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Cross characteristic representations of symplectic and unitary groups

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1. Introduction

In [LS], Landazuri and Seitz gave lower bounds for irreducible representations of Chevalley groups in nondefining characteristic (when referring to irreducible representations for quasi-simple groups G, we will assume that the modules are nontrivial on $F^*(G)$). See also [SZ,GPPS,HF] for some improvements on these bounds. These results have proved to be useful in many applications. In particular, they have been used to classify the maximal subgroups of classical groups containing an element of prime order acting irreducibly on a subspace

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of large dimension (cf. [GPPS]), and to show that low-dimensional modules in characteristic p for groups with no normal p-subgroup are semisimple (see [Gu]).

It is also important to identify the modules which have dimension close to the smallest possible dimension and to prove that there are no irreducible modules with dimension in a certain range above it. This was done in [GPPS,GT1] for $SL_n(q)$. Further improvements were obtained by Brundan and Kleshchev [BrK]. Hiss and Malle [HM] have obtained results similar to [GT1] for unitary groups.

In this paper, we consider the groups $G = Sp_{2n}(q)$ with $n \geqslant 2$ and $q = p^f$ odd and $G = SU_n(q)$ with $n \geqslant 3$. Throughout the paper, r is a prime not dividing q and k is algebraically closed of characteristic r. Landazuri and Seitz [LS] had already shown that the minimal dimension d of any irreducible module in the nondefining characteristic r is $(q^n - 1)/2$ for the symplectic case, and $[(q^n - 1)/(q + 1)]$ for the unitary case. It was proved in [GPPS] that (aside from some small exceptions) every irreducible kG-module in a nondefining characteristic has dimension d, $d + 1$ or at least dimension $2d$. In characteristic 0, Tiep and Zalesskii [TZ1] (using Deligne–Lusztig theory) obtained much stronger results about the gap between possible dimensions for all the classical groups. Similar results for complex representations of exceptional groups were obtained by Lübeck [Lu]. Here we show that a similar result is true in all characteristics other than the defining characteristic. The gap we obtain is essentially the same as in characteristic 0. The smallest modules are the Weil modules described below.

The families $SL_n(q)$, $Sp_{2n}(q)$ with q odd, and $SU_n(q)$ all have Weil modules which are much smaller than the other irreducible modules. The differences between the small modules and other modules for the other Chevalley groups are not as dramatic. This makes it much more difficult and requires new methods to analyze those other groups. In particular, the family of $Sp_{2n}(q)$ with q even has recently been handled in [GT2].

The methods we use are different for the two families. If V is a kH-module, we denote by τ_V the Brauer character associated to V. Although τ_V is a priori only defined on elements whose order is coprime to r, we can extend τ_V to H by declaring that $\tau_V(g) = \tau_V(g')$ where $g = g'h = hg'$ with g' of order coprime to r (clearly, such g' is unique). For the symplectic case, our main method is to analyze modules with various local properties and by restricting to various families of subgroups which contain a conjugate of every element of the group we can determine the Brauer character of the module. Thus, we obtain results which characterize the Weil modules by several different properties (see Section 2 for statements of the main results and more details). Observe that it is not known whether the decomposition matrix in this case has unitriangular shape or not.

For the unitary group, we start from the deep results of Hiss and Malle [HM] which depend on Deligne–Luzstig theory and knowledge of the decomposition matrices. We can improve their bounds. Indeed, we obtain the correct bound for the dimension of an irreducible cross characteristic module (other than the Weil modules) for the unitary groups. We also obtain more detailed information for some of the low rank unitary groups which depend upon the results of Broué and Michel [BM] on unions of r-blocks and the results of Fong–Srinivasan [FS] on basic sets of Brauer characters (cf. also [GH]).

The Weil modules for the symplectic groups $G = Sp_{2n}(q)$ with q odd are constructed in a very natural way. Let E be an extra-special group of order pq^{2n} of exponent p (i.e. $[E, E] = \Phi(E) = Z(E)$ has order p). For each nontrivial linear character χ of $Z(E)$, the group E has a unique irreducible module M of dimension q^n over any algebraically closed field of characteristic $r \neq p$ that affords the $Z(E)$-character $q^n\chi$. Now G acts faithfully on E and trivially on $Z(E)$, and one can extend M to the semidirect product EG. If we restrict M to G, then $M = [t, M] \oplus C_M(t)$ where t is the central involution in G. If $r \neq 2$, these are irreducible modules; if $r = 2$, then $[t, M]$ is irreducible and $M/C_M(t) \simeq [t, M]$. It turns out that there are only two possible isomorphism types for M as kG-modules. We call the irreducible kG-modules obtained in such a manner *Weil modules*. Observe that the modules in characteristic $r > 0$ are just the reductions of the corresponding characteristic 0 modules (since M itself is the reduction of the irreducible EG-module which as noted is unique given the central character). If r is odd, there are two irreducible modules of each dimension $(q^n \pm 1)/2$; if $r = 2$ we get two irreducible modules of dimension $(q^n - 1)/2$.

A similar but slightly more complicated construction [S] leads to the complex Weil modules of the special unitary groups $SU_n(q)$ (here q may be even as well); there is one such a module of dimension $(q^n + q(-1)^n)/(q + 1)$ and q such of dimension $(q^n - (-1)^n)/(q + 1)$. All of them extend to $U_n(q)$ if $n \geqslant 3$, see [TZ2, Lemma 4.7]. Furthermore, any nontrivial irreducible constituent of the reduction modulo any cross characteristic r of a complex Weil module of $SU_n(q)$ or $U_n(q)$ lifts to characteristic 0, cf. for instance [HM]. Abusing language, we will refer to any such irreducible constituent a *Weil module* in characteristic r.

There is an extensive literature on the Weil modules. We summarize some of the known results in Section 5 and give some references in the bibliography.

We will then apply our results to the classification of quadratic modules and to answer some questions about minimal polynomials of elements of prime order in cross characteristic representations of Chevalley groups. We also indicate how one can use our results to find the modulo 2 structure of the rank 3 permutation module M of $Sp_{2n}(q)$ on 1-spaces of the natural module $\mathbb{F}_q^{2n}$, cf. Example 10.2; $M \pmod r$ for $r \neq 2$ was considered by Liebeck [Li], and Zalesskii and Suprunenko [ZS].

The paper is organized as follows. Sections 2 and 3 contain the formulation of our main theorems. Section 4 collects some general results that we will need in the sequel. Section 5 describes Weil modules of finite symplectic groups and some of their properties. In Sections 6 and 7 we study the modules with certain properties $(\mathcal{R}_1)$ (cf. Theorem 2.2) and property $(\mathcal{R}_2)$ (cf. Theorem 2.3). In Sections 8 and 9 we prove Theorem 2.2 for $n \geqslant 3$ and $r \neq 2$, respectively $r = 2$. In Section 10 we finish the proof of Theorem 2.2, and give proofs of Theorems 2.1 and 2.3.

In Sections 11 and 12 we study cross characteristic representations of finite unitary groups of low dimension, and prove Theorems 2.5–2.7. Theorems 3.1 and 3.2 about the minimal polynomial problem are proved in Section 13. Finally, Theorem 3.3 is proved in Section 14.

2. Low-dimensional representations of finite symplectic and unitary groups

In this section we state our results about low-dimensional cross characteristic representations of finite symplectic and unitary groups. Recall that we assume throughout the paper that r is a prime not dividing q and k is algebraically closed of characteristic r.

Theorem 2.1. *Let $G = Sp_{2n}(q)$ with $n \geqslant 2$ and $q = p^f$ odd. Let V be an irreducible kG-module of dimension less than $(q^n - 1)(q^n - q)/2(q + 1)$. Then V is either the trivial module, or a Weil module of dimension $(q^n \pm 1)/2$.*

Observe that G has a unique irreducible complex character ρ of degree $(q^n - 1)(q^n - q)/2(q + 1)$, and ρ is irreducible modulo r, cf. Lemma 7.4, so the bound given in this theorem is the best possible.

Theorem 2.2. *Let $G = Sp_{2n}(q)$ with $n \geqslant 2$ and $q = p^f$ odd. Let V be an irreducible kG-module with property*

($\mathcal{R}_1$) *a long root subgroup has at most $(q - 1)/2$ nontrivial linear characters on V.*

Then V is either trivial or a Weil module.

If V is a Weil module and Z is a long root subgroup, then the set of nontrivial linear characters of Z occurring on V is one of the two sets Ω_i, $i = 1, 2$, defined in Section 5, both of cardinality $(q - 1)/2$. Accordingly V is said to have *type i*.

Theorem 2.3. *Let $G = Sp_{2n}(q)$ with $n \geqslant 3$ and $q = p^f$ odd. Let V be an irreducible kG-module satisfying at least one of the following conditions.*

($\mathcal{R}_2$) *If $Y = Y_1 \times Y_2$ is a commuting pair of (distinct) long root subgroups, then all nontrivial linear characters of Y on V are of the form $\alpha \otimes \beta$, where either $\alpha, \beta \in \Omega_1$ or $\alpha, \beta \in \Omega_2$.*

($\mathcal{W}$) *For some j with $2 \leqslant j \leqslant n - 1$, the restriction of V to a standard subgroup $Sp_{2j}(q)$ involves only irreducible Weil modules and maybe the trivial modules.*

($\mathcal{Q}$) *Any P_n-orbit of nontrivial linear Q_n-characters on V is of length less than $(q^n-1)(q^n-q)/2(q+1)$.*

Then V is either trivial or a Weil module.

By a standard subgroup $Sp_{2j}(q)$ in $Sp_{2n}(q)$ we mean the pointwise stabilizer of a nondegenerate $(2n-2j)$-dimensional subspace of the natural module. Also, P_j is the stabilizer of a j-dimensional totally isotropic subspace in the natural module, and $Q_j = O_p(P_j)$.

Theorem 2.2 immediately yields the following consequence.

Corollary 2.4. *Let $G = Sp_{2n}(q)$ with $n \geqslant 2$ and $q = p^f$ odd. Let V be an irreducible kG-module such that the restriction of V to a standard subgroup $SL_2(q)$ involves only Weil modules of a given type and maybe the trivial module. Then V is either trivial or a Weil module.*

The example $G = Sp_{2n}(3)$ with $r = 2$ shows that one cannot remove the words "of a given type" from Corollary 2.4: all irreducible modules of $SL_2(3)$ in characteristic 2 are either Weil module or the trivial module.

Throughout the paper, $U_n(q)$ stands for the general unitary group $GU_n(\mathbb{F}_{q^2})$. By a standard subgroup $SU_j(q)$ in $SU_n(q)$ or $U_n(q)$ we mean the pointwise stabilizer in $SU_n(q)$ of a nondegenerate $(n-j)$-dimensional subspace of the natural module. Furthermore, P_j is the stabilizer in $SU_n(q)$ of a j-dimensional totally isotropic subspace in the natural module, and $Q_j = O_p(P_j)$. As an analogue of Theorem 2.3, we have the following results.

Theorem 2.5. *Let $G = SU_n(q)$ or $U_n(q)$, and let $n \geqslant 4$. Let V be an irreducible kG-module with the following property*:

($\mathcal{W}$) *For some j, $3 \leqslant j \leqslant n-1$, the restriction of V to a standard subgroup $SU_j(q)$ involves only irreducible Weil modules and maybe the trivial modules.*

Then V is either of dimension 1 or a Weil module.

Theorem 2.6. *Let $S = SU_n(q)$, $n \geqslant 5$, and $m = [n/2]$. Suppose that V is an irreducible kS-module such that any P_m-orbit of nontrivial linear characters of $Z(Q_m)$ on V is of length less than $(q^n-1)(q^{n-1}-q)/(q^2-1)(q+1)$ if n is even, and $(q^{n-1}-1)(q^{n-2}-q)/(q^2-1)(q+1)$ if n is odd. Then V is either trivial or a Weil module.*

Theorem 2.6 is also true for $n=4$ and $S=U_4(q)$. If $S=SU_4(q)$, then we need to replace the bound $q(q-1)(q^2+1)$ by $q(q-1)(q^2+1)/\gcd(2,q-1)$, cf. Proposition 11.7.

Hiss and Malle [HM] have shown that any irreducible $SU_n(q)$-module V in cross characteristic r is either trivial or a Weil module, if

$$\dim(V) < q^{n-2}(q-1)\left[\frac{q^{n-2}-1}{q+1}\right],$$

$n \geqslant 6$ and $(n,q) \neq (6,3)$. We will improve this "gap" result by establishing the following theorem, in which we define

$$\kappa_n(q,r) = \begin{cases} 1, & \text{if } \mathrm{char}(k) = r \text{ divides } \dfrac{q^{2[n/2]}-1}{q^2-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 2.7. *Let $G = SU_n(q)$ and $n \geqslant 5$. Suppose that* $\mathrm{char}(k) = r$ *and V is an irreducible kG-module of dimension less than*

$$\mathfrak{d}(n,q,r) := \begin{cases} \dfrac{(q^n-1)(q^{n-1}-q)}{(q^2-1)(q+1)}, & \text{if } 2 \mid n \text{ and } q=2, \\ \dfrac{(q^n-1)(q^{n-1}+1)}{(q^2-1)(q+1)} - 1 - \kappa_n(q,r), & \text{if } 2 \mid n \text{ and } q>2, \\ \dfrac{(q^n+1)(q^{n-1}-q^2)}{(q^2-1)(q+1)} - \kappa_n(q,r), & \text{if } n \geqslant 7 \text{ is odd}, \\ \dfrac{(q^n+1)(q^{n-1}-q^2)}{(q^2-1)(q+1)} - 1, & \text{if } n = 5. \end{cases}$$

Then V is either trivial or a Weil module.

If $n \geqslant 6$ is even and $q=2$, then $SU_n(q)$ has an irreducible complex character ϑ of degree equal to $\mathfrak{d}(n,q,r)$, cf. [TZ1, Corollary 4.2]. By Theorem 2.6, $\vartheta \pmod r$ is irreducible in any characteristic r. In general, if $n \geqslant 5$ then $SU_n(q)$ has an irreducible complex character of degree

$$\begin{cases} \dfrac{(q^n-1)(q^{n-1}+1)}{(q^2-1)(q+1)}, & \text{if } n \geqslant 6 \text{ is even and } q>2, \\ \dfrac{(q^n+1)(q^{n-1}-q^2)}{(q^2-1)(q+1)}, & \text{if } n \geqslant 5 \text{ is odd} \end{cases}$$

(which is at most $\mathfrak{d}(n,q,r)+2$), cf. [TZ1, Corollary 4.2]. If q is odd, then the reduction modulo $r=2$ of the complex unipotent character $\chi_{(n-2,2)}$ of $SU_n(q)$ labeled by the partition $(n-2,2)$ has an irreducible constituent of degree $\mathfrak{d}(n,q,r)$ if $n \geqslant 5$, cf. [HM]. More generally, if $n \geqslant 5$ and $r \mid (q+1)$, then

$\chi_{(n-2,2)} \pmod r$ contains an irreducible constituent of degree $\mathfrak{d}(n,q,r)$, cf. [ST]. Hence the bound $\mathfrak{d}(n,q,r)$ given in Theorem 2.7 is the correct bound.

If $G = SU_4(q)$ and $q > 2$, then we need to replace the bound $\mathfrak{d}(4,q,r)$ in Theorem 2.7 by (see [HM])

$$\frac{(q^2+1)(q^2-q+1)}{\gcd(2,q-1)} - 1.$$

3. Minimal polynomials and quadratic modules

In this section we state our results concerning the *minimal polynomial problem* and the *quadratic module* problem.

If Θ is a kG-representation and $g \in G$ then $d_\Theta(g)$ stands for the degree of the minimal polynomial of $\Theta(g)$; similarly for $d_V(g)$ where V is a kG-module. For $g \in G$, $o(g)$ is the order of g modulo $Z(G)$. In generic position one expects that $d_V(g) = o(g)$; so the minimal polynomial problem is to classify all triples (G, V, g), where G is a finite group, V an irreducible G-module, and $g \in G$ an element such that $1 < d_V(g) < o(g)$. This is a problem with long history, different instances, and numerous results; for a brief account of it see [Z2].

Important results on the minimal polynomial problem in the case where G is a finite Lie-type group of simply connected type in characteristic p, g is a unipotent element of order p, and V is an irreducible G-module in characteristic $r \neq p$, have been proved by Zalesskii [Z1,Z2]. In particular, he has determined all possible pairs (G, g), see Theorem 13.1. It remains to classify the modules V for each of these pairs (G, g). This task has been done in [TZ2] in the case $r = 0$. Here we complete the classification of possible modules V in any characteristic $r \neq p$.

Theorem 3.1. *Let G be a finite quasi-simple group of Lie type of characteristic $p > 0$ of simply connected type, and suppose $g \in G$ is of order p. Let Θ be a nontrivial absolutely irreducible representation of G in characteristic $r \neq p$ such that $d_\Theta(g) < p$. Then $p > 2$ and one of the following holds*:

- (i) *$G = SU_3(p)$, g is a transvection, and Θ is the reduction modulo r of the (unique) complex representation of degree $p(p-1)$.*
- (ii) *$G = SL_2(p)$, and Θ is either a Weil representation or a representation of degree $p-1$.*
- (iii) *$G = SL_2(p^2)$, and Θ is a Weil representation.*
- (iv) *$G = Sp_4(p)$, and Θ is either a Weil representation, or the unique representation of degree $p(p-1)^2/2$.*
- (v) *$G = Sp_{2n}(p)$, $n \geqslant 3$, g is a transvection, and Θ is a Weil representation.*

Moreover, in each of these cases there exists a representation Θ and an element g satisfying the above conditions.

Another interesting instance of the minimal polynomial problem is to study the case where G is a finite classical group in characteristic p, g is a semisimple element, and V is an irreducible G-module in characteristic $r \neq p$. In this case, all possible pairs (G, g) have been identified by DiMartino and Zalesskii in [DZ], see Theorem 13.2 (see also [FLZ,Z3] for results on somewhat different but related configurations of the problem). The possible modules V for each of these pairs (G, g) in the case $r = 0$ have been classified in [TZ2]. Here we complete the classification of possible modules V in any characteristic $r \neq p$.

Theorem 3.2. *Let $G = Sp_{2n}(q)$ with $n > 1$ and $(n, q) \neq (2, 3)$, or $G = U_n(q)$ with $n > 2$. Let s be a prime not dividing q and let $g \in G$ be a noncentral element such that g belongs to a proper parabolic subgroup of G and $o(g)$ is a power of s. Let V be a nontrivial absolutely irreducible G-module in characteristic coprime to q such that $d_V(g) < o(g)$. Then V is a Weil module.*

In the case $(n, q) = (2, 3)$ there exists one more possibility for V, cf. Remark 13.3.

Theorem 3.2 and the following theorem complete the problem of classifying quadratic modules in characteristic s for finite groups G with $F^*(G)$ being quasi-simple but not of Lie type in the same characteristic s. See Section 14 for a detailed discussion of the quadratic module problem and a classification which follows from [Ch] and Theorems 3.2, 3.3.

Theorem 3.3. *Each of the groups $2Sp_6(2)$, $2\Omega_8^+(2)$, $2J_2$, $2G_2(4)$, $2Sz$, and $2Co_1$, has a unique irreducible quadratic $\mathbb{F}_3$-module V. In the first two cases V can be obtained by reducing the root lattice of type E_8 modulo 3, and in the last four cases V can be obtained by reducing the Leech lattice modulo 3.*

4. Preliminary results and notation

Let k be a field (usually assumed to be algebraically closed for simplicity) of characteristic $r \geqslant 0$. Let G be a finite group and V be a finite-dimensional kG-module. If H is a subgroup of G, we denote by $[H, V]$ the subspace generated by all elements of the form $(h - 1)v$ with $h \in H$ and $v \in V$, and by $C_V(H)$ the subspace of V consisting of all vectors fixed by H.

Let $\mathrm{soc}(V)$ denote the socle of V and consider the socle series of V. Thus $\mathrm{soc}_0(V) = 0$ and $\mathrm{soc}_i(V)$ is defined by $\mathrm{soc}_i(V)/\mathrm{soc}_{i-1}(V) = \mathrm{soc}(V/\mathrm{soc}_{i-1}(V))$.

Suppose that S is a composition factor of V. Let $j(S)$ denote the smallest i so that S is a composition factor of $\mathrm{soc}_i(V)$. We say S is a *level* $j(S)$ *composition factor of* V.

Lemma 4.1. *Let S be a composition factor of a kG-module V of level $j = j(S)$. Let e denote the multiplicity of S in* $\mathrm{soc}_j(V)$. *There exists a unique submodule $\Gamma = \Gamma_V(S)$ of V with the following properties*:

(i) $\Gamma/\mathrm{rad}(\Gamma)$ *is a direct sum of e copies of S*;
(ii) $\Gamma \subseteq \mathrm{soc}_j(V)$.

Proof. We induct on the dimension of V. If V is semisimple the result is clear. Next, $\Gamma_V(S) = \Gamma_W(S)$ with $W = \mathrm{soc}_j(V)$ and so we may assume $V = \mathrm{soc}_j(V)$. Similarly, we may assume that $V/\mathrm{rad}(V)$ involves only the composition factor S (since $\Gamma(S)$ is contained in the preimage of the S-homogeneous component of the map $V \to V/\mathrm{rad}(V)$).

Suppose that $V = A \oplus B$ and $0 \neq B$ does not involve S. Then $\Gamma_A(S)$ exists by induction, and any such module Γ is contained in A (because if $\phi \in \mathrm{Hom}(\Gamma, B)$, then $\ker(\phi) + \mathrm{rad}(\Gamma) = \Gamma$). So we may assume that every indecomposable summand of V involves S and modulo its radical involves only S.

At this point, V satisfies the conditions for Γ. We claim that no proper submodule does. If a proper submodule U did satisfy the conditions, then $U + \mathrm{rad}(V) = V$, whence $U = V$. $\square$

We state the next result in more generality than we need. We will be applying this in the situation where L is a Levi subgroup (or normal in a Levi subgroup) and U is the unipotent radical of the corresponding parabolic subgroup, with g an element conjugating P to the opposite parabolic.

Lemma 4.2. *Let k be an algebraically closed field of characteristic $r \geqslant 0$. Let V be a kG-module with $C_V(G) = 0$. Assume that $P = LU$ is a subgroup of G with U a normal r'-subgroup of P, $g \in N_G(L)$ with $G = \langle U, U^g\rangle$. Then the following statements hold*:

(i) $V = [U, V] \oplus C_V(U)$.
(ii) *If V is irreducible and $[U, V]$ is a semisimple L-module, then V is a semisimple L-module.*
(iii) *If S is an L-composition factor of $C_V(U)$, then either S or $S^{g^{-1}}$ is an L-composition factor of $[U, V]$.*
(iv) *If S is an L-composition factor of $C_V(U)$ of level i, then either S is a composition factor of $[U, V]$ of level less than i or $S^{g^{-1}}$ is an L-composition factor of $[U, V]$ of level at most i. In particular, if g centralizes L, then S is an L-composition factor of $[U, V]$ of level at most i.*

Proof. (i) is clear since U is an r'-group.

(ii) since $[U,V]$ is a semisimple L-module, so is $g([U,V]) = [U^g,V]$. Thus, $\mathrm{soc}_L(V)$ is U- and U^g-invariant (as any subspace containing $[U,V]$ is U-invariant). Since $G = \langle U, U^g\rangle$, $\mathrm{soc}_L(V)$ is G-invariant and so is equal to V by irreducibility.

