

Variational regularisation for inverse problems with imperfect forward operators and general noise models

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Received 12 June 2020, revised 23 October 2020

Accepted for publication 27 October 2020

Published 30 November 2020



CrossMark

Abstract

We study variational regularisation methods for inverse problems with imperfect forward operators whose errors can be modelled by order intervals in a partial order of a Banach lattice. We carry out analysis with respect to existence and convex duality for general data fidelity terms and regularisation functionals. Both for *a priori* and *a posteriori* parameter choice rules, we obtain convergence rates of the regularised solutions in terms of Bregman distances. Our results apply to fidelity terms such as Wasserstein distances, φ -divergences, norms, as well as sums and infimal convolutions of those.

Keywords: imperfect forward models, f -divergences, Kullback–Leibler divergence, Wasserstein distances, Bregman distances, discrepancy principle, Banach lattices

1. Introduction

We consider linear inverse problems

$$Au = \bar{f}, \quad (1.1)$$

where $A : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear bounded operator (referred to as the forward operator or the forward model) acting between two Banach spaces \mathcal{X} and \mathcal{Y} . The exact measurement \bar{f} is typically

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not available and only a noisy version of it f_δ is known along with an estimate of the noise level δ . Since the inversion of (1.1) is often unstable with respect to noise and hence ill-posed, it requires regularisation. Variational regularisation replaces solving (1.1) by the following optimisation problem

$$\min_{u \in \mathcal{X}} \frac{1}{\alpha} \mathcal{H}(Au|f_\delta) + \mathcal{J}(u), \quad (1.2)$$

where $\mathcal{H}(\cdot|f)$ is a so-called *data fidelity* function that models statistical properties of the noise in f and $\mathcal{J}(\cdot)$ is a *regularisation* functional that stabilises the inversion. The *regularisation parameter* $\alpha > 0$ balances the influence of the data fidelity and the regularisation. The amount of noise δ in the measurement f_δ is assumed to be such that

$$\mathcal{H}(\bar{f}|f_\delta) \leq \delta. \quad (1.3)$$

The fidelity function often depends only on the difference of the arguments, i.e. $\mathcal{H}(v|f) = h(v - f)$ for some function h . The most common example is $\mathcal{H}(v|f) = \frac{1}{2} \|v - f\|^2$. There are, however, cases when the fidelity function depends on its arguments in a more complicated manner; an example is the Kullback–Leibler divergence that is used to model Poisson noise [1], where $\mathcal{H}(v|f) = \int (v \log \frac{v}{f} - (v - f)) dx$ (see also the review paper [2]). Problems with general fidelity functions were analysed in [3, 4].

To guarantee convergence of the minimisers of (1.2) to a solution of (1.1) as the noise level δ decreases, the regularisation parameter α needs to be chosen as a function of the measurement noise $\alpha = \alpha(\delta)$ (*a priori* parameter choices) or of the measurement itself and of measurement noise $\alpha = \alpha(f_\delta, \delta)$ (*a posteriori* parameter choices). For *a priori* parameter choice rules, convergence rates for solutions of (1.2) in different scenarios have been obtained, e.g., in [5–9]. A classical *a posteriori* parameter choice rule is the so-called discrepancy principle, originally introduced in [10] and later studied in, e.g., [11–13]. Roughly speaking, it consists in choosing $\alpha = \alpha(f_\delta, \delta)$ such that the following equation is satisfied

$$\mathcal{H}(Au^\alpha|f_\delta) = \delta,$$

where u^α is the solution of (1.2) corresponding to the regularisation parameter α .

In many applications, not only the measurement f_δ is noisy, but also the forward operator A that generated the data is not precisely known. Errors in the operator may come from the uncertainty in some model-related parameters such as the point-spread function of a microscope, simplified model geometry and/or discretisation. A classical approach to modelling errors in the forward operator assumes an error estimate in the operator norm, i.e.

$$\|A_h - A\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq h, \quad (1.4)$$

where $A_h : \mathcal{X} \rightarrow \mathcal{Y}$ is a linear bounded operator that we have numerical access to and $h \geq 0$ describes the approximation error (e.g., [14–17]). To guarantee convergence in this setting, the parameter α needs to be chosen as the function of δ and h (*a priori* choice rules) or of δ , h , f_δ and A_h (*a posteriori* choice rules). Generalisations of the discrepancy principle to this setting are available [18–20], but they usually rely on a triangle inequality that $\mathcal{H}(\cdot|f)$ needs to satisfy.

