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Relationship between causality of stochastic processes and zero blocks of their joint innovation transfer matrices

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Abstract: We consider output processes which are realizable by stochastic linear time-invariant (LTI) systems. Such processes can always be realized by LTI systems in forward innovation form, and we study the transfer matrices of such LTI realizations. We show that such a transfer matrix is consistent with an acyclic directed graph if and only if the edges of this graph represent Granger-causality relations among the components of the output process. By consistency we mean that if there is no edge between two vertices of the graph, then the corresponding block of the transfer matrix is zero. Under this assumption, conditional Granger non-causality between the components of the process is equivalent with a zero block in the transfer matrix.

1. INTRODUCTION

Many complex systems arise by interconnecting several smaller subsystems which communicate with each other. The resulting network structure and its consequences for the global behavior of the system are of interest both for control and analysis of such systems. In addition, reverse engineering of this network structure is a major challenge in several applications. Understanding the relationship between the network structure and the global observed behavior is essential for addressing all these problems. Unfortunately, this relationship is far from obvious: even if the number of subsystems is known, and each subsystem generates observations, it is not clear if the interaction between any two subsystems induces an intrinsic relationship between their observed behavior. If the observed behavior is modelled as a collection of stochastic processes, then various notions from probability theory can be used to formalize the interactions among them. For example, the notion of *(conditional) Granger causality* Granger (1963) can be used. Informally, a process $\mathbf{y}_1$ does not conditionally Granger cause a process $\mathbf{y}_2$ with respect to $\mathbf{y}_3$, if using the past values of $\mathbf{y}_1$, $\mathbf{y}_2$ and $\mathbf{y}_3$ do not allow to predict the future values of $\mathbf{y}_2$ with a higher accuracy then using only the past values of $\mathbf{y}_2$ and $\mathbf{y}_3$. The concept of Granger causality has been used in systems biology, neuroscience and economics Roebroeck et al. (2011); Valdes-Sosa et al. (2011). Although there are several ways to represent a stochastic process (auto-regressive, moving average, state-space models), the relationship between Granger causalities and the network structure of these representations is not evident.

In this paper, we consider discrete-time multivariate stochastic processes with a proper rational spectrum, i.e., stochastic processes which can be interpreted as outputs of linear-time time-invariant stochastic state-space representations, shortly LTI state-space representation, driven by a white noise process. Consider such a process $\mathbf{y}$. It is well known that there exists an LTI state-space representation whose output is $\mathbf{y}$ and whose noise process is the innovation process of $\mathbf{y}$. Furthermore, if this state-space representation is minimal then it is unique up to isomorphism, and hence its transfer matrix is uniquely determined by $\mathbf{y}$. We will call this transfer matrix the *innovation transfer matrix* of $\mathbf{y}$. It is well known that the LTI state-space representation with innovation noise (and hence the innovation transfer matrix) can be computed from the covariances of $\mathbf{y}$, or estimated from a sample path of $\mathbf{y}$ using subspace identification methods (Lindquist and Picci (2015)).

Contribution. We show that the innovation transfer matrix of a process $\mathbf{y}$ is consistent with a transitive acyclic graph, if and only if the components of $\mathbf{y}$ are related by conditional Granger non-causality in a way determined by that graph. By consistency with a graph we mean that the edges of the graph correspond to potentially non-zero blocks of the innovation transfer matrix. That is, *we relate the graph structure of the innovation transfer matrix with a graph formed by conditional Granger non-causality relations of the components of* $\mathbf{y}$. Note that each block of the innovation transfer matrix can be viewed as a transfer matrix of a subsystem. Hence, the graph with which the innovation transfer matrix is consistent can be interpreted as a description of interconnections among various subsystems. That is, the results of the paper relate *intrinsic properties of a process* with the interconnection structure of a finite representation (innovation transfer matrix) of this process. In addition to providing insights into fundamental theoretical problems, the result of the paper could serve as a starting point for testing (conditional) Granger causality.

Related work. Reverse engineering of the network structure of deterministic linear systems has been investigated in i.e.,

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Yuan et al. (2015, 2011); Nordling and Jacobsen (2011) and the references therein. In contrast to the cited papers, we consider stochastic systems and we relate their network structure to Granger causality of their outputs. In Dufour and Renault (1998); Caines (1976); Gevers and Anderson (1982) the relationship between Granger causality of two processes and their Wold decomposition was investigated. Granger causality for state-space representation was studied by using transfer matrix approach in Barnett and Seth (2015). Contrary to those papers, we consider a more general graph of Granger causality relations, which involves more than two processes, and we relate it to the zero blocks of the innovation transfer matrix of the joint process. The notion of conditional Granger causality is a type of spurious causality in Hsiao (1982), where Hsiao related causality relation with representations but did not discuss multiple causality conditions. More complex causality structure was also studied for state space representation, see Caines et al. (2003); Caines and Wynn (2007); Caines et al. (2009). In comparison with Caines and Wynn (2007); Caines et al. (2009) we consider conditional Granger non-causality, while in the cited papers (non-conditional) Granger causality [2] and conditional orthogonality has been studied where the conditional orthogonality condition has no trivial translation into (conditional) Granger non-causality. Regarding innovation transfer matrices, the results in Caines and Wynn (2007) and Caines et al. (2009) form special cases of the results in this paper since we have arbitrary covariance of the innovation process. For representing different kinds of causality relations Eichler (2005) worked on the approach of causality graphs, however Eichler (2005) did not link the causality properties of the process with its LTI representations.

Outline. Before presenting our results, in §2 we introduce the terminology and the basic tools, such as the stochastic processes of interest, Hilbert spaces generated by stochastic processes and transfer matrices. Then, in §3.1 we characterize Granger non-causality between two components of a process with the help of transfer matrices. As a generalization, in §3.2 we present our main result for conditional Granger non-causalities between several components of a process. In §3.3, we provide an example for our main result.