(iii) follows from (iv).

Finally, we prove (iv). Suppose the claim is false. If $S^{g^{-1}}$ is not an L-composition factor of $[U,V]$ of level at most i, then S is not an L-composition factor of $[U^g,V]$ of level at most i; it then follows that $\Gamma_V(S)$ is a submodule of $C_V(U^g)$. On the other hand, if S is not a composition factor of $[U,V]$ of level less than i, then $\Gamma_V(S)\cap C_V(U)\neq 0$. Thus,

$$0\neq \Gamma_V(S)\cap C_V(U)\subseteq C_V(U^g)\cap C_V(U)=C_V(G)=0,$$

a contradiction. □

Lemma 4.3. *Let R be a ring and V a finite length R-module. Let $\mathcal{X}$ be a family of isomorphism classes of simple R-modules.*

(i) *There exists a unique submodule $V(\mathcal{X})$ of V which is maximal with respect to all composition factors of $V(\mathcal{X})$ belonging to $\mathcal{X}$.*
(ii) *$V(\mathcal{X})$ is the minimal submodule of V such that $V(\mathcal{X})$ has all composition factors in $\mathcal{X}$ and* $\mathrm{soc}(V/V(\mathcal{X}))$ *has no composition factors in $\mathcal{X}$.*
(iii) *If $V=V_1\oplus V_2$, then $V(\mathcal{X})=V_1(\mathcal{X})\oplus V_2(\mathcal{X})$.*

Proof. Note that if M_1 and M_2 are submodules involving only composition factors in $\mathcal{X}$, then so does M_1+M_2 (since it is a homomorphic image of $M_1\oplus M_2$). This shows (i). Clearly, (ii) holds and (iii) follows from (i) and (ii). □

Throughout the paper until Section 9, we fix $G = Sp_{2n}(q)$ with $n\geqslant 1$ and $q=p^f$ for p an odd prime. We assume that $r\neq p$ and k is an algebraically closed field of characteristic r. If $n\leqslant 2$, then all irreducible kG-modules are well known (see [Bu,Wh1,Wh2,Wh3]).

Let B be a Borel subgroup of G. We consider the maximal parabolic subgroups containing B. Let P_j denote the stabilizer of a totally isotropic j-subspace in the natural representation of G. Let $Q_j = O_p(P_j)$ and let $Z_j = Z(Q_j)$. Let P'_j denote the subgroup of P_j generated by the root subgroups of P_j (which is usually the commutator subgroup of P_j).

In particular, let $Z = Z_1 = Z(P'_1)$, so that Z is a long root subgroup, say $\{x_\alpha(t)\mid t\in\mathbb{F}_q^*\}$, of G, and $P_1 = N_G(Z)$. Throughout the paper, every long root subgroup will be considered as $\{x_\beta(t)\mid t\in\mathbb{F}_q^*\}$; in particular, $x_\beta(t)$ is conjugate to $x_\alpha(t)$. Let L_j denote a Levi subgroup of P_j, so $L_j = GL_j(q)\times Sp_{2(n-j)}(q)$. Let L'_j denote the subgroup of L_j generated by the root subgroups in L_j (and so $L'_j = SL_j(q)\times Sp_{2(n-j)}(q)$). We can identify Z_j with the $GL_j(q)$-module of symmetric $j \times j$ matrices over $\mathbb{F}_q$. Note that $Sp_{2(n-j)}(q)$ acts trivially on Z_j and that Q_j/Z_j is just the tensor product of the natural modules for the components of L_j. For $1 \leqslant d \leqslant n-1$, let $H_d \simeq Sp_{2d}(q) \times Sp_{2n-2d}(q)$ be the stabilizer of a nondegenerate $2d$-dimensional subspace of the natural module of G.

The next result gives a family of subgroups which contain a conjugate of every element in G. Note that $SL_2(q^n)$ naturally embeds in $Sp_{2n}(q)$ by viewing the natural 2-dimensional module over $\mathbb{F}_{q^n}$ as a $2n$-dimensional vector space over $\mathbb{F}_q$.

Lemma 4.4. *If $g \in G$, then a conjugate of g is contained in at least one of the following subgroups: P_1, P_n, P_j', H_d, and $SL_2(q^n)$.*

Proof. Write $g = su$, where s is semisimple and u is unipotent and $[s, u] = 1$. Let V be the natural module for G. Suppose W is an irreducible s-submodule of V of dimension e. If W is not self-dual, then the homogeneous component $H(W)$ of V corresponding to W is totally singular as is $H(W^*)$. If $\dim(H(W)) = n$, then g is conjugate to an element of P_n. Otherwise, g stabilizes the nonsingular subspace $H(W) \oplus H(W^*)$, whence g is conjugate to an element of H_d for $d = 2\dim(H(W))$.

So assume every irreducible component of s is self-dual. Then $H(W)$ is nonsingular (since $H(W)$ is orthogonal to all other homogeneous components—pass to the algebraic closure to see this). If $H(W) \neq V$, then g is conjugate to an element of some H_d.

Now assume that $V = H(W)$. If $V = W$, then $g = s$ and g is contained in the centralizer of a cyclic Sylow l-subgroup where l is a primitive prime divisor of $q^{2n} - 1$ (if $n = 1$ or $(n, q) = (3, 2)$, this l does not exist, but the result follows by inspection)—this centralizer has order $q^n + 1$ which is the same as the order of the centralizer in $SL_2(q^n)$. Thus, by Sylow's Theorem, g is conjugate to an element of $SL_2(q^n)$.

Suppose that W is a proper subspace of V. If W is nonsingular, then s is conjugate to an element of H_e. If W is totally singular and n-dimensional, then s is conjugate to an element of P_n. If $e < n$ and W is totally singular, then s leaves invariant a subspace of the form $W \oplus W'$ where W' is a complement to $W^{\perp}$ and so s is conjugate to an element of H_{2e}. So we have proved the result for the case $g = s$.

Thus, we may assume that $u \neq 1$. Note that $C_V(u) \cap [u, V]$ is a nontrivial totally singular g-invariant subspace. So we may assume that it contains W which is therefore g-invariant. Thus, we may assume $g \in P_e$. Let $\alpha = \det(s|_W)$. As W is self-dual, $\alpha = \pm 1$. If $\alpha = 1$, then $\det(g|_W) = 1$ and so a conjugate of g lies in P_e'.

We claim that $\alpha = -1$ implies that g stabilizes a maximal totally singular subspace and so g is conjugate to an element of P_n. We induct on n. If $n = 1$, the result is clear. Since $\det(s) = 1 = \alpha^{2n/e}$, we see that e divides n. Pass to $W^{\perp}/W$. The inductive hypothesis still holds, whence g leaves invariant a totally singular subspace U/W in $W^{\perp}/W$. Then, g stabilizes the maximal totally singular subspace U as desired. $\square$

We will also need the following well-known fact about pairs of long root subgroups in G. It follows from the fact that $P_1 = N_G(Z_1)$ and that G is a rank 3 permutation group on the cosets of P_1 (when $n \geqslant 2$).

Lemma 4.5. *If $n \geqslant 2$ then $Sp_{2n}(q)$ has 2 orbits on pairs of distinct long root subgroups. Either the long root subgroups commute or they generate an $SL_2(q)$* (*which acts trivially on a nondegenerate subspace of codimension* 2). *In particular, any two commuting pairs of distinct long root subgroups are conjugate.*

Next we make the following observation about the Jordan canonical form $\mathrm{Jord}(J_s \otimes J_t)$ of $J_s \otimes J_t$, where J_j is the Jordan block of size j with eigenvalue 1 over a field k of characteristic r.

Lemma 4.6. (i) *Suppose that $1 \leqslant s, t \leqslant r-1$ and $s+t > r$. Then* $\mathrm{Jord}(J_s \otimes J_t)$ *contains a block of size r.*

(ii) *Suppose that $r = 2$, let $s \geqslant 2^n - 1$ and $t \geqslant 2$. Then* $\mathrm{Jord}(J_s \otimes J_t)$ *contains a block of size $\geqslant 2^n$.*

Proof. (i) follows from [F, Theorem 8.2.7].

(ii) It suffices to prove that the minimal polynomial of $J_s \otimes J_2$ has degree $\geqslant 2^n$ for $s = 2^n - 1$. Let an operator g act on a k-space $\langle e_1, \dots, e_s\rangle$, respectively $\langle f_1, f_2\rangle$, via the matrix J_s, respectively J_2. Then direct computation shows that $(g-1)^{2^n-1}(e_s \otimes f_2) = e_1 \otimes f_1$, and so we are done. $\square$

The following two lemmas are obvious in characteristic 0.

Lemma 4.7. *Let V and W be kG-modules with Brauer characters $\sum_{i=1}^{s} m_i\varphi_i$ and $\sum_{i=1}^{s} n_i\varphi_i$, where φ_i are absolutely irreducible and pairwise different and $m_i, n_i \in \mathbb{Z}$. Then* $\dim \mathrm{Hom}_{kG}(V, W) \leqslant \sum_{i=1}^{s} m_i n_i$.

Proof. Induction on $\dim(W)$. The statement is obvious if W is irreducible (indeed, $\mathrm{Hom}_{kG}(V, W) = \mathrm{Hom}_{kG}(V/\mathrm{rad}(V), W)$ and so we are in the semisimple case). For the induction step, assume that W has a simple submodule U. From the exact sequence $0 \to \mathrm{Hom}_{kG}(V, U) \to \mathrm{Hom}_{kG}(V, W) \to \mathrm{Hom}_{kG}(V, W/U)$ it follows that $\dim \mathrm{Hom}_{kG}(V, W) \leqslant \dim \mathrm{Hom}_{kG}(V, U) + \dim \mathrm{Hom}_{kG}(V, W/U)$, and we may apply the induction hypothesis. $\square$

In the notation of Lemma 4.7, we use $[V, V]_G$ to denote $\sum_{i=1}^{s} m_i^2$.

Lemma 4.8. *Let $H \leqslant G$, let V be an irreducible kG-module and U any kH-module. Then*

$$\dim \operatorname{Hom}_{kH}(U, V|_H) \cdot \dim \operatorname{Hom}_{kH}(V|_H, U) \leqslant \dim \operatorname{Hom}_{kG}\bigl(U^G, U^G\bigr).$$

Proof. Since V is irreducible, we have

$$\begin{aligned} &\dim \operatorname{Hom}_{kG}\bigl(U^G, U^G\bigr) \\ &\quad \geqslant \dim \operatorname{Hom}_{kG}\bigl(U^G, V\bigr) \cdot \dim \operatorname{Hom}_{kG}\bigl(V, U^G\bigr) \\ &\quad = \dim \operatorname{Hom}_{kH}(U, V|_H) \cdot \dim \operatorname{Hom}_{kH}(V|_H, U). \qquad \square \end{aligned}$$

Corollary 4.9. *Let H be a subgroup of G and let U, V be kH-modules. For $a \in G$, let $H_a = H \cap aHa^{-1}$, $U_a = U|_{H_a}$, $V_a = V|_{H_a}$, V^a the kH_a-module obtained from V with the action $x \circ v = (a^{-1}xa)(v)$, and $V'_a = V^a|_{H_a}$. Assume that either $a \in N_G(H_a)$ or $a^2 \in N_G(H)$. Then*

$$\dim \operatorname{Hom}_{kH_a}\bigl(V'_a, U_a\bigr) \leqslant \sqrt{[U_a, U_a]_{H_a} \cdot [V_a, V_a]_{H_a}}.$$

Proof. Observe that if $x \in H_a$ then $a^{-1}xa \in H_a$. (It is so if $x \in N_G(H_a)$. If $a^2 \in N_G(H)$, then $x \in H \cap aHa^{-1}$ implies $a^{-1}xa \in a^{-1}Ha \cap H = aHa^{-1} \cap H = H_a$ since $a^{-2}Ha^2 = H$.) Thus the map $x \mapsto a^{-1}xa$ is an automorphism of H_a. From this it follows that $[V_a, V_a]_{H_a} = [V'_a, V'_a]_{H_a}$. On the other hand, Lemma 4.7 and the Schwartz inequality imply that $\dim \operatorname{Hom}_{kH_a}(V'_a, U_a) \leqslant \sqrt{[U_a, U_a]_{H_a} \cdot [V'_a, V'_a]_{H_a}}$, so we are done. $\square$

Lemma 4.10. *Let G be a finite group with a subgroup H. Let α and β be two Brauer characters of H in characteristic other than p, and let g be a p-element of G. Suppose that either*

(i) *$\alpha = \beta$ on $O^{p'}(H)$, or*
(ii) *$\alpha(h) = \beta(h)$ whenever $h \in H$ and $|h| = |g|$.*

Then $\alpha^G(g) = \beta^G(g)$.

Proof. Clearly, (i) implies (ii). So we assume (ii) holds. In this case

$$\alpha^G(g) - \beta^G(g) = \frac{1}{|H|} \sum_{\substack{x \in G \\ h = xgx^{-1} \in H}} \bigl(\alpha(h) - \beta(h)\bigr) = 0$$

because of (ii). $\square$

5. Weil modules

In this section we provide background material concerning Weil modules. Most of this is well known and is contained in some of the papers listed in the references. Much can be proved inductively, using the techniques in the next section.

Let E be a group with the following properties:

(a) $|E| = q^{1+2n}$, $n \geqslant 1$;
(b) $Z(E) = [E, E]$ has order q; and
(c) E has exponent p.

Then $G = Sp_{2n}(q)$ acts on E as a group of automorphisms. Indeed, let $G_0 = CSp_{2n}(q)$, the group which preserves up to scalar multiples the alternating form preserved by G. So G_0/G is cyclic of order $q-1$ and GC has index 2 in G_0, where C is the group of scalars. Then G_0 acts as a group of automorphisms on E and G is the normal subgroup which centralizes $Z(E)$.

Let H be the semidirect product EG and $H_0 = EG_0$.

Fix a nontrivial irreducible character χ of $Z(E)$. Then E has a unique irreducible representation over k of dimension q^n where $Z(E)$ acts via χ. Since this character is invariant under H, it is not difficult to see that we obtain an irreducible kH-module $M(\chi)$ which restricts to the irreducible kE-module as given. This extension $M(\chi)$ is unique if $(n,q) \neq (1,3)$, cf. [Ge]. Moreover, since EG_0 permutes the $M(\chi)$ and has precisely 2 orbits of size $(q-1)/2$, we see that as kG-modules either all $M(\chi)$ are isomorphic or are of two different isomorphism types (we will see that in fact the latter holds). Note that EG_0 interchanges these two orbits. Thus, the two possible isomorphism classes are interchanged by the outer diagonal automorphism of G.

Note that this module $M(\chi)$ exists and is irreducible for all characteristics $r \neq p$ as a kH-module.

We will need the following property of the modules $M(\chi)$.

Lemma 5.1. *Let k be an algebraically closed field of characteristic $r \geqslant 0$. Let $G = Sp_{2n}(q)$, $n \geqslant 1$, with q odd and not a multiple of r, and $H = EG$. Let P_1' be the subgroup of G which is the derived subgroup of the stabilizer of an 1-space. Let χ, χ' be any two nontrivial irreducible characters of $Z(E)$, and let $M(\chi)$ denote the kH-module described above.*

(i) *$M(\chi) \otimes M(\chi)^*$ is a rank one free $E/Z(E)$-module and is isomorphic to $k \oplus k_{P_1'}^G$ as kG-modules; and*
(ii) $\mathrm{Ext}^1_H(M(\chi), M(\chi')) = 0$.

Proof. Observe (ii) is clear in the case $\chi \neq \chi'$, since any such extension splits uniquely as a module over E. So assume $\chi = \chi'$ and write $M = M(\chi)$.

Note that $W = M \otimes M^* \cong \mathrm{Hom}(M, M)$ is a free rank one $E/Z(E)$-module. This is because $\tau_M(x) = 0$ for all $x \in E \setminus Z(E)$ and so $\tau_W(x) = 0$ for all such x. Since $Z(E)$ is trivial on W, this implies that W is a free module. Since $\dim(W) = q^{2n} = |E/Z(E)|$, it must be of rank one.

Therefore G permutes transitively the nontrivial characters of $E/Z(E)$. So $W = C_W(E) \oplus [E, W]$ with $[E, W]$ irreducible for H. Since $C_W(E) = \mathrm{Hom}_E(M, M) = \mathrm{Hom}_{EG}(M, M)$, it follows that $C_W(E) \simeq k$ as EG-modules. Now $[E, W]$ is a direct sum of 1-dimensional eigenspaces for E that are permuted transitively by G. Since P_1' is the stabilizer of a nontrivial character of $E/Z(E)$, it follows that $[E, W] \simeq \lambda_{P_1'}^G$ as G-modules for some character λ.

If $n \geqslant 2$, then P_1' is perfect (unless $(n, q) = (2, 3)$), and so λ is trivial as desired.

If $n = 1$, since P_1' has $(q-1)/2$ nontrivial eigenvalues with multiplicity 2 and 1 trivial eigenvalue on $M(\chi)$, it follows that $C_W(P_1')$ has dimension $2q - 1$. On the other hand, if λ is nontrivial a straightforward computation (using Frobenius reciprocity and Mackey's Theorem) shows that $\mathrm{Hom}_{P_1'}(k, \lambda_{P_1'}^G)$ is the number of double cosets $P_1' \backslash G / P_1'$ not contained in the normalizer of P_1'. The number of such double cosets is $q - 1$. If $(n, q) = (2, 3)$, one argues similarly. This completes the proof of (i).

Clearly, $H^1(H, [E, W]) = 0$, since E is a normal r'-subgroup and it has no fixed points on $[E, W]$. So $H^1(H, W) = H^1(H, k) = \mathrm{Hom}_H(H, k) = 0$. It follows that $\mathrm{Ext}_H^1(M(\chi), M(\chi)) = H^1(H, W) = 0$. $\square$

We now define the Weil modules. Denote $M = M(\chi)$.

First consider the case $r \neq 2$. Then $M = C_M(t) \oplus [t, M]$ where t is the central involution in $G = Sp_{2n}(q)$. It is well known that these G-submodules are irreducible of dimensions $(q^n \pm 1)/2$. (This also follows from our proof: we will see by induction on n that G has no trivial constituents on M—now apply the [LS] bound.) We will call these the Weil modules.

As we remarked above, there are either one or two Weil modules for each dimension. In fact, it also follows by induction that there are precisely two Weil modules for each dimension and that the (Brauer) characters can be distinguished by their values on long root elements. So for $r \neq 2$, there are two Weil modules for each dimension.

The Weil modules are self-dual if and only if $q \equiv 1 \pmod 4$ (if z is a long root element in G (a transvection), then z and z^{-1} are conjugate in G precisely when $q \equiv 1 \pmod 4$). If r is odd, it is then straightforward to see that the module of dimension $(q^n + 1)/2$ is orthogonal (as a P_n-module, this Weil module is a direct sum of 2 irreducible modules, one of dimension 1). It is not too difficult (using induction to reduce to the case of $SL_2(q)$) to see that the module is symplectic if it has dimension $(q^n - 1)/2$.

If $r=2$, then $[t,M]\leqslant C_M(t)$ and is of dimension $(q^n-1)/2$. We call this a Weil module. Again, there are two choices interchanged by the outer diagonal automorphism (we make the same choice as above for $q=3$, $n=1$). Note that $M/C_M(t)$ is isomorphic to $[t,M]$ (the isomorphism is given by $m\mapsto(t-1)m$). Thus M has 2 isomorphic composition factors which are Weil modules and a trivial composition factor (this is also true for $(q,n)=(3,1)$ given our definition of Weil modules).

It is easy to see from what we have said above that the field of definition for the Weil modules in positive characteristic r is $\mathbb{F}_r$ or $\mathbb{F}_{r^2}$. The former holds precisely when z is conjugate to z^r. If the module is not self-dual, then this shows that either it is defined over $\mathbb{F}_r$ or is contained in the unitary group.

We need a few more facts about the modules $M(\chi)$. Keep notation as in Lemma 5.1. If $r\neq 2$, then the G-module $M(\chi)$ is a direct sum of 2 different Weil modules. If $r=2$, there are 3 composition factors. We need a bit more information on the structure in this case.

Lemma 5.2. *If $r=2$, then $M(\chi)$ is a uniserial G-module with socle series $W,1,W$ with W a Weil module.*

Proof. It suffices to show that $M:=M(\chi)$ has no trivial G-submodule in its socle (and by passing to the dual, no trivial quotient). For if we have shown this, then the socle must be simple and, similarly, modulo the radical the module is simple (and both simple modules are isomorphic to the same Weil module, W). Thus, the socle series is as claimed.

It suffices to prove this for $SL_2(q^n)$, because $Sp_{2n}(q)$ contains $SL_2(q^n)$ and if the subgroup has no fixed points, of course the full group does not either.

Suppose that it did and consider $V:=M(\chi)\otimes M(\chi)^*$. As we noted above, M contains W. Hence V contains W^* in its G-socle. On the other hand, by Lemma 5.1, $V=k\oplus k_{P_1'}^G$. Thus, $\mathrm{Hom}_G(W^*,V)\simeq\mathrm{Hom}_{P_1'}(W^*,k\oplus k)$. However, P_1' has no fixed points on a Weil module (note that $\dim(W^*)=(q^n-1)/2$ since $r=2$) and so this term is 0, a contradiction. Thus, $C_M(G)=0$ as claimed. □

Corollary 5.3. *Suppose that $r=2$, $G=Sp_4(q)$, and $q+1=2^a$. Let $h\in L_1'\simeq SL_2(q)$ be an element of order $q+1$. Let V be a Weil module of G of dimension $(q^2-1)/2$, and consider any P_1'-submodule of type $M(\chi)$ (of dimension q) in V. Then h has exactly one Jordan block (of size q) on $M(\chi)$.*

Proof. Since all $M(\chi)$ are conjugate, it suffices to prove the claim for any particular χ. Assume the contrary: h has $t\geqslant 2$ Jordan blocks on $M:=M(\chi)$, of size $k_1\geqslant\cdots\geqslant k_t\geqslant 1$.

Let $z=h^{(q+1)/2}$. Then z is the central involution of L_1'. We claim that $\dim(C_M(z))\leqslant(q+1)/2$. Indeed, $C_M(z)$ is an L_1'-submodule of $M(\chi)$. So, if $\dim(C_M(z))>(q+1)/2$, then by Lemma 5.2, $\dim(C_M(z))=q$, i.e. z acts trivially on each $M(\chi)$. Since $C_V(Q_1)$ is an irreducible Weil module in characteristic 2 of L_1', z also acts trivially on $C_V(Q_1)$. Thus z acts trivially on V. This is a contradiction, since G acts nontrivially on V and G is generated by all conjugates of z.

