Geometric Algebra and its Application to Mathematical Physics

Chris J. L. Doran
Sidney Sussex College

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Summary

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MRAO, Cavendish Laboratory, Madingley Road, Cambridge.

Clifford algebras have been studied for many years and their algebraic properties are well known. In particular, all Clifford algebras have been classified as matrix algebras over one of the three division algebras. But Clifford Algebras are far more interesting than this classification suggests; they provide the algebraic basis for a unified language for physics and mathematics which offers many advantages over current techniques. This language is called *geometric algebra* — the name originally chosen by Clifford for his algebra — and this thesis is an investigation into the properties and applications of Clifford's geometric algebra. The work falls into three broad categories:

- The formal development of geometric algebra has been patchy and a number of important subjects have not yet been treated within its framework. A principle feature of this thesis is the development of a number of new algebraic techniques which serve to broaden the field of applicability of geometric algebra. Of particular interest are an extension of the geometric algebra of spacetime (the spacetime algebra) to incorporate multiparticle quantum states, and the development of a multivector calculus for handling differentiation with respect to a linear function.

- A central contention of this thesis is that geometric algebra provides the natural language in which to formulate a wide range of subjects from modern mathematical physics. To support this contention, reformulations of Grassmann calculus, Lie algebra theory, spinor algebra and Lagrangian field theory are developed. In each case it is argued that the geometric algebra formulation is computationally more efficient than standard approaches, and that it provides many novel insights.

- The ultimate goal of a reformulation is to point the way to new mathematics and physics, and three promising directions are developed. The first is a new approach to relativistic multiparticle quantum mechanics. The second deals with classical models for quantum spin-1/2. The third details an approach to gravity based on gauge fields acting in a flat spacetime. The Dirac equation forms the basis of this gauge theory, and the resultant theory is shown to differ from general relativity in a number of its features and predictions.

Preface

This dissertation is the result of work carried out in the Department of Applied Mathematics and Theoretical Physics between October 1990 and October 1993. Sections of the dissertation have appeared in a series of collaborative papers [1] — [10]. Except where explicit reference is made to the work of others, the work contained in this dissertation is my own.

Acknowledgements

Many people have given help and support over the last three years and I am grateful to them all. I owe a great debt to my supervisor, Nick Manton, for allowing me the freedom to pursue my own interests, and to my two principle collaborators, Anthony Lasenby and Stephen Gull, whose ideas and inspiration were essential in shaping my research. I also thank David Hestenes for his encouragement and his company on an arduous journey to Poland. Above all, I thank Julie Cooke for the love and encouragement that sustained me through to the completion of this work. Finally, I thank Stuart Rankin and Margaret James for many happy hours in the Mill, Mike and Rachael, Tim and Imogen, Paul, Alan and my other colleagues in DAMTP and MRAO.

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To my parents

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Chapter 1

Introduction

This thesis is an investigation into the properties and applications of Clifford's geometric algebra. That there is much new to say on the subject of Clifford algebra may be a surprise to some. After all, mathematicians have known how to associate a Clifford algebra with a given quadratic form for many years [11] and, by the end of the sixties, their algebraic properties had been thoroughly explored. The result of this work was the classification of all Clifford algebras as matrix algebras over one of the three associative division algebras (the real, complex and quaternion algebras) [12]–[16]. But there is much more to geometric algebra than merely Clifford algebra. To paraphrase from the introduction to "*Clifford Algebra to Geometric Calculus*" [24], Clifford algebra provides the *grammar* from which geometric algebra is constructed, but it is only when this grammar is augmented with a number of secondary definitions and concepts that one arrives at a true geometric algebra. In fact, the algebraic properties of a geometric algebra are very simple to understand, they are those of Euclidean vectors, planes and higher-dimensional (hyper)surfaces. It is the computational power brought to the manipulation of these objects that makes geometric algebra interesting and worthy of study. This computational power does not rest on the construction of explicit matrix representations, and very little attention is given to the matrix representations of the algebras used. Hence there is little common ground between the work in this thesis and earlier work on the classification and study of Clifford algebras.

There are two themes running through this thesis: that geometric algebra is the natural language in which to formulate a wide range of subjects in modern mathematical physics, and that the reformulation of known mathematics and physics in terms of geometric algebra leads to new ideas and possibilities. The development of new mathematical formulations has played an important role in the progress of physics. One need only consider the benefits of Lagrange's and Hamilton's reformulations of classical mechanics, or Feynman's path integral (re)formulation of quantum mechanics, to see how important the process of reformulation can be. Reformulations are often interesting simply for the novel and unusual insights they can provide. In other cases, a new mathematical approach can lead to significant computational advantages, as with the use of quaternions for combining rotations in three dimensions. At the back of any programme of reformulation, however, lies the hope that it will lead to new mathematics or physics. If this turns out to be the case, then the new formalism will usually be adopted and employed by the wider community. The new results and ideas contained in this thesis should support the claim that geometric algebra offers distinct advantages over more conventional techniques, and so deserves to be taught and used widely.

The work in this thesis falls broadly into the categories of formalism, reformulation and results. Whilst the foundations of geometric algebra were laid over a hundred years ago, gaps in the formalism still remain. To fill some of these gaps, a number of new algebraic techniques are developed within the framework of geometric algebra. The process of reformulation concentrates on the subjects of Grassmann calculus, Lie algebra theory, spinor algebra and Lagrangian field theory. In each case it is argued that the geometric algebra formulation is computationally more efficient than standard approaches, and that it provides many novel insights. The new results obtained include a real approach to relativistic multiparticle quantum mechanics, a new classical model for quantum spin-1/2 and an approach to gravity based on gauge fields acting in a flat spacetime. Throughout, consistent use of geometric algebra is maintained and the benefits arising from this approach are emphasised.

This thesis begins with a brief history of the development of geometric algebra and a review of its present state. This leads, inevitably, to a discussion of the work of David Hestenes [17]–[34], who has done much to shape the modern form of the subject. A number of the central themes running through his research are described, with particular emphasis given to his ideas on mathematical design. Geometric algebra is then introduced, closely following Hestenes' own approach to the subject. The central axioms and definitions are presented, and a notation is introduced which is employed consistently throughout this work. In order to avoid introducing too much formalism at once, the material in this thesis has been split into two halves. The first half, Chapters 1 to 4, deals solely with applications to various algebras employed in mathematical physics. Accordingly, only the required algebraic concepts are introduced in Chapter 1. The second half of the thesis deals with applications of geometric algebra to problems in mechanics and field theory. The essential new concept required here is that of the differential with respect to variables defined in a geometric algebra. This topic is known as *geometric calculus*, and is introduced in Chapter 5.

Chapters 2, 3 and 4 demonstrate how geometric algebra embraces a number of algebraic structures essential to modern mathematical physics. The first of these is Grassmann algebra, and particular attention is given to the Grassmann "calculus" introduced by Berezin [35]. This is shown to have a simple formulation in terms of the properties of non-orthonormal frames and examples are given of the algebraic advantages offered by this new approach. Lie algebras and Lie groups are considered in Chapter 3. Lie groups underpin many structures at the heart of modern particle physics, so it is important to develop a framework for the study of their properties within geometric algebra. It is shown that all (finite dimensional) Lie algebras can be realised as bivector algebras and it follows that all matrix Lie groups can be realised as spin groups. This has the interesting consequence that every linear transformation can be represented as a monomial of (Clifford) vectors. General methods for constructing bivector representations of Lie algebras are given, and explicit constructions are found for a number of interesting cases.

The final algebraic structures studied are spinors. These are studied using the *spacetime algebra* — the (real) geometric algebra of Minkowski spacetime. Explicit maps are constructed between Pauli and Dirac column spinors and spacetime multivectors, and it is shown that the role of the scalar unit imaginary of quantum mechanics is played by a fixed spacetime bivector. Changes of representation are discussed, and the Dirac equation is presented in a form in which it can be analysed and solved without requiring the construction of an explicit matrix representation. The concept of the multiparticle spacetime algebra is then introduced and is used to construct both non-relativistic and relativistic two-particle states. Some relativistic two-particle wave equations are considered and a new equation, based solely in the multiparticle spacetime algebra, is proposed. In a final application, the multiparticle spacetime algebra is used to reformulate aspects of the 2-spinor calculus developed by Penrose & Rindler [36, 37].

The second half of this thesis deals with applications of geometric calculus. The essential techniques are described in Chapter 5, which introduces the concept of the *multivector derivative* [18, 24]. The multivector derivative is the natural extension of calculus for functions mapping between geometric algebra elements (multivectors). Geometric calculus is shown to be ideal for studying Lagrangian mechanics and two new ideas are developed — multivector Lagrangians and multivector-parameterised transformations. These ideas are illustrated by detailed application to two models for spinning point particles. The first, due to Barut & Zanghi [38], models an electron by a classical spinor equation. This model suffers from a number of defects, including an incorrect prediction for the precession of the spin axis in a magnetic field. An alternative model is proposed which removes many of these defects and hints strongly that, at the classical level, spinors are the generators of rotations. The second model is taken from pseudoclassical mechanics [39], and has the interesting property that the Lagrangian is no longer a scalar but a bivector-valued function. The equations of motion are solved exactly and a number of conserved quantities are derived.

Lagrangian field theory is considered in Chapter 6. A unifying framework for vectors, tensors and spinors is developed and applied to problems in Maxwell and Dirac theory. Of particular interest here is the construction of new conjugate currents in the Dirac theory, based on continuous transformations of multivector spinors which have no simple counterpart in the column spinor formalism. The chapter concludes with the development of an extension of multivector calculus appropriate for multivector-valued linear functions.

The various techniques developed throughout this thesis are brought together in Chapter 7, where a theory of gravity based on gauge transformations in a flat spacetime is presented. The motivation behind this approach is threefold: (1) to introduce gravity through a similar route to the other interactions, (2) to eliminate passive transformations and base physics solely in terms of active transformations and (3) to develop a theory within the framework of the spacetime algebra. A number of consequences of this theory are explored and are compared with the predictions of general relativity and spin-torsion theories. One significant consequence is the appearance of time-reversal asymmetry in radially-symmetric (point source) solutions. Geometric algebra offers numerous advantages over conventional tensor calculus, as is demonstrated by some remarkably compact formulae for the Riemann tensor for various field configurations. Finally, it is suggested that the consistent employment of geometric algebra opens up possibilities for a genuine multiparticle theory of gravity.

1.1 Some History and Recent Developments

There can be few pieces of mathematics that have been re-discovered more often than Clifford algebras [26]. The earliest steps towards what we now recognise as a geometric algebra were taken by the pioneers of the use of complex numbers in physics. Wessel, Argand and Gauss all realised the utility of complex numbers when studying 2-dimensional problems and, in particular, they were aware that the exponential of an imaginary number is a useful means of representing rotations. This is simply a special case of the more general method for performing rotations in geometric algebra.

The next step was taken by Hamilton, whose attempts to generalise the complex numbers to three dimensions led him to his famous quaternion algebra (see [40] for a detailed history of this subject). The quaternion algebra is the Clifford algebra of 2-dimensional anti-Euclidean space, though the quaternions are better viewed as a subalgebra of the Clifford algebra of 3-dimensional space. Hamilton's ideas exerted a strong influence on his contemporaries, as can be seen form the work of the two people whose names are most closely associated with modern geometric algebra — Clifford and Grassmann.

Grassmann is best known for his algebra of extension. He defined hypernumbers e_i, which he identified with unit directed line segments. An arbitrary vector was then written as $a^i e_i$, where the a^i are scalar coefficients. Two products were assigned to these hypernumbers, an inner product

$$e_i \cdot e_j = e_j \cdot e_i = \delta_{ij} \tag{1.1}$$

and an outer product

$$e_i \wedge e_j = -e_j \wedge e_i. \tag{1.2}$$

The result of the outer product was identified as a directed plane segment and Grassmann extended this concept to include higher-dimensional objects in arbitrary dimensions. A fact overlooked by many historians of mathematics is that, in his later years, Grassmann combined his interior and exterior products into a single, *central* product [41]. Thus he wrote

$$ab = a \cdot b + a \wedge b, \tag{1.3}$$

though he employed a different notation. The central product is precisely Clifford's product of vectors, which Grassmann arrived at independently from (and slightly prior to) Clifford. Grassmann's motivation for introducing this new product was to show that Hamilton's quaternion algebra could be embedded within his own extension algebra. It was through attempting to unify the quaternions and Grassmann's algebra into a single mathematical system that Clifford was also led to his algebra. Indeed, the paper in which Clifford introduced his algebra is entitled "*Applications of Grassmann's extensive algebra*" [42].

Despite the efforts of these mathematicians to find a simple unified geometric algebra (Clifford's name for his algebra), physicists ultimately adopted a hybrid system, due largely to Gibbs. Gibbs also introduced two products for vectors. His scalar (inner) product was essentially that of Grassmann, and his vector (cross) product was abstracted from the quaternions. The vector product of two vectors was a third, so his algebra was closed and required no additional elements. Gibbs' algebra proved to be well suited

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The end result of these events was that Clifford's algebra was lost amongst the wealth of new algebras being created in the late 19th century [40]. Few realised its great promise and, along with the quaternion algebra, it was relegated to the pages of pure algebra texts. Twenty more years passed before Clifford algebras were re-discovered by Dirac in his theory of the electron. Dirac arrived at a Clifford algebra through a very different route to the mathematicians before him. He was attempting to find an operator whose square was the Laplacian and he hit upon the matrix operator $\gamma^\mu \partial_\mu$, where the γ-matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2I\eta^{\mu\nu}. \tag{1.4}$$

Sadly, the connection with vector geometry had been lost by this point, and ever since the γ-matrices have been thought of as operating on an internal electron spin space.

There the subject remained, essentially, for a further 30 years. During the interim period physicists adopted a wide number of new algebraic systems (coordinate geometry, matrix algebra, tensor algebra, differential forms, spinor calculus), whilst Clifford algebras were thought to be solely the preserve of electron theory. Then, during the sixties, two crucial developments dramatically altered the perspective. The first was made by Atiyah and Singer [43], who realised the importance of Dirac's operator in studying manifolds which admitted a global spin structure. This led them to their famous index theorems, and opened new avenues in the subjects of geometry and topology. Ever since, Clifford algebras have taken on an increasingly more fundamental role and a recent text proclaimed that Clifford algebras "*emerge repeatedly at the very core of an astonishing variety of problems in geometry and topology*" [15].

Whilst the impact of Atiyah's work was immediate, the second major step taken in the sixties has been slower in coming to fruition. David Hestenes had an unusual training as a physicist, having taken his bachelor's degree in philosophy. He has often stated that this gave him a different perspective on the role of language in understanding [27]. Like many theoretical physicists in the sixties, Hestenes worked on ways to incorporate larger multiplets of particles into the known structures of field theory. During the course of these investigations he was struck by the idea that the Dirac matrices could be interpreted as vectors, and this led him to a number of new insights into the structure and meaning of the Dirac equation and quantum mechanics in general [27].

The success of this idea led Hestenes to reconsider the wider applicability of Clifford algebras. He realised that a Clifford algebra is no less than a system of directed numbers and, as such, is the natural language in which to express a number of theorems and results from algebra and geometry. Hestenes has spent many years developing Clifford algebra into a complete language for physics, which he calls geometric algebra. The reason for preferring this name is not only that it was Clifford's original choice, but also that it serves to distinguish Hestenes' work from the strictly algebraic studies of many contemporary texts.

During the course of this development, Hestenes identified an issue which has been to problems in electromagnetism, and quickly became popular. This was despite the clear deficiencies of the vector product — it is not associative and cannot be generalised to higher dimensions. Though special relativity was only a few years off, this lack of generalisability did not appear to deter physicists and within a few years Gibbs' vector algebra had become practically the exclusive language of vector analysis.

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coordinate geometry	spinor calculus
complex analysis	Grassmann algebra
vector analysis	Berezin calculus
tensor analysis	differential forms
Lie algebras	twistors
Clifford algebra	

Table 1.1: Some algebraic systems employed in modern physics

paid little attention — that of mathematical design. Mathematics has grown into an enormous group undertaking, but few people concern themselves with how the results of this effort should best be organised. Instead, we have a situation in which a vast range of disparate algebraic systems and techniques are employed. Consider, for example, the list of algebras employed in theoretical (and especially particle) physics contained in Table 1.1. Each of these has their own conventions and their own methods for proving similar results. These algebras were introduced to tackle specific classes of problem, and each is limited in its overall scope. Furthermore, there is only a limited degree of integrability between these systems. The situation is analogous to that in the early years of software design. Mathematics has, in essence, been designed "bottom-up". What is required is a "top-down" approach — a familiar concept in systems design. Such an approach involves identifying a single algebraic system of maximal scope, coherence and simplicity which encompasses all of the narrower systems of Table 1.1. This algebraic system, or language, must be sufficiently general to enable it to formulate any result in any of the sub-systems it contains. But it must also be efficient, so that the interrelations between the subsystems can be clearly seen. Hestenes' contention is that geometric algebra is precisely the required system. He has shown how it incorporates many of the systems in Table 1.1, and part of the aim of this thesis is to fill in some of the remaining gaps.

This "top-down" approach is contrary to the development of much of modern mathematics, which attempts to tackle each problem with a system which has the minimum number of axioms. Additional structure is then handled by the addition of further axioms. For example, employing geometric algebra for problems in topology is often criticised on the grounds that geometric algebra contains redundant structure for the problem (in this case a metric derived from the inner product). But there is considerable merit to seeing mathematics the other way round. This way, the relationships between fields become clearer, and generalisations are suggested which could not be seen form the perspective of a more restricted system. For the case of topology, the subject would be seen in the manner that it was originally envisaged — as the study of properties of manifolds that are unchanged under deformations. It is often suggested that the geniuses of mathematics are those who can see beyond the symbols on the page to their deeper significance. Atiyah, for example, said that a good mathematician sees analogies between proofs, but a great mathematician sees analogies between analogies[1]. Hestenes takes this as evidence that these people understood the issues of design and saw mathematics "top-down", even if it

[1] I am grateful to Margaret James for this quote.

was not formulated as such. By adopting good design principles in the development of mathematics, the benefits of these insights would be available to all. Some issues of what constitutes good design are debated at various points in this introduction, though this subject is only in its infancy.

In conclusion, the subject of geometric algebra is in a curious state. On the one hand, the algebraic structures keeps reappearing in central ideas in physics, geometry and topology, and most mathematicians are now aware of the importance of Clifford algebras. On the other, there is far less support for Hestenes' contention that geometric algebra, built on the framework of Clifford algebra, provides a unified language for much of modern mathematics. The work in this thesis is intended to offer support for Hestenes' ideas.

1.2 Axioms and Definitions

The remaining sections of this chapter form an introduction to geometric algebra and to the conventions adopted in this thesis. Further details can be found in "*Clifford algebra to geometric calculus*" [24], which is the most detailed and comprehensive text on geometric algebra. More pedagogical introductions are provided by Hestenes [25, 26] and Vold [44, 45], and [30] contains useful additional material. The conference report on the second workshop on "*Clifford algebras and their applications in mathematical physics*" [46] contains a review of the subject and ends with a list of recommended texts, though not all of these are relevant to the material in this thesis.

In deciding how best to define geometric algebra we arrive at our first issue of mathematical design. Modern mathematics texts (see "*Spin Geometry*" by H.B Lawson and M.-L. Michelsohn [15], for example) favour the following definition of a Clifford algebra. One starts with a vector space V over a commutative field k, and supposes that q is a quadratic form on V. The tensor algebra of V is defined as

$$\mathcal{T}(V) = \sum_{r=0}^{\infty} \otimes^r V, \tag{1.5}$$

where $\otimes$ is the tensor product. One next defines an ideal $\mathcal{I}_q(V)$ in $\mathcal{T}(V)$ generated by all elements of the form $v \otimes v + q(v)1$ for $v \in V$. The Clifford algebra is then defined as the quotient

$$Cl(V, q) \equiv \mathcal{T}(V)/\mathcal{I}_q(V). \tag{1.6}$$

This definition is mathematically correct, but has a number of drawbacks:

1. The definition involves the tensor product, $\otimes$, which has to be defined initially.

2. The definition uses two concepts, tensor algebras and ideals, which are irrelevant to the properties of the resultant geometric algebra.

3. Deriving the essential properties of the Clifford algebra from (1.6) requires further work, and none of these properties are intuitively obvious from the axioms.

4. The definition is completely useless for introducing geometric algebra to a physicist or an engineer. It contains too many concepts that are the preserve of pure mathematics.

Clearly, it is desirable to find an alternative axiomatic basis for geometric algebra which does not share these deficiencies. The axioms should be consistent with our ideas of what constitutes good design. The above considerations lead us propose the following principle:

The axioms of an algebraic system should deal directly with the objects of interest.

That is to say, the axioms should offer some intuitive feel of the properties of the system they are defining.

The central properties of a geometric algebra are the grading, which separates objects into different types, and the associative product between the elements of the algebra. With these in mind, we adopt the following definition. A geometric algebra $\mathcal{G}$ is a graded linear space, the elements of which are called multivectors. The grade-0 elements are called scalars and are identified with the field of real numbers (we will have no cause to consider a geometric algebra over the complex field). The grade-1 elements are called vectors, and can be thought of as directed line segments. The elements of $\mathcal{G}$ are defined to have an addition, and each graded subspace is closed under this. A product is also defined which is associative and distributive, though non-commutative (except for multiplication by a scalar). The final axiom (which distinguishes a geometric algebra from other associative algebras) is that the square of any vector is a scalar.

Given two vectors, a and b, we find that

$$\begin{aligned}(a+b)^2 &= (a+b)(a+b) \\ &= a^2+(ab+ba)+b^2. \end{aligned} \tag{1.7}$$

It follows that

$$ab+ba = (a+b)^2 - a^2 - b^2 \tag{1.8}$$

and hence that $(ab+ba)$ is also a scalar. The geometric product of 2 vectors a, b can therefore be decomposed as

$$ab = a\cdot b + a\wedge b, \tag{1.9}$$

where

$$a\cdot b \equiv \tfrac{1}{2}(ab+ba) \tag{1.10}$$

is the standard scalar, or inner, product (a real scalar), and

$$a\wedge b \equiv \tfrac{1}{2}(ab-ba) \tag{1.11}$$

is the antisymmetric outer product of two vectors, originally introduced by Grassmann. The outer product of a and b anticommutes with both a and b,

$$\begin{aligned} a(a\wedge b) &= \tfrac{1}{2}(a^2b-aba) \\ &= \tfrac{1}{2}(ba^2-aba) \\ &= -\tfrac{1}{2}(ab-ba)a \\ &= -(a\wedge b)a, \end{aligned} \tag{1.12}$$

so $a\wedge b$ cannot contain a scalar component. The axioms are also sufficient to show that $a\wedge b$ cannot contain a vector part. If we supposed that $a\wedge b$ contained a vector part c, then the symmetrised product of $a \wedge b$ with c would necessarily contain a scalar part. But $c(a \wedge b) + (a \wedge b)c$ anticommutes with any vector d satisfying $d\cdot a = d\cdot b = d\cdot c = 0$, and so cannot contain a scalar component. The result of the outer product of two vectors is therefore a new object, which is *defined* to be grade-2 and is called a *bivector*. It can be thought of as representing a directed plane segment containing the vectors a and b. The bivectors form a linear space, though not all bivectors can be written as the exterior product of two vectors.

The definition of the outer product is extended to give an inductive definition of the grading for the entire algebra. The procedure is illustrated as follows. Introduce a third vector c and write

$$\begin{aligned} c(a\wedge b) &= \tfrac{1}{2}c(ab-ba) \\ &= (a\cdot c)b-(b\cdot c)a-\tfrac{1}{2}(acb-bca) \\ &= 2(a\cdot c)b-2(b\cdot c)a+\tfrac{1}{2}(ab-ba)c, \end{aligned} \tag{1.13}$$

so that

$$c(a\wedge b)-(a\wedge b)c = 2(a\cdot c)b-2(b\cdot c)a. \tag{1.14}$$

The right-hand side of (1.14) is a vector, so one decomposes $c(a\wedge b)$ into

$$c(a\wedge b) = c\cdot(a\wedge b)+c\wedge(a\wedge b) \tag{1.15}$$

where

$$c\cdot(a\wedge b) \equiv \tfrac{1}{2}\left[c(a\wedge b)-(a\wedge b)c\right] \tag{1.16}$$

and

$$c\wedge(a\wedge b) \equiv \tfrac{1}{2}\left[c(a\wedge b)+(a\wedge b)c\right]. \tag{1.17}$$

The definitions (1.16) and (1.17) extend the definitions of the inner and outer products to the case where a vector is multiplying a bivector. Again, (1.17) results in a new object, which is assigned grade-3 and is called a trivector. The axioms are sufficient to prove that the outer product of a vector with a bivector is associative:

$$\begin{aligned} c\wedge(a\wedge b) &= \tfrac{1}{2}\left[c(a\wedge b)+(a\wedge b)c\right] \\ &= \tfrac{1}{4}\left[cab-cba+abc-bac\right] \\ &= \tfrac{1}{4}\left[2(c\wedge a)b+acb+abc+2b(c\wedge a)-bca-cba\right] \\ &= \tfrac{1}{2}\left[(c\wedge a)b+b(c\wedge a)+a(b\cdot c)-(b\cdot c)a\right] \\ &= (c\wedge a)\wedge b. \end{aligned} \tag{1.18}$$

The definitions of the inner and outer products are extended to the geometric product of a vector with a grade-r multivector A_r as,

$$aA_r = a\cdot A_r + a\wedge A_r \tag{1.19}$$

where the inner product

$$a\cdot A_r \equiv \langle aA_r\rangle_{r-1} = \tfrac{1}{2}(aA_r-(-1)^r A_r a) \tag{1.20}$$

lowers the grade of A_r by one and the outer (exterior) product

$$a\wedge A_r \equiv \langle aA_r\rangle_{r+1} = \tfrac{1}{2}(aA_r + (-1)^r A_r a) \tag{1.21}$$

raises the grade by one. We have used the notation $\langle A\rangle_r$ to denote the result of the operation of taking the grade-r part of A (this is a projection operation). As a further abbreviation we write the scalar (grade 0) part of A simply as $\langle A\rangle$.

The entire multivector algebra can be built up by repeated multiplication of vectors. Multivectors which contain elements of only one grade are termed *homogeneous*, and will usually be written as A_r to show that A contains only a grade-r component. Homogeneous multivectors which can be expressed purely as the outer product of a set of (independent) vectors are termed *blades*.

The geometric product of two multivectors is (by definition) associative, and for two homogeneous multivectors of grade r and s this product can be decomposed as follows:

$$A_r B_s = \langle AB\rangle_{r+s} + \langle AB\rangle_{r+s-2} \ldots + \langle AB\rangle_{|r-s|}. \tag{1.22}$$

The "$\cdot$" and "$\wedge$" symbols are retained for the lowest-grade and highest-grade terms of this series, so that

$$A_r\cdot B_s \equiv \langle AB\rangle_{|s-r|} \tag{1.23}$$

$$A_r\wedge B_s \equiv \langle AB\rangle_{s+r}, \tag{1.24}$$

which we call the interior and exterior products respectively. The exterior product is associative, and satisfies the symmetry property

$$A_r\wedge B_s = (-1)^{rs} B_s\wedge A_r. \tag{1.25}$$

An important operation which can be performed on multivectors is *reversion*, which reverses the order of vectors in any multivector. The result of reversing the multivector A is written $\tilde{A}$, and is called the *reverse* of A. The reverse of a vector is the vector itself, and for a product of multivectors we have that

$$(AB)\tilde{} = \tilde{B}\tilde{A}. \tag{1.26}$$

It can be checked that for homogeneous multivectors

$$\tilde{A}_r = (-1)^{r(r-1)/2} A_r. \tag{1.27}$$

It is useful to define two further products from the geometric product. The first is the scalar product

$$A*B \equiv \langle AB\rangle. \tag{1.28}$$

This is commutative, and satisfies the useful cyclic-reordering property

$$\langle A\ldots BC\rangle = \langle CA\ldots B\rangle. \tag{1.29}$$

In positive definite spaces the scalar product defines the modulus function

$$|A| \equiv (A*A)^{1/2}. \tag{1.30}$$

The second new product is the commutator product, defined by

$$A \times B \equiv \tfrac{1}{2}(AB - BA). \tag{1.31}$$

The associativity of the geometric product ensures that the commutator product satisfies the Jacobi identity

$$A \times (B \times C) + B \times (C \times A) + C \times (A \times B) = 0. \tag{1.32}$$

Finally, we introduce an operator ordering convention. *In the absence of brackets, inner, outer and scalar products take precedence over geometric products.* Thus $a \cdot bc$ means $(a \cdot b)c$ and not $a \cdot (bc)$. This convention helps to eliminate unruly numbers of brackets. Summation convention is also used throughout this thesis.

One can now derive a vast number of properties of multivectors, as is done in Chapter 1 of [24]. But before proceeding, it is worthwhile stepping back and looking at the system we have defined. In particular, we need to see that the axioms have produced a system with sensible properties that match our intuitions about physical space and geometry in general.

1.2.1 The Geometric Product

Our axioms have led us to an associative product for vectors, $ab = a \cdot b + a \wedge b$. We call this the *geometric product*. It has the following two properties:

- Parallel vectors (*e.g.* a and αa) commute, and the the geometric product of parallel vectors is a scalar. Such a product is used, for example, when finding the length of a vector.

- Perpendicular vectors (a, b where $a \cdot b = 0$) anticommute, and the geometric product of perpendicular vectors is a bivector. This is a directed plane segment, or directed area, containing the vectors a and b.

Independently, these two features of the algebra are quite sensible. It is therefore reasonable to suppose that the product of vectors that are neither parallel nor perpendicular should contain both scalar and bivector parts.

But what does it mean to add a scalar to a bivector?

This is the point which regularly causes the most confusion (see [47], for example). Adding together a scalar and a bivector doesn't seem right — they are different types of quantities. But this is exactly what you do want addition to do. The result of adding a scalar to a bivector is an object that has both scalar and bivector parts, in exactly the same way that the addition of real and imaginary numbers yields an object with both real and imaginary parts. We call this latter object a "complex number" and, in the same way, we refer to a (scalar + bivector) as a "multivector", accepting throughout that we are combining objects of different types. The addition of scalar and bivector does not result in a single new quantity in the same way as $2 + 3 = 5$; we are simply keeping track of separate components in the symbol $ab = a \cdot b + a \wedge b$ or $z = x + iy$. This type of addition, of objects from separate linear spaces, could be given the symbol $\oplus$, but it should be evident from our experience of complex numbers that it is harmless, and more convenient, to extend the definition of addition and use the ordinary + sign.

Further insights are gained by the construction of explicit algebras for finite dimensional spaces. This is achieved most simply through the introduction of an orthonormal frame of vectors $\{\sigma_i\}$ satisfying

$$\sigma_i \cdot \sigma_j = \delta_{ij} \tag{1.33}$$

or

$$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}. \tag{1.34}$$

This is the conventional starting point for the matrix representation theory of finite Clifford algebras [13, 48]. It is also the usual route by which Clifford algebras enter particle physics, though there the $\{\sigma_i\}$ are thought of as operators, and not as orthonormal vectors. The geometric algebra we have defined is associative and any associative algebra can be represented as a matrix algebra, so why not define a geometric algebra as a matrix algebra? There are a number of flaws with this approach, which Hestenes has frequently drawn attention to [26]. The approach fails, in particular, when geometric algebra is used to study projectively and conformally related geometries [31]. There, one needs to be able to move freely between different dimensional spaces. Matrix representations are too rigid to achieve this satisfactorily. An example of this will be encountered shortly.

There is a further reason for preferring not to introduce Clifford algebras via their matrix representations. It is related to our second principle of good design, which is that

the axioms af an algebraic system should not introduce redundant structure.

The introduction of matrices is redundant because all geometrically meaningful results exist independently of any matrix representations. Quite simply, matrices are irrelevant for the development of geometric algebra.

The introduction of a basis set of n independent, orthonormal vectors $\{\sigma_i\}$ defines a basis for the entire algebra generated by these vectors:

$$1, \quad \{\sigma_i\}, \quad \{\sigma_i \wedge \sigma_j\}, \quad \{\sigma_i \wedge \sigma_j \wedge \sigma_k\}, \quad \ldots, \quad \sigma_1 \wedge \sigma_2 \ldots \wedge \sigma_n \equiv I. \tag{1.35}$$

Any multivector can now be expanded in this basis, though one of the strengths of geometric algebra is that it possible to carry out many calculations in a basis-*free* way. Many examples of this will be presented in this thesis,

The highest-grade blade in the algebra (1.35) is given the name "pseudoscalar" (or directed volume element) and is of special significance in geometric algebra. Its unit is given the special symbol I (or i in three or four dimensions). It is a pure blade, and a knowledge of I is sufficient to specify the vector space over which the algebra is defined (see [24, Chapter 1]). The pseudoscalar also defines the duality operation for the algebra, since multiplication of a grade-r multivector by I results in a grade-$(n-r)$ multivector.

1.2.2 The Geometric Algebra of the Plane

A 1-dimensional space has insufficient geometric structure to be interesting, so we start in two dimensions, taking two orthonormal basis vectors σ_1 and σ_2. These satisfy the relations

$$(\sigma_1)^2 = 1 \tag{1.36}$$
$$(\sigma_2)^2 = 1 \tag{1.37}$$

and

$$\sigma_1 \cdot \sigma_2 = 0. \tag{1.38}$$

The outer product $\sigma_1 \wedge \sigma_2$ represents the directed area element of the plane and we assume that σ_1, σ_2 are chosen such that this has the conventional right-handed orientation. This completes the geometrically meaningful quantities that we can make from these basis vectors:

$$\underset{\text{scalar}}{1,} \qquad \underset{\text{vectors}}{\{\sigma_1, \sigma_2\},} \qquad \underset{\text{bivector}}{\sigma_1 \wedge \sigma_2.} \tag{1.39}$$

Any multivector can be expanded in terms of these four basis elements. Addition of multivectors simply adds the coefficients of each component. The interesting expressions are those involving products of the bivector $\sigma_1 \wedge \sigma_2 = \sigma_1\sigma_2$. We find that

$$\begin{aligned} (\sigma_1\sigma_2)\sigma_1 &= -\sigma_2\sigma_1\sigma_1 = -\sigma_2, \\ (\sigma_1\sigma_2)\sigma_2 &= \sigma_1 \end{aligned} \tag{1.40}$$

and

$$\begin{aligned} \sigma_1(\sigma_1\sigma_2) &= \sigma_2 \\ \sigma_2(\sigma_1\sigma_2) &= -\sigma_1. \end{aligned} \tag{1.41}$$

The only other product to consider is the square of $\sigma_1 \wedge \sigma_2$,

$$(\sigma_1 \wedge \sigma_2)^2 = \sigma_1\sigma_2\sigma_1\sigma_2 = -\sigma_1\sigma_1\sigma_2\sigma_2 = -1. \tag{1.42}$$

These results complete the list of the products in the algebra. In order to be completely explicit, consider how two arbitrary multivectors are multiplied. Let

$$A = a_0 + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_1 \wedge \sigma_2 \tag{1.43}$$
$$B = b_0 + b_1\sigma_1 + b_2\sigma_2 + b_3\sigma_1 \wedge \sigma_2, \tag{1.44}$$

then we find that

$$AB = p_0 + p_1\sigma_1 + p_2\sigma_2 + p_3\sigma_1 \wedge \sigma_2, \tag{1.45}$$

where

$$\begin{aligned} p_0 &= a_0b_0 + a_1b_1 + a_2b_2 - a_3b_3, \\ p_1 &= a_0b_1 + a_1b_0 + a_3b_2 - a_2b_3, \\ p_2 &= a_0b_2 + a_2b_0 + a_1b_3 - a_3b_1, \\ p_3 &= a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1. \end{aligned} \tag{1.46}$$

Calculations rarely have to be performed in this detail, but this exercise does serve to illustrate how geometric algebras can be made intrinsic to a computer language. One can even think of (1.46) as generalising Hamilton's concept of complex numbers as ordered pairs of real numbers.

The square of the bivector $\sigma_1 \wedge \sigma_2$ is -1, so the even-grade elements $z = x + y\sigma_1\sigma_2$ form a natural subalgebra, equivalent to the *complex numbers*. Furthermore, $\sigma_1 \wedge \sigma_2$ has the geometric effect of rotating the vectors $\{\sigma_1, \sigma_2\}$ in their own plane by 90° clockwise when multiplying them on their left. It rotates vectors by 90° anticlockwise when multiplying on their right. (This can be used to define the orientation of σ_1 and σ_2).

The equivalence between the even subalgebra and complex numbers reveals a new interpretation of the structure of the Argand diagram. From any vector $r = x\sigma_1 + y\sigma_2$ we can form an even multivector z by

$$z \equiv \sigma_1 r = x + Iy, \tag{1.47}$$

where

$$I \equiv \sigma_1\sigma_2. \tag{1.48}$$

There is therefore a one-to-one correspondence between points in the Argand diagram and vectors in two dimensions,

$$r = \sigma_1 z, \tag{1.49}$$

where the vector σ_1 defines the real axis. Complex conjugation,

$$z^* \equiv \tilde{z} = r\sigma_1 = x - Iy, \tag{1.50}$$

now appears as the natural operation of reversion for the even multivector z. Taking the complex conjugate of z results in a new vector r^* given by

$$\begin{aligned} r^* &= \sigma_1 \tilde{z} \\ &= (z\sigma_1)\tilde{} \\ &= (\sigma_1 r \sigma_1)\tilde{} \\ &= \sigma_1 r \sigma_1 \\ &= -\sigma_2 r \sigma_2. \end{aligned} \tag{1.51}$$

We will shortly see that equation (1.51) is the geometric algebra representation of a reflection in the σ_1 axis. This is precisely what one expects for complex conjugation.

This identification of points on the Argand diagram with (Clifford) vectors gives additional operational significance to complex numbers of the form $\exp(i\theta)$. The even multivector equivalent of this is $\exp(I\theta)$, and applied to z gives

$$\begin{aligned} e^{I\theta} z &= e^{I\theta}\sigma_1 r \\ &= \sigma_1 e^{-I\theta} r. \end{aligned} \tag{1.52}$$

But we can now remove the σ_1, and work entirely in the (real) Euclidean plane. Thus

$$r' = e^{-I\theta} r \tag{1.53}$$

rotates the vector r anticlockwise through an angle θ. This can be verified from the fact that

$$e^{-I\theta}\sigma_1 = (\cos\theta - \sin\theta I)\sigma_1 = \cos\theta\,\sigma_1 + \sin\theta\sigma_2 \tag{1.54}$$

and

$$e^{-I\theta}\sigma_2 = \cos\theta\,\sigma_2 - \sin\theta\sigma_1. \tag{1.55}$$

Viewed as even elements in the 2-dimensional geometric algebra, exponentials of "imaginaries" generate rotations of real vectors. Thinking of the unit imaginary as being a directed plane segment removes much of the mystery behind the usage of complex numbers. Furthermore, exponentials of bivectors provide a very general method for handling rotations in geometric algebra, as is shown in Chapter 3.

1.2.3 The Geometric Algebra of Space

If we now add a third orthonormal vector σ_3 to our basis set, we generate the following geometric objects:

$$\begin{array}{cccc} 1, & \{\sigma_1, \sigma_2, \sigma_3\}, & \{\sigma_1\sigma_2, \sigma_2\sigma_3, \sigma_3\sigma_1\}, & \sigma_1\sigma_2\sigma_3. \\ \text{scalar} & \text{3 vectors} & \text{3 bivectors} & \text{trivector} \\ & & \text{area elements} & \text{volume element} \end{array} \tag{1.56}$$

From these objects we form a linear space of $(1 + 3 + 3 + 1) = 8 = 2^3$ dimensions. Many of the properties of this algebra are shared with the 2-dimensional case since the subsets $\{\sigma_1, \sigma_2\}$, $\{\sigma_2, \sigma_3\}$ and $\{\sigma_3, \sigma_1\}$ generate 2-dimensional subalgebras. The new geometric products to consider are

$$\begin{aligned} (\sigma_1\sigma_2)\sigma_3 &= \sigma_1\sigma_2\sigma_3, \\ (\sigma_1\sigma_2\sigma_3)\sigma_k &= \sigma_k(\sigma_1\sigma_2\sigma_3) \end{aligned} \tag{1.57}$$

and

$$(\sigma_1\sigma_2\sigma_3)^2 = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3 = \sigma_1\sigma_2\sigma_1\sigma_2\sigma_3^2 = -1. \tag{1.58}$$

These relations lead to new geometric insights:

- A simple bivector rotates vectors in its own plane by 90°, but forms trivectors (volumes) with vectors perpendicular to it.
- The trivector $\sigma_1\wedge\sigma_2\wedge\sigma_3$ commutes with all vectors, and hence with all multivectors.

The trivector (pseudoscalar) $\sigma_1\sigma_2\sigma_3$ also has the algebraic property of squaring to -1. In fact, of the eight geometrical objects, four have negative square, $\{\sigma_1\sigma_2,\ \sigma_2\sigma_3,\ \sigma_3\sigma_1\}$ and $\sigma_1\sigma_2\sigma_3$. Of these, the pseudoscalar $\sigma_1\sigma_2\sigma_3$ is distinguished by its commutation properties and in view of these properties we give it the special symbol i,

$$i \equiv \sigma_1\sigma_2\sigma_3. \tag{1.59}$$

It should be quite clear, however, that the symbol i is used to stand for a pseudoscalar and therefore cannot be used for the commutative scalar imaginary used, for example, in quantum mechanics. Instead, the symbol j is used for this uninterpreted imaginary, consistent with existing usage in engineering. The definition (1.59) will be consistent with our later extension to 4-dimensional spacetime.

The algebra of 3-dimensional space is the Pauli algebra familiar from quantum mechanics. This can be seen by multiplying the pseudoscalar in turn by σ_3, σ_1 and σ_2 to find

$$\begin{aligned}(\sigma_1\sigma_2\sigma_3)\sigma_3 = \sigma_1\sigma_2 &= i\sigma_3, \\ \sigma_2\sigma_3 &= i\sigma_1, \\ \sigma_3\sigma_1 &= i\sigma_2,\end{aligned} \tag{1.60}$$

which is immediately identifiable as the algebra of Pauli spin matrices. But we have arrived at this algebra from a totally different route, and the various elements in it have very different meanings to those assigned in quantum mechanics. Since 3-dimensional space is closest to our perception of the world, it is worth emphasising the geometry of this algebra in greater detail. A general multivector M consists of the components

$$M = \underset{\text{scalar}}{\alpha} + \underset{\text{vector}}{\boldsymbol{a}} + \underset{\text{bivector}}{i\boldsymbol{b}} + \underset{\text{pseudoscalar}}{i\beta} \tag{1.61}$$

where $\boldsymbol{a} \equiv a_k\sigma_k$ and $\boldsymbol{b} \equiv b_k\sigma_k$. The reason for writing spatial vectors in bold type is to maintain a visible difference between spatial vectors and spacetime 4-vectors. This distinction will become clearer when we consider relativistic physics. The meaning of the $\{\sigma_k\}$ is always unambiguous, so these are not written in bold type.

Each of the terms in (1.61) has a separate geometric significance:

1. *scalars* are physical quantities with magnitude but no spatial extent. Examples are mass, charge and the number of words in this thesis.

2. *vectors* have both a magnitude and a direction. Examples include relative positions, displacements and velocities.

3. *bivectors* have a magnitude and an orientation. They do not have a shape. In Figure 1.1 the bivector $\boldsymbol{a}\wedge\boldsymbol{b}$ is represented as a parallelogram, but any other shape could have been chosen. In many ways a circle is more appropriate, since it suggests the idea of sweeping round from the $\boldsymbol{a}$ direction to the $\boldsymbol{b}$ direction. Examples of bivectors include angular momentum and any other object that is usually represented as an "axial" vector.

4. *trivectors* have simply a handedness and a magnitude. The handedness tells whether the vectors in the product $\boldsymbol{a}\wedge\boldsymbol{b}\wedge\boldsymbol{c}$ form a left-handed or right-handed set. Examples include the scalar triple product and, more generally, alternating tensors.

These four objects are represented pictorially in Figure 1.1. Further details and discussions are contained in [25] and [44].

The space of even-grade elements of the Pauli algebra,

$$\psi = \alpha + i\boldsymbol{b}, \tag{1.62}$$

is closed under multiplication and forms a representation of the quarternion algebra. Explicitly, identifying $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ with $i\sigma_1$, $-i\sigma_2$, $i\sigma_3$ respectively, the usual quarternion relations are recovered, including the famous formula

$$\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = \boldsymbol{ijk} = -1. \tag{1.63}$$

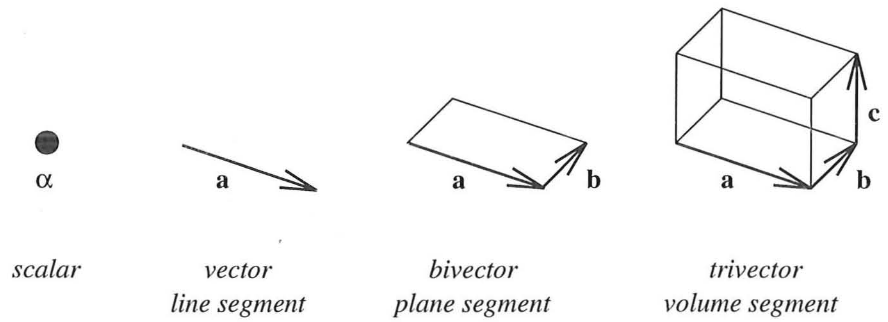

Figure 1.1: Pictorial representation of the elements of the Pauli algebra.

The quaternion algebra sits neatly inside the geometric algebra of space and, seen in this way, the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ do indeed generate 90° rotations in three orthogonal directions. Unsurprisingly, this algebra proves to be ideal for representing arbitrary rotations in three dimensions.

Finally, for this section, we recover Gibbs' cross product. Since the $\times$ and $\wedge$ symbols have already been assigned meanings, we will use the $\perp$ symbol for the Gibbs' product. This notation will not be needed anywhere else in this thesis. The Gibbs' product is given by an outer product together with a duality operation (multiplication by the pseudoscalar),

$$\boldsymbol{a} \perp \boldsymbol{b} \equiv -i\boldsymbol{a}\wedge\boldsymbol{b}. \tag{1.64}$$

The duality operation in three dimensions interchanges a plane with a vector orthogonal to it (in a right-handed sense). In the mathematical literature this operation goes under the name of the Hodge dual. Quantities like $\boldsymbol{a}$ or $\boldsymbol{b}$ would conventionally be called "polar vectors", while the "axial vectors" which result from cross-products can now be seen to be disguised versions of *bivectors*. The vector triple product $\boldsymbol{a} \perp (\boldsymbol{b} \perp \boldsymbol{c})$ becomes $-\boldsymbol{a}\cdot(\boldsymbol{b}\wedge\boldsymbol{c})$, which is the 3-dimensional form of an expression which is now legitimate in arbitrary dimensions. We therefore drop the restriction of being in 3-dimensional space and write

$$\begin{aligned} a\cdot(b\wedge c) &= \tfrac{1}{2}(ab\wedge c - b\wedge ca) && (1.65)\\ &= a\cdot bc - a\cdot cb && (1.66) \end{aligned}$$

where we have recalled equation (1.14).

1.2.4 Reflections and Rotations

One of the clearest illustrations of the power of geometric algebra is the way in which it deals with reflections and rotations. The key to this approach is that, given any unit vector n ($n^2 = 1$), an arbitrary vector a can be resolved into parts parallel and perpendicular to n,

$$\begin{aligned} a &= n^2 a \\ &= n(n\cdot a + n\wedge a) \\ &= a_{\|} + a_{\perp}, && (1.67) \end{aligned}$$

where

$$a_{\|} = a\cdot nn \tag{1.68}$$
$$a_{\perp} = nn\wedge a. \tag{1.69}$$

The result of reflecting a in the hyperplane orthogonal to n is the vector $a_{\perp} - a_{\|}$, which can be written as

$$\begin{aligned} a_{\perp} - a_{\|} &= nn\wedge a - a\cdot nn \\ &= -n\cdot an - n\wedge an \\ &= -nan. \end{aligned} \tag{1.70}$$

This formula for a reflection extends to arbitrary multivectors. For example, if the vectors a and b are both reflected in the hyperplane orthogonal to n, then the bivector $a\wedge b$ is reflected to

$$\begin{aligned} (-nan)\wedge(-nbn) &= \tfrac{1}{2}(nannbn - nbnnan) \\ &= na\wedge bn. \end{aligned} \tag{1.71}$$

In three dimensions, the sign difference between the formulae for vectors and bivectors accounts for the different behaviour of "polar" and "axial" vectors under reflections.

Rotations are built from pairs of reflections. Taking a reflection first in the hyperplane orthogonal to n, and then in the hyperplane orthogonal to m, leads to the new vector

$$\begin{aligned} -m(-nan)m &= mnanm \\ &= Ra\tilde{R} \end{aligned} \tag{1.72}$$

where

$$R \equiv mn. \tag{1.73}$$

The multivector R is called a *rotor*. It contains only even-grade elements and satisfies the identity

$$R\tilde{R} = \tilde{R}R = 1. \tag{1.74}$$

Equation (1.74) ensures that the scalar product of two vectors is invariant under rotations,

$$\begin{aligned} (Ra\tilde{R})\cdot(Rb\tilde{R}) &= \langle Ra\tilde{R}Rb\tilde{R}\rangle \\ &= \langle a\tilde{R}Rb\tilde{R}R\rangle \\ &= \langle ab\rangle \\ &= a\cdot b. \end{aligned} \tag{1.75}$$

As an example, consider rotating the unit vector a into another unit vector b, leaving all vectors perpendicular to a and b unchanged. This is accomplished by a reflection perpendicular to the unit vector half-way between a and b (see Figure 1.2)

$$n \equiv (a+b)/|a+b|. \tag{1.76}$$

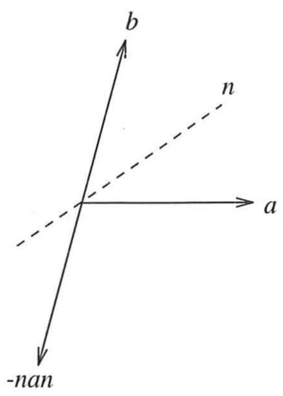

Figure 1.2: A rotation composed of two reflections.

This reflects a into $-b$. A second reflection is needed to then bring this to b, which must take place in the hyperplane perpendicular to b. Together, these give the rotor

$$R = bn = \frac{1+ba}{|a+b|} = \frac{1+ba}{\sqrt{2(1+b\cdot a)}}, \tag{1.77}$$

which represents a simple rotation in the $a\wedge b$ plane. The rotation is written

$$b = Ra\tilde{R}, \tag{1.78}$$

and the inverse transformation is given by

$$a = \tilde{R}bR. \tag{1.79}$$

The transformation $a \mapsto Ra\tilde{R}$ is a very general way of handling rotations. In deriving this transformation the dimensionality of the space of vectors was at no point specified. As a result, the transformation law works for *all* spaces, *whatever dimension.* Furthermore, it works for *all* types of geometric object, *whatever grade.* We can see this by considering the image of the product ab when the vectors a and b are both rotated. In this case, ab is rotated to

$$Ra\tilde{R}Rb\tilde{R} = Rab\tilde{R}. \tag{1.80}$$

In dimensions higher than 5, an arbitrary even element satisfying (1.74) does not necessarily map vectors to vectors and will not always represent a rotation. The name "rotor" is then retained only for the even elements that do give rise to rotations. It can be shown that all (simply connected) rotors can be written in the form

$$R = \pm e^{B/2}, \tag{1.81}$$

where B is a bivector representing the plane in which the rotation is taking place. (This representation for a rotor is discussed more fully in Chapter 3.) The quantity

$$b = e^{\alpha B/2} a e^{-\alpha B/2} \tag{1.82}$$

is seen to be a pure vector by Taylor expanding in α,

$$b = a + \alpha B \cdot a + \frac{\alpha^2}{2!} B \cdot (B \cdot a) + \cdots. \tag{1.83}$$

The right-hand side of (1.83) is a vector since the inner product of a vector with a bivector is always a vector (1.14). This method of representing rotations directly in terms of the plane in which they take place is very powerful. Equations (1.54) and (1.55) illustrated this in two dimensions, where the quantity $\exp(-I\theta)$ was seen to rotate vectors anticlockwise through an angle θ. This works because in two dimensions we can always write

$$e^{-I\theta/2} r e^{I\theta/2} = e^{-I\theta} r. \tag{1.84}$$

In higher dimensions the double-sided (bilinear) transformation law (1.78) is required. This is much easier to use than a one-sided rotation matrix, because the latter becomes more complicated as the number of dimensions increases. This becomes clearer in three dimensions. The rotor

$$R \equiv \exp(-i\boldsymbol{a}/2) = \cos(|\boldsymbol{a}|/2) - i\frac{\boldsymbol{a}}{|\boldsymbol{a}|}\sin(|\boldsymbol{a}|/2) \tag{1.85}$$

represents a rotation of $|\boldsymbol{a}| = (\boldsymbol{a}^2)^{1/2}$ radians about the axis along the direction of $\boldsymbol{a}$. This is already simpler to work with than 3×3 matrices. In fact, the representation of a rotation by (1.85) is precisely how rotations are represented in the quaternion algebra, which is well-known to be advantageous in three dimensions. In higher dimensions the improvements are even more dramatic.

Having seen how individual rotors are used to represent rotations, we must look at their composition law. Let the rotor R transform the unit vector a into a vector b,

$$b = Ra\tilde{R}. \tag{1.86}$$

Now rotate b into another vector b', using a rotor R'. This requires

$$b' = R'b\tilde{R}' = (R'R)a(R'R)\tilde{} \tag{1.87}$$

so that the transformation is characterised by

$$R \mapsto R'R, \tag{1.88}$$

which is the (left-sided) group combination rule for rotors. It is immediately clear that the product of two rotors is a third rotor,

$$R'R(R'R)\tilde{} = R'R\tilde{R}\tilde{R}' = R'\tilde{R}' = 1, \tag{1.89}$$

so that the rotors do indeed form a (Lie) group.

The usefulness of rotors provides ample justification for adding up terms of different grades. The rotor R on its own has no geometric significance, which is to say that no meaning should be attached to the individual scalar, bivector, 4-vector ... parts of R. When R is written in the form $R = \pm e^{B/2}$, however, the bivector B has clear geometric significance, as does the vector formed from $Ra\tilde{R}$. This illustrates a central feature of geometric algebra, which is that both geometrically meaningful objects (vectors, planes ...) and the elements that act on them (rotors, spinors ...) are represented in the same algebra.

1.2.5 The Geometric Algebra of Spacetime

As a final example, we consider the geometric algebra of spacetime. This algebra is sufficiently important to deserve its own name — spacetime algebra — which we will usually abbreviate to STA. The square of a vector is no longer positive definite, and we say that a vector x is timelike, lightlike or spacelike according to whether $x^2 > 0$, $x^2 = 0$ or $x^2 < 0$ respectively. Spacetime consists of a single independent timelike direction, and three independent spacelike directions. The spacetime algebra is then generated by a set of orthonormal vectors $\{\gamma_\mu\}$, $\mu = 0\ldots3$, satisfying

$$\gamma_\mu\cdot\gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+\ -\ -\ -). \tag{1.90}$$

(The significance of the choice of metric signature will be discussed in Chapter 4.) The full STA is 16-dimensional, and is spanned by the basis

$$1, \qquad \{\gamma_\mu\} \qquad \{\sigma_k,\ i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i. \tag{1.91}$$

The spacetime bivectors $\{\sigma_k\}$, $k = 1\ldots3$ are defined by

$$\sigma_k \equiv \gamma_k\gamma_0. \tag{1.92}$$

They form an orthonormal frame of vectors in the space relative to the γ_0 direction. The spacetime pseudoscalar i is defined by

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \tag{1.93}$$

and, since we are in a space of even dimension, i anticommutes with all odd-grade elements and commutes with all even-grade elements. It follows from (1.92) that

$$\sigma_1\sigma_2\sigma_3 = \gamma_1\gamma_0\gamma_2\gamma_0\gamma_3\gamma_0 = \gamma_0\gamma_1\gamma_2\gamma_3 = i. \tag{1.94}$$

The following geometric significance is attached to these relations. An inertial system is completely characterised by a future-pointing timelike (unit) vector. We take this to be the γ_0 direction. This vector/observer determines a map between spacetime vectors $a = a^\mu\gamma_\mu$ and the even subalgebra of the full STA via

$$a\gamma_0 = a_0 + \boldsymbol{a} \tag{1.95}$$

where

$$a_0 = a\cdot\gamma_0 \tag{1.96}$$

$$\boldsymbol{a} = a\wedge\gamma_0. \tag{1.97}$$

The even subalgebra of the STA is isomorphic to the Pauli algebra of space defined in Section 1.2.3. This is seen from the fact that the $\sigma_k = \gamma_k\gamma_0$ all square to $+1$,

$$\sigma_k{}^2 = \gamma_k\gamma_0\gamma_k\gamma_0 = -\gamma_k\gamma_k\gamma_0\gamma_0 = +1, \tag{1.98}$$

and anticommute,

$$\sigma_j\sigma_k = \gamma_j\gamma_0\gamma_k\gamma_0 = \gamma_k\gamma_j\gamma_0\gamma_0 = -\gamma_k\gamma_0\gamma_j\gamma_0 = -\sigma_k\sigma_j \quad (j \neq k). \tag{1.99}$$

There is more to this equivalence than simply a mathematical isomorphism. The way we think of a vector is as a line segment existing for a period of time. It is therefore sensible that what we perceive as a vector should be represented by a spacetime bivector. In this way the algebraic properties of space are determined by those of spacetime.

As an example, if x is the spacetime (four)-vector specifying the position of some point or event, then the "spacetime split" into the γ_0-frame gives

$$x\gamma_0 = t + \boldsymbol{x}, \tag{1.100}$$

which defines an observer time

$$t = x\cdot\gamma_0 \tag{1.101}$$

and a relative position vector

$$\boldsymbol{x} = x\wedge\gamma_0. \tag{1.102}$$

One useful feature of this approach is the way in which it handles Lorentz-scalar quantities. The scalar x^2 can be decomposed into

$$\begin{aligned} x^2 &= x\gamma_0\gamma_0 x \\ &= (t+\boldsymbol{x})(t-\boldsymbol{x}) \\ &= t^2 - \boldsymbol{x}^2, \end{aligned} \tag{1.103}$$

which must also be a scalar. The quantity $t^2 - \boldsymbol{x}^2$ is now seen to be automatically Lorentz-invariant, without needing to consider a Lorentz transformation.

The split of the six spacetime bivectors into relative vectors and relative bivectors is a frame/observer-dependent operation. This can be illustrated with the Faraday bivector $F = \frac{1}{2}F^{\mu\nu}\gamma_\mu\wedge\gamma_\nu$, which is a full, 6-component spacetime bivector. The spacetime split of F into the γ_0-system is achieved by separating F into parts which anticommute and commute with γ_0. Thus

$$F = \boldsymbol{E} + i\boldsymbol{B}, \tag{1.104}$$

where

$$\boldsymbol{E} = \tfrac{1}{2}(F - \gamma_0 F\gamma_0) \tag{1.105}$$

$$i\boldsymbol{B} = \tfrac{1}{2}(F + \gamma_0 F\gamma_0). \tag{1.106}$$

Here, both $\boldsymbol{E}$ and $\boldsymbol{B}$ are spatial vectors, and $i\boldsymbol{B}$ is a spatial bivector. This decomposes F into separate electric and magnetic fields, and the explicit appearance of γ_0 in the formulae for $\boldsymbol{E}$ and $\boldsymbol{B}$ shows that this split is observer-dependent. In fact, the identification of spatial vectors with spacetime bivectors has always been implicit in the physics of electromagnetism through formulae like $E_k = F_{k0}$.

The decomposition (1.104) is useful for constructing relativistic invariants from the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. Since F^2 contains only scalar and pseudoscalar parts, the quantity

$$\begin{aligned} F^2 &= (\boldsymbol{E}+i\boldsymbol{B})(\boldsymbol{E}+i\boldsymbol{B}) \\ &= \boldsymbol{E}^2 - \boldsymbol{B}^2 + 2i\boldsymbol{E}\cdot\boldsymbol{B} \end{aligned} \tag{1.107}$$

is Lorentz-invariant. It follows that both $\boldsymbol{E}^2 - \boldsymbol{B}^2$ and $\boldsymbol{E}\cdot\boldsymbol{B}$ are observer-invariant quantities.

Equation (1.94) is an important geometric identity, which shows that relative space and spacetime share the same pseudoscalar i. It also exposes the weakness of the matrix-based approach to Clifford algebras. The relation

$$\sigma_1\sigma_2\sigma_3 = i = \gamma_0\gamma_1\gamma_2\gamma_3 \tag{1.108}$$

cannot be formulated in conventional matrix terms, since it would need to relate the 2×2 Pauli matrices to 4×4 Dirac matrices. Whilst we borrow the symbols for the Dirac and Pauli matrices, it must be kept in mind that the symbols are being used in a quite different context — they represent a frame of orthonormal vectors rather than representing individual components of a single isospace vector.

The identification of relative space with the even subalgebra of the STA necessitates developing a set of conventions which articulate smoothly between the two algebras. This problem will be dealt with in more detail in Chapter 4, though one convention has already been introduced. Relative (or spatial) vectors in the γ_0-system are written in bold type to record the fact that in the STA they are actually bivectors. This distinguishes them from spacetime vectors, which are left in normal type. No problems can arise for the $\{\sigma_k\}$, which are unambiguously spacetime bivectors, so these are also left in normal type. The STA will be returned to in Chapter 4 and will then be used throughout the remainder of this thesis. We will encounter many further examples of its utility and power.

1.3 Linear Algebra

We have illustrated a number of the properties of geometric algebra, and have given explicit constructions in two, three and four dimensions. This introduction to the properties of geometric algebra is now concluded by developing an approach to the study of linear functions and non-orthonormal frames.

1.3.1 Linear Functions and the Outermorphism

Geometric algebra offers many advantages when used for developing the theory of linear functions. This subject is discussed in some detail in Chapter 3 of "*Clifford algebra to geometric calculus*" [24], and also in [2] and [30]. The approach is illustrated by taking a linear function $f(a)$ mapping vectors to vectors in the same space. This function in extended via *outermorphism* to act linearly on multivectors as follows,

$$\underline{f}(a\wedge b\wedge \ldots \wedge c) \equiv f(a)\wedge f(b) \ldots \wedge f(c). \tag{1.109}$$

The underbar on $\underline{f}$ shows that $\underline{f}$ has been constructed from the linear function f. The definition (1.109) ensures that $\underline{f}$ is a grade-preserving linear function mapping multivectors to multivectors.

An example of an outermorphism was encountered in Section 1.2.4, where we considered how multivectors behave under rotations. The action of a rotation on a vector a was written as

$$R(a) = e^{B/2}ae^{-B/2}, \tag{1.110}$$

Equation (1.94) is an important geometric identity, which shows that relative space and spacetime share the same pseudoscalar i. It also exposes the weakness of the matrix-based approach to Clifford algebras. The relation

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$$R(a) = e^{B/2}ae^{-B/2}, \tag{1.110}$$

where B is the plane(s) of rotation. The outermorphism extension of this is simply

$$\underline{R}(A) = e^{B/2} A e^{-B/2}. \tag{1.111}$$

An important property of the outermorphism is that the outermorphism of the product of two functions in the product of the outermorphisms,

$$\begin{aligned} f[g(a)]\wedge f[g(b)]\ldots\wedge f[g(c)] &= \underline{f}[g(a)\wedge g(b)\ldots\wedge g(c)] \\ &= \underline{f}[\underline{g}(a\wedge b\wedge\ldots\wedge c)]. \end{aligned} \tag{1.112}$$

To ease notation, the product of two functions will be written simply as $\underline{f}\,\underline{g}(A)$, so that (1.112) becomes

$$fg(a)\wedge fg(b)\ldots\wedge fg(c) = \underline{f}\,\underline{g}(a\wedge b\wedge\ldots\wedge c). \tag{1.113}$$

The pseudoscalar of an algebra is unique up to a scale factor, and this is used to define the determinant of a linear function via

$$\det(f) \equiv \underline{f}(I)I^{-1}, \tag{1.114}$$

so that

$$\underline{f}(I) = \det(f)I. \tag{1.115}$$

This definition clearly illustrates the role of the determinant as the volume scale factor. The definition also serves to give a very quick proof of one of the most important properties of determinants. It follows from (1.113) that

$$\begin{aligned} \underline{f}\,\underline{g}(I) &= \underline{f}(\det(g)I) \\ &= \det(g)\underline{f}(I) \\ &= \det(f)\det(g)I \end{aligned} \tag{1.116}$$

and hence that

$$\det(fg) = \det(f)\det(g). \tag{1.117}$$

This proof of the product rule for determinants illustrates our third (and final) principle of good design:

> *Definitions should be chosen so that the most important theorems can be proven most economically.*

The definition of the determinant clearly satisfies this criteria. Indeed, it is not hard to see that all of the main properties of determinants follow quickly from (1.115).

The adjoint to f, written as $\overline{f}$, is defined by

$$\overline{f}(a) \equiv e^i\langle f(e_i)a\rangle \tag{1.118}$$

where $\{e_i\}$ is an arbitrary frame of vectors, with reciprocal frame $\{e^i\}$. A frame-invariant definition of the adjoint can be given using the vector derivative, but we have chosen not to introduce multivector calculus until Chapter 5. The definition (1.118) ensures that

$$\begin{aligned} b\cdot\overline{f}(a) &= a\cdot(b\cdot e^i \underline{f}(e_i)) \\ &= a\cdot\underline{f}(b). \end{aligned} \tag{1.119}$$

A symmetric function is one for which $\underline{f} = \overline{f}$.