2. PRELIMINARIES

2.1 Notation and terminology

We will use standard terminology from theory of stochastic processes, see for example Lindquist and Picci (2015). In particular, we consider discrete-time stochastic processes whose values are vectors with real entries, i.e. by a stochastic process $\mathbf{z}$ taking values in $\mathbb{R}^k$ we mean a sequence $\{\mathbf{z}(t)\}_{t\in\mathbb{Z}}$ of random variables taking values in $\mathbb{R}^k$, where $\mathbf{z}(t)$ is referred as the value of $\mathbf{z}$ at time t. Here, as usual, $\mathbb{Z}$ denotes the set of integers. If ν is a random variable with values in $\mathbb{R}^k$, then by the coordinates of ν we will mean the random variables ν^i, $i = 1,\ldots,k$ taking values in $\mathbb{R}$, such that $\nu = [\nu^1,\ldots,\nu^k]^T$. We denote by $E[\nu]$ the mathematical expectation of a random variable ν. For standard notions of stochastic processes, such as wide-sense stationarity, spectral density, etc. we refer to Lindquist and Picci (2015). The class of processes studied in this paper is defined below:

Definition 1. [ZMSIR] A stochastic process $\mathbf{z}$ is called zero-mean square-integrable with rational spectrum (abbreviated by ZMSIR), if it is a zero mean, square-integrable, wide-sense stationary, purely non-deterministic and full rank process whose spectral density is rational and strictly positive definite on the unit circle.

In the sequel, we will use various Hilbert-spaces generated by stochastic processes. The zero-mean square-integrable random variables taking values in $\mathbb{R}$ form a Hilbert space with the covariance as the inner product, Lindquist and Picci (2015). We denote this Hilbert-space by $\mathscr{H}$. By the Hilbert-space generated by a set S of elements of $\mathscr{H}$ we will mean the smallest (with respect to set inclusion) closed subspace of $\mathscr{H}$ which contains S. Consider a ZMSIR process $\mathbf{z}$ taking values in $\mathbb{R}^k$. Then for each $\ell \in \mathbb{R}^k$, $t\in\mathbb{Z}$, $\ell^T\mathbf{z}(t)$ is an element of $\mathscr{H}$ and we denote by $\mathcal{H}^{\mathbf{z}}$, $\mathcal{H}^{\mathbf{z}}_{t-}$, $\mathcal{H}^{\mathbf{z}}_{t+}$, $\mathcal{H}^{\mathbf{z}}_{t}$, $t\in\mathbb{Z}$ respectively the Hilbert-spaces generated by the sets $\{\ell^T\mathbf{z}(s) \mid s\in\mathbb{Z}\}$, $\{\ell^T\mathbf{z}(s) \mid s\in\mathbb{Z}, s\le t-1\}$, $\{\ell^T\mathbf{z}(s) \mid s\in\mathbb{Z}, s\ge t\}$, and $\{\ell^T\mathbf{z}(t)\}$. Informally, $\mathcal{H}^{\mathbf{z}}$ is the Hilbert-space generated by the coordinates of all the values (past and future) of $\mathbf{z}$, $\mathcal{H}^{\mathbf{z}}_{t-}$ is the Hilbert-space generated by the coordinates of the past values $\{\mathbf{z}(s)\}_{s=-\infty}^{t-1}$ of $\mathbf{z}$ up to time $t-1$, $\mathcal{H}^{\mathbf{z}}_{t+}$ is the Hilbert-space generated by the coordinates of the future values $\{\mathbf{z}(s)\}_{s=t}^{\infty}$ of $\mathbf{z}$ starting from the time instance t, and $\mathcal{H}^{\mathbf{z}}_{t}$ is the Hilbert-space generated by the coordinates of $\mathbf{z}(t)$. If $\mathbf{z}_1,\ldots,\mathbf{z}_n$ are vector valued processes then $\mathbf{z} = \left[\mathbf{z}_1^T,\ldots,\mathbf{z}_n^T\right]^T$ will denote the process defined by $\mathbf{z}(t) = \left[\mathbf{z}_1^T(t),\ldots,\mathbf{z}_n^T(t)\right]^T$, $t\in\mathbb{Z}$. In this case, sometimes we will denote $\mathcal{H}^{\mathbf{z}}_{t-}$, $\mathcal{H}^{\mathbf{z}}_{t+}$, $\mathcal{H}^{\mathbf{z}}$ by $\mathcal{H}^{\mathbf{z}_1,\ldots,\mathbf{z}_n}_{t-}$, $\mathcal{H}^{\mathbf{z}_1,\ldots,\mathbf{z}_n}_{t+}$, $\mathcal{H}^{\mathbf{z}_1,\ldots,\mathbf{z}_n}_{t}$ respectively.

If $\eta\in\mathscr{H}$ and $\mathcal{H}$ is a closed subspace of $\mathscr{H}$, then we denote by $E_l[\eta\,|\,\mathcal{H}]$ the orthogonal projection of η onto $\mathcal{H}$. The orthogonal projection onto $\mathcal{H}$ of a random variable ν taking values in $\mathbb{R}^k$ is defined as follows. Assume that $\nu = [\nu^1,\ldots,\nu^k]^T$, i.e. $\nu^i\in\mathscr{H}$, $i=1,\ldots,k$ are the coordinates of ν. Then the orthogonal projection of ν onto $\mathcal{H}$, denoted by $E_l[\nu|\mathcal{H}]$, is defined as $E_l[\nu|\mathcal{H}] := [\hat{\nu}^1,\ldots,\hat{\nu}^k]^T$, where $\hat{\nu}^i = E_l[\nu^i|\mathcal{H}]$, $i=1,\ldots,k$. That is, $E_l[\nu|\mathcal{H}]$ is the random variable with values in $\mathbb{R}^k$ obtained by projecting the coordinates of ν onto $\mathcal{H}$. By a slight abuse of terminology and notation, we will say that a random variable ν taking values in $\mathbb{R}^k$ belongs to a closed subspace $\mathcal{H}$ of $\mathscr{H}$, denoted by $\nu\in\mathcal{H}$, if every coordinate of ν is an element of $\mathcal{H}$. This is equivalent to saying that $\ell^T\nu$ is an element of $\mathcal{H}$ for all $\ell\in\mathbb{R}^k$. A random variable ν is said to be orthogonal to a closed subspace $\mathcal{H}$ of $\mathscr{H}$, denoted by $\nu\perp\mathcal{H}$, if $E[\nu\eta]=0$ for all $\eta\in\mathcal{H}$.