An alternative approach to modelling operator errors using order intervals in Banach lattices was proposed in [21–23]. It assumes that the spaces \mathcal{X} and \mathcal{Y} have a lattice structure [24] and, instead of (1.4), lower and upper bounds for the operator are available

$$A^l \leq A \leq A^u, \quad (1.5)$$

where the inequalities are understood in the sense of a partial order for linear operators, i.e.

$$A^1 u \leq Au \leq A^u u \quad \text{for all } u \geq 0. \quad (1.6)$$

The inequalities in (1.6) are understood in an abstract sense of a Banach lattice; which for L^p spaces means inequality almost everywhere. In order for the partial order bounds (1.5) to be well-defined, we assume that $A : \mathcal{X} \rightarrow \mathcal{Y}$ is a regular operator [24], i.e. that it can be written as a difference of two positive operators, $A = A_1 - A_2$, where for any $u \geq 0$ it holds that $A_{1,2}u \geq 0$. Some examples of regular operators will be given later.

The approach (1.6) to describing errors in the forward operator was studied in the context of the residual method in the case $\mathcal{Y} = L^\infty$ when the data fidelity is a characteristic function of a norm ball

$$\mathcal{H}(\cdot|f_\delta) = \chi_{\|\cdot - f\|_\infty \leq \delta}. \quad (1.7)$$

In this case, one solves the following problem

$$\min_u \mathcal{J}(u) \quad \text{s.t. } A^1 u \leq f^u, \quad A^u u \geq f^1, \quad (1.8)$$

where $f^1 := f_\delta - \delta \mathbb{1}$ and $f^u := f_\delta + \delta \mathbb{1}$ are pointwise (a.e.) lower and upper bounds for the exact data \bar{f} in (1.1) such that $f^1 \leq \bar{f} \leq f^u$ and $\mathbb{1}$ is the constant one-function. For comparison, with the data term (1.7) and without an operator error, (1.2) translates into

$$\min_u \mathcal{J}(u) \quad \text{s.t. } f^1 \leq Au \leq f^u, \quad (1.9)$$

where the constraint is equivalent to $\|Au - f_\delta\|_\infty \leq \delta$. (In [25], a connection is made between the lower and upper bounds f^1, f^u and confidence intervals.)

One can show that the partial order based condition (1.5) implies the norm based condition (1.4). Indeed, given A^1, A^u as in (1.5), one defines

$$A_h := \frac{A^u + A^1}{2}, \quad h := \frac{\|A^u - A^1\|}{2}.$$

It can be readily verified that the so defined A_h satisfies (1.4). The opposite implication is, in general, wrong. Hence, if an estimate (1.5) is available, it allows one to describe the operator error more precisely and one may expect better reconstructions. Indeed, it was found in [23] that solving (1.2) with $\mathcal{H}(Au|f_\delta) = \|Au - f_\delta\|_\infty$ and α chosen according to a generalised discrepancy principle [18] based on (1.4) produces overregularised solutions compared to (1.8), i.e. the generalised discrepancy principle tends to overestimate the regularisation parameter. One of the reasons for this is the use of the triangle inequality to account for (1.4), which makes the estimates not sharp, in general.

The motivation for this paper is two-fold. First, we want to extend the approach (1.5) and (1.8) to a broader class of fidelity terms than the characteristic function of a ball and more general data spaces than L^∞ . We also aim at a unified analysis of problems with fidelities that do not satisfy a triangle-type inequality, which is interesting in its own right. Our proofs mostly rely on convex analysis and duality.

Setup. We consider the inverse problem (1.1), where $\mathcal{X} = \mathcal{U}^*$ and $\mathcal{Y} = \mathcal{V}^*$ are duals of Banach lattices \mathcal{U} and \mathcal{V} , respectively. We assume that the partial order on \mathcal{Y} is induced by the partial order in \mathcal{V} as follows: $y \geq 0 \iff \langle y, v \rangle \geq 0 \quad \forall v \in \mathcal{V}, \quad v \geq 0$ (cf lemma A.4 in the appendix).