Now if all k_i are at most $(q+1)/2$, then by [SS, Lemma 1.3] all Jordan blocks of z on $M(\chi)$ are of size 1, and so z acts trivially on $M(\chi)$, a contradiction. Hence we may assume that $k_1 = (q+1)/2 + b$ with $1 \leqslant b \leqslant (q-3)/2$. By [SS, Lemma 1.3], z has b Jordan blocks of size 2 and $((q+1)/2 - b) + (q - k_1) = q - 2b$ blocks of size 1 on $M(\chi)$. Thus $\dim(C_M(z)) = b + (q-2b) = q - b \geqslant (q+3)/2$, again a contradiction. □

Let $\varepsilon = \exp(2\pi \mathrm{i}/p)$. If $Y = \{x_\gamma(t) \mid t \in \mathbb{F}_q^*\}$ is a long root subgroup of $G = Sp_{2n}(q)$, we will denote by Ω_1 the set of linear characters of Y of the form

$$\lambda_a : x_\gamma(t) \mapsto \varepsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(at)},$$

where $a \in \mathbb{F}_q^*$ and a is a square. Similarly, Ω_2 is the set of all λ_a where $a \in \mathbb{F}_q^*$ and a is any nonsquare. Let W be a Weil module for G. We see (by restricting to P_1) that Z_1 has precisely $(q-1)/2$ nontrivial characters on W. Since Y is conjugate to Z_1, it follows that $\mathrm{Spec}^*(Y, W)$ (the set of all nontrivial linear characters of Y that occur on W) is either Ω_1 or Ω_2. From the above discussion it follows that in characteristic 2, the Weil module is determined by the i for which $\mathrm{Spec}^*(Y, W) = \Omega_i$ and in characteristic not 2 by i and by dimension (or by the kernel). In this case we will also say that W has *type* i. Observe that the Weil modules occurring in each $M(\chi)$ are of the same type, cf. [TZ2, Lemma 2.6(iii)]. If $A = Sp_{2m}(q)$ is a standard subgroup of G, then we can also define the type for Weil modules of A in a consistent way—i.e. the Weil modules are determined by the set of nontrivial eigenvalues for a long root subgroup of A—since all the long root subgroups are G-conjugate.

Applying this observation to a commuting pair of long root subgroups, we obtain the following key property of Weil modules of G.

Lemma 5.4. *Assume that $n \geqslant 2$. Let (Y_1, Y_2) be a commuting pair of long root subgroups. If W is a Weil module of G, then the only nontrivial linear characters of $Y_1 \times Y_2$ occurring on W are of the form $\alpha \otimes \beta$ with either $\alpha, \beta \in \Omega_1$ or $\alpha, \beta \in \Omega_2$.*

(By a nontrivial linear character of Y we mean a linear character whose restriction to both Y_1 and Y_2 is nontrivial.)

Lemma 5.5. *Assume that $(n, q) \neq (1, 3)$. Let X be any kG-module on which G acts nontrivially and let $M(\chi)$ be the above kG-module of dimension q^n. Then $M(\chi) \otimes X$ affords all nontrivial linear characters of Z_1.*

Proof. Without loss we may assume that $\mathrm{Spec}(Z_1, M(\chi))$, the set of all linear characters of Z_1 occurring on $M(\chi)$, is $\Omega_1 \cup \{1\}$. Since $(n,q) \neq (1,3)$ and G acts nontrivially on X, we may assume that $\mathrm{Spec}(Z_1, X)$ contains either Ω_1 or Ω_2 (and has at least two characters). Now the statement is obvious if $q = 3, 5$. When $q > 5$, the statement boils down to the following: if F^+, respectively F^-, denotes the set of all (nonzero) squares, respectively nonsquares, in $\mathbb{F}_q$, then $F := F^+ \cup (F^+ + F^\epsilon) \supseteq F^+ \cup F^-$ for any $\epsilon = \pm$.

First observe that the equation $x^2 - y^2 = a$ has nonzero solutions $(x, y) = ((a+1)/2, (a-1)/2))$ if $a \neq 0, \pm 1$. Hence $|F| \geqslant (q-3) \geqslant (q+1)/2$ and we are done if $q \equiv \epsilon \pmod 4$. Suppose $q \equiv -\epsilon \pmod 4$. In this case, fix $a \in F^\epsilon$ and observe that $ax^2 + 1 \neq 0$ for any $x \in \mathbb{F}_q$. If $ax^2 + 1 \in F^+$ for any $0 \neq x \in \mathbb{F}_q$, then the polynomial $(at^2 + 1)^{(q-1)/2} - 1$ would have q distinct roots in $\mathbb{F}_q$, a contradiction. Hence $(F^+ + F^\epsilon) \cap F^- \neq \emptyset$. Thus $F \cap F^- \neq \emptyset$, but $F \supseteq F^+$ and so we are done. $\square$

6. Spectra of long root subgroups

Let $G = Sp_{2n}(q)$, $n \geqslant 2$, with $q = p^f$ with p odd. Let k be an algebraically closed field of characteristic $r \neq p$ and V a nontrivial irreducible kG-module. In this and the next sections, we consider a few different properties which force the module to be special. We say that V has property $(\mathcal{R}_1)$ if Z_1 has at most (and therefore exactly) $(q-1)/2$ nontrivial linear characters on V.

Lemma 6.1. *Let V be any* (*nontrivial*) *irreducible kG-module with property* $(\mathcal{R}_1)$. *Assume that* $(n,q) \neq (2,3)$. *Then*

(i) $C_V(Z_1) = C_V(Q_1)$, *and*
(ii) *the P_1'-module* $[Z_1, V]$ *is a direct sum of some* $M(\chi)$.

Proof. (i) Assume the contrary: $U := [Q_1, C_V(Z_1)] \neq 0$. Consider a long root subgroup Z_2 inside L_1' and take any nontrivial linear character α of Q_1, which is not fixed by any nontrivial element of Z_2. Then for any nonzero vector v in the α-eigenspace of Q_1 on U, v^{Z_2} generates the regular Z_2-module R. Thus V affords all linear characters of Z_2, contrary to $(\mathcal{R}_1)$.

(ii) Each χ-eigenspace W_χ of Z_1 on $[Z_1, V]$ has the form $M(\chi) \otimes X$, where X is a certain L_1'-module. Let Z_2 be a long root subgroup inside L_1'. If L_1' acts nontrivially on X, then Lemma 5.5 implies that $M(\chi) \otimes X$ affords all nontrivial linear characters of Z_2, contrary to $(\mathcal{R}_1)$. Hence L_1' acts trivially on X, whence W_χ is a direct sum of some copies of $M(\chi)$. $\square$

It turns out that the following converse of Lemma 6.1 is true.

Lemma 6.2. *Let $n \geqslant 2$ and let V be an irreducible kG-module such that $C_V(Z_1) = C_V(Q_1)$ and such that the P_1'-module $[Z_1, V]$ is a direct sum of some $M(\chi)$. Then the following statements hold.*

(i) *V has property $(\mathcal{R}_1)$.*
(ii) *If $r \neq 2$ and* $\dim(V) > 1$, *then the L_1'-module V is semisimple, with all irreducible summands being Weil modules.*

Proof. (i) By Lemma 4.2, all composition factors of the L_1'-module $C := C_V(Q_1)$ are Weil modules or are trivial. Denote $W = [Z_1, V]$ and let $\mathcal{X}_i$ be the family of simple L_1'-modules consisting of Weil modules of type i (including also trivial modules if $r = 2$).

If $r \neq 2$, each simple P_1'-module $M(\chi)$ is semisimple as an L_1'-module and indeed is a sum of two Weil modules of different dimension but of the same type. It follows that $W = W_1 \oplus W_2$ where W_i, $i = 1, 2$, is a direct sum of Weil modules of type i for L_1'. Each W_i is P_1'-invariant, because this is precisely the sum of Z_1-eigenspaces corresponding to one orbit on the weights of Z_1. Also, $W_i = W(\mathcal{X}_i)$, cf. Lemma 4.3.

If $r = 2$ then by Lemma 5.2 we also have $W = W_1 \oplus W_2$ where all L_1'-composition factors of W_i are trivial modules and Weil modules of type i. Moreover, by Lemma 5.2 $\operatorname{soc}(W_i)$ is a direct sum of Weil modules of type i (and in particular contains no trivial modules). This implies that $W_i = W(\mathcal{X}_i)$. Also, W_i is P_1'-invariant, because this module is precisely the sum of Z_1-eigenspaces corresponding to one orbit on the weights of Z_1.

Now $V(\mathcal{X}_i) = W_i \oplus C(\mathcal{X}_i)$ for $i = 1, 2$. Since Q_1 acts trivially on C, $C(\mathcal{X}_i)$ is P_1'-invariant, whence $V(\mathcal{X}_i)$ is invariant under P_1'. Clearly, it is also invariant under $C_G(L_1') \simeq SL_2(q)$. Since $G = \langle P_1', SL_2(q)\rangle$, it follows that $V = V(\mathcal{X}_i)$ for $i = 1$ or 2 (and the other term is 0). The result follows.

(ii) Consider the L_1'-submodule V' of the socle of V which consists of Weil modules. This is P_1'-invariant, since V' is precisely the direct sum of W plus the corresponding submodule in C. On the other hand, V' is clearly invariant under $C_G(L_1') = SL_2(q)$, hence $V' = V$. $\square$

7. Spectra of commuting pairs of long root subgroups

Recall that by a commuting pair of long root subgroups we mean any pair (Y, Y'), where $Y = \{x_\beta(t) \mid t \in \mathbb{F}_q^*\}$ and $Y' = \{x_{\beta'}(t) \mid t \in \mathbb{F}_q^*\}$, where (β, β') is any orthogonal pair of long roots.

In this section we study kG-modules V with the following property:

$(\mathcal{R}_2)$ $\operatorname{Spec}^*(Y \times Y', V) \subseteq \{\alpha \otimes \beta \mid \text{either } \alpha, \beta \in \Omega_1 \text{ or } \alpha, \beta \in \Omega_2\}$.

Clearly, $(\mathcal{R}_1)$ implies $(\mathcal{R}_2)$. Also, any Weil module satisfies $(\mathcal{R}_2)$ by Lemma 5.4.

Proposition 7.1. *Let $n \geqslant 3$ and let V be any irreducible kG-module with property $(\mathcal{R}_2)$. Then the following statements hold.*

(i) $V = C_V(Q_1) \oplus [Z_1, V]$.
(ii) *The P_1'-module $[Z_1, V]$ is a direct sum of some $M(\chi)$.*
(iii) *All composition factors of the L_1'-module V are Weil modules or trivial.*
(iv) *V has property $(\mathcal{R}_1)$.*

Proof. (i) Assume the contrary: $U := [Q_1, C_V(Z_1)] \neq 0$. Write $U = \bigoplus_\alpha U_\alpha$, where U_α is the α-eigenspace for Q_1 on U. Observe that L_1' acts transitively on the nontrivial linear characters of Q_1, hence the sum runs over all nontrivial α. Let $(e_1, \dots, e_n, f_1, \dots, f_n)$ be a symplectic basis of the natural module for G. We may assume that P_1' fixes e_1. Since $n \geqslant 3$, we may consider the following commuting product of long root subgroups:

$$Y = \left\{ \begin{pmatrix} I_n & D \\ 0 & I_n \end{pmatrix} \,\middle|\, D = \mathrm{diag}(0, a, b, 0, \dots, 0),\ a, b \in \mathbb{F}_q \right\}.$$

View Q_1/Z_1 as the additive group $\langle e_i, f_j \mid i, j > 1\rangle_{\mathbb{F}_q}$ and let α be the character corresponding to the vector $f_2 + f_3$. Observe that no nontrivial element of Y fixes α. Hence, if $0 \neq v \in U_\alpha$, then u^Y generates the regular Y-module R. It follows that V affords all linear characters of Y, contrary to $(\mathcal{R}_2)$.

(ii) Consider any nonzero Z_1-eigenspace V_χ of Z_1 on $[Z_1, V]$. Then $V_\chi \simeq M(\chi) \otimes X$ for some L_1'-module X. We need to show that L_1' acts trivially on X. Assume the contrary. Pick a long root subgroup $Y < L_1'$. Then by Lemma 5.5, $\mathrm{Spec}(Y, V_\chi)$ contains every nontrivial linear character λ of Y. Thus V affords every character of the form $\chi \otimes \lambda$ for the group $Z_1 \times Y$, again contrary to $(\mathcal{R}_2)$. Observe that this argument also works when $n = 2$.

(iii) and (iv) follow from (i), (ii), and Lemmas 4.2 and 6.2. □

Lemma 7.2. *Assume that $n \geqslant 2$. Then P_n acts on the set of nontrivial linear characters of Q_n with two orbits of length $(q^n - 1)/2$. These two orbits occur in the restriction of Weil modules of dimension $(q^n - 1)/2$ to Q_n. All other orbits have length at least $(q^n - 1)(q^n - q)/2(q + 1)$.*

Proof. One can identify Q_n with the space of symmetric $(n \times n)$-matrices over $\mathbb{F}_q$, and then any $A \in L_n \simeq GL_n(q)$ acts on Q_n via $X \mapsto {}^tAXA$. Any linear character of Q_n now has the form

$$X \mapsto \varepsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$$

for some $B \in Q_n$. Thus every L_n-orbit on nontrivial linear characters of Q_n is just an orbit of L_n on $Q_n \setminus \{1\}$. If the latter orbit contains a matrix X of rank j, then the stabilizer of X in L_n is

$$\left[q^{j(n-j)}\right] \cdot \left(O_j(q) \times GL_{n-j}(q)\right).$$

So the length of this orbit is $(q^n - 1)/2$ if $j = 1$ (there are exactly two orbits of this kind; they correspond to squares and nonsquares in $\mathbb{F}_q$), or at least $(q^n - 1)(q^n - q)/2(q + 1)$ if $j \geqslant 2$. The two Weil characters of degree $(q^n - 1)/2$ when restricted to Q_n give us the orbits of smallest length. □

Theorem 7.3. *Let V be any irreducible kG-module. Suppose that either $n \geqslant 3$ and any P_n-orbit of Q_n-characters on V is of length less than $(q^n - 1)(q^n - q)/2(q+1)$, or $n = 2$ and $\dim(V) < (q^n - 1)(q^n - q)/2(q + 1)$. Then all conclusions of Proposition* 7.1 *hold*; *in particular, V has property $(\mathcal{R}_1)$.*

Proof. First restrict V to the parabolic subgroup P_n. By Lemma 7.2, the condition on V implies that there is a (formal) sum V' of Weil and trivial modules of G such that $V|_{Q_n} \simeq V'|_{Q_n}$. Since $n \geqslant 2$, Q_n contains a commuting pair (Y, Y') of long root subgroups. By Lemma 5.4, V', and so V, has property $(\mathcal{R}_2)$ for the pair (Y, Y') (and so for any commuting pair as well).

If $n \geqslant 3$, we are done by Proposition 7.1. Assume that $n = 2$. Then conclusion (ii) of Proposition 7.1 holds as well, as we have observed in its proof. Thus we may write $[Z_1, V]$ as the sum of $M(\chi)$, and each $M(\chi)$ occurs with multiplicity s_i if $\chi \in \Omega_i$, $i = 1, 2$.

It remains to establish conclusion (i). Assume the contrary, that $U := [Q_1, C_V(Z_1)] \neq 0$. Consider the commuting product $Y = Z_1 \times Z_2$, where $Z_2 < L_1'$. Observe that the fixed point subspace of Z_2 on $M(\chi)$ has dimension 1, whence the multiplicity of the Y-character $\chi \otimes 1$ on V is s_i. On the other hand, Z_2 acts on nontrivial linear characters of Q_1 with $q - 1$ fixed points and $q - 1$ regular orbits. It follows that the multiplicity of the Y-character $1 \otimes \chi$ on V is at least $q - 1$. Since the pairs (Z_1, Z_2) and (Z_2, Z_1) are conjugate in V, we come to the conclusion that $s_i \geqslant q - 1$. Thus

$$\begin{aligned}\dim(V) &\geqslant \dim(U) + \dim\left([Z_1, V]\right) \geqslant \left(q^2 - 1\right) + (q - 1)q(q - 1)\\ &= \left(q^2 + 1\right)(q - 1),\end{aligned}$$

contrary to the assumption that $\dim(V) < q(q - 1)^2/2$. □

Corollary 7.4. *Suppose that $n \geqslant 2$. Then the* (*unique*) *irreducible complex character ρ of $Sp_{2n}(q)$ of degree $(q^n - 1)(q^n - q)/2(q + 1)$ is irreducible modulo any prime r different from p.*

Proof. The statement is well known for $n = 2$, cf. [Wh1,Wh2,Wh3], hence we may assume $n \geqslant 3$. The existence and uniqueness of such ρ follow from

[TZ1, Theorem 5.2]. Assume that $\rho|_{Q_n}$ contains more than one L_n-orbit of linear characters of Q_n. By Lemma 7.2, there is a character μ of G such that $\rho|_{Q_n} = \mu|_{Q_n}$ and μ is a sum of Weil and trivial characters of G. Thus ρ satisfies $(\mathcal{R}_2)$. By Proposition 7.1, ρ satisfies $(\mathcal{R}_1)$ and $(\mathcal{W})$, contrary to [TZ2, Theorem 1.1]. Hence $\rho|_{Q_n}$ consists of exactly one L_n-orbit. By Clifford's Theorem, $(\rho \pmod r))|_{P_n}$ is irreducible. $\square$

8. Proof of Theorem 2.2: $r \neq 2$ and $n > 2$

We keep notation as in Sections 6 and 7.

Theorem 8.1. *Assume that $r \neq 2$ and $n \geqslant 3$. Suppose that V is a nontrivial irreducible kG-module such that Z_1 has only $(q-1)/2$ nontrivial characters on V. Then V is a Weil module* (*and in particular has dimension* $(q^n \pm 1)/2$).

We will prove this result by showing that the Brauer character τ_V of V is the same as that of a Weil module. We prove the result in a series of lemmas. For definiteness we assume that the nontrivial Z_1-characters occurring on V belong to Ω_1. Let W_n^- and W_n^+ denote the 2 Weil modules for G corresponding to the set Ω_1 of Z_1-characters, of dimension $(q^n-1)/2$ and $(q^n+1)/2$, respectively.

We will use the notation $C = A + B$ to indicate that this is true in the Grothendieck group $G_0(X)$ of a group X.

By Lemmas 6.1 and 6.2, $V = C_V(Q_1) \oplus [Z_1, V]$, $[Z_1, V] = s\sum_{\chi\in\Omega_1} M(\chi)$ as P_1'-module, and $C_V(Q_1) = aW_{n-1}^- + bW_{n-1}^+$ as L_1'-module, for some integers $a, b, s \geqslant 0$.

First we observe that $s = a + b$. For, if $t \in Z_1$ is a transvection, then

$$\tau_V(t) = a\frac{q^{n-1}-1}{2} + b\frac{q^{n-1}+1}{2} + sq^{n-1}\frac{-1+\sqrt{\epsilon q}}{2},$$

where $\epsilon = (-1)^{(q-1)/2}$. On the other hand, for a G-conjugate t' of t which is contained in L_1' we have

$$\tau_V(t') = a\frac{-1+q^{n-2}\sqrt{\epsilon q}}{2} + b\frac{1+q^{n-2}\sqrt{\epsilon q}}{2} + s\frac{(q-1)q^{n-2}\sqrt{\epsilon q}}{2}.$$

Since $\tau_V(t) = \tau_V(t')$, we obtain $s = a + b$. Therefore,

$$V = aW_n^- + bW_n^+ \quad \text{as } P_1'\text{-modules.} \tag{1}$$

We next consider the subgroups $H_d := Sp_{2d}(q) \times Sp_{2(n-d)}(q)$ which are the stabilizer of nondegenerate $2d$-subspaces, $1 \leqslant d \leqslant n-1$. It is well known that

$$\begin{aligned} W_n^-\big|_{H_d} &= W_d^- \otimes W_{n-d}^+ + W_d^+ \otimes W_{n-d}^-, \\ W_n^+\big|_{H_d} &= W_d^- \otimes W_{n-d}^- + W_d^+ \otimes W_{n-d}^+. \end{aligned}$$

We may assume that $B := Sp_{2(n-d)}(q)$ is contained in L_1', whence it follows from (1) that

$$V|_B = \big(a\big(q^d+1\big)/2+b\big(q^d-1\big)/2\big)W_{n-d}^- + \big(a\big(q^d-1\big)/2 + b\big(q^d+1\big)/2\big)W_{n-d}^+. \tag{2}$$

Since $A := Sp_{2d}(q)$ is G-conjugate to a subgroup of L_1', (1) also implies that

$$V|_A = \big(a\big(q^{n-d}+1\big)/2+b\big(q^{n-d}-1\big)/2\big)W_d^- + \big(a\big(q^{n-d}-1\big)/2+b\big(q^{n-d}+1\big)/2\big)W_d^+. \tag{3}$$

Thus all the composition factors of $V|_{H_d}$ are of form $W_d^i \otimes W_{n-d}^j$, where $i, j = 1, 2$. Note that the central involution z of G acts as a scalar on V, and it acts as $-\epsilon^n$ on W_n^- and as ϵ^n on W_n^+, where $\epsilon = (-1)^{(q-1)/2}$. Matching the action of z on different composition factors of H_d, we arrive at one of the following two possibilities:

$$V|_{H_d} = xW_d^- \otimes W_{n-d}^+ + yW_d^+ \otimes W_{n-d}^- \quad \text{or} \tag{4}$$

$$V|_{H_d} = xW_d^- \otimes W_{n-d}^- + yW_d^+ \otimes W_{n-d}^+. \tag{5}$$

Suppose we are in the case of (4). Then Eqs. (2)–(4) have only one solution $b = 0$, $x = y = a$. This means that $V = aW_n^-$ in $G_0(P_1')$ and $G_0(H_d)$.

Suppose we are in the case of (5) and $d \neq n/2$. Such d exists since $n \geqslant 3$. Then Eqs. (2), (3), and (5) have only one solution $a = 0$, $x = y = b$. This means that $V = bW_n^+$ in $G_0(P_1')$ and in $G_0(H_d)$ for all $d \neq n/2$. Now for $d = n/2$, Eqs. (2), (3), and (5) imply $x = y = b$ as well, since we already know that $a = 0$. Thus $V = bW_n^+$ in $G_0(H_d)$ for $d = n/2$.

So we now have the following lemma.

Lemma 8.2. *There is a Weil module W of G and $s \in \mathbb{N}$ such that $V = sW$ for all the subgroups H_d and P_1'. In particular, $\tau_V(x) = s\tau_W(x)$ for x in a conjugate of one of these subgroups.*

We need to consider the other families of subgroups given in Lemma 4.4.

Lemma 8.3. *$V = sW$ as $SL_2(q^n)$-modules.*

Proof. Let $H = SL_2(q^n)$. Let Q denote a maximal unipotent subgroup of H. Since $Q \leqslant P_1'$, $V = sW$ as Q-modules by Lemma 8.2. Also, since the central involution z of H is contained in H_1, $\tau_V(z) = s\tau_W(z)$. It follows by inspection of the irreducible modules for H that $V = sW$ as H-modules. □

Lemma 8.4. *$V = sW$ as P_1-modules.*

Proof. Let $\chi \in \Omega_1$ and let V_χ and W_χ denote the χ-eigenspaces for Z_1 on V and W, respectively, and let J be the stabilizer of χ in P_1. Then $J = C \times P_1'$ where $C = Z(G)$. So by Lemma 8.2, $V_\chi = sW_\chi$ as J-modules. Since $[Z_1, V] = (V_\chi)_J^{P_1}$ (and similarly for W), it follows that $[Z_1, V] = s[Z_1, W]$ as P_1-modules. In particular, $[Z_1, V] = s[Z_1, W]$ as L_1-modules. But $L_1 < Sp_2(q) \times L_1' = H_1$ and $V = sW$ as H_1-modules by Lemma 8.2, hence $C_V(Z_1) = sC_W(Z_1)$ as L_1-modules. Since Q_1 acts trivially on $C_V(Z_1)$ and $C_W(Z_1)$, it follows that $C_V(Z_1) = sC_W(Z_1)$ as P_1-modules. □

We now consider P_j for $j > 1$. We first need the following lemma.

