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ON THE STRUCTURE OF A TRIANGLE-FREE INFINITE-CHROMATIC GRAPH OF GYARFAS

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ABSTRACT. Gyárfás has recently constructed an elegant new example of a triangle-free infinite graph G with infinite chromatic number. We analyze its structure by studying the properties of a nested family of subgraphs G_n whose union is G.

KEY WORDS AND PHRASES: Triangle-free, infinite-chromatic

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1. INTRODUCTION.

Gyárfás [1] described a new example of a triangle-free infinite-chromatic graph G as follows: the vertices of G form an $\infty \times \infty$ matrix, i.e., $V = \left\{v_{i,j},;\ i,\ j = 1,2,\ldots\right\}$, and the vertex $v_{i,j}$ is adjacent to every vertex of the $(i + j)$-th column, i.e., the set E of edges of G is given by $E = \left\{v_{i,j}v_{k,i+j};\ i,j,k = 1,2,\ldots\right\}$. It is easy to see that G is triangle-free for if u,v,w, were vertices of a triangle with u having the smallest column index, then the fact that uv and uw are edges would mean v and w are adjacent vertices in the same column, which is impossible. That G requires infinitely many colors follows from Theorem 2 below, although it also follows directly from the fact that, for $j > i$, the i-th column contains a vertex adjacent to all vertices of the j-th column.

In what follows we will accomplish two things. We first describe an augmenting sequence of finite graphs, which has G as its limit, and determine the structure of these graphs. This gives a deeper insight into the actual structure of G. Unless otherwise specified, we follow the graph theoretic notation and terminology of Harary [2].

2. AN ANALYSIS OF G.

In this section we will define a sequence G_n of graphs converging to G. We will give some results on the structure of each G_n, and an alternative way of constructing G_n which gives a different perspective on its structure. Finally, we will note that the desired properties of G_n can be demonstrated by considering subgraphs H_n (which are roughly half of G_n).

DEFINITION. For any positive integer n, let G_n be the subgraph of G obtained by removing all vertices $v_{i,j}$ with i and j greater than n, i.e., G_n is the induced subgraph $\left\langle\left\{v_{i,j};\ 1 \le i,j \le n\right\}\right\rangle$ of G.

First we prove a theorem on the degrees of the vertices of G_n.

THEOREM 1. For $0 \le k \le 2n - 2$, the number of vertices of G_n of degree k is $n - |n - k - 1|$, while there are no vertices of degree greater than $2n - 2$.

PROOF. Consider G_n as an $n \times n$ matrix. Then it is easy to see that

$$\deg(v_{i,j}) = \begin{cases} n + j - 1, & \text{if } 1 \le i \le n - j \\ \\ j - 1, & \text{if } n - j + 1 \le i \le n. \end{cases}$$

Thus by setting $k = n + j - 1$ or $k = j - 1$, we see there are either $2n - k - 1$ or $k + 1$ vertices, respectively, of order k, as claimed.

In the next theorem we give the chromatic number $\chi(G_n)$ of each G_n.

THEOREM 2. For $k \geq 1$, G_n is k-colorable if $n < 2^k$, while G_{2^k} has chromatic number $k + 1$, that is, $\chi(G_n) = 1 + [\log_2 n]$.

PROOF. Since G_{n-1} is a subgraph of G_n, it suffices to show that G_{2^k-1} is k colorable whereas G_{2^k} is not. To show the latter, suppose on the contrary that G_{2^k} is colored in k colors, and let N_j denote the set of colors used on the vertices in the j-th column of G_{2^k}. Now for $i < j$, $v_{j-i,i}$ is adjacent to every vertex in column j so $N_i \not\subseteq N_j$. The sets N_j thus form a collection of 2^k distinct nonempty subsets of a k-element set, which is impossible.

To show that G_{2^k-1} is k-colorable, let C be a set of k colors and let N_j, $j = 1,2,\ldots,2^k-1$, be an enumeration of the nonempty subsets of C which is non-increasing in order of size. For example, such an enumeration when $k = 3$ and $C = \{c_1,c_2,c_3\}$ is: $\{c_1,c_2,c_3\}$, $\{c_1,c_2\}$, $\{c_1,c_3\}$, $\{c_2,c_3\}$, $\{c_1\}$, $\{c_2\}$ $\{c_3\}$. This enumeration provides that if $j > i$, then there is a color in N_i which is not in N_j. Therefore, color the vertex $v_{r,i}$ with a color in N_i which is not in N_{r+i}; if $r + i > 2^k - 1$, then use any color in N_i. This clearly yields a k-coloring of G_{2^k-1}.

