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Twistor description of spinning particles in AdS

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Abstract: The two-twistor formulation of particle mechanics in D-dimensional anti-de Sitter space for $D = 4, 5, 7$, which linearises invariance under the AdS isometry group $\mathrm{Sp}(4;\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$, is generalized to the massless N-extended "spinning particle". The twistor variables are gauge invariant with respect to the initial N local worldline supersymmetries; this simplifies aspects of the quantum theory such as implications of global gauge anomalies. We also give details of the two-supertwistor form of the superparticle, in particular the massive superparticle on AdS_5.

Keywords: Extended Supersymmetry, Field Theories in Higher Dimensions, Sigma Models, Space-Time Symmetries

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1 Introduction

A general feature of particle, string or brane dynamics is that isometries of the background spacetime become symmetries of the particle, string or brane action. In the case of anti-de Sitter (AdS) backgrounds, the AdS isometry group is usually realized non-linearly on the worldline, worldsheet or worldvolume fields, which complicates the extraction of physical consequences of these symmetries. This is especially true for the supersymmetries of superparticles, superstrings or superbranes in the supersymmetric "AdS×S" vacua of string/M-theory.

There are various ways to linearize AdS (super)isometries. One, exploited in [1, 2], is to regard AdS_D as a hypersurface in $\mathbb{E}^{2,d}$ for $d = D-1$. An alternative is to exploit the fact that the AdS_D isometry group is also the conformal isometry group of its d-dimensional Minkowski boundary; twistor methods [3, 4] are then available for some spacetime dimensions (as are supertwistor methods [5, 6]). This idea inspired a construction by Claus et al. of an action for a massive spin-zero particle in AdS_5 for which the AdS_5 isometries are realized linearly on twistor variables [7].

Twistor methods are available for $d = 3, 4, 6, 10$ (and supertwistors for $d = 3, 4, 6$) [8, 9]. This is because these are the Minkowski space dimensions for which the Lorentz group is $\mathrm{SL}(2;\mathbb{K})$ [10–13] and the conformal group is $\mathrm{Sp}(4;\mathbb{K})$ [14, 15], where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ (the normed division algebras). This fact suggests that the Claus et al. construction might be applicable more generally, and in particular to $\mathrm{AdS}_{4,7}$ as well as AdS_5. The suggestion is attractive because the maximally supersymmetric "AdS×S" vacua of string/M-theory have an AdS_D factor precisely for $D = 4, 5, 7$.

A geometrical approach to this problem was formulated by Cederwall [16, 17], who used the observation that an AdS_D geodesic is the intersection of a plane in $\mathbb{E}^{2,d}$ with the AdS_D hypersurface, the plane being specified by a 2-form on $\mathbb{E}^{2,d}$. He then showed that a pair of twistor variables could be used to parametrize the 2-form associated to a null or timelike geodesic of AdS_5, and a null geodesic of $\mathrm{AdS}_{4,7}$, but more than two twistors would be needed for a timelike geodesic in $\mathrm{AdS}_{4,7}$. The combined results of [7, 16, 17] can be summarized as follows: a two-twistor formulation of the classical mechanics of a free spin-zero particle of mass m in AdS_D is possible for $D = 5$, with a known action, and it is also possible for $D = 4, 7$ but only if $m = 0$.

It is important to appreciate here that $m = 0$ implies, and is implied by, a null worldline, but it does *not* imply that the mass parameter M of the quantum wave equation in AdS_D is zero. For $m = 0$, the standard zero-spin particle action is invariant under all *conformal isometries* of AdS_D which implies, assuming preservation of conformal invariance upon quantization, that the mass parameter of the quantum wave equation is $M = M_c$

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with $(M_c R/\hbar)^2 = -D(D-2)/4$ [18]. More generally, $M^2 = M_c^2 + (m/\hbar)^2$, and the Breitenlohner-Freedman bound [19, 20] is $(mR/\hbar)^2 \geq -1/4$, for which we gave a simple uncertainty-principle interpretation in [21].

The main result of [21] was a variant of the Claus et al. construction that applies uniformly to (super)particle mechanics in AdS_D for $D = 4, 5, 7$. It leads (in the "bosonic" case) to a manifestly $\mathrm{Sp}(4;\mathbb{K})$-invariant action with canonical variables that are the entries of a 4×4 matrix over $\mathbb{K}$, which transforms linearly with respect to both $\mathrm{Sp}(4;\mathbb{K})$ and an $O(2;\mathbb{K})$ gauge group (defined to preserve a $\mathbb{K}$-hermitian quadratic form). As the **4** of $\mathrm{Sp}(4;\mathbb{K})$ is a twistor for d-dimensional Minkowski space with $d = 2+\dim\mathbb{K}$, these canonical variables constitute a "two-twistor"; i.e. a twistor doublet.[1]

A peculiar feature of this construction is that it involves a mass-dependent change of variables that has the effect of eliminating the mass-dependence from the action, which suggests that it is actually valid only for $m = 0$. This was verified in [21] by a comparison of the Noether charges for the manifest $\mathrm{Sp}(4;\mathbb{K})$ symmetry with the Noether charges for invariance under the AdS isometries; they turn out to coincide only for $m = 0$. While this does not explain how the restriction to $m = 0$ comes about (we postpone discussion of this point) it does confirm that the two-twistor action of [21] indeed describes a particle (albeit massless) in AdS_D for $D = 4, 5, 7$.

The AdS_5 case is special because $O(2;\mathbb{C}) \cong \mathrm{U}(2)$ has an 'extra' $\mathrm{U}(1)$ factor unrelated to spin, in contrast to $O(2;\mathbb{K}) \cong \mathrm{Spin}(1+\dim\mathbb{K})$ for $\mathbb{K} = \mathbb{R}, \mathbb{H}$. This (and the fact that $\mathbb{K} = \mathbb{C}$ for $D = 5$) makes possible an m-dependent *complex* redefinition of the twistor variables that 'realigns' the AdS isometry group $\mathrm{SU}(2,2)_{\mathrm{AdS}}$ with the manifest $\mathrm{Sp}(4;\mathbb{C}) \cong \mathrm{U}(2,2)$ symmetry group; i.e. $\mathrm{U}(2,2) \supset \mathrm{SU}(2,2)_{\mathrm{AdS}}$. Its only other effect is to re-introduce the mass m into the 'extra' $\mathrm{U}(1) \subset \mathrm{U}(2,2)$, which coincides with the gauged $\mathrm{U}(1) \subset \mathrm{U}(2)$; the results of [7] for the massive particle in AdS_5 are thereby recovered.[2]

In [21] we extended these results to the superparticle [25–27]. We showed that 'supersymmetrization' on Mink_d 'slices' of AdS_D suffices because the resulting action has 'hidden' supersymmetries. Here we confirm this explicitly for any m when $D = 5$ and for $m = 0$ when $D = 4, 7$. In [21] we identified the particular $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ cases of relevance to the 'AdS×S' vacua of String/M-theory. Here we verify that the manifest $\mathrm{OSp}(\mathcal{N}|4;\mathbb{K})$ invariance supergroup coincides with the AdS isometry supergroup, again for any m when $D = 5$ and for $m = 0$ when $D = 4, 7$.

Even in the exceptional AdS_5 case for which a (super)particle mass is compatible with its (super)twistor formulation, there is a quantum constraint on the mass coming from the possibility of a global $\mathrm{U}(2)$ anomaly; we show that the absence of this gauge anomaly requires the quantization condition $mR/\hbar \in \mathbb{Z}$, where R is the AdS_5 radius. This result depends on fact that $\mathrm{U}(2)$ is a quotient of $\mathrm{U}(1)\times\mathrm{SU}(2)$ by Z_2; if the gauge group were $\mathrm{U}(1)\times\mathrm{SU}(2)$ then the quantization condition would be $2mR/\hbar \in \mathbb{Z}$. This weaker quantization condition is implicit in the results of [28], which makes use of earlier results in [29].

[1] In [21] we called this a "bi-twistor", in accord with some earlier usage (e.g. [22]), but as a "bi-spinor" is generally taken to mean a tensor represented as the sum of products of spinors we now prefer the terminology "two-twistor", which is also in accord with some earlier usage (e.g [23]).

[2] They may also be recovered from the co-adjoint orbit approach to particle dynamics [24]; this approach was applied to massless particle dynamics in Mink_d for $d = 3, 4, 6$ in [15].

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The main purpose of this paper, however, is to generalise the construction of [21] to one that applies to the N-extended "spinning particle" in AdS_D, again for $D = 4, 5, 7$. Like the superparticle, spin is incorporated via the addition of anticommuting worldline variables but now these are spacetime vectors and scalars rather than spinors, and they are introduced according to the requirement of *local worldline supersymmetry* rather than rigid spacetime supersymmetry.

The original "spinning particle" action was a generalization of the standard action for a point particle in a Minkowski background to one incorporating $N = 1$ local worldline supersymmetry [30]; it provides a classical (or pre-quantum) description of a spin-$\frac{1}{2}$ particle in the sense that its quantization yields the Dirac equation [31]. The further generalization to N-extended local worldline supersymmetry for $N > 1$, and a local $\mathrm{SO}(N)$ gauge invariance, leads to an action that describes (at least for a four-dimensional Minkowski background) a "classical" particle of spin $N/2$ [32–34].

Here we are interested in an AdS background for the generic N-extended spinning particle. For $N \leq 2$ the relevant action is just the specialization to AdS of the action given in [33] for an arbitrary spacetime background, but it was erroneously claimed there that $N > 2$ allows only flat backgrounds. This was corrected by Kuzenko and Yarevskaya [35], who showed that maximally symmetric background spacetime metrics are also allowed, in particular AdS; quantum aspects were subsequently explored by Bastianelli et al. [36]. The first task that we set ourselves in this paper is to summarize the status of the classical N-extended spinning particle in an AdS background using a formalism and notation differing from [35] and [36] but suited to our purposes.

A crucial input to our subsequent construction of a two-twistor action for a spinning particle in AdS_D for $D = 4, 5, 7$ is the two-twistor action found in [37] for a massive N-extended spinning particle in a d-dimensional Minkowski background for $d = 3, 4, 6$. A significant feature of that action (which carries over to the AdS_D case) is that the twistor variables, and the new anticommuting variables required for non-zero spin, are all *gauge invariant with respect to the original local worldline supersymmetry*. The only remaining gauge-invariances other than time-reparametrization invariance are the local $\mathrm{SO}(N)$ (for $N > 1$) and those generated by the $O(2;\mathbb{K})$ "spin-shell" constraints (which determine the Pauli-Lubanski 3-form [38]). Here we rederive these results using the $\mathrm{SL}(2;\mathbb{K})$ and $\mathrm{Sp}(4;\mathbb{K})$ notation to express them in a uniform way for $d = 3, 4, 6$, and we take this opportunity to explain details of the new notation.

With this Mink_d result in hand, we proceed to the AdS_D case. As for the spin-zero particle, a comparison of the $\mathrm{Sp}(4;\mathbb{K})$ Noether charges with the AdS_D isometry charges shows that they coincide only for zero mass, so the two-twistor action must again be interpreted as describing a massless spinning particle in AdS_D. Moreover, the complex redefinition that can be used in the AdS_5 case to circumvent this obstruction to non-zero mass for the zero-spin particle, and the (zero-superspin) superparticle, no longer does so for the spinning particle. Because of this, we limit our subsequent discussion of the quantum theory to the massless spinning particle.

A general feature of N-extended spinning particle actions is that N anticommuting variables become redundant in the $m \to 0$ limit, in the sense that they are not required

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by the N-extended local supersymmetry [37]. This is also a feature of our two-twistor action for the N-extended spinning particle in an $AdS_{4,5,7}$, as is to be expected from our conclusion that it describes a *massless* particle in these background spacetimes. Omission of the redundant anticommuting variables leaves us with a "reduced" two-twistor action and for $N = 1$ we find that quantization yields results consistent with expectations derived from the standard action for the $N = 1$ massless spinning particle.

A complete discussion of the quantum theory might require something like a generalization of the results of [28]. We leave this to the future. Here we restrict ourselves, as we did for the superparticle in [21], to an analysis of some quantum implications of the classical anticommuting variables. By omitting the redundant anticommuting variables we recover expected results for $D = 4$ (a massless particle of spin $N/2$) and for $D = 5, 7$ when $N = 1$. The $D = 5, 7$ cases with $N > 1$ are complicated by global $SO(N)$ anomalies; we postpone a summary of our conclusions for these cases. We close with a discussion of some issues raised by our results.

The $N = 2$ spinning particle is special because of the possibility of including an $SO(2)$ worldline Chern-Simons (WCS) term [34]. It turns out that the WCS term leads to a mismatch between the AdS isometry Noether charges and those of the manifest $Sp(4;\mathbb{K})$ symmetry of its would-be twistor formulation. The WCS term obstructs the twistor construction of the spinning particle in a way that is similar to the inclusion of non-zero mass for $D = 5$ when $N > 0$; the details are left to an appendix.

2 The N-extended spinning particle

We begin with a review of the status of the N-extended spinning particle, but in a notation that requires only the introduction of a background spacetime metric rather than a vielbein. We assume a spacetime (of unspecified dimension) with metric $g_{mn}(x)$ in coordinates x^m. The phase-space action for the N-extended spinning particle in this spacetime is a functional of maps from the particle's worldline to a phase superspace with "bosonic" coordinates $\{x^m, p_m\}$ and $SO(N)$ N-plets of anticommuting (but Lorentz vector plus Lorentz scalar) coordinates $\{\psi_i^m, \xi_i\}$ $(i = 1, \ldots, N)$. The reparametrization invariant action takes the form

$$S = \int dt \, \{L_{\rm geom} + L_{\rm constraint}\} \ , \tag{2.1}$$

for arbitrary worldline time t. The "geometrical" part of the Lagrangian is

$$L_{\rm geom} = \dot{x}^m p_m + \frac{1}{2}\psi_i^m \dot{\psi}_i^n g_{mn} + \frac{1}{2}\xi_i \dot{\xi}_i \, , \tag{2.2}$$

where a sum over the index i is implicit, and we choose conventions for which the product of two 'real' anticommuting variables is 'real' *without* the customary additional imaginary unit factor. An equivalent alternative expression is

$$L_{\rm geom} = \dot{x}^m \pi_m + \frac{1}{2}\psi_i^m (D_t\psi_i)^n g_{mn}(x) + \frac{1}{2}\xi_i \dot{\xi}_i \, , \tag{2.3}$$

where

$$(D_t\psi_i)^p = \dot{\psi}_i^p + \dot{x}^m \Gamma_{mn}{}^p \psi_i^n \, , \qquad \Gamma_{mn}{}^p = \frac{1}{2}\left(g_{pm,n} + g_{pn,m} - g_{mn,p}\right) \ , \tag{2.4}$$

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and

$$\pi_m = p_m + \frac{1}{2}\Gamma_{mpq}\psi_i^p\psi_i^q\,, \qquad \Gamma_{mpq} = \Gamma_{mp}{}^n g_{nq}\,. \tag{2.5}$$

The geometrical Lagrangian $L_{\rm geom}$ is the pullback to the worldline of a one-form on the phase superspace whose exterior derivative is the orthosymplectic 2-form

$$\Omega = dp_m dx^m + \frac{1}{2}d\psi_i^m d\psi_i^n g_{mn} - \frac{1}{2}dx^m d\psi_i^p g_{pq,m}\psi_i^q + d\xi_i d\xi_i\,. \tag{2.6}$$

The inverse of Ω yields the canonical Poisson bracket (PB) relations:

$$\{x^m, p_n\}_{PB} = \delta_n^m\,, \quad \{\psi_i^m, \psi_i^n\}_{PB} = g^{mn}\delta_{ij}\,, \quad \{\xi_i, \xi_j\}_{PB} = \delta_{ij}\,, \tag{2.7}$$

and

$$\{p_m, \psi_i^n\}_{PB} = \frac{1}{2}g^{nq}g_{qp,m}\psi_i^p\,, \qquad \{p_m, p_n\}_{PB} = -\frac{1}{4}g^{pq}g_{pr,m}g_{qs,n}\psi_i^r\psi_i^s\,. \tag{2.8}$$

These PB relations imply that

$$\{\pi_m, \psi_i^n\}_{PB} = \Gamma_{mp}{}^n\psi_i^p\,, \qquad \{\pi_m, \pi_n\}_{PB} = \frac{1}{2}R_{mnrs}\psi_i^r\psi_i^s\,, \tag{2.9}$$

where R_{mnrs} is the Riemann curvature tensor:

$$R^m{}_{nrs} = 2\partial_{[r}\Gamma_{s]n}{}^m + 2\Gamma_{p[r}{}^m\Gamma_{s]n}{}^p\,. \tag{2.10}$$

The constraint part of the Lagrangian is

$$L_{\rm constraint} = -e\mathcal{H} - \chi_i\mathcal{Q}_i - \frac{1}{2}f_{ij}\mathcal{J}_{ij}\,, \tag{2.11}$$

where e, χ_i, f_{ij} are Lagrange multiplers for the phase space constraints. For a particle of mass m,

$$\mathcal{Q}_i = \lambda_i^m\pi_m + m\xi_i\,, \qquad \mathcal{J}_{ij} = \psi_i^m\psi_j^n g_{mn} + \xi_i\xi_j\,. \tag{2.12}$$

We leave open for the moment the precise form of the Hamiltonian constraint function $\mathcal{H}$. It must be chosen such that the set of constraint functions is first-class, since they will then generate the gauge invariances of the action that are needed to allow the elimination of unphysical variables.

Using the PB relations given above one finds that

$$\{\mathcal{J}_{ij}, \mathcal{J}_{kl}\}_{PB} = 2\left(\delta_{k[i}\mathcal{J}_{j]l} - \delta_{l[j}\mathcal{J}_{i]k}\right)\,, \tag{2.13}$$

which shows that $\mathcal{J}_{ij}$ is a generator of SO(N), and

$$\begin{aligned}\{\mathcal{J}_{ij}, \mathcal{Q}_k\}_{PB} &= \left\{\psi_i^p\psi_j^q g_{pq}, \psi_k^m\pi_m\right\}_{PB} + \mu\left\{\xi_i\xi_j, \xi_k\right\}_{PB} \\ &= 2\delta_{k[j}\psi_{i]}^m\pi_m - 2\psi_k^m\psi_i^p\psi_j^q\Gamma_{mqp} + \psi_k^m\psi_i^p\psi_j^q g_{pq,m} + 2\mu\delta_{k[j}\xi_{i]} \\ &= 2\delta_{k[j}\mathcal{Q}_{i]}\,,\end{aligned} \tag{2.14}$$

which shows that the N supercharges are the components of an N-vector of SO(N). One also finds that

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = \left(g^{mn}\pi_m\pi_n + m^2\right)\delta_{ij} + \frac{1}{2}\psi_i^m\psi_j^n\psi_k^r\psi_k^s R_{mnrs}\,. \tag{2.15}$$

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For $N = 1$ the last term on the right-hand side is zero, so

$$\{\mathcal{Q}, \mathcal{Q}\}_{PB} = 2\mathcal{H}\,, \qquad 2\mathcal{H} = g^{mn}\pi_m\pi_n + m^2\,, \qquad (N = 1). \tag{2.16}$$

In this case $\mathcal{H}$ is fixed by the requirement that the set of constraints is first class, and the algebra of constraint functions is then that of $N = 1$ worldline supersymmetry.

