

Ph.D. 12053

THE CALCULATION OF INSTANTON DETERMINANTS

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Dissertation submitted for the degree of Doctor of Philosophy  
at the University of Cambridge.

September 1981

Summary of a Dissertation Entitled  
"The Calculation of Instanton Determinants"

by

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This dissertation deals with successive elucidations of the form and structure of functional determinants of operators acting in the background field of Yang-Mills instantons.

In the first chapter a general review is given of the way in which instanton effects arise in field theory calculations, and how the principal technique of semi-classical approximation of relevant functional integrals leads naturally to a consideration of instanton determinants. A brief outline of the construction of Atiyah, Drinfeld, Hitchin and Manin - of central importance in such calculations - is appended, together with the forms taken by the Green functions (including those for tensor products) in this formalism.

The second chapter employs zeta-function renormalisation (as used by a number of authors) to obtain an expression for the variation of the determinant with respect to its parameters; this leads to a discussion of the vacuum polarisation current due to instantons, an extension of the work of Brown and Creamer being presented, and then compared with the work of Corrigan, Goddard, Osborn and Templeton.

The third chapter deals with the efforts of various authors (Osborn, Berg and Lüscher) to remove the variation from the determinant obtained by the methods above; Jack's generalisation of this work to tensor products is introduced, and its implications for  $SU(2)$  discussed along with explicit forms for the 't Hooft instanton solutions.

Next an ansatz due to Osborn for the form of the determinant in the case of  $SU(2)$  is presented, with an investigation of its limiting and conformal properties; details of numerical checks on its accuracy are given for  $k=2$  and  $k=3$ .

Using results from this calculation, and employing conformal properties of various integrals involved, an exact form for the determinant in the case of general two-instanton 't Hooft ( $SU(2)$ ) solutions is obtained.

A final chapter briefly reviews the progress made in these investigations and possible future developments.



## PREFACE

The work contained in this dissertation is original except where otherwise indicated. No part of it has been, or is being, submitted for any degree, diploma or other qualification at any other University.

I should like to thank my supervisor, Dr P. Goddard, for his continual help and encouragement throughout this work; I am also grateful to Dr H. Osborn for many useful discussions.

I acknowledge a Science Research Council Studentship.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration.

G. P. M. Jones

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## CHAPTER 1: Basic Results and Formalism

### 1. Introduction

It is generally believed that the most likely candidate at present for a theory of Nature will be one in which the strong interactions are modelled by Quantum Chromodynamics, a non-Abelian gauge theory of a type first introduced by Yang and Mills.<sup>1</sup> In an attempt to elucidate the detailed structure of this theory, standard perturbative techniques have been used; but the situation is complicated by the occurrence of non-perturbative effects.

The first of these arises from the presence of non-trivial local minima found by Belavin, Polyakov, Schwarz and Tyupkin<sup>2</sup> in the Euclidean domain of the action functional of such non-Abelian gauge theories. A direct consequence of this is the dependence of the corresponding quantum field theories on an additional parameter  $\Theta$ <sup>3</sup> (at least in the absence of any coupled massless fermion fields or scalar fields which realise  $U(1)$  chiral symmetry). Even though  $\Theta$  is presumably zero in Quantum Chromodynamics, and coupled quark fields are involved as well as gluons,  $\mathcal{E}(\Theta)$ , the vacuum energy density, contains information of interest:  $\mathcal{E}''(0)$  can be related to the mass of the  $U(1)$  singlet pseudo-scalar Goldstone boson (insofar as the  $1/N$  expansion provides a good approximation).<sup>4</sup>

Further, the fact that these local minima are characterised by an integer  $k$  (the Pontryagin index), which labels the topologically inequivalent classes of such field configurations, leads to a resolution of the  $U(1)$  problem associated with this supposed Goldstone boson, and provides perhaps the main phenomenological consequence of these non-perturbative ideas so far.<sup>5</sup>

## 2. The Semi-Classical Approximation

Otherwise, when investigating these effects, one has recourse to semi-classical methods.<sup>6,7</sup> Typically one is dealing with a Euclidean functional integral of the form

$$Z = \frac{1}{Z_0} \int d[\phi] e^{-\frac{1}{g^2} S_\phi} \Phi(\phi) \quad (1.1)$$

where  $S_\phi$  will be the gauge-invariant Euclidean action. By a suitable choice of  $\Phi$ ,  $Z$  generates all the (Euclidean) Green functions of the theory (which themselves effectively define that theory). For small values of  $g$  the integral may be approximated to leading order in this parameter by a sum of Gaussian integrals centred at the minima of the action  $S_\phi$ .

Belavin et al<sup>2</sup> first investigated these minima and exhibited an explicit form for one of them. As noted above, it was shown how they could be characterised by their Pontryagin index  $k$  (an integer), which labels topologically inequivalent classes of field configurations. Within each of these classes the action is bounded by a constant multiple of  $|k|$  and, furthermore, this bound is saturated by values of the gauge potential for which the field strength  $F_{\mu\nu} = \pm {}^*F_{\mu\nu}^a$ , where  ${}^*F_{\mu\nu}^a = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} F^{a\rho\sigma}$  is the dual of  $F_{\mu\nu}^a$ .

The self-dual solution found by Belavin et al<sup>2</sup> with  $k = 1$ , generally called an instanton ('t Hooft's terminology) depends on five parameters: four co-ordinates of position and a scale which corresponds to the instanton's "size". The calculation of the semi-classical contribution to the functional integration measure in terms of integrals over these solution parameters was first obtained by 't Hooft for the gauge group  $SU(2)$ ,<sup>8</sup> calculations

further analysed<sup>9</sup> and subsequently extended to  $\int U(n)$ .<sup>10</sup> Use has been made of these results for a variety of purposes,<sup>11</sup> generally in the "dilute gas" approximation.

In this one assumes that the set of minima can be represented tolerably faithfully by an arbitrary superposition of arbitrary numbers of single instanton and anti-instanton fields; the corresponding contribution to the functional measure is then taken to be the appropriate product of single instanton measures together with a statistical weight factor  $1/(n_+!n_-!)$ , where  $n_+$ ,  $n_-$  are respectively the numbers of instantons and anti-instantons. In this form the functional measure corresponds precisely (in statistical mechanics terms) to a free gas of two types of bosons, although interactions of some kind need to be introduced subsequently between instantons and anti-instantons since arbitrary configurations of these will not, in general, be stationary points of the action.<sup>12,13</sup> But even neglecting these problems, and calculating  $\mathcal{E}(\theta) = K(1 - \cos \theta)$ ,<sup>5,12</sup> one finds  $K$  infinite from a divergent integral over the instanton scale size. This highlights the crucial difficulty with the dilute gas approximation: the formalism itself is weighted towards large scale sizes, but if the instanton scales become comparable with their separations the initial superposed configurations are no longer even an approximate stationary point of the action.

There is a further problem with this approach, for it is unclear to what extent one may be over-counting in the functional measure by virtue of the overlapping superpositions. Indeed, Witten<sup>14</sup> arguing from calculations based on the  $1/N$  expression, has questioned the whole basis of the approximation, though exact calculations<sup>15,16</sup> in the closely-related two-dimensional

$CP^N$ -model suggest no fundamental conflict between the  $1/N$  expansion and instanton methods as such. As a further indication of the doubtful nature of the dilute gas approximation, calculations by Frolov and Schwarz<sup>17</sup> on the  $O(3)$   $\sigma$ -model and Berg and Lüscher<sup>18</sup> for the  $CP^N$  generalisation; suggest that the instantons behave as a Coulomb gas in its dense phase (see also Belavin, Fateev, Schwarz and Tyupkin<sup>19</sup>).

Thus it would clearly be desirable to apply the semi-classical procedure systematically to gauge theories, making use of a well-defined, complete set of classical solutions about which one can expand the functional integral measure. Witten<sup>20</sup>, 't Hooft<sup>21</sup> and Jackiw, Nohl and Rebbi<sup>22</sup> succeeded in progressively generalising the instanton solutions of Belavin *et al*<sup>2</sup> to one depending on  $5k + 4$  parameters, and having Pontryagin index  $k$ . These results were later subsumed and extended by the construction of Atiyah, Hitchin, Drinfeld and Manin (referred to as ADHM hereinafter).<sup>23</sup> In this, the general self-dual solution for arbitrary compact classical group is exhibited. The work of these authors is of such importance in what follows that it is given in some detail below. Although all self-dual solutions are produced by this technique and Atiyah and Jones<sup>24</sup> have shown that the space of self-dual instanton solutions largely exhausts the topological structure of the full space of field configurations, it remains only a conjecture<sup>25</sup> that the functional integrals occurring can be well-approximated by the semi-classical approach of above just using these configurations for arbitrary  $k$ .

### 3. Asymptotic Expansions of Functional Integrals

In using this approach to calculate (1.1), it is instructive<sup>26</sup> to consider the finite-dimension analogue:

$$I = \int d^n x f(x) e^{-\frac{1}{g^2} S(x)}. \quad (1.2)$$

If the minimum of  $S(x)$  occurs on a  $k$ -dimensional set of points  $M$ , parametrised by  $x(t_1, \dots, t_k)$  with  $S(M) = S_0$ ,

$$\left. \frac{\partial S}{\partial x_i} \right|_M = 0, \quad \left. \frac{\partial^2 S}{\partial x_i \partial x_j} \right|_M = C_{ij}(t_1, \dots, t_k) \quad (1.3)$$

then as  $g \rightarrow 0$  the leading contribution to  $I$  is an integral over  $M$ :

$$I \underset{g \rightarrow 0}{\sim} g^{n-k} (2\pi)^{\frac{n-k}{2}} \int \prod_1^k dt_i \sqrt{n} f_0 e^{-\frac{1}{g^2} S_0} (\det' C)^{-\frac{1}{2}} \quad (1.4)$$

$$\text{where } n = \det \left( \sum_i \frac{\partial x_i}{\partial t_i} \frac{\partial x_i}{\partial t_m} \right) \quad (1.5)$$

and  $f_0$  is the restriction of  $f(x)$  to  $M$ . The prime on  $\det C$  indicates that only non-zero eigenvalues are to be taken.

Returning to the field theory version of (1.4) a similar result is obtained, but care must be taken that the measure has been suitably normalised to ensure no factors of  $(\sqrt{2\pi g})^n$ ,  $n \rightarrow \infty$ , occur, and that only determinants of dimensionless quantities are computed. The latter is achieved by the introduction of a parameter  $\mu$ , of dimension length<sup>-1</sup>. Thus a more appropriate finite dimension analogy is

$$\int d^n x \left( \frac{\mu^2}{2\pi g^2} \right)^{n/2} e^{-\frac{1}{g^2} S} \sim \left( \frac{\mu}{g\sqrt{2\pi}} \right)^k e^{-\frac{1}{g^2} S_0} \int \prod_1^k dt_i \sqrt{n} f_0 (\det' C)^{-\frac{1}{2}} \quad (1.6)$$

which can then be taken over directly to the field theories under discussion since the set of parameters describing the minima of the action in this case come to be finite-dimensional.

In the general situation under consideration, a Yang-Mills gauge theory, with gauge group  $G$ , is described by a vector potential  $A_\mu$  and a field strength  $F_{\mu\nu}$ , where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu] \quad (1.7)$$

both taking values in the lie algebra at  $G$  and transforming under elements of  $G$  as

$$A_\mu \xrightarrow{g} g(x)^{-1} A_\mu g(x) + g(x)^{-1} \partial_\mu g(x), \quad (1.8)$$

$$F_{\mu\nu} \xrightarrow{g} g(x)^{-1} F_{\mu\nu} g(x). \quad (1.9)$$

Then an appropriate gauge-invariant Euclidean action  $S$  is

$$S = -\frac{1}{2} \int d^4x \operatorname{Tr} (F_{\mu\nu} F_{\mu\nu}). \quad (1.10)$$

The investigations of Belavin et al<sup>2</sup> concerned vector potentials which are pure gauges at Euclidean  $\infty$ . Then (as stated above)

$$-\frac{1}{2} \int d^4x \operatorname{Tr} (F_{\mu\nu}^* F_{\mu\nu}) = 8\pi^2 k, \quad (1.11)$$

$$k = 0, \pm 1, \pm 2, \dots$$

And since

$$S = -\frac{1}{4} \int d^4x \operatorname{Tr} [(F \pm F^*)^2 \pm 2 F_{\mu\nu}^* F_{\mu\nu}],$$



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self-dual on anti-self-dual field strengths clearly saturate the lower bound of the action, and are minima.

The gauge theory analogue of the argument leading to (1.4) in the finite dimension case can then be carried through<sup>26</sup>.

It is convenient to split an arbitrary potential  $A_\mu$  into three pieces

$$A_\mu = A_\mu^\circ + D_\mu^\circ \phi + a_\mu. \quad (1.12)$$

Here  $A_\mu^\circ$  is an instanton potential depending on a number  $N(k)$  of parameters  $t_i$ ,  $D_\mu^\circ$  is the covariant derivative formed from this defined by

$$D_\mu^\circ \phi = \partial_\mu \phi + [A_\mu^\circ, \phi] \quad (1.13)$$

and  $a_\mu$ ,  $D_\mu^\circ \phi$ ,  $\frac{\partial A_\mu^\circ}{\partial t_i}$  ( $i=1, \dots, N(k)$ ) are chosen to be mutually orthogonal, i.e.

$$\int a_\mu D_\mu^\circ \phi d^4x = 0.$$

In this,  $a_\mu$  represent quantum fluctuations about the classical background field  $A_\mu^\circ$ , while  $D_\mu^\circ \phi$  are essentially gauge transformations, contributing only a volume term in the calculation (albeit infinite, as the group of gauge transformations is infinite-dimensional) which is divided out by  $Z_0$  in (1.1).

The expansion of the action up to terms quadratic in  $a_\mu$  is

$$S = 8\pi^2 |k| + \int d^4x \text{Tr}(a_\mu \Delta_{\mu\nu} a_\nu) + O(a^3) \quad (1.14)$$

where

$$\Delta_{\mu\nu} a_\nu = (D_\nu^\circ)^2 a_\mu + 2[F_{\mu\nu}^\circ, a_\nu] - D_\nu^\circ D_\mu^\circ a_\nu. \quad (1.15)$$

The Jacobian  $\sqrt{\eta'}$  corresponding to  $\sqrt{\eta}$  in (1.4) has two parts: one from the finite-dimensional set of parameters  $t_i$ ,

$$\sqrt{\eta'} = \left[ \det \left\{ \int d^4x \operatorname{Tr} \frac{\partial A_\mu^\circ}{\partial t_i} \cdot \frac{\partial A_\mu^\circ}{\partial t_j} \right\} \right]^{\frac{1}{2}} \quad (1.16)$$

and another from the functional integral over  $\phi$ ,

$$\left[ \det \left( -(D^\circ)^2 / \mu^2 \right) \right]^{\frac{1}{2}}. \quad (1.17)$$

The analogue of  $\left[ \det' (C / \mu^2) \right]^{-\frac{1}{2}}$  is

$$\left[ \det' (-\Delta_{\mu\nu} / \mu^2) \right]^{-\frac{1}{2}}.$$

It may then be shown<sup>26</sup>, by relating

$$\left[ \det' (-\Delta_{\mu\nu} / \mu^2) \right]^{\frac{1}{2}} \text{ to } \left[ \det' (\Delta_1 / \mu^2) \right]^{\frac{1}{2}}$$

where  $(\Delta_1)_{\mu\nu} a_\nu = -D_\mu^\circ a_\nu - 2[F_{\mu\nu}^\circ, a_\nu]$  (1.18)

using the self-duality of  $F_{\mu\nu}^\circ$ , that the leading contribution to the asymptotic expansion as  $g \rightarrow 0$  for each  $k$  is given by

$$\left( \frac{\mu}{\sqrt{2\pi}g} \right)^{N(k)} e^{-\frac{8\pi^2 k}{g}} \int \prod_i^{N(k)} dt_i \frac{\left[ \det' (-(D^\circ)^2 / \mu^2) \right]^{\frac{1}{2}}}{\left[ \det' (\Delta_1 / \mu^2) \right]^{\frac{1}{2}}} \sqrt{\eta'} \Phi. \quad (1.19)$$

This provides an expansion in terms of the functional determinants of operators in background fields of classical instantons, which are seen to enter crucially in this approach.

#### 4. The ADHM Construction

The construction of Atiyah, Hitchin, Drinfeld and Manin<sup>23</sup> mentioned above has played a central rôle in the subsequent investigations of instantons and their properties (see 26,27 for full discussion in this context).

The techniques employed have their origins in twistor methods<sup>28</sup>. Atiyah and Ward<sup>29</sup> used these to reduce the problem of constructing all self-dual solutions of the Yang-Mills equations to one of complex algebraic geometry; then building on the work of Barth<sup>30</sup> and Horrocks<sup>31</sup>, ADHM obtained the general method of construction outlined below (following the treatment and notation of 27 ).

For a general compact lie group the self-dual solutions are obtained by adding together the relevant constructions for each component simple lie algebra occurring in the decomposition of the lie algebra of the original group. Quite simple descriptions of the solutions exist for each of the four sequences of compact groups  $(SU(n+1), O(2n+1), O(2n), Sp(n))$  but only  $Sp(n)$  will be treated here, since the formalism is simplest and the others may be obtained by suitable embeddings.

The instanton gauge potential can be written in this formalism as

$$A_\mu = v^\dagger \partial_\mu v \quad (1.20)$$

where (for the case of the symplectic group  $Sp(n)$  <sup>27</sup>)  $v(x)$  is an  $(n+k) \times n$  matrix of quaternions subject to

$$v^\dagger v = 1_n \quad (1.21)$$

and

$$v^\dagger(x) \Delta(x) = 0 \quad (1.22)$$

Here

$$\Delta_{\lambda i}(x) = a_{\lambda i} + b_{\lambda i}x,$$

$$1 \leq i \leq k, \quad 1 \leq \lambda \leq n+k,$$

so  $\Delta$ ,  $a$  and  $b$  are  $(n+k) \times k$  matrices of quaternions

( $x = x_0 - i\vec{x} \cdot \vec{\sigma}$  : the quaternionic representation);

$k$  is the instanton number. For (1.20) to yield a self-dual field strength,  $a^+a$ ,  $b^+b$  and  $a^+b$  are constrained to be symmetric as  $k \times k$  quaternionic matrices. This, in its turn, forces  $\Delta^+\Delta$  to be the real and symmetric for all  $x$ ; it must also be non-singular. Thus the following quantities may be defined:

$$f = (\Delta^+\Delta)^{-1}, \quad (1.23)$$

$$\mu = a^+a, \quad (1.24)$$

$$\nu = b^+b. \quad (1.25)$$

It is then straightforward to show that the resultant  $F_{\mu\nu}$  is self-dual, and that  $k$  is indeed the instanton number<sup>27</sup>.

In terms of this construction the Green function of the covariant Laplacian transforming under the fundamental representation takes a particularly elegant form:

$$G(x, y) = \frac{v^+(x) v(y)}{4\pi^2 |x-y|^2}, \quad (1.25)$$

this being, in fact, the simplest possible generalisation of the ordinary Green function

$$G_0(x, y) = \frac{1}{4\pi^2 |x-y|^2} \quad (1.27)$$

that transforms correctly under the gauge group, i.e.

$$G(x, y) \xrightarrow{g} g(x)^{-1} G(x, y) g(y) \quad (1.28)$$

It is fairly straightforward<sup>27</sup>, using standard techniques of this construction, to verify that (1.26) does indeed satisfy the remaining condition

$$D^2 G(x, y) = 0 \quad , \quad x \neq y \quad (1.29)$$

(since clearly  $G(x, y) \underset{x \rightarrow y}{\sim} G_0(x, y)$   $\infty$  is also required).

On the other hand, to derive an equivalent form for the adjoint representation — which will be of importance in evaluating the determinants arising in (1.19) — is very much more involved<sup>32</sup>.

Using  $q$  to denote the fundamental representation of a gauge group  $G$ , the adjoint representation can be obtained by decomposing  $q \otimes \bar{q}$ ; this is then regarded as a 2-index object, one index transforming according to the fundamental representation and the other as its complex conjugate. The appropriate covariant derivative is

$$D_\mu = 1 \otimes 1 \partial_\mu + A_\mu \otimes 1 + 1 \otimes \bar{A}_\mu \quad (1.30)$$

Naively one might hope that the obvious extension of (1.26)

$$G(x, y) = \frac{v(x)^\dagger v(y) \otimes \overline{v(x)^\dagger v(y)}}{4\pi^2 |x-y|^2} \quad (1.31)$$

provides the correct form for the Green function; but as Brown, Carlitz, Creamer and Lee<sup>33</sup> at first pointed out, a further non-singular term has to be added.

Considering the general case of a direct product of two simple groups  $G_1$  and  $G_2$  with covariant derivative

$$D_\mu = 1 \otimes 1 \partial_\mu + A_{1\mu} \otimes 1 + 1 \otimes A_{2\mu} , \quad (1.32)$$

the Green function is found after some analysis<sup>32</sup> to be

$$G_1(x, y) = \frac{[v_1(x) \otimes v_2(x)]^\dagger (1 - \mathcal{M}) [v_1(y) \otimes v_2(y)]}{4\pi^2 |x - y|^2} \quad (1.33)$$

where  $\mathcal{M}$  is a square matrix of dimension  $4(n_1 + k_1)(n_2 + k_2)$ . It is defined with reference to another matrix,  $M$ , which is  $(k_1 k_2 \times k_1 k_2)$ -dimensional, and constant (as is  $\mathcal{M}$ ) (see 32 for details); both are conformally invariant. The latter matrix, which acts on the tensor product  $W_1 \otimes W_2$  of a  $k_1$ -dimensional space  $W_1$  and  $k_2$ -dimensional space  $W_2$ , enters crucially into a number of calculations that follow, particularly in the related forms  $M_S$  and  $M_A$  derived from it, respectively the restrictions to the  $\frac{1}{2}k(k+1)$ -dimensional symmetric and  $\frac{1}{2}k(k-1)$ -dimensional anti-symmetric subspaces of  $W \otimes W$ .

Utilising the fundamental results and working in the formalism outlined above, expressions may now be sought for the instanton determinants occurring in (1.19). This is attempted in the work below according to the following scheme:

Chapter two employs zeta-function renormalisation (as used by a number of authors) to obtain an expression for the variation of the determinants

with respect to its parameters; this leads to a discussion of the vacuum polarisation current due to instantons, an extension of the work of Brown and Creamer<sup>34</sup> being presented, and then compared with the work of Corrigan, Goddard, Osborn and Templeton<sup>35</sup>.

The third chapter deals with the effects of various authors (Osborn<sup>36</sup>, Berg and Lüscher<sup>37</sup>) to remove the variation from the determinant obtained by the methods above; Jack's<sup>38</sup> generalisation of this work to tensor products is introduced, and its implications for  $SU(2)$  discussed with explicit forms for the 't Hooft instanton solutions.

Next an ansatz due to Osborn<sup>39</sup> for the form of the determinant in the case of  $SU(2)$  is presented, with an investigation of its limiting and conformal properties; details of numerical checks on its accuracy are given for  $k=2$  and  $k=3$ .

Using results from this calculation, and employing conformal properties of various integrals involved, an exact form for the determinants in the case of the general two-instanton 't Hooft ( $SU(2)$ ) solution is obtained.

A final chapter briefly reviews the progress made in these investigations and possible future developments.



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## CHAPTER 2: Zeta-Function Regularisation of Determinants

In this chapter a method of defining and regularising functional determinants is discussed, and then applied to the case in hand, namely that of the covariant Laplacian in the background field of instantons, the variation of this determinant with respect to the instanton parameters being obtained.

Arising naturally in this context is the vacuum polarisation current induced by these field configurations. In section 2 an extension of the first work by Brown and Creamer is presented and then compared with the later calculations of Corrigan, Goddard, Osborn and Templeton; the latter form the basis of subsequent investigations in the following chapter.

### 1. Zeta-Function Methods

There have been two principal means of defining functional determinants developed by field theorists in instanton calculations, namely a Pauli-Villars technique<sup>1</sup> and a zeta-function method<sup>2-5</sup>. The latter seems to possess a number of advantages in this context, particularly for discussing conformal properties of the determinants<sup>6,7</sup>, although its part in a consistent scheme for defining and evaluating Green functions of the theory has not yet been shown to all orders in the coupling constant.

As in the introduction to the semi-classical approximation, it is instructive to consider a finite-dimensional analogue (following<sup>7,8</sup> in this and what follows). For a finite  $n \times n$  hermitian matrix  $A$ , positive definite with eigenvalues  $\lambda_i$ ,  $1 \leq i \leq n$  (not necessarily distinct), one may set

$$\zeta_A(s) = \sum_{i=1}^n \lambda_i^{-s} \quad (2.1)$$

which defines a function analytic in  $s$  with the following properties:

$$\zeta_A(0) = n, \quad (2.2)$$

$$\zeta'_A(0) = -\ln \det A. \quad (2.3)$$

Similarly for a differential operator (such as  $-D^2$ , which is positive definite if one works on the sphere  $S^4$ , conformally related to the flat Euclidean space  $R^4$ ) with an infinite set of eigenvalues, define

$$\zeta_{-D^2}(s) = \sum_1^\infty \lambda_n^{-s}. \quad (2.4)$$

But this leads to difficulties with  $\zeta_{-D^2}(0)$ . In fact the series in (2.4) is typically only defined for  $\text{Re}(s) > 2$ : to continue analytically beyond this to  $s=0$  a technique from the analysis of the Riemannian zeta function<sup>9</sup> can be employed, where

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \frac{1}{e^t - 1}. \quad (2.5)$$

The integral in (2.5) is then suitable for evaluating the analytic continuation of  $\zeta$  to all complex  $s$ , revealing a pole at  $s=1$  (and 2).

Analogously we can define

$$\zeta_A = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} \text{Tr}(e^{-At}) \quad (2.6)$$

and further generalise this to  $-D^2$  by noting that the equivalent of  $e^{-At}$  in this case is " $e^{D^2 t}$ " or, more properly,  $\mathcal{G}(x, y; t)$  satisfying

$$\frac{\partial \mathcal{G}}{\partial t} = -D^2 \mathcal{G} \quad (2.7)$$

which defines a function analytic in  $s$  with the following properties:

$$\zeta_A(0) = n \quad , \quad (2.2)$$

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$$\frac{\partial \mathcal{G}}{\partial t} = -D^2 \mathcal{G} \quad (2.7)$$

and  $\mathcal{G}(x, y; 0) = \delta(x-y)$ . (2.8)

These define the heat kernel<sup>10</sup> in the case where  $-D^2$  is a second-order elliptic operator on a compact manifold (see also 11).

Then  $\text{Tr}(e^{D^2 t}) = \text{Tr} \mathcal{G}(t) = \int \text{tr} \mathcal{G}(x, x; t) d^n x$  (2.9)

( $\text{tr}$  referring to internal indices). The asymptotic properties of

$\mathcal{G}(x, x; t)$  as  $t \downarrow 0$  show that  $\zeta_{-D^2}(s)$  is regular for  $\text{Re}(s) > \frac{1}{2}n$  and there are poles (as above) at  $s=2$  and  $1$ ;  $\zeta_{-D^2}(s)$  is regular at  $s=0$ .

