \arrow[r, "\mathrm{LLC}", "\sim"'] & \mathrm{Hom}_{\mathrm{cts}}(T(k), \mathbb{C}^\times)
\end{tikzcd}$$

and the fact that the left vertical arrow is a surjection by Lemma 21, it suffices to show that the unramified characters of $T(k)$ are precisely the restrictions of the unramified characters of $S(k)$. Clearly restriction preserves unramification, and every unramified character of $T(k)$ extends to an unramified character of $S(k)$ because $X_*(T)^{\Gamma_k}$ is a summand of $X_*(S)^{\Gamma_k}$. □

The universal cover exact sequence for $\widehat{T}$ expands, by Lemma 21, to the exact sequence

$$\mathrm{H}^1(W_k, X^*(T)_{\mathbb{C}}) \longrightarrow \mathrm{H}^1(W_k, \widehat{T}) \longrightarrow \mathrm{H}^2(W_k, X^*(T)) \longrightarrow 0.$$

At the same time, there is a canonical identification

$$\mathrm{coker}\big(\mathrm{Hom}(X_*(T)^{\Gamma_k}_{\mathbb{R}}, \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cts}}(T(k), \mathbb{C}^\times)\big) \simeq \mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times).$$

Combining these two facts yields a cohomological description of the character group of $T(k)^1$.

Corollary 24 *Let T be a k-torus. The local Langlands correspondence and universal cover coboundary furnish an isomorphism (in Borel cohomology if k is archimedean)*

$$H^2(W_k, X^*(T)) \simeq \mathrm{Hom}_{cts}(T(k)^1, \mathbb{C}^\times).$$

5.3 Evaluation at an involution

Let T be a k-torus. Every Galois-equivariant homomorphism $X^*(T) \to \{\pm 1\}$ induces a map

$$\mathrm{H}^2(W_k, X^*(T)) \to \mathrm{H}^2(W_k, \{\pm 1\}).$$

The source is the character group of $T(k)^1$ and the target is a group of order two. It follows by Pontryagin duality that this map is evaluation of characters at an involution. The goal of this subsection is to show that when $T = \mathbb{G}_m$ and the map $X^*(T) = \mathbb{Z} \to \{\pm 1\}$ is nontrivial, this involution is nontrivial. Although the claim is quite weak, it is all that is needed for our computation of $e_{r,*}\varphi^*(c_G)$.

When k is nonarchimedean of characteristic zero we can describe $\mathrm{H}^2(W_k, X^*(T))$ using the cup-product pairing of Tate duality, as in Sect. 1, and the claim follows immediately from the naturality of that pairing. But in general there is no such naturality statement compatible with our coboundary description of $\mathrm{H}^2(W_k, X^*(T))$. Instead, we prove nontriviality using an argument with long exact sequences that rests on the following computation.

Lemma 25 *Let A be a W_k-module which is finitely generated as an abelian group.*

1. *If k is archimedean then $H^3_{Borel}(W_k, A)$ is torsionfree.*
2. *If k is nonarchimedean then $H^3(W_k, A)$ is a p-group.*

Recall that in the nonarchimedean case, p is the residue characteristic of k. A (possibly infinite) group is a *p-group* if each of its elements has order a power of p.

Proof When k is archimedean, we can argue as in the proof of Lemma 21. For $k = \mathbb{C}$ we conclude that $\mathrm{H}^3_{\mathrm{Borel}}(W_{\mathbb{C}}, A) = 0$ and for $k = \mathbb{R}$ we conclude, using the calculation on page 310 of [3], that $\mathrm{H}^3_{\mathrm{Borel}}(W_{\mathbb{R}}, A)$ is a free $\mathbb{Z}$-module of finite type.

Next, assume k is nonarchimedean. When $\operatorname{char} k = 0$, Karpuk proved [36, Section 3.2] the stronger statement that $\mathrm{H}^i(W_k, A) = 0$ for all $i \geq 3$. A slight modification of his argument, which we outline below, proves the Lemma 25.