For $N = 2$ we may use the standard algebraic identities

$$R_{mnrs} \equiv R_{rsmn}\,, \qquad R_{m[nrs]} \equiv 0\,, \tag{2.17}$$

to deduce that

$$\psi_i^m\psi_j^n\psi_k^r\psi_k^s R_{mnrs} \equiv \frac{1}{2}\delta_{ij}\psi_l^m\psi_l^n\psi_k^r\psi_k^s R_{mnrs}\,. \tag{2.18}$$

Using this identity, we can rewrite (2.15) as

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = 2\delta_{ij}\mathcal{H}\,, \tag{2.19}$$

where [33]

$$2\mathcal{H} = g^{mn}\pi_m\pi_n + m^2 + \frac{1}{4}\psi_l^m\psi_l^n\psi_k^r\psi_k^s R_{mnrs} \qquad (N = 1, 2)\,. \tag{2.20}$$

This result also applies to $N = 1$ because in that case the quartic fermion term is identically zero.

For $N > 2$ it may appear from (2.15) that we must insist on a flat background in order to have a first-class set of constraints, but this is slightly too strong a condition; any maximally symmetric background is also possible [35], as we now review for AdS.

2.1 $N > 2$ and AdS

For an AdS background,

$$R_{mnrs} = -2R^{-2}g_{m[r}g_{s]n}\,, \tag{2.21}$$

where R is the constant AdS radius. In this case (2.15) reduces to

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = \left(g^{mn}\pi_m\pi_n + m^2\right)\delta_{ij} + R^{-2}J_{ik}J_{jk}\,, \tag{2.22}$$

where

$$J_{ij} = \psi_i^m\psi_j^n g_{mn}\,. \tag{2.23}$$

For $N = 2$ we have the identity

$$J_{ik}J_{jk} \equiv \delta_{ij}J^2\,, \qquad \left(J^2 = \frac{1}{2}J_{ij}J_{ij}\right)\,. \tag{2.24}$$

This allows us to rewrite (2.22) as

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = 2\mathcal{H}\delta_{ij}\,, \qquad 2\mathcal{H} = g^{mn}\pi_m\pi_n + m^2 + R^{-2}J^2\,. \tag{2.25}$$

This is the specialization to AdS of (2.19) and (2.20).

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For $N > 2$ the $O(N)$ tensor $J_{ik}J_{jk}$ is not proportional to δ_{ij}, so we must proceed differently. From the expression for $\mathcal{J}_{ij}$ in (2.12) we see that

$$J_{ij} = \mathcal{J}_{ij} - \xi_i \xi_j \,, \tag{2.26}$$

which gives us

$$J_{ik}J_{jk} = \mathcal{J}_{ik}\mathcal{J}_{jk} + 2R^{-2}\xi_k \xi_{(i}\mathcal{J}_{j)k} \,, \tag{2.27}$$

and hence

$$J^2 = \mathcal{J}^2 - R^{-2}\xi_i\xi_j\mathcal{J}_{ij} \,, \qquad \left(\mathcal{J}^2 = \frac{1}{2}\mathcal{J}_{ij}\mathcal{J}_{ij}\right) . \tag{2.28}$$

We may now rewrite (2.22) as

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = 2\mathcal{H}(a)\,\delta_{ij} + K_{ij}(a) \,, \tag{2.29}$$

where, for arbitrary parameter a (related to b of [36]),

$$2\mathcal{H}(a) = g^{mn}\pi_m\pi_n + m^2 + aR^{-2}J^2 \,, \tag{2.30}$$

and

$$K_{ij}(a) = R^{-2}\left[\mathcal{J}_{ik}\mathcal{J}_{jk} - a\delta_{ij}\mathcal{J}^2\right] - R^{-2}\left[2\xi_{(i}\mathcal{J}_{j)k} - a\delta_{ij}\xi_l\mathcal{J}_{lk}\right]\xi_k \,. \tag{2.31}$$

This result is consistent with the requirement of first-class constraints for any value of the constant a. We should find that $\mathcal{H}(a)$ is an SO(N) singlet for any value of a, and we do because

$$\{\mathcal{J}_{ij}, \mathcal{H}(a)\}_{PB} = \pi^m\left\{\psi_i^p\psi_j^q g_{pq}, \pi_m\right\}_{PB} + \frac{a}{2R^2}\left\{J_{ij}, J^2\right\}_{PB} = 0 \,. \tag{2.32}$$

The first term on the right hand side is zero because of a cancelation between the terms coming from the PB of π_m with g_{pq} and the PB of π_m with $\psi_i^p\psi_j^q$. The second, a-dependent, term is obviously zero. The only other PB of relevance is

$$\{\mathcal{Q}_i, \mathcal{H}(a)\}_{PB} = \frac{(a-1)}{R^2}\left[\mathcal{J}_{ij}\psi_j^m\pi_m - \xi_i\xi_j\mathcal{Q}_j\right] \,. \tag{2.33}$$

We see that the requirement of first-class constraints still allows arbitrary a, but the $a = 1$ case is special. For $a = 1$ we have, for any N, both

$$\{\mathcal{Q}_i, \mathcal{H}(1)\}_{PB} = 0 \,, \tag{2.34}$$

and

$$\{\mathcal{Q}_i, \mathcal{Q}_j\}_{PB} = 2\mathcal{H}(1)\,\delta_{ij} + K_{ij}(1) \,, \qquad \mathrm{tr}\, K(1) = 0 \,. \tag{2.35}$$

For $N = 2$ the traceless matrix $K(1)$ is zero and $\mathcal{H}(1)$ coincides with the $N = 2$ Hamiltonian constraint function of (2.25).

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2.1.1 Killing vectors and Noether charges

Every Killing vector field k of the background spacetime corresponds to a symmetry of the spin-zero particle action, with Noether charge $k^m p_m$. How does this generalize to the spinning particle? Using the PB relations given previously, one may verify that for any killing vector field k the quantity

$$K = k^m \pi_m - \frac{1}{2} (\partial_m k_n) \psi_i^m \psi_i^n \tag{2.36}$$

is a constant of the motion. This shows that K is the extension to the spinning particle of the Noether charge $k^m p_m$ of the spin-zero particle associated to invariance of the particle action under an isometry of the spacetime metric. As such, K should be gauge invariant, which requires

$$\{K, \mathcal{Q}_i\}_{PB} = 0 \,. \tag{2.37}$$

A PB calculation confirms that this condition is satisfied.

3 Minkowski spinning particle in SL(2; $\mathbb{K}$) notation

A twistor action for the massive spinning particle in d-dimensional Minkowsi spacetime for $d = 3, 4, 6$ was found in [37]. Our aim in this section is to rederive that result using the SL(2; $\mathbb{K}$) spinor notation of [21]. This allows us to consider together the $d = 3, 4, 6$ cases, and to present further details of the notation.

Let $\{x^\mu; \mu = 0, 1, \ldots, d-1\}$ be Minkowski coordinates for a d-dimensional Minkowski space. The associated light-cone coordinates are

$$x^\pm = x^0 \pm x^{d-1} \,, \qquad x = (x^1, \ldots, x^{d-2}) \,. \tag{3.1}$$

For $d = 3, 4, 6, 10$, we may view the transverse position in $\mathbb{R}^{d-2}$ as an element of, respectively, $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are the four normed division algebras over $\mathbb{R}$; recall that for $x \in \mathbb{K}$ the norm-squared of x is $\bar{x}x$ where $\bar{x}$ is the $\mathbb{K}$-conjugate of x. We may also represent a point in Minkowski space by the following 2×2 hermitian[3] matrix

$$\mathbb{X} = \begin{pmatrix} x^+ & x \\ \bar{x} & x^- \end{pmatrix} . \tag{3.2}$$

The Lorentz group, with an element $\mathbb{L}$, acts on $\mathbb{X}$ by

$$\mathbb{X} \to \mathbb{L}\mathbb{X}\mathbb{L}^\dagger \,, \qquad \det\left(\mathbb{L}\mathbb{L}^\dagger\right) = 1 \,. \tag{3.3}$$

Let us examine this transformation separately for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

- $\mathbb{K} = \mathbb{R}$. In this case the condition $\det(\mathbb{L}\mathbb{L}^\dagger) = 1$ is equivalent to $|\det \mathbb{L}| = 1$. The subgroup for which $\det \mathbb{L} = 1$ is SL(2; $\mathbb{R}$), the $d = 3$ Lorentz group.

[3] Or "$\mathbb{K}$-hermitian in the terminology of [21]; we henceforth simplify this to "hermitian", it being understood that complex conjugation is generalized to conjugation with respect to $\mathbb{K}$.

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- $\mathbb{K} = \mathbb{C}$. In this case the condition on $\mathbb{L}$ defines the group $\mathrm{SL}_2(2;\mathbb{C})$ whereas the Lorentz group is $\mathrm{SL}_1(2;\mathbb{C})$, for which $\det \mathbb{L} = 1$ (see e.g. [39]); in other words, $\mathrm{SL}(2;\mathbb{C}) \cong \mathrm{SL}_2(2;\mathbb{C})$, by definition here, and the group action of (3.3) therefore includes an additional U(1), which is such that the "rotation" subgroup is U(2) rather than SU(2).

- $\mathbb{K} = \mathbb{H}$. Although the determinant of an arbitrary quaternionic matrix cannot be intrinsically defined (i.e. without recourse to a matrix representation of the quaternion algebra) the determinant of an *hermitian* quaternionic matrix has an intrinsic definition,[4] which is such that $\mathbb{L} \in \mathrm{SL}(2;\mathbb{H}) \cong \mathrm{Spin}(1,5)$, the $d = 6$ Lorentz group, when $\det(\mathbb{L}\mathbb{L}^\dagger) = 1$.

- $\mathbb{K} = \mathbb{O}$. In this case matrix multiplication becomes non-associative. In addition, the number of real independent parameters of $\mathbb{L}$ is now only $8 \times 4 - 1 = 31$, which is 14 less than the 45 required for $\mathrm{Spin}(1,9)$. However, these two problems 'cancel' because the non-associativity of the octonions introduces another 14 real parameters, this being the dimension of its G_2 group of automorphisms; it is therefore possible to interpret $\mathrm{SL}(2;\mathbb{O})$ as the $d = 10$ Lorentz group [11–13].

In all these cases, a Lorentz vector is expressed as a bi-spinor, a spinor being equivalent to a 2-component $\mathbb{K}$-valued column vector, but in this paper we focus exclusively on the $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ cases.

3.1 The spin-zero particle and SL(2; $\mathbb{K}$) spinors

Consider now a free point particle of mass m and zero spin in a d-dimensional Minkowski spacetime, with metric $\eta = \mathrm{diag}(-1, 1, \ldots, 1)$. The standard reparametrisation-invariant phase-space action is

$$S = \int dt \left\{ \dot{x}^\mu p_\mu - \frac{1}{2} e \left(p^2 + m^2 \right) \right\}, \qquad p^2 = \eta^{\mu\nu} p_\mu p_\nu \,, \tag{3.4}$$

where p_μ is the particle's d-momentum. For $d = 3, 4, 6$ we may represent p by the 2×2 matrix

$$\mathbb{P} = \begin{pmatrix} p_+ & p \\ \bar{p} & p_- \end{pmatrix}, \qquad p_\pm = p_0 \pm p_1 \,, \qquad p \in \mathbb{K} \,. \tag{3.5}$$

The Lorentz group acts on this matrix as follows

$$\mathbb{P} \to (\mathbb{L}^\dagger)^{-1} \mathbb{P}\, \mathbb{L}^{-1} \,. \tag{3.6}$$

This transformation is such that

$$\mathrm{tr}(\dot{\mathbb{X}}\mathbb{P}) \to \mathrm{tr}(\mathbb{L}\dot{\mathbb{X}}\mathbb{P}\mathbb{L}^{-1}) \,, \tag{3.7}$$

which is invariant for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ by the cyclic property of the trace. This property fails for $\mathbb{K} = \mathbb{H}$ but is still true for the real part of the trace (which we denote by $\mathrm{tr}_\mathbb{R}$). In fact

$$\mathrm{tr}_\mathbb{R}(\dot{\mathbb{X}}\mathbb{P}) = 2\dot{x}^m p_m \,. \tag{3.8}$$

[4] A useful reference on determinants of quaternionic matrices is [40].

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Actually, we have not yet used the fact that $\mathbb{L}$ represents a Lorentz transformation; we have only used the existence of $\mathbb{L}^{-1}$ because no metric is needed for the contraction of $\dot{x}$ with p. Observe now that

$$-p^2 \equiv \det \mathbb{P} \to \det \left[(\mathbb{L}^\dagger)^{-1} \mathbb{P}\, \mathbb{L}^{-1} \right] = \det \mathbb{P}\,. \tag{3.9}$$

For $\mathbb{K} = \mathbb{R}, \mathbb{C}$ the equality follows from the usual properties of determinants that allow us to rewrite the transformed determinant as $\det \mathbb{P} / \det(\mathbb{L}\mathbb{L}^\dagger)$. This equality is not obvious for $\mathbb{K} = \mathbb{H}$ but still true, as is easily verified explicitly for 2×2 matrices. We now see that the action (3.4) may be rewritten as

$$S = \int dt \left\{ \frac{1}{2} \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{X}}\mathbb{P}) + \frac{1}{2} e \left(\det \mathbb{P} - m^2 \right) \right\}. \tag{3.10}$$

So far we have seen how to write the Lorentz scalars $\dot{x}^m p_m$ and p^2 in $\mathrm{SL}(2;\mathbb{K})$ notation, but any scalar product of Lorentz vectors or co-vectors may be expressed in this notation. For any 2×2 hermitian matrix $\mathbb{X}$, the Cayley-Hamilton theorem states that

$$\mathbb{X}^2 - \mathrm{tr}_{\mathbb{R}}(\mathbb{X})\, \mathbb{X} + (\det \mathbb{X})\, \mathbb{I}_2 \equiv 0\,, \tag{3.11}$$

which we may rewrite as [41]

$$\mathbb{X}\tilde{\mathbb{X}} \equiv -(\det \mathbb{X}) \mathbb{I}_2 = x^2 \mathbb{I}_2\,, \tag{3.12}$$

where

$$\tilde{\mathbb{X}} = \mathbb{X} - (\mathrm{tr}_{\mathbb{R}} \mathbb{X}) \mathbb{I}_2 \qquad \left[\Leftrightarrow \ \mathbb{X} = \tilde{\mathbb{X}} - (\mathrm{tr}_{\mathbb{R}} \tilde{\mathbb{X}}) \mathbb{I}_2 \right]. \tag{3.13}$$

Taking the trace yields

$$\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\mathbb{X}}) = -2 \det \mathbb{X} = 2x^2\,, \tag{3.14}$$

and replacing $\mathbb{X}$ by $\mathbb{X} + \mathbb{Y}$ in this relation yields

$$2x \cdot y = \mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\mathbb{Y}}) = \mathrm{tr}_{\mathbb{R}}(\tilde{\mathbb{X}}\mathbb{Y})\,. \tag{3.15}$$

This tells us that if x^m is represented by $\mathbb{X}$ then x_m is represented by $\tilde{\mathbb{X}}$. In other words, in the $\mathrm{SL}(2;\mathbb{K})$ spinor formalism the raising and lowering of Lorentz vector indices becomes the process of "trace reversal".

3.2 $N = 1$ spinning particle

We now generalize to the $N = 1$ massive spinning particle, in a d-dimensional Minkowski background. The standard reparametrisation-invariant phase-space action is

$$S = \int dt \left\{ \dot{x}^\mu p_\mu + \frac{1}{2} \lambda^\mu \dot{\lambda}^\nu \eta_{\mu\nu} + \frac{1}{2} \xi \dot{\xi} - \frac{1}{2} e \left(p^2 + m^2 \right) - \chi \left(\lambda^\mu p_\mu + m\xi \right) \right\}. \tag{3.16}$$

For accord with the analysis to follow of the spinning particle in an AdS_D background with $D = d + 1$, we are now calling the anticommuting vector variable λ^μ.

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In addition to representing x^μ and p_μ by the 2×2 hermitian matrices $\mathbb{X}$ and $\mathbb{P}$, we can represent λ^μ by the hermitian matrix

$$\Lambda = \begin{pmatrix} \lambda^+ & \lambda \\ \bar{\lambda} & \lambda^- \end{pmatrix}, \qquad \lambda^\pm = \lambda^0 \pm \lambda^1 , \tag{3.17}$$

where λ is anticommuting for $\mathbb{K} = \mathbb{R}, \mathbb{C}$ and of the form $\lambda_0 + \mathbf{i}\cdot\boldsymbol{\lambda}$ for $\mathbb{K} = \mathbb{H}$, where λ_0 and $\boldsymbol{\lambda}$ are anticommuting (and $\mathbf{i}$ is the standard triplet of unit imaginary quaternions). We shall use the term "fermionic" to cover all three cases.

We may now rewrite the action (3.16) as

$$\begin{aligned} S = \int dt \Big\{ & \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{X}}\mathbb{P}) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}(\tilde{\Lambda}\dot{\Lambda}) + \frac{1}{2}\xi\dot{\xi} \\ & + \frac{1}{2}e\left(\det\mathbb{P} - m^2\right) - \frac{1}{2}\chi\left[\mathrm{tr}_{\mathbb{R}}\left(\Lambda\mathbb{P}\right) + 2m\xi\right]\Big\} . \end{aligned} \tag{3.18}$$

This action is invariant under the following worldline supertranslation gauge transformations:

$$\delta\mathbb{X} = \alpha\tilde{\mathbb{P}} + \epsilon\Lambda , \quad \delta\Lambda = -\tilde{\mathbb{P}}\epsilon , \quad \delta\xi = -m\epsilon \tag{3.19}$$

and

$$\delta e = \dot{\alpha} + 2\epsilon\chi , \qquad \delta\chi = \dot{\epsilon} , \tag{3.20}$$

where $\alpha(t)$ is a commuting parameter and $\epsilon(t)$ an anticommuting parameter.