To calculate  $\zeta'_{-D^2}(0)$ , this asymptotic expansion of the heat kernel must be investigated in greater depth. Setting<sup>10</sup>

$$\mathcal{G}(x, y; t) \underset{t \downarrow 0}{\sim} \frac{1}{16\pi^2 t} \exp\left\{-\frac{1}{4t}|x-y|^2\right\} \sum_0^\infty a_n(x, y) t^n \quad (2.10)$$

the co-efficients  $a_n$  may be evaluated iteratively from (2.7), (2.8) by equating powers of  $t$ <sup>3, 12</sup>:

$$(x-y)_\mu D_\mu a_0(x, y) = 0, \quad a_0(x, x) = 1, \quad (2.11)$$

$$n a_n(x, y) + (x-y)_\mu D_\mu a_n(x, y) = D^2 a_{n-1}(x, y), \quad n \geq 1. \quad (2.12)$$

Apart from infra-red problems (cf. below), the residues of  $\zeta(s)$  at  $s=1, 2$  and its value at  $s=0$  are controlled by the small- $t$  behaviour of  $\mathcal{G}(x, x; t)$ ,

$$\text{Res}_{s=2} \zeta(s) = \frac{1}{16\pi^2} \int \text{tr} a_0(x, x) d^4 x, \quad (2.13)$$

$$\text{Res}_{s=1} \zeta(s) = \frac{1}{16\pi^2} \int \text{tr} a_1(x, x) d^4 x, \quad (2.14)$$

$$\zeta(0) = \frac{1}{16\pi^2} \int \text{tr} a_2(x, x) d^4 x. \quad (2.15)$$

(2.11) is solved by the standard path-ordered exponential (taken along the straight-line path from  $x$  to  $y$ )

$$a_0(x, y) = P \exp \left( - \int_y^x A_\mu dx_\mu \right) \quad (2.16)$$

which can then be used with (2.12) to give

$$a_1(x, x) = \left[ D^2 a_0(x, y) \right]_{x=y} = 0 \quad (2.17)$$

and

$$a_2(x, x) = \left[ \frac{1}{6} D^2 D^2 a_0(x, y) \right]_{x=y} = \frac{1}{12} F_{\mu\nu} F_{\mu\nu}. \quad (2.18)$$

So the residue of  $\zeta(s)$  at  $s=2$  is infra-red divergent, at  $s=1$  it vanishes and

$$\zeta(0) = \frac{1}{12 \cdot 16\pi^2} \int \text{tr} F_{\mu\nu} F_{\mu\nu} d^4x = -\frac{1}{12} k \quad (2.19)$$

for a self-dual solution.

To calculate  $\zeta'(0)$ , using the fact that

$$\text{Res}_{s=0} (\Gamma(s) \zeta(s)) = \text{Res}_{s=0} \int_0^\infty dt t^{s-1} \text{Tr} e^{tD^2} = \frac{1}{16\pi^2} \int \text{tr} a_2(x, x) d^4x \quad (2.20)$$

one obtains from differentiation

$$\zeta'(0) = \frac{d}{ds} \left[ \frac{1}{s \Gamma(s)} \right] \Big|_{s=0} \cdot \frac{1}{16\pi^2} \int d^4x \text{tr} a_2(x, x) \quad (2.21)$$

$$+ \left[ \int_0^\infty dt \text{Tr} (e^{tD^2}) - \frac{1}{16\pi^2 s} \int d^4x \text{tr} a_2(x, x) \right] \Big|_{s=0}$$

and

$$\frac{d}{ds} \left[ \frac{1}{s \Gamma(s)} \right] \Big|_{s=0} = \gamma \quad (\text{Euler's constant}). \quad (2.22)$$



Aside from the difficulty of analytically continuing the right-hand side of (2.21), it is not apparent how it could be evaluated without detailed knowledge of the eigenvalues of  $-D^2$ .

These difficulties, and the problems of infra-red divergences (which arise only in the determinant as a multiplicative factor independent of the instanton parameters) can be obviated if one considers  $\delta \zeta(s)$  - the variation in  $\zeta(s)$  induced by a change  $\delta A_\mu$  of the potential. As  $\alpha_0(x, x) = 1$ ,  $\delta \zeta(s)$  is regular at both  $s=1$  and 2 (2.13), and further  $\delta \zeta(0) = 0$  if  $A_\mu$  satisfies the equations of motion, (2.19) then being proportional to the action, a constant.

In these circumstances

$$\delta \zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty dt t^s \text{Tr} [e^{tD^2} \delta D^2] \quad (2.23)$$

$$\text{and so } \delta \zeta'(0) = \left[ \int_0^\infty dt t^s \text{Tr} [e^{tD^2} \delta D^2] \right]_{s=0} \quad (2.24)$$

the integrals defined by analytic continuation.

Integrating by parts, and denoting the inverse of  $D^2$  by its Green function,

$$D^2 G(x, y) = -\delta(x-y) \quad , \quad (2.25)$$

$$\text{then } \delta \zeta'(0) = \left[ \int_0^\infty dt t^{s-1} \text{Tr} [e^{tD^2} G \delta D^2] \right]_{s=0} \quad (2.26)$$

$$\text{Now } \delta D^2 = D_\mu \delta A_\mu + \delta A_\mu D_\mu \quad (2.27)$$

so (2.26) becomes

$$\delta \mathcal{Z}'(0) = \left[ s \int_0^\infty dt t^{s-1} \text{Tr} \left[ e^{\epsilon D^2} \delta A_\mu \vec{D}_\mu G + e^{\epsilon D^2} G \overleftarrow{D}_\mu \delta A_\mu \right] \right]_{s=0} \quad (2.28)$$

with the notation

$$\vec{D}_\mu G(x, y) = \frac{\partial}{\partial x^\mu} G(x, y) + A_\mu(x) G(x, y) \quad (2.29)$$

and  $G(x, y) \overleftarrow{D}_\mu = -\frac{\partial}{\partial y^\mu} G(x, y) + G(x, y) A_\mu(y)$  . (2.30)

Thus the residue at  $s=0$  in

$$\int dt t^{s-1} \text{Tr} \left[ \delta A_\mu (\vec{D}_\mu G e^{\epsilon D^2} + e^{\epsilon D^2} G \overleftarrow{D}_\mu) \right] \quad (2.31)$$

which is controlled by small- $t$  behaviour will provide  $\delta \mathcal{Z}'(0)$  . In fact it is the constant term in the asymptotic expansion of

$$\int d^4x d^4y t_r \left[ \delta A_\mu (\vec{D}_\mu G(x, y) \mathcal{G}(y, x; t) + \mathcal{G}(x, y; t) G(y, x) \overleftarrow{D}_\mu) \right] \quad (2.32)$$

that is required; this is obtained from consideration of the expansion of

$\mathcal{G}(x, y; t)$  . An obvious choice like

$$\mathcal{G}(x, y; t) = \bar{\Phi}(x, y) \delta(x-y) + O(t) \quad (2.33)$$

where  $\bar{\Phi}(x, y) = \alpha_0(x, y)$  , the path-ordered exponential of (2.16), will reproduce the expression obtained by Brown and Creamer<sup>13</sup> in their investigation of the vacuum polarisation current created by instantons (see below). In their work, a point-splitting approach was adopted that led to ill-defined expressions whose ambiguities were resolved by rather ad hoc means.

Noting with these authors that

$$G(x, y) = \frac{1}{4\pi^2} \left( \frac{\Phi(x, y)}{|x-y|^2} + R(x, y) \right), \quad (2.34)$$

where  $R(x, y)$  is non-singular at  $x=y$ , one may further apply the zeta-function method to obtain rigorously their end result<sup>7</sup>.

For Brown and Creamer found that only this regular part of  $G(x, y)$  contributes to the constant term sought in (2.32). This will occur if

$$\frac{1}{4\pi^2} \int d^4x d^4y \operatorname{tr} \left[ \delta A_\mu \vec{D}_\mu \left( \frac{\Phi(x, y)}{|x-y|^2} \right) \mathcal{G}(y, x; t) \right] \quad (2.35)$$

has no such term as  $t \downarrow 0$ .

Now

$$\frac{1}{t^2} \int d^4x x^{\mu_1} \dots x^{\mu_N} (x^2)^{-M} e^{-x^2/4t} \quad (2.36)$$

is finite and (changing variables) of order  $t^{N/2-M}$ , vanishing by anti-symmetry if  $N$  is odd, provided  $\frac{1}{2}N - M > 2$ .

Thus the only terms of relevance are

$$\frac{1}{t^2} \int d^4x d^4y \operatorname{tr} \left[ \delta A_\mu \vec{D}_\mu \left( \frac{\Phi(x, y)}{|x-y|^2} \right) \left( a_0(y, x) + t a_1(y, x) \right) e^{-\frac{|x-y|^2}{4t}} \right]. \quad (2.37)$$

Expanding

$$\vec{D}_\mu \left( \frac{\Phi(x, y)}{|x-y|^2} \right) = |x-y|^{-2} \vec{D}_\mu \Phi(x, y) - 2|x-y|^{-4} (x-y)_\mu \Phi(x, y)$$

multiplied by  $a_0(y, x) + t a_1(y, x)$  in a Taylor series about  $x=y$ , one uses<sup>3,14</sup>

$$\begin{aligned} \vec{D}_\mu \Phi(x, y) &= \frac{1}{2} F_{\mu\nu}(x) (y-x)^\nu \\ &+ \frac{1}{3} D_\lambda F_{\mu\nu}(x) (y-x)^\lambda (y-x)^\nu + O(|x-y|^2) \end{aligned} \quad (2.38)$$

and  $\Phi(x, y) \Phi(y, x) = 1$  , (2.39)

$$a_1(y, x) = O(|x-y|^2) . \quad (2.40)$$

Then with

$$\frac{1}{16\pi^2 t^2} \int d^4x \frac{x^\mu x^\nu}{x^2} e^{-x^2/4t} = \delta^{\mu\nu}/4 \quad (2.41)$$

the contribution of the singular part of  $G(x, y)$  to (2.33) is

$$\frac{1}{48\pi^2} \int d^4x \operatorname{tr} [\delta A_\mu(x) D_\nu F_{\mu\nu}(x)] \quad (2.42)$$

which vanishes in the case under consideration, since  $A_\mu$  satisfies the equation of motion  $D_\nu F_{\mu\nu} = 0$  . So finally only  $R(x, y)$  remains, and using

$$\frac{1}{16\pi^2 t^2} \int d^4x e^{-x^2/4t} = 1 \quad (2.43)$$

Brown and Creamer's expression <sup>13</sup> is achieved:

$$\delta \mathcal{J}'(0) = -\delta \ln \det(-D^2) = \int d^4x \operatorname{tr} [\delta A_\mu(x) \mathcal{J}_\mu(x)] , \quad (2.44)$$

where  $\mathcal{J}_\mu(x) = \frac{1}{4\pi^2} \left[ \vec{D}_\mu R(x, y) + R(x, y) \overleftarrow{D}_\mu \right] \Big|_{x=y}$  (2.45)

is the vacuum polarisation current induced by the presence of the instantons (see 13 for a full discussion of this aspect).

## 2. Calculation of Vacuum Polarisation Current

This current thus enters critically in the calculation of instanton determinants. The basic technique employed in its evaluation is the ex-

traction of the regular component of the Green function  $G(x, y)$

$$\frac{1}{4\pi^2} R(x, y) = G(x, y) - \frac{\Phi(x, y)}{4\pi^2 |x-y|^2} \quad (2.46)$$

where  $\Phi(x, y)$  is the standard path-ordered exponential;  $\mathcal{T}_\mu$  is then calculated via (2.45). This method was developed by Brown and Creamer<sup>13</sup> and first applied by them in the case of the extended 't Hooft solution<sup>15,16</sup>: in what follows we treat the general situation in the formalism of Atiyah, Drinfeld, Hitchin and Manin (cf. infra). Simple Taylor expansions are used to this end.

Expanding about  $y$ , only first-order in  $x-y$  need be considered, since higher-order terms in (2.45) will vanish as  $x \rightarrow y$ . With

$$G(x, y) = \frac{M(x, y)}{4\pi^2 |x-y|^2} \quad (2.47)$$

by (1.26)  $M(x, y) = v^\dagger(x) v(y)$ ; the expansion is straightforward:

$$\begin{aligned} M \underset{x \rightarrow y}{\sim} & 1 + \xi_\mu \partial_\mu v^\dagger(y) v(y) + \frac{\xi_\mu \xi_\nu}{2!} \partial_{\mu\nu} v^\dagger(y) v(y) \\ & + \frac{\xi_\mu \xi_\nu \xi_\lambda}{3!} \partial_{\mu\nu\lambda} v^\dagger(y) v(y) + O(\xi^4) \end{aligned} \quad (2.48)$$

where  $\xi_\mu = (x-y)_\mu$ .

Similarly we may expand

$$\Phi(x, y) = P \exp \left( - \int_y^x A_\mu dx_\mu \right)$$

$$= 1 - \int_y^x A_\mu dx^\mu + \int_y^x A_\mu dx_1^\mu \int_y^{x_1} A_\nu dx_2^\nu \quad (2.49)$$

$$- \int_y^x A_\mu dx_1^\mu \int_y^{x_1} A_\nu dx_2^\nu \int_y^{x_2} A_\lambda dx_3^\lambda + \dots$$

$$= 1 - \int_y^x A_\mu dx^\mu + X + Y + \dots \quad (2.50)$$

To third order in  $\xi$  (since we have a factor  $|x-y|^2$  in the denominator)

$$\int_y^x A_\mu dx^\mu = 0 + A_\mu(y) \xi_\mu + \frac{\xi_\mu \xi_\nu}{2!} \partial_\nu A_\mu$$

$$+ \frac{\xi_\mu \xi_\nu \xi_\lambda}{3!} \partial_{\mu\nu} A_\lambda + O(\xi^4). \quad (2.51)$$

Also  $X = \int_y^x dx_1^\mu A_\mu(x_1) f(x_1)$

where  $f(x_1) = \xi'_\mu A_\mu(y) + \frac{\xi'_\mu \xi'_\nu}{2!} \partial_\nu A_\mu(y)$

$$+ \frac{\xi'_\mu \xi'_\nu \xi'_\lambda}{3!} \partial_{\mu\nu} A_\lambda(y) + O(\xi^4),$$

$$\xi'_\mu = x_\mu - y_\mu.$$

So  $X = 0 + \frac{\xi_\mu \xi_\nu}{2!} [A_\mu A_\nu]$

$$+ \frac{\xi_\mu \xi_\nu \xi_\lambda}{3!} [\partial_\nu A_\mu A_\lambda + \partial_\lambda A_\mu A_\nu$$

$$+ \frac{1}{2} A_\mu (\partial_\nu A_\lambda + \partial_\lambda A_\nu)] + O(\xi^4). \quad (2.52)$$

Let 
$$h_{\mu\nu\lambda}(y) = \partial_\nu A_\mu A_\lambda + \partial_\lambda A_\mu A_\nu$$

$$+ \frac{1}{2} A_\mu (\partial_\nu A_\lambda + \partial_\lambda A_\nu) .$$

Then

$$Y = \int_y^\infty A_\mu(x) dx^\mu g(x_i) \quad (2.53)$$

where 
$$g(x_i) = \frac{\xi'_\mu \xi'_\nu}{2} [A_\mu(y) A_\nu(y)] + \frac{\xi'_\mu \xi'_\nu \xi'_\lambda}{3!} h_{\mu\nu\lambda}(y) ,$$

giving 
$$Y = \frac{\xi'_\mu \xi'_\nu \xi'_\lambda}{3!} \cdot \frac{A_\mu}{2} (A_\lambda A_\nu + A_\nu A_\lambda) + O(\xi^4) . \quad (2.54)$$

So 
$$\Phi(x, y) \underset{x \rightarrow y}{\sim} 1 - A_\mu \xi_\mu$$

$$+ \frac{\xi_\mu \xi_\nu}{2!} [-\partial_\nu A_\mu + A_\mu A_\nu] \quad (2.55)$$

$$+ \frac{\xi_\mu \xi_\nu \xi_\lambda}{3!} [-A_\mu A_\lambda A_\nu + \partial_\nu A_\mu A_\lambda$$

$$+ \partial_\lambda A_\mu A_\nu - \partial_{\mu\nu} A_\lambda$$

$$+ \frac{A_\mu}{2} (\partial_\nu A_\lambda + \partial_\lambda A_\nu)] + O(\xi^4) .$$

Since  $v^\dagger v = 1$  (cf. 1.21)

we have  $\partial_\mu v^\dagger v = -v^\dagger \partial_\mu v$

and so 
$$\begin{aligned} \xi_\mu \partial_\mu v^\dagger v &= -\xi_\mu v^\dagger \partial_\mu v \\ &= -A_\mu \xi_\mu \end{aligned} \quad (2.56)$$

causing the first-order terms in (2.48) and (2.55) to cancel in (2.46),

leaving

$$4\pi^2 R(x,y) = B_{\mu\nu} \frac{\xi_\mu \xi_\nu}{2! \xi^2} + C_{\mu\nu\lambda} \frac{\xi_\mu \xi_\nu \xi_\lambda}{3! \xi^3} \quad (2.57)$$

where

$$B_{\mu\nu} = \partial_{\mu\nu} v^\dagger v - (A_\mu A_\nu - \partial_\nu A_\mu) \quad (2.58)$$

and

$$C_{\mu\nu\lambda} = \partial_{\mu\nu\lambda} v^\dagger v \quad (2.59)$$

$$\begin{aligned} &- \left[ -A_\mu A_\nu A_\lambda + \partial_\nu A_\mu A_\lambda + \partial_\lambda A_\mu A_\nu \right. \\ &\quad \left. - \partial_{\mu\nu} A_\lambda + \frac{A_\mu}{2} (\partial_\nu A_\lambda + \partial_\lambda A_\nu) \right]. \end{aligned}$$

In order to compute  $\mathcal{T}_\mu$  of (2.45) it is necessary to adopt some convention for the limiting value of  $\xi_\mu \xi_\nu / \xi^2$  as  $\xi \rightarrow 0$ . Naively one might take this as  $\delta_{\mu\nu}/4$ , the symmetric limit; but in fact there are two limiting processes involved here, and it is important that the orders be strictly preserved.

$D_\mu$  acts on  $R$  in two ways: differentiation by  $\partial_\mu$ , and via multiplication by  $A_\mu$ . Clearly, in any sensible limiting scheme, the latter will only contribute a term from  $B_{\mu\nu} \frac{\xi_\mu \xi_\nu}{2! \xi^2}$ ; but the pieces obtained



by  $\partial_\mu$  acting on R must be considered more carefully.

Now

$$\begin{aligned} \partial_\alpha \left( \frac{\xi_\mu \xi_\nu \xi_\lambda}{\xi^2} \right) \\ = \frac{\delta_{\alpha\mu} \xi_\nu \xi_\lambda + \xi_\mu \xi_\lambda \delta_{\alpha\nu} + \xi_\mu \xi_\nu \delta_{\alpha\lambda}}{\xi^2} \quad (2.60) \\ - 2 \frac{\xi_\mu \xi_\nu \xi_\lambda \xi_\alpha}{\xi^4} \end{aligned}$$

Having performed this limiting process of differentiation we may now take

$$\frac{\xi_\mu \xi_\nu}{\xi^2} \xrightarrow{\xi \rightarrow 0} \frac{\delta_{\mu\nu}}{4} \quad (2.61)$$

and

$$\frac{\xi_\mu \xi_\nu \xi_\lambda \xi_\alpha}{\xi^4} \xrightarrow{\xi \rightarrow 0} \frac{1}{24} \left( \delta_{\mu\nu} \delta_{\lambda\alpha} + \delta_{\mu\alpha} \delta_{\nu\lambda} + \delta_{\mu\lambda} \delta_{\nu\alpha} \right), \quad (2.62)$$

so (2.60) becomes

$$\partial_\alpha \left( \frac{\xi_\mu \xi_\nu \xi_\lambda}{\xi^2} \right) \xrightarrow{\xi \rightarrow 0} \frac{1}{6} \left( \delta_{\alpha\mu} \delta_{\nu\lambda} + \delta_{\mu\lambda} \delta_{\nu\alpha} + \delta_{\mu\nu} \delta_{\lambda\alpha} \right) \quad (2.63)$$

not, as might be expected,  $\frac{1}{3}$  of the symmetrised sum of (2.61).

Similarly

$$\partial_\alpha \left( \frac{\xi_\mu \xi_\nu}{\xi^2} \right) = \frac{\delta_{\alpha\mu} \xi_\nu + \delta_{\alpha\nu} \xi_\mu}{\xi^2} - 2 \frac{\xi_\alpha \xi_\mu \xi_\nu}{\xi^4} \quad (2.64)$$

The latter term  $\rightarrow 0$  as  $\xi \rightarrow 0$ ; but the first is ill-defined.

For present purposes it will be taken to be zero (as  $J(x)$  is regular). This point will be returned to later.

$\partial_\alpha$  acting on  $C_{\mu\nu\lambda}$  produces nothing in the limit  $x \rightarrow y$ , but  $\partial_\alpha B_{\mu\nu}$  will contribute.

The term produced is proportional to  $\partial_\alpha B_{\mu\nu} \cdot \frac{\xi_\mu \xi_\nu}{\xi^2} \Big|_{\xi \rightarrow 0}$ , which becomes  $\partial_\alpha B_{\mu\nu} \cdot \frac{\delta_{\mu\nu}}{4}$ . Here there is no ambiguity about limiting processes and the  $\delta_{\mu\nu}$  may be taken within the differentiation, producing  $\partial_\alpha B_{\mu\mu}/4$ .

$$\text{Now from (2.58)} \quad B_{\mu\mu} = \partial^2 v^\dagger \cdot v + (-A_\mu A_\mu + \partial \cdot A) \quad (2.65)$$

As is usual in instanton contexts (e.g. 't Hooft's solutions<sup>15</sup>) we work in the gauge  $\partial \cdot A = 0$ , so

$$B_{\mu\mu} = \partial^2 v^\dagger \cdot v - A^2 \quad (2.66)$$

$$\text{and} \quad \frac{\partial_\alpha B_{\mu\mu}}{4} = \frac{1}{4} (\partial_\alpha \partial^2 v^\dagger \cdot v + \partial^2 v^\dagger \partial_\alpha v - \partial_\alpha A_\lambda A_\lambda - A_\lambda \partial_\alpha A_\lambda) \quad (2.67)$$

So using (2.65) and putting together the component parts obtained above, we have

$$\begin{aligned} \mathcal{J}_\mu = \frac{1}{4g_{\pi^2}} & \left\{ \partial_\mu \partial^2 v^\dagger \cdot v \right. \\ & \left. + \partial^2 A_\mu + \frac{1}{3} [A_\mu A^2 + A_\lambda A_\mu A_\lambda + A^2 A_\mu] \right\} \end{aligned}$$

$$-\frac{1}{3} [2\partial_\mu A_\lambda A_\lambda + A_\lambda \partial_\mu A_\lambda$$

$$2\partial_\lambda A_\mu A_\lambda + A_\lambda \partial_\lambda A_\mu]$$

$$-\frac{3}{2} [\partial_\mu \partial^2 v^\dagger \cdot v + \partial^2 v^\dagger \partial_\mu v - \partial_\mu A_\lambda \cdot A_\lambda - A_\lambda \partial_\mu A_\lambda]$$

$$+ 3A_\mu (\partial^2 v^\dagger \cdot v - A^2) \} \quad (2.68)$$

### 3. Vacuum Polarisation Current for 't Hooft Solutions

The vacuum polarisation current was first obtained by Brown and Creamer<sup>13</sup> for the case of the 't Hooft  $SU(2)$  solutions<sup>15</sup>; (2.68) can be checked against their result.

In this case

$$A_\mu = i\eta_{\mu\nu} \frac{\partial_\nu \ln \pi}{2} \quad (2.69)$$

where<sup>17</sup>

$$\eta_{\mu\nu} = (\epsilon_{0a\mu\nu} + \delta_{a\mu} \delta_{0\nu} - \delta_{a\nu} \delta_{0\mu}) \sigma^a \quad (2.70)$$

( $\sigma^a$  - Pauli matrices),

$$\text{and} \quad \pi = \sum_0^k \frac{\lambda_i^2}{|x - y_i|^2} \quad (2.71)$$

the instanton superpotential<sup>18</sup>;  $\lambda_i$ ,  $y_i$  respectively the instanton strengths and positions.

Then regarding  $v$  as a column of  $k+1$  quaternions

$$v_s = \frac{\lambda_s x_s^+}{x_s^2} \Pi^{-\frac{1}{2}}, \quad 0 \leq s \leq k \quad (2.72)$$

where  $x_s = x - y_s$ , in the quaternionic representation. Using (2.72)

$\partial^2 \partial_\mu v^+ \cdot v$  may be calculated.

First

$$\partial_\mu v_s^+ = \left( \frac{e_\mu^+ \lambda_s}{x_s^2} - 2 \frac{x_s^+ x_s^\mu \lambda_s}{x_s^4} \right) \Pi^{-\frac{1}{2}} - \frac{1}{2} \Pi^{-\frac{3}{2}} \partial_\mu \Pi \frac{x_s^+ \lambda_s}{x_s^2} \quad (2.73)$$

with

$$e_\mu^+ = \partial_\mu x.$$

$$\text{So } \partial_{\mu\lambda} v_s^+ = \frac{3}{4} \Pi^{-\frac{1}{2}} \partial_\mu \Pi \partial_\lambda \Pi \frac{\lambda_s x_s^+}{x_s^2} - \frac{1}{2} \Pi^{-\frac{3}{2}} \partial_{\mu\lambda} \Pi \frac{\lambda_s x_s^+}{x_s^2}$$

$$- \frac{1}{2} \Pi^{-\frac{3}{2}} \partial_\mu \Pi \lambda_s \frac{(i\eta_{\lambda\nu} x_s^\nu - x_s^\lambda) x_s^+}{x_s^4}$$

$$- \frac{1}{2} \Pi^{-\frac{3}{2}} \partial_\lambda \Pi \lambda_s \frac{(i\eta_{\mu\nu} x_s^\nu - x_s^\mu) x_s^+}{x_s^4}$$

$$+ \Pi^{-\frac{1}{2}} \lambda_s \frac{(i\eta_{\mu\lambda} - \delta_{\mu\lambda}) x_s^+}{x_s^4} \quad (2.74)$$

$$+ \Pi^{-\frac{1}{2}} \lambda_s \frac{(i\eta_{\mu\nu} x_s^\nu - x_s^\mu) (i\eta_{\lambda\kappa} x_s^\kappa + x_s^\lambda)}{x_s^4 x_s^2} x_s^+$$

$$- 4 \Pi^{-\frac{1}{2}} \lambda_s \frac{(i\eta_{\mu\nu} x_s^\nu - x_s^\mu) x_s^+ x_s^\lambda}{x_s^6}.$$

and

$$\begin{aligned} \partial^2 v_s^+ &= \frac{3}{4} \pi^{-\frac{f}{2}} (\partial_\mu \pi)^2 \lambda_s \frac{x_s^+}{x_s^2} - \pi^{-\frac{f}{2}} \partial_\mu \pi \frac{(i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) x_s^+}{x_s^4} \\ &\quad - 2 \pi^{-\frac{f}{2}} \lambda_s \frac{x_s^+}{x_s^4} \end{aligned} \quad (2.75)$$

Giving

$$\begin{aligned} \partial_\lambda \partial^2 v_s^+ &= -\frac{15}{8} \pi^{-\frac{f}{2}} (\partial_\mu \pi)^2 \partial_\lambda \pi \frac{x_s^+}{x_s^2} + \frac{3}{2} \pi^{-\frac{f}{2}} \partial_{\mu\lambda} \pi \partial_\mu \pi \frac{x_s^+}{x_s^2} \\ &\quad + \frac{3}{4} \pi^{-\frac{f}{2}} (\partial_\mu \pi)^2 (i\gamma_{\lambda\nu} x_s^\nu - x_s^\lambda) \frac{x_s^+}{x_s^4} \\ &\quad + \frac{3}{2} \pi^{-\frac{f}{2}} \partial_\mu \pi \partial_\lambda \pi (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) \frac{x_s^+}{x_s^4} \\ &\quad - \pi^{-\frac{f}{2}} \partial_{\mu\lambda} \pi (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) \frac{x_s^+}{x_s^4} \\ &\quad - \pi^{-\frac{f}{2}} \partial_\mu \pi (i\gamma_{\mu\lambda} - \delta_{\mu\lambda}) \frac{x_s^+}{x_s^4} \\ &\quad - \pi^{-\frac{f}{2}} \partial_\mu \pi (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) \cdot (i\gamma_{\lambda\alpha} x_s^\alpha - 3x_s^\lambda) \frac{x_s^+}{x_s^6} \\ &\quad + 2 \frac{x_s^+}{x_s^4} \pi^{-\frac{f}{2}} \partial_\lambda \pi - 4 \pi^{-\frac{f}{2}} (i\gamma_{\lambda\alpha} x_s^\alpha - 3x_s^\lambda) \frac{x_s^+}{x_s^6} \end{aligned} \quad (2.76)$$

and

$$\begin{aligned}
\partial_\lambda \partial^2 v^\dagger \cdot v &= \sum_s \partial_\lambda \partial^2 v_s^\dagger \cdot v_s \\
&= -\frac{15}{8} \pi^{-3} \partial_\lambda \pi (\partial_\mu \pi)^2 + \frac{3}{2} \pi^{-2} \partial_{\mu\lambda} \pi \partial_\mu \pi \\
&\quad + \frac{3}{4} \pi^{-3} (\partial_\mu \pi)^2 \sum_s (i\gamma_{\lambda\nu} x_s^\nu - x_s^\lambda) \frac{\lambda_s^2}{x_s^4} \\
&\quad + \frac{3}{2} \pi^{-3} \partial_\mu \pi \partial_\lambda \pi \sum_s (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) \frac{\lambda_s^2}{x_s^4} \\
&\quad - \pi^{-2} \partial_{\mu\lambda} \pi \sum_s (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) \frac{\lambda_s^2}{x_s^4} \\
&\quad - \pi^{-2} \partial_\mu \pi (i\gamma_{\mu\lambda} - \delta_{\mu\lambda}) \sum_s \frac{\lambda_s^2}{x_s^4} \quad (2.77) \\
&\quad - \pi^{-2} \partial_\mu \pi \sum_s (i\gamma_{\mu\nu} x_s^\nu - x_s^\mu) (i\gamma_{\lambda\alpha} x_s^\alpha - 3x_s^\lambda) \frac{\lambda_s^2}{x_s^6} \\
&\quad + 2\pi^{-2} \partial_\lambda \pi \sum_s \frac{\lambda_s^2}{x_s^4} \\
&\quad - 4\pi^{-1} \sum_s (i\gamma_{\lambda\alpha} x_s^\alpha - 3x_s^\lambda) \frac{\lambda_s^2}{x_s^6} .
\end{aligned}$$

Then considering the other terms in (2.68)

$$\begin{aligned}
&\frac{1}{3} [A_\mu A^2 + A_\lambda A_\mu A_\lambda + A^2 A_\mu] \\
&= \frac{i^2}{3} \left[ \frac{3}{2} \frac{(\partial_\lambda \pi)^2}{\pi^2} \cdot A_\mu + \frac{1}{4} \frac{(\partial_\lambda \pi)^2}{\pi^2} \sigma^a A_\mu \sigma^a \right] \\
&= -\frac{5i}{24} \gamma_{\mu\nu} \partial_\nu \pi (\partial_\lambda \pi)^2 \pi^{-3}, \quad (2.78)
\end{aligned}$$

and

$$\frac{1}{3} \partial^2 A_\mu = \frac{i}{3} \gamma_{\mu\nu} \left( (\partial_\lambda \pi)^2 \partial_\nu \pi \pi^{-3} - \partial_{\lambda\nu} \pi \partial_\lambda \pi \pi^{-2} \right) \quad (2.79)$$

by direct calculation.