The Hochschild–Serre spectral sequence, mentioned in the proof of Lemma 21, gives the vanishing for $i \geq 4$ and can be used to show, for $i = 3$, that $\mathrm{H}^2(W_k, P) \simeq \mathrm{H}^3(W_k, M)$, where $P \stackrel{\text{def}}{=} M_{\mathbb{Q}}/M$. Let $P[n]$ denote the n-torsion subgroup of P. The cohomology of W_k in P can be computed as a direct limit of the cohomologies in the torsion subgroups:

$$\mathrm{H}^2(W_k, P) \simeq \varinjlim_{n} \mathrm{H}^2(W_k, P[n]).$$

Since $P[n]$ decomposes as a direct sum of its Sylow subgroups, we can rewrite the direct limit as

$$\mathrm{H}^2(W_k, P) \simeq \mathrm{H}^2(W_k, P[p^\infty]) \oplus \varinjlim_{p\nmid n} \mathrm{H}^2(W_k, P[n])$$

where $P[p^\infty] \stackrel{\text{def}}{=} \bigcup_{n\geq 0} P[p^n]$. Since the group $P[p^\infty]$ is a p-group, the cohomology group $\mathrm{H}^2(W_k, P[p^\infty])$ is a p-group as well. It therefore suffices to show that the second summand above vanishes.

By Tate duality (see Theorem 30), if n is prime to p then $\mathrm{H}^2(W_k, P[n])$ is dual to $\mathrm{Hom}_{W_k}(P[n], \mu)$ where μ is the group of roots of unity in $\bar{k}$. It follows that the second term above, $\varinjlim_{p\nmid n} \mathrm{H}^2(W_k, P[n])$, is dual to

$$\mathrm{Hom}_{W_k}(P/P[p^\infty], \mu).$$

Since $P[p^\infty]$ is a summand of P, it suffices to show that $\mathrm{Hom}_{W_k}(P, \mu) = 0$. Karpuk has an argument for the vanishing that works just as well when $\operatorname{char} k > 0$. $\square$

Lemma 25 together with a long exact sequence argument proves our desired nontriviality.

Corollary 26 *Let ℓ be a finite separable extension of k, let $M = \mathrm{Ind}_{\ell/k}\,\mathbb{Z}$, and let $f\colon M \to \{\pm 1\}$ be the unique nontrivial homomorphism. If $\operatorname{char} k \neq 2$ then the* *induced map*

$$f_* : H^2(W_k, M) \to H^2(W_k, \{\pm 1\})$$

(taken in Borel cohomology if k is archimedean) is nontrivial.

Proof Since f factors through the augmentation map $M \to \mathbb{Z}$, it suffices to prove the corollary in the case where $\ell = k$ and $M = \mathbb{Z}$. The long exact sequence for the short exact sequence $1 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \{\pm 1\} \to 1$ yields a coboundary map

$$\mathrm{H}^2(W_k, \{\pm 1\}) \to \mathrm{H}^3(W_k, \mathbb{Z})$$

whose cokernel measures the failure of surjectivity. But by Lemma 25 the coboundary map must be trivial: there are no nontrivial homomorphisms between $\{\pm 1\}$ and a torsionfree group or a p-group with $p \neq 2$. ◻

5.4 Involutions from representations of *L*-groups

In this subsection, we explain how to construct from an orthogonal representation r of ${}^L G$ a central involution $z_r \in Z(k)$, generalizing Gross and Reeder's canonical involution. We then relate this involution to our description of the character group of $Z(k)^1$.

The involution z_r depends only on the restriction of r to $\widehat{G}$. This restriction is an algebraic representation of a complex reductive group and can thus be understood through its weights. We compute the weights of the representation with respect to a fixed, Galois-stable maximal torus $\widehat{T}$ of $\widehat{G}$, which is dual to a minimal Levi T of the quasi-split group G. Since $r|_{\widehat{G}}$ is the restriction of a representation of the L-group, the multiset of its weights is Galois-stable. Moreover, each weight can be interpreted as a coweight of T.