3.3 Twistors and $O(2;\mathbb{K})$ gauge invariance

The hermitian matrix $\mathbb{P}$ may be written in the form

$$\mathbb{P} = \mp\mathbb{U}\mathbb{U}^\dagger , \tag{3.21}$$

where the top sign applies if $p^0 > 0$ and the bottom sign applies if $p^0 < 0$. The 2×2 matrix $\mathbb{U}$ has the $\mathrm{SL}(2;\mathbb{K})$ transformation $\mathbb{U} \to \mathbb{L}\mathbb{U}$. In other words, the two columns of $\mathbb{U}$ are $\mathrm{SL}(2;\mathbb{K})$ spinors; they constitute a doublet of $O(2;\mathbb{K})$, which acts on $\mathbb{U}$ from the right:

$$\mathbb{U} \to \mathbb{U}\mathbb{N} , \qquad \mathbb{N}\mathbb{N}^\dagger = \mathbb{I}_2 = \mathbb{N}^\dagger\mathbb{N} . \tag{3.22}$$

The expression for $\mathbb{P}$ in terms of $\mathbb{U}$ is therefore $O(2;\mathbb{K})$ invariant. This expression solves the mass-shell constraint $\det\mathbb{P} = m^2$ provided that

$$\det(\mathbb{U}\mathbb{U}^\dagger) = m^2 , \tag{3.23}$$

which becomes a new mass-shell constraint. This shows that the matrix $\mathbb{U}$ is invertible provided $m^2 > 0$. It has more independent components than $\mathbb{P}$ but these will be unphysical provided the $O(2;\mathbb{K})$ transformations of $\mathbb{U}$ are realized as a local symmetry on the particle's worldline. In this case, $\mathbb{U}(t)$ can be interpreted as a choice of spatial frame at each point of the worldline; this is because $O(2;\mathbb{K})$ is essentially the group of space rotations. Let us examine this assertion separately for $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

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- $\mathbb{K} = \mathbb{R}$. In this case $\mathbb{N}$ is an orthogonal 2×2 matrix and hence an element of $O(2)$, which contains the SO(2) subgroup of rotations in $\mathbb{E}^2$.
- $\mathbb{K} = \mathbb{C}$. In this case $\mathbb{N}$ is a complex unitary 2×2 matrix and hence an element of U(2). This contains the $\mathrm{SU}(2) \cong \mathrm{Spin}(3)$ subgroup of rotations in $\mathbb{E}^3$, but also an additional U(1) factor (as anticipated in our earlier discussion of the Lorentz groups).
- $\mathbb{K} = \mathbb{H}$. In this case $\mathbb{N}$ is a "quaternionic unitary" 2×2 matrix. Such matrices are elements of the $Sp_2 \cong \mathrm{USp}(4) \cong \mathrm{Spin}(5)$ subgroup of rotations in $\mathbb{E}^5$.
- $\mathbb{K} = \mathbb{O}$. We include this case only for completeness. The number of real parameters of $\mathbb{N}$ is now $8 \times 4 - 10 = 22$, which is the right number to reduce the 32 components of $\mathbb{U}$ to the 10 components of $\mathbb{P}$. It is also 14 short of the 36 needed for Spin(9), and we presume that $O(2;\mathbb{O})$ *could* be interpreted as Spin(9). However, generators of a *gauge* invariance, when incorporated into a phase-space action need not form a Lie algebra; they need only be first class.

Given that $\mathbb{U}$ is invertible, we may introduce its inverse $\mathbb{V} \equiv \mathbb{U}^{-1}$ (the left and right inverses are equal even in the quaternionic case [42]). Then

$$\tilde{\mathbb{P}} = \pm \det(\mathbb{U}\mathbb{U}^\dagger)\mathbb{V}^\dagger\mathbb{V} \qquad \left(\mathbb{V} = \mathbb{U}^{-1}\right) , \tag{3.24}$$

which follows from a verification of the identity (3.12) with $\mathbb{X}$ replaced by $\mathbb{P}$; i.e.

$$\mathbb{P}\tilde{\mathbb{P}} \equiv -(\det \mathbb{P})\mathbb{I}_2 . \tag{3.25}$$

This expression for $\tilde{\mathbb{P}}$ is a special case of a more general result, valid under the assumption that $\det(\mathbb{U}\mathbb{U}^\dagger)$ is non-zero. Given any hermitian 2×2 matrix $\mathbb{A}$ transforming by conjugation under $O(2;\mathbb{K})$, we can construct from it the hermitian matrix

$$\mathbb{T} = \sqrt{\det(\mathbb{U}\mathbb{U}^\dagger)}\, \mathbb{V}^\dagger \mathbb{A} \mathbb{V} , \tag{3.26}$$

which has the transformation properties of a Lorentz vector. The hermitian matrix for the corresponding Lorentz covector is then

$$\tilde{\mathbb{T}} = \frac{1}{\sqrt{\det(\mathbb{U}\mathbb{U}^\dagger)}} \mathbb{U}\tilde{\mathbb{A}}\mathbb{U}^\dagger . \tag{3.27}$$

This follows by verification of the identity (3.12) for $\mathbb{T}$, assuming its validity for $\mathbb{A}$:

$$\mathbb{T}\tilde{\mathbb{T}} = \mathbb{V}^\dagger \mathbb{A}\tilde{\mathbb{A}}\mathbb{U}^\dagger = -(\det \mathbb{A})\mathbb{I}_2 = -(\det \mathbb{T})\mathbb{I}_2 . \tag{3.28}$$

The choice $\mathbb{A} = \pm\sqrt{\det(\mathbb{U}\mathbb{U}^\dagger)}\, \mathbb{I}_2$ yields $\mathbb{T} = \tilde{\mathbb{P}}$ and $\tilde{\mathbb{T}} = \mathbb{P}$.

Now we solve the local supersymmetry constraint by writing Λ ($\tilde{\Lambda}$) in the form

$$\Lambda = m\mathbb{V}^\dagger \left(\Psi \pm \xi \mathbb{I}_2\right) \mathbb{V} , \qquad \left(\tilde{\Lambda} = \frac{1}{m} \mathbb{U} \left(\Psi \mp \xi \mathbb{I}_2\right) \mathbb{U}^\dagger\right) , \tag{3.29}$$

where

$$\Psi^\dagger = \Psi , \qquad \mathrm{tr}_{\mathbb{R}} \Psi = 0 . \tag{3.30}$$

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As indicated, the fermionic hermitian matrix Ψ has zero real-trace. It is $\mathrm{SL}(2;\mathbb{K})$-inert but transforms by conjugation under the $O(2;\mathbb{K})$ gauge group:

$$\Psi \to \mathbb{N}^\dagger \Psi \mathbb{N}\,. \tag{3.31}$$

Substitution yields the Lagrangian

$$L = \frac{1}{4}\mathrm{tr}_{\mathbb{R}}\Psi\dot{\Psi} + \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger)\,, \tag{3.32}$$

where

$$\mathbb{W} = \pm\left[\mathbb{X}\mathbb{U} + \xi\mathbb{V}^\dagger\Psi\right] + \frac{1}{2}\mathbb{V}^\dagger\Psi^2\,. \tag{3.33}$$

The fermionic nature of Ψ implies that

$$\mathrm{tr}_{\mathbb{R}}\Psi^2 \equiv 0\,, \tag{3.34}$$

but $\Psi^2 \not\equiv 0$. For example, for $\mathbb{K} = \mathbb{H}$ we may write

$$\Psi = \begin{pmatrix} \varrho & \varsigma + \mathbf{i}\cdot\boldsymbol{\psi} \\ \varsigma - \mathbf{i}\cdot\boldsymbol{\psi} & -\varrho \end{pmatrix}\,, \tag{3.35}$$

for anticommuting singlets ϱ and ς and anticommuting triplet $\boldsymbol{\psi}$, and then

$$\Psi^2 = \begin{pmatrix} -\mathbf{i}\cdot(2\varsigma\boldsymbol{\psi} + \boldsymbol{\psi}\times\boldsymbol{\psi}) & 2\varrho(\varsigma + \mathbf{i}\cdot\boldsymbol{\psi}) \\ -2\varrho(\varsigma - \mathbf{i}\cdot\boldsymbol{\psi}) & \mathbf{i}\cdot(2\varsigma\boldsymbol{\psi} - \boldsymbol{\psi}\times\boldsymbol{\psi}) \end{pmatrix}\,. \tag{3.36}$$

Notice that Ψ^2 is *anti*-hermitian, since we are using a convention such that hermitian conjugation does not change the order of anticommuting factors.

For the $\mathbb{K} = \mathbb{C}$ case we have

$$\Psi = \begin{pmatrix} \varrho & \varsigma + i\psi \\ \varsigma - i\psi & -\varrho \end{pmatrix}\,, \quad \Psi^2 = \begin{pmatrix} -2i\varsigma\psi & 2\varrho(\varsigma + i\psi) \\ -2\varrho(\varsigma - i\psi) & 2i\varsigma\psi \end{pmatrix}\,, \tag{3.37}$$

and the $\mathbb{K} = \mathbb{R}$ case is found by setting $\psi = 0$. A special feature of these cases is that Ψ^2 is traceless. This statement is equivalent to (3.34) for $\mathbb{K} = \mathbb{R}$ but (3.34) leaves open the possibility of a non-zero imaginary part of the trace for $\mathbb{K} = \mathbb{C}, \mathbb{H}$. Indeed, it *is* non-zero for $\mathbb{K} = \mathbb{H}$, but for $\mathbb{K} = \mathbb{C}$ it is zero. This fact will become important in section 5.

The "incidence relation" (3.33) implies the identity

$$\mathbb{G} := \mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U} - \Psi^2 \equiv 0\,. \tag{3.38}$$

This becomes a constraint in the action with independent phase space variables $(\mathbb{U}, \mathbb{W}, \Psi)$, imposed by an anti-hermitian Lagrange multiplier $\mathbb{S}$:

$$S = \int dt\left\{\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}(\Psi\dot{\Psi}) - \mathrm{tr}_{\mathbb{R}}(\mathbb{S}\mathbb{G}) + \frac{\ell}{2}\left(\det\mathbb{U}\mathbb{U}^\dagger - m^2\right)\right\}\,. \tag{3.39}$$

The new mass-shell constraint generates the new gauge-transformation

$$\delta_\alpha\mathbb{W} = m^2\mathbb{V}^\dagger\alpha\,, \qquad \delta_\alpha\ell = \dot{\alpha} + 2\alpha\,\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{V})\,, \tag{3.40}$$

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for which invariance may be verified by means of the identity

$$\frac{d}{dt}\left[\det(\mathbb{U}\mathbb{U}^\dagger)\right] \equiv 2\det(\mathbb{U}\mathbb{U}^\dagger)\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{V})\,. \tag{3.41}$$

However, this gauge invariance is equivalent to time reparametrization invariance (for reasons explained in [43] in the context of the $d = 6$ superparticle).

Notice that ξ has dropped out of the above action. This is because the new twistor variables are inert under the original worldline supersymmetry gauge transformations; more precisely, $\mathbb{U}$ is inert and $\mathbb{W}$ is inert modulo a gauge transformation generated by the new mass-shell constraint, as we now explain. Using the incidence relation (3.33) to compute the local supersymmetry transformation of $\mathbb{W}$ from those of the initial variables $\mathbb{X}$ and ξ, we find that

$$\delta_\epsilon\mathbb{W} = \pm\left[\delta_\epsilon\mathbb{X}\mathbb{U} + \delta_\epsilon\xi\mathbb{V}^\dagger\Psi\right] = \pm\epsilon\left[\Lambda\mathbb{U} - m\mathbb{V}^\dagger\Psi\right] = m\epsilon\xi\mathbb{V}^\dagger\,, \tag{3.42}$$

where the last equality uses (3.29). We see that $\mathbb{W}$ is *not* strictly inert under the initial local worldline supersymmetry, but its transformation is just an α-gauge transformation of (3.40) with parameter

$$\alpha = m^{-1}\epsilon\xi\,, \tag{3.43}$$

as originally found in [37].

3.4 Twistors as Sp(4; $\mathbb{K}$) spinors

Notice that

$$\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) - \frac{d}{dt}\left[\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left(\mathbb{W}\mathbb{U}^\dagger\right)\right] = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger - \dot{\mathbb{W}}\mathbb{U}^\dagger)\,. \tag{3.44}$$

Introducing the 4×2 matrix

$$\mathbb{Z} = \begin{pmatrix}\mathbb{U}\\ \mathbb{W}\end{pmatrix}, \tag{3.45}$$

and the 4×4 antisymmetric matrix

$$\Omega = \begin{pmatrix}0 & \mathbb{I}_2\\ -\mathbb{I}_2 & 0\end{pmatrix}, \tag{3.46}$$

we may rewrite this as

$$\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\Omega\,\dot{\mathbb{Z}}\mathbb{Z}^\dagger) + \frac{d}{dt}\left[\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left(\mathbb{W}\mathbb{U}^\dagger\right)\right]. \tag{3.47}$$

In this notation, and omitting a total derivative, the action (3.39) becomes

$$S = \int dt\left\{\frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\Omega\,\dot{\mathbb{Z}}\mathbb{Z}^\dagger) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}(\Psi\dot{\Psi}) - \mathrm{tr}_{\mathbb{R}}(\mathbb{S}\mathbb{G}) - \frac{\ell}{2}\left(\det\mathbb{U}\mathbb{U}^\dagger - m^2\right)\right\}. \tag{3.48}$$

The spin-shell constraint functions $\mathbb{G}$ may also be written in terms of $\mathbb{Z}$:

$$\mathbb{G} = -\mathbb{Z}^\dagger\Omega\mathbb{Z} - \Psi^2\,. \tag{3.49}$$

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The advantage of rewriting the action in this way is that it makes manifest the fact that only the mass-shell constraint breaks what would otherwise be an invariance under the following linear transformation of $\mathbb{Z}$ with 4×4 matrix parameter $\mathbb{M}$ with entries in $\mathbb{K}$:

$$\mathbb{Z} \to \mathbb{M}\mathbb{Z}\,, \qquad \mathbb{M}^\dagger \Omega \mathbb{M} = \Omega\,. \tag{3.50}$$

This transformation defines the group that, following [11], we call $\mathrm{Sp}(4;\mathbb{K})$. Let us examine the $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ cases in turn.

- $\mathbb{K} = \mathbb{R}$. In this case the 4×4 real matrix $\mathbb{M}$ is an element of $\mathrm{Sp}(4;\mathbb{R}) \cong \mathrm{Spin}(2,3)$, the $d = 3$ conformal group.

- $\mathbb{K} = \mathbb{C}$. The 4×4 real antisymmetric matrix Ω is diagonalizable over $\mathbb{C}$, with doubly-degenerate eigenvalues $\pm i$, so $\mathbb{M}$ is now an element of $\mathrm{U}(2,2)$. Ignoring discrete factors, this is equivalent to the product of $\mathrm{U}(1)$ with $\mathrm{SU}(2,2) \cong \mathrm{Spin}(2,4)$, the $d = 4$ conformal group. In other words, the group $\mathrm{SL}(4;\mathbb{C})$ is almost equivalent to the $d = 4$ conformal group but, as for the rotation and Lorentz groups in four spacetime dimensions, it includes an additional $\mathrm{U}(1)$ factor.

- $\mathbb{K} = \mathbb{H}$. In this case the 4×4 quaternionic matrix $\mathbb{M}$ is an element of $\mathrm{Sp}(4;\mathbb{H}) \cong \mathrm{Spin}(2,6)$, the $d = 6$ conformal group.

- $\mathbb{K} = \mathbb{O}$. We comment on this case only for the sake of completeness. The 4×4 hermitian octonionic matrices that one might expect to span $\mathrm{Sp}(4;\mathbb{O})$ have only 52 real parameters, which is 14 short of the 66 needed for the $d = 10$ conformal group $\mathrm{Spin}(2,10)$. However a version of the "add 14 rule" summarized earlier is again applicable, so one may interpret $\mathrm{Sp}(4;\mathbb{O})$ as $\mathrm{Spin}(2,10)$ [14].

In summary, $\mathrm{Sp}(4;\mathbb{K})$ is (essentially) the conformal group of d-dimensional Minkowski spacetime for $d = 2 + \dim \mathbb{K}$. A conformal group spinor is a twistor, which means that the 4×2 matrix $\mathbb{Z}$ is a "two-twistor"; i.e. a twistor doublet acted upon from the left by $\mathrm{Sp}(4;\mathbb{K})$ and from the right by the gauge group $O(2;\mathbb{K})$:

$$\mathbb{Z} \to \mathbb{M}\mathbb{Z}\mathbb{N}\,. \tag{3.51}$$

Returning to the action (3.39), we see that only the mass-shell constraint breaks the conformal invariance. This is apparently true even if we set $m^2 = 0$, but in that case the mass-shell constraint tells us that $\mathbb{U}$ is no longer invertible, and this implies that there are additional gauge invariances, which implies that the action is no longer in canonical form (despite appearances). One may expect that when these additional gauge invariances are taken into account, the phase space action will be the standard one-twistor action for a massless particle in Mink_d for $d = 3, 4, 6$ with manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance [8, 15], as has been verified for the $d = 3$ case in [37].

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3.4.1 $N > 1$

The extension to $N > 1$ is almost immediate: Ψ becomes Ψ_i with $i = 1, \ldots, N$, so the spin-shell constraint function is now

$$\mathbb{G} = -\mathbb{Z}^\dagger \Omega \mathbb{Z} - \Psi_i \Psi_i \,, \tag{3.52}$$

and there is now an $\mathrm{SO}(N)$ constraint with constraint function

$$\mathcal{J}_{ij} = \frac{1}{2} \mathrm{tr}_{\mathbb{R}} \left(\Psi_i \Psi_j \right) \,. \tag{3.53}$$

The action for the N-extended spinning particle of mass m in a Mink_d background is

$$\begin{aligned} S = \int dt \Big\{ & \frac{1}{2} \mathrm{tr}_{\mathbb{R}} (\Omega \dot{\mathbb{Z}} \mathbb{Z}^\dagger) + \frac{1}{4} \mathrm{tr}_{\mathbb{R}} (\Psi_i \dot{\Psi}_i) - \mathrm{tr}_{\mathbb{R}} (\mathbb{S}\mathbb{G}) - \frac{1}{2} f_{ij} \mathcal{J}_{ij} \\ & - \frac{\ell}{2} \left(\det \mathbb{U}\mathbb{U}^\dagger - m^2 \right) \Big\} \,. \end{aligned} \tag{3.54}$$

In the above formulae, a sum over repeated $\mathrm{SO}(N)$ vector indices is implicit.