Also

$$\begin{aligned} & -(\partial_\mu A_\lambda \cdot A_\lambda + A_\lambda \partial_\mu A_\lambda) \\ &= \frac{1}{4} \gamma_{\lambda\alpha\alpha} \partial_{\alpha\mu} \ln \pi \gamma_{\lambda\beta\beta} \partial_\beta \ln \pi (\sigma_\alpha \sigma_\beta + \sigma_\beta \sigma_\alpha) \\ &= \frac{1}{2} \gamma_{\lambda\alpha\alpha} \gamma_{\lambda\beta\alpha} \partial_{\alpha\mu} \ln \pi \partial_\beta \ln \pi \\ &= \frac{3}{2} \partial_{\alpha\mu} \ln \pi \partial_\alpha \ln \pi \\ &= \frac{3}{2} (\partial_{\alpha\mu} \pi \cdot \partial_\alpha \pi \cdot \pi^{-2} - (\partial_\alpha \pi)^2 \partial_\mu \pi \cdot \pi^{-3}); \quad (2.80) \end{aligned}$$

$$\begin{aligned} & -\partial_\mu A_\lambda \cdot A_\lambda \\ &= \frac{1}{4} \gamma_{\lambda\alpha\alpha} \partial_{\alpha\mu} \ln \pi \gamma_{\lambda\beta\beta} \partial_\beta \ln \pi (\delta_{ab} + i \epsilon_{abc} \sigma^c) \\ &= \frac{1}{4} [\gamma_{\lambda\alpha\alpha} \gamma_{\lambda\beta\alpha} + i \epsilon_{abc} \sigma^c \gamma_{\lambda\alpha\alpha} \gamma_{\lambda\beta\beta}] \partial_{\alpha\mu} \ln \pi \partial_\beta \ln \pi. \end{aligned}$$

Then using standard combination formulae for the  $\gamma$ 's (for which see Appendix of first reference in 1),

and

$$\gamma_{\lambda\alpha\alpha} \gamma_{\lambda\beta\alpha} = 3 \delta_{\alpha\beta}$$

$$\epsilon_{cab} \gamma_{\lambda\alpha\alpha} \gamma_{\lambda\beta\beta} = 2 \gamma_{cab},$$

we have

$$\begin{aligned}
 & -\partial_\mu A_\lambda A_\lambda \\
 &= \frac{1}{4} \left( 3\partial_\mu \alpha \Pi \partial_\alpha \Pi \Pi^{-2} - 3(\partial_\alpha \Pi)^2 \partial_\mu \Pi \Pi^{-3} \right. \\
 & \quad \left. - 2i\gamma_{\alpha\beta} \partial_\alpha \Pi \partial_\beta \Pi \Pi^{-2} \right) . \quad (2.81)
 \end{aligned}$$

Similarly

$$\begin{aligned}
 & -(\partial_\lambda A_\mu A_\lambda + A_\lambda \partial_\lambda A_\mu) \\
 &= \frac{1}{4} \gamma_{\mu\alpha\alpha} \partial_\lambda \alpha \ln \Pi \gamma_{\lambda\beta\beta} \partial_\beta \ln \Pi (\sigma_a \sigma_b + \sigma_b \sigma_a) \\
 &= \frac{1}{2} \gamma_{\mu\alpha\alpha} \gamma_{\lambda\beta\beta} \partial_\lambda \alpha \ln \Pi \partial_\beta \ln \Pi \\
 &= \frac{1}{2} \partial_\mu \alpha \Pi \partial_\alpha \Pi \Pi^{-2} , \quad (2.82)
 \end{aligned}$$

using

$$\gamma_{\mu\alpha\alpha} \gamma_{\lambda\beta\beta} = \delta_{\mu\lambda} \delta_{\alpha\beta} - \delta_{\mu\beta} \delta_{\alpha\lambda} - \epsilon_{\mu\alpha\lambda\beta} .$$

Also

$$\begin{aligned}
 & -\partial_\lambda A_\mu A_\lambda \\
 &= \frac{1}{4} \gamma_{\mu\alpha\alpha} \partial_\lambda \alpha \ln \Pi \gamma_{\lambda\beta\beta} \partial_\beta \ln \Pi \sigma_a \sigma_b \\
 &= \frac{1}{4} \gamma_{\mu\alpha\alpha} \gamma_{\lambda\beta\beta} \partial_\lambda \alpha \ln \Pi \partial_\beta \ln \Pi [\delta_{ab} + i\epsilon_{abc} \sigma^c]
 \end{aligned}$$

which, using (2.82) and

$$\epsilon_{abc} \gamma_{\mu\alpha\alpha} \gamma_{\lambda\beta\beta} = \delta_{\mu\lambda} \gamma_{\alpha\beta c} - \delta_{\mu\beta} \gamma_{\alpha\lambda c} - \delta_{\alpha\lambda} \gamma_{\mu\beta c} + \delta_{\alpha\beta} \gamma_{\mu\lambda c} ,$$



becomes

$$\frac{1}{4} \partial_{\mu\alpha} \Pi \partial_{\alpha} \Pi \Pi^{-2} + \frac{i}{4} \gamma_{\alpha\beta} \partial_{\mu\alpha} \Pi \partial_{\beta} \Pi \Pi^{-2} + \frac{i}{4} \gamma_{\mu\lambda} \partial_{\lambda\alpha} \Pi \partial_{\alpha} \Pi \Pi^{-2} \quad (2.83)$$

From (2.75)

$$\partial^2 v^\dagger v = -4 \sum_f \frac{\lambda_f^2}{x_f^4} \Pi^{-1} + \frac{1}{4} (\partial_{\mu} \Pi)^2 \Pi^{-2} \quad (2.84)$$

and 
$$A^2 = -\frac{3}{4} (\partial_{\mu} \Pi)^2 \Pi^{-2}; \quad (2.85)$$

So  $B_{\mu\mu}$  of (2.66) is

$$\left( \Pi^{-2} (\partial_{\mu} \Pi)^2 - 4 \Pi^{-1} \sum_f \frac{\lambda_f^2}{x_f^4} \right), \quad (2.86)$$

i.e. real.

Now  $J_{\mu}$  enters in (2.44) as  $\text{tr} [\delta A_{\mu} J_{\mu}]$ , so only those parts proportional to  $\sigma^a$  are relevant. Since  $B_{\mu\mu}$  is real,

$$\partial_{\alpha} B_{\mu\mu} = (\partial_{\alpha} \partial^2 v^\dagger v + \partial^2 v \partial_{\alpha} v - \partial_{\alpha} A_{\lambda} A_{\lambda} - A_{\lambda} \partial_{\alpha} A_{\lambda})$$

is also real in (2.68) and hence contributes nothing to the effective polarisation currents. Similarly all other terms in the component parts of (2.68)

without this factor of  $\sigma^a$  may be discarded.

Thus gathering together the relevant terms (i.e. those proportional to  $\eta$ ) we have

$$\left\{ \frac{1}{3} (A_\lambda A^2 + A_\mu A_\lambda A_\mu + A^2 A_\lambda) + \frac{1}{3} \partial^2 A_\lambda \right.$$

$$\left. - \frac{1}{3} [2\partial_\lambda A_\mu A_\mu + A_\mu \partial_\lambda A_\mu \right.$$

$$\left. + 2\partial_\mu A_\lambda A_\lambda + A_\mu \partial_\mu A_\lambda] \right\}$$

$$\stackrel{=}{=}_{\text{eff.}} \left\{ -\frac{5i}{24} \eta_{\lambda\nu} \partial_\nu \pi (\partial_\mu \pi)^2 \pi^{-3} \right.$$

$$\left. + \frac{1}{3} [i\eta_{\lambda\nu} ((\partial_\mu \pi)^2 \partial_\nu \pi \pi^{-3} - \partial_{\mu\nu} \pi \partial_\mu \pi \pi^{-2}) \right] \quad (2.87)$$

$$\left. + \frac{3}{4} i\eta_{\alpha\beta} \partial_{\lambda\alpha} \pi \partial_\beta \pi \pi^{-2} + \frac{1}{4} i\eta_{\lambda\mu} \partial_{\mu\alpha} \pi \partial_\alpha \pi \pi^{-2} \right\}$$

$$= \left\{ i\eta_{\lambda\nu} \partial_\nu \pi (\partial_\mu \pi)^2 \pi^{-3} \left(\frac{1}{8}\right) \right.$$

$$\left. + i\eta_{\mu\nu} \partial_\nu \pi \partial_{\mu\lambda} \pi \pi^{-2} \left(-\frac{1}{4}\right) \right. \quad (2.88)$$

$$\left. + i\eta_{\alpha\beta} \partial_{\alpha\lambda} \pi \partial_\beta \pi \pi^{-2} \left(\frac{1}{4}\right) \right\}.$$

Also (from (2.77), keeping only the relevant terms)

$$\partial_\lambda \partial^2 v^{\dagger, \nu} \stackrel{=}{=}_{\text{eff.}} \left\{ i\eta_{\lambda\nu} \partial_\nu \pi (\partial_\mu \pi)^2 \pi^{-3} \left(-\frac{3}{8}\right) \right.$$

$$\left. + i\eta_{\mu\nu} \partial_\nu \pi \partial_{\mu\lambda} \pi \pi^{-2} \left(\frac{1}{4}\right) \right.$$

(2.89)

$$\left. + i\eta_{\lambda\mu} \partial_{\mu\alpha} \pi \partial_\alpha \pi \pi^{-2} \left(\frac{1}{4}\right) \right.$$

$$\left. + \pi^{-1} i\eta_{\lambda\alpha} \partial_\alpha \left( \sum_f \frac{\lambda_f^2}{x_f^2} \right) \right\}.$$

So 
$$J_\lambda = \frac{1}{4\pi^2 \cdot 12} \left\{ i\eta_{\lambda\nu} \partial_\nu \Pi (\partial_\mu \Pi)^2 \Pi^{-3} \left( -\frac{3}{8} + \frac{1}{8} \right) \right. \\ + i\eta_{\mu\nu} \partial_\nu \Pi \partial_\mu \lambda \Pi \Pi^{-2} \left( \frac{1}{4} + \frac{1}{4} \right) \\ + i\eta_{\lambda\mu} \partial_\mu \Pi \partial_\alpha \Pi \Pi^{-2} \left( \frac{1}{4} - \frac{1}{4} \right) \\ + i\eta_{\lambda\alpha} \partial_\alpha \left( \sum_f \frac{\dot{\lambda}_f^2}{x_f^4} \right) \quad (2.90) \\ + \frac{3}{2} i\eta_{\lambda\nu} \partial_\nu \Pi (\partial_\mu \Pi)^2 \Pi^{-3} \\ \left. - 6 i\eta_{\lambda\nu} \partial_\nu \Pi \sum_f \frac{\dot{\lambda}_f^2}{x_f^4} \Pi^{-2} \right\}$$

$$= \frac{i}{96\pi^2} \left\{ \frac{1}{2} \eta_{\lambda\nu} \partial_\nu \Pi (\partial_\mu \Pi)^2 \Pi^{-3} \right. \\ - \eta_{\mu\nu} \partial_\nu \Pi \partial_\mu \lambda \Pi \Pi^{-2} \\ - 2 \eta_{\lambda\alpha} \partial_\alpha \left( \sum_f \frac{\dot{\lambda}_f^2}{x_f^4} \right) \quad (2.91) \\ - 3 \eta_{\lambda\nu} \partial_\nu \Pi (\partial_\mu \Pi)^2 \Pi^{-3} \\ \left. + 12 \eta_{\lambda\nu} \partial_\nu \Pi \sum_f \frac{\dot{\lambda}_f^2}{x_f^4} \Pi^{-2} \right\} .$$

Defining (with 13 )

$$\sigma = -\frac{1}{4\pi^2} \cdot \frac{i}{12} \left[ \frac{1}{4} (\partial_\lambda \Pi)^2 \Pi^{-2} - \sum_f \frac{\dot{\lambda}_f^2}{x_f^4} \Pi^{-1} \right] \quad (2.92)$$

then

$$\eta_{\lambda\nu} D_\nu \sigma$$

$$\begin{aligned}
&= -\frac{i}{48\pi^2} \left[ \frac{1}{2} \eta_{\lambda\nu} \partial_{\mu\nu} \Pi \partial_\mu \Pi \Pi^{-2} + 3 \eta_{\lambda\nu} \partial_\nu \Pi \sum_f \frac{\lambda_f^2}{x_f^2} \Pi^{-2} \right. \\
&\quad \left. - \eta_{\lambda\nu} \partial_\nu \left( \sum_f \frac{\lambda_f^2}{x_f^2} \right) \Pi^{-1} - \eta_{\lambda\mu} \partial_\mu \Pi (\partial_\nu \Pi)^2 \Pi^{-3} \right],
\end{aligned}
\tag{2.93}$$

and we can write

$$\mathcal{T}_\lambda = \eta_{\lambda\nu} D_\nu \sigma + \tilde{\mathcal{T}}_\lambda. \tag{2.94}$$

Consider  $D_\lambda \eta_{\lambda\nu} D_\nu \sigma$ .

Now

$$\begin{aligned}
D_\lambda D_\nu \eta_{\lambda\nu} &= \frac{1}{2} [D_\lambda, D_\nu] \eta_{\lambda\nu} \\
&= \frac{1}{2} F_{\lambda\nu} \eta_{\lambda\nu} \\
&= 0,
\end{aligned}$$

since  $F_{\mu\nu}$  is self-dual (by construction) and  $\eta_{\mu\nu}$  anti-self-dual.

Equally,  $D_\nu \sigma$  contributes nothing in (2.44), for

$$\begin{aligned}
&\int d^4x \operatorname{tr} \{ \delta A_\mu D_\nu \sigma \eta_{\mu\nu} \} \\
&= \int d^4x \operatorname{tr} \{ \sigma \eta_{\mu\nu} D_\mu \delta A_\nu \} \\
&= \int d^4x \operatorname{tr} \{ \sigma \eta_{\mu\nu} \delta F_{\mu\nu} \}
\end{aligned}$$

which again vanishes because of the opposite dualities of  $\eta$  and  $F$ .

Thus we obtain Brown and Creamer's results<sup>13</sup> for the effective vacuum polarisation current

$$\begin{aligned} \mathcal{J}_\lambda &\stackrel{\text{eff.}}{=} \tilde{\mathcal{J}}_\lambda \\ &= \frac{-i}{96\pi^2} \left( \eta_{\alpha\beta} \partial_\alpha \pi \partial_\beta \pi \pi^{-2} - \eta_{\lambda\nu} \partial_\alpha \pi \partial_\nu \pi \pi^{-2} \right. \\ &\quad \left. + \eta_{\lambda\nu} (\partial_\alpha \pi)^2 \partial_\nu \pi \pi^{-3} \right) \end{aligned} \quad (2.95)$$

and  $\tilde{\mathcal{J}}_\lambda$  is conserved, as is easily checked.

This is true for the general current found in (2.68) though the lack of any essential simplicity in the result has unfortunately prevented direct verification. Similarly  $\mathcal{J}_\mu$  is anti-hermitian (as  $A_\mu$  is), but in the derivation of above there is a manifest lack of symmetry under  $\mathcal{J}_\mu \rightarrow \mathcal{J}_\mu^\dagger$ .

It has already been noted that the initial calculations of Brown and Creamer<sup>13</sup> were plagued by problems of limiting behaviour, solved by the approach of Corrigan *et al*<sup>7</sup>. Similarly the further problems of the generalisation of the earlier work were also avoided by these authors.

In the expansion of  $\Phi(x,y)$  (cf. (2.49)) they arrived at the following ansatz on the basis of gauge covariance and Euclidean transformation properties:

$$\Phi(x,y) = v^\dagger(x) \left( 1 + |x-y| b H(x,y) b^\dagger \right) v(y); \quad (2.96)$$

where the use of the ADHM construction allows structural mimicking of the Green function

$$G(x,y) = \frac{v^\dagger(x) v(y)}{4\pi^2 |x-y|^2}$$

from which it is subtracted in (2.34).

Then using the defining equations for  $\Phi(x, y)$ ,

$$(x-y)_\mu \vec{D}_\mu \Phi(x, y) = 0 = \Phi(x, y) \overleftarrow{D}_\mu (x-y)_\mu ,$$

a power series in  $(x-y)$  of  $H$  about  $\bar{x} = \frac{x+y}{2}$  can be obtained (see 7 for details).

With  $\bar{x} = \frac{x+y}{2}$  these authors obtained

$$H(x, y) = \frac{1}{2} f(\bar{x}) + \frac{1}{12} f(\bar{x}) \left[ (x-y) \Delta(\bar{x})^\dagger b - b^\dagger \Delta(\bar{x}) (x-y)^\dagger \right] f(\bar{x}) + O(|x-y|^2) . \quad (2.97)$$

Then using this via (2.96) in (2.45) gives

$$J_\mu = \frac{1}{12\pi^2} v^\dagger b f (e_\mu \Delta^\dagger b - b^\dagger \Delta e_\mu^\dagger) f b^\dagger v \quad (2.98)$$

which is manifestly anti-hermitian ( $e_\mu = \partial_\mu x$ ,  $x$  in the quaternionic representation); it is then straightforward to show that its covariant derivative is zero (notation introduced in 19 is helpful in this context). Using the various forms for  $v$ ,  $b$ ,  $f$  etc. in the case of  $SU(2)$ , (2.98) may be shown, after some algebra, to reproduce (2.95), further confirming this result.

Clearly the elegance and simplicity of (2.98) coupled with its full generality, make this form the obvious choice for further investigations, and in particular for seeking to remove the variation from  $\delta A_\mu$  in (2.45). This will be carried out in the next chapter.

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### CHAPTER 3: Integral Expressions for Instanton Determinants

In this chapter the efforts of various authors to undo the variation present in (2.44) is reported, following principally Osborn and the more complete work of Berg and Lüscher. In this, using the current of (2.98),  $\text{tr}(\delta A_\mu J_\mu)$  is re-written in terms of various variables,  $A_\mu, J_\mu$  having been transformed to equivalent quantities in another, larger space. Then after some manipulations and re-grouping of terms, it is possible to extract the variation from (2.44). In the process of doing so, a five-dimensional integral is introduced.

In section 2, Jack's extension of this to tensor products is discussed, together with the implications of this work of computation of instanton determinants in the particular case of  $SU(2)$ . These are further considered in section 3 with the particulation of the above to the 't Hooft form in preparation for the following chapter.

#### 1. Basic Techniques

In the previous chapter it was shown how the determinant of an elliptic operator (such as the covariant Laplacian in the background field of instantons) could be obtained from the zeta-function of that operator. It was further shown that under a variation of the parameters in the general solution  $A_\mu$ , the corresponding change in  $\mathcal{J}'(0)$ , where

$$\det\left(-\frac{D^2}{\mu^2}\right) = \exp\left[-\ln\mu^2 \mathcal{J}(0) - \mathcal{J}'(0)\right] \quad (3.1)$$

( $\mu$  a regularisation mass-scale),  
was given by

$$\delta \mathcal{I}'(0) = \frac{1}{12\pi^2} \int d^4x \operatorname{tr} [\delta A_\mu \mathcal{J}_\mu] . \quad (3.2)$$

Here  $\mathcal{J}_\mu$ , the vacuum polarisation current due to the presence of the instanton gauge field, is (in the notation of Chapter 1 and 1 )

$$\mathcal{J}_\mu = v^\dagger b f (e_\mu \Delta^\dagger b - b^\dagger \Delta \bar{e}_\mu) f^\dagger b^\dagger v \quad (3.3)$$

and 
$$A_\mu = v^\dagger \partial_\mu v .$$

The removal of the variation in (3.2) was first effected by Berg and Lüscher<sup>2</sup> and Osborn<sup>3</sup>; in the main, the latter's treatment is presented here.

In this, the formalism of Drinfeld and Manin<sup>4</sup> is used, writing

$$\Delta(x) = a + bx = \begin{pmatrix} \lambda^+ \\ B + Cx \end{pmatrix} \quad (3.4)$$

where B and C are square  $2k \times 2k$ -dimensional matrices and  $\lambda^+$  acts from a space W to N-dimensional representation of the gauge group.

Then a solution to  $v^\dagger \Delta = 0$  and  $v^\dagger v = 1_N$  is given by

$$v^\dagger = U^\dagger (1_N, -u^\dagger) , \quad (3.5)$$

$$u(x)^\dagger = \lambda^+ (B + Cx)^{-1} , \quad (3.6)$$

where

$$(UU^\dagger)^{-1} = 1_N + u^\dagger u .$$

So

$$A_\mu = -U^\dagger \lambda^+ G^{-1} \bar{e}_\mu C^\dagger u U + U^{-1} \partial_\mu U \quad (3.7)$$

with

$$G = (B + Cx)^\dagger (B + Cx); \text{ and defining } \tilde{A}_\mu \text{ a gauge}$$

transform of  $A_\mu$  by

$$A_\mu = U^{-1} \tilde{A}_\mu U + U^{-1} \partial_\mu U , \quad (3.8)$$

then 
$$\tilde{A}_\mu = -U U^\dagger \lambda^\dagger G^{-1} \bar{e}_\mu C^\dagger u . \quad (3.9)$$

Now 
$$\begin{aligned} (U U^\dagger)^{-1} &= 1_n + u^\dagger u \\ &= 1_n + \lambda^\dagger G^{-1} \lambda \\ &= \lambda^\dagger G^{-1} f^{-1} \lambda^{-1} , \end{aligned} \quad (3.10)$$

where 
$$f^{-1} = \Delta^\dagger \Delta = \lambda \lambda^\dagger + G ,$$

so 
$$\tilde{A}_\mu = -\lambda^\dagger f^{-1} \bar{e}_\mu C^\dagger u . \quad (3.11)$$

Since 
$$\delta A_\mu = U^{-1} \delta \tilde{A}_\mu U + D_\mu(A) U^{-1} \delta U$$

and  $\tilde{J}_\mu$  is covariantly conserved ,

$$\int d^4x \operatorname{tr} \{ \delta A_\mu \tilde{J}_\mu \} = \int d^4x \operatorname{tr} \{ \delta \tilde{A}_\mu \tilde{\tilde{J}}_\mu \}$$

where

$$\begin{aligned} \tilde{\tilde{J}}_\mu &= U \tilde{J}_\mu U^{-1} \\ &= \lambda^\dagger f P^\dagger f (e_\mu P^\dagger - P \bar{e}_\mu) f C^\dagger u \end{aligned} \quad (3.12)$$

(here  $P(x) = b^\dagger \Delta(x) = C^\dagger (B + Cx)$ ).