The adjoint also extends via outermorphism and we find that, for example,

$$\begin{aligned}
\overline{f}(a \wedge b) &= \overline{f}(a) \wedge \overline{f}(b) \\
&= e^i \wedge e^j a \cdot \underline{f}(e_i) b \cdot \underline{f}(e_j) \\
&= \tfrac{1}{2} e^i \wedge e^j \left(a \cdot \underline{f}(e_i) b \cdot \underline{f}(e_j) - a \cdot \underline{f}(e_j) b \cdot \underline{f}(e_i) \right) \\
&= \tfrac{1}{2} e^i \wedge e^j (a \wedge b) \cdot \underline{f}(e_j \wedge e_i).
\end{aligned} \tag{1.120}$$

By using the same argument as in equation (1.119), it follows that

$$\langle \underline{f}(A) B \rangle = \langle A \overline{f}(B) \rangle \tag{1.121}$$

for all multivectors A and B. An immediate consequence is that

$$\begin{aligned}
\det \overline{f} &= \langle I^{-1} \overline{f}(I) \rangle \\
&= \langle \underline{f}(I^{-1}) I \rangle \\
&= \det f.
\end{aligned} \tag{1.122}$$

Equation (1.121) turns out to be a special case of the more general formulae,

$$\begin{aligned}
A_r \cdot \overline{f}(B_s) &= \overline{f}[\underline{f}(A_r) \cdot B_s] & r \le s \\
\underline{f}(A_r) \cdot B_s &= \underline{f}[A_r \cdot \overline{f}(B_s)] & r \ge s,
\end{aligned} \tag{1.123}$$

which are derived in [24, Chapter 3].

As an example of the use of (1.123) we find that

$$\overline{f}(\underline{f}(AI) I^{-1}) = A I \overline{f}(I^{-1}) = A \det f, \tag{1.124}$$

which is used to construct the inverse functions,

$$\begin{aligned}
\underline{f}^{-1}(A) &= \det(f)^{-1} \overline{f}(AI) I^{-1} \\
\overline{f}^{-1}(A) &= \det(f)^{-1} I^{-1} \underline{f}(IA).
\end{aligned} \tag{1.125}$$

These equations show how the inverse function is constructed from a double-duality operation. They are also considerably more compact and efficient than any matrix-based formula for the inverse.

Finally, the concept of an eigenvector is generalized to that of an eigenblade A_r, which is an r-grade blade satisfying

$$\underline{f}(A_r) = \alpha A_r, \tag{1.126}$$

where α is a *real* eigenvalue. Complex eigenvalues are in general not considered, since these usually loose some important aspect of the geometry of the function f. As an example, consider a function f satisfying

$$\begin{aligned}
f(a) &= b \\
f(b) &= -a,
\end{aligned} \tag{1.127}$$

for some pair of vectors a and b. Conventionally, one might write

$$f(a+jb) = -j(a+jb) \tag{1.128}$$

and say that $a+bj$ is an eigenvector with eigenvalue $-j$. But in geometric algebra one can instead write

$$\underline{f}(a\wedge b) = b\wedge(-a) = a\wedge b, \tag{1.129}$$

which shows that $a\wedge b$ is an eigenblade with eigenvalue $+1$. This is a geometrically more useful result, since it shows that the $a\wedge b$ plane is an invariant plane of f. The unit blade in this plane generates its own complex structure, which is the more appropriate object for considering the properties of f.

1.3.2 Non-Orthonormal Frames

At various points in this thesis we will make use of non-orthonormal frames, so a number of their properties are summarised here. From a set of n vectors $\{e_i\}$, we define the pseudoscalar

$$E_n = e_1\wedge e_2\wedge\ldots\wedge e_n. \tag{1.130}$$

The set $\{e_i\}$ constitute a (non-orthonormal) frame provided $E_n \neq 0$. The reciprocal frame $\{e^i\}$ satisfies

$$e^i\cdot e_j = \delta^i_j, \tag{1.131}$$

and is constructed via [24, Chapter 1]

$$e^i = (-1)^{i-1} e_1\wedge\ldots\check{e}_i\ldots\wedge e_n E^n, \tag{1.132}$$

where the check symbol on $\check{e}_i$ signifies that this vector is missing from the product. E^n is the pseudoscalar for the reciprocal frame, and is defined by

$$E^n = e^n\wedge e^{n-1}\wedge\ldots\wedge e^1. \tag{1.133}$$

The two pseudoscalars E_n and E^n satisfy

$$E_n E^n = 1, \tag{1.134}$$

and hence

$$E^n = E_n/(E_n)^2. \tag{1.135}$$

The components of the vector a in the e^i frame are given by $a\cdot e_i$, so that

$$a = (a\cdot e_i)e^i, \tag{1.136}$$

from which we find that

$$\begin{aligned} 2a &= 2a\cdot e_i e^i \\ &= e_i a e^i + a e_i e^i \\ &= e_i a e^i + na. \end{aligned} \tag{1.137}$$

The fact that $e_ie^i = n$ follows from (1.131) and (1.132). From (1.137) we find that

$$e_iae^i = (2-n)a, \tag{1.138}$$

which extends for a multivector of grade r to give the useful results:

$$\begin{aligned} e_iA_re^i &= (-1)^r(n-2r)A_r, \\ e_i(e^i\cdot A_r) &= rA_r, \\ e_i(e^i\wedge A_r) &= (n-r)A_r. \end{aligned} \tag{1.139}$$

For convenience, we now specialise to positive definite spaces. The results below are easily extended to arbitrary spaces through the introduction of a metric indicator function [28]. A symmetric metric tensor g can be defined by

$$g(e^i) = e_i, \tag{1.140}$$

so that, as a matrix, it has components

$$g_{ij} = e_i\cdot e_j. \tag{1.141}$$

Since

$$\underline{g}(E^n) = \tilde{E}_n, \tag{1.142}$$

it follows from (1.115) that

$$\det(g) = E_n\tilde{E}_n = |E_n|^2. \tag{1.143}$$

It is often convenient to work with the *fiducial frame* $\{\sigma_k\}$, which is the orthonormal frame determined by the $\{e_i\}$ via

$$e_k = h(\sigma_k) \tag{1.144}$$

where h is the *unique, symmetric* fiducial tensor. The requirement that h be symmetric means that the $\{\sigma_k\}$ frame must satisfy

$$\sigma_k\cdot e_j = \sigma_j\cdot e_k, \tag{1.145}$$

which, together with orthonormality, defines a set of n^2 equations that determine the σ_k (and hence h) uniquely, up to permutation. These permutations only alter the labels for the frame vectors, and do not re-define the frame itself. From (1.144) it follows that

$$e^j\cdot e_k = h(e^j)\cdot\sigma_k = \delta^j_k \tag{1.146}$$

so that

$$h(e^j) = \sigma^j = \sigma_j. \tag{1.147}$$

(We are working in a positive definite space, so $\sigma_j = \sigma^j$ for the orthonormal frame $\{\sigma_j\}$.) It can now be seen that h is the "square-root" of g,

$$g(e^j) = e_j = h(\sigma_j) = h^2(e^j). \tag{1.148}$$

It follows that

$$\det(h) = |E_n|. \tag{1.149}$$

The fiducial tensor, together with other non-symmetric square-roots of the metric tensor, find many applications in the geometric calculus approach to differential geometry [28]. We will also encounter a similar object in Chapter 7.

We have now seen that geometric algebra does indeed offer a natural language for encoding many of our geometric perceptions. Furthermore, the formulae for reflections and rotations have given ample justification to the view that the Clifford product is a fundamental aspect of geometry. Explicit construction in two, three and four dimensions has shown how geometric algebra naturally encompasses the more restricted algebraic systems of complex and quaternionic numbers. It should also be clear from the preceding section that geometric algebra encompasses both matrix and tensor algebra. The following three chapters are investigations into how geometric algebra encompasses a number of further algebraic systems.

Chapter 2

Grassmann Algebra and Berezin Calculus

This chapter outlines the basis of a translation between Grassmann calculus and geometric algebra. It is shown that geometric algebra is sufficient to formulate all of the required concepts, thus integrating them into a single unifying framework. The translation is illustrated with two examples, the "Grauss integral" and the "Grassmann Fourier transform". The latter demonstrates the full potential of the geometric algebra approach. The chapter concludes with a discussion of some further developments and applications. Some of the results presented in this chapter first appeared in the paper "*Grassmann calculus, pseudoclassical mechanics and geometric algebra*" [1].

2.1 Grassmann Algebra versus Clifford Algebra

The modern development of mathematics has led to the popularly held view that Grassmann algebra is more fundamental than Clifford algebra. This view is based on the idea (recall Section 1.2) that a Clifford algebra is the algebra of a quadratic form. But, whilst it is true that every (symmetric) quadratic form defines a Clifford algebra, it is certainly not true that the usefulness of geometric algebra is restricted to metric spaces. Like all mathematical systems, geometric algebra is subject to many different interpretations, and the inner product need not be related to the concepts of metric geometry. This is best illustrated by a brief summary of how geometric algebra is used in the study of projective geometry.

In projective geometry [31], points are labeled by vectors, a, the magnitude of which is unimportant. That is, points in a projective space of dimension $n-1$ are identified with rays in a space of dimension n which are solutions of the equation $x \wedge a = 0$. Similarly, lines are represented by bivector blades, planes by trivectors, and so on. Two products (originally defined by Grassmann) are needed to algebraically encode the principle concepts of projective geometry. These are the progressive and regressive products, which encode the concepts of the join and the meet respectively. The progressive product of two blades is simply the outer product. Thus, for two points a and b, the line joining them together is represented projectively by the bivector $a \wedge b$. If the grades of A_r and B_s sum to more than n and the vectors comprising A_r and B_s span n-dimensional space, then

the join is the pseudoscalar of the space. The regres... to [39]. More
the progressive product and duality. Duality is ... [51, 52], as
pseudoscalar, and is denoted A_r^*. For two blades ...

... of Grass-
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$$(A_r \vee B_s)^* = A_r^* \wedge B ...$$

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example, in two-dimensional projective geometry (performed i... (2.6)
space) the point of intersection of the lines given by $\boldsymbol{A}$ and $\boldsymbol{B}$, w...

... t of
... les

$$\begin{aligned} \boldsymbol{A} &= \boldsymbol{a}i \\ \boldsymbol{B} &= \boldsymbol{b}i, \end{aligned}$$

is given by the point

$$\boldsymbol{A} \vee \boldsymbol{B} = -\boldsymbol{a} \cdot \boldsymbol{B} = -i\boldsymbol{a} \wedge \boldsymbol{b}.$$

The definition of the meet shows clearly that it is most simply formulated in ... of the inner product, yet no metric geometry is involved. It is probably unsurprisin... learn that geometric algebra is ideally suited to the study of projective geometry [3... It is also well suited to the study of determinants and invariant theory [24], which are also usually thought to be the preserve of Grassmann algebra [49, 50]. For these reasons there seems little point in maintaining a rigid division between Grassmann and geometric algebra. The more fruitful approach is to formulate the known theorems from Grassmann algebra in the wider language of geometric algebra. There they can be compared with, and enriched by, developments from other subjects. This program has been largely completed by Hestenes, Sobczyk and Ziegler [24, 31]. This chapter addresses one of the remaining subjects — the "calculus" of Grassmann variables introduced by Berezin [35].

Before reaching the main content of this chapter, it is necessary to make a few comments about the use of complex numbers in applications of Grassmann variables (particularly in particle physics). We saw in Sections 1.2.2 and 1.2.3 that within the 2-dimensional and 3-dimensional *real* Clifford algebras there exist multivectors that naturally play the rôle of a unit imaginary. Similarly, functions of several complex variables can be studied in a real $2n$-dimensional algebra. Furthermore, in Chapter 4 we will see how the Schrödinger, Pauli and Dirac equations can all be given real formulations in the algebras of space and spacetime. This leads to the speculation that a scalar unit imaginary may be unnecessary for fundamental physics. Often, the use of a scalar imaginary disguises some more interesting geometry, as is the case for imaginary eigenvalues of linear transformations. However, there are cases in modern mathematics where the use of a scalar imaginary is entirely superfluous to calculations. Grassmann calculus is one of these. Accordingly, the unit imaginary is dropped in what follows, and an entirely real formulation is given.

2.2 The Geometrisation of Berezin Calculus

The basis of Grassmann/Berezin calculus is described in many sources. Berezin's "*The method of second quantisation*" [35] is one of the earliest and most cited texts, and a the join is the pseudoscalar of the space. The regressive product, denoted $\vee$, is built from the progressive product and duality. Duality is defined as (right)-multiplication by the pseudoscalar, and is denoted A_r^*. For two blades A_r and B_s, the meet is then defined by

$$(A_r \vee B_s)^* = A_r^* \wedge B_s^* \tag{2.1}$$

$$\Rightarrow A_r \vee B_s = A_r^* \cdot B_s. \tag{2.2}$$

It is implicit here that the dual is taken with respect to the join of A_r and B_s. As an example, in two-dimensional projective geometry (performed in the geometric algebra of space) the point of intersection of the lines given by $\boldsymbol{A}$ and $\boldsymbol{B}$, where

$$\boldsymbol{A} = \boldsymbol{a}i \tag{2.3}$$

$$\boldsymbol{B} = \boldsymbol{b}i, \tag{2.4}$$

is given by the point

$$\boldsymbol{A} \vee \boldsymbol{B} = -\boldsymbol{a} \cdot \boldsymbol{B} = -i\boldsymbol{a} \wedge \boldsymbol{b}. \tag{2.5}$$

The definition of the meet shows clearly that it is most simply formulated in terms of the inner product, yet no metric geometry is involved. It is probably unsurprising to learn that geometric algebra is ideally suited to the study of projective geometry [31]. It is also well suited to the study of determinants and invariant theory [24], which are also usually thought to be the preserve of Grassmann algebra [49, 50]. For these reasons there seems little point in maintaining a rigid division between Grassmann and geometric algebra. The more fruitful approach is to formulate the known theorems from Grassmann algebra in the wider language of geometric algebra. There they can be compared with, and enriched by, developments from other subjects. This program has been largely completed by Hestenes, Sobczyk and Ziegler [24, 31]. This chapter addresses one of the remaining subjects — the "calculus" of Grassmann variables introduced by Berezin [35].

Before reaching the main content of this chapter, it is necessary to make a few comments about the use of complex numbers in applications of Grassmann variables (particularly in particle physics). We saw in Sections 1.2.2 and 1.2.3 that within the 2-dimensional and 3-dimensional *real* Clifford algebras there exist multivectors that naturally play the rôle of a unit imaginary. Similarly, functions of several complex variables can be studied in a real $2n$-dimensional algebra. Furthermore, in Chapter 4 we will see how the Schrödinger, Pauli and Dirac equations can all be given real formulations in the algebras of space and spacetime. This leads to the speculation that a scalar unit imaginary may be unnecessary for fundamental physics. Often, the use of a scalar imaginary disguises some more interesting geometry, as is the case for imaginary eigenvalues of linear transformations. However, there are cases in modern mathematics where the use of a scalar imaginary is entirely superfluous to calculations. Grassmann calculus is one of these. Accordingly, the unit imaginary is dropped in what follows, and an entirely real formulation is given.

2.2 The Geometrisation of Berezin Calculus

The basis of Grassmann/Berezin calculus is described in many sources. Berezin's "*The method of second quantisation*" [35] is one of the earliest and most cited texts, and a useful summary of the main results from this is contained in the Appendices to [39]. More recently, Grassmann calculus has been extended to the field of superanalysis [51, 52], as well as in other directions [53, 54].

The basis of the approach adopted here is to utilise the natural embedding of Grassmann algebra within geometric algebra, thus reversing the usual progression from Grassmann to Clifford algebra via quantization. We start with a set of n Grassmann variables $\{\zeta_i\}$, satisfying the anticommutation relations

$$\{\zeta_i, \zeta_j\} = 0. \tag{2.6}$$

The Grassmann variables $\{\zeta_i\}$ are mapped into geometric algebra by introducing a set of n independent Euclidean vectors $\{e_i\}$, and replacing the product of Grassmann variables by the exterior product,

$$\zeta_i\zeta_j \quad \leftrightarrow \quad e_i \wedge e_j. \tag{2.7}$$

Equation (2.6) is now satisfied by virtue of the antisymmetry of the exterior product,

$$e_i \wedge e_j + e_j \wedge e_i = 0. \tag{2.8}$$

In this way any combination of Grassmann variables can be replaced by a multivector. Nothing is said about the interior product of the e_i vectors, so the $\{e_i\}$ frame is completely arbitrary.

In order for the above scheme to have computational power, we need a translation for for the calculus introduced by Berezin [35]. In this calculus, differentiation is defined by the rules

$$\frac{\partial \zeta_j}{\partial \zeta_i} = \delta_{ij}, \tag{2.9}$$

$$\zeta_j \frac{\overleftarrow{\partial}}{\partial \zeta_i} = \delta_{ij}, \tag{2.10}$$

together with the "graded Leibnitz rule",

$$\frac{\partial}{\partial \zeta_i}(f_1 f_2) = \frac{\partial f_1}{\partial \zeta_i} f_2 + (-1)^{[f_1]} f_1 \frac{\partial f_2}{\partial \zeta_i}, \tag{2.11}$$

where $[f_1]$ is the parity of f_1. The parity of a Grassmann variable is determined by whether it contains an even or odd number of vectors. Berezin differentiation is handled within the algebra generated by the $\{e_i\}$ frame by introducing the reciprocal frame $\{e^i\}$, and replacing

$$\frac{\partial}{\partial \zeta_i}(\quad \leftrightarrow \quad e^i \cdot (\tag{2.12}$$

so that

$$\frac{\partial \zeta_j}{\partial \zeta_i} \quad \leftrightarrow \quad e^i \cdot e_j = \delta^i_j. \tag{2.13}$$

It should be remembered that upper and lower indices are used to distinguish a frame from its reciprocal frame, whereas Grassmann algebra only uses these indices to distinguish metric signature.

The graded Leibnitz rule follows simply from the axioms of geometric algebra. For example, if f_1 and f_2 are grade-1 and so translate to vectors a and b, then the rule (2.11) becomes

$$e^i \cdot (a \wedge b) = e^i \cdot ab - ae^i \cdot b, \tag{2.14}$$

which is simply equation (1.14) again.

Right differentiation translates in a similar manner,

$$)\frac{\overleftarrow{\partial}}{\partial \zeta_i} \quad \leftrightarrow \quad) \cdot e^i, \tag{2.15}$$

and the standard results for Berezin second derivatives [35] can also be verified simply. For example, given that F is the multivector equivalent of the Grassmann variable $f(\zeta)$,

$$\begin{aligned} \frac{\partial}{\partial \zeta_i}\frac{\partial}{\partial \zeta_j} f(\zeta) \quad \leftrightarrow \quad e^i \cdot (e^j \cdot F) &= (e^i \wedge e^j) \cdot F \\ &= -e^j \cdot (e^i \cdot F) \end{aligned} \tag{2.16}$$

shows that second derivatives anticommute, and

$$\left(\frac{\partial f}{\partial \zeta_i}\right) \frac{\overleftarrow{\partial}}{\partial \zeta_j} \quad \leftrightarrow \quad (e^i \cdot F) \cdot e^j = e^i \cdot (F \cdot e^j) \tag{2.17}$$

shows that left and right derivatives commute.

The final concept needed is that of integration over a Grassmann algebra. In Berezin calculus, this is defined to be the same as right differentiation (apart perhaps from some unimportant extra factors of j and 2π [52]), so that

$$\int f(\zeta) d\zeta_n d\zeta_{n-1} \dots d\zeta_1 \equiv f(\zeta) \frac{\overleftarrow{\partial}}{\partial \zeta_n} \frac{\overleftarrow{\partial}}{\partial \zeta_{n-1}} \cdots \frac{\overleftarrow{\partial}}{\partial \zeta_1}. \tag{2.18}$$

These translate in exactly the same way as the right derivative (2.12). The only important formula is that for the total integral

$$\int f(\zeta) d\zeta_n d\zeta_{n-1} \dots d\zeta_1 \quad \leftrightarrow \quad (\dots((F \cdot e^n) \cdot e^{n-1}) \dots) \cdot e^1 = \langle F E^n \rangle, \tag{2.19}$$

where again F is the multivector equivalent of $f(\zeta)$, as defined by (2.6). Equation (2.19) picks out the coefficient of the pseudoscalar part of F since, if $\langle F \rangle_n$ is given by αE_n, then

$$\langle F E^n \rangle = \alpha. \tag{2.20}$$

Thus the Grassman integral simply returns the coefficient α.

A change of variables is performed by a linear transformation f, say, with

$$e_i' = f(e_i) \tag{2.21}$$

$$\Rightarrow E_n' = \underline{f}(E_n) = \det(f) E_n. \tag{2.22}$$

But the $\{e^i\}$ must transform under $\overline{f}^{-1}$ to preserve orthonormality, so

$$e^{i\prime} = \overline{f}^{-1}(e^i) \tag{2.23}$$

$$\Rightarrow E^{n\prime} = \det(f)^{-1} E^n, \tag{2.24}$$

which recovers the usual result for a change of variables in a Grassmann multiple integral. That $E_n' E^{n\prime} = 1$ follows from the definitions above.

In the above manner all the basic formulae of Grassmann calculus can be derived in geometric algebra, and often these derivations are simpler. Moreover, they allow for the results of Grassmann algebra to be incorporated into a wider scheme, where they may find applications in other fields. As a further comment, this translation also makes it clear why no measure is associated with Grassmann integrals: nothing is being added up!

2.2.1 Example I. The "Grauss" Integral

The Grassmann analogue of the Gaussian integral [35],

$$\int \exp\{\tfrac{1}{2}a^{jk}\zeta_j\zeta_k\}\, d\zeta_n \ldots d\zeta_1 = \det(a)^{1/2}, \tag{2.25}$$

where a^{jk} is an antisymmetric matrix, is one of the most important results in applications of Grassmann algebra. This result is used repeatedly in fermionic path integration, for example. It is instructive to see how (2.25) is formulated and proved in geometric algebra. First, we translate

$$\tfrac{1}{2}a^{jk}\zeta_j\zeta_k \quad \leftrightarrow \quad \tfrac{1}{2}a^{jk}e_j \wedge e_k = A, \text{ say}, \tag{2.26}$$

where A is a general bivector. The integral now becomes

$$\int \exp\{\tfrac{1}{2}a^{jk}\zeta_j\zeta_k\}\, d\zeta_n \ldots d\zeta_1 \quad \leftrightarrow \quad \langle (1 + A + \frac{A\wedge A}{2!} + \ldots) E^n \rangle. \tag{2.27}$$

It is immediately clear that (2.27) is only non-zero for even n ($= 2m$ say), in which case (2.27) becomes

$$\langle (1 + A + \frac{A\wedge A}{2!} + \ldots) E^n \rangle = \frac{1}{m!}\langle (A)^m E^n \rangle. \tag{2.28}$$

This type of expression is considered in Chapter 3 of [24] in the context of the eigenvalue problem for antisymmetric functions. This provides a good illustration of how the systematic use of a unified language leads to analogies between previously separate results.

In order to prove that (2.28) equals $\det(a)^{1/2}$ we need the result that, in spaces with Euclidean or Lorentzian signature, any bivector can be written, not necessarily uniquely, as a sum of orthogonal commuting blades. This is proved in [24, Chapter 3]. Using this result, we can write A as

$$A = \alpha_1 A_1 + \alpha_2 A_2 + \ldots \alpha_m A_m, \tag{2.29}$$

where

$$\begin{aligned} A_i \cdot A_j &= -\delta_{ij} & (2.30)\\ [A_i, A_j] &= 0 & (2.31)\\ A_1 A_2 \ldots A_m &= I. & (2.32) \end{aligned}$$

But the $\{e^i\}$ must transform under $\overline{f}^{-1}$ to preserve orthonormality, so

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where

$$A_i{\cdot}A_j = -\delta_{ij} \tag{2.30}$$

$$[A_i, A_j] = 0 \tag{2.31}$$

$$A_1 A_2 \ldots A_m = I. \tag{2.32}$$

Equation (2.28) now becomes,

$$\langle(\alpha_1\alpha_2\ldots\alpha_m)IE^n\rangle = \det(g)^{-1/2}\alpha_1\alpha_2\ldots\alpha_m, \tag{2.33}$$

where g is the metric tensor associated with the $\{e_i\}$ frame (1.140).

If we now introduce the function

$$f(a) = a\cdot A, \tag{2.34}$$

we find that [24, Chapter 3]

$$\begin{aligned} \underline{f}(a\wedge b) &= (a\cdot A)\wedge(b\cdot A) \\ &= \tfrac{1}{2}(a\wedge b)\cdot(A\wedge A) - (a\wedge b)\cdot AA. \end{aligned} \tag{2.35}$$

It follows that the A_i blades are the eigenblades of f, with

$$\underline{f}(A_i) = \alpha_i^2 A_i, \tag{2.36}$$

and hence

$$\underline{f}(I) = \underline{f}(A_1\wedge A_2\wedge\ldots A_m) = (\alpha_1\alpha_2\ldots\alpha_m)^2 I \tag{2.37}$$

$$\Rightarrow \det(f) = (\alpha_1\alpha_2\ldots\alpha_m)^2. \tag{2.38}$$

In terms of components, however,

$$\begin{aligned} f_j^k &= e_j\cdot f(e^k) \\ &= g_{jl}a^{lk}, \end{aligned} \tag{2.39}$$

$$\Rightarrow \det(f) = \det(g)\det(a). \tag{2.40}$$

Inserting (2.40) into (2.33), we have

$$\frac{1}{m!}\langle(A)^m E^n\rangle = \det(a)^{1/2}, \tag{2.41}$$

as required.

This result can be derived more succinctly using the fiducial frame $\sigma_i = h^{-1}(e_i)$ to write (2.27) as

$$\frac{1}{m!}\langle(A')^m I\rangle, \tag{2.42}$$

where $A' = \frac{1}{2}a^{jk}\sigma_j\sigma_k$. This automatically takes care of the factors of $\det(g)^{1/2}$, though it is instructive to note how these appear naturally otherwise.

2.2.2 Example II. The Grassmann Fourier Transform

Whilst the previous example did not add much new algebraically, it did serve to demonstrate that notions of Grassmann calculus were completely unnecessary for the problem. In many other applications, however, the geometric algebra formulation does provide for important algebraic simplifications, as is demonstrated by considering the Grassmann Fourier transform.

In Grassmann algebra one defines Fourier integral transformations between anticommuting spaces $\{\zeta_k\}$ and $\{\rho_k\}$ by [39]

$$\begin{aligned} G(\zeta) &= \int \exp\{j\sum \zeta_k\rho^k\}H(\rho)d\rho^n\ldots d\rho^1 \\ H(\rho) &= \epsilon^n \int \exp\{-j\sum \zeta_k\rho^k\}G(\zeta)d\zeta_n\ldots d\zeta_1, \end{aligned} \tag{2.43}$$

where $\epsilon^n = 1$ for n even and j for n odd. The factors of j are irrelevant and can be dropped, so that (2.43) becomes

$$\begin{aligned} G(\zeta) &= \int \exp\{\sum \zeta_k\rho^k\}H(\rho)d\rho^n\ldots d\rho^1 \\ H(\rho) &= (-1)^n \int \exp\{-\sum \zeta_k\rho^k\}G(\zeta)d\zeta_n\ldots d\zeta_1. \end{aligned} \tag{2.44}$$

These expressions are translated into geometric algebra by introducing a pair of anticommuting copies of the same frame, $\{e_k\}$, $\{f_k\}$, which satisfy

$$e_j\cdot e_k = f_j\cdot f_k \tag{2.45}$$

$$e_j\cdot f_k = 0. \tag{2.46}$$

The full set $\{e_k, f_k\}$ generate a $2n$-dimensional Clifford algebra. The translation now proceeds by replacing

$$\begin{aligned} \zeta_k &\leftrightarrow e_k, \\ \rho^k &\leftrightarrow f^k, \end{aligned} \tag{2.47}$$

where the $\{\rho^k\}$ have been replaced by elements of the reciprocal frame $\{f^k\}$. From (2.45), the reciprocal frames must also satisfy

$$e^j\cdot e^k = f^j\cdot f^k. \tag{2.48}$$

We next define the bivector (summation convention implied)

$$J = e_j\wedge f^j = e^j\wedge f_j. \tag{2.49}$$

The equality of the two expressions for J follows from (2.45),

$$\begin{aligned} e_j\wedge f^j &= (e_j\cdot e_k)e^k\wedge f^j \\ &= (f_j\cdot f_k)e^k\wedge f^j \\ &= e^k\wedge f_k. \end{aligned} \tag{2.50}$$

The bivector J satisfies

$$\begin{aligned} e_j\cdot J &= f_j \qquad & f_j\cdot J &= -e_j, \\ e^j\cdot J &= f^j \qquad & f^j\cdot J &= -e^j, \end{aligned} \tag{2.51}$$

and it follows that

$$(a\cdot J)\cdot J = -a, \tag{2.52}$$

for any vector a in the $2n$-dimensional algebra. Thus J generates a complex structure, which on its own is sufficient reason for ignoring the scalar j. Equation (2.52) can be extended to give

$$e^{-J\theta/2}ae^{J\theta/2} = \cos\theta\, a + \sin\theta\, a\cdot J, \tag{2.53}$$

from which it follows that $\exp\{J\pi/2\}$ anticommutes with all vectors. Consequently, this quantity can only be a multiple of the pseudoscalar and, since $\exp\{J\pi/2\}$ has unit magnitude, we can define the orientation such that

$$e^{J\pi/2} = I. \tag{2.54}$$

This definition ensures that

$$E_n F^n = E^n F_n = I. \tag{2.55}$$

Finally, we introduce the notation

$$C_k = \frac{1}{k!}\langle J^k\rangle_{2k}. \tag{2.56}$$

The formulae (2.44) now translate to

$$\begin{aligned} G(e) &= \sum_{j=0}^{n}(C_j\wedge H(f))\cdot F_n \\ H(f) &= (-1)^n\sum_{j=0}^{n}(\tilde{C}_j\wedge G(e))\cdot E^n, \end{aligned} \tag{2.57}$$

where the convention is adopted that terms where $C_j\wedge H$ or $\tilde{C}_j\wedge G$ have grade less than n do not contribute. Since G and H only contain terms purely constructed from the $\{e_k\}$ and $\{f^k\}$ respectively, (2.57) can be written as

$$\begin{aligned} G(e) &= \sum_{j=0}^{n}(C_{n-j}\wedge\langle H(f)\rangle_j)\cdot F_n \\ H(f) &= \sum_{j=0}^{n}(-1)^j(\langle G(e)\rangle_j\wedge C_{n-j})\cdot E^n. \end{aligned} \tag{2.58}$$

So far we have only derived a formula analogous to (2.44), but we can now go much further. By using

$$e^{J\theta} = \cos^n\theta + \cos^{n-1}\theta\,\sin\theta\, C_1 + \ldots + \sin^n\theta\, I \tag{2.59}$$

to decompose $e^{J(\theta+\pi/2)} = e^{J\theta}I$ in two ways, it can be seen that

$$C_{n-r} = (-1)^r C_r I = (-1)^r I C_r, \tag{2.60}$$

and hence (using some simple duality relations) (2.58) become

$$\begin{aligned} G(e) &= \sum_{j=0}^{n} C_j\cdot H_j E_n \\ H(f) &= (-1)^n\sum_{j=0}^{n} G_j\cdot C_j F^n. \end{aligned} \tag{2.61}$$

Finally, since G and H are pure in the $\{e_k\}$ and $\{f^k\}$ respectively, the effect of dotting with C_k is simply to interchange each e_k for an $-f_k$ and each f_k for an e_k. For vectors this is achieved by dotting with J. But, from (2.53), the same result is achieved by a rotation through $\pi/2$ in the planes of J. Rotations extend simply via outermorphism, so we can now write

$$\begin{aligned} C_j \cdot H_j &= e^{J\pi/4} H_j e^{-J\pi/4} \\ G_j \cdot C_j &= e^{-J\pi/4} G_j e^{J\pi/4}. \end{aligned} \tag{2.62}$$

We thus arrive at the following equivalent expressions for (2.57):

$$\begin{aligned} G(e) &= e^{J\pi/4} H(f) e^{-J\pi/4} E_n \\ H(f) &= (-1)^n e^{-J\pi/4} G(e) e^{J\pi/4} F^n. \end{aligned} \tag{2.63}$$

The Grassmann Fourier transformation has now been reduced to a rotation through $\pi/2$ in the planes specified by J, followed by a duality transformation. Proving the "inversion" theorem (*i.e.* that the above expressions are consistent) amounts to no more than carrying out a rotation, followed by its inverse,

$$\begin{aligned} G(e) &= e^{J\pi/4} \left((-1)^n e^{-J\pi/4} G(e) e^{J\pi/4} F^n \right) e^{-J\pi/4} E_n \\ &= G(e) E^n E_n = G(e). \end{aligned} \tag{2.64}$$

This proof is considerably simpler than any that can be carried out in the more restrictive system of Grassmann algebra.

2.3 Some Further Developments

We conclude this chapter with some further observations. We have seen how most aspects of Grassmann algebra and Berezin calculus can be formulated in terms of geometric algebra. It is natural to expect that other fields involving Grassmann variables can also be reformulated (and improved) in this manner. For example, many of the structures studied by de Witt [52] (super-Lie algebras, super-Hilbert spaces) have natural multivector expressions, and the cyclic cohomology groups of Grassmann algebras described by Coquereaux, Jadczyk and Kastler [53] can be formulated in terms of the multilinear function theory developed by Hestenes & Sobczyk [24, Chapter 3]. In Chapter 5 the formulation of this chapter is applied Grassmann mechanics and the geometric algebra approach is again seen to offer considerable benefits. Further applications of Grassmann algebra are considered in Chapter 3, in which a novel approach to the theory of linear functions is discussed. A clear goal for future research in this subject is to find a satisfactory geometric algebra formulation of supersymmetric quantum mechanics and field theory. Some preliminary observations on how such a formulation might be achieved are made in Chapter 5, but a more complete picture requires further research.

As a final comment, it is instructive to see how a Clifford algebra is traditionally built from the elements of Berezin calculus. It is well known [35] that the operators

$$\hat{Q}_k = \zeta_k + \frac{\partial}{\partial \zeta_k}, \tag{2.65}$$

satisfy the Clifford algebra generating relations

$$\{\hat{Q}_j, \hat{Q}_k\} = 2\delta_{jk}, \tag{2.66}$$

and this has been used by Sherry to provide an alternative approach to quantizing a Grassmann system [55, 56]. The geometric algebra formalism offers a novel insight into these relations. By utilising the fiducial tensor, we can write

$$\begin{aligned} \hat{Q}_k a(\zeta) \quad \leftrightarrow \quad e_k \wedge A + e^k \cdot A &= h(\sigma_k) \wedge A + h^{-1}(\sigma_k) \cdot A \\ &= \underline{h}(\sigma_k \wedge \underline{h}^{-1}(A)) + \underline{h}(s_k \cdot \underline{h}^{-1}(A)) \\ &= \underline{h}[\sigma_k \underline{h}^{-1}(A)], \end{aligned} \tag{2.67}$$

where A is the multivector equivalent of $a(\zeta)$ and we have used (1.123). The operator $\hat{Q}_k$ thus becomes an orthogonal Clifford vector (now Clifford multiplied), sandwiched between a symmetric distortion and its inverse. It is now simple to see that

$$\{\hat{Q}_j, \hat{Q}_k\} a(\zeta) \quad \leftrightarrow \quad \underline{h}(2\sigma_j \cdot \sigma_k \underline{h}^{-1}(A)) = 2\delta_{jk} A. \tag{2.68}$$

The above is an example of the ubiquity of the fiducial tensor in applications involving non-orthonormal frames. In this regard it is quite surprising that the fiducial tensor is not more prominent in standard expositions of linear algebra.

Berezin [35] defines dual operators to the $\hat{Q}_k$ by

$$\hat{P}_k = -j(\zeta_k - \frac{\partial}{\partial \zeta_k}), \tag{2.69}$$

though a more useful structure is derived by dropping the j, and defining

$$\hat{P}_k = \zeta_k - \frac{\partial}{\partial \zeta_k}. \tag{2.70}$$

These satisfy

$$\{\hat{P}_j, \hat{P}_k\} = -2\delta_{jk} \tag{2.71}$$

and

$$\{\hat{P}_j, \hat{Q}_k\} = 0, \tag{2.72}$$

so that the $\hat{P}_k, \hat{Q}_k$ span a $2n$-dimensional balanced algebra (signature n, n). The $\hat{P}_k$ can be translated in the same manner as the $\hat{Q}_k$, this time giving (for a homogeneous multivector)

$$\hat{P}_k a(\zeta) \quad \leftrightarrow \quad e_k \wedge A_r - e^k \cdot A_r = (-1)^r \underline{h}[\underline{h}^{-1}(A_r)\sigma_k]. \tag{2.73}$$

The $\{\sigma_k\}$ frame now sits to the right of the multivector on which it operates. The factor of $(-1)^r$ accounts for the minus sign in (2.71) and for the fact that the left and right multiples anticommute in (2.72). The $\hat{Q}_k$ and $\hat{P}_k$ can both be given right analogues if desired, though this does not add anything new. The $\{\hat{Q}_k\}$ and $\{\hat{P}_k\}$ operators are discussed more fully in Chapter 4, where they are related to the theory of the general linear group.

Chapter 3

Lie Groups and Spin Groups

This chapter demonstrates how geometric algebra provides a natural arena for the study of Lie algebras and Lie groups. In particular, it is shown that every matrix Lie group can be realised as a spin group. Spin groups consist of even products of unit magnitude vectors, and arise naturally from the geometric algebra treatment of reflections and rotations (introduced in Section 1.2.4). The generators of a spin group are bivectors, and it is shown that every Lie algebra can be represented by a bivector algebra. This brings the computational power of geometric algebra to applications involving Lie groups and Lie algebras. An advantage of this approach is that, since the rotors and bivectors are all elements of the same algebra, the discussion can move freely between the group and its algebra. The spin version of the general linear group is studied in detail, revealing some novel links with the structures of Grassmann algebra studied in Chapter 2. An interesting result that emerges from this work is that every linear transformation can be represented as a (geometric) product of vectors. Some applications of this result are discussed. A number of the ideas developed in this chapter appeared in the paper "*Lie groups as spin groups*" [2].

Throughout this chapter, the geometric algebra generated by p independent vectors of positive norm and q of negative norm is denoted as $\Re_{p,q}$. The grade-k subspace of this algebra is written as $\Re^k_{p,q}$ and the space of vectors, $\Re^1_{p,q}$, is abbreviated to $\Re^{p,q}$. The Euclidean algebra $\Re_{n,0}$ is abbreviated to $\Re_n$, and the vector space $\Re^1_n$ is written as $\Re^n$. Lie groups and their algebras are labeled according to the conventions of J.F. Cornwell's "*Group Theory in Physics*", Vol. 2 [57]. (A useful table of these conventions is found on page 392).

3.1 Spin Groups and their Generators

In this chapter we are interested in spin groups. These arise from the geometric algebra representation of orthogonal transformations — linear functions on $\Re^{p,q}$ which preserve inner products. We start by considering the case of the Euclidean algebra $\Re_n$. The simplest orthogonal transformation of $\Re^n$ is a reflection in the hyperplane perpendicular to some unit vector n,

$$\underline{n}(a) = -nan, \tag{3.1}$$

where we have recalled equation (1.70). (A convenient feature of the underbar/overbar notation for linear functions is that a function can be written in terms of the multivector that determines it.) The function $\underline{n}$ satisfies

$$\underline{n}(a)\cdot\underline{n}(b) = \langle nannbn\rangle = a\cdot b, \tag{3.2}$$

and so preserves the inner product. On combining $\underline{n}$ with a second reflection $\underline{m}$, where

$$\underline{m}(a) = -mam, \tag{3.3}$$

the function

$$\underline{m}\,\underline{n}(a) = mnanm \tag{3.4}$$

is obtained. This function also preserves inner products, and in Section 1.2.4 was identified as a rotation in the $m\wedge n$ plane. The group of even products of unit vectors is denoted spin(n). It consists of all even multivectors (rotors) satisfying

$$R\tilde{R} = 1 \tag{3.5}$$

and such that the quantity $Ra\tilde{R}$ is a vector for all vectors a. The double-sided action of a rotor R on a vector a is written as

$$\underline{R}(a) = Ra\tilde{R} \tag{3.6}$$

and the $\underline{R}$ form the group of rotations on $\Re^n$, denoted SO(n). The rotors afford a spin-1/2 description of rotations, hence rotor groups are referred to as *spin groups*.

In spaces with mixed signature the situation is slightly more complicated. In order to take care of the fact that a unit vector can now have $n^2 = \pm 1$, equation (3.1) must be modified to

$$\underline{n}(a) = -nan^{-1}. \tag{3.7}$$

Taking even combinations of reflections now leads to functions of the type

$$\underline{M}(a) = MaM^{-1}, \tag{3.8}$$

as opposed to $Ma\tilde{M}$. Again, the spin group spin(p,q) is defined as the group of even products of unit vectors, but its elements now satisfy $M\tilde{M} = \pm 1$. The term "rotor" is retained for elements of spin(p,q) satisfying $R\tilde{R} = 1$. The subgroup of spin(p,q) containing just the rotors is called the rotor group (this is sometimes written as spin$^+(p,q)$ in the literature). The action of a rotor on a vector a is always defined by (3.6). Spin groups and rotor groups are both Lie groups and, in a space with mixed signature, the spin group differs from the rotor group only by a direct product with an additional subgroup of discrete transformations.

The generators of a spin group are found by adapting the techniques found in any of the standard texts of Lie group theory (see [57], for example). We are only interested in the subgroup of elements connected to the identity, so only need to consider the rotor group. We introduce a one-parameter set of rotors $R(t)$, so that

$$R(t)a\tilde{R}(t) = \langle R(t)a\tilde{R}(t)\rangle_1 \tag{3.9}$$

for all vectors a and for all values of the parameter t. On differentiating with respect to t, we find that the quantity

$$\begin{aligned} R'a\tilde{R} + Ra\tilde{R}' &= R'\tilde{R}(Ra\tilde{R}) + (Ra\tilde{R})R\tilde{R}' \\ &= R'\tilde{R}(Ra\tilde{R}) - (Ra\tilde{R})R'\tilde{R} \end{aligned} \tag{3.10}$$

must be a vector, where we have used $R\tilde{R} = 1$ to deduce that

$$R'\tilde{R} = -R\tilde{R}'. \tag{3.11}$$

The commutator of $R'\tilde{R}$ with an arbitrary vector therefore results in a vector, so $R'\tilde{R}$ can only contain a bivector part. ($R'\tilde{R}$ cannot contain a scalar part, since $(R'\tilde{R})\tilde{} = -R'\tilde{R}$.) The generators of a rotor group are therefore a set of bivectors in the algebra containing the rotors.

A simple application of the Jacobi identity gives, for vectors a, b, c, and d,

$$(a\wedge b)\times(c\wedge d) = [(a\wedge b)\cdot c]\wedge d - [(a\wedge b)\cdot d]\wedge c, \tag{3.12}$$

so the commutator product of two bivector blades results in a third bivector. It follows that the space of bivectors is closed under the commutator product, and hence that the bivectors (together with the commutator product) form the Lie algebra of a spin group. It should be noted that the commutator product, $\times$, in equation (3.12) differs from the commutator bracket by a factor of 1/2. The commutator product is simpler to use, since it is the bivector part of the full geometric product of two bivectors A and B:

$$AB = A\cdot B + A\times B + A\wedge B \tag{3.13}$$

where

$$A\cdot B + A\wedge B = \tfrac{1}{2}(AB + BA), \tag{3.14}$$

$$A\times B = \tfrac{1}{2}(AB - BA). \tag{3.15}$$

For this reason the commutator product will be used throughout this chapter.

Since the Lie algebra of a spin group is generated by the bivectors, it follows that all rotors simply connected to the identity can be written in the form

$$R = e^{B/2}, \tag{3.16}$$

which ensures that

$$\tilde{R} = e^{-B/2} = R^{-1}. \tag{3.17}$$

The form of a rotor given by equation (3.16) was found in Section 1.2.4, where rotations in a single Euclidean plane were considered. The factor of 1/2 is included because rotors provide a half-angle description of rotations. In terms of the Lie algebra, the factor of 1/2 is absorbed into our use of the commutator product, as opposed to the commutator bracket.

It can be shown that, in positive definite spaces, all rotors can be written in the form of (3.16). The bivector B is not necessarily unique, however, as can be seen by considering the power series definition of the logarithm,

$$\ln X = 2[H + \frac{H^3}{3} + \frac{H^5}{5} + \cdots] \tag{3.18}$$

where

$$H = \frac{X-1}{X+1}. \tag{3.19}$$

It is implicit in this formula that $1+X$ is invertible, and the logarithm will not be well-defined if this is not the case. For example, the pseudoscalar I in $\Re_{4,0}$ is a rotor ($I\tilde{I} = 1$), the geometric effect of which is to reverse the sign of all vectors. But $1+I$ is not invertible, since $(1+I)^2 = 2(1+I)$. This manifests itself as a non-uniqueness in the logarithm of I — given *any* bivector blade B satisfying $B^2 = -1$, I can be written as

$$I = \exp\{B(1-I)\frac{\pi}{2}\}. \tag{3.20}$$

Further problems can arise in spaces with mixed signature. In the spacetime algebra, for example, whilst the rotor

$$R = (\gamma_0 + \gamma_1 - \gamma_2)\gamma_2 = 1 + (\gamma_0 + \gamma_1)\gamma_2 \tag{3.21}$$

can be written as

$$R = \exp\{(\gamma_0 + \gamma_1)\gamma_2\}, \tag{3.22}$$

the rotor

$$-R = \exp\{\gamma_1\gamma_2\frac{\pi}{2}\}R = -1 - (\gamma_0 + \gamma_1)\gamma_2 \tag{3.23}$$

cannot be written as the exponential of a bivector. The problem here is that the series for $\ln(-X)$ is found by replacing H by H^{-1} in equation (3.18) and, whilst $1 + R = 2 + (\gamma_0 + \gamma_1)\gamma_2$ is invertible, $1 - R = -(\gamma_0 + \gamma_1)\gamma_2$ is null and therefore not invertible.

Further examples of rotors with no logarithm can be constructed in spaces with other signatures. Near the identity, however, the Baker-Campbell-Hausdorff formula ensures that, for suitably small bivectors, one can always write

$$e^{A/2}e^{B/2} = e^{C/2}. \tag{3.24}$$

So, as is usual in Lie group theory, the bulk of the properties of the rotor (and spin) groups are transferred to the properties of their bivector generators.

In the study of Lie groups and their algebras, the adjoint representation plays a particularly important role. The adjoint representation of a spin group is formed from functions mapping the Lie algebra to itself,

$$Ad_M(B) \equiv MBM^{-1} = \underline{M}(B). \tag{3.25}$$

The adjoint representation is therefore formed by the outermorphism action of the linear functions $\underline{M}(a) = MaM^{-1}$. For the rotor subgroup, we have

$$Ad_R(B) = \underline{R}(B) = RB\tilde{R}. \tag{3.26}$$

It is immediately seen that the adjoint representation satisfies

$$Ad_{M_1}[Ad_{M_2}(B)] = Ad_{M_1M_2}(B). \tag{3.27}$$

The adjoint representation of the Lie group induces a representation of the Lie algebra as

$$ad_{A/2}(B) = A\times B, \tag{3.28}$$

or

$$ad_A(B) = 2A\times B. \tag{3.29}$$

The Jacobi identity ensures that

$$\begin{aligned}\tfrac{1}{2}(ad_A ad_B - ad_B ad_A)(C) &= 2[A\times(B\times C) - B\times(A\times C)] \\ &= 2(A\times B)\times C \\ &= ad_{A\times B}(C).\end{aligned} \tag{3.30}$$

The Killing form is constructed by considering ad_A as a linear operator on the space of bivectors, and defining

$$K(A,B) = \mathrm{Tr}(ad_A ad_B). \tag{3.31}$$

For the case where the Lie algebra is the set of all bivectors, we can define a basis set of bivectors as $B_K = e_i\wedge e_j$ $(i<j)$ with reciprocal basis $B^K = e^j\wedge e^i$. Here, the index K is a simplicial index running from 1 to $n(n-1)/2$ over all combinations of i and j with $i<j$. A matrix form of the adjoint representation is now given by

$$(ad_A)^K{}_J = 2(A\times B_J)\cdot B^K \tag{3.32}$$

so that the Killing form becomes

$$\begin{aligned}K(A,B) &= 4\sum_{J,K=1}^{n(n-1)/2}(A\times B_J)\cdot B^K(B\times B_K)\cdot B^J \\ &= 2[A\times(B\times(e_i\wedge e_j))]\cdot(e^j\wedge e^i) \\ &= \langle ABe_i\wedge e_j e^j\wedge e^i - Ae_i\wedge e_j Be^j\wedge e^i\rangle\end{aligned} \tag{3.33}$$

Now,

$$\begin{aligned}e_i\wedge e_j e^j\wedge e^i &= e_i e_j e^j\wedge e^i \\ &= n(n-1)\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}e_i\wedge e_j Be^j\wedge e^i &= e_i e_j Be^j\wedge e^i \\ &= e_i e_j Be^j e^i - e_i e_j e^i\cdot e^j B \\ &= [(n-4)^2 - n]B\end{aligned} \tag{3.35}$$

where we have used equations (1.139). On recombining (3.34) and (3.35), the Killing form on a bivector algebra becomes

$$K(A,B) = 8(n-2)\langle AB\rangle \tag{3.36}$$

$$\begin{array}{|lcll|}\hline E_{ij} & = & e_i \wedge e_j & (i<j \quad i,j=1\ldots p) \\ F_{ij} & = & f_i \wedge f_j & (i<j \quad i,j=1\ldots q) \\ G_{ij} & = & e_i \wedge f_j & (i=1\ldots p,\ j=1\ldots q). \\ \hline \end{array}$$

Table 3.1: Bivector Basis for so(p,q)

and so is given by the scalar part of the geometric product of two bivectors. The constant is irrelevant, and will be ignored. The same form will be inherited by all sub-algebras of the bivector algebra, so we can write

$$K(A,B) \equiv A \cdot B \tag{3.37}$$

as the Killing form for any bivector (Lie) algebra. This product is clearly symmetric, and is invariant under the adjoint action of any of the group elements. The fact that both the scalar and bivector parts of the geometric product of bivectors now have roles in the study of Lie algebras is a useful unification — rather than calculate separate commutators and inner products, one simply calculates a geometric product and reads off the parts of interest.

As an example, the simplest of the spin groups is the full rotor group spin(p,q) in some $\Re_{p,q}$. The Lie algebra of spin(p,q) is the set of bivectors $\Re^2_{p,q}$. By introducing a basis set of p positive norm vectors $\{e_i\}$ and q negative norm vectors $\{f_i\}$, a basis set for the full Lie algebra is given by the generators in Table 3.1. These generators provide a bivector realisation of the Lie algebra so(p,q). When the $\{e_i\}$ and $\{f_i\}$ are chosen to be orthonormal, it is immediately seen that the Killing form has $(p(p-1)+q(q-1))/2$ bivectors of negative norm and pq of positive norm. The sum of these is $n(n-1)/2$, where $n=p+q$. The algebra is unaffected by interchanging the signature of the space from $\Re_{p,q}$ to $\Re_{q,p}$. Compact Killing metrics arise from bivectors in positive (or negative) definite vector spaces.

We now turn to a systematic study of the remaining spin groups and their bivector generators. These are classified according to their invariants which, for the classical groups, are non-degenerate bilinear forms. In the geometric algebra treatment, bilinear forms are determined by certain multivectors, and the groups are studied in terms of these invariant multivectors.

3.2 The Unitary Group as a Spin Group

It has already been stressed that the use of a unit scalar imaginary frequently hides useful geometric information. This remains true for the study of the unitary groups. The basic idea needed to discuss the unitary groups was introduced in Section 2.2.2. One starts in an n-dimensional space of arbitrary signature, and introduces a second (anticommuting) copy of this space. Thus, if the set $\{e_i\}$ form a frame for the first space, the second space is generated by a frame $\{f_i\}$ satisfying equations (2.45) and (2.46). The spaces are related by the "doubling" bivector J, defined as (2.49)

$$J = e_j \wedge f^j = e^j \wedge f_j. \tag{3.38}$$

We recall from Section 2.2.2 that J satisfies

$$(a\cdot J)\cdot J = -a \tag{3.39}$$

for all vectors a in the $2n$-dimensional space. From J the linear function $\underline{J}$ is defined as

$$\underline{J}(a) \equiv a\cdot J = e^{-J\pi/4} a e^{J\pi/4}. \tag{3.40}$$

The function $\underline{J}$ satisfies

$$\underline{J}^2(a) = -a \tag{3.41}$$

and provides the required complex structure — the action of $\underline{J}$ being equivalent to multiplication of a complex vector by j.

An important property of J is that it is independent of the frame from which it was constructed. To see this, consider a transformation $\underline{h}$ taking the $\{e_i\}$ to a new frame

$$e_i' = \underline{h}(e_i) \tag{3.42}$$

$$\Rightarrow \quad e^{i\prime} = \overline{h}^{-1}(e^i) \tag{3.43}$$

so that the transformed J is

$$\begin{aligned} J' &= \underline{h}(e_j)\wedge\overline{h}^{-1}(f^j) \\ &= (e_k e^k\cdot\underline{h}(e_j))\wedge\overline{h}^{-1}(f^j). \end{aligned} \tag{3.44}$$

But $\underline{h}(e_j)$ remains in the space spanned by the $\{e_i\}$, so

$$\begin{aligned} e^k\cdot\underline{h}(e_j) &= f^k\cdot\underline{h}(f_j) \\ &= f_j\cdot\overline{h}(f^k), \end{aligned} \tag{3.45}$$

and now

$$\begin{aligned} J' &= e_k\wedge\big(f_j\cdot\overline{h}(f^k)\overline{h}^{-1}(f^j)\big) \\ &= e_k\wedge\overline{h}^{-1}\overline{h}(f^k) \\ &= J. \end{aligned} \tag{3.46}$$

We now turn to a study of the properties of the outermorphism of $\underline{J}$. A simple application of the Jacobi identity yields

$$\begin{aligned} (a\wedge b)\times J &= (a\cdot J)\wedge b + a\wedge(b\cdot J) \\ &= \underline{J}(a)\wedge b + a\wedge\underline{J}(b) \end{aligned} \tag{3.47}$$

and, using this result again, we derive

$$\begin{aligned} [(a\wedge b)\times J]\times J &= \underline{J}^2(a)\wedge b + \underline{J}(a)\wedge\underline{J}(b) + \underline{J}(a)\wedge\underline{J}(b) + a\wedge\underline{J}^2(b) \\ &= 2(\underline{J}(a\wedge b) - a\wedge b). \end{aligned} \tag{3.48}$$

It follows that

$$\underline{J}(B) = B + \tfrac{1}{2}(B\times J)\times J, \tag{3.49}$$

We recall from Section 2.2.2 that J satisfies

$$(a\cdot J)\cdot J = -a \tag{3.39}$$

for all vectors a in the $2n$-dimensional space. From J the linear function $\underline{J}$ is defined as

$$\underline{J}(a) \equiv a\cdot J = e^{-J\pi/4} a e^{J\pi/4}. \tag{3.40}$$

The function $\underline{J}$ satisfies

$$\underline{J}^2(a) = -a \tag{3.41}$$

and provides the required complex structure — the action of $\underline{J}$ being equivalent to multiplication of a complex vector by j.

An important property of J is that it is independent of the frame from which it was constructed. To see this, consider a transformation $\underline{h}$ taking the $\{e_i\}$ to a new frame

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so that the transformed J is

$$\begin{aligned} J' &= \underline{h}(e_j)\wedge\overline{h}^{-1}(f^j) \\ &= (e_k e^k\cdot\underline{h}(e_j))\wedge\overline{h}^{-1}(f^j). \end{aligned} \tag{3.44}$$

But $\underline{h}(e_j)$ remains in the space spanned by the $\{e_i\}$, so

$$\begin{aligned} e^k\cdot\underline{h}(e_j) &= f^k\cdot\underline{h}(f_j) \\ &= f_j\cdot\overline{h}(f^k), \end{aligned} \tag{3.45}$$

and now

$$\begin{aligned} J' &= e_k\wedge\big(f_j\cdot\overline{h}(f^k)\overline{h}^{-1}(f^j)\big) \\ &= e_k\wedge\overline{h}^{-1}\overline{h}(f^k) \\ &= J. \end{aligned} \tag{3.46}$$

We now turn to a study of the properties of the outermorphism of $\underline{J}$. A simple application of the Jacobi identity yields

$$\begin{aligned} (a\wedge b)\times J &= (a\cdot J)\wedge b + a\wedge(b\cdot J) \\ &= \underline{J}(a)\wedge b + a\wedge\underline{J}(b) \end{aligned} \tag{3.47}$$

and, using this result again, we derive

$$\begin{aligned} [(a\wedge b)\times J]\times J &= \underline{J}^2(a)\wedge b + \underline{J}(a)\wedge\underline{J}(b) + \underline{J}(a)\wedge\underline{J}(b) + a\wedge\underline{J}^2(b) \\ &= 2(\underline{J}(a\wedge b) - a\wedge b). \end{aligned} \tag{3.48}$$

It follows that

$$\underline{J}(B) = B + \tfrac{1}{2}(B\times J)\times J, \tag{3.49}$$

for all bivectors B. If the bivector B commutes with J, then we see that

$$\underline{J}(B) = B, \tag{3.50}$$

so that B is an eigenbivector of $\underline{J}$ with eigenvalue $+1$. The converse is also true — all eigenbivectors of $\underline{J}$ with eigenvalue $+1$ commute with J. This result follows by using

$$\underline{J}(B) = B \tag{3.51}$$

to write the eigenbivector B as

$$B = \tfrac{1}{2}(B + \underline{J}(B)). \tag{3.52}$$

But, for a blade $a\wedge b$,

$$\begin{aligned} [a\wedge b + \underline{J}(a\wedge b)]\times J &= \underline{J}(a)\wedge b + a\wedge\underline{J}(b) + \underline{J}^2(a)\wedge\underline{J}(b) + \underline{J}(a)\wedge\underline{J}^2(b) \\ &= 0, \end{aligned} \tag{3.53}$$

and the same must be true for all sums of blades. All bivectors of the form $B + \underline{J}(B)$ therefore commute with J, from which it follows that all eigenbivectors of $\underline{J}$ also commute with J. In fact, since the action of $\underline{J}$ on bivectors satisfies

$$\underline{J}^2(a\wedge b) = \underline{J}^2(a)\wedge\underline{J}^2(b) = (-a)\wedge(-b) = a\wedge b, \tag{3.54}$$

any bivector of the form $B + \underline{J}(B)$ is an eigenbivector of $\underline{J}$.

The next step in the study of the unitary group is to find a representation of the Hermitian inner product. If we consider a pair of complex vectors u and v with components $\{u_k\}$ and $\{v_k\}$, where

$$\begin{aligned} u_k &= x_k + jy_k \\ v_k &= r_k + js_k, \end{aligned} \tag{3.55}$$

then

$$\epsilon(u, v) = u_k^\dagger v_k = x_k r_k + y_k s_k + j(x_k s_k - y_k r_k). \tag{3.56}$$

Viewed as a pair of real products, (3.56) contains a symmetric and a skew-symmetric term. The symmetric part is the inner product in our $2n$-dimensional vector space. Any skew-symmetric inner product can be written in the form $(a\wedge b)\cdot B$, where B is some bivector. For the Hermitian inner product this bivector is J, which follows immediately from considering the real part of the inner product of $\epsilon(ja, b)$. The form of the Hermitian inner product in our $2n$-dimensional vector space is therefore

$$\epsilon(a, b) = a\cdot b + (a\wedge b)\cdot Jj. \tag{3.57}$$

This satisfies

$$\epsilon(b, a) = a\cdot b - (a\wedge b)\cdot Jj = \epsilon(a, b)^*, \tag{3.58}$$

as required. The introduction of the j disguises the fact that the Hermitian product contains two separate bilinear forms, both of which are invariant under the action of the unitary group. All orthogonal transformations leave $a\cdot b$ invariant, but only a subset will leave $(a\wedge b)\cdot J$ invariant as well. These transformations must satisfy

$$\big(\underline{f}(a)\wedge\underline{f}(b)\big)\cdot J = (a\wedge b)\cdot\overline{f}(J) = (a\wedge b)\cdot J \tag{3.59}$$

for all vectors a and b. The invariance group therefore consists of all orthogonal transformations whose outermorphism satisfies

$$\underline{f}(J) = J. \tag{3.60}$$

This requirement excludes all discrete transformations, since a vector n will only generate a symmetry if

$$\begin{aligned} \underline{n}(J) = nJn^{-1} &= J \\ \Rightarrow \qquad n\cdot J &= 0, \end{aligned} \tag{3.61}$$

and no such vector n exists. It follows that the symmetry group is constructed entirely from the double sided action of the elements of the spin group which satisfy

$$MJ = JM. \tag{3.62}$$

These elements afford a spin group representation of the unitary group.

Equation (3.62) requires that, for a rotor R simply connected to the identity, the bivector generator of R commutes with J. The Lie algebra of a unitary group is therefore realised by the set of bivectors commuting with J, which we have seen are also eigenbivectors of $\underline{J}$. Given an arbitrary bivector B, therefore, the bivector

$$B_J = B + \underline{J}(B) \tag{3.63}$$

is contained in the bivector algebra of u(p,q). This provides a quick method for writing down a basis set of generators. It is convenient at this point to introduce an orthonormal frame of vectors $\{e_i, f_i\}$ satisfying

$$e_i\cdot e_j = f_i\cdot f_j = \eta_{ij} \tag{3.64}$$

$$e_i\cdot f_j = 0, \tag{3.65}$$

where $\eta_{ij} = \eta_i\delta_{jk}$ (no sum) and η_i is the metric indicator ($= 1$ or -1). This frame is used to write down a basis set of generators which are orthogonal with respect to the Killing form. Such a basis for u(p,q) is contained in Table 3.2. This basis has dimension

$$\tfrac{1}{2}n(n-1) + \tfrac{1}{2}n(n-1) + n = n^2. \tag{3.66}$$

Of these, $p^2 + q^2$ bivectors have negative norm, and $2pq$ have positive norm.

The algebra of Table 3.2 contains the bivector J, which commutes with all other elements of the algebra and generates a U(1) subgroup. This is factored out to give the basis for su(p,q) contained in Table 3.3. The H_i are written in the form given to take care of the metric signature of the vector space. When working in $\Re_{2n}$ one can simply write

$$H_i = J_i - J_{i+1}. \tag{3.67}$$

$$\boxed{\begin{array}{rcll} E_{ij} & = & e_ie_j + f_if_j & (i<j=1\dots n) \\ F_{ij} & = & e_if_j - f_ie_j & \quad '' \\ J_i & = & e_if_i & (i=1\dots n). \end{array}}$$

Table 3.2: Bivector Basis for u(p,q)

$$\boxed{\begin{array}{rcll} E_{ij} & = & e_ie_j + f_if_j & (i<j=1\dots n) \\ F_{ij} & = & e_if_j - f_ie_j & \quad '' \\ H_i & = & e_if^i - e_{i+1}f^{i+1} & (i=1\dots n-1). \end{array}}$$

Table 3.3: Bivector Basis for su(p,q)

The use of Hermitian forms hides the properties of J in the imaginary j, which makes it difficult to relate the unitary groups to other groups. In particular, the group of linear transformations on $\Re^{2n}$ whose outermorphism leaves J invariant form the symplectic group Sp(n,R). Since U(n) leaves $a \cdot b$ invariant as well as J, we obtain the group relation

$$\mathrm{U}(n) \cong \mathrm{O}(2n) \cap \mathrm{Sp}(n,\mathrm{R}). \tag{3.68}$$

More generally, we find that

$$\mathrm{U}(p,q) \cong \mathrm{O}(2p,2q) \cap \mathrm{Sp}(p,q,\mathrm{R}), \tag{3.69}$$

where Sp(p,q,R) is group of linear transformations leaving J invariant in the mixed-signature space $\Re_{2p,2q}$. The geometric algebra approach to Lie group theory makes relations such as (3.69) quite transparent. Furthermore, the doubling bivector J appears in many other applications of geometric algebra — we saw one occurrence in Section 2.2.2 in the discussion of the Grassmann-Fourier transform. Other applications include multiparticle quantum mechanics and Hamiltonian mechanics [32]. Consistent use of geometric algebra can reveal these (often hidden) similarities between otherwise disparate fields.

3.3 The General Linear Group as a Spin Group

The development of the general linear group as a spin group parallels that of the unitary groups. Again, the dimension of the space is doubled by introducing a second space, but this time the second space has opposite signature. This leads to the development of a Grassmann structure, as opposed to a complex structure. Vectors in $\Re^{p,q}$ are then replaced by null vectors in $\Re^{n,n}$, where $n = p + q$. Since a (positive) dilation of a null vector can also be given a rotor description, it becomes possible to build up a rotor description of the entire general linear group from combinations of dilations and orthogonal transformations.

The required construction is obtained by starting with a basis set of vectors $\{e_i\}$ in $\Re^{p,q}$, and introducing a second space of opposite signature. The second space is generated by a set of vectors $\{f_i\}$ satisfying

$$e_i\cdot e_j = -f_i\cdot f_j \tag{3.70}$$
$$e_i\cdot f_j = 0, \tag{3.71}$$

and the full set $\{e_i, f_i\}$ form a basis set for $\Re^{n,n}$. The vector space $\Re^{n,n}$ is split into two null spaces by introducing the bivector K defined by

$$K = e^j\wedge f_j = -e_j\wedge f^j. \tag{3.72}$$

Again, K is independent of the initial choice of the $\{e_i\}$ frame. The bivector K determines the linear function $\underline{K}$ by

$$\underline{K}(a) \equiv a\cdot K. \tag{3.73}$$

The function $\underline{K}$ satisfies

$$\begin{aligned} \underline{K}(e_i) &= f_i & \underline{K}(f_i) &= e_i \\ \underline{K}(e^i) &= -f^i & \underline{K}(f^i) &= -e^i, \end{aligned} \tag{3.74}$$

and

$$\underline{K}^2(a) = (a\cdot K)\cdot K = a, \tag{3.75}$$

for all vectors a.

Proceeding as for the complexification bivector J we find that, for an arbitrary bivector B,

$$\underline{K}(B) = -B + \tfrac{1}{2}(B\times K)\times K. \tag{3.76}$$

Any bivector commuting with K is therefore an eigenbivector of $\underline{K}$, but now with eigenvalue -1.