4 Twistors and the spinning particle in AdS_D

We now return to the N-extended spinning particle in AdS_D. Recall that the action takes the form

$$S = \int dt \left\{ \dot{x}^m p_m + \frac{1}{2} \psi_i^m \dot{\psi}_i^n g_{mn} + \frac{1}{2} \xi_i \dot{\xi}_i - e \mathcal{H} - \chi_i \mathcal{Q}_i - \frac{1}{2} f_{ij} \mathcal{J}_{ij} \right\} \,. \tag{4.1}$$

There was some freedom in the choice of constraint functions, represented by the constant a. Choosing $a = 0$ we have

$$\mathcal{H} = g^{mn} \pi_m \pi_n + m^2 \,, \qquad \mathcal{Q}_i = \psi_i^m \pi_m + m \xi_i \,. \tag{4.2}$$

4.1 Poincaré patch coordinates

We shall now choose coordinates $x^m = \{x^\mu, z\}$ adapted to the foliation of AdS_D by Minkowski hypersurfaces. The metric is

$$ds^2 = \left(\frac{R}{z} \right)^2 \left(dx^\mu dx^\nu \eta_{\mu\nu} + dz^2 \right) \,. \tag{4.3}$$

The geometric part of the Lagrangian becomes

$$L_{\mathrm{geom}} = \dot{x}^\mu p_\mu + \dot{z} p_z + \frac{1}{2} \eta_{\mu\nu} \lambda_i^\mu \dot{\lambda}_i^\nu + \frac{1}{2} \zeta_i \dot{\zeta}_i + \frac{1}{2} \xi_i \dot{\xi}_i \,, \tag{4.4}$$

where

$$\lambda_i^\mu = \left(\frac{R}{z} \right) \psi_i^\mu \,, \qquad \zeta_i = \left(\frac{R}{z} \right) \psi_i^z \,. \tag{4.5}$$

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Now we turn to the constraints. The non-zero components of the Levi-Civita affine connection are

$$\Gamma_{zz}{}^{z} = -\frac{1}{z}\,, \quad \Gamma_{\mu\nu}{}^{z} = \frac{1}{z}\eta_{\mu\nu}\,, \qquad \Gamma_{\mu z}{}^{\nu} = \Gamma_{z\mu}{}^{\nu} = -\frac{1}{z}\delta^{\nu}_{\mu}\,. \tag{4.6}$$

Using this we find that

$$\pi_z = p_z\,, \qquad \pi_\mu = p_\mu + z^{-1}\lambda^i_\mu\zeta_i\,, \tag{4.7}$$

and using these relations we find that

$$\mathcal{Q}_i = \left(\frac{z}{R}\right)\left[p\cdot\lambda_i + \zeta_i p_z + \left(\frac{mR}{z}\right)\xi_i\right] + R^{-1}\lambda_i\cdot\lambda_j\zeta_j\,, \tag{4.8}$$

and that

$$2\mathcal{H} = \left(\frac{z}{R}\right)^2\left[p^2 + p_z^2 + \left(\frac{mR}{z}\right)^2\right] + 2\frac{z}{R^2}p\cdot\lambda_i\zeta_i - R^{-2}\lambda_i\cdot\lambda_j\zeta_i\zeta_j\,, \tag{4.9}$$

where

$$p\cdot\lambda_i \equiv p_\mu\lambda^\mu_i\,, \qquad \lambda_i\cdot\lambda_j \equiv \lambda^\mu_i\lambda^\nu_j\eta_{\mu\nu}\,. \tag{4.10}$$

The SO(N) constraint functions in the new variables are

$$\mathcal{J}_{ij} = \lambda_i\cdot\lambda_j + \zeta_i\zeta_j + \xi_i\xi_j\,, \tag{4.11}$$

and the constraint $\mathcal{J}_{ij} = 0$ may be used to eliminate $\lambda_i\cdot\lambda_j$ in the expression for the supersymmetry constraint functions; the result for $\mathcal{Q}_i$ is

$$\mathcal{Q}_i = \left(\frac{z}{R}\right)\left[p\cdot\lambda_i + \zeta_i p_z + \left(\frac{mR+\zeta_j\xi_j}{z}\right)\xi_i\right]\,. \tag{4.12}$$

We may now use both $\mathcal{J}_{ij} = 0$ and $\mathcal{Q}_i = 0$ to simplify the expression for $\mathcal{H}$ to

$$\mathcal{H} = \left(\frac{z}{R}\right)\left[p^2 + p_z^2 + \left(\frac{mR+\zeta_j\xi_j}{z}\right)^2\right]\,. \tag{4.13}$$

Finally, we can absorb the overall factors of z/R in these expressions by a redefinition of the Lagrange multipliers, after which the action in the new variables becomes

$$S = \int dt\left\{L_{\rm geom} - \frac{1}{2}\tilde{e}\,\tilde{\mathcal{H}} - \tilde{\chi}_i\,\tilde{\mathcal{Q}}_i - \frac{1}{2}f_{ij}\mathcal{J}_{ij}\right\}\,, \tag{4.14}$$

where

$$\tilde{\mathcal{H}} = p^2 + \Delta^2\,, \qquad \tilde{\mathcal{Q}}_i = p\cdot\lambda_i + \Xi_i\,, \tag{4.15}$$

with

$$\Xi_i = p_z\zeta_i + \left(\frac{mR+\zeta_j\xi_j}{z}\right)\xi_i\,, \qquad \Delta^2 = p_z^2 + \left(\frac{mR+\zeta_j\xi_j}{z}\right)^2\,. \tag{4.16}$$

From the geometrical part of the Lagrangian, given by (4.4), we may read off the Poisson brackets of the new canonical variables. The non-zero canonical Poisson brackets are

$$\{x^\mu, p_\nu\}_{PB} = \delta^\mu_\nu\,, \qquad \{z, p_z\}_{PB} = 1\,, \tag{4.17}$$

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and

$$\left\{\lambda_i^\mu, \lambda_j^\nu\right\}_{PB} = \eta^{\mu\nu}\delta_{ij}\,, \qquad \{\zeta_i, \zeta_j\}_{PB} = \delta_{ij}\,, \qquad \{\xi_i, \xi_j\}_{PB} = \delta_{ij}\,. \tag{4.18}$$

Using these relations we find that

$$\{\Xi_i, \Xi_j\}_{PB} = \Delta^2\,\delta_{ij}\,, \qquad \left\{\Xi_i, \Delta^2\right\}_{PB} = 0\,, \tag{4.19}$$

and hence that

$$\left\{\tilde{\mathcal{Q}}_i, \tilde{\mathcal{Q}}_j\right\}_{PB} = 2\tilde{\mathcal{H}}\,\delta_{ij}\,, \tag{4.20}$$

which is the expected N-extended worldline supersymmetry algebra.

4.1.1 The AdS isometries

The Noether charges corresponding to the AdS isometries in the Poincaré patch coordinates are

$$\begin{aligned} P_\mu &= p_\mu \\ L^{\mu\nu} &= 2x^{[\mu}p^{\nu]} - \lambda_i^\mu\lambda_i^\nu\,, \qquad D = x\cdot p + zp_z \\ K^\mu &= x^2p^\mu + z^2p^\mu - 2x^\mu(x\cdot p + zp_z) + 2\lambda_i^\mu(x\cdot\lambda_i + z\zeta_i)\,. \end{aligned} \tag{4.21}$$

In $\mathrm{SL}(2;\mathbb{K})$ bi-spinor notation

$$\begin{aligned} P_\mu &\to \mathbb{P} \\ L^\mu{}_\nu + D\delta^\mu_\nu &\to \mathbb{D} \equiv \mathbb{P}\mathbb{X} + \frac{1}{2}\tilde{\Lambda}_i\Lambda_i + zp_z\mathbb{I}_2 \\ K^\mu &\to \mathbb{K}\,, \end{aligned} \tag{4.22}$$

where[5]

$$\mathbb{K} \equiv \frac{1}{2}\tilde{\mathbb{P}}\left[\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\mathbb{X}}) + 2z^2\right] - \mathbb{X}\left[\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\mathbb{P}) + 2zp_z\right] + \Lambda_i\left[\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\Lambda}_i) + 2z\zeta_i\right]\,. \tag{4.23}$$

Using the $\mathrm{SL}(2;\mathbb{K})$ matrix identities

$$\begin{aligned} \mathbb{X}\mathbb{P}\mathbb{X} &\equiv \mathbb{X}\,\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\mathbb{P}) - \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\mathbb{X}})\tilde{\mathbb{P}}\,, \\ \Lambda_i\tilde{\Lambda}_i\mathbb{X} - \mathbb{X}\tilde{\Lambda}_i\Lambda_i &\equiv 2\Lambda_i\mathrm{tr}_{\mathbb{R}}(\tilde{\Lambda}_i\mathbb{X})\,, \end{aligned} \tag{4.24}$$

we can write $\mathbb{K}$ in the following alternative form:

$$\mathbb{K} = \mathbb{X}\mathbb{P}\mathbb{X} + z^2\tilde{\mathbb{P}} - 2\mathbb{X}zp_z + \frac{1}{2}\left(\Lambda_i\tilde{\Lambda}_i\mathbb{X} - \mathbb{X}\tilde{\Lambda}_i\Lambda_i\right) + 2\Lambda_i z\zeta_i\,. \tag{4.25}$$

[5]We trust that this use of $\mathbb{K}$ to denote the matrix of Noether charges associated to K^μ will not be confused with its use elsewhere to denote one of the four division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$.

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4.2 A change of anticommuting variables

The first step in the passage to a two-twistor version of the action for a spinning particle in AdS_D for $D = 4, 5, 7$ is to make a redefinition of the scalar[6] anticommuting variables. First we define

$$Z_i = p_z \xi_i - \left(\frac{mR + \zeta_j \xi_j}{z} \right) \zeta_i \,. \tag{4.26}$$

These phase-space functions satisfy PB relations analogous to those of (4.19):

$$\{Z_i, Z_j\}_{PB} = \Delta^2 \, \delta_{ij} \ , \qquad \{Z_i, \Delta^2\}_{PB} = 0 \,. \tag{4.27}$$

In addition,

$$\{Z_i, \Xi_j\}_{PB} = 0 \,. \tag{4.28}$$

Next, we define the new variables

$$\xi'_i = \Xi_i / \Delta \,, \qquad \zeta'_i = Z_i / \Delta \,. \tag{4.29}$$

These primed variables satisfy the canonical PB relations

$$\{\xi'_i, \xi'_j\}_{PB} = \delta_{ij} = \{\zeta'_i, \zeta'_j\}_{PB} \ , \qquad \{\xi'_i, \zeta'_j\}_{PB} = 0 \,. \tag{4.30}$$

The primed anticommuting variables are related to the unprimed ones by a rotation:

$$\begin{pmatrix} \xi'_i \\ \zeta'_i \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix} \begin{pmatrix} \zeta_i \\ \xi_i \end{pmatrix} \,, \tag{4.31}$$

where the angle φ is such that

$$p_z = \Delta \cos\varphi \,, \qquad \frac{mR + \zeta_i \xi_i}{z} = \Delta \sin\varphi \,. \tag{4.32}$$

Now we make use of the following **Key Identity**:

$$\dot{z} p_z + \frac{1}{2}(\zeta_i \dot{\zeta}_i + \xi_i \dot{\xi}_i) \equiv -z p_z \Delta^{-1} \dot{\Delta} + \frac{1}{2}(\zeta'_i \dot{\zeta}'_i + \xi'_i \dot{\xi}'_i) + \frac{d}{dt}(z p_z - mR\varphi) \tag{4.33}$$

Provided that we can ignore the total derivative term, this identity allows us to view Δ as a canonical variable (with conjugate variable $z p_z / \Delta$). On AdS, in distinction to its universal cover, timelike geodesics are closed paths on which φ increases by 2π on each traversal [21]. This means that the integral of $mR\dot{\varphi}$ is only defined modulo a multiple of $2\pi mR$, which suggests that the path integral will be well-defined for $m \neq 0$ only if $mR \in \mathbb{Z}$; we return to this issue in section 5.

We shall proceed on the assumption that the total derivative term on the right hand side of our "key identity" may be ignored. We may then use this identity to rewrite the action (4.14) so that

$$L_{\mathrm{geom}} = \dot{x}^\mu p_\mu - z p_z \Delta^{-1} \dot{\Delta} + \frac{1}{2}\left(\lambda_i \cdot \dot{\lambda}_i + \zeta'_i \dot{\zeta}'_i + \xi'_i \dot{\xi}'_i \right) \,. \tag{4.34}$$

[6] By "scalar" we mean here with respect to the d-dimensional Lorentz group.

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The constraint functions are now

$$\tilde{\mathcal{Q}}_i = p \cdot \lambda_i + \Delta \xi_i' \,, \qquad \tilde{\mathcal{H}} = \frac{1}{2}\left(p^2 + \Delta^2\right) \,, \tag{4.35}$$

where the expression for Δ^2 in terms of the primed variables is

$$\Delta^2 = p_z^2 + \left(\frac{mR + \xi_i' \zeta_i'}{z}\right)^2 \,. \tag{4.36}$$

For $N > 1$ we also have

$$\mathcal{J}_{ij} = \lambda_i \cdot \lambda_j + \xi_i' \xi_j' + \zeta_i' \zeta_j' \,. \tag{4.37}$$

4.2.1 Conversion to SL(2; $\mathbb{K}$) notation

In SL(2; $\mathbb{K}$) matrix notation for the Lorentz d-vectors, eq. (4.34) becomes

$$L_{\rm geom} = \mathrm{tr}_{\mathbb{R}}\left(\frac{1}{2}\dot{\mathbb{X}}\mathbb{P} + \frac{1}{4}\tilde{\Lambda}_i \dot{\Lambda}_i\right) + \frac{1}{2}(\zeta_i' \dot{\zeta}_i' + \xi_i' \dot{\xi}_i') - z p_z \Delta^{-1} \dot{\Delta} \,. \tag{4.38}$$

In addition, the constraints are now

$$\tilde{\mathcal{H}} = \frac{1}{2}\left(-\det \mathbb{P} + \Delta^2\right) \,, \qquad \tilde{\mathcal{Q}}_i = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\Lambda_i \mathbb{P}) + \Delta \xi_i' \tag{4.39}$$

and

$$\mathcal{J}_{ij} = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left(\tilde{\Lambda}_i \Lambda_j\right) + \zeta_i' \zeta_j' + \xi_i' \xi_j' \,. \tag{4.40}$$

If Δ were a constant, and if we could omit the ζ_i' variables, then the action would reduce to the action for a spinning particle of mass Δ in Mink$_d$ for $d = 3, 4, 6$, with ξ_i' in place of ξ_i. This observation allows us to pass to a new two-twistor form of the action for the spinning particle in AdS$_D$ for $D = d + 1$ by using the results of the previous section for the spinning particle in Mink$_d$.

4.3 Two-twistor action

We now write $\mathbb{P} = \mp \mathbb{U}\mathbb{U}^\dagger$ as we did for the particle in Minkowski space. The constraint $\det \mathbb{P} = \Delta^2$ becomes

$$\Delta^2 = \det(\mathbb{U}\mathbb{U}^\dagger) \,. \tag{4.41}$$

We write the constraint in this way because we no longer interpret it as a mass-shell constraint on $\mathbb{U}$; instead, we interpret it as a constraint that determines Δ in terms of $\mathbb{U}$. Recalling the definition of $\tilde{\mathbb{P}}$ in (3.24), we now have

$$\tilde{\mathbb{P}} = \pm \Delta^2 \mathbb{V}^\dagger \mathbb{V} \,. \tag{4.42}$$

In addition, it follows from (4.41) that

$$\Delta^{-1}\dot{\Delta} = \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{V}) \,. \tag{4.43}$$

We next solve the local supersymmetry constraints by introducing N traceless fermionic hermitian matrix variables Ψ_i by

$$\Lambda_i = \Delta \mathbb{V}^\dagger(\Psi_i \pm \xi_i' \mathbb{I}_2)\mathbb{V} \qquad \left[\Leftrightarrow \tilde{\Lambda}_i = \Delta^{-1}\mathbb{U}(\Psi_i \mp \xi_i' \mathbb{I}_2)\mathbb{U}^\dagger\right] \,, \tag{4.44}$$

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where Δ is now shorthand for $\sqrt{\det(\mathbb{U}\mathbb{U}^\dagger)}$. Upon substitution for Δ and Λ_i, the variable ξ'_i drops out, leaving us with the new lagrangian

$$L = \mathrm{tr}_{\mathbb{R}}\left(\dot{\mathbb{U}}\mathbb{W}^\dagger\right) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\dot{\Psi}_i\right) + \frac{1}{2}\zeta'_i\dot{\zeta}'_i - \frac{1}{2}f_{ij}\mathcal{J}_{ij}\,, \tag{4.45}$$

where now

$$\mathcal{J}_{ij} = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\Psi_j\right) + \zeta'_i\zeta'_j\,, \tag{4.46}$$

and the variable $\mathbb{W}$ conjugate to $\mathbb{U}$ is found to be

$$\mathbb{W} = \pm(\mathbb{X}\mathbb{U} + \xi'_i\mathbb{V}^\dagger\Psi_i) + \frac{1}{2}\mathbb{V}^\dagger\Psi_i\Psi_i - zp_z\mathbb{V}^\dagger\,. \tag{4.47}$$

This incidence relation implies the same identity as in the d-dimensional Minkowski case:

$$\mathbb{G} := \mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U} - \Psi_i\Psi_i \equiv 0\,. \tag{4.48}$$

As before, we may view $\mathbb{W}$ as an independent canonical variable in the action by using a Lagrange multipler to impose $\mathbb{G} = 0$ as a new phase-space constraint. This yields the action

$$S = \int dt\left\{\mathrm{tr}_{\mathbb{R}}\left(\dot{\mathbb{U}}\mathbb{W}^\dagger\right) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\dot{\Psi}_i\right) + \frac{1}{2}\zeta'_i\dot{\zeta}'_i - \mathrm{tr}_{\mathbb{R}}\left(\mathbb{S}\mathbb{G}\right) - \frac{1}{2}f_{ij}\mathcal{J}_{ij}\right\}. \tag{4.49}$$

All constraints are first-class, with $\mathbb{G}$ generating an $O(2;\mathbb{K})$ gauge invariance. As for the Minkowski case of section 3, the absence of any fermionic constraints implies that the two-twistor variables must be gauge invariant with respect to the initial N-extended local supersymmetries, and a calculation using the new incidence relation (4.47) confirms this.

Let us now pause to consider how the action (4.49) differs from the action (3.39). One difference is that (4.49) involves N additional anticommuting variables (ζ'_i) that serve no obvious purpose, but we postpone discussion of this point. The most important difference is that the mass-shell constraint of (3.39) is absent from (4.49). This has two immediate implications. One is that the phase space dimension has increased by 2, which is consistent with the fact that we now have a particle in a spacetime of dimension $D = d+1$. The second is that the action is now $\mathrm{Sp}(4;\mathbb{K})$ invariant because the mass-shell constraint of (3.39) is the only term in that action that is not $\mathrm{Sp}(4;\mathbb{K})$ invariant. If we rewrite (4.49) in terms of the two-twistor $\mathbb{Z}$ introduced in subsection 3.4 then we arrive at the *manifestly* $\mathrm{Sp}(4;\mathbb{K})$ invariant action

$$\begin{aligned} S = \int dt\Big\{&\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left(\Omega\dot{\mathbb{Z}}\mathbb{Z}^\dagger + \frac{1}{2}\Psi_i\dot{\Psi}_i\right) + \frac{1}{2}\zeta'_i\dot{\zeta}'_i + \mathrm{tr}_{\mathbb{R}}\left[\mathbb{S}\left(\mathbb{Z}^\dagger\Omega\mathbb{Z} + \Psi_i\Psi_i\right)\right] \\ &- \frac{1}{2}f_{ij}\left[\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\Psi_j\right) + 2\zeta'_i\zeta'_j\right]\Big\}\,. \end{aligned} \tag{4.50}$$

A peculiar feature of this action is that it is independent of the mass parameter m. This is due to the m-dependence of the change of variables that we made but it is still puzzling: if there is no m-dependence in the action, how can it describe anything other than a *massless* particle?

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We shall return to this question later, but a point to appreciate here is that there could be more than one way to embed $\mathrm{Sp}(4;\mathbb{K})$ into the full symmetry group of the action (which must be infinite-dimensional since any product of constants of motion is another constant of motion). There is, therefore, no guarantee that the linearly-realized $\mathrm{Sp}(4;\mathbb{K})$ invariance group of the above action coincides with the $\mathrm{Sp}(4;\mathbb{K})$ group of AdS_D isometries. In fact, as we shall now explain, this correspondence holds only if $m = 0$.