Furthermore, we assume that (1.1) possesses a non-negative \mathcal{J} -minimising solution $u_{\mathcal{J}}^{\dagger}$, i.e.

$$Au_{\mathcal{J}}^{\dagger} = \bar{f}, \quad u_{\mathcal{J}}^{\dagger} \geq 0 \quad \text{and} \quad \mathcal{J}(u_{\mathcal{J}}^{\dagger}) \leq \mathcal{J}(u) \quad \text{for all } u \text{ such that } Au = \bar{f}. \quad (1.10)$$

We propose the following extension of (1.2) to the case when the forward operator is known only up to the order interval given in (1.5)

$$\min_{\substack{u \in \mathcal{X} \\ v \in \mathcal{Y}}} \frac{1}{\alpha} \mathcal{H}(v|f_{\delta}) + \mathcal{J}(u) \quad \text{s.t. } A^{\dagger}u \leq v \leq A^u u, \quad (1.11)$$

where $\mathcal{J} : \mathcal{X} \rightarrow \overline{\mathbb{R}}_+$ and $\mathcal{H}(\cdot|f) : \mathcal{Y} \rightarrow \overline{\mathbb{R}}_+$ (as a function of its first argument) are assumed proper, convex and weakly-* lower semicontinuous (cf assumption 1).

Main contribution. In this work we study convergence of solutions of (1.11) to a \mathcal{J} -minimising solution of (1.1) as the noise in data and operators decreases, and obtain convergence rates in one-sided Bregman distances with respect to \mathcal{J} . We also give conditions when (1.11) admits strong duality, in which case the convergence rates translate to symmetric Bregman distances. Furthermore, we analyse an *a posteriori* parameter choice rule based on a discrepancy principle for (1.11).

Our results apply inter alia to general φ -divergences, as for instance the Kullback–Leibler divergence, and coercive fidelities such as powers of norms or Wasserstein distances from optimal transport. In addition, we also obtain rates for sums and infimal convolutions of different fidelities, as used for instance in mixed-noise removal. Even for exact operators, our analysis goes beyond the state of the art in problems with fidelity terms that lack a triangle-type inequality.

Structure of the paper. In section 2 we study existence of solutions of the problem (1.11) and its dual and establish sufficient conditions for strong duality. In section 3 we derive convergence rates for *a priori* parameter choice rules. In section 4 we formulate a discrepancy principle for the problem (1.11) and also obtain convergence rates. For readers' convenience, we present some background material on Banach lattices in the appendix.

1.1. Examples of regular operators

Below, we give some examples of regular operators and discuss how lower and upper bounds in the sense of (1.5)–(1.6) can be obtained.

Example 1.1. If \mathcal{Y} is an abstract maximum space (a generalisation of L^{∞}) or if \mathcal{X} is an abstract Lebesgue space (a generalisation of L^1) then all linear bounded operators are regular, i.e. they can be written as a difference of two positive operators. More details can be found in the appendix.

Example 1.2 (Integral operators—perturbations of the kernel). Let $A : L^p(\Omega) \rightarrow L^q(\Omega)$ ($\Omega \subset \mathbb{R}^d$ bounded, $p, q \geq 1$) be an integral operator with a (p, q) -bounded kernel k [26],

$$Au(x) := \int_{\Omega} k(x, \xi)u(\xi) \, d\xi. \quad (1.12)$$

The operator A can be written as

$$A = A_+ - A_-, \quad A_{\pm} := \int_{\Omega} k(x, \xi)_{\pm} u(\xi) \, d\xi,$$

where k_+ and k_- are the positive and the negative parts of k (in the a.e. sense in $\Omega \times \Omega$). Clearly, A_{\pm} are positive and A is regular.

Suppose that the kernel is corrupted by an unknown (p, q) -bounded perturbation such that we only know pointwise lower and upper bounds for k ,

$$k^l(x, \xi) \leq k(x, \xi) \leq k^u(x, \xi) \quad \text{a.e. in } \Omega \times \Omega. \quad (1.13)$$

Then lower and upper operators in the sense of (1.5) are given by

$$A^l u(x) := \int_{\Omega} k^l(x, \xi) u(\xi) \, d\xi, \quad A^u u(x) := \int_{\Omega} k^u(x, \xi) u(\xi) \, d\xi.$$

It should be noted that the bounds (1.13) are of a deterministic nature. They could arise, for example, if the kernel depends on additional parameters $\theta \in \Theta$, i.e. $k(x, \xi) = k_{\theta}(x, \xi)$. If reconstructing the unknown parameter θ is not of independent interest, the dependence on it can be eliminated by defining

$$k^l(x, \xi) := \inf_{\theta \in \Theta} k_{\theta}(x, \xi), \quad k^u(x, \xi) := \sup_{\theta \in \Theta} k_{\theta}(x, \xi),$$

provided the suprema and infima are finite for a.e. x, ξ and $k^{l,u}$ are (p, q) -bounded.