An arbitrary vector a in $\Re^{n,n}$ can be decomposed into a pair of null vectors,

$$a = a_+ + a_-, \tag{3.77}$$

where

$$a_+ = \tfrac{1}{2}(a + \underline{K}(a)), \tag{3.78}$$
$$a_- = \tfrac{1}{2}(a - \underline{K}(a)). \tag{3.79}$$

That a_+ is null follows from

$$\begin{aligned} (a_+)^2 &= \tfrac{1}{4}\big(a^2 + 2a\cdot(a\cdot K) + (a\cdot K)\cdot(a\cdot K)\big) \\ &= \tfrac{1}{4}(a^2 - [(a\cdot K)\cdot K]\cdot a) \\ &= \tfrac{1}{4}(a^2 - a^2) \\ &= 0, \end{aligned} \tag{3.80}$$

and the same holds for a_-. The scalar product between a_+ and a_- is, of course, non-zero:

$$a_+\cdot a_- = \tfrac{1}{4}(a^2 - (a\cdot K)^2) = \tfrac{1}{2}a^2. \tag{3.81}$$

This construction decomposes $\Re^{n,n}$ into two separate null spaces, $\mathcal{V}^n$ and $\mathcal{V}^{n*}$, defined by

$$\begin{aligned} \underline{K}(a) &= a \qquad \forall a \in \mathcal{V}^n \\ \underline{K}(a) &= -a \qquad \forall a \in \mathcal{V}^{n*}, \end{aligned} \tag{3.82}$$

so that

$$\Re^{n,n} = \mathcal{V}^n \oplus \mathcal{V}^{n*}. \tag{3.83}$$

A basis is defined for each of $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ by

$$w_i = \tfrac{1}{2}(e_i + \underline{K}(e_i)) \tag{3.84}$$

$$w^{*i} = \tfrac{1}{2}(e^i - \underline{K}(e^i)), \tag{3.85}$$

respectively. These basis vectors satisfy

$$w_i \cdot w_j = w^{*i} \cdot w^{*j} = 0 \tag{3.86}$$

and

$$w^{*i} \cdot w_j = \tfrac{1}{2}\delta^i_j. \tag{3.87}$$

In conventional accounts, the space $\mathcal{V}^n$ would be recognised as a Grassmann algebra (all vector generators anticommute), with $\mathcal{V}^{n*}$ identified as the dual space of functions acting on $\mathcal{V}^n$. In Chapter 2 we saw how both $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ can be represented in terms of functions in a single n-dimensional algebra. Here, a different construction is adopted in which the $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ spaces are kept maximally distinct, so that they generate a $2n$-dimensional vector space. This is the more useful approach for the study of the Lie algebra of the general linear group. We shall shortly see how these two separate approaches are reconciled by setting up an isomorphism between operations in the two algebras.

We are interested in the group of orthogonal transformations which keep the $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ spaces separate. For a vector a in $\mathcal{V}^n$, the orthogonal transformation $\underline{f}$ must then satisfy

$$\underline{f}(a) = \underline{f}(a) \cdot K. \tag{3.88}$$

But, since $a = a \cdot K$ and $\underline{f}^{-1} = \overline{f}$, equation (3.88) leads to

$$\begin{aligned} a \cdot K &= \overline{f}[\underline{f}(a) \cdot K] \\ &= a \cdot \overline{f}(K), \end{aligned} \tag{3.89}$$

which must hold for all a. It follows that

$$\underline{f}(K) = K \tag{3.90}$$

and we will show that the $\underline{f}$ satisfying this requirement form the general linear group GL(n,R). The orthogonal transformations satisfying (3.90) can each be given a spin description, which enables the general linear group to be represented by a spin group. The elements of this spin group must satisfy

$$MK = KM. \tag{3.91}$$

$$
\boxed{\begin{array}{rcll} E_{ij} & = & e_i e_j - \hat{e}_i \hat{e}_j & (i < j = 1 \ldots n) \\ F_{ij} & = & e_i \hat{e}_j - \hat{e}_i e_j & \quad '' \\ J_i & = & e_i \hat{e}_i & (i = 1 \ldots n). \end{array}}
$$

Table 3.4: Bivector Basis for gl(n,R)

$$
\boxed{\begin{array}{rcll} E_{ij} & = & e_i e_j - \hat{e}_i \hat{e}_j & (i < j = 1 \ldots n) \\ F_{ij} & = & e_i \hat{e}_j - \hat{e}_i e_j & \quad '' \\ H_i & = & e_i \hat{e}_i - e_{i+1} \hat{e}_{i+1} & (i = 1 \ldots n-1). \end{array}}
$$

Table 3.5: Bivector Basis for sl(n,R)

The generators of the rotor part of the spin group are therefore the set of bivectors which commute with K, which are eigenbivectors of $\underline{K}$ with eigenvalue -1.

Before writing down an orthogonal basis for the Lie algebra, it is useful to introduce some further notation. We now take $\{e_i\}$ to be an orthonormal basis for the Euclidean algebra $\Re^n$, and $\{\hat{e}_i\}$ to be the corresponding basis for the anti-Euclidean algebra $\Re^{0,n}$. These basis vectors satisfy

$$
\begin{gathered} e_i \cdot e_j = \delta_{ij} = -\hat{e}_i \cdot \hat{e}_j \\ e_i \cdot \hat{e}_j = 0. \end{gathered} \tag{3.92}
$$

The hat also serves as a convenient abbreviation for the action of $\underline{K}$ on a vector a,

$$
\hat{a} \equiv \underline{K}(a). \tag{3.93}
$$

Since all bivectors in the Lie algebra of GL(n,R) are of the form $B - \underline{K}(B)$, an orthogonal basis for the Lie algebra can now be written down easily. Such a basis is contained in Table 3.4. The algebra in Table 3.4 includes K, which generates an abelian subgroup. This is factored out to leave the Lie algebra sl(n,R) contained in Table 3.5.

The form of the Lie algebra for the group GL(n,R) is clearly very close to that for U(n) contained in Table 3.2. The reason can be seen by considering the bilinear form generated by the bivector K,

$$
\epsilon(a, b) = a \cdot \underline{K}(b). \tag{3.94}
$$

If we decompose a and b in the orthonormal basis of (3.92),

$$
a = x^i e_i + y^i \hat{e}_i \tag{3.95}
$$

$$
b = r^i e_i + s^i \hat{e}_i, \tag{3.96}
$$

we find that

$$
\epsilon(a, b) = x^i s^i - y^i r^i, \tag{3.97}
$$

which is the component form of the symplectic norm in $\Re^{2n}$. We thus have the group relation

$$
\mathrm{GL}(n,\mathrm{R}) \cong \mathrm{O}(n,n) \cap \mathrm{Sp}(n,\mathrm{R}), \tag{3.98}
$$

which is to be compared with (3.68) and (3.69). The differences between the Lie algebras of GL(n,R) and U(n) are due solely to the metric signature of the underlying vector space which generates the bivector algebra. It follows that both Lie algebras have the same complexification, since complexification removes all dependence on signature. In the theory of the classification of the semi-simple Lie algebras, the complexification of the su(n) and sl(n,R) algebras is denoted A_{n-1}.

An alternative basis for sl(n,R) can be given in terms of the $\{w_i\}$ and $\{w_i^*\}$ null frames, which are now defined as

$$\begin{aligned} w_i &= \tfrac{1}{2}(e_i + \hat{e}_i) \\ w_i^* &= \tfrac{1}{2}(e_i - \hat{e}_i). \end{aligned} \tag{3.99}$$

The $\{w_i\}$ and $\{w_i^*\}$ frames satisfy

$$w_i \cdot w_j = w_i^* \cdot w_j^* = 0 \tag{3.100}$$

and

$$w_i w_j^* + w_j^* w_i = \delta_{ij}, \tag{3.101}$$

which are identifiable as the relations of the algebra of fermionic creation and annihilation operators. The pseudoscalars for the the $\mathcal{V}^n$ and $\mathcal{V}^{n*}$ spaces are defined by

$$\begin{aligned} W_n &= w_1 w_2 \ldots w_n \\ W_n^* &= w_1^* w_2^* \ldots w_n^* \end{aligned} \tag{3.102}$$

respectively. If we now define

$$\begin{aligned} I_{ij}^+ &= \tfrac{1}{2}(E_{ij} + F_{ij}) \\ &= \tfrac{1}{2}(e_i - \hat{e}_i)(e_j + \hat{e}_j) \\ &= 2w_i^* w_j \end{aligned} \tag{3.103}$$

and

$$\begin{aligned} I_{ij}^- &= \tfrac{1}{2}(E_{ij} - F_{ij}) \\ &= \tfrac{1}{2}(e_i + \hat{e}_i)(e_j - \hat{e}_j) \\ &= -2w_j^* w_i, \end{aligned} \tag{3.104}$$

we see that a complete basis for sl(n,R) is defined by the set $\{I_{ij}^+, I_{ij}^-, H_i\}$. This corresponds to the *Chevalley basis* for A_{n-1}. Furthermore, a complete basis set of generators for GL(n,R) is given by the set $\{w_i^* \wedge w_j\}$, defined over all i, j. This is perhaps the simplest of the possible basis sets for the Lie algebra, though it has the disadvantage that it is not orthogonal with respect to the Killing form.

We now turn to a proof that the subgroup of the spin group which leaves K invariant does indeed form a representation of GL(n,R). With a vector a in $\Re^n$ represented by the null vector $a_+ = (a + \hat{a})$ in $\Re^{n,n}$, we must prove that an arbitrary linear transformation of a, $a \mapsto \underline{f}(a)$, can be written in $\Re_{n,n}$ as

$$a_+ \mapsto M a_+ M^{-1}, \tag{3.105}$$

where M is a member of the spin group spin(n,n) which commutes with K. We start by considering the polar decomposition of an arbitrary matrix $\boldsymbol{M}$. Assuming that det $\boldsymbol{M} \neq 0$, the matrix $\boldsymbol{M}\bar{\boldsymbol{M}}$ can be written (not necessarliy uniquely) as

$$\boldsymbol{M}\bar{\boldsymbol{M}} = \boldsymbol{S}\boldsymbol{\Lambda}\bar{\boldsymbol{S}} \tag{3.106}$$

where $\boldsymbol{S}$ is an orthogonal transformation (which can be arranged to be a rotation matrix), and $\boldsymbol{\Lambda}$ is a diagonal matrix with positive entries. One can now write

$$\boldsymbol{M} = \boldsymbol{S}\boldsymbol{\Lambda}^{1/2}\boldsymbol{R} \tag{3.107}$$

where $\boldsymbol{\Lambda}^{1/2}$ is the diagonal matrix of positive square roots of the entries of $\boldsymbol{\Lambda}$ and $\boldsymbol{R}$ is a matrix defined by

$$\boldsymbol{R} = \boldsymbol{\Lambda}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{M}. \tag{3.108}$$

The matrix $\boldsymbol{R}$ satisfies

$$\begin{aligned} \boldsymbol{R}\bar{\boldsymbol{R}} &= \boldsymbol{\Lambda}^{-1/2}\bar{\boldsymbol{S}}\boldsymbol{M}\bar{\boldsymbol{M}}\boldsymbol{S}\boldsymbol{\Lambda}^{-1/2} \\ &= \boldsymbol{\Lambda}^{-1/2}\boldsymbol{\Lambda}\boldsymbol{\Lambda}^{-1/2} \\ &= \boldsymbol{I} \end{aligned} \tag{3.109}$$

and so is also orthogonal. It follows from (3.107) that an arbitrary non-singular matrix can be written as a diagonal matrix with positive entries sandwiched between a pair of orthogonal matrices. As a check, this gives n^2 degrees of freedom. To prove the desired result, we need only show that orthogonal transformations and positive dilations can be written in the form of equation (3.105).

We first consider rotations. The E_{ij} generators in Table 3.4 produce rotors of the form

$$R = \exp\{(E - \hat{E})/2\}, \tag{3.110}$$

where

$$E = \alpha^{ij}E_{ij} \tag{3.111}$$

and the α^{ij} are a set of scalar coefficients. The effect of the rotor R on a_+ generates

$$\begin{aligned} \underline{R}(a_+) &= R(a+\hat{a})\tilde{R} \\ &= Ra\tilde{R} + (Ra\tilde{R})\cdot K \\ &= e^{E/2}ae^{-E/2} + (e^{E/2}ae^{-E/2})\cdot K \end{aligned} \tag{3.112}$$

and so accounts for all rotations of the vector a in $\Re^n$. To complete the set of orthogonal transformations, a representation for reflections must be found as well. A reflection in the hyperplane orthogonal to the vector n in $\Re_n$ is represented by the element $n\hat{n}$ in $\Re_{n,n}$. Since $n\hat{n}\hat{n}n = -1$, $n\hat{n}$ is not a rotor and belongs to the disconnected part of spin(n,n). That $n\hat{n}$ commutes with K, and so is contained in spin(n,n), is verified as follows,

$$\begin{aligned} n\hat{n}K &= 2n\hat{n}\cdot K + nK\hat{n} \\ &= 2n^2 + 2n\cdot K\hat{n} + Kn\hat{n} \\ &= 2(n^2+\hat{n}^2) + Kn\hat{n} \\ &= Kn\hat{n}. \end{aligned} \tag{3.113}$$

The action of $n\hat{n}$ on a vector is determined by (3.8), and gives

$$\begin{aligned} n\hat{n}a_+n\hat{n} &= -n\hat{n}a\hat{n}n - (n\hat{n}a\hat{n}n)\cdot K \\ &= -nan - (nan)\cdot K, \end{aligned} \tag{3.114}$$

as required.

Finally, we need to see how positive dilations are given a rotor description. A dilation in the n direction by an amount e^λ is generated by the rotor

$$R = e^{-\lambda n\hat{n}/2}, \tag{3.115}$$

where the generator $-\lambda n\hat{n}/2$ is built from the K_i in Table 3.4. Acting on the null vector $n_+ = n + \hat{n}$, the rotor (3.115) gives

$$\begin{aligned} Rn_+\tilde{R} &= e^{-\lambda n\hat{n}/2}n_+e^{+\lambda n\hat{n}/2} \\ &= e^{-\lambda n\hat{n}}(n+\hat{n}) \\ &= (\cosh\lambda - n\hat{n}\sinh\lambda)(n+\hat{n}) \\ &= (\cosh\lambda + \sinh\lambda)(n+\hat{n}) \\ &= e^\lambda n_+. \end{aligned} \tag{3.116}$$

In addition, for vectors perpendicular to n in $\Re^n$, the action of R on their null vector equivalents has no effect. These are precisely the required properties for a dilation in the n direction. This concludes the proof that the general linear group is represented by the subgroup of spin(n,n) consisting of elements commuting with K. As an aside, this construction has led us to the E_{ij} and K_i generators in Table (3.4). Commutators of the E_{ij} and K_i give the remaining F_{ij} generators, which are sufficient to close the algebra.

The determinant of a linear function on $\Re^n$ is easily represented in $\Re_{n,n}$ since

$$\underline{f}(e_1)\wedge\underline{f}(e_2)\wedge\ldots\wedge\underline{f}(e_n) = \det\underline{f}\,E_n \tag{3.117}$$

becomes

$$MW_nM^{-1} = \det\underline{f}W_n, \tag{3.118}$$

in the null space of $\mathcal{V}^n$. Here M is the spin group element representing the linear function $\underline{f}$. From the definitions of W_n and W_n^* (3.102), we can write

$$\det\underline{f} = 2^n\langle\tilde{W}_n^*MW_nM^{-1}\rangle, \tag{3.119}$$

from which many of the standard properties of determinants can be derived.

3.3.1 Endomorphisms of $\Re_n$

We now turn to a second feature of $\Re_{n,n}$, which is its effectiveness in discussing endomorphisms of $\Re_n$. These are maps of $\Re_n$ onto itself, and the set of all such maps is denoted end$(\Re_n)$. Since the algebra $\Re_n$ is 2^n-dimensional, the endomorphism algebra is isomorphic to the algebra of real $2^n \times 2^n$ matrices,

$$\text{end}(\Re_n) \cong \mathrm{R}(2^n). \tag{3.120}$$

But the Clifford algebra $\Re_{n,n}$ is also isomorphic to the algebra of $2^n \times 2^n$ matrices, so every endomorphism of $\Re_n$ can be represented by a multivector in $\Re_{n,n}$[1]. Our first task is therefore to find how to construct each multivector equivalent of a given endomorphism.

Within $\Re_n$, endomorphisms are built up from the the primitive operations of the inner and outer products with the $\{e_i\}$. It is more useful, however, to adopt the following basis set of functions,

$$\begin{aligned} \underline{e}_i(A) &\equiv e_i \cdot A + e_i \wedge A = e_i A & (3.121) \\ \underline{\hat{e}}_i(A) &\equiv -e_i \cdot A + e_i \wedge A = \hat{A} e_i, & (3.122) \end{aligned}$$

where the hat (parity) operation in $\Re_n$ is defined by

$$\hat{A}_r \equiv (-1)^r A_r \tag{3.123}$$

and serves to distinguish even-grade and odd-grade multivectors. The reason for the use of the hat in both $\Re_n$ and $\Re_{n,n}$ will become apparent shortly. The $\{\underline{e}_i\}$ and $\{\underline{\hat{e}}_i\}$ operations are precisely those found in Section 2.3 in the context of Berezin calculus, though with the fiducial tensor $\underline{h}$ now set to the identity. They satisfy the relations

$$\begin{aligned} \underline{e}_i \underline{e}_j + \underline{e}_j \underline{e}_i &= 2\delta_{ij} \\ \underline{\hat{e}}_i \underline{\hat{e}}_j + \underline{\hat{e}}_j \underline{\hat{e}}_i &= -2\delta_{ij} & (3.124) \\ \underline{e}_i \underline{\hat{e}}_j + \underline{\hat{e}}_j \underline{e}_i &= 0, & (3.125) \end{aligned}$$

which are the defining relations for a vector basis in $\Re_{n,n}$. This establishes the isomorphism between elements of end($\Re_n$) and multivectors in $\Re_{n,n}$. Any element of end($\Re_n$) can be decomposed into sums and products of the $\{\underline{e}_i\}$ and $\{\underline{\hat{e}}_i\}$ functions, and so immediately specifies a multivector in $\Re_{n,n}$ built from the same combinations of the $\{e_i\}$ and $\{\hat{e}_i\}$ basis vectors.

To complete the construction, we must find a 2^n-dimensional subspace of $\Re_{n,n}$ on which endomorphisms of $\Re_n$ are faithfully represented by (left) multiplication by elements of $\Re_{n,n}$. The required subspace is a *minimal left ideal* of $\Re_{n,n}$ and is denoted $\mathcal{I}^n$. It is constructed as follows. We define a set of bivector blades by

$$K_i \equiv e_i \hat{e}_i. \tag{3.126}$$

Here, and in the remainder of this section, we have dropped the summation convention. The K_i satisfy

$$\begin{aligned} K_i \cdot K_j &= \delta_{ij} & (3.127) \\ K_i \times K_j &= 0 & (3.128) \end{aligned}$$

and the bivector K is can be written as

$$K = \sum_i K_i. \tag{3.129}$$

[1] I am grateful to Frank Sommen and Nadine Van Acker for pointing out the potential usefulness of this result.

A family of commuting idempotents are now defined by

$$I_i \equiv \tfrac{1}{2}(1 + K_i) = w_i^* w_i, \tag{3.130}$$

and have the following properties:

$$I_i^2 = I_i \tag{3.131}$$
$$I_i I_j = I_j I_i \tag{3.132}$$
$$e_i I_i = w_i = \hat{e}_i I_i \tag{3.133}$$
$$I_i e_i = w_i^* = -I_i \hat{e}_i \tag{3.134}$$
$$K_i I_i = I_i. \tag{3.135}$$

From the I_i the idempotent I is defined by

$$I \equiv \prod_{i=1}^{n} I_i = I_1 I_2 \dots I_n = w_1^* w_1 w_2^* w_2 \dots w_n^* w_n = W_n^* \tilde{W}_n. \tag{3.136}$$

I has the following properties:

$$I^2 = I \tag{3.137}$$
$$e_i I = \hat{e}_i I \tag{3.138}$$

and

$$E_n I = \hat{E}_n I = W_n I = W_n, \tag{3.139}$$

where E_n is the pseudoscalar for the Euclidean algebra $\Re_n$ and $\hat{E}_n$ is the pseudoscalar for the anti-Euclidean algebra $\Re_{0,n}$. The relationships in (3.139) establish an equivalence between the $\Re^n$, $\Re^{0,n}$ and $\mathcal{V}^n$ vector spaces.

Whilst the construction of I has made use of an orthonormal frame, the form of I is actually independent of this choice. This can be seen by writing I in the form

$$I = \frac{1}{2^n}\left(1 + K + \frac{K \wedge K}{2!} + \dots + \frac{K \wedge K \wedge \dots \wedge K}{n!}\right) \tag{3.140}$$

and recalling that K is frame-independent. It is interesting to note that the bracketed term in (3.140) is of the same form as the Grassmann exponential considered in Section 2.2.1.

The full 2^n-dimensional space $\mathcal{I}^n$ is generated by left multiplication of I by the entire algebra $\Re_{n,n}$,

$$\mathcal{I}^n = \Re_{n,n} I. \tag{3.141}$$

Since multiplication of I by e_i and $\hat{e}_i$ are equivalent, every occurrence of an $\hat{e}_i$ in a multivector in $\Re_{n,n}$ can be replaced by an e_i, so that there is a simple $1 \leftrightarrow 1$ equivalence between elements of $\Re_n$ and $\mathcal{I}^n$. The action of an element of end$(\Re_n)$ can now be represented in $\Re_{n,n}$ by left multiplication of $\mathcal{I}^n$ by the appropriate multivector. For a multivector A_r in $\Re_n$ the equivalence between the basic operators (3.122) is seen from

$$e_i A_r I \quad \leftrightarrow \quad e_i A_r \tag{3.142}$$

and

$$\hat{e}_i A_r I \quad \leftrightarrow \quad \hat{A}_r e_i. \tag{3.143}$$

The parity operation on the right-hand side of (3.143) arises because the $\hat{e}_i$ vector must be anticommuted through each of the vectors making up the A_r multivector. This is the reason for the different uses of the overhat notation for the $\Re_n$ and $\Re_{n,n}$ algebras. Symbolically, we can now write

$$e_i \mathcal{I}^n \quad \leftrightarrow \quad e_i \Re_n \tag{3.144}$$

$$\hat{e}_i \mathcal{I}^n \quad \leftrightarrow \quad \hat{\Re}_n e_i. \tag{3.145}$$

Also, from the definitions of w_i and w_i^* (3.99), we find the equivalences

$$w_i \mathcal{I}^n \quad \leftrightarrow \quad e_i \wedge \Re_n \tag{3.146}$$

$$w_i^* \mathcal{I}^n \quad \leftrightarrow \quad e_i \cdot \Re_n, \tag{3.147}$$

which establishes contact with the formalism of Grassmann/Berezin calculus given in Chapter 2. We can now move easily between the formalism with dot and wedge products used in Chapter 2 and the null-vector formalism adopted here. The chosen application should dictate which is the more useful.

We next consider the quantity $n\hat{n}$, where n is a unit vector. The action of this on $\mathcal{I}^n$ gives

$$n\hat{n}\mathcal{I}^n \quad \leftrightarrow \quad n\hat{\Re}_n n. \tag{3.148}$$

The operation on the right-hand side is the outermorphism action of a reflection in the hyperplane perpendicular to n. In the previous section we used a double-sided application of $n\hat{n}$ on null vectors to represent reflections in $\Re^n$. We now see that the same object can be applied single-sidedly in conjunction with the idempotent I to also produce reflections. The same is true of products of reflections. For example, the rotor (3.110) gives

$$e^{(E-\hat{E})/2} M I = e^{E/2} M e^{-\hat{E}/2} I \quad \leftrightarrow \quad e^{E/2} M e^{-E/2}, \tag{3.149}$$

demonstrating how the two-bladed structure of the E_{ij} generators is used to represent concurrent left and right multiplication in $\Re_n$.

The operation $\Re_n \mapsto \hat{\Re}_n$ is performed by successive reflections in each of the e_i directions. We therefore find the equivalence

$$e_1\hat{e}_1 e_2\hat{e}_2 \ldots e_n\hat{e}_n \mathcal{I}^n \quad \leftrightarrow \quad \hat{\Re}_n. \tag{3.150}$$

But

$$e_1\hat{e}_1 e_2\hat{e}_2 \ldots e_n\hat{e}_n = e_n \ldots e_2 e_1 \hat{e}_1 \hat{e}_2 \ldots \hat{e}_n = \tilde{E}_n \hat{E}_n = E_{n,n} \tag{3.151}$$

is the unit pseudoscalar in $\Re_{n,n}$, so multiplication of an element of $\mathcal{I}^n$ by $E_{n,n}$ corresponds to the parity operation in $\Re_n$. As a check, $(E_{n,n})^2$ is always $+1$, so the result of two parity operations is always the identity.

The correspondence between the single-sided and double-sided forms for a dilation are not quite so simple. If we consider the rotor $\exp\{-\lambda n\hat{n}/2\}$ again, we find that, for the vector n,

$$e^{-\lambda n\hat{n}/2} n I = e^{\lambda/2} n I \quad \leftrightarrow \quad e^{\lambda/2} n \tag{3.152}$$

For vectors perpendicular to n, however, we find that

$$e^{-\lambda n\hat{n}/2}n_{\perp}I = n_{\perp}e^{-\lambda/2n\hat{n}}I \quad \leftrightarrow \quad e^{-\lambda/2}n_{\perp}, \tag{3.153}$$

so the single-sided formulation gives a stretch along the n direction of $\exp\{\lambda\}$, but now combined with an overall dilation of $\exp\{-\lambda/2\}$. This overall factor can be removed by an additional boost with the exponential of a suitable multiple of K. It is clear, however, that both single-sided and double-sided application of elements of the spin group which commute with K can be used to give representations of the general linear group.

Finally, we consider even products of the null vectors w_i and w_i^*. These generate the operations

$$\begin{aligned} w_i w_i^* \mathcal{I}^n &\leftrightarrow e_i\cdot(e_i\wedge\Re_n) \\ w_i^* w_i \mathcal{I}^n &\leftrightarrow e_i\wedge(e_i\cdot\Re_n) \end{aligned} \tag{3.154}$$

which are *rejection* and *projection* operations in $\Re_n$ respectively. For a vector a in $\Re_n$, the operation of projecting a onto the e_i direction is performed by

$$P_i(a) = e_i e_i\cdot a, \tag{3.155}$$

and for a general multivector,

$$P_i(A) = e_i\wedge(e_i\cdot A). \tag{3.156}$$

This projects out the components of A which contain a vector in the e_i direction. The projection onto the orthogonal complement of e_i (the rejection) is given by

$$P_i^{\perp}(A) = e_i\cdot(e_i\wedge A). \tag{3.157}$$

Projection operations correspond to singular transformations, and we now see that these are represented by products of null multivectors in $\Re_{n,n}$. This is sufficient to ensure that singular transformations can also be represented by an even product of vectors, some of which may now be null.

Two results follow from these considerations. Firstly, every matrix Lie group can be represented by a spin group — every matrix Lie group can be defined as a subgroup of GL(n,R) and we have shown how GL(n,R) can be represented as a spin group. It follows that every Lie algebra can be represented by a bivector algebra, since all Lie algebras have a matrix representation via the adjoint representation. The discussion of the unitary group has shown, however, that subgroups of GL(n,R) are not, in general, the best way to construct spin-group representations. Other, more useful, constructions are given in the following Sections. Secondly, every linear transformation on $\Re^n$ can be represented in $\Re_{n,n}$ as an even product of vectors, the result of which commutes with K. It is well known that quaternions are better suited to rotations in three dimensions than 3×3 matrices. It should now be possible to extend these advantages to arbitrary linear functions. A number of other applications for these results can be envisaged. For example, consider the equation

$$\boldsymbol{u}'(s) = \boldsymbol{M}(s)\boldsymbol{u}(s), \tag{3.158}$$

For vectors perpendicular to n, however, we find that

$$e^{-\lambda n\hat{n}/2} n_\perp I = n_\perp e^{-\lambda/2 n\hat{n}} I \quad \leftrightarrow \quad e^{-\lambda/2} n_\perp, \tag{3.153}$$

so the single-sided formulation gives a stretch along the n direction of $\exp\{\lambda\}$, but now combined with an overall dilation of $\exp\{-\lambda/2\}$. This overall factor can be removed by an additional boost with the exponential of a suitable multiple of K. It is clear, however, that both single-sided and double-sided application of elements of the spin group which commute with K can be used to give representations of the general linear group.

Finally, we consider even products of the null vectors w_i and w_i^*. These generate the operations

$$\begin{aligned} w_i w_i^* \mathcal{I}^n &\leftrightarrow e_i \cdot (e_i \wedge \Re_n) \\ w_i^* w_i \mathcal{I}^n &\leftrightarrow e_i \wedge (e_i \cdot \Re_n) \end{aligned} \tag{3.154}$$

which are *rejection* and *projection* operations in $\Re_n$ respectively. For a vector a in $\Re_n$, the operation of projecting a onto the e_i direction is performed by

$$P_i(a) = e_i e_i \cdot a, \tag{3.155}$$

and for a general multivector,

$$P_i(A) = e_i \wedge (e_i \cdot A). \tag{3.156}$$

This projects out the components of A which contain a vector in the e_i direction. The projection onto the orthogonal complement of e_i (the rejection) is given by

$$P_i^\perp(A) = e_i \cdot (e_i \wedge A). \tag{3.157}$$

Projection operations correspond to singular transformations, and we now see that these are represented by products of null multivectors in $\Re_{n,n}$. This is sufficient to ensure that singular transformations can also be represented by an even product of vectors, some of which may now be null.

Two results follow from these considerations. Firstly, every matrix Lie group can be represented by a spin group — every matrix Lie group can be defined as a subgroup of GL(n,R) and we have shown how GL(n,R) can be represented as a spin group. It follows that every Lie algebra can be represented by a bivector algebra, since all Lie algebras have a matrix representation via the adjoint representation. The discussion of the unitary group has shown, however, that subgroups of GL(n,R) are not, in general, the best way to construct spin-group representations. Other, more useful, constructions are given in the following Sections. Secondly, every linear transformation on $\Re^n$ can be represented in $\Re_{n,n}$ as an even product of vectors, the result of which commutes with K. It is well known that quaternions are better suited to rotations in three dimensions than 3×3 matrices. It should now be possible to extend these advantages to arbitrary linear functions. A number of other applications for these results can be envisaged. For example, consider the equation

$$\boldsymbol{u}'(s) = \boldsymbol{M}(s)\boldsymbol{u}(s), \tag{3.158}$$

where $\boldsymbol{u}(s)$ and $\boldsymbol{M}(s)$ are vector and matrix functions of the parameter s and the prime denotes the derivative with respect to s. By replacing the vector $\boldsymbol{u}$ by the null vector u in $\Re^{n,n}$, equation (3.158) can be written in the form

$$u' = B(s)\cdot u, \tag{3.159}$$

where $B(s)$ is a bivector. If we now write $u = Ru_0\tilde{R}$, where u_0 is a constant vector, then equation (3.158) reduces to the rotor equation

$$R' = \tfrac{1}{2}BR, \tag{3.160}$$

which may well be easier to analyse (a similar rotor reformulation of the Lorentz force law is discussed in [20]).

3.4 The Remaining Classical Groups

We now turn attention to some of the remaining matrix Lie groups. Again, all groups are realised as subgroups of the orthogonal group and so inherit a spin-group representation. The various multivectors and linear functions which remain invariant under the group action are discussed, and simple methods are given for writing down the Bivector generators which form the Lie algebra. The results from this chapter are summarised in Section 3.5.

3.4.1 Complexification — so(n,C)

Complexification of the Orthogonal groups $\mathrm{O}(p,q)$ leads to a single, non-compact, Lie group in which all reference to the underlying metric is lost. With the u_k and v_k defined as in Equation (3.55), the invariant bilinear form is

$$\epsilon(u,v) = u_k v_k = x_k r_k - y_k s_k + j(x_k s_k + y_k r_k). \tag{3.161}$$

This is symmetric, and the real part contains equal numbers of positive and negative norm terms. The Lie group $\mathrm{O}(n,\mathrm{C})$ will therefore be realised in the "balanced" algebra $\Re_{n,n}$. To construct the imaginary part of (3.161), however, we need to find a symmetric function which squares to give minus the identity. This is in contrast to the $\underline{K}$ function, which is antisymmetric, and squares to $+1$. The solution is to introduce the "star" function

$$a^* \equiv (-1)^{n+1} E_n a E_n^{-1}, \tag{3.162}$$

so that

$$\begin{aligned} e_i^* &= e_i \\ \hat{e}_i^* &= -\hat{e}_i. \end{aligned} \tag{3.163}$$

The use of the $*$ notation is consistent with the definitions of $\{w_i\}$ and $\{w_i^*\}$ bases (3.99). The star operator is used to define projections into the Euclidean and anti-Euclidean subspaces of $\Re^{n,n}$:

$$\begin{aligned} \underline{E}_n(a) = \tfrac{1}{2}(a + a^*) = a\cdot E_n E_n^{-1} \\ \underline{\hat{E}}_n(a) = \tfrac{1}{2}(a - a^*) = a\wedge E_n E_n^{-1}. \end{aligned} \tag{3.164}$$

E_{ij}	$=$	$e_i e_j - \hat{e}_i \hat{e}_j$	$(i < j = 1 \ldots n)$
F_{ij}	$=$	$e_i \hat{e}_j + \hat{e}_i e_j.$	"

Table 3.6: Bivector Basis for so(n,C)

The Euclidean pseudoscalar E_n anticommutes with K, so the star operator anticommutes with the $\underline{K}$ function. It follows that the combined function

$$\underline{K}^*(a) \equiv \underline{K}(a^*) \tag{3.165}$$

satisfies

$$\begin{aligned}\underline{K}^{*2}(a) &= \underline{K}[\underline{K}(a^*)^*] \\ &= -\underline{K}[\underline{K}(a^{**})] \\ &= -a\end{aligned} \tag{3.166}$$

and

$$\begin{aligned}\overline{K^*}(a) &= -[\underline{K}(a)]^* \\ &= \underline{K}^*(a),\end{aligned} \tag{3.167}$$

and so has the required properties. The complex symmetric norm can now be written on $\Re^{n,n}$ as

$$\epsilon(a,b) = a\cdot b + ja\cdot\underline{K}^*(b), \tag{3.168}$$

which can verified by expanding in the $\{e_i, \hat{e}_i\}$ basis of (3.92).

An orthogonal transformation $\underline{f}$ will leave $\epsilon(a,b)$ invariant provided that

$$\underline{K}^*\underline{f}(a) = \underline{f}\,\underline{K}^*(a), \tag{3.169}$$

which defines the group O(n,C). Each function $\underline{f}$ in O(n,C) can be constructed from the corresponding elements of spin(n,n), which defines the spin-group representation. The bivector generators must satisfy

$$\underline{K}^*[e^{\lambda B/2}ae^{-\lambda B/2}] = e^{\lambda B/2}\underline{K}^*(a)e^{-\lambda B/2}, \tag{3.170}$$

which reduces to the requirement

$$\underline{K}^*(B\cdot a) = B\cdot\underline{K}^*(a) \tag{3.171}$$

$$\Rightarrow \quad B\cdot a = -\underline{K}^*[B\cdot\underline{K}^*(a)] = -\underline{K}^*(B)\cdot a \tag{3.172}$$

$$\Rightarrow \quad \underline{K}^*(B) = -B. \tag{3.173}$$

Since $\underline{K}^{*2}(B) = B$ for all bivectors B, the generators which form the Lie algebra so(n,C) are all of the form $B - \underline{K}^*(B)$. This is used to write down the bivector basis in Table 3.6. Under the commutator product, the E_{ij} form a closed sub-algebra which is isomorphic to so(n). The F_{ij} fulfil the role of "jE_{ij}". The Killing metric has $n(n-1)/2$ entries of positive and signature and the same number of negative signature.

3.4.2 Quaternionic Structures — sp(n) and so*($2n$)

The quaternionic unitary group (usually denoted Sp(n) or HU(n)) is the invariance group of the Hermitian-symmetric inner product of quaternion-valued vectors. By analogy with the unitary group, the quaternionic structure is introduced by now quadrupling the real space $\Re^n$ or $\Re^{p,q}$ to $\Re^{4n}$ or $\Re^{4p,4q}$. We deal with the Euclidean case first and take $\{e_i\}$ to be an orthonormal basis set for $\Re^n$. Three further copies of $\Re^n$ are introduced, so that $\{e_i, e_i^1, e_i^2, e_i^3\}$ form an orthonormal basis for $\Re^{4n}$. Three "doubling" bivectors are now defined as

$$\begin{aligned} J_1 &= e_i e_i^1 + e_i^2 e_i^3 \\ J_2 &= e_i e_i^2 + e_i^3 e_i^1 \\ J_3 &= e_i e_i^3 + e_i^1 e_i^2, \end{aligned} \tag{3.174}$$

which define the three functions

$$\underline{J}_i(a) = a \cdot J_i. \tag{3.175}$$

(The introduction of an orthonormal frame is not essential since each of the J_i are independent of the intial choice of frame. Orthonormal frames do ease the discussion of the properties of the J_i, however, so will be used frequently in this and the following sections).

The combined effect of $\underline{J}_1$ and $\underline{J}_2$ on a vector a produces

$$\begin{aligned} \underline{J}_1\underline{J}_2(a) &= \underline{J}_1(a \cdot e_i e_i^2 - a \cdot e_i^2 e_i + a \cdot e_i^3 e_i^1 - a \cdot e_i^1 e_i^3) \\ &= a \cdot e_i e_i^3 - a \cdot e_i^2 e_i^1 - a \cdot e_i^3 e_i + a \cdot e_i^1 e_i^2 \\ &= \underline{J}_3(a). \end{aligned} \tag{3.176}$$

The $\underline{J}_i$ functions therefore generate the quaternionic structure

$$\underline{J}_1^2 = \underline{J}_2^2 = \underline{J}_3^2 = \underline{J}_1\underline{J}_2\underline{J}_3 = -1. \tag{3.177}$$

The Hermitian-symmetric quaternion inner product can be realised in $\Re_{4n}$ by

$$\epsilon(a,b) = a \cdot b + a \cdot \underline{J}_1(b)\boldsymbol{i} + a \cdot \underline{J}_2(b)\boldsymbol{j} + a \cdot \underline{J}_3(b)\boldsymbol{j}, \tag{3.178}$$

where $\{\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}\}$ are a basis set of quaterions (see Section 1.2.3). The inner product (3.178) contains four separate terms, each of which must be preserved by the invariance group. This group therefore consists of orthogonal transformations satisfying

$$\underline{f}(J_i) = J_i \qquad i = 1 \ldots 3 \tag{3.179}$$

and the spin group representation consists of the elements of spin($4n$) which commute with all of the J_i. The bivector generators of the invariance group therefore also commute with the J_i. The results established in Section 3.2 apply for each of the J_i in turn, so an arbitrary bivector in the Lie algebra of Sp(n) must be of the form

$$B_{HU} = B + \underline{J}_1(B) + \underline{J}_2(B) + \underline{J}_3(B). \tag{3.180}$$

This result is used to write down the orthogonal basis set in Table 3.7. The algebra has dimension $2n^2 + n$ and rank n.

E_{ij}	$=$	$e_ie_j + e_i^1e_j^1 + e_i^2e_j^2 + e_i^3e_j^3$	$(i < j = 1 \ldots n)$
F_{ij}	$=$	$e_ie_j^1 - e_i^1e_j - e_i^2e_j^3 + e_i^3e_j^2$	"
G_{ij}	$=$	$e_ie_j^2 - e_i^2e_j - e_i^3e_j^1 + e_i^1e_j^3$	"
H_{ij}	$=$	$e_ie_j^3 - e_i^3e_j - e_i^1e_j^2 + e_i^2e_j^1$	"
F_i	$=$	$e_ie_i^1 - e_i^2e_i^3$	$(i = 1 \ldots n)$
G_i	$=$	$e_ie_i^2 - e_i^3e_i^1$	"
H_i	$=$	$e_ie_i^3 - e_i^1e_i^2$	"

Table 3.7: Bivector Basis for sp(n)

The above extends easily to the case of sp(p,q) by working in the algebra $\Re_{4p,4q}$. With $\{e_i\}$ now a basis for $\Re^{p,q}$, the doubling bivectors are defined by

$$J_1 = e_ie^{1^i} + e_i^2e^{3^i} \quad etc \tag{3.181}$$

and the quaternion relations (3.177) are still satisfied. The Lie algebra is then generated in exactly the same may. The resultant algebra has a Killing metric with $2(p^2+q^2)+p+q$ negative entries and $4pq$ positive entries.

The properties of the $\underline{K}^*$ function found in Section 3.4.1 suggests that an alternative quaternionic structure could be found in $\Re_{2n,2n}$ by introducing anticommuting $\underline{K}^*$ and $\underline{J}$ functions. This is indeed the case. With $\{e_i\}$ and $\{f_i\}$ a pair of anticommuting orthonormal bases for $\Re^n$, a basis for $\Re^{2n,2n}$ is defined by $\{e_i, f_i, \hat{e}_i, \hat{f}_i\}$. The hat operation is now defined by

$$\hat{a} = \underline{K}(a) = a\cdot K \tag{3.182}$$

with

$$K = e_i\hat{e}_i + f_i\hat{f}_i. \tag{3.183}$$

A complexification bivector is defined by

$$J = e_if^i + \hat{e}_i\hat{f}^i = e_if_i - \hat{e}_i\hat{f}_i \tag{3.184}$$

and additional doubling bivectors are defined by

$$\begin{aligned} K_1 &= e_i\hat{e}_i - f_i\hat{f}_i \\ K_1 &= e_i\hat{f}_i + f_i\hat{e}_i. \end{aligned} \tag{3.185}$$

The set $\{J, K_1, K_2\}$ form a set of three bivectors, no two of which commute.

With pseudoscalars E_n and F_n defined by

$$\begin{aligned} E_n &= e_1e_2\ldots e_n \\ F_n &= f_1f_2\ldots f_n, \end{aligned} \tag{3.186}$$

the star operation is defined by

$$a^* = -E_nF_na\tilde{F}_n\tilde{E}_n. \tag{3.187}$$

$$
\boxed{\begin{array}{rcll}
E_{ij} & = & e_ie_j + f_if_j - \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j & (i<j=1\ldots n) \\
F_{ij} & = & e_if_j - f_ie_j + \hat{e}_i\hat{f}_j - \hat{f}_i\hat{e}_j & \quad '' \\
G_{ij} & = & e_i\hat{e}_j + \hat{e}_ie_j + f_i\hat{f}_j + \hat{f}_if_j & \quad '' \\
H_{ij} & = & e_i\hat{f}_j + \hat{f}_ie_j - f_i\hat{e}_j - \hat{e}_if_j & \quad '' \\
H_i & = & e_if_i + \hat{e}_i\hat{f}_i & (i=1\ldots n)
\end{array}}
$$

Table 3.8: Bivector Basis for so*(n)

$\underline{K}_i^*$ operations are now defined by

$$\underline{K}_i^*(a) = \underline{K}_i(a^*) = a^* \cdot K_i. \tag{3.188}$$

These satisfy

$$\underline{K}_i^{*2}(a) = -a \tag{3.189}$$

and

$$\begin{aligned}
\underline{K}_1^*\underline{K}_2^*(a) &= \underline{K}_1[\underline{K}_2(a^*)^*] \\
&= -\underline{K}_1\underline{K}_2(a) \\
&= \underline{J}(a).
\end{aligned} \tag{3.190}$$

The $\underline{J}$ and $\underline{K}_i^*$ therefore form a quaternionic set of linear functions satisfying

$$\underline{K}_1^{*2} = \underline{K}_2^{*2} = \underline{J}^2 = \underline{K}_1^*\underline{K}_2^*\underline{J} = -1. \tag{3.191}$$

Orthogonal functions commuting with each of the $\underline{J}$ and $\underline{K}_i^*$ functions will therefore leave a quaternionic inner product invariant. This inner product can be written as

$$\epsilon(a,b) = a \cdot \underline{J}(b) + \boldsymbol{i}a \cdot b + \boldsymbol{j}a \cdot \underline{K}_1^*(b) + \boldsymbol{k}a \cdot \underline{K}_2^*(b), \tag{3.192}$$

which expansion in the $\{e_i, f_i, \hat{e}_i, \hat{f}_i\}$ frame shows to be equivalent to the skew-Hermitian quaternionic inner product

$$\epsilon(u,v) = u_k^\dagger \boldsymbol{i} v_k. \tag{3.193}$$

The invariance group of (3.193) is denoted SO*$(2n)$ (or Sk$(n$,H)). The bivector generators of the invariance group must satisfy $\underline{J}(B) = B$ and $\underline{K}_i^*(B) = -B$ and so are of the form

$$B_{H^*} = B + \underline{J}(B) - \underline{K}_1^*(B) - \underline{K}_2^*(B). \tag{3.194}$$

This leads to the orthogonal set of basis generators in Table 3.8.

The bivector algebra so*(n) has dimension $n(2n-1)$ and a Killing metric with n^2 negative entries and $n^2 - n$ positive entries. This algebra is one of the possible real forms of the complexified algebra D_n. Some of the properties of so*$(2n)$, including its representation theory, have been discussed by Barut & Bracken [58].

$$
\begin{array}{|rcll|}
\hline
E_{ij} & = & e_ie_j + f_if_j - \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j & (i<j=1\ldots n) \\
F_{ij} & = & e_if_j - f_ie_j - \hat{e}_i\hat{f}_j + \hat{f}_i\hat{e}_j & \quad '' \\
G_{ij} & = & e_i\hat{e}_j - \hat{e}_ie_j + f_i\hat{f}_j - \hat{f}_if_j & \quad '' \\
H_{ij} & = & e_i\hat{f}_j + \hat{f}_ie_j - f_i\hat{e}_j - \hat{e}_if_j & \quad '' \\
J_i & = & e_if_i - \hat{e}_i\hat{f}_i & (i=1\ldots n) \\
K_i & = & e_i\hat{e}_i + f_i\hat{f}_i & \quad '' \\
\hline
\end{array}
$$

Table 3.9: Bivector Basis for gl(n,C)

3.4.3 The Complex and Quaternionic General Linear Groups

The general linear group over the complex field, GL(n,C), is constructed from linear functions in the $2n$-dimensional space $\Re^{2n}$ which leave the complex structure intact,

$$\underline{h}(a)\cdot J = \underline{h}(a\cdot J). \tag{3.195}$$

These linear functions can be represented by orthogonal functions in $\Re^{2n,2n}$ using the techniques introduced in Section 3.3. Thus, using the conventions of Section 3.4.2, a vector a in $\Re^{2n}$ is represented in $\Re^{2n,2n}$ by the null vector $a_+ = a + \hat{a}$, and the complex structure is defined by the bivector J of equation (3.184). These definitions ensure that the $\underline{J}$ function keeps null vectors in the same null space,

$$\underline{K}\,\underline{J}(a) = \underline{J}\,\underline{K}(a) \tag{3.196}$$

$$\Rightarrow (a\cdot J)\cdot K - (a\cdot K)\cdot J = a\cdot(J\times K) = 0, \tag{3.197}$$

which is satisfied since $J\times K = 0$. The spin group representation of GL(n,C) consists of all elements of spin$(2n, 2n)$ which commute with both J and K and hence preserve both the null and complex structures. The bivector generators of the Lie algebra gl(n,C) are therefore of the form

$$B_C = B + \underline{J}(B) - \underline{K}(B) - \underline{K}\,\underline{J}(B) \tag{3.198}$$

which yields the set of generators in Table 3.9. This algebra has $2n^2$ generators, as is to be expected. The two abelian subgroups are removed in the usual manner to yield the Lie algebra for sl(n,C) given in Table 3.10. The Killing metric gives $n^2 - 1$ terms of both positive and negative norm.

The general linear group with quaternionic entries (denoted U*($2n$) or GL(n,H)) is constructed in the same manner as the above, except that now the group is contained in the algebra $\Re_{4n,4n}$. Thus we start in the algebra $\Re_{4n}$ and introduce a quaternionic structure through the J_i bivectors of equations (3.174). The $\Re_{4n}$ algebra is then doubled to a $\Re_{4n,4n}$ algebra with the introduction of a suitable K bivector, and the J_i are extended to new bivectors

$$J_i' = J_i - \hat{J}_i. \tag{3.199}$$

The spin-group representation of U*($2n$) then consists of elements of spin$(4n, 4n)$ which commute with all of the J_i' and with K. The bivectors generators are all of the form

$$B_H = B + \underline{J}_1'(B) + \underline{J}_2'(B) + \underline{J}_3'(B) - \underline{K}[B + \underline{J}_1'(B) + \underline{J}_2'(B) + \underline{J}_3'(B)]. \tag{3.200}$$

E_{ij}	$=$	$e_ie_j + f_if_j - \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j$	$(i < j = 1 \ldots n)$
F_{ij}	$=$	$e_if_j - f_ie_j - \hat{e}_i\hat{f}_j + \hat{f}_i\hat{e}_j$	"
G_{ij}	$=$	$e_i\hat{e}_j - \hat{e}_ie_j + f_i\hat{f}_j - \hat{f}_if_j$	"
H_{ij}	$=$	$e_i\hat{f}_j + \hat{f}_ie_j - f_i\hat{e}_j - \hat{e}_if_j$	"
G_i	$=$	$J_i - J_{i+1}$	$(i = 1 \ldots n-1)$
H_i	$=$	$K_i - K_{i+1}$	"

Table 3.10: Bivector Basis for sl(n,C)

The result is a $(4n^2)$-dimensional algebra containing the single abelian factor K. This is factored out in the usual way to yield the bivector Lie algebra su*$(2n)$.

3.4.4 The symplectic Groups Sp(n,R) and Sp(n,C)

The symplectic group Sp(n,R) consists of all linear functions $\underline{h}$ acting on $\Re^{2n}$ satisfying $\overline{h}(J) = J$, where J is the doubling bivector from the $\Re_n$ algebra to the $\Re_{2n}$ algebra. A spin-group representation is achieved by doubling to $\Re_{2n,2n}$ and constructing Sp(n,R) as a subgroup of GL($2n$,R). In $\Re_{2n}$, the symplectic inner product is given by $(a \wedge b) \cdot J$. In $\Re_{2n,2n}$, with K defined as in Equation (3.183), the vectors a and b are replaced by the null vectors a_+ and b_+. Their symplectic inner product is given by

$$(a_+ \wedge b_+) \cdot J_S = [\underline{K}(a_+) \wedge \underline{K}(b_+)] \cdot J_S = (a_+ \wedge b_+) \cdot \underline{K}(J_S). \qquad (3.201)$$

The symplectic bivector in $\Re_{2n,2n}$ satisfies

$$\underline{K}(J_S) = J_S \qquad (3.202)$$

and so is defined by

$$J_S = J + \hat{J} = e_if_i + \hat{e}_i\hat{f}_i. \qquad (3.203)$$

(This differs from the J defined in equation (3.184), so generates an alternative complex structure). The group Sp(n,R) is the subgroup of orthogonal transformations on $\Re^{2n,2n}$ which leave both J_S and K invariant. The spin-group representation consists of all elements which commute with both J_S and K. The bivector generators of Sp(n,R) are all of the form

$$B_{Sp} = B + \underline{J}_S(B) - \underline{K}(B) - \underline{K}\,\underline{J}_S(B). \qquad (3.204)$$

An orthogonal basis for the algebra sp(n,R) is contained in Table 3.11. This has dimension $n(2n+1)$ and a Killing metric with n^2 negative entries and n^2+n positive entries. The same construction can be used to obtain the algebras for sp(p,q,R) by starting from $\Re_{p,q}$ and doubling this to $\Re_{2p,2q}$.

The group Sp(n,C) consists of functions on $\Re_{4n}$ satisfying $\overline{h}(J_1) = J_1$ and which also preserve the complex structure,

$$\underline{h}(a \cdot J_3) = \underline{h}(a) \cdot J_3. \qquad (3.205)$$

E_{ij}	$=$	$e_ie_j + f_if_j - \hat{e}_i\hat{e}_j - \hat{f}_i\hat{f}_j$	$(i < j = 1 \ldots n)$
F_{ij}	$=$	$e_if_j - f_ie_j - \hat{e}_i\hat{f}_j + \hat{f}_i\hat{e}_j$	"
G_{ij}	$=$	$e_i\hat{e}_j - \hat{e}_ie_j - f_i\hat{f}_j + \hat{f}_if_j$	"
H_{ij}	$=$	$e_i\hat{f}_j - \hat{f}_ie_j + f_i\hat{e}_j - \hat{e}_if_j$	"
F_i	$=$	$e_if_i - \hat{e}_i\hat{f}_i$	$(i = 1 \ldots n)$
G_i	$=$	$e_i\hat{e}_i - f_i\hat{f}_i$	"
H_i	$=$	$e_i\hat{f}_i + f_i\hat{e}_i$	"

Table 3.11: Bivector Basis for sp(n,R)

The complex and symplectic structures satisfy $\underline{J}_3(J_1) = -J_1$, so J_3 and J_1 do not commute. Instead they are two-thirds of the quaternionic set of bivectors introduced in Section 3.4.2. The C-skew inner product on $\Re^{4n}$ is written

$$\epsilon(a,b) = a \cdot \underline{J}_1(b) - ja \cdot \underline{J}_1\underline{J}_3(b) = a \cdot \underline{J}_1(b) + ja \cdot \underline{J}_2(b). \tag{3.206}$$

By analogy with Sp(n,R), a spin-group representation of Sp(n,C) is constructed as a subgroup of GL($2n$,C)in $\Re_{4n,4n}$. With the null structure defined by K, the symplectic structure is now determined by

$$J_S = J_1 + \underline{K}(J_1) \tag{3.207}$$

and the complex structure by

$$J = J_2 - \underline{K}(J). \tag{3.208}$$

The Lie algebra sp(n,C) is formed from the set of bivectors in $\Re^2_{4n,4n}$ which commute with all of the K, J and J_S bivectors. With this information, it is a simple matter to write down a basis set of generators.

3.5 Summary

In the preceding sections we have seen how many matrix Lie groups can be represented as spin groups, and how all (finite dimensional) Lie algebras can be realised as bivector algebras. These results are summarised in Tables 3.12 and 3.13. Table 3.12 lists the classical bilinear forms, their invariance groups, the base space in which the spin group representation is constructed and the general form of the bivector generators. The remaining general linear groups are listed in Table 3.13. Again, their invariant bivectors and the general form of the generators are listed. For both tables, the conventions for the various functions and bivectors used are those of the section where the group was discussed.

A number of extensions to this work can be considered. It is well known, for example, that the Lie group G_2 can be constructed in $\Re_{0,7}$ as the invariance group of a particular trivector (which is given in [46]). This suggests that the techniques explored in this chapter can be applied to the exceptional groups. A geometric algebra is a graded space and in Chapter 5 we will see how this can be used to define a multivector bracket which satisfies

Type	Form of $\epsilon(a,b)$	Group	Base Space	Form of Bivector Generators
R-symmetric	$a\cdot b$	$\mathrm{SO}(p,q)$	$\Re^{p,q}$	B
R-skew	$a\cdot \underline{J}(b)$	Sp(n,R)	$\Re^{2n,2n}$	$B+\underline{J}_S(B)-\underline{K}(B+\underline{J}_S(B))$
C-symmetric	$a\cdot b + ja\cdot \underline{K}^*(b)$	SO(n,C)	$\Re^{n,n}$	$B-\underline{K}^*(B)$
C-skew	$a\cdot \underline{J}_1(b) + ja\cdot \underline{J}_2(b)$	Sp(n,C)	$\Re^{4n,4n}$	$B+\underline{J}(B)+\underline{J}_S(B)+\underline{J}\,\underline{J}_S(B)$
				$-\underline{K}($ $''$ $)$
C-Hermitian	$a\cdot b + ja\cdot \underline{J}(b)$	$\mathrm{U}(p,q)$	$\Re^{2p,2q}$	$B+\underline{J}(B)$
H-Hermitian	$a\cdot b + a\cdot \underline{J}_1(b)\boldsymbol{i} +$ $a\cdot \underline{J}_2(b)\boldsymbol{j} + a\cdot \underline{J}_3(b)\boldsymbol{j}$	Sp(n)	$\Re^{4n}$	$B+\underline{J}_1(B)+\underline{J}_2(B)+\underline{J}_3(B)$
H-Skew	$a\cdot \underline{J}(b) + a\cdot \underline{K}_1^*(b) +$ $a\cdot b\boldsymbol{i} + a\cdot \underline{K}_2^*(b)\boldsymbol{k}$	$\mathrm{SO}^*(2n)$	$\Re^{2n,2n}$	$B+\underline{J}(B)-\underline{K}_1^*(B)-\underline{K}_2^*(B)$

Table 3.12: The Classical Bilinear Forms and their Invariance Groups

the super-Jacobi identities. This opens up the possibility of further extending the work of this chapter to include super-Lie algebras. Furthermore, we shall see in Chapter 4 that the techniques developed for doubling spaces are ideally suited to the study of multiparticle quantum theory. Whether some practical benefits await the idea that all general linear transformations can be represented as even products of vectors remains to be seen.

Group	Base Space	Invariants	Form of Bivector Generators
GL(n,R)	$\Re^{n,n}$	K	$B-\underline{K}(B)$
GL(n,C)	$\Re^{2n,2n}$	K, J	$B+\underline{J}(B)-\underline{K}(B+\underline{J}(B))$
GL(n,H) / SU*(n)	$\Re^{4n,4n}$	K, J_1', J_2', J_3'	$B+\underline{J}_1'(B)+\underline{J}_2'(B)+\underline{J}_3'(B)$
			$-\underline{K}($ $''$ $)$

Table 3.13: The General Linear Groups

Chapter 4

Spinor Algebra

This chapter describes a translation between conventional matrix-based spinor algebra in three and four dimensions [59, 60], and an approach based entirely in the (real) geometric algebra of spacetime. The geometric algebra of Minkowski spacetime is called the *spacetime algebra* or, more simply, the STA. The STA was introduced in Section 1.2.5 as the geometric algebra generated by a set of four orthonormal vectors $\{\gamma_\mu\}$, $\mu = 0\ldots 3$, satisfying

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu} = \mathrm{diag}(+\ -\ -\ -). \tag{4.1}$$

Whilst the $\{\gamma_\mu\}$ satisfy the Dirac algebra generating relations, they are to be thought of as an orthonormal frame of independent vectors and not as components of a single "isospace" vector. The full STA is spanned by the basis

$$1, \qquad \{\gamma_\mu\} \qquad \{\sigma_k,\, i\sigma_k\}, \qquad \{i\gamma_\mu\}, \qquad i, \tag{4.2}$$

where

$$i \equiv \gamma_0\gamma_1\gamma_2\gamma_3 \tag{4.3}$$

and

$$\sigma_k \equiv \gamma_k\gamma_0. \tag{4.4}$$

The meaning of these equation was discussed in Section 1.2.5.

The aim of this chapter is to express both spinors and matrix operators within the real STA. This results in a very powerful language in which all algebraic manipulations can be performed without ever introducing a matrix representation. The Pauli matrix algebra is studied first, and an extension to multiparticle systems is introduced. The Dirac algebra and Dirac spinors are then considered. The translation into the STA quickly yields the Dirac equation in the form first found by Hestenes [17, 19, 21, 27]. The concept of the multiparticle STA is introduced, and is used to formulate a number of two-particle relativistic wave equations. Some problems with these are discussed and a new equation, which has no spinorial counterpart, is proposed. The chapter concludes with a discussion of the 2-spinor calculus of Penrose & Rindler [36]. Again, it is shown how a scalar unit imaginary is eliminated by the use of the real multiparticle STA. Some sections of this chapter appeared in the papers "*States and operators in the spacetime algebra*" [6] and "*2-Spinors, twistors and supersymmetry in the spacetime algebra* [4].

4.1 Pauli Spinors

This section establishes a framework for the study of the Pauli operator algebra and Pauli spinors within the geometric algebra of 3-dimensional space. The geometric algebra of space was introduced in Section 1.2.3 and is spanned by

$$1, \qquad \{\sigma_k\}, \qquad \{i\sigma_k\}, \qquad i. \tag{4.5}$$

Here the $\{\sigma_k\}$ are a set of three relative vectors (spacetime bivectors) in the γ_0-system. Vectors in this system are written in bold type to distinguish them from spacetime vectors. There is no possible confusion with the $\{\sigma_k\}$ symbols, so these are left in normal type. When working non-relativistically within the even subalgebra of the full STA some notational modifications are necessary. Relative vectors $\{\sigma_k\}$ and relative bivectors $\{i\sigma_k\}$ are both bivectors in the full STA, so spatial reversion and spacetime reversion have different effects. To distinguish these, we define the operation

$$A^\dagger = \gamma_0 \tilde{A} \gamma_0, \tag{4.6}$$

which defines reversion in the Pauli algebra. The presence of the γ_0 vector in the definition of Pauli reversion shows that this operation is dependent on the choice of spacetime frame. The dot and wedge symbols also carry different meanings dependent on whether their arguments are treated as spatial vectors or spacetime bivectors. The convention adopted here is that the meaning is determined by whether their arguments are written in bold type or not. Bold-type objects are treated as three-dimensional multivectors, whereas normal-type objects are treated as belonging to the full STA. This is the one potentially confusing aspect of our conventions, though in practice the meaning of all the symbols used is quite unambiguous.

The Pauli operator algebra [59] is generated by the 2×2 matrices

$$\hat{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_2 = \begin{pmatrix} 0 & -j \\ j & 0 \end{pmatrix}, \quad \hat{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{4.7}$$

These operators act on 2-component complex spinors

$$|\psi\rangle = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \tag{4.8}$$

where ψ_1 and ψ_2 are complex numbers. We have adopted a convention by which standard quantum operators appear with carets, and quantum states are written as kets and bras. We continue to write the unit scalar imaginary of conventional quantum mechanics as j, which distinguishes it from the geometric pseudoscalar i.

To realise the Pauli operator algebra within the algebra of space, the column Pauli spinor $|\psi\rangle$ is placed in one-to-one correspondence with the even multivector ψ (which satisfies $\psi = \gamma_0 \psi \gamma_0$) through the identification[1]

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k i\sigma_k. \tag{4.9}$$

[1]This mapping was first found by Anthony Lasenby.

In particular, the basis spin-up and spin-down states become

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \leftrightarrow \quad 1 \tag{4.10}$$

and

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \leftrightarrow \quad -i\sigma_2. \tag{4.11}$$

The action of the four quantum operators $\{\hat{\sigma}_k, j\}$ can now be replaced by the operations

$$\hat{\sigma}_k \left|\psi\right\rangle \quad \leftrightarrow \quad \sigma_k \psi \sigma_3 \qquad (k = 1, 2, 3) \tag{4.12}$$

and

$$j \left|\psi\right\rangle \quad \leftrightarrow \quad \psi i\sigma_3. \tag{4.13}$$

Verifying these relations is a matter of routine computation, for example

$$\hat{\sigma}_1 \left|\psi\right\rangle = \begin{pmatrix} -a^2 + ja^1 \\ a^0 + ja^3 \end{pmatrix} \quad \leftrightarrow \quad \begin{matrix} -a^2 + a^1 i\sigma_3 \\ -a^0 i\sigma_2 + a^3 i\sigma_1 \end{matrix} = \sigma_1 \left(a^0 + a^k i\sigma_k\right) \sigma_3. \tag{4.14}$$

With these definitions, the action of complex conjugation of a Pauli spinor translates to

$$\left|\psi\right\rangle^* \quad \leftrightarrow \quad \sigma_2 \psi \sigma_2. \tag{4.15}$$

The presence of a fixed spatial vector on the *left*-hand side of ψ shows that complex conjugation is a frame-dependent concept.

As an illustration, the Pauli equation (in natural units),

$$j\partial_t \left|\psi\right\rangle = \frac{1}{2m}\left((-j\boldsymbol{\nabla} - e\boldsymbol{A})^2 - e\hat{\sigma}_k B^k\right)\left|\psi\right\rangle + eV\left|\psi\right\rangle, \tag{4.16}$$

can be written (in the Coulomb gauge) as [22]

$$\partial_t \psi i\sigma_3 = \frac{1}{2m}(-\boldsymbol{\nabla}^2\psi + 2e\boldsymbol{A}\cdot\boldsymbol{\nabla}\psi i\sigma_3 + e^2\boldsymbol{A}^2\psi) - \frac{e}{2m}\boldsymbol{B}\psi\sigma_3 + eV\psi, \tag{4.17}$$

where $\boldsymbol{B}$ is the magnetic field vector $B^k\sigma_k$. This translation achieves two important goals. The scalar unit imaginary is eliminated in favour of right-multiplication by $i\sigma_3$, and all terms (both operators and states) are now real-space multivectors. Removal of the distinction between states and operators is an important conceptual simplification.

We next need to find a geometric algebra equivalent of the spinor inner product $\langle\psi\,|\phi\rangle$. In order to see how to handle this, we need only consider its real part. This is given by

$$\Re\langle\psi\,|\phi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\phi\rangle, \tag{4.18}$$

so that, for example,

$$\begin{aligned} \langle\psi\,|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\psi\rangle &= \langle(a^0 - ia^j\sigma_j)(a^0 + ia^k\sigma_k)\rangle \\ &= (a^0)^2 + a^k a^k. \end{aligned} \tag{4.19}$$

In particular, the basis spin-up and spin-down states become

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and

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and

$$j\,|\psi\rangle \quad \leftrightarrow \quad \psi i\sigma_3. \tag{4.13}$$

Verifying these relations is a matter of routine computation, for example

$$\hat{\sigma}_1 \,|\psi\rangle = \begin{pmatrix} -a^2 + ja^1 \\ a^0 + ja^3 \end{pmatrix} \quad \leftrightarrow \quad \begin{matrix} -a^2 + a^1 i\sigma_3 \\ -a^0 i\sigma_2 + a^3 i\sigma_1 \end{matrix} \quad = \sigma_1 \left(a^0 + a^k i\sigma_k\right) \sigma_3. \tag{4.14}$$

With these definitions, the action of complex conjugation of a Pauli spinor translates to

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$$j\partial_t \,|\psi\rangle = \frac{1}{2m}\left((-j\boldsymbol{\nabla} - e\boldsymbol{A})^2 - e\hat{\sigma}_k B^k\right)|\psi\rangle + eV\,|\psi\rangle\,, \tag{4.16}$$

can be written (in the Coulomb gauge) as [22]

$$\partial_t \psi i\sigma_3 = \frac{1}{2m}(-\boldsymbol{\nabla}^2\psi + 2e\boldsymbol{A}\cdot\boldsymbol{\nabla}\psi i\sigma_3 + e^2\boldsymbol{A}^2\psi) - \frac{e}{2m}\boldsymbol{B}\psi\sigma_3 + eV\psi, \tag{4.17}$$

where $\boldsymbol{B}$ is the magnetic field vector $B^k\sigma_k$. This translation achieves two important goals. The scalar unit imaginary is eliminated in favour of right-multiplication by $i\sigma_3$, and all terms (both operators and states) are now real-space multivectors. Removal of the distinction between states and operators is an important conceptual simplification.