4.4 The Sp(4; $\mathbb{K}$) Noether charges

The Noether charges associated to the manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance of the action (4.50) are (passing over the one exception for $D = 5$ that we return to later) the entries of the 4×4 matrix

$$\mathbb{J} \equiv \mathbb{Z}\mathbb{Z}^\dagger \,. \tag{4.51}$$

Following [21], we split $\mathbb{J}$ into its three independent 2×2 blocks and evaluate them in spacetime using the incidence relations (4.44) and (4.47). First we have

$$\mp\, \mathbb{U}\mathbb{U}^\dagger = \mathbb{P}\,, \qquad \mathbb{U}\mathbb{W}^\dagger = -\mathbb{P}\mathbb{X} - zp_z\mathbb{I}_2 - \frac{1}{2}\tilde{\Lambda}_i\Lambda_i\,, \tag{4.52}$$

which coincide (passing over the $D = 5$ exception alluded to above) with the Noether charges of the Weyl subgroup of isometries of d-dimensional Minkowski spacetime. Then we have

$$\begin{aligned}\pm\mathbb{W}\mathbb{W}^\dagger \;=\; & -\mathbb{X}\mathbb{P}\mathbb{X} - 2zp_z\mathbb{X} + \left[z^2 - \frac{(mR + \xi'_k\zeta'_k)^2}{\Delta^2}\right]\tilde{\mathbb{P}} + \frac{1}{2}(\Lambda_i\tilde{\Lambda}_i\mathbb{X} - \mathbb{X}\tilde{\Lambda}_i\Lambda_i) \\ & - \frac{1}{2}\frac{zp_z}{\Delta^2}(\Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}} - \tilde{\mathbb{P}}\tilde{\Lambda}_i\Lambda_i) - \frac{1}{4\Delta^2}\Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}}\tilde{\Lambda}_j\Lambda_j\,.\end{aligned} \tag{4.53}$$

We may simplify this expression by means of the identities

$$\begin{aligned}\Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}} - \tilde{\mathbb{P}}\tilde{\Lambda}_i\Lambda_i \;&\equiv\; 2\Lambda_i\,\mathrm{tr}_{\mathbb{R}}(\Lambda_i\mathbb{P}) \\ \Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}}\tilde{\Lambda}_j\Lambda_j \;&\equiv\; \frac{1}{2}\tilde{\mathbb{P}}\,\mathrm{tr}_{\mathbb{R}}(\Lambda_i\tilde{\Lambda}_j)\,\mathrm{tr}_{\mathbb{R}}(\Lambda_i\tilde{\Lambda}_j) - 2\Lambda_i\,\mathrm{tr}_{\mathbb{R}}(\Lambda_j\mathbb{P})\,\mathrm{tr}_{\mathbb{R}}(\Lambda_i\tilde{\Lambda}_j)\,.\end{aligned} \tag{4.54}$$

Combining these two identities with the constraints $\mathcal{Q}_i = 0$ and $\mathcal{J}_{ij} = 0$ in the form

$$\mathrm{tr}_{\mathbb{R}}(\Lambda_i\mathbb{P}) = -2\Delta\xi'_i\,, \qquad \mathrm{tr}_{\mathbb{R}}(\Lambda_i\tilde{\Lambda}_j) = -2(\zeta'_i\zeta'_j + \xi'_i\xi'_j)\,, \tag{4.55}$$

we deduce that

$$\begin{aligned}\Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}} - \tilde{\mathbb{P}}\tilde{\Lambda}_i\Lambda_i \;&=\; -4\Delta\Lambda_i\xi'_i\,, \\ \Lambda_i\tilde{\Lambda}_i\tilde{\mathbb{P}}\tilde{\Lambda}_j\Lambda_j \;&=\; -4(\xi'_i\zeta'_i)^2\tilde{\mathbb{P}} + 8\Delta(\Lambda_j\zeta'_j)(\xi'_i\zeta'_i)\,.\end{aligned} \tag{4.56}$$

Finally, using (4.56) in (4.53) we deduce that

$$\pm\,\mathbb{W}\mathbb{W}^\dagger = \mathbb{K} - \left(\frac{mR}{\Delta}\right)^2\tilde{\mathbb{P}} + 2\left(\frac{mR}{\Delta}\right)(\Lambda_i - \Delta^{-1}\xi'_i)\,\zeta'_i\,, \tag{4.57}$$

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on the constraint surface, where $\mathbb{K}$ is the Hermitian matrix of (4.25) that represents the AdS Noether charge K^μ, but now expressed in terms of the new variables:

$$\begin{aligned}\mathbb{K} = \mathbb{X}\mathbb{P}\mathbb{X} + z^2\tilde{\mathbb{P}} - 2\mathbb{X}zp_z + \frac{1}{2}\left(\Lambda_i\tilde{\Lambda}_i\mathbb{X} - \mathbb{X}\tilde{\Lambda}_i\Lambda_i\right) + 2\Lambda_i z\zeta_i \\ +\Lambda_i\left[2\left(\frac{zp_z}{\Delta}\right)\xi_i' - 2\left(\frac{mR+\xi_k'\zeta_k'}{\Delta}\right)\zeta_i'\right].\end{aligned} \tag{4.58}$$

The m-dependence of this expression is purely the result of the m-dependence of our change of variables. By means of the formulae

$$\tilde{\mathbb{P}} = \pm\Delta^2\mathbb{V}^\dagger\mathbb{V}\,, \qquad \Lambda_i = \Delta\mathbb{V}^\dagger\left(\Psi_i \pm \xi_i'\,\mathbb{I}_2\right)\mathbb{V}\,, \tag{4.59}$$

we may further simplify the expression for $\pm\mathbb{W}\mathbb{W}^\dagger$ to

$$\pm\,\mathbb{W}\mathbb{W}^\dagger = \mathbb{K} \mp \mathbb{V}^\dagger\left[(mR)^2 \mp 2(mR)\Psi_i\zeta_i'\right]\mathbb{V}\,. \tag{4.60}$$

This confirms, incidentally, the invariance of $\mathbb{W}\mathbb{W}^\dagger$ with respect to the original local worldline supersymmetries. However, it also shows that there is a discrepancy between the $\mathrm{Sp}(4;\mathbb{K})$ Noether charges and the AdS_D Noether charges *unless* $m = 0$.

4.4.1 Redundant anticommuting variables

We observed above that the anticommuting variables ζ_i' of the two-twistor action (4.49) serve no obvious purpose. They are absent from the spin-shell constraints. For $N > 1$ they appear in the $\mathrm{SO}(N)$ constraint, but this is just because they form an N-vector of $\mathrm{SO}(N)$. None of the essential features of the action (4.49) would change if these variables were absent; they are, in this sense, redundant. It appears that we could omit them but is there any other justification for doing so?

There is, because we have just seen that (4.49) describes a *massless* particle in AdS_D and inspection of the action (4.1) from which we started shows that the variables ξ_i of that action are similarly redundant when $m = 0$ (in fact, for the $N = 1$ case, $\zeta' = \xi$ when $m = 0$). The massive spinning particle in a Minkowski background also has anticommuting variables that become redundant in a massless limit [37] and, unless omitted, they lead to a reducible space of polarisation states; we should expect the same to be true for an AdS background.

Omitting the variables ζ_i' is equivalent to imposing $\zeta_i' = 0$ as additional, but second-class, constraints. We could implement this in the two-twistor action (4.49) by means of additional Lagrange multipliers but it is obviously simpler to directly set $\zeta_i' = 0$ to get

$$S = \int dt\left\{\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left[\Omega\dot{\mathbb{Z}}\mathbb{Z}^\dagger + \frac{1}{2}\Psi_i\dot{\Psi}_i + \mathbb{S}\left(\mathbb{Z}^\dagger\Omega\mathbb{Z} + \Psi_i\Psi_i\right)\right] - \frac{1}{2}f_{ij}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\Psi_j\right)\right\}. \tag{4.61}$$

When the ζ_i' variables are similarly omitted from (4.60), this formula simplifies to

$$\pm\,\mathbb{W}\mathbb{W}^\dagger = \mathbb{K} \mp (mR)^2\mathbb{V}^\dagger\mathbb{V}\,. \tag{4.62}$$

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5 Non-zero mass for AdS_5

For the AdS_5 case the matrices $\mathbb{U}$ and $\mathbb{W}$ are complex, rather than real or quaternionic. In this case, we may replace $\mathbb{W}$ by a new independent complex matrix variable $\breve{\mathbb{W}}$ by setting

$$\mathbb{W} = \breve{\mathbb{W}} + imR\,\mathbb{V}^\dagger\,. \tag{5.1}$$

Substitution yields

$$\begin{aligned}\mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\mathbb{W}^\dagger) &= \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\breve{\mathbb{W}}^\dagger) + mR\,\mathrm{tr}_{\mathbb{R}}(i\dot{\mathbb{U}}\mathbb{V})\\ &= \mathrm{tr}_{\mathbb{R}}(\dot{\mathbb{U}}\breve{\mathbb{W}}^\dagger) + \frac{d}{dt}\left[(mR)\arg(\det\mathbb{U}^\dagger)\right]\,,\end{aligned} \tag{5.2}$$

so the geometric part of the Lagrangian is unchanged if we discard the total derivative.

In fact, the entire action (4.50) has the same form in terms of $\breve{\mathbb{W}}$ as it did in terms of $\mathbb{W}$, *except for the trace of the spin-shell constraint.* To see this we observe that

$$\mathbb{U}\mathbb{W}^\dagger = \mathbb{U}\breve{\mathbb{W}}^\dagger - imR\,\mathbb{I}_2\,, \tag{5.3}$$

from which it follows that

$$\mathbb{G} = \mathbb{U}^\dagger\breve{\mathbb{W}} - \breve{\mathbb{W}}^\dagger\mathbb{U} - \Psi_i\Psi_i - 2imR\,\mathbb{I}_2\,. \tag{5.4}$$

The parameter mR contributes only to the trace of $\mathbb{G}$, which is the 'extra' U(1) part.

The $\mathrm{Sp}(4;\mathbb{C}) \cong \mathrm{U}(2,2)$ invariance of this new action in terms of $\mathbb{U}$ and $\breve{\mathbb{W}}$ may be made manifest by writing it in terms of the new twistor variables

$$\breve{\mathbb{Z}} = \begin{pmatrix}\mathbb{U}\\ \breve{\mathbb{W}}\end{pmatrix}\,. \tag{5.5}$$

The result, if we omit the redundant anticommuting variables ζ_i', is the action

$$S = \int dt\left\{\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left[\Omega\dot{\breve{\mathbb{Z}}}\breve{\mathbb{Z}}^\dagger + \frac{1}{2}\Psi_i\dot{\Psi}_i - \mathbb{S}\mathbb{G}\right] - \frac{1}{2}f_{ij}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\Psi_j\right)\right\}\,, \tag{5.6}$$

where

$$\mathbb{G} = -\breve{\mathbb{Z}}^\dagger\Omega\breve{\mathbb{Z}} - 2imR - \Psi_i\Psi_i\,. \tag{5.7}$$

The Noether charges implied by the manifest $\mathrm{Sp}(4;\mathbb{C}) \cong \mathrm{U}(2,2)$ symmetry are now contained in the 4×4 hermitian matrix

$$\breve{\mathbb{J}} = \breve{\mathbb{Z}}\breve{\mathbb{Z}}^\dagger\,. \tag{5.8}$$

The action (5.6) is formally the same as (4.61) except that the mass now appears in the U(1) constraint imposed by the trace of the antihermitian Lagrange multiplier $\mathbb{S}$. For $N = 0$, it is precisely the action for a spin-zero particle of mass m in AdS_5 of [7]. For $N > 0$ we appear to have a "spinning" extension of this massive particle action, but we have still to check whether the Noether charges $\breve{\mathbb{J}}$ are those implied by invariance under AdS_5 isometries.

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Strictly speaking, what we have to check is that the AdS_5 Noether charges are those combinations of the components of $\tilde{\mathbb{J}}$ that generate the $SU(2,2)$ subgroup of $U(2,2)$, so the status of the 'extra' $U(1)$ factor requires clarification. Its generator is the imaginary part of the trace of $\mathbb{U}\mathbb{W}^\dagger$ or, equivalently, the real trace of $i\mathbb{U}\mathbb{W}^\dagger$. However,

$$\mathrm{tr}_{\mathbb{R}}(i\mathbb{U}\mathbb{W}^\dagger) = -\frac{i}{2}\mathrm{tr}\left(\mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U}\right) = -\frac{i}{2}\mathrm{tr}\,\mathbb{G}\,. \tag{5.9}$$

The last of these equalities relies on the fact that $\Psi_i\Psi_i$ has zero trace for $\mathbb{K} = \mathbb{C}$ (and not merely zero real trace); this was noted for $N = 1$ in section 3.3 but the result extends immediately to $N > 1$. What this equality shows is that not all components of $\mathbb{J}$ are Noether charges (for $D = 5$) because one combination is the constraint function for the 'extra' $U(1)$ factor in the $U(2)$ gauge group,[7] and the same is true of $\breve{\mathbb{J}}$.

We are now in a position to return to the problem of the $(mR)^2$ term in the expression (4.60) for $\pm\mathbb{W}\mathbb{W}^\dagger$. The new Noether charges are

$$\begin{aligned}
\pm\breve{\mathbb{W}}\breve{\mathbb{W}}^\dagger &= \pm\mathbb{W}\mathbb{W}^\dagger \pm imR\left(\mathbb{W}\mathbb{V} - \mathbb{V}^\dagger\mathbb{W}^\dagger\right) \pm (mR)^2\mathbb{V}^\dagger\mathbb{V} \\
&= \mathbb{K} \mp i(mR)\,\mathbb{V}^\dagger\left(\mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U}\right)\mathbb{V} \\
&= \mathbb{K} \mp i(mR)\,\mathbb{V}^\dagger(\Psi_i\Psi_i)\mathbb{V}\,,
\end{aligned} \tag{5.10}$$

where the second line uses (4.62) and the last line uses the spin-shell constraint $\mathbb{G} = 0$. For the spin-zero ($N = 0$) particle the last term is absent, so there is no longer a discrepancy between the $SU(2,2)$ Noether charges and the AdS_5 isometry Noether charges.

For $N \geq 1$ we have merely replaced the original discrepancy for $m \neq 0$ by another one that cannot be eliminated in a similar way. The 'discrepancy' is itself a conserved charge having the same PB relations with the other AdS isometry charges as does $\mathbb{K}$, except that it has zero PB with $\mathbb{P}$ and hence no effect on the algebra. In other words, the conserved charges $\mathbb{P}, \mathbb{D}$ and

$$\breve{\mathbb{K}} = \mathbb{K} \mp i(mR)\,\mathbb{V}^\dagger(\Psi_i\Psi_i)\mathbb{V} \tag{5.11}$$

span an algebra that is isomorphic to the AdS isometry algebra, but it is not the AdS isometry algebra because the extra term in $\breve{\mathbb{K}}$ is

$$\mp i(mR)\mathbb{V}^\dagger(\Psi_i\Psi_i)\mathbb{V} = -(i/\Delta^2)\left[\mathbb{P}\tilde{\Lambda}_i\Lambda_i + 2\Delta\xi'_i\Lambda_i\right]\,, \tag{5.12}$$

which is incompatible with the general form (2.36) because of the inverse Δ factors.

5.1 The U(2) gauge anomaly and mass quantization

We now consider some implications of the quantum theory for the Claus et al. action [7], which in our notation is the $N = 0$ case of (5.6):

$$S = \int dt\left\{\frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left[\Omega\dot{\breve{\mathbb{Z}}}\breve{\mathbb{Z}}^\dagger + \mathbb{S}\left(\breve{\mathbb{Z}}^\dagger\Omega\breve{\mathbb{Z}} + 2imR\right)\right]\right\}\,. \tag{5.13}$$

[7] For presentational simplicity we ignore the distinction between $U(2)$ and $U(1)\times SU(2)$ here, although it will become important below.

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We may rewrite this as $S = S_0 + S_m$, where S_0 is the action for $m = 0$ and

$$S_m = mR \int dt \, \mathrm{tr}_{\mathbb{R}}(i\mathbb{S}) \,. \tag{5.14}$$

This term is essentially a worldline Chern-Simons term for the U(1) gauge group contained in U(2). The qualification "essentially" could have been omitted if the gauge group were $\mathrm{U}(1) \times \mathrm{SU}(2)$ because then S_m would be a WCS term for the U(1) factor and we could ignore the SU(2) factor. However, it is important for the quantum theory that

$$\mathrm{U}(2) = [\mathrm{U}(1) \times \mathrm{SU}(2)]/\mathbb{Z}_2 \,, \tag{5.15}$$

because the quotient by $\mathbb{Z}_2$ makes a difference. The (finite) U(2) gauge transformations are

$$\check{\mathbb{Z}} \to \check{\mathbb{Z}} G \,, \qquad \mathbb{S} \to G^{-1}\mathbb{S}G + G^{-1}\dot{G} \,, \tag{5.16}$$

where the parameter $G(t)$ is a map from the worldline to the U(2) gauge group. The action is invariant if $m = 0$, but for $m \neq 0$ we have

$$S_m \to S_m + mR \int dt \, \mathrm{tr}_{\mathbb{R}}(i\dot{G}G^{-1}) \,. \tag{5.17}$$

In the context of the Euclidean path integral[8] we must consider closed worldlines, in which case the maps $G(t)$ may have a non-zero winding number, specified by an integer since $\pi_1(\mathrm{U}(2)) = \mathbb{Z}$. For example, if we make the identification $t \sim t + 1$ then representative maps in these integer homotopy classes are

$$G_n(t) = [\exp\{-i\pi(\mathbb{I}_2 + \sigma_3)\, t\}]^n \qquad n \in \mathbb{Z} \,. \tag{5.18}$$

This implies $\mathrm{tr}_{\mathbb{R}}(i\dot{G}G^{-1}) = 2n\pi$, and using this in (5.17) we deduce that

$$e^{iS_m} \to \left[e^{(mR)2\pi i}\right]^n e^{iS_m} \,. \tag{5.19}$$

It follows that U(2) invariance of the path integral requires the quantization condition

$$mR \in \mathbb{Z} \,. \tag{5.20}$$

We must remind the reader here that the mass m in the classical action is not the same as the mass parameter M appearing the Klein-Gordon equation in an AdS_5 background, but is related to it by the $D = 5$ case of the relation given in the introduction, i.e. $M^2 = M_c^2 + m^2$ (in units for which $\hbar = 1$) where $M_c^2 = -15/4$ (for $D = 5$). At the level of classical particle mechanics, a particle of zero mass has a null worldline, and this corresponds to $m = 0$ even in AdS.

If the gauge group were $\mathrm{U}(1) \times \mathrm{SU}(2)$ then $G(t)$ would be replaced by the composition of an SU(2) transformation with a U(1) transformation. Since $\pi_1(\mathrm{SU}(2))$ is trivial

[8] We may pass over the question of how the analytic continuation from Lorentzian to Euclidean spacetime metric is accomplished in the two-twistor formulation because the mass term in the action is independent of the spacetime metric.