Let $m \colon X_*(T) \to \mathbb{N}$ be a Galois-invariant multiset of weights. In our application m will be the multiset of weights of a representation of ${}^L G$, but the greater generality is convenient for some proofs. As in Sect. 4.1, choose a gauge $X^*(\widehat{T}) \setminus \{0\} \to \{\pm 1\}$. This time we require the gauge to be Galois-invariant, and the requirement can be met because the Galois action preserves some pinning containing $\widehat{T}$. Define

$$z_m \stackrel{\text{def}}{=} \prod_{0 < \varpi \in X_*(T)} \varpi(-1)^{m(\varpi)}.$$

The involution z_m is independent of the choice of gauge. For r a representation of ${}^L G$ we set $z_r \stackrel{\text{def}}{=} z_m$ where $m(\varpi)$ is the multiplicity of ϖ in $r|_{\widehat{T}}$.

In this setting, the generalization of the spin character of Sect. 4.3 is the character $e_m \colon \pi_1(\widehat{G}) \simeq X^*(T) \to \{\pm 1\}$ defined by the formula

$$e_m(\lambda) = \prod_{0 < \varpi \in X_*(T)} (-1)^{m(\varpi)\langle \lambda, \varpi \rangle}.$$

Lemma 27 *Let $m\colon X_*(T) \to \mathbb{N}$ be a Galois-invariant multiset of weights. If G has connected center then the composite map*

$$\mathrm{Hom}_{cts}(Z(k)^1, \mathbb{C}^\times) \xrightarrow{\simeq} H^2(W_k, \pi_1(\widehat{G})) \xrightarrow{e_{m,*}} H^2(W_k, \{\pm 1\}) \xrightarrow{\mathrm{sgn}} \{\pm 1\}$$

(taken in Borel cohomology if k is archimedean) is evaluation at z_m.

Proof By additivity it suffices to consider the case where m is a (multiplicity-one) Galois orbit Ω. After choosing a gauge, we can factor $e_m\colon X^*(T) \to \{\pm 1\}$ as the composition

$$X^*(T) \longrightarrow \mathbb{Z}[\Omega] \longrightarrow \{\pm 1\}$$

of the map

$$\lambda \mapsto \sum_{\varpi \in \Omega} \langle \lambda, \varpi \rangle [\varpi]$$

and the mod-two augmentation map $\mathbb{Z}[\Omega] \to \{\pm 1\}$.

Apply the functor $\mathrm{H}^2(W_k, -)$ to this composition. By naturality of the coboundary, the image of the first map is isomorphic to the restriction map

$$\mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times) \to \mathrm{Hom}_{\mathrm{cts}}(S(k)^1, \mathbb{C}^\times)$$

where S is the k-torus with character group $\mathbb{Z}[\Omega]$. The image of the second map is a homomorphism

$$\mathrm{Hom}_{\mathrm{cts}}(S(k)^1, \mathbb{C}^\times) \to \{\pm 1\}$$

which is nontrivial by Corollary 26. And by Pontryagin duality this map must be evaluation at an element of order two. Since $S(k) \simeq \ell^\times$ where ℓ is the fixed field of any element of Ω, the group $S(k)$ has a unique element of order two, namely -1. The lemma follows. $\square$

Let χ_φ denote Langlands's central character, the construction of which is summarized in Borel's Corvallis article [7, 10.1],

Theorem 28 *Let $\varphi\colon W_k \to {}^L G$ be an L-parameter, let $r\colon {}^L G \to O(V)$ be a complex representation of ${}^L G$, and let $c_G \in H^2_{Borel}({}^L G, \pi_1(\widehat{G}))$ classify the extension $\widehat{G}_{univ} \rtimes W_k$. Then*

$$\mathrm{sgn}(e_{r,*}\varphi^*(c_G)) = \chi_\varphi(z_r).$$

Proof First, assume the center of G is connected. Let $\varphi_0\colon W_k \to \widehat{G}$ denote the 1-cocycle corresponding to φ, so that $\varphi(w) = \varphi_0(w)w$. The morphism of short exact sequences