Lemma 8.5. *$V = sW$ as Q_j-modules and $C_V(Z_j) = C_V(Q_j)$.*

Proof. If $j = 1$, this has already been proved. Since $Q_j \leqslant P_1'$, the first statement holds by Lemma 8.2. Since the second statement holds for W, the first statement implies the second. □

Lemma 8.6. *$V = sW$ as P_j-modules for all j.*

Proof. (1) Induction on j. The case $j = 1$ is just Lemma 8.4. For the induction step let $j > 1$. Write $L_j = A \times B$, where $A = GL_j(q)$ and $B = Sp_{2(n-j)}(q)$.

Let V_α be a weight space for Z_j in $[Z_j, V]$. The weights α that occur are precisely those occurring on W. In particular, V_α is a direct sum of irreducible homogeneous Q_j-modules and P_j is transitive on this collection of weights. Also, if we identify Z_j with the space of symmetric $(j \times j)$-matrices over $\mathbb{F}_q$, then α corresponds to a symmetric matrix of rank 1. Hence, $J := \mathrm{Stab}_{P_j}(\alpha)$ is contained in a conjugate of P_{j-1} (and contains $Q_j B$).

Since P_j transitively permutes the Z_j weight spaces, we see that that $V \simeq (V_\alpha)_J^{P_j} \oplus C_V(Z_j)$ as P_j-modules. We have noticed that $J \leqslant P_{j-1}$. In particular, this implies by the induction hypothesis that $V_\alpha = sW_\alpha$ as J-modules (where W_α is the corresponding weight space for Z_1 on W). Thus, $[Z_j, V] = s[Z_j, W]$ as P_j-modules.

(2) Assume $j < n$. Since $L_j < Sp_{2j}(q) \times Sp_{2(n-j)}(q) = H_j$, $V = sW$ as L_j-modules by Lemma 8.2. On the other hand, $[Z_j, V] = s[Z_j, W]$ as L_j-modules by the previous paragraph. It follows that $C_V(Z_j) = sC_W(Z_j)$ as L_j-modules and so as P_j-modules, since Q_j acts trivially on $C_V(Z_j)$ and $C_W(Z_j)$ by Lemma 8.5.

(3) Now assume that $j = n$. As we explained in (2), it suffices to show that $V = sW$ as L_n-modules. Let $g \in L_n$ be any r'-element. Consider the (faithful) action of g on the maximal totally isotropic subspace M fixed by P_n, and write $g = su$, with s the semisimple part and u the unipotent part. If g fixes a proper subspace $M' \neq 0$ of M, then g lies in a conjugate of P_i with $i = \dim(M') < j$, whence $V = sW$ as $\langle g\rangle$-modules by induction hypothesis. Now assume that g

is irreducible on M. If $u \neq 1$, then $C_M(u) \neq 0$ is a g-invariant proper subspace of M, a contradiction. Hence $u = 1$, and $g = s$ is irreducible on M. If the $\langle g\rangle$-module M is not self-dual, then g is contained in a torus $T \simeq \mathbb{Z}_{q^n-1}$ of L_n, and moreover one can embed T in a standard subgroup $SL_2(q^n)$ of G. According to Lemma 8.3, $V = sW$ as $\langle g\rangle$-modules. If the $\langle g\rangle$-module M is self-dual, then one can show that n is even and g stabilizes a nondegenerate subspace of dimension n, whence a conjugate of g is contained in $H_{n/2}$ and so $V = sW$ as $\langle g\rangle$-modules by Lemma 8.2. Consequently, $V = sW$ as L_n-modules. □

This completes the proof of Theorem 8.1.

9. Proof of Theorem 2.2: $r = 2$ and $n > 2$

Here we prove Theorem 2.2 for the case of characteristic $r = 2$ and $n > 2$. Let V be an irreducible kG-module with property $(\mathcal{R}_1)$, say $\mathrm{Spec}^*(Z_1, V) = \Omega_1$. We will denote by W_n the irreducible Weil module in characteristic 2 of G such that $\mathrm{Spec}^*(Z_1, W_n) = \Omega_1$. Let $\epsilon = (-1)^{(q-1)/2}$ and let Z_2 be a long root subgroup inside $L_1' = Sp_{2n-2}(q)$.

By Lemma 6.1, $V = C_V(Q_1) \oplus [Z_1, V]$ and $[Z_1, V] = m\sum_{\chi\in\Omega_1} M(\chi)$ as P_1'-modules for some $m \in \mathbb{N}$. By Lemma 4.2, $C_V(Q_1) = aW_{n-1} + b\cdot 1$ as L_1'-modules for some integers $a, b \geqslant 0$. Thus

$$V|_{L_1'} = m(q-1)/2\cdot(2W_{n-1}+1) + aW_{n-1} + b\cdot 1.$$

First we observe that $a = m$. Indeed, let t be a transvection in Z_1. Then we may assume that

$$\tau_V(t) = mq^{n-1}\bigl(-1+\sqrt{\epsilon q}\,\bigr)/2 + a\bigl(q^{n-1}-1\bigr)/2 + b.$$

Now let $t' \in Z_2$ be G-conjugate to t. Then

$$\tau_V(t') = \frac{m(q-1)}{2}\left(1 + 2\frac{-1+q^{n-2}\sqrt{\epsilon q}}{2}\right) + a\bigl(-1+q^{n-2}\sqrt{\epsilon q}\,\bigr)/2 + b.$$

Since $\tau_V(t) = \tau_V(t')$, it follows that $(m-a)(q^{n-1} - q^{n-2}\sqrt{\epsilon q}\,) = 0$, i.e. $a = m$, as stated.

We will prove that $V = mW_n + b\cdot 1$. The above discussion shows that this holds for V considered as a P_1'-module.

Next we proceed to prove this equality for V as an H_d-module, where $1 \leqslant d \leqslant n-1$. First we can view the component $B := Sp_{2n-2d}(q)$ of H_d as a standard subgroup of L_1' and get $W_n|_B = (q^d+1)/2\cdot W_{n-d} + (q^d-1)/2\cdot(W_{n-d}+1)$, (recall that $r = 2$). We can get only W_{n-d}, but not its algebraic conjugate, in this restriction, because of the condition on the spectrum of a G-conjugate of Z_1 lying in B. Since $V = mW_n + b\cdot 1$ in $G_0(L_1')$, one has

$$V|_B = mq^dW_{n-d} + \bigl(b + m\bigl(q^d-1\bigr)/2\bigr)\cdot 1.$$

On the other hand, the first component A of H_d is G-conjugate to a standard subgroup of type $Sp_{2d}(q)$ inside L_1', hence

$$V|_A = mq^{n-d}W_d + \bigl(b + m\bigl(q^{n-d}-1\bigr)/2\bigr)\cdot 1.$$

The shape of $V|_A$ and of $V|_B$ implies that

$$V|_{A\times B} = xW_d \otimes W_{n-d} + y\cdot W_d \otimes 1_B + z\cdot 1_A \otimes W_{n-d} + s\cdot 1,$$

where $x \in \mathbb{Z}$ and

$$\begin{aligned} y &= mq^{n-d} - x\bigl(q^{n-d}-1\bigr)/2, \qquad z = mq^d - x\bigl(q^d-1\bigr)/2,\\ s &= b + (x-2m)\bigl(q^d-1\bigr)\bigl(q^{n-d}-1\bigr)/4. \end{aligned}$$

In order to determine x, we compute $\tau_V(g)$ in two ways, where $g = tt''$, $t \in Z_1 \leqslant A$ is the abovementioned transvection, and $t'' \in B$ is L_1'-conjugate to $t' \in Z_2$. The formula for $V|_{P_1'}$ tells us that

$$\begin{aligned} \tau_V(g) &= mq^{n-2}\sqrt{\epsilon q}\bigl(-1+\sqrt{\epsilon q}\bigr)/2 + m\bigl(-1+q^{n-2}\sqrt{\epsilon q}\bigr)/2 + b\\ &= m\bigl(-1+\epsilon q^{n-1}\bigr)/2 + b, \end{aligned}$$

since t acts scalarly on each $M(\chi)$ which is an L_1'-module of type $(W_{n-1}, 1, W_{n-1})$, and trivially on the rest. On the other hand, the shape of $V|_{H_d}$ yields $\tau_V(g)$ equal to

$$\begin{aligned} &\frac{x(-1+q^{d-1}\sqrt{\epsilon q})(-1+q^{n-d-1}\sqrt{\epsilon q})}{4}\\ &\quad + \frac{y(-1+q^{d-1}\sqrt{\epsilon q})}{2} + \frac{z(-1+q^{n-d-1}\sqrt{\epsilon q})}{2} + s\\ &\quad = \frac{xq^{n-2}(\sqrt{\epsilon q}-q)^2}{4} + \frac{m(2q^{n-1}\sqrt{\epsilon q}-q^n-1)}{2} + b. \end{aligned}$$

From this it follows that $(x-2m)q^{n-2}(\sqrt{\epsilon q}-q)^2 = 0$, i.e. $x = 2m$. Hence $y = z = m$, $s = b$, and so

$$V|_{H_d} = 2mW_d \otimes W_{n-d} + m\cdot W_d \otimes 1_B + m\cdot 1_A \otimes W_{n-d} + b\cdot 1,$$

i.e. V and $mW_n + b\cdot 1$ agree on H_d.

Next consider the subgroup $H = SL_2(q^n)$ of G. Let J be a maximal unipotent subgroup of H. Since $J \leqslant P_1'$, $V = mW_n + b\cdot 1$ as J-modules. Again by inspecting the irreducible modules for H we see that $V = mW_n + b\cdot 1$ as H-modules.

It remains to deal with P_j. At this point, the argument given in Section 8 as for the case r odd goes through unchanged and thus we have shown:

Theorem 9.1. *Assume that $r = 2$ and $n \geqslant 3$. Suppose that V is a nontrivial irreducible kG-module such that Z_1 has only $(q-1)/2$ nontrivial linear characters on V. Then V is a Weil module (and in particular has dimension $(q^n-1)/2$).*

10. Proofs of Main Theorems for symplectic groups

Lemma 10.1. *Let $S = Sp_4(q)$ with $q = p^f$ odd. Suppose that V is an irreducible kS-module in cross characteristic r which does not lift to zero characteristic. Then the following statements hold.*

(i) *V does not have property $(\mathcal{R}_1)$.*
(ii) $(p, \dim(V)) = 1$. *In particular, if $q = p$ then* $\mathrm{Spec}(g, V) \ni 1$ *for any transvection $g \in S$.*
(iii) *Let $q = p$ and $g \in S$ be a nontrivial product of two commuting transvections. Then $d_V(g) = p$.*

Proof. The r-Brauer characters of S are described in [Wh1,Wh2,Wh3]. Using this description, one can readily check (i) and that $p \nmid \dim(V)$. If $q = p$ and $\mathrm{Spec}(g, V) \not\ni 1$ for a transvection $g \in S$, then we may choose g to be a generator of Z_1 and see that $C_V(Z_1) = 0$, whence the dimension of $V = [Z_1, V]$ is divisible by $\dim(M(\chi)) = p$, a contradiction.

Under the assumptions in (iii), assume that $d_V(g) < p$. The case $p = 3$ can be checked directly, so we will assume that $p > 3$. Choose $g = zt$ where $1 \neq z \in Z_1$ and t is a transvection in $L_1' \simeq SL_2(p)$. First observe that $U := [Q_1, C_V(Z_1)] = 0$. For if $U \neq 0$, then since t has a regular orbit on the natural module for L_1', it follows that t has a regular orbit on the set of linear Q_1-characters occurring on U. Thus U contains a regular $k\langle g\rangle$-module, contrary to the condition $d_V(g) < p$. Next consider the χ-eigenspace $M(\chi) \otimes X$ for Z_1 on V for any nontrivial linear character χ of Z_1. We claim that L_1' acts trivially on X. If not, then $\mathrm{Spec}(t, M(\chi) \otimes X)$ contains all nontrivial pth roots ϵ^i of unity by Lemma 5.5. We may assume that $\chi(z) = \epsilon$. Thus $\mathrm{Spec}(g, V) \supseteq \mathrm{Spec}(g, M(\chi) \otimes X) \ni \epsilon^i$ for all $i \in \{0, 1, \ldots, p-1\} \setminus \{1\}$. Doing the same thing with another χ (recall $p > 3$), we come to the conclusion that $\mathrm{Spec}(g, V) = \{\epsilon^i \mid 0 \leqslant i \leqslant p-1\}$, i.e., $d_V(g) = p$, again a contradiction. Consequently, V satisfies the hypothesis of Lemma 6.2 and therefore V has property $(\mathcal{R}_1)$ by that lemma. But this contradicts (i). □

Proof of Theorem 2.2. The case $n \geqslant 3$ has been completed in Sections 8 and 9. Assume that $n = 2$. If V is liftable to characteristic 0, then the statement follows from [TZ2]. If V is not liftable, then we may apply Lemma 10.1. □

Proof of Theorem 2.1. Let V be an irreducible kG-module of dimension less than $(q^n - 1)(q^n - q)/2(q + 1)$. By Theorem 7.3, V enjoys $(\mathcal{R}_1)$. It remains to apply Theorem 2.2. □

Proof of Theorem 2.3. By Lemma 5.4, $(\mathcal{W})$ implies $(\mathcal{R}_2)$. By Theorem 7.3, $(\mathcal{Q})$ implies $(\mathcal{R}_2)$. Finally, $(\mathcal{R}_2)$ implies $(\mathcal{R}_1)$ by Proposition 7.1, so we are done by Theorem 2.2. □

Example 10.2. Let $n \geqslant 2$ and q be odd. The group $Sp_{2n}(q)$ acts as a rank 3 permutation group on the set of 1-spaces of the natural module $\mathbb{F}_q^{2n}$. The submodule structure of the corresponding permutation module M was determined by Liebeck in [Li] in any cross characteristic $r \neq 2$; and the composition factors of M (mod p) were found by Zalesskii and Suprunenko in [ZS]. Using our results one can also determine the structure of $M \pmod 2$.

It is known that the character of $Sp_{2n}(q)$ on M is $1 + \alpha_n + \beta_n$, where α_n and β_n are irreducible characters of degree $(q^n - 1)(q^n + q)/2(q - 1)$ and $(q^n + 1)(q^n - q)/2(q - 1)$, respectively. As we mentioned in the proof of Lemma 7.2, each linear character of Q_n has the form

$$\lambda_B : X \mapsto \varepsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$$

for some symmetric matrix B. Some of P_n-orbits on $\mathrm{Irr}(Q_n)$ are: $\mathcal{O}_1$ and $\mathcal{O}_2$ of length $(q^n - 1)/2$ (corresponding to those B of rank 1), $\mathcal{O}_3$ and $\mathcal{O}_4$ of length $(q^n - 1)(q^n - q)/2(q + 1)$, respectively $(q^n - 1)(q^n - q)/2(q - 1)$ (corresponding to those B of rank 2, which define a quadratic form of type $-$, respectively $+$). One can show that

$$\alpha_n|_{Q_n} = \sum_{\lambda \in \mathcal{O}_1} \lambda + \sum_{\lambda \in \mathcal{O}_2} \lambda + \sum_{\lambda \in \mathcal{O}_4} \lambda + \frac{q^n - 1}{q - 1} \cdot 1_{Q_n},$$

$$\beta_n|_{Q_n} = \sum_{\lambda \in \mathcal{O}_4} \lambda + \frac{q^n - q}{q - 1} \cdot 1_{Q_n}.$$

Let η_n and $\bar{\eta}_n$ be the reduction modulo 2 of the two complex irreducible Weil characters of degree $(q^n - 1)/2$. Define $\kappa = 1$ if n is even and 0 otherwise. We claim that there is an irreducible Brauer character γ such that

$$\alpha_n \pmod 2 = (1 + \kappa) + \eta_n + \bar{\eta}_n + \gamma, \qquad \beta_n \pmod 2 = \kappa + \gamma.$$

Indeed, the case $n = 2$ was done in [Wh1]. Suppose $n \geqslant 3$ and let γ be the composition factor of $\beta_n \pmod 2$ whose restriction to Q_n involves $\mathcal{O}_4$. Since $\beta_n \pmod 2 - \gamma$ is trivial on Q_n, it is a multiple of 1_{S_n}. Now $L_n' = SL_n(q)$ cannot act trivially on the Q_n-fixed points inside γ (otherwise P_n' would have too many fixed points). Hence the formula for $\beta_n \pmod 2$ follows. One can show that all composition factors of $\beta_n \pmod 2$ appear in $\alpha_n \pmod 2$. Each composition factor of $(\alpha_n - \beta_n) \pmod 2$ restricted to Q_n involves only $\mathcal{O}_1$, $\mathcal{O}_2$ (and maybe 1_{Q_n}), hence it is trivial or a Weil module by Theorem 2.3, whence the formula for $\alpha_n \pmod 2$ follows. Detailed argument will be given in [LST]. Other rank 3 permutation modules of finite classical groups will be handled in [ST].

11. Representations of small unitary groups

Let $G = U_n(q)$, $q = p^f$, and k be an algebraically closed field of characteristic r coprime to q. Weil modules of G are discussed in detail in [TZ2]. In particular, if $n \geqslant 3$ then there are $(q+1)^2$ complex modules, with character ζ_n^{ij}, $0 \leqslant i, j \leqslant q$, where ζ_n^{ij} is obtained from $\zeta_n^i = \zeta_n^{i0}$ via multiplying by a linear character, and ζ_n^i is calculated in [TZ2, Lemma 4.1]. Reduction modulo r of complex Weil modules is discussed in [DT,HM].

Let P_1 be the first parabolic subgroup of G, $Q_1 = O_p(P_1)$, $Z_1 = Z(Q_1)$. We may think of P_1 as $\mathrm{Stab}_G(\langle e\rangle_{\mathbb{F}_{q^2}})$, where e is a nonzero isotropic vector in the natural module $W = \mathbb{F}_{q^2}^n$ for G. Let $S := SU_n(q)$, $P_1' = \mathrm{Stab}_S(e)$, $P_1'' = \mathrm{Stab}_G(e)$. Then $P_1'' = Q_1 \cdot L$ with $L \simeq U_{n-2}(q)$ and $P_1' = Q_1 \cdot K$ with $K \simeq SU_{n-2}(q)$. For each nontrivial linear character χ of Z_1, there is an irreducible module of dimension q^{n-2} of Q_1 whose restriction to Z_1 is $q^{n-2}\chi$ and which extends to an irreducible module $M(\chi)$ of P_1''. Furthermore, if U is any kP_1''-module whose restriction to Z_1 involves only χ, then $U \simeq M(\chi) \otimes X$ for some kL-module X. The last two claims can be proved using Lemma 2.1 in the preprint version of [MT].

We say that a kS-module V has property $(\mathcal{W})$ if for some k, $3 \leqslant j \leqslant n-1$, the restriction of V to a standard subgroup $SU_j(q)$ involves only irreducible Weil and trivial modules. Our argument will particularly rely on analyzing the behavior of the subgroup $R_3 := O_p(P)$, where P is the first parabolic subgroup of a standard subgroup $SU_3(q)$ in $SU_n(q)$ if n is odd, and the subgroup $R_4 := O_p(P)$, where P is the second parabolic subgroup of a standard subgroup $SU_4(q)$ in $SU_n(q)$ if n is even. Note that R_3 is of extra-special type of order q^3, and R_4 is elementary abelian of order q^4. A key role, similar to the role of property $(\mathcal{R}_2)$ in the case of symplectic groups, is played by the following two observations.

Lemma 11.1. *Let V be a Weil module or a trivial module of $SU_3(q)$. Then the restriction of V to R_3 contains no nontrivial linear character of R_3.*

Proof. The claim follows from the formula for ζ_n^i given in [TZ2, Lemma 4.1]. See also Table 3.2 of [Geck]. $\square$

Let $A = U_4(q)$ and $W := \langle e_1, e_2, f_1, f_2\rangle_{\mathbb{F}_{q^2}}$ be the natural module of A, and let the hermitian form have the matrix $\left(\begin{smallmatrix} 0 & I_2 \\ I_2 & 0 \end{smallmatrix}\right)$. Let $P = \mathrm{Stab}_A(\langle e_1, e_2\rangle_{\mathbb{F}_{q^2}})$ and $R_4 = O_p(P)$.

Lemma 11.2. *In the above notation, P has two orbits, say $\mathcal{O}_1$ and $\mathcal{O}_2$, on the set of nontrivial linear characters of R_4, of length $(q^4-1)/(q+1)$ and $q(q^4-1)/(q+1)$, respectively. The first orbit occurs on any of Weil modules of A. Furthermore, both $\mathcal{O}_1 \cdot \mathcal{O}_1$ and $\mathcal{O}_1 \cdot \mathcal{O}_2$ intersect $\mathcal{O}_2$. Finally, $\mathcal{O}_1$ is also a P'-orbit, and $\mathcal{O}_2$ splits into* $\gcd(2, q-1)$ *P'-orbits of equal length, where $P' = P \cap SU_4(q)$.*

Proof. Fix a nonzero element $\theta \in \mathbb{F}_{q^2}$ such that $\theta^{q-1} = -1$. Then

$$R_4 = \left\{ \begin{pmatrix} I_2 & \theta X \\ 0 & I_2 \end{pmatrix} \;\middle|\; X = \begin{pmatrix} a & c \\ c^q & b \end{pmatrix},\ a, b \in \mathbb{F}_q,\ c \in \mathbb{F}_{q^2} \right\}.$$

Thus we may identify R_4 with the space of hermitian (2×2)-matrices over $\mathbb{F}_{q^2}$. Next, $P = R_4 \cdot C$, where $C \simeq GL_2(q^2)$. Any linear character of R_4 now has the form

$$X \mapsto \varepsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$$

for some $B \in R_4$. Thus every C-orbit on nontrivial linear characters of R_4 is just a C-orbit on $R_4 \setminus \{0\}$. The latter orbits are $\mathcal{O}_1$, that of those X of rank 1, and $\mathcal{O}_2$, that of rank 2. Clearly, $|\mathcal{O}_1| = (q^4 - 1)/(q + 1)$ and $|\mathcal{O}_2| = q(q^4 - 1)/(q + 1)$. Since the dimension of any Weil module V is less than $|\mathcal{O}_2|$, $\mathcal{O}_1$ occurs on $V|_Q$. The claim about $\mathcal{O}_1 \cdot \mathcal{O}_1$ and $\mathcal{O}_1 \cdot \mathcal{O}_2$ follows from the observation that one can find hermitian matrices $X, Y, Z \in R_4$, where X, Y are of rank 1 and Z is of rank 2 such that the rank of $X + Y$ and of $X + Z$ is 2. The last claim of the lemma can be seen by direct computation. □

Proposition 11.3. *Let $G = U_3(q)$ or $SU_3(q)$ with $q = p^f$ and Q be a p-Sylow subgroup of G. Let V be any irreducible kG-module such that the restriction $V|_Q$ contains no nontrivial linear character of Q. Then V is a Weil module or a module of dimension* 1.