Defining  $a_\mu$  and  $j_\mu$  by

$$\tilde{A}_\mu = \lambda^\dagger a_\mu \lambda , \quad (3.13)$$

$$\tilde{\tilde{J}}_\mu = \lambda^\dagger j_\mu \lambda , \quad (3.14)$$

then the gauge field and current for the space  $W$  may be defined thus:

$$\hat{A}_\mu = a_\mu \lambda \lambda^\dagger \quad (3.15)$$

$$\hat{J}_\mu = j_\mu \lambda \lambda^\dagger . \quad (3.16)$$

Using standard techniques in the context of the general ADHM solution (see 5 for details) it may be shown that

$$\partial_\mu \{ f(e_\mu p^+ - p e_\mu) f \} = 0 \quad (3.17)$$

and so 
$$\partial_\mu j_\mu = -a_\mu \lambda \lambda^\dagger j_\mu + j_\mu \lambda \lambda^\dagger a_\mu; \quad (3.18)$$

which ensures  $D_\mu(\tilde{A}) \tilde{\mathcal{J}}_\mu = 0$ ,  $D_\mu(\hat{A}) \hat{\mathcal{J}}_\mu = 0$  and also

$$\begin{aligned} & \int d^4x \operatorname{tr} \{ \delta \tilde{A}_\mu \tilde{\mathcal{J}}_\mu \} - \int d^4x \operatorname{tr}_w \{ \delta \hat{A}_\mu \hat{\mathcal{J}}_\mu \} \\ &= \int d^4x \operatorname{tr} \{ \lambda \delta \lambda^\dagger (a_\mu \lambda \lambda^\dagger j_\mu^\dagger - j_\mu \lambda \lambda^\dagger a_\mu) \} \\ &= 0 \end{aligned} \quad (3.19)$$

as

$$a_\mu = \mathcal{O}(x^{-3})$$

and

$$j_\mu = \mathcal{O}(x^{-5}).$$

Thus finally one obtains

$$\delta \mathcal{I}'(0) = \frac{1}{12\pi^2} \int d^4x \operatorname{tr}_w \{ \delta \hat{A}_\mu \hat{\mathcal{J}}_\mu \} \quad (3.20)$$

where

$$\hat{A}_\mu = -f \bar{e}_\mu v,$$

$$\hat{\mathcal{J}}_\mu = f p^+ f (e_\mu p^+ - p \bar{e}_\mu) f v, \quad (3.21)$$

$$v = C^\dagger u \lambda^\dagger.$$

After fairly lengthy manipulations (details in 3) it can be shown that

$$\begin{aligned} & \operatorname{tr} [\delta \hat{A}_\mu \hat{\mathcal{J}}_\mu] \\ &= -\partial_\mu \operatorname{tr} [p^{+\dagger} f^\dagger \delta(f p^+) f (e_\mu p^+ - p \bar{e}_\mu) f v] \\ &+ 6 \operatorname{tr} [\delta(f v) f p^+ f p] - \operatorname{tr} [\delta(f p) f p^\dagger f (e_\mu p^\dagger f p \bar{e}_\mu + 2 p f p^\dagger)]. \end{aligned} \quad (3.22)$$

The derivative vanishes as a surface term in (3.20) as  $\delta(fP^+) = O(x^{-2})$  and  $f^2 = O(x^{-2})$ . Then defining<sup>3</sup>

$$X = fP, \quad (3.23)$$

$$\bar{X} = f\bar{P}, \quad (3.24)$$

$$Y = f^2 I_2, \quad (3.25)$$

(3.22) is written more succinctly in (3.20) as

$$\delta J'(0) = \frac{1}{12\pi^2} \int d^4x \left\{ 6 \operatorname{tr} [\delta Y X \bar{X}] - \operatorname{tr} [\delta X \bar{X} (e_\mu \bar{X} \times \bar{e}_\mu + 2 X \bar{X})] \right\} \quad (3.26)$$

Further simplifications may be achieved via integration by parts and suitable combinations and terms to obtain

$$\begin{aligned} \delta J'(0) = \frac{1}{12\pi^2} \delta \int d^4x \left\{ -2 \operatorname{tr} [Y \bar{X} X] + \operatorname{tr} [\bar{e}_\mu X \bar{e}_\mu \bar{X} X] \right. \\ \left. - \frac{3}{2} \operatorname{tr} [\bar{X} X \bar{X} X] \right\} + \delta \Theta, \end{aligned} \quad (3.27)$$

$$\text{where } \delta \Theta = -\frac{1}{12\pi^2} \int d^4x \operatorname{tr} [(\delta X \bar{X} - X \delta \bar{X}) X \bar{X}], \quad (3.28)$$

showing  $\delta J'(0)$  as a variation of an integral plus a further less explicit term. The former may be further re-written as

$$\frac{1}{12\pi^2} \delta \int \left\{ \frac{5}{2} \operatorname{tr} [Y Y] - \frac{1}{2} \operatorname{tr} [\bar{X} X \bar{X} X] \right\} d^4x \quad (3.29)$$

and the latter as

$$\frac{1}{12\pi^2} \int d^4x \phi,$$

where  $\phi = \epsilon_{\alpha\beta\gamma\delta} \text{tr} [\delta k_\alpha k_\beta k_\gamma k_\delta]$  (3.30)

and  $k_\alpha = f \partial_\alpha f^{-1}$  (3.31)

$\delta\theta$  was successfully re-expressed as the variation of an integral by Berg and Lüscher<sup>2</sup>.

They considered the properties of a function  $q(\xi)$  defined by

$$q(\xi) = \frac{\epsilon_{\alpha\beta\gamma\delta\lambda}}{5} \text{tr} \{ M^{-1} \partial_\alpha M M^{-1} \partial_\beta M M^{-1} \partial_\gamma M M^{-1} \partial_\delta M M^{-1} \partial_\lambda M \} \quad (3.32)$$

where  $M(\xi) \in GL(k, \mathbb{C})$  an arbitrary function of 5 real variables  $\xi_0, \xi_1, \xi_2, \xi_3, \xi_4$ .

Then

$$\delta q = \partial_\alpha \epsilon_{\alpha\beta\gamma\delta\lambda} \text{tr} \{ M^{-1} \delta M M^{-1} \partial_\beta M \dots M^{-1} \partial_\lambda M \}. \quad (3.33)$$

Introducing a parameter  $t$ , an integral form of this equation is

$$q = \partial_\alpha \int_0^1 dt \epsilon_{\alpha\beta\gamma\delta\lambda} \text{tr} \{ K^{-1} \partial_t K K^{-1} \partial_\beta K \dots K^{-1} \partial_\lambda K \} \quad (3.34)$$

where  $K(t, \xi)$ ,  $0 \leq t \leq 1$ , is any curve of invertible matrices (in  $GL(k, \mathbb{C})$ ), such that  $K(0, \xi)$  is diagonal and  $K(1, \xi) = M(\xi)$ .

Then taking  $\xi_\mu = x_\mu$  ( $\mu = 0, 1, 2, 3$ ) and  $\xi_4$

an instanton parameter with respect to which the variation is made, and putting  $M = f^{-1}$ ,

$$\phi = \epsilon_{\mu\nu\rho\sigma} \text{tr} \{ f \delta f^{-1} f \partial_\mu f^{-1} f \partial_\nu f^{-1} f \partial_\rho f^{-1} f \partial_\sigma f^{-1} \}$$

( $\mu, \nu, \rho, \sigma$  from 0 to 3)

$$= \epsilon_{\mu\nu\rho\sigma} \text{tr} \{ f \partial_\mu f^{-1} f \partial_\nu f^{-1} \dots f \partial_\sigma f^{-1} \} \delta \xi_4$$

( $\mu, \nu, \rho, \sigma$  from 0 to 4)

$$= \frac{1}{5} \epsilon_{\mu\nu\rho\sigma} \text{tr} \{ f \partial_\mu f^{-1} f \partial_\nu f^{-1} \dots f \partial_\sigma f^{-1} \} \delta \xi_4 \quad (3.35)$$

So by (3.34) with  $K = \epsilon f^{-1} + (1-\epsilon)(1+x^2)$

$$\phi = \partial_\alpha \int_0^1 dt \epsilon_{\mu\rho\sigma\lambda} \text{tr} \{ K^{-1} \partial_\epsilon K K^{-1} \partial_\beta K \dots K^{-1} \partial_\lambda K \} \delta \xi_4$$

$$= \delta \xi_4 \cdot \frac{\partial}{\partial \xi_4} \int_0^1 dt \epsilon_{\mu\rho\sigma\lambda} \text{tr} \{ K^{-1} \partial_\epsilon K \dots K^{-1} \partial_\lambda K \}$$

$$+ \frac{\partial}{\partial x_\mu} \left[ \delta \xi_4 \int_0^1 dt \epsilon_{\mu\rho\sigma\lambda} \text{tr} \{ K^{-1} \partial_\epsilon K \dots K^{-1} \partial_\lambda K \} \right]$$

( $\mu = 0, 1, 2, 3$ )

( $\beta, \sigma, \delta, \lambda = 0, 1, 2, 3, 4$ )

$$= \delta \int_0^1 dt \epsilon_{\mu\rho\sigma\lambda} \text{tr} \{ K^{-1} \partial_\epsilon K \dots K^{-1} \partial_\lambda K \}$$

$$+ \partial_\mu \Sigma_\mu, \quad \text{some } \Sigma_\mu. \quad (3.36)$$

$$\delta \Theta = \frac{1}{12\pi^2} \int d^4x \phi = \frac{1}{12\pi^2} \delta \int d^4x \int dt \epsilon_{\mu\rho\sigma\lambda} \text{tr} \{ K^{-1} \partial_\epsilon K \dots K^{-1} \partial_\lambda K \}$$

where the surface term from  $\partial_\mu \Sigma_\mu$  in (3.36) vanishes.

Writing  $\epsilon = \xi'_4$ ,  $x_\mu = \xi'_\mu$  ( $\mu = 0, 1, 2, 3$ )

$$\delta\theta = \frac{1}{12\eta^2} \cdot \frac{1}{5} \delta \int d^4x \int_0^1 d\xi_4 \epsilon_{\alpha\beta\sigma\delta\lambda} \text{tr} \{ K^{-1} \partial_\alpha K \dots K^{-1} \partial_\lambda K \}. \quad (3.37)$$

Thus  $\int d^4x \text{tr} \{ \delta A_\mu J_\mu \}$  may now be written as the total variation of four- and five-dimensional integrals:

$$\begin{aligned} & \int d^4x \text{tr} \{ \delta A_\mu J_\mu \} \\ &= \delta \left\{ \frac{1}{48\pi^2} \int d^4x (20 \text{tr} [f_\nu f_\nu] - \text{tr} [K^2 K^2]) \right. \\ & \quad \left. + \frac{1}{12\eta^2} \cdot \frac{1}{5} \int d^4x \int_0^1 d\xi_4 \epsilon_{\alpha\beta\sigma\delta\lambda} \text{tr} [k_\alpha k_\beta k_\gamma k_\delta k_\lambda] \right\} \quad (3.38) \end{aligned}$$

with  $k_\alpha = K^{-1} \partial_\alpha K$ .

So finally, removing the variation, one has

$$\begin{aligned} \mathcal{D}_k &= -\ln \left\{ \det(-D^2/\mu^2) / \det(-D_0^2/\mu^2) \right\} \\ &= \frac{1}{12} (I + \Theta - k \ln \mu^2) + F(k) \quad (3.39) \end{aligned}$$

Here  $D_{0\mu}$  is the trivial covariant derivative ( $k=0$ ) and is inserted in (3.39) to divide out the common divergent factor on flat space. Removing the variation from (3.38) introduces problems of divergence; thus  $I$  has to be regularised:

$$I = \lim_{k^2 \rightarrow \infty} \left[ \frac{1}{\pi^2} \int_{k^2 < k^2} d^4x \left\{ 5 \text{tr} (f_\nu f_\nu) - \frac{1}{4} \text{tr} (k^2 k^2) - k \ln k^2 \right\} \right] \quad (3.40)$$



$F(k)$  is independent of the parameters of the general instanton solution, and was found by Berg and Lüscher<sup>2</sup> to be  $-(\alpha(\frac{1}{2}) + \frac{7}{36})k$  (i.e. linear in  $k$ , as conjectured by Osborn<sup>3</sup>), where

$$\alpha(\frac{1}{2}) = -2 \mathcal{J}'(-1) - \frac{1}{6} \ln 2 - \frac{5}{72} . \quad (3.41)$$

## 2. Extension to Tensor Products

The above results all pertain to the case of  $A_\mu$  in the fundamental representation of the gauge group; using similar techniques Jack<sup>6</sup> was able to extend this work to  $A_\mu$  a tensor product of two independent self-dual gauge fields.

Thus defining

$$\tilde{D}_\mu = \partial_\mu \tilde{I} + \tilde{A}_\mu , \quad (3.42)$$

$$\tilde{A}_\mu = 1_1 \otimes A_{2\mu} + A_{1\mu} \otimes 1_2 , \quad (3.43)$$

$$\tilde{I} = 1_1 \otimes 1_2 , \quad (3.44)$$

the analysis of Chapter 2 goes through in an exactly analogous fashion and

$$-\delta \ln \det(-\tilde{D}^2) = \int d^4x \operatorname{tr} \{ \delta \tilde{A}_\mu \tilde{J}_\mu \} \quad (3.45)$$

where  $\tilde{J}_\mu = 1_1 \otimes J_{2\mu} + J_{1\mu} \otimes 1_2 + \frac{1}{4\pi^2} \tilde{K}_\mu , \quad (3.46)$

$J_{1,2\mu}$  obtained as in individual variations of  $\ln(-D_{1,2}^2)$  and

$$\tilde{K}_\mu = \left[ \vec{\tilde{D}}_\mu K(x,y) + K(x,y) \overleftarrow{\tilde{D}}_\mu \right] \Big|_{x=y} \quad (3.47)$$

with  $K(x, y)$  defined by the tensor product Green function

$$\tilde{C}_1(x, y) = \frac{1}{4\pi^2} \left\{ \frac{v_1(x)^\dagger v_1(y) \otimes v_2(x)^\dagger v_2(y)}{|x-y|^2} + K(x, y) \right\} \quad (3.48)$$

(see Chapter 1 eqn. (1.33)).

$J_1$  and  $J_2$  in (3.46) contribute their respective determinants in (3.45), and  $\tilde{K}$  a further term  $\tilde{I}$ :

$$\ln \det \left( -\frac{\tilde{D}^2}{\mu^2} \right) = N_1 \ln \det \left( -\frac{D_1^2}{\mu^2} \right) + N_2 \ln \det \left( -\frac{D_2^2}{\mu^2} \right) - \tilde{I} + \text{const.}, \quad (3.49)$$

where  $N_1, N_2$  are the dimensions of  $A_{1, \mu}$  and

$$\tilde{I} = \ln \det \{ M(v \otimes v) \} - \frac{1}{16\pi^2} \int d^4x \ln \det f_1 v \partial^2 \partial^2 \ln \det f_2 v_2. \quad (3.50)$$

Here  $M$  is the matrix in the extension of the ADHM construction to tensor products (see 6 for references).

In Chapter 1, it was seen (eqn. (1.19)) how the adjoint representation enters into the calculation of the semi-classical approximation, and the determinants for the former have been examined in some detail by Jack<sup>6</sup> applying the above results.

For the results for the adjoint representation can be obtained by judicious selection of  $A_1$  and  $A_2$ . Thus for  $SU(n)$ , taking  $A_\mu$  and its conjugate  $A_\mu^*$  the adjoint is obtained directly.

Then with  $A_\mu = A_\mu \otimes 1 + 1 \otimes A_\mu^*$ ,

$$\ln (-D^{a^2}) = 2n \ln \det (-D^2) - \ln \det [M^a(v \otimes v^\dagger)] + \frac{1}{16\pi^2} \int d^4x \ln \det f_0 \partial^2 \partial^2 \ln \det f_1 v + \text{const.} \quad (3.51)$$

Similarly for  $S_P(r)$ , though with more work<sup>6</sup>,

$$\begin{aligned} \ln \det(-D^2) &= (2r+6) \ln \det(-D^2) - \ln \det[M_A(v \otimes v)] \\ &\quad + \frac{1}{32\pi^2} \int d^4x \ln \det f_v \partial^2 \partial^2 \ln \det f_v + \text{const.} \end{aligned} \quad (3.52)$$

$M_A$  was defined on  $(W \otimes W)_A$  the anti-symmetric part of  $W \otimes W$ .

For the particular case of  $SU(2) = S_P(1)$ , (3.51) and (3.52)

may be combined, and, using  $M^2 = M$ ,  $\det M = \det M_r \det M_A$ ,

give (cf. (3.39), 3.41)

$$\begin{aligned} \mathcal{D}_k &= - \ln \left\{ \det(-D^2/\mu^2) / \det(-D_0^2/\mu^2) \right\} \\ &= \frac{1}{6} \left( \ln \det \{ M_r(v \otimes v) \} + k \ln 2 \right) - \left( d\left(\frac{1}{2}\right) + \frac{\varepsilon}{\pi} \right) k \\ &\quad + \frac{1}{192\pi^2} \int d^4x \ln \det f_v \partial^2 \partial^2 \ln \det f_v - \frac{k \ln \mu^2}{12} \end{aligned} \quad (3.53)$$

where the undetermined constants above were obtained by Osborn<sup>7</sup> by taking the case of  $S_P(k)$  and considering  $k$  commuting  $S_P(1)$  factors when the eigenvalues of  $-D^2$  can be determined, as in 2, but in the context of zeta-function regularisation.

### 3. 't Hooft Solutions

In what follows, we shall be mainly concerned with evaluating  $\mathcal{D}_k$  for the case of the background instanton field given by the 't Hooft solutions<sup>8</sup>. In terms of the ADHM parameters, these are described by<sup>9</sup>

$$a_{0i} = y_0 \lambda_i, \quad b_{0i} = -\lambda_i, \quad 1 \leq i \leq k \quad (3.54)$$

$$a_{ij} = y_i \lambda_0 \delta_{ij}, \quad b_{ij} = -\lambda_0 \delta_{ij}, \quad 1 \leq i \leq k.$$

$$\text{Then with } \phi = \sum_0^k \frac{\lambda_i^2}{x_i^2}, \quad x_i = x - y_i, \quad (3.55)$$

(the superpotential)

$$A_\alpha = \frac{1}{2} i \gamma_{\alpha\beta} \partial_\beta \ln \phi. \quad (3.56)$$

In terms of the matrices <sup>3</sup>

$$X = \begin{bmatrix} x_1 & & & \\ & \ddots & & \\ & & \bigcirc & \\ \bigcirc & & & \ddots \\ & & & & x_k \end{bmatrix}, \quad (3.57)$$

$$\lambda = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_k \end{bmatrix}, \quad (3.58)$$

$$f^{-1} = \Delta^+ \Delta = X^2 \lambda_0^2 + x_0^2 \lambda \lambda^T \quad (3.59)$$

and

$$v = b^+ b = \lambda_0^2 \mathbb{1}_k + \lambda \lambda^T. \quad (3.60)$$

$$\text{Then, using } \phi = \lambda^T (X^2)^{-1} \lambda + \frac{\lambda_0^2}{x_0^2}, \quad (3.61)$$

$$f = (X^2)^{-1} \lambda_0^{-2} - \frac{1}{\phi} (X^2)^{-1} \frac{\lambda \lambda^T}{\lambda_0^2} (X^2)^{-1}, \quad (3.62)$$

$$f_v = (X^2)^{-1} - \frac{1}{\phi} (X^2)^{-1} \lambda \lambda^T (X^2)^{-1} - \frac{1}{x_0^2} I_k \quad (3.63)$$

and

$$k_\alpha = \frac{1}{2} f \partial_\alpha f^{-1} = (X^2)^{-1} X_\alpha - \frac{1}{\phi} (X^2)^{-1} \lambda \lambda^T \left( (X^2)^{-1} X_\alpha - \frac{x_{0\alpha}}{x_0^2} I_k \right). \quad (3.64)$$

Finally, using

$$\phi^{(n)} = \lambda^T (X^2)^{-n} \lambda + \frac{\lambda_0^2}{x_0^{2n}}, \quad (3.65)$$

it can be shown<sup>3</sup>

$$t_r[f_v f_v] = \sum_0^k \frac{1}{x_i^4} - 2 \frac{\phi^{(1)}}{\phi} + \frac{\phi^{(2)^2}}{\phi^2}, \quad (3.66)$$

$$\frac{1}{16} t_r[k^2 k^2] = \sum_0^k \frac{1}{x_i^4} - 4 \frac{\phi^{(1)}}{\phi} + \frac{1}{2} \frac{\partial \phi \cdot \partial \phi}{\phi^2} + \frac{\phi^{(2)^2}}{\phi^2}$$

$$+ \frac{1}{16} \frac{(\partial \phi \cdot \partial \phi)^2}{\phi^4} - \frac{1}{2} \frac{\partial \phi \cdot \partial \phi}{\phi^2} \phi^{(2)}$$

$$+ \frac{1}{\phi^2} \pi_{\alpha\beta} \pi_{\alpha\beta} - \frac{1}{2} \frac{\partial_\alpha \phi \partial_\beta \phi \pi_{\alpha\beta}}{\phi^2}, \quad (3.67)$$

where  $\pi_{\alpha\beta} = \sum_i \frac{\lambda_i^2 x_i^\alpha x_i^\beta}{(x_i^2)^3}$

$$= \frac{1}{8} \partial_\alpha \partial_\beta \phi + \frac{1}{4} \delta_{\alpha\beta} \phi^{(2)} \quad (3.68)$$

So noting  $\phi^{(2)} = \frac{1}{8} \partial^2 \phi^{(2)}$  one obtains<sup>3</sup>

$$\begin{aligned} 5 \epsilon_r [f_\nu f_\nu] - \frac{1}{4} \epsilon_r [k^2 k^2] \\ = \sum_i \frac{1}{x_i^4} - \frac{1}{16} \frac{(\partial\phi \cdot \partial\phi)^2}{\phi^4} + \frac{\pi^2}{2} \sum_i \delta^r(x_i) + \partial_\alpha T_\alpha, \end{aligned} \quad (3.69)$$

where

$$T_\alpha = \left\{ \frac{3}{4} \frac{1}{\phi} \partial_\alpha \phi^{(2)} - \frac{5}{4} \frac{\partial_\alpha \phi}{\phi^2} \phi^{(2)} - \frac{1}{32} \frac{1}{\phi^2} \partial_\alpha (\partial\phi \cdot \partial\phi) + \frac{1}{16} \partial_\alpha \phi \frac{\partial\phi \cdot \partial\phi}{\phi^2} \right\}. \quad (3.70)$$

The derivative term contributes nothing in the integral (3.38), and that of  $\delta\Theta$  vanishes in this case (cf. below, Chapter 5). So for  $SU(2)$  the final result is

$$\delta\mathcal{F}(0) = \frac{1}{12\pi^2} \delta \int d^4x \left( \sum_i \frac{1}{x_i^4} - \frac{1}{16} \frac{(\partial\phi \cdot \partial\phi)^2}{\phi^4} \right) + \frac{k}{24} \quad (3.71)$$

and

$$\mathcal{D}_k = \frac{1}{12\pi^2} \int_{\text{reg}} d^4x \left\{ \sum_i \frac{1}{x_i^4} - \frac{1}{16} \frac{(\partial\phi \cdot \partial\phi)^2}{\phi^4} \right\} + \frac{k}{24} \quad (3.72)$$

$$- \left( d\left(\frac{1}{2}\right) + \frac{7}{36} \right) k - \frac{k}{12} \ln \mu^2.$$

Here the regularisation is necessary since the first term of (3.71) removes the singularities as  $x_i \rightarrow 0$  but introduces a divergence  $\sim k \ln R^2$  as  $R \rightarrow \infty$ , which is, however, independent of the parameters  $a$  and  $b$ . (3.72) presents the determinant as essentially a four-dimensional integral, whose properties will be further considered in the following chapter.

Chapter 3: References

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## CHAPTER 4: The Osborn Ansatz

In this chapter attention is focussed on determinants for the case of instantons described by the 't Hooft solutions (cf. supra) and in particular on elucidating the structure of the integral occurring in (3.72) of the previous chapter. Having considered its limiting properties, an ansatz modelling these suggested by Osborn is described and examined; in section 2 the conformal behaviour of both is investigated. This is followed by a detailed numerical comparison of its behaviour against the exact function for two and three instantons. Various appendices and tables provide further computational information, programs and results.

### 1. Limiting Properties of Determinants

As a first step to evaluating (3.39) for the general case, attempts have been made to elucidate its structure for the simpler and more explicit 't Hooft solutions (cf. supra, Chapter 3). By considering the various limiting and conformal properties of (3.71), Osborn<sup>1</sup> sought to formulate an ansatz that would reproduce these and the known form for  $k=1$  (see below).

Following 1 consider the behaviour of  $I[\phi_k]$ , where

$$I[\phi_k] = \frac{1}{\pi^2} \int_{\text{reg}} d^4x \left\{ \sum_0^k \frac{1}{|x_n|^4} - \frac{1}{16} \partial^2 \ln \phi_k \partial^2 \ln \phi_k \right\} + \frac{1}{2} k \quad (4.1)$$

is the form taken by  $I + \Theta$  ( $\Theta = 0$  in this case) in (3.39) with

$$\phi_k = \sum_0^k \frac{\lambda_n^2}{|x_n|^2}, \quad x_n = x - y_n$$

(cf. (3.72)).

In particular, we investigate the case where the instanton configuration degenerates to one corresponding to a lower topological index; that is  $y_i \rightarrow y_j$  or  $\lambda_i \rightarrow 0$  (and equivalently  $y_i^2 \rightarrow \infty$ ). In the first limit let

$$\phi_i^{(ij)} = \frac{\lambda_i^2}{x_i^2} + \frac{\lambda_j^2}{x_j^2}, \quad (4.2)$$

then (4.1) can be written as

$$\begin{aligned} I[\phi_k] = \frac{1}{\pi^2} \int_{\text{reg}} d^2x \left\{ \sum_{i \neq j} \frac{1}{|x_i|^4} + \frac{1}{16} \partial^2 \ln \phi_i^{(ij)} \partial^2 \ln \phi_j^{(ij)} \right. \\ \left. - \frac{1}{16} \partial^2 \ln \phi_k \partial^2 \ln \phi_k \right\} \\ + I[\phi_i^{(ij)}] + \frac{1}{2}(k-1). \end{aligned} \quad (4.3)$$

$I[\phi_i^{(ij)}]$  is just the case of  $k=1$  (in the conformally extended form<sup>2</sup>) and can be evaluated

$$I[\phi_i^{(ij)}] = -\ln \frac{\lambda_i^2 \lambda_j^2 |y_i - y_j|^2}{(\lambda_i^2 + \lambda_j^2)^2} + \frac{1}{3}. \quad (4.4)$$

In the integrand of (4.3), there are now no divergences at  $x_j$  as  $y_i \rightarrow y_j$ , and so the limit can be taken inside the integral, together with

$$\frac{1}{4} \partial^2 \ln \phi_i^{(ij)} \underset{y_i \rightarrow y_j}{\sim} -\frac{1}{|x_j|^2} \quad (4.5)$$

which leads to

$$I[\phi_k] \underset{y_i \rightarrow y_j}{\sim} -\ln \frac{\lambda_i^2 \lambda_j^2 |y_i - y_j|^2}{(\lambda_i^2 + \lambda_j^2)^2} + \frac{1}{3} + I[\bar{\phi}_{k-1}] \quad (4.6)$$

where  $\bar{\phi}_{k-1}$  is the obvious limit of  $\phi_k$  as  $y_i \rightarrow y_j$ , viz.

$$\bar{\phi}_{k-1} = \sum_{\substack{n=1 \\ n \neq i}}^k \frac{\bar{\lambda}_n^2}{x_n^2}, \quad \bar{\lambda}_n = \lambda_n, \quad n \neq j, \quad \bar{\lambda}_j = \lambda_i + \lambda_j. \quad (4.7)$$

Consider now the case of  $\lambda_i \rightarrow 0$ . Then writing

$$\phi_k = \phi'_{k-1} + \frac{\lambda_i^2}{x_i^2} \quad (4.8)$$

where  $\phi'_{k-1} = \sum_{\substack{n=1 \\ n \neq i}}^k \frac{\lambda_n^2}{x_n^2},$  (4.9)

then

$$\begin{aligned} \partial^2 \ln \phi_k &= - \frac{(\partial \phi_k)^2}{\phi_k^2} \\ &= - \frac{(\partial \phi'_{k-1})^2}{\phi_k^2} + \frac{1}{\phi_k^2} \left\{ \frac{4\lambda_i^4}{|x_i|^6} - 4 \frac{\partial_\mu \phi'_{k-1} 2x_{i\mu} \lambda_i^2}{|x_i|^4} \right\}. \end{aligned} \quad (4.10)$$

So in (4.1) there arises a divergent term

$$\int \frac{1}{\phi_k^4} \cdot \frac{16\lambda_i^8}{|x_i|^2} d^4x \quad \text{as } x_i \rightarrow 0;$$

for the remainder,  $\lambda_i \rightarrow 0$  without problem. So

$$I[\phi_k] \underset{\lambda_i \rightarrow 0}{\sim} \frac{1}{\pi^2} \int_{\text{reg}} \left\{ \frac{1}{|x_i|^4} - \frac{1}{\phi_k^4} \cdot \frac{\lambda_i^8}{|x_i|^2} \right\} d^4x + \frac{1}{2} + I[\phi'_{k-1}] \quad (4.11)$$

where the regularisation refers to the divergence at  $\infty$ .