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(\widehat{G}) \arrow[r] \arrow[d, "\simeq"] & \widehat{G}_{\mathrm{univ}} \arrow[r] \arrow[d] & \widehat{G} \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & X^*(Z) \arrow[r] & \hat{\mathfrak{z}} \arrow[r] & \widehat{Z} \arrow[r] & 1 \end{tikzcd}$$

gives rise to a commutative square

$$\begin{tikzcd} \mathrm{H}^1(W_k, \widehat{G}) \arrow[r] \arrow[d] & \mathrm{H}^1(W_k, \widehat{Z}) \arrow[d] \\ \mathrm{H}^2(W_k, \pi_1(\widehat{G})) \arrow[r, "\simeq"] & \mathrm{H}^2(W_k, X^*(Z)). \end{tikzcd}$$

Pass φ_0 around this square. By Lemma 7, the left arrow maps φ_0 to $\varphi^*(c_G)$. The top arrow maps φ_0 to the cocycle representing Langlands's character $\chi_\varphi\colon Z(k) \to \mathbb{C}^\times$, and the image of this cocycle under the right arrow corresponds to the restriction of χ_φ to $Z(k)^1$. Hence this restriction corresponds to $\varphi^*(c_G)$. The Lemma now follows from Lemma 27.

For a general G, whose center need not be connected, Langlands's construction of χ_φ starts with a choice of embedding of G into a group G_1 whose center Z_1 is connected. It turns out that every L-parameter $\varphi\colon W_k \to {}^LG$ lifts to an L-parameter $\varphi_1\colon W_k \to {}^LG_1$ [37, Théorème 7.1]. One then defines χ_φ as the restriction to $Z(k)$ of the character χ_{φ_1} of $Z_1(k)$. The resulting character χ_φ is independent of the choice of lift and the choice of G_1.

The projection $\widehat{G}_1 \to \widehat{G}$ induces the diagram

$$\begin{tikzcd} & \pi_1(G_1) \arrow[r, hook] \arrow[dl, "e_{r\circ h}"'] \arrow[dd, "f"] & \widehat{G}_{1,\mathrm{univ}} \rtimes W_k \arrow[r, two heads] \arrow[dd, "g"] & {}^LG_1 \arrow[dd, "h"] & & (c_1) \\ \{\pm 1\} & & & & W_k \arrow[ul, "\varphi_1"'] \arrow[dl, "\varphi"] & \\ & \pi_1(G) \arrow[r, hook] \arrow[ul, "e_r"] & \widehat{G}_{\mathrm{univ}} \rtimes W_k \arrow[r, two heads] & {}^LG & & (c_G) \end{tikzcd}$$

with exact rows. Let $c_1 \overset{\mathrm{def}}{=} c_{G_1} \in \mathrm{H}^2({}^LG_1, \pi_1(G_1))$ classify the upper extension. On the one hand, the existence of g implies that $h^*(c_G) = f_*(c_1)$, from which we conclude that

$$e_{r,*}\varphi^*(c_G) = e_{r,*}\varphi_1^*\, h^*(c_G) = e_{r,*}\varphi_1^*\, f_*(c_1) = e_{r\circ h,*}\varphi_1^*(c_1).$$

On the other hand, its easy to see from the definition of z_r that

$$\chi_\varphi(z_r) = \chi_{\varphi_1}(z_{r\circ h}).$$

These calculations reduce the general case to the previous one. $\square$

6 Synthesis

In this section, we complete the proof of Theorem A and explain its relationship to the Plancherel measure.