Example 1.3 (Integral operators—discretisation). Let the operator A be as defined in example 1.2 on an interval $\Omega \subset \mathbb{R}$ and consider its approximation by Riemann sums. In particular, let $S_n^l(x)$ and $S_n^u(x)$ denote the lower and upper Riemann sums in (1.12) obtained using an n -point discretisation. Then these sums define lower and upper operators in the sense of (1.5),

$$A_n^l u(x) := S_n^l(x), \quad A_n^u u(x) := S_n^u(x).$$

As we refine the discretisation (i.e. $n \rightarrow \infty$), these bounds converge pointwise to $Au(x)$.

Example 1.4 (Integration with respect to a vector-valued measure). Example 1.2 can be generalised as follows. Let $\mu \in \mathcal{M}(\Omega; Y)$ be a vector-valued Radon measure [27], where Ω is a compact metric space and Y is a Banach lattice with the Radon–Nikodým property. Define partial order on $\mathcal{M}(\Omega; Y)$ as follows

$$\mu \geq_{\mathcal{M}} 0 \iff \mu(E) \geq_Y 0 \quad \text{for all } \mu\text{-measurable subsets } E \subset \Omega. \quad (1.14)$$

Let $A : \mathcal{C}(\Omega) \rightarrow Y$ be defined as follows

$$Au := \int_{\Omega} u \, d\mu.$$

Since Y is a lattice, it is clear that A is regular. Lower and upper bounds $\mu^l \leq_{\mathcal{M}} \mu, \mu \leq_{\mathcal{M}} \mu^u$ in the sense of (1.14) define lower and upper operators $A^{l,u}$ in the sense of (1.5).

Example 1.5 (1D source identification). We consider the operator $A : \mathcal{M}([0, 1]) \rightarrow \mathcal{C}([0, 1])$, $A : u \mapsto \varphi$, where φ solves

$$\begin{cases} -(a\varphi)' = u, & \text{on } (0, 1), \\ \varphi(0) = 0, \\ \varphi'(0) = 0. \end{cases}$$

Here $a : [0, 1] \rightarrow \mathbb{R}$ is a continuous function which meets $a \geq a_0 > 0$ on $[0, 1]$ and $u \in \mathcal{M}([0, 1])$ is a Radon measure with integrable antiderivative $U(x) := \int_0^x du$. Integrating the equation yields

$$Au(x) = \varphi(x) = - \int_0^x \frac{U(y)}{a(y)} dy.$$

Clearly, $A \leq 0$ and hence regular. Hence, if $\underline{a}, \bar{a} : [0, 1] \rightarrow \mathbb{R}$ are continuous functions such that $\underline{a} \leq a \leq \bar{a}$ on $[0, 1]$ and $\underline{a} \geq a_0 > 0$ on $[0, 1]$, we can define operators

$$\begin{aligned} A^l u(x) &= - \int_0^x \frac{U(y)}{\underline{a}(y)} dy, \\ A^u u(x) &= - \int_0^x \frac{U(y)}{\bar{a}(y)} dy, \end{aligned}$$

which meet $A^l u \leq Au \leq A^u u$ for $u \geq 0$ (and hence $U \geq 0$). If $\|\bar{a} - \underline{a}\|_C \rightarrow 0$, then $A^{l,u}$ converge to A in the operator norm.

If one defines the operator A on $L^1((0, 1))$ instead of $\mathcal{M}([0, 1])$, the antiderivative U is continuous and one can approximate the integrals in A^l and A^u with lower and upper Riemann sums, respectively. This gives rise to operators A_n^l and A_n^u such that $A_n^l \leq A^l \leq A \leq A^u \leq A_n^u$. If then additionally $n \rightarrow \infty$, the operators $A_n^{l,u}$ converge to A .

Note that a similar approach can be used for estimating the diffusivity a for a given source term. In this case, however, the forward operator A becomes non-linear. This would require an extension of our theory.

Example 1.6 (Conditional expectations). Let Ω be a separable metric space and (Ω, Σ, μ) be a probability space. Let $B \subset \Sigma$ be a sub- σ -algebra of Σ and let $\{E_i\}_{i=1}^\infty$ be its minimal generator (which exists, since Ω is separable). The *conditional expectation operator* $A : L_\mu^p(\Omega) \rightarrow L_\mu^p(\Omega)$ is defined as follows

$$Au := \sum_{i=1}^{\infty} \frac{\int_{E_i} u d\mu}{\mu(E_i)} \chi_{E_i},$$

under the convention $0/0 = 0$. Clearly, $A \geq 0$ and hence regular.

