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AN ITERATIVE WIENER–HOPF METHOD FOR TRIANGULAR MATRIX FUNCTIONS WITH EXPONENTIAL FACTORS*

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Abstract. This paper introduces a new method for constructing approximate solutions to a class of Wiener–Hopf equations. This is particularly useful since exact solutions of this class of Wiener–Hopf equations currently cannot be obtained. The proposed method could be considered as a generalization of the "pole removal" technique and Schwarzschild's series. The criteria for convergence is proved. The error in the approximation is explicitly estimated, and by a sufficient number of iterations it could be made arbitrarily small. Typically only a few iterations are required for practical purposes. The theory is illustrated by numerical examples that demonstrate the advantages of the proposed procedure. This method was motivated by and successfully applied to problems in acoustics.

Key words. Wiener–Hopf equations, Riemann–Hilbert problem, iterative methods

AMS subject classifications. 45E10, 47A68, 35Q15

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1. Introduction.

Many boundary value problems in mathematical physics can be approached by the Wiener–Hopf method. Classically the Wiener–Hopf technique was used for linear PDEs with semi-infinite boundary conditions such as the Sommerfeld half-plane problem. This was later extended to boundary conditions on multiple semi-infinite lines [30]. Although the reduction to the Wiener–Hopf equation is still straightforward, finding a solution for the resulting equation has been challenging [6, 10, 12, 14, 16, 39]. This is due to the fact that the Wiener–Hopf factorization of a matrix (rather than a scalar) function is now needed. Hence the ability to solve such equations is crucial to extending the classical use of Wiener–Hopf techniques to more realistic and complicated settings. Also, such types of matrix Wiener–Hopf equations are associated with convolution-type operators on a finite interval [8] and arise in a number of applications [20, 44]. The aim of this paper is to develop an algorithmic iterative method of solution for some equations of this type.

More precisely, we construct an approximate solution of a Wiener–Hopf equation with triangular matrix functions containing exponential factors. The aim is to find functions $\Phi_-^{(0)}(\alpha)$, $\Phi_-^{(L)}(\alpha)$, $\Psi_+^{(0)}(\alpha)$, and $\Psi_+^{(L)}(\alpha)$ that are analytic in respective half-planes and satisfying the relationship

$$\begin{pmatrix}\Phi_-^{(0)}(\alpha)\\ \Phi_-^{(L)}(\alpha)\end{pmatrix}=\begin{pmatrix}A(\alpha) & B(\alpha)e^{i\alpha L}\\ C(\alpha)e^{-i\alpha L} & 0\end{pmatrix}\begin{pmatrix}\Psi_+^{(0)}(\alpha)\\ \Psi_+^{(L)}(\alpha)\end{pmatrix}+\begin{pmatrix}f_1(\alpha)\\ f_2(\alpha)\end{pmatrix} \tag{1}$$

on the strip $a\le\Im(\alpha)\le b$. The functions $A(\alpha)$, $B(\alpha)$ and $C(\alpha)$ are known and L is a positive constant. The conditions on the matrix functions are specified in section 2. The existence of such a factorization under certain assumptions was addressed in [31, p. 150] and [11].

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Wiener–Hopf equations of type (1) have been the topic of previous research, for example, in the case of meromorphic matrix entries [4, 5] and in the framework of almost periodic functions [22, 23]. Different factorizations of the matrix in (1) have been studied; for example, [9] considers the generalized factorization, which has weaker conditions on the factors. In [15] triangular matrix functions are considered but without the exponential off diagonal factors. However, currently no exact solution of (1) is known. Furthermore, the general question of constructive Wiener–Hopf factorization is widely open [30, 39].

We note that there is an additional difficulty in finding a factorization (see section 2.1) of the matrix in (1) because of the presence of analytic functions $e^{i\alpha L}$ and $e^{-i\alpha L}$, which have exponential growth in one of the half-planes. The first step of the procedure proposed here is a partial factorization that has at most polynomial growth in the respective half-planes. The classes for which this is a complete factorization are also discussed. The next step is the Wiener–Hopf additive splitting of the remainder term that hinders the application of Liouville's theorem. The additive Wiener–Hopf splitting is routine, unlike the multiplicative one, which is also utilized in other novel methods [33, 34]. After the application of the analytic continuation there are still some unknowns in the formula—those are approximated by an iterative procedure. The presence of the exponential terms speeds up convergence. At each step a scalar Wiener–Hopf equation is solved. The criteria for convergence is formulated and proved. It is shown that the iterations converge quickly to the exact solution in numerical examples.

The procedure can be summarized as follows (see section 3 for details):

1. A partial factorization with exponential factors in the desired half-planes.
2. Additive splitting of some terms.
3. Application of Liouville's theorem.
4. Iterative procedure to determine the remaining unknowns.

This procedure bypasses the need to construct a multiplicative matrix factorization. So, in particular, partial indices (which are known to be linked to stability [27, 31]) are not obtained. Instead the growth at infinity of certain terms plays a role. In this paper we will treat the base case with no growth at infinity. Some other cases will be treated in the forthcoming paper [29].

This work was motivated by certain problems in acoustics which are presented in [29] and briefly discussed in section 2.2. An equation of type (1) appears when there are different boundary conditions in physical space $(-\infty, 0)$, $(0, L)$, and (L, ∞). In the Fourier space this results in the presence of $e^{i\alpha L}$ and $e^{-i\alpha L}$ in the off-diagonal elements. The convergence and speed of convergence of the proposed iterative procedure depend on the distance L between the changes in boundary conditions. The other entries in the Wiener–Hopf matrix are linked to the Fourier transform of the known boundary conditions. The functions $\Phi_{\pm}^{(0)}(\alpha)$ and $\Psi_{\pm}^{(0)}(\alpha)$ are the unknown half-range Fourier transform and/or their derivatives. $f_1(\alpha)$ and $f_2(\alpha)$ are derived from the Fourier transform of the forcing, for example, the incoming acoustic wave.

Problems of the type described in the preceding paragraph have been studied extensively by an array of different methods. In connection to the proposed procedure there are two approaches which are relevant, the Schwarzschild's series and the "pole removal," which will be discussed in turn. In the original paper [40] Schwarzschild studied the diffraction of a normally incident plane wave by a slit of finite length in a perfectly conducting screen. This was done by considering the diffractions from the half-planes as a sequence of excitations from each other. This was later extended to

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near-normal incidence in [24] and to all angles in [19, 32]. Recently, the Schwarzschild series were examined in connection with other methods in [41, 42]. Schwarzschild's series relate to the more general framework of the geometric theory of diffraction [25] and the physical theory of diffraction [43, section 8.7.3]. What is common among all those approaches and the one proposed in this paper is the idea of solving parts of the problem and then bringing the parts together in an iterative manner. Some other iterative methods were also proposed, for example, in [46] and are different from the ones described above.

In the literature concerning applications of the Wiener–Hopf technique, one of the most widely used methods is the so-called "pole removal" [38, sections 4.4, 5.3] or "singularities matching" [13, section 4.4.2]. It has a severe limitation that certain functions have to be rational or meromorphic. One way to extend the use of this method is by employing rational approximation [2, 26], which was successfully used in [1, 3, 27, 44]. However, even with this extension the class of functions which can be solved is rather limited. In this paper we propose a different development of the pole removal technique for functions that have arbitrary singularities.