2.2 Innovation transfer matrix

Consider a transfer matrix $G(z)$ of a finite-dimensional discrete-time stable deterministic LTI system (Anderson and Moore, 1979, Appendix C & D). Consider its Laurent series expansion, i.e. let $G_k\in\mathbb{R}^{n\times m}$, $k\ge 0$ be such that $G(z)=\sum_{k=0}^{\infty}G_k z^{-k}$ for all $z\in\mathbb{C}$ such that $|z|\ge 1$. If $\mathbf{y}$ is a ZMSIR process, then the expression $\sum_{k=0}^{\infty}G_k\mathbf{y}(t-k)$ converges in the topology of $\mathcal{H}^{\mathbf{y}}_{t-}$ (Anderson and Moore, 1979, Theorem 4.1). In the sequel, we will write

$$G(z)\mathbf{y}(t) = \sum_{k=0}^{\infty} G_k\mathbf{y}(t-k).$$

That is, $G(z)$ can be interpreted as a causal linear filter which transforms $\mathbf{y}$ to the process $\{G(z)\mathbf{y}(t)\}_{t=-\infty}^{\infty}$.

[2] Granger non-causality from $\mathbf{y}_1$ to $\mathbf{y}_2$ is equivalent with (weak) feedback freeness of $(\mathbf{y}_1,\mathbf{y}_2)$.

Below we define what we mean by the innovation transfer matrix. To this end, consider a ZMSIR process $\mathbf{y}$ and let $\Lambda_k = E[\mathbf{y}(t+k)\mathbf{y}^T(t)]$ for all $k \in \mathbb{Z}$. Then $\mathbf{y}$ has a spectral density function $\Phi_{\mathbf{y}}(z) = \sum_{k=-\infty}^{+\infty} \Lambda_k z^k$ defined for all $z \in \mathbb{C}$ such that $|z| = 1$. It is well known Lindquist and Picci (2015); Anderson and Moore (1979) that $\Phi_{\mathbf{y}}(z)$ admits a unique decomposition

$$\Phi_{\mathbf{y}}(z) = P(z)\Omega P^T(z^{-1}),\ |z| = 1$$

with the following conditions: Ω is a positive definite symmetric matrix, $P(z)$ and its inverse [3] $Q(z) := P^{-1}(z)$ are transfer matrices of a finite-dimensional deterministic exponentially stable discrete-time LTI system and $\lim_{z\to\infty} P(z) = I$ (Anderson and Moore, 1979, Section 9.4,Theorem 4.1). We call $P(z)$ *the innovation transfer matrix of* $\mathbf{y}$.

The reason for this terminology is as follows. Define the *forward innovation process* of a ZMSIR process $\mathbf{y}$ (that we call shortly as innovation process) as

$$\mathbf{e}(t) := \mathbf{y}(t) - E_l[\mathbf{y}(t)|\mathcal{H}^{\mathbf{y}}_{t-}].$$

It is known (Lindquist and Picci, 2015, Section 4.1.3) that $\mathbf{e}$ is a white noise ZMSIR process for which $\mathcal{H}^{\mathbf{e}}_{t-} = \mathcal{H}^{\mathbf{y}}_{t-}$. From the properties of $P(z)$ and $Q(z)$ the expressions $P(z)\mathbf{e}(t) = \sum_{k=0}^{\infty} P_k\mathbf{e}(t-k)$ and $Q(z)\mathbf{y}(t) = \sum_{k=0}^{\infty} Q_k\mathbf{y}(t-k)$ are well defined, where $P(z) = \sum_{k=0}^{\infty} P_k z^{-k}$ and $Q(z) = \sum_{k=0}^{\infty} Q_k z^{-k}$ are the Laurent series expansions of $P(z)$ and $Q(z)$. Moreover, for all $t \in \mathbb{Z}$,

$$\mathbf{y}(t) = P(z)\mathbf{e}(t), \text{ and } \mathbf{e}(t) = Q(z)\mathbf{y}(t).$$

That is, the innovation transfer matrix $P(z)$ is a causal linear filter which transforms the innovation process $\mathbf{e}$ into $\mathbf{y}$ and $Q(z)$ is a causal linear filter which transforms $\mathbf{y}$ into the innovation process $\mathbf{e}$.

The innovation transfer matrix of $\mathbf{y}$ has the following interpretation in terms of state-space representations. Consider a stable stochastic LTI state-space representation

$$\begin{aligned} \mathbf{x}(t+1) &= A\mathbf{x}(t) + B\mathbf{v}(t) \\ \mathbf{y}(t) &= C\mathbf{x}(t) + D\mathbf{v}(t) \end{aligned} \tag{1}$$

of $\mathbf{y}$, where $\mathbf{x}$ is the state process, $\mathbf{v}$ is a white noise process and A, B, C, D are matrices with real entries of suitable dimensions. In addition, $\mathbf{v}(t)$ is uncorrelated with $\mathbf{x}(t-k)$, $k \geq 0$. Note that the transfer matrix $G(z) = C(zI - A)^{-1}B + D$ of (1) satisfies, $\Phi_{\mathbf{y}}(z) = G(z)E[\mathbf{v}(t)\mathbf{v}^T(t)]G^T(z^{-1})$. Moreover, $G(z)$ has Laurent series expansion $G(z) = \sum_{k=0}^{\infty} G_k z^{-k}$ for any $z \in \mathbb{C}$, $\|z\| \geq 1$, where $G_0 = D$ and $G_k = CA^{k-1}B$, $k \geq 1$, thus

$$\mathbf{y}(t) = G(z)\mathbf{v}(t) = \sum_{k=1}^{\infty} CA^{k-1}B\mathbf{v}(t-k) + D\mathbf{v}(t).$$