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so that, for example,

$$\begin{aligned} \langle\psi\,|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\psi\rangle \quad &= \quad \langle(a^0 - ia^j\sigma_j)(a^0 + ia^k\sigma_k)\rangle \\ &= \quad (a^0)^2 + a^k a^k. \end{aligned} \tag{4.19}$$

Since

$$\langle\psi|\phi\rangle = \Re\langle\psi|\phi\rangle - j\Re\langle\psi|j\phi\rangle, \tag{4.20}$$

the full inner product becomes

$$\langle\psi\,|\phi\rangle \quad \leftrightarrow \quad (\psi,\phi)_S \equiv \langle\psi^\dagger\phi\rangle - \langle\psi^\dagger\phi i\sigma_3\rangle i\sigma_3. \tag{4.21}$$

The right hand side projects out the $\{1, i\sigma_3\}$ components from the geometric product $\psi^\dagger\phi$. The result of this projection on a multivector A is written $\langle A\rangle_S$. For Pauli-even multivectors this projection has the simple form

$$\langle A\rangle_S = \tfrac{1}{2}(A - i\sigma_3 A i\sigma_3). \tag{4.22}$$

As an example of (4.21), consider the expectation value

$$\langle\psi|\hat{\sigma}_k\,|\psi\rangle \quad \leftrightarrow \quad \langle\psi^\dagger\sigma_k\psi\sigma_3\rangle - \langle\psi^\dagger\sigma_k\psi i\rangle i\sigma_3 = \sigma_k\cdot\langle\psi\sigma_3\psi^\dagger\rangle_1, \tag{4.23}$$

which gives the mean value of spin measurements in the k direction. The STA form indicates that this is the component of the spin vector $\boldsymbol{s} = \psi\sigma_3\psi^\dagger$ in the σ_k direction, so that $\boldsymbol{s}$ is the coordinate-free form of this vector. Since $\psi\sigma_3\psi^\dagger$ is both Pauli-odd and Hermitian-symmetric (reverse-symmetric in the Pauli algebra), $\boldsymbol{s}$ contains only a vector part. (In fact, both spin and angular momentum are better viewed as bivector quantities, so it is usually more convenient to work with $i\boldsymbol{s}$ instead of $\boldsymbol{s}$.)

Under an active rotation, the spinor ψ transforms as

$$\psi \mapsto \psi' = R_0\psi, \tag{4.24}$$

where R_0 is a constant rotor. The quantity ψ' is even, and so is a second spinor. (The term "spinor" is used in this chapter to denote any member of a linear space which is closed under left-multiplication by a rotor R_0.) The corresponding transformation law for $\boldsymbol{s}$ is

$$\boldsymbol{s} \mapsto \boldsymbol{s}' = R_0\boldsymbol{s}R_0^\dagger, \tag{4.25}$$

which is the standard double-sided rotor description for a rotation, introduced in Section 1.2.4.

The definitions (4.9), (4.12) and (4.13) have established a simple translation from the language of Pauli operators and spinors into the geometric algebra of space. But the STA formulation can be taken further to afford new insights into the role of spinors in the Pauli theory. By defining

$$\rho = \psi\psi^\dagger \tag{4.26}$$

the spinor ψ can be written

$$\psi = \rho^{1/2}R, \tag{4.27}$$

where R is defined as

$$R = \rho^{-1/2}\psi. \tag{4.28}$$

R satisfies

$$RR^\dagger = 1 \tag{4.29}$$

and is therefore a spatial rotor. The spin vector can now be written

$$s = \rho R\sigma_3 R^\dagger, \tag{4.30}$$

which demonstrates that the double-sided construction of the expectation value (4.23) contains an instruction to rotate and dilate the fixed σ_3 axis into the spin direction. The original states of quantum mechanics have now become operators in the STA, acting on vectors. The decomposition of the spinor ψ into a density term ρ and a rotor R suggests that a deeper substructure underlies the Pauli theory. This is a subject which has been frequently discussed by David Hestenes [19, 22, 23, 27]. As an example of the insights afforded by this decomposition, it is now clear "why" spinors transform single-sidedly under active rotations of fields in space. If the vector $\boldsymbol{s}$ is to be rotated to a new vector $R_0\boldsymbol{s}R_0^\dagger$ then, according to the rotor group combination law, R must transform to R_0R. This produces the spinor transformation law (4.24).

We should now consider the status of the fixed $\{\sigma_k\}$ frame. The form of the Pauli equation (4.17) illustrates the fact that, when forming covariant expressions, the $\{\sigma_k\}$ only appear explicitly on the right-hand side of ψ. In an expression like

$$A^k\hat{\sigma}_k|\psi\rangle \quad \leftrightarrow \quad \boldsymbol{A}\psi\sigma_3, \tag{4.31}$$

for example, the quantity $\boldsymbol{A}$ is a spatial vector and transforms as

$$\boldsymbol{A} \mapsto \boldsymbol{A}' = R_0\boldsymbol{A}R_0^\dagger. \tag{4.32}$$

The entire quantity therefore transforms as

$$\boldsymbol{A}\psi\sigma_3 \mapsto R_0\boldsymbol{A}R_0^\dagger R_0\psi\sigma_3 = R_0\boldsymbol{A}\psi\sigma_3, \tag{4.33}$$

so that $\boldsymbol{A}\psi\sigma_3$ is another spinor, as required. Throughout this derivation, the σ_3 sits on the right-hand side of ψ and does not transform — it is part of a fixed frame in space. A useful analogy is provided by rigid-body dynamics, in which a rotating frame $\{\boldsymbol{e}_k\}$, aligned with the principal axes of the body, can be related to a fixed laboratory frame $\{\sigma_k\}$ by

$$\boldsymbol{e}_k = R\sigma_k R^\dagger. \tag{4.34}$$

The dynamics is now completely contained in the rotor R. The rotating frame $\{\boldsymbol{e}_k\}$ is unaffected by the choice of laboratory frame. A different fixed laboratory frame,

$$\sigma_k' = R_1\sigma_k R_1^\dagger, \tag{4.35}$$

simply requires the new rotor

$$R' = RR_1^\dagger \tag{4.36}$$

to produce the same rotating frame. Under an active rotation, the rigid body is rotated about its centre of mass, whilst the laboratory frame is fixed. Such a rotation takes

$$\boldsymbol{e}_k \mapsto \boldsymbol{e}_k' = R_0\boldsymbol{e}_k R_0^\dagger \tag{4.37}$$

which is enforced by the rotor transformation $R \mapsto R_0R$. The fixed frame is shielded from this rotation, and so is unaffected by the active transformation. This is precisely what happens in the Pauli theory. The spinor ψ contains a rotor, which shields vectors on the right-hand side of the spinor from active rotations of spatial vectors.

Since multiplication of a column spinor by j is performed in the STA by right-sided multiplication by $i\sigma_3$, a U(1) gauge transformation is performed by

$$\psi \mapsto \psi' = \psi e^{\phi i\sigma_3}. \tag{4.38}$$

This right-sided multiplication by the rotor $R = \exp\{\phi i\sigma_3\}$ is equivalent to a rotation of the initial (fixed) frame to the new frame $\{R\sigma_k R^\dagger\}$. Gauge invariance can therefore now be interpreted as the requirement that physics is unaffected by the position of the σ_1 and σ_2 axes in the $i\sigma_3$ plane. In terms of rigid-body dynamics, this means that the body behaves as a symmetric top. These analogies between rigid-body dynamics and the STA form of the Pauli theory are quite suggestive. We shall shortly see how these analogies extend to the Dirac theory.

4.1.1 Pauli Operators

In our geometric algebra formalism, an arbitrary operator $\hat{M}\,|\psi\rangle$ is replaced by a linear function $M(\psi)$ acting on even multivectors in the algebra of space. The function $M(\psi)$ is an example of the natural extension of linear algebra to encompass linear operators acting on multivectors. The study of such functions is termed "multilinear function theory" and some preliminary results in this field, including a new approach to the Petrov classification of the Riemann tensor, have been given by Hestenes & Sobczyk [24]. Since ψ is a 4-component multivector, the space of functions $M(\psi)$ is 16-dimensional, which is the dimension of the group GL(4,R). This is twice as large as the 8-dimensional Pauli operator algebra (which forms the group GL(2,C)). The subset of multilinear functions which represent Pauli operators is defined by the requirement that $M(\psi)$ respects the complex structure,

$$\begin{aligned} j\hat{M}(j\,|\psi\rangle) &= -\hat{M}\,|\psi\rangle \\ \Rightarrow M(\psi i\sigma_3)i\sigma_3 &= -M(\psi). \end{aligned} \tag{4.39}$$

The set of $M(\psi)$ satisfying (4.39) is 8-dimensional, as required.

The Hermitian operator adjoint is defined by

$$\langle\psi\,\big|\hat{M}\phi\big\rangle = \langle\hat{M}^\dagger\psi\,|\phi\rangle\,. \tag{4.40}$$

In terms of the function $M(\psi)$, this separates into two equations

$$\langle\psi^\dagger M(\phi)\rangle = \langle M^\dagger_{HA}(\psi)\phi\rangle \tag{4.41}$$

and

$$\langle\psi^\dagger M(\phi)i\sigma_3\rangle = \langle M^\dagger_{HA}(\psi)\phi i\sigma_3\rangle \tag{4.42}$$

where the subscript on M_{HA} labels the STA representation of the Pauli operator adjoint. The imaginary equation (4.42) is automatically satisfied by virtue of (4.41) and (4.39).

The adjoint of a multilinear function is defined in the same way as that of a linear function (Section 1.3), so that

$$\langle \bar{M}(\psi)\phi\rangle = \langle \psi M(\phi)\rangle. \tag{4.43}$$

The Pauli operator adjoint is therefore given by the combination of a reversion, the geometric adjoint, and a second reversion,

$$M_{HA}(\psi) = \bar{M}^\dagger(\psi^\dagger). \tag{4.44}$$

For example, if $M(\psi) = A\psi B$, then

$$\bar{M}(\psi) = B\psi A \tag{4.45}$$

and

$$\begin{aligned} M_{HA}(\psi) &= (B\psi^\dagger A)^\dagger \\ &= A^\dagger \psi B^\dagger \end{aligned} \tag{4.46}$$

Since the STA action of the $\hat{\sigma}_k$ operators takes ψ into $\sigma_k\psi\sigma_3$, it follows that these operators are, properly, Hermitian. Through this approach, the Pauli operator algebra can now be fully integrated into the wider subject of multilinear function theory.

4.2 Multiparticle Pauli States

In quantum theory, 2-particle states are assembled from direct products of single-particle states. For example, a basis for the outer-product space of two spin-1/2 states is given by the set

$$\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix},\quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix},\quad \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix},\quad \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix}. \tag{4.47}$$

To represent these states in the STA, we must consider forming copies of the STA itself. We shall see shortly that, for relativistic states, multiparticle systems are constructed by working in a $4n$-dimensional configuration space. Thus, to form two-particle relativistic states, we work in the geometric algebra generated by the basis set $\{\gamma_\mu^1, \gamma_\mu^2\}$, where the basis vectors from different particle spacetimes *anticommute*. (The superscripts label the particle space.) If we wanted to adopt the same procedure when working non-relativistically, we would set up a space spanned by $\{\sigma_i^1, \sigma_i^2\}$, where the basis vectors from different particle spaces also anticommute. This construction would indeed suffice for an entirely non-relativistic discussion. The view adopted throughout this thesis, however, is that the algebra of space is derived from the more fundamental relativistic algebra of spacetime. The construction of multiparticle Pauli states should therefore be consistent with the construction of relativistic multiparticle states. It follows that the spatial vectors from two separate copies of spacetime are given by

$$\sigma_i^1 = \gamma_i^1\gamma_0^1 \tag{4.48}$$

$$\sigma_i^2 = \gamma_i^2\gamma_0^2 \tag{4.49}$$

and so satisfy

$$\sigma_i^1\sigma_j^2 = \gamma_i^1\gamma_0^1\gamma_j^2\gamma_0^2 = \gamma_i^1\gamma_j^2\gamma_0^2\gamma_0^1 = \gamma_j^2\gamma_0^2\gamma_i^1\gamma_0^1 = \sigma_j^2\sigma_i^1. \tag{4.50}$$

In constructing multiparticle Pauli states, we must therefore take the basis vectors from different particle spaces as commuting. In fact, for the non-relativistic discussion of this section, it does not matter whether these vectors are taken as commuting or anticommuting. It is only when we come to consider relativistic states, and in particular the 2-spinor calculus, that the difference becomes important.

Since multiparticle states are ultimately constructed in a subalgebra of the geometric algebra of relativistic configuration space, the elements used all inherit a well-defined Clifford multiplication. There is therefore no need for the tensor product symbol $\otimes$, which is replaced by simply juxtaposing the elements. Superscripts are used to label the single-particle algebra from which any particular element is derived. As a further abbreviation $i^1\sigma_1^1$ is written, wherever possible, as $i\sigma_1^1$ *etc.* This helps to remove some of the superscripts. The unit element of either space is written simply as 1.

The full 2-particle algebra generated by commuting basis vectors is $8 \times 8 = 64$ dimensional. The spinor subalgebra is $4 \times 4 = 16$ dimensional, which is twice the dimension of the direct product of two 2-component complex spinors. The dimensionality has doubled because we have not yet taken the complex structure of the spinors into account. While the role of j is played in the two single-particle spaces by right multiplication by $i\sigma_3^1$ and $i\sigma_3^2$ respectively, standard quantum mechanics does not distinguish between these operations. A projection operator must therefore be included to ensure that right multiplication by $i\sigma_3^1$ or $i\sigma_3^2$ reduces to the same operation. If a two-particle spin state is represented by the multivector ψ, then ψ must satisfy

$$\psi i\sigma_3^1 = \psi i\sigma_3^2 \tag{4.51}$$

from which we find that

$$\psi = -\psi i\sigma_3^1 i\sigma_3^2 \tag{4.52}$$

$$\Rightarrow \quad \psi = \psi\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2). \tag{4.53}$$

On defining

$$E = \tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2), \tag{4.54}$$

it is seen that

$$E^2 = E \tag{4.55}$$

so right multiplication by E is a projection operation. It follows that the two-particle state ψ must contain a factor of E on its right-hand side. We can further define

$$J = Ei\sigma_3^1 = Ei\sigma_3^2 = \tfrac{1}{2}(i\sigma_3^1 + i\sigma_3^2) \tag{4.56}$$

so that

$$J^2 = -E. \tag{4.57}$$

Right-sided multiplication by J takes over the role of j for multiparticle states.

The STA representation of a 2-particle Pauli spinor is now given by $\psi^1\phi^2 E$, where ψ^1 and ϕ^2 are spinors (even multivectors) in their own spaces. A complete basis for 2-particle spin states is provided by

$$\begin{aligned}
\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\quad\leftrightarrow\quad E \\
\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} &\quad\leftrightarrow\quad -i\sigma_2^1 E \\
\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\quad\leftrightarrow\quad -i\sigma_2^2 E \\
\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} &\quad\leftrightarrow\quad i\sigma_2^1 i\sigma_2^2 E.
\end{aligned} \tag{4.58}$$

This procedure extends simply to higher multiplicities. All that is required is to find the "quantum correlator" E_n satisfying

$$E_n i\sigma_3^j = E_n i\sigma_3^k = J_n \quad \text{for all } j,\, k. \tag{4.59}$$

E_n can be constructed by picking out the $j=1$ space, say, and correlating all the other spaces to this, so that

$$E_n = \prod_{j=2}^{n} \tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^j). \tag{4.60}$$

The form of E_n is independent of which of the n spaces is singled out and correlated to. The complex structure is defined by

$$J_n = E_n i\sigma_3^j, \tag{4.61}$$

where $i\sigma_3^j$ can be chosen from any of the n spaces. To illustrate this consider the case of $n=3$, where

$$\begin{aligned}
E_3 &= \tfrac{1}{4}(1 - i\sigma_3^1 i\sigma_3^2)(1 - i\sigma_3^1 i\sigma_3^3) && (4.62)\\
&= \tfrac{1}{4}(1 - i\sigma_3^1 i\sigma_3^2 - i\sigma_3^1 i\sigma_3^3 - i\sigma_3^2 i\sigma_3^3) && (4.63)
\end{aligned}$$

and

$$J_3 = \tfrac{1}{4}(i\sigma_3^1 + i\sigma_3^2 + i\sigma_3^3 - i\sigma_3^1 i\sigma_3^2 i\sigma_3^3). \tag{4.64}$$

Both E_3 and J_3 are symmetric under permutations of their indices.

A significant feature of this approach is that all the operations defined for the single-particle STA extend naturally to the multiparticle algebra. The reversion operation, for example, still has precisely the same definition — it simply reverses the order of vectors in any given multivector. The spinor inner product (4.21) also generalises immediately, to

$$(\psi, \phi)_S = \langle E_n\rangle^{-1}[\langle \psi^\dagger \phi E_n\rangle E_n - \langle \psi^\dagger \phi J_n\rangle J_n]. \tag{4.65}$$

The factor of $\langle E_n\rangle^{-1}$ is included so that the operation

$$P(M) = \langle E_n\rangle^{-1}[\langle M E_n\rangle E_n - \langle M J_n\rangle J_n] \tag{4.66}$$

is a projection operation (*i.e.* $P(M)$ satisfies $P^2(M) = P(M)$). The fact that $P(M)$ is a projection operation follows from the results

$$\begin{aligned} P(E_n) &= \langle E_n \rangle^{-1} [\langle E_n E_n \rangle E_n - \langle E_n J_n \rangle J_n] \\ &= \langle E_n \rangle^{-1} [\langle E_n \rangle E_n - \langle E_n i\sigma_3^j \rangle J_n] \\ &= E_n \end{aligned} \tag{4.67}$$

and

$$\begin{aligned} P(J_n) &= \langle E_n \rangle^{-1} [\langle J_n E_n \rangle E_n - \langle J_n J_n \rangle J_n] \\ &= J_n. \end{aligned} \tag{4.68}$$

4.2.1 The Non-Relativistic Singlet State

As an application of the formalism outlined above, consider the 2-particle singlet state $|\epsilon\rangle$, defined by

$$|\epsilon\rangle = \frac{1}{\sqrt{2}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}. \tag{4.69}$$

This is represented in the two-particle STA by the multivector

$$\epsilon = \frac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2). \tag{4.70}$$

The properties of ϵ are more easily seen by writing

$$\epsilon = \tfrac{1}{2}(1 + i\sigma_2^1 i\sigma_2^2)\tfrac{1}{2}(1 + i\sigma_3^1 i\sigma_3^2)\sqrt{2} i\sigma_2^1, \tag{4.71}$$

which shows how ϵ contains the commuting idempotents $\frac{1}{2}(1 + i\sigma_2^1 i\sigma_2^2)$ and $\frac{1}{2}(1 + i\sigma_3^1 i\sigma_3^2)$. The normalisation ensures that

$$\begin{aligned} (\epsilon, \epsilon)_S &= 2\langle \epsilon^\dagger \epsilon \rangle E_2 \\ &= 4\langle \tfrac{1}{2}(1 + i\sigma_2^1 i\sigma_2^2)\tfrac{1}{2}(1 + i\sigma_3^1 i\sigma_3^2) \rangle E_2 \\ &= E_2. \end{aligned} \tag{4.72}$$

The identification of the idempotents in ϵ leads immediately to the results that

$$i\sigma_2^1 \epsilon = \tfrac{1}{2}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 + i\sigma_3^1 i\sigma_3^2)\sqrt{2} i\sigma_2^1 = -i\sigma_2^2 \epsilon \tag{4.73}$$

and

$$i\sigma_3^1 \epsilon = -i\sigma_3^2 \epsilon, \tag{4.74}$$

and hence that

$$i\sigma_1^1 \epsilon = i\sigma_3^1 i\sigma_2^1 \epsilon = -i\sigma_2^2 i\sigma_3^1 \epsilon = i\sigma_2^2 i\sigma_3^2 \epsilon = -i\sigma_1^2 \epsilon. \tag{4.75}$$

If M^1 is an arbitrary even element in the Pauli algebra ($M = M^0 + M^k i\sigma_k^1$), then it follows that ϵ satisfies

$$M^1 \epsilon = M^{2\dagger} \epsilon. \tag{4.76}$$

This provides a novel demonstration of the rotational invariance of ϵ. Under a joint rotation in 2-particle space, a spinor ψ transforms to $R^1 R^2 \psi$, where R^1 and R^2 are copies of the same rotor but acting in the two different spaces. The combined quantity R^1R^2 is a rotor acting in 6-dimensional space, and its generator is of the form of the E_{ij} generators for SU(n) (Table 3.3). From equation (4.76) it follows that, under such a rotation, ϵ transforms as

$$\epsilon \mapsto R^1R^2\epsilon = R^1R^{1\dagger}\epsilon = \epsilon, \tag{4.77}$$

so that ϵ is a genuine 2-particle scalar.

4.2.2 Non-Relativistic Multiparticle Observables

Multiparticle observables are formed in the same way as for single-particle states. Some combination of elements from the fixed $\{\sigma_k^j\}$ frames is sandwiched between a multiparticle wavefunction ψ and its spatial reverse $\psi^\dagger$. An important example of this construction is provided by the multiparticle spin current. The relevant operator is

$$\begin{aligned} S_k(\psi) &= \sigma_k^1\psi\sigma_3^1 + \sigma_k^2\psi\sigma_3^2 + \cdots + \sigma_k^n\psi\sigma_3^n \\ &= -[i\sigma_k^1\psi i\sigma_3^1 + i\sigma_k^2\psi i\sigma_3^2 + \cdots + i\sigma_k^n\psi i\sigma_3^n] \end{aligned} \tag{4.78}$$

and the corresponding observable is

$$\begin{aligned} (\psi, S_k(\psi))_S &= -\langle E_n\rangle^{-1}\langle\psi^\dagger(i\sigma_k^1\psi i\sigma_3^1 + \cdots + i\sigma_k^n\psi i\sigma_3^n)E_n\rangle E_n \\ &\quad +\langle E_n\rangle^{-1}\langle\psi^\dagger(i\sigma_k^1\psi i\sigma_3^1 + \cdots + i\sigma_k^n\psi i\sigma_3^n)J_n\rangle J_n \\ &= -2^{n-1}[\langle(i\sigma_k^1 + \cdots + i\sigma_k^n)\psi J\psi^\dagger\rangle E_n + \langle(i\sigma_k^1 + \cdots + i\sigma_k^n)\psi\psi^\dagger\rangle J_n] \\ &= -2^{n-1}(i\sigma_k^1 + \cdots + i\sigma_k^n){*}(\psi J\psi^\dagger)E_n. \end{aligned} \tag{4.79}$$

The multiparticle spin current is therefore defined by the bivector

$$\boldsymbol{S} = 2^{n-1}\langle\psi J\psi^\dagger\rangle_2 \tag{4.80}$$

where the right-hand side projects out from the full multivector $\psi J\psi^\dagger$ the components which are pure bivectors in each of the particle spaces. The result of projecting out from the multivector M the components contained entirely in the ith-particle space will be denoted $\langle M\rangle^i$, so we can write

$$\boldsymbol{S}^i = 2^{n-1}\langle\psi J\psi^\dagger\rangle_2^i. \tag{4.81}$$

Under a joint rotation in n-particle space, ψ transforms to $R_1\ldots R_n\psi$ and $\boldsymbol{S}$ therefore transforms to

$$R^1\ldots R^n\boldsymbol{S}R^{n\dagger}\ldots R^{1\dagger} = R^1\boldsymbol{S}^1R^{1\dagger} + \cdots + R^n\boldsymbol{S}^nR^{n\dagger}. \tag{4.82}$$

Each of the single-particle spin currents is therefore rotated by the same amount in its own space. That the definition (4.80) is sensible can be checked with the four basis states (4.58). The form of $\boldsymbol{S}$ for each of these is contained in Table 4.1.

Other observables can be formed using different fixed multivectors. For example, a two-particle invariant is generated by sandwiching a constant multivector Σ between the singlet state ϵ,

$$M = \epsilon\Sigma\epsilon^\dagger. \tag{4.83}$$

Pauli State	Multivector Form	Spin Current
$\vert\uparrow\uparrow\rangle$	E_2	$i\sigma_3^1 + i\sigma_3^2$
$\vert\uparrow\downarrow\rangle$	$-i\sigma_2^2 E_2$	$i\sigma_3^1 - i\sigma_3^2$
$\vert\downarrow\uparrow\rangle$	$-i\sigma_2^1 E_2$	$-i\sigma_3^1 + i\sigma_3^2$
$\vert\downarrow\downarrow\rangle$	$i\sigma_2^1 i\sigma_2^2 E_2$	$-i\sigma_3^1 - i\sigma_3^2$

Table 4.1: Spin Currents for 2-Particle Pauli States

Taking $\Sigma = 1$ yields

$$M = \epsilon\epsilon^\dagger = 2\tfrac{1}{2}(1 + i\sigma_2^1 i\sigma_2^2)\tfrac{1}{2}(1 + i\sigma_3^1 i\sigma_3^2) = \tfrac{1}{2}(1 + i\sigma_1^1 i\sigma_1^2 + i\sigma_2^1 i\sigma_2^2 + i\sigma_3^1 i\sigma_3^2) \tag{4.84}$$

and $\Sigma = i^1 i^2$ gives

$$M = \epsilon i^1 i^2 \epsilon^\dagger = \tfrac{1}{2}(i^1 i^2 + \sigma_1^1 \sigma_1^2 + \sigma_2^1 \sigma_2^2 + \sigma_3^1 \sigma_3^2). \tag{4.85}$$

This shows that both $i\sigma_k^1 i\sigma_k^2$ and $\sigma_k^1 \sigma_k^2$ are invariants under two-particle rotations. In standard quantum mechanics these invariants would be thought of as arising from the "inner product" of the spin vectors $\hat{\sigma}_i^1$ and $\hat{\sigma}_i^2$. Here, we have seen that the invariants arise in a completely different way by looking at the full multivector $\epsilon\epsilon^\dagger$. It is interesting to note that the quantities $i\sigma_k^1 i\sigma_k^2$ and $\sigma_k^1 \sigma_k^2$ are similar in form to the symplectic (doubling) bivector J introduced in Section 3.2.

The contents of this section should have demonstrated that the multiparticle STA approach is capable of reproducing most (if not all) of standard multiparticle quantum mechanics. One important result that follows is that the unit scalar imaginary j can be completely eliminated from quantum mechanics and replaced by geometrically meaningful quantities. This should have significant implications for the interpretation of quantum mechanics. The main motivation for this work comes, however, from the extension to relativistic quantum mechanics. There we will part company with operator techniques altogether, and the multiparticle STA will suggest an entirely new approach to relativistic quantum theory.

4.3 Dirac Spinors

We now extend the procedures developed for Pauli spinors to show how Dirac spinors can be understood in terms of the geometry of *real* spacetime. This reveals a geometrical role for spinors in the Dirac theory (a role which was first identified by Hestenes [19, 21]). Furthermore, this formulation is *representation-free*, highlighting the intrinsic content of the Dirac theory.

We begin with the γ-matrices in the standard Dirac-Pauli representation [59],

$$\hat{\gamma}_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_k = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}. \tag{4.86}$$

A Dirac column spinor $|\psi\rangle$ is placed in one-to-one correspondence with an 8-component even element of the STA via [4, 61]

$$|\psi\rangle = \begin{pmatrix} a^0 + ja^3 \\ -a^2 + ja^1 \\ -b^3 + jb^0 \\ -b^1 - jb^2 \end{pmatrix} \quad \leftrightarrow \quad \psi = a^0 + a^k i\sigma_k + i(b^0 + b^k i\sigma_k). \tag{4.87}$$

With the spinor $|\psi\rangle$ now replaced by an even multivector, the action of the operators $\{\hat{\gamma}_\mu, \hat{\gamma}_5, j\}$ (where $\hat{\gamma}_5 = \hat{\gamma}^5 = -j\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$) becomes

$$\begin{array}{rcl} \hat{\gamma}_\mu |\psi\rangle & \leftrightarrow & \gamma_\mu \psi \gamma_0 \quad (\mu = 0, \ldots, 3) \\ j\,|\psi\rangle & \leftrightarrow & \psi\, i\sigma_3 \\ \hat{\gamma}_5 |\psi\rangle & \leftrightarrow & \psi\sigma_3, \end{array} \tag{4.88}$$

which are verified by simple computation; for example

$$\hat{\gamma}_5 |\psi\rangle = \begin{pmatrix} -b^3 + jb^0 \\ -b^1 - jb^2 \\ a^0 + ja^3 \\ -a^2 + ja^1 \end{pmatrix} \quad \leftrightarrow \quad \begin{array}{l} -b^3 + b^0\sigma_3 + b^1 i\sigma_2 - b^2 i\sigma_1 \\ +a^0\sigma_3 + a^3 i - a^2\sigma_1 + a^1\sigma_2 \end{array} = \psi\sigma_3. \tag{4.89}$$

Complex conjugation in this representation becomes

$$|\psi\rangle^* \quad \leftrightarrow \quad -\gamma_2 \psi \gamma_2, \tag{4.90}$$

which picks out a preferred direction on the left-hand side of ψ and so is not a Lorentz-invariant operation.

As a simple application of (4.87) and (4.88), the Dirac equation

$$\hat{\gamma}^\mu (j\partial_\mu - eA_\mu)\,|\psi\rangle = m\,|\psi\rangle \tag{4.91}$$

becomes, upon postmultiplying by γ_0,

$$\nabla \psi i\sigma_3 - eA\psi = m\psi\gamma_0 \tag{4.92}$$

which is the form first discovered by Hestenes [17]. Here $\nabla = \gamma^\mu \partial_\mu$ is the vector derivative in spacetime. The properties of ∇ will be discussed more fully in Chapter 6. This translation is direct and unambiguous, leading to an equation which is not only coordinate-free (since the vectors $\nabla = \gamma^\mu \partial_\mu$ and $A = \gamma^\mu A_\mu$ no longer refer to any frame) but is also representation-free. In manipulating (4.92) one needs only the algebraic rules for multiplying spacetime multivectors, and the equation can be solved completely without ever having to introduce a matrix representation. Stripped of the dependence on a matrix representation, equation (4.92) expresses the intrinsic geometric content of the Dirac equation.

To discuss the spinor inner product, it is necessary to distinguish between the Hermitian and Dirac adjoint. These are written as

$$\begin{array}{lcl} \langle\bar{\psi}| & - & \text{Dirac adjoint} \\ \langle\psi| & - & \text{Hermitian adjoint,} \end{array} \tag{4.93}$$

which translate as follows,

$$\begin{array}{ccc} \langle\bar{\psi}| & \leftrightarrow & \tilde{\psi} \\ \langle\psi| & \leftrightarrow & \psi^\dagger = \gamma_0\tilde{\psi}\gamma_0. \end{array} \tag{4.94}$$

This makes it clear that the Dirac adjoint is the natural frame-invariant choice. The inner product is handled in the same manner as in equation (4.21), so that

$$\langle\bar{\psi}\,|\phi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\phi\rangle - \langle\tilde{\psi}\phi i\sigma_3\rangle i\sigma_3 = \langle\tilde{\psi}\phi\rangle_S, \tag{4.95}$$

which is also easily verified by direct calculation. In Chapters 6 and 7 we will be interested in the STA form of the Lagrangian for the Dirac equation so, as an illustration of (4.95), this is given here:

$$\mathcal{L} = \langle\bar{\psi}|(\hat{\gamma}_\mu(j\partial^\mu - eA^\mu) - m)|\psi\rangle \quad \leftrightarrow \quad \langle\nabla\psi i\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle. \tag{4.96}$$

By utilising (4.95) the STA forms of the Dirac spinor bilinear covariants [60] are readily found. For example,

$$\langle\bar{\psi}|\hat{\gamma}_\mu\,|\psi\rangle \quad \leftrightarrow \quad \langle\tilde{\psi}\gamma_\mu\psi\gamma_0\rangle - \langle\tilde{\psi}\gamma_\mu\psi i\gamma_3\rangle i\sigma_3 = \gamma_\mu\!\cdot\!\langle\psi\gamma_0\tilde{\psi}\rangle_1 \tag{4.97}$$

identifies the vector $\psi\gamma_0\tilde{\psi}$ as the coordinate-free representation of the Dirac current. Since $\psi\tilde{\psi}$ is even and reverses to give itself, it contains only scalar and pseudoscalar terms. We can therefore define

$$\rho e^{i\beta} \equiv \psi\tilde{\psi}. \tag{4.98}$$

Assuming $\rho \neq 0$, ψ can now be written as

$$\psi = \rho^{1/2}e^{i\beta/2}R \tag{4.99}$$

where

$$R = (\rho e^{i\beta})^{-1/2}\psi. \tag{4.100}$$

The even multivector R satisfies $R\tilde{R} = 1$ and is therefore a spacetime rotor. Double-sided application of R on a vector a produces a Lorentz transformation. The STA equivalents of the full set of bilinear covariants [33] can now be written as

$$\begin{array}{rrcl} \text{Scalar} & \langle\bar{\psi}\,|\psi\rangle & \leftrightarrow & \langle\psi\tilde{\psi}\rangle = \rho\cos\beta \\ \text{Vector} & \langle\bar{\psi}|\hat{\gamma}_\mu\,|\psi\rangle & \leftrightarrow & \psi\gamma_0\tilde{\psi} = \rho v \\ \text{Bivector} & \langle\bar{\psi}|j\hat{\gamma}_{\mu\nu}\,|\psi\rangle & \leftrightarrow & \psi i\sigma_3\tilde{\psi} = \rho e^{i\beta}S \\ \text{Pseudovector} & \langle\bar{\psi}|\hat{\gamma}_\mu\hat{\gamma}_5\,|\psi\rangle & \leftrightarrow & \psi\gamma_3\tilde{\psi} = \rho s \\ \text{Pseudoscalar} & \langle\bar{\psi}|j\hat{\gamma}_5\,|\psi\rangle & \leftrightarrow & \langle\psi\tilde{\psi}i\rangle = -\rho\sin\beta, \end{array} \tag{4.101}$$

where

$$\begin{array}{rcl} v & = & R\gamma_0\tilde{R} \\ s & = & R\gamma_3\tilde{R} \end{array} \tag{4.102}$$

and

$$S = isv. \tag{4.103}$$

These are summarised neatly by the equation

$$\psi(1+\gamma_0)(1+i\gamma_3)\tilde{\psi} = \rho\cos\beta + \rho v + \rho e^{i\beta}S + i\rho s + i\rho\sin\beta. \tag{4.104}$$

The full Dirac spinor ψ contains (in the rotor R) an instruction to carry out a rotation of the fixed $\{\gamma_\mu\}$ frame into the frame of observables. The analogy with rigid-body dynamics discussed in Section 4.1 therefore extends immediately to the relativistic theory. The single-sided transformation law for the spinor ψ is also "understood" in the same way that it was for Pauli spinors.

Once the spinor bilinear covariants are written in STA form (4.101) they can be manipulated far more easily than in conventional treatments. For example the Fierz identities, which relate the various observables (4.101), are simple to derive [33]. Furthermore, reconstituting ψ from the observables (up to a gauge transformation) is now a routine exercise, carried out by writing

$$\begin{aligned}\langle\psi\rangle_S &= \tfrac{1}{4}(\psi + \gamma_0\psi\gamma_0 - i\sigma_3(\psi+\gamma_0\psi\gamma_0)i\sigma_3)\\ &= \tfrac{1}{4}(\psi + \gamma_0\psi\gamma_0 + \sigma_3\psi\sigma_3 + \gamma_3\psi\gamma_3),\end{aligned} \tag{4.105}$$

so that

$$\psi\langle\tilde{\psi}\rangle_S = \tfrac{1}{4}\rho(e^{i\beta} + v\gamma_0 - e^{i\beta}Si\sigma_3 + s\gamma_3). \tag{4.106}$$

The right-hand side of (4.106) can be found directly from the observables, and the left-hand side gives ψ to within a complex multiple. On defining

$$Z = \tfrac{1}{4}\rho(e^{i\beta} + v\gamma_0 - e^{i\beta}Si\sigma_3 + s\gamma_3) \tag{4.107}$$

we find that, up to an arbitrary phase factor,

$$\psi = (\rho e^{i\beta})^{1/2}Z(Z\tilde{Z})^{-1/2}. \tag{4.108}$$

An arbitrary Dirac operator $\hat{M}\,|\psi\rangle$ is replaced in the STA by a multilinear function $M(\psi)$, which acts linearly on the entire even subalgebra of the STA. The 64 real dimensions of this space of linear operators are reduced to 32 by the constraint (4.39)

$$M(\psi i\sigma_3) = M(\psi)i\sigma_3. \tag{4.109}$$

Proceeding as at (4.44), the formula for the Dirac adjoint is

$$M_{DA}(\psi) = \tilde{\bar{M}}(\tilde{\psi}). \tag{4.110}$$

Self-adjoint Dirac operators satisfy $\tilde{M}(\psi) = \bar{M}(\tilde{\psi})$ and include the $\hat{\gamma}_\mu$. The Hermitian adjoint, M_{HA}, is derived in the same way:

$$M_{HA}(\psi) = \bar{M}^\dagger(\psi^\dagger), \tag{4.111}$$

in agreement with the non-relativistic equation (4.44).

Two important operator classes of linear operators on ψ are projection and symmetry operators. The particle/antiparticle projection operators are replaced by

$$\frac{1}{2m}(m \mp \hat{\gamma}_\mu p^\mu)|\psi\rangle \quad\leftrightarrow\quad \frac{1}{2m}(m\psi \mp p\psi\gamma_0), \tag{4.112}$$

and the spin-projection operators become

$$\tfrac{1}{2}(1 \pm \hat{\gamma}_\mu s^\mu \hat{\gamma}_5)|\psi\rangle \quad \leftrightarrow \quad \tfrac{1}{2}(\psi \pm s\psi\gamma_3). \tag{4.113}$$

Provided that $p \cdot s = 0$, the spin and particle projection operators commute.

The three discrete symmetries C, P and T translate equally simply (following the convention of Bjorken & Drell [59]):

$$\begin{array}{lcl} \hat{P}\,|\psi\rangle & \leftrightarrow & \gamma_0\psi(\bar{x})\gamma_0 \\ \hat{C}\,|\psi\rangle & \leftrightarrow & \psi\sigma_1 \\ \hat{T}\,|\psi\rangle & \leftrightarrow & i\gamma_0\psi(-\bar{x})\gamma_1, \end{array} \tag{4.114}$$

where $\bar{x} = \gamma_0 x \gamma_0$ is (minus) a reflection of x in the time-like γ_0 axis.

The STA representation of the Dirac matrix algebra will be used frequently throughout the remainder of this thesis. In particular, it underlies much of the gauge-theory treatment of gravity discussed in Chapter 7.

4.3.1 Changes of Representation — Weyl Spinors

In the matrix theory, a change of representation is performed with a 4×4 complex matrix $\hat{S}$. This defines new matrices

$$\hat{\gamma}'_\mu = \hat{S}\hat{\gamma}_\mu\hat{S}^{-1}, \tag{4.115}$$

with a corresponding spinor transformation $|\psi\rangle \mapsto \hat{S}\,|\psi\rangle$. For the Dirac equation, it is also required that the transformed Hamiltonian be Hermitian, which restricts (4.115) to a unitary transformation

$$\hat{\gamma}'_\mu = \hat{S}\hat{\gamma}_\mu\hat{S}^\dagger, \quad \hat{S}\hat{S}^\dagger = 1. \tag{4.116}$$

The STA approach to handling alternative matrix representations is to find a suitable analogue of the Dirac-Pauli map (4.87) which ensures that the effect of the matrix operators is still given by (4.88). The relevant map is easy to construct once the $\hat{S}$ is known which relates the new representation to the Dirac-Pauli representation. One starts with a column spinor $|\psi\rangle'$ in the new representation, constructs the equivalent Dirac-Pauli spinor $\hat{S}^\dagger|\psi\rangle'$, then maps this into its STA equivalent using (4.87). This technique ensures that the action of j and the $\{\hat{\gamma}_\mu, \hat{\gamma}_5\}$ matrices is still given by (4.88), and the $\hat{C}$, $\hat{P}$ and $\hat{T}$ operators are still represented by (4.114). The STA form of the Dirac equation is always given by (4.92) and so is a truly representation-free expression.

The STA from of the Dirac and Hermitian adjoints is always given by the formulae (4.110) and (4.111) respectively. But the separate transpose and complex conjugation operations retain some dependence on representation. For example, complex conjugation in the Dirac-Pauli representation is given by (4.90)

$$|\psi\rangle^* \quad \leftrightarrow \quad -\gamma_2\psi\gamma_2. \tag{4.117}$$

In the Majorana representation, however, we find that the action of complex conjugation on the Majorana spinor produces a different effect on its STA counterpart,

$$|\psi\rangle^*_{\text{Maj}} \quad \leftrightarrow \quad \psi\sigma_2. \tag{4.118}$$

In the operator/matrix theory complex conjugation is a representation-dependent concept. This limits its usefulness for our representation-free approach. Instead, we think of $\psi \mapsto -\gamma_2\psi\gamma_2$ and $\psi \mapsto \psi\sigma_2$ as distinct operations that can be performed on the multivector ψ. (Incidentally, equation 4.118 shows that complex conjugation in the Majorana representation does indeed coincide with our STA form of the charge conjugation operator (4.114), up to a conventional phase factor.)

To illustrate these techniques consider the Weyl representation, which is defined by the matrices [60]

$$\hat{\gamma}_0' = \begin{pmatrix} 0 & -I \\ -I & 0 \end{pmatrix} \quad \text{and} \quad \hat{\gamma}_k' = \begin{pmatrix} 0 & -\hat{\sigma}_k \\ \hat{\sigma}_k & 0 \end{pmatrix}. \tag{4.119}$$

The Weyl representation is obtained from the Dirac-Pauli representation by the unitary matrix

$$\hat{u} = \frac{1}{\sqrt{2}} \begin{pmatrix} I & I \\ -I & I \end{pmatrix}. \tag{4.120}$$

A spinor in the Weyl representation is written as

$$|\psi\rangle' = \begin{pmatrix} |\chi\rangle \\ |\bar{\eta}\rangle \end{pmatrix}, \tag{4.121}$$

where $|\chi\rangle$ and $|\bar{\eta}\rangle$ are 2-component spinors. Acting on $|\psi\rangle'$ with $\hat{u}^\dagger$ gives

$$\hat{u}^\dagger|\psi\rangle' = \tfrac{1}{\sqrt{2}} \begin{pmatrix} |\chi\rangle - |\bar{\eta}\rangle \\ |\chi\rangle + |\bar{\eta}\rangle. \end{pmatrix} \tag{4.122}$$

Using equation (4.87), this is mapped onto the even element

$$\hat{u}^\dagger|\psi\rangle' = \tfrac{1}{\sqrt{2}} \begin{pmatrix} |\chi\rangle - |\bar{\eta}\rangle \\ |\chi\rangle + |\bar{\eta}\rangle \end{pmatrix} \quad \leftrightarrow \quad \psi = \chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) - \bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3), \tag{4.123}$$

where χ and $\bar{\eta}$ are the Pauli-even equivalents of the 2-component complex spinors $|\chi\rangle$ and $|\bar{\eta}\rangle$, as defined by equation (4.9). The even multivector

$$\psi = \chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) - \bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3) \tag{4.124}$$

is therefore our STA version of the column spinor

$$|\psi\rangle' = \begin{pmatrix} |\chi\rangle \\ |\bar{\eta}\rangle \end{pmatrix}, \tag{4.125}$$

where $|\psi\rangle'$ is acted on by matrices in the Weyl representation. As a check, we observe that

$$\hat{\gamma}_0'|\psi\rangle' = \begin{pmatrix} -|\bar{\eta}\rangle \\ -|\chi\rangle \end{pmatrix} \quad \leftrightarrow \quad -\bar{\eta}\tfrac{1}{\sqrt{2}}(1+\sigma_3) + \chi\tfrac{1}{\sqrt{2}}(1-\sigma_3) = \gamma_0\psi\gamma_0 \tag{4.126}$$

and

$$\hat{\gamma}_k\,|\psi\rangle = \begin{pmatrix} -\hat{\sigma}_k|\bar{\eta}\rangle \\ \hat{\sigma}_k|\chi\rangle \end{pmatrix} \quad \leftrightarrow \quad -\sigma_k\bar{\eta}\sigma_3\tfrac{1}{\sqrt{2}}(1+\sigma_3) - \sigma_k\chi\sigma_3\tfrac{1}{\sqrt{2}}(1-\sigma_3) = \gamma_k\psi\gamma_0. \tag{4.127}$$

(We have used equation (4.12) and the fact that γ_0 commutes with all Pauli-even elements.) The map (4.123) does indeed have the required properties.

4.4 The Multiparticle Spacetime Algebra

We now turn to the construction of the relativistic multiparticle STA. The principle is simple. We introduce a set of four (anticommuting) basis vectors $\{\gamma_\mu^i\}$, $\mu = 0\ldots3$, $i = 1\ldots n$ where n is the number of particles. These vectors satisfy

$$\gamma_\mu^i \cdot \gamma_\nu^j = \delta^{ij}\eta_{\mu\nu} \tag{4.128}$$

and so span a $4n$-dimensional space. We interpret this as n-particle configuration space. The construction of such a space is a standard concept in classical mechanics and non-relativistic quantum theory, but the construction is rarely extended to relativistic systems. This is due largely to the complications introduced by a construction involving multiple times. In particular, Hamiltonian techniques appear to break down completely if a strict single-time ordering of events is lost. But we shall see that the multiparticle STA is ideally suited to the construction of relativistic states. Furthermore, the two-particle current no longer has a positive-definite timelike component, so can describe antiparticles without the formal requirement for field quantisation.

We will deal mainly with the two-particle STA. A two-particle quantum state is represented in this algebra by the multivector $\psi = \Psi E$, where $E = E_2$ is the two-particle correlator (4.54) and Ψ is an element of the 64-dimensional direct product space of the two even sub-algebras of the one-dimensional algebras. This construction ensures that ψ is 32-dimensional, as is required for a real equivalent of a 16-component complex column vector. Even elements from separate algebras automatically commute (recall (4.50)) so a direct product state has $\psi = \psi^1\psi^2 E = \psi^2\psi^1 E$. The STA equivalent of the action of the two-particle Dirac matrices $\hat{\gamma}_\mu^i$ is defined by the operators

$$\beta_\mu^i(\psi) = \gamma_\mu^i \psi \gamma_0^i. \tag{4.129}$$

These operators satisfy

$$\beta_\mu^1\beta_\mu^2(\psi) = \gamma_\mu^1\gamma_\mu^2\psi\gamma_0^2\gamma_0^1 = \gamma_\mu^2\gamma_\mu^1\psi\gamma_0^1\gamma_0^2 = \beta_\mu^2\beta_\mu^1(\psi) \tag{4.130}$$

and so, despite introducing a set of anticommuting vectors, the β_μ^i from different particle spaces commute. In terms of the matrix theory, we have the equivalences

$$\hat{\gamma}_\mu \otimes I|\psi\rangle \quad \leftrightarrow \quad \beta_\mu^1(\psi), \tag{4.131}$$

$$I \otimes \hat{\gamma}_\mu|\psi\rangle \quad \leftrightarrow \quad \beta_\mu^2(\psi). \tag{4.132}$$

Conventional treatments (*e.g.* Corson [62]) usually define the operators

$$\beta_\mu(\psi) = \tfrac{1}{2}[\beta_\mu^1(\psi) + \beta_\mu^2(\psi)], \tag{4.133}$$

which generate the well-known Duffin-Kemmer ring

$$\beta_\mu\beta_\nu\beta_\rho + \beta_\rho\beta_\nu\beta_\mu = \eta_{\nu\rho}\beta_\mu + \eta_{\nu\mu}\beta_\rho. \tag{4.134}$$

This relation is verified by first writing

$$\begin{aligned}
\beta_\nu\beta_\rho(\psi) &= \tfrac{1}{4}[(\gamma_\nu\gamma_\rho)^1\psi + (\gamma_\nu\gamma_\rho)^2\psi + \gamma_\nu^1\gamma_\rho^2\psi\gamma_0^2\gamma_0^1 + \gamma_\nu^2\gamma_\rho^1\psi\gamma_0^1\gamma_0^2] && (4.135)\\
\Rightarrow \beta_\mu\beta_\nu\beta_\rho(\psi) &= \tfrac{1}{8}[\gamma_{\mu\nu\rho}^1\psi\gamma_0^1 + \gamma_{\mu\nu\rho}^2\psi\gamma_0^2 + \gamma_{\nu\rho}^1\gamma_\mu^2\psi\gamma_0^2 + \gamma_{\nu\rho}^2\gamma_\mu^1\psi\gamma_0^1 + \\
&\quad \gamma_{\mu\nu}^1\gamma_\rho^2\psi\gamma_0^2 + \gamma_{\mu\nu}^2\gamma_\rho^1\psi\gamma_0^1 + \gamma_{\mu\rho}^1\gamma_\nu^2\psi\gamma_0^2 + \gamma_{\mu\rho}^2\gamma_\nu^1\psi\gamma_0^1] && (4.136)
\end{aligned}$$

where $\gamma_{\mu\nu} = \gamma_\mu \gamma_\nu$ *etc.* In forming $\beta_\mu \beta_\nu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu$ we are adding a quantity to its reverse, which simply picks up the vector part of the products of vectors sitting on the left-hand side of ψ in (4.136). We therefore find that

$$\begin{aligned}
(\beta_\mu \beta_\mu \beta_\rho + \beta_\rho \beta_\nu \beta_\mu)\psi &= \tfrac{1}{4}[\langle \gamma^1_{\mu\nu\rho} + \gamma^2_{\nu\rho}\gamma^1_\mu + \gamma^2_{\mu\nu}\gamma^1_\rho + \gamma^2_{\mu\rho}\gamma^1_\nu \rangle_1 \psi \gamma^1_0 + \\
&\quad \langle \gamma^2_{\mu\nu\rho} + \gamma^1_{\nu\rho}\gamma^2_\mu + \gamma^1_{\mu\nu}\gamma^2_\rho + \gamma^1_{\mu\rho}\gamma^2_\nu \rangle_1 \psi \gamma^2_0] \\
&= \tfrac{1}{2}[\eta_{\mu\nu}\gamma^1_\rho + \eta_{\nu\rho}\gamma^1_\mu]\psi\gamma^1_0 + \tfrac{1}{2}[\eta_{\mu\nu}\gamma^2_\rho + \eta_{\nu\rho}\gamma^2_\mu]\psi\gamma^2_0 \\
&= \eta_{\mu\nu}\beta_\rho(\psi) + \eta_{\nu\rho}\beta_\mu(\psi).
\end{aligned} \tag{4.137}$$

The realisation of the Duffin-Kemmer algebra demonstrates that the multiparticle STA contains the necessary ingredients to formulate the relativistic two-particle equations that have been studied in the literature.

The simplest relativistic two-particle wave equation is the Duffin-Kemmer equation (see Chapter 6 of [62]), which takes the form

$$\partial_\mu \beta_\mu(\psi) J = m\psi. \tag{4.138}$$

Here, ψ is a function of a single set of spacetime coordinates x^μ, and $\partial_\mu = \partial_{x^\mu}$. Equation (4.138) describes a non-interacting field of spin $0 \oplus 1$. Since the wavefunction is a function of one spacetime position only, (4.138) is not a genuine two-body equation. Indeed, equation (4.138) has a simple one-body reduction, which is achieved by replacing ψ by a 4×4 complex matrix [62, 63].

The first two-particle equation to consider in which ψ is a genuine function of position in configuration space is the famous Bethe-Salpeter equation [64]

$$(j\hat{\nabla}^1 - m^1)(j\hat{\nabla}^2 - m^2)|\psi(x^1, x^2)\rangle = jI|\psi(x^1, x^2)\rangle \tag{4.139}$$

where $\hat{\nabla}^1 = \hat{\gamma}^1_\mu \partial_{x^{\mu 1}}$ *etc.* and I is an integral operator describing the inter-particle interaction (Bethe & Salpeter [64] considered a relativistic generalisation of the Yukawa potential). The STA version of (4.139) is

$$\nabla^1 \nabla^2 \psi \gamma^2_0 \gamma^1_0 + [m^1 \nabla^2 \psi \gamma^2_0 + m^2 \nabla^1 \psi \gamma^1_0 - I(\psi)]J = m^1 m^2 \psi, \tag{4.140}$$

where ∇^1 and ∇^2 are vector derivatives in the particle 1 and particle 2 spaces respectively.

An alternative approach to relativistic two-body wave equations was initiated by Breit [65] in 1929. Breit wrote down an approximate two-body equation based on an equal-time approximation and applied this approximation to the fine structure of orthohelium. Breit's approach was developed by Kemmer [66] and Fermi & Yang [67], who introduced more complicated interactions to give phenomenological descriptions of the deuteron and pions respectively. More recently, this work has been extended by a number of authors (see Koide [68] and Galeõa & Leal Ferriara [63] and references therein). These approaches all make use of an equation of the type (in STA form)

$$E\psi + (\gamma^1_0 \wedge \nabla^1 + \gamma^2_0 \wedge \nabla^2)\psi J - m^1 \gamma_0 \psi \gamma^1_0 - m^2 \gamma^2_0 \psi \gamma^2_0 - I(\psi) = 0, \tag{4.141}$$

where $\psi = \psi(x^1, x^2)$ is a function of position in configuration space and $I(\psi)$ again describes the inter-particle interaction. Equation (4.141) can be seen to arise from an equal-time approximation to the STA equation

$$\gamma^1_0(\nabla^1 \psi J + m^1 \psi \gamma^1_0) + \gamma^2_0(\nabla^2 \psi J + m^1 \psi \gamma^2_0) - I(\psi) = 0. \tag{4.142}$$

In the case where the interaction is turned off and ψ is a direct-product state,

$$\psi = \psi^1(x^1)\psi^2(x^2)E, \tag{4.143}$$

equation (4.142) recovers the single-particle Dirac equations for the two separate particles. (This is also the case for the Bethe-Salpeter equation (4.139).) The presence of the γ_0^1 and γ_0^2 on the left-hand side mean that equation (4.142) is not Lorentz covariant, however, so can at best only be an approximate equation. From the STA form (4.139), one can immediately see how to proceed to a fully covariant equation. One simply removes the γ_0's from the left. The resultant equation is

$$(\nabla^1\psi\gamma_0^1 + \nabla^2\psi\gamma_0^2)J - I(\psi) = (m^1 + m^2)\psi, \tag{4.144}$$

and indeed such an equation has recently been proposed by Krolikowski [69, 70] (who did not use the STA).

These considerations should make it clear that the multiparticle STA is entirely sufficient for the study of relativistic multiparticle wave equations. Furthermore, it removes the need for either matrices or an (uninterpreted) scalar imaginary. But, in writing down (4.144), we have lost the ability to recover the single-particle equations. If we set $I(\psi)$ to zero and use (4.143) for ψ, we find that

$$\left(\psi^2(\nabla\psi i\gamma_3)^1 + \psi^1(\nabla\psi i\gamma_3)^2 - (m^1 + m^2)\psi^1\psi^2\right)E = 0. \tag{4.145}$$

On dividing through by $\psi^1\psi^2$ we arrive at the equation

$$(\psi^1)^{-1}(\nabla\psi i\gamma_3)^1 + (\psi^2)^{-1}(\nabla\psi i\gamma_3)^2 - m^1 - m^2 = 0, \tag{4.146}$$

and there is now no way to ensure that the correct mass is assigned to the appropriate particle.

There is a further problem with the equations discussed above. A multiparticle action integral will involve integration over the entire $4n$-dimensional configuration space. In order that boundary terms can be dealt with properly (see Chapter 6) such an integral should make use of the configuration space vector derivative $\nabla = \nabla^1 + \nabla^2$. This is not the case for the above equations, in which the ∇^1 and ∇^2 operators act separately. We require a relativistic two-particle wave equation for particles of different masses which is derivable from an action integral and recovers the individual one-particle equations in the absence of interactions. In searching for such an equation we are led to an interesting proposal — one that necessitates parting company with conventional approaches to relativistic quantum theory. To construct a space on which the full ∇ can act, the 32-dimensional spinor space considered so far is insufficient. We will therefore extend our spinor space to the the entire 128-dimensional even subalgebra of the two-particle STA. Right multiplication by the correlator E then reduces this to a 64-dimensional space, which is now sufficient for our purposes. With ψ now a member of this 64-dimensional space, a suitable wave equation is

$$\left(\frac{\nabla^1}{m^1} + \frac{\nabla^2}{m^2}\right)\psi J - \psi(\gamma_0^1 + \gamma_0^2) - I(\psi) = 0. \tag{4.147}$$

The operator $(\nabla^1/m^1 + \nabla^2/m^2)$ is formed from a dilation of ∇, so can be easily incorporated into an action integral (this is demonstrated in Chapter 6). Furthermore, equation (4.147) is manifestly Lorentz covariant. In the absence of interactions, and with ψ taking the form of (4.143), equation (4.147) successfully recovers the two single-particle Dirac equations. This is seen by dividing through by $\psi^1\psi^2$ to arrive at

$$\left(\frac{1}{\psi^1}\frac{\nabla^1}{m^1}\psi^1 i\sigma_3^1 + \frac{1}{\psi^2}\frac{\nabla^2}{m^2}\psi^2 i\sigma_3^2 - \gamma_0^1 - \gamma_0^2\right) E = 0. \tag{4.148}$$

The bracketed term contains the sum of elements from the two separate spaces, so both terms must vanish identically. This ensures that

$$\begin{aligned}\frac{1}{\psi^1}\frac{\nabla^1}{m^1}\psi^1 i\sigma_3^1 &= \gamma_0^1 \\ \Rightarrow \nabla^1\psi^1 i\sigma_3^1 &= m^1\psi^1\gamma_0^1,\end{aligned} \tag{4.149}$$

with the same result holding in the space of particle two. The fact that the particle masses are naturally attached to their respective vector derivatives is interesting, and will be mentioned again in the context of the STA gauge theory of gravity (Chapter 7).

No attempt at solving the full equation (4.147) for interacting particles will be made here (that would probably require a thesis on its own). But it is worth drawing attention to a further property of the equation. The current conjugate to gauge transformations is given by

$$j = \frac{j^1}{m^1} + \frac{j^2}{m^2} \tag{4.150}$$

where j^1 and j^2 are the projections of $\langle\psi(\gamma_0^1+\gamma_0^2)\tilde{\psi}\rangle_1$ into the individual particle spaces. The current j satisfies the conservation equation

$$\nabla\cdot j = 0 \tag{4.151}$$

or

$$\left(\frac{\nabla^1}{m^1} + \frac{\nabla^2}{m^2}\right)\cdot\langle\psi(\gamma_0^1+\gamma_0^2)\tilde{\psi}\rangle_1 = 0. \tag{4.152}$$

For the direct-product state (4.143) the projections of j into the single-particle spaces take the form

$$\begin{aligned} j^1 &= \langle\psi^2\tilde{\psi}^2\rangle(\psi^1\gamma_0^1\tilde{\psi}^1) \\ j^2 &= \langle\psi^1\tilde{\psi}^1\rangle(\psi^2\gamma_0^2\tilde{\psi}^2). \end{aligned} \tag{4.153}$$

But the quantity $\langle\psi\tilde{\psi}\rangle$ is not positive definite, so the individual particle currents are no longer necessarily future-pointing. These currents can therefore describe *antiparticles.* (It is somewhat ironic that most of the problems associated with the single-particle Dirac equation can be traced back to the fact that the timelike component of the current is positive definite. After all, producing a positive-definite density was part of Dirac's initial triumph.) Furthermore, the conservation law (4.151) only relates to the total current in configuration space, so the projections onto individual particle spaces have the potential for very strange behaviour. For example, the particle 1 current can turn round in space-time, which would be interpreted as an annihilation event. The interparticle correlations induced by the configuration-space current j also afford insights into the non-local aspects of quantum theory. Equation (4.147) should provide a fruitful source of future research, as well as being a useful testing ground for our ideas of quantum behaviour.

4.4.1 The Lorentz Singlet State

Returning to the 32-dimensional spinor space of standard two-particle quantum theory, our next task is to find a relativistic analogue of the Pauli singlet state discussed in Section 4.2.1. Recalling the definition of ϵ (4.70), the property that ensured that ϵ was a singlet state was that

$$i\sigma_k^1\epsilon = -i\sigma_k^2\epsilon, \qquad k = 1\ldots 3. \tag{4.154}$$

In addition to (4.154), a relativistic singlet state, which we will denote as η, must satisfy

$$\sigma_k^1\eta = -\sigma_k^2\eta, \qquad k = 1\ldots 3. \tag{4.155}$$

It follows that η satisfies

$$i^1\eta = \sigma_1^1\sigma_2^1\sigma_3^1\eta = -\sigma_3^2\sigma_2^2\sigma_1^2\eta = i^2\eta \tag{4.156}$$

so that

$$\begin{aligned} \eta &= -i^1 i^2\eta & (4.157)\\ \Rightarrow \eta &= \tfrac{1}{2}(1 - i^1 i^2)\eta. & (4.158)\end{aligned}$$

Such a state can be formed by multiplying ϵ by the idempotent $(1-i^1i^2)/2$. We therefore define

$$\eta \equiv \epsilon\tfrac{1}{2}(1 - i^1 i^2) = \tfrac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2)\tfrac{1}{2}(1 - i^1 i^2). \tag{4.159}$$

This satisfies

$$i\sigma_k^1\eta = i\sigma_k^1\epsilon\tfrac{1}{2}(1 - i^1 i^2) = -i\sigma_k^2\eta \qquad k = 1\ldots 3 \tag{4.160}$$

and

$$\sigma_k^1\eta = -\sigma_k^1 i^1 i^2\eta = i^2 i\sigma_k^2\eta = -\sigma_k^2\eta \qquad k = 1\ldots 3. \tag{4.161}$$

These results are summarised by

$$M^1\eta = \tilde{M}^2\eta, \tag{4.162}$$

where M is an even multivector in either the particle 1 or particle 2 STA. The proof that η is a relativistic invariant now reduces to the simple identity

$$R^1R^2\eta = R^1\tilde{R}^1\eta = \eta, \tag{4.163}$$

where R is a relativistic rotor acting in either particle-one or particle-two space.

Equation (4.162) can be seen as arising from a more primitive relation between vectors in the separate spaces. Using the result that $\gamma_0^1\gamma_0^2$ commutes with η, we can derive

$$\begin{aligned} \gamma_\mu^1\eta\gamma_0^1 &= \gamma_\mu^1\gamma_0^1\gamma_0^2\eta\gamma_0^2\gamma_0^1\gamma_0^1 \\ &= \gamma_0^2(\gamma_\mu\gamma_0)^1\eta\gamma_0^2 \\ &= \gamma_0^2\gamma_0^2\gamma_\mu^2\eta\gamma_0^2 \\ &= \gamma_\mu^2\eta\gamma_0^2, & (4.164)\end{aligned}$$

and hence we find that, for an arbitrary vector a,

$$a^1\eta\gamma_0^1 = a^2\eta\gamma_0^2. \tag{4.165}$$

Equation (4.162) follows immediately from (4.165) by writing

$$\begin{aligned}
(ab)^1\eta &= a^1b^1\eta\gamma_0^1\gamma_0^1 \\
&= a^1b^2\eta\gamma_0^2\gamma_0^1 \\
&= b^2a^1\eta\gamma_0^1\gamma_0^2 \\
&= b^2a^2\eta\gamma_0^2\gamma_0^2 \\
&= (ba)^2\eta. \qquad (4.166)
\end{aligned}$$

Equation (4.165) can therefore be viewed as the fundamental property of the relativistic invariant η.

From η a number of Lorentz-invariant two-particle multivectors can be constructed by sandwiching arbitrary multivectors between η and $\tilde{\eta}$. The simplest such object is

$$\begin{aligned}
\eta\tilde{\eta} &= \epsilon\tfrac{1}{2}(1 - i^1 i^2)\tilde{\epsilon} \\
&= \tfrac{1}{2}(1 + i\sigma_1^1 i\sigma_1^2 + i\sigma_2^1 i\sigma_2^2 + i\sigma_3^1 i\sigma_3^2)\tfrac{1}{2}(1 - i^1 i^2) \\
&= \tfrac{1}{4}(1 - i^1 i^2) - \tfrac{1}{4}(\sigma_k^1\sigma_k^2 - i\sigma_k^1 i\sigma_k^2). \qquad (4.167)
\end{aligned}$$

This contains a scalar + pseudoscalar term, which is obviously invariant, together with the invariant grade-4 multivector $(\sigma_k^1\sigma_k^2 - i\sigma_k^1 i\sigma_k^2)$. The next simplest object is

$$\begin{aligned}
\eta\gamma_0^1\gamma_0^2\tilde{\eta} &= \tfrac{1}{2}(1 + i\sigma_1^1 i\sigma_1^2 + i\sigma_2^1 i\sigma_2^2 + i\sigma_3^1 i\sigma_3^2)\tfrac{1}{2}(1 - i^1 i^2)\gamma_0^1\gamma_0^2 \\
&= \tfrac{1}{4}(\gamma_0^1\gamma_0^2 + i^1 i^2\gamma_k^1\gamma_k^2 - i^1 i^2\gamma_0^1\gamma_0^2 - \gamma_k^1\gamma_k^2) \\
&= \tfrac{1}{4}(\gamma_0^1\gamma_0^2 - \gamma_k^1\gamma_k^2)(1 - i^1 i^2). \qquad (4.168)
\end{aligned}$$

On defining the symplectic (doubling) bivector

$$J \equiv \gamma_\mu^1\gamma^{\mu 2} \qquad (4.169)$$

and the two-particle pseudoscalar

$$I \equiv i^1 i^2 = i^2 i^1 \qquad (4.170)$$

the invariants from (4.168) are simply J and IJ. As was disussed in Section (3.2), the bivector J is independent of the choice of spacetime frame, so is unchanged by the two-sided application of the rotor $R = R^1R^2$. It follows immediately that the 6-vector IJ is also invariant.