Proof. (1) Because of the factorization $U_3(q) = N_{U_3(q)}(Q)\,SU_3(q)$ and because any Weil module of $SU_3(q)$ is extendible to $U_3(q)$, it suffices to prove the proposition for $G = U_3(q)$. Also, the statement is known in the case of characteristic 0, cf. [Geck, Table 3.2]. Hence we may assume that V does not lift to characteristic 0. The case $q = 2$ can be checked directly, so we will assume $q > 2$. A theorem of Broué and Michel [BM] asserts

$$\mathcal{E}_r\big(G, (s)\big) := \bigcup_{\substack{t \in C_G(s) \\ t \text{ an } r\text{-element}}} \mathcal{E}\big(G, (st)\big) \tag{6}$$

is a union of r-blocks, where $s \in G$ is a semisimple r'-element and $\mathcal{E}(G, (st))$ is the Lusztig series [DM] of irreducible complex characters of G corresponding to the G-conjugacy class of the semisimple element st. (Note that we have identified G with the dual group G^*.) Abusing notation, we also denote by $\mathcal{E}_r(G, (s))$ the set of irreducible r-Brauer characters that belong to this union of r-blocks. Assume V belongs to $\mathcal{E}_r(G, (s))$. According to [FS], $\{\hat{\chi} \mid \chi \in \mathcal{E}(G, (s))\}$ forms a basic set for the Brauer characters in $\mathcal{E}_r(G, (s))$, where $\hat{\chi}$ denotes the restriction of χ to r'-classes. Let φ be the Brauer character of V, $1 \neq x \in Z(Q)$ and $y \in Q \setminus Z(Q)$. Recall we are assuming that $V|_Q$ contains no nontrivial linear character of Q, and

$r \neq p$. Since $\theta(1) + (q-1)\theta(x) - q\theta(y) = 0$ for any $\theta \in \mathrm{Irr}(Q)$ except for the case θ is a nontrivial linear character, it follows that

$$\varphi(1) + (q-1)\varphi(x) - q\varphi(y) = 0. \tag{7}$$

(2) Now $C_G(s)$ is $(U_1(q))^3$, $GL_1(q^2) \times U_1(q)$, $U_1(q^3)$, $U_2(q) \times U_1(q)$, or $s = 1$.

We claim that in the first three cases φ lifts to characteristic 0. Indeed, a result of Hiss and Malle [HM, Proposition 1] states that the degree of any Brauer character in $\mathcal{E}_r(G, (s))$, in particular $\varphi(1)$, is divisible by $(G : C_G(s))_{p'}$. In these three cases, $C_G(s)$ is a maximal torus. For any t as in (6), s is a power of st and $t \in C_G(s)$, hence $C_G(st) = C_G(s)$. Thus unipotent characters of $C_G(st)$ have degree 1, whence Lusztig's parameterization [DM] of irreducible complex characters of G implies that $\psi(1) = (G : C_G(s))_{p'}$ for any irreducible complex character ψ in $\mathcal{E}_r(G, (s))$. Therefore, all irreducible characters in $\mathcal{E}_r(G, (s))$, no matter complex or Brauer, have the same degree. It follows that $\varphi = \hat{\psi}$ for some irreducible complex character $\psi \in \mathcal{E}_r(G, (s))$, as stated.

Since we assume V does not lift to characteristic 0, none of the first three cases can occur. In the last case we may write $\varphi = a + b\hat{\rho} + c\hat{\chi}$, where $a, b, c \in \mathbb{Z}$ and ρ, χ are unipotent characters of G of degree $q(q-1)$ and q^3, respectively. The condition (7) implies that $c = 0$. It is well known that $\hat{\rho}$ is irreducible, hence the irreducibility of φ implies that $\varphi = 1_G$ or $\hat{\rho}$, and so we are done as ρ is a Weil character. The fourth case can be treated similarly. □

Lemma 11.4. *Let $A = SU_3(q)$, and let W be an irreducible Weil module of A over k and X any kA-module. Suppose that $(W \otimes X)|_{R_3}$ contains no nontrivial linear character of R_3. Then $Z(R_3)$ acts trivially on X.*

Proof. Observe that $W|_{R_3}$ contains all $q-1$ irreducible characters α_i, $1 \leqslant i \leqslant q-1$, of degree q of R_3. Assume that $Z(R_3)$ acts nontrivially on X. Then $X|_{R_3}$ contains $\bar{\alpha}_i$ for some i. It follows that $(W \otimes X)|_{R_3}$ contains $\alpha_i\bar{\alpha}_i$, which is the sum of all linear characters of R_3, contrary to the assumption. □

Let $G = SU_n(q)$ or $U_n(q)$ with $n \geqslant 4$ and V be an irreducible kG-module. We say that V *has property* $(\mathcal{R}_3)$ if the restriction $V|_{R_3}$ of V to the subgroup R_3 of a standard subgroup $SU_3(q)$ of G does not contain any nontrivial linear character of R_3. Similarly, we say that V *has property* $(\mathcal{R}_4)$ if the restriction $V|_{R_4}$ of V to the subgroup R_4 of a standard subgroup $SU_4(q)$ of G contains only linear characters of R_4 that belong to the orbit $\mathcal{O}_1$ (defined in Lemma 11.2) and maybe the trivial character.

Proposition 11.5. *Let $S = SU_n(q)$, $n \geqslant 5$, $(n, q) \neq (5, 2)$. Let V be any kS-module either with property $(\mathcal{W})$ or with property $(\mathcal{R}_3)$. Then $C_V(Q_1) = C_V(Z_1)$ and the P_1'-module $[Z_1, V]$ is a direct sum of $M(\chi)$'s.*

Proof. The property $(\mathcal{W})$ for V implies that $V|_A$ involves only Weil and trivial modules, where $A = SU_3(q)$ is any standard subgroup, and that $\mathrm{Spec}(R_3, V)$ contains no nontrivial linear characters of R_3 by Lemma 11.1. So we may assume that $(\mathcal{R}_3)$ holds.

If W_χ is the χ-eigenspace for Z_1 on V, where χ is any nontrivial linear character of Z_1, then $W_\chi = M(\chi) \otimes X$ for some K-module X. By Lemma 11.4, $Z(R_3)$ acts trivially on X. But the condition on (n, q) implies that $K = SU_{n-2}(q)$ is quasi-simple. Hence K acts trivially on X, and so $[Z_1, V]$ is a direct sum of some $M(\chi)$.

Next assume that $U := [Q_1, C_V(Z_1)] \neq 0$, and consider a K-orbit $\mathcal{O}$ of nontrivial linear characters of Q_1 occurring on U. We may identify $\mathcal{O}$ with the set of all vectors of fixed norm $\mu = 0$ or 1 in the natural module $W = \mathbb{F}_{q^2}^{n-2}$ for K. Choose a basis $(e_1, \dots, e_{n-2})$ of W in which the Gram matrix of the hermitian form is

$$\mathrm{diag}\left(\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, I_{n-5}\right).$$

In the case $\mu = 0$, $\mathcal{O}$ contains $\alpha = te_1 + e_3$, where $0 \neq t \in \mathbb{F}_{q^2}$ and $t + t^q = 0$. In the case $\mu = 1$, $\mathcal{O}$ contains $\alpha = e_2 + e_3$. Choose a standard subgroup $A = SU_3(q)$ inside K as the pointwise stabilizer of the subspace $\langle e_4, \dots, e_{n-2}\rangle_{\mathbb{F}_{q^2}}$, and let $R_3 = \mathrm{Stab}_A(e_1)$. Then

$$R_3 = \left\{ \mathrm{diag}\left(\begin{pmatrix} 1 & a & b \\ 0 & 1 & -a^q \\ 0 & 0 & 1 \end{pmatrix}, I_{n-5}\right) \,\middle|\, a, b \in \mathbb{F}_{q^2},\ a^{q+1} + b + b^q = 0 \right\},$$

and so no nontrivial element of R_3 fixes α. Thus if v is a nonzero α-eigenvector for Q_1 in U, then v^{R_3} generates a regular kR_3-module. In particular, V affords all nontrivial linear characters of R_3, a contradiction. □

Next we determine the kG-modules V with property $(\mathcal{W})$, $(\mathcal{R}_3)$, or $(\mathcal{R}_4)$, for $G = U_4(q)$. We use the notation of [N1] for conjugacy classes and complex characters of G. For any irreducible r-Brauer character φ of G, we define

$$\varphi[3] = \varphi(1) + (q-1)\varphi(x) - q\varphi(y),$$
$$\varphi[4] = \varphi(1) + (q-1)\varphi(x) - q\varphi(z),$$

where x, respectively y, z, is an element of class $A_2(0)$, respectively $A_3(0)$, $A_4(0)$, of G.

Lemma 11.6. *Let $G = U_4(q)$ and $\varphi \in \mathrm{IBr}_r(G)$ be an irreducible character with* (*at least one of the properties*) $(\mathcal{W})$, $(\mathcal{R}_3)$, *or* $(\mathcal{R}_4)$. *Then*

(i) $\varphi[3] = \varphi[4] = 0$.
(ii) *If* φ *lifts to characteristic* 0 *then* φ *is either of degree* 1 *or a Weil character.*

Proof. (1) As we have already mentioned above, $(\mathcal{W})$ implies $(\mathcal{R}_3)$. Assume that φ has property $(\mathcal{R}_3)$. Consider a subgroup R_3 inside a standard subgroup $A = SU_3(q)$ of G. Since the central nontrivial elements of R_3 belong to class $A_2(0)$ and noncentral elements belong to class $A_4(0)$ of G, $(\mathcal{R}_3)$ implies that $\varphi[4] = 0$.

Next we restrict φ to the parabolic subgroup $P_1 = q^{1+4} : (U_2(q) \times \mathbb{Z}_{q^2-1})$ of G and let ψ be any irreducible constituent of $\varphi|_{P_1}$. Assume that $\psi|_{Q_1}$ contains a nontrivial linear character of Q_1/Z_1, where $Q_1 = O_p(Q_1)$ and $Z_1 = Z(Q_1)$. Observe that P_1 has exactly two orbits on nontrivial linear characters of Q_1/Z_1, $\mathcal{C}_1$ of length $(q+1)(q^2-1)$ and $\mathcal{C}_2$ of length $q(q-1)(q^2-1)$. Moreover, $\mathcal{C}_1$ is afforded by the complex character $\gamma_4(0)$ of P_1 (in the notation of [N1]), and $\mathcal{C}_2$ is afforded by $\gamma_7(0)$. By Clifford's Theorem, we may assume that $\psi|_{Q_1} = \gamma_j(0)|_{Q_1}$ with $j = 4$ or 7. Now we may choose R_3 to be contained in Q_1, with a nontrivial central element x belonging to class $A_2(0)$ and a noncentral element z belonging to class $A_6(0)$ of P_1. Since

$$\rho(1) + (q-1)\rho(x) - q\rho(z) > 0 \quad \text{for } \rho = \gamma_4(0) \text{ and } \rho = \gamma_7(0),$$

we see that $\psi|_{R_3}$ contains nontrivial linear characters of R_3, contrary to $(\mathcal{R}_3)$.

Next assume that $\psi|_{Z_1}$ contains the trivial character 1_{Z_1}. The result we have just proved above implies that $Q_1 \leqslant \mathrm{Ker}(\psi)$ in this case. Thus ψ is actually a representation of P_1/Q_1. Since all r-modular representations of P_1/Q_1 lift to characteristic 0, we may assume that ψ is a complex representation of P_1/Q_1, i.e. one of the representations $\gamma_i(k,l)$ listed in [N1] with $i = 1, 2, 3$, or 8. Choose an element $y \in Q_1$ of class $A_4(0)$ and $z \in Q_1$ of class $A_6(0)$ of P_1. Then $\gamma_i(k,l)$ takes the same value at y and z for $i = 1, 2, 3$, and 8. Hence $\psi(y) = \psi(z)$.

Finally, assume that $\psi|_{Z_1}$ does not contain 1_{Z_1}. Then each irreducible constituent of $\psi|_{Q_1}$ is of degree q^2 and vanishes at both y and z, as they are not central in Q_1. Thus we again have $\psi(y) = \psi(z)$.

We have shown that $\varphi(y) = \varphi(z)$. Note that y belongs to class $A_3(0)$ of G and z belongs to class $A_4(0)$ of G. Hence $\varphi[4] = 0$ implies $\varphi[3] = 0$.

(2) Assume φ has property $(\mathcal{R}_4)$. We restrict φ to the parabolic subgroup $P_2 = R_4 : GL_2(q^2)$ of G and let ψ be any irreducible constituent of $\varphi|_{P_2}$. Here R_4 contains some element x from class $A_2(0)$ of G and some element y from class $A_3(0)$ of G. By assumption, either $\psi|_{R_4}$ is trivial or it yields the short orbit $\mathcal{O}_1$ of R_4-characters. Since $\mathcal{O}_1$ is afforded by the character $\chi_{19}(0,1)$ of G (in the notation of [N1]), we easily check that $\varphi[3] = 0$.

Assume that $\psi|_{R_4}$ contains the trivial character 1_{R_4}. Then ψ is actually a representation of P_2/R_4. Since all r-modular representations of P_2/R_4 lift to characteristic 0, we may assume that ψ is a complex representation of P_2/R_4, i.e. one of the representations $\beta_i(k,l)$ listed in [N1] with $i = 1, 2, 3$, or 8. Choose an element $y \in P_2$ of class $A_4(0)$ and $z \in P_2$ of class $A_5(0)$ of P_2. Then $\beta_i(k,l)$ takes the same value at y and z for $i = 1, 2, 3$ and 8. Hence $\psi(y) = \psi(z)$.

Assume that $\psi|_{R_4}$ yields the orbit $\mathcal{O}_1$. Then $\psi = \lambda^{P_2}$, where λ is an irreducible Brauer character of the inertia group $I = R_4([q^2] : (U_1(q) \times GL_1(q^2)))$ of an R_4-character $\alpha \in \mathcal{O}_1$. Since I is solvable, λ lifts to characteristic 0 by the Fong–Swan Theorem. Hence we may assume that ψ is a complex representation of P_2 yielding only $\mathcal{O}_1$, i.e. one of the representations $\beta_i(k,l)$ listed in [N1] with $i = 4$ or 5. One can check that $\beta_i(k,l)$ takes the same value at y and z for $i = 4$ and 5. Hence $\psi(y) = \psi(z)$.

We have shown that $\varphi(y) = \varphi(z)$. Note that y belongs to class $A_3(0)$ of G and z belongs to class $A_4(0)$ of G. Hence $\varphi[3] = 0$ implies $\varphi[4] = 0$.

(3) Now assume $\varphi[4] = 0$ and φ lifts to characteristic 0. Then φ is either Weil or of degree 1, according to [TZ2, Lemma 4.10]. □

Proposition 11.7. *Let $G = U_4(q)$ or $SU_4(q)$ and $\varphi \in \mathrm{IBr}_r(G)$ be an irreducible Brauer character with* (*at least one of the properties*) $(\mathcal{W})$, $(\mathcal{R}_3)$, *or* $(\mathcal{R}_4)$. *Then φ is either of degree* 1 *or a Weil character.*

Proof. (1) Clearly, it suffices to prove the statement for $U_4(q)$, so we will assume that $G = U_4(q)$. The case $q = 2$ can be checked directly using [Atlas,JLPW], hence we assume $q > 2$. By Lemma 11.6, we may assume that $\varphi[3] = \varphi[4] = 0$ and φ does not lift to characteristic 0. Using the result of Broué and Michel [BM], we assume that φ belongs to $\mathcal{E}_r(G,(s))$, where s is a semisimple r'-element. Again according to [FS], $\{\hat{\chi} \mid \chi \in \mathcal{E}(G,(s))\}$ forms a basic set for the Brauer characters in $\mathcal{E}_r(G,(s))$.

(2) Arguing as in part (2) of the proof of Proposition 11.3, one can show that if $C_G(s)$ is any of the tori $GL_1(q^4)$, $(GL_1(q^2))^2$, $GL_1(q^2) \times (U_1(q))^2$, $U_1(q) \times U_1(q^3)$, and $(U_1(q))^4$, then φ lifts to characteristic 0. So we may assume that $C_G(s)$ is none of those tori.

Assume that $C_G(s)$ is $GL_2(q^2)$, $GL_1(q^2) \times U_2(q)$, or $U_2(q) \times (U_1(q))^2$. In each of these cases, we can find two characters $\alpha, \beta \in \mathcal{E}(G,(s))$ and $a, b \in \mathbb{Z}$ such that $\varphi = a\hat{\alpha} + b\hat{\beta}$. The equations $\varphi[3] = \varphi[4] = 0$ imply that $a = b = 0$, a contradiction.

Suppose that $C_G(s) = (U_2(q))^2$. In this case, we can choose 4 characters $\alpha, \beta, \gamma, \delta \in \mathcal{E}(G,(s))$ (they are certain $\chi_i(k,l)$ with $i = 22$, 21, 21, and 20, respectively, in the notation of [N1]), and $a, b, c, d \in \mathbb{Z}$, such that $\varphi = a\hat{\alpha} + b\hat{\beta} + c\hat{\gamma} + d\hat{\delta}$. The conditions $\varphi[3] = \varphi[4] = 0$ imply that $a = -d$ and $b + c = d(1-q)$. Since $\varphi(1) = d(q^2+1)(q^2-q+1)(q-1)$, we have $d > 0$. But in this case the multiplicity of 1_{R_4} in $\varphi|_{R_4}$ is $-d(q^2+1)(q-1) < 0$, a contradiction.

Suppose that $C_G(s) = U_3(q) \times U_1(q)$. In this case, we can choose 3 characters $\alpha, \beta, \gamma \in \mathcal{E}(G,(s))$ (they are certain $\chi_i(k,l)$ with $i = 19$, 17, and 18, respectively, in the notation of [N1]), and $a, b, c \in \mathbb{Z}$, such that $\varphi = a\hat{\alpha} + b\hat{\beta} + c\hat{\gamma}$. The conditions $\varphi[3] = \varphi[4] = 0$ imply that $b = c = 0$. Since α is a Weil character and $\varphi = a\hat{\alpha}$, φ is also a Weil character.

(3) Finally, suppose that $s = 1$, i.e., φ belongs to a unipotent block. The decomposition matrix

$$D = \begin{pmatrix} 1 & & & & \\ a_1 & 1 & & & \\ a_2 & b_2 & 1 & & \\ a_3 & b_3 & c_3 & 1 & \\ a_4 & b_4 & c_4 & d_4 & 1 \end{pmatrix}$$

of the block (in the standard ordering of the unipotent characters, which are $\chi_i(0)$, with $i = 1$, 14, 12, 13, and 11, respectively, in the notation of [N1]), is approximated by

$$D = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 1 & 1 & & \\ 1 & 0 & 1 & 1 & \\ 1 & 1 & 1 & q & 1 \end{pmatrix},$$

cf. [HM, Proposition 6]. Writing φ as a $\mathbb{Z}$-combination of $\hat{\chi}_i(0)$ and using the condition $\varphi[3] = \varphi[4] = 0$, we see that φ must be a linear character, a Weil character, or the last Brauer character in the block. In the first two cases we are done. In the third case, $\varphi[3] = q^4(q^2 - c_4 - d_4(q - c_3)) > 0$, as can be seen using the above approximation of D. □

Proposition 11.8. *Let $G = U_5(q)$ or $SU_5(q)$ and $\varphi \in \mathrm{IBr}_r(G)$ be an irreducible Brauer character with* (*at least one of the properties*) $(\mathcal{W})$, $(\mathcal{R}_3)$, *or* $(\mathcal{R}_4)$. *Then φ is either of degree* 1 *or a Weil character.*

Proof. (1) Clearly, it suffices to prove the statement for $U_5(q)$, so we will assume that $G = U_5(q)$. The case $q = 2$ can be checked directly from [Atlas,JLPW], hence we assume $q > 2$.

Note that $(\mathcal{R}_4)$ implies $(\mathcal{W})$. For, if φ has property $(\mathcal{R}_4)$, then every constituent ψ of $\varphi|_A$ also satisfies $(\mathcal{R}_4)$, where $A \simeq SU_4(q)$ is a standard subgroup of G. By Proposition 11.7, ψ is either trivial or Weil character, whence φ satisfies $(\mathcal{W})$.

Let V be a kG-module affording φ and φ as in the proposition. By Proposition 11.5, $V = C_V(Q_1) \oplus [Z_1, V]$, and the P_1'-module $[Z_1, V]$ is a direct sum of $M(\chi)$. Here $Q_1 = q^{1+6}$, $Z_1 = Z(Q_1)$. Let s be the r'-part of $q + 1$. By Lemma 4.2, every constituent of $C_V(Q_1)$ is either trivial or a Weil module for $L_1' = SU_3(q)$. According to [DT, Theorem 7.2], there are some integers $a, b_i, c \in \mathbb{Z}$ such that

$$\varphi|_{P_1'} = a \sum_{1_{Z_1} \neq \chi \in \mathrm{IBr}_r(Z_1)} M(\chi) + \sum_{i=0}^{s-1} b_i \zeta_3^i + c \cdot 1_{P_1'}. \tag{8}$$

Here ζ_m^i, $0 \leqslant i \leqslant q$, are Weil characters of $SU_m(q)$. In particular, $\zeta_3^i(1) = (q^2-q+1)-\delta_{i,0}$. Let $t \in Z_1$ be a transvection. Then (8) yields

$$\varphi(t) = -aq^3 + \bigl(q^2-q\bigr)b_0 + \bigl(q^2-q+1\bigr)\sum_{i=1}^{s-1} b_i + c.$$

Next, let $t' \in L_1'$ be a transvection. Then

$$\varphi(t') = -aq^2(q-1) - qb_0 - (q-1)\sum_{i=1}^{s-1} b_i + c,$$

cf. [TZ2, Lemma 4.1]. Since t and t' are conjugate in G, $\varphi(t) = \varphi(t')$, whence

$$a = \sum_{i=0}^{s-1} b_i. \tag{9}$$

On the other hand, branching formula for Weil characters [T1] yields

$$\zeta_5^i\big|_{P_1'} = \sum_{1_{Z_1} \neq \chi \in \mathrm{IBr}_r(Z_1)} M(\chi) + \zeta_3^i. \tag{10}$$

Altogether (8)–(10) imply that the restrictions of φ and $\sum_{i=1}^{s-1} b_i\zeta_5^i + c$ to P_1', and so to a p-Sylow subgroup T of G, are the same.

(2) For $i = 1, \ldots, 7$, let $x_i \in T$ be an element of class $A_{1i}(0)$ in G (in the notation of [N2]). For any Brauer character ϕ of G, let

$$\phi[j] = \begin{cases} \phi(x_1) + (q-1)\phi(x_2) - q\phi(x_j), & j = 3, 4, \\ \phi(x_2) + (q-1)\phi(x_3) - q\phi(x_j), & j = 5, 6, \\ \phi(x_4) + (q-1)\phi(x_6) - q\phi(x_7), & j = 7. \end{cases}$$

Observe that $\zeta_5^i[j] = 0$ for any i and j and clearly $\rho[j] = 0$ for the trivial character ρ. Hence the result of part (1) implies that

$$\varphi[j] = 0, \quad 3 \leqslant j \leqslant 7. \tag{11}$$

If φ lifts to characteristic 0, then already the two relations $\varphi[3] = \varphi[4] = 0$ imply that φ is either of degree 1 or Weil, cf. [TZ2, Lemma 4.10]. Therefore we will assume that φ does not lift to characteristic 0.