6.1 Proof of Theorem A

Let G be a quasi-split reductive k-group, let $\varphi\colon \mathrm{WD}_k \to {}^L G$ be a tempered parameter, and let $r\colon {}^L G \to \mathrm{O}(V)$ be a complex orthogonal representation. Since root numbers of orthogonal Weil–Deligne representations are unaffected by semisimplification, we replace φ by its restriction $\varphi\colon W_k \to {}^L G$ to the Weil group. The natural action of the Weil group on $\widehat{G}$ lifts to an action on its universal cover. Using this action, apply Lemma B to the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\widehat{G}) & \longrightarrow & \widehat{G}_{\mathrm{univ}} & \longrightarrow & \widehat{G} & \longrightarrow & 1 \\ & & \big\downarrow{\scriptstyle e_r} & & \big\downarrow & & \big\downarrow{\scriptstyle r|_{\widehat{G}}} & & \\ 1 & \longrightarrow & \{\pm 1\} & \longrightarrow & \mathrm{Pin}(V) & \longrightarrow & \mathrm{O}(V) & \longrightarrow & 1 \qquad (c_{\mathrm{pin}}), \end{array}$$

taking $W = W_k$. Here $c_{\mathrm{pin}} \in \mathrm{H}^2_{\mathrm{Borel}}(\mathrm{O}(V), \{\pm 1\})$ classifies the (bottom) pin extension and $c_G \in \mathrm{H}^2_{\mathrm{Borel}}({}^L G, \pi_1(\widehat{G}))$ classifies the extension $\widehat{G}_{\mathrm{univ}} \rtimes W_k$. We conclude from the lemma that

$$r^*(c_{\mathrm{pin}}) = e_{r,*}(c_G) \cdot p^* r|_{W_k}^*(c_{\mathrm{pin}}) \tag{3}$$

where $p\colon {}^L G \to W$ is the projection. Hence

$$\varphi^* r^*(c_{\mathrm{pin}}) = \varphi^* e_{r,*}(c_G) \cdot \varphi^* p^* r|_{W_k}^*(c_{\mathrm{pin}}),$$

taking values in $\mathrm{H}^2(W_k, \{\pm 1\})$ for k nonarchimedean and in $\mathrm{H}^2_{\mathrm{Borel}}(W_k, \{\pm 1\})$ for k archimedean. By our formulation of Deligne's theorem, Theorem 16,

$$\frac{\omega(\varphi, r)}{\omega(\varphi, \det r)} = \mathrm{sgn}\big(\varphi^* r^*(c_{\mathrm{pin}})\big).$$

At the same time, letting $\varphi_{\mathrm{prin}}\colon \mathrm{WD}_k \to {}^L G$ denote the principal parameter, the composition $r|_{W_k} \circ p \circ \varphi$ is the restriction of $r \circ \varphi_{\mathrm{prin}}$ to the Weil group. Deligne's theorem again implies that

$$\frac{\omega(\varphi_{\mathrm{prin}}, r)}{\omega(\varphi_{\mathrm{prin}}, \det r)} = \mathrm{sgn}\big(\varphi^* p^* r|_{W_k}^*(c_{\mathrm{pin}})\big).$$

Since $\widehat{G}$ is connected and r is orthogonal, $\det r$ restricts trivially to $\widehat{G}$. Hence

$$\omega(\varphi_{\mathrm{prin}}, \det r) = \omega(\varphi, \det r).$$

All in all, then, (3) simplifies to

$$\frac{\omega(\varphi, r)}{\omega(\varphi_{\mathrm{prin}}, r)} = \mathrm{sgn}\big(\varphi^* e_{r,*}(c_G)\big).$$

Finally, Theorem 28 identifies the righthand side with $\chi_\varphi(z_r)$.

6.2 Application to Plancherel measure

Let G be a reductive k-group and let M be a Levi subgroup of G. From this data we can form an orthogonal representation $r_{G,M}$ of ${}^L M$ which one might call the relative adjoint representation for $G \supseteq M$, namely, the representation

$$\mathrm{Lie}(\widehat{G})/\,\mathrm{Lie}(Z(\widehat{M})^{\Gamma_k}).$$

The set of nonzero weights of this $\widehat{M}$-representation is the root system of $\widehat{G}$.