From the definition of J (4.169), we find that

$$\begin{aligned}
J\wedge J &= -2\gamma_0^1\gamma_0^2\gamma_k^1\gamma_k^2 + (\gamma_k^1\gamma_k^2)\wedge(\gamma_j^1\gamma_j^2) \\
&= 2(\sigma_k^1\sigma_k^2 - i\sigma_k^1 i\sigma_k^2), \qquad (4.171)
\end{aligned}$$

which recovers the 4-vector invariant from (4.167). The complete set of two-particle invariants can therefore be constructed from J alone, and these are summarised in Table 4.2. These invariants are well-known and have been used in constructing phenomenological models of interacting particles [63, 68]. The STA derivation of the invariants is quite new, however, and the role of the doubling bivector J has not been previously noted.

Invariant	Type of Interaction	Grade
1	Scalar	0
J	Vector	2
$J\wedge J$	Bivector	4
IJ	Pseudovector	6
I	Pseudoscalar	8

Table 4.2: Two-Particle Relativistic Invariants

4.5 2-Spinor Calculus

We saw in Section 4.3.1 how spinors in the Weyl representation are handled within the (single-particle) STA. We now turn to a discussion of how the 2-spinor calculus developed by Penrose & Rindler [36, 37] is formulated in the multiparticle STA. From equation (4.87), the chiral projection operators $\frac{1}{2}(1 \pm \hat{\gamma}_5')$ result in the STA multivectors

$$\begin{aligned} \tfrac{1}{2}(1+\hat{\gamma}_5')|\psi\rangle' &\quad\leftrightarrow\quad \psi\tfrac{1}{2}(1+\sigma_3) = \chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) \\ \tfrac{1}{2}(1-\hat{\gamma}_5')|\psi\rangle' &\quad\leftrightarrow\quad \psi\tfrac{1}{2}(1-\sigma_3) = -\bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3). \end{aligned} \tag{4.172}$$

The 2-spinors $|\chi\rangle$ and $|\bar{\eta}\rangle$ can therefore be given the STA equivalents

$$\begin{aligned} |\chi\rangle &\quad\leftrightarrow\quad \chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) \\ |\bar{\eta}\rangle &\quad\leftrightarrow\quad -\bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3). \end{aligned} \tag{4.173}$$

These differ from the representation of Pauli spinors, and are closer to the "minimal left ideal" definition of a spinor given by some authors (see Chapter 2 of [13], for example). Algebraically, the $(1 \pm \sigma_3)$ projectors ensure that the 4-dimensional spaces spanned by elements of the type $\chi\frac{1}{\sqrt{2}}(1+\sigma_3)$ and $\bar{\eta}\frac{1}{\sqrt{2}}(1-\sigma_3)$ are closed under left multiplication by a relativistic rotor. The significance of the $(1 \pm \sigma_3)$ projectors lies not so much in their algebraic properties, however, but in the fact that they are the γ_0-space projections of the null vectors $\gamma_0 \pm \gamma_3$. This will become apparent when we construct some 2-spinor "observables".

Under a Lorentz transformation the spinor ψ transforms to $R\psi$, where R is a relativistic rotor. If we separate the rotor R into Pauli-even and Pauli-odd terms,

$$R = R_+ + R_- \tag{4.174}$$

where

$$R_+ = \tfrac{1}{2}(R + \gamma_0 R \gamma_0) \tag{4.175}$$

$$R_- = \tfrac{1}{2}(R - \gamma_0 R \gamma_0), \tag{4.176}$$

then we can write

$$\begin{aligned} R\chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) &= R_+\chi\tfrac{1}{\sqrt{2}}(1+\sigma_3) + R_-\chi\sigma_3\tfrac{1}{\sqrt{2}}(1+\sigma_3) \\ R\bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3) &= R_+\bar{\eta}\tfrac{1}{\sqrt{2}}(1-\sigma_3) - R_-\bar{\eta}\sigma_3\tfrac{1}{\sqrt{2}}(1-\sigma_3). \end{aligned} \tag{4.177}$$

The transformation laws for the Pauli-even elements χ and $\bar{\eta}$ are therefore

$$\chi \mapsto R_+\chi + R_-\chi\sigma_3 \tag{4.178}$$

$$\bar{\eta} \mapsto R_+\bar{\eta} - R_-\bar{\eta}\sigma_3, \tag{4.179}$$

which confirms that $|\chi\rangle$ transforms under the operator equivalent of R, but that $|\bar{\eta}\rangle$ transforms under the equivalent of

$$R_+ - R_- = \gamma_0 R\gamma_0 = (\gamma_0\tilde{R}\gamma_0)^{\sim} = (R^{-1})^{\dagger}. \tag{4.180}$$

This split of a Lorentz transformations into two distinct operations is an unattractive feature of the 2-spinor formalism, but it is an unavoidable consequence of attempting to perform relativistic calculations within the Pauli algebra of 2×2 matrices. The problem is that the natural anti-involution operation is Hermitian conjugation. This operation is dependent on the choice of a relativistic timelike vector, which breaks up expressions in a way that disguises their frame-independent meaning.

The 2-spinor calculus attempts to circumvent the above problem by augmenting the basic 2-component spinor with a number of auxilliary concepts. The result is a language which has proved to be well-suited to the study of spinors in a wide class of problems and it is instructive to see how some features of the 2-spinor are absorbed into the STA formalism. The central idea behind the 2-spinor calculus is that a two-component complex spinor $|\kappa\rangle$, derived form the Weyl representation (4.121), is replaced by the complex "vector" κ^A. Here the A is an abstract index labeling the fact that κ^A is a single spinor belonging to some complex, two-dimensional linear space. We represent this object in the STA as

$$\kappa^A \leftrightarrow \kappa\tfrac{1}{2}(1+\sigma_3). \tag{4.181}$$

(The factor of $1/2$ replaces $1/\sqrt{2}$ simply for convenience in some of the manipulations that follow.) The only difference now is that, until a frame is chosen in spin-space, we have no direct mapping between the components of κ^A and κ. Secifying a frame in spin space also picks out a frame in spacetime (determined by the null tetrad). If this spacetime frame is identified with the $\{\gamma^\mu\}$ frame, then the components $\kappa^{\mathbf{A}}$ of κ^A specify the Pauli-even multivector κ via the identification of equation (4.9). A second frame in spin-space produces different components $\kappa^{\mathbf{A}}$, and will require a different identification to equation (4.9), but will still lead to the same multivector $\kappa\frac{1}{2}(1+\sigma_3)$. 2-Spinors are equipped with a Lorentz-invariant inner product derived from a metric tensor ϵ_{AB}. This is used to lower indices so, for every 2-spinor κ^A, there is a corresponding κ_A. Both of these must have the same multivector equivalent, however, in the same way that a^μ and a_μ both have the STA equivalent a.

To account for the second type of relativistic 2-spinor, $|\bar{\eta}\rangle$ (4.121), a second linear space (or module) is introduced and elements of this space are labeled with bars and primed indices. Thus an abstract element of this space is written as $\bar{\omega}^{A'}$. In a given basis, the components of $\bar{\omega}^{\mathbf{A}'}$ are related to those of $\omega^{\mathbf{A}}$ by complex conjugation,

$$\bar{\omega}^{0'} = \overline{\omega^0}, \qquad \bar{\omega}^{1'} = \overline{\omega^1}. \tag{4.182}$$

To construct the STA equivalent of $\bar{\omega}^{A'}$ we need a suitable equivalent for this operation. Our equivalent operation should satisfy the following criteria:

1. The operation can only affect the right-hand side $\omega(1+\sigma_3)/2$, so that Lorentz invariance is not compromised;

2. From equation (4.173), the STA equivalent of $\bar{\omega}^{A'}$ must be a multivector projected by the $(1-\sigma_3)/2$ idempotent, so the conjugation operation must switch idempotents;

3. The operation must square to give the identity;

4. The operation must anticommute with right-multiplication by $i\sigma_3$.

The only operation satisfying all of these criteria is right-multiplication by some combination of σ_1 and σ_2. Choosing between these is again a matter of convention, so we will represent 2-spinor complex conjugation by right-multiplication by $-\sigma_1$. It follows that our representation for the abstract 2-spinor $\bar{\omega}^{A'}$ is

$$\bar{\omega}^{A'} \quad \leftrightarrow \quad -\omega\tfrac{1}{2}(1+\sigma_3)\sigma_1 = -\omega i\sigma_2\tfrac{1}{2}(1-\sigma_3). \tag{4.183}$$

Again, once a basis is chosen, ω is constructed using the identification of equation (4.9) with the components $\omega^0 = \overline{\bar{\omega}^{0'}}$ and $\omega^1 = \overline{\bar{\omega}^{1'}}$.

4.5.1 2-Spinor Observables

Our next step in the STA formulation of 2-spinor calculus is to understand how to represent quantities constructed from pairs of 2-spinors. The solution is remarkably simple. One introduces a copy of the STA for each spinor, and then simply multiplies the STA elements together, incorporating suitable correlators as one proceeds. For example, the quantity $\kappa^A\bar{\kappa}^{A'}$ becomes

$$\kappa^A\bar{\kappa}^{A'} \quad \leftrightarrow \quad -\kappa^1\tfrac{1}{2}(1+\sigma_3^1)\kappa^2 i\sigma_2^2\tfrac{1}{2}(1-\sigma_3^2)\tfrac{1}{2}(1-i\sigma_3^1 i\sigma_3^2). \tag{4.184}$$

To see how to manipulate the right-hand side of (4.184) we return to the relativistic two-particle singlet η (4.159). The essential property of η under multiplication by even elements was equation (4.162). This relation is unaffected by further multiplication of η on the right-hand side by an element that commutes with E. We can therefore form the object

$$\epsilon = \eta\tfrac{1}{2}(1+\sigma_3^1) \tag{4.185}$$

(not to be confused with the non-relativistic Pauli singlet state) which will still satisfy

$$M^1\epsilon = \tilde{M}^2\epsilon \tag{4.186}$$

for all even multivectors M. The 2-particle state ϵ is still a relativistic singlet in the sense of equation (4.163). From (4.185) we see that ϵ contains

$$
\begin{aligned}
\tfrac{1}{2}(1 - i^1 i^2)\tfrac{1}{2}(1 + \sigma_3^1)\tfrac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2) &= \tfrac{1}{2}(1 - i\sigma_3^1 i^2)\tfrac{1}{2}(1 + \sigma_3^1)E \\
&= \tfrac{1}{2}(1 - i\sigma_3^2 i^2)\tfrac{1}{2}(1 + \sigma_3^1)E \\
&= \tfrac{1}{2}(1 + \sigma_3^2)\tfrac{1}{2}(1 + \sigma_3^1)E,
\end{aligned}
\tag{4.187}
$$

so we can write

$$
\epsilon = \tfrac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 + \sigma_3^2)\tfrac{1}{2}(1 + \sigma_3^1)E. \tag{4.188}
$$

A second invariant is formed by right-sided multiplication by $(1 - \sigma_3^1)/2$, and we define

$$
\bar{\epsilon} = \eta\tfrac{1}{2}(1 - \sigma_3^1). \tag{4.189}
$$

Proceeding as above, we find that

$$
\bar{\epsilon} = \tfrac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 - \sigma_3^2)\tfrac{1}{2}(1 - \sigma_3^1)E. \tag{4.190}
$$

This split of the full relativistic invariant η into ϵ and $\bar{\epsilon}$ lies at the heart of much of the 2-spinor calculus. To see why, we return to equation (4.184) and from this we extract the quantity $\frac{1}{2}(1 + \sigma_3^1)\frac{1}{2}(1 - \sigma_3^2)\frac{1}{2}(1 - i\sigma_3^1 i\sigma_3^2)$. This can be manipulated as follows:

$$
\begin{aligned}
\tfrac{1}{2}(1 + \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^2)E &= \gamma_0^1\tfrac{1}{2}(1 - \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^2)E\gamma_0^1 \\
&= \gamma_0^1 i\sigma_2^2\tfrac{1}{2}(1 - \sigma_3^1)(-i\sigma_2^2)\tfrac{1}{2}(1 - \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^2)E\gamma_0^1 \\
&= \gamma_0^1 i\sigma_2^2\tfrac{1}{2}(1 - \sigma_3^1)(i\sigma_2^1 - i\sigma_2^2)\tfrac{1}{2}(1 - \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^2)E\gamma_0^1 \\
&= \gamma_0^1 i\sigma_2^2\tfrac{1}{\sqrt{2}}(1 - \sigma_3^1)\bar{\epsilon}\gamma_0^1 \\
&= -\tfrac{1}{\sqrt{2}}(\gamma_0^1 + \gamma_3^1)i\sigma_2^1\bar{\epsilon}\gamma_0^1,
\end{aligned}
\tag{4.191}
$$

which shows how an $\bar{\epsilon}$ arises naturally in the 2-spinor product. This $\bar{\epsilon}$ is then used to project everything to its left back down to a single-particle space. We continue to refer to each space as a "particle space" partly to stress the analogy with relativistic quantum states, but also simply as a matter of convenience. In 2-spinor calculus there is no actual notion of a particle associated with each copy of spacetime.

Returning to the example of $\kappa^A\bar{\kappa}^{A'}$ (4.184), we can now write

$$
\begin{aligned}
-\kappa^1\tfrac{1}{2}(1 + \sigma_3^1)\kappa^2 i\sigma_2^2\tfrac{1}{2}(1 - \sigma_3^2)E &= -\kappa^1\kappa^2 i\sigma_2^2\tfrac{1}{2}(1 + \sigma_3^1)\tfrac{1}{2}(1 - \sigma_3^2)E \\
&= \kappa^1\kappa^2\tfrac{1}{\sqrt{2}}(\gamma_0^1 + \gamma_3^1)\bar{\epsilon}\gamma_0^1 \\
&= [\kappa\tfrac{1}{\sqrt{2}}(\gamma_0 + \gamma_3)\tilde{\kappa}]^1\bar{\epsilon}\gamma_0^1.
\end{aligned}
\tag{4.192}
$$

The key part of this expression is the null vector $\kappa(\gamma_0 + \gamma_3)\tilde{\kappa}/\sqrt{2}$, which is formed in the usual STA manner by a rotation/dilation of the fixed null vector $(\gamma_0 + \gamma_3)/\sqrt{2}$ by the even multivector κ. The appearance of the null vector $(\gamma_0 + \gamma_3)/\sqrt{2}$ can be traced back directly to the $(1 + \sigma_3)/2$ idempotent, justifying the earlier comment that these idempotents have a clear geometric origin.

There are three further manipulations of the type performed in equation (4.191) and the results of these are summarised in Table 4.3. These results can be used to find a single-particle equivalent of any expression involving a pair of 2-spinors. We will see shortly how these reductions are used to construct a null tetrad, but first we need to find an STA formulation of the 2-spinor inner product.

$\frac{1}{2}(1+\sigma_3^1)\frac{1}{2}(1-\sigma_3^2)E$	$=$	$-\frac{1}{\sqrt{2}}(\gamma_0^1+\gamma_3^1)i\sigma_2^1\bar{\epsilon}\gamma_0^1$
$\frac{1}{2}(1-\sigma_3^1)\frac{1}{2}(1+\sigma_3^2)E$	$=$	$-\frac{1}{\sqrt{2}}(\gamma_0^1-\gamma_3^1)i\sigma_2^1\epsilon\gamma_0^1$
$\frac{1}{2}(1+\sigma_3^1)\frac{1}{2}(1+\sigma_3^2)E$	$=$	$-\frac{1}{\sqrt{2}}(\sigma_1^1+i\sigma_2^1)\epsilon$
$\frac{1}{2}(1-\sigma_3^1)\frac{1}{2}(1-\sigma_3^2)E$	$=$	$-\frac{1}{\sqrt{2}}(-\sigma_1^1+i\sigma_2^1)\bar{\epsilon}$

Table 4.3: 2-Spinor Manipulations

4.5.2 The 2-spinor Inner Product

Spin-space is equipped with an anti-symmetric inner product, written as either $\kappa^A\omega_A$ or $\kappa^A\omega^B\epsilon_{AB}$. In a chosen basis, the inner product is calculated as

$$\kappa^A\omega_A = \kappa^{\mathbf{A}}\omega_{\mathbf{A}} = \kappa^0\omega^1 - \kappa^1\omega^0, \tag{4.193}$$

which yields a Lorentz-invariant complex scalar. The antisymmetry of the inner product suggests forming the STA expression

$$\begin{aligned}\tfrac{1}{2}(\kappa^A\omega^B - \omega^A\kappa^B) \quad \leftrightarrow \quad & \tfrac{1}{2}(\kappa^1\omega^2 - \kappa^2\omega^1)\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)E \\ &= -\tfrac{1}{2}\left(\kappa\tfrac{1}{\sqrt{2}}(\sigma_1+i\sigma_2)\tilde{\omega} - \tfrac{1}{\sqrt{2}}\omega(\sigma_1+i\sigma_2)\tilde{\kappa}\right)^1\epsilon \\ &= -\tfrac{1}{\sqrt{2}}\langle\kappa(\sigma_1+i\sigma_2)\tilde{\omega}\rangle_{0,4}^1\epsilon.\end{aligned} \tag{4.194}$$

The antisymmetric product therefore picks out the scalar and pseudoscalar parts of the quantity $\kappa(\sigma_1^1+i\sigma_2^1)\tilde{\omega}$. This is sensible, as these are the two parts that are invariant under Lorentz transformations. Fixing up the factor suitably, our STA representation of the 2-spinor inner product will therefore be

$$\kappa^A\omega_A \quad \leftrightarrow \quad -\langle\kappa(\sigma_1+i\sigma_2)\tilde{\omega}\rangle_{0,4} = -\langle\kappa i\sigma_2\tilde{\omega}\rangle + i\langle\kappa i\sigma_1\tilde{\omega}\rangle. \tag{4.195}$$

That this agrees with the 2-spinor form in a given basis can be checked simply by expanding out the right-hand side of (4.195).

A further insight into the role of the 2-spinor inner product is gained by assembling the full even multivector (an STA spinor)

$$\psi = \kappa\tfrac{1}{2}(1+\sigma_3) + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3). \tag{4.196}$$

The 2-spinor inner product can now be written as

$$\begin{aligned}\psi\tilde{\psi} &= [\kappa\tfrac{1}{2}(1+\sigma_3) + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)][-\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\omega} + \tfrac{1}{2}(1-\sigma_3)\tilde{\kappa}] \\ &= -\kappa\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\omega} + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)\tilde{\kappa} \\ &= -\langle\kappa(\sigma_1+i\sigma_2)\tilde{\omega}\rangle_{0,4}\end{aligned} \tag{4.197}$$

which recovers (4.195). The 2-spinor inner product is therefore seen to pick up both the scalar and pseudoscalar parts of a full Dirac spinor product $\psi\tilde{\psi}$. Interchanging κ and ω in ψ (4.196) is achieved by right-multiplication by σ_1, which immediately reverses the sign

$$
\boxed{\begin{array}{rcl}
\frac{1}{2}(1+\sigma_3^1)\frac{1}{2}(1-\sigma_3^2)E & = & -\frac{1}{\sqrt{2}}(\gamma_0^1+\gamma_3^1)i\sigma_2^1\bar{\epsilon}\gamma_0^1 \\
\frac{1}{2}(1-\sigma_3^1)\frac{1}{2}(1+\sigma_3^2)E & = & -\frac{1}{\sqrt{2}}(\gamma_0^1-\gamma_3^1)i\sigma_2^1\epsilon\gamma_0^1 \\
\frac{1}{2}(1+\sigma_3^1)\frac{1}{2}(1+\sigma_3^2)E & = & -\frac{1}{\sqrt{2}}(\sigma_1^1+i\sigma_2^1)\epsilon \\
\frac{1}{2}(1-\sigma_3^1)\frac{1}{2}(1-\sigma_3^2)E & = & -\frac{1}{\sqrt{2}}(-\sigma_1^1+i\sigma_2^1)\bar{\epsilon}
\end{array}}
$$

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which yields a Lorentz-invariant complex scalar. The antisymmetry of the inner product suggests forming the STA expression

$$\begin{aligned}
\tfrac{1}{2}(\kappa^A\omega^B - \omega^A\kappa^B) \quad \leftrightarrow \quad & \tfrac{1}{2}(\kappa^1\omega^2 - \kappa^2\omega^1)\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)E \\
&= -\tfrac{1}{2}\left(\kappa\tfrac{1}{\sqrt{2}}(\sigma_1+i\sigma_2)\tilde{\omega} - \tfrac{1}{\sqrt{2}}\omega(\sigma_1+i\sigma_2)\tilde{\kappa}\right)^1 \epsilon \\
&= -\tfrac{1}{\sqrt{2}}\langle\kappa(\sigma_1+i\sigma_2)\tilde{\omega}\rangle_{0,4}^1\epsilon.
\end{aligned} \tag{4.194}$$

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The 2-spinor inner product can now be written as

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\psi\tilde{\psi} &= [\kappa\tfrac{1}{2}(1+\sigma_3) + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)][-\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\omega} + \tfrac{1}{2}(1-\sigma_3)\tilde{\kappa}] \\
&= -\kappa\tfrac{1}{2}(1+\sigma_3)i\sigma_2\tilde{\omega} + \omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)\tilde{\kappa} \\
&= -\langle\kappa(\sigma_1+i\sigma_2)\tilde{\omega}\rangle_{0,4}
\end{aligned} \tag{4.197}$$

which recovers (4.195). The 2-spinor inner product is therefore seen to pick up both the scalar and pseudoscalar parts of a full Dirac spinor product $\psi\tilde{\psi}$. Interchanging κ and ω in ψ (4.196) is achieved by right-multiplication by σ_1, which immediately reverses the sign of $\psi\tilde{\psi}$. An important feature of the 2-spinor calculus has now emerged, which is that the unit scalar imaginary is playing the role of the spacetime pseudoscalar. This is a point in favour of 2-spinors over Dirac spinors, but it is only through consistent employment of the STA that this point has become clear.

The general role of the ϵ_{AB} tensor when forming contractions is also now clear. In the STA treatment, ϵ_{AB} serves to antisymmetrise on the two particle indices carried by its STA equivalent. (It also introduces a factor of $\sqrt{2}$, which is a result of the conventions we have adopted.) This antisymmetrisation always results in a scalar + pseudoscalar quantity, and the pseudoscalar part can always be pulled down to an earlier copy of spacetime. In this manner, antisymmetrisation always removes two copies of spacetime, as we should expect from the contraction operation.

4.5.3 The Null Tetrad

An important concept in the 2-spinor calculus is that of a *spin-frame*. This consists of a pair of 2-spinors, κ^A and ω^A say, normalised such that $\kappa^A\omega_A = 1$. In terms of the full spinor ψ (4.196), this normalisation condition becomes $\psi\tilde{\psi} = 1$. But this is simply the condition which ensures that ψ is a spacetime rotor! Thus the role of a "normalised spin-frame" in 2-spinor calculus is played by a spacetime rotor in the STA approach. This is a considerable conceptual simplification. Furthermore, it demonstrates how elements of abstract 2-spinor space can be represented in terms of geometrically meaningful objects — a rotor, for example, being simply a product of an even number of unit vectors.

Attached to the concept of a spin-frame is that of a null tetrad. Using κ^A and ω^A as the generators of the spin frame, the null tetrad is defined as follows:

$$
\begin{aligned}
l^a = \kappa^A\bar{\kappa}^{A'} \quad \leftrightarrow \quad & -\kappa^1\kappa^2 \tfrac{1}{2}(1+\sigma_3^1) i\sigma_2^2 \tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\kappa(\gamma_0+\gamma_3)\tilde{\kappa}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_0+\gamma_3)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.198}
$$

$$
\begin{aligned}
n^a = \omega^A\bar{\omega}^{A'} \quad \leftrightarrow \quad & -\omega^1\omega^2 \tfrac{1}{2}(1+\sigma_3^1) i\sigma_2 \tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\omega(\gamma_0+\gamma_3)\tilde{\omega}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_0-\gamma_3)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.199}
$$

$$
\begin{aligned}
m^a = \kappa^A\bar{\omega}^{A'} \quad \leftrightarrow \quad & -\kappa^1\omega^2 \tfrac{1}{2}(1+\sigma_3^1) i\sigma_2 \tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\kappa(\gamma_0+\gamma_3)\tilde{\omega}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_1+i\gamma_2)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.200}
$$

and

$$
\begin{aligned}
\bar{m}^a = \omega^A\bar{\kappa}^{A'} \quad \leftrightarrow \quad & -\omega^1\kappa^2 \tfrac{1}{2}(1+\sigma_3^1) i\sigma_2 \tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\omega(\gamma_0+\gamma_3)\tilde{\kappa}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_1-i\gamma_2)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1.
\end{aligned} \tag{4.201}
$$

of $\psi\tilde{\psi}$. An important feature of the 2-spinor calculus has now emerged, which is that the unit scalar imaginary is playing the role of the spacetime pseudoscalar. This is a point in favour of 2-spinors over Dirac spinors, but it is only through consistent employment of the STA that this point has become clear.

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$$
\begin{aligned}
l^a = \kappa^A\bar{\kappa}^{A'} \quad \leftrightarrow \quad & -\kappa^1\kappa^2\tfrac{1}{2}(1+\sigma_3^1)i\sigma_2^2\tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\kappa(\gamma_0+\gamma_3)\tilde{\kappa}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_0+\gamma_3)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.198}
$$

$$
\begin{aligned}
n^a = \omega^A\bar{\omega}^{A'} \quad \leftrightarrow \quad & -\omega^1\omega^2\tfrac{1}{2}(1+\sigma_3^1)i\sigma_2\tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\omega(\gamma_0+\gamma_3)\tilde{\omega}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_0-\gamma_3)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.199}
$$

$$
\begin{aligned}
m^a = \kappa^A\bar{\omega}^{A'} \quad \leftrightarrow \quad & -\kappa^1\omega^2\tfrac{1}{2}(1+\sigma_3^1)i\sigma_2\tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\kappa(\gamma_0+\gamma_3)\tilde{\omega}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_1+i\gamma_2)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1,
\end{aligned} \tag{4.200}
$$

and

$$
\begin{aligned}
\bar{m}^a = \omega^A\bar{\kappa}^{A'} \quad \leftrightarrow \quad & -\omega^1\kappa^2\tfrac{1}{2}(1+\sigma_3^1)i\sigma_2\tfrac{1}{2}(1-\sigma_3^2)E \\
&= \tfrac{1}{\sqrt{2}}[\omega(\gamma_0+\gamma_3)\tilde{\kappa}]^1\bar{\epsilon}\gamma_0^1 \\
&= [\psi\tfrac{1}{\sqrt{2}}(\gamma_1-i\gamma_2)\tilde{\psi}]^1\bar{\epsilon}\gamma_0^1.
\end{aligned} \tag{4.201}
$$

The key identity used to arrive at the final two expression is

$$
\begin{aligned}
\psi(\gamma_1 + i\gamma_2)\tilde{\psi} &= \psi(1+\sigma_3)\gamma_1\tilde{\psi} \\
&= \kappa(1+\sigma_3)\gamma_1\tilde{\psi} \\
&= -\kappa\gamma_1(1+\sigma_3)i\sigma_2\tilde{\omega} \\
&= -\kappa\gamma_1(1+\sigma_3)\sigma_1\tilde{\omega} \\
&= \kappa(\gamma_0+\gamma_3)\tilde{\omega}.
\end{aligned}
\tag{4.202}
$$

The simplest spin frame is formed when $\psi = 1$. In this case we arrive at the following comparison with page 120 of Penrose & Rindler [36];

$$
\begin{array}{rcl}
l^a = \frac{1}{\sqrt{2}}(t^a + z^a) & \leftrightarrow & \frac{1}{\sqrt{2}}(\gamma_0 + \gamma_3) \\
n^a = \frac{1}{\sqrt{2}}(t^a - z^a) & \leftrightarrow & \frac{1}{\sqrt{2}}(\gamma_0 - \gamma_3) \\
m^a = \frac{1}{\sqrt{2}}(x^a - jy^a) & \leftrightarrow & \frac{1}{\sqrt{2}}(\gamma_1 + i\gamma_2) \\
\bar{m}^a = \frac{1}{\sqrt{2}}(x^a + jy^a) & \leftrightarrow & \frac{1}{\sqrt{2}}(\gamma_1 - i\gamma_2).
\end{array}
\tag{4.203}
$$

The significant feature of this translation is that the "complex vectors" m^a and $\bar{m}^a$ have been replaced by vector + trivector combinations. This agrees with the observation that the imaginary scalar in the 2-spinor calculus plays the role of the spacetime pseudoscalar. We can solve (4.203) for the Minkowski frame $\{t^a, x^a, y^a, z^a\}$ (note how the abstract indices here simply record the fact that the $t \ldots z$ are vectors). The only subtlety is that, in recovering the vector y^a from our expression for jy^a, we must post-multiply our 2-particle expression by $i\sigma_3^1$. The factor of $(1+\sigma_3^1)$ means that at the one-particle level this operation reduces to right-multiplication by i. We therefore find that

$$
\begin{array}{ccccccc}
t^a & \leftrightarrow & \gamma_0 & \quad & y^a & \leftrightarrow & -\gamma_2 \\
x^a & \leftrightarrow & \gamma_1 & \quad & z^a & \leftrightarrow & \gamma_3.
\end{array}
\tag{4.204}
$$

The only surprise here is the sign of the y-vector γ_2. This sign can be traced back to the fact that Penrose & Rindler adopt an usual convention for the σ_2 Pauli matrix (page 16). This is also reflected in the fact that they identify the quaternions with vectors (page 22), and we saw in Section 1.2.3 that the quaternion algebra is generated by the spatial bivectors $\{i\sigma_1, -i\sigma_2, i\sigma_3\}$.

An arbitrary spin-frame, encoded in the rotor R, produces a new null tetrad simply by Lorentz rotating the vectors in (4.203), yielding

$$
\begin{array}{rclcrcl}
l &=& R\frac{1}{\sqrt{2}}(\gamma_0+\gamma_3)\tilde{R}, & \quad & m &=& R\frac{1}{\sqrt{2}}(\gamma_1+i\gamma_2)\tilde{R}, \\
n &=& R\frac{1}{\sqrt{2}}(\gamma_0-\gamma_3)\tilde{R}, & \quad & \bar{m} &=& R\frac{1}{\sqrt{2}}(\gamma_1-i\gamma_2)\tilde{R}.
\end{array}
\tag{4.205}
$$

In this manner, the (abstract) null tetrad becomes a set of four arbitrary vector/trivector combinations in (4.205), satisfying the anticommutation relations [4]

$$
\tfrac{1}{2}\{l,n\} = 1, \quad \tfrac{1}{2}\{m,\bar{m}\} = 1, \quad \text{all others} = 0. \tag{4.206}
$$

4.5.4 The $\nabla^{A'A}$ Operator

The final 2-spinor object that we need a translation of is the differential operator $\nabla^{A'A}$. The translation of $\nabla^{A'A}$ will clearly involve the vector derivative $\nabla = \gamma^\mu \partial_{x^\mu}$ and this must appear in such a way that it picks up the correct transformation law under a rotation in two-particle space. These observations lead us to the object

$$\nabla^{A'A} \quad \leftrightarrow \quad \nabla^1 \epsilon \gamma_0^1, \tag{4.207}$$

so that, under a rotation,

$$\begin{aligned} \nabla^1 \epsilon \gamma_0^1 \mapsto R^1 R^2 \nabla^1 \epsilon \gamma_0^1 &= R^1 \nabla R^2 \epsilon \gamma_0^1 \\ &= (R \nabla \tilde{R})^1 \epsilon \gamma_0^1, \end{aligned} \tag{4.208}$$

and the ∇ does indeed inherit the correct vector transformation law. In this chapter we are only concerned with the "flat-space" vector derivative ∇; a suitable formulation for "curved-space" derivatives will emerge in Chapter 7. A feature of the way that the multiparticle STA is employed here is that each spinor $\kappa(1+\sigma_3)/2$ is a function of position in its own spacetime,

$$\kappa^j \tfrac{1}{2}(1+\sigma_3^j) = \kappa^j(x^j) \tfrac{1}{2}(1+\sigma_3^j). \tag{4.209}$$

When such an object is projected into a different copy of spacetime, the position dependence must be projected as well. In this manner, spinors can be "pulled back" into the same spacetime as the differential operator ∇.

We are now in a position to form the contraction $\nabla^{A'A}\kappa^B\epsilon_{AB}$. We know that the role of the ϵ_{AB} is to antisymmetrise on the relevant particle spaces (in this case the 2 and 3 spaces), together with introducing a factor of $\sqrt{2}$. Borrowing from the 2-spinor notation, we denote this operation as $\epsilon_{2,3}$. We can now write

$$\nabla^{A'A}\kappa_A = \nabla^{A'A}\kappa^B\epsilon_{AB} \quad \leftrightarrow \quad \nabla^1 \epsilon^{12} \gamma_0^1 \kappa^3 \tfrac{1}{2}(1+\sigma_3^3) E_3 \epsilon_{2,3}, \tag{4.210}$$

where we have introduced the notation ϵ^{ij} for the ϵ invariant (singlet state) under joint rotations in the ith and jth copies of spacetime. Equation (4.210) is manipulated to give

$$\begin{aligned} &\nabla^1 \epsilon^{12} \gamma_0^1 \kappa^3 \tfrac{1}{2}(1+\sigma_3^3) E_3 \epsilon_{2,3} \\ &\quad = \nabla^1 \tfrac{1}{\sqrt{2}}(i\sigma_2^1 - i\sigma_2^2)\kappa^3 \tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)\tfrac{1}{2}(1+\sigma_3^3) E_3 \epsilon_{2,3} \gamma_0^1 \\ &\quad = \nabla^1 \tfrac{1}{\sqrt{2}} \left(-i\sigma_2^1 \langle (\sigma_1 + i\sigma_2)\tilde{\kappa}\rangle_{0,4}^2 + \langle i\sigma_2(\sigma_1 + i\sigma_2)\tilde{\kappa}\rangle_{0,4}^2 \right) \tfrac{1}{2}(1+\sigma_3^1)\epsilon^{23}\gamma_0^1 E_3 \end{aligned} \tag{4.211}$$

and projecting down into particle-one space, the quantity that remains is

$$\nabla^{A'A}\kappa_A \quad \leftrightarrow \quad \nabla \tfrac{1}{\sqrt{2}}[i\sigma_2\langle\kappa(\sigma_1 + i\sigma_2)\rangle_{0,4} + \langle\kappa(\sigma_1 + i\sigma_2)i\sigma_2\rangle_{0,4}]\tfrac{1}{2}(1+\sigma_3)\gamma_0. \tag{4.212}$$

We now require the following rearrangement:

$$\begin{aligned} &[i\sigma_2\langle\kappa(\sigma_1 + i\sigma_2)\rangle_{0,4} + \langle\kappa(\sigma_1 + i\sigma_2)i\sigma_2\rangle_{0,4}]\tfrac{1}{2}(1+\sigma_3) \\ &\qquad = [i\sigma_2(\langle\kappa i\sigma_2\rangle - i\langle\kappa i\sigma_1\rangle) - \langle\kappa\rangle + i\langle\kappa i\sigma_3\rangle]\tfrac{1}{2}(1+\sigma_3) \\ &\qquad = [-\langle\kappa\rangle + i\sigma_k\langle\kappa i\sigma_k\rangle]\tfrac{1}{2}(1+\sigma_3) \\ &\qquad = -\kappa\tfrac{1}{2}(1+\sigma_3). \end{aligned} \tag{4.213}$$

Using this, we find that

$$\nabla^{A'A}\kappa_A \quad \leftrightarrow \quad -\tfrac{1}{\sqrt{2}}\nabla\kappa\tfrac{1}{2}(1+\sigma_3)\gamma_0 = -\tfrac{1}{\sqrt{2}}\nabla\gamma_0\kappa\tfrac{1}{2}(1-\sigma_3), \tag{4.214}$$

where pulling the γ_0 across to the left-hand side demonstrates how the $\nabla^{A'A}$ switches between idempotents (modules). Equation (4.214) essentially justifies the procedure described in this section, since the translation (4.214) is "obvious" from the Weyl representation of the Dirac algebra (4.119). The factor of $1/\sqrt{2}$ in (4.214) is no longer a product of our conventions, but is an unavoidable aspect of the 2-spinor calculus. It remains to find the equivalent to the above for the expression $\nabla^{AA'}\bar{\omega}_{A'}$. The translation for $\nabla^{AA'}$ is obtained from that for $\nabla^{A'A}$ by switching the "particle" indices, so that

$$\nabla^{AA'} \quad \leftrightarrow \quad -\nabla^2\epsilon\gamma_0^2 = -\nabla^1\bar{\epsilon}\gamma_0^1. \tag{4.215}$$

Then, proceeding as above, we find that

$$\nabla^{AA'}\bar{\omega}_{A'} \quad \leftrightarrow \quad -\tfrac{1}{\sqrt{2}}\nabla\omega i\sigma_2\tfrac{1}{2}(1+\sigma_3)\gamma_0. \tag{4.216}$$

4.5.5 Applications

The above constitutes the necessary ingredients for a complete translation of the 2-spinor calculus into the STA. We close this chapter by considering two important applications. It should be clear from both that, whilst the steps to achieve the STA form are often quite complicated, the end result is nearly always more compact and easier to understand than the original 2-spinor form. An objective of future research in this subject is to extract from 2-spinor calculus the techniques which are genuinely powerful and useful. These can then be imported into the STA, which will suitably enriched by so doing. The intuitive geometric nature of the STA should then make these techniques available to a wider audience of physicists than currently employ the 2-spinor calculus.

The Dirac Equation

The Dirac equation in 2-spinor form is given by the pair of equations [36, page 222]

$$\begin{aligned} \nabla^{A'A}\kappa_A &= \mu\bar{\omega}^{A'} \\ \nabla^{AA'}\bar{\omega}_{A'} &= \mu\kappa^A. \end{aligned} \tag{4.217}$$

The quantity μ is defined to be $m/\sqrt{2}$, where m is the electron mass. The factor of $1/\sqrt{2}$ demonstrates that such factors are intrinsic to the way that the $\nabla^{A'A}$ symbol encodes the vector derivative. The equations (4.217) translate to the pair of equations

$$\begin{aligned} \nabla\kappa\tfrac{1}{2}(1+\sigma_3)\gamma_0 &= m\omega i\sigma_2\tfrac{1}{2}(1-\sigma_3) \\ -\nabla\omega i\sigma_2\tfrac{1}{2}(1-\sigma_3)\gamma_0 &= m\kappa\tfrac{1}{2}(1+\sigma_3). \end{aligned} \tag{4.218}$$

If we now define the full spinor ψ by

$$\psi = \kappa\tfrac{1}{2}(1+\sigma_3) + \omega\sigma_2\tfrac{1}{2}(1-\sigma_3) \tag{4.219}$$

we find that

$$\begin{aligned}\nabla\psi\gamma_0 &= m[\omega\sigma_2\tfrac{1}{2}(1-\sigma_3) - m\kappa\tfrac{1}{2}(1+\sigma_3)]i \\ &= -m\psi i\sigma_3.\end{aligned} \tag{4.220}$$

We thus recover the STA version of the Dirac equation (4.92)

$$\nabla\psi i\sigma_3 = m\psi\gamma_0. \tag{4.221}$$

Of the pair of equations (4.217), Penrose & Rindler write "*an advantage of the 2-spinor description is that the γ-matrices disappear completely – and complicated γ-matrix identities simply evaporate!*" [36, page 221]. Whilst this is true, the comment applies even more strongly to the STA form of the Dirac equation (4.221), in which complicated 2-spinor identities are also eliminated!

Maxwell's Equations

In the 2-spinor calculus the real, antisymmetric tensor F^{ab} is written as

$$F^{ab} = \psi^{AB}\epsilon^{A'B'} + \epsilon^{AB}\psi^{A'B'}, \tag{4.222}$$

where ψ^{AB} is symmetric on its two indices. We first need the STA equivalent of ψ^{AB}. Assuming initially that ψ^{AB} is arbitrary, we can write

$$\psi^{AB} \quad \leftrightarrow \quad \psi\tfrac{1}{2}(1+\sigma_3^1)\tfrac{1}{2}(1+\sigma_3^2)E = \psi\tfrac{1}{2}(1+\sigma_3^1)\sigma_1^1\epsilon, \tag{4.223}$$

where ψ is an arbitrary element of the product space of the two single-particle Pauli-even algebras. A complete basis for ψ is formed by all combinations of the 7 elements $\{1, i\sigma_k^1, i\sigma_k^2\}$. The presence of the singlet ϵ allows all elements of second space to be projected down into the first space, and it is not hard to see that this accounts for all possible even elements in the one-particle STA. We can therefore write

$$\psi\tfrac{1}{2}(1+\sigma_3^1)\sigma_1^1\epsilon = M^1\epsilon, \tag{4.224}$$

where M is an arbitrary even element. The condition that ψ^{AB} is symmetric on its two indices now becomes (recalling that ϵ is antisymmetric on its two particle indices)

$$M^1\epsilon = -M^2\epsilon = -\tilde{M}^1\epsilon \tag{4.225}$$

$$\Rightarrow M = -\tilde{M}. \tag{4.226}$$

This condition projects out from M the components that are bivectors in particle-one space, so we can write

$$\psi^{AB} \quad \leftrightarrow \quad F^1\epsilon \tag{4.227}$$

where F is now a bivector. For the case of electromagnetism, F is the Faraday bivector, introduced in Section (1.2.5). The complete translation of F^{ab} is therefore

$$F^{ab} \quad \leftrightarrow \quad F^1\epsilon + F^1\epsilon\sigma_1^1\sigma_1^2 = F^1\eta \tag{4.228}$$

where η is the full relativistic invariant.

The 2-spinor form of the Maxwell equations an be written

$$\nabla^{A'B}\psi^{AC}\epsilon_{BC} = -J^{AA'} \tag{4.229}$$

where $J^{AA'}$ is a "real" vector (*i.e.* it has no trivector components). Recalling the convention that ϵ^{ij} denotes the singlet state in coupled $\{i,j\}$-space, the STA version of equation (4.229) is

$$\nabla^1\epsilon^{12}\gamma_0^1 F^3\epsilon^{34}\epsilon_{2,4} = -J^1\epsilon^{13}\gamma_0^1. \tag{4.230}$$

This is simplified by the identity

$$\epsilon^{12}\epsilon^{34}\epsilon_{2,4} = \epsilon^{13}, \tag{4.231}$$

which is proved by expanding the left-hand side and then performing the antisymmetrisation. The resultant equation is

$$\nabla^1 F^3\epsilon^{13} = -J^1\epsilon^{13}, \tag{4.232}$$

which has a one-particle reduction to

$$\nabla F = J. \tag{4.233}$$

This recovers the STA form of the Maxwell equations [17]. The STA form is remarkably compact, makes use solely of spacetime quantities and has a number of computational advantages over second-order wave equations [8]. The 2-spinor calculus also achieves a first-order formulation of Maxwell's equations, but at the expense of some considerable abstractions. We will return to equation (4.233) in Chapters 6 and 7.

Chapter 5

Point-particle Lagrangians

In this chapter we develop a multivector calculus as the natural extension of the calculus of functions of a single parameter. The essential new tool required for such a calculus is the multivector derivative, and this is described first. It is shown how the multivector derivative provides a coordinate-free language for manipulating linear functions (forming contractions *etc.*). This supersedes the approach used in earlier chapters, where such manipulations were performed by introducing a frame.

The remainder of this chapter then applies the techniques of multivector calculus to the analysis of point-particle Lagrangians. These provide a useful introduction to the techniques that will be employed in the study of field Lagrangians in the final two chapters. A novel idea discussed here is that of a multivector-valued Lagrangian. Such objects are motivated by the pseudoclassical mechanics of Berezin & Marinov [39], but can only be fully developed within geometric algebra. Forms of Noether's theorem are given for both scalar and multivector-valued Lagrangians, and for transformations parameterised by both scalars and multivectors. This work is applied to the study of two semi-classical models of electron spin. Some aspects of the work presented in this chapter appeared in the papers "*Grassmann mechanics, multivector derivatives and geometric algebra*" [3] and "*Grassmann calculus, pseudoclassical mechanics and geometric algebra*" [1].

5.1 The Multivector Derivative

The idea of a *vector derivative* was partially introduced in Chapter 4, where it was seen that the STA form of the Dirac equation (4.92) required the operator $\nabla = \gamma^\mu \partial_{x^\mu}$, where $x^\mu = \gamma^\mu \cdot x$. The same operator was later seen to appear in the STA form of the Maxwell equations (4.233), $\nabla F = J$. We now present a more formal introduction to the properties of the vector and multivector derivatives. Further details of these properties are contained in [18] and [24, Chapter 2], the latter of which is particularly detailed in its treatment.

Let X be a mixed-grade multivector

$$X = \sum_r X_r, \tag{5.1}$$

and let $F(X)$ be a general multivector-valued function of X. The grades of $F(X)$ need not be the same as those of its argument X. For example, the STA representation of a

Dirac spinor as $\psi(x)$ is a map from the vector x onto an arbitrary even element of the STA. The derivative of $F(X)$ in the A direction, where A has the same grades as X, is defined by

$$A{*}\partial_X F(X) \equiv \lim_{\tau\to 0} \frac{F(X+\tau A) - F(X)}{\tau}. \tag{5.2}$$

(It is taken as implicit in this definition that the limit exists.) The operator $A{*}\partial_X$ satisfies all the usual properties for partial derivatives. To define the multivector derivative ∂_X, we introduce an arbitrary frame $\{e_j\}$ and extend this to define a basis for the entire algebra $\{e_J\}$, where J is a general (simplicial) index. The multivector derivative is now defined by

$$\partial_X = \sum_J e^J e_J{*}\partial_X. \tag{5.3}$$

The directional derivative $e_J{*}\partial_X$ is only non-zero when e_J is of the same grade(s) as X, so ∂_X inherits the multivector properties of its argument X. The contraction in (5.3) ensures that the quantity ∂_X is independent of the choice of frame, and the basic properties of ∂_X can be formulated without any reference to a frame.

The properties of ∂_X are best understood with the aid of some simple examples. The most useful result for the multivector derivative is

$$\partial_X \langle XA \rangle = P_X(A), \tag{5.4}$$

where $P_X(A)$ is the projection of A on to the grades contained in X. From (5.4) it follows that

$$\begin{aligned} \partial_X \langle \tilde{X} A \rangle &= P_X(\tilde{A}) \\ \partial_{\tilde{X}} \langle \tilde{X} A \rangle &= P_X(A). \end{aligned} \tag{5.5}$$

Leibniz' rule can now be used in conjunction with (5.4) to build up results for the action of ∂_X on more complicated functions. For example,

$$\partial_X \langle X\tilde{X} \rangle^{k/2} = k \langle X\tilde{X} \rangle^{(k-2)/2} \tilde{X}. \tag{5.6}$$

The multivector derivative acts on objects to its immediate right unless brackets are present, in which case ∂_X acts on the entire bracketed quantity. If ∂_X acts on a multivector that is not to its immediate right, we denote this with an overdot on the ∂_X and its argument. Thus $\dot{\partial}_X A\dot{B}$ denotes the action of ∂_X on B,

$$\dot{\partial}_X A\dot{B} = e^J A e_J{*}\partial_X B. \tag{5.7}$$

The overdot notation is an invaluable aid to expressing the properties of the multivector derivative. In particular, it neatly encodes the fact that, since ∂_X is a multivector, it does not necessarily commute with other multivectors and often acts on functions to which it is not adjacent. As an illustration, Leibniz' rule can now be given in the form

$$\partial_X(AB) = \dot{\partial}_X \dot{A} B + \dot{\partial}_X A\dot{B}. \tag{5.8}$$

The only drawback with the overdot notation comes in expressions which involve time derivatives. It is usually convenient to represent these with overdots as well, and in such instances the overdots on multivector derivatives will be replaced by overstars.

The most useful form of the multivector derivative is the derivative with respect to a vector argument, ∂_a or ∂_x. Of these, the derivative with respect to position x is particularly important. This is called the vector derivative, and is given special the symbol

$$\partial_x = \nabla = \nabla_x. \tag{5.9}$$

The operator ∇ sometimes goes under the name of the Dirac operator, though this name is somewhat misleading since ∂_x is well-defined in all dimensions and is in no way tied to quantum-mechanical notions. In three dimensions, for example, $\partial_{\boldsymbol{x}} = \boldsymbol{\nabla}$ contains all the usual properties of the div, grad and curl operators. There are a number of useful formulae for derivatives with respect to vectors, a selection of which is as follows:

$$\begin{aligned}
\partial_a a \cdot b &= b \\
\partial_a a^2 &= 2a \\
\partial_a \cdot a &= n \\
\partial_a \wedge a &= 0 \\
\partial_a a \cdot A_r &= r A_r \\
\partial_a a \wedge A_r &= (n-r) A_r \\
\partial_a A_r a &= (-1)^r (n-2r) A_r,
\end{aligned} \tag{5.10}$$

where n is the dimension of the space. The final three equations in (5.10) are the frame-free forms of formulae given in Section (1.3.2).

Vector derivatives are very helpful for developing the theory of linear functions, as introduced in Section (1.3). For example, the adjoint to the linear function $\underline{f}$ can be defined as

$$\overline{f}(a) \equiv \partial_b \langle a \underline{f}(b) \rangle. \tag{5.11}$$

It follows immediately that

$$b \cdot \overline{f}(a) = b \cdot \partial_c \langle a \underline{f}(c) \rangle = \langle \underline{f}(b \cdot \partial_c c) a \rangle = \underline{f}(b) \cdot a. \tag{5.12}$$

Invariants can be constructed equally simply. For example, the trace of $\underline{f}(a)$ is defined by

$$\mathrm{Tr}\underline{f} \equiv \partial_a \cdot \underline{f}(a) \tag{5.13}$$

and the "characteristic bivector" of $\underline{f}(a)$ is defined by

$$B = \tfrac{1}{2} \partial_a \wedge \underline{f}(a). \tag{5.14}$$

An anti-symmetric function $\underline{f} = -\overline{f}$ can always be written in the form

$$\underline{f}(a) = a \cdot B \tag{5.15}$$

and it follows from equation (5.10) that B is the characteristic bivector.

Many other aspects of linear algebra, including a coordinate-free proof of the Cayley-Hamilton theorem, can be developed similarly using combinations of vector derivatives [24, Chapter 3].

5.2 Scalar and Multivector Lagrangians

As an application of the multivector derivative formalism just outlined, we consider Lagrangian mechanics. We start with a scalar-valued Lagrangian $L = L(X_i, \dot{X}_i)$, where the X_i are general multivectors, and $\dot{X}_i$ denotes differentiation with respect to time. We wish to find the $X_i(t)$ which extremise the action

$$S = \int_{t_1}^{t_2} dt\, L(X_i, \dot{X}_i). \tag{5.16}$$

The solution to this problem can be found in many texts (see *e.g.* [71]). We write

$$X_i(t) = X_i^0(t) + \epsilon Y_i(t), \tag{5.17}$$

where Y_i is a multivector containing the same grades as X_i and which vanishes at the endpoints, ϵ is a scalar and X_i^0 represents the extremal path. The action must now satisfy $\partial_\epsilon S = 0$ when $\epsilon = 0$, since $\epsilon = 0$ corresponds to $X_i(t)$ taking the extremal values. By applying the chain rule and integrating by parts, we find that

$$\begin{aligned}
\partial_\epsilon S &= \int_{t_1}^{t_2} dt\, \Big((\partial_\epsilon X_i)*\partial_{X_i} L + (\partial_\epsilon \dot{X}_i)*\partial_{\dot{X}_i} L\Big) \\
&= \int_{t_1}^{t_2} dt\, \Big(Y_i*\partial_{X_i} L + \dot{Y}_i*\partial_{\dot{X}_i} L\Big) \\
&= \int_{t_1}^{t_2} dt\, Y_i*\Big(\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L)\Big).
\end{aligned} \tag{5.18}$$

Setting ϵ to zero now just says that X_i is the extremal path, so the extremal path is defined by the solutions to the Euler-Lagrange equations

$$\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L) = 0. \tag{5.19}$$

The essential advantage of this derivation is that it employs genuine derivatives in place of the less clear concept of an infinitessimal. This will be exemplified when we study Lagrangians containing spinor variables.

We now wish to extend the above argument to a multivector-valued Lagrangian L. Taking the scalar product of L with an arbitrary constant multivector A produces a scalar Lagrangian $\langle LA \rangle$. This generates its own Euler-Lagrange equations,

$$\partial_{X_i}\langle LA \rangle - \partial_t(\partial_{\dot{X}_i}\langle LA \rangle) = 0. \tag{5.20}$$

A "permitted" multivector Lagrangian is one for which the equations from each A are mutually consistent, so that each component of the full L is capable of simultaneous extremisation.

By contracting equation (5.20) on the right-hand side by ∂_A, we find that a necessary condition on the dynamical variables is

$$\partial_{X_i} L - \partial_t(\partial_{\dot{X}_i} L) = 0. \tag{5.21}$$

For a permitted multivector Lagrangian, equation (5.21) is also *sufficient* to ensure that equation (5.20) is satisfied for all A. This is taken as part of the definition of a multivector Lagrangian. We will see an example of how these criteria can be met in Section 5.3.

5.2.1 Noether's Theorem

An important technique for deriving consequences of the equations of motion resulting from a given Lagrangian is the study of the symmetry properties of the Lagrangian itself. The general result needed for this study is Noether's theorem. We seek a form of this theorem which is applicable to both scalar-valued and multivector-valued Lagrangians. There are two types of symmetry to consider, depending on whether the transformation of variables is governed by a scalar or by a multivector parameter. We will look at these separately.

It is important to recall at this point that all the results obtained here are derived in the *coordinate-free* language of geometric algebra. Hence all the symmetry transformations considered are *active*. Passive transformations have no place in this scheme, as the introduction of an arbitrary coordinate system is an unnecessary distraction.

5.2.2 Scalar Parameterised Transformations

Given a Lagrangian $L = L(X_i, \dot{X}_i)$, which can be either scalar-valued or multivector-valued, we wish to consider variations of the variables X_i controlled by a single scalar parameter, α. We write this as

$$X_i' = X_i'(X_i, \alpha), \tag{5.22}$$

and assume that $X_i'(\alpha = 0) = X_i$. We now define the new Lagrangian

$$L'(X_i, \dot{X}_i) = L(X_i', \dot{X}_i'), \tag{5.23}$$

which has been obtained from L by an active transformation of the dynamical variables. Employing the identity $L' = \langle L'A \rangle \partial_A$, we proceed as follows:

$$\begin{aligned} \partial_\alpha L' &= (\partial_\alpha X_i')*\partial_{X_i'}\langle L'A\rangle\partial_A + (\partial_a \dot{X}_i')*\partial_{\dot{X}_i'}\langle L'A\rangle\partial_A \\ &= (\partial_a X_i')*\left(\partial_{X_i'}\langle L'A\rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'A\rangle)\right)\partial_A + \partial_t\left((\partial_a X_i')*\partial_{\dot{X}_i'}L'\right). \end{aligned} \tag{5.24}$$

The definition of L' ensures that it has the same functional form of L, so the quantity

$$\partial_{X_i'}\langle L'A\rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'A\rangle)L' \tag{5.25}$$

is obtained by taking the Euler-Lagrange equations in the form (5.20) and replacing the X_i by X_i'. If we now assume that the X_i' satisfy the same equations of motion (which must be checked for any given case), we find that

$$\partial_\alpha L' = \partial_t\left((\partial_a X_i')*\partial_{\dot{X}_i'}L'\right) \tag{5.26}$$

and, if L' is independent of α, the corresponding quantity $(\partial_a X_i') * \partial_{\dot{X}_i'}L'$ is conserved. Alternatively, we can set α to zero so that (5.25) becomes

$$[\partial_{X_i'}\langle L'A\rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'A\rangle)L']\Big|_{\alpha=0} = \partial_{X_i}\langle LA\rangle - \partial_t(\partial_{\dot{X}_i}\langle LA\rangle) \tag{5.27}$$

which vanishes as a consequence of the equations of motion for X_i. We therefore find that

$$\partial_a L'|_{\alpha=0} = \partial_t\left((\partial_a X_i')*\partial_{\dot{X}_i'}L'\right)\Big|_{\alpha=0}, \tag{5.28}$$

which is probably the most useful form of Noether's theorem, in that interesting consequences follow from (5.28) regardless of whether or not L' is independent of α. A crucial step in the derivation of (5.28) is that the Euler-Lagrange equations for a multivector-valued Lagrangian are satisfied in the form (5.20). Hence the consistency of the equations (5.20) for different A is central to the development of the theory of multivector Lagrangians.

To illustrate equation (5.28), consider time translation

$$\begin{aligned} X_i'(t,\alpha) &= X_i(t+\alpha) && (5.29)\\ \Rightarrow \partial_a X_i'|_{\alpha=0} &= \dot{X}_i. && (5.30)\end{aligned}$$

Assuming there is no explicit time-dependence in L, equation (5.28) gives

$$\partial_t L = \partial_t(\dot{X}_i*\partial_{\dot{X}_i}L), \tag{5.31}$$

from which we define the conserved Hamiltonian by

$$H = \dot{X}_i*\partial_{\dot{X}_i}L - L. \tag{5.32}$$

If L is multivector-valued, then H will be a multivector of the same grade(s).

5.2.3 Multivector Parameterised Transformations

The most general single transformation for the variables X_i governed by a multivector M can be written as

$$X_i' = f(X_i, M), \tag{5.33}$$

where f and M are time-independent functions and multivectors respectively. In general f need not be grade-preserving, which provides a route to deriving analogues for supersymmetric transformations.

To follow the derivation of (5.26), it is useful to employ the differential notation [24],

$$\underline{f}_M(X_i, A) \equiv A*\partial_M f(X_i, M). \tag{5.34}$$

The function $\underline{f}_M(X_i, A)$ is a linear function of A and an arbitrary function of M and X_i. With L' defined as in equation (5.23), we derive

$$\begin{aligned} A*\partial_M L' &= \underline{f}_M(X_i, A)*\partial_{X_i'}L' + \underline{f}_M(\dot{X}_i, M)*\partial_{\dot{X}_i'}L' \\ &= \underline{f}_M(X_i, A)*\left(\partial_{X_i'}\langle L'B\rangle - \partial_t(\partial_{\dot{X}_i'}\langle L'B\rangle)\right)\partial_B + \partial_t\left(\underline{f}_M(X_i, A)*\partial_{\dot{X}_i'}L'\right) \\ &= \partial_t\left(\underline{f}_M(X_i, A)*\partial_{\dot{X}_i'}L'\right), && (5.35)\end{aligned}$$

where again it is necessary to assume that the equations of motion are satisfied for the transformed variables. We can remove the A-dependence by differentiating, which yields

$$\partial_M L' = \partial_t\left(\partial_A \underline{f}_M(X_i, A)*\partial_{\dot{X}_i'}L'\right) \tag{5.36}$$

and, if L' is independent of M, the corresponding conserved quantity is

$$\partial_A \underline{f}_M(X_i, A)*\partial_{\dot{X}_i'}L' = \overset{*}{\partial}_M f(X_i, \overset{*}{M})*\partial_{\dot{X}_i'}L', \tag{5.37}$$

where the overstar on $\overset{*}{M}$ denote the argument of $\overset{*}{\partial}_M$.

It is not usually possible to set M to zero in (5.35), but it is interesting to see that conserved quantities can be found regardless. This shows that standard treatments of Lagrangian symmetries [71] are unnecessarily restrictive in only considering infinitesimal transformations. The subject is richer than this suggests, though without multivector calculus the necessary formulae are hard to find.

In order to illustrate (5.37), consider reflection symmetry applied to the harmonic oscillator Lagrangian

$$L(x, \dot{x}) = \tfrac{1}{2}(\dot{x}^2 - \omega^2 x^2). \tag{5.38}$$

The equations of motion are

$$\ddot{x} = -\omega^2 x \tag{5.39}$$

and it is immediately seen that, if x is a solution, then so to is x', where

$$x' = -nxn^{-1}. \tag{5.40}$$

Here n is an arbitrary vector, so x' is obtained from x by a reflection in the hyperplance orthogonal to n. Under the reflection (5.40) the Lagrangian is unchanged, so we can find a conserved quantity from equation (5.37). With $f(x, n)$ defined by

$$f(x, n) = -nxn^{-1} \tag{5.41}$$

we find that

$$\underline{f}_n(x, a) = -axn^{-1} + nxn^{-1}an^{-1}. \tag{5.42}$$

Equation (5.37) now yields the conserved quantity

$$\begin{aligned} \partial_a(-axn^{-1} + nxn^{-1}an^{-1})*(-n\dot{x}n^{-1}) &= \partial_a\langle ax\dot{x}n^{-1} - a\dot{x}xn^{-1}\rangle \\ &= \langle x\dot{x}n^{-1} - \dot{x}xn^{-1}\rangle_1 \\ &= 2(x\wedge\dot{x})\cdot n^{-1}. \end{aligned} \tag{5.43}$$

This is conserved for all n, from which it follows that the angular momentum $x \wedge \dot{x}$ is conserved. This is not a surprise, since rotations can be built out of reflections and different reflections are related by rotations. It is therefore natural to expect the same conserved quantity from both rotations and reflections. But the derivation does show that the multivector derivative technique works and, to my knowledge, this is the first time that a classical conserved quantity has been derived conjugate to transformations that are not simply connected to the identity.

5.3 Applications — Models for Spinning Point Particles

There have been numerous attempts to construct classical models for spin-half particles (see van Holten [72] for a recent review) and two such models are considered in this section. The first involves a scalar point-particle Lagrangian in which the dynamical variables include spinor variables. The STA formalism of Chapter 4 is applied to this Lagrangian and used to analyse the equations of motion. Some problems with the model are discussed, and a more promising model is proposed. The second example is drawn from pseudoclassical mechanics. There the dynamical variables are Grassmann-valued entities, and the formalism of Chapter 2 is used to represent these by geometric vectors. The resulting Lagrangian is multivector-valued, and is studied using the techniques just developed. The equations of motion are found and solved, and again it is argued that the model fails to provide an acceptable picture of a classical spin-half particle.

1. The Barut-Zanghi Model

The Lagrangian of interest here was introduced by Barut & Zanghi [38] (see also [7, 61]) and is given by

$$L = \tfrac{1}{2}j(\dot{\bar{\Psi}}\Psi - \bar{\Psi}\dot{\Psi}) + p_\mu(\dot{x}^\mu - \bar{\Psi}\gamma^\mu\Psi) + qA_\mu(x)\bar{\Psi}\gamma^\mu\Psi \tag{5.44}$$

where Ψ is a Dirac spinor. Using the mapping described in Section (4.3), the Lagrangian (5.45) can be written as

$$L = \langle\dot{\psi}i\sigma_3\tilde{\psi} + p(\dot{x} - \psi\gamma_0\tilde{\psi}) + qA(x)\psi\gamma_0\tilde{\psi}\rangle. \tag{5.45}$$

The dynamical variables are x, p and ψ, where ψ is an even multivector, and the dot denotes differentiation with respect to some arbitrary parameter τ.

The Euler-Lagrange equation for ψ is

$$\begin{aligned} \partial_\psi L &= \partial_\tau(\partial_{\dot{\psi}}L) \\ \Rightarrow \quad \partial_\tau(i\sigma_3\tilde{\psi}) &= -i\sigma_3\dot{\tilde{\psi}} - 2\gamma_0\tilde{\psi}p + 2q\gamma_0\tilde{\psi}A \\ \Rightarrow \quad \dot{\psi}i\sigma_3 &= P\psi\gamma_0, \end{aligned} \tag{5.46}$$

where

$$P \equiv p - qA. \tag{5.47}$$

In deriving (5.46) there is no pretence that ψ and $\tilde{\psi}$ are independent variables. Instead they represent two occurrences of the same variable ψ and all aspects of the variational principle are taken care of by the multivector derivative.

The p equation is

$$\dot{x} = \psi\gamma_0\tilde{\psi} \tag{5.48}$$

but, since $\dot{x}^2 = \rho^2$ is not, in general, equal to 1, τ cannot necessarily be viewed as the proper time for the particle. The x equation is

$$\begin{aligned} \dot{p} &= q\nabla A(x)\cdot(\psi\gamma_0\tilde{\psi}) \\ &= q(\nabla\wedge A)\cdot\dot{x} + q\dot{x}\cdot\nabla A \\ \Rightarrow \dot{P} &= qF\cdot\dot{x}. \end{aligned} \tag{5.49}$$

We now use (5.28) to derive some consequences for this model. The Hamiltonian is given by

$$\begin{aligned} H &= \dot{x}*\partial_{\dot{x}}L + \dot{\psi}*\partial_{\dot{\psi}}L - L \\ &= P\cdot\dot{x}, \end{aligned} \tag{5.50}$$

and is conserved absolutely. The 4-momentum and angular momentum are only conserved if $A=0$, in which case (5.45) reduces to the free-particle Lagrangian

$$L_0 = \langle \dot{\psi} i\sigma_3 \tilde{\psi} + p(\dot{x} - \psi\gamma_0\tilde{\psi})\rangle. \tag{5.51}$$

The 4-momentum is found from translation invariance,

$$x' = x + \alpha a, \tag{5.52}$$

and is simply p. The component of p in the $\dot{x}$ direction gives the energy (5.50). The angular momentum is found from rotational invariance, for which we set

$$\begin{aligned} x' &= e^{\alpha B/2} x e^{-\alpha B/2} \\ p' &= e^{\alpha B/2} p e^{-\alpha B/2} \\ \psi' &= e^{\alpha B/2}\psi. \end{aligned} \tag{5.53}$$

It is immediately apparent that L_0' is independent of α, so the quantity

$$(B{\cdot}x)*\partial_{\dot{x}}L_0 + \tfrac{1}{2}(B\psi)*\partial_{\dot{\psi}}L_0 = B{\cdot}(x\wedge p + \tfrac{1}{2}\psi i\sigma_3\tilde{\psi}) \tag{5.54}$$

is conserved for arbitrary B. The angular momentum is therefore defined by

$$J = p\wedge x - \tfrac{1}{2}\psi i\sigma_3\tilde{\psi}, \tag{5.55}$$

which identifies $-\psi i\sigma_3\tilde{\psi}/2$ as the internal spin. The factor of $1/2$ clearly originates from the transformation law (5.53). The free-particle model defined by (5.51) therefore does have some of the expected properties of a classical model for spin, though there is a potential problem with the definition of J (5.55) in that the spin contribution enters with the opposite sign to that expected from field theory (see Chapter 6).