(3) Assume that φ belongs to $\mathcal{E}_r(G,(s))$, where s is a semisimple r'-element. We may also assume that $C_G(s)$ is none of the tori $U_1(q^5)$, $GL_1(q^4) \times U_1(q)$, $(GL_1(q^2))^2 \times U_1(q)$, $GL_1(q^2) \times U_1(q^3)$, $GL_1(q^2) \times (U_1(q))^3$, $U_1(q^3) \times (U_1(q))^2$, and $(U_1(q))^5$, since in any of these cases φ would lift to characteristic 0, as one can see by arguing as in part (2) of the proof of Proposition 11.3.

Assume that $C_G(s)$ is $GL_2(q^2) \times U_1(q)$, $GL_1(q^2) \times U_2(q) \times U_1(q)$, $U_1(q^3) \times U_2(q)$, $U_2(q) \times (U_1(q))^3$, $GL_1(q^2) \times U_3(q)$, $U_3(q) \times (U_1(q))^2$, or $(U_2(q))^2 \times U_1(q)$. In each of these cases, we can find $t = 2, 3$, or 4 characters $\alpha_k \in \mathcal{E}(G,(s))$

and $a_k \in \mathbb{Z}$ such that $\varphi = \sum_{k=1}^{t} a_k \hat{\alpha}_k$. Equations (11) imply that $a_k = 0$ for all k, a contradiction.

Assume that $C_G(s)$ is $U_3(q) \times U_2(q)$. In this case we may write $\varphi = \sum_{k=1}^{6} a_k \hat{\beta}_k$ for certain characters $\beta_k \in \mathcal{E}(G,(s))$ (they are labeled as $A_{3k}(i,j)$ in [N2], with $1 \leqslant k \leqslant 6$) and $a_k \in \mathbb{Z}$. Equations (11) imply that $a_1 = a_6 = 0$ and $-a_2 = a_3 = a_4 = a_5$, whence $\varphi(1) = 0$, a contradiction.

Assume that $C_G(s)$ is $U_4(q) \times U_1(q)$. In this case we may write $\varphi = \sum_{k=1}^{5} a_k \hat{\gamma}_k$ for certain characters $\gamma_k \in \mathcal{E}(G,(s))$ (they are labeled as $A_{2k}(i,j)$ in [N2], with $1 \leqslant k \leqslant 5$) and $a_k \in \mathbb{Z}$. Equations (11) imply that $a_k = 0$ for $k \leqslant 4$, whence $\varphi(1) = a_5 \hat{\gamma}_5$. Since γ_5 is a Weil character, φ is a Weil character.

Finally, assume that $s = 1$, i.e. φ is a unipotent block. In this case we may write $\varphi = \sum_{k=1}^{7} a_k \hat{\delta}_k$ for certain unipotent characters δ_k (they are labeled as $A_{1k}(i,j)$ in [N2], with $1 \leqslant k \leqslant 7$) and $a_k \in \mathbb{Z}$. Equations (11) imply that $a_k = 0$ for $k \leqslant 5$, whence $\varphi(1) = a_6 \hat{\delta}_6 + a_7 \hat{\delta}_7$. Since δ_6 is a Weil character and δ_7 is the trivial character, we are done. □

12. Representations of large unitary groups

First we give an upper bound for the dimension of any module V satisfying the conclusion of Proposition 11.5.

Lemma 12.1. *Let $S = SU_n(q)$, $n \geqslant 6$, and $M := M(\chi)$ be the afore described irreducible kP_1'-module of dimension q^{n-2}. Then*

$$\dim \mathrm{Hom}_{kS}\big(M^S, M^S\big) \leqslant \begin{cases} 2q^4 + 3q^3 + 3q^2 - q - 2, & \text{if } n > 6 \text{ is odd},\\ q^4 + 3q^3 + 4q^2 - q - 2, & \text{if } n > 6 \text{ is even},\\ q^4 + 3q^3 + 4q^2 - q - 2, & \text{if } n = 6 \text{ but } 2 \mid q,\\ q^4 + 3q^3 + 5q^2 + q - 1, & \text{if } n = 6 \text{ and } 2 \nmid q. \end{cases}$$

Proof. To ease the notation, denote $H = P_1'$. Let A be a set of representatives of $H\backslash S/H$. For any $a \in A$, define H_a, M_a, M_a' as in Corollary 4.9. By Frobenius reciprocity and Mackey's Theorem,

$$\begin{aligned} \mathrm{Hom}_{kS}\big(M^S, M^S\big) &\simeq \mathrm{Hom}_{kH}\big(\big(M^S\big)\big|_H, M\big) \simeq \mathrm{Hom}_{kH}\Big(\bigoplus_{a\in A}\big(M_a'\big)^H, M\Big)\\ &\simeq \bigoplus_{a\in A} \mathrm{Hom}_{kH}\big(\big(M_a'\big)^H, M\big) \quad (\text{since } M \text{ is irreducible})\\ &\simeq \bigoplus_{a\in A} \mathrm{Hom}_{kH_a}\big(M_a', M_a\big). \end{aligned}$$

The dimension of each hom-space in the latter sum will be bounded using Corollary 4.9.

Let $W = \mathbb{F}_{q^2}^n$ be the natural module for S, with hermitian form $u \circ v$ and a basis $(e_1, \dots, e_n)$ with Gram matrix

$$\operatorname{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, I_{n-4}\right).$$

Then we may assume that $H = \mathrm{Stab}_S(e_1)$. The double cosets of H in S correspond to $(2q^2 - 1)$ H-orbits on nonzero isotropic vectors in W, which are $\{\lambda e_1\}$, $\{v \in W \mid e_1 \circ v = \lambda,\ v \circ v = 0\}$, and $\{0 \neq v \in W \mid v \circ v = e_1 \circ v = 0\}$, where $\lambda \in \mathbb{F}_{q^2}^*$.

For the first kind of double cosets, we may choose $a = \operatorname{diag}(\lambda, \lambda^{-q}, I_{n-3}, \lambda^{q-1})$ (in the chosen basis), and observe that a normalizes $H_a = H$. Since M_a is irreducible in this case, $[M_a, M_a]_{H_a} = 1$.

For the second kind, choose

$$a = \operatorname{diag}\left(\begin{pmatrix} 0 & \lambda^{-q} \\ \lambda & 0 \end{pmatrix}, I_{n-3}, -\lambda^{q-1}\right),$$

and note that a normalizes $H_a = K = \mathrm{Stab}_S(e_1, e_2) \simeq SU_{n-2}(q)$. The character of M_a is $\sum_{i=0}^{q} \zeta_{n-2}^i$. From [DT, §7] it follows that $[M_a, M_a]_{H_a} \leqslant (q+1)^2 + q^2$ if n is odd, and $\leqslant 1 + (q+1)^2$ if n is even.

For the last orbit, choose

$$a = \operatorname{diag}\left(\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, I_{n-4}\right),$$

and observe that $a^2 = 1$. Here $H_a = \mathrm{Stab}_S(e_1, e_3)$. We consider the subgroup $J := \mathrm{Stab}_S(e_1, e_2, e_3)$ of H_a, which plays the rôle of P_1' for $K = \mathrm{Stab}_S(e_1, e_2) \simeq SU_{n-2}(q)$. The character of $M|_K$ has just been described above. Next, the restriction of ζ_{n-2}^i to J is the sum of ζ_{n-4}^i (a Weil character of $SU_{n-4}(q) < J$ inflated to J), and $(q-1)$ pairwise distinct irreducible modules, which are the analogues of $M(\chi)$ for J. It follows that

$$[M_a, M_a]_{H_a} \leqslant \begin{cases} (q+1)^2 + q^2 + (q-1)(q+1)^2 & \text{if } n \text{ is odd,} \\ (q+1)^2 + 1 + (q-1)(q+1)^2 & \text{if either } n > 6 \text{ is even} \\ & \text{or } n = 6 \text{ but } q \text{ is even,} \\ 2(q+1)^2 + 1 + (q-1)(q+1)^2 & \text{if } n = 6 \text{ and } q \text{ is odd.} \end{cases}$$

As we have mentioned above, the chosen representatives a satisfy the hypothesis of Corollary 4.9, whence $\dim \mathrm{Hom}_{kH_a}(M_a', M_a) \leqslant [M_a, M_a]_{H_a}$. Thus

$$\dim \mathrm{Hom}_{kS}\big(M^S, M^S\big) \leqslant \sum_{a \in A} [M_a, M_a]_{H_a}$$

$$\leqslant \begin{cases} q^4+3q^3+4q^2-q-2 & \text{if } n>6 \text{ is even or} \\ & \text{if } n=6 \text{ but } q \text{ is even,} \\ q^4+3q^3+5q^2+q-1 & \text{if } n=6 \text{ and } q \text{ is odd,} \\ 2q^4+3q^3+3q^2-q-2 & \text{if } n>6 \text{ is odd.} \end{cases} \qquad \square$$

Corollary 12.2. *Let $G = U_n(q)$ or $SU_n(q)$ with $n \geqslant 6$. Let V be an irreducible kG-module such that $C_V(Z_1) = C_V(Q_1)$ and that the P_1'-module $[Z_1, V]$ is a direct sum of some $M(\chi)$. Then for any composition factor V' of the kS-module V, where $S = SU_n(q)$, we have* $\dim(V') \leqslant 2q^{n-2}(q-1)\kappa$, *where*

$$\kappa \leqslant \begin{cases} \lfloor (q^4+3q^3+4q^2-q-2)^{1/2} \rfloor, & \text{if } n>6 \text{ is even or} \\ & \text{if } n=6 \text{ but } 2 \mid q, \\ \lfloor (q^4+3q^3+5q^2+q-1)^{1/2} \rfloor, & \text{if } n=6 \text{ and } 2 \nmid q, \\ \lfloor (2q^4+3q^3+3q^2-q-2)^{1/2} \rfloor, & \text{if } n>6 \text{ is odd.} \end{cases}$$

Proof. If $G = U_n(q)$, we still have $V' = C_{V'}(Q_1) \oplus [Z_1, V']$, and the kP_1'-module $[Z_1, V']$ is a direct sum of some $M(\chi)$, since $P_1' < S$. Let W_χ be the χ-eigenspace for Z_1 on V', where χ is a nontrivial linear character of Z_1. Then W_χ is a direct sum of say κ copies of $M(\chi)$. By Lemma 4.8, $\kappa^2 \leqslant \dim \mathrm{Hom}_{kS}(M(\chi)^S, M(\chi)^S)$, whence the bound on κ follows from Lemma 12.1. This is true for any χ, hence $\dim(V') \leqslant 2\dim([Z_1, V']) \leqslant 2q^{n-2}(q-1)\kappa$. $\square$

Theorem 12.3. *Let $G = U_n(q)$ or $SU_n(q)$ with $n \geqslant 3$. Let V be an irreducible kG-module such that $C_V(Z_1) = C_V(Q_1)$ and that the P_1'-module $[Z_1, V]$ is a direct sum of some $M(\chi)$. Then V is either a Weil module or a module of dimension* 1.

Proof. (1) If $n \leqslant 5$ then we may choose a subgroup R_3 inside Q_1 and containing $Z(Q_1)$, hence the assumption on V implies $(\mathcal{R}_3)$, and so we are done by Propositions 11.3, 11.7, and 11.8. So we may assume $n \geqslant 6$. First we assume, in addition, that $(n, q) \neq (6, 2), (6, 3), (7, 2)$. Let V' be any composition factor of the $SU_n(q)$-module V. The statement is clear if $\dim(V') = 1$, so we assume that $\dim(V') > 1$. Then Corollary 12.2 and the assumption on (n, q) imply that

$$\dim(V') < \begin{cases} q^{n-2}\big(q^{n-2}-q\big)(q-1)/(q+1) & \text{if } n \text{ is odd,} \\ q^{n-2}\big(q^{n-2}-1\big)(q-1)/(q+1) & \text{if } n \text{ is even,} \end{cases}$$

where $\mathrm{char}(k) = r$. By [HM], V' is an irreducible Weil module of S. Since any Weil module is extendible to $U_n(q)$ and since V is irreducible, we conclude that $V = V'$ and V is a Weil module.

(2) Suppose that $(n, q) = (6, 2)$. The assumption $C_V(Z_1) = C_V(Q_1)$ implies that $V|_{Q_1}$ does not contain any nontrivial linear character of $Q_1 = 2_+^{1+8}$. Hence $(\phi(1) + \phi(z))/2 = \phi(x) = \phi(y)$, where ϕ is the Brauer character of V, z is the central involution of Q_1 (of class $2A$ of G, in the notation of [Atlas]),

$x \in Q_1 \setminus Z(Q_1)$ is of order 2 (of class $2B$ of G), and $y \in Q_1 \setminus Z(Q_1)$ is of order 4 (of class $4A$ of G). Inspecting the r-Brauer character table of G [JLPW] with $r \neq 2$, we see that the only irreducible Brauer characters satisfying this last condition are Weil or trivial.

For the remaining cases we may assume that $G = U_n(q)$ and denote $S = SU_n(q)$. We will identify G with G^* and use some results of [HM].

(3) Suppose that $(n,q) = (7,2)$. Since $G \simeq S \times \mathbb{Z}_3$, any irreducible G-module restricts irreducibly to S. By Corollary 12.2, $\dim(V) \leqslant 520$. Thus $\dim(V)$ is less than the third (nontrivial) complex degree of G, which is 860 according to [TZ1, Table V]. Hence [HM, Proposition 1] implies that V belongs to $\mathcal{E}_r(G,(s))$, where $s = 1$ or s is such that $C_G(s) = U_{n-1}(q) \times U_1(q)$. The assumption on V also implies that V satisfies the conclusion of [HM, Lemma 10], therefore we may apply [HM, Lemma 14] to V. Thus V is a modular constituent of either a unipotent character χ_λ labeled by the partition $\lambda = (6,1)$, $(4,3)$, $(4,2,1)$, $(4,1^3)$ of 7, or a complex character $\chi_{s,\lambda}$ labeled by $s \neq 1$ and $\lambda = (6)$, $(5,1)$, $(4,2)$, $(4,1^2)$, $(3,3)$, $(3,2,1)$. In the former case, the fragment of the decomposition matrix of the principal r-block of G corresponding to all partitions of 7 which are larger or equal to $(4,1^3)$ is approximated by [HM, Proposition 8]. This information is enough to show that either $\dim(V) \geqslant 858$ or V is Weil. In the latter case, the fragment of the decomposition matrix for $\mathcal{E}(G,(s))$ corresponding to all partitions of 6 which are larger or equal to $(3,2,1)$ is approximated by [HM, Proposition 7]. Again, this information allows us to show that either $\dim(V) \geqslant 43 \cdot 21$ or V is Weil. Thus we conclude that V is a Weil module.

(4) Finally, assume that $(n,q) = (6,3)$. Let V' be an irreducible constituent of the S-module V. By Corollary 12.2, $\dim(V') \leqslant 4536$. Observe that the third complex degree of S is 5551 (cf. [TZ1, Table V]), and the complex characters of the first two degrees extend to G. Hence, an easy argument using [HM, Proposition 1] shows that V belongs to $\mathcal{E}_r(G,(s))$, where $s = 1$ or s is such that $C_G(s) = U_{n-1}(q) \times U_1(q)$. Since $G \simeq (S * Z(G)) \cdot \mathbb{Z}_2$, $\dim(V) \leqslant 2\dim(V') \leqslant 9072$. The assumption on V also implies that V satisfies the conclusion of [HM, Lemma 10], therefore we may apply [HM, Lemma 14] to V. Thus V is a modular constituent of either a unipotent character χ_λ labeled by the partition $\lambda = (5,1)$, $(3,3)$, $(3,2,1)$, $(3,1^3)$ of 6, or a complex character $\chi_{s,\lambda}$ labeled by $s \neq 1$ and $\lambda = (5)$, $(4,1)$, $(3,2)$, $(3,1^2)$, $(2,2,1)$, $(2,1^3)$. In the former case, the fragment of the decomposition matrix of the principal r-block of G corresponding to all partitions of 6 which are larger or equal to $(3,1^3)$ is approximated by [HM, Proposition 7]. Using this, we can show that either $\dim(V) \geqslant 10735$, or V is Weil, or V is labeled by $(4,2)$. But in the last case $\dim(V) \geqslant 5547$ and $V|_S$ is irreducible, as shown in the proof of Theorem 16 of [HM], so $\dim(V') = \dim(V) \geqslant 5547$, a contradiction. Suppose $V \in \mathcal{E}_r(G,(s))$ with $s \neq 1$. In this case, the fragment of the decomposition matrix for $\mathcal{E}(G,(s))$ corresponding to all partitions of 5 which are larger or equal to $(2,1^3)$ is approximated by [HM,

Proposition 6]. Again, this information allows us to show that either $\dim(V) \geqslant 182 \cdot 60$ or V is Weil. Thus we conclude that V is a Weil module. $\square$

Proof of Theorem 2.5. It follows from Propositions 11.7, 11.8 when $n = 4, 5$, and from Proposition 11.5 and Theorem 12.3 when $n > 5$.

This theorem yields the following surprising consequence.

Corollary 12.4. *Let $G = SU_n(q)$ or $U_n(q)$, $n \geqslant 3$, and V be an irreducible kG-module such that $V|_{Q_1}$ contains no nontrivial linear character of Q_1. Then V is either of degree* 1 *or a Weil module.*

Proof. We may embed a subgroup R_3 in such a way that $Z(R_3) = Z(Q_1)$. The assumption on V now implies that $V|_{R_3}$ contains no nontrivial linear character of R_3, that is V has property $(\mathcal{R}_3)$. Let $A \simeq SU_3(q)$ be a standard subgroup containing R_3. By Proposition 11.3, all composition factors of $V|_A$ are trivial or Weil, whence V has property $(\mathcal{W})$ and so V is either of degree 1 or Weil by Theorem 2.5. $\square$

If $n = 2m$, then Q_m is abelian. If $n = 2m + 1$, then we may identify Q_m with the set

$$\left\{[X, a] \,\middle|\, X \in M_m(\mathbb{F}_{q^2}),\ a \in \mathbb{F}_{q^2}^m,\ X + {}^tX^{(q)} + a \cdot {}^ta^{(q)} = 0\right\},$$

with the group operation $[X, a] \cdot [Y, b] = [X + Y - a \cdot {}^tb^{(q)}, a + b]$. Then $Z(Q_m)$ consists of all elements of the form $[X, 0]$.

Lemma 12.5. *Let $S = SU_n(q)$ with $n \geqslant 5$. Set $m = [n/2]$. Then P_m acts on the set of nontrivial linear characters of $Z(Q_m)$ with one orbit of length $(q^{2m} - 1)/(q + 1)$, and one orbit of length $l_2 := (q^{2m} - 1)(q^{2m-1} - q)/(q^2 - 1)(q + 1)$. The first orbit occurs on any Weil module of S. All the remaining orbits have length greater than $(q^{2m} - 1)(q^{2m-1} + 1)/(q^2 - 1)(q + 1)$.*

Proof. One can identify $Z(Q_m)$ with the space of skew-hermitian $(m \times m)$-matrices over $\mathbb{F}_{q^2}$, and then the action of P_m on $Z(Q_m)$ reduces to the action of $L_m := GL_m(q^2)$ if n is odd, and $L_m := SL_m(q^2) \cdot \mathbb{Z}_{q-1}$ if n is even, via $X \mapsto {}^tA^{(q)}XA$ for $X \in Z(Q_m)$ and $A \in L_m$. Here ${}^{(q)}$ is the qth Frobenius map. Any linear character of $Z(Q_m)$ now has the form $X \mapsto \varepsilon^{\mathrm{tr}_{\mathbb{F}_q/\mathbb{F}_p}(\mathrm{Tr}(BX))}$ for some $B \in Z(Q_m)$. Thus every L_m-orbit on nontrivial linear characters of $Z(Q_m)$ is just an orbit of L_m on $Z(Q_m) \setminus \{1\}$. Assume the latter orbit contains a matrix X of rank j. If $j = m \geqslant 3$ then the $SL_m(q^2)$-orbit of X has length equal to $(SL_m(q^2) : U_m(q))$, which is clearly larger than $(q^{2m} - 1)(q^{2m-1} + 1)/(q^2 - 1)(q + 1)$. If $j = m = 2$ then $n = 5$ is odd, and if $j \leqslant m - 1$ then the $SL_m(q^2)$-orbit and the

$GL_m(q^2)$-orbit of X are the same. Thus in the remaining cases we may assume that $L_m = GL_m(q^2)$. Then the stabilizer of X in L_m is $[q^{2j(m-j)}] \cdot (U_j(q) \times GL_{m-j}(q^2))$. So the length of this orbit is $(q^{2m}-1)/(q+1)$ if $j=1$ (there is exactly one orbit of this kind), $(q^{2m}-1)(q^{2m-1}-q)/(q^2-1)(q+1)$ if $j=2$, or larger than $(q^{2m}-1)(q^{2m-1}+1)/(q^2-1)(q+1)$ if $j \geqslant 3$. The Weil characters of S when restricted to Q_n give us the orbit of smallest length. □

Lemma 12.6. *Let $n = 2m+1$ and ϕ be an irreducible character of Q_m. Suppose that $\phi|_{Z(Q_m)}$ contains a linear character α corresponding to a matrix B of rank j (in the notation of the proof of Lemma* 12.5). *Then $\phi(1) = q^j$.*

Proof. Again we identify $Z(Q_m)$ with the skew-hermitian $(m \times m)$-matrices over $\mathbb{F}_{q^2}$. Let $N = \{X \in Z(Q_m) \mid \mathrm{Tr}(BX) = 0\}$. Then $N \lhd Q_m$ since $N \leqslant Z(Q_m)$, and $N \leqslant \mathrm{Ker}(\alpha) \leqslant \mathrm{Ker}(\phi)$. Moreover, $Q_m/N \simeq C_1 \times C_2$, where C_1 is of extra-special type of order q^{1+2j} with $Z(C_1) \nsubseteq \mathrm{Ker}(\alpha)$, and C_2 is elementary abelian of order q^{2m-2j}. Hence the claim follows. □

Proof of Theorem 2.6 (even n).

(1) Assume that $n = 2m \geqslant 6$ and V as in the theorem. By Lemma 12.5, there is a formal sum V' of Weil modules and maybe trivial modules of S such that $V|_{Q_m} = V'|_{Q_m}$. Let $W := \langle e_1, \dots, e_m, f_1, \dots, f_m\rangle_{\mathbb{F}_{q^2}}$ be the natural module of S, and let the hermitian form have the matrix $\left(\begin{smallmatrix} 0 & I_m \\ I_m & 0 \end{smallmatrix}\right)$. We may assume $P_m = \mathrm{Stab}_S(\langle e_1, \dots, e_m\rangle_{\mathbb{F}_{q^2}})$ and $P_1 = \mathrm{Stab}_S(\langle e_m\rangle_{\mathbb{F}_{q^2}})$.