If we allow μ to be a finite signed measure, then we can generalise the definition as follows

$$Au := \sum_{i=1}^{\infty} \frac{\int_{E_i} u d\mu}{|\mu|(E_i)} \chi_{E_i},$$

where $|\mu|$ is the total variation of μ . Clearly,

$$A = A_+ - A_-, \quad A_\pm u := \sum_{i=1}^{\infty} \frac{\int_{E_i} u d\mu_\pm}{|\mu|(E_i)} \chi_{E_i}$$

and $A_\pm \geq 0$, hence A is regular. In contrast to example 1.4, partial order bounds on μ in the sense of (1.14) do not translate into lower and upper bounds (1.6) for A since A is not an integral operator (in particular, it is not linear in μ).

2. Primal and dual problems

In this section we establish existence of solutions to (1.11) using the direct method, where standard assumptions on the forward operators, the regularisation, and fidelity function will guarantee coercivity and lower semicontinuity. Subsequently, we derive the dual maximisation problem and prove existence and strong duality under the additional assumption that the data space \mathcal{Y} is an abstract maximum space.

2.1. Existence of a primal solution

We make the following standard assumptions on the regularisation functional \mathcal{J} , the fidelity function \mathcal{H} , and the operators $A^{1,u}$.

Assumption 1. The regularisation functional $\mathcal{J} : \mathcal{X} \rightarrow \overline{\mathbb{R}}_+$ is

- Proper, convex and weakly-* lower semicontinuous;
- Its non-empty sublevel sets $\{u \in \mathcal{X} : \mathcal{J}(u) \leq C\}$ are weakly-* sequentially compact.

The fidelity function $\mathcal{H}(\cdot|\cdot) : \mathcal{Y} \times \mathcal{Y} \rightarrow \overline{\mathbb{R}}_+$ is

- Proper, convex in its first argument and weakly-* lower semicontinuous jointly in both arguments;
- $\mathcal{H}(v|f) = 0$ if and only if $v = f$.

Assumption 2. The operators $A, A^{1,u} : \mathcal{X} \rightarrow \mathcal{Y}$ are weak-* to weak-* continuous.

A sufficient condition for assumption 2 to hold is given in lemma A.5 in the appendix.

Theorem 2.1. *Suppose that assumptions 1 and 2 hold true. Then (1.11) has a solution.*

Proof. Consider a minimising sequence (u_k, v_k) . Due to assumption 1 there exists a convergent subsequence u_k (that we don't relabel) such that

$$u_k, \rightharpoonup^* u_\infty.$$

Then assumption 2 yields

$$A^{1,u} u_k, \rightharpoonup^* A^{1,u} u_\infty.$$

From (1.11) we get that for all k

$$0 \leq v_k - A^1 u_k \leq (A^u - A^1) u_k,$$

hence

$$\|v_k - A^1 u_k\| \leq \|(A^u - A^1) u_k\|$$

and

$$\|v_k\| \leq \|A^1 u_k\| + \|(A^u - A^1) u_k\| \leq C,$$

since weakly-* convergent sequences are bounded.

Since \mathcal{Y} is a dual of a separable Banach space \mathcal{V} , by the sequential Banach–Alaoglu theorem the sequence v^k contains a weakly-* convergent subsequence v_k (that we do not relabel) such that

$$v_k, \rightharpoonup^* v_\infty.$$

Since both $A^{1,u}u_k$ and v_k converge weakly-* and order intervals in \mathcal{Y} are weakly-* closed due to lemma A.4, we obtain that

$$A^1u_\infty \leq v_\infty \leq A^u u_\infty.$$

Hence (u_∞, v_∞) is feasible for (1.11). Furthermore, since $\mathcal{J}(\cdot)$ and $\mathcal{H}(\cdot|f)$ are weakly-* lower semicontinuous, we get that

$$\begin{aligned} \frac{1}{\alpha} \mathcal{H}(v_\infty|f) + \mathcal{J}(u_\infty) &\leq \liminf_{k \rightarrow \infty} \frac{1}{\alpha} \mathcal{H}(v_k|f) + \mathcal{J}(u_k) \\ &= \inf_{\substack{u \in \mathcal{X} \\ v \in \mathcal{Y} \\ A^1u \leq v \leq A^u u}} \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u). \end{aligned}$$