The structure of the paper is as follows. In section 2 the required classes of functions are introduced and their essential properties are listed. We also provide some motivation behind those Wiener–Hopf systems. In section 3 the proposed iterative procedure is described in detail, and its convergence is examined in section 4. Section 5 presents numerical results of two examples (graphically illustrated) to compare the iterative procedure to the exact solution in a variety of cases. Lastly we describe possible future work.

2. Preliminaries.

In order to formulate the problem (1) we have to specify the suitable class of functions for all the terms in the equation. We will also need some properties of this class of function denoted $\{\!\{a, b\}\!\}$, which will ensure we stay in the desired class of functions after each iteration. These will be discussed in the appendix. We will also briefly describe the motivation behind this method.

2.1. Wiener–Hopf factorization for functions in $\{\!\{a, b\}\!\}$.

We recall the additive and multiplicative scalar splitting of functions belonging to $\{\!\{a, b\}\!\}$ (see the appendix). A convention will be used to distinguish the additive and the multiplicative splitting by using superscript and subscript notation, e.g., $F^{\pm}$ for additive and $K_{\pm}$ for multiplicative.

THEOREM 2.1 (additive splitting [17]). *On the real line a function $F(t) \in \{0\}$ is given. There exist two functions $F^{\pm}(z)$ analytic in the upper and lower half-planes, respectively, with boundary functions on the real line belonging to the classes $\{\!\{0, \infty\}\!\}$ and $\{\!\{-\infty, 0\}\!\}$ and satisfying*

$$F(t) = F^{+}(t) + F^{-}(t)$$

on the real line.

Next, the multiplicative splitting or factorization problem is examined. The *index* of a continuous nonzero function $K(t)$ on the real line is the winding number of the curve $(\mathrm{Re}\, K(t), \mathrm{Im}\, K(t))$, $t \in \mathbb{R}$.

THEOREM 2.2 (multiplicative splitting [17]). *Let a nonzero function $K(t)$ such that $K(t) - 1 \in \{0\}$ and $\mathrm{ind}\, K(t) = 0$ be given. There exist two functions $K_{\pm}(z)$ analytic in the upper and lower half-planes, respectively, with boundary functions $K_{\pm}(t) - 1$*

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on the real line belonging to the classes $\{\!\{0,\infty\}\!\}$ *and* $\{\!\{-\infty,0\}\!\}$*, satisfying*

$$K(t)=K_+(t)K_-(t),$$

on the real line.

Remark 2.3. Note that if the original function in Theorem 2.1 or Theorem 2.2 is from the class $\{\!\{a\}\!\}$, then the components belong to the classes $\{\!\{a,\infty\}\!\}$ and $\{\!\{-\infty,a\}\!\}$.

With these definitions we can specify the classes of functions in (1). The given functions $A(\alpha)-1$, $C(\alpha)-1$, $B(\alpha)-1$ are required to be in $\{\!\{a,b\}\!\}$ and $A(\alpha)$, $C(\alpha)$, $B(\alpha)$ to have zero index. We also need $C(\alpha)$ and $B(\alpha)$ to have no zeros on the strip—this corresponds to the determinant of the original Wiener–Hopf matrix (1) being nonzero. We will also require that $A(\alpha)$ has no zeros on the strip—this corresponds to the subproblem of (1) (when L is very large) being nondegenerate. This will also ensure that we are dividing by nonzero functions. The "forcing terms" $f_1(\alpha)$ and $f_2(\alpha)$ should be in $\{\!\{a,b\}\!\}$. We look for $\Phi_-^{(0)}(\alpha)$ and $\Phi_-^{(L)}(\alpha)$ in $\{\!\{-\infty,b\}\!\}$ and $\Psi_+^{(0)}(\alpha)$ and $\Psi_+^{(L)}(\alpha)$ in $\{\!\{a,\infty\}\!\}$.

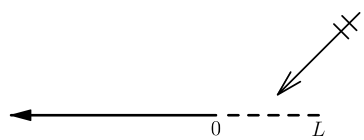

FIG. 1. *A plane-wave scattering by a rigid half-line plate* $(-\infty,0)$ *and a poroelastic edge* $(0,L)$.

2.2. Motivation. The initial motivation for this work came from the following acoustics problem: investigate the effect of a finite poroelastic trailing edge on noise production [7, 21]. The situation is modeled with a plane-wave scattering by a rigid half-line plate $(-\infty,0)$ and a poroelastic edge $(0,L)$ with the poroelastic to rigid plate transition at $x=0$ (see Figure 1). The matrix Wiener–Hopf problem is obtained from consideration of the Fourier transform with respect to the ends of the poroelastic plate at $x=0$ and $x=L$. Define the half-range and full-range Fourier transform with respect to $x=0$:

$$\Phi(\alpha,y)=\int_{-\infty}^{0}\phi(\xi,y)e^{i\alpha\xi}d\xi+\int_{0}^{\infty}\phi(\xi,y)e^{i\alpha\xi}d\xi, \tag{2}$$

$$=\Phi_-^{(0)}(\alpha,y)+\Phi_+^{(0)}(\alpha,y). \tag{3}$$

In the case when there is only one transition point $x=0$, the unknown functions would be $\Phi_-^{(0)}(\alpha,y)$, $\Phi_+^{(0)}(\alpha,y)$ (or their derivatives) and we would only need to solve a scalar Wiener–Hopf equation [38]. Since there is a change of boundary conditions at $x=L$ as well, we will also define the Fourier transforms with respect to the point $x=L$:

$$\Phi^{(L)}(\alpha,y)=\int_{-\infty}^{L}\phi(\xi,y)e^{i\alpha(\xi-L)}d\xi+\int_{L}^{\infty}\phi(\xi,y)e^{i\alpha(\xi-L)}d\xi, \tag{4}$$

$$=\Phi_-^{(L)}(\alpha,y)+\Phi_+^{(L)}(\alpha,y). \tag{5}$$

The relation between the transforms is $\Phi^{(L)}(\alpha,y)=\Phi(\alpha,y)e^{i\alpha L}$. The next step is to write down the relationship between different half-range transforms by using the

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boundary conditions [13]. These relations can be combined to form a matrix Wiener–Hopf equation. The resulting matrix, which motivated the present method, is (see [29] for a detailed discussion)

$$\begin{pmatrix} \Phi_-^{(0)}(\alpha) \\ \Phi_-^{(L)}(\alpha) \end{pmatrix} = -\begin{pmatrix} \frac{1-\gamma(\alpha)P}{\gamma(\alpha)} & Pe^{i\alpha L} \\ \frac{1}{\gamma(\alpha)}e^{-i\alpha L} & 0 \end{pmatrix} \begin{pmatrix} \Phi_+^{'(0)}(\alpha) \\ \Phi_+^{'(L)}(\alpha) \end{pmatrix} + \begin{pmatrix} f_1(\alpha) \\ f_2(\alpha) \end{pmatrix},$$

where $\gamma(\alpha) = \sqrt{\alpha^2 - k_0^2}$, k_0 is the acoustic wave number and, in the simplest case, P is a constant. The exponentials are due to the boundary conditions since the half-range Fourier transform with respect to different points is needed.

We note that on the left-hand side the unknown functions are minus half-transforms $\Phi_-^{(0)}(\alpha)$ and $\Phi_-^{(L)}(\alpha)$, and on the right-hand side they are the derivatives of the plus half-transforms, which in this paper are denoted by $\Psi_+^{(0)}(\alpha)$ and $\Psi_+^{(L)}(\alpha)$.