Consider the standard subgroup $A' \simeq SU_4(q)$ as the pointwise stabilizer of $\langle e_j, f_j \mid 3 \leqslant j \leqslant m\rangle_{\mathbb{F}_{q^2}}$. Adding a torus of order $q+1$ to A', we get a subgroup $A \simeq U_4(q)$ of S that induces the full unitary group on $\langle e_1, e_2, f_1, f_2\rangle_{\mathbb{F}_{q^2}}$. Then the afore defined subgroup $R_4 := \mathrm{Stab}_A(e_1, e_2)$ of A is contained in Q_m. Since $V'|_A$ involves only Weil and trivial modules of A, Lemma 11.2 implies that $\mathrm{Spec}(R_4, V)$ contains only $(q^4-1)/(q+1)$ nontrivial linear characters of R_4 (namely, the ones in $\mathcal{O}_1$).

(2) Here we show that the P_1'-module $[Z_1, V]$ is a direct sum of some $M(\chi)$. Again, if χ is a nontrivial linear character of Z_1, then the χ-eigenspace of Z_1 on V is $M(\chi) \otimes X$ for some K-module X, where $K \simeq SU_{n-2}(q)$. By Lemma 11.2, $\mathrm{Spec}(R_4, M(\chi)) \supset \mathcal{O}_1$. If R_4 acts nontrivially on X, then the last statement of Lemma 11.2 implies that $\mathrm{Spec}(R_4, V)$ contains a nontrivial linear character from $\mathcal{O}_2$, contrary to the conclusion of part (1). Hence R_4 acts trivially on X, whence K also acts trivially on X, since $K = SU_{n-2}(q)$ is quasi-simple.

(3) Next we show that $C_V(Z_1) = C_V(Q_1)$. Assume the contrary: $U := [Q_1, C_V(Z_1)] \neq 0$, and consider a K-orbit $\mathcal{O}$ on nontrivial linear Q_1-characters occurring on U. Then we may identify $\mathcal{O}$ either with the set of all nonzero isotropic vectors in $W' := \langle e_j, f_j \mid 1 \leqslant j \leqslant m-1\rangle_{\mathbb{F}_{q^2}}$, or with the set of all vectors of norm, say, 1 in W'. In the former case, choose $\alpha \in \mathcal{O}$ to be f_1. In the

latter case, choose $\alpha \in \mathcal{O}$ to be $te_1 + f_1$, where $t \in \mathbb{F}_{q^2}$ and $t + t^q = 1$. In either case, $R := \mathrm{Stab}_{R_4}(\alpha)$ is of order q. Let U_α be the α-eigenspace of Z_1 on U. Since R fixes U_α, R fixes an 1-subspace $\langle v\rangle_k$ in U. Let λ be the character of R on this 1-subspace. Then λ has exactly q^3 different extensions to R_4, and the sum of them is exactly λ^{R_4}. Since λ^{R_4} is the character of the kR_4-submodule generated by v, we have shown that U affords at least q^3 distinct linear characters of R_4. This contradicts the conclusion of part (2), because $q^3 - 1 > (q^4 - 1)/(q + 1)$.

From parts (2) and (3) and Lemma 4.2 it follows that $V|_K$ involves only Weil and trivial modules of $K = SU_{n-2}(q)$. We will need this consequence for the proof of the theorem in the case n is odd.

(4) The results of parts (2) and (3) imply that V satisfies the hypothesis of Theorem 12.3, and so we are done. $\Box$

Proof of Theorem 2.6 (odd n).

Assume that $n = 2m + 1 \geqslant 5$ and V as in the theorem. By Lemma 12.5, there is a formal sum V' of Weil modules and maybe trivial modules of S such that $V|_{Z(Q_m)} = V'|_{Z(Q_m)}$. Let $W := \langle e_1, \dots, e_m, f_1, \dots, f_m, g\rangle_{\mathbb{F}_{q^2}}$ be the natural module of S, and let the hermitian form have the matrix

$$\begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We may assume $P_m = \mathrm{Stab}_S(\langle e_1, \dots, e_m\rangle_{\mathbb{F}_{q^2}})$ and $P_1 = \mathrm{Stab}_S(\langle e_m\rangle_{\mathbb{F}_{q^2}})$. Then $Z(Q_m) = \mathrm{Stab}_S(e_1, \dots, e_m, g)$ and so it plays the rôle of the subgroup Q_m for $T := \mathrm{Stab}_S(g) \simeq SU_{2m}(q)$. Since the restriction of V' to T involves only Weil and trivial modules of T, we see that every composition factor of the T-module V satisfies the hypothesis and therefore also the conclusion of part (1) of the proof of Theorem 2.6 for even n (as we mentioned at the end of the proof of Theorem 2.6 for even n, the restriction of V to the standard subgroup $M := SU_{n-3}(q)$ involves only Weil and trivial modules of M). Thus V has property $(\mathcal{W})$, and so we are done by Theorem 2.5. $\Box$

To prove Theorem 2.7, we compare the Brauer character in question to an irreducible complex character ϑ of degree $(q^n - 1)(q^{n-1} + 1)/(q + 1)(q^2 - 1)$ if n is even, and $(q^n + 1)(q^{n-1} - q^2)/(q + 1)(q^2 - 1)$ if n is odd. Such a character exists by [TZ1, Corollary 4.2]. As shown in [T2], ϑ is a constituent of the permutation character ω of $SU_n(q)$ on the natural module $\mathbb{F}_{q^2}^n$.

For a finite group of Lie type L, let $d_r(L)$ be the smallest degree > 1 of an irreducible representation of L in cross characteristic r. We let $m = [n/2]$ and consider the subgroup $P'_m = Q_m : L'_m$ of P_m, where $L'_m \simeq SL_m(q^2)$.

Lemma 12.7. *Let $S = SU_n(q)$ with $n \geqslant 4$. Let ω be the above permutation character of S. Then the multiplicity of $1_{P'_m}$ in $\omega|_{P'_m}$ is at most $q^2 + q + 1$.*

Proof. The multiplicity in question is exactly the number of P'_m-orbits on the vectors of the natural module V. First assume that $n = 2m$ and consider a symplectic basis $(e_1, \dots, e_m, f_1, \dots, f_m)$ of V. Then P_m may be identified with $\mathrm{Stab}_S(U)$, where $U = \langle e_1, \dots, e_m\rangle_{\mathbb{F}_{q^2}}$. Clearly, L'_m acts transitively on nonzero elements of U and V/U, and Q_m acts on the coset $f_1 + U$ with q orbits. Thus the number of P'_m-orbits on V is at most $q + 2$. Next assume $n = 2m + 1$. Then we may write $V = \langle e_1, \dots, f_m, g\rangle_{\mathbb{F}_{q^2}}$, with g orthogonal to all e_i, f_i. Let $U' = \langle e_1, \dots, e_m, g\rangle_{\mathbb{F}_{q^2}}$. Clearly, L'_m acts transitively on nonzero elements of U and V/U'. Furthermore, Q_m acts on the coset $f_1 + U'$ with q orbits, and L'_m acts transitively on the coset $g + U$. Thus the number of P'_m-orbits on V is at most $1 + 1 + q + (q^2 - 1) = q^2 + q + 1$. $\square$

Proof of Theorem 2.7 (even n).

Let V be as in the theorem and $n = 2m \geqslant 6$. If $q = 2$ or if all P_m-orbits of nontrivial linear characters of Q_m occurring on V are of length less than $l_2 := (q^n - 1)(q^{n-1} - q)/(q + 1)(q^2 - 1)$, then the statement follows directly from Theorem 2.6. Hence we will assume that $q > 2$ and at least one of P_m-orbit of Q_m-characters on V has length at least l_2. Since $\dim(V) < \mathfrak{d}(n, q, r) < (q^n - 1)(q^{n-1} + 1)/(q + 1)(q^2 - 1)$, this orbit is exactly the (unique) P_m-orbit of length l_2 by Lemma 12.5. Since $\dim(V) - l_2 < (q^n - 1)/(q + 1)$, all the remaining Q_m-characters on V are trivial. Let W be the complex module of S affording the character ϑ. The same argument as above but applied to W shows that the Q_m-module W yields the above P_m-orbit of length l_2 and $\dim(W) - l_2$ times the trivial character. Thus we may write

$$V|_{Q_m} = V_1 \oplus C_V(Q_m), \qquad W|_{Q_m} = W_1 \oplus C_W(Q_m), \tag{12}$$

where V_1 and W_1 afford the same Q_m-character.

Let τ be the Brauer character of V. Let $g \in L'_m \simeq SL_m(q^2)$ be a transvection (in L'_m). Then g is $U_n(q^2)$-conjugate to an element $g' \in Q_m$ (one may choose g' to have the matrix $\left(\begin{smallmatrix} I_m & X \\ 0 & I_m \end{smallmatrix}\right)$ in some symplectic basis of the natural module, where $X \in M_m(\mathbb{F}_{q^2})$ is diagonal skew-hermitian of rank 2). Since $m \geqslant 3$, we see that $C_{U_n(q^2)}(g') \cdot S = U_n(q^2)$, whence g' and g are S-conjugate. From (12) it now follows that

$$\tau(g) - \vartheta(g) = \tau(g') - \vartheta(g') = \dim(V) - \dim(W) = \tau(1) - \vartheta(1). \tag{13}$$

Clearly, L'_m acts on V_1 and W_1, with (Brauer) characters say τ_1 and ϑ_1. Since $m \geqslant 3$, the proof of Lemma 12.5 shows that L'_m acts transitively on the linear characters of Q_m occurring on V_1 and W_1, with stabilizer

$$H = \left[q^{4(m-2)}\right] \cdot \left(SU_2(q) \times SL_{m-2}(q^2)\right) \cdot \mathbb{Z}_{q+1}.$$

Thus $\tau_1 = \alpha^{L'_m}$ and $\vartheta_1 = \beta^{L'_m}$ for some linear (Brauer) characters α and β of H.

Claim that H/H' is a p'-group, where $H' = [H, H]$. For, if $m \geqslant 4$ then the normal subgroup $Q := [q^{4(m-2)}]$ of H is the sum of two natural modules for $SL_{m-2}(q^2)$. If $m = 3$ then Q is the sum of two natural modules for $SU_2(q) \simeq SL_2(q)$. In either case, we then have $Q \leqslant H'$. Next, $SL_{m-2}(q^2)$ is perfect. Also, $SU_2(q)$ is perfect if $q \geqslant 4$ and $SU_2(3)/[SU_2(3), SU_2(3)] \simeq \mathbb{Z}_3$. Hence the claim follows.

Now we have $O^{p'}(H) \leqslant H'$, and so $\alpha = \beta$ on $O^{p'}(H)$. By Lemma 4.10,

$$\tau_1(g) = \vartheta_1(g). \tag{14}$$

Let τ_2, respectively ϑ_2, be the L'_m-character of $C_V(Q_m)$, respectively of $C_W(Q_m)$. From (13) and (14) it follows that

$$\tau_2(g) - \vartheta_2(g) = \tau_2(1) - \vartheta_2(1). \tag{15}$$

Observe that

$$\begin{aligned}\tau_2(1) &= \dim(V) - l_2 < \mathfrak{d}(n, q, r) - l_2 = \frac{q^n - 1}{q^2 - 1} - 1 - \kappa_n(q, r)\\ &= d_r(L'_m),\end{aligned}$$

since $L'_m = SL_m(q^2)$ and $m \geqslant 3$, $q \geqslant 3$, cf. [GT1]. It follows that L'_m acts trivially on $C_V(Q_m)$, whence $\tau_2(g) = \tau_2(1)$. But in this case (15) implies that $\vartheta_2(g) = \vartheta_2(1)$. Since L'_m is generated by transvections, we come to the conclusion that L'_m acts trivially on $C_W(Q_m)$. Thus $C_W(P'_m)$ equals $C_W(Q_m)$ and so has dimension $(q^n - 1)/(q^2 - 1) \geqslant q^4 + q^2 + 1$. This last inequality contradicts Lemma 12.7, since ϑ is a constituent of ω $\Box$.

The proof of Theorem 2.7 in the odd case is slightly more complicated. We begin with the following lemma, in which I is the stabilizer of a linear character of $Z(Q_m)$ from the P_m-orbit of length l_2, cf. Lemma 12.5. We are particularly interested in irreducible kI-representations which extend a given irreducible representation of degree q^2 of Q_m, cf. Lemma 12.6.

Lemma 12.8. *Let $S = SU_5(2)$ and α, β be two irreducible kI-representations, which both extend a given irreducible representation ϕ of degree 4 of Q_m. Then $\alpha(x) = \beta(x)$ for all involutions $x \in I$.*

Proof. Recall that $I = Q_m : U_2(2)$ and $Q_m = 2^{4+4}$. This group and its character table can be constructed using GAP. In particular, I has 6 involution classes, 3 irreducible complex characters of degree 2 and 15 of degree 4. Since the statement is obviously true for $x \in Q$, we only need to look at the involutions outside of Q. Observe that all involutions $y \in I \setminus Q$ form a single conjugacy class in I. (Indeed, consider an irreducible complex character of $U_2(2)$ of degree 2 and inflate it to a character, say, μ of I. Clearly, $\mu(y) = 0$. Inspecting the character table of I, we see that μ takes value 2 at 5 involution classes and vanishes at the last class. Thus the last class consists of the involutions $y \in I \setminus Q$.)

One can show that $\beta = \alpha \otimes \lambda$, where λ is a linear character of $U_2(2)$. In order to prove $\alpha(y) = \beta(y)$, it is therefore enough to show that $\alpha(y) = 0$. Since ϕ is irreducible and lifts to a complex representation of Q, we see that α also lifts to a complex representation of I. Without loss we may assume that α is a complex representation. It is clear that ϕ vanishes at some involutions of Q, and so the same is true for α. Inspecting the character table of I, we see that this property eliminates six characters of degree 4 of I, and all the nine others vanish on the involutions $y \in I \setminus Q$. $\square$

Next we extend Lemma 12.8 to the general case.

Lemma 12.9. *Let $S = SU_n(q)$, $n = 2m + 1 \geqslant 5$, and α, β be two irreducible kI-representations, which both extend a given irreducible representation ϕ of degree q^2 of Q_m. Then $\alpha(x) = \beta(x)$ for all elements $x \in I$ of order p.*

Proof. (1) Fix a basis $(e_1, \dots, e_m, f_1, \dots, f_m, g)$ of the natural module of S, in which the Gram matrix of the hermitian form is

$$\begin{pmatrix} 0 & I_m & 0 \\ I_m & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we may choose

$$P_m = \mathrm{Stab}_S\big(\langle e_1, \dots, e_m\rangle_{\mathbb{F}_{q^2}}\big), \qquad Q_m = \mathrm{Stab}_S(e_1, \dots, e_m).$$

One can identify $Z(Q_m) = \mathrm{Stab}_S(e_1, \dots, e_m, g)$ with the skew-hermitian $(m \times m)$-matrices over $\mathbb{F}_{q^2}$. According to Lemma 12.6, we may assume that $\phi|_{Z(Q_m)} = q^2\lambda$, where the character λ corresponds to the matrix $X = \mathrm{diag}(a, a, 0, 0, \dots, 0)$ with $0 \neq a \in \mathbb{F}_{q^2}$ and $a + a^q = 0$. Then $I = \mathrm{Stab}_{P_m}(\lambda) = Q_m : J$, where $J = [q^{4(m-2)}] : (U_2(q) \times GL_{m-2}(q^2))$.

(2) Let $K := \mathrm{Ker}(\alpha)$. In the proof of Lemma 12.6 we defined a certain subgroup N of $Z(Q_m)$ and showed that $N \leqslant \mathrm{Ker}(\phi)$ and $Q_m/N = C_1 \times C_2$, where C_1 is of extra-special type of order q^{1+4} and C_2 is elementary abelian of order q^{2m-4}. Note that J normalizes N, whence $N \lhd I$. Clearly, $N \leqslant K$. Next, C_2 is also J-stable, and direct computation shows that the only J-stable linear character of C_2 is the trivial one. But C_2 centralizes C_1 and $\phi|_{C_1}$ is irreducible, hence $C_2 \leqslant K$.

(3) We have shown that α is actually an irreducible representation of $C_1 \cdot J$. The same holds for β, and $\alpha|_{C_1} = \beta|_{C_1}$ is irreducible. In this case, $\beta = \alpha \otimes \mu$, where μ is a linear representation of J (inflated to I). Observe that the normal subgroup $R := [q^{4(m-2)}]$ of J is the sum of two copies of the natural module for

the subgroup $T := GL_{m-2}(q^2)$ of J. Since T acts transitively on the nontrivial elements of each copy, we have $[T, R] = R$, whence $R \leqslant J'$.

Now assume that $q \geqslant 3$. Since $GL_{m-2}(q^2)' = SL_{m-2}(q^2)$ and $U_2(q)' = SU_2(q)$, we see that J/J' is a p'-group. Therefore, if $x \in I$ is of order p then $\mu(x) = 1$, whence $\alpha(x) = \beta(x)$ and we are done in the case $q \geqslant 3$.

(4) From now on we assume that $q = 2$. Observe that RT' centralizes C_1 (modulo C_2). Hence by Schur's Lemma RT' acts scalarly on α. If $m \geqslant 4$, then $T' = SL_{m-2}(q^2)$ also acts transitively on the nontrivial elements of each copy of its natural module in R, whence $[R, T'] = R$, RT' is perfect, and so $RT' \leqslant \mathrm{Ker}(\alpha)$. If $m = 2$ then $RT' = 1$. Assume $m = 3$. Then $T' = 1$. In this case, the subgroup $U := U_2(q)$ acts on R (of order q^4) as on its natural module. Hence the only U-stable linear character of R is the trivial one. Thus $R \leqslant \mathrm{Ker}(\alpha)$ in this case as well.

We have shown that RT' is contained in the kernel of α and β. Let M be the subgroup of I generated by N, C_2, R, and T'. Then $M \leqslant \mathrm{Ker}(\alpha)$, and $O^{2'}(I/M) \simeq C_1U' = 2^{1+4} : SU_2(2)$.

(5) Consider the standard subgroup

$$S^* = \mathrm{Stab}_S(e_3, \ldots, e_m, f_3, \ldots, f_m) \simeq SU_5(2)$$

of S. For any subgroup X of S, let $X^* = X \cap S^*$. Then one can show that I^*, Q_m^*, and M^* play respectively the rôles of I, Q_m, and M for S^*, and, moreover, $O^{2'}(I/M) \simeq O^{2'}(I^*/M^*)$. Consequently, the lemma in the case $q = 2$ follows from Lemma 12.8. $\square$

Proof of Theorem 2.7 (odd n).

Let V be as in the theorem and $n = 2m + 1 \geqslant 5$. If all P_m-orbits of nontrivial linear characters of $Z(Q_m)$ occurring on V are of length less than $l_2 := (q^{n-1} - 1)(q^{n-2} - q)/(q + 1)(q^2 - 1)$, then the statement follows directly from Theorem 2.6. Hence we will assume that at least one of P_m-orbit of $Z(Q_m)$-characters on V has length $l \geqslant l_2$. If we identify $Z(Q_m)$ and its linear characters with skew-hermitian $(m \times m)$-matrices over $\mathbb{F}_{q^2}$, then each character α from this orbit corresponds to some matrix of rank $j \geqslant 2$ by Lemma 12.5. If ϕ is an irreducible character of Q_m such that $\phi|_{Z(Q_m)}$ contains α, then $\phi(1) = q^j$ and $\phi|_{Z(Q_m)} = q^j\alpha$ by Lemma 12.6. This is true for each α, hence $\dim(V) \geqslant q^j l$. Since

$$\dim(V) < \mathfrak{d}(n, q, r) < q^3\big(q^{n-1} - 1\big)\big(q^{n-2} + 1\big)\big/(q + 1)\big(q^2 - 1\big),$$

we have $j = 2$ and $l = l_2$, i.e., this orbit is exactly the (unique) P_m-orbit of length l_2, cf. Lemma 12.5. Since

$$\dim(V) - q^2 l_2 \leqslant \big(q^{n-1} - q^2\big)\big/\big(q^2 - 1\big),$$

all the remaining $Z(Q_m)$-characters on V are trivial. Observe that $Q_m/Z(Q_m)$ is the natural module for $P_m/Q_m \simeq GL_m(q^2)$. Hence any P_m-orbit on nontrivial linear characters of $Q_m/Z(Q_m)$ has length at least $(q^{n-1}-1)/(q^2-1)$. This in turn implies that Q_m acts trivially on $C_V(Z(Q_m))$. Let W be the complex module of S affording the character ϑ. The same argument as above but applied to W shows that the Q_m-module W yields the above P_m-orbit of length l_2 (of $Z(Q_m)$-characters) and $\dim(W) - q^2 l_2$ times the trivial character. Thus we may write

$$V|_{Q_m} = V_1 \oplus C_V(Q_m), \qquad W|_{Q_m} = W_1 \oplus C_W(Q_m), \tag{16}$$

where V_1 and W_1 afford the same Q_m-character.

Let τ be the Brauer character of V. Let $g \in L_m \simeq GL_m(q^2)$ be a transvection (in L_m). Then g is $U_n(q^2)$-conjugate to an element $g' \in Z(Q_m)$ (one may choose g' to have the matrix

$$\begin{pmatrix} I_m & X & 0 \\ 0 & I_m & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in some basis of the natural module that we used in the proof of Lemma 12.9, where $X \in M_m(\mathbb{F}_{q^2})$ is diagonal skew-hermitian of rank 2). Since $n \geqslant 5$, we see that $C_{U_n(q^2)}(g') \cdot S = U_n(q^2)$, whence g' and g are S-conjugate. From (16) it now follows that

$$\tau(g) - \vartheta(g) = \tau(g') - \vartheta(g') = \dim(V) - \dim(W) = \tau(1) - \vartheta(1). \tag{17}$$

Clearly, P_m acts on V_1 and W_1, with (Brauer) characters say τ_1 and ϑ_1. By Lemma 12.5, P_m acts transitively on the linear characters of $Z(Q_m)$ occurring on V_1 and W_1, with stabilizer $I = Q_m : J$, and

$$J = \left[q^{4(m-2)}\right] \cdot \left(U_2(q) \times GL_{m-2}\left(q^2\right)\right).$$

Thus $\tau_1 = \alpha^{P_m}$ and $\vartheta_1 = \beta^{P_m}$ for some (Brauer) characters α and β of I of degree q^2. Also note that $\alpha|_{Q_m} = \beta|_{Q_m}$ is irreducible.

By Lemma 12.9, $\alpha(x) = \beta(x)$ on every element $x \in I$ of order p. Hence, we may apply Lemma 4.10 to conclude that

$$\tau_1(g) = \vartheta_1(g). \tag{18}$$

Let τ_2, respectively ϑ_2, be the P_m-character of $C_V(Q_m)$, respectively of $C_W(Q_m)$. From (17) and (18) it follows that

$$\tau_2(g) - \vartheta_2(g) = \tau_2(1) - \vartheta_2(1). \tag{19}$$

Observe that if $m \geqslant 3$ then

$$\begin{aligned} \tau_2(1) &= \dim(V) - q^2 l_2 < \mathfrak{d}(n,q,r) - q^2 l_2 = \frac{q^{n-1}-q^2}{q^2-1} - \kappa_n(q,r) \\ &= d_r(L_m) \end{aligned}$$

since $L_m = GL_m(q^2)$, cf. [GT1]. The same is true for $m = 2$, with $\kappa_n(q,r)$ replaced by 1. From this it follows that $L'_m = SL_m(q^2)$ acts trivially on $C_V(Q_m)$, whence $\tau_2(g) = \tau_2(1)$. But in this case (19) implies that $\vartheta_2(g) = \vartheta_2(1)$. Since L'_m is generated by transvections, we come to the conclusion that L'_m acts trivially on $C_W(Q_m)$. Thus $C_W(P'_m)$ equals $C_W(Q_m)$ and so has dimension $(q^{n-1} - q^2)/(q^2 - 1)$. If $m \geqslant 3$, then clearly $\dim(C_W(P'_m)) \geqslant q^4 + q^2 + 1$, contrary to Lemma 12.7, since ϑ is a constituent of ω.