From stochastic realization theory Lindquist and Picci (2015) it is known that the LTI state-space representation (1) can be chosen to be in the so called *forward innovation form*, for which $\mathbf{v}$ is the innovation process $\mathbf{e}$ of $\mathbf{y}$, $D = I$ is the identity matrix and B is the static Kalman gain. The transfer matrix $G(z)$ of the LTI state-space representation in forward innovation form coincides with the innovation transfer matrix $P(z)$ of $\mathbf{y}$. Conversely, if we consider any LTI state-space representation (1) such that its transfer matrix $G(z)$ has a stable rational proper inverse (it is minimum phase), then it can be shown that there exists an invertible matrix M, such that $\mathbf{v}(t) = M\mathbf{e}(t)$ for all $t \in \mathbb{Z}$. Therefore, $G(z)$ can be obtained from the innovation transfer matrix by multiplying it with M^{-1} from the right. That is, up to a right multiplication by a nonsingular matrix, the innovation transfer matrix can be viewed as the transfer matrix of any LTI state-space representation of $\mathbf{y}$ with a stable causal inverse.

[3] $P(z)Q(z) = Q(z)P(z) = I$ for all z in the domain of definition of P and Q

3. NON-CAUSALITY IN TRANSFER MATRICES

In this section we present our main result which deals with the following question: what form of a transfer matrix in an input-output system can mean (conditional) Granger non-causality.

3.1 Granger non-causality

To begin with, we define the notion of Granger non-causality introduced in Granger (1963). Granger non-causality is also known as feedback freeness in the literature. In addition, it can be defined via conditional orthogonality of certain spaces. More precisely, Definition 2 is equivalent with (weak) feedback freeness of $(\mathbf{y}_1, \mathbf{y}_2)$ in Gevers and Anderson (1982); Caines and Chan (1975); Caines (1976) and with conditional orthogonality of the spaces $\mathcal{H}^{\mathbf{y}_2}_{t+}$ and $\mathcal{H}^{\mathbf{y}_1}_{t-}$ with respect to $\mathcal{H}^{\mathbf{y}_2}_{t-}$ (Lindquist and Picci, 2015, Section 2.6.5).

Definition 2. Consider a ZMSIR process $\mathbf{y} = \left[\mathbf{y}_1^T, \mathbf{y}_2^T\right]^T$. We say that $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ if for all $t, k \in \mathbb{Z}$, $k \geq 0$

$$E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}^{\mathbf{y}_2}_{t-}] = E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}^{\mathbf{y}}_{t-}].$$

It is well known that Granger non-causality is equivalent to the innovation transfer matrix being in block triangular form Caines (1976); Caines and Chan (1975); Gevers and Anderson (1982), or in other words, the Wold decomposition being in block triangular form. In the next theorem we recall this result.

Theorem 1. Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T]^T$. Then $\mathbf{y}_1$ does not Granger cause $\mathbf{y}_2$ if and only if the innovation transfer matrix $P(z)$ of $\mathbf{y}$ has the form of

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) \\ 0 & P_{22}(z) \end{bmatrix}, \tag{2}$$

where $P_{ij}(z)$ is an $(n_i \times n_j)$ block of $P(z)$, for $i, j = 1, 2$, and n_i is such that $\mathbf{y}_i$ takes values in $\mathbb{R}^{n_i}$, $i = 1, 2$.

Let $\mathbf{e}$ be the innovation process of $\mathbf{y}$, and let $\mathbf{e}_i$ be the $\mathbb{R}^{n_i}$ valued process, $i = 1, 2$ such that $\mathbf{e} = [\mathbf{e}_1^T, \mathbf{e}_2^T]^T$. Then (2) is equivalent to

$$\begin{aligned} \mathbf{y}_1(t) &= P_{11}(z)\mathbf{e}_1(t) + P_{12}(z)\mathbf{e}_2(t) \\ \mathbf{y}_2(t) &= P_{22}(z)\mathbf{e}_2(t) \end{aligned}$$

In the next subsection, we generalize Theorem 1 for ZMSIR processes partitioned into $n \geq 2$ components. The result is about the equivalence of conditional non-causality relations among the components and the zero blocks of the innovation transfer matrix.

3.2 Conditional Granger non-causality

Our goal is to generalize Theorem 1 for a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$ with more complex causality structure, correspondingly, for a more complex zero structure of the transfer matrices. For this, we need to introduce the notion of conditional Granger non-causality.

Definition 3. Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$. We say that $\mathbf{y}_1$ conditionally does not Granger cause $\mathbf{y}_2$ with respect to $\mathbf{y}_3$, if for all $t, k \in \mathbb{Z}$, $k \geq 0$

$$E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}_2,\mathbf{y}_3}] = E_l[\mathbf{y}_2(t+k) \mid \mathcal{H}_{t-}^{\mathbf{y}}].$$

Intuitively, conditional Granger non-causality from $\mathbf{y}_1$ to $\mathbf{y}_2$ with respect to $\mathbf{y}_3$ means that the past values of $\mathbf{y}_1$ does not help to predict $\mathbf{y}_2$ if we already have the information about the past values of $\mathbf{y}_2$ and $\mathbf{y}_3$.