This problem is not treated in this paper further since the purpose of this paper is to present the iterative procedure in the simplest and most general case in order to make it easy to apply to many situations. In [29] it is found that convergence occurs for all L as long as the porosity is not very large (which is needed for the model to be relevant). Physically this is linked to the fact that the change in the boundary conditions between a rigid and a not very porous plate is not that large, but nevertheless the noise reduction is significant at low frequencies.

The paper [29] also discusses other examples, e.g., the diffraction with a finite rigid plate of length L, leading to the following Wiener–Hopf equation:

$$\begin{pmatrix} \Phi_-^{'(0)}(\alpha) \\ \Phi_-^{(L)}(\alpha) \end{pmatrix} = -\begin{pmatrix} \gamma(\alpha) & e^{i\alpha L} \\ -e^{-i\alpha L} & 0 \end{pmatrix} \begin{pmatrix} \Phi_+^{(0)}(\alpha) \\ \Phi_+^{'(L)}(\alpha) \end{pmatrix} - \begin{pmatrix} g_1(\alpha) \\ 0 \end{pmatrix}.$$

The advantage of considering the following example is that, although the exact matrix factorization cannot be constructed, the results of the new iterative procedure can be compared to the exact solution obtained by other methods (such as Mathieu functions). It is found that the procedure converges unless L is very small, and when it does converge it is very close to the exact solution in only 2–3 iterations.

3. Iterative Wiener–Hopf factorization.

The most characteristic aspect of the Wiener–Hopf method is the application of Liouville's theorem in order to obtain two separate equations from one equation. In order for Liouville's theorem to be used, two conditions have to be satisfied: analyticity and (at most) polynomial growth at infinity. These two conditions will be treated here in turn. First, a partial factorization is considered that has the exponential functions in the right place and some of the required analyticity, that is,

$$\begin{pmatrix} \frac{-e^{-i\alpha L}}{B_-(\alpha)} & \frac{A(\alpha)}{C(\alpha)B_-(\alpha)} \\ \frac{1}{B_-(\alpha)} & 0 \end{pmatrix} \begin{pmatrix} \Phi_-^{(0)}(\alpha) \\ \Phi_-^{(L)}(\alpha) \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -B_+(\alpha) \\ \frac{A(\alpha)}{B_-(\alpha)} & B_+(\alpha)e^{i\alpha L} \end{pmatrix} \begin{pmatrix} \Psi_+^{(0)}(\alpha) \\ \Psi_+^{(L)}(\alpha) \end{pmatrix} + \begin{pmatrix} f_3 \\ f_4 \end{pmatrix}, \tag{6}$$

on a strip $a \leq \Im(\alpha) \leq b$, where $f_3(\alpha) = \frac{-e^{-i\alpha L}}{B_-(\alpha)} f_1 + \frac{A(\alpha)}{C(\alpha)B_-(\alpha)} f_2$ and $f_4(\alpha) = \frac{f_1}{B_-(\alpha)}$.

Note that $f_3(\alpha)$ and $f_4(\alpha)$ are still in $\{\!\{a, b\}\!\}$ and that $B(\alpha)$ and $C(\alpha)$ have no zeros on the strip; see section 2.1. There are two cases which would allow us to solve

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the above equation exactly. The first is when $\frac{A(\alpha)}{C(\alpha)B_-(\alpha)}$ is in $\{\!\{-\infty, b\}\!\}$ and $\frac{A(\alpha)}{B_-(\alpha)}$ is in $\{\!\{a, \infty\}\!\}$. Then the matrix factorization has been already achieved in (6). In particular, this is true for matrices that have the form

$$\begin{pmatrix} kB_-(\alpha)C_+(\alpha) & B(\alpha)e^{i\alpha L} \\ C(\alpha)e^{-i\alpha L} & 0 \end{pmatrix},$$

where k is a constant. The second important case is when $\left(\frac{A(\alpha)}{C(\alpha)B_-(\alpha)}\right)^+$ and $\left(\frac{A(\alpha)}{B_-(\alpha)}\right)^-$ are rational functions. Then the pole removal method [38, sections 4.4, 5.3], [13, section 4.4.2] could be employed to obtain the factorization.

In the generic case we start with the partial factorization (6) and then use the additive Wiener–Hopf splitting. We present the detailed description. Equation (6) can be rearranged as

$$\begin{pmatrix} \frac{-e^{-i\alpha L}}{B_-(\alpha)} & \left(\frac{A(\alpha)}{C(\alpha)B_-(\alpha)}\right)^- \\ \frac{1}{B_-(\alpha)} & 0 \end{pmatrix} \begin{pmatrix} \Phi_-^{(0)}(\alpha) \\ \Phi_-^{(L)}(\alpha) \end{pmatrix} + \begin{pmatrix} \left(\frac{A(\alpha)}{C(\alpha)B_-(\alpha)}\right)^+ \Phi_-^{(L)}(\alpha) \\ -\left(\frac{A(\alpha)}{B_-(\alpha)}\right)^- \Psi_+^{(0)}(\alpha) \end{pmatrix} \tag{7}$$

$$= \begin{pmatrix} 0 & -B_+(\alpha) \\ \left(\frac{A(\alpha)}{B_-(\alpha)}\right)^+ & B_+(\alpha)e^{i\alpha L} \end{pmatrix} \begin{pmatrix} \Psi_+^{(0)}(\alpha) \\ \Psi_+^{(L)}(\alpha) \end{pmatrix} + \begin{pmatrix} f_3 \\ f_4 \end{pmatrix} \tag{8}$$

on a strip $a \leq \Im(\alpha) \leq b$. Recall that the additive splitting is denoted by superscripts $\pm$ and the multiplicative splitting by subscripts $\pm$. As the next step we make the additive splittings of the second term of (7)—which are possible since it is in the class $\{\!\{a, b\}\!\}$—in the same way as for the second term of (8). Then, Liouville's theorem could be applied because the exponential functions are in the correct place and all the functions have the desired analyticity. Thus, we can apply the Wiener–Hopf procedure as usual, and the four equations (defined for all α in the complex plane) then become

$$\begin{aligned} \frac{-e^{-i\alpha L}}{B_-}\Phi_-^{(0)} + \left(\frac{A}{CB_-}\right)^- \Phi_-^{(L)} + \left(\left(\frac{A}{CB_-}\right)^+ \Phi_-^{(L)}\right)^- - f_3^- &= 0, \\ -B_+\Psi_+^{(L)} - \left(\left(\frac{A}{CB_-}\right)^+ \Phi_-^{(L)}\right)^+ + f_3^+ &= 0, \\ \frac{1}{B_-}\Phi_-^{(0)} - \left(\left(\frac{A}{B_-}\right)^- \Psi_+^{(0)}\right)^- - f_4^- &= 0, \\ \left(\frac{A}{B_-}\right)^+ \Psi_+^{(0)} + B_+e^{i\alpha L}\Psi_+^{(L)} + \left(\left(\frac{A}{B_-}\right)^- \Psi_+^{(0)}\right)^+ + f_4^+ &= 0. \end{aligned}$$