Now let $\varphi\colon \mathrm{WD}_k \to {}^L M$ be a tempered discrete L-parameter for M and π a representation of $G(k)$ in the L-packet of φ. It is expected that the Weil–Deligne representation $r_{G,M} \circ \varphi$ encodes in its adjoint γ-factor the Plancherel measure on the part of the tempered dual of $G(k)$ coming from parabolic inductions of unramified twists of π. This expectation is recorded in Hiraga, Ichino, and Ikeda's article on formal degrees [4, Conjecture 1.5] but originates in Harish–Chandra's description of the Plancherel measure and Langlands's conjecture on normalization of intertwining operators. Theorem A largely computes the root number of this Weil–Deligne representation:

$$\omega(\varphi, r_{G,M}) = \omega(\varphi_{\mathrm{prin}}, r_{G,M}) \cdot \pi(z_{\mathrm{Ad}})$$

where z_{Ad} is Gross and Reeder's canonical involution for G (not M).

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Appendix A: Characters via Weil–Tate duality

Let T be a finite type k-group of multiplicative type. Our description of $\mathrm{H}^2(W_k, X^*(T))$ in Sect. 5 when T is a torus brings to mind a related and more classical result, Tate duality, which in this case describes $\mathrm{H}^2(\Gamma_k, X^*(T))$ as the character group of the profinite completion $T(k)_{\mathrm{pro}}$ of $T(k)$. This conclusion holds more generally whenever T is reduced. By analogy, one would hope that our description of $\mathrm{H}^2(W_k, X^*(T))$ from Sect. 5.2 as the character group of $T(k)^1$ could be strengthened by relaxing the hypothesis that T is a torus to the hypothesis that T is reduced.

In this appendix we provide partial evidence, in Theorem 30, for this hope by giving such a description in two cases, both for nonarchimedean k: when char k is arbitrary and T is finite and reduced, and when char $k = 0$ and T is arbitrary. Our starting point is Karpuk's thesis [36], which constructs a cup-product pairing generalizing Tate duality to the Weil group, provided that char $k = 0$. After explaining how this works, we will have two identifications of $\mathrm{H}^2(W_k, X^*(T))$ with a character group when T is a torus, one by coboundary and one by cup product. We then show that the two identifications are inverses of each other.

To compare the Harish–Chandra subgroup with Karpuk's formulation of Weil–Tate duality, we need a small lemma on (metric) completions. Let $\ell = \check{k}$ be the completion of the maximal unramified extension of k and let $\bar{\ell}$ be a separable closure of ℓ. By Krasner's lemma, $\Gamma_\ell = I_k$.

Lemma 29 *An element $a \in \bar{\ell}$ is algebraic over k if and only if its Galois orbit is finite.*

Proof The forward implication is clear. For the reverse implication, consider the polynomial $f(x) \in \bar{\ell}[x]$ whose set of roots is the Galois orbit of a. Since the coefficients of f are Galois-invariant, it suffices to show that k is the set of Galois-invariant elements of $\bar{\ell}$. This claim is a theorem of Tate [38, Theorem 1] which holds also when char $k > 0$ [1, Chapter XIII, Sect. 5, Lemma 1]. □

Theorem 30 *Let T be a finite type k-group of multiplicative type. If* char $k = 0$ *or T is finite and reduced then the cup-product pairing*

$$H^2(W_k, X^*(T)) \otimes H^0\big(W_k, \mathrm{Hom}(X^*(T), \mathcal{O}_{\bar{\ell}}^\times)\big) \to \mathbb{Q}/\mathbb{Z}$$

induces a canonical identification

$$H^2(W_k, X^*(T)) \simeq \mathrm{Hom}_{cts}(T(k)^1, \mathbb{C}^\times).$$

Here we interpret $\mathbb{Q}/\mathbb{Z}$ as a subgroup of $\mathbb{C}^\times$ via the exponential map $t \mapsto \exp(2\pi i t)$.