Therefore, (u_∞, v_∞) is a solution of (1.11). \square

2.2. Dual problem

To simplify our notation, we introduce an operator $B : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{Y}$

$$Bu = \begin{pmatrix} A^1u \\ -A^u u \end{pmatrix} \quad (2.1)$$

and an operator $E : \mathcal{Y} \rightarrow \mathcal{Y} \times \mathcal{Y}$

$$Ev = \begin{pmatrix} v \\ -v \end{pmatrix}. \quad (2.2)$$

With this notation we can rewrite (1.11) as follows

$$\min_{\substack{u \in \mathcal{X} \\ v \in \mathcal{Y} \\ Bu \leq Ev}} \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u). \quad (2.3)$$

Proposition 2.2. *The (Lagrangian) dual problem of (2.3) is given by*

$$\sup_{\substack{\mu \in \mathcal{Y}^* \times \mathcal{Y}^* \\ \mu \geq 0}} -\frac{1}{\alpha} \mathcal{H}^*(\alpha E^* \mu|f) - \mathcal{J}^*(-B^* \mu). \quad (2.4)$$

Proof. The Lagrangian of (2.3) is given by

$$\mathcal{L}(u, v, \mu) = \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u) + \langle \mu, Bu - Ev \rangle,$$

where $\mu \in \mathcal{Y}^* \times \mathcal{Y}^*$, $\mu \geq 0$. Minimising the Lagrangian in u and v , we obtain

$$\begin{aligned} \inf_{u,v} \mathcal{L}(u, v, \mu) &= \inf_{u,v} \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u) + \langle \mu, Bu - Ev \rangle \\ &= \inf_u [\mathcal{J}(u) - \langle -B^* \mu, u \rangle] + \frac{1}{\alpha} \inf_v [\mathcal{H}(v|f) - \langle \alpha E^* \mu, v \rangle] \\ &= -\mathcal{J}^*(-B^* \mu) - \frac{1}{\alpha} \mathcal{H}^*(\alpha E^* \mu|f). \end{aligned}$$

Taking a supremum over $\mu \geq 0$ gives (2.4). \square

It is well known (e.g., [28]) that

$$\inf_{\substack{u \in \mathcal{X} \\ v \in \mathcal{Y} \\ Bu \leq Ev}} \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u) \geq \sup_{\substack{\mu \in \mathcal{Y}^* \times \mathcal{Y}^* \\ \mu \geq 0}} -\frac{1}{\alpha} \mathcal{H}^*(\alpha E^* \mu|f) - \mathcal{J}^*(-B^* \mu),$$

which is referred to as *weak duality*.

Remark 2.3. If the fidelity function depends only on the difference of its arguments, i.e. $\mathcal{H}(\cdot|f) = h(\cdot - f)$, then

$$\mathcal{H}^*(\alpha E^* \mu|f) = h^*(\alpha E^* \mu) + (\alpha E^* \mu, f)$$

and problem (2.4) becomes

$$\sup_{\mu \geq 0} -\frac{1}{\alpha} h^*(\alpha E^* \mu) - (\alpha E^* \mu, f) - \mathcal{J}^*(-B^* \mu). \quad (2.5)$$

If $h(\cdot) = \frac{1}{2} \|\cdot\|_{\mathcal{Y}}^2$, we have $h^*(\cdot) = \frac{1}{2} \|\cdot\|_{\mathcal{Y}^*}^2$ and hence we obtain the standard form (e.g., [29])

$$\begin{aligned} \sup_{\mu \geq 0} -\frac{\alpha}{2} \|E^* \mu\|_{\mathcal{Y}^*}^2 - (E^* \mu, f) - \mathcal{J}^*(-B^* \mu) \\ = -\inf_{\mu \geq 0} \frac{\alpha}{2} \|E^* \mu\|_{\mathcal{Y}^*}^2 + (E^* \mu, f) + \mathcal{J}^*(-B^* \mu). \end{aligned}$$

2.3. Existence of a dual solution and strong duality

The goal of this section is to study the relationship between the primal problem (2.3) and its dual (2.4), establishing strong duality and existence of a dual solution, and obtaining complementarity conditions for Lagrange multipliers associated with constraints in (2.3).

We will need the following result from [28, theorem 2.165].