To interpret our generalization of Theorem 1 we introduce a class of graphs: the *transitive acyclic directed graphs (TADG)*. Consider a graph $G = (V, E)$, with set of nodes $V = \{1, \ldots, n\}$ and set of directed edges $E \subseteq V \times V$ and denote the edge from node i to node j by (i, j). The graph G is called *acyclic* if there is no cycle, i.e., closed directed path. Furthermore, it is *transitive* if for $i, j, k \in V$ $(i, j), (j, k) \in E \Longrightarrow (i, k) \in E$.

Let $P(z)$ be the innovation transfer matrix of a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$. Assume that $\mathbf{y}_i$ takes values in $\mathbb{R}^{n_i}$, $i = 1, \ldots, n$ and consider the following decomposition of $P(z)$

$$P(z) = \begin{bmatrix} P_{11}(z) & P_{12}(z) & \ldots, & P_{1n}(z) \\ P_{21}(z) & P_{22}(z) & \ldots, & P_{2n}(z) \\ \vdots & \vdots & \ldots & \vdots \\ P_{n1}(z) & P_{n2}(z) & \ldots, & P_{nn}(z) \end{bmatrix}, \tag{3}$$

where $P_{ij}(z)$ is a $(n_i \times n_j)$ block of $P(z)$. In other words, if $\mathbf{e} = [\mathbf{e}_1^T, \ldots, \mathbf{e}_n^T]^T$ is the innovation process of $\mathbf{y}$ such that e_i takes values in $\mathbb{R}^{n_i}$ then, for $i = 1, \ldots, n$

$$\mathbf{y}_i(t) = \sum_{j=1}^{n} P_{ij}(z)\mathbf{e}_j(t).$$

We say that the transfer matrix $P(z)$ *has G-zero structure* if for all $i, j = 1, \ldots, n$, $i \neq j$ the block transfer matrix $P_{ij}(z)$ is zero if and only if $(j, i) \notin E$. Roughly speaking, an (i, j)-block of the transfer matrix $P(z)$ is zero if the (j, i)-element of the adjacency matrix of G is zero.

Take a TADG graph $G = (V, E)$ with the set of nodes $V = \{1, 2, \ldots, n\}$ and define I_j as the index containing node j and its *parent nodes*, $I_j := \{j\} \cup \{i | \exists (i, j) \in E\}$. For simplicity, if we have a set $I = \{i_1, \ldots, i_k\}$ then we write $\mathbf{y}_I$ for the process $[\mathbf{y}_{i_1}^T, \ldots, \mathbf{y}_{i_k}^T]^T$.

The next theorem is the main result of this paper. It is a generalization of Theorem 1 for transfer matrices with TADG graph-zero structure.

Theorem 2. Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$ and a TADG graph $G = (V, E)$ where $V = \{1, \ldots, n\}$. The innovation transfer matrix $P(z)$ of $\mathbf{y}$ has a G-zero structure if and only if $\mathbf{y}_i$ conditionally does not Granger cause $\mathbf{y}_j$ with respect to $\mathbf{y}_{I_j}$ for every $(i, j) \notin E$.

Informally, Theorem 2 can be explained as follows: knowing which components help to predict which component in a process $\mathbf{y}$, is equivalent with knowing how the information flows between $\mathbf{y}$ and its innovation process. If the transfer matrix is not an innovation transfer matrix or its graph-zero structure is non-TADG then in general, the zero blocks of the transfer matrix do not define non-causal relations.

In the proof of Theorem 2 we rely on the property of TADG-zero structures that they are closed to multiplication and inversion.

Lemma 1. Consider a ZMSIR process $\mathbf{y} = [\mathbf{y}_1^T, \ldots, \mathbf{y}_n^T]^T$ and a TADG graph $G = (V, E)$, $V = \{1, \ldots, n\}$. If the innovation transfer matrix $P(z)$ of $\mathbf{y}$ has G-zero structure then the inverse transfer matrix $Q(z) := P^{-1}(z)$ has also G-zero structure.

Proof. We start with proving that if a transfer matrix $P(z)$ has G-zero structure then any power of $P(z)$ has G-zero structure. Take $(P^2(z))_{ij} = \sum_{k \in \{l \in V | (j,l),(l,i) \in E\}} P_{ik}(z) P_{kj}(z)$ and notice that from transitivity $(j, l), (l, i) \in E \Rightarrow (j, i) \in E$. It then follows that if $(j, i) \notin E$ then the set $\{l \in V | (j, l), (l, i) \in E\} = \emptyset$ and hence $(P^2(z))_{ij} = 0$. Consequently, $P^2(z)$ has G-zero structure. If we assume that $P^r(z)$ has G-zero structure for $r = 1, \ldots, m$ then it follows that $P^{m+1}(z) = P^m(z)P(z)$ is a product of two transfer matrices with G-zero structure thus $(P^{m+1}(z))_{ij} = \sum_{k \in \{l \in V | (j,l),(l,i) \in E\}} P_{ik}^m(z) P_{kj}(z)$ is zero if $(j, i) \notin E$. In this way we proved that $P^r(z)$ has G-zero structure for all $r \geq 1$. Since $P^{-1}(z)$ can be expressed as a polynomial of $P(z)$, and taking linear combinations preserves the G-zero structure, the statement follows. □

Proof. [Proof of Theorem 2] We start with the proof of sufficiency. From Lemma 1 we know that the inverse of the innovation transfer matrix $P(z) = \sum_{k=0}^{\infty} P_k z^{-k}$, denoted by $Q(z) = \sum_{k=0}^{\infty} Q_k z^{-k}$ has G-zero structure. Take the decomposition of $P(z)$ and $Q(z)$ as in (3). From the G-zero structure of $P(z)$ and $Q(z)$ we obtain that $P_{ji}(z) = Q_{ji}(z) = 0$ for all $(i, j) \notin E$, or equivalently $i \notin I_j$. Therefore, $\mathbf{y}_j(t)$ and $\mathbf{e}_j(t)$, $j \in \{1, \ldots, n\}$ can be written as follows

$$\begin{aligned} \mathbf{y}_j(t) &= \mathbf{e}_j(t) + \sum_{i \in I_j} \underbrace{\sum_{k=1}^{\infty} (P_k)_{ji} \mathbf{e}_i(t-k)}_{P_{ij}(z)\mathbf{e}_i} \\ \mathbf{e}_j(t) &= \mathbf{y}_j(t) + \sum_{i \in I_j} \underbrace{\sum_{k=1}^{\infty} (Q_k)_{ji} \mathbf{y}_i(t-k)}_{Q_{ij}(z)\mathbf{y}_i}. \end{aligned} \tag{4}$$