Returning to the interacting model (5.45), further useful properties can be derived from transformations in which the spinor is acted on from the right. These correspond to gauge transformations, though a wider class is now available than for the standard column-spinor formulation. From the transformation

$$\psi' = \psi e^{\alpha i\sigma_3} \tag{5.56}$$

we find that

$$\partial_\tau\langle\psi\tilde{\psi}\rangle = 0, \tag{5.57}$$

and the transformation

$$\psi' = \psi e^{\alpha\sigma_3} \tag{5.58}$$

yields

$$\partial_\tau\langle i\psi\tilde{\psi}\rangle = -2P{\cdot}(\psi\gamma_3\tilde{\psi}). \tag{5.59}$$

Equations (5.57) and (5.59) combine to give

$$\partial_\tau(\psi\tilde{\psi}) = 2iP{\cdot}(\psi\gamma_3\tilde{\psi}). \tag{5.60}$$

Finally, the duality transformation

$$\psi' = \psi e^{\alpha i} \tag{5.61}$$

yields

$$2\langle\dot{\psi}\sigma_3\tilde{\psi}\rangle = 0. \tag{5.62}$$

A number of features of the Lagrangian (5.45) make it an unsatisfactory model a classical electron. We have already mentioned that the parameter τ cannot be identified with the proper time of the particle. This is due to the lack of reparameterisation invariance in (5.45). More seriously, the model predicts a zero gyromagnetic moment [61]. Furthermore, the $\dot{P}$ equation (5.49) cannot be correct, since here one expects to see $\dot{p}$ rather than $\dot{P}$ coupling to $F\cdot x$. Indeed, the definition of P (5.47) shows that equation (5.49) is not gauge invariant, which suggests that it is a lack of gauge invariance which lies behind some of the unacceptable features of the model.

2. Further Spin-half Models

We will now see how to modify the Lagrangian (5.45) to achieve a suitable set of classical equations for a particle with spin. The first step is to consider gauge invariance. Under the local gauge transformation

$$\psi \mapsto \psi \exp\{-i\sigma_3\phi(\tau)\} \tag{5.63}$$

the "kinetic" spinor term $\langle\dot{\psi}i\sigma_3\tilde{\psi}\rangle$ transforms as

$$\langle\dot{\psi}i\sigma_3\tilde{\psi}\rangle \mapsto \langle\dot{\psi}i\sigma_3\tilde{\psi}\rangle + \langle\psi\tilde{\psi}\dot{\phi}\rangle. \tag{5.64}$$

The final term can be written as

$$\langle\psi\tilde{\psi}\dot{\phi}\rangle = \langle\psi\tilde{\psi}\dot{x}\cdot(\nabla\phi)\rangle, \tag{5.65}$$

and, when $\nabla\phi$ is generalised to an arbitrary gauge field qA, (5.64) produces the interaction term

$$L_I = q\langle\psi\tilde{\psi}\dot{x}\cdot A\rangle. \tag{5.66}$$

This derivation shows clearly that the A field must couple to $\dot{x}$ and not to $\psi\gamma_0\tilde{\psi}$, as it is not until after the equations of motion are found that $\psi\gamma_0\tilde{\psi}$ is set equal to $\dot{x}$. That there should be an $\dot{x}\cdot A$ term in the Lagrangian is natural since this is the interaction term for a classical point particle, and a requirement on any action that we construct is that it should reproduce classical mechanics in the limit where spin effects are ignored (*i.e.* as $\hbar \mapsto 0$). But a problem still remains with (5.66) in that the factor of $\psi\tilde{\psi}$ is unnatural and produces an unwanted term in the ψ equation.. To remove this, we must replace the $\langle\dot{\psi}i\sigma_3\tilde{\psi}\rangle$ term by

$$L_0 = \langle\dot{\psi}i\sigma_3\psi^{-1}\rangle, \tag{5.67}$$

where, for a spinor $\psi = (\rho e^{i\beta})^{1/2}R$,

$$\psi^{-1} = (\rho e^{i\beta})^{-1/2}\tilde{R}. \tag{5.68}$$

In being led the term (5.67), we are forced to break with conventional usage of column spinors. The term (5.67) now suggests what is needed for a suitable classical model. The quantity $\langle\dot{\psi}i\sigma_3\psi^{-1}\rangle$ is unchanged by both dilations and duality transformations of ψ and so is only dependent on the rotor part of ψ. It has been suggested that the rotor part of ψ encodes the dynamics of the electron field and that the factor of $(\rho\exp\{i\beta\})^{1/2}$ is a quantum-mechanical statistical term [29]. Accepting this, we should expect that our classical model should involve only the rotor part of ψ and that the density terms should have no effect. Such a model requires that the Lagrangian be invariant under local changes of $\rho\exp\{i\beta\}$, as we have seen is the case for L_0 (5.67). The remaining spinorial term is the current term $\psi\gamma_0\tilde{\psi}$ which is already independent of the duality factor β. It can be made independent of the density ρ as well by dividing by ρ. From these observations we are led to the Lagrangian

$$L = \langle \dot{\psi} i\sigma_3 \psi^{-1} + p(\dot{x} - \psi\gamma_0\tilde{\psi}/\rho) - q\dot{x}\cdot A\rangle. \tag{5.69}$$

The p equation from (5.69) recovers

$$\dot{x} = \psi\gamma_0\tilde{\psi}/\rho = R\gamma_0\tilde{R}, \tag{5.70}$$

so that $\dot{x}^2 = 1$ and τ is automatically the affine parameter. This removes one of the defects with the Barut-Zanghi model. The x equation gives, using standard techniques,

$$\dot{p} = qF\cdot\dot{x}, \tag{5.71}$$

which is now manifestly gauge invariant and reduces to the Lorentz force law when the spin is ignored and the velocity and momentum are collinear, $p = m\dot{x}$. Finally, the ψ equation is found by making use of the results

$$\partial_\psi\langle\psi M\psi^{-1}\rangle = M\psi^{-1} + \dot{\partial}_\psi\langle\psi M\dot{\psi}^{-1}\rangle = 0 \tag{5.72}$$

$$\Rightarrow \partial_\psi\langle M\psi^{-1}\rangle = -\psi^{-1}M\psi^{-1} \tag{5.73}$$

and

$$\begin{aligned}\partial_\psi\rho &= \frac{1}{2\rho}\partial_\psi(\psi\gamma_0\tilde{\psi}\psi\gamma_0\tilde{\psi}) \\ &= \frac{2}{\rho}\gamma_0\rho e^{i\beta}\gamma_0\tilde{\psi} \\ &= 2\rho\psi^{-1}\end{aligned} \tag{5.74}$$

to obtain

$$-\psi^{-1}\dot{\psi}i\sigma_3\psi^{-1} - \frac{1}{\rho}(2\gamma_0\tilde{\psi}p - 2\psi^{-1}\langle p\psi\gamma_0\tilde{\psi}\rangle) = \partial_\tau(i\sigma_3\psi^{-1}). \tag{5.75}$$

By multiplying equation (5.75) with ψ, one obtains

$$\tfrac{1}{2}\dot{S} = p\wedge\dot{x}, \tag{5.76}$$

where

$$S \equiv \psi i\sigma_3\psi^{-1} = Ri\sigma_3\tilde{R}. \tag{5.77}$$

Thus the ψ variation now leads directly to the precession equation for the spin. The complete set of equations is now

$$\dot{S} = 2p\wedge\dot{x} \tag{5.78}$$
$$\dot{x} = R\gamma_0\tilde{R} \tag{5.79}$$
$$\dot{p} = qF\cdot\dot{x} \tag{5.80}$$

which are manifestly Lorentz covariant and gauge invariant. The Hamiltonian is now $p\cdot\dot{x}$ and the free-particle angular momentum is still defined by J (5.55), though now the spin bivector S is always of unit magnitude.

A final problem remains, however, which is that we have still not succeeded in constructing a model which predicts the correct gyromagnetic moment. In order to achieve the correct coupling between the spin and the Faraday bivector, the Lagrangian (5.69) must be modified to

$$L = \langle \dot{\psi} i\sigma_3\psi^{-1} + p(\dot{x} - \psi\gamma_0\tilde{\psi}/\rho) + q\dot{x}\cdot A - \frac{q}{2m}F\psi i\sigma_3\psi^{-1}\rangle. \tag{5.81}$$

The equations of motion are now

$$\begin{aligned}\dot{S} &= 2p\wedge\dot{x} + \frac{q}{m}F\times S \\ \dot{x} &= R\gamma_0\tilde{R} \\ \dot{p} &= qF\cdot\dot{x} - \frac{q}{2m}\nabla F(x)\cdot S,\end{aligned} \tag{5.82}$$

which recover the correct precession formulae in a constant magnetic field. When p is set equal to $m\dot{x}$, the equations (5.82) reduce to the pair of equations studied in [72].

3. A Multivector Model — Pseudoclassical Mechanics Reconsidered

Pseudoclassical mechanics [39, 73, 74] was originally introduced as the classical analogue of quantum spin one-half (*i.e.* for particles obeying Fermi statistics). The central idea is that the "classical analogue" of the Pauli or Dirac algebras is an algebra where all inner products vanish, so that the dynamical variables are Grassmann variables. From the point of view of this thesis, such an idea appears fundamentally flawed. Furthermore, we have already seen how to construct sensible semi-classical approximations to Dirac theory. But once the Grassmann variables have been replaced by vectors through the procedure outlined in Chapter 2, pseudoclassical Lagrangians do become interesting, in that they provide examples of acceptable multivector Lagrangians. Such a Lagrangian is studied here, from a number of different perspectives. An interesting aside to this work is a new method of generating super-Lie algebras, which could form the basis for an alternative approach to their representation theory.

The Lagrangian we will study is derived from a pseudoclassical Lagrangian introduced by Berezin & Marinov [39]. This has become a standard example in non-relativistic pseudoclassical mechanics [73, Chapter 11]. With a slight change of notation, and dropping an irrelevant factors of j, the Lagrangian can be written as

$$L = \tfrac{1}{2}\zeta_i\dot{\zeta}_i - \tfrac{1}{2}\epsilon_{ijk}\omega_i\zeta_j\zeta_k, \tag{5.83}$$

where the $\{\zeta_i\}$ are formally Grassmann variable and the $\{\omega_i\}$ are a set of three scalar constants. Here, i runs from 1 to 3 and, as always, the summation convention is implied. Replacing the set of Grassmann variables $\{\zeta_i\}$ with a set of three (Clifford) vectors $\{e_i\}$, the Lagrangian (5.83) becomes [1]

$$L = \tfrac{1}{2} e_i \wedge \dot{e}_i - \omega, \tag{5.84}$$

where

$$\omega = \tfrac{1}{2}\epsilon_{ijk}\omega_i e_j e_k = \omega_1(e_2 \wedge e_3) + \omega_2(e_3 \wedge e_1) + \omega_3(e_1 \wedge e_2). \tag{5.85}$$

The equations of motion from (5.84) are found by applying equation (5.21)

$$\begin{aligned} \partial_{e_i} \tfrac{1}{2}(e_j \wedge \dot{e}_j - \omega) &= \partial_t[\partial_{\dot{e}_i} \tfrac{1}{2}(e_j \wedge \dot{e}_j - \omega)] \\ \Rightarrow \quad \dot{e}_i + 2\epsilon_{ijk}\omega_j e_k &= -\partial_t e_i \\ \Rightarrow \quad \dot{e}_i &= -\epsilon_{ijk}\omega_j e_k. \end{aligned} \tag{5.86}$$

We have used the 3-dimensional result

$$\partial_a a \wedge b = 2b, \tag{5.87}$$

and we stress again that this derivation uses a genuine *calculus*, so that each step is well-defined.

We are now in a position to see how the Lagrangian (5.84) satisfies the criteria to be a "permitted" multivector Lagrangian. If B is an arbitrary bivector, then the scalar Lagrangian $\langle LB \rangle$ produces the equations of motion

$$\begin{aligned} \partial_{e_i}\langle LB \rangle - \partial_t(\partial_{\dot{e}_i}\langle LB \rangle) &= 0 \\ \Rightarrow \quad (\dot{e}_i + \epsilon_{ijk}\omega_j e_k) \cdot B &= 0. \end{aligned} \tag{5.88}$$

For this to be satisfied for all B, we simply require that the bracketed term vanishes. Hence equation (5.86) is indeed sufficient to ensure that each scalar component of L is capable of simultaneous extremisation. This example illustrates a further point. For a fixed B, equation (5.88) does not lead to the full equations of motion (5.86). It is only by allowing B to vary that we arrive at (5.86). It is therefore an essential feature of the formalism that L is a multivector, and that (5.88) holds for all B.

The equations of motion (5.86) can be written out in full to give

$$\begin{aligned} \dot{e}_1 &= -\omega_2 e_3 + \omega_3 e_2 \\ \dot{e}_2 &= -\omega_3 e_1 + \omega_1 e_3 \\ \dot{e}_3 &= -\omega_1 e_2 + \omega_2 e_1, \end{aligned} \tag{5.89}$$

which are a set of three coupled first-order vector equations. In terms of components, this gives nine scalar equations for nine unknowns, which illustrates how multivector Lagrangians have the potential to package up large numbers of equations into a single, highly compact entity. The equations (5.89) can be neatly combined into a single equation by introducing the reciprocal frame $\{e^i\}$ (1.132),

$$e^1 = e_2 \wedge e_3 E_n^{-1} \quad etc. \tag{5.90}$$

where

$$E_n \equiv e_1 \wedge e_2 \wedge e_3. \tag{5.91}$$

With this, the equations (5.89) become

$$\dot{e}_i = e^i \cdot \omega, \tag{5.92}$$

which shows that potentially interesting geometry underlies this system, relating the equations of motion of a frame to its reciprocal.

We now proceed to solve equation (5.92). On feeding (5.92) into (5.85), we find that

$$\dot{\omega} = 0, \tag{5.93}$$

so that the ω plane is constant. We next observe that (5.89) also imply

$$\dot{E}_n = 0, \tag{5.94}$$

which is important as it shows that, if the $\{e_i\}$ frame initially spans 3-dimensional space, then it will do so for all time. The constancy of E_n means that the reciprocal frame (5.90) satisfies

$$\dot{e}^1 = -\omega_2 e^3 + \omega_3 e^2 \quad etc. \tag{5.95}$$

We now introduce the symmetric metric tensor $\underline{g}$, defined by

$$\underline{g}(e^i) = e_i. \tag{5.96}$$

This defines the reciprocal bivector

$$\begin{aligned} \omega^* &\equiv \underline{g}^{-1}(\omega) \\ &= \omega_1(e^2 \wedge e^3) + \omega_2(e^3 \wedge e^1) + \omega_3(e^1 \wedge e^2), \end{aligned} \tag{5.97}$$

so that the reciprocal frame satisfies the equations

$$\dot{e}^i = e_i \cdot \omega^*. \tag{5.98}$$

But, from (1.123), we have that

$$e_i \cdot \omega^* = e_i \cdot \underline{g}^{-1}(\omega) = \underline{g}^{-1}(e^i \cdot \omega). \tag{5.99}$$

Now, using (5.92), (5.98) and (5.99), we find that

$$\underline{g}(\dot{e}^i) = e^i \cdot \omega = \dot{e}_i = \partial_t \underline{g}(e^i) \tag{5.100}$$

$$\Rightarrow \dot{\underline{g}} = 0. \tag{5.101}$$

Hence the metric tensor is constant, even though its matrix coefficients are varying. The variation of the coefficients of the metric tensor is therefore purely the result of the time variation of the frame, and is not a property of the frame-independent tensor. It follows that the fiducial tensor (1.144) is also constant, and suggests that we should look at the equations of motion for the fiducial frame $\sigma_i = \underline{h}^{-1}(e_i)$. For the $\{\sigma_i\}$ frame we find that

$$\begin{aligned} \dot{\sigma}_i &= \underline{h}^{-1}(\dot{e}_i) \\ &= \underline{h}^{-1}(\underline{h}^{-1}(\sigma_i)\cdot\omega) \\ &= \sigma_i\cdot\underline{h}^{-1}(\omega). \end{aligned} \tag{5.102}$$

If we define the bivector

$$\Omega = \underline{h}^{-1}(\omega) = \omega_1\sigma_2\sigma_3 + \omega_2\sigma_3\sigma_1 + \omega_3\sigma_1\sigma_2 \tag{5.103}$$

(which must be constant, since both $\underline{h}$ and ω are), we see that the fiducial frame satisfies the equation

$$\dot{\sigma}_i = \sigma_i\cdot\Omega. \tag{5.104}$$

The underlying fiducial frame simply rotates at a constant frequency in the Ω plane. If $\sigma_i(0)$ denotes the fiducial frame specified by the initial setup of the $\{e_i\}$ frame, then the solution to (5.104) is

$$\sigma_i(t) = e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}, \tag{5.105}$$

and the solution for the $\{e_i\}$ frame is

$$\begin{aligned} e_i(t) &= \underline{h}(e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}) \\ e^i(t) &= \underline{h}^{-1}(e^{-\Omega t/2}\sigma_i(0)e^{\Omega t/2}). \end{aligned} \tag{5.106}$$

Ultimately, the motion is that of an orthonormal frame viewed through a constant (symmetric) distortion. The $\{e_i\}$ frame and its reciprocal representing the same thing viewed through the distortion and its inverse. The system is perhaps not quite as interesting as one might have hoped, and it has not proved possible to identify the motion of (5.106) with any physical system, except in the simple case where $\underline{h} = I$. On the other hand, we did start with a very simple Lagrangian and it is reassuring to recover a rotating frame from an action that was motivated by the pseudoclassical mechanics of spin.

Some simple consequences follow immediately from the solution (5.106). Firstly, there is only one frequency in the system, ν say, which is found via

$$\begin{aligned} \nu^2 &= -\Omega^2 \\ &= {\omega_1}^2 + {\omega_2}^2 + {\omega_3}^2. \end{aligned} \tag{5.107}$$

Secondly, since

$$\Omega = i(\omega_1\sigma_1 + \omega_2\sigma_2 + \omega_3\sigma_3), \tag{5.108}$$

the vectors

$$u \equiv \omega_1 e_1 + \omega_2 e_2 + \omega_3 e_3, \tag{5.109}$$

and

$$u^* = \underline{g}^{-1}(u), \tag{5.110}$$

are conserved. This also follows from

$$u = -E_n\omega^* \tag{5.111}$$

$$u^* = E^n\omega. \tag{5.112}$$

Furthermore,

$$\begin{aligned} e_i e_i &= \underline{h}(\sigma_i)\underline{h}(\sigma_i) \\ &= \sigma_i \underline{g}(\sigma_i) \\ &= \mathrm{Tr}(\underline{g}) \end{aligned} \tag{5.113}$$

must also be time-independent (as can be verified directly from the equations of motion). The reciprocal quantity $e^i e^i = \mathrm{Tr}(\underline{g}^{-1})$ is also conserved. We thus have the set of four standard rotational invariants, $\sigma_i \sigma_i$, the axis, the plane of rotation and the volume scale-factor, each viewed through the pair of distortions $\underline{h}$, $\underline{h}^{-1}$. This gives the following set of 8 related conserved quantities:

$$\{e_i e_i,\, e^i e^i,\, u,\, u^*,\, \omega,\, \omega^*,\, E_n,\, E^n\}. \tag{5.114}$$

Lagrangian Symmetries and Conserved Quantities

We now turn to a discussion of the symmetries of (5.84). Although we have solved the equations of motion exactly, it is instructive to derive some of their consequences directly from the Lagrangian. We only consider continuous symmetries parameterised by a single scalar, so the appropriate form of Noether's theorem is equation (5.28), which takes the form

$$\partial_a L'|_{\alpha=0} = \partial_t \left(\tfrac{1}{2} e_i \wedge (\partial_a e_i')\right)\Big|_{\alpha=0}. \tag{5.115}$$

In writing this we are explicitly making use of the equations of motion and so are finding "on-shell" symmetries. The Lagrangian could be modified to extend these symmetries off-shell, but this will not be considered here.

We start with time translation. From (5.32), the Hamiltonian is

$$H = \tfrac{1}{2} e_i \wedge \dot{e}_i - L = \omega, \tag{5.116}$$

which is a constant bivector, as expected. The next symmetry to consider is a dilation,

$$e_i' = e^{\alpha} e_i. \tag{5.117}$$

For this transformation, equation (5.115) gives

$$2L = \partial_t \left(\tfrac{1}{2} e_i \wedge e_i\right) = 0, \tag{5.118}$$

so dilation symmetry shows that the Lagrangian vanishes along the classical path. This is quite common for first-order systems (the same is true of the Dirac Lagrangian), and is important in deriving other conserved quantities.

The final "classical" symmetry to consider is a rotation,

$$e_i' = e^{\alpha B/2} e_i e^{-\alpha B/2}. \tag{5.119}$$

Equation (5.115) now gives

$$B \times L = \partial_t \left(\tfrac{1}{2} e_i \wedge (B \cdot e_i)\right) \tag{5.120}$$

but, since $L = 0$ when the equations of motion are satisfied, the left hand side of (5.120) vanishes, and we find that the bivector $e_i \wedge (B \cdot e_i)$ in conserved. Had our Lagrangian been a scalar, we would have derived a scalar-valued function of B at this point, from which a single conserved bivector — the angular momentum — could be found. Here our Lagrangian is a bivector, so we find a conserved bivector-valued function of a bivector — a set of $3 \times 3 = 9$ conserved scalars. The quantity $e_i \wedge (B \cdot e_i)$ is a symmetric function of B, however, so this reduces to 6 independent conserved scalars. To see what these are we introduce the dual vector $b = iB$ and replace the conserved bivector $e_i \wedge (B \cdot e_i)$ by the equivalent vector-valued function,

$$\underline{f}(b) = e_i \cdot (b \wedge e_i) = e_i \cdot b e_i - b e_i e_i = \underline{g}(b) - b\mathrm{Tr}(\underline{g}). \tag{5.121}$$

This is conserved for all b, so we can contract with ∂_b and observe that $-2\mathrm{Tr}(\underline{g})$ is constant. It follows that $\underline{g}(b)$ is constant for all b, so rotational symmetry implies conservation of the metric tensor — a total of 6 quantities, as expected.

Now that we have derived conservation of $\underline{g}$ and ω, the remaining conserved quantities can be found. For example, $E_n = \det(\underline{g})^{1/2} i$ shows that E_n is constant. One interesting scalar-controlled symmetry remains, however, namely

$$e_i' = e_i + \alpha \omega_i a, \tag{5.122}$$

where a is an arbitrary constant vector. For this symmetry (5.115) gives

$$\tfrac{1}{2} a \wedge \dot{u} = \partial_t \left(\tfrac{1}{2} e_i \wedge (\omega_i a) \right) \tag{5.123}$$

$$\Rightarrow a \wedge \dot{u} = 0, \tag{5.124}$$

which holds for all a. Conservation of u therefore follows directly from the symmetry transformation (5.122). This symmetry transformation bears a striking resemblance to the transformation law for the fermionic sector of a supersymmetric theory [75]. Although the geometry behind (5.122) is not clear, it is interesting to note that the pseudoscalar transforms as

$$E_n' = E_n + \alpha a \wedge \omega, \tag{5.125}$$

and is therefore not invariant.

Poisson Brackets and the Hamiltonian Formalism

Many of the preceding results can also be derived from a Hamiltonian approach. As a by-product, this reveals a new and remarkably compact formula for a super-Lie bracket. We have already seen that the Hamiltonian for (5.84) is ω, so we start by looking at how the Poisson bracket is defined in pseudoclassical mechanics [39]. Dropping the j and adjusting a sign, the Poisson bracket is defined by

$$\{a(\zeta), b(\zeta)\}_{PB} = a \frac{\overleftarrow{\partial}}{\partial \zeta_k} \frac{\partial}{\partial \zeta_k} b. \tag{5.126}$$

The geometric algebra form of this is

$$\{A, B\}_{PB} = (A \cdot e^k) \wedge (e^k \cdot B), \tag{5.127}$$

where A and B are arbitrary multivectors. We will consider the consequences of this definition in arbitrary dimensions initially, before returning to the Lagrangian (5.84). Equation (5.127) can be simplified by utilising the fiducial tensor,

$$\begin{aligned}(A\cdot\underline{h}^{-1}(\sigma_k))\wedge(\underline{h}^{-1}(\sigma_k)\cdot B) &= \underline{h}[\underline{h}^{-1}(A)\cdot\sigma_k]\wedge\underline{h}[\sigma_k\cdot\underline{h}^{-1}(B)] \\ &= \underline{h}[(\underline{h}^{-1}(A)\cdot\sigma_k)\wedge(\sigma_k\cdot\underline{h}^{-1}(B))]. \end{aligned} \tag{5.128}$$

If we now assume that A and B are homogeneous, we can employ the rearrangement

$$\begin{aligned}(A_r\cdot\sigma_k)\wedge(\sigma_k\cdot B_s) &= \tfrac{1}{4}\langle(A_r\sigma_k-(-1)^r\sigma_kA_r)(\sigma_kB_s-(-1)^sB_s\sigma_k)\rangle_{r+s-2} \\ &= \tfrac{1}{4}\langle nA_rB_s-(n-2r)A_rB_s-(n-2s)A_rB_s \\ &\quad +[n-2(r+s-2)]A_rB_s\rangle_{r+s-2} \\ &= \langle A_rB_s\rangle_{r+s-2}\end{aligned} \tag{5.129}$$

to write the Poisson bracket as

$$\{A_r,B_s\}_{PB}=\underline{h}\langle\underline{h}^{-1}(A_r)\underline{h}^{-1}(B_s)\rangle_{r+s-2}. \tag{5.130}$$

This is a very neat representation of the super-Poisson bracket. The combination rule is simple, since the $\underline{h}$ always sits outside everything:

$$\{A_r,\{B_s,C_t\}_{PB}\}_{PB}=\underline{h}\left\langle\underline{h}^{-1}(A_r)\langle\underline{h}^{-1}(B_s)\underline{h}^{-1}(C_t)\rangle_{s+t-2}\right\rangle_{r+s+t-4}. \tag{5.131}$$

Clifford multiplication is associative and

$$\langle A_rB_s\rangle_{r+s-2}=-(-1)^{rs}\langle B_sA_r\rangle_{r+s-2}, \tag{5.132}$$

so the bracket (5.130) generates a super-Lie algebra. This follows from the well-known result [76] that a graded associative algebra satisfying the graded commutator relation (5.132) automatically satisfies the super-Jacobi identity. The bracket (5.130) therefore provides a wonderfully compact realisation of a super-Lie algebra. We saw in Chapter 3 that any Lie algebra can be represented by a bivector algebra under the commutator product. We now see that this is a special case of the more general class of algebras closed under the product (5.130). A subject for future research will be to use (5.130) to extend the techniques of Chapter 3 to include super-Lie algebras.

Returning to the system defined by the Lagrangian (5.84), we can now derive the equations of motion from the Poisson bracket as follows,

$$\begin{aligned}\dot{e}_i &= \{e_i,H\}_{PB} \\ &= \underline{h}(\sigma_i\cdot\Omega) \\ &= e^i\cdot\omega.\end{aligned} \tag{5.133}$$

It is intersting to note that, in the case where $\underline{h}=I$, time derivatives are determined by (one-half) the commutator with the (bivector) Hamiltonian. This suggests an interesting comparison with quantum mechanics, which has been developed in more detail elsewhere [1].

Similarly, some conservation laws can be derived, for example

$$\{E_n, H\}_{PB} = \underline{h}\langle i\Omega\rangle_3 = 0 \tag{5.134}$$

and

$$\{\omega, H\}_{PB} = \underline{h}\langle\Omega\Omega\rangle_2 = 0 \tag{5.135}$$

show that E_n and ω are conserved respectively. The bracket (5.130) gives zero for any scalar-valued functions, however, so is no help in deriving conservation of $e_i e_i$. Furthermore, the bracket only gives the correct equations of motion for the $\{e_i\}$ frame, since these are the genuine dynamical variables.

This concludes our discussion of pseudoclassical mechanics and multivector Lagrangians in general. Multivector Lagrangians have been shown to possess the capability to package up large numbers of variables in a single action principle, and it is to be hoped that further, more interesting applications can be found. Elsewhere [1], the concept of a bivector-valued action has been used to give a new formulation of the path integral for pseudoclassical mechanics. The path integrals considered involved genuine Riemann integrals in parameter space, though it has not yet proved possible to extend these integrals beyond two dimensions.

Chapter 6

Field Theory

We now extend the multivector derivative formalism of Chapter 5 to encompass field theory. The multivector derivative is seen to provide great formal clarity by allowing spinors and tensors to be treated in a unified way. The relevant form of Noether's theorem is derived and is used to find new conjugate currents in Dirac theory. The computational advantages of the multivector derivative formalism are further exemplified by derivations of the stress-energy and angular-momentum tensors for Maxwell and coupled Maxwell-Dirac theory. This approach provides a clear understanding of the role of antisymmetric terms in the stress-energy tensor, and the relation of these terms to spin. This chapter concludes with a discussion of how the formalism of multivector calculus is extended to incorporate differentiation with respect to a multilinear *function*. The results derived in this section are crucial to the development of an STA-based theory of gravity, given in Chapter 7. Many of the results obtained in this chapter appeared in the paper "*A multivector derivative approach to Lagrangian field theory*" [7].

Some additional notation is useful for expressions involving the vector derivative ∇. The left equivalent of ∇ is written as $\overleftarrow{\nabla}$ and acts on multivectors to its immediate left. (It is not always necessary to use $\overleftarrow{\nabla}$, as the overdot notation can be used to write $A\overleftarrow{\nabla}$ as $\dot{A}\dot{\nabla}$.) The operator $\overleftrightarrow{\nabla}$ acts both to its left and right, and is taken as acting on everything within a given expression, for example

$$A\overleftrightarrow{\nabla}B = \dot{A}\dot{\nabla}B + A\dot{\nabla}\dot{B}. \tag{6.1}$$

Transformations of spacetime position are written as

$$x' = f(x). \tag{6.2}$$

The differential of this is the linear function

$$\underline{f}(a) = a\cdot\nabla f(x) = \underline{f}_x(a), \tag{6.3}$$

where the subscript labels the position dependence. A useful result for vector derivatives is that

$$\begin{aligned}
\nabla_x &= \partial_a a\cdot\nabla_x \\
&= \partial_a (a\cdot\nabla_x x')\cdot\nabla_{x'} \\
&= \partial_a \underline{f}(a)\cdot\nabla_{x'} \\
&= \overline{f}_x(\nabla_{x'}).
\end{aligned} \tag{6.4}$$

6.1 The Field Equations and Noether's Theorem

In what follows, we restrict attention to the application of multivector calculus to relativistic field theory. The results are easily extended to include the non-relativistic case. Furthermore, we are only concerned with scalar-valued Lagrangian densities. It has not yet proved possible to construct a multivector-valued field Lagrangian with interesting properties.

We start with a scalar-valued Lagrangian density

$$\mathcal{L} = \mathcal{L}(\psi_i, a\cdot\nabla\psi_i), \tag{6.5}$$

where $\{\psi_i\}$ are a set of multivector fields. The Lagrangian (6.5) is a functional of ψ_i and the directional derivatives of ψ_i. In many cases it is possible to write $\mathcal{L}$ as a functional of ψ and $\nabla\psi$, and this approach was adopted in [7]. Our main application will be to gravity, however, and there we need the more general form of (6.5).

The action is defined as

$$S = \int |d^4x|\, \mathcal{L}, \tag{6.6}$$

where $|d^4x|$ is the invariant measure. Proceeding as in Chapter 5, we write

$$\psi_i(x) = \psi_i^0(x) + \epsilon\phi_i(x), \tag{6.7}$$

where ϕ_i contains the same grades as ψ_i, and ψ_i^0 is the extremal path. Differentiating, and using the chain rule, we find that

$$\begin{aligned} \partial_\epsilon S &= \int |d^4x|\, [(\partial_\epsilon\psi_i)*\partial_{\psi_i}\mathcal{L} + (\partial_\epsilon\psi_{i,\mu})*\partial_{\psi_{i,\mu}}\mathcal{L}] \\ &= \int |d^4x|\, [\phi_i*\partial_{\psi_i}\mathcal{L} + (\phi_{i,\mu})*\partial_{\psi_{i,\mu}}\mathcal{L}]. \end{aligned} \tag{6.8}$$

Here, a fixed frame $\{\gamma^\mu\}$ has been introduced, determining a set of coordinates $x^\mu \equiv \gamma^\mu\cdot x$. The derivative of ψ_i with respect to x^μ is denoted as $\psi_{i,\mu}$. The multivector derivative $\partial_{\psi_{i,\mu}}$ is defined in the same way as ∂_{ψ_i}. The frame can be eliminated in favour of the multivector derivative by defining

$$\partial_{\psi_{i,a}} \equiv a_\mu \partial_{\psi_{i,\mu}}, \tag{6.9}$$

where $a_\mu = \gamma_\mu\cdot a$, and writing

$$\partial_\epsilon S = \int |d^4x|\, [\phi_i*\partial_{\psi_i}\mathcal{L} + (\partial_a\cdot\nabla\phi_i)*\partial_{\psi_{i,a}}\mathcal{L}]. \tag{6.10}$$

It is now possible to perform all manipulations without introducing a frame. This ensures that Lorentz invariance is manifest throughout the derivation of the field equations.

Assuming that the boundary term vanishes, we obtain

$$\partial_\epsilon S = \int |d^4x|\, \phi_i*[\partial_{\psi_i}\mathcal{L} - \partial_a\cdot\nabla(\partial_{\psi_{i,a}}\mathcal{L})]. \tag{6.11}$$

Setting $\epsilon = 0$, so that the ψ_i takes their extremal values, we find that the extremal path is defined by the solutions of the Euler-Lagrange equations

$$\partial_{\psi_i}\mathcal{L} - \partial_a \cdot \nabla(\partial_{\psi_{i,a}}\mathcal{L}) = 0. \tag{6.12}$$

The multivector derivative allows for vectors, tensors and spinor variables to be handled in a single equation — a useful and powerful unification.

Noether's theorem for field Lagrangians is also be derived in the same manner as in Chapter 5. We begin by considering a general multivector-parameterised transformation,

$$\psi_i' = f(\psi_i, M), \tag{6.13}$$

where f and M are position-independent functions and multivectors respectively. With $\mathcal{L}' \equiv \mathcal{L}(\psi_i', a\cdot\nabla\psi_i')$, we find that

$$\begin{aligned} A*\partial_M\mathcal{L}' &= \underline{f}_M(\psi_i, A)*\partial_{\psi_i'}\mathcal{L}' + \underline{f}_M(\partial_a\cdot\nabla\psi_i, A)*\partial_{\psi'_{i,a}}\mathcal{L}' \\ &= \nabla\cdot[\partial_a\underline{f}_M(\psi_i, A)*\partial_{\psi'_{i,a}}\mathcal{L}'] + \underline{f}_M(\psi_i, A)*[\partial_{\psi_i'}\mathcal{L}' - \partial_a\cdot\nabla(\partial_{\psi'_{i,a}}\mathcal{L}')]. \end{aligned} \tag{6.14}$$

If we now assume that the ψ_i' satisfy the same field equations as the ψ_i (which must again be verified) then we find that

$$\partial_M\mathcal{L}' = \partial_A\nabla\cdot[\partial_a\underline{f}_M(\psi_i, A)*\partial_{\psi'_{i,a}}\mathcal{L}']. \tag{6.15}$$

This is a very general result, applying even when ψ_i' is evaluated at a different spacetime point from ψ_i,

$$\psi_i'(x) = f[\psi_i(h(x)), M]. \tag{6.16}$$

By restricting attention to a scalar-parameterised transformation, we can write

$$\partial_\alpha\mathcal{L}'|_{\alpha=0} = \nabla\cdot[\partial_b(\partial_\alpha\psi_i')*\partial_{\psi_{i,b}}\mathcal{L}]\Big|_{\alpha=0}, \tag{6.17}$$

which holds provided that the ψ_i satisfy the field equations (6.12) and the transformation is such that $\psi_i'(\alpha=0) = \psi_i$. Equation (6.17) turns out, in practice, to be the most useful form of Noether's theorem.

From (6.17) we define the conjugate current

$$j = \partial_b\,(\partial_\alpha\psi_i'|_{\alpha=0})*\partial_{\psi_{i,b}}\mathcal{L}. \tag{6.18}$$

If $\mathcal{L}'$ is independent of α, j satisfies the conservation equation

$$\nabla\cdot j = 0. \tag{6.19}$$

An inertial frame relative to the constant time-like velocity γ_0 then sees the charge

$$Q = \int |d^3x|\, j\cdot\gamma_0 \tag{6.20}$$

as conserved with respect to its local time.

6.2 Spacetime Transformations and their Conjugate Tensors

In this section we use Noether's theorem in the form (6.17) to analyse the consequences of Poincaré and conformal invariance. These symmetries lead to the identification of stress-energy and angular-momentum tensors, which are used in the applications to follow.

1. Translations

A translation of the spacetime fields ψ_i is achieved by defining new spacetime fields ψ_i' by

$$\psi_i'(x) = \psi_i(x'), \tag{6.21}$$

where

$$x' = x + \alpha n. \tag{6.22}$$

Assuming that $\mathcal{L}'$ is only x-dependent through the fields $\psi_i(x)$, equation (6.17) gives

$$n\cdot\nabla\mathcal{L} = \nabla\cdot[\partial_a(n\cdot\nabla\psi_i)*\partial_{\psi_{i,a}}\mathcal{L}] \tag{6.23}$$

and from this we define the canonical stress-energy tensor by

$$\underline{T}(n) = \partial_a(n\cdot\nabla\psi_i)*\partial_{\psi_{i,a}}\mathcal{L} - n\mathcal{L}. \tag{6.24}$$

The function $\underline{T}(n)$ is linear on n and, from (6.23), $\underline{T}(n)$ satisfies

$$\nabla\cdot\underline{T}(n) = 0. \tag{6.25}$$

To write down a conserved quantity from $\underline{T}(n)$ it is useful to first construct the adjoint function

$$\begin{aligned}
\overline{T}(n) &= \partial_b\langle n\underline{T}(b)\rangle \\
&= \partial_b\langle n\cdot\partial_a(b\cdot\nabla\psi_i)*\partial_{\psi_{i,a}}\mathcal{L} - n\cdot b\mathcal{L}\rangle \\
&= \dot{\nabla}\langle\dot{\psi}_i\partial_{\psi_{i,n}}\mathcal{L}\rangle - n\mathcal{L}.
\end{aligned} \tag{6.26}$$

This satisfies the conservation equation

$$\dot{\overline{T}}(\dot{\nabla}) = 0, \tag{6.27}$$

which is the adjoint to (6.25). In the γ_0 frame the field momentum is defined by

$$p = \int |d^3x|\,\overline{T}(\gamma_0) \tag{6.28}$$

and, provided that the fields all fall off suitably at large distances, the momentum p is conserved with respect to local time. This follows from

$$\begin{aligned}
\gamma_0\cdot\nabla p &= \int |d^3x|\,\dot{\overline{T}}(\gamma_0\gamma_0\cdot\dot{\nabla}) \\
&= -\int |d^3x|\,\dot{\overline{T}}(\gamma_0\gamma_0\wedge\dot{\nabla}) \\
&= 0.
\end{aligned} \tag{6.29}$$

The total field energy, as seen in the γ_0 frame, is

$$E = \int |d^3x|\gamma_0\cdot\overline{T}(\gamma_0). \tag{6.30}$$

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&= 0.
\end{aligned} \tag{6.29}$$

The total field energy, as seen in the γ_0 frame, is

$$E = \int |d^3x|\gamma_0\cdot\overline{T}(\gamma_0). \tag{6.30}$$

2. Rotations

If we assume initially that all fields ψ_i transform as spacetime vectors, then a rotation of these fields from one spacetime point to another is performed by

$$\psi_i'(x) = e^{\alpha B/2}\psi_i(x')e^{-\alpha B/2}, \tag{6.31}$$

where

$$x' = e^{-\alpha B/2}xe^{\alpha B/2}. \tag{6.32}$$

This differs from the point-particle transformation law (5.53) in the relative directions of the rotations for the position vector x and the fields ψ_i. The result of this difference is a change in the relative sign of the spin contribution to the total angular momentum. In order to apply Noether's theorem (6.17), we use

$$\partial_\alpha \psi_i'|_{\alpha=0} = B\times\psi_i - (B\cdot x)\cdot\nabla\psi_i \tag{6.33}$$

and

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = -(B\cdot x)\cdot\nabla\mathcal{L} = \nabla\cdot(x\cdot B\mathcal{L}). \tag{6.34}$$

Together, these yield the conjugate vector

$$\underline{J}(B) = \partial_a[B\times\psi_i - (B\cdot x)\cdot\nabla\psi_i]*\partial_{\psi_{i,a}}\mathcal{L} + B\cdot x\mathcal{L}, \tag{6.35}$$

which satisfies

$$\nabla\cdot\underline{J}(B) = 0. \tag{6.36}$$

The adjoint to the conservation equation (6.36) is

$$\begin{aligned} \dot{\overline{J}}(\dot{\nabla})\cdot B &= 0 \quad \text{for all } B \\ \Rightarrow \quad \dot{\overline{J}}(\dot{\nabla}) &= 0. \end{aligned} \tag{6.37}$$

The adjoint function $\overline{J}(n)$ is a position-dependent bivector-valued linear function of the vector n. We identify this as the canonical angular-momentum tensor. A conserved bivector in the γ_0-system is constructed in the same manner as for $\overline{T}(n)$ (6.28). The calculation of the adjoint function $\overline{J}(n)$ is performed as follows:

$$\begin{aligned} \overline{J}(n) &= \partial_B\langle \underline{J}(B)n\rangle \\ &= \partial_B\langle (B\times\psi_i - B\cdot(x\wedge\nabla)\psi_i)*\partial_{\psi_{i,n}}\mathcal{L} + B\cdot x\mathcal{L}n\rangle \\ &= -x\wedge[\dot{\nabla}\dot{\psi}_i*\partial_{\psi_{i,n}}\mathcal{L} - n\mathcal{L}] + \langle \psi_i\times\partial_{\psi_{i,n}}\mathcal{L}\rangle_2 \\ &= \overline{T}(n)\wedge x + \langle \psi_i\times\partial_{\psi_{i,n}}\mathcal{L}\rangle_2. \end{aligned} \tag{6.38}$$

If one of the fields ψ, say, transforms single-sidedly (as a spinor), then $\overline{J}(n)$ contains the term $\langle \frac{1}{2}\psi\partial_{\psi,n}\mathcal{L}\rangle_2$.

The first term in $\overline{J}(n)$ (6.38) is the routine $p\wedge x$ component, and the second term is due to the spin of the field. The general form of $\overline{J}(n)$ is therefore

$$\overline{J}(n) = \overline{T}(n)\wedge x + S(n). \tag{6.39}$$

By applying (6.37) to (6.39) and using (6.27), we find that

$$\overline{T}(\dot{\nabla})\wedge\dot{x} + \dot{S}(\dot{\nabla}) = 0. \tag{6.40}$$

The first term in (6.40) returns (twice) the characteristic bivector of $\overline{T}(n)$. Since the antisymmetric part of $\overline{T}(n)$ can be written in terms of the characteristic bivector B as

$$\overline{T}_{-}(a) = \tfrac{1}{2}B\cdot a, \tag{6.41}$$

equation (6.40) becomes

$$B = -\dot{S}(\dot{\nabla}). \tag{6.42}$$

It follows that, in any Poincaré-invariant theory, the antisymmetric part of the stress-energy tensor is a total divergence. But, whilst $T_{-}(n)$ is a total divergence, $x\wedge T_{-}(n)$ certainly is not. So, in order for (6.37) to hold, the antisymmetric part of $T(n)$ must be retained since it cancels the divergence of the spin term.

3. Dilations

While all fundamental theories should be Poincaré-invariant, an interesting class go beyond this and are invariant under conformal transformations. The conformal group contains two further symmetry transformations, dilations and special conformal transformations. Dilations are considered here, and the results below are appropriate to any scale-invariant theory.

A dilation of the spacetime fields is achieved by defining

$$\psi_i'(x) = e^{d_i\alpha}\psi_i(x') \tag{6.43}$$

where

$$x' = e^{\alpha}x \tag{6.44}$$

$$\Rightarrow \quad \nabla\psi_i'(x) = e^{(d_i+1)\alpha}\nabla_{x'}\psi_i(x'). \tag{6.45}$$

If the theory is scale-invariant, it is possible to assign the "conformal weights" d_i in such a way that the left-hand side of (6.17) becomes

$$\partial_\alpha\mathcal{L}'|_{\alpha=0} = \nabla\cdot(x\mathcal{L}). \tag{6.46}$$

In this case, equation (6.17) takes the form

$$\nabla\cdot(x\mathcal{L}) = \nabla\cdot[\partial_a(d_i\psi_i + x\cdot\nabla\psi_i)*\partial_{\psi_{i,a}}\mathcal{L}], \tag{6.47}$$

from which the conserved current

$$j = d_i\partial_a\psi_i*\partial_{\psi_{i,a}}\mathcal{L} + \underline{T}(x) \tag{6.48}$$

is read off. Conservation of j (6.48) implies that

$$\nabla\cdot\underline{T}(x) = \partial_a\underline{T}(a) = -\nabla\cdot(d_i\partial_a\psi_i*\partial_{\psi_{i,a}}\mathcal{L}) \tag{6.49}$$

so, in a scale-invariant theory, the trace of the canonical stress-energy tensor is a total divergence. By using the equations of motion, equation (6.49) can be written, in four dimensions, as

$$d_i\langle\psi_i\partial_{\psi_i}\mathcal{L}\rangle + (d_i+1)(\partial_a\cdot\nabla\psi_i)*\partial_{\psi_{i,a}}\mathcal{L} = 4\mathcal{L}, \tag{6.50}$$

which can be taken as an alternative definition for a scale-invariant theory.

4. Inversions

The remaining generator of the conformal group is inversion,

$$x' = x^{-1}. \tag{6.51}$$

As it stands, this is not parameterised by a scalar and so cannot be applied to (6.17). In order to derive a conserved tensor, the inversion (6.51) is combined with a translation to define the special conformal transformation [77]

$$x' = h(x) \equiv (x^{-1} + \alpha n)^{-1} = x(1 + \alpha n x)^{-1}. \tag{6.52}$$

From this definition, the differential of $h(x)$ is given by

$$\underline{h}(a) = a\cdot\nabla h(x) = (1 + \alpha x n)^{-1} a (1 + \alpha n x)^{-1} \tag{6.53}$$

so that $\underline{h}$ defines a spacetime-dependent rotation/dilation. It follows that $\underline{h}$ satisfies

$$\underline{h}(a)\cdot\underline{h}(b) = \lambda(x) a\cdot b \tag{6.54}$$

where

$$\lambda(x) = (1 + 2\alpha n\cdot x + \alpha^2 x^2 n^2)^{-2}. \tag{6.55}$$

That the function $\underline{h}(a)$ satisfies equation (6.54) demonstrates that it belongs to the conformal group.

The form of $\underline{h}(a)$ (6.53) is used to postulate transformation laws for all fields (including spinors, which transform single-sidedly) such that

$$\mathcal{L}' = (\det \underline{h})\mathcal{L}(\psi_i(x'), \underline{h}(a)\cdot\nabla_{x'}\psi_i(x')), \tag{6.56}$$

which implies that

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = \partial_\alpha \det \underline{h}|_{\alpha=0}\, \mathcal{L} + (\partial_\alpha x'|_{\alpha=0})\cdot\nabla\mathcal{L}. \tag{6.57}$$

Since

$$\det \underline{h} = (1 + 2\alpha n\cdot x + \alpha^2 x^2 n^2)^{-4}, \tag{6.58}$$

it follows that

$$\partial_\alpha \det \underline{h}|_{\alpha=0} = -8x\cdot n. \tag{6.59}$$

We also find that

$$\partial_\alpha x'|_{\alpha=0} = -(xnx), \tag{6.60}$$

and these results combine to give

$$\partial_\alpha \mathcal{L}'|_{\alpha=0} = -8x\cdot n\mathcal{L} - (xnx)\cdot\nabla\mathcal{L} = -\nabla\cdot(xnx\mathcal{L}). \tag{6.61}$$

Special conformal transformations therefore lead to a conserved tensor of the form

$$\begin{aligned} \underline{T}_{SC}(n) &= \partial_a\langle(-(xnx)\cdot\nabla\psi_i + \partial_\alpha\psi_i'(x))*\partial_{\psi_{i,a}}\mathcal{L} + xnx\mathcal{L}\rangle_{\alpha=0} \\ &= -\underline{T}(xnx) + \partial_a\langle(\partial_\alpha\psi_i'(x))*\partial_{\psi_{i,a}}\mathcal{L}\rangle_{\alpha=0}. \end{aligned} \tag{6.62}$$

The essential quantity here is the vector $-xnx$, which is obtained by taking the constant vector n and reflecting it in the hyperplane perpendicular to the chord joining the point where n is evaluated to the origin. The resultant vector is then scaled by a factor of x^2.

In a conformally-invariant theory, both the antisymmetric part of $\underline{T}(n)$ and its trace are total divergences. These can therefore be removed to leave a new tensor $T'(n)$ which is symmetric and traceless. The complete set of divergenceless tensors is then given by

$$\{T'(x),\, T'(n),\, xT'(n)x,\, J'(n) \equiv T'(n)\wedge x\} \tag{6.63}$$

This yields a set of $1+4+4+6=15$ conserved quantities — the dimension of the conformal group. All this is well known, of course, but it is the first time that geometric algebra has been systematically applied to this problem. It is therefore instructive to see how geometric algebra is able to simplify many of the derivations, and to generate a clearer understanding of the results.

6.3 Applications

We now look at a number of applications of the formalism established in the preceding sections. We start by illustrating the techniques with the example of electromagnetism. This does not produce any new results, but does lead directly to the STA forms of the Maxwell equations and the electromagnetic stress-energy tensor. We next consider Dirac theory, and a number of new conjugate currents will be identified. A study of coupled Maxwell-Dirac theory then provides a useful analogue for the discussion of certain aspects of a gauge theory of gravity, as described in Chapter 7. The final application is to a two-particle action which recovers the field equations discussed in Section 4.4.

The essential result needed for what follows is

$$\begin{aligned}\partial_{\psi,a}\langle b\cdot\nabla\psi M\rangle &= a_\mu\partial_{\psi,\mu}\langle(b^\nu\psi_{,\nu}M\rangle \\ &= a\cdot bP_\psi(M)\end{aligned} \tag{6.64}$$

where $P_\psi(M)$ is the projection of M onto the grades contained in ψ. It is the result (6.64) that enables all calculations to be performed without the introduction of a frame. It is often the case that the Lagrangian can be written in the form $\mathcal{L}(\psi_i, \nabla\psi_i)$, when the following result is useful:

$$\begin{aligned}\partial_{\psi,a}\langle\nabla\psi M\rangle &= \partial_{\psi,a}\langle b\cdot\nabla\psi M\partial_b\rangle \\ &= a\cdot bP_\psi(M\partial_b) \\ &= P_\psi(Ma).\end{aligned} \tag{6.65}$$

1. Electromagnetism

The electromagnetic Lagrangian density is given by

$$\mathcal{L} = -A\cdot J + \tfrac{1}{2}F\cdot F, \tag{6.66}$$

where A is the vector potential, $F = \nabla \wedge A$, and A couples to an external current J which is not varied. To find the equations of motion we first write $F\cdot F$ as a function of ∇A,

$$\begin{aligned} F\cdot F &= \tfrac{1}{4}\langle(\nabla A - (\nabla A)\tilde{\ })^2\rangle \\ &= \tfrac{1}{2}\langle\nabla A \nabla A - \nabla A(\nabla A)\tilde{\ }\rangle. \end{aligned} \tag{6.67}$$

The field equations therefore take the form

$$\begin{aligned} -J - \partial_b\cdot\nabla\tfrac{1}{2}\langle\nabla A b - (\nabla A)\tilde{\ } b\rangle_1 &= 0 \\ \Rightarrow \quad -J - \partial_b\cdot\nabla F\cdot b &= 0 \\ \Rightarrow \quad \nabla\cdot F &= J. \end{aligned} \tag{6.68}$$

This is combined with the identity $\nabla\wedge F = 0$ to yield the full set of Maxwell's equations, $\nabla F = J$.

To calculate the free-field stress-energy tensor, we set $J = 0$ in (6.66) and work with

$$\mathcal{L}_0 = \tfrac{1}{2}\langle F^2\rangle. \tag{6.69}$$

Equation (6.26) now gives the stress-energy tensor in the form

$$\overline{T}(n) = \dot{\nabla}\langle\dot{A}F\cdot n\rangle - \tfrac{1}{2}n\langle F^2\rangle. \tag{6.70}$$

This expression is physically unsatisfactory as is stands, because it is not gauge-invariant. In order to find a gauge-invariant form of (6.70), we write [60]

$$\begin{aligned} \dot{\nabla}\langle\dot{A}F\cdot n\rangle &= (\nabla\wedge A)\cdot(F\cdot n) + (F\cdot n)\cdot\nabla A \\ &= F\cdot(F\cdot n) - (F\cdot\dot{\nabla})\cdot n\dot{A} \end{aligned} \tag{6.71}$$

and observe that, since $\nabla\cdot F = 0$, the second term is a total divergence and can therefore be ignored. What remains is

$$\begin{aligned} \overline{T}_{\mathrm{em}}(n) &= F\cdot(F\cdot n) - \tfrac{1}{2}nF\cdot F \\ &= \tfrac{1}{2}Fn\tilde{F}, \end{aligned} \tag{6.72}$$

which is the form of the electromagnetic stress-energy tensor obtained by Hestenes [17]. The tensor (6.72) is gauge-invariant, traceless and symmetric. The latter two properties follow simultaneously from the identity

$$\partial_a\overline{T}_{\mathrm{em}}(a) = \partial_a\tfrac{1}{2}Fa\tilde{F} = 0. \tag{6.73}$$

The angular momentum is obtained from (6.38), which yields

$$\overline{J}(n) = (\dot{\nabla}\langle\dot{A}Fn\rangle - \tfrac{1}{2}n\langle F^2\rangle)\wedge x + A\wedge(F\cdot n), \tag{6.74}$$

where we have used the stress-energy tensor in the form (6.70). This expression suffers from the same lack of gauge invariance, and is fixed up in the same way, using (6.71) and

$$-(F\cdot n)\wedge A + x\wedge[(F\cdot\dot{\nabla})\cdot n\dot{A}] = x\wedge[(F\cdot\overleftrightarrow{\nabla})\cdot nA], \tag{6.75}$$

which is a total divergence. This leaves simply

$$\overline{J}(n) = \overline{T}_{\mathrm{em}}(n)\wedge x. \tag{6.76}$$

By redefining the stress-energy tensor to be symmetric, the spin contribution to the angular momentum is absorbed into (6.72). For the case of electromagnetism this has the advantage that gauge invariance is manifest, but it also suppresses the spin-1 nature of the field. Suppressing the spin term in this manner is not always desirable, as we shall see with the Dirac equation.

The free-field Lagrangian (6.69) is not only Poincaré-invariant; it is invariant under the full conformal group of spacetime [7, 77]. The full set of divergenceless tensors for free-field electromagnetism is therefore $\overline{T}_{\mathrm{em}}(x)$, $\overline{T}_{\mathrm{em}}(n)$, $x\overline{T}_{\mathrm{em}}(n)x$, and $\overline{T}_{\mathrm{em}}(n)\wedge x$. It is a simple matter to calculate the modified conservation equations when a current is present.

2. Dirac Theory[1]

The multivector derivative is particularly powerful when applied to the STA form of the Dirac Lagrangian. We recall from Chapter 5 that the Lagrangian for the Dirac equation can be written as (4.96)

$$\mathcal{L} = \langle \nabla\psi i\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{6.77}$$

where ψ is an even multivector and A is an external gauge field (which is not varied). To verify that (6.77) does give the Dirac equation we use the Euler-Lagrange equations in the form

$$\partial_\psi\mathcal{L} = \partial_a\cdot\nabla(\partial_{\psi,a}\mathcal{L}) \tag{6.78}$$

to obtain

$$\begin{aligned}(\nabla\psi i\gamma_3)\tilde{} - 2e\gamma_0\tilde{\psi}A - 2m\tilde{\psi} &= \partial_a\cdot\nabla(i\gamma_3\tilde{\psi}a) \\ &= i\gamma_3\tilde{\psi}\overleftarrow{\nabla}.\end{aligned} \tag{6.79}$$

Reversing this equation, and postmultiplying by γ_0, we obtain

$$\nabla\psi i\sigma_3 - eA\psi = m\psi\gamma_0, \tag{6.80}$$

as found in Chapter 4 (4.92). Again, it is worth stressing that this derivation employs a genuine calculus, and does not resort to treating ψ and $\tilde{\psi}$ as independent variables.

We now analyse the Dirac equation from the viewpoint of the Lagrangian (6.77). In this Section we only consider position-independent transformations of the spinor ψ. Spacetime transformations are studied in the following section. The transformations we are interested in are of the type

$$\psi' = \psi e^{\alpha M}, \tag{6.81}$$

where M is a general multivector and α and M are independent of position. Operations on the right of ψ arise naturally in the STA formulation of Dirac theory, and can be

[1]The basic idea developed in this section was provided by Anthony Lasenby.

thought of as generalised gauge transformations. They have no such simple analogue in the standard column-spinor treatment. Applying (6.17) to (6.81), we obtain

$$\nabla\cdot\langle\psi M i\gamma_3\tilde{\psi}\rangle_1 = \partial_\alpha \mathcal{L}'|_{\alpha=0}, \tag{6.82}$$

which is a result that we shall exploit by substituting various quantities for M. If M is odd both sides vanish identically, so useful information is only obtained when M is even. The first even M to consider is a scalar, λ, so that $\langle\psi M i\gamma_3\tilde{\psi}\rangle_1$ is zero. It follows that

$$\begin{aligned} \partial_\alpha\left(e^{2\alpha\lambda}\mathcal{L}\right)\Big|_{\alpha=0} &= 0 \\ \Rightarrow \quad \mathcal{L} &= 0, \end{aligned} \tag{6.83}$$

and hence that, when the equations of motion are satisfied, the Dirac Lagrangian vanishes.

Next, setting $M = i$, equation (6.82) gives

$$\begin{aligned} -\nabla\cdot(\rho s) &= -m\partial_\alpha\langle e^{2i\alpha}\rho e^{i\beta}\rangle\Big|_{\alpha=0}, \\ \Rightarrow \nabla\cdot(\rho s) &= -2m\rho\sin\beta, \end{aligned} \tag{6.84}$$

where $\rho s = \psi\gamma_3\tilde{\psi}$ is the spin current. This equation is well-known [33], though it is not usually observed that the spin current is the current conjugate to duality rotations. In conventional versions, these would be called "axial rotations", with the role of i taken by γ_5. In the STA approach, however, these rotations are identical to duality transformations for the electromagnetic field. The duality transformation generated by $e^{i\alpha}$ is also the continuous analogue of the discrete symmetry of mass conjugation, since $\psi \mapsto \psi i$ changes the sign of the mass term in $\mathcal{L}$. It is no surprise, therefore, that the conjugate current, ρs, is conserved for massless particles.

Finally, taking M to be an arbitrary bivector B yields

$$\begin{aligned} \nabla\cdot(\psi B\cdot(i\gamma_3)\tilde{\psi}) &= 2\langle\nabla\psi i B\cdot\gamma_3\tilde{\psi} - eA\psi B\cdot\gamma_0\tilde{\psi}\rangle \\ &= \left\langle eA\psi(\sigma_3 B\sigma_3 - B)\gamma_0\tilde{\psi}\right\rangle, \end{aligned} \tag{6.85}$$

where the Dirac equation (6.80) has beed used. Both sides of(6.85) vanish for $B = i\sigma_1, i\sigma_2$ and σ_3, with useful equations arising on taking $B = \sigma_1, \sigma_2$ and $i\sigma_3$. The last of these, $B = i\sigma_3$, corresponds to the usual $U(1)$ gauge transformation of the spinor field, and gives

$$\nabla\cdot J = 0, \tag{6.86}$$

where $J = \psi\gamma_0\tilde{\psi}$ is the current conjugate to phase transformations, and is strictly conserved. The remaining transformations generated by $e^{\alpha\sigma_1}$ and $e^{\alpha\sigma_2}$ give

$$\begin{aligned} \nabla\cdot(\rho e_1) &= 2e\rho A\cdot e_2 \\ \nabla\cdot(\rho e_2) &= -2e\rho A\cdot e_1, \end{aligned} \tag{6.87}$$

where $\rho e_\mu = \psi\gamma_\mu\tilde{\psi}$. Although these equations have been found before [33], the role of ρe_1 and ρe_2 as currents conjugate to right-sided $e^{\alpha\sigma_2}$ and $e^{\alpha\sigma_1}$ transformations has not been noted. Right multiplication by σ_1 and σ_2 generates charge conjugation, since the transformation $\psi \mapsto \psi' \equiv \psi\sigma_1$ takes (6.80) into

$$\nabla\psi' i\sigma_3 + eA\psi' = m\psi'\gamma_0. \tag{6.88}$$

It follows that the conjugate currents are conserved exactly if the external potential vanishes, or the particle has zero charge.

3. Spacetime Transformations in Maxwell-Dirac Theory

The canonical stress-energy and angular-momentum tensors are derived from spacetime symmetries. In considering these it is useful to work with the full coupled Maxwell-Dirac Lagrangian, in which the free-field term for the electromagnetic field is also included. This ensures that the Lagrangian is Poincaré-invariant. The full Lagrangian is therefore

$$\mathcal{L} = \langle \nabla\psi i\gamma_3\tilde{\psi} - eA\psi\gamma_0\tilde{\psi} - m\psi\tilde{\psi} + \tfrac{1}{2}F^2\rangle, \tag{6.89}$$

in which both ψ and A are dynamical variables.

From the definition of the stress-energy tensor (6.26) and the fact that the Dirac part of the Lagrangian vanishes when the field equations are satisfied (6.83), $\overline{T}(n)$ is given by

$$\overline{T}(n) = \dot{\nabla}\langle\dot{\psi} i\gamma_3\tilde{\psi}n\rangle + \dot{\nabla}\langle\dot{A}Fn\rangle - \tfrac{1}{2}nF\cdot F. \tag{6.90}$$

Again, this is not gauge-invariant and a total divergence must be removed to recover a gauge-invariant tensor. The manipulations are as at (6.71), and now yield

$$\overline{T}_{\mathrm{md}}(n) = \dot{\nabla}\langle\dot{\psi} i\gamma_3\tilde{\psi}n\rangle - n\cdot JA + \tfrac{1}{2}\tilde{F}nF, \tag{6.91}$$

where $J = \psi\gamma_0\tilde{\psi}$. The tensor (6.91) is now gauge-invariant, and conservation can be checked simply from the field equations. The first and last terms are the free-field stress-energy tensors and the middle term, $-n\cdot JA$, arises from the coupling. The stress-energy tensor for the Dirac theory in the presence of an external field A is conventionally defined by the first two terms of (6.91), since the combination of these is gauge-invariant.

Only the free-field electromagnetic contribution to $\overline{T}_{\mathrm{md}}$ (6.91) is symmetric; the other terms each contain antisymmetric parts. The overall antisymmetric contribution is

$$\begin{aligned} T_-(n) &= \tfrac{1}{2}[\overline{T}_{\mathrm{md}}(n) - \underline{T}_{\mathrm{md}}(n)] \\ &= \tfrac{1}{2}n\cdot[A\wedge J - \dot{\nabla}\wedge\langle\dot{\psi} i\gamma_3\tilde{\psi}\rangle_1] \\ &= \tfrac{1}{2}n\cdot\langle AJ - \nabla\psi i\gamma_3\tilde{\psi} + \dot{\nabla}\langle\dot{\psi} i\gamma_3\tilde{\psi}\rangle_3\rangle_2 \\ &= n\cdot(\nabla\cdot(\tfrac{1}{4}i\rho s)) \\ &= n\cdot(-i\nabla\wedge(\tfrac{1}{4}\rho s)), \end{aligned} \tag{6.92}$$

and is therefore completely determined by the exterior derivative of the spin current [78].

The angular momentum is found from (6.39) and, once the total divergence is removed, the gauge-invariant form is

$$\overline{J}(n) = \overline{T}_{\mathrm{md}}(n)\wedge x + \tfrac{1}{2}i\rho s\wedge n. \tag{6.93}$$

The ease of derivation of $\overline{J}(n)$ (6.93) compares favourably with traditional operator-based approaches [60]. It is crucial to the identification of the spin contribution to the angular momentum that the antisymmetric component of $\overline{T}_{\mathrm{md}}(n)$ is retained. In (6.93) the spin term is determined by the trivector is, and the fact that this trivector can be dualised to the vector s is a unique property of four-dimensional spacetime.