To conclude this section, we describe an alternative way to construct G_n which we feel gives some insight into its structure and chromatic number. In accordance with established terminology, we will say that a point covers a set S of points if it is adjacent to every point of S. A set T of points smothers S if exactly one point in T covers S, and T smothers a finite sequence $S_1, S_2, \ldots, S_k$ of sets of points if there are distinct points $t_1, t_2,\ldots,t_k$ in T such that t_j covers S_j for $j = 1,2,\ldots,k$.

We now describe how to construct G_n using this idea and the join operation +, where H + H' is the graph obtained from the union of H and H' by joining every point of H to every point of H'; see [2, p. 21]. We describe how to build G_n in three stages. Here the notation H + H' + H" stands for the union of two joins H + H' and H' + H", and similarly for more summands each of which will be a complete graph K_n or its complement, the totally disconnected graph $\overline{K}_n$.

STAGE 1: Build $\overline{K}_2 + K_1 + \overline{K}_{n-2}$. (Label the vertex K_1 by r.)

STAGE 2: Replace the j-th point in $\overline{K}_{n-2}$, numbering from bottom to top, by $S_j = K_1 + \overline{K}_n + K_1$, for $j = 1,2,\ldots,n-2$. (Label the left K_1 by a_j and the right K_1 by b_j.)

STAGE 3: Replace the $\overline{K}_n$ in S_j by the set of n points T^j where the adjacency in S_j is preserved, but T_n^j smothers $T_n^1, T_n^2, \ldots, T_n^{j-1}$, for $j = 1,2,\ldots,n-2$. (Suppose that t_{jk} in T_n^j covers T_n^{j-k} for $k = 1,2,\ldots, j-1$.)

Figure 1 shows how the construction progresses when n = 5. This resulting graph, when a single isolated point (corresponding to $v_{n,1}$) is added, is isomorphic to G_n. We will not formally prove this, though it is easy to see that an isomorphism is obtained by mapping r to $v_{1,1}$, b_j to $v_{n-j,1}$, a_j to $v_{n-1-j,2}$ (and the two vertices of degree 1 to $v_{i,2}$, $i = n-1,n$), and T_n^j to the vertices in column n+1-j with $t_{j,k}$ mapping to $v_{k,n+1-j}$. It is also easy to see that another minimal coloring (besides the one given in the proof of Theorem 2) is obtained by only using colors $c_1, c_2, \ldots, c_i$ to color T_n^j for $i \leq j < 2^i$ and $i = 1,2,\ldots,k$.

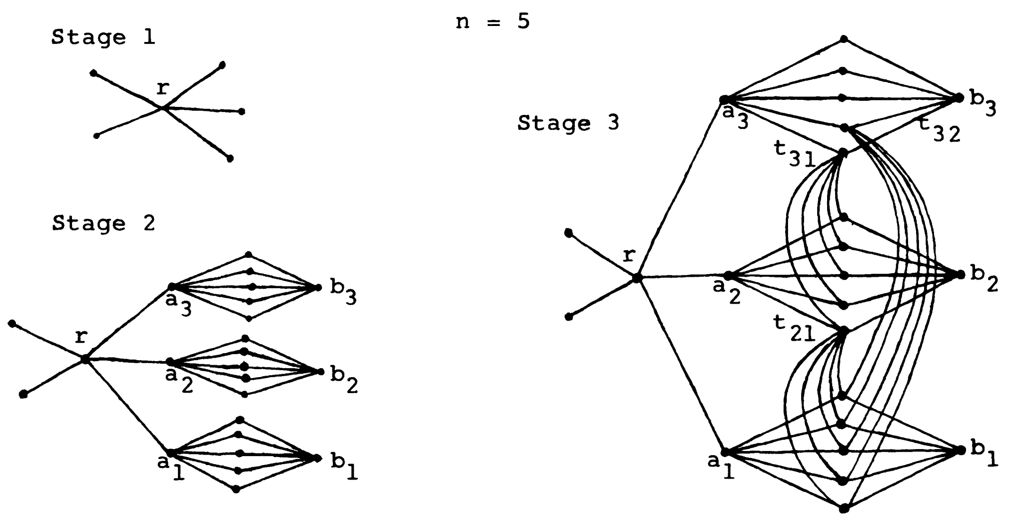

Figure 1.

Finally, let H_n be the subgraph of G_n with the $\frac{1}{2}(n^2-3n+6)$ vertices

$$\{v_{1,1}, v_{1,n}\} \cup \{v_{i,j};\ j = 2,\ldots,n-1,\ i = 1,\ldots,n-j\}$$

It is clear that H_n is triangle-free and for $n = 2^k$ applying the argument in the proof of Theorem 2 to columns 2 through n shows that H_n is not k colorable. Although H_n is simpler then G_n while still retaining the cascading appearance illustrated by Figure 1, H_n is still not critical.

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