Note that since $\mathbf{e}$ is the innovation process of $\mathbf{y}$ it follows that $\mathcal{H}_{t-}^{\mathbf{y}} = \mathcal{H}_{t-}^{\mathbf{e}}$. Consider that $\mathbf{e}(t) \perp \mathcal{H}_{t-}^{\mathbf{y}}$ and $\mathbf{e}(t-k) \in \mathcal{H}_{t-}^{\mathbf{y}}$ for $k > 0$. Then, taking the projection of $\mathbf{y}_j(t)$ onto $\mathcal{H}_{t-}^{\mathbf{y}}$ we obtain that

$$E_l[\mathbf{y}_j(t) | \mathcal{H}_{t-}^{\mathbf{y}}] = \sum_{i \in I_j} \sum_{k=1}^{\infty} (P_k)_{ji} \mathbf{e}_i(t-k) \in \mathcal{H}_{t-}^{\mathbf{e}_{I_j}}.$$

We mention here that $\mathbf{e}_{I_j}(t) := \mathbf{y}_{I_j}(t) - E_l[\mathbf{y}_{I_j}(t) | \mathcal{H}_{t-}^{\mathbf{y}}]$ is not the innovation process of $\mathbf{y}_{I_j}$ unless (conditional) Granger causality conditions hold. Notice that from (4) we have that $\mathbf{e}_j(t) \in \mathcal{H}_{(t+1)-}^{\mathbf{y}_{I_j}}$. From transitivity, if $i \in I_j$ then $I_i \subseteq I_j$ and thus $\mathbf{e}_i(t-1) \in H_{t-}^{\mathbf{y}_{I_j}}$ for all $i \in I_j$. It leads to the following:

$$E_l[\mathbf{y}_j(t) | \mathcal{H}_{t-}^{\mathbf{y}}] \in \mathcal{H}_{t-}^{\mathbf{e}_{I_j}} \subseteq \mathcal{H}_{t-}^{\mathbf{y}_{I_j}}.$$

As a consequence, $E_l[\mathbf{y}_j(t) | \mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_j(t) | \mathcal{H}_{t-}^{\mathbf{y}_{I_j}}]$ for $j \in \{1, \ldots, n\}$. Likewise, the k-step prediction equals

$$E_l[\mathbf{y}_j(t+k) | \mathcal{H}_{t-}^{\mathbf{y}}] = \sum_{i \in I_j} \sum_{l=k+1}^{\infty} (P_l)_{ji} \mathbf{e}_i(t+k-l) \in \mathcal{H}_{t-}^{\mathbf{y}_{I_j}}.$$

It implies that $E_l[\mathbf{y}_j(t+k)|\mathcal{H}_{t-}^{\mathbf{y}}] = E_l[\mathbf{y}_j(t+k)|\mathcal{H}_{t-}^{\mathbf{y}_{I_j}}]$, thus that $\mathbf{y}_i$ conditionally does not Granger cause $\mathbf{y}_j$ with respect to $\mathbf{y}_{I_j}$ for $i \notin I_j$, or equivalently, for $(i,j) \notin E$.

It remains to see that if the conditional Granger non-causality conditions hold then the innovation transfer matrix has G-zero structure. From the conditions it follows that for any $(i,j) \notin E$, $E_l[\mathbf{y}_j(t)|H_{t-}^{\mathbf{y}_{I_j}}] = E[\mathbf{y}_j(t)|H_{t-}^{\mathbf{y}_{I_j},\mathbf{y}_i}]$. Since for every i, either $(i,j) \in E$ and hence $i \in I_j$, or $(i,j) \notin E$ and hence $i \notin I_j$ by transitivity, it then follows that $E_l[\mathbf{y}_j(t)|\mathcal{H}_{t-}^{\mathbf{y}_{I_j}}] = E_l[\mathbf{y}_j(t)|\mathcal{H}_{t-}^{\mathbf{y}}]$, thus $\mathbf{e}_{I_j}$ is the innovation process of $\mathbf{y}_{I_j}$. From $\mathbf{y}_j(t) = \mathbf{e}_j(t) + \sum_{i=1}^n \sum_{k=1}^\infty (P_k)_{ji}\mathbf{e}_i(t-k)$ we can deduce that

$$E_l[\mathbf{e}_j(t) + \sum_{i=1}^{n}\sum_{k=1}^{\infty}(P_k)_{ji}\mathbf{e}_i(t-k)|\mathcal{H}_{t-}^{\mathbf{y}}] =$$

$$\sum_{i=1}^{n}\sum_{k=1}^{\infty}(P_k)_{ji}\mathbf{e}_i(t-k) \in \mathcal{H}_{t-}^{\mathbf{y}_{I_j}} = \mathcal{H}_{t-}^{\mathbf{e}_{I_j}}.$$