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we would have been able to focus exclusively on the U(1) gauge invariance by setting $G(t) = g(t)\mathbb{I}_2$ for $g(t) \in \mathrm{U}(1)$, in which case the global gauge anomaly is of the simpler U(1) type discussed in [44]. The maps $g(t)$ from a closed worldline to U(1) fall into the integer homotopy classes of $\pi_1(\mathrm{U}(1)) = \mathbb{Z}$, and we may choose $g_n(t) = \exp[-2n\pi i t]$ as their representatives. In this case

$$G(t) = e^{-2n\pi i t}\mathbb{I}_1 \quad \Rightarrow \quad \mathrm{tr}_{\mathbb{R}}(i\dot{G}G^{-1}) = 4n\pi t\,, \tag{5.21}$$

and using this in (5.17) we deduce that

$$e^{iS_m} \to \left[e^{(2mR)2\pi i}\right]^n e^{iS_m}\,, \tag{5.22}$$

and hence that U(1) gauge invariance of the path integral requires the quantization condition

$$2mR \in \mathbb{Z}\,. \tag{5.23}$$

This quantization condition is implicit in the results of [28], but it is weaker than the quantization condition $mR \in \mathbb{Z}$ required for U(2) gauge invariance of the quantum path integral. The stronger quantization condition is also needed for the total derivative term proportional to mR in the key identity (4.33) to be an exact differential, as we have already remarked.

6 The superparticle

The starting point in [21] for the construction of a supertwistor action for the superparticle in AdS_D was the following action:

$$S = \int dt \left\{ \frac{1}{2}\mathrm{tr}_{\mathbb{R}}\left[(\dot{\mathbb{X}} + \Theta_i^\dagger\dot{\Theta}^i - \dot{\Theta}_i^\dagger\Theta^i)\mathbb{P}\right] + \dot{z}p_z - \frac{1}{2}e(p^2 + \Delta^2)\right\}\,, \tag{6.1}$$

where

$$\Delta^2 = p_z^2 + \left(\frac{mR}{z}\right)^2\,, \tag{6.2}$$

and Θ^i is an $\mathcal{N}$-plet ($i = 1, \ldots, \mathcal{N}$) of $\mathbb{K}$-valued matrices of *anticommuting* variables, acted upon from the left by the R-symmetry group $O(\mathcal{N}; \mathbb{K})$ and on the right by $\mathrm{SL}(2; \mathbb{K})$. If the anticommuting variables are omitted then we recover the action, in Poincaré-patch coordinates, for a spin-zero particle of mass m in AdS_D.

The effect of the anticommuting variables is to enlarge the Poincaré invariance on Mink_d slices to a super-Poincaré invariance. The above action is therefore invariant *by construction* under the action of a super-Poincaré group on the variables $\{\mathbb{X}, \mathbb{P}, \Theta^i\}$, under which the variables $\{z, p_z\}$ are inert. There is also a linearly realized $O(\mathcal{N}; \mathbb{K})$ R-symmetry, which acts only on the anticommuting variables, and a scale invariance with respect to which Θ^i has dimension $-\frac{1}{2}$ if we assign dimension -1 to the AdS coordinates $\mathbb{X}$ and z.

The motivation for this action comes from the observation of [45] that a massive superparticle in a Mink_d background has additional "hidden" supersymmetries. As AdS_D

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is conformal to Mink_D, we may expect the action (6.1) to have additional "hidden" supersymmetries for $m = 0$, and this is indeed the case. In fact, for $m = 0$ the following constants of motion are Noether charges for an $\text{OSp}(\mathcal{N}|4;\mathbb{K})$ invariance:

$$\begin{aligned}
\mathbb{P} &= \mathbb{P}\,,\\
\mathbb{Q}^i &= \Theta^i\mathbb{P}\,,\\
\mathbb{D} &= \mathbb{P}(\mathbb{X} + \Theta^\dagger_k\Theta^k) + zp_z\mathbb{I}_2\,, \qquad \mathbb{R}^i{}_j = \Theta^i\mathbb{P}\Theta^\dagger_j\,,\\
\mathbb{S}^i &= \Theta^i\left[\mathbb{P}\left(\mathbb{X} + \Theta^\dagger_j\Theta^j\right) + zp_z\right]\,,\\
\mathbb{K} &= -(\mathbb{X} - \Theta^\dagger_k\Theta^k)\mathbb{P}(\mathbb{X} + \Theta^\dagger_l\Theta^l) - 2zp_z\mathbb{X} + z^2\tilde{\mathbb{P}}\,.
\end{aligned} \tag{6.3}$$

These are constants of motion irrespective of whether m is zero or non-zero but the PBs of the $\mathbb{S}^i$ charges close on $\mathbb{K}$ (the Noether charge) to yield the expected algebra of $\text{OSp}(\mathcal{N}|4;\mathbb{K})$ only if $p^2 + p_z^2 = 0$, which is the mass-shell constraint for $m = 0$.

We conclude from this result that for $m = 0$ the action (6.1) is an action for the massless superparticle in AdS_D. One should appreciate here that this action is *much simpler* than the standard one for a massless superparticle in an AdS background, because that action has a hidden fermionic gauge invariance [27] that is generally called "kappa-symmetry". The results of [45] for the Minkowski background case strongly suggest that the action (6.1) is, at least for $m = 0$, a gauge-fixed version of the kappa-symmetric action. We make no attempt here to verify this as none of the results to follow depend on its validity.

For $m \neq 0$ the PBs of the $\mathbb{S}^i$ charges close on $\mathbb{K}' = \mathbb{K} - (mR/\Delta)^2\tilde{\mathbb{P}}$ but the PB of $\mathbb{K}'$ with $\mathbb{S}^i$ is non-zero, so one is led to a superalgebra with more generators than that of $\text{OSp}(\mathcal{N}|4;\mathbb{K})$; we suspect that it is infinite dimensional (which would not be not surprising for a free particle). We may attempt to rectify this problem by modifying $\mathbb{S}^i$ in addition to $\mathbb{K}$. Although there is no such modification that resolves the problem for the general case, we may replace $\mathbb{S}^i$ and $\mathbb{K}$ in the $D = 5$ (complex) case by

$$\breve{\mathbb{S}}^i = \mathbb{S}^i - i(mR)\Theta^i\,, \qquad \breve{\mathbb{K}} = \mathbb{K} - 2i(mR)\Theta^\dagger_i\Theta^i\,. \tag{6.4}$$

These are again constants of the motion that generate symmetries of the action, and the PBs of the $\breve{\mathbb{S}}^i$ charges close on $\breve{\mathbb{K}}$, as a consequence of the mass-shell constraint $p^2 + \Delta^2 = 0$. The factors of i are crucial to this result, and hence to existence of Noether charges spanning the algebra of $\text{OSp}(\mathcal{N}|4;\mathbb{C}) \cong \text{U}(2,2|N)$ for arbitrary m.

6.1 Supertwistor formulation

The construction of a two-supertwistor form of the action again starts by setting $\mathbb{P} = \mp\mathbb{U}\mathbb{U}^\dagger$. The subsequent steps for $m = 0$ were explained in [21]; they involve the introduction of the new variables

$$\mathbb{W} = \pm\left(\mathbb{X} - \Theta^\dagger_i\Theta^i\right)\mathbb{U} - zp_z\mathbb{V}^\dagger\,, \qquad \Xi^i = \Theta^i\mathbb{U}\,, \tag{6.5}$$

which satisfy an identity $\mathbb{G} \equiv 0$ for

$$\mathbb{G} = \mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U} \pm 2\Xi^\dagger_i\Xi^i\,. \tag{6.6}$$

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This identity becomes a constraint in the action, imposed by an anti-hermitian Lagrange multipler $\mathbb{L}$. This action is

$$S = \int dt\, \mathrm{tr}_{\mathbb{R}} \{ \dot{\mathbb{U}}\mathbb{W}^\dagger \mp \Xi_i^\dagger \dot{\Xi}^i - \mathbb{L}\mathbb{G} \} \,. \tag{6.7}$$

The anticommuting variables are now scalars (with respect to the Mink_d Lorentz group) appearing in the $\mathcal{N}$ $\mathbb{K}$-valued matrices Ξ^i which are now acted upon from the left by the R-symmetry group $O(\mathcal{N};\mathbb{K})$ and on the right by the $O(2;\mathbb{K})$ gauge group, for which $\mathbb{G}$ is the generator.

Collectively, the new canonical variables are the components of a two-supertwistor, i.e. an $O(2;\mathbb{K})$ doublet of spinors of an $\mathrm{OSp}(\mathcal{N}|4;\mathbb{K})$ symmetry supergroup, whose generators are gauge-invariant supertwistor bilinears:

$$\begin{aligned} \mathbb{P} &= \mp \mathbb{U}\mathbb{U}^\dagger \,, \\ \mathbb{Q}^i &= \mp \Xi^i \mathbb{U}^\dagger \,, \\ \mathbb{D} &= -\mathbb{U}\mathbb{W}^\dagger \,, \qquad \mathbb{R}^i{}_j = \mp \Xi^i \Xi_j^\dagger \,, \\ \mathbb{S}^i &= -\Xi^i \mathbb{W}^\dagger \,, \\ \mathbb{K} &= = \pm \mathbb{W}\mathbb{W}^\dagger \,. \end{aligned} \tag{6.8}$$

By using the relations (6.5) to rewrite these two-supertwistor bilinears in terms of the variables $\{\mathbb{X}, \mathbb{P}, \Theta^i\}$ we confirm that they are indeed the AdS superisometry generators (6.3) *provided that* $m = 0$. We could not have hoped for more than this because, as explained above, the action from which we started is only invariant under the AdS superisometries when $m = 0$, *unless* $D = 5$, which requires a separate analysis that we now present.

For $D = 5$ and $m \neq 0$ we saw in section 5 that we should rewrite the (super)twistor action in terms of the new matrix variable

$$\breve{\mathbb{W}} = \mathbb{W} - imR\, \mathbb{V}^\dagger \,. \tag{6.9}$$

Omitting a total time derivative from the Lagrangian, the action (6.7) becomes

$$S = \int dt\, \mathrm{tr}_{\mathbb{R}} \{ \dot{\mathbb{U}}\breve{\mathbb{W}}^\dagger \mp \Xi_i^\dagger \dot{\Xi}^i - \mathbb{L}\mathbb{G} \} \,, \tag{6.10}$$

where $\mathbb{G}$, when written in terms of $\breve{\mathbb{W}}$, becomes

$$\mathbb{G} = \mathbb{U}^\dagger \breve{\mathbb{W}} - \breve{\mathbb{W}}^\dagger \mathbb{U} \pm 2\Xi_i^\dagger \Xi^i + 2imR\, \mathbb{I}_2 \,. \tag{6.11}$$

As we saw in section 5 for the bosonic particle, the only change is a constant term proportional to mR in the U(1) constraint.

The $\mathrm{OSp}(\mathcal{N}|4;\mathbb{C})$ generators are now formally the same as those in (6.8) but with $\breve{\mathbb{W}}$ in place of $\mathbb{W}$. This adds a constant term to the $\mathbb{D}$ generator but, for reasons explained in section 5, this is just equivalent to the change in the U(1) constraint. That leaves $\mathbb{S}^i$ and $\mathbb{K}$ which are replaced by

$$\breve{\mathbb{S}}^i = -\Xi^i \breve{\mathbb{W}}^\dagger \,, \qquad \breve{\mathbb{K}} = \pm \breve{\mathbb{W}}\breve{\mathbb{W}}^\dagger \,. \tag{6.12}$$

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We now need to check that these definitions accord with those of (6.4). First we consider

$$-\Xi^i\breve{\mathbb{W}}^\dagger = -\Theta^i\mathbb{U}\mathbb{W}^\dagger - imR\,\Theta^i = \mathbb{S}^i - imR\,\Theta^i\,, \tag{6.13}$$

which is indeed the expression for $\breve{\mathbb{S}}^i$ in (6.4). Next we have

$$\pm\,\breve{\mathbb{W}}\breve{\mathbb{W}}^\dagger = \pm\mathbb{W}\mathbb{W}^\dagger \pm imR\,\mathbb{V}^\dagger\left(\mathbb{U}^\dagger\mathbb{W} - \mathbb{W}^\dagger\mathbb{U}\right)\mathbb{V} \pm (mR)^2\mathbb{V}^\dagger\mathbb{V}\,. \tag{6.14}$$

The $m = 0$ equality $\pm\mathbb{W}\mathbb{W}^\dagger = \mathbb{K}$ relies on the $m = 0$ mass-shell constraint; for $m \neq 0$ we find, as in section 5, that

$$\pm\,\mathbb{W}\mathbb{W}^\dagger = \mathbb{K} \mp (mR)^2\mathbb{V}^\dagger\mathbb{V}\,. \tag{6.15}$$

Using this, and the superparticle spin-shell constraint $\mathbb{G} = 0$, in (6.14) we have

$$\pm\,\breve{\mathbb{W}}\breve{\mathbb{W}}^\dagger = \mathbb{K} - 2imR\,\mathbb{V}^\dagger\Xi_i^\dagger\Xi^i\mathbb{V} = \mathbb{K} - 2imR\Theta_i^\dagger\Theta^i\,, \tag{6.16}$$

which is exactly $\breve{\mathbb{K}}$ of (6.4).

We end with a comment on the sign of the fermion 'kinetic' term in (6.7), or in (6.10). Recall that the upper sign is for positive energy and the lower sign for negative energy. As first pointed out in [46], and further discussed in the context of twistor-type actions in [37], this correlation is a required feature of *spacetime* supersymmetry in the context of the mechanics of particles, or strings and branes, because it is needed for compatibility of spacetime supersymmetry with the existence of the negative energy states that are inevitable in relativistic quantum mechanics.

7 Quantum spinning particle

In the context of phase-space actions for particle mechanics with first-class phase-space constraints, the passage from classical to quantum mechanics involves the steps spelled out by Dirac. First, the canonical variables are replaced by operators acting on some Hilbert space of quantum states, and their canonical Poisson bracket relations are replaced by canonical (anti)commutation relations. In this step the classical constraint functions become operators; ordering ambiguities may arise but in the cases that we consider a choice of ordering exists such that the quantum constraints are also first-class (any remaining ambiguity is then intrinsic to the quantum theory). Next, "physical" states are taken to be those annihilated by the constraint operators. In the case of reparametrization invariant phase-space actions with first-class constraints, as considered here, these physical state conditions encode all physical properties of the quantum particle.

In our case, spin degrees of freedom arise from fermionic variables in the classical action, and a simplifying feature of the twistor formulation is that all these variables are physical because there are no longer any fermionic constraints. In the quantum theory these variables become operators that act on a finite dimensional space of polarisation states. Assuming a minimal realization of their canonical anticommutation relations, one can determine the dimension of this polarisation state space.

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This was done for the massless superparticle in [21]. Here we perform a similar analysis for the massless N-extended spinning particle. Our starting point will be the "reduced" action (4.61), which we recall here:

$$S = \int dt \left\{ \frac{1}{2} \mathrm{tr}_{\mathbb{R}} \left(\Omega \dot{\mathbb{Z}} \mathbb{Z}^\dagger + \frac{1}{2} \Psi_i \dot{\Psi}_i \right) + \mathrm{tr}_{\mathbb{R}} \left[\mathbb{S} \left(\mathbb{Z}^\dagger \Omega \mathbb{Z} + \Psi_i \Psi_i \right) \right] - \frac{1}{2} f_{ij} \mathrm{tr}_{\mathbb{R}} \left(\Psi_i \Psi_j \right) \right\} . \quad (7.1)$$

Classically, this action governs the dynamics of a massless particle in $\mathrm{AdS}_{4,5,7}$ with additional anticommuting variables.

7.1 Canonical anticommutation relations

We begin by choosing a convenient parametrization for the matrices Ψ_i in terms of 'real' anticommuting variables. It will be sufficient to consider in detail the quaternionic case, for which we may write

$$\Psi_i = \begin{pmatrix} \rho_i & \varsigma_i + \mathbf{i} \cdot \boldsymbol{\psi}_i \\ \varsigma_i - \mathbf{i} \cdot \boldsymbol{\psi}_i & -\rho_i \end{pmatrix} , \quad (7.2)$$

and we replace $\mathbf{i} \cdot \boldsymbol{\psi}_i$ by $i\psi_i$ for $\mathbb{K} = \mathbb{C}$ and omit it for $\mathbb{K} = \mathbb{R}$.

As the matrices Ψ_i transform by conjugation under $O(2; \mathbb{K})$, which is the transverse rotation group in $D = 3 + \dim \mathbb{K}$ (with an additional U(1) factor for $D = 5$) we should expect the components of Ψ_i to transform (for each $i = 1, \ldots, N$) as a $(D-2)$ vector. To make this manifest, we define anticommuting variables $\{\vartheta^I; I = 1, \ldots, D-2\}$ such that (for $D = 7$)

$$\vartheta_i^1 = \rho_i \, , \quad \vartheta_i^2 = \varsigma_i \, , \quad \boldsymbol{\vartheta}_i = \boldsymbol{\psi}_i \, . \quad (7.3)$$

We find that

$$\frac{1}{4} \mathrm{tr}_{\mathbb{R}} (\Psi_i \dot{\Psi}_i) = \frac{1}{2} \vartheta_i^I \dot{\vartheta}_i^I \, , \qquad J_{ij} = \vartheta_i^I \vartheta_j^I \, . \quad (7.4)$$

These are manifestly Spin(5) invariant expressions, and from the first of them we may read off the canonical Poisson bracket relations, which become the following canonical anticommutation relations for the corresponding hermitian operators $\hat{\vartheta}_i^I$ of the quantum theory:

$$\left\{ \hat{\vartheta}_i^I, \hat{\vartheta}_j^J \right\} = \delta_{ij} \delta^{IJ} \, . \quad (7.5)$$

To include the real and complex cases we take $I = 1, \ldots, D-2$, where $D = 4, 5, 7$.

7.1.1 The global SO(N) anomaly

For $N > 1$ real anticommuting worldline variables, with standard canonical PB relations, in the N-vector representation of a gauged SO(N), the path integral measure is ill-defined because of a global gauge anomaly [44]. This is easily verified for $N = 3$ (the simplest case for which SO(N) is non-abelian): the components of the anticommuting 3-vector become the Pauli matrices times a constant factor, so the polarisation states are 2-component SU(2) spinors, but the SO(3) generators that must annihilate these states are also Pauli matrices, so there are no physical states.

This anomaly cancels if we have an *even* number of N-plets of real anticommuting variables [44]. In the context of the "reduced" action (7.1) the number of such N-plets

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is $D-2$, as we have just seen, so the anomaly cancels for $D=4$ but *not* for $D=5,7$. For $N=2$ and odd D, the anomaly can be cancelled by the inclusion of a WCS term, in principle, but this option is not available in the two-twistor form of the action, for the reasons explained in the appendix.

To summarize, the "reduced" action (7.1) will yield a consistent anomaly-free quantum theory for $D=4$, and for $D=5,7$ if $N=1$. This conclusion is confirmed by the detailed analysis to follow.

For $D=5,7$ and $N>1$ the reduced action leads to an inconsistent quantum theory because of the global $\mathrm{SO}(N)$ gauge anomaly. This anomaly can be cancelled by re-instating the redundant N-plet of anticommuting variables; i.e. by reverting to the action (4.49). However, we expect this "unreduced" action to lead to a reducible polarisation state space. This is confirmed for our analysis below of the $N=2$ case; we do not attempt a detailed analysis of the $D=5,7$ cases for $N>2$.