Note that Liouville's theorem is applied before any approximations are made. The

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four equations can be rearranged as

$$\left(\frac{A}{CB_-}\right)^-\Phi_-^{(L)} = f_3^- - \left(\left(\frac{A}{CB_-}\right)^+\Phi_-^{(L)}\right)^- + \frac{e^{-i\alpha L}}{B_-}\Phi_-^{(0)}, \tag{9}$$

$$B_+\Psi_+^{(L)} = f_3^+ - \left(\left(\frac{A}{CB_-}\right)^+\Phi_-^{(L)}\right)^+, \tag{10}$$

$$-\left(\frac{A}{B_-}\right)^+\Psi_+^{(0)} = \left(\left(\frac{A}{B_-}\right)^-\Psi_+^{(0)}\right)^+ + f_4^+ + B_+e^{i\alpha L}\Psi_+^{(L)}, \tag{11}$$

$$\frac{\Phi_-^{(0)}}{B_-} = \left(\left(\frac{A}{B_-}\right)^-\Psi_+^{(0)}\right)^- + f_4^-. \tag{12}$$

When the equations are written in this form it is clear that if $\Phi_-^{(L)}$ is known, then it could be used to calculate $\Psi_+^{(L)}$, and this in turn produces $\Psi_+^{(0)}$ followed by the calculation of $\Phi_-^{(0)}$ and so on. To avoid cumbersome notation, we will label coefficients in (9)–(12) as $K_i^{\pm}$ and obtain the following system:

$$K_1^-\Phi_-^{(L)} = -\left(K_1^+\Phi_-^{(L)}\right)^- + f_3^- + K_4^-e^{-i\alpha L}\Phi_-^{(0)}, \tag{13}$$

$$K_2^+\Psi_+^{(L)} = -\left(K_1^+\Phi_-^{(L)}\right)^+ + f_3^+, \tag{14}$$

$$K_3^+\Psi_+^{(0)} = -\left(K_3^-\Psi_+^{(0)}\right)^+ + f_4^+ + K_2^+e^{i\alpha L}\Psi_+^{(L)}, \tag{15}$$

$$K_4^-\Phi_-^{(0)} = -\left(K_3^-\Psi_+^{(0)}\right)^- + f_4^-, \tag{16}$$

where $K_1 = \frac{A}{CB_-}$, $K_2 = B$, $K_3 = -\frac{A}{B_-}$, and $K_4 = \frac{1}{B}$ in the original notation. Using (14) and (16), we eliminate $\Psi_+^{(L)}$ and $\Phi_-^{(0)}$ from (13) and (15), respectively:

$$K_1^-\Phi_-^{(L)} = f_3^- - \left(K_1^+\Phi_-^{(L)}\right)^- + e^{-i\alpha L}\left(-\left(K_3^-\Psi_+^{(0)}\right)^- + f_4^-\right), \tag{17}$$

$$K_3^+\Psi_+^{(0)} = f_4^+ - \left(K_3^-\Psi_+^{(0)}\right)^+ + e^{i\alpha L}\left(-\left(K_1^+\Phi_-^{(L)}\right)^+ + f_3^+\right). \tag{18}$$

The functions $\Psi_+^{(L)}$ and $\Phi_-^{(0)}$ can be found from (14) and (16) once (17) and (18) are solved.

So far the equations are exact, but in order to make progress an approximation will be used. In order to solve approximately we will describe an iterative procedure, where the nth iteration is denoted by $\Phi_-^{(L)n}$ and $\Psi_+^{(0)n}$. If $\Phi_-^{(L)n}$ is known, it could be substituted into (18) to calculate $\Psi_+^{(0)n+1}$, and then the function $\Psi_+^{(0)n+1}$ can be used in (17) to find $\Phi_-^{(L)n+1}$ and so on.

Hence, for this iterative procedure it is enough to choose an initial value of $\Phi_-^{(L)0}$. Since (17) can be considered on a horizontal line $\Im(\alpha) = a < 0$, on that line the term with $e^{-i\alpha L}$ will be small (especially for L large). This justifies neglecting the term with $e^{-i\alpha L}$ as a first approximation. Hence, to find $\Phi_-^{(L)0}$ the aim is to solve

$$K_1^-\Phi_-^{(L)0} = f_3^- - \left(K_1^+\Phi_-^{(L)0}\right)^-. \tag{19}$$

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The above equation can be rearranged as a scalar Wiener–Hopf equation in the following manner:

$$\left(K_1\Phi_-^{(L)0}\right)^- = f_3^-. \tag{20}$$

Introduce an unknown function D^+ defined by

$$D^+ = \left(K_1\Phi_-^{(L)0}\right)^+. \tag{21}$$

Then, combining (20) and (21) we obtain a Wiener–Hopf equation

$$K_1\Phi_-^{(L)0} = D^+ + f_3^-. \tag{22}$$

This equation has the form discussed in the appendix, so we will have $\Phi_-^{(L)0} \in \{\!\{-\infty, a\}\!\}$ as desired. The solution of this equation is

$$\Phi_-^{(L)0} = \frac{1}{(K_1)_-}\left(\frac{f_3^-}{(K_1)_+}\right)^-.$$

This can be taken as the initial approximation of the solution, which is used in our iterative procedure.

To compute the next step we will need to solve (18) for $\Psi_+^{(0)1}$ on a horizontal line $\Im(\alpha) = b > 0$. Note that $\Phi_-^{(L)1}$ is defined on this line and can be evaluated numerically. Exactly in the same manner as for (19), the solution of

$$K_3^+\Psi_+^{(0)1} = \left(-\left(K_3^-\Psi_+^{(0)1}\right)^+ + f_4^+\right) + e^{i\alpha L}\left(-\left(K_1^+\Phi_-^{(L)0}\right)^+ + f_3^+\right)$$

will lead to the scalar Wiener–Hopf equation now on the line $\Im(\alpha) = b$. It can be found that

$$\Psi_+^{(0)1} = \frac{1}{(K_3)_+}\left(\frac{f_4^+ + e^{i\alpha L}\left(-\left(K_1^+\Phi_-^{(L)0}\right)^+ + f_3^+\right)}{(K_3)_-}\right)^+. \tag{23}$$

It is easy to see that this formula holds for all iterations with a trivial change of $\Psi_+^{(0)1}$ to $\Psi_+^{(0)n}$ and $\Phi_-^{(L)0}$ to $\Phi_-^{(L)n-1}$. Similarly, the general recurrence formula for $\Phi_-^{(L)n}$ is

$$\Phi_-^{(L)n} = \frac{1}{(K_1)_-}\left(\frac{f_3^- + e^{-i\alpha L}\left(-\left(K_3^-\Psi_+^{(0)n}\right)^- + f_4^-\right)}{(K_1)_+}\right)^-. \tag{24}$$

The convergence of this procedure is examined in the next section. Numerical examples of this procedure are given in section 5 and are compared to exact solutions (which are known for these special cases).