The sole term breaking conformal invariance in (6.89) is the mass term $\langle m\psi\tilde{\psi}\rangle$, and it is useful to consider the currents conjugate to dilations and special conformal transformations, and to see how their non-conservation arises from this term. For dilations, the conformal weight of a spinor field is $\frac{3}{2}$, and equation (6.48) yields the current

$$j_d = \underline{T}_{\mathrm{md}}(x) \tag{6.94}$$

(after subtracting out a total divergence). The conservation equation is

$$\begin{aligned} \nabla\cdot j_d &= \partial_a\cdot\underline{T}_{\mathrm{md}}(a) \\ &= \langle m\psi\tilde{\psi}\rangle. \end{aligned} \tag{6.95}$$

Under a spacetime transformation the A field transforms as

$$A(x) \mapsto A'(x) \equiv \overline{f}[A(x')], \tag{6.96}$$

where $x' = f(x)$. For a special conformal transformation, therefore, we have that

$$A'(x) = (1+anx)^{-1}A(x')(1+axn)^{-1}. \tag{6.97}$$

Since this is a rotation/dilation, we postulate for ψ the single-sided transformation

$$\psi'(x) = (1+anx)^{-2}(1+axn)^{-1}\psi(x'). \tag{6.98}$$

In order to verify that the condition (6.56) can be satisfied, we need the neat result that

$$\nabla\left((1+anx)^{-2}(1+axn)^{-1}\right) = 0. \tag{6.99}$$

This holds for all vectors n, and the bracketed term is immediately a solution of the massless Dirac equation (it is a *monogenic* function on spacetime). It follows from (6.56) that the conserved tensor is

$$\underline{T}_{SC}(n) = -\underline{T}_{\mathrm{md}}(xnx) - n\cdot(ix\wedge(\rho s)). \tag{6.100}$$

and the conservation equation is

$$\nabla\cdot\underline{T}_{SC}(xnx) = -2\langle m\psi\tilde{\psi}\rangle n\cdot x. \tag{6.101}$$

In both (6.95) and (6.101) the conjugate tensors are conserved as the mass goes to zero, as expected.

4. The Two-Particle Dirac Action

Our final application is to see how the two-particle equation (4.147) can be obtained from an action integral. The Lagrangian is

$$\mathcal{L} = \langle(\frac{\nabla^1}{m^1} + \frac{\nabla^2}{m^2})\psi J(\gamma_0^1 + \gamma_0^2)\tilde{\psi} - 2\psi\tilde{\psi}\rangle, \tag{6.102}$$

where ψ is a function of position in the 8-dimensional configuration space, and ∇^1 and ∇^2 are the vector derivatives in their respective spaces. The action is

$$S = \int |d^8x| \mathcal{L}. \tag{6.103}$$

If we define the function $\underline{h}$ by

$$\underline{h}(a) = i^1 \frac{a \cdot i^1}{m^1} + i^2 \frac{a \cdot i^2}{m^2}, \tag{6.104}$$

where i^1 and i^2 are the pseudoscalars for particle-one and particle-two spaces respectively, then we can write the action as

$$S = \int |d^8x| \langle \underline{h}(\partial_b) b \cdot \nabla \psi J(\gamma_0^1 + \gamma_0^2)\tilde{\psi} - 2\psi\tilde{\psi}\rangle. \tag{6.105}$$

Here $\nabla = \nabla^1 + \nabla^2$ is the vector derivative in 8-dimensional configuration space. The field equation is

$$\partial_\psi \mathcal{L} = \partial_a \cdot \nabla(\partial_{\psi,a} \mathcal{L}), \tag{6.106}$$

which leads to

$$[\underline{h}(\partial_a) a \cdot \nabla \psi J(\gamma_0^1 + \gamma_0^2)]^{\sim} - 4\tilde{\psi} = \partial_a \cdot \nabla [J(\gamma_0^1 + \gamma_0^2)\tilde{\psi}\underline{h}(a)]. \tag{6.107}$$

The reverse of this equation is

$$\underline{h}(\partial_a) a \cdot \nabla \psi J(\gamma_0^1 + \gamma_0^2) = 2\psi \tag{6.108}$$

and post-multiplying by $(\gamma_0^1 + \gamma_0^2)$ obtains

$$(\frac{\nabla^1}{m^1} + \frac{\nabla^2}{m^2})\psi J = \psi(\gamma_0^1 + \gamma_0^2), \tag{6.109}$$

as used in Section 4.4.

The action (6.102) is invariant under phase rotations in two-particle space,

$$\psi \mapsto \psi' \equiv \psi e^{-\alpha J}, \tag{6.110}$$

and the conserved current conjugate to this symmetry is

$$\begin{aligned} j &= \partial_a(-\psi J) * \partial_{\psi,a} \mathcal{L} \\ &= \partial_a \langle \psi E(\gamma_0^1 + \gamma_0^2)\tilde{\psi}\underline{h}(a)\rangle \\ &= \overline{h}\langle \psi E(\gamma_0^1 + \gamma_0^2)\tilde{\psi}\rangle_1. \end{aligned} \tag{6.111}$$

This current satisfies the conservation equation

$$\nabla \cdot j = 0 \tag{6.112}$$

or, absorbing the factor of E into ψ,

$$(\frac{\nabla^1}{m^1} + \frac{\nabla^2}{m^2}) \cdot \langle \psi(\gamma_0^1 + \gamma_0^2)\tilde{\psi}\rangle_1 = 0. \tag{6.113}$$

Some properties of this current were discussed briefly in Section 4.4.

6.4 Multivector Techniques for Functional Differentiation

We have seen how the multivector derivative provides a natural and powerful extension to the calculus of single-variable functions. We now wish to extend these techniques to encompass the derivative with respect to a linear function $\underline{h}(a)$. We start by introducing a fixed frame $\{e_i\}$, and define the scalar coefficients

$$h_{ij} \equiv \underline{h}(e_i)\cdot e_j. \tag{6.114}$$

Each of the scalars h_{ij} can be differentiated with respect to, and we seek a method of combining these into a single multivector operator. If we consider the derivative with respect to h_{ij} on the scalar $\underline{h}(b)\cdot c$, we find that

$$\begin{aligned}\partial_{h_{ij}}\underline{h}(b)\cdot c &= \partial_{h_{ij}}(b^k c^l h_{kl}) \\ &= b^i c^j.\end{aligned} \tag{6.115}$$

If we now multiply both sides of this expression by $a\cdot e_i e_j$ we obtain

$$a\cdot e_i e_j \partial_{h_{ij}}\underline{h}(b)\cdot c = a\cdot bc. \tag{6.116}$$

This successfully assembles a frame-independent vector on the right-hand side, so the operator $a\cdot e_i e_j \partial_{h_{ij}}$ must also be frame-independent. We therefore define the vector functional derivative $\partial_{\underline{h}(a)}$ by

$$\partial_{\underline{h}(a)} \equiv a\cdot e_i e_j \partial_{h_{ij}}, \tag{6.117}$$

where all indices are summed over and h_{ij} is given by (6.114).

The essential property of $\partial_{\underline{h}(a)}$ is, from (6.116),

$$\partial_{\underline{h}(a)}\underline{h}(b)\cdot c = a\cdot bc \tag{6.118}$$

and this result, together with Leibniz' rule, is sufficient to derive all the required properties of the $\partial_{\underline{h}(a)}$ operator. The procedure is as follows. With B a fixed bivector, we write

$$\begin{aligned}\partial_{\underline{h}(a)}\langle\underline{h}(b\wedge c)B\rangle &= \dot{\partial}_{\underline{h}(a)}\langle\dot{\underline{h}}(b)\underline{h}(c)B\rangle - \dot{\partial}_{\underline{h}(a)}\langle\dot{\underline{h}}(c)\underline{h}(b)B\rangle \\ &= a\cdot b\underline{h}(c)\cdot B - a\cdot c\underline{h}(b)\cdot B \\ &= \underline{h}[a\cdot(b\wedge c)]\cdot B\end{aligned} \tag{6.119}$$

which extends, by linearity, to give

$$\partial_{\underline{h}(a)}\langle\underline{h}(A)B\rangle = \underline{h}(a\cdot A)\cdot B, \tag{6.120}$$

where A and B are both bivectors. Proceeding in this manner, we find the general formula

$$\partial_{\underline{h}(a)}\langle\underline{h}(A)B\rangle = \sum_r \langle\underline{h}(a\cdot A_r)B_r\rangle_1. \tag{6.121}$$

For a fixed grade-r multivector A_r, we can now write

$$\begin{aligned}
\partial_{\underline{h}(a)}\underline{h}(A_r) &= \partial_{\underline{h}(a)}\langle\underline{h}(A_r)X_r\rangle\partial_{X_r} \\
&= \langle\underline{h}(a\cdot A_r)X_r\rangle_1\partial_{X_r} \\
&= (n-r+1)\underline{h}(a\cdot A_r), \qquad (6.122)
\end{aligned}$$

where n is the dimension of the space and a result from page 58 of [24] has been used.

Equation (6.121) can be used to derive formulae for the functional derivative of the adjoint. The general result is

$$\begin{aligned}
\partial_{\underline{h}(a)}\overline{h}(A_r) &= \partial_{\underline{h}(a)}\langle\underline{h}(X_r)A_r\rangle\partial_{X_r} \\
&= \langle\underline{h}(a\cdot\dot{X}_r)A_r\rangle_1\dot{\partial}_{X_r}. \qquad (6.123)
\end{aligned}$$

When A is a vector, this admits the simpler form

$$\partial_{\underline{h}(a)}\overline{h}(b) = ba. \qquad (6.124)$$

If $\underline{h}$ is a symmetric function then $\underline{h} = \overline{h}$, but this cannot be exploited for functional differentiation, since $\underline{h}$ and $\overline{h}$ are independent for the purposes of calculus.

As two final applications, we derive results for the functional derivative of the determinant (1.115) and the inverse function (1.125). For the determinant, we find that

$$\begin{aligned}
\partial_{\underline{h}(a)}\underline{h}(I) &= \underline{h}(a\cdot I) \\
\Rightarrow \quad \partial_{\underline{h}(a)}\det(\underline{h}) &= \underline{h}(a\cdot I)I^{-1} \\
&= \det(\underline{h})\overline{h}^{-1}(a), \qquad (6.125)
\end{aligned}$$

where we have recalled the definition of the inverse (1.125). This coincides with standard formulae for the functional derivative of the determinant by its corresponding tensor. The proof given here, which follows directly from the definitions of the determinant and the inverse, is considerably more concise than any available to conventional matrix/tensor methods. The result for the inverse function is found from

$$\partial_{\underline{h}(a)}\langle\underline{h}(B_r)\overline{h}^{-1}(A_r)\rangle = \langle\underline{h}(a\cdot B_r)\overline{h}^{-1}(A_r)\rangle_1 + \dot{\partial}_{\underline{h}(a)}\langle\dot{\overline{h}}^{-1}(A_r)\underline{h}(B_r)\rangle = 0 \qquad (6.126)$$

from which it follows that

$$\begin{aligned}
\partial_{\underline{h}(a)}\langle\overline{h}^{-1}(A_r)B_r\rangle &= -\langle\underline{h}[a\cdot\underline{h}^{-1}(B_r)]\overline{h}^{-1}(A_r)\rangle_1 \\
&= -\langle\overline{h}^{-1}(a)\cdot B_r\overline{h}^{-1}(A_r)\rangle_1, \qquad (6.127)
\end{aligned}$$

where use has been made of results for the adjoint (1.123).

We have now assembled most of the necessary formalism and results for the application of geometric calculus to field theory. In the final chapter we apply this formalism to develop a gauge theory of gravity.

For a fixed grade-r multivector A_r, we can now write

$$\begin{aligned}\partial_{\underline{h}(a)}\underline{h}(A_r) &= \partial_{\underline{h}(a)}\langle\underline{h}(A_r)X_r\rangle\partial_{X_r}\\ &= \langle\underline{h}(a\cdot A_r)X_r\rangle_1\partial_{X_r}\\ &= (n-r+1)\underline{h}(a\cdot A_r),\end{aligned}\tag{6.122}$$

where n is the dimension of the space and a result from page 58 of [24] has been used.

Equation (6.121) can be used to derive formulae for the functional derivative of the adjoint. The general result is

$$\begin{aligned}\partial_{\underline{h}(a)}\overline{h}(A_r) &= \partial_{\underline{h}(a)}\langle\underline{h}(X_r)A_r\rangle\partial_{X_r}\\ &= \langle\underline{h}(a\cdot\dot{X}_r)A_r\rangle_1\dot{\partial}_{X_r}.\end{aligned}\tag{6.123}$$

When A is a vector, this admits the simpler form

$$\partial_{\underline{h}(a)}\overline{h}(b) = ba.\tag{6.124}$$

If $\underline{h}$ is a symmetric function then $\underline{h} = \overline{h}$, but this cannot be exploited for functional differentiation, since $\underline{h}$ and $\overline{h}$ are independent for the purposes of calculus.

As two final applications, we derive results for the functional derivative of the determinant (1.115) and the inverse function (1.125). For the determinant, we find that

$$\begin{aligned}\partial_{\underline{h}(a)}\underline{h}(I) &= \underline{h}(a\cdot I)\\ \Rightarrow\quad \partial_{\underline{h}(a)}\det(\underline{h}) &= \underline{h}(a\cdot I)I^{-1}\\ &= \det(\underline{h})\overline{h}^{-1}(a),\end{aligned}\tag{6.125}$$

where we have recalled the definition of the inverse (1.125). This coincides with standard formulae for the functional derivative of the determinant by its corresponding tensor. The proof given here, which follows directly from the definitions of the determinant and the inverse, is considerably more concise than any available to conventional matrix/tensor methods. The result for the inverse function is found from

$$\partial_{\underline{h}(a)}\langle\underline{h}(B_r)\overline{h}^{-1}(A_r)\rangle = \langle\underline{h}(a\cdot B_r)\overline{h}^{-1}(A_r)\rangle_1 + \dot{\partial}_{\underline{h}(a)}\langle\dot{\overline{h}}^{-1}(A_r)\underline{h}(B_r)\rangle = 0\tag{6.126}$$

from which it follows that

$$\begin{aligned}\partial_{\underline{h}(a)}\langle\overline{h}^{-1}(A_r)B_r\rangle &= -\langle\underline{h}[a\cdot\underline{h}^{-1}(B_r)]\overline{h}^{-1}(A_r)\rangle_1\\ &= -\langle\overline{h}^{-1}(a)\cdot B_r\overline{h}^{-1}(A_r)\rangle_1,\end{aligned}\tag{6.127}$$

where use has been made of results for the adjoint (1.123).

We have now assembled most of the necessary formalism and results for the application of geometric calculus to field theory. In the final chapter we apply this formalism to develop a gauge theory of gravity.

Chapter 7

Gravity as a Gauge Theory

In this chapter the formalism described throughout the earlier chapters of this thesis is employed to develop a gauge theory of gravity. Our starting point is the Dirac action, and we begin by recalling how electromagnetic interactions arise through right-sided transformations of the spinor field ψ. We then turn to a discussion of Poincaré invariance, and attempt to introduce gravitational interactions in a way that closely mirrors the introduction of the electromagnetic sector. The new dynamical fields are identified, and an action is constructed for these. The field equations are then found and the derivation of these is shown to introduce an important consistency requirement. In order that the correct minimally-coupled Dirac equation is obtained, one is forced to use the simplest action for the gravitational fields — the Ricci scalar. Some free-field solutions are obtained and are compared with those of general relativity. Aspects of the manner in which the theory employs only active transformations are then illustrated with a discussion of extended-matter distributions.

By treating gravity as a gauge theory of active transformations in the (flat) spacetime algebra, some important differences from general relativity emerge. Firstly, coordinates are unnecessary and play an entirely secondary role. Points are represented by vectors, and all formulae given are coordinate-free. The result is a theory in which spacetime does not play an active role, and it is meaningless to assign physical properties to spacetime. The theory is one of forces, not geometry. Secondly, the gauge-theory approach leads to a first-order set of equations. Despite the fact that the introduction of a set of coordinates reproduces the matter-free field equations of general relativity, the requirement that the first-order variables should exist globally has important consequences. These are illustrated by a discussion of point-source solutions.

There has, of course, been a considerable discussion of whether and how gravity can be formulated as a gauge theory. The earliest attempt was by Utiyama [79], and his ideas were later refined by Kibble [80]. This led to the development of what is now known as the Einstein-Cartan-Kibble-Sciama (ECKS) theory of gravity. A detailed review of this subject was given in 1976 by Hehl *et al.* [81]. More recently, the fibre-bundle approach to gauge theories has been used to study general relativity [82]. All these developments share the idea that, at its most fundamental level, gravity is the result of spacetime curvature (and, more generally, of torsion). Furthermore, many of these treatments rely on an uncomfortable mixture of passive coordinate transformations and active tetrad transformations. Even when active transformations are emphasised, as by Hehl *et al.*, the transformations are still viewed as taking place on an initially curved spacetime manifold. Such ideas are rejected here, as is inevitable if one only discusses the properties of spacetime fields, and the interactions between them.

7.1 Gauge Theories and Gravity

We prepare for a discussion of gravity by first studying how electromagnetism is introduced into the Dirac equation. We start with the Dirac action in the form

$$S_D = \int |d^4x| \langle \nabla\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.1}$$

and recall that, on defining the transformation

$$\psi \mapsto \psi' \equiv \psi e^{-i\sigma_3\phi}, \tag{7.2}$$

the action is the same whether viewed as a function of ψ or ψ'. This is global phase invariance. The transformation (7.2) is a special case of the more general transformation

$$\psi' = \psi\tilde{R}_0, \tag{7.3}$$

where R_0 is a constant rotor. We saw in Chapter 4 that a Dirac spinor encodes an instruction to rotate the $\{\gamma_\mu\}$ frame onto a frame of observables $\{e_\mu\}$. ψ' is then the spinor that generates the same observables from the rotated initial frame

$$\gamma_\mu' = R_0\gamma_\mu\tilde{R}_0. \tag{7.4}$$

It is easily seen that (7.3) and (7.4) together form a symmetry of the action, since

$$\begin{aligned}\langle \nabla\psi' i\gamma_3'\tilde{\psi}' - m\psi'\tilde{\psi}'\rangle &= \langle \nabla\psi\tilde{R}_0 iR_0\gamma_3\tilde{R}_0R_0\tilde{\psi} - m\psi\tilde{R}_0R_0\tilde{\psi}\rangle \\ &= \langle \nabla\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle. \end{aligned}\tag{7.5}$$

The phase rotation (7.2) is singled out by the property that it leaves both the time-like γ_0-axis and space-like γ_3-axis unchanged.

There is a considerable advantage in introducing electromagnetic interactions through transformations such as (7.3). When we come to consider spacetime rotations, we will find that the only difference is that the rotor R_0 multiplies ψ from the left instead of from the right. We can therefore use an identical formalism for introducing both electromagnetic and gravitational interactions.

Following the standard ideas about gauge symmetries, we now ask what happens if the phase ϕ in (7.2) becomes a function of position. To make the comparison with gravity as clear as possible, we write the transformation as (7.3) and only restrict $\tilde{R}_0$ to be of the form of (7.2) once the gauging has been carried out. To study the effect of the transformation (7.3) we first write the vector derivative of ψ as

$$\nabla\psi = \partial_a(a\cdot\nabla\psi) \tag{7.6}$$

which contains a coordinate-free contraction of the directional derivatives of ψ with their vector directions. Under the transformation (7.3) the directional derivative becomes

$$\begin{aligned} a\cdot\nabla\psi' &= a\cdot\nabla(\psi\tilde{R}_0) \\ &= a\cdot\nabla\psi\tilde{R}_0 + \psi a\cdot\nabla\tilde{R}_0 \\ &= a\cdot\nabla\psi\tilde{R}_0 - \tfrac{1}{2}\psi\tilde{R}_0\chi(a), \end{aligned} \tag{7.7}$$

where

$$\chi(a) \equiv -2R_0 a\cdot\nabla\tilde{R}_0. \tag{7.8}$$

From the discussion of Lie groups in Section (3.1), it follows that $\chi(a)$ belongs to the Lie algebra of the rotor group, and so is a (position-dependent) bivector-valued linear function of the vector a.

It is now clear that, to maintain local invariance, we must replace the directional derivatives $a\cdot\nabla$ by covariant derivatives D_a, where

$$D_a\psi \equiv a\cdot\nabla\psi + \tfrac{1}{2}\psi\Omega(a). \tag{7.9}$$

We are thus forced to introduce a new set of dynamical variables — the bivector field $\Omega(a)$. The transformation properties of $\Omega(a)$ must be the same as those of $\chi(a)$. To find these, replace ψ' by ψ and consider the new transformation

$$\psi \mapsto \psi' = \psi\tilde{R}, \tag{7.10}$$

so that the rotor $\tilde{R}_0$ is transformed to $\tilde{R}_0\tilde{R} = (RR_0)\tilde{\ }$. The bivector $\chi(a)$ then transforms to

$$\begin{aligned} \chi'(a) &= -2(RR_0)a\cdot\nabla(RR_0)\tilde{\ } \\ &= R\chi(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}. \end{aligned} \tag{7.11}$$

It follows that the transformation law for $\Omega(a)$ is

$$\Omega(a) \mapsto \Omega'(a) \equiv R\Omega(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}, \tag{7.12}$$

which ensures that

$$\begin{aligned} D_a{}'(\psi') &= a\cdot\nabla(\psi\tilde{R}) + \tfrac{1}{2}\psi\tilde{R}\Omega'(a) \\ &= a\cdot\nabla(\psi\tilde{R}) + \tfrac{1}{2}\psi\tilde{R}(R\Omega(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}) \\ &= a\cdot\nabla\psi\tilde{R} + \tfrac{1}{2}\psi\Omega(a)\tilde{R} \\ &= D_a(\psi)\tilde{R}. \end{aligned} \tag{7.13}$$

The action integral (7.1) is now modified to

$$S' = \int |d^4x| \langle \partial_a D_a\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.14}$$

from which the field equations are

$$\nabla\psi i\gamma_3 + \tfrac{1}{2}\partial_a\psi\Omega(a)\cdot(i\gamma_3) = m\psi. \tag{7.15}$$

which contains a coordinate-free contraction of the directional derivatives of ψ with their vector directions. Under the transformation (7.3) the directional derivative becomes

$$\begin{aligned} a\cdot\nabla\psi' &= a\cdot\nabla(\psi\tilde{R}_0) \\ &= a\cdot\nabla\psi\tilde{R}_0 + \psi a\cdot\nabla\tilde{R}_0 \\ &= a\cdot\nabla\psi\tilde{R}_0 - \tfrac{1}{2}\psi\tilde{R}_0\chi(a), \end{aligned} \tag{7.7}$$

where

$$\chi(a) \equiv -2R_0 a\cdot\nabla\tilde{R}_0. \tag{7.8}$$

From the discussion of Lie groups in Section (3.1), it follows that $\chi(a)$ belongs to the Lie algebra of the rotor group, and so is a (position-dependent) bivector-valued linear function of the vector a.

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$$\begin{aligned} \chi'(a) &= -2(RR_0)a\cdot\nabla(RR_0)\tilde{} \\ &= R\chi(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}. \end{aligned} \tag{7.11}$$

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$$\Omega(a) \mapsto \Omega'(a) \equiv R\Omega(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}, \tag{7.12}$$

which ensures that

$$\begin{aligned} D_a{}'(\psi') &= a\cdot\nabla(\psi\tilde{R}) + \tfrac{1}{2}\psi\tilde{R}\Omega'(a) \\ &= a\cdot\nabla(\psi\tilde{R}) + \tfrac{1}{2}\psi\tilde{R}(R\Omega(a)\tilde{R} - 2Ra\cdot\nabla\tilde{R}) \\ &= a\cdot\nabla\psi\tilde{R} + \tfrac{1}{2}\psi\Omega(a)\tilde{R} \\ &= D_a(\psi)\tilde{R}. \end{aligned} \tag{7.13}$$

The action integral (7.1) is now modified to

$$S' = \int |d^4x| \langle \partial_a D_a\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.14}$$

from which the field equations are

$$\nabla\psi i\gamma_3 + \tfrac{1}{2}\partial_a\psi\Omega(a)\cdot(i\gamma_3) = m\psi. \tag{7.15}$$

For the case of electromagnetism, the rotor R is restricted to the form of (7.2), so the most general form that $\Omega(a)$ can take is

$$\Omega(a) = 2a\cdot(eA)i\sigma_3. \tag{7.16}$$

The "minimally-coupled" equation is now

$$\nabla\psi i\gamma_3 - eA\psi\gamma_0 = m\psi, \tag{7.17}$$

recovering the correct form of the Dirac equation in the presence of an external A field.

7.1.1 Local Poincaré Invariance

Our starting point for the introduction of gravity as a gauge theory is the Dirac action (7.1), for which we study the effect of local Poincaré transformations of the spacetime fields. We first consider translations

$$\psi(x) \mapsto \psi'(x) \equiv \psi(x'), \tag{7.18}$$

where

$$x' = x + a \tag{7.19}$$

and a is a constant vector. To make these translations local, the vector a must become a function of position. This is achieved by replacing (7.19) with

$$x' = f(x), \tag{7.20}$$

where $f(x)$ is now an arbitrary mapping between spacetime positions. We continue to refer to (7.20) as a translation, as this avoids overuse of the word "transformation". It is implicit in what follows that all translations are local and are therefore determined by completely arbitrary mappings. The translation (7.20) has the interpretation that the field ψ has been physically moved from the old position x' to the new position x. The same holds for the observables formed from ψ, for example the current $J(x) = \psi\gamma_0\tilde{\psi}$ is transformed to $J'(x) = J(x')$.

As it stands, the translation defined by (7.20) is not a symmetry of the action, since

$$\begin{aligned} \nabla\psi'(x) &= \nabla\psi(f(x)) \\ &= \overline{f}(\nabla_{x'})\psi(x'), \end{aligned} \tag{7.21}$$

and the action becomes

$$S' = \int |d^4x'|\,(\det \underline{f})^{-1}\langle\overline{f}(\nabla_{x'})\psi' i\gamma_3\tilde{\psi}' - m\psi'\tilde{\psi}'\rangle. \tag{7.22}$$

To recover a symmetry from (7.20), one must introduce an arbitrary, position-dependent linear function $\overline{h}$. The new action is then written as

$$S_h = \int |d^4x|(\det \underline{h})^{-1}\,\langle\overline{h}(\nabla)\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle. \tag{7.23}$$

Under the translation

$$\psi(x) \mapsto \psi'(x) \equiv \psi(f(x)), \tag{7.24}$$

the action S_h transforms to

$$S'_h = \int |d^4x'|(\det \underline{f})^{-1}(\det \underline{h}')^{-1}\langle \overline{h}'\overline{f}(\nabla_{x'})\psi' i\gamma_3\tilde{\psi}' - m\psi'\tilde{\psi}'\rangle \tag{7.25}$$

and the original action is recovered provided that $\overline{h}$ has the transformation law

$$\overline{h}_x \mapsto \overline{h}'_x \equiv \overline{h}_{x'}\overline{f}_x^{-1} \qquad \text{where } x' = f(x). \tag{7.26}$$

This is usually the most useful form for explicit calculations, though alternatively we can write

$$\overline{h}_x \mapsto \overline{h}'_x \equiv \overline{h}_{x'}\overline{f}_{x'} \qquad \text{where } x = f(x'), \tag{7.27}$$

which is sometimes more convenient to work with.

In arriving at (7.23) we have only taken local translations into account — the currents are being moved from one spacetime position to another. To arrive at a gauge theory of the full Poincaré group we must consider rotations as well. (As always, the term "rotation" is meant in the sense of Lorentz transformation.) In Chapter 6, rotational invariance of the Dirac action (7.1) was seen by considering transformations of the type $\psi(x) \mapsto R_0\psi(\tilde{R}_0 x R_0)$, where R_0 is a constant rotor. By writing the action in the form of (7.23), however, we have already allowed for the most general type of transformation of position dependence. We can therefore translate back to x, so that the rotation takes place at a point. In doing so, we completely decouple rotations and translations. This is illustrated by thinking in terms of the frame of observables $\{e_\mu\}$. Given this frame at a point x, there are two transformations that we can perform on it. We can either move it somewhere else (keeping it in the same orientation with respect to the $\{\gamma_\mu\}$ frame), or we can rotate it at a point. These two operations correspond to different symmetries. A suitable physical picture might be to think of "experiments" in place of frames. We expect that the physics of the experiment will be unaffected by moving the experiment to another point, or by changing its orientation in space.

Active rotations of the spinor observables are driven by rotations of the spinor field,

$$\psi \mapsto \psi' \equiv R_0\psi. \tag{7.28}$$

Since $\overline{h}(a)$ is a spacetime vector field, the corresponding law for $\overline{h}$ must be

$$\overline{h}(a) \quad\mapsto\quad \overline{h}'(a) \equiv R_0\overline{h}(a)\tilde{R}_0. \tag{7.29}$$

By writing the action (7.23) in the form

$$S_h = \int |d^4x|(\det \underline{h})^{-1}\,\langle \overline{h}(\partial_a)a\cdot\nabla\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.30}$$

we observe that it is now invariant under the rotations defined by (7.28) and (7.29). All rotations now take place at a point, and the presence of the $\overline{h}$ field ensures that a rotation at a point has become a global symmetry. To make this symmetry local, we need only replace the directional derivative of ψ by a covariant derivative, with the property that

$$D'_a(R\psi) = RD_a\psi, \tag{7.31}$$

where R is a position-dependent rotor. But this is precisely the problem that was tackled at the start of this section, the only difference being that the rotor R now sits to the left of ψ. Following the same arguments, we immediately arrive at the definition:

$$D_a\psi \equiv (a\cdot\nabla + \tfrac{1}{2}\Omega(a))\psi, \tag{7.32}$$

where $\Omega(a)$ is a (position-dependent) bivector-valued linear function of the vector a. Under local rotations $\psi \mapsto R\psi$, $\Omega(a)$ transforms to

$$\Omega(a) \mapsto \Omega(a)' \equiv R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}. \tag{7.33}$$

Under local translations, $\Omega(a)$ must transform in the same manner as the $a\cdot\nabla R\tilde{R}$ term, so

$$\begin{array}{lcll} \Omega_x(a) & \mapsto & \Omega_{x'}\underline{f}_x(a) & \text{if } x' = f(x), \\ \Omega_x(a) & \mapsto & \Omega_{x'}\underline{f}^{-1}_{x'}(a) & \text{if } x = f(x'). \end{array} \tag{7.34}$$

(The subscript x on $\Omega_x(a)$ labels the position dependence.)

The action integral

$$S = \int |d^4x| (\det \underline{h})^{-1} \langle \overline{h}(\partial_a) D_a\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.35}$$

is now invariant under both local translations and rotations. The field equations derived from S will have the property that, if $\{\psi, \overline{h}, \Omega\}$ form a solution, then so too will any new fields obtained from these by local Poincaré transformations. This local Poincaré symmetry has been achieved by the introduction of two gauge fields, $\overline{h}$ and Ω, with a total of $(4\times4) + (4\times6) = 40$ degrees of freedom. This is precisely the number expected from gauging the 10-dimensional Poincaré group.

Before turning to the construction of an action integral for the gauge fields, we look at how the covariant derivative of (7.32) must extend to act on the physical observables derived from ψ. These observables are all formed by the double-sided action of the spinor field ψ on a constant multivector Γ (formed from the $\{\gamma_\mu\}$) so that

$$A \equiv \psi\Gamma\tilde{\psi}. \tag{7.36}$$

The multivector A therefore transforms under rotations as

$$A \mapsto RA\tilde{R}, \tag{7.37}$$

and under translations as

$$A(x) \mapsto A(f(x)). \tag{7.38}$$

We refer to objects with the same transformation laws as A as being *covariant* (an example is the current, $J = \psi\gamma_0\tilde{\psi}$). If we now consider directional derivatives of A, we see that these can be written as

$$a\cdot\nabla A = (a\cdot\nabla\psi)\Gamma\tilde{\psi} + \psi\Gamma(a\cdot\nabla\psi)\tilde{}, \tag{7.39}$$

at a point has become a global symmetry. To make this symmetry local, we need only replace the directional derivative of ψ by a covariant derivative, with the property that

$$D'_a(R\psi) = RD_a\psi, \tag{7.31}$$

where R is a position-dependent rotor. But this is precisely the problem that was tackled at the start of this section, the only difference being that the rotor R now sits to the left of ψ. Following the same arguments, we immediately arrive at the definition:

$$D_a\psi \equiv (a\cdot\nabla + \tfrac{1}{2}\Omega(a))\psi, \tag{7.32}$$

where $\Omega(a)$ is a (position-dependent) bivector-valued linear function of the vector a. Under local rotations $\psi \mapsto R\psi$, $\Omega(a)$ transforms to

$$\Omega(a) \mapsto \Omega(a)' \equiv R\Omega(a)\tilde{R} - 2a\cdot\nabla R\tilde{R}. \tag{7.33}$$

Under local translations, $\Omega(a)$ must transform in the same manner as the $a\cdot\nabla R\tilde{R}$ term, so

$$\begin{array}{rcll} \Omega_x(a) & \mapsto & \Omega_{x'}\underline{f}_x(a) & \text{if } x' = f(x), \\ \Omega_x(a) & \mapsto & \Omega_{x'}\underline{f}_{x'}^{-1}(a) & \text{if } x = f(x'). \end{array} \tag{7.34}$$

(The subscript x on $\Omega_x(a)$ labels the position dependence.)

The action integral

$$S = \int |d^4x| (\det \underline{h})^{-1} \langle \overline{h}(\partial_a) D_a\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle, \tag{7.35}$$

is now invariant under both local translations and rotations. The field equations derived from S will have the property that, if $\{\psi, \overline{h}, \Omega\}$ form a solution, then so too will any new fields obtained from these by local Poincaré transformations. This local Poincaré symmetry has been achieved by the introduction of two gauge fields, $\overline{h}$ and Ω, with a total of $(4\times 4) + (4\times 6) = 40$ degrees of freedom. This is precisely the number expected from gauging the 10-dimensional Poincaré group.

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The multivector A therefore transforms under rotations as

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and under translations as

$$A(x) \mapsto A(f(x)). \tag{7.38}$$

We refer to objects with the same transformation laws as A as being *covariant* (an example is the current, $J = \psi\gamma_0\tilde{\psi}$). If we now consider directional derivatives of A, we see that these can be written as

$$a\cdot\nabla A = (a\cdot\nabla\psi)\Gamma\tilde{\psi} + \psi\Gamma(a\cdot\nabla\psi)\tilde{}, \tag{7.39}$$

which immediately tells us how to turn these into covariant derivatives. Rotations of A are driven by single-sided rotations of the spinor field ψ, so to arrive at a covariant derivative of A we simply replace the spinor directional derivatives by covariant derivatives, yielding

$$\begin{aligned}
&(D_a\psi)\Gamma\tilde{\psi} + \psi\Gamma(D_a\psi)\tilde{} \\
&= (a\cdot\nabla\psi)\Gamma\tilde{\psi} + \psi\Gamma(a\cdot\nabla\psi)\tilde{} + \tfrac{1}{2}\Omega(a)\psi\Gamma\tilde{\psi} - \tfrac{1}{2}\psi\Gamma\tilde{\psi}\Omega(a) \\
&= a\cdot\nabla(\psi\Gamma\tilde{\psi}) + \Omega(a)\times(\psi\Gamma\tilde{\psi}).
\end{aligned} \tag{7.40}$$

We therefore define the covariant derivative for "observables" by

$$\mathcal{D}_a A \equiv a\cdot\nabla A + \Omega(a)\times A. \tag{7.41}$$

This is applicable to all multivector fields which transform double-sidedly under rotations. The operator $\mathcal{D}_a$ has the important property of satisfying Leibniz' rule,

$$\mathcal{D}_a(AB) = (\mathcal{D}_a A)B + A(\mathcal{D}_a B), \tag{7.42}$$

so that $\mathcal{D}_a$ is a *derivation*. This follows from the identity

$$\Omega(a)\times(AB) = (\Omega(a)\times A)B + A(\Omega(a)\times B). \tag{7.43}$$

For notational convenience we define the further operators

$$D\psi \equiv \overline{h}(\partial_a)D_a\psi \tag{7.44}$$

$$\mathcal{D}A \equiv \overline{h}(\partial_a)\mathcal{D}_a A, \tag{7.45}$$

and for the latter we write

$$\mathcal{D}A = \mathcal{D}\cdot A + \mathcal{D}\wedge A, \tag{7.46}$$

where

$$\mathcal{D}\cdot A_r \equiv \langle \mathcal{D}A\rangle_{r-1} \tag{7.47}$$

$$\mathcal{D}\wedge A_r \equiv \langle \mathcal{D}A\rangle_{r+1}. \tag{7.48}$$

The operator $\mathcal{D}$ can be thought of as a covariant vector derivative. D and $\mathcal{D}$ have the further properties that

$$a\cdot D = D_{\underline{h}(a)} \tag{7.49}$$

$$a\cdot\mathcal{D} = \mathcal{D}_{\underline{h}(a)}. \tag{7.50}$$

7.1.2 Gravitational Action and the Field Equations

Constructing a form of the Dirac action that is invariant under local Poincaré transformations has required the introduction of $\overline{h}$ and Ω fields, with the transformation properties (7.26), (7.29), (7.33) and (7.34). We now look for invariant scalar quantities that can be formed from these. We start by defining the field-strength $R(a\wedge b)$ by

$$\tfrac{1}{2}R(a\wedge b)\psi \equiv [D_a, D_b]\psi, \tag{7.51}$$

so that

$$R(a\wedge b) = a\cdot\nabla\Omega(b) - b\cdot\nabla\Omega(a) + \Omega(a)\times\Omega(b). \tag{7.52}$$

$R(a\wedge b)$ is a bivector-valued function of its bivector argument $a\wedge b$. For an arbitrary bivector argument we define

$$R(a\wedge c + c\wedge d) = R(a\wedge b) + R(c\wedge d) \tag{7.53}$$

so that $R(B)$ is now a bivector-valued linear function of its bivector argument B. The space of bivectors is 6-dimensional so $R(B)$ has, at most, 36 degrees of freedom. (The counting of degrees of freedom is somewhat easier than in conventional tensor calculus, since two of the symmetries of $R^{\alpha\beta\gamma\delta}$ are automatically incorporated.) $R(B)$ transforms under local translations as

$$R_x(B) \mapsto R_{x'}\underline{f}_x(B) \qquad \text{where } x' = f(x), \tag{7.54}$$

and under local rotations as

$$R(B) \mapsto R_0 R(B)\tilde{R}_0. \tag{7.55}$$

The field-strength is contracted once to form the linear function

$$R(b) = \overline{h}(\partial_a)\cdot R(a\wedge b), \tag{7.56}$$

which has the same transformation properties as $R(B)$. We use the same symbol for both functions and distinguish between them through their argument, which is either a bivector (B or $a\wedge b$) or a vector (a).

Contracting once more, we arrive at the ("Ricci") scalar

$$\mathcal{R} = \overline{h}(\partial_b)\cdot R(b) = \overline{h}(\partial_b\wedge\partial_a)\cdot R(a\wedge b). \tag{7.57}$$

$\mathcal{R}$ transforms as a scalar function under both rotations and translations. As an aside, it is interesting to construct the analogous quantity to $\mathcal{R}$ for the electromagnetic gauge sector. For this we find that

$$[D_a^{em}, D_b^{em}]\psi = e(b\wedge a)\cdot F\psi i\sigma_3 \tag{7.58}$$

$$\Rightarrow \overline{h}(\partial_b\wedge\partial_a)[D_a^{em}, D_b^{em}]\psi = -2e\overline{h}(F)\psi i\sigma_3. \tag{7.59}$$

Interestingly, this suggests that the bivector $\overline{h}(F)$ has a similar status to the Ricci scalar, and not to the field-strength tensor.

Since the Ricci scalar $\mathcal{R}$ is a covariant scalar, the action integral

$$S_G = \int |d^4x|(\det\underline{h})^{-1}\,\mathcal{R}/2 \tag{7.60}$$

is invariant under all local Poincaré transformations. The choice of action integral (7.60) is the same as that of the Hilbert-Palatini principle in general relativity, and we investigate the consequences of this choice now. Once we have derived both the gravitational and matter equations, we will return to the subject of whether this choice is unique.

From (7.60) we write the Lagrangian density as

$$\mathcal{L}_G = \tfrac{1}{2}\mathcal{R}\det\underline{h}^{-1} = \mathcal{L}_G(\overline{h}(a), \Omega(a), b\cdot\nabla\Omega(a)). \tag{7.61}$$

The action integral (7.60) is over a region of flat spacetime, so all the variational principle techniques developed in Chapter 6 hold without modification. The only elaboration needed is to define a calculus for $\Omega(a)$. Such a calculus can be defined in precisely the same way as the derivative $\partial_{\underline{h}(a)}$ was defined (6.117). The essential results are:

$$\partial_{\Omega(a)}\langle\Omega(b)B\rangle = a\cdot bB \tag{7.62}$$
$$\partial_{\Omega(b),a}\langle c\cdot\nabla\Omega(d)B\rangle = a\cdot cb\cdot dB, \tag{7.63}$$

where B is an arbitrary bivector.

We assume that the overall action is of the form

$$\mathcal{L} = \mathcal{L}_G - \kappa\mathcal{L}_M, \tag{7.64}$$

where $\mathcal{L}_M$ describes the matter content and $\kappa = 8\pi G$. The first of the field equations is found by varying with respect to $\overline{h}$, producing

$$\begin{aligned}\kappa\partial_{\overline{h}(a)}\mathcal{L}_M &= \tfrac{1}{2}\partial_{\overline{h}(a)}(\langle\overline{h}(\partial_b\wedge\partial_c)R(c\wedge b)\rangle\det\underline{h}^{-1})\\ &= (R(a) - \tfrac{1}{2}\underline{h}^{-1}(a)\mathcal{R})\det\underline{h}^{-1}.\end{aligned} \tag{7.65}$$

The functional derivative with respect to $\overline{h}(a)$ of the matter Lagrangian is taken to define the stress-energy tensor of the matter field through

$$\mathcal{T}\underline{h}^{-1}(a)\det\underline{h}^{-1} \equiv \partial_{\overline{h}(a)}\mathcal{L}_M, \tag{7.66}$$

so that we arrive at the field equations in the form

$$R(a) - \tfrac{1}{2}\underline{h}^{-1}(a)\mathcal{R} = \kappa\mathcal{T}\underline{h}^{-1}(a). \tag{7.67}$$

It is now appropriate to define the functions

$$\mathcal{R}(a\wedge b) \equiv R\underline{h}(a\wedge b) \tag{7.68}$$
$$\mathcal{R}(a) \equiv R\underline{h}(a) = \partial_a\cdot\mathcal{R}(a\wedge b) \tag{7.69}$$
$$\mathcal{G} \equiv \mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R}. \tag{7.70}$$

These are covariant under translations (they simply change their position dependence), and under rotations they transform as *e.g.*

$$\mathcal{R}(B) \mapsto R_0\mathcal{R}(\tilde{R}_0BR_0)\tilde{R}_0. \tag{7.71}$$

Equation (7.71) is the defining rule for the transformation properties of a *tensor,* and we hereafter refer to (7.68) through to (7.70) as the Riemann, Ricci and Einstein tensors respectively. We can now write (7.67) in the form

$$\mathcal{G}(a) = \kappa\mathcal{T}(a), \tag{7.72}$$

which is the (flat-space) gauge-theory equivalent of Einstein's field equations.

In the limit of vanishing gravitational fields ($\overline{h}(a) \mapsto a$ and $\Omega(a) \mapsto 0$) the stress-energy tensor defined by (7.66) agrees with the canonical stress-energy tensor (6.24), up to a total divergence. When $\Omega(a)$ vanishes, the matter action is obtained from the free-field $\mathcal{L}(\psi_i, a\cdot\nabla\psi_i)$ through the introduction of the transformation defined by $x' = h(x)$. Denoting the transformed fields by ψ_i', we find that

$$\partial_{\overline{h}(a)}[(\det \underline{h})^{-1}\mathcal{L}']\Big|_{\underline{h}=I} = \partial_{\overline{h}(a)}\mathcal{L}(\psi_i', \underline{h}(a)\cdot\nabla\psi_i')\Big|_{\underline{h}=I} - a\mathcal{L} \tag{7.73}$$

and

$$\begin{aligned}
&\partial_{\overline{h}(a)}\mathcal{L}(\psi_i', \underline{h}(b)\cdot\nabla\psi_i')\Big|_{\underline{h}=I} \\
&= \partial_{\overline{h}(a)}[\psi_i'*\partial_{\psi_i}\mathcal{L} + (\partial_b\cdot\overline{h}(\nabla)\psi_i')*\partial_{\psi_{i,b}}\mathcal{L}]_{\underline{h}=I} \\
&= \partial_b(a\cdot\nabla\psi_i)*\partial_{\psi_{i,b}}\mathcal{L} + \partial_{\overline{h}(a)}\dot{\psi}_i'*[\partial_{\psi_i}\mathcal{L} + \underline{h}(\partial_b)\cdot\dot{\nabla}\partial_{\psi_{i,b}}\mathcal{L}]_{\underline{h}=I}.
\end{aligned} \tag{7.74}$$

When the field equations are satisifed, the final term in (7.74) is a total divergence, and we recover the stress-energy tensor in the form (6.24). This is unsurprising, since the derivations of the functional and canonical stress-energy tensors are both concerned with the effects of moving fields from one spacetime position to another.

The definition (7.66) differs from that used in general relativity, where the functional derivative is taken with respect to the metric tensor [83]. The functional derivative with respect to the metric ensures that the resultant stress-energy tensor is symmetric. This is not necessarily the case when the functional derivative is taken with respect to $\overline{h}(a)$. This is potentially important, since we saw in Chapter 6 that the antisymmetric contribution to the stress-energy tensor is crucial for the correct treatment of the spin of the fields.

We next derive the field equations for the $\Omega(a)$ field. We write these in the form

$$\begin{aligned}
&\partial_{\Omega(a)}\mathcal{L}_G - \partial_b\cdot\nabla(\partial_{\Omega(a),b}\mathcal{L}_G) \\
&\quad = \kappa\left\{\partial_{\Omega(a)}\mathcal{L}_M - \partial_b\cdot\nabla(\partial_{\Omega(a),b}\mathcal{L}_M)\right\} \equiv \kappa\mathcal{S}(a)\det\underline{h}^{-1},
\end{aligned} \tag{7.75}$$

where the right-hand side defines the function $\mathcal{S}(a)$. Performing the derivatives on the left-hand side of the equation gives

$$\det\underline{h}^{-1}\Omega(b)\times\left(\overline{h}(\partial_b)\wedge\overline{h}(a)\right) + \partial_b\cdot\nabla\left(\overline{h}(b)\wedge\overline{h}(a)\det\underline{h}^{-1}\right) = \kappa\mathcal{S}(a)\det\underline{h}^{-1}. \tag{7.76}$$

On contracting (7.76) with $\underline{h}^{-1}(\partial_a)$, we find that

$$\begin{aligned}
&\kappa\underline{h}^{-1}(\partial_a)\cdot\mathcal{S}(a)\det\underline{h}^{-1} \\
&= \partial_a\cdot[\Omega(b)\times(\overline{h}(\partial_b)\wedge a)]\det\underline{h}^{-1} + \underline{h}^{-1}(\partial_a)\cdot\left\{\partial_b\cdot\nabla\left(\overline{h}(b)\wedge\overline{h}(a)\det\underline{h}^{-1}\right)\right\} \\
&= 2\overline{h}(\partial_a)\cdot\Omega(a)\det\underline{h}^{-1} - 3\overline{h}(\overleftrightarrow{\nabla}\det\underline{h}^{-1}) - \overline{h}(\nabla)\underline{h}^{-1}(\partial_a)\cdot\dot{\overline{h}}(a)\det\underline{h}^{-1} \\
&\quad + \dot{\overline{h}}(\dot{\nabla})\det\underline{h}^{-1}.
\end{aligned} \tag{7.77}$$

If we now make use of the result that

$$\begin{aligned}
\langle a\cdot\nabla\overline{h}(b)\underline{h}^{-1}(\partial_b)\det\underline{h}^{-1}\rangle &= -(a\cdot\nabla\overline{h}(\partial_b))*\partial_{\overline{h}(b)}\det\underline{h}^{-1} \\
&= -a\cdot\nabla\det\underline{h}^{-1} \quad \text{(chain rule)},
\end{aligned} \tag{7.78}$$

from which it follows that

$$\bar{h}(\dot{\nabla})\langle\dot{\bar{h}}(a)\underline{h}^{-1}(\partial_a)\det\underline{h}^{-1}\rangle = -\bar{h}(\nabla\det\underline{h}^{-1}), \tag{7.79}$$

we find that

$$\begin{aligned}\kappa\underline{h}^{-1}(\partial_a)\cdot\mathcal{S}(a)\det\underline{h}^{-1} &= 2\bar{h}(\partial_a)\cdot\Omega(a)\det\underline{h}^{-1} - 2\bar{h}(\overleftrightarrow{\nabla}\det\underline{h}^{-1}) \\ &= -2\mathcal{D}_a\bar{h}(\partial_a\det\underline{h}^{-1}).\end{aligned} \tag{7.80}$$

We will see shortly that it is crucial to the derivation of the correct matter field equations that

$$\mathcal{D}_a\bar{h}(\partial_a\det\underline{h}^{-1}) = 0. \tag{7.81}$$

This places a strong restriction on the form of $\mathcal{L}_M$, which must satisfy

$$\underline{h}^{-1}(\partial_a)\cdot\big(\partial_{\Omega(a)}\mathcal{L}_M - \partial_b\cdot\nabla(\partial_{\Omega(a),b}\mathcal{L}_M)\big) = 0. \tag{7.82}$$

This condition is satisfied for our gauge theory based on the Dirac equation, since the bracketed term in equation (7.82) is

$$\begin{aligned}(\partial_{\Omega(a)} - \partial_b\cdot\overrightarrow{\nabla}\,\partial_{\Omega(a),b})\langle D\psi i\gamma_3\tilde{\psi} - m\psi\tilde{\psi}\rangle &= \tfrac{1}{2}\bar{h}(a)\cdot(\psi i\gamma_3\tilde{\psi}) \\ &= -\tfrac{1}{2}i\bar{h}(a)\wedge s.\end{aligned} \tag{7.83}$$

It follows immediately that the contraction in (7.82) vanishes, since

$$\underline{h}^{-1}(\partial_a)\cdot(i\bar{h}(a)\wedge s) = -i\partial_a\wedge a\wedge s = 0. \tag{7.84}$$

We define $\mathcal{S}$ by

$$\mathcal{S} \equiv \tfrac{1}{2}\psi i\gamma_3\tilde{\psi}, \tag{7.85}$$

so that we can now write

$$\mathcal{S}(a) = \bar{h}(a)\cdot\mathcal{S}. \tag{7.86}$$

Given that (7.81) does hold, we can now write (7.76) in the form

$$\begin{aligned}\kappa\mathcal{S}(a) &= \dot{\bar{h}}(\nabla)\wedge\dot{\bar{h}}(a) + \Omega(b)\times\big(\bar{h}(\partial_b)\wedge\bar{h}(a)\big) - \big(\Omega(b)\cdot\bar{h}(\partial_b)\big)\wedge\bar{h}(a) \\ &= \bar{h}(\partial_b)\wedge\big(b\cdot\nabla\bar{h}(a) + \Omega(b)\cdot\bar{h}(a)\big) \\ &= \mathcal{D}\wedge\bar{h}(a).\end{aligned} \tag{7.87}$$

The right-hand side of this equation could be viewed as the torsion though, since we are working in a flat spacetime, it is preferable to avoid terminology borrowed from Riemann-Cartan geometry. When the left-hand side of (7.87) vanishes, we arrive at the simple equation

$$\mathcal{D}\wedge\bar{h}(a) = 0, \tag{7.88}$$

valid for all constant vectors a. All differential functions $\underline{f}(a) = a\cdot\nabla f(x)$ satisfy $\nabla\wedge\bar{f}(a) = 0$, and (7.88) can be seen as the covariant generalisation of this result. Our gravitational field equations are summarised as

$$\mathcal{G}(a) = \kappa\mathcal{T}(a) \tag{7.89}$$

$$\mathcal{D}\wedge\bar{h}(a) = \kappa\mathcal{S}(a), \tag{7.90}$$

which hold for all constant vectors a.

7.1.3 The Matter-Field Equations

We now turn to the derivation of the matter-field equations. We start with the Dirac equation, and consider the electromagnetic field equations second.

The Dirac Equation

We have seen how the demand for invariance under local Poincaré transformations has led to the action

$$S = \int |d^4x| (\det \underline{h})^{-1} \langle \overline{h}(\partial_a) D_a \psi i\gamma_3 \tilde{\psi} - m\psi\tilde{\psi} \rangle. \tag{7.91}$$

Applying the Euler-Lagrange equations (6.12) to this, and reversing the result, we find that

$$\det \underline{h}^{-1} \left(\overline{h}(\partial_a) a \cdot \nabla \psi i\gamma_3 + \overline{h}(\partial_a) \wedge \Omega(a) \psi i\gamma_3 - 2m\psi \right) = -\partial_a \cdot \nabla (\overline{h}(a) \psi i\gamma_3 \det \underline{h}^{-1}), \tag{7.92}$$

which can be written as

$$D\psi i\gamma_3 = m\psi - \tfrac{1}{2} \mathcal{D}_a \overline{h}(\partial_a \det \underline{h}^{-1}) \psi i\gamma_3. \tag{7.93}$$

We now see why it is so important that $\mathcal{D}_a \overline{h}(\partial_a \det \underline{h}^{-1})$ vanishes. Our point of view throughout has been to start from the Dirac equation, and to introduce gauge fields to ensure local Poincaré invariance. We argued initially from the point of view of the Dirac action, but we could equally well have worked entirely at the level of the equation. By starting from the Dirac equation

$$\nabla \psi i\gamma_3 = m\psi, \tag{7.94}$$

and introducing the $\overline{h}$ and $\Omega(a)$ fields in the same manner as in Section 2.1, we find that the correct minimally coupled equation is

$$D\psi i\gamma_3 = m\psi. \tag{7.95}$$

If we now make the further restriction that our field equations are derivable from an action principle, we must demand that (7.93) reduces to (7.95). We are therefore led to the constraint that $\mathcal{D}_a \overline{h}(\partial_a \det \underline{h}^{-1})$ vanishes. To be consistent, this constraint must be derivable from the gravitational field equations. We have seen that the usual Hilbert-Palatini action satisfies this requirement, but higher-order contributions to the action would not. This rules out, for example, the type of "$R + R^2$" Lagrangian often considered in the context of Poincaré gauge theory [84, 85, 86]. Satisfyingly, this *forces* us to a theory which is first-order in the derivatives of the fields. The only freedom that remains is the possible inclusion of a cosmological constant, though such a term would obviously violate our intention that gravitational forces should result directly from interactions between particles.

The full set of equations for Dirac matter coupled to gravity is obtained from the action

$$S = \int |d^4x| (\det \underline{h})^{-1} \left(\tfrac{1}{2} \mathcal{R} - \kappa \langle \overline{h}(\partial_a) D_a \psi i\gamma_3 \tilde{\psi} - m\psi\tilde{\psi} \rangle \right), \tag{7.96}$$

and the field equations are

$$\begin{aligned}\mathcal{G}(a) &= \kappa\langle a\cdot D\psi i\gamma_3\tilde{\psi}\rangle_1 && (7.97)\\ \mathcal{D}\wedge\overline{h}(a) &= \kappa\overline{h}(a)\cdot(\tfrac{1}{2}\psi i\gamma_3\tilde{\psi}) = \kappa\overline{h}(a)\cdot(\tfrac{1}{2}is) && (7.98)\\ D\psi i\sigma_3 &= m\psi\gamma_0. && (7.99)\end{aligned}$$

It is not clear that self-consistent solutions to these equations could correspond to any physical situation, as such a solution would describe a self-gravitating Dirac fluid. Self-consistent solutions have been found in the context of cosmology, however, and the solutions have the interesting property of forcing the universe to be at critical density [10].

The Electromagnetic Field Equations

We now return to the introduction of the electromagnetic field. From the action (7.91), and following the procedure of the start of this chapter, we arrive at the action

$$S_{D+EM} = \int |d^4x|(\det \underline{h})^{-1}\,\langle\overline{h}(\partial_a)(D_a\psi i\gamma_3\tilde{\psi} - ea\cdot A\psi\gamma_0\tilde{\psi}) - m\psi\tilde{\psi}\rangle. \qquad (7.100)$$

The field equation from this action is

$$D\psi i\sigma_3 - e\mathcal{A}\psi = m\psi\gamma_0, \qquad (7.101)$$

where we have introduced the notation

$$\mathcal{A} = \overline{h}(A). \qquad (7.102)$$

It is to be expected that $\mathcal{A}$ should appear in the final equation, rather than A. The vector potential A originated as the generalisation of the quantity $\nabla\phi$. If we examine what happens to this under the translation $\phi(x) \mapsto \phi(x')$, with $x' = f(x)$, we find that

$$\nabla\phi \;\mapsto\; \overline{f}(\nabla_{x'}\phi(x')). \qquad (7.103)$$

It follows that A must also pick up a factor of $\overline{f}$ as it is moved from x' to x,

$$A(x) \;\mapsto\; \overline{f}(A(x')), \qquad (7.104)$$

so it is the quantity $\mathcal{A}$ that is Poincaré-covariant, as are all the other quantities in equation (7.101). However, $\mathcal{A}$ is not invariant under local $U(1)$ transformations. Instead, we must construct the Faraday bivector

$$F = \nabla\wedge A. \qquad (7.105)$$

It could be considered a weakness of conventional spin-torsion theory that, in order to construct the gauge-invariant quantity F, one has to resort to the use of the flat-space vector derivative. Of course, in our theory background spacetime has never gone away, and we are free to exploit the vector derivative to the full.

The conventional approach to gauge theories of gravity (as discussed in [81], for example) attempts to define a minimal coupling procedure for all matter fields, preparing the way for a true curved-space theory. The approach here has been rather different, in that everything is derived from the Dirac equation, and we are attempting to put electromagnetic and gravitational interactions on as similar a footing as possible. Consequently, there is no reason to expect that the gravitational field should "minimally couple" into the electromagnetic field. Instead, we must look at how F behaves under local translations. We find that

$$\begin{aligned} F(x) \mapsto \nabla\wedge\overline{f}A(x') &= \overline{f}(\nabla_{x'}\wedge A(x')) && (7.106) \\ &= \overline{f}F(x'), && (7.107) \end{aligned}$$

so the covariant form of F is

$$\mathcal{F} \equiv \overline{h}(F). \tag{7.108}$$

$\mathcal{F}$ is covariant under local Poincaré transformations, and invariant under $U(1)$ transformations. The appropriate action for the electromagnetic field is therefore

$$S_{EM} = \int |d^4x| (\det \underline{h})^{-1} \langle \tfrac{1}{2}\mathcal{F}\mathcal{F} - A\cdot J\rangle, \tag{7.109}$$

which reduces to the standard electromagnetic action integral in the limit where $\overline{h}$ is the identity. To find the electromagnetic field equations, we write

$$\mathcal{L}_{EM} = \det \underline{h}^{-1} \langle \tfrac{1}{2}\mathcal{F}\mathcal{F} - A\cdot J\rangle = \mathcal{L}(A, a\cdot\nabla A), \tag{7.110}$$

and treat the $\overline{h}$ and J fields as external sources. There is no Ω-dependence in (7.109), so $\mathcal{L}_{EM}$ satisfies the criteria of equation (7.82).

Variation of $\mathcal{L}_{EM}$ with respect to A leads to the equation

$$\partial_a\cdot\nabla(a\cdot\underline{h}(\mathcal{F})\det\underline{h}^{-1}) = \nabla\cdot\left(\underline{h}\overline{h}(\nabla\wedge A)\det\underline{h}^{-1}\right) = \det\underline{h}^{-1}J, \tag{7.111}$$

which combines with the identity

$$\nabla\wedge F = 0 \tag{7.112}$$

to form the Maxwell equations in a gravitational background. Equation (7.111) corresponds to the standard second-order wave equation for the vector potential A used in general relativity. It contains only the functions $\underline{h}\overline{h} \equiv g^{-1}$ and $\det\underline{h}^{-1} = (\det g)^{1/2}$, where g is the symmetric "metric" tensor. The fact that equation (7.111) only involves $\underline{h}$ through the metric tensor is usually taken as evidence that the electromagnetic field does not couple to torsion.

So far, we only have the Maxwell equations as two separate equations (7.111) and (7.112). It would be very disappointing if our STA approach did not enable us to do better since one of the many advantages of the STA is that, in the absence of a gravitational field, Maxwell's equations

$$\nabla\cdot F = J \qquad \nabla\wedge F = 0 \tag{7.113}$$

can be combined into a single equation

$$\nabla F = J. \tag{7.114}$$

This is more than a mere notational convenience. The ∇ operator is invertible, and can be used to develop a first-order propagator theory for the F-field [8]. This has the advantages of working directly with the physical field, and of correctly predicting the obliquity factors that have to be put in by hand in the second-order approach (based on a wave equation for A). It would undermine much of the motivation for pursuing first-order theories if this approach cannot be generalised to include gravitational effects. Furthermore, if we construct the stress-energy tensor, we find that

$$\begin{aligned}\mathcal{T}_{EM}\underline{h}^{-1}(a)\det\underline{h}^{-1} &= \tfrac{1}{2}\partial_{\overline{h}(a)}\langle\overline{h}(F)\overline{h}(F)\det\underline{h}^{-1}\rangle \\ &= \left(\overline{h}(a\cdot F)\cdot\mathcal{F} - \tfrac{1}{2}\underline{h}^{-1}(a)\mathcal{F}\cdot\mathcal{F}\right)\det\underline{h}^{-1},\end{aligned} \tag{7.115}$$

which yields

$$\begin{aligned}\mathcal{T}_{EM}(a) &= -(\mathcal{F}\cdot a)\cdot\mathcal{F} - \tfrac{1}{2}a\mathcal{F}\cdot\mathcal{F} \\ &= -\tfrac{1}{2}\mathcal{F}a\mathcal{F}.\end{aligned} \tag{7.116}$$

This is the covariant form of the tensor found in Section (6.2). It is intersting to see how the definition of $\mathcal{T}_{EM}$ as the functional derivative of $\mathcal{L}$ with respect to $\partial_{\overline{h}(a)}$ automatically preserves gauge invariance. For electromagnetism this has the effect of forcing $\mathcal{T}_{EM}$ to be symmetric. The form of $\mathcal{T}_{EM}$ (7.116) makes it clear that it is $\mathcal{F}$ which is the genuine physical field, so we should seek to express the field equations in terms of this object. To achieve this, we first write the second of the field equations (7.90) in the form

$$\mathcal{D}\wedge\overline{h}(a) = \overline{h}(\nabla\wedge a) + \kappa\overline{h}(a)\cdot\mathcal{S}, \tag{7.117}$$

which holds for all a. If we now define the bivector $B = a\wedge b$, we find that

$$\begin{aligned}\mathcal{D}\wedge\overline{h}(B) &= [\mathcal{D}\wedge\overline{h}(a)]\wedge\overline{h}(b) - \overline{h}(a)\wedge\mathcal{D}\wedge\overline{h}(b) \\ &= \overline{h}(\nabla\wedge a)\wedge\overline{h}(b) - \overline{h}(a)\wedge\overline{h}(\nabla\wedge b) + \kappa(\overline{h}(a)\cdot\mathcal{S})\wedge\overline{h}(b) \\ &\quad -\kappa\overline{h}(a)\wedge(\overline{h}(b)\cdot\mathcal{S}) \\ &= \overline{h}(\nabla\wedge B) - \kappa\overline{h}(B)\times\mathcal{S},\end{aligned} \tag{7.118}$$

which is used to write equation (7.112) in the form

$$\mathcal{D}\wedge\mathcal{F} - \kappa\mathcal{S}\times\mathcal{F} = \overline{h}(\nabla\wedge F) = 0. \tag{7.119}$$

Next, we use a double-duality transformation on (7.111) to write the left-hand side as

$$\begin{aligned}\nabla\cdot(\underline{h}(\mathcal{F})\det\underline{h}^{-1}) &= i\nabla\wedge(i\underline{h}(\mathcal{F})\det\underline{h}^{-1}) \\ &= i\nabla\wedge(\overline{h}^{-1}(i\mathcal{F})) \\ &= i\overline{h}^{-1}\left(\mathcal{D}\wedge(i\mathcal{F}) + \kappa(i\mathcal{F})\times\mathcal{S}\right),\end{aligned} \tag{7.120}$$

so that (7.111) becomes

$$\mathcal{D}\cdot\mathcal{F} - \kappa\mathcal{S}\cdot\mathcal{F} = i\overline{h}(Ji)\det\underline{h}^{-1} = \underline{h}^{-1}(J). \tag{7.121}$$

Writing

$$\mathcal{J} = \underline{h}^{-1}(J) \tag{7.122}$$

we can now combine (7.119) and (7.121) into the single equation

$$\mathcal{D}\mathcal{F} - \kappa\mathcal{S}\mathcal{F} = \mathcal{J}, \tag{7.123}$$

which achieves our objective. The gravitational background has led to the vector derivative ∇ being generalised to $\mathcal{D} - \kappa\mathcal{S}$. Equation (7.123) surely deserves considerable study. In particular, there is a clear need for a detailed study of the Green's functions of the $\mathcal{D} - \kappa\mathcal{S}$ operator. Furthermore, (7.123) makes it clear that, even if the A equation does not contain any torsion term, the $\mathcal{F}$ equation certainly does. This may be of importance in studying how $\mathcal{F}$ propagates from the surface of an object with a large spin current.