Note that $\mathcal{H}_{t-}^{\mathbf{y}_{I_j}} = \mathcal{H}_{t-}^{\mathbf{e}_{I_j}}$ holds since $\mathbf{e}_{I_j}$ is the innovation process of $\mathbf{y}_{I_j}$. Considering that $\sum_{i\in I_j}\sum_{k=1}^{\infty}(P_k)_{ji}\mathbf{e}_i(t–k) \in \mathcal{H}_{t-}^{\mathbf{e}_{I_j}}$ it follows that $\sum_{i\notin I_j}\sum_{k=1}^{\infty}(P_k)_{ji}\mathbf{e}_i(t-k) \in \mathcal{H}_{t-}^{\mathbf{e}_{I_j}}$. Consequently, there exist $\{\alpha_k\}_{k=0}^{\infty}$ such that [4]

$$\sum_{i\notin I_j}\sum_{k=1}^{\infty}(P_k)_{ji}\mathbf{e}_i(t-k) = \sum_{k=1}^{\infty}\alpha_k \mathbf{e}_{I_j}(t-k). \tag{5}$$

Denote the complementary index set of I_j by $\bar{I}_j := \{1,\dots,n\} \setminus I_j$ and for $\bar{I}_j = \{i_1,\dots,i_m\}$ denote the transfer matrix $[(P_k)_{ji_1},\dots,(P_k)_{ji_m}]$ from $\mathbf{e}_{\bar{I}_j}$ to $\mathbf{y}_j$ by $(P_k)_{j\bar{I}_j}$. Notice that (5) can be written as

$$\sum_{k=1}^{\infty}\left[(P_k)_{j\bar{I}_j}, -\alpha_k\right]\mathbf{e}(t-k) = 0.$$

If we take the variance of the equation above and consider that $\mathbf{e}$ is a white noise process, we have that

$$\sum_{k=1}^{\infty}[(P_k)_{j\bar{I}_j}, -\alpha_k]E[\mathbf{e}(t)\mathbf{e}^T(t)][(P_k)_{j\bar{I}_j}, -\alpha_k]^T = 0.$$

Since $\mathbf{y}$ is a weakly stationary full rank process, the variance of its innovation process at any time $t \in \mathbb{Z}$, $E[\mathbf{e}(t)\mathbf{e}^T(t)]$ is strictly positive definite. It then follows that $\alpha_k = 0$ and $(P_k)_{j\bar{I}_j} = 0$ for all $k \geq 0$ which proofs the G-zero structure of P. □

Remark 1. Notice that if $\mathbf{y}_i$ is a root node in the TADG graph then none of the other components causes $\mathbf{y}_i$ (simple consequence of $I_i = \{i\}$). In this case the conditional Granger non-causality becomes Granger non-causality.

Remark 2. It is worth to mention that we started from a fixed partitioning of the output process and investigated the conditional non-causality structure between the chosen components. In fact, we can also choose the components of the output process for which the transfer matrices have TADG-zero structure. The more detailed partitioning we choose for the block transfer matrix, the more information we have about the causality structure.

[4] Considering that the one dimensional components of $\{E[\mathbf{e}_{I_j}(t)\mathbf{e}_{I_j}^T(t)]^{-1}\mathbf{e}_{I_j}(t-k)\}_{k=0}^{\infty}$ form an orthonormal basis for $\mathcal{H}_{t-}^{\mathbf{e}_{I_j}}$, the existence of $\{\alpha_k\}_{k=0}^{\infty}$ follows.

3.3 Example for non-TADG and TADG-zero structures

In this subsection we give an example for Theorem 2 and explain Remark 2 in more details. In Remark 2 we mentioned that the components of the process for which we observe the causality relations can be chosen in several ways. As an example, take the simplest non-transitive directed graph $G = (\{1,2,3\}, \{(1,2),(2,3)\})$. Suppose that the transfer matrices between a process $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$ and its innovation process $\mathbf{e}$ has G-zero structure as follows:

$$P(z) = \begin{bmatrix} P_{11}(z) & 0 & 0 \\ P_{21}(z) & P_{22}(z) & 0 \\ 0 & P_{43}(z) & P_{44}(z) \end{bmatrix}. \tag{6}$$

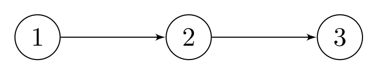
If $\mathbf{y}_2$ can be decomposed into $[\mathbf{y}_{21}^T, \mathbf{y}_{22}^T]^T$, it can happen that the transfer matrix above has a hidden TADG-zero structure for $[\mathbf{y}_1^T, \mathbf{y}_{21}^T, \mathbf{y}_{22}^T, \mathbf{y}_3^T]$ e.g.,

$$P(z) = \left[\begin{array}{c|cc|c} P_{11}(z) & 0 & 0 & 0 \\ \hline P_{21}(z) & P_{22}(z) & P_{23}(z) & 0 \\ 0 & 0 & P_{33}(z) & 0 \\ \hline 0 & 0 & P_{43}(z) & P_{44}(z) \end{array}\right] \tag{7}$$

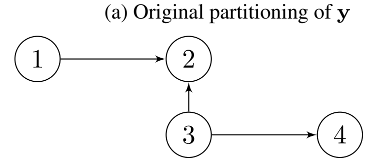
or that the transfer matrix has non-TADG structure (there might be for other partitioning) for $[\mathbf{y}_1^T, \mathbf{y}_{21}^T, \mathbf{y}_{22}^T, \mathbf{y}_3^T]$ e.g.,

$$P(z) = \left[\begin{array}{c|cc|c} P_{11}(z) & 0 & 0 & 0 \\ \hline P_{21}(z) & P_{22}(z) & 0 & 0 \\ 0 & P_{32}(z) & P_{33}(z) & 0 \\ \hline 0 & 0 & P_{43}(z) & P_{44}(z) \end{array}\right]. \tag{8}$$

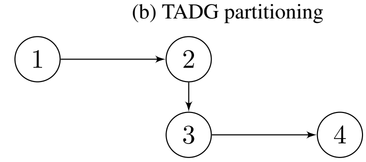
Fig 1a–1b and 1c illustrate the three possible zero-structures (6) – (7) and (8) of the innovation transfer matrix, respectively.