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4. Convergence of the method. The convergence of iterations relies on consideration of (17) and (18) on different lines $\Im(\alpha) = a < 0$ and $\Im(\alpha) = b > 0$ within the strip of analyticity $a \leq \Im(a) \leq b$, in a similar way as in [28]. We will employ the notation

$$D_n^+ = \left(K_1^+(\Phi_-^{(L)n} - \Phi_-^{(L)n-1})\right)^+, \quad E_n^- = \left(K_3^-(\Psi_+^{(0)n} - \Psi_+^{(0)n-1})\right)^- .$$

From (23)–(24), the difference of the values of the function after $n+1$ and n times is

$$\Phi_-^{(L)n+1} - \Phi_-^{(L)n} = -\frac{\left(\frac{e^{-i\alpha L}E_{n+1}^-}{(K_1)_+}\right)^-}{(K_1)_-}, \quad \Psi_+^{(0)n+1} - \Psi_+^{(0)n} = -\frac{\left(\frac{e^{i\alpha L}D_n^+}{(K_3)_-}\right)^+}{(K_3)_+}. \tag{25}$$

Note that the forcing terms $f_i^\pm$ do not influence the convergence. In order to estimate the differences (25) in magnitude we need some inequalities for the Wiener–Hopf additive decomposition. This has been addressed in [26] for all L_p spaces; here we will need a very special case of that result, which has a particularly simple form. We will use the fact that if $F(t) = F^+(t) + F^-(t)$, as in Theorem 2.1, then

$$\|F^\pm\|_2 \leq \|F\|_2. \tag{26}$$

The same inequality can be observed from the fact that the map $F \to F_\pm$ in $L_2(\mathbb{R})$ is the Szegö orthoprojector with the norm 1. We will be looking at the L_2 norm on different lines within the strip of analyticity.

As well as employing (25) to assert convergence we will need to derive a relationship for E_{n+1}^- and E_n^-. This is done by a similar procedure to that used to derive (25) but considering the other unknown in the scalar Wiener–Hopf equation. In other words the expressions of the unknown plus and minus functions are obtained. This is done explicitly below. Write (17) and (18) as they are used in the nth iteration:

$$K_1^-\Phi_-^{(L)n} = f_3^- - \left(K_1^+\Phi_-^{(L)n}\right)^- + e^{-i\alpha L}\left(-\left(K_3^-\Psi_+^{(0)n}\right)^- + f_4^-\right), \tag{27}$$

$$K_3^+\Psi_+^{(0)n} = f_4^+ - \left(K_3^-\Psi_+^{(0)n}\right)^+ + e^{i\alpha L}\left(-\left(K_1^+\Phi_-^{(L)n-1}\right)^+ + f_3^+\right). \tag{28}$$

We can add $K_1^+\Phi_-^{(L)n}$ to both sides of (27) and $K_3^-\Psi_+^{(0)n}$ to both sides of (28) to obtain

$$K_1\Phi_-^{(L)n} = f_3^- + \left(K_1^+\Phi_-^{(L)n}\right)^+ + e^{-i\alpha L}\left(-\left(K_3^-\Psi_+^{(0)n}\right)^- + f_4^-\right), \tag{29}$$

$$K_3\Psi_+^{(0)n} = f_4^+ + \left(K_3^-\Psi_+^{(0)n}\right)^- + e^{i\alpha L}\left(-\left(K_1^+\Phi_-^{(L)n-1}\right)^+ + f_3^+\right). \tag{30}$$

If we consider the difference between the consecutive iterations, we derive

$$K_1(\Phi_-^{(L)n} - \Phi_-^{(L)n-1}) = D_n^+ + e^{-i\alpha L}E_n^-, \tag{31}$$

$$K_3(\Psi_+^{(0)n+1} - \Psi_+^{(0)n}) = E_{n+1}^- + e^{i\alpha L}D_n^+. \tag{32}$$

This makes the coupling between the equations explicit. Note that

$$D_n^+ = (K_1)_+\left(\frac{e^{-i\alpha L}E_n^-}{(K_1)_+}\right)^+, \quad E_{n+1}^- = (K_3)_-\left(\frac{e^{i\alpha L}D_n^+}{(K_3)_-}\right)^-,$$

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where D_n^+ can be derived by solving the scalar Wiener–Hopf equation (31) with the unknown minus function $\Phi_-^{(L)n} - \Phi_-^{(L)n-1}$ and unknown plus function D_n^+, and $e^{-i\alpha L}E_n^-$ is assumed to be known from the previous iterations. Similarly, E_{n+1}^- is obtained by solving the scalar Wiener–Hopf equation (32).

We will obtain estimates of the size of $\|D_n^+\|_2^a$. Let $\max_{x\in\mathbb{R}} |(K_1)_+(x+ai)| = d_1$ and $\max_{x\in\mathbb{R}} |(K_1)_+^{-1}(x+ai)e^{-iL(x+ai)}| = \epsilon_1$. Note that K_1 is bounded since it is in $L_2(\mathbb{R}) \cap$ Hölder, and note that K_1 is nonzero. Then using (26) we have

$$\|D_n^+\|_2^a \le d_1 \left\| \left(\frac{e^{-i\alpha L}E_n^-}{(K_1)_+} \right)^+ \right\|_2^a \le d_1 \left\| \frac{e^{-i\alpha L}E_n^-}{(K_1)_+} \right\|_2^a \le d_1\epsilon_1 \|E_n^-\|_2^a. \tag{33}$$

Similarly, defining $\max_{x\in\mathbb{R}}(K_3)_-(x+bi) = d_2$ and $\max_{x\in\mathbb{R}} e^{iL(x+bi)}(K_3)_-^{-1}(x+bi) = \epsilon_2$, we obtain

$$\|E_{n+1}^-\|_2^b \le d_2 \left\| \left(\frac{e^{i\alpha L}D_n^+}{(K_3)_-} \right)^- \right\|_2^b \le d_2 \left\| \left(\frac{e^{i\alpha L}D_n^+}{(K_3)_-} \right) \right\|_2^a \le d_2\epsilon_2 \|D_n^+\|_2^a. \tag{34}$$

Next, we note that the following is true:

$$\|E_{n+1}^-\|_2^a \le \|E_{n+1}^-\|_2^b, \qquad \|D_n^+\|_2^b \le \|D_n^+\|_2^a. \tag{35}$$

This is intuitively clear since E_{n+1}^- is further from singularities on the line $\Im(\alpha) = a$ than on $\Im(\alpha) = b$ and the opposite for D_n^+. The inequalities follow from the Poisson formula for the real line [37, Cor 6.4.1] and Hölder inequality. Combining (35) with inequalities (33) and (34) we obtain the key result for demonstrating convergence:

$$\|E_{n+1}^-\|_2^a \le d_1 d_2 \epsilon_2 \epsilon_1 \|E_n^-\|_2^a, \qquad \|D_{n+1}^+\|_2^b \le d_1 d_2 \epsilon_2 \epsilon_1 \|D_n^+\|_2^a. \tag{36}$$

The convergence of the procedure is shown in the next theorem.

Theorem 4.1. *For sufficiently large L there exists a constant $q < 1$ such that*

$$\|\Phi_-^{(L)n+1} - \Phi_-^{(L)n}\|_2^a \le q \|\Phi_-^{(L)n} - \Phi_-^{(L)n-1}\|_2^a$$

for all n, and hence the error at the nth iteration is

$$\|\Phi_-^{(L)n} - \Phi_-^{(L)}\|_2^a \le \frac{q^n}{1-q} \|\Phi_-^{(L)0} - \Phi_-^{(L)1}\|_2^a.$$

Analogous statements are true for $\Psi_+^{(0)n}$.