Theorem 2.4 ([28]). Consider the following optimisation problem

$$\inf_{x \in X} g(x) \quad \text{s.t. } Lx \in K, \quad (\mathcal{P})$$

and its dual

$$\sup_{y^* \in Y^*} -\chi_K^*(y^*) - g^*(-L^* y^*), \quad (\mathcal{D})$$

where X and Y are Banach spaces, $L: X \rightarrow Y$ is a linear bounded operator, L^* its adjoint, $K \subset Y$ a closed convex set, and $g: X \rightarrow \mathbb{R}$ a proper convex lower semicontinuous function with convex conjugate $g^*: X^* \rightarrow \mathbb{R}$. The characteristic function of K is denoted by $\chi_K(\cdot)$ and its convex conjugate (i.e. the support function of K) by $\chi_K^*(\cdot)$. Suppose that the following regularity condition is satisfied

$$0 \in \text{int}(L(\text{dom } g) - K). \quad (2.6)$$

Then there is no duality gap between problems (\mathcal{P}) and (\mathcal{D}) . If the optimal value of (\mathcal{P}) is finite, then the dual problem (\mathcal{D}) has at least one solution $\bar{y}^* \in Y^*$.

The regularity condition (2.6) is due to Robinson [30] and plays an important role in the stability of optimisation problems under perturbations of the feasible set [28].

To ensure that (2.6) is satisfied in the primal problem (2.3), we will need to assume that the positive cone in \mathcal{Y} has a non-empty interior. This naturally leads to the concept of abstract maximum spaces [24] which are a generalisation of $L^\infty(\Omega)$.

Definition 2.5. A Banach lattice \mathcal{Y} with norm $\|\cdot\|$ is called an AM-space (abstract maximum space) if

$$\|x \vee y\| = \|x\| \vee \|y\|, \quad \forall x, y \geq 0.$$

An element $\mathbb{1} \in \mathcal{Y}$ which meets

$$\mathbb{1} \geq 0, \quad \|\mathbb{1}\| = 1, \quad \|x\| \leq 1 \implies |x| \leq \mathbb{1},$$

is called *unit* of \mathcal{Y} . Here $x \vee y$ and $|x|$ denote the usual supremum and absolute value of elements in a Banach lattice (cf appendix).

Theorem 2.6. Let \mathcal{Y} be an AM-space with unit $\mathbb{1}$ and suppose that there exist $u_0 \in \text{dom}(\mathcal{J})$ and $v_0 \in \text{dom}(\mathcal{H}(\cdot|f))$ such that

$$A^1 u_0 + \varepsilon \mathbb{1} \leq v_0 \leq A^u u_0 - \varepsilon \mathbb{1},$$

where $\varepsilon > 0$ is a constant. Then Robinson's condition (2.6) is satisfied in the primal problem.

Proof. In the notation of theorem 2.4, we have $X = \mathcal{X} \times \mathcal{Y}$, $g(u, v) := \frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u)$, $L := (B, -E)$ and $K = \mathcal{Y}_- \times \mathcal{Y}_-$ (where \mathcal{Y}_- denotes the negative cone in \mathcal{Y}).

Take an arbitrary $y = (y^1, y^2) \in \mathcal{Y} \times \mathcal{Y}$ with $\|y\| \leq \varepsilon$. Without loss of generality we can choose the norm on $\mathcal{Y} \times \mathcal{Y}$ to be $\|y\| = \max(\|y^1\|, \|y^2\|)$. Hence, the definition of the unit implies

$$-\varepsilon \mathbb{1} \leq y^{1,2} \leq \varepsilon \mathbb{1}.$$

To show Robinson regularity, we need to write y as

$$y = Bu - Ev - z \tag{2.7}$$

for some $u \in \text{dom}(\mathcal{J})$, $v \in \text{dom}(\mathcal{H}(\cdot|f))$ and $z = (z^1, z^2) \in \mathcal{Y} \times \mathcal{Y}$, $z^{1,2} \leq 0$. Writing this in terms of A^1 and A^u , we get

$$y^1 = A^1 u - v - z^1, \quad y^2 = v - A^u u - z^2.$$

Take $u = u_0$ and $v = v_0$. Then

$$\begin{aligned} z^1 &= A^1 u_0 - v_0 - y^1 \leq -\varepsilon \mathbb{1} - y^1 \leq 0, \\ z^2 &= v_0 - A^u u_0 - y^2 \leq -\varepsilon \mathbb{1} - y^2 \leq 0, \end{aligned}$$

and we can take $z^{1,2}$ as above to represent y as in (2.7). Hence, the Robinson condition (2.6) is satisfied. \square

Corollary 2.7. Since the optimal value of the primal problem (2.3) is finite, using theorem 2.4 we conclude that there exists a solution μ of the dual problem (2.4) and there is no duality gap, i.e.