7.1.4 Comparison with Other Approaches

We should now compare our theory with general relativity and the ECKS theory. In Sections 7.2 and 7.3 a number of physical differences are illustrated, so here we concentrate on the how the mathematics compares. To simplify the comparison, we will suppose initially that spin effects are neglible. In this case equation (7.90) simplifies to $\mathcal{D}\wedge\bar{h}(a) = 0$. This equation can be solved to give $\Omega(a)$ as a function of $\bar{h}(a)$. This is achieved by first "protracting" with $\underline{h}^{-1}(\partial_a)$:

$$\begin{aligned}\underline{h}^{-1}(\partial_a)\wedge(\mathcal{D}\wedge\bar{h}(a)) &= \underline{h}^{-1}(\partial_a)\wedge\left[\bar{h}(\nabla)\wedge\bar{h}(a) + \bar{h}(\partial_b)\wedge(\Omega(b)\cdot\bar{h}(a))\right] \\ &= \underline{h}^{-1}(\partial_b)\wedge\bar{h}(\nabla)\wedge\bar{h}(b) + 2\bar{h}(\partial_b)\wedge\Omega(b) = 0.\end{aligned} \tag{7.124}$$

Contracting this expression with the vector $\underline{h}^{-1}(a)$ and rearranging yields

$$\begin{aligned}2\Omega(a) &= -2\bar{h}(\partial_b)\wedge(\Omega(b)\cdot\underline{h}^{-1}(a)) - \underline{h}^{-1}(a)\cdot\underline{h}^{-1}(\partial_b)\bar{h}(\nabla)\wedge\bar{h}(b) \\ &\quad +\underline{h}^{-1}(\partial_b)\wedge(a\cdot\nabla\bar{h}(b)) - \underline{h}^{-1}(\partial_b)\wedge\bar{h}(\dot{\nabla})\dot{\bar{h}}(b)\cdot\underline{h}^{-1}(a) \\ &= -2\bar{h}(\nabla\wedge g(a)) + \bar{h}(\nabla)\wedge\underline{h}^{-1}(a) - \bar{h}(\dot{\nabla})\wedge\dot{\bar{h}}g(a) \\ &\quad +\underline{h}^{-1}(\partial_b)\wedge(a\cdot\nabla\bar{h}(b)) \\ &= -\bar{h}(\nabla\wedge g(a)) + \underline{h}^{-1}(\partial_b)\wedge(a\cdot\nabla\bar{h}(b)),\end{aligned} \tag{7.125}$$

where

$$g(a) \equiv \bar{h}^{-1}\underline{h}^{-1}(a). \tag{7.126}$$

The quantity $g(a)$ is the gauge-theory analogue of the metric tensor. It is symmetric, and arises naturally when forming inner products,

$$\underline{h}^{-1}(a)\cdot\underline{h}^{-1}(b) = a\cdot g(b) = g(a)\cdot b. \tag{7.127}$$

Under translations $g(a)$ transforms as

$$g_x(a) \mapsto \bar{f}_x g_{x'}\underline{f}_x(a), \quad \text{where } x' = f(x), \tag{7.128}$$

and under an active rotation $g(a)$ is unchanged. The fact that $g(a)$ is unaffected by active rotations limits its usefulness, and this is a strong reason for not using the metric tensor as the foundation of our theory.

The comparison with general relativity is clarified by the introduction of a set of 4 coordinate functions over spacetime, $x^\mu = x^\mu(x)$. From these a coordinate frame is defined by

$$e_\mu = \partial_\mu x, \tag{7.129}$$

where $\partial_\mu = \partial_{x^\mu}$. The reciprocal frame is defined as

$$e^\mu = \nabla x^\mu \tag{7.130}$$

and satisfies

$$e^\nu \cdot e_\mu = (\partial_\mu x)\cdot \nabla x^\nu = \partial_{x^\mu} x^\nu = \delta^\nu_\mu. \tag{7.131}$$

From these we define a frame of "contravariant" vectors

$$g_\mu = \underline{h}^{-1}(e_\mu) \tag{7.132}$$

and a dual frame of "covariant" vectors

$$g^\mu = \overline{h}(e^\mu). \tag{7.133}$$

These satisfy (no torsion)

$$\begin{aligned} g_\mu \cdot g^\nu &= \delta^\nu_\mu, & (7.134)\\ \mathcal{D}\wedge g^\mu &= 0 & (7.135)\end{aligned}$$

and

$$g_\mu \cdot \mathcal{D} g_\nu - g_\nu \cdot \mathcal{D} g_\mu = 0. \tag{7.136}$$

The third of these identities is the flat-space equivalent of the vanishing of the Lie bracket for a coordinate frame in Riemannian geometry.

From the $\{g_\mu\}$ frame the metric coefficients are defined by

$$g_{\mu\nu} = g_\mu \cdot g_\nu, \tag{7.137}$$

which enables us to now make contact with Riemannian geometry. Writing Ω_μ for $\Omega(e_\mu)$, we find from (7.125) that

$$2\Omega_\mu = g_\alpha \wedge (\partial_\mu g^\alpha) + g^\alpha \wedge g^\beta \partial_\beta g_{\alpha\mu}. \tag{7.138}$$

The connection is defined by

$$\Gamma^\mu_{\nu\lambda} = g^\mu \cdot (\mathcal{D}_\nu g_\lambda) \tag{7.139}$$

so that, with $a_\lambda \equiv a \cdot g_\lambda$,

$$\begin{aligned} \partial_\nu a_\lambda - \Gamma^\mu_{\nu\lambda} a_\mu &= \partial_\nu (a\cdot g_\lambda) - a\cdot(\mathcal{D}_\nu g_\lambda) \\ &= g_\lambda \cdot (\mathcal{D}_\nu a) & (7.140)\end{aligned}$$

as required — the connection records the fact that, by writing $a_\lambda = a\cdot g_\lambda$, additional x-dependence is introduced through the g_λ.

By using (7.138) in (7.139), $\Gamma^\mu_{\nu\lambda}$ is given by

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2}g^{\mu\alpha}(\partial_\nu g_{\alpha\lambda} + \partial_\lambda g_{\alpha\nu} - \partial_\alpha g_{\nu\lambda}), \tag{7.141}$$

which is the conventional expression for the Christoffel connection. In the absence of spin, the introduction of a coordinate frame unpackages our equations to the set of scalar equations used in general relativity. The essential difference is that in GR the quantity $g_{\mu\nu}$ is fundamental, and can only be defined locally, whereas in our theory the fundamental variables are the $\overline{h}$ and Ω fields, which are defined globally throughout spacetime. One might expect that the only differences that could show up from this shift would be due to global, topological considerations. In fact, this is not the case, as is shown in the following sections. The reasons for these differences can be either physical, due to the different understanding attached to the variables in the theory, or mathematical, due often to the constraint that the metric must be generated from a suitable $\overline{h}$ function. It is not always the case that such an $\overline{h}$ function can be found, as is demonstrated in Section 7.2.1.

The ability to develop a coordinate-free theory of gravity offers a number of advantages over approaches using tensor calculus. In particular, the physical content of the theory is separated from the artefacts of the chosen frame. Thus the $\overline{h}$ and Ω fields only differ from the identity and zero in the presence of matter. This clarifies much of the physics involved, as well as making many equations easier to manipulate.

Many of the standard results of classical Riemannian geometry have particularly simple expressions in this STA-based theory. Similar expressions can be found in Chapter 5 of Hestenes & Sobczyk [24], who have developed Riemannian geometry from the viewpoint of geometric calculus. All the symmetries of the Riemann tensor are summarised in the single equation

$$\partial_a\wedge\mathcal{R}(a\wedge b) = 0. \tag{7.142}$$

This says that the trivector $\partial_a\wedge\mathcal{R}(a\wedge b)$ vanishes for all values of the vector b, and so represents a set of 16 scalar equations. These reduce the 36-component tensor $\mathcal{R}(B)$ to a function with only 20 degrees of freedom — the correct number for Riemannian geometry. Equation (7.142) can be contracted with ∂_b to yield

$$\partial_a\wedge\mathcal{R}(a) = 0, \tag{7.143}$$

which says that the Ricci tensor is symmetric. The Bianchi identity is also compactly written:

$$\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(B) = 0, \tag{7.144}$$

where the overdot notation is defined via

$$\dot{\mathcal{D}}\dot{T}(M) \equiv \mathcal{D}T(M) - \partial_a T(a\cdot\mathcal{D}M). \tag{7.145}$$

Equation (7.144) can be contracted with $\partial_b\wedge\partial_a$ to yield

$$\begin{aligned}(\partial_b\wedge\partial_a)\cdot\big(\dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(a\wedge b)\big) &= \partial_b\cdot\big(\dot{\mathcal{R}}(\dot{\mathcal{D}}\wedge b) - \dot{\mathcal{D}}\wedge\dot{\mathcal{R}}(b)\big) \\ &= -2\dot{\mathcal{R}}(\dot{\mathcal{D}}) + \mathcal{D}\mathcal{R} = 0. \end{aligned} \tag{7.146}$$

It follows that

$$\dot{\mathcal{G}}(\dot{\mathcal{D}}) = 0 \tag{7.147}$$

which, in conventional terms, represents conservation of the Einstein tensor. Many other results can be written equally compactly.

The inclusion of torsion leads us to a comparison with the ECKS theory, which is certainly closest to the approach adopted here. The ECKS theory arose from attempts to develop gravity as a gauge theory, and modern treatments do indeed emphasise active transformations [81]. However, the spin-torsion theories ultimately arrived at all involve a curved-space picture of gravitational interactions, even if they started out as a gauge theory in flat space. Furthermore, the separation into local translations and rotations is considerably cleaner in the theory developed here, as all transformations considered are finite, rather than infinitessimal. The introduction of a coordinate frame can be used to reproduce the equations of a particular type of spin-torsion theory (one where the torsion is generated by Dirac matter) but again differences result from our use of a flat background spacetime. The inclusion of torsion alters equations (7.142) to (7.147). For example, equation (7.142) becomes

$$\partial_a \wedge \mathcal{R}(a \wedge b) = -\kappa b \cdot \mathcal{D}\mathcal{S} + \tfrac{1}{2}\kappa \mathcal{D} \wedge \mathcal{S}\, b, \tag{7.148}$$

equation (7.143) becomes

$$\partial_a \wedge \mathcal{R}(a) = -\kappa \mathcal{D} \cdot \mathcal{S} \tag{7.149}$$

and equation (7.144) becomes

$$\dot{\mathcal{D}} \wedge \dot{\overline{\mathcal{R}}}(B) + \kappa \mathcal{S} \times \overline{\mathcal{R}}(B) = 0. \tag{7.150}$$

The presence of torsion destroys many of the beautiful results of Riemannian geometry and, once the connection between the gauge theory quantities and their counterparts in Riemannian geometry is lost, so too is much of the motivation for adopting a curved-space viewpoint.

Finally, it is important to stress that there is a difference between the present gauge theory of gravity and Yang-Mills gauge theories. Unlike Yang-Mills theories, the Poincaré gauge transformations do not take place in an internal space, but in real spacetime — they transform between physically distinct situations. The point is not that all physical observables should be gauge invariant, but that the fields should satisfy the same equations, regardless of their state. Thus an accelerating body is subject to the same physical laws as a static one, even though it may be behaving quite differently (it could be radiating away electromagnetic energy, for example).

7.2 Point Source Solutions

In this section we seek solutions to the field equations in the absence of matter. In this case, the stress-energy equation (7.67) is

$$\mathcal{R}(a) - \tfrac{1}{2}a\mathcal{R} = 0, \tag{7.151}$$

which contracts to give

$$\mathcal{R} = 0. \tag{7.152}$$

Our field equations are therefore

$$\begin{aligned} \mathcal{D}\wedge\bar{h}(a) &= \bar{h}(\nabla\wedge a) \\ \mathcal{R}(a) &= 0. \end{aligned} \tag{7.153}$$

As was discussed in the previous section, if we expand in a basis then the equations for the coordinates are the same as those of general relativity. It follows that any solution to (7.153) will generate a metric which solves the Einstein equations. But the converse does not hold — the additional physical constraints at work in our theory rule out certain solutions that are admitted by general relativity. This is illustrated by a comparison of the Schwarzschild metric used in general relativity with the class of radially-symmetric static solutions admitted in the present theory. Throughout the following sections we use units with $G = 1$.

7.2.1 Radially-Symmetric Static Solutions

In looking for radially-symmetric solutions to (7.153), it should be clear that we are actually finding possible field configurations around a δ-function source (a point of matter). That is, we are studying the analog of the Coulomb problem in electrostatics. In general, specifying the matter and spin densities specifies the $\bar{h}$ and Ω fields completely via the field equations (7.89) and (7.90). Applying an active transformation takes us to a different matter configuration and solves a different (albeit related) problem. This is not the case when symmetries are present, in which case a class of gauge transformations exists which do not alter the matter and field configurations. For the case of point-source solutions, practically all gauge transformations lead to new solutions. In this case the problem is simplified by imposing certain symmetry requirements at the outset. By this means, solutions can be classified into equivalence classes. This is very natural from the point of view of a gauge theory, though it should be borne in mind that in our theory gauge transformations can have physical consequences.

Here we are interested in the class of radially-symmetric static solutions. This means that, if we place the source at the origin in space, we demand that the $\bar{h}$ and Ω fields only show dependence on x through the spatial radial vector (spacetime bivector)

$$\mathbf{x} = x\wedge\gamma_0. \tag{7.154}$$

Here γ_0 is a fixed time-like direction. We are free to choose this as we please, so that a global symmetry remains. This rigid symmetry can only be removed with further physical assumptions; for example that the matter is comoving with respect to the Hubble flow of galaxies (*i.e.* it sees zero dipole moment in the cosmic microwave background anisotropy).

To facilitate the discussion of radially-symmetric solutions, it is useful to introduce a set of polar coordinates

$$\begin{aligned} t &= \gamma_0\cdot x & -\infty < t < \infty \\ r &= |x\wedge\gamma_0| & 0 \leq r < \infty \\ \cos\theta &= -\gamma_3\cdot x/r & 0 \leq \theta \leq \pi \\ \tan\phi &= \gamma_2\cdot x/(\gamma_1\cdot x) & 0 \leq \phi < 2\pi, \end{aligned} \tag{7.155}$$

where the $\{\gamma_1, \gamma_2, \gamma_3\}$ frame is a fixed, arbitrary spatial frame. From these coordinates, we define the coordinate frame

$$\begin{aligned} e_t &= \partial_t x = \gamma_0 \\ e_r &= \partial_r x = \sin\theta \cos\phi\gamma_1 + \sin\theta \sin\phi\gamma_2 + \cos\theta\gamma_3 \\ e_\theta &= \partial_\theta x = r(\cos\theta \cos\phi\gamma_1 + \cos\theta \sin\phi\gamma_2 - \sin\theta\gamma_3) \\ e_\phi &= \partial_\phi x = r \sin\theta(-\sin\phi\gamma_1 + \cos\phi\gamma_2). \end{aligned} \tag{7.156}$$

The best-known radially-symmetric solution to the Einstein equations is given by the Schwarzschild metric,

$$ds^2 = (1 - 2M/r)dt^2 - (1 - 2M/r)^{-1}dr^2 - r^2(d\theta^2 + sin^2\theta\, d\phi^2), \tag{7.157}$$

from which the components of $g_{\mu\nu} = g_\mu \cdot g_\nu$ can be read straight off. Since $g_\mu = \underline{h}^{-1}(e_\nu)$, we need to "square root" $g_{\mu\nu}$ to find a suitable $\underline{h}^{-1}$ (and hence $\overline{h}$) that generates it. This $\underline{h}^{-1}$ is only unique up to rotations. If we look for such a function we immediately run into a problem — the square roots on either side of the horizon (at $r = 2M$) have completely different forms. For example, the simplest forms have

$$\left.\begin{array}{ll} g_t = (1 - 2M/r)^{1/2} e_t & g_\theta = e_\theta \\ g_r = (1 - 2M/r)^{-1/2} e_r & g_\phi = e_\phi \end{array}\right\} \quad \text{for } r > 2M \tag{7.158}$$

and

$$\left.\begin{array}{ll} g_t = (2M/r - 1)^{1/2} e_r & g_\theta = e_\theta \\ g_r = (2M/r - 1)^{-1/2} e_t & g_\phi = e_\phi \end{array}\right\} \quad \text{for } r < 2M. \tag{7.159}$$

These do not match at $r = 2M$, and there is no rotation which gets round this problem. As we have set out to find the fields around a δ-function source, it is highly undesirable that these fields should be discontinuous at some finite distance from the source. Rather than resort to coordinate transformations to try and patch up this problem, we will postulate a suitably general form for $\overline{h}$ and Ω, and solve the field equations for these. Once this is done, we will return to the subject of the problems that the Schwarzschild metric presents.

We postulate the following form for $\overline{h}(a)$

$$\begin{array}{rclrcl} \overline{h}(e_t) &=& f_1 e_t + f_2 e_r & \overline{h}(e_\theta) &=& e_\theta \\ \overline{h}(e_r) &=& g_1 e_r + g_2 e_t & \overline{h}(e_\phi) &=& e_\phi, \end{array} \tag{7.160}$$

where f_i and g_i are functions of r only. We can write $\overline{h}$ in the more compact form

$$\overline{h}(a) = a + a \cdot e_t ((f_1 - 1)e_t + f_2 e_r) - a \cdot e_r ((g_1 - 1)e_r + g_2 e_t), \tag{7.161}$$

and we could go further and replace e_r and r by the appropriate functions of x. This would show explicitly how $\overline{h}(a)$ is a linear function of a and a non-linear function of $x \wedge \gamma_0$. We also postulate a suitable form for $\Omega(a)$, writing Ω_μ for $\Omega(e_\mu)$,

$$\begin{array}{rclrcl} \Omega_t &=& \alpha e_r e_t & \Omega_\theta &=& (\beta_1 e_r + \beta_2 e_t) e_\theta / r \\ \Omega_r &=& 0 & \Omega_\phi &=& (\beta_1 e_r + \beta_2 e_t) e_\phi / r, \end{array} \tag{7.162}$$

with α and β_i functions of r only. More compactly, we can write

$$\Omega(a) = \alpha a \cdot e_t e_r e_t - a \wedge (e_r e_t)(\beta_1 e_t + \beta_2 e_r)/r. \tag{7.163}$$

We could have used (7.138) to solve for $\Omega(a)$ in terms of the f_i and g_i, but this vastly complicates the problem. The second-order form of the equations immediately introduces unpleasant non-linearities, and the equations are far less easy to solve. The better approach is to use (7.138) to see what a suitable form for $\Omega(a)$ looks like, but to then leave the functions unspecified. Equations (7.160) and (7.162) do not account for the most general type of radially-symmetric static solution. The trial form is chosen to enable us to find a single solution. The complete class of solutions can then be obtained by gauge transformations, which will be considered presently.

The first of the field equations (7.153) can be written as

$$\mathcal{D}\wedge g^{\mu} = \bar{h}(\nabla)\wedge g^{\mu} + g^{\nu}\wedge(\Omega_{\nu}\cdot g_{\nu}) = 0, \tag{7.164}$$

which quickly yields the four equations

$$g_1 f_1' - g_2 f_2' + \alpha({f_1}^2 - {f_2}^2) = 0 \tag{7.165}$$
$$g_1 g_2' - g_1' g_2 + \alpha(f_1 g_2 - f_2 g_1) = 0 \tag{7.166}$$
$$g_1 = \beta_1 + 1 \tag{7.167}$$
$$g_2 = \beta_2, \tag{7.168}$$

where the primes denote differentiation with respect to r. We immediately eliminate β_1 and β_2 using (7.167) and (7.168). Next, we calculate the field strength tensor. Writing $R_{\mu\nu}$ for $R(e_\mu\wedge e_\nu)$, we find that

$$\begin{array}{rcl} R_{tr} &=& -\alpha' e_r e_t \\ R_{t\theta} &=& \alpha(g_1 e_t + g_2 e_r)e_\theta/r \\ R_{t\phi} &=& \alpha(g_1 e_t + g_2 e_r)e_\phi/r \\ R_{r\theta} &=& (g_1' e_r + g_2' e_t)e_\theta/r \\ R_{r\phi} &=& (g_1' e_r + g_2' e_t)e_\phi/r \\ R_{\theta\phi} &=& ({g_1}^2 - {g_2}^2 - 1)e_\theta e_\phi/r^2. \end{array} \tag{7.169}$$

Contracting with g^μ and setting the result equal to zero gives the final four equations

$$2\alpha + \alpha' r = 0 \tag{7.170}$$
$$2g_1' + f_1\alpha' r = 0 \tag{7.171}$$
$$2g_2' + f_2\alpha' r = 0 \tag{7.172}$$
$$\alpha r(f_1 g_1 - f_2 g_2) + r(g_1 g_1' - g_2 g_2') + {g_1}^2 - {g_2}^2 - 1 = 0. \tag{7.173}$$

The first of these (7.170) can be solved for α immediately,

$$\alpha = \frac{M}{r^2}, \tag{7.174}$$

where M is the (positive) constant of integration and represents the mass of the source. Equations (7.171) and (7.172) now define the f_i in terms of the g_i

$$\alpha f_1 = g_1' \tag{7.175}$$
$$\alpha f_2 = g_2'. \tag{7.176}$$

These are consistent with (7.166), and substituted into (7.165) yield

$$(f_1 g_1 - f_2 g_2)' = 0. \tag{7.177}$$

But the quantity $f_1 g_1 - f_2 g_2$ is simply the determinant of $\overline{h}$, so we see that

$$\det \underline{h} = f_1 g_1 - f_2 g_2 = \text{constant}. \tag{7.178}$$

We expect the effect of the source to fall away to zero at large distances, so $\overline{h}$ should tend asymptotically to the identity function. It follows that the constant $\det \underline{h}$ should be set to 1. All that remains is the single differential equation (7.173)

$$\tfrac{1}{2} r \partial_r ({g_1}^2 - {g_2}^2) + {g_1}^2 - {g_2}^2 = 1 - M/r, \tag{7.179}$$

to which the solution is

$${g_1}^2 - {g_2}^2 = 1 - 2M/r, \tag{7.180}$$

ensuring consistency with $\det \underline{h} = 1$.

We now have a set of solutions defined by

$$\begin{aligned} \alpha &= M/r^2 \\ {g_1}^2 - {g_2}^2 &= 1 - 2M/r \\ M f_1 &= r^2 g_1' \\ M f_2 &= r^2 g_2'. \end{aligned} \tag{7.181}$$

The ease of derivation of this solution set compares very favourably with the second-order metric-based approach. A particularly pleasing feature of this derivation is the direct manner in which α is found. This is the coefficient of the Ω_t bivector, which accounts for the radial acceleration of a test particle. We see that it is determined simply by the Newtonian formula!

The solutions (7.181) are a one-parameter set. We have a free choice of the g_2 function, say, up to the constraints that

$${g_2}^2(r) \geq 2M/r - 1, \tag{7.182}$$

and

$$\left. \begin{aligned} f_1, g_1 &\to 1 \\ f_2, g_2 &\to 0 \end{aligned} \right\} \quad \text{as} \quad r \to \infty. \tag{7.183}$$

As an example, which will be useful shortly, one compact form that the solution can take is

$$\begin{aligned} g_1 &= \cosh(M/r) - e^{M/r} M/r & \qquad f_1 &= \cosh(M/r) + e^{M/r} M/r \\ g_2 &= -\sinh(M/r) + e^{M/r} M/r & \qquad f_2 &= -\sinh(M/r) - e^{M/r} M/r. \end{aligned} \tag{7.184}$$

The solution (7.181) can be substituted back into (7.169) and the covariant field strength tensor (Riemann tensor) is found to be

$$\begin{aligned} \mathcal{R}(B) &= -2\frac{M}{r^3} B + 3\frac{M}{r^3} B \times (e_r e_t) e_r e_t \\ &= -\frac{M}{2r^3}(B + 3 e_r e_t B e_r e_t). \end{aligned} \tag{7.185}$$

It can now be confirmed that $\partial_a \cdot \mathcal{R}(a\wedge b) = 0$. Indeed, one can simultaneously check both the field equations and the symmetry properties of $\mathcal{R}(B)$, since $\mathcal{R}(a) = 0$ and $\partial_a \wedge \mathcal{R}(a\wedge b) = 0$ combine into the single equation

$$\partial_a \mathcal{R}(a\wedge b) = 0. \tag{7.186}$$

This equation greatly facilitates the study of the Petrov classification of vacuum solutions to the Einstein equations, as is demonstrated in Chapter 3 of Hestenes & Sobczyk [24]. There the authors refer to $\partial_a \cdot \mathcal{R}(a\wedge b)$ as the *contraction* and $\partial_a \wedge \mathcal{R}(a\wedge b)$ as the *protraction*. The combined quantity $\partial_a \mathcal{R}(a\wedge b)$ is called simply the *traction*. These names have much to recommend them, and are adopted wherever necessary.

Verifying that (7.185) satisfies (7.186) is a simple matter, depending solely on the result that, for an arbitrary bivector B,

$$\begin{aligned} \partial_a(a\wedge b + 3Ba\wedge bB^{-1}) &= \partial_a(a\wedge b + 3BabB^{-1} - 3Ba\cdot bB^{-1}) \\ &= \partial_a(a\wedge b - 3a\cdot b) \\ &= \partial_a(ab - 4a\cdot b) \\ &= 0. \end{aligned} \tag{7.187}$$

The compact form of the Riemann tensor (7.185), and the ease with which the field equations are verified, should serve to demonstrate the power of the STA approach to relativistic physics.

Radially-Symmetric Gauge Transformations

From a given solution in the set (7.181) we can generate further solutions via radially-symmetric gauge transformations. We consider Lorentz rotations first. All rotations leave the metric terms $g_{\mu\nu} = g_\mu \cdot g_\mu$ unchanged, since these are defined by invariant inner products, so ${g_1}^2 - {g_2}^2$, ${f_1}^2 - {f_2}^2$, $f_1 g_2 - f_2 g_1$ and $\det \underline{h}$ are all invariant. Since the fields are a function of $x\wedge e_t$ only, the only Lorentz rotations that preserve symmetry are those that leave $x\wedge e_t$ unchanged. It is easily seen that these leave the Riemann tensor (7.185) unchanged as well. There are two such transformations to consider; a rotation in the $e_\theta \wedge e_\phi$ plane and a boost along the radial axis. The rotors that determine these are as follows:

$$\text{Rotation:} \quad R = \exp(\chi(r) i e_r e_t/2); \tag{7.188}$$

$$\text{Radial Boost:} \quad R = \exp(\chi(r) e_r e_t/2). \tag{7.189}$$

Both rotations leave Ω_t untransformed, but introduce an Ω_r and transform the Ω_θ and Ω_ϕ terms.

If we take the solution in the form (7.184) and apply a radial boost determined by the rotor

$$R = \exp\left(\frac{M}{2r} e_r e_t\right), \tag{7.190}$$

we arrive at the following, highly compact solution

$$\begin{aligned} \overline{h}(a) &= a + \frac{M}{r} a\cdot e_- e_- \\ \Omega(a) &= \frac{M}{r^2}(e_-\wedge a + 2e_-\cdot a e_r e_t) \end{aligned} \tag{7.191}$$

where

$$e_- = e_t - e_r. \qquad (7.192)$$

Both the forms (7.191) and (7.184) give a metric which, in GR, is known as the (advanced-time) Eddington-Finkelstein form of the Schwarzschild solution,

$$ds^2 = (1 - 2M/r)dt^2 - (4M/r)dr\,dt - (1 + 2M/r)dr^2 - r^2(d\theta^2 + sin^2\theta\, d\phi^2). \qquad (7.193)$$

There are also two types of transformation of position dependence to consider. The first is a (radially-dependent) translation up and down the e_t-axis,

$$x^\dagger = f(x) = x + u(r)e_t. \qquad (7.194)$$

(We use the dagger to denote the transformed position, since we have already used a prime to denote the derivative with respect to r.) From (7.194) we find that

$$\underline{f}(a) = a - u'a\cdot e_r e_t \qquad (7.195)$$

$$\overline{f}^{-1}(a) = a + u'a\cdot e_t e_r, \qquad (7.196)$$

and that

$$x^\dagger\wedge e_t = x\wedge e_t. \qquad (7.197)$$

Since all x-dependence enters $\overline{h}$ through $x\wedge e_t$ it follows that $\overline{h}_{x^\dagger} = \overline{h}_x$ and $\Omega_{x^\dagger} = \Omega_x$. The transformed functions therefore have

$$\begin{aligned}
\overline{h}^\dagger(e_t) &= (f_1 + u'g_2)e_t + (f_2 + u'g_1)e_r && (7.198)\\
\overline{h}^\dagger(e_r) &= \overline{h}(e_r) && (7.199)\\
\Omega^\dagger(e_t) &= \Omega(e_t) && (7.200)\\
\Omega^\dagger(e_r) &= (Mu'/r^2)e_r e_t, && (7.201)
\end{aligned}$$

with all other terms unchanged. The f_i's transform, but the g_i's are fixed. A time translation can be followed by a radial boost to replace the $\Omega^\dagger(e_r)$ term by $\Omega(e_r)$, and so move between solutions in the one-parameter set of (7.181).

The final transformation preserving radial symmetry is a radial translation, where the fields are stretched out along the radial vector. We define

$$x^\dagger = f(x) = x\cdot e_t e_t + u(r)e_r \qquad (7.202)$$

so that

$$\begin{aligned}
r^\dagger &= |x^\dagger\wedge e_t| = u(r) && (7.203)\\
e_r^\dagger &= \frac{x^\dagger\wedge e_t}{|x^\dagger\wedge e_t|}e_t = e_r. && (7.204)
\end{aligned}$$

The differential of this transformation gives

$$\begin{aligned}
\underline{f}(a) &= a\cdot e_t e_t - u'a\cdot e_r e_r + \frac{u}{r}a\wedge(e_r e_t)e_r e_t && (7.205)\\
\overline{f}^{-1}(a) &= a\cdot e_t e_t - \frac{1}{u'}a\cdot e_r e_r + \frac{r}{u}a\wedge(e_r e_t)e_r e_t && (7.206)
\end{aligned}$$

and

$$\det \underline{f} = u'(u/r)^2. \tag{7.207}$$

The new function $\overline{h}^{\dagger} = \overline{h}_{x^\dagger}\overline{f}^{-1}$ has an additional dilation in the $e_\theta e_\phi$ plane, and the behaviour in the $e_r e_t$ plane is defined by

$$f_i^\dagger(r) = f_i(r^\dagger) \tag{7.208}$$

$$g_i^\dagger(r) = \frac{1}{u'} g_i(r^\dagger). \tag{7.209}$$

The horizon has now moved from $r = 2M$ to $r^\dagger = 2M$, as is to be expected for an active radial dilation. The physical requirements of our theory restrict the form that the transformation (7.202) can take. The functions r and $u(r)$ both measure the radial distance from a given origin, and since we do not want to start moving the source around (which would change the problem) we must have $u(0) = 0$. The function $u(r)$ must therefore be monotomic-increasing to ensure that the map between r and r' is 1-to-1. Furthermore, $u(r)$ must tend to r at large r to ensure that the field dies away suitably. It follows that

$$u'(r) > 0, \tag{7.210}$$

so the transformation does not change the sign of $\det \underline{h}$.

We have now found a 4-parameter solution set, in which the elements are related via the rotations (7.188) and (7.189) and the transformations (7.194) and (7.202). The fields are well-defined everywhere except at the origin, where a point mass is present. A second set of solutions is obtained by the discrete operation of time-reversal, defined by

$$f(x) = -e_t x e_t \tag{7.211}$$

$$\Rightarrow f(x)\wedge e_t = -(e_t x e_t)\wedge e_t = x\wedge e_t. \tag{7.212}$$

This translation on its own just changes the signs of the f_i functions, and so reverses the sign of $\det \underline{h}$. The translation therefore defines fields whose effects do not vanish at large distances. To correct this, the $\overline{h}$ and Ω fields must also be time-reversed, so that the new solution has

$$\begin{aligned} \overline{h}^T(a) &= -e_t \overline{h}_{f(x)}(-e_t a e_t) e_t \\ &= e_t \overline{h}(e_t a e_t) e_t \end{aligned} \tag{7.213}$$

and

$$\begin{aligned} \Omega^T(a) &= e_t \Omega_{f(x)}(-e_t a e_t) e_t \\ &= -e_t \Omega(e_t a e_t) e_t. \end{aligned} \tag{7.214}$$

For example, the result of time-reversal on the solution defined by (7.191) is the new solution

$$\begin{aligned} \overline{h}^T(a) &= e_t[e_t a e_t + \frac{M}{r}(e_t a e_t)\cdot e_- e_-] e_t \\ &= a + \frac{M}{r} a\cdot e_+ e_+ \end{aligned} \tag{7.215}$$

and

$$
\begin{aligned}
\Omega^T(a) &= -\frac{M}{r^2} e_t (e_- \wedge (e_t a e_t) + 2e_- \cdot (e_t a e_t) e_r e_t) e_t \\
&= \frac{M}{r^2} (2a \cdot e_+ e_r e_t - e_+ \wedge a), \qquad (7.216)
\end{aligned}
$$

where $e_+ = e_t + e_r$. This new solution reproduces the metric of the retarded-time Eddington-Finkelstein form of the Schwarzschild solution. Time reversal has therefore switched us from a solution where particles can cross the horizon on an inward journey, but cannot escape, to a solution where particles can leave, but cannot enter. Covariant quantities, such as the field strength (7.169), are, of course, unchanged by time reversal. From the gauge-theory viewpoint, it is natural that the solutions of the field equations should fall into sets which are related by discrete transformations that are not simply connected to the identity. The solutions are simply reproducing the structure of the Poincaré group on which the theory is constructed.

Behaviour near the Horizon

For the remainder of this section we restrict the discussion to solutions for which $\det \underline{h} = 1$. For these the line element takes the form

$$
\begin{aligned}
ds^2 &= (1 - 2M/r)dt^2 - (f_1 g_2 - f_2 g_1) 2dr\, dt - ({f_1}^2 - {f_2}^2) dr^2 \\
&\quad - r^2 (d\theta^2 + \sin^2\theta\, d\phi^2). \qquad (7.217)
\end{aligned}
$$

The horizon is at $r = 2M$, and at this distance we must have

$$
g_1 = \pm g_2. \qquad (7.218)
$$

But, since $\det \underline{h} = f_1 g_1 - f_2 g_2 = 1$, we must also have

$$
f_1 g_2 - f_2 g_1 = \pm 1 \quad \text{at } r = 2M, \qquad (7.219)
$$

so an off-diagonal term must be present in the metric at the horizon. The assumption that this term can be transformed away everywhere does not hold in our theory. This resolves the problem of the Schwarzschild discontinuity discussed at the start of this section. The Schwarzschild metric does not give a solution that is well-defined everywhere, so lies outside the set of metrics that are derivable from (7.181). Outside the horizon, however, it is always possible to transform to a solution that reproduces the Schwarzschild line element, and the same is true inside. But the transformations required to do this do not mesh at the boundary, and their derivatives introduce δ-functions there. Because the Schwarzschild line element is valid on either side of the horizon, it reproduces the correct Riemann tensor (7.185) on either side. Careful analysis shows, however, that the discontinuities in the Ω_θ and Ω_ϕ fields required to reproduce the Schwarzschild line element lead to δ-functions at the horizon in $\mathcal{R}(a \wedge b)$.

The fact that the $f_1 g_2 - f_2 g_1$ term must take a value of ± 1 at the horizon is interesting, since this term changes sign under time-reversal (7.213). Once a horizon has formed, it is therefore no longer possible to find an $\overline{h}$ such that the line element derived from it is invariant under time reversal. This suggests that the $f_1 g_2 - f_2 g_1$ term retains information about the process by which the horizon formed — recording the fact that at some earlier time matter was falling in radially. Matter infall certainly picks out a time direction, and knowledge of this is maintained after the horizon has formed. This irreversibility is apparent from the study of test particle geodesics [9]. These can cross the horizon to the inside in a finite external coordinate time, but can never get back out again, as one expects of a black hole.

The above conclusions differ strongly from those of GR, in which the ultimate form of the Schwarzschild solution is the Kruskal metric. This form is arrived at by a series of coordinate transformations, and is motivated by the concept of "maximal extension" — that all geodesics should either exist for all values of their affine parameter, or should terminate at a singularity. None of the solutions presented here have this property. The solution (7.191), for example, has a pole in the proper-time integral for outgoing radial geodesics. This suggests that particles following these geodesics would spend an infinite coordinate time hovering just inside the horizon. In fact, in a more physical situation this will not be the case — the effects of other particles will tend to sweep all matter back to the centre. The solutions presented here are extreme simplifications, and there is no compelling physical reason why we should look for "maximal" solutions. This is important, as the Kruskal metric is time-reverse symmetric and so must fail to give a globally valid solution in our theory. There are a number of ways to see why this happens. For example, the Kruskal metric defines a spacetime with a different global topology to flat spacetime. We can reach a similar conclusion by studying how the Kruskal metric is derived from the Schwarzschild metric. We assume, for the time being, that we are outside the horizon so that a solution giving the Schwarzschild line element is

$$\begin{aligned} g_1 &= \Delta^{1/2} \qquad & g_2 &= 0 \\ f_1 &= \Delta^{-1/2} \qquad & f_2 &= 0 \end{aligned} \tag{7.220}$$

where

$$\Delta = 1 - 2M/r. \tag{7.221}$$

The first step is to re-interpret the coordinate transformations used in general relativity as active local translations. For example, the advanced Eddington-Finkelstein metric is reached by defining

$$t^\dagger - r^\dagger = t - (r + 2M\ln(r - 2M)) \tag{7.222}$$

$$r^\dagger = r \tag{7.223}$$

or

$$x^\dagger = x - 2M\ln(r - 2M)e_t, \tag{7.224}$$

which is now recognisable as a translation of the type of equation (7.194). The result of this translation is the introduction of an $f_2^\dagger$ function

$$f_2^\dagger = -\frac{2M}{r}\Delta^{-1/2}, \tag{7.225}$$

which now ensures that $f_1^\dagger g_2^\dagger - f_2^\dagger g_1^\dagger = 1$ at the horizon. The translation (7.224), which is only defined outside the horizon, has produced a form of solution which at least has a chance of being extended across the horizon. In fact, an additional boost is still required to remove some remaining discontinuities. A suitable boost is defined by

$$R = \exp(e_r e_t \chi/2), \tag{7.226}$$

where

$$\sinh\chi = \tfrac{1}{2}(\Delta^{-1/2} - \Delta^{1/2}) \tag{7.227}$$

and so is also only defined outside the horizon. The result of this pair of transformations is the solution (7.191), which now extends smoothly down to the origin.

In a similar manner, it is possible to reach the retarted-time Eddington-Finkelstein metric by starting with the translation defined by

$$t^\dagger + r^\dagger = t + (r + 2M\ln(r - 2M)) \tag{7.228}$$

$$r^\dagger = r. \tag{7.229}$$

The Kruskal metric, on the other hand, is reached by combining the advance and retarded coordinates and writing

$$t^\dagger - r^\dagger = t - (r + 2M\ln(r - 2M)) \tag{7.230}$$

$$t^\dagger + r^\dagger = t + (r + 2M\ln(r - 2M)), \tag{7.231}$$

which defines the translation

$$x^\dagger = x\cdot e_t e_t + (r + 2M\ln(r - 2M))e_r. \tag{7.232}$$

This translation is now of the type of equation (7.202), and results in a completely different form of solution. The transformed solution is still only valid for $r > 2M$, and the transformation (7.232) has not introduced the required $f_1 g_2 - f_2 g_1$ term. No additional boost or rotation manufactures a form which can then be extended to the origin. The problem can still be seen when the Kruskal metric is written in the form

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(dw^2 - dz^2) - r^2(d\theta^2 + sin^2\theta\, d\phi^2), \tag{7.233}$$

where

$$z^2 - w^2 = \frac{1}{2M}(r - 2M)e^{-r/2M} \tag{7.234}$$

$$\frac{w}{z} = \tanh\left(\frac{t}{4M}\right), \tag{7.235}$$

which is clearly only defined for $r > 2M$. The loss of the region with $r < 2M$ does not present a problem in GR, since the r-coordinate has no special significance. But it is a problem if r is viewed as the distance from the source of the fields, as it is in the present theory, since then the fields must be defined for all r. Even in the absence of torsion, the flat-space gauge-theory approach to gravity produces physical consequences that clearly differ from general relativity, despite the formal mathematical similarities between the two theories.

7.2.2 Kerr-Type Solutions

We now briefly discuss how the Kerr class of solutions fit into the present theory. The detailed comparisons of the previous section will not be reproduced, and we will simply illustrate a few novel features. Our starting point is the Kerr line element in Boyer-Lindquist form [87]

$$ds^2 = dt^2 - \rho^2(\frac{dr^2}{\Delta} + d\theta^2) - (r^2 + L^2)\sin^2\theta\, d\phi^2 - \frac{2Mr}{\rho^2}(L\sin^2\theta\, d\phi - dt)^2, \tag{7.236}$$

where

$$\begin{aligned} \rho^2 &= r^2 + L^2\cos^2\theta & (7.237) \\ \Delta &= r^2 - 2Mr + L^2. & (7.238) \end{aligned}$$

The coordinates all have the same meaning (as functions of spacetime position x) as defined in the preceding section (7.155), and we have differed from standard notation in labelling the new constant by L as opposed to the more popular a. This avoids any confusion with our use of a as a vector variable and has the added advantage that the two constants, L and M, are given similar symbols. It is assumed that $|L| < M$, as is expected to be the case in any physically realistic situation.

The solution (7.236) has two horizons (at $\Delta = 0$) where the line element is singular and, as with the Schwarzschild line element, no continuous function $\overline{h}$ exists which generates (7.236). However, we can find an $\overline{h}$ which is well-behaved outside the outer horizon, and a suitable form is defined by

$$\begin{aligned} \overline{h}(e_t) &= \frac{r^2 + L^2}{\rho\Delta^{1/2}} e_t - \frac{L}{r\rho} e_\phi \\ \overline{h}(e_r) &= \frac{\Delta^{1/2}}{\rho} e_r \\ \overline{h}(e_\theta) &= \frac{r}{\rho} e_\theta \\ \overline{h}(e_\phi) &= \frac{r}{\rho} e_\phi - \frac{Lr^2\sin^2\theta}{\rho\Delta^{1/2}} e_t. \end{aligned} \tag{7.239}$$

The Riemann tensor obtained from (7.236) has the remarkably compact form

$$\mathcal{R}(B) = -\frac{M}{2(r - iL\cos\theta)^3}(B + 3e_r e_t B e_r e_t). \tag{7.240}$$

(This form for $\mathcal{R}(B)$ was obtained with the aid of the symbolic algebra package *Maple*.) To my knowledge, this is the first time that the Riemann tensor for the Kerr solution has been cast in such a simple form.

Equation (7.240) shows that the Riemann tensor for the Kerr solution is algebraically very similar to that of the Schwarzschild solution, differing only in that the factor of $(r - iL\cos\theta)^3$ replaces r^3. The quantity $r - iL\cos\theta$ is a scalar + pseudoscalar object and so commutes with the rest of $\mathcal{R}(B)$. It follows that the field equations can be verified in precisely the same manner as for the Schwarzschild solution (7.187). It has been known for many years that the Kerr metric can be obtained from the Schwarzschild metric via a complex coordinate transformation [88, 89]. This "trick" works by taking the Schwarzschild metric in a null tetrad formalism and carrying out the coordinate transformation

$$r \mapsto r - jL\cos\theta. \tag{7.241}$$

Equation (7.240) shows that there is more to this trick than is usually supposed. In particular, it demonstrates that the unit imaginary in (7.241) is better thought of as a spacetime pseudoscalar. This is not a surprise, since we saw in Chapter 4 that the role of the unit imaginary in a null tetrad is played by the spacetime pseudoscalar in the STA formalism.

The Riemann tensor (7.240) is clearly defined for all values of r (except $r = 0$). We therefore expect to find an alternative form of $\overline{h}$ which reproduces (7.240) and is also defined globally. One such form is defined by

$$\begin{aligned}
\overline{h}(e_t) &= e_t + \frac{1}{2\rho^2}(2Mr + L^2\sin^2\theta)e_- - \frac{L}{r\rho}e_\phi \\
\overline{h}(e_r) &= e_r + \frac{1}{2\rho^2}(2Mr - L^2\sin^2\theta)e_- \\
\overline{h}(e_\theta) &= \frac{r}{\rho}e_\theta \\
\overline{h}(e_\phi) &= \frac{r}{\rho}e_\phi - \frac{Lr^2\sin^2\theta}{\rho^2}e_-,
\end{aligned} \tag{7.242}$$

with

$$e_- = (e_t - e_r). \tag{7.243}$$

This solution can be shown to lead to the Riemann tensor in the form (7.240). The solution (7.242) reproduces the line element of the advanced-time Eddington-Finkelstein form of the Kerr solution. Alternatives to (7.242) can be obtained by rotations, though at the cost of complicating the form of $\mathcal{R}(B)$. One particular rotation is defined by the rotor

$$R = \exp\left\{\frac{L}{2r\rho}e_\theta\wedge(e_t - e_r)\right\}, \tag{7.244}$$

which leads to the compact solution

$$\overline{h}(a) = a + \frac{Mr}{\rho^2}a\cdot e_- e_- - \frac{L}{r\rho}a\cdot e_r e_\phi + (\frac{r}{\rho} - 1)a\wedge(e_r e_t)e_r e_t. \tag{7.245}$$

None of these solutions correspond to the original form found by Kerr [90]. Kerr's solution is most simply expressed as

$$\overline{h}(a) = a - \alpha a\cdot n n \tag{7.246}$$

where α is a scalar-valued function and $n^2 = 0$. The vector n can be written in the form

$$n = (e_t - \boldsymbol{n}e_t) \tag{7.247}$$

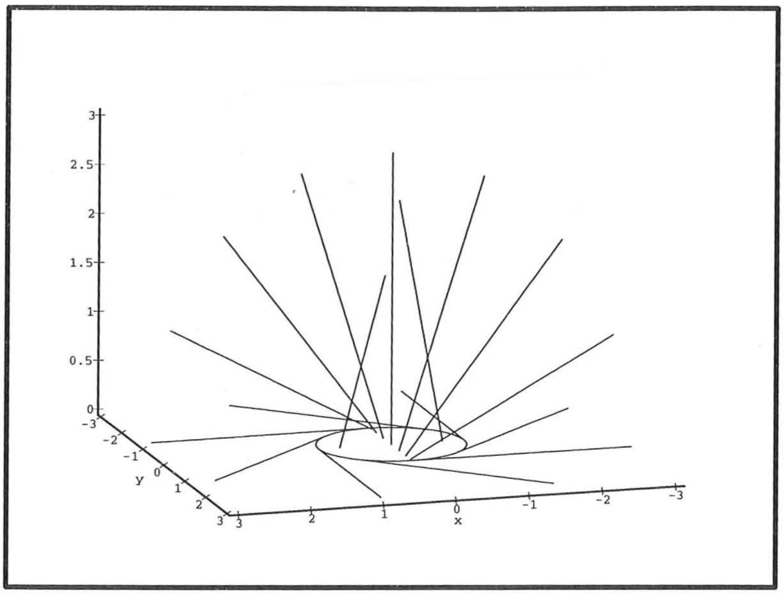

Figure 7.1: Incoming light paths for the Kerr solution I — view from above. The paths terminate over a central disk in the $z = 0$ plane.

where $\boldsymbol{n}$ is a spatial vector. The explicit forms of $\boldsymbol{n}$ and α can be found in Schiffer *et al.* [89] and in Chapter 6 of "*The mathematical theory of black holes*" by S. Chandrasekhar [91]. These forms will not be repeated here. From the field equations it turns out that $\boldsymbol{n}$ satisfies the equation [89]

$$\boldsymbol{n}\cdot\boldsymbol{\nabla}\boldsymbol{n} = 0. \tag{7.248}$$

The integral curves of $\boldsymbol{n}$ are therefore straight lines, and these represent the possible paths for incoming light rays. These paths are illustrated in figures (7.1) and (7.2). The paths terminate over a central disk, where the matter must be present. The fact that the solution (7.246) must represent a disk of matter was originally pointed out by Kerr in a footnote to the paper of Newman and Janis [88]. This is the paper that first gave the derivation of the Kerr metric via a complex coordinate transformation. Kerr's observation is ignored in most modern texts (see [91] or the popular account [92]) where it is claimed that the solution (7.246) represents not a disk but a ring of matter — the ring singularity, where the Riemann tensor is infinite.

The transformations taking us from the solution (7.246) to solutions with a point singularity involve the translation

$$f(x) = x' \equiv x - \frac{L}{r} x\cdot(i\sigma_3), \tag{7.249}$$

which implies that

$$(r')^2 = r^2 + L^2 \cos^2\theta. \tag{7.250}$$

Only the points for which r satisfies

$$r \geq |L\cos\theta| \tag{7.251}$$

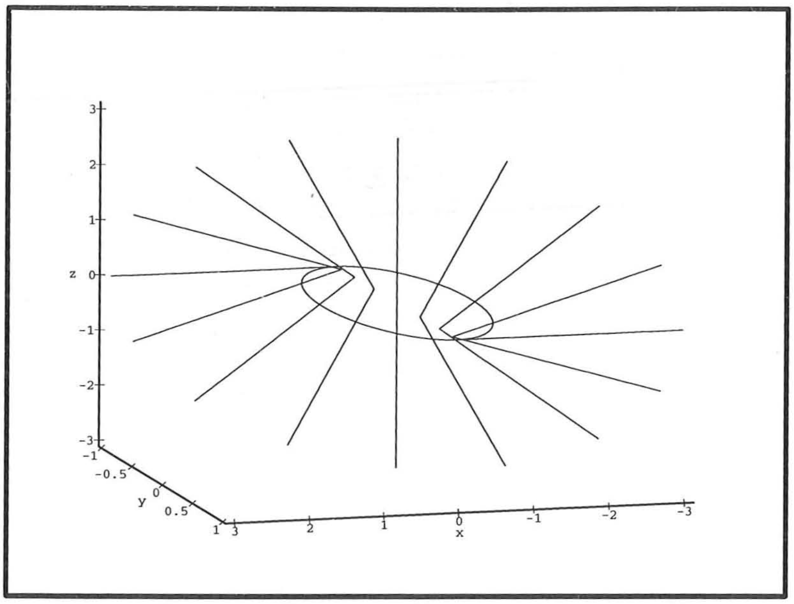

Figure 7.2: Incoming null geodesics for the Kerr solution II — view from side on.

are mapped onto points in the transformed solution, and this has the effect of cutting out the central disk and mapping it down to a point. Curiously, the translation achieves this whilst keeping the total mass fixed (*i.e.* the mass parameter M is unchanged). The two types of solution (7.242) and (7.246) represent very different matter configurations, and it is not clear that they can really be thought of as equivalent in anything but an abstract mathematical sense.

7.3 Extended Matter Distributions

As a final application of our flat-space gauge theory of gravity, we study how extended matter distributions are handled. We do so by concentrating on gravitational effects in and around stars. This is a problem that is treated very successfully by general relativity (see [93, Chapter 23] for example) and, reassuringly, much of the mathematics goes through unchanged in the theory considered here. This is unsurprising, since we will assume that all effects due spin are neglible and we have already seen that, when this is the case, the introduction of a coordinate frame will reproduce the field equations of GR. It will be clear, however, that the *physics* of the situation is quite different and the central purpose of this section is to highlight the differences. Later in this section we discuss some aspects of rotating stars, which remains an active source of research in general relativity. Again, we work in units where $G = 1$.

We start by assuming the simplest distribution of matter — that of an ideal fluid. The matter stress-energy tensor then takes the form

$$\mathcal{T}(a) = (\rho + p)a{\cdot}uu - pa, \tag{7.252}$$

where ρ is the energy density, p is the pressure and u is the 4-velocity field of the fluid. We now impose a number of physical restrictions on $\mathcal{T}(a)$. We first assume that the matter distribution is radially symmetric so that ρ and p are functions of r only, where r is the (flat-space!) radial distance from the centre of the star, as defined by (7.155). (We use translational invariance to choose the centre of the star to coincide with the spatial point that we have labelled as the origin). Furthermore, we will assume that the star is non-accelerating and can be taken as being at rest with respect to the cosmic frame (we can easily boost our final answer to take care of the case where the star is moving at a constant velocity through the cosmic microwave background). It follows that the velocity field u is simply e_t, and $\mathcal{T}$ now takes the form

$$\mathcal{T}(a) = (\rho(r) + p(r))a\cdot e_t e_t - p(r)a. \tag{7.253}$$

This must equal the gravitational stress-energy tensor (the Eintein tensor), which is generated by the $\overline{h}$ and Ω gauge fields. Radial symmetry means that $\overline{h}$ will have the general form of (7.160). Furthermore, the form of $\mathcal{G}(a)$ derived from (7.160) shows that f_2 and g_2 must be zero, and hence that $\overline{h}$ is diagonal. This conclusion could also have been reached by considering the motions of the underlying particles making up the star. If these follow worldlines $x(\tau)$, where τ is the affine parameter, then u is defined by

$$u = \underline{h}^{-1}(\dot{x}), \tag{7.254}$$
$$\Rightarrow \quad \dot{x} = \underline{h}(e_t). \tag{7.255}$$

A diagonal $\overline{h}$ ensures that $\dot{x}$ is also in the e_t direction, so that the consituent particles are also at rest in the 3-space relative to e_t. That this should be so could have been introduced as an additional physical requirement. Either way, by specifying the details of the matter distribution we have restricted $\overline{h}$ to be of the form

$$\overline{h}(a) = (f(r) - 1)a\cdot e_t e_t - (g(r) - 1)a\cdot e_r e_r + a. \tag{7.256}$$

The ansätz for the gravitational fields is completed by writing

$$\begin{array}{rclcrcl} \Omega_t & = & \alpha(r)e_r e_t & \quad & \Omega_\theta & = & (g(r)-1)e_r e_\theta/r \\ \Omega_r & = & 0 & & \Omega_\phi & = & (g(r)-1)e_r e_\phi/r, \end{array} \tag{7.257}$$

where again it is convenient to keep $\alpha(r)$ as a free variable, rather than solving for it in terms of f and g. The problem can now be solved by using the field equations on their own, but it is more convenient to supplement the equations with the additional condition

$$\dot{\mathcal{T}}(\dot{\mathcal{D}}) = 0, \tag{7.258}$$

which reduces to the single equation

$$p'(r) = \frac{\alpha f}{g}(\rho + p). \tag{7.259}$$

Solving the field equations is now routine. One can either follow the method of Section 3.1, or can simply look up the answer in any one of a number of texts. The solution is that

$$g(r) = (1 - 2m(r)/r)^{1/2} \tag{7.260}$$

where

$$m(r) = \int_0^r 4\pi r'^2 \rho(r')\, dr'. \tag{7.261}$$

The pressure is found by solving the Oppenheimer-Volkov equation

$$p' = -\frac{(\rho+p)(m(r)+4\pi r^3 p)}{r(r-2m(r))}, \tag{7.262}$$

subject to the condition that $p(R)=0$, where R is the radius of the star. The remaining term in $\overline{h}$ is then found by solving the differential equation

$$\frac{f'(r)}{f(r)} = -\frac{m(r)+4\pi r^3 p}{r(r-2m(r))} \tag{7.263}$$

subject to the constraint that

$$f(R) = (1-2m(R)/R)^{-1/2}. \tag{7.264}$$

Finally, $\alpha(r)$ is given by

$$\alpha(r) = (fg)^{-1}(m(r)/r^2 + 4\pi r p). \tag{7.265}$$

The complete solution leads to a Riemann tensor of the form

$$\begin{aligned}\mathcal{R}(B) &= 4\pi\left[(\rho+p)B\cdot e_t e_t - \rho B\times(e_r e_t)e_r e_t\right] \\ &\quad -\frac{m(r)}{2r^3}(B + 3e_r e_t B e_r e_t)\end{aligned} \tag{7.266}$$

which displays a neat split into a surface term, due to the local density and pressure, and a (tractionless) volume term, due to the matter contained inside the shell of radius r.

The remarkable feature of thie solution is that (7.261) is quite clearly a flat-space integral! The importance of this integral is usually downplayed in GR, but in the context of a flat-space theory it is entirely natural — it shows that the field outside a sphere of radius r is determined completely by the energy density within the shell. It follows that the field outside the star is associated with a "mass" M given by

$$M = \int_0^R 4\pi r'^2 \rho(r')\, dr'. \tag{7.267}$$

We can understand the meaning of the definition of $m(r)$ by considering the covariant integral of the energy density

$$\begin{aligned}E_0 &= e_t i \int \underline{h}^{-1}(d^3x)\rho \\ &= \int_0^R 4\pi r'^2 (1-2m(r')/r')^{-1/2}\rho(r')\, dr'.\end{aligned} \tag{7.268}$$

This integral is invariant under active spatial translations of the energy density. That is to say, E_0 is independent of where that matter actually is. In particular, E_0 could be evaluated with the matter removed to a sufficiently great distance that each particle making up the star can be treated in isolation. It follows that E_0 must be the sum of the individual mass-energies of the component particles of the star — E_0 contains no contribution from the interaction between the particles. If we now expand (7.268) we find that

$$
\begin{aligned}
E_0 &\approx \int_0^R 4\pi r'^2 (\rho(r') + \rho(r')m(r')/r')\, dr' \\
&= M - \text{Potential Energy}. \qquad (7.269)
\end{aligned}
$$

The external mass M is therefore the sum of the mass-energy E_0 (which ignored interactions) and a potential energy term. This is entirely what one expects. Gravity is due to the presence of energy, and not just (rest) mass. The effective mass seen outside a star is therefore a combination of the mass-energies of the constituent particles, together with the energy due to their interaction with the remaining particles that make up the star. This is a very appealing physical picture, which makes complete sense within the context of a flat-space gauge theory. Furthermore, it is now clear why the definition of M is not invariant under radial translations. Interpreted actively, a radial translation changes the matter distribution within the star, so the component particles are in a new configuration. It follows that the potential energy will have changed, and so too will the total energy. An external observer sees this as a change in the strength of the gravitational attraction of the star.

An important point that the above illustrates is that, given a matter distribution in the form of $\mathcal{T}(a)$ and (more generally) $\mathcal{S}(a)$, the field equations are sufficient to tie down the gauge fields uniquely. Then, given any solution of the field equation $\mathcal{G}(a) = 8\pi\mathcal{T}(a)$, a new solution can always be reached by an active transformation. But doing so alters $\mathcal{T}(a)$, and the new solution is appropriate to a *different* matter distribution. It is meaningless to continue talking about covariance of the equations once the matter distribution is specified.

Whilst a non-vanishing $\mathcal{T}(a)$ does tie down the gauge fields, the vacuum raises a problem. When $\mathcal{T}(a) = 0$ any gauge transformation can be applied, and we seem to have no way of specifying the field outside a star, say. The resolution of this problem is that matter (energy) must always be present in some form, whether it be the sun's thermal radiation, the solar wind or, ultimately, the cosmic microwave background. At some level, matter is always available to tell us what the $\overline{h}$ and Ω fields are doing. This fits in with the view that spacetime itself does not play an active role in physics and it is the presence of matter, not spacetime curvature, that generates gravitational interactions.

Since our theory is based on active transformations in a flat spacetime, we can now use local invariance to gain some insights into what the fields inside a rotating star might be like. To do this we rotate a static solution with a boost in the e_ϕ direction. The rotor that achieves this is

$$R = \exp\{\omega(r,\theta)\hat{\phi}e_t\} \qquad (7.270)$$

where

$$\hat{\phi} \equiv e_\phi/(r\sin\theta). \qquad (7.271)$$

The new matter stress-energy tensor is

$$\mathcal{T}(a) = (\rho + p)a\cdot(\cosh\omega\, e_t + \sinh\omega\,\hat{\phi})(\cosh\omega e_t + \sinh\omega\,\hat{\phi}) - pa, \qquad (7.272)$$

and the Einstein tensor is similarly transformed. The stress-energy tensor (7.272) can only properly be associated with a rotating star if it carries angular momentum. The definitions of momentum and angular momentum are, in fact, quite straight-forward. The flux of momentum through the 3-space defined by a time-like vector a is $\mathcal{T}(a)$ and the angular momentum bivector is defined by

$$\mathcal{J}(a) = x \wedge \mathcal{T}(a). \tag{7.273}$$

Once gravitational interactions are turned on, these tensors are no longer conserved with respect to the vector derivative,

$$\dot{\mathcal{T}}(\dot{\nabla}) \neq 0, \tag{7.274}$$

and instead the correct law is (7.258). This situation is analogous to that of coupled Dirac-Maxwell theory (see Section 6.3). Once the fields are coupled, the individual (free-field) stress-energy tensors are no longer conserved. To recover a conservation law, one must either replace directional derivatives by covariant derivatives, or realise that it is only the total stress-energy tensor that is conserved. The same is true for gravity. Once gravitational effects are turned on, the only quantity that one expects to be conserved is the sum of the individual matter and gravitational stress-energy tensors. But the field equations ensure that this sum is always zero, so conservation of total energy-momentum ceases to be an issue.

If, however, a global time-like symmetry is present, one can still sensibly separate the total (zero) energy into gravitational and matter terms. Each term is then separately conserved with respect to this global time. For the case of the star, the total 4-momentum is the sum of the individual fluxes of 4-momentum in the e_t direction. We therefore define the conserved momentum P by

$$P = \int d^3x\, \mathcal{T}(e_t) \tag{7.275}$$

and the total angular momentum J by

$$J = \int d^3x\, x \wedge \mathcal{T}(e_t). \tag{7.276}$$

Concentrating on P first, we find that

$$P = M_{rot} e_t \tag{7.277}$$

where

$$M_{rot} = 2\pi \int_0^R dr \int_0^\pi d\theta\, r^2 \sin\theta \left[\rho(r)\cosh^2\omega(r,\theta) + p(r)\sinh^2\omega(r,\theta)\right]. \tag{7.278}$$

The effective mass M_{rot} reduces to M when the rotation vanishes, and rises with the magnitude of ω, showing that the internal energy of the star is rising. The total 4-momentum is entirely in the e_t direction, as it should be. Performing the J integral next, we obtain

$$J = -i\sigma_3\, 2\pi \int_0^R dr \int_0^\pi d\theta\, r^3 \sin^2\theta\, (\rho(r) + p(r)) \sinh\omega(r,\theta)\, \cosh\omega(r,\theta), \tag{7.279}$$

so the angular momentum is contained in the spatial plane defined by the $\hat{\phi} e_t$ direction. Performing an active radial boost has generated a field configuration with suitable momentum and angular momentum properties for a rotating star.

Unfortunately, this model cannot be physical, since it does not tie down the shape of the star — an active transformation can always be used to alter the shape to any desired configuration. The missing ingredient is that the particles making up the star must satisfy their own geodesic equation for motion in the fields due to the rest of the star. The simple rotation (7.270) does not achieve this.

Attention is drawn to these points for the following reason. The boost (7.270) produces a Riemann tensor at the surface of the star of

$$\mathcal{R}(B) = -\frac{M_{rot}}{2r^3}\left(B + 3e_r(\cosh\omega\, e_t + \sinh\omega\, \hat{\phi})Be_r(\cosh\omega\, e_t + \sinh\omega\, \hat{\phi})\right), \qquad (7.280)$$

which is that for a rotated Schwarzschild-type solution, with a suitably modified mass. This form is very different to the Riemann tensor for the Kerr solution (7.240), which contains a complicated duality rotation. Whilst a physical model will undoubtedly require additional modifications to the Riemann tensor (7.280), it is not at all clear that these modifications will force the Riemann tensor to be of Kerr type. Indeed, the differences between the respective Riemann tensors would appear to make this quite unlikely. The suggestion that a rotating star does not couple onto a Kerr-type solution is strengthened by the fact that, in the 30 or so years since the discovery of the Kerr solution [90], no-one has yet found a solution for a rotating star that matches onto the Kerr geometry at its boundary.

7.4 Conclusions

The gauge theory of gravity developed from the Dirac equation has a number of interesting and surprising features. The requirement that the gravitational action should be consistent with the Dirac equation leads to a unique choice for the action integral (up to the possible inclusion of a cosmological constant). The result is a set of equations which are first-order in the derivatives of the fields. This is in contrast to general relativity, which is a theory based on a set of second-order partial differential equations for the metric tensor. Despite the formal similarities between the theories, the study of point-source solutions reveals clear differences. In particular, the first-order theory does not admit solutions which are invariant under time-reversal.

The fact that the gauge group consists of active Poincaré transformations of spacetime fields means that gauge transformations relate physically distinct situations. It follows that observations can determine the nature of the $\overline{h}$ and Ω fields. This contrasts with Yang-Mills theories based on internal gauge groups, where one expects that all observables should be gauge-*invariant*. In this context, an important open problem is to ascertain how the details of radial collapse determine the precise nature of the $\overline{h}$ and Ω fields around a black hole.

A strong point in favour of the approach developed here is the great formal clarity that geometric algebra brings to the study of the equations. This is illustrated most clearly in the compact formulae for the Riemann tensor for the Schwarzschild and Kerr solutions and for radially-symmetric stars. No rival method (tensor calculus, differential forms, Newman-Penrose formalism) can offer such concise expressions.

For 80 years, general relativity has provided a successful framework for the study of gravitational interactions. Any departure from it must be well-motivated by sound physical and mathematical reasons. The mathematical arguments in favour of the present approach include the simple manner in which transformations are handled, the algebraic compactness of many formulae and the fact that torsion is perhaps better viewed as a spacetime field than as a geometric effect. Elsewhere, a number of authors have questioned whether the view that gravitational interactions are the result of spacetime geometry is correct (see [94], for example). The physical motivation behind the present theory is provided by the identification of the $\overline{h}$ and Ω fields as the dynamical variables. The physical structure of general relativity is very much that of a classical field theory. Every particle contributes to the curvature of spacetime, and every particle moves on the resultant curved manifold. The picture is analogous to that of electromagnetism, in which all charged particles contribute to an electromagnetic field (a kind of global ledger). Yet an apparently crucial step in the development of Q.E.D. was Feynman's realisation (together with Wheeler [95, 96]) that the electromagnetic field can be eliminated from classical electrodynamics altogether. A similar process may be required before a quantum multiparticle theory of gravity can be constructed. In the words of Einstein [97]

> *...the energy tensor can be regarded only as a provisional means of representing matter. In reality, matter consists of electrically charged particles ...*

The status of the $\overline{h}$ and Ω fields can be regarded as equally provisional. They may simply represent the aggregate behaviour of a large number of particles, and as such would not be of fundamental significance. In this case it would be wrong to attach too strong a physical interpretation to these fields (*i.e.* that they are the result of spacetime curvature and torsion).

An idea of how the $\overline{h}$ field could arise from direct interparticle forces is provided by the two-particle Dirac action constructed in Section 6.3. There the action integral involved the differential operator $\nabla^1/m^1 + \nabla^2/m^2$, so that the vector derivatives in each particle space are weighted by the mass of the particle. This begins to suggest a mechanism by which, at the one-particle level, the operator $\overline{h}(\nabla)$ encodes an inertial drag due to the other particle in the universe. This is plausible, as the $\overline{h}$ field was originally motivated by considering the effect of translating a field. The theory presented here does appear to contain the correct ingredients for a generalisation to a multiparticle quantum theory, though only time will tell if this possibility can be realised.

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