Proof. Consider first the formula in (25) on the line $x+ai$. Let $\max_{x\in\mathbb{R}} |(K_1)_-^{-1}(x+ai)| = c_1$, and note that K_1 is nonzero on the strip so the constant is well defined. Then, using (26), we have

$$\|\Phi_-^{(L)n+1} - \Phi_-^{(L)n}\|_2^a \le c_1 \left\| \left(\frac{e^{-i\alpha L}E_{n+1}^-}{(K_1)_+} \right)^- \right\|_2^a \le c_1\epsilon_1 \|E_{n+1}^-\|_2^a. \tag{37}$$

This implies that

$$\|\Phi_-^{(L)n+1} - \Phi_-^{(L)n}\|_2^a \le \frac{\|E_{n+1}^-\|_2^a}{\|E_n^-\|_2^a} \|\Phi_-^{(L)n} - \Phi_-^{(L)n-1}\|_2^a. \tag{38}$$

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Hence we obtained the desired result with $q = d_1 d_2 \epsilon_2 \epsilon_1$ using (36). The contraction mapping theorem can be applied if $q < 1$ to show that the iterations converge to the exact solution $\Phi_-^{(L)}$. Note that we can make ϵ_1 and ϵ_2 (and hence q) arbitrarily small by taking L sufficiently large.

Similarly, we consider the second equation in (25) on the line $x + bi$. In the same manner we derive

$$\|\Psi_+^{(0)n+1} - \Psi_+^{(0)n}\|_2^b \leq q \|\Psi_+^{(0)n} - \Psi_+^{(0)n-1}\|_2^b. \tag{39}$$

□

Note that the procedure converges for all possible initial functions. But the accuracy of the initial step determines how quickly the small desired error will be achieved.

5. Examples. In this section the method proposed in this paper will be illustrated numerically. Two examples will be considered. In both cases the exact solutions are known and will be compared with the outcomes of the iterative procedure. The first example is of the type (1) and is chosen to be as simple as possible. The second example has been studied by other researchers in connection with integral equations. The resulting Wiener–Hopf system is more general than (1), but it can also be reduced to solving equations similar to (17) and (18), and hence the derivations in this paper apply. A more involved example of application of the proposed method is found in [29] and is used in the setting of acoustics (see section 2.2 for more details). Note that in the latter case no exact solution is known.

Example 5.1. The first numerical example will be the simplest possible in order to illustrate the theory. Consider (1) with

$$\begin{pmatrix} \frac{0.5}{(\alpha - i\lambda)(\alpha + i\lambda)} + 1 & B_+(\alpha) e^{i\alpha L} \\ e^{-i\alpha L} & 0 \end{pmatrix},$$

where λ is a complex parameter and $B_+(\alpha)$ is an arbitrary function satisfying the conditions stated in section 2.1. For the forcing terms take

$$f_4^-(\alpha) = f_3^-(\alpha) = \frac{1}{\alpha - i} \quad \text{and} \quad f_4^+(\alpha) = f_3^+(\alpha) = \frac{1}{\alpha + 2i}.$$

In this example $a = b = \Re(\lambda)$ and so we require $\Re(\lambda) > 0$. We have that

$$K_3(\alpha) = K_1(\alpha) = \frac{0.5}{(\alpha - i\lambda)(\alpha + i\lambda)} + 1.$$

Substituting into (17) and (18) and using partial fractions gives

$$K_1(\alpha)\Phi_-^{(L)}(\alpha) = \frac{1}{\alpha - i} + \frac{k_2}{\alpha + i\lambda} + e^{-i\alpha L}\left(\frac{k_1}{\alpha - i\lambda} + \frac{1}{\alpha - i}\right), \tag{40}$$

$$K_1(\alpha)\Psi_+^{(0)}(\alpha) = \frac{1}{\alpha + 2i} - \frac{k_1}{\alpha - i\lambda} + e^{i\alpha L}\left(\frac{1}{\alpha + 2i} - \frac{k_2}{\alpha + i\lambda}\right), \tag{41}$$

and we are required to find $\Phi_-^{(L)}(\alpha)$ and $\Psi_+^{(0)}(\alpha)$. In this simple case we can solve these coupled equations exactly and find constants k_1 and k_2 explicitly. The constants can be found by removing the possible singularities at $\pm i\lambda$ of $\Phi_-^{(L)}(\alpha)$ and $\Psi_+^{(0)}(\alpha)$ to ensure correct analyticity. Define

$$p = e^{-\lambda L}, \quad c = 2i\lambda f_1^-(-i\lambda), \quad \text{and} \quad d = 2i\lambda f_1^+(i\lambda);$$

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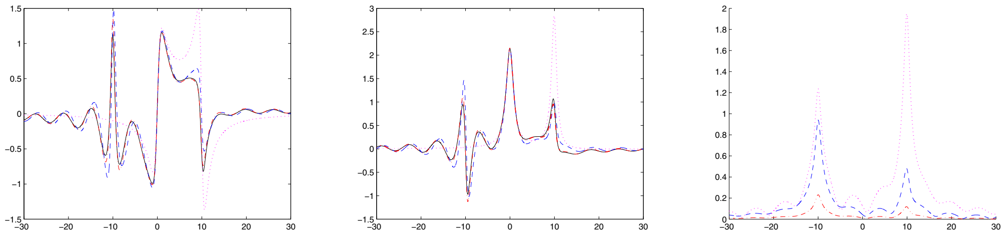

FIG. 2. *(Example 5.1.) The real part (left figure) and imaginary part (middle figure) of $\Phi_-^{(L)}$ (solid black line), $\Phi_-^{(L)0}$ (dotted line), $\Phi_-^{(L)1}$ (blue dashed line), and $\Phi_-^{(L)2}$ (red dashed line). The right figure shows the decrease in the absolute value of $\Phi_-^{(L)} - \Phi_-^{(L)n}$. The parameters are $\lambda = 0.7+10i$ and $L = 1$. (See online version for color.)*

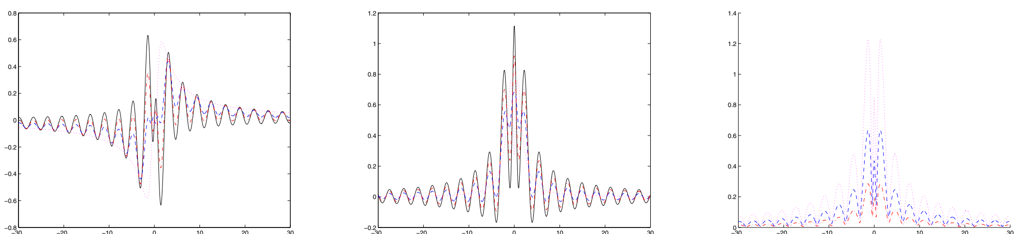

FIG. 3. *(Example 5.1.) The real part (left figure) and imaginary part (middle figure) of $\Phi_-^{(L)}$ (solid black line), $\Phi_-^{(L)0}$ (dotted line), $\Phi_-^{(L)1}$ (blue dashed line), and $\Phi_-^{(L)2}$ (red dashed line). The right figure shows the decrease in the absolute value of $\Phi_-^{(L)} - \Phi_-^{(L)n}$. The parameters are $\lambda = 0.2$ and $L = 2$. (See online version for color.)*

then the constants are given by

$$k_1 = \frac{d - pc}{1 - p} \quad \text{and} \quad k_2 = \frac{c - pd}{1 - p}.$$

Next, we will solve these coupled Wiener–Hopf equations using the iterative procedure described in section 3. The first step is to neglect the term with $e^{-i\alpha L}$ in (40) and hence uncouple the equations and obtain a value for $k_2^{(0)}$. This value is then substituted into (41) to obtain $k_1^{(0)}$. The iterative procedure leads to the values

$$k_1^{(n)} = d + pd - pk_2^{(n)}, \quad k_2^{(n)} = c + pc - pk_1^{(n-1)}, \quad \text{with} \quad k_2^{(0)} = c.$$

This converges to the actual solution as long as $p < 1$. In most cases the convergence is very fast and the line of the first iteration is indistinguishable from the approximate solution. In the cases when the convergence is slow (small L and small $\Re(\lambda)$) the first iteration still retains some features of the solution; see Figures 2 and 3. The first guess has the correct overall shape, and all the consecutive iterations have the maxima and the minima in the right places even when the magnitude is quite different. Also note that even in the cases of slow convergence, the second iteration is already close to the actual solution.