Proof Let $\bar{k}$ be the separable closure of k in $\bar{\ell}$. First, assume char $k = 0$. Karpuk showed that the topological groups $\mathrm{H}^2(W_k, X^*(T))$ and $\mathrm{Hom}(X^*(T), \mathcal{O}_{\bar{\ell}}^\times)^{W_k}$ are Pontryagin dual to each other [36, Proposition 3.3.5]. The split short exact sequence for the valuation on $\bar{\ell}^\times$ gives rise to an exact sequence describing Karpuk's group as a kernel:

$$1 \longrightarrow \mathrm{Hom}(X^*(T), \mathcal{O}_{\bar{\ell}}^\times)^{W_k} \longrightarrow \mathrm{Hom}(X^*(T), \bar{\ell}^\times)^{W_k} \longrightarrow \mathrm{Hom}(X^*(T), \mathbb{Z})^{W_k}.$$

Since the action of W_k on $X^*(T)$ has finite orbits, any W_k-equivariant homomorphism $X^*(T) \to \bar{\ell}^\times$ factors through $\bar{k}^\times$ by Theorem 29. Hence we may rewrite the sequence as

$$1 \longrightarrow \mathrm{Hom}(X^*(T), \mathcal{O}_{\bar{\ell}}^\times)^{W_k} \longrightarrow T(k) \longrightarrow X_*(T).$$

So the lefthand group is the Harish–Chandra subgroup of $T(k)$.

When T is finite and reduced, the fact [3, Lemma 4.2.1] that the inflation map

$$\mathrm{H}^2(\Gamma_k, X^*(T)) \to \mathrm{H}^2(W_k, X^*(T))$$

is an isomorphism reduces the problem to Tate duality for finite modules. □

Our next goal is to compare the description of Theorem 30 with our earlier coboundary description when T is a torus. Tate duality is the intermediary in the comparison. Let $T(k)_{\mathrm{pro}}$ denote the profinite completion of $T(k)$. Since $T(k)^1$ is compact, it is a subgroup of $T(k)_{\mathrm{pro}}$.

Lemma 31 *Let T be a k-torus. The following square commutes, where the horizontal arrows are the cup-product pairings and the vertical arrows are restriction.*

$$\begin{array}{ccc} H^2(\Gamma_k, X^*(T)) & \xrightarrow{\simeq} & \mathrm{Hom}_{cts}(T(k)_{pro}, \mathbb{C}^\times) \\ \downarrow & & \twoheaddownarrow \\ H^2(W_k, X^*(T)) & \xrightarrow{\simeq} & \mathrm{Hom}_{cts}(T(k)^1, \mathbb{C}^\times). \end{array}$$

Proof It suffices to observe that the subgroup inclusion $T(k)^1 \hookrightarrow T(k)_{\mathrm{pro}}$ can be identified with the map

$$\mathrm{H}^0(W_k, \mathrm{Hom}(X^*(T), \mathcal{O}_{\bar{\ell}}^\times)) \to \mathrm{H}^0(\Gamma_k, \mathrm{Hom}(X^*(T), \bar{\ell}^\times))$$

induced by inclusion $\mathcal{O}_{\bar{\ell}}^\times \hookrightarrow \bar{\ell}^\times$. Lemma 29 identifies the righthand group with $T(k)$. □

Tate duality describes $\mathrm{H}^2(\Gamma_k, X^*(T))$ as the character group of $T(k)_{\mathrm{pro}}$. Moreover, the extension of Artin reciprocity to tori describes $\mathrm{H}^1(\Gamma_k, \widehat{T})$ as the same character group. The two comparisons are related by the coboundary map for the exponential exact sequence

$$1 \longrightarrow X^*(T) \longrightarrow X^*(T)_{\mathbb{C}} \longrightarrow \widehat{T} \longrightarrow 1.$$

There is a twist, however: the composite isomorphism below is inversion [5, Section 8.2].

$$\mathrm{Hom}_{\mathrm{cts}}(T(k)_{\mathrm{pro}}) \simeq \mathrm{H}^1(\Gamma_k, \widehat{T}) \to \mathrm{H}^2(\Gamma_k, X^*(T)) \simeq \mathrm{Hom}_{\mathrm{cts}}(T(k)_{\mathrm{pro}})$$