7.1.2 Conformal invariance

As the action (7.1) describes a particle of zero mass (classically) we should expect it to be invariant under the conformal isometries of AdS_D, as is the case for a Minkowski background [47]. Given this, and assuming that conformal invariance is preserved upon quantization, we should expect to find quantum wave equations in AdS_D that are conformally invariant.

While the two-twistor formulation linearizes invariance under the $\mathrm{Sp}(4;\mathbb{K})$ group of AdS_D isometries, we cannot expect it to do the same for the conformal isometries. However, as explained in [21], the larger conformal isometry group is also linearly realized for $D=4$. We should therefore expect to find that conformal invariance is preserved by the quantum theory at least in this case.

7.2 $N=1$

We first consider the quantum theory for $N=1$. In terms of the operators

$$\gamma^I = \sqrt{2}\,\hat{\vartheta}^I\,, \tag{7.6}$$

the canonical anticommutation relations are

$$\{\gamma^I,\gamma^J\} = 2\delta^{IJ} \qquad (I,J=1,2,\ldots D-2)\,. \tag{7.7}$$

These relations can be realized by Dirac matrices for $\mathbb{E}^{D-2}$. Let us consider $D=4,5,7$ in turn:

- $D=4$. In this case we may choose $\gamma^1=\sigma_1$ and $\gamma^2=\sigma_3$. The state space is a 2-dimensional real vector space, so there are two polarisation states. This is what we should expect from a massless Dirac equation for a minimal spinor field in AdS_4; since the equation is conformally invariant and AdS_4 is conformal to Mink_4, the number of linearly independent polarisation states of the particle should be the same in AdS_4 as in Mink_4.

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- $D = 5$. In this case we may choose to realize the anticommutation relations in terms of the three 2×2 Pauli matrices, but these are complex so the state space is now a 2-dimensional complex vector space. This is equivalent to a 4-dimensional real vector space, so the number of independent polarisation states is now 4. This is exactly what we should expect of a massless Dirac equation for a minimal spinor in AdS_5. The minimal spinor has 4 complex, or 8 real, components but only half are propagating; this is a standard result for the Dirac equation in $Mink_5$ but AdS_5 is conformal to $Mink_5$.

- $D = 7$. In this case we may choose three of the 5 Dirac matrices to be $\boldsymbol{\sigma} \otimes \sigma_1$, where $\boldsymbol{\sigma}$ is the triplet of Pauli matrices, and then choose the other two to be $\mathbb{I}_2 \otimes \sigma_2$ and $\mathbb{I}_2 \otimes \sigma_3$. This gives a realization in terms of 4×4 complex matrices, which is equivalent to a realization in terms of 8×8 real matrices, so the state space is 8-dimensional. Again, this is exactly what we should expect of a massless Dirac equation for a minimal spinor in AdS_7; in this case the minimal spinor has 16 real components, implying 8 independent propagating modes.

Notice that the polarisation state space has dimension $2 \dim \mathbb{K}$ for $D = 3 + \dim \mathbb{K}$. This is consistent with our expectation that the action (7.1) describes a massless particle, with spin $\frac{1}{2}$ (or the higher-dimensional equivalent) when $N = 1$.

7.3 $N = 2$

For $N = 2$ we may replace the $(D-2)$ pairs of 'real' anticommuting variables ϑ_i^I by the complex anticommuting variables

$$\chi_I = \frac{1}{\sqrt{2}} \left(\vartheta_1^I + i \vartheta_2^I \right) \qquad \left[\Rightarrow \ \bar{\chi}_I = \frac{1}{\sqrt{2}} \left(\vartheta_1^I - i \vartheta_2^I \right) \right] . \tag{7.8}$$

This gives us $J = i \chi_I \bar{\chi}_I$ for the classical SO(2) constraint function.

The canonical anticommutation relations of the quantum theory are now those of $(D-2)$ fermi oscillators:

$$\left\{ \hat{\chi}_I, \hat{\chi}_J^\ddagger \right\} = \delta_{IJ} \, , \tag{7.9}$$

where $\ddagger$ indicates hermitian conjugation within the polarisation state space. With standard operator ordering, the operator version of J is[9]

$$\hat{J} = \frac{1}{2} \sum_{I=1}^{D-2} \left[\hat{\chi}_I^\ddagger, \hat{\chi}_I \right] = \sum_{I=1}^{D-2} \hat{\nu}_I - \frac{D-2}{2} \, , \tag{7.10}$$

where the $\hat{\nu}_I$ are the $(D-2)$ fermi number operators $\hat{\nu}_I = \hat{\chi}_I^\ddagger \hat{\chi}_I$ (no sum), with eigenvalues ν_I that are either 0 or 1. In a basis for which the operators $\hat{\nu}_I$ are diagonal, the SO(2) constraint becomes

$$\sum_{I=1}^{D-2} \nu_I = \frac{D-2}{2} \qquad (\text{even } D) \, . \tag{7.11}$$

[9] This step requires multiplication by i in order to get an hermitian operator; recall that the factor of i that is usually required for 'reality' of products of 'real' anticommuting variables is absent in our conventions.

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The restriction to even D arises because there is no solution of this equation for odd D. This is the well-known problem of a global $\mathrm{SO}(2) \cong \mathrm{U}(1)$ anomaly for an odd number of fermi oscillators, although the anomaly is actually the clash between U(1) gauge invariance and discrete symmetries that are broken for a non-zero WCS term [34, 44]. The result of integrating out the anticommuting variables, in the context of a path-integral quantization, is a WCS term with a coefficient $c = \pm\frac{1}{2}$. If this is taken into account the SO(2) constraint in the form of (7.11) is modified to

$$\sum_{I=1}^{D-2} \nu_I = \frac{D-2}{2} \pm \frac{1}{2} \qquad (\text{odd } D)\,, \tag{7.12}$$

where one must choose one sign or the other.

Although there is no fundamental reason to exclude the WCS term, we show in the appendix that its inclusion creates a mismatch between the AdS isometry group and the manifest $\mathrm{Sp}(4;\mathbb{K})$ symmetry group of the action (7.1), so its inclusion in this context is problematic. However, the global U(1) gauge anomaly can still be avoided by restoring the redundant anticommuting variables ζ'_i to the action, i.e. by reverting to the action (4.50). In this case the SO(2) constraint becomes

$$\sum_{I=1}^{D-2} \nu_I + \nu_{\zeta'} = \frac{D-2}{2} - \frac{1}{2}\,, \tag{7.13}$$

where $\nu_{\zeta'}$ is the extra fermion occupation number, and the modification on the right hand side takes into account the zero point contribution of the additional fermi oscillator. After allowing for both possible values of $\nu_{\zeta'}$, this reduces to the constraint (7.12) on ν_I, but now *both signs are allowed*. Thus, relative to the $c \neq 0$ resolution of the global U(1) anomaly, the $\zeta'_i \neq 0$ resolution leads to a doublet degeneracy in the polarisation state space.

Let us now consider in turn the implications of these observations for $D = 4, 5, 7$.

- $D = 4$. In this case the SO(2) constraint is $\hat{\nu}_1 + \hat{\nu}_2 = 1$, which has the two solutions

$$(\nu_1, \nu_2) = \{(1,0)\,, \quad (0,1)\}\,. \tag{7.14}$$

  There are therefore two polarisation states, which is what we should expect for a massless particle of spin 1, which is consistent with conformal invariance.

- $D = 5$. In this case we must use (7.12), so the SO(2) constraint is

$$\nu_1 + \nu_2 + \nu_3 = \frac{3 \pm 1}{2}\,. \tag{7.15}$$

  For either choice of sign there are three solutions (either one state is empty and the other two full, or vice versa). Apart from the doublet degeneracy if we take both sets of three states, this is the expected result for a massless 2-form or 3-form gauge potential, but neither of these possibilities is consistent with conformal invariance.

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- $D = 7$. This case is similar to $D = 5$. The SO(2) constraint is

$$\nu_1 + \nu_2 + \nu_3 + \nu_4 + \nu_5 = \frac{5 \pm 1}{2} \,. \tag{7.16}$$

  For either choice of sign there are 10 polarisation states, which is the number expected for massless 3-form or 4-form gauge potential; again, neither of these possibilities is consistent with conformal invariance.

7.4 $N > 2$ for $D = 4$

For $D = 4$ we should expect the "reduced" action (7.1) to describe a massless particle of spin $N/2$ in AdS_4. We shall consider in detail only $N = 3$ and $N = 4$, from which the generalization to higher N should be clear.

- $N = 3$. Recalling the choice $\gamma^1 = \sigma_1$ and $\gamma^2 = \sigma_3$ of Mink_3 Dirac matrices from the $N = 1$ case, we have the following 8×8 matrix realization of the $N = 3$ anticommutation relations for γ_i^I:

$$\begin{aligned} \gamma_1^I &= \gamma^I \otimes \sigma_2 \otimes \mathbb{I}_2 \\ \gamma_2^I &= \mathbb{I}_2 \otimes \gamma^I \otimes \sigma_2 \\ \gamma_3^I &= \sigma_2 \otimes \mathbb{I}_2 \otimes \gamma^I \end{aligned} \tag{7.17}$$

  These matrices are not real but they are all pure imaginary, which implies that the SO(3) generators are *real*:

$$\begin{aligned} \hat{J}_{12} &= -\frac{1}{2}\varepsilon_{IJ}\, \gamma^I \otimes \gamma^J \otimes i\sigma_2 \\ \hat{J}_{23} &= -\frac{1}{2}\varepsilon_{IJ}\, i\sigma_2 \otimes \gamma^I \otimes \gamma^J \\ \hat{J}_{31} &= \frac{1}{2}\varepsilon_{IJ}\, \gamma^I \otimes i\sigma_2 \otimes \gamma^J \,. \end{aligned} \tag{7.18}$$

  Because these matrices are real, we may consistently suppose that the state space is real.[10] Without the SO(3) constraint we have an 8-dimensional phase space, but now we must consider the effects of the constraints.

  The 4×4 matrix $\varepsilon_{IJ}\gamma^I \otimes \gamma^J$ has eigenvalues $0, 4$, both doubly degenerate. The zero-eigenvalue eigenvectors, i.e. zero modes, take the form $u \oplus i\sigma_2 u$ for 2-component column vector u. A zero mode of J_{12} is therefore $(u \oplus i\sigma_2 u) \oplus (v \oplus i\sigma_2 v)$ for two 2-component column vectors (u, v). Requiring that this (real) 4-vector also be a zero mode of J_{23} leads to the restriction $v = i\sigma_2 u$, so we now have a two-parameter space of zero mode 8-vectors of the form $u \oplus i\sigma_2 u \oplus i\sigma_2 u \oplus (-u)$. These states are also annihilated by $J_{31} = -i[J_{12}, J_{23}]$, so the SO(3) constraints reduce the initial 8-dimensional state space to a 2-dimensional subspace.

[10] This amounts to an assumption that the worldline time reversal invariance of the particle action becomes a discrete gauge invariance of the quantum theory [34]; see [48] for a recent discussion.

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- $N = 4$. We now have the following *real* 16×16 matrix realization of the anticommutation relations:

$$\begin{aligned}\gamma_1^I &= \left[\gamma^I \otimes \sigma_2 \otimes \mathbb{I}_2\right] \otimes \sigma_2 \\ \gamma_2^I &= \left[\mathbb{I}_2 \otimes \gamma^I \otimes \sigma_2\right] \otimes \sigma_2 \\ \gamma_3^I &= \left[\sigma_2 \otimes \mathbb{I}_2 \otimes \gamma^I\right] \otimes \sigma_2 \\ \gamma_4^I &= \left[\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2\right] \otimes \gamma^I \end{aligned} \tag{7.19}$$

  where the brackets are included merely to emphasize that the enclosed 8×8 matrices are those of (7.17), and the 8×8 identity matrix. The SO(4) generators are

$$\hat{J}_{ij} = \hat{J}_{ij}^{(N=3)} \otimes \mathbb{I}_2 \qquad i, j = 1, 2, 3 \tag{7.20}$$

  which generate an SO(3) subgroup, and

$$\begin{aligned}\hat{J}_{14} &= -\frac{1}{2}\varepsilon_{IJ}\left[\gamma^I \otimes i\sigma_2 \otimes \mathbb{I}_2\right] \otimes \gamma^J \\ \hat{J}_{24} &= -\frac{1}{2}\varepsilon_{IJ}\left[\mathbb{I}_2 \otimes \gamma^I \otimes i\sigma_2\right] \otimes \gamma^J \\ \hat{J}_{34} &= -\frac{1}{2}\varepsilon_{IJ}\left[i\sigma_2 \otimes \mathbb{I}_2 \otimes \gamma^I\right] \otimes \gamma^J . \end{aligned} \tag{7.21}$$

  The $N = 3$ results tell us that those 16-vectors annihilated by the generators of the SO(3) subgroup take the form

$$V = [u \oplus i\sigma_2 u \oplus i\sigma_2 u \oplus (-u)] \oplus [v \oplus i\sigma_2 v \oplus i\sigma_2 v \oplus (-v)] , \tag{7.22}$$

  where u and v are both two-component column vectors. These are annihilated by $\hat{J}_{14}$ if $v = i\sigma_2 u$, and then also by $\hat{J}_{24}$ and $\hat{J}_{34}$ as a consequence of SO(3) invariance. The SO(4) constraints therefore reduce the initial 16-dimensional state space to a 2-dimensional subspace.

Notice that the analysis for $N = 4$ is essentially an iteration of that for $N = 3$. Further iteration leads to the same conclusion for $N > 2$: the physical subspace of polarisation states is two-dimensional, exactly as one expects for a massless particle of any spin in any conformally flat 4D spacetime, in particular AdS_4.

8 Discussion

This paper is the continuation of a previous one [21], inspired by results of Claus et al. [7] and Cederwall [16, 17], on a two-twistor formulation of relativistic particle and superparticle mechanics in a D-dimensional anti-de Sitter background, for $D = 4, 5, 7$. A novelty of [21] was the use of $SL(2;\mathbb{K})$-spinor notation where $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ (the associative division algebras) to arrive at a simple two-twistor action for a massless particle in AdS_D, for $D = 3 + \dim \mathbb{K}$, with a manifest $Sp(4;\mathbb{K})$ invariance. For these spacetime dimensions the AdS isometry group is $Sp(4;\mathbb{K})$ (for an appropriate definition [11, 14]) and we verified that the $Sp(4;\mathbb{K})$ Noether charges coincide with those associated to invariance under AdS isometries.

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The starting point of the construction is the observation that AdS_D can be foliated by Mink_d hypersurfaces, where $d = D - 1$. If the action for a massive particle in AdS_D is expressed in the "Poincaré-patch" coordinates adapted to this foliation then its motion within any given hypersurface is that of a particle in Mink_d, with a mass Δ that is a particular constant of motion. The crucial step introduced in [21] is to take Δ to be a new canonical variable and then, by means of a "key identity", rewrite the action in terms of a new set of canonical variables that include Δ. At this point one may observe that the action for constant Δ *is* the action for a particle of mass Δ in Mink_d, which may be expressed in two-twistor form for $d = 3, 4, 6$. Subsequent elimination of Δ by means of the mass-shell constraint leads to an action with manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance, and a local $O(2;\mathbb{K})$ gauge invariance associated to "spin-shell" constraints.

The AdS isometry group is essentially $\mathrm{Sp}(4;\mathbb{K})$ (it contains an additional U(1) factor for $\mathbb{K} = \mathbb{C}$) and the canonical variables constitute a pair of 4-plets of this group. However, the new action is independent of the mass m originally assumed, and the Noether charges of the manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance coincide with those of the AdS isometry group only if $m = 0$, so only in this case do we have a manifestly linear realization of the AdS isometry group, with canonical variables that we can identify as two-twistor variables.

This result raises the following question: at which point in the construction does the restriction to zero mass arise? The answer is not entirely clear to us but there is an obvious potential problem with the choice of Poincaré-patch coordinates because these coordinates do not cover the whole of AdS. They cover only half of AdS (if we restrict to AdS rather than its covering space) with the two halves separated by a Killing horizon. Massive particle geodesics are curves that pass through both halves, so there is a global issue here for non-zero mass that we passed over. At the computational level, one may observe that the "key identity" of (4.33) involves a total time derivative of $mR\varphi$, where (as explained in [21]) φ is an angular parameter on timelike geodesics. Strictly speaking, this term is not a total time derivative because $d\varphi$ is not an exact 1-form.

The construction of [7] also uses a metric adapted to the foliation of AdS by Minkowski hypersurfaces, but it then proceeds differently in a way that is specific to AdS_5, in which case $O(2;\mathbb{C}) \cong \mathrm{U}(2)$, and the mass appears in the final result (in the combination mR, where R is the AdS radius) as a contribution to the U(1) charge. For $m = 0$ the action agrees with the AdS_5 case of that found in [21], and although it differs for $m \neq 0$ there is a complex redefinition of the two-twistor variables of the $m = 0$ action for an AdS_5 background that changes only the U(1) constraint, such that the result of [7] is recovered. No such redefinition is possible for AdS_4 or AdS_7, so only the massless particle has a two-twistor formulation in these cases, as originally shown by a very different method in [16, 17].

Even for AdS_5, the two-twistor formulation of particle mechanics is incompatible with an *arbitrary* mass when we pass to the quantum theory. We have shown that the absence of a global U(2) gauge anomaly requires the quantization condition $mR \in \mathbb{Z}$ (which also ensures that the ambiguity in the action due to the $mRd\varphi$ term we neglected in our application of the "key identity" does not lead to an ambiguity in the path integral). This U(2) anomaly is similar to the global U(1) gauge anomaly in quantum mechanics with anticommuting variables discussed in Elitzur et al. [44] because it is essentially an anomaly in a U(1) sub-

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group of U(2), but because U(2) is a $\mathbb{Z}_2$ quotient of $\mathrm{U}(1)\times\mathrm{SU}(2)$, the quantization condition for U(2) is stronger by a factor of 2 than it would be for the U(1) factor of $\mathrm{U}(1)\times\mathrm{SU}(2)$.

However, the main new results of this paper are contained in an extension of the construction described above to the N-extended "spinning particle", which generalizes the zero-spin particle action to one that has N local worldline supersymmetries. For a four-dimensional Minkowski background, this action is known to describe, upon quantization, a particle of spin $N/2$, [30, 32, 33]. Although the background spacetime is restricted to be conformally flat for $N > 2$, this still allows the choice of an AdS background [35, 36].

A crucial input is the two-twistor action for the *massive* N-extended spinning particle in a Mink_d background for $d = 3, 4, 6$ [37]. The output is then a two-twistor action for a *massless* N-extended spinning particle in an AdS_D background. For a massless particle one may expect an N-plet of anticommuting variables to be "redundant", as happens in the $m \to 0$ limit of the standard N-extended spinning particle action, and this is indeed the case. This has implications for the quantum theory because redundant anticommuting variables imply a degeneracy of polarisation states beyond that necessary for a massless particle of definite spin. Omitting such redundant variables leaves us with a "reduced" action, and we have verified that the $N = 1$ version of it leads to a polarisation state space of the expected dimension for a spin-$\frac{1}{2}$ particle in AdS_D.