(a) Original partitioning of $\mathbf{y}$



(b) TADG partitioning



(c) non-TADG partitioning

Fig. 1. Graph-zero structures of the innovation transfer matrix

To illustrate the results of Theorem 2, we give an example for the case when there is a hidden TADG-zero structure for $[\mathbf{y}_1^T, \mathbf{y}_{21}^T, \mathbf{y}_{22}^T, \mathbf{y}_3^T]$. We define the following state space representation of a $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$ process ($\mathbf{y}_2 = [\mathbf{y}_{21}^T, \mathbf{y}_{22}^T]^T$):

$$\mathbf{x}(t+1) = \underbrace{\left[\begin{array}{c|cc|c} 0.7 & 0 & 0 & 0 \\ \hline 0 & 0.6 & 0 & 0 \\ 0 & 0 & 0.7 & 0 \\ \hline 0 & 0 & 0 & 0.8 \end{array}\right]}_{A}\mathbf{x}(t) + \underbrace{\left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{B}\mathbf{e}(t)$$

$$\mathbf{y}(t) = \underbrace{\left[\begin{array}{c|cc|c} 0.8 & 0 & 0 & 0 \\ \hline 0.3 & 0.5 & 0.4 & 0 \\ 0 & 0 & 0.5 & 0 \\ \hline 0 & 0 & 0.7 & 0.9 \end{array}\right]}_{C}\mathbf{x}(t) + \underbrace{\left[\begin{array}{c|cc|c} 1 & 0 & 0 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]}_{D}\mathbf{e}(t).$$

Calculating the innovation transfer matrix [5] of the process $\mathbf{y}$ as $P(z) = C(Iz - A)^{-1}B + D$ we obtain that

$$P(z) = \left[\begin{array}{c|cc|c} \frac{z+0.1}{z-0.7} & 0 & 0 & 0 \\ \hline \frac{0.3}{z-0.7} & \frac{z-0.1}{z-0.6} & \frac{0.4}{z-0.7} & 0 \\ 0 & 0 & \frac{z-0.2}{z-0.7} & 0 \\ \hline 0 & 0 & \frac{0.7}{z-0.7} & \frac{z+0.1}{z-0.8} \end{array}\right]$$

Notice that $P(z)$ has G-zero structure for the TADG graph $G = (\{1, 2, 3, 4\}, \{(1, 2), (3, 2), (3, 4)\})$ and the partitioning $[\mathbf{y}_1^T, \mathbf{y}_{12}^T, \mathbf{y}_{22}^T, \mathbf{y}_3^T]^T$ of $\mathbf{y}$. Note that from Lemma 1 we also know that the inverse transfer matrix $Q(z) = P(z)^{-1}$ has also G-zero structure. Indeed,

$$Q(z) = \left[\begin{array}{c|cc|c} \frac{z-0.7}{z+0.1} & 0 & 0 & 0 \\ \hline \frac{-0.3z+0.18}{z^2-0.01} & \frac{z-0.6}{z-0.1} & \frac{-0.4z+0.24}{z^2-0.3z+0.02} & 0 \\ 0 & 0 & \frac{z-0.7}{z-0.2} & 0 \\ \hline 0 & 0 & \frac{-0.7z+0.56}{z^2-0.1z-0.02} & \frac{z-0.8}{z+0.1} \end{array}\right].$$

From Theorem 2 the G-zero structure of $P(z)$ is equivalent with the following (conditional) Granger causality conditions:

(i) $[\mathbf{y}_{12}^T, \mathbf{y}_{22}^T, \mathbf{y}_3^T]^T$ does not Granger cause $\mathbf{y}_1$

(ii) $\mathbf{y}_3$ conditionally does not Granger cause $\mathbf{y}_{21}$ with respect to $[\mathbf{y}_1^T, \mathbf{y}_{22}^T]^T$

(iii) $\mathbf{y}_3$ does not Granger cause $\mathbf{y}_{22}$

(iv) $[\mathbf{y}_1^T, \mathbf{y}_{12}^T, \mathbf{y}_3^T]^T$ does not Granger cause $\mathbf{y}_{22}$

(v) $[\mathbf{y}_1^T, \mathbf{y}_{12}^T]^T$ conditionally does not Granger cause $\mathbf{y}_3$ with respect to $\mathbf{y}_{22}$.

From the conditions above we can also infer for the original partitioning of $\mathbf{y} = [\mathbf{y}_1^T, \mathbf{y}_2^T, \mathbf{y}_3^T]^T$. Accordingly, from (i) it follows that $[\mathbf{y}_2^T, \mathbf{y}_3^T]^T$ does not Granger cause $\mathbf{y}_1$; from (ii)-(iii) we can derive that $\mathbf{y}_3$ conditionally does not Granger cause $\mathbf{y}_2$ with respect to $\mathbf{y}_1$ and finally, from (v) we have that $\mathbf{y}_1$ conditionally does not Granger cause $\mathbf{y}_3$ with respect to $\mathbf{y}_2$. Note that without partitioning $\mathbf{y}_2$ into $[\mathbf{y}_{12}^T, \mathbf{y}_{22}^T]^T$, the two latter conditions do not follow from the zero-structure of the (3×3 block) transfer matrix. In addition, condition (i) and (iii) can also be seen from Theorem 1.

CONCLUSION

This paper formulated results on Granger non-causality and graph structures of transfer matrices. We showed that Granger non-causality and conditional Granger non-causality can appear in a forward innovation representation as a zero block of the transfer matrix. We introduced the class of transitive acyclic directed graphs (TADG) to which we restricted ourself in terms of zero structure of transfer matrices. We can conclude that if the zero structure of the innovation transfer matrix of a process has TADG-zero structure then it characterizes conditional Granger non-causalities among the components of the process. This opens up the possibility to test and identify conditional Granger non-causalities. Future work will be directed to including inputs, translating the results to state-space representations and to introducing quantitative characterizations of Granger causality.

[5] For minimum phase systems where $D = I$ the innovation transfer matrix is the transfer matrix.

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