In this special case it is possible to say more about the convergence of the solution

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compared to the general case (section 4). It is easy to see that

$$\Phi_{-}^{(L)n+1} - \Phi_{-}^{(L)n} = \frac{1}{K_1}\left(\frac{k_2^{(n+1)} - k_2^{(n)}}{\alpha + i\lambda} + e^{-i\alpha L}\frac{k_1^{(n+1)} - k_1^{(n)}}{\alpha - i\lambda}\right).$$

Hence,

$$\Phi_{-}^{(L)n+1} - \Phi_{-}^{(L)n} = \frac{(k_2^{(n+1)} - k_2^{(n)})}{K_1}\left(\frac{1}{\alpha + i\lambda} + e^{-i\alpha L}\frac{-p}{\alpha - i\lambda}\right).$$

This means that

$$\|\Phi_{-}^{(L)n+1} - \Phi_{-}^{(L)n}\|_2 \le p^2 \|\Phi_{-}^{(L)n} - \Phi_{-}^{(L)n-1}\|_2.$$

This can be verified numerically and is illustrated in Figure 4.

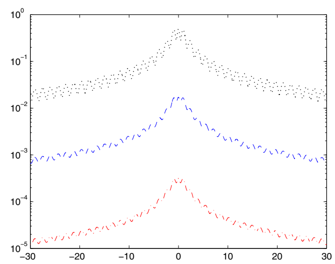

FIG. 4. *(Example 5.1.) The decrease in the absolute value of $\Phi_{-}^{(L)} - \Phi_{-}^{(L)n}$ for $n = 0, 1, 2$. The parameters are $\lambda = 0.5$ and $L = 4$.*

Example 5.2. The next example is more complicated and arises from an integral equation. The ability to solve integral equations is important in many applications [36, 45]. Consider the (one-sided) integral equation

$$u(x) = \lambda \int_0^\infty k(x - t)u(t)dt + f(x), \quad 0 < x < \infty,$$

with the matrix kernel given by

$$k(x) = \begin{pmatrix} e^{-|x|} & e^{-|x-L|} \\ e^{-|x+L|} & e^{-|x|} \end{pmatrix},$$

with $f(x)$ a forcing function, λ, L real parameters, and $u(x) = (u^{(1)}(x), u^{(2)}(x))^T$ to be determined. This system was considered in [20] and more recently in [5]. This integral equation can be reduced to a Wiener–Hopf equation by extending the range of x to the whole real line and applying the Fourier transform. The resulting Wiener–Hopf equation is of a more general type than (1). It has been shown [5] that the solution could be reduced to finding two constants C_1 and C_2 by the Wiener–Hopf method. In this example it is possible to obtain the exact solution, which provides a good way to test the ideas that are introduced in this paper. Once the solution to the Wiener–Hopf equation is obtained, the inverse Fourier transform will provide the solution to the integral equation. First we will need to define some functions. Let us

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take two constants $\lambda_0 = \sqrt{1-2\lambda}$ and $\lambda_1 = \sqrt{1-4\lambda}$ for some $\lambda \in (-\infty, 0.25]$; then we define

$$\begin{aligned} M_-(\alpha) =& \frac{\alpha - i\lambda_1}{\alpha - i\lambda_0}, \quad M_+(\alpha) = \frac{\alpha + i\lambda_0}{\alpha + i\lambda_1}, \\ K_-(\alpha) =& \frac{\alpha - i\lambda_0}{\alpha - i}, \quad K_+(\alpha) = \frac{\alpha + i}{\alpha + i\lambda_0}. \end{aligned}$$

The forcing is taken as $F_1^+(\alpha) = \frac{1}{\alpha - 2i}$ and $F_2^+ = 0$, and we use the following additive splittings

$$\begin{aligned} L_1^+(\alpha) - L_1^-(\alpha) &= \frac{\alpha - i}{(\alpha - i\lambda_0)(\alpha - 2i)}, \\ L_2^+(\alpha) - L_2^-(\alpha) &= \frac{2\lambda e^{-i\alpha L}}{(\alpha - 2i)(\alpha + i\lambda_0)(\alpha - i\lambda_1)}. \end{aligned}$$

The Wiener–Hopf method reduces the solution to equations similar to (17) and (18). The unknown functions $U_-^{(2)}(\alpha)$ and $U_+^{(1)}(\alpha)$ are the half-range Fourier transforms (3) of $u^{(2)}(\alpha)$ and $u^{(1)}(\alpha)$. They are given by

$$\begin{aligned} U_-^{(2)}(\alpha) =& M_-(\alpha)\left(L_2^-(\alpha) + \frac{C_2}{\alpha + i\lambda_0} + \frac{2\lambda K_-(\alpha)e^{-i\alpha L}}{(\alpha + i\lambda_1)(\alpha - i\lambda_0)}\left(L_1^-(\alpha) + \frac{C_1}{\alpha - i\lambda_0}\right)\right), \\ U_+^{(1)}(\alpha) =& K_+(\alpha)\left(L_1^+(\alpha) + \frac{C_1}{\alpha - i\lambda_0} + \frac{2\lambda M_+(\alpha)e^{i\alpha L}}{(\alpha + i)(\alpha - i\lambda_0)}\left(L_2^+(\alpha) + \frac{C_2}{\alpha + i\lambda_0}\right)\right). \end{aligned}$$

The constants can be found explicitly as

$$C_1 = \frac{d_1 + d_2 b}{b^2 + 1}, \qquad C_2 = \frac{d_2 - d_1 b}{b^2 + 1},$$

where

$$b = \frac{2\lambda e^{iL\lambda_0}}{(\lambda_0 + 1)(\lambda_0 + \lambda_1)}, \qquad d_1 = 2ib\lambda_0 L_2^+(i\lambda_0), \qquad d_2 = 2ib\lambda_0 L_1^-(-i\lambda_0).$$

By employing the iterative procedure we obtain

$$\begin{aligned} C_1^{(0)} &= d_1, & C_2^{(1)} &= d_2 - d_1 b, \\ C_1^{(1)} &= d_1 + d_2 b - d_1 b^2, & C_2^{(1)} &= d_2 - d_1 b - d_2 b^2 + d_1 b^3. \\ C_1^{(2)} &= d_1 + d_2 b - d_1 b^2 - d_2 b^3 + d_1 b^4. \end{aligned}$$

In fact, each iteration adds two more terms in the Taylor expansion of the constants. For example, for $\lambda = 0.1$ and $L = 0.0001$ the maximum error for $U_-^{(2)0}$ is 10^{-4} and for $U_-^{(2)1}$ is 10^{-6}. Note that the convergence speed of iterations depends on b^2, in the same way as in the previous example. It fact, it is small for all values of λ and L and for $b^2 \le 0.18$. This shows that in this example the convergence is extremely fast for all values of the parameters. Another factor, which influences the error of the iterative solution, is suitability of the initial value. For example, with $\lambda = -15$ and $L = 0.04$ the initial guess $U_+^{(1)0}$ is very bad; see the red dotted line in Figure 5. But since $b^2 = 0.0747$, even the first iteration $U_+^{(1)0}$ is already very accurate; see the blue dashed line in Figure 5.