As mentioned above, a discrepancy for non-zero mass between the AdS Noether charges and those of the manifest $\mathrm{Sp}(4;\mathbb{C})$ invariance arising from our construction of the two-twistor form of the spin-zero particle in AdS_5 can be eliminated by a change of variables that yields the action of [7]. For $N \geq 1$ it is not possible to eliminate this discrepancy in the same way, even for AdS_5 (and a similar problem arises for the massless $N = 2$ spinning particle if a worldline Chern-Simons (WCS) term is included, as we show in the appendix). Curiously, no similar difficulty arises for the superparticle, for any number of spacetime supersymmetries; we have shown that the manifest OSp invariance of the supertwistor action corresponds precisely with the expected AdS superisometries, even for the massive superparticle in AdS_5.

One advantage of the twistor formulation of the spinning particle is that the anticommuting variables are all physical in the sense that they are not subject to gauge transformations (in contrast to the standard formulation in which they are subject to local worldline supersymmetry transformations). This simplifies an investigation into those aspects of the quantum theory that arise from the presence of anticommuting variables in the classical theory. Such an investigation was carried out in [21] for the massless superparticle. Here we have presented the results of a similar investigation for the massless N-extended spinning particle; an important consideration in this case is the possibility of a global $\mathrm{SO}(N)$ gauge anomaly of the type analysed in detail in [44].

The implications of this quantum anomaly for the $N = 2$ spinning particle in a Minkowski background were investigated in [34] (and there are also implications for the massless superparticle [48]). The anomaly is really a clash between the $\mathrm{SO}(2) \cong \mathrm{U}(1)$ gauge invariance and discrete symmetries that would be broken by a WCS term. If one demands preservation of the U(1) gauge invariance then each worldline fermi-oscillator makes a contribution to the U(1) charge that is equivalent to a contribution of $\pm 1/2$ to

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the coefficient of an effective WCS term. It is therefore impossible to maintain both U(1) gauge invariance and a zero WCS term if the number of relevant fermi oscilators is odd. For the "reduced" $N = 2$ two-twistor action this number is even when $D = 4$ but odd for $D = 5, 7$. As we are limited to zero WCS term in the twistor formulation (for reasons explained in the appendix) we can avoid the global gauge anomaly in these $D = 5, 7$ cases only by re-instating the redundant anticommuting variables (which amount to one fermi oscillator for $N = 2$), but this comes at the cost of a reducible polarisation state space.

For $N > 2$ there is a potential global SO(N) anomaly [44], and we have determined its implications for the quantum mechanics of the $N > 2$ spinning particle. As for $N = 2$ case, the "reduced" two-twistor action for $N > 2$ leads to an inconsistent quantum theory for $D = 5, 7$, which can again be remedied by the re-instatement of a redundant N-plet of anticommuting variables but, again, at the likely cost of a reducible polarisation state space.

There are no global gauge anomalies for $N = 1$, however. In particular, for AdS_5 it should be possible to extend the $N = 0$ quantum results of [28] to $N = 1$. It should also be possible to make contact with the work of Adamo et al. on a twistor formulation of free field equations in AdS_5 [49], as well as other related work [50] and perhaps other approaches to particle mechanics in AdS backgrounds [51].

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A N=2 and the WCS term

The $N = 2$ case of the spinning particle action (4.1) is special because the Lagrange multiplier $f_{ij} = \varepsilon_{ij} f$ for the SO(2) $\cong$ U(1) constraint is then a U(1) gauge potential and we may add to the action the WCS term

$$S_{WCS} = \hbar c \int f dt \,. \tag{A.1}$$

The factor of $\hbar$ multiplying the constant c ensures that this term has the dimensions of action, and is needed anyway (unless one chooses units for which $\hbar = 1$) because the WCS term should be considered as a possible local one-loop addition to the action.

For $N = 2$ we may write $\mathcal{J}_{ij} = \varepsilon_{ij}\mathcal{J}$, in which case

$$\frac{1}{2}\mathcal{J}_{ij}\mathcal{J}_{ij} = \mathcal{J}^2 \,, \qquad \mathcal{J} = \frac{1}{2}\varepsilon^{ij}\mathcal{J}_{ij} \,. \tag{A.2}$$

Notice that this is consistent with our earlier definition of $\mathcal{J}^2$ in (2.24) (and this is also true for J^2 if we write $J_{ij} = \varepsilon_{ij} J$). In this notation, the f-dependent term in the Lagrangian is

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now

$$L_f = -f\,(\mathcal{J} - \hbar c)\ . \tag{A.3}$$

Variation of f therefore yields the modified SO(2) constraint $\mathcal{J} = \hbar c$. This sets a bilinear in anticommuting variables equal to a real constant, which is an equation without solutions, but the factor of $\hbar$ tells us that we should not be looking for classical solutions.

The U(1) gauge transformations associated to the SO(2) constraint are

$$(\xi_1 + i\xi_2) \to g(t)(\xi_1 + i\xi_2) \qquad \textit{et cetera}\,, \qquad f \to f - ig^{-1}\dot{g}\,, \tag{A.4}$$

where $g(t) \in \mathrm{U}(1)$ and "*et cetera*" stands for similar transformations of all other fermionic variables. As discussed in section 5 in the context of the other WCS that is possible for a (super)spin-zero (super)particle in AdS_5, the integral of $f dt$ is shifted by $2\pi w$ for a "large" U(1) gauge transformation for which $g(t)$ has winding number w, and this means that the path-integral is U(1) gauge-invariant only if $c \in \mathbb{Z}$, although this conclusion may be changed when "fermionic" variables are present.

However, the task we set ourselves here is to determine how our results of section 4 are affected, in the $N = 2$ case, by the addition of a WCS term. Our earlier derivation of the simplified constraints in Poincaré-patch coordinates applies only for $c = 0$, when specialized to $N = 2$, because the *unmodified* SO(2) constraint was used to simplify the other constraints. We should therefore expect some changes for $c \neq 0$. For simplicity we now set $\hbar = 1$.

A.1 Poincaré-patch coordinates redux

It will suffice to focus on the effects of the WCS term on the supersymmetry generators $\mathcal{Q}_i$ as given by (4.8). Previously, we simplified this expression for $\mathcal{Q}_i$ using the SO(N) constraint; we can now do the same again for $N = 2$ but we must use the modified constraint, which can be writen as

$$\lambda_i \cdot \lambda_j = c\,\varepsilon_{ij} - \zeta_i\zeta_j - \xi_i\xi_j\,. \tag{A.5}$$

As before, this yields

$$\mathcal{Q}_i = \left(\frac{z}{R}\right)\tilde{\mathcal{Q}}_i\,, \qquad \tilde{\mathcal{Q}}_i = p\cdot\lambda_i + \Xi_i\,, \tag{A.6}$$

but now

$$\Xi_i = p_z\zeta_i + \left(\frac{mR + \zeta_j\xi_j}{z}\right)\xi_i + \frac{c}{z}\varepsilon_{ij}\zeta_j\,. \tag{A.7}$$

Previously, we used both the supersymetry and SO(N) constraints to simplify the expression for $\mathcal{H} \equiv \mathcal{H}(0)$ in (4.9) but closure of the constraint algebra for $c \neq 0$ eliminates the ambiguity in the Hamiltonian constraint function, which is necessarily $\mathcal{H}(1)$. Making this choice and then proceeding as before we find that

$$2\mathcal{H}(1) = \left(\frac{z}{R}\right)^2\tilde{\mathcal{H}}\,, \qquad 2\tilde{\mathcal{H}} = p^2 + \Delta^2\,, \tag{A.8}$$

where now

$$\Delta^2 = p_z^2 + \left(\frac{mR + \zeta_k\xi_k}{z}\right)^2 + \left(\frac{c^2 + 2c(\zeta^2 - \xi^2)}{z^2}\right)\,, \tag{A.9}$$

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with

$$\zeta^2 \equiv \frac{1}{2}\varepsilon_{ij}\zeta_i\zeta_j\,, \qquad \xi^2 \equiv \frac{1}{2}\varepsilon_{ij}\xi_i\xi_j\,. \tag{A.10}$$

A Poisson bracket computation shows, as before but now for $N = 2$ with a WCS term, that

$$\left\{\tilde{\mathcal{Q}}_i, \tilde{\mathcal{Q}}_j\right\}_{PB} = 2\tilde{\mathcal{H}}\delta_{ij}\,, \tag{A.11}$$

and hence that the constraint functions generate a local $N = 2$ worldline supersymmetry. However, the supersymmetry transformations are modified as a result of the modifications of the constraints. For any function ϕ of the phase-space variables,

$$\delta_\epsilon\phi = \left\{\epsilon_i\tilde{\mathcal{Q}}_i, \phi\right\}_{PB}\,. \tag{A.12}$$

For the canonical variables this yields the c-dependent transformations

$$\delta_\epsilon p_z = \frac{mR}{z^2}\xi_i\epsilon_i - \frac{(c-\xi^2)}{z^2}\varepsilon_{ij}\zeta_i\epsilon_j\,, \qquad \delta_\epsilon\zeta_i = p_z\epsilon_i - \frac{(c-\xi^2)}{z}\varepsilon_{ij}\epsilon_j\,, \tag{A.13}$$

along with the unchanged, and hence c-independent, transformations

$$\begin{aligned} &\delta_\epsilon x^\mu = \lambda_i^\mu\epsilon_i\,, && \delta_\epsilon z = \zeta_i\epsilon_i\,, \\ &\delta_\epsilon\lambda_i^\mu = \eta^{\mu\nu}p_\nu\epsilon_i && \delta_\epsilon\xi_i = \left(\frac{mR+\zeta_j\xi_j}{z}\right)\epsilon_i - z^{-1}\zeta_i\xi_j\epsilon_j\,. \end{aligned} \tag{A.14}$$

Using these results one may verify that Δ^2, and hence $\tilde{\mathcal{H}}$, is still invariant under local supersymmetry, as required by the $N = 2$ supersymmetry algebra (A.11).

By analogy with the definition (A.7), which extends to $c \neq 0$ the definition Ξ_i of (4.16), we can also extend to $c \neq 0$ the definition Z_i of (4.26), which becomes

$$Z_i = p_z\xi_i - \left(\frac{mR+\zeta_j\xi_j}{z}\right)\zeta_i - \frac{c}{z}\varepsilon_{ij}\xi_j\,. \tag{A.15}$$

A PB calculation shows that we again have

$$\{Z_i, Z_j\}_{PB} = \Delta^2\delta_{ij}\,, \qquad \{Z_i, \Xi_j\}_{PB} = 0\,, \tag{A.16}$$

where Δ^2 is now given by (A.9). If we proceed as before to set

$$\Xi_i = \Delta\,\xi_i'\,, \qquad Z_i = \Delta\,\zeta_i'\,, \tag{A.17}$$

then we find, as before, that the new primed anticommuting variables satisfy the simple PB relations

$$\left\{\xi_i', \xi_j'\right\}_{PB} = \delta_{ij} = \left\{\zeta_i', \zeta_j'\right\}_{PB}\,, \qquad \left\{\xi_i', \zeta_j'\right\}_{PB} = 0\,. \tag{A.18}$$

The relation between the old anticommuting variables and the new, primed, ones is now significantly more complicated. A detailed analysis, which we omit, shows that it is possible to rewrite the action in terms of the primed variables but the "key identity" relating the geometrical part of the action in the two sets of variables allows us to take the next step towards a two-twistor action only if either $c = 0$ or $m = 0$. Since we are interested in $c \neq 0$ we shall proceed on the assumption that $m = 0$.

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A.2 Zero mass

For $m = 0$ the expression (A.9) simplifies to

$$\Delta^2 = p_z^2 + \left(\frac{c + \zeta^2 - \xi^2}{z}\right)^2 , \tag{A.19}$$

which suggests that we define an angle φ such that

$$p_z = \Delta \cos\varphi \,, \qquad \left(\frac{c + \zeta^2 - \xi^2}{z}\right) = \Delta \sin\varphi \ . \tag{A.20}$$

We then have

$$\xi_i' = \zeta_i \cos\varphi + \varepsilon_{ij}\zeta_j \sin\varphi \,, \qquad \zeta_i' = \xi_i \cos\varphi - \varepsilon_{ij}\xi_j \sin\varphi \,, \tag{A.21}$$

and these relations imply

$$\xi^2 = (\zeta')^2 \,, \qquad \zeta^2 = (\xi')^2 \quad \Rightarrow \quad \Delta^2 = p_z^2 + \left(\frac{c + (\xi')^2 - (\zeta')^2}{z}\right)^2 . \tag{A.22}$$

Proceeding as we did for the generic N case in section 4 now leads to the identity

$$\dot{z}p_z + \frac{1}{2}\left(\zeta_i\dot{\zeta}_i + \xi_i\dot{\xi}_i\right) \equiv -\frac{zp_z}{\Delta}\dot{\Delta} + \frac{1}{2}\left(\xi_i'\dot{\xi}_i' + \zeta_i'\dot{\zeta}_i'\right) + \frac{d}{dt}\left(zp_z + c\varphi\right) . \tag{A.23}$$

Using this result we may proceed as before. After translating to $\mathrm{SL}(2;\mathbb{K})$-spinor notation we arrive at a Lagrangian that is formally identical to (4.38) with local worldline supersymmetry constraints that are formally identical to those of (4.39), but we are now restricted to $N = 2$ and the SO(2) constraint is $\mathcal{J} = c$ with[11]

$$\mathcal{J} = \frac{1}{4}\mathrm{tr}_{\mathbb{R}}\left(\varepsilon^{ij}\tilde{\Lambda}_i\Lambda_j\right) + (\xi')^2 + (\zeta')^2 \,. \tag{A.24}$$

We may now follow the steps of subsection 4.3. We write $\mathbb{P} = \mp\mathbb{U}\mathbb{U}^\dagger$ and substitute, solving the $\tilde{\mathcal{H}} = 0$ constraint for Δ in terms of $\mathbb{U}$. we then solve the constraints $\mathcal{Q}_i = 0$ for Λ_i in terms of the new matrix variables Ψ_i, exactly as in (4.44) athough the Δ factor that appears in that expression for Λ_i is now different. This results in the new Lagrangian

$$L = \mathrm{tr}_{\mathbb{R}}\left(\dot{\mathbb{U}}\mathbb{W}^\dagger\right) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}\left(\Psi_i\dot{\Psi}_i\right) + \frac{1}{2}\zeta_i'\dot{\zeta}_i' - f(\mathcal{J} - c) \,, \tag{A.25}$$

where now

$$\mathcal{J} = \frac{1}{2}\varepsilon^{ij}\mathrm{tr}_{\mathbb{R}}(\Psi_i\Psi_j) + \zeta_i'\zeta_j' \,, \tag{A.26}$$

and $\mathbb{W}$ is exactly as given in the incidence relation (4.47), although now restricted to $N = 2$. Because this incidence relation is unchanged, the identity of (4.48) still holds, with the same expression for $\mathbb{G}$. As before, this must be included as a constraint imposed by a

[11] Recall that $\tilde{\Lambda}$ is the "trace-reverse" of the matrix Λ.

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Lagrange multiplier in the action with $\mathbb{U}$ and $\mathbb{W}$ as independent variables. After combining $\mathbb{U}$ and $\mathbb{W}$ into the two-twistor $\mathbb{Z}$ we thus arrive at the Lagrangian

$$L_c = \frac{1}{2}\mathrm{tr}_{\mathbb{R}}(\Omega\dot{\mathbb{Z}}\mathbb{Z}^\dagger) + \frac{1}{4}\mathrm{tr}_{\mathbb{R}}(\Psi_i\dot{\Psi}_i) + \zeta'_i\dot{\zeta}'_i + \mathrm{tr}_{\mathbb{R}}\left[\mathbb{S}\left(\mathbb{Z}^\dagger\Omega\mathbb{Z} + \Psi_i\Psi_i\right)\right] - f(\mathcal{J}-c)\,. \tag{A.27}$$

Of course, we could have deduced this result more directly by simply restricting the two-twistor action of (4.49) to $N=2$ and then modifying the SO(2) constraint to include the WCS term. However, in order to check whether the manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance of the two-twistor Lagrangian L_c is the invariance inherited from the isometry group of the AdS background, we need to rewrite the manifest $\mathrm{Sp}(4;\mathbb{K})$ Noether charges in terms of the variables of the original action. As this step involves details of the expression for Δ^2, which differs from the expression used in subsection 4.3, it was necessary to arrive at (A.27) by the longer route.

As we have seen in subsection 4.4, the only Noether charges of the manifest $\mathrm{Sp}(4;\mathbb{K})$ invariance that are not guaranteed to arise from AdS isometries are those contained in $\mathbb{W}\mathbb{W}^\dagger$. Our earlier discussion of this issue for the generic N case implies, when restricted to $N=2$, that $\pm\mathbb{W}\mathbb{W}^\dagger = \mathbb{K}$ when both $m=0$ and $c=0$, where $\mathbb{K}$ is the AdS isometry matrix-charge of (4.58); in terms of the new, primed, variables this becomes

$$\begin{aligned}\mathbb{K} = {}& \frac{1}{2}\tilde{\mathbb{P}}\left(\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\tilde{\mathbb{X}}) + 2z^2\right) - \mathbb{X}\left(\mathrm{tr}_{\mathbb{R}}(\mathbb{X}\mathbb{P}) + zp_z\right) + 2\Lambda_i\xi'_i\frac{zp_z}{\Delta} \\ & + \Lambda_i\mathrm{tr}_{\mathbb{R}}\left(\mathbb{X}\tilde{\Lambda}_i\right) - 2\Lambda_i\left(\frac{c+(\xi')^2-(\zeta')^2}{\Delta}\right)\varepsilon_{ij}\xi'_j\,.\end{aligned} \tag{A.28}$$

As we are now considering only $m=0$, any discrepancy between $\pm\mathbb{W}\mathbb{W}^\dagger$ and $\mathbb{K}$ must be zero when $c=0$. This is confirmed by a direct calculation, which shows that

$$\pm\,\mathbb{W}\mathbb{W}^\dagger = \mathbb{K} \mp \left[c-(\zeta')^2\right]^2\mathbb{V}^\dagger\mathbb{V}\,. \tag{A.29}$$

The discrepancy between $\pm\mathbb{W}\mathbb{W}^\dagger$ and $\mathbb{K}$ reduces to $\mp c\mathbb{V}^\dagger\mathbb{V}$ if we omit the ζ'_i variables on the grounds that they are redundant for $m=0$. Comparison with (4.60) shows that the constant c now replaces the constant mR; as in that case, the c-dependent term can be removed by a complex change of variables when $D=5$ but this leads to (5.11) with (mR) replaced by c, and since $\Psi_i\Psi_i$ is non-zero for $N=2$ there is still a discrepancy.

We should stress that there is no fundamental obstruction to the inclusion of a WCS term in the action for the $N=2$ spinning particle, just as there is no fundamental obstruction to the inclusion of a mass for $N>0$, but both are obstructions to the existence of a two-twistor action, even in the special case of AdS_5.

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