Setting  $u = \frac{x_i}{\lambda_i}$ , ( $x = y_i + u\lambda_i$ ),

$$\begin{aligned}\phi_k &\sim \frac{1}{u} + \sum_{j=0}^{k-1} \frac{\lambda_j^2}{|y_i - y_j|^2} + O(\lambda_i^2) \\ &= \frac{1}{u} + B + O(\lambda_i^2)\end{aligned}\quad (4.12)$$

and 
$$\int_{\text{reg}} \left\{ \frac{1}{x_i^4} - \frac{1}{\phi_k^4} \cdot \frac{\lambda_i^8}{x_i^{12}} \right\} d^4x$$

$$= \int_{\text{reg}} d^4u \lambda_i^4 \left\{ \frac{1}{\lambda_i^4 u^4} - \frac{1}{\bar{\phi}^4 \lambda_i^4 u^{12}} \right\} \quad (4.13)$$

$$(\bar{\phi} = \frac{1}{u^2} + B)$$

$$\begin{aligned}&= \lim_{R^2 \rightarrow \infty} \left\{ \int_{x^2 < R^2} \frac{d^4u}{u^4} \left( 1 - \frac{1}{\bar{\phi}^4 u^8} \right) - \pi^2 \ln R^2 \right\} \\ &= \pi^2 \lim_{R^2 \rightarrow \infty} \left\{ \frac{11}{6} + \ln \frac{B}{\lambda_i^2} + \ln \left( R^2 + \frac{\lambda_i^2}{B} \right) - \ln R^2 \right\} \\ &= \pi^2 \left( \frac{11}{6} + \ln \frac{B}{\lambda_i^2} \right).\end{aligned}\quad (4.14)$$

Thus 
$$I[\phi_k] \sim_{\lambda_i \rightarrow 0} -\ln \frac{\lambda_i^2}{\phi_{k-1}'(y_i)} + \frac{7}{3} + I[\phi_{k-1}'] \quad (4.15)$$

Similarly for  $y_i \rightarrow \infty$  with  $x = y_i + u\lambda_i$ ,

$$\phi_k \sim \frac{1}{u^2} + \sum_{j=0}^{k-1} \frac{\lambda_j^2}{|y_i - y_j|^2} + O\left(\frac{1}{y_i^2}\right),$$

the same term as before contributes (as  $\alpha, \alpha_i \rightarrow 0$ ) and the same result (4.15) obtains.

Osborn<sup>1</sup> suggested the following ansatz for  $I$  that satisfies the limiting relations (4.6) and (4.15)

$$I^0 = 2 \left( \ln \sum_0^k \tilde{\lambda}_n - \ln \prod_0^k \lambda_n^2 \right) + \ln \det p_k + \frac{2}{3} k. \quad (4.16)$$

The verification of this depends on the detailed properties of  $\det p_k$ , where

$$(p_k)_{nm} = \sum_{l=0}^k t_{nl} \delta_{nm} - t_{nm} \quad (4.17)$$

$$t_{nm} = \frac{\lambda_m^2 \lambda_n^2}{|y_m - y_n|^2} \quad m \neq n, \quad t_{nn} = 0. \quad (4.18)$$

The rule for evaluating  $\det p_k$  was first given by Sylvester<sup>4</sup> — whose name it bears (as Sylvester's unisigant); the symmetric case relevant here was treated by Borchardt<sup>5</sup> (see also 3 for further discussion).

By the complete symmetry of  $\det p_k$  under permutations of the  $t_{mn}$ , the special case of  $t_{0i} \rightarrow \infty$  may be considered<sup>1</sup>.

$$\det p_k = \det \begin{pmatrix} \Sigma_{11} & -t_{12} & -t_{13} & \dots \\ -t_{12} & \Sigma_{22} & -t_{23} & \dots \\ -t_{13} & -t_{23} & \Sigma_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (4.19)$$

where  $\sum_{i=1}^k \epsilon_{ii} = \sum_{l=0}^k \epsilon_{il}$ .

$\epsilon_{0i}$  occurs only in  $\sum_{ii}$ , so abstracting this term

$$\det p_k = \epsilon_{10} \det \begin{pmatrix} 1 + \frac{\sum_{ii}}{\epsilon_{10}} & -\frac{\epsilon_{12}}{\epsilon_{10}} & -\frac{\epsilon_{13}}{\epsilon_{10}} & \dots \\ -\epsilon_{12} & \sum_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}, \quad (4.20)$$

then expanding by the first row

$$\det p_k \underset{\epsilon_{10} \rightarrow \infty}{\sim} \epsilon_{10} \det \bar{p}_{k-1} + O(1) \quad (4.21)$$

where  $\bar{p}_{k-1}$  is the  $(k-1) \times (k-1)$  matrix obtained from  $p_k$  by eliminating the first row and column, and combining terms such that  $\bar{\epsilon}_{ii} = \epsilon_{ii} + \epsilon_{0i}$ ,  $1 \leq i \leq k$ .

Considering now the behaviour of (4.16) under  $y_0 \rightarrow y_1$  (that is  $\epsilon_{10} \rightarrow \infty$ )

$\ln \sum_n \lambda_n^2$  is unchanged (but  $\bar{\lambda}_1^2 = \lambda_0^2 + \lambda_1^2$ ),

and  $\ln \prod_{i=1}^k \lambda_i^2 \underset{\epsilon_{10} \rightarrow \infty}{\sim} \ln \prod_{i=1}^k \bar{\lambda}_i^2 - \ln \frac{\lambda_1^2 + \lambda_0^2}{\lambda_1^2 \lambda_0^2}$ ;

and from (4.21)

$$\ln \det p_k \sim \ln \epsilon_{10} + \ln \det \bar{p}_{k-1}, \quad \text{where}$$

$$\ln t_{10} = \ln \frac{\lambda_1^2 \lambda_0^2}{|y_1 - y_0|^2}.$$

So under  $t_{10} \rightarrow \infty$ ,

$$\begin{aligned} I_k^0 &\underset{t_{10} \rightarrow \infty}{\sim} \bar{I}_{k-1}^0 + 2 \ln \frac{\lambda_1^2 + \lambda_0^2}{\lambda_1^2 \lambda_0^2} + \ln \frac{\lambda_1^2 \lambda_0^2}{|y_1 - y_0|^2} + \frac{7}{3} \\ &\sim \bar{I}_{k-1}^0 + \frac{7}{3} - \ln \frac{|y_1 - y_0|^2 \lambda_1^2 \lambda_0^2}{(\lambda_1^2 + \lambda_0^2)^2}, \end{aligned} \quad (4.22)$$

viz. precisely the behaviour of (4.6).

To consider the limit  $\lambda_i \rightarrow 0$  ( $\sigma y_i^2 \rightarrow \infty$ ) it is convenient to take  $t_{0i} \rightarrow 0$  ( $\forall i$ ) (again by the symmetry of the situation this is permissible).

Then

$$\det p_k = \det \begin{pmatrix} \Sigma_{11} & -t_{12} & -t_{13} & \dots \\ -t_{12} & \Sigma_{22} & & \\ \vdots & & \ddots & \end{pmatrix}.$$

Adding columns 2 to  $k$  on to the first, using

$$\Sigma_{11} = t_{10} + t_{12} + \dots + t_{1n},$$

$$\det p_k = \det \begin{pmatrix} t_{01} & -t_{12} & -t_{13} & \dots \\ t_{02} & \Sigma_{22} & & \\ \vdots & & \ddots & \end{pmatrix}$$

and adding the 2nd to kth rows to the first

$$\det p_k = \det \begin{pmatrix} \sum_{i=2}^k t_{0i} & t_{02} & t_{03} & \dots \\ t_{02} & \Sigma_{22} & & \\ \vdots & & \ddots & \end{pmatrix}$$

and then expanding by the first row we have

$$\det p_k \sim \sum_i t_{0i} \cdot \det p'_{k-1} + O(\epsilon_{0i}^2) \quad (4.23)$$

since each column other than the first has an element of order  $\epsilon_{0i}$ , and multiplies a matrix with a column of similar order.  $p'_{k-1}$  is defined simply by deleting the first row and column.

Examining (4.16) again as  $\lambda_0 \rightarrow 0$  (or  $y_0^2 \rightarrow \infty$ )

$$\ln \sum_0^k \lambda_n^2 \sim_{\lambda_0 \rightarrow 0} \ln \sum_1^k \lambda_n^2$$

and

$$\ln \prod_0^k \lambda_n^2 \sim_{\lambda_0 \rightarrow 0} \ln \prod_1^k \lambda_n^2 + \ln \lambda_0^2$$



By (4.23)

$$\ln \det p_k \underset{\epsilon_{0i} \rightarrow 0}{\sim} \ln \det p'_{k-1} + \ln \sum_0^k \epsilon_{0i};$$

but  $\sum_{i=0}^k \epsilon_{0i} = \lambda_0^2 \phi'_{k-1}(y_0) \quad (\text{cf. (4.9)})$ .

So  $I_k^0 \underset{\epsilon_{0i} \rightarrow 0}{\sim} I_{k-1}^0 + \frac{7}{3} - \ln \frac{1}{\lambda_0^2 \phi'_{k-1}(y_0)} - 2 \ln \lambda_0^2$

$$\sim I_{k-1}^0 + \frac{7}{3} - \ln \frac{\lambda_0^2}{\phi'_{k-1}(y_0)}, \quad (4.24)$$

reproducing (4.13), and confirming the parallel limiting behaviour of (4.16 and (4.1).

## 2. Conformal Properties

Having shown that (4.16) satisfies the various limiting relations of  $I[\phi_k]$ , it is necessary to ensure that its conformal properties are compatible. From the earliest days of instantons<sup>2</sup>, conformal techniques have been a recurrent idea in the development of the subject<sup>3</sup>; they will also be of cardinal importance in the following chapter.

The properties of  $I$  are most easily investigated via the relation<sup>1</sup>

$$I + \Theta = \frac{\mathcal{J}}{6\pi^2} + 2 \ln \det \{M, (\gamma \otimes \gamma)\} + (2 \ln 2 + \frac{3}{2}k) \quad (4.25)$$

where for the case under consideration  $(SU(2))$ ,  $\Theta = 0$ ;

here  $\mathcal{J} = - \int d^4x \ln \det f_\nu \partial^2 \partial^2 \ln \det f_\nu$ , (4.26)

and  $M_j$  was defined in Chapter 1 through equation (1.33).

Expressed in quaternionic form a conformal transformation may be written

$$x \rightarrow x' = (\alpha x + \beta)(\gamma x + \chi)^{-1}. \quad (4.27)$$

$$\text{Since } \Delta(x) \rightarrow \Delta(x') = \Delta'(x)(\gamma x + \chi)^{-1} \quad (4.28)$$

for a self-dual gauge field  $A_\mu$  given by the Atiyah, Drinfeld, Hitchin and Manin construction, this change corresponds to one in parameters of

$$a \rightarrow a' = a\chi + b\beta \quad (4.29)$$

$$b \rightarrow b' = a\gamma + b\alpha. \quad (4.30)$$

$$\text{Also } f \rightarrow f' \frac{\kappa}{\Omega} \quad (4.31)$$

$$\text{where } \Omega = \kappa|\gamma x + \chi|^{-2} \quad (4.32)$$

$$\kappa^2 = \det \begin{pmatrix} \alpha & \beta \\ \gamma & \chi \end{pmatrix} = |\alpha\gamma' - \beta\gamma|^2. \quad (4.33)$$

$$\text{Since } {}^6 \partial^2 \partial^2 \ln \det f = -\text{tr}[F_{\mu\nu} F_{\mu\nu}], \quad (4.34)$$

then under a conformal change

$$\partial^2 \partial^2 \ln \det f \rightarrow \Omega^{-4} \partial^2 \partial^2 \ln \det f'. \quad (4.35)$$

So denoting the change induced by this conformal transformation

by  $\Delta_c$ , and letting it act on  $J$  of (4.26), then

$$\begin{aligned}\Delta_c \frac{J}{16\pi^2} &= -k \Delta_c \ln \det v + \frac{k}{16\pi^2} \int d^4x \ln \frac{k}{\Omega} \partial^2 \partial^2 \ln \det f' \\ &= -k \Delta_c \ln \det v - k \ln \det v' \\ &\quad + k^2 \ln |\gamma|^2 + \frac{k}{16\pi^2} \int d^4x \partial^2 \partial^2 \ln \frac{k}{\Omega} \ln \det f'\end{aligned}\tag{4.36}$$

integrating by parts.

Using (4.32)

$$\partial^2 \partial^2 \ln \frac{k}{\Omega} = -16\pi^2 \delta^4(x + \gamma^{-1}x)$$

and

$$f'(-\gamma^{-1}x) = v|\Lambda|^2$$

where

$$|\Lambda| = \frac{k}{|\gamma|}, \quad \Lambda = \alpha \gamma^{-1}x - \beta.$$

$$\Delta_c \frac{J}{16\pi^2} = -2k \Delta_c \ln \det v + k^2 \ln k^2.\tag{4.37}$$

The ansatz (4.16) can be naturally extended to the complete solution via <sup>1</sup>

$$I^0 + \Theta^0 = \ln \det \{M_S(v \otimes v)\} - \ln \det \{M_A(v \otimes v)\} + (\ln 2 + \frac{7}{3})k\tag{4.38}$$

which leads to the corresponding form for  $J^0$ :

$$\frac{J^0}{16\pi^2} = -\ln \det \{M(v \otimes v)\} + \left(\frac{5}{6} - \ln 2\right)k \quad (4.39)$$

So 
$$\Delta_c \frac{J^0}{16\pi^2} = -2k \Delta_c \ln \det v - \Delta_c \ln \det M.$$

But it is shown in 3 that  $\Delta_c \ln \det M = -k^2 \ln k^2$ ,

so 
$$\Delta_c \frac{J^0}{16\pi^2} = -2k \Delta_c \ln \det v + k^2 \ln k^2, \quad \text{reproducing (4.37)}$$

Thus (4.38) (or equivalently (4.39)) is found to model correctly both the leading singular behaviour and conformal properties of  $I$  (and  $J$ ).

### 3. Numerical Computation for $k=2$

To investigate this ansatz more fully, it was checked numerically on a computer for  $k=2$  and  $k=3$  with collinear instantons in the case of the restricted 't Hooft solutions.

For  $k=2$  the starting-point is (4.3):

$$\begin{aligned} I[\phi_2] &= \frac{1}{\pi^2} \int_{\text{reg}} d^4x \left\{ \sum_1^2 \frac{1}{|x_1|^4} - \frac{1}{16} \partial^2 \ln \phi_2 \partial^2 \ln \phi_2 \right\} + \frac{1}{2} \cdot 2 \\ &= \frac{1}{\pi^2} \int_{\text{reg}} d^4x \left\{ \frac{1}{16} \partial^2 \ln \phi_{12} \partial^2 \ln \phi_{12} - \frac{1}{16} \partial^2 \ln \phi_2 \partial^2 \ln \phi_2 \right\} \\ &\quad + I[\phi_{12}] + \frac{1}{2} \end{aligned} \quad (4.40)$$

as in (4.3),

where  $\phi_{12} = \frac{\lambda_1^2}{x_1^2} + \frac{\lambda_2^2}{x_2^2}$  ,  $\phi_2 = 1 + \phi_{12}$

and  $I[\phi_{12}] = -\ln \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2)^2} + \frac{7}{3}$

by direct calculation.

To regularise the integral of (4.40) (which diverges as  $\ln R$  as  $R \rightarrow \infty$ ) we subtract off  $I[\phi'_2]$  where

$$\phi'_2 = \phi_2|_{y_1=y_2=0} ,$$

the latter having the same behaviour at infinity.

That is, we subtract off the unregularised

$$\frac{1}{\pi^2} \int d^4x \left\{ \frac{1}{x^4} - \frac{1}{16} \partial^2 \ln \phi'_2 \partial^2 \ln \phi'_2 \right\}. \quad (4.41)$$

To obtain the finite parts of this consider

$$\begin{aligned} I[\tilde{\phi}_{12}] &= \frac{1}{\pi^2} \int_{\text{reg}} d^4x \left( \frac{1}{x_1^4} + \frac{1}{x_2^4} - \frac{1}{16} \partial^2 \ln \tilde{\phi}_{12} \partial^2 \ln \tilde{\phi}_{12} \right) + \frac{1}{2} \\ &= -\ln \frac{\tilde{\lambda}_1^2 \tilde{\lambda}_2^2 |\tilde{y}_1 - \tilde{y}_2|^2}{(\tilde{\lambda}_1^2 + \tilde{\lambda}_2^2)^2} + \frac{7}{3} \end{aligned}$$

as before. Now let  $\tilde{y}_2 \rightarrow \infty$ ,  $\tilde{\lambda}_2 \rightarrow \infty$ ,  $\tilde{\lambda}_2/\tilde{y}_2 \rightarrow 1$ ,

so

$$\begin{aligned} I[\tilde{\phi}_{12}] &\rightarrow I[\phi'_2] = \frac{1}{\pi^2} \int_{\text{reg}} d^4x \left( \frac{1}{x_1^4} - \frac{1}{16} \partial^2 \ln \phi'_2 \partial^2 \ln \phi'_2 \right) + \frac{1}{2} \\ &= -\ln \tilde{\lambda}_1^2 + \frac{7}{3} ; \end{aligned}$$

if we set  $\tilde{\lambda}_1^2 = \lambda_1^2 + \lambda_2^2$ , and  $\tilde{y}_1 = 0$ , then

$$\frac{1}{\pi^2} \int_{\text{reg}} d^4x \left\{ \frac{1}{x^4} - \frac{1}{16} \partial^2 \ln \phi'_2 \partial^2 \ln \phi'_2 \right\} = -\ln(\lambda_1^2 + \lambda_2^2) + \frac{7}{3} - \frac{1}{2}; \quad (4.42)$$

so

$$I[\phi_2] = \frac{1}{\pi^2} \int d^4x \left\{ \frac{1}{16} \partial^2 \ln \phi_{12} \partial^2 \ln \phi_{12} - \frac{1}{16} \partial^2 \ln \phi_2 \partial^2 \ln \phi_2 \right\}$$

$$- \frac{1}{\pi^2} \int d^4x \left\{ \frac{1}{x^4} - \frac{1}{16} \partial^2 \ln \phi'_2 \partial^2 \ln \phi'_2 \right\} + \frac{1}{2}$$

$$- \ln \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2)^2} + \frac{7}{3} + \frac{7}{3} - \frac{1}{2} - \ln(\lambda_1^2 + \lambda_2^2)$$

and

$$\int_{\text{reg}} d^4x (\partial^2 \ln \phi_{12} \partial^2 \ln \phi_{12} - \partial^2 \ln \phi_2 \partial^2 \ln \phi_2) \quad (4.43)$$

$$= 16\pi^2 \left\{ I[\phi_2] + \ln \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2)^2} - \frac{14}{3} \right\}.$$

Now the ansatz for the general  $k=2$  situation is (cf. supra)

$$I_\omega^0 = 2 \left\{ \ln \sum_n \lambda_n^2 - \ln \prod_n \lambda_n^2 \right\} + \ln \det p_k + \frac{7}{3} \cdot 2$$

$$= 2 \ln \frac{\lambda_1^2 \lambda_2^2 \lambda_0^2}{(\lambda_1^2 + \lambda_2^2 + \lambda_0^2)} + \frac{14}{3} + \ln \det p_k, \quad (4.44)$$

where

$$p_k = \begin{pmatrix} \epsilon_{01} + \epsilon_{12} & -\epsilon_{12} \\ -\epsilon_{12} & \epsilon_{02} + \epsilon_{12} \end{pmatrix} \quad (4.45)$$

$$(\epsilon_{ij} = \frac{\lambda_i^2 \lambda_j^2}{|y_i - y_j|^2})$$

and

$$\ln \det p_k = \ln \lambda_1^2 \lambda_2^2 \left( 1 + \frac{\lambda_1^2 + \lambda_2^2}{|y_1 - y_2|^2} \right). \quad (4.46)$$

Letting now  $y_0 \rightarrow \infty$ ,  $\lambda_0 \rightarrow \infty$ ,  $\lambda_0/y_0 \rightarrow 1$ , we have

$$\begin{aligned} I_{(1)}^0 &= -2 \ln \lambda_1^2 \lambda_2^2 + \frac{14}{3} + \ln \lambda_1^2 \lambda_2^2 + \ln \left( 1 + \frac{\lambda_1^2 + \lambda_2^2}{|y_1 - y_2|^2} \right) \\ &= \frac{14}{3} - \ln \lambda_1^2 \lambda_2^2 + \ln \left( 1 + \frac{\lambda_1^2 + \lambda_2^2}{|y_1 - y_2|^2} \right) \end{aligned} \quad (4.47)$$

and the ansatz's value for (4.43) is

$$\begin{aligned} &16\pi^2 \left\{ I_{(1)}^0 + \ln \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2)^2} - \frac{14}{3} \right\} \\ &= 16\pi^2 \ln \left( \frac{|y_1 - y_2|^2}{\lambda_1^2 + \lambda_2^2} + 1 \right). \end{aligned} \quad (4.48)$$

In fact one can go slightly further than this. For we know that  $I_{(1)}^0$  accurately reproduces the conformal properties of  $I[\phi_2]$ ; thus

$$I[\phi_2] = I_{(1)}^0 + f(c) \quad (4.49)$$

for some function of the conformal invariants of the instanton parameters.

But for  $k=2$  this is unique:

$$c = \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2 + |y_1 - y_2|^2)^3} \quad (4.50)$$

By carefully expanding  $I[\phi_2]$  to  $O(\lambda_1^2)$  it has been shown<sup>1</sup> that  $f'(0) = 1$ . Thus to (4.48) is added  $(6\pi^2 c$  - being the first term in the Taylor expansion of  $f(c)$ ; it is then this modified form of (4.48) that is compared numerically with (4.43) in Tables I to IV.

Table I provides sample values of configurations in which the instantons have equal strengths ( $\lambda_1^2 = \lambda_2^2$ ).  $c$  is the conformal invariant,  $I$  the numerical value of (4.43) and  $A$  the calculated value of the modified ansatz. It can be seen how to the two decimal places given (dictated by absolute accuracies within the computation) the results are remarkably good. As might be expected, the agreement improves with decreasing  $c$ . Tables II and III provide respectively small and large unequal instanton strengths, with varying separation; the agreement is again excellent (usually better than 0.1%).

To establish whether the inconsistency can be attributed solely to computational error, it is possible to investigate further the accuracy by a series of consistency checks. By virtue of  $f$  being strictly a function of  $c$ , holding the latter constant should ensure a constancy of deviation between the integral and the ansatz.

This can be done for example simply by interchanging  $\lambda^2$  and  $s^2$  (there is no obvious symmetry between them in the integral) as in Table I,



where the error remains approximately the same even though  $I$  varies considerably. Alternatively, the formula for  $c$  (eqn. (4.50)) can be solved as a cubic in  $|y_1 - y_2|^2$  given  $c, \lambda_1^2, \lambda_2^2$ . This was done for  $c = 1/37.5, 1/75$  and  $1/150$  for various  $\lambda_1^2$  and results displayed in Table IV. As can be seen, even for widely-varying  $\lambda_1^2, \lambda_2^2, S$  and  $I$ , the errors within each conformal group are remarkably constant. This seems to confirm that the computation reflects the behaviour of the integral sufficiently faithfully and that the modified ansatz for  $k=2$  provides an excellent approximation.

#### 4. Numerical Computation for $k=3$

For the case of  $k=3$ , it proves more convenient (and more accurate) to investigate the equivalent ansatz for  $J$  (cf. (4.26), (4.39)).

$J$  is given by

$$J = - \int d^4x \partial^2 \partial^2 \ln \chi \cdot \ln \chi$$

where, for 3 collinear instantons in 4-dimensional radial co-ordinates,

$$\begin{aligned} \chi = & r^2 (r^2 + p^2 - 2pr \cos \Theta) (r^2 + q^2 + 2qr \cos \Theta) \\ & + \lambda_1^2 (r^2 + q^2 + 2qr \cos \Theta) r^2 \\ & + \lambda_2^2 (r^2 + p^2 - 2pr \cos \Theta) (r^2 + q^2 + 2qr \cos \Theta) \\ & + \lambda_3^2 (r^2 + p^2 - 2pr \cos \Theta) r^2, \end{aligned} \tag{4.51}$$

$\lambda_i$  the instanton strengths,  $\rho, z$  the separations. For reasons of numerical convergence the logarithmic factor in  $J$  must be removed.

This is effected integrating by parts:

$$\begin{aligned}
 -J &= \int_R d^4x \partial^2 \partial^2 \ln \chi \cdot \ln \chi = \int_R \partial_\mu \partial^2 \ln \chi \ln \chi d\Omega_\mu \\
 &\quad - \int_R \partial^2 \ln \chi \partial_\mu \ln \chi d\Omega_\mu \\
 &\quad + \int_R (\partial^2 \ln \chi)^2 d^4x
 \end{aligned} \tag{4.52}$$

$$= -48\pi^2 (3\ln R^2 + 3) + \int_R (\partial^2 \ln \chi)^2 d^4x. \tag{4.53}$$

The logarithmic factor is now absent from the integral but at the expense of an overall divergence being introduced (signalled by the presence of the cancelling  $-144\pi^2 \ln R^2$  in (4.53):  $J$  itself is finite); this must be removed by hand.

To do this we seek the highest-order term in  $\partial^2 \ln \chi$  as a function of  $r$ .

Now  $\chi$  is a sextic, so we can write

$$\partial^2 \chi = 48r^4 + a \tag{4.54}$$

and  $\partial \chi = (6r^5 + b, c)$

in  $(r, \theta)$   $\sigma$ -ordinates, where  $a, b, c$  are polynomials whose exact

forms are of no immediate importance except insofar as they are of lower order than the leading terms.

$$\text{So } (\partial\chi)^2 = 36r^{10} + 12r^5b + c^2 + b^2$$

and

$$\begin{aligned}\partial^2 \ln \chi &= \frac{\partial^2 \chi}{\chi} - \frac{(\partial\chi)^2}{\chi^2} = \frac{48r^4 + a}{\chi} - \frac{(36r^{10} + 12r^5b + b^2 + c^2)}{\chi^2} \\ &= \frac{12r^4}{\chi} + \left[ \frac{a}{\chi} - \frac{b^2 + c^2 + 12r^5b - 36r^4\psi}{\chi^2} \right]\end{aligned}\quad (4.55)$$

writing  $r^{10} = r^4(\chi - \psi)$ , where  $\psi = \chi - r^6$ ;

$$\text{or more compactly, } \partial^2 \ln \chi = \frac{12r^4}{\chi} + D. \quad (4.56)$$

$$\text{Then } (\partial^2 \ln \chi)^2 = \frac{144r^8}{\chi^2} + \frac{24r^4D}{\chi} + D^2, \quad (4.57)$$

giving rise to a term

$$\int_R \frac{144r^8}{\chi^2} d^4x \sim 144\pi^2 \ln R^2, \quad \text{cancelling}$$

the divergence in (4.53).

To extract this divergence explicitly, we use

$$\frac{r^8 \cdot r^3}{\chi^2} = r^5 \frac{(\chi - \psi)}{\chi^2} \quad (r^3 \text{ from } d^4x = 4\pi r^3 \sin^2\theta d\theta dr) \quad (4.58)$$

and

$$\begin{aligned}\frac{r'}{\chi} &= \frac{1}{6\chi} \left[ \frac{\partial\chi}{\partial r} - \left( \frac{\partial\chi}{\partial r} - 6r^5 \right) \right] \\ &= \frac{1}{6\chi} \left[ \frac{\partial\chi}{\partial r} - \phi \right]\end{aligned}\quad (4.59)$$

where  $\phi = \frac{\partial \chi}{\partial r} - 6r^5$ .

Then with these

$$\begin{aligned} \int_k \frac{144r^8}{\chi^2} d^4x &= 144 \int_k \frac{r^5}{\chi} 4\pi \sin^2\theta d\theta dr - 144 \int_k \frac{r^2\psi}{\chi^2} d^4x \\ &= \frac{144}{6} \int \frac{\partial \chi}{\partial r} \cdot \frac{1}{\chi} 4\pi \sin^2\theta d\theta dr \\ &\quad - \frac{144}{6} \int \frac{\phi}{\chi} 4\pi \sin^2\theta d\theta dr - 144 \int \frac{r^2\psi}{\chi^2} d^4x \end{aligned} \quad (4.60)$$

$$= 48\pi^2 (3\ln R^2 - \ln \lambda_2^2 p^2 q^2) \quad (4.61)$$

$$- \frac{144}{6} \int \frac{\phi}{\chi} 4\pi \sin^2\theta d\theta dr - 144 \int \frac{r^2\psi}{\chi^2} d^4x$$

exhibiting the divergence.