Assume that $m = 2$. It is shown in [T2] that ω contains 1_S (with multiplicity $q+1$), ϑ, and some irreducible character ρ of degree $q^3(q^2+1)(q^2-q+1)$ (and some other characters). Since $\dim(C_W(P'_m)) = q^2$, it follows from Lemma 12.7 that $\rho|_{P'_m}$ does not contain the trivial character $1_{P'_m}$. On the other hand, one can show using [N2] that ρ is an irreducible constituent of $(1_{P_m})^S$ and so ρ contains 1_{P_m}, again a contradiction. $\Box$

13. Minimal polynomial problem

As we mentioned in Section 3, the following theorem concerning the minimal polynomial problem for unipotent elements of finite groups of Lie type was proved by Zalesskii.

Theorem 13.1 [Z1,Z2]. *Let G be a universal quasi-simple finite group of Lie type of characteristic $p > 0$, and suppose $g \in G$ is of order p. Let Θ be a nontrivial absolutely irreducible representation of G in characteristic $r \neq p$ such that $d_\Theta(g) < p$. Then $p > 2$ and one of the following holds.*

(i) *$G = SU_3(p)$ and g is a transvection.*
(ii) *$G = SL_2(p^2)$.*
(iii) *$G = Sp_4(p)$.*
(iv) *$G = Sp_{2n}(p)$, $n \geqslant 1$, g is a transvection.*

Proof of Theorem 3.1. According to Theorem 13.1, (G, g) has to be as listed in (i)–(v). Furthermore, the emerging Θ in the case $r = 0$ have been classified in [TZ2, Theorem 3.2]. Here we complete the case $r > 0$.

(1) First we consider the case (i): $G = SU_3(p)$, g is a transvection and $1 < d_V(g) < p$. Observe that $Q = O_p(C_G(g))$ is extra-special of order p^3. The condition $d_\Theta(g) < p$ implies that g does not have eigenvalue 1 on Θ, whence $\Theta|_Q$ contains no nontrivial linear character of Q. Now we may apply Proposition 11.3.

(2) The cases (ii) and (iii) follow easily by inspecting the Brauer characters of $SL_2(p)$ and $SL_2(p^2)$.

(3) Suppose we are in case (iv): $G = Sp_4(p)$ and $d_\Theta(g) < p$. Because of [TZ2, Theorem 3.2], we may assume that Θ is not liftable to zero characteristic. If g is a transvection, then either Θ has property $(\mathcal{R}_1)$ or $1 \notin \mathrm{Spec}(g, \Theta)$, and both cases are impossible by Lemma 10.1(i), (ii). If g is not a transvection, then g is

a nontrivial product of two commuting transvections (cf. [Z2]), and so we may apply Lemma 10.1(iii).

(4) Consider the main case (v): $G = Sp_{2n}(p)$ and g is a transvection. By [Z2, Theorem 3], $\Theta(g)$ does not have eigenvalue 1. Hence the condition $d_\Theta(g) < p$ implies that $d_\Theta(g) = (p-1)/2$, i.e. Θ satisfies condition $(\mathcal{R}_1)$. It remains to apply Theorem 2.2. $\square$

The following result concerning the minimal polynomial problem for semisimple elements of finite classical groups has been proved by DiMartino and Zalesskii.

Theorem 13.2 [DZ]. *Let G be a finite classical group in characteristic p with G' being quasi-simple. Let $s \neq p$ be a prime and $g \in G$ be a noncentral element such that g belongs to a proper parabolic subgroup of G and $o(g)$ is a power of s. Let V be any absolutely irreducible module of G of dimension > 1 over a field k of characteristic $r \neq p$. Then either $d_V(g) = o(g)$ or $d_V(g) = o(g) - 1$. Moreover, if $d_V(g) = o(g) - 1$, then for some $z \in Z(G)$ one of the following holds.*

(i) $G = Sp_{2n}(p)$, $p > 2$, $n \geqslant 2$, $o(g) = p + 1$, *and* $\operatorname{rank}(g - z) = 2$.
(ii) $G \leqslant U_n(p)$, $p > 2$, $n > 2$, $o(g) = p + 1$, *and* $\operatorname{rank}(g - z) = 1$.
(iii) $G \leqslant U_n(q)$, $p = 2$, $n > 2$, $o(g) = s = q + 1$, *and* $\operatorname{rank}(g - z) = 1$.
(iv) $G \leqslant U_n(8)$, $n > 2$, $o(g) = 9$, *and* $\operatorname{rank}(g - z) = 1$.
(v) $G \leqslant U_n(2)$, $n > 4$, $o(g) = 9$, *and* $\operatorname{rank}(g - z) = 3$.

If $r = 0$ (and $G \neq Sp_4(3)$), then it is shown in [TZ2, Theorem 5.2] that V is a Weil module of G. The rest of this section is to prove Theorem 3.2, which produces a similar result in cross characteristic case.

Proof of Theorem 3.2 (the symplectic group case). Let $G = Sp_{2n}(q) \neq Sp_4(3)$ and (V, g) satisfy the hypotheses of Theorem 3.2. Applying Theorem 13.2 and replacing g by gz, we may assume that g is an element of order $q+1$ in a standard subgroup $SL_2(q)$ of L_1' and let $A = \langle g \rangle$.

(1) First we show that $C_V(Z_1) = C_V(Q_1)$. Assume the contrary: $U := [Q_1, C_V(Z_1)] \neq 0$. Then $U = \sum_{\alpha \in \mathcal{O}} U_\alpha$ is the direct sum of Q_1-eigenspaces, and $\mathcal{O}$ is the set of all nontrivial linear characters of Q_1. As usual, the action of $L_1' \simeq Sp_{2n-2}(q)$ on $\mathcal{O}$ is similar to the action of L_1' on $\mathbb{F}_q^{2n-2} \setminus \{0\}$. Choosing g to be contained in L_1', we see that $\mathcal{O}$ contains a regular A-orbit. It follows that U contains a regular kA-orbit, contrary to the condition $d_V(g) \leqslant q$.

(2) Here we consider the case $r \neq 2$. Consider the χ-eigenspace $M(\chi) \otimes X$ of Z_1 on $[Z_1, V]$. Then direct computation shows that $\operatorname{Spec}(g, M(\chi))$ contains all $(q+1)$th roots of unity but -1. Therefore, if g has more than one eigenvalue on X, then $\operatorname{Spec}(g, M(\chi) \otimes X)$ contains all $(q+1)$th roots of unity, contrary to the condition $d_V(g) \leqslant q$. Thus g acts scalarly on X, which implies that L_1' acts

trivially on X (since $L_1' = Sp_{2n-2}(q)$ and $(n,q) \neq (2,3)$). By Lemma 6.2, V has property $(\mathcal{R}_1)$, and so V is a Weil module by Theorem 2.2.

(3) Finally we consider the case $r = 2$.

First assume that $n = 2$ if $q > 3$ and $n = 3$ if $q = 3$. If $n = 2$, then by Corollary 5.3, g has a single Jordan block J_q on $M(\chi)$. If $n = 3$, then by Corollary 5.3, g has a Jordan block of size $q = 3$ on at least one of the composition factors of $M(\chi)$, so g has a Jordan block of size at least q on $M(\chi)$. If g acts nontrivially on X, then g has a block J_t with $t \geqslant 2$ on X. In this case, g has a block of size $q+1$ on $M(\chi) \otimes X$ due to Lemma 4.6, which contradicts the condition $d_V(g) \leqslant q$. Hence g acts trivially on X, and so does L_1'. Now we may apply Lemma 6.2 and Theorem 2.2 to conclude that V is a Weil module.

In the case where $n \geqslant 3$ and $(n,q) \neq (3,3)$, the result we have just proved shows that the restriction of V to any standard subgroup of type $Sp_4(q)$ if $q > 3$ and of type $Sp_6(3)$ if $q = 3$ involves only Weil and trivial modules. Hence V has property $(\mathcal{W})$ and so V is a Weil module by Theorem 2.3. □

Remark 13.3. Let (G, V, g) be as in Theorem 3.2, and assume that $(n,q) = (2,3)$. Then V is either a Weil module, or the (unique) unipotent representation ρ of degree 6 (this additional possibility for V was missing in [TZ2, Theorem 3.4]). Indeed, if $r \neq 2$ then we can verify the claim just by looking at $\mathrm{Spec}(g, \phi)$ of the element g (of class $4A$, in the notation of [Atlas]) for any $\phi \in \mathrm{IBr}_r(G)$. If $r = 2$, then part (1) of the above proof of Theorem 3.2 shows that $V|_{Q_1}$ contains no nontrivial linear characters of $Q_1 = 3_+^{1+2}$. This implies that, if ϕ is the Brauer character of V and $\psi = \phi + \bar{\phi}$, then $\psi(1) + 2\psi(z) - 3\psi(x) = 0$, where $z \in Z(Q_1)$ is an element of order 3 (of class $3A$ of G), and $x \in Q_1 \setminus Z(Q_1)$ (of class $3D$ of G). Checking the 2-Brauer characters of G for this property using [JLPW], we see that V is one of the listed modules.

Next we proceed to consider the case of unitary groups. Fix an element $\delta \in \mathbb{F}_{q^2}$ of order $q+1$. By a *pseudoreflection* in $U_n(q)$ we mean an element g with matrix $\mathrm{diag}(\delta, 1, \dots, 1)$ in an orthonormal basis of the natural module for $U_n(q)$. Replacing g by gz^{-1}, we see that the elements g mentioned in case (ii)–(iv) of Theorem 13.2 are pseudoreflections. Let ξ be a primitive $(q+1)$th root of unity in $\mathbb{C}$. Also, we let $G = U_n(q)$ and k be an algebraically closed field of characteristic r coprime to q, and keep the notation P_1', L, K as in Section 11. We begin with the following observation.

Lemma 13.4. *Let $n \geqslant 3$, and assume that $q+1$ is a prime power. Let V be any kG-module such that $d_V(g) \leqslant q$ for any pseudoreflection $g \in G$. Then $C_V(Q_1) = C_V(Z_1)$.*

Proof. Consider a pseudoreflection $g' \in L = U_{n-2}(q)$ and let $A = \langle g' \rangle$. Assume that $U := [Q_1, C_V(Z_1)] \neq 0$. Let U_α be a nonzero eigenspace for Q_1 on U, and let $\mathcal{O}$ be the L-orbit of α. We may identify $\mathcal{O}$ either with the set of nonzero isotropic vectors of the natural module $W' = \mathbb{F}_{q^2}^{n-2}$ for L, or with the set of vectors of a fixed nonzero norm in W'. Then $\mathcal{O}$ has a regular A-orbit. From this it follows that U contains the regular kA-module, contrary to the assumption that $d_V(g') \leqslant q$. Thus $C_V(Z_1) = C_V(Q_1)$. □

Proof of Theorem 3.2 (the unitary group case). Assume that $G = U_n(q)$ and (V, g) satisfies the hypothesis of Theorem 3.2. Applying Theorem 13.2 and replacing g by gz^{-1}, we arrive at the following two cases.

(1) $o(g) = q + 1$ is a prime power, and g is a pseudoreflection. Then Lemma 13.4 implies that V satisfies the hypothesis of Corollary 12.4, whence V is a Weil module.

(2) $q = 2$, $o(g) = 9$, and g belongs to a standard subgroup $U_3(2)$.

First we consider the case $n = 5$. We may assume that $g \in L < P_1''$, $L \simeq U_3(2)$. Observe that $\langle g\rangle$ acts regularly on the nonzero vectors of the natural module $\mathbb{F}_4^3$ of L and on the nontrivial linear characters of Q_1 as well. Since $d_V(g) < |g| = 9$, it follows that $C_V(Q_1) = C_V(Z_1)$. By Corollary 12.4, V is a Weil module.

The above argument shows that the restriction of V to any standard subgroup $SU_5(2)$ involves only Weil or trivial modules. Thus V has property $(\mathcal{W})$. By Theorem 2.5, V is a Weil module. □

14. Quadratic modules in characteristic 3

Let $G = U_n(q)$ and (V, g) be as in Theorem 3.2. If $q = 2$ and $r = o(g) = 3$ then V is a *quadratic module* in characteristic 3, i.e. G is generated by the set of all elements $g \in G$ for which $[g, g, V] = 0$. Quadratic pairs (G, V) with $F^*(G)$ being quasi-simple were studied by Thompson and Ho, cf. [Th,Ho2,Ho1] (without using the classification of finite simple groups). The groups G admitting a quadratic module have been classified by Timmesfeld [Ti] under certain mild conditions. Using the classification of finite simple groups, Meierfrankenfeld (private communication) and Chermak [Ch] showed the following result.

Theorem 14.1 [Ch]. *Let G be a finite group with $F^*(G)$ quasi-simple, $s > 2$ a prime, and let V be a faithful irreducible $\mathbb{F}_sG$-module. Suppose that there is an elementary abelian s-subgroup A such that $G = \langle A^G\rangle$ and $[A, A, V] = 0$. Then one of the following holds.*

(a) *$F^*(G)/Z(F^*(G))$ is a group of Lie type in characteristic s.*
(b) *$s = 3$, $|A| = 3$, and either*
  (i) *$G = PU_n(2)$, $n \geqslant 5$;*
  (ii) *$G = 2\mathbb{A}_n$, $n \geqslant 5$, $n \neq 6$; or*

(iii) *$Z(G)$ is a nontrivial 2-group and $G/Z(G)$ is $Sp_6(2)$, $\Omega_8^+(2)$, $G_2(4)$, Co_1, Sz, J_2.*

We are interested in classifying *the modules V* for the groups listed in the theorem. The case (a) was considered by Premet and Suprunenko in [PS]. The case (b)(ii) was completed by Meierfrankenfeld in [Me], where he showed that V is a basic spin module of G.

In the case (b)(i), if $1 \neq h \in A$ then $|h| = 3$. By [Ch, Lemma 5.8], h lifts to an element g of order 3 in $U_n(2)$. Multiplying g by a suitable central element of $U_n(2)$, we may assume that g fixes a 2-dimensional subspace (in the natural module) pointwise, whence g fixes a nonzero isotropic vector. Thus g satisfies the hypothesis of Theorem 13.2 and therefore V is a Weil module by Theorem 3.2.

Finally, we classify the quadratic modules emerging in the case (b)(iii) of Theorem 14.1 by proving Theorem 3.3.

Proof of Theorem 3.3. One can check that the above groups act faithfully on the root lattice of type E_8, respectively the Leech lattice Λ_{24}. If χ is the corresponding character, then one can find an element g of order 3 such that $\chi(g) = -\chi(1)/2$. It follows that $g^2 + g + 1 = 0$ on the lattice, whence the lattice reduced modulo 3 is a (faithful) quadratic $\mathbb{F}_3$-module.

From now on we assume that k is an algebraically closed field of characteristic 3, G is one of the above groups, V is a faithful irreducible kG-module, for which there is an element $g \in G$ of order 3 such that $[g, g, V] = 0$. We keep the notation for conjugate classes of G as in [Atlas], and refer to irreducible Brauer characters as given in [JLPW]. Observe that G is quasi-simple. In what follows, "irreducible" means absolutely irreducible, and any modular representation is in characteristic 3 (except in part (2)). Let φ be the Brauer character of V.

We will frequently use the following observation: if X is any insoluble subgroup of G that contains a conjugate of g, then X has an irreducible quadratic k-module of dimension > 1; moreover, any composition factor of the X-module V is quadratic.

(1) First we consider the case $G = 2Sp_6(2)$.

First observe that g cannot be of class $3B$ in G. Otherwise a conjugate of g is contained in a subgroup $L := {}^2G_2(3)$ of G, but one can check that L has no irreducible quadratic k-modules of dimension > 1.

Next, note that G contains a subgroup $H := Sp_4(3)$. Since $(G : H) = 28$, we may assume that $g \in H$. Since elements of class $3A$ and $3B$ of H belong to class $3B$ in G, g has to be of class $3C$ or $3D$ in H. In turn, H has a subgroup $K \simeq SL_2(9)$, which meets the classes $3C$ and $3D$ of H. Thus we may assume that $g \in K$.

Since H and K both contain the central involution of G, any constituent of $\varphi|_H$, respectively of $\varphi|_K$, is faithful. Let ψ be a constituent of $\varphi|_H$. It is easy to show that K has only two faithful irreducible quadratic k-modules, both of dimension 2. Now if $\psi(1) \neq 4$, then $\psi(1) = 16$ or 40, in which cases $\psi|_K$ has constituents of dimension 6, a contradiction. Hence ψ is the unique faithful irreducible Brauer character of degree 4 of H. In particular, if $x \in H$ is of order 5, then $\psi(h) = -\psi(1)/4$. This is true for any constituent of $\varphi|_H$, therefore $\varphi(h) = -\varphi(1)/4$. G has only one irreducible Brauer character satisfying the last equality, namely the one of degree 8, and this one can be obtained by reducing the root lattice of type E_8 modulo 3.

(2) Here we consider the case $G = 2\Omega_8^+(2)$.

It is more convenient to work with the full covering group $\widehat{G} := 2^2 \cdot \overline{G}$ of $\overline{G} := \Omega_8^+(2)$. Let ρ be the Brauer character of the natural module $W := \mathbb{F}_2^8$ of $\overline{G}$. Then $\overline{G}$ has 5 classes of elements of order 3, and ρ takes value 5, -4, -4, -1, 2, on these five classes, respectively, and the triality automorphism of $\overline{G}$ permutes the first three classes. Hence, without loss of generality, we may assume that $\rho(\bar{g}) \geqslant -1$, where $\bar{g}$ is the image of g in $\overline{G}$. This implies that the fixed point subspace of $\bar{g}$ on W has dimension $\geqslant 2$, whence $\bar{g}$ fixes a nonisotropic vector of W and so $\bar{g}$ belongs to a subgroup $\overline{A} \simeq Sp_6(2)$ of $\overline{G}$. Let A be the complete inverse image of $\overline{A}$ in $\widehat{G}$ and let $B = A^{(\infty)}$. Since $Sp_6(2)$ has no nontrivial quadratic modules, cf. [Ch], $B = 2Sp_6(2)$. Restricting φ to B and using the result of part (1), we see that $\varphi(x) = -\varphi(1)/4$ for some element $x \in \widehat{G}$ of order 5. This property excludes all but the 3-Brauer character of degree 8 of G.

(3) Next we consider the case $G = 2J_2$.

G has a subgroup $H = SU_3(3)$ of index 200, hence we may assume that $g \in H$. Observe that g belongs to class $3A$ of H, for otherwise a conjugate of g would be contained in a Frobenius subgroup of order 21 of H, contrary to [Ch, Lemma 3.1]. Thus g is a root element of H. Let ψ be any constituent of $\varphi|_H$. Then we can lift ψ to a representation Ψ of $\mathcal{H} := SL_3(k)$, and $\Psi(h)$ is quadratic for any root element $h \in \mathcal{H}$. Let the highest weight of Ψ be $a\omega_1 + b\omega_2$, where ω_1, ω_2 are the fundamental weights of $\mathcal{H}$ and $0 \leqslant a, b \leqslant 2$. If $a = 2$ for instance, then by Smith's Theorem the restriction of Ψ to the first fundamental subgroup $SL_2(k)$ has a direct summand of dimension 3, which is the basic Steinberg module of $SL_2(k)$ and on which root elements of $SL_2(k)$ act freely; a contradiction. Thus $0 \leqslant a, b \leqslant 1$, i.e. $\psi(1) = 1$, 3, 3, or 7. It follows that $\hat{\psi}(1) - 7\hat{\psi}(x) - 8\hat{\psi}(y) + 14\hat{\psi}(z) = 0$, where x, respectively y, z, is an element of class $4C$, respectively $7A$, $8A$, of H and $\hat{\psi} = \psi + \bar{\psi}$. One can show that x, respectively y, z, belongs to class $4A$, respectively $7A$, $8A$, in J_2. This implies that $\hat{\varphi}(1) - 7\hat{\varphi}(x) - 8\hat{\varphi}(y) + 14\hat{\varphi}(z) = 0$, where $\hat{\varphi} = \varphi + \overline{\varphi}$. The only faithful irreducible 3-Brauer characters of G with this property are the 2 characters of degree 6, and the one of degree 14. The first two occur in the reduction modulo 3 of the Leech lattice. The restriction of the last one to H contains constituents of degree 6 which are not quadratic as we have already shown, so we are done.

(4) Here we consider the case $G = 2G_2(4)$.

G has a subgroup $J = 2J_2$ of index 416, hence we may assume that $g \in J$. Let ψ be any constituent of $\varphi|_J$. The result of part (3) shows that $\psi(1) = 6$, and

$\mathbb{Q}(\psi) = \mathbb{Q}(\sqrt{5})$. Let ψ^* be the conjugate of ψ under the Galois automorphism of $\mathbb{Q}(\sqrt{5})$, and let $\hat{\psi} = \psi + \psi^*$. Then $2\hat{\psi}(1) - 7\hat{\psi}(x) + 5\hat{\psi}(y) = 0$, where x, respectively y, is an element of class $5A$, respectively $7A$, of J. It follows that $2\hat{\varphi}(1) - 7\hat{\varphi}(x) + 5\hat{\varphi}(y) = 0$, where $\hat{\varphi} = \varphi + \varphi^*$. The only faithful irreducible 3-Brauer characters of G with this property is the (unique) character of degree 12, and this one occurs in the reduction modulo 3 of the Leech lattice.

(5) Here we consider the case $G = 2Sz$.

According to [Ch], $C_G(g)$ has a composition factor isomorphic to $PSU_4(3)$. Hence g is of class $3A$ in G and so a conjugate of g is contained in a subgroup $M \simeq 2G_2(4)$ of G. We will assume that $g \in M$. Let ψ be any constituent of $\varphi|_J$. The result of part (4) shows that $\psi(1) = 12$, and $2\psi(1) - 7\psi(x) + 5\psi(y) = 0$, where x, respectively y, is an element of class $5A$, respectively $7A$, of M. It follows that $2\varphi(1) - 7\varphi(x) + 5\varphi(y) = 0$. The only faithful irreducible 3-Brauer characters of G with this property is the (unique) character of degree 12, and this one occurs in the reduction modulo 3 of the Leech lattice.

(6) Finally, let $G = 2Co_1$.

Observe that G has a subgroup $S = 6 \cdot Sz \cdot 2$. The result of part (5) implies that $\varphi|_S$ involves only the two irreducible 3-Brauer characters of S of degree 12. Based on partial information available at present about 3-Brauer characters of G, Hiss and Müller (private communication) have been able to show that there is exactly one φ satisfying this condition; namely, the one obtained by reducing the Leech lattice modulo 3. $\square$

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