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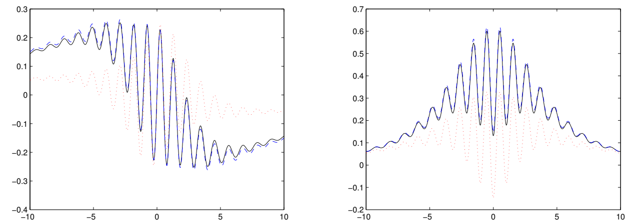

FIG. 5. *(Example 5.2.) The real part (left figure) and imaginary part (right figure) of* $U_+^{(1)}$ *(solid black line),* $U_+^{(1)0}$ *(red dotted line), and* $U_+^{(1)1}$ *(blue dashed line). The parameters are* $\lambda = -15$ *and* $L = 0.04$. *(See online version for color.)*

6. Conclusion. The Wiener–Hopf method is a powerful tool for solving boundary value problems and has been applied in an impressive array of situations. In the case of scalar Wiener–Hopf equations the solution is algorithmic. In the matrix case, the matrix Wiener–Hopf factorization cannot, in general, be obtained and hinders the use of the method. The present paper presents an algorithmic way of bypassing this step for a class of matrix functions (1). Only scalar Wiener–Hopf splittings are used in an iterative procedure. We provide the conditions for the convergence of iterations to the exact solution. This also enables us to estimate the error at each iteration. The numerical examples show that in most cases only a few iterations are required.

It is clear that this method could be applied to a wider class of Wiener–Hopf systems than (1). For instance, the matrix in Example 5.2 is not triangular. However, it still can be solved using our methods. Further work could be done to extend this method to nontriangular matrix functions with exponential factors.

This method has been motivated by applications and was already used in [29]. There is scope for using this method in a variety of boundary value problems with finite geometries or more than one transition point in the boundary conditions. Possible applications could be in different areas such as electromagnetism [13], fracture mechanics [35], and economics [18]. It has also been shown in numerical Example 5.2 that some integral equations can also be solved, which opens a large scope of applications.

Appendix. In this appendix we will outline the definition and properties of the class of functions $\{\!\{[a,b]\}\!\}$. The Hölder continuous functions on the compactified real line are defined as functions $F(x)$, which are Hölder continuous at all points of the real line including "infinity." A Hölder continuous function on the real line produces well-defined boundary values $G_\pm(t) = \lim_{y\searrow 0} G(t \pm iy)$ of its Cauchy-type integral $G(t+iy)$ at every point t of the compactified real line [36]. The class $L_2(\mathbb{R})$ is also very useful since the Fourier transform is an isometry of $L_2(\mathbb{R})$ due to Plancherel's theorem. Thus, the intersection [17, section 1.2]

$$\{\!\{[0]\}\!\} = L_2(\mathbb{R}) \cap \text{ Hölder}$$

turns out to be convenient for Wiener–Hopf problems. The preimage of $\{\!\{[0]\}\!\}$ under the Fourier transform is denoted $\{[0]\}$.

Given a function $f \in \{[0]\}$ on the real line we can define the splitting

$$f_+(t) = \begin{cases} f(t) & \text{if } t > 0, \\ 0 & \text{if } t < 0, \end{cases} \qquad f_-(t) = \begin{cases} 0 & \text{if } t > 0, \\ -f(t) & \text{if } t < 0. \end{cases} \tag{42}$$

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Using this splitting, we define the class $\{\![0,\infty]\!\}$ to contain functions $f_+(t)$ and the class $\{\![-\infty,0]\!\}$ to contain functions $f_-(t)$ for all $f \in \{\![0]\!\}$.

In this paper we will need to refer to functions that are analytic on strips or (shifted) half-planes. Following [17, section 13], we define $f \in \{\![a]\!\}$ if $e^{-ax}f \in \{\![0]\!\}$, that is, a shift in the Fourier space. Finally, $f = f_+ + f_- \in \{\![a,b]\!\}$ if $f_+ \in \{\![a]\!\}$ and $f_- \in \{\![b]\!\}$. From the definition of f_+ and f_- it is clear that also $f_+ \in \{\![a,\infty]\!\}$ and $f_- \in \{\![-\infty,b]\!\}$.

The Fourier transform of functions in the class $\{\![a,b]\!\}$ is denoted $\{\!\{\![a,b]\!\}\!\}$ (including the case $a=-\infty$ and/or $b=\infty$). The celebrated Paley–Wiener theorem states the following:

$$F_+(z) \in \{\!\{\![a,\infty]\!\}\!\} \Longrightarrow F_+(z) \text{ analytic in } \mathrm{Im} z > a, \tag{43}$$

$$F_-(z) \in \{\!\{\![-\infty,b]\!\}\!\} \Longrightarrow F_-(z) \text{ analytic in } \mathrm{Im} z < b, \tag{44}$$

$$F(z) \in \{\!\{\![a,b]\!\}\!\} \Longrightarrow F(z) \text{ analytic in } a < \mathrm{Im}\, z < b. \tag{45}$$

For the remainder of the paper we will assume that $a < 0$ and $b > 0$; i.e., the strip encloses the real axis.

We briefly describe some of the properties of $\{\!\{\![a,b]\!\}\!\}$ that will ensure that the solution we seek is in the desired class. We will need the fact that if $G \in \{\!\{\![a,b]\!\}\!\}$, $F \in \{\!\{\![\alpha,\beta]\!\}\!\}$ ($a<b$, $\alpha<\beta$) and $H(z) = F(z)G(z)$. Then $H(z) \in \{\!\{\![\max(a,\alpha),\min(b,\beta)]\!\}\!\}$; i.e., $H(z)$ is analytic in the respective strip. This is a simple but still fundamental property for the rest of the derivation. Note that a similar statement does not hold for the class $L_2(\mathbb{R})$.

We will also use the fact that if $G-1 \in \{\!\{\![a,b]\!\}\!\}$, $\operatorname{ind} G(t) = 0$ and $G(t)$ nonzero on the strip $a \le \Im(\alpha) \le b$, then $G^{-1}-1 \in \{\!\{\![a,b]\!\}\!\}$.

Note that in the proposed method we only solve scalar Wiener–Hopf equations of a form to those detailed in [17, section 3.2]. That is, for a scalar equation on the real line

$$K(t)\Phi_-(t) = \Phi_+(t) + F(t), \tag{46}$$

we need that K is a nonzero function, such that $K(t)-1 \in \{\![0]\!\}$ and $\operatorname{ind} K(t) = 0$. If also $F(t) \in \{\![0]\!\}$, then we can find Φ_+ and Φ_- in $\{\!\{\![0,\infty]\!\}\!\}$ and $\{\!\{\![-\infty,0]\!\}\!\}$ on the real axis. We will need the extension of those results for the case when (46) holds on a strip $a \le \Im(\alpha) \le b$. This has been obtained in [28].

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