So  $J = 48\pi^2 \lambda_2^2 p^2 q^2 + 144\pi^2$

$$+ 24 \int \frac{\phi}{\chi} 4\pi \sin^2\theta d\theta dr$$

$$+ 144 \int \frac{\psi}{\chi^2} 4\pi r^5 dr \sin^2\theta d\theta$$

$$- 24 \int \frac{D r^7}{\chi} 4\pi dr \sin^2\theta d\theta - \int D^2 4\pi r^3 dr d\theta \sin^2\theta.$$

(4.62)

Or re-writing this

$$\begin{aligned} J &= 48\pi^2 (3 + \ln \lambda_2^2 p^2 q^2) \\ &+ \int FA d\theta dr - \int F d\theta dr \end{aligned} \quad (4.63)$$

where

$$FA = 96\pi s \dot{u}^2 \theta \left( \frac{\Phi}{\chi} + 6r^s \frac{\Psi}{\chi} \right), \quad (4.64)$$

$$F = 4\pi r^3 s \dot{u}^2 \theta \left( \frac{24Dr^4}{\chi} + D^2 \right), \quad (4.65)$$

one obtains the form occurring in the numerical computation.

Now considering the form of the ansatz for  $k=3$  we have from (4.39) and using results on  $M^3$

$$\begin{aligned} \frac{J^0}{16\pi^2} &= -\ln \det \{M(v \otimes v)\} + \left(\frac{5}{6} - \ln 2\right) 3 \\ &= -2 \cdot 3 \cdot \left( \ln \sum_0^3 \lambda_n^2 - \ln \prod_0^3 \lambda_n^2 \right) + \ln \det p_k \\ &\quad - 2 \sum_{\substack{m \neq n \\ 0}}^3 t_{mn} - \frac{5}{2} \\ &= 6 \ln \lambda_1^2 \lambda_2^2 \lambda_3^2 + \ln \det p_k - 6 \ln \lambda_1^2 \lambda_2^2 \lambda_3^2 + 2 \ln p^2 q^2 (p+q)^2 - \frac{5}{2} \\ &= 2 \ln p^2 q^2 (p+q)^2 + \ln \det p_k - \frac{5}{2} \end{aligned} \quad (4.66)$$

in the limit  $\lambda_0^2 \rightarrow \infty$ ,  $y_0^2 \rightarrow \infty$ ,  $\lambda_0/y_0 \rightarrow 1$ .

Also in this case from (4.17)

$$\det p_K = \det \begin{pmatrix} \lambda_1^2 + \epsilon_{12} + \epsilon_{13} & -\epsilon_{12} & -\epsilon_{13} \\ -\epsilon_{12} & \lambda_2^2 + \epsilon_{21} + \epsilon_{23} & -\epsilon_{23} \\ -\epsilon_{13} & -\epsilon_{23} & \lambda_3^2 + \epsilon_{31} + \epsilon_{32} \end{pmatrix} \quad (4.67)$$

$$= \det \begin{pmatrix} \lambda_1^2 & -\epsilon_{12} & -\epsilon_{13} \\ \lambda_2^2 & \lambda_2^2 + \epsilon_{21} + \epsilon_{23} & -\epsilon_{23} \\ \lambda_3^2 & -\epsilon_{23} & \lambda_3^2 + \epsilon_{31} + \epsilon_{32} \end{pmatrix}$$

adding the second and third columns to the first,

$$= \lambda_1^2 \lambda_2^2 \lambda_3^2 \det \begin{pmatrix} 1 & -\frac{\lambda_2^2}{p^2} & -\frac{\lambda_3^2}{q^2} \\ 1 & 1 + \frac{\lambda_1^2}{p^2} + \frac{\lambda_3^2}{q^2} & -\frac{\lambda_3^2}{q^2} \\ 1 & -\frac{\lambda_2^2}{q^2} & 1 + \frac{\lambda_1^2}{(p+q)^2} + \frac{\lambda_3^2}{q^2} \end{pmatrix}$$

$$= \lambda_1^2 \lambda_2^2 \lambda_3^2 \left\{ 1 + \frac{\lambda_1^2 + \lambda_2^2}{p^2} + \frac{\lambda_3^2 + \lambda_2^2}{q^2} + \frac{\lambda_3^2 + \lambda_1^2}{(p+q)^2} \right.$$

$$\left. + (\lambda_1^2 + \lambda_2^2 + \lambda_3^2) \left( \frac{\lambda_2^2}{p^2 q^2} + \frac{\lambda_1^2}{p^2 (p+q)^2} + \frac{\lambda_3^2}{(p+q)^2 q^2} \right) \right\} \quad (4.68)$$

$$= \lambda_1^2 \lambda_2^2 \lambda_3^2 K$$

So

$$J^0 = 16\pi^2 \left( \ln \lambda_1^2 \lambda_2^2 \lambda_3^2 + \frac{5}{2} + 2 \ln \rho^2 q^2 (\rho + q)^2 + \ln \kappa \right) \quad (4.69)$$

which is the value of the function  $H$  compared against  $J$  in the computation below.

Tables V and VI provide sample values in the cases of equidistant instantons and symmetrical cases  $(\lambda_1 = \lambda_3)$ ; Table VII presents a few general (collinear) configurations. As can be seen the results are usually better than 0.1%; although six figures are given for completeness the absolute accuracy is about 0.01. It is perhaps pertinent to note that the two integrals from FA and F in (4.63) are generally quite close: the leading contribution is from  $48\pi^2 \ln \lambda_2^2 \rho^2 q^2$ .

Since the form for the general conformal invariants is not known for  $k=3$ , no check is possible as it was for  $k=2$  (cf. infra). A consistency check was however obtained, by letting  $\lambda_1 = 0$  or  $\rho = 0$ , reproducing a two-instanton configuration; these were found to be in good agreement with the previous computation for  $k=2$ .

Clearly the ansatz models the behaviour of these integrals remarkably well. For the case of  $k=2$ , further investigation was attempted by a variety of polynomial and logarithmic fits to the error as a function of  $c$ , but without success. The fact that such good results were obtained with relatively simple programs and low absolute accuracies suggests the possibility of more refined calculations enabling the first few terms of the series expansion of  $f(c)$  of (4.49) to be obtained; this approach has in its favour the small value of  $c$  ( $\leq \frac{1}{27}$ ) in this context.

Further evidence in support of the high degree of accuracy to which this ansatz models the behaviour of the 't Hooft solution is provided by the work of Chakrabarti and Comtet<sup>7</sup>. In this they considered a particular class of multi-instanton configurations in which the parameters of collinear instantons are completely constrained by the index of the solution.

Using standard superpotential formalism<sup>8</sup> with

$$A_\mu = i\eta_{\mu\nu}^+ \partial_\nu \ln \rho \quad (4.70)$$

their  $(\alpha-1)$ -index solution is

$$\rho(x) = \sum_{k=0}^{\alpha-1} \frac{5e^{2\frac{k\pi}{\alpha}}}{\left\{ \left( t - \tan \frac{k\pi}{\alpha} \right)^2 + r^2 \right\}} \quad (4.71)$$

For this special class, it is possible to obtain an explicit form for the instanton determinant of the covariant Laplacian in that field, as a function of  $\alpha$ <sup>7</sup>; the result may then be compared with that for Osborn's construction in this particular case.

This Chakrabarti and Comtet do, and some of their results are reproduced in Table VIII; here  $\alpha-1$  is the index and  $J-J_0$  the error. As before the high degree of accuracy for  $k=2$  and 3 is confirmed, and the ansatz is also seen to work excellently for higher indices. The authors of 7 estimate that  $J-J_0 \sim 0.05\alpha$  for large  $\alpha$ , to be compared with the asymptotically leading term of  $2\alpha \ln \alpha$  in  $J$ . That this approximation should be so good and yet clearly only approximate is intriguing; in the next chapter an exact calculation is presented.



## APPENDIX A: Details of Computation

To evaluate  $J$  numerically, a routine from the National Algorithm Group's Fortran Library was chosen: DO1DAF; double precision was used throughout.

In this, a double integral is calculated to specified absolute accuracy by repeated applications of the method described by Patterson<sup>9</sup>.

The integral

$$I = \int_a^b \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dx dy \quad (4.72)$$

is expressed as

$$I = \int_a^b F(y) dy$$

where  $F(y) = \int_{\phi_1(y)}^{\phi_2(y)} f(x,y) dy ;$

both integrals are then evaluated by the method of the optimum addition of points to Gauss quadrature formulae, as described by Patterson. An interlacing common-point technique is used: starting from the 3-point Gauss rule, further evaluations are added (but retaining the points of the earlier formulae) to obtain respectively 7, 15, 31, 63, 127 and 255 point rules.

Each integral is calculated by successive applications of these formulae until two results are obtained which differ by less than the specified absolute accuracy.

The integration range of the  $r$  variable ( $Y$  in the program) was split up into ten regions, whose boundaries were determined by fixed multiples of the scale set by the instanton separations.

An attempt was made to distribute the integration evenly: thus the ranges were compressed near the instantons and expanded far from them (where little contribution was made to the total). Suitable accuracies (typically 0.0001) were then set for each region, and adjusted after trial runs.

APPENDIX B: Program for k=2

```

C      INTEGRATION TEST K=2
C
C      VARIABLE ASSIGNMENTS
C      INTEGER NOUT,IFAIL,NPTS,I
C      REAL*8 YA,YB,S,AC(10),ANS,PHI1,PHI2,F,E,K,D,X01AAF,
2L1,L2,L3,AL,P,O,T,G,H,P2,Q2,PO,FC,YU(10),YL(10)
C      EXTERNAL F,PHI1,PHI2
C      DATA NOUT /6/
C      COMMON/PARS/L1,L2,L3,P,O,P2,Q2,PO
C      WRITE (NOUT,99999)

C      OUTER INTEGRATION IN R
C
C      PARAMETER VALUES
C      INSTANTON STRENGTHS
C      L1=20.0
C      L2=0.5
C      INSTANTON SEPARATION
C      O=0.25
C      P2=P*P
C      Q2=O*O
C      PO=P*O
C
C      ABSOLUTE ACCURACIES
C      AC(1)=0.0001
C      AC(2)=0.0001
C      AC(3)=0.0001
C      AC(4)=0.0001
C      AC(5)=0.0001
C      AC(6)=0.00001
C      AC(7)=0.00001
C      AC(8)=0.00001
C      AC(9)=0.00001
C      AC(10)=0.00001
C
C      INTEGRATION RANGES
C      YA=0.0
C      YB=0.5*O
C      YC=1.0*O
C      YD=2.0*O
C      YE=5.0*O
C      YF=15.0*O
C      YG=30.0*O
C      YH=50.0*O
C      YI=150.0*O
C      YJ=500.0*O
C      YK=1000.0*O
C
C      UPPER LIMITS
C      YU(1)=YA
C      YU(2)=YB
C      YU(3)=YC
C      YU(4)=YD
C      YU(5)=YE
C      YU(6)=YF
C      YU(7)=YG
C      YU(8)=YH
C      YU(9)=YI

```

```

C      YU(10)=YJ
C
C      LOWER LIMITS
      YL(1)=YB
      YL(2)=YC
      YL(3)=YD
      YL(4)=YE
      YL(5)=YF
      YL(6)=YG
      YL(7)=YH
      YL(8)=YI
      YL(9)=YJ
      YL(10)=YK
C
C      MAIN CALCULATION
      WRITE (NOUT,99970) L1,L2,L3,P,Q
      IFAIL=1
      S=0.0
      WRITE (NOUT,99950)
      DO 5 I=1,10
      IFAIL=1.
      NAGLIBRARY ROUTINE
      CALL D01DAF(YU(1),YL(1),PHI1,PHI2,F,AC(1),ANS,NPTS,IFAIL)
      IF (IFAIL) 10,10,15
15  WRITE (NOUT,99997) IFAIL
10  WRITE (NOUT,99998) I,ANS,AC(1),NPTS,YU(1),YL(1)
C      WRITE INTEGRAL VALUE
      5  S=+ANS+S
      WRITE (NOUT,99995) S
      WRITE ANSATZ VALUE
      D=H(0.0)
      WRITE (NOUT,99996) D
C      WRITE CONFORMAL INVARIANT
      G=L1*L2*4*02/((L1+L2+4*02)**3)
      WRITE (NOUT,99955) G
      40 STOP
C
C      FORMAT STATEMENTS
99999  FORMAT (4(1X/),31H INVESTIGATION FOR LOGDET K = 2/(1X)
99998  FORMAT (/13,1H ,E13.6,2H ,E13.6,2H ,I6,7H ,E13.6,2H ,
2  E13.6)
99997  FORMAT (/36H CONVERGENCE NOT OBTAINED IFAIL= ,I4)
99996  FORMAT (8H TEST = ,E13.6/)
99995  FORMAT (/18H TOTAL INTEGRAL = ,E13.6/)
99993  FORMAT (/19H LEADING INTEGRAL/)
99970  FORMAT (/18H PARAMETERS L1 = ,E13.6,
2  6H L2 = ,E13.6,6H L3 = ,E13.6/5H P = ,E13.6,5H Q = ,E13.6/)
99960  FORMAT (/)
99955  FORMAT (/15H C-INVARIANT = ,E13.6)
99950  FORMAT (/33H MAIN INTEGRAL ABSACC
2  41H NO. OF EVAL. LOWER LIMIT UPPER LIMIT/)
      END
C
C      SET LOWER LIMIT OF INNER THETA INTEGRAL
      FUNCTION PHI1(Y)
      REAL*8 Y
      PHI1=0
      RETURN
      END
C

```

```

C      SET UPPER LIMIT OF INNER THETA INTEGRAL
      FUNCTION PHI2(Y)
      REAL*8 Y
      REAL*8 X01AAF
      PHI2=(1.0)*X01AAF(0.0)
      RETURN
      END

```

```

C      CALCULATION OF INTEGRAND
      FUNCTION F(X,Y)
      REAL*8 X,Y,J,K,G,H,L1,L2,L3,M,N,P,O
      REAL*8 X01AAF,Y2,P2,O2,P0,FC,CS,SN,CS2,SN2
      COMMON/PARS/L1,L2,L3,P,O,P2,O2,P0
      Y2=Y*Y
      CS=DCOS(X)
      SN=DSIN(X)
      CS2=CS*CS
      SN2=SN*SN
      R1=Y*Y+O*O-2*O*Y*CS
      R2=Y*Y+O*O+2*O*Y*CS
      F1=L1/R1+L2/R2
      D1=-2*(L1*(Y-O*CS)/(R1*R1))+L2*(Y+O*CS)/(R2*R2)
      D2=-2*(L1*O*SN/(R1*R1))-L2*O*SN/(R2*R2)
      BX=D1*D1+D2*D2
      BX=BX*BX
      F2=1+F1
      F1=F1*F1
      F1=F1*F1
      F2=F2*F2
      F2=F2*F2
      K=L1*R2+L2*R1
      J=R1*R2+K
      R1=R1*R1
      R2=R2*R2
      H=(R1*R1*R2*R2)/(K*K*K*K)
      G=(R1*R1*R2*R2)/(J*J*J*J)
      Z=16*(1-((L1+L2)/(L1+L2+Y2))**4)/(Y2*Y2)
      F=4*X01AAF(0.0)*SN2*Y*Y2*((H-G)*BX-Z)
      RETURN
      END

```

```

C      CALCULATION OF ANSATZ
      FUNCTION H(D)
      REAL*8 D,K,L1,L2,L3,P,O,P2,O2,P0
      REAL*8 X01AAF
      COMMON/PARS/L1,L2,L3,P,O,P2,O2,P0
      H=16*(DLOG(1+4*O2/(L1+L2))+L1*L2*4*O2/((L1+L2+4*O2)**3))
      2*X01AAF(0.0)*X01AAF(0.0)
      RETURN
      END

```

## APPENDIX C: Program for k=3

```

C      INTEGRATION TEST K=3
C
C      VARIABLE ASSIGNMENTS
C      INTEGER NOUT,IFAIL,NPTS,I
C      REAL*8 YA,YB,S,U,AC(10),ANS,PHI1,PHI2,F,FA,FB,E,K,D,X01AAF,
2L1,L2,L3,P,Q,T,P2,Q2,PO,FC,YU(10),YL(10)
C
C      EXTERNAL F,FA,PHI1,PHI2
C      DATA NOUT /6/
C      COMMON/PARS/L1,L2,L3,P,Q,P2,Q2,PO
C      WRITE (NOUT,99999)
C
C      OUTER INTEGRATION IN R
C
C      PARAMETER VALUES
C      INSTANTON STRENGTHS
C      L1=1.0
C      L2=4.0
C      L3=8.0
C      INSTANTON SEPARATIONS
C      P=1.0
C      Q=1.0
C      P2=P*P
C      Q2=Q*Q
C      PO=P*Q
C
C      ABSOLUTE ACCURACIES
C      AC(1)=0.0001
C      AC(2)=0.001
C      AC(3)=0.001
C      AC(4)=0.001
C      AC(5)=0.01
C      AC(6)=0.00001
C      AC(7)=0.001
C      AC(8)=0.0001
C      AC(9)=0.00001
C      AC(10)=0.00001
C
C      INTEGRATION RANGES
C      YA=0.0
C      YB=P/4.0
C      YC=3*(P)/4.0
C      YD=(P+Q)/2
C      YF=3*(P+Q)/4
C      YF=5.0*Q
C      YG=10.0*Q
C      YH=50.0*Q
C      YI=100.0*Q
C      YJ=1000.0*Q
C      YK=50000.0*Q
C
C      UPPER LIMITS
C      YU(1)=YA
C      YU(2)=YB
C      YU(3)=YC
C      YU(4)=YD
C      YU(5)=YF
C      YU(6)=YF
C      YU(7)=YG
C      YU(8)=YH
C      YU(9)=YI
C      YU(10)=YJ
C
C      LOWER LIMITS
C      YL(1)=YB
C      YL(2)=YC
C      YL(3)=YD

```

```

YL(4)=YE
YL(5)=VF
YL(6)=VG
YL(7)=YH
YL(8)=VI
YL(9)=YJ
YL(10)=YK

```

C  
C

```

MAIN CALCULATION

```

```

WRITE (NOUT,99970) L1,L2,L3,P,Q

```

```

IFAIL=1

```

```

S=0.0

```

```

WRITE (NOUT,99950)

```

```

DO 5 I=1,10

```

```

IFAIL=1

```

```

CALL D01DAF(YU(I),YL(I),PHI1,PHI2,F,AC(I),ANS,NPTS,IFAIL)

```

```

IF (IFAIL) 10,10,15

```

```

15 WRITE (NOUT,99997) IFAIL

```

```

10 WRITE (NOUT,99998) I,ANS,AC(I),NPTS,YU(I),YL(I)

```

C

```

WRITE F

```

```

5 S=-ANS+S

```

```

T=0.0

```

```

WRITE (NOUT,99993)

```

```

DO 30 I=1,10

```

```

IFAIL=1

```

```

CALL D01DAF(YU(I),YL(I),PHI1,PHI2,FA,AC(I),ANS,NPTS,IFAIL)

```

```

IF (IFAIL) 20,20,25

```

```

25 WRITE (NOUT,99997) IFAIL

```

```

20 WRITE (NOUT,99998) I,ANS,AC(I),NPTS,YU(I),YL(I)

```

C

```

WRITE FA

```

```

30 T=ANS+T

```

```

S=-S

```

```

WRITE (NOUT,99995) S,T

```

```

WRITE SURFACE TERMS

```

```

U=(48*(X01AAF(0.0)**2))*C

```

```

2+3+DLOG(L2*P2*02))

```

```

WRITE (NOUT,99994) U

```

```

WRITE INTEGRAL VALUE

```

```

U=-S+T+U

```

```

WRITE (NOUT,99992) U

```

C

```

WRITE ANSATZ VALUE

```

```

D=H(0.0)

```

```

WRITE (NOUT,99996) D

```

```

40 STOP

```

C

```

FORMAT STATEMENTS

```

C

```

99999 FORMAT (4(1X/),29H INTEGRATION FOR LOGDET K = 3/1X)

```

C

```

99998 FORMAT (/13,1H ,E13.6,2H ,E13.6,2H ,I6,7H ,E13.6,2H ,

```

```

2 E13.6)

```

C

```

99997 FORMAT (/36H CONVERGENCE NOT OBTAINED IFAIL= ,I4)

```

C

```

99996 FORMAT (29H TEST (CONJECTURAL RESULT) = ,E13.6/)

```

C

```

99995 FORMAT (/20H MAIN INTEGRAL = ,E13.6//20H LEADING INTEGRAL = ,

```

```

2 E13.6/)

```

C

```

99994 FORMAT (20H SURFACE TERMS = ,E13.6/)

```

C

```

99993 FORMAT (/19H LEADING INTEGRAL/)

```

C

```

99992 FORMAT (/29H -MAIN+LEADING+SURFACE = ,E13.6/)

```

C

```

99970 FORMAT (/18H PARAMETERS L1 = ,E13.6,

```

```

2 6H L2 = ,E13.6,6H L3 = ,E13.6/5H P = ,E13.6/5H Q = ,E13.6/)

```

C

```

99960 FORMAT (/)

```

C

```

99950 FORMAT (/33H MAIN INTEGRAL ABSACC ,

```



2 41H NO. OF EVAL. LOWER LIMIT UPPER LIMIT/)

END

SET LOWER LIMIT OF INNER THETA INTEGRAL

FUNCTION PHI1(Y)

REAL\*8 Y

PHI1=0

RETURN

END

SET UPPER LIMIT OF THETA INTEGRAL

FUNCTION PHI2(Y)

REAL\*8 Y

REAL\*8 X01AAF

PHI2=(1.0)\*X01AAF(0.0)

RETURN

END

CALCULATION OF FIRST INTEGRAND

FUNCTION F(X,Y)

REAL\*8 X,Y,K,G,L1,L2,L3,M,N,P,Q

REAL\*8 X01AAF,Y2,P2,Q2,P0,FC,CS,SN,CS2,SN2

COMMON/PARS/L1,L2,L3,P,Q,P2,Q2,P0

Y2=Y\*Y

CS=DCOS(X)

SN=DSIN(X)

CS2=CS\*CS

SN2=SN\*SN

FC=Q\*L2+Q\*L1-P\*L2-P\*Q2-P\*L3+Q\*P2

A=-64\*Y2\*PQ\*CS2-8\*PQ\*(L2+Y2)

2+(24\*Y\*(FC)

3+64\*Y2\*Y\*(Q-P))\*CS

4+8\*(Q2\*(L2+L1)+P2\*(L2+L3)+P2\*Q2)

5+24\*Y2\*(L2+L1+L3+Q2+P2)

K=-4\*Y2\*PQ\*(L2+Y2)\*CS2

2-(2\*Y\*P\*Q2\*L2-2\*Y\*P2\*Q\*L2+2\*Y2\*Y\*(-FC-Q\*Y2+P\*Y2))\*CS

4+P2\*Q2\*L2+Y2\*(Q2\*L2+Q2\*L1+P2\*L2+P2\*Q2

5+P2\*L3+Y2\*L2+Y2\*L1+Y2\*L3+Y2\*Q2+Y2\*P2)

C=8\*Y\*PQ\*(L2+Y2)\*SN\*CS

2+(2\*PQ\*L2\*(Q-P)+2\*Y2\*(-FC-Q\*Y2+P\*Y2))\*SN

B=-8\*Y\*PQ\*(L2+2\*Y2)\*CS2

2-(2\*P\*Q2\*L2-2\*P2\*Q\*L2+6\*Y2\*(-FC)+10\*Y2\*Y2\*(P-Q))\*CS

4+2\*Y\*(Q2\*L2+Q2\*L1+P2\*L2+P2\*Q2+P2\*L3

5+2\*Y2\*L2+2\*Y2\*L1+2\*Y2\*Q2+2\*Y2\*L3+2\*Y2\*P2)

N=(Y2+L2)\*(Y2+P2-2\*P\*Y\*CS)

2\*(Y2+Q2+2\*Q\*Y\*CS)+L1\*Y2

3\*(Y2+Q2+2\*Q\*Y\*CS)+L3\*Y2

4\*(Y2+P2-2\*P\*Y\*CS)

D=A/N-(P\*B+12\*B\*Y\*5+C\*C-36\*K\*Y\*4)/(N\*N)

G=(24\*D\*Y\*4)/N+D\*D

F=4\*(Y\*3)\*(SN2)\*G\*(X01AAF(0.0))

RETURN

END

CALCULATION OF SECOND INTEGRAND

FUNCTION FA(X,Y)

REAL\*8 X,Y,L1,L2,L3,M,N,K,P,Q

REAL\*8 X01AAF,Y2,P2,Q2,P0,FC,CS,SN,CS2,SN2

COMMON/PARS/L1,L2,L3,P,Q,P2,Q2,P0

Y2=Y\*Y



```

CS=DCOS(X)
SN=DSIN(X)
CS2=CS*CS
SN2=SN*SN
FC=0*L2+0*L1-P*L2-P*Q2-P*L3+0*P2
N=(Y2+L2)*(Y2+P2-2*P*Y*CS)
2*(Y2+Q2+2*Q*Y*CS)+L1*Y2
3*(Y2+Q2+2*Q*Y*CS)+L3*Y2
4*(Y2+P2-2*P*Y*CS)
M=-4*Y2*PQ*(L2+Y2)*CS2
2-(2*Y*P*Q2*L2-2*Y*P2*Q*L2+2*Y2*Y*(-FC-0*Y2+P*Y2))*CS
4+(P2*Q2*L2+Y2*(Q2*L2+Q2*L1+P2*L2+P2*Q2
5+P2*L3+Y2*L2+Y2*L1+Y2*Q2+Y2*L3+Y2*P2))
K=-8*Y*PQ*(L2+2*Y2)*CS2
2-(2*P*Q2*L2-2*P2*Q*L2+6*Y2*(-FC)+10*Y2*Y2*(P-Q))*CS
4+(2*Y*(Q2*L2+Q2*L1+P2*L2+P2*Q2+P2*L3
5+2*Y2*L2+2*Y2*L1+2*Y2*Q2+2*Y2*L3+2*Y2*P2))
FA=96*X01AAF(0.0)*(SN2)*(K/N+6*Y2*Y2*Y*M/N**2)
RETURN
END

```

C  
C

```

CALCULATION OF ANSATZ
FUNCTION H(D)
REAL*8 D,K,L1,L2,L3,P,Q,P2,Q2,PQ
REAL*8 X01AAF
COMMON/PARS/L1,L2,L3,P,Q,P2,Q2,PQ
K=1+(L3+L1)/(P+Q)**2+(L2+L3)/Q2+(L1+L2)/P2
2+(L1+L2+L3)*(L2/(PQ*PQ)+L1/(P*(P+Q))**2
3+L3/(Q*(P+Q))**2)
H=(DLOG(L1*L2*L3)/12.0-3.0/8.0+7.0/12.0+DLOG(PQ*(P+Q)))/3
2+DLOG(K)/12)*(192*X01AAF(0.0)**2)
RETURN
END

```

$\lambda_1^2$	$\lambda_2^2$	$S^2$	c	I	A
1	1	1	$0.370370 \times 10^{-1}$	68.06	69.88
1	1	4	$0.185185 \times 10^{-1}$	175.75	176.41
1	4	1	$0.185185 \times 10^{-1}$	31.06	31.71
1	1	16	$0.274348 \times 10^{-2}$	347.37	347.41
1	16	1	$0.274349 \times 10^{-2}$	9.426	9.460
1	1	64	$0.222612 \times 10^{-3}$	552.18	552.18
1	1	256	$0.149067 \times 10^{-4}$	767.43	767.43
1	1	1024	$0.948108 \times 10^{-6}$	985.41	985.43