Lemma 32 *Let T be a k-torus. The following square anticommutes, where the top arrow is the (profinitely completed) Langlands correspondence, the bottom arrow is from Theorem 30, the left arrow is the exponential coboundary, and the right arrow is restriction.*

$$\begin{array}{ccc} H^1(W_k, \widehat{T}) & \xrightarrow{\simeq} & \mathrm{Hom}_{cts}(T(k), \mathbb{C}^\times) \\ \downarrow & (-1) & \twoheaddownarrow \\ H^2(W_k, X^*(T)) & \xrightarrow{\simeq} & \mathrm{Hom}_{cts}(T(k)^1, \mathbb{C}^\times). \end{array}$$

Proof Consider the following extension of the given diagram in which the left square commutes by the compatibility of inflation and coboundary.

$$\begin{tikzcd} \mathrm{H}^1(\Gamma_k, \widehat{T}) \arrow[r, hook] \arrow[d, "\simeq"] & \mathrm{H}^1(W_k, \widehat{T}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Hom}_{\mathrm{cts}}(T(k), \mathbb{C}^\times) \arrow[d, two heads] \\ \mathrm{H}^2(\Gamma_k, X^*(T)) \arrow[r, two heads] & \mathrm{H}^2(W_k, X^*(T)) \arrow[r] & \mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times). \end{tikzcd}$$

By Lemma 31 and the compatibility between Tate duality and the local Langlands correspondence, the right square becomes commutative after restriction along the upper left arrow. We'll use this to deduce commutativity of the right square by a diagram chase.

The bottom left arrow is surjective by Lemma 31 and exactness of Pontryagin duality. Starting with an element $c \in \mathrm{H}^1(W_k, \widehat{T})$, move it clockwise around the left square, choosing a lift along the bottom left arrow. We thereby produce an element $c' \in \mathrm{H}^1(W_k, \widehat{T})$ in the image of $\mathrm{H}^1(\Gamma_k, \widehat{T})$. The elements c and c' have the same image in $\mathrm{H}^2(W_k, X^*(T))$, hence differ by an element in the kernel of the middle vertical arrow, or in other words, by an element in the image of $\mathrm{H}^1(W_k, X^*(T)_{\mathbb{C}})$. To prove commutativity of the right square, it therefore suffices to show that the following composition $f_T : \mathrm{H}^1(W_k, X^*(T)) \to \mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times)$ vanishes:

$$\mathrm{H}^1(W_k, X^*(T)_{\mathbb{C}}) \longrightarrow \mathrm{H}^1(W_k, \widehat{T}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathrm{cts}}(T(k), \mathbb{C}^\times) \twoheadrightarrow \mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times).$$

For this, we use the same strategy as in the proof of Lemma 23. When $T = \mathbb{G}_{\mathrm{m}}$, the image of the first map in $\mathrm{Hom}_{\mathrm{cts}}(k^\times, \mathbb{C}^\times)$ is the group of unramified characters and the claim is clear. It follows from Shapiro's lemma that f_T vanishes for any torus T that is induced, in other words, a product of Weil restrictions of split tori. For a general torus T, choose an embedding $T \hookrightarrow S$ into an induced torus S and let R denote the cokernel, a third torus. This embedding yields the commutative square

$$\begin{tikzcd} \mathrm{H}^1(W_k, X^*(S)_{\mathbb{C}}) \arrow[r, "f_S"] \arrow[d] & \mathrm{Hom}_{\mathrm{cts}}(S(k)^1, \mathbb{C}^\times) \arrow[d] \\ \mathrm{H}^1(W_k, X^*(T)_{\mathbb{C}}) \arrow[r, "f_T"] & \mathrm{Hom}_{\mathrm{cts}}(T(k)^1, \mathbb{C}^\times). \end{tikzcd}$$

Since $\mathrm{H}^2(W_k, X^*(R)_{\mathbb{C}}) = 0$ by Lemma 21, the left arrow is surjective. We can thus deduce the vanishing of f_T from the vanishing of f_S. □

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