TABLE I: k=2 Symmetric Cases

$\lambda_1^2$	$\lambda_2^2$	$r^2$	c	I	A
0.25	0.5	1	$0.233236 \times 10^{-1}$	136.56	137.48
0.1	0.5	1	$0.127070 \times 10^{-1}$	156.47	156.81
0.05	0.25	1	$0.568958 \times 10^{-2}$	232.35	232.45
0.01	0.25	1	$0.124977 \times 10^{-2}$	249.40	249.41
1	0.25	0.01	$0.124977 \times 10^{-2}$	1.446	1.456
0.001	0.1	1	$0.749270 \times 10^{-4}$	377.24	377.24
0.0001	0.1	1	$0.751110 \times 10^{-5}$	378.51	378.52

TABLE II:    k=2 Small Instanton Strengths

$\lambda_1^2$	$\lambda_2^2$	$S^2$	c	I	A
1	4	256	$0.575942 \times 10^{-4}$	624.57	624.57
1	4	1024	$0.375936 \times 10^{-5}$	841.18	841.19
1	4	4096	$0.237548 \times 10^{-6}$	1059.52	1059.53
8	4	4096	$0.189068 \times 10^{-5}$	921.54	921.56
16	4	4096	$0.375936 \times 10^{-5}$	841.18	841.19
16	25	4096	$0.231400 \times 10^{-4}$	728.62	728.64
50	25	1024	$0.964310 \times 10^{-3}$	424.08	424.10

TABLE III:    k=2 Large Instanton Separations

$$c = 1/37.5$$

$\lambda_1^2$	$\lambda_2^2$	$\rho^2$	I	A
1	1	0.342980	28.0771	29.2056
3	1	0.646496	33.8995	35.0286
3	1	3.24070	118.750	119.880

$$c = 1/75$$

$\lambda_1^2$	$\lambda_2^2$	$\rho^2$	I	A
4	1	0.57847	19.6030	19.4012
6	1	1.24600	27.5762	27.9746
2	1	7.31347	196.704	197.103

$$c = 1/150$$

$\lambda_1^2$	$\lambda_2^2$	$\rho^2$	I	A
3	3	0.174358	5.44221	5.57627
0.5	3	9.29250	205.464	205.601
12	3	47.3531	230.890	231.028

TABLE IV: Constant Conformal Invariant Groups

$\lambda_1^2$	$\lambda_2^2$	$\lambda_3^2$	$\rho$	$z$	I	A
1	4	8	1	1	2113.88	2109.47
2	8	16	1	1	2523.69	2421.36
32	4	64	1	1	3519.97	3516.09
32	4	64	10	10	6769.73	6767.31
1	4	8	10	10	5772.49	5772.09
100	50	150	10	10	7679.20	7670.10
100	50	10	0.1	0.1	1121.70	1120.77
1	5	10	0.1	0.1	-696.04	-697.01
1	5	0.1	0.1	0.1	-1636.93	-1637.00

TABLE V:  $k=3$  Equidistant Instantons

$\lambda_1^2$	$\lambda_2^2$	$\lambda_3^2$	$p$	$q$	I	A
1	4	1	1	8	3614.37	3611.94
4	50	4	1	8	4886.64	4885.41
100	50	200	1	10	6417.94	6413.54
1	4	1	1	25	4991.82	4989.37
4	50	4	1	25	6188.63	6188.35
150	50	150	1	25	7593.41	7590.88
0.1	10	0.1	1	25	4507.53	4505.03
0.1	5	0.1	1	10	3184.70	3184.35
0.1	5	0.1	1	0.01	-980.76	-981.08

TABLE VI:  $k=3$  Symmetric Instanton Strengths

$\lambda_1^2$	$\lambda_2^2$	$\lambda_3^2$	$\rho$	$z$	I	A
1	4	8	2	6	4011.97	4007.90
1	50	10	3	20	6265.89	6265.26
100	10	500	3	20	7582.84	7578.95
100	1	2	3	10	5471.49	5471.12
1	0.5	2	0.1	0.3	-1370.95	-1375.29
2	5	10	0.01	20	3766.35	3766.28
100	50	1	0.01	10	4131.43	4131.28
10	15	50	5	12	6191.53	6186.91
0.01	0.1	0.5	5	12	3571.97	3571.95

TABLE VII: k=3 Unequal Instanton Parameters



$\alpha$	$J-J_0$
2	$-0.277 \times 10^{-8}$
3	$0.255 \times 10^{-1}$
4	$0.639 \times 10^{-1}$
5	0.108
6	0.155
9	0.305
12	0.460
15	0.617
18	0.775

TABLE VIII: Comparisons of Osborn Ansatz with Exact Results  
of Chakrabarti and Comtet

Chapter 4: References

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3. E. Corrigan, P. Goddard, H. Osborn and S. Templeton, Nucl. Phys. B159 (1979) 469.
4. J.J. Sylvester, Quarterly J. Maths 1 (1855) 42.
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6. H. Osborn, Nucl. Phys. B159 (1979) 497.
7. A. Chakrabarti and A. Comtet, IPNO/TH 81-20.
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## CHAPTER 5: Exact Calculation for k=2

In this chapter the instanton determinant for the general k=2 't Hooft solution is calculated. After a first section introducing key conformal properties relating the general case to that of the symmetric version (equal instanton strengths), the calculation for the latter is presented in detail. A brief conclusion follows.

### 1. Use of Conformal Properties

It was shown in Chapter 3 how for the particular case of  $SU(2)$  Jack's work led to the following expression for the determinant of the covariant Laplacian in the background field of instantons (cf. (3.53)).

$$\mathcal{D}_k = \frac{1}{6} \ln \det \{ M_s(v \otimes v) \} + \frac{1}{192\pi^2} \mathcal{J} - \left( d\left(\frac{1}{2}\right) + \frac{5}{72} - 2\ln 2 + \frac{\ln \mu^2}{12} \right) k ,$$

where  $\mathcal{J} = - \int d^4x \ln \det f \partial \bar{\partial} \ln \det f \bar{\partial} \partial$  . (5.1)

Calculation of  $\mathcal{J}$  thus provides the determinant; this is carried out below for k=2, relating first the general case to the symmetric configuration, which proves more readily calculable.

To this end we use the fact that the ansatz  $\mathcal{J}_0$  of the previous chapter (i.e. (4.39)), defined by

$$\frac{\mathcal{J}_0(\underline{\lambda})}{16\pi^2} = - \ln \det \{ M(v \otimes v) \} + \left( \frac{5}{6} - \ln 2 \right) k , \quad (5.2)$$

where  $\underline{\lambda}$  is the set of instanton parameters, reproduces the leading singular behaviour and conformal properties of  $\mathcal{J}(\underline{\lambda})$  for  $SU(2)$ . In

particular,  $\mathcal{T} - \mathcal{T}_0$  is conformally invariant, and must therefore only be a function of combinations of instanton parameters that are also conformally invariant. For  $k=2$  this is unique:

$$c(\underline{\lambda}) = \frac{\lambda_1^2 \lambda_2^2 |y_1 - y_2|^2}{(\lambda_1^2 + \lambda_2^2 + |y_1 - y_2|^2)^3}, \quad (5.3)$$

$\lambda_i$  the instanton strengths,  $y_i$  the positions;

so 
$$\mathcal{T}(\underline{\lambda}) = \mathcal{T}_0(\underline{\lambda}) + f(c(\underline{\lambda})). \quad (5.4)$$

In what follows,  $\lambda_1$  and  $\lambda_2$  are set equal, to  $a$ , say, greatly simplifying the evaluation of the integral (5.1) by virtue of resultant symmetries and cancellations. This provides  $\mathcal{T}(\underline{\lambda}_0)$  where

$$c(\underline{\lambda}_0) = \frac{a^4 s^2}{(2a^2 + s^2)^3}$$

( $s$  being the instanton separation). To obtain  $\mathcal{T}(\underline{\lambda})$  for general

$\underline{\lambda}$ , a restricted set  $\underline{\lambda}_0$  with  $\lambda_1 = \lambda_2$  is found such that

$c(\underline{\lambda}_0) = c(\underline{\lambda})$ . Then, using this set,

$$\begin{aligned} \mathcal{T}(\underline{\lambda}) &= \mathcal{T}_0(\underline{\lambda}) + f(c(\underline{\lambda})) \\ &= \mathcal{T}_0(\underline{\lambda}) + f(c(\underline{\lambda}_0)) \\ &= \mathcal{T}(\underline{\lambda}_0) + [\mathcal{T}_0(\underline{\lambda}) - \mathcal{T}_0(\underline{\lambda}_0)] \end{aligned} \quad (5.5)$$

by (5.4), giving  $\mathcal{T}(\underline{\lambda})$  in terms of calculable quantities.

To see that it is always possible to find such a set  $\underline{\lambda}_0$ , it is helpful to consider the properties of (5.3). Writing this as a cubic in

$|y_1 - y_2|^2$ , we have:

$$f(\lambda_1^2, \lambda_2^2, x) = cx^3 + 3c(\lambda_1^2 + \lambda_2^2)x^2 + x(3c(\lambda_1^2 + \lambda_2^2)^2 - \lambda_1^2 \lambda_2^2) + c(\lambda_1^2 + \lambda_2^2)^3 = 0. \quad (5.6)$$

For a given value of  $(\lambda_1^2 + \lambda_2^2)$ , this always has one (unphysical) negative root, and two others that are either both imaginary or real; the possible situations are indicated in the diagram below:

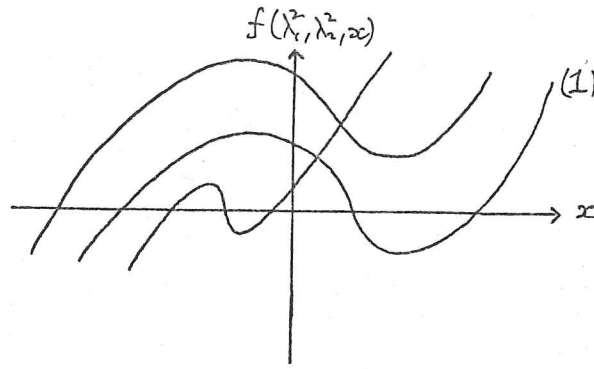


Fig. 1

If there exists one real (positive) root, there must exist another (i.e. curve (1) above). Consider seeking a solution  $x$  for  $c$  a value obtained from a known set of possible parameters  $(\lambda_1^2, \lambda_2^2, x')$ , and taking  $\lambda_1^2 = \lambda_2^2 = \bar{\lambda}^2$ .

Then from (5.6)

$$\begin{aligned} f(\bar{\lambda}^2, \bar{\lambda}^2, x) &= cx^3 + 3c(2\bar{\lambda}^2)x^2 + x(3c(2\bar{\lambda}^2)^2 - \bar{\lambda}^4) + c(2\bar{\lambda}^2)^3 \\ &= f(\lambda_1^2, \lambda_2^2, x) + x(\lambda_1^2 \lambda_2^2 - \bar{\lambda}^4) \end{aligned} \quad (5.7)$$

as  $2\bar{\lambda}^2 = \lambda_1^2 + \lambda_2^2$ .

So the solutions of  $f(\bar{\lambda}^2, \bar{\lambda}^2, x) = 0$  are given by those of

$$f(\lambda_1^2, \lambda_2^2, x) = x(\bar{\lambda}^4 - \lambda_1^2 \lambda_2^2). \quad (5.8)$$

Now the arithmetic mean is greater than or equal to the geometric mean

$$\frac{\lambda_1^2 + \lambda_2^2}{2} = \bar{\lambda}^2 \geq \sqrt{\lambda_1^2 \lambda_2^2}$$

so 
$$\bar{\lambda}^4 - \lambda_1^2 \lambda_2^2 \geq 0 \quad (5.9)$$

Since we are considering values of  $c$  and  $\lambda_1^2, \lambda_2^2$  for which one positive root exists,  $f(\lambda_1^2, \lambda_2^2, x)$  has the form of curve (1) in Fig. 1; the roots of (5.5) are therefore given by the points of intersection of this curve and the straight line  $f = x(\bar{\lambda}^4 - \lambda_1^2 \lambda_2^2)$ ; (see Fig. 2).

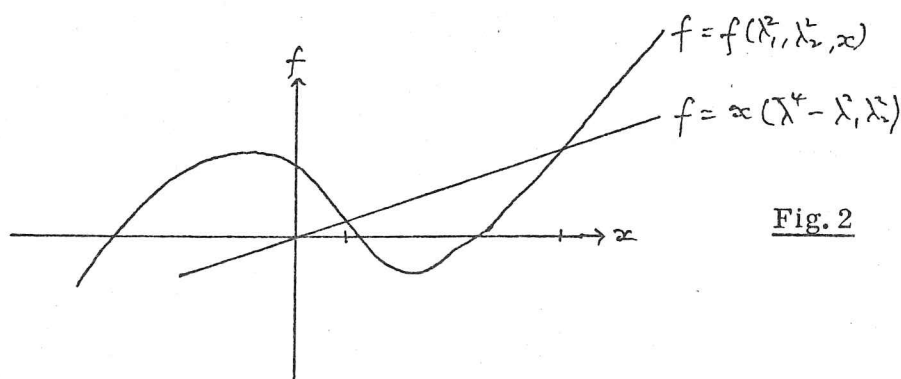


Fig. 2

So two positive values of  $x$ , that is  $|y_1 - y_2|^2$ , exist which furnish, with  $\bar{\lambda}$ , the required parameters for (5.5).

## 2. Computational Details

The integral  $-J(\underline{\lambda})$  can be re-written more symmetrically (integrating by parts) as

$$\begin{aligned}
\int_{x^2 < R^2} \partial^2 \partial^2 \ln \chi \ln \chi d^4 x &= \int_{x^2 = R^2} \partial^2 \partial_\mu \ln \chi \ln \chi dS_\mu \\
&- \int_{x^2 = R^2} \partial^2 \ln \chi \partial_\mu \ln \chi dS_\mu \\
&+ \int (\partial^2 \ln \chi)^2 d^4 x,
\end{aligned} \tag{5.10}$$

putting  $\ln \chi = -\ln \det f_v$ .

In the case of 't Hooft's solution<sup>1</sup> and  $k=2$ , with instantons of strength  $a$  and positions  $y_i$ ,

$$-\ln \det f_v = \ln \left\{ x_1^2 x_2^2 \left( 1 + \frac{a^2}{x_1^2} + \frac{a^2}{x_2^2} \right) \right\} \tag{5.11}$$

where  $x_i^2 = (x - y_i)^2$ ,

and so

$$\begin{aligned}
-J(\lambda) &= \lim_{R \rightarrow \infty} \int_{x^2 < R^2} \partial^2 \partial^2 \ln \chi \ln \chi d^4 x \\
&= \lim_{R \rightarrow \infty} \left\{ 32\pi^2 \ln R^4 - 64\pi^2 + \int_{x^2 < R^2} (\partial^2 \ln \chi)^2 d^4 x \right\}. \tag{5.12}
\end{aligned}$$

Taking the origin of four-dimensional polar co-ordinates midway between the instantons, and  $\Theta$  measured from the line joining them, the  $\phi$  and  $\psi$  angular dependence may be integrated out (so

$d^4 x = 4\pi \sin^2 \Theta d\Theta r^3 dr$ ) and the integral becomes even in  $r$ .

Then with  $x = r^2$

$$\partial^2 \ln \chi = \frac{8(x + s^2/4 + 2a^2)}{\chi} - \frac{16a^4 x^2}{\chi^2} + \frac{16a^2 s^2 x^2 \cos^2 \theta}{\chi^2} \quad (5.13)$$

$$= \frac{\mu}{\chi} - \frac{\nu}{\chi^2} + \lambda \frac{\cos^2 \theta}{\chi^2} \quad (5.14)$$

In the integral of (5.12),  $\frac{\mu^2}{\chi^2}$  contributes

$$128\pi \int_0^\pi \int_0^{K^2} \frac{\mu^2 x \sin^2 \theta d\theta dx}{\chi^2} = 64\pi \int_0^\pi \int_0^{K^2} \frac{\sin^2 \theta}{\chi} \cdot \frac{\partial \chi}{\partial x} dx d\theta$$

$$+ I_1 + I_2 + K_1 + K_2$$

where  $I_1 = 128\pi \int_0^\pi \int_0^{K^2} \frac{s^2}{2} \cdot \frac{\cos^2 \theta \sin^2 \theta d\theta dx}{\chi}$ , (5.15a)

$$I_2 = -128\pi \int_0^\pi \int_0^{K^2} \frac{(s^2/4 + a^2) \sin^2 \theta d\theta dx}{\chi}$$
, (5.15b)

$$K_1 = -256\pi a^2 \int_0^\pi \int_0^{K^2} \frac{(x^2 + x(s^2/4 + 2a^2)) \sin^2 \theta d\theta dx}{\chi^2}$$
, (5.15c)

$$K_2 = 128\pi s^2 \int_0^\pi \int_0^{K^2} \frac{x^2 \cos^2 \theta \sin^2 \theta d\theta dx}{\chi^2}$$
. (5.15d)



Then

$$64\pi \int_0^\pi \int_0^{R^2} \frac{\sin^2 \theta d\theta}{\chi} \cdot \frac{\partial \chi}{\partial x} dx = 32\pi^2 (\ln R^4 - \ln k^2) \quad (5.16)$$

where  $k^2 = s^2/4 (2a^2 + s^2/4)$ ,

which exactly cancels the divergent surface term in (5.12); thus all upper limits may be set to infinity.

$I_1$  and  $I_2$  are dealt with in Appendix B.

The evaluation of  $K_1$  and  $K_2$  in (5.11) will be given in some detail, as they illustrate the principal techniques used in all subsequent calculations.

Consider  $K_2$ .

$$\text{Now } \int_0^\pi \frac{\cos^2 \theta \sin^2 \theta d\theta}{(A - B \cos^2 \theta)} = \frac{\pi}{B^2} \left[ \frac{2A-B}{2\sqrt{A(A-B)}} - 1 \right]$$

where  $\chi = A - B \cos^2 \theta$  (see Appendix A),

$$\text{so } K_2 = 128\pi \int_0^\infty \int_0^\pi \frac{x^2 s^2 \cos^2 \theta \sin^2 \theta dx d\theta}{\chi^2} = \frac{128\pi^2}{s^2} \int_0^\infty \left( \frac{2A-B}{2\sqrt{A(A-B)}} - 1 \right) dx$$

converges, though each part of the integrand separately diverges.

$$\text{As } \frac{d}{dx} \left\{ x \sqrt{\frac{A}{A-B}} + x \sqrt{\frac{A-B}{A}} \right\}$$

$$= \sqrt{\frac{A}{A-B}} + \sqrt{\frac{A-B}{A}}$$

(5.17)

$$+ \frac{((2a^2 + s^2/2)x + 2k^2)x s^2}{2\sqrt{A(A-B)^3}} - \frac{((2a^2 - s^2/2)x + 2k^2)x s^2}{2\sqrt{A^3(A-B)^3}},$$

So 
$$\int_0^{R^2} \frac{(2A-B)}{2\sqrt{A(A-B)}} - R^2 = \frac{1}{2} \left[ x \sqrt{\frac{A}{A-B}} + x \sqrt{\frac{A-B}{A}} \right]_0^{R^2} - R^2$$

$$- \int_0^{R^2} \frac{s^2}{4} \cdot \frac{((2a^2 + s^2/2)x^2 + 2k^2x) dx}{\sqrt{A(A-B)^3}} \quad (5.18)$$

$$+ \int_0^{R^2} \frac{s^2}{4} \cdot \frac{((2a^2 - s^2/2)x^2 + 2k^2x) dx}{\sqrt{A^3(A-B)}}$$

and the first two terms of the right-hand side cancel in the limit  $R^2 \rightarrow \infty$ , leaving convergent integrals.

Making the substitution  $x = \frac{t-1}{t+1} \cdot k$

in 
$$\int_0^\infty \frac{s^2/4 \cdot ((2a^2 + s^2/2)x^2 + 2k^2x) dx}{\sqrt{A(A-B)^3}} \quad \text{and using}$$

the fact that

$$\begin{aligned} \int_0^\infty I(x) dx &\rightarrow \int_1^\infty I'(t) dt + \int_{-\infty}^{-1} I'(t) dt \\ &= 2 \int_1^\infty \text{even part} \{ I'(t) \} dt \end{aligned}$$

we obtain

$$\frac{s^2}{4} \int_1^\infty \frac{4 \left[ (2a^2 + s^2/2)(t^2+1) + 2k(t^2-1) \right] k^3 dt}{\alpha \beta^3 \sqrt{(t^2+p^2)(t^2+q^2)^3}} \quad (5.19)$$

$$= \frac{s^2}{4} \int_1^\infty \frac{4 \left[ (2a^2 + s^2/2 + 2k)t^2 + (2a^2 + s^2/2 - 2k) \right] k^3 dt}{\alpha \beta^3 \sqrt{(t^2+p^2)(t^2+q^2)^3}} \quad (5.20)$$

where  $\alpha^2 = 2k^2 + k(2a^2 + s^2/2)$  , (5.21a)

$\beta^2 = 2k^2 + k(2a^2 - s^2/2)$  , (5.21b)

$p^2+1 = 4k^2/\alpha^2$  , (5.21c)

$q^2+1 = 4k^2/\beta^2$  . (5.21d)

Put  $\frac{t}{\sqrt{t^2+q^2}} = v$

then (5.20) transforms to

$$\frac{s^2 k^3}{\alpha \beta^3 q^3} \int_{\frac{1}{\sqrt{1+q^2}}}^1 dv \frac{(2a^2 + s^2/2 + 2k)q^2 v^2 + (2a^2 + s^2/2 - 2k)(1-v^2)}{\sqrt{(1-v^2)(\kappa'^2 + \kappa^2 v^2)}} \quad (5.22)$$

where  $\kappa'^2 = 1 - \beta^2/z^2$  ,  $\kappa^2 = 1 - \kappa'^2 = \beta^2/z^2$  .

Now  $-\int \frac{v^m dv}{\sqrt{(1-v^2)(k'^2 + k^2 v^2)}}$  is a standard

elliptic integral <sup>2</sup> (m even)

and

$$-\int \frac{v^m dv}{\sqrt{(1-v^2)(k'^2 + k^2 v^2)}} = \int \frac{\cos^m \phi d\phi}{\sqrt{1 - k^2 \sin^2 \phi}} = \int \operatorname{cn}^m u du = C_m$$

with  $v = \cos \phi = \operatorname{cn} u$ ;

here  $C_0 = u = F(\phi, k)$ , the elliptic function of the first kind; (5.23a)

$$C_2 = \frac{1}{k^2} [E(u) + k'^2 u], \quad \text{where } E(u) = E(\phi, k),$$

the elliptic function of the second kind; (5.23b)

$$C_4 = \frac{1}{3k^4} [(2 - k^2) k'^2 u + 2(2k^2 - 1) E(u) + k^2 \operatorname{sn} u \operatorname{cn} u \operatorname{dn} u]$$

(5.23c)

(See 2 for further details).

So (5.22) gives

$$-\frac{s^2 k^3}{\alpha \beta^3 \gamma^3} \left\{ (2\alpha^2 + s^2/2 + 2k) \gamma^2 C_2 + (2\alpha^2 + s^2/2 + 2k) (C_0 - C_2) \right\} \quad (5.24)$$

evaluated at the limits  $v = 1$  and  $\frac{1}{\sqrt{1+q^2}}$

Similarly

$$\int_0^\infty \frac{s^2}{4} \cdot \frac{(2a^2 - s^2/2)x^2 + 2k^2x}{\sqrt{A^3(A-B)}} dx \quad (5.25)$$

via the transformation

$$x = \frac{t-1}{t+1} \cdot k$$

becomes

$$\frac{4 \cdot s^2/4}{a^3 \beta} \int_1^\infty \frac{[(2a^2 - s^2/2 + 2k)t^2 + (2a^2 - s^2/2 - 2k)] k^3 dt}{\sqrt{(t^2 + p^2)^3 (t^2 + q^2)}} \quad (5.26)$$

Then setting

$$\frac{t}{\sqrt{t^2 + p^2}} = v, \quad (5.26) \text{ transforms to}$$

$$\frac{s^2 k^3}{a^3 \beta p^2 q} \int_{\frac{1}{\sqrt{1+p^2}}}^1 \frac{(2a^2 - s^2/2 + 2k)p^2 v^2 + (2a^2 - s^2/2 - 2k)(1-v^2) dv}{\sqrt{(1-v^2)(1-k^2 v^2)}} \quad (5.27)$$

Integrals of the form

$$\int \frac{v^r dv}{\sqrt{(1-v^2)(1-k^2 v^2)}}$$

(r even)

$$= \int \frac{\sin^r \phi d\phi}{\sqrt{1-k^2 \sin^2 \phi}} = \int \sin^r u du = A_r$$

$$(v = \sin \phi = \sin u)$$

are also elliptic in structure; the first few are

$$A_0 = u = F(\phi, k) \quad , \quad (5.28a)$$

$$A_2 = \frac{1}{k^2} [u - E(u)] \quad (5.28b)$$

$$A_4 = -\frac{1}{3k^4} [(2+k^2)u - 2(1+k^2)E(u) + k^2 \operatorname{sn} u \cdot \operatorname{cn} u \operatorname{dn} u] \quad (5.28c)$$

$$A_6 = \frac{1}{5k^2} [\operatorname{sn}^3 u \cdot \operatorname{cn} u \cdot \operatorname{dn} u + 4(1+k^2)A_4 - 3A_2] \quad (5.28d)$$

So (5.27) gives rise to a term

$$\frac{s^2 k^3}{\alpha^3 \beta^3 q^3} \left\{ (2\alpha^2 - s^2/2 + 2k) p^2 A_2 + (2\alpha^2 - s^2/2 - 2k)(A_0 - A_2) \right\} \quad (5.29)$$

evaluated at both limits. (5.24) and (5.29) provide  $K_2$  ,

$$K_2 = \frac{128\pi^2 k^3}{\alpha^3 \beta^3 q^3} \left\{ (2\alpha^2 - s^2/2 + 2k) p^2 A_2 - (2k - (2\alpha^2 - s^2/2))(A_0 - A_2) \right\} \quad (5.30)$$

$$- \frac{128\pi^2 k^3}{\alpha^3 \beta^3 q^3} \left\{ (2\alpha^2 + s^2/2 + 2k) q^2 C_2 - (2k - (2\alpha^2 + s^2/2))(C_0 - C_2) \right\} .$$

The same procedures may be applied to  $K_1$ .

$$K_1 = 286\pi\alpha^2 \int_0^\pi \int_0^\infty \frac{(x^2 + x(s^2/4 + 2a^2)) \sin^2\theta d\theta dx}{x^2} \quad (5.31)$$

$$= 128\pi^2\alpha^2 \int_0^\infty \frac{(x^2 + x(s^2/4 + 2a^2)) dx}{\sqrt{A^2(A-B)}} \quad (5.32)$$

(using Appendix A).

As before set  $x = \frac{t-1}{t+1} \cdot k$ , then

$$K_1 = \frac{512\pi^2\alpha^2 k^2}{\alpha^3\beta} \int_1^\infty \frac{(k + s^2/4 + 2a^2)t^2 + (k - (s^2/4 + 2a^2)) dt}{\sqrt{(E+p)^2(E+q^2)}} \quad (5.33)$$

and with  $\frac{t}{\sqrt{E+p^2}} = v$  this becomes

$$\begin{aligned} & \frac{512\pi^2\alpha^2 k^2}{\alpha^3\beta p^2 q} \int_1^\infty \frac{dw}{\sqrt{1+p^2}} \frac{(k + s^2/4 + 2a^2)p^2 v^2 + (k - (s^2/4 + 2a^2))(1-v^2)}{\sqrt{(1-v^2)(1-k^2 v^2)}} \\ &= \frac{512\pi^2\alpha^2 k^2}{\alpha^3\beta p^2 q} \left\{ (k + s^2/4 + 2a^2)p^2 A_2 + (k - (s^2/4 + 2a^2))(A_0 - A_2) \right\} . \end{aligned} \quad (5.34)$$

The cross-terms in  $(\partial^2 \ln \chi)^2$  are (cf. (5.14))

$$-2 \frac{\mu^2}{\chi^3} + 2 \frac{\mu \lambda \cos^2 \theta}{\chi^3} \quad (5.35)$$

which via Appendix A lead to the following  $x$ -integral:

$$64 \pi^2 a^2 \int_0^\infty dx x^2 (x + s^2/4 + 2a^2) \left[ \frac{(s^2 - a^2)}{A^{3/2} (A-B)^{1/2}} - \frac{3a^2}{A^{3/2} (A-B)^{3/2}} \right]. \quad (5.36)$$

The first term

$$L_1 = 64 \pi^2 a^2 (s^2 - a^2) \int_0^\infty \frac{x^2 (x + s^2/4 + 2a^2) dx}{A^{3/2} (A-B)^{1/2}} \quad (5.37)$$

can be evaluated directly. Putting  $x = \frac{t-1}{t+1} \cdot k$

and then  $\frac{t}{\sqrt{t^2 + p^2}} = v$ , we find

$$L_1 = \frac{256 \pi^2 a^2 (s^2 - a^2)}{\alpha^\beta} \int_1^\infty dt k^3 \left\{ \frac{k(t^4 - 1) + (s^2/4 + 2a^2)(t^2 - 1)^2}{\sqrt{(t^2 + p^2)^5 (t^2 + 2)}} \right\} \quad (5.38)$$

$$= \frac{256 \pi^2 a^2 (s^2 - a^2) k^2}{\alpha^\beta p^4 2} \int_{\frac{1}{\sqrt{1+p^2}}}^1 dv \frac{k(p^4 v^4 - (1-v^2)) + (s^2/4 + 2a^2)(v^2 p^2 + 1)^2}{\sqrt{(1-v^2)(1-k^2 v^2)}} \quad (5.39)$$



$$= \frac{256\pi^2 a^2 (s^2 - a^2) k^3}{\alpha^3 \beta^4 q} \left\{ k [A_4 (p^4 - 1) + 2A_2 - A_0] \right. \\ \left. + (s^2/4 + 2a^2) [(p^2 + 1)^2 A_4 - 2(p^2 + 1) A_2 + A_0] \right\}.$$

(5.40)

Then noting  $x = \frac{1}{s^2} (A - (A - B))$

$$L_2 = -192\pi^2 a^4 \int_0^\infty \frac{x^2 (x + s^2/4 + 2a^2) dx}{A^{3/2} (A - B)^{3/2}}$$

may be re-written as

$$- \frac{192\pi^2 a^4}{s^2} \left\{ \int_0^\infty \frac{x (x^2 + s^2/4 + 2a^2) dx}{A^{1/2} (A - B)^{3/2}} - \int_0^\infty \frac{x (x + s^2/4 + 2a^2) dx}{A^{3/2} (A - B)^{1/2}} \right\}.$$

(5.41)

The first term of (5.41)

$$- \frac{192\pi^2 a^4}{s^2} \int_0^\infty \frac{x (x + s^2/4 + 2a^2) dx}{A^{1/2} (A - B)^{3/2}} \quad (5.42)$$

$$= - \frac{4 \cdot 192\pi^2 a^4 k^2}{\alpha \beta^3 s^2} \int_1^\infty \frac{dk [(k + s^2/4 + 2a^2) \ell^2 + (k - (s^2/4 + 2a^2))] }{\sqrt{(\ell^2 + p^2)(\ell^2 + q^2)^3}} \quad (5.43)$$

$$= - \frac{768\pi^2 a^4 k^2}{s^2 \alpha \beta^3 q^3} \int_{\frac{1}{\sqrt{1+q^2}}}^1 \frac{dv [(k + s^2/4 + 2a^2) q^2 v^2 + (k - (s^2/4 + 2a^2))(1 - v^2)] }{\sqrt{(1 - v^2)(k'^2 + k^2 v^2)^3}} \quad (5.44)$$

$$= \frac{768\pi^2 a^4 k^2}{s^2 \alpha \beta^3 q^3} \left\{ (k + s^2/4 + 2a^2) q^2 C_2 + (k - (s^2/4 + 2a^2))(C_0 - C_2) \right\} \quad (5.45)$$

proceeding as above.

Similarly

$$\frac{192\pi^2 a^4}{s^2} \int_0^\infty \frac{x(x + s^2/4 + 2a^2) dx}{A^{3/2} (A - B)^{1/2}} \quad (5.46)$$

$$= \frac{768\pi^2 a^4 k^2}{\alpha^3 \beta s^2} \int_1^\infty dt \frac{[(k + s^2/4 + 2a^2)t^2 + (k - (s^2/4 + 2a^2))]}{\sqrt{(t^2 + p^2)^3 (t^2 + q^2)}} \quad (5.47)$$

$$= \frac{768\pi^2 a^4 k^2}{s^2 \alpha^3 \beta p^2 q} \int_{\frac{1}{\sqrt{1+p^2}}}^1 dv \frac{(k + s^2/4 + 2a^2)p^2 v^2 + (k - (s^2/4 + 2a^2))(1 - v^2)}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} \quad (5.48)$$

$$= \frac{768\pi^2 a^4 k^2}{s^2 \alpha^3 \beta p^2 q} \left\{ (k + s^2/4 + 2a^2) p^2 A_2 + (k - (s^2/4 + 2a^2))(A_0 - A_2) \right\} \quad (5.49)$$

So putting these results together

$$\begin{aligned} L_2 = & \frac{768\pi^2 a^4 k^2}{s^2 \alpha^3 \beta p^2 q} \left\{ (k + s^2/4 + 2a^2) p^2 A_2 + (k - (s^2/4 + 2a^2))(A_0 - A_2) \right\} \\ & + \frac{768\pi^2 a^4 k^2}{s^2 \alpha \beta^3 q^3} \left\{ (k + s^2/4 + 2a^2) q^2 C_2 + (k - (s^2/4 + 2a^2))(C_0 - C_2) \right\}, \end{aligned} \quad (5.50)$$

The remaining terms in  $(\partial^2 \ln \chi)^2$ ,

$$\frac{v^2}{\chi^4} - \frac{2v\lambda \cos^2 \theta}{\chi^4} + \frac{\lambda^2 \cos^4 \theta}{\chi^4} \quad (5.51)$$

give rise to the integrals  $M_1 + M_2 + M_3$

$$\text{where } M_1 = 32 a^4 (a^2 - s^2)^2 \pi^2 \int_0^\infty \frac{x^3 dx}{A^{3/2} (A-B)^{5/2}}, \quad (5.52)$$

$$M_2 = -64 (a^2 - s^2) a^6 \pi^2 \int_0^\infty \frac{x^3 dx}{A^{5/2} (A-B)^{3/2}}, \quad (5.53)$$

$$M_3 = 160 a^8 \pi^2 \int_0^\infty \frac{x^3 dx}{A^{7/2} (A-B)^{1/2}} \quad (5.54)$$

using Appendix A once more.

$$\text{So } M_3 = 160 a^8 \pi^2 \int_0^\infty \frac{x^3 dx}{A^{7/2} (A-B)^{1/2}} \quad (5.55)$$

$$= \frac{640 a^8 \pi^2 k^4}{a^7 \beta} \int_1^\infty \frac{dt (t^2 - 1)^3}{\sqrt{(t^2 + p^2)(t^2 + q^2)}} \quad (5.56)$$

$$= \frac{640 a^8 \pi^2 k^4}{a^7 \beta p^6 q} \int_1^1 \frac{((p^2 + 1)v^2 - 1)^3 dv}{\sqrt{(1 - v^2)(1 - k^2 v^2)}} \quad (5.57)$$

$$= \frac{640 a^8 \pi^2 k^4}{a^7 \beta p^6 q} \left\{ (p^2 + 1)^3 A_6 - 3(p^2 + 1)^2 A_4 + 3(p^2 + 1) A_2 - A_0 \right\} \quad (5.58)$$

$M_1$  and  $M_2$  may be re-written using  $x = \frac{1}{s^2}(A - (A-B))$ ,

so 
$$\int_0^\infty \frac{x^3 dx}{A^{1/2}(A-B)^{3/2}} = -\frac{1}{s^2} \int_0^\infty \frac{x^2 dx}{A^{3/2}(A-B)^{1/2}} \quad (5.59)$$

$$+ \frac{1}{s^4} \int_0^\infty \frac{x dx}{A^{1/2}(A-B)^{3/2}} - \frac{1}{s^4} \int_0^\infty \frac{x dx}{A^{3/2}(A-B)^{1/2}}$$

$$= -\frac{4k^3}{s^2 \alpha^3 \beta} \int_1^\infty \frac{(t^2-1)^2 dt}{\sqrt{(t^2+p^2)(t^2+q^2)}} \quad (5.60)$$

$$+ \frac{4k^2}{s^4 \alpha^3 \beta} \int_1^\infty \frac{(t^2-1) dt}{\sqrt{(t^2+p^2)(t^2+q^2)^3}} - \frac{4k^2}{s^4 \alpha^3 \beta} \int_1^\infty \frac{(t^2-1) dt}{\sqrt{(t^2+p^2)^3(t^2+q^2)}}$$

$$= -\frac{4k^3}{p^4 q s^2 \alpha^3 \beta} \int_{\frac{1}{\sqrt{1+p^2}}}^1 \frac{((1+p^2)v^2-1)^2 dv}{\sqrt{(1-v^2)(1-k^2 v^2)}} \quad (5.61)$$

$$+ \frac{4k^2}{q^3 s^4 \alpha^3 \beta} \int_{\frac{1}{\sqrt{1+q^2}}}^1 \frac{((1+q^2)v^2-1) dv}{\sqrt{(1-v^2)(k^2 + k^2 v^2)}} - \frac{4k^2}{p^2 q s^4 \alpha^3 \beta} \int_{\frac{1}{\sqrt{1+p^2}}}^1 \frac{((1+p^2)v^2-1) dv}{\sqrt{(1-v^2)(1-k^2 v^2)}}.$$

So  $M_2 = 256\pi^2 \alpha^6 (a^2 - s^2).$

$$\left\{ \frac{k^3}{\alpha^3 \beta q p^4 s^2} [(1+p^2) A_4 - 2(1+p^2) A_2 + A_0] \right.$$

$$+ \frac{k^2}{\alpha^3 \beta^3 s^4} [(1+q^2) C_2 - C_0]$$

$$+ \left. \frac{k^2}{\alpha^3 \beta p^2 s^4} [(1+p^2) A_2 - A_0] \right\}. \quad (5.62)$$

Similarly

$$\int_0^\infty \frac{x^3 dx}{A^{3/2}(A-B)^{3/2}} = \frac{1}{s^2} \int_0^\infty \frac{x^2 dx}{A^{1/2}(A-B)^{3/2}} - \frac{1}{s^4} \int_0^\infty \frac{x dx}{A^{1/2}(A-B)^{3/2}} + \frac{1}{s^4} \int_0^\infty \frac{x dx}{A^{3/2}(A-B)^{1/2}} \quad (5.63)$$

$$= \frac{4k^3}{s^2 \alpha \beta^3} \int_1^\infty \frac{(t^2-1)^2 dt}{\sqrt{(t^2+p^2)(t^2+q^2)^3}} - \frac{4k^2}{s^4 \alpha \beta^3} \int_1^\infty \frac{(t^2-1) dt}{\sqrt{(t^2+p^2)(t^2+q^2)^3}} + \frac{4k}{s^4 \alpha \beta^3} \int_1^\infty \frac{(t^2-1) dt}{\sqrt{(t^2+p^2)^3(t^2+q^2)}} \quad (5.64)$$

$$= \frac{4k^3}{s^2 \alpha \beta^3 q^5} \int_1^1 \frac{((1+q^2)v^2-1)^2 dv}{\sqrt{(1-v^2)(k^2+k^2v^2)^3}} \quad (5.65)$$

$$- \frac{4k^2}{s^4 \alpha \beta^3 q^3} \int_1^1 \frac{((1+q^2)v^2-1) dv}{\sqrt{(1-v^2)(k^2+k^2v^2)^3}} + \frac{4k}{s^4 \alpha \beta^3 p^2} \int_1^1 \frac{((1+p^2)v^2-1) dv}{\sqrt{(1-v^2)(1-k^2v^2)^3}}$$

So .  $M_1 = 128\pi^2 a^4 (a^2 - s^2)^2.$

$$\left\{ \frac{k^3}{\alpha \beta^3 q^5 s^2} \left[ -(1+q^2)^2 C_4 + 2(1+q^2) C_2 - C_0 \right] + \frac{k^2}{\alpha \beta^3 q^3 s^4} \left[ (1+q^2) C_2 - C_0 \right] + \frac{k^2}{\alpha^2 \beta p^2 q s^4} \left[ (1+p^2) A_2 - A_0 \right] \right\} . \quad (5.66)$$

### 3. Results

Using the methods described above and putting together the various component parts we have

$$\begin{aligned}
 \mathcal{D}_k &= -\ln \left\{ \det(-D^2/\mu^2) / \det(-D_0^2/\mu^2) \right\} \\
 &= \frac{1}{6} \ln \det [M_s(v \otimes v)] - \left( \alpha\left(\frac{1}{2}\right) + \frac{5}{72} - 2\ln 2 + \frac{\ln \mu^2}{12} \right) \\
 &\quad + \frac{1}{192\pi^2} \left\{ J(\lambda_0) + [J_0(\lambda) - J_0(\lambda_0)] \right\} \quad (5.67)
 \end{aligned}$$

where  $J_0(\lambda) = 16\pi^2 \left\{ -\ln \det [M(v \otimes v)] + \left(\frac{5}{6} - \ln 2\right) 2 \right\},$

$\lambda_0$  is the symmetric set described in section 1 and

$$\begin{aligned}
 -J(\lambda_0) &= -32\pi^2 \left[ 2 + \ln \left( s^2/4 (2\alpha^2 + s^2/4) \right) \right] \\
 &\quad + I_1 + I_2 + K_1 + K_2 + M_1 + M_2 + M_3. \quad (5.68)
 \end{aligned}$$

$I_1$  and  $I_2$  are given in Appendix B and

$$\begin{aligned}
 K_1 &= \frac{512\pi^2 \alpha^2 k^2}{\alpha^3 \beta p^2 q} \left\{ (k + s^2/4 + 2\alpha^2) p^2 A_2 + (k - (s^2/4 + 2\alpha^2)) (A_0 - A_2) \right\}, \\
 K_2 &= 128\pi^2 k^3 \left\{ (2\alpha^2 - s^2/2 + 2k) p^2 A_2 - (2k - (2\alpha^2 - s^2/2)) (A_0 - A_2) \right\} \quad (5.69)
 \end{aligned}$$

$$= \frac{128\pi^2 k^3}{\alpha \beta^3 q^3} \left\{ (2a^2 + s^2/2 + 2k) q^2 C_2 - (2k - (2a^2 + s^2/2)) (C_0 - C_2) \right\}; \quad (5.70)$$

$$L_1 = \frac{64\pi^2 a^2 k^3 (s^2 - a^2)}{\alpha^3 \beta^4 q} \left\{ k [A_4 (p^4 - 1) + 2A_2 - A_0] + (s^2/4 + 2a^2) [(p^2 + 1)^2 A_4 - 2(p^2 + 1) A_2 + A_0] \right\}, \quad (5.71)$$

$$L_2 = \frac{768\pi^2 a^4 k^2}{s^2 \alpha^3 \beta^2 q} \left\{ (k + s^2/4 + 2a^2) p^2 A_2 + (k - (s^2/4 + 2a^2)) (A_0 - A_2) \right\} + \frac{768\pi^2 a^4 k^2}{s^2 \alpha \beta^3 q^3} \left\{ (k + s^2/4 + 2a^2) q^2 C_2 + (k - (s^2/4 + 2a^2)) (C_0 - C_2) \right\}; \quad (5.72)$$

$$M_1 = 128\pi^2 a^4 (a^2 - s^2)^2 \cdot \left\{ \frac{k^3}{\alpha \beta^3 q s^2} [-(1 + q^2)^2 C_4 + 2(1 + q^2) C_2 - C_0] + \frac{k^2}{\alpha \beta^2 q^2 s^4} [(1 + q^2) C_2 - C_0] + \frac{k^2}{\alpha^2 \beta p^2 q^2 s^4} [(1 + p^2) A_2 - A_0] \right\}, \quad (5.73)$$

$$M_2 = 256\pi^2 a^6 (a^2 - s^2) \cdot \left\{ \frac{k^3}{\alpha^3 \beta p^2 q^2 s^4} [(1 + p^2) A_4 - 2(1 + p^2) A_2 + A_0] + \frac{k^2}{\alpha \beta^2 q^2 s^4} [(1 + q^2) C_2 - C_0] + \frac{k^2}{\alpha^2 \beta p^2 q^2 s^4} [(1 + p^2) A_2 - A_0] \right\}, \quad (5.74)$$

$$M_3 = \frac{640\pi^2 a^8 k^4}{\alpha^2 \beta p^6 q} \left\{ (p^2+1)^3 A_6 - 3(p^2+1)^2 A_4 + 3(p^2+1) A_2 - A_0 \right\}. \quad (5.75)$$

$$\text{Here } A_n = A_n(u) \Big|_{u=1} - A_n(u) \Big|_{u=\frac{1}{\sqrt{1+p^2}}},$$

$$C_n = C_n(u) \Big|_{u=1} - C_n(u) \Big|_{u=\frac{1}{\sqrt{1+q^2}}},$$

with the  $A_n$ 's and  $C_n$ 's as defined above.

These results provide the component parts for the evaluation of  $J$  and thus  $D_k$ ; unfortunately it has not proved possible to bring all these terms together in a way that explicitly exhibits an underlying simplicity of structure. In particular we have not been able to write the result showing explicitly the known conformal invariance properties by constructing the function  $f(c)$  of (5.4). And this for an instanton configuration that Berg and Lüscher<sup>3</sup> rightly emphasise as atypical: for the  $\Theta$ -term of (3.37) is identically zero for  $k=2$  and for 't Hooft's solutions generally, both of which obtain here. The implied complexity of other high-index determinants suggests the need for more natural variables (perhaps involving complex parametrisation, cf. infra), in terms of which the results take on more compact forms.



# APPENDIX A: Evaluation of $\Theta$ -integrals

For the  $\Theta$ -integrations, the basic result<sup>4</sup> is

$$\int_0^\pi \frac{d\Theta}{(A - B \cos^2 \Theta)} = \frac{1}{A} \beta\left(\frac{1}{2}, \frac{1}{2}\right) F\left(\frac{1}{2}, 1; 1; \frac{B}{A}\right) \quad (5.76)$$

$$= \frac{\pi}{\sqrt{A(A-B)}} \quad (5.77)$$

where  $\beta(x, y)$  is the beta function.

Whence, by judicious differentiation (treating A and B as independent variables) one obtains the following results:

$$\text{defining } T_r^{n,m} = \int_0^\pi \frac{\sin^n \Theta \cos^m \Theta d\Theta}{\chi^r}, \quad (5.78)$$

$$\text{then } T_2^{2,2} = \frac{\pi}{B^2} \left[ \frac{2A-B}{\sqrt{A(A-B)}} - 1 \right], \quad (5.79)$$

$$T_2^{2,0} = \frac{\pi}{2\sqrt{A^3(A-B)}}, \quad (5.80)$$

$$T_3^{2,2} = \frac{\pi}{8\sqrt{A^3(A-B)^3}}, \quad (5.81)$$

$$T_3^{2,0} = \frac{\pi(4A-3B)}{8\sqrt{A^5(A-B)^3}}, \quad (5.82)$$

$$T_4^{2,4} = \frac{\pi}{16\sqrt{A^3(A-B)^5}}, \quad (5.83)$$

$$T_4^{2,2} = \frac{\pi(2A-B)}{16\sqrt{A}(A-B)^{5/2}}, \quad (5.84)$$

$$T_4^{3,0} = \frac{\pi(8A^2 - 12AB + 5B^2)}{16\sqrt{A}(A-B)^{5/2}}. \quad (5.85)$$

# APPENDIX B: Evaluation of $I_1$ and $I_2$

It has so far proved impossible to find expressions in closed form for the integral

$$2\pi \int \frac{(s^2/2 \cos^2 \theta - (s^2/4 + a^2)) \sin^2 \theta d\theta dx}{X} \quad (5.86)$$

but representations in terms of infinite series can be obtained .

$$\int_0^\pi \frac{\cos^2 \theta \sin^2 \theta d\theta}{(A - B \cos^2 \theta)} \quad (A > B \quad \forall x)$$

$$= \int_0^\pi \frac{\sin^2 \theta}{A} \sum_{n=0}^{\infty} \left(\frac{B}{A}\right)^n (\cos^2 \theta)^{n+1} d\theta \quad (5.87)$$

$$= \frac{\pi}{A} \cdot \sum_{n=0}^{\infty} \frac{(2n+2)!}{2^{2n+2} ((n+1)!)^2 (n+2)} \cdot \left(\frac{B}{A}\right)^n \quad (5.88)$$

So we have integrals of type  $\int_0^\infty \frac{B^n}{A^{n+1}} dx$ ,  $n \geq 0$

$$B = xs^2, \quad A = x^2 + x(s^2/2 + 2a^2) + k^2.$$

Now  $\int_0^\infty \frac{dx}{(x^2 + 2bx + c)} = \frac{1}{\sqrt{c-b^2}} \cot^{-1} \frac{b}{\sqrt{c-b^2}}, \quad (5.89)$

so  $\int_0^\infty \frac{x^n dx}{(x^2 + 2bx + c)^{n+1}} = \frac{(-1)^n}{2^n} \cdot \frac{1}{n!} \frac{\partial^n}{\partial b^n} \left\{ \frac{1}{\sqrt{c-b^2}} \cot^{-1} \frac{b}{\sqrt{c-b^2}} \right\} \quad (5.90)$

with  $b = s^2/4 + a^2$ ,  $c - b^2 = a^2(a^2 + s^2/4)$ .

Thus

$$\begin{aligned}
 I_1 &= 64\pi s^2 \int \frac{\cos\theta \sin^2\theta \, dx \, d\theta}{x} \\
 &= 64\pi^2 \cdot \sum_0^{\infty} (-1)^n \frac{(2n+2)!(n+1) s^{2n+2}}{2^{3n+3} (n+1)!^3 (n+2)} \cdot \frac{\partial^n}{\partial b^n} \left\{ \frac{1}{\sqrt{c-b^2}} \cot^{-1} \frac{b}{\sqrt{c-b^2}} \right\}.
 \end{aligned}
 \tag{5.91}$$

Similarly

$$\begin{aligned}
 I_2 &= -64 \cdot 2\pi (s^2/4 + a^2) \int \frac{\sin^2\theta \, d\theta \, dx}{x} \\
 &= -64\pi^2 (s^2/2 + 2a^2) \sum_0^{\infty} (-1)^n \frac{2n! s^{2n}}{2^{3n+1} (n!)^3} \cdot \frac{\partial^n}{\partial b^n} \left\{ \frac{1}{\sqrt{c-b^2}} \cot^{-1} \frac{b}{\sqrt{c-b^2}} \right\}.
 \end{aligned}
 \tag{5.92}$$

Chapter 5: References

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## CHAPTER 6: Conclusion

The calculation of the previous chapter obtains an expression for the functional determinant of the covariant Laplacian in the background field of the  $k=2$   $SU(2)$  't Hooft instanton; to achieve this, it drew on the various techniques and ideas reported in the preceding chapters. It was shown how each of the component parts of the integral  $J$  (4.26) may be evaluated; unfortunately, however, it has not proved possible to bring all these terms together in a way that exhibits an underlying simplicity of structure, and the result serves to emphasise the complexity of the situation. This is borne out in a number of other ways.

Apart from the generally involved nature of such determinant calculations - from 't Hooft's pioneering calculation<sup>1</sup> through to later work - no clear sense of computational direction has emerged. Although the ADHM construction has provided an obvious and convenient framework in which to discuss such matters (though even this has some difficulties: see below), no clear-cut set of technical procedures has been established.

Thus conformal properties proved of great importance in the previous chapters; a number of authors<sup>2</sup> have investigated the rôle of conformal invariants in this context. But it is soon found that the relevant equations become intractable.

Similarly, in investigating the properties of Osborn's ansatz (cf. *supra*) use was made of the simplifying properties of instantons on a line. And the first extension (by Witten) of the one-instanton solution of Belavin, Polyakov, Schwarz and Tyupkin<sup>5</sup> was that of  $n$ -instantons arranged along

a line<sup>4</sup>. This suggests a possibly fruitful avenue for further investigation, using perhaps complex variable techniques (setting  $z = r + it$  for example). Indeed recently, Boutaleb-Joutei, Chakrabarti and Comtet<sup>5</sup> have considered a particular class of  $SU(2)$  multi-instanton configurations along a line, in which the sizes and separations are constrained in a special way, with resultant simplifications. In particular, using complex variable techniques, this has enabled them to obtain completely explicit forms (with arbitrary  $k$ ) for the instanton determinants (see above and 6). They have expressed the hope that a hierarchy of such solutions might be generated, thus providing further explicit forms. But the success of their scheme serves in part to emphasize how restricted (with no free parameters in each  $k$ -instanton solution) a class of solutions it is necessary to consider in order to obtain compact forms for instanton determinants.

An attempt at a deeper understanding of instantons in the context of functional integrals was made by Belavin, Fateev, Schwarz and Tyupkin<sup>7</sup>. From analogies with two-dimensional  $CP^{n-1}$  models (see 8 and a pertinent short review in 9), in which the leading contribution of the  $k$ -instanton to the functional integral is the partition function (at unit temperature) for a classical neutral Coulomb gas of  $2k$  particles, each of mass  $m$  (the renormalisation group invariant mass),  $k$  of which are positively charged, the remainder negatively, they conjectured that instantons be considered as composed of instanton quarks. Thus for  $SU(n)$  the  $4nk$  instantons parameters correspond to  $n$  species of instanton quarks with multiplicity  $k$ , each having a freely-varying Euclidean position in four-dimensional space. An important aspect of the two-dimensional Coulomb gas is its critical

point at  $T=1$  at which the pressure diverges<sup>10</sup>; this indicates that the dilute (i.e. non-interacting or weakly-interacting) gas approximation is inappropriate: the corresponding statement for four dimensions would be that the system of instantons quarks is in the plasma phase. Thus this conjecture has important consequences for the vexed question of dilute gas approximations; unfortunately little progress has been made beyond the initial conjecture<sup>8</sup>.

Instanton determinants arose in the use of the semi-classical approach to approximating functional integrals; to employ them in this context requires a form in which the explicit dependence on the instanton parameters is manifest. As the above calculations and comments have shown, even in the most complete general case to date, that for the  $k=2$   $SU(2)$  solution, the lack of succinctness and computational manageability renders it less suitable for insertion into functional integrals.

Nevertheless, calculations have already begun on the next stage of evaluation, investigating the other essential ingredient of this semi-classical approach, namely the functional measure to be used in the integration. Goddard, Mansfield and Osborn<sup>11</sup> have obtained the relevant form for  $k=2$ , as well as discussing zero modes and associated topics, equally vital for a full understanding (see 12 for a detailed review of these and related matters).

But here arises another problem. The cornerstone of much of the work outlined in the preceding chapters, the ADHM construction, while elegant and compact, does not provide an unconstrained parametrisation for the multi-instanton solutions with the full quota of variables, except for  $k=1$  and  $2$  and (though with complications)  $k=3$ .



Further, the very basis of the semi-classical expansion - expanding about a restricted set of pure instanton and anti-instanton configurations - though sampling all topological sectors of gauge equivalence classes of index  $k$ , is not self-obviously sufficient for a sensible theory (but see 13 ). What additional field configurations should be added, if any, remains unclear.

These problems notwithstanding, much progress has been made in the calculation of instanton determinants as part of the broader programme of semi-classical approximation to functional integrals; and the tantalising elegances and simplicities that arise in diverse but related fields hold out to the optimist the prospect of a deeper underlying structure one day being found.

Chapter 6: References

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