$$\frac{1}{\alpha} \mathcal{H}(v|f) + \mathcal{J}(u) = -\frac{1}{\alpha} \mathcal{H}^*(\alpha E^* \mu|f) - \mathcal{J}^*(-B^* \mu), \tag{2.8}$$

where (u, v) is a primal optimal solution. Moreover, from [28, theorem 3.6] we conclude that μ is a Lagrange multiplier for the constraint $Bu \leq Ev$ in (2.3) and the following complementarity condition holds

$$\langle \mu, Bu - Ev \rangle = 0. \quad (2.9)$$

Theorem 2.8. Let μ be an optimal solution of (2.4) and (u, v) be an optimal solution of (2.3). Then under the assumptions of theorem 2.6 we have the following relations

$$-B^* \mu \in \partial \mathcal{J}(u), \quad \alpha E^* \mu \in \partial \mathcal{H}(v|f).$$

Proof. Using the Fenchel–Young inequality, strong duality (2.8) and the feasibility of (u, v) , we obtain

$$\begin{aligned} \langle -B^* \mu, u \rangle &\leq \mathcal{J}(u) + \mathcal{J}^*(-B^* \mu) \\ &= -\frac{1}{\alpha} \mathcal{H}^*(\alpha E^* \mu|f) - \frac{1}{\alpha} \mathcal{H}(v|f) \\ &= -\frac{1}{\alpha} [\mathcal{H}^*(\alpha E^* \mu|f) + \mathcal{H}(v|f)] \\ &\leq -\frac{1}{\alpha} \langle \alpha E^* \mu, v \rangle \\ &= -\langle \mu, Ev \rangle \\ &\leq -\langle \mu, Bu \rangle \\ &= \langle -B^* \mu, u \rangle. \end{aligned}$$

Hence, equality holds everywhere and we get that $-B^* \mu \in \partial \mathcal{J}(u)$ and $\alpha E^* \mu \in \partial \mathcal{H}(v|f)$. \square

3. Convergence analysis

Having investigated well-posedness of the primal and dual problems, we can now prove convergence rates of solutions as the noise in the data and the operator tends to zero. To this end we consider sequences

$$A_n^l, A_n^u: \quad A_n^l \leq A \leq A_n^u \quad \forall n, \quad (3.1a)$$

$$\|A_n^u - A_n^l\| \leq \eta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.1b)$$

$$f_n, \delta_n: \quad \mathcal{H}(\bar{f}|f_n) \leq \delta_n \quad \forall n, \quad (3.1c)$$

$$\delta_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.1d)$$

$$\alpha_n: \quad \alpha_n \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad (3.1e)$$

and corresponding sequences (u_n, v_n) and μ_n which solve problems (2.3) and (2.4), respectively. We are interested in studying the behaviour of (u_n, v_n) as $n \rightarrow \infty$ and would like to prove that u_n converges to a \mathcal{J} -minimizing solution $u_{\mathcal{J}}^\dagger$ (cf (1.10)) whereas v_n approaches the exact data \bar{f} .

Remark 3.1. If the fidelity function depends on the difference of the arguments, i.e. $\mathcal{H}(\bar{f}|f_n) = h(\bar{f} - f_n)$, then it does not matter if we choose $\mathcal{H}(\bar{f}|f_n)$ or $\mathcal{H}(f_n|\bar{f})$ in (3.1c). For

asymmetric fidelities such as the Kullback–Leibler divergence it does. If we think of the Kullback–Leibler divergence $D_{\text{KL}}(p|q)$ as the amount of information lost by using q instead of p (see [31]), then it actually makes sense to choose $\mathcal{H}(\bar{f}|f_n)$ in (3.1c), i.e. to measure the amount of information lost by using the noisy measurement f_n instead of the exact one \bar{f} .

We start with results that do not require the existence of a dual solution and are valid under general assumptions (cf theorem 2.1).

3.1. Convergence of primal solutions

We consider a sequence of primal problems (2.3)

$$\min_{\substack{u,v \\ B_n u \leq Ev}} \frac{1}{\alpha_n} \mathcal{H}(v|f_n) + \mathcal{J}(u),$$

where $B_n : \mathcal{X} \rightarrow \mathcal{Y} \times \mathcal{Y}$ is defined as follows

$$B_n := \begin{pmatrix} A_n^l \\ -A_n^u \end{pmatrix}.$$

Under assumptions 1 and 2, we obtain the following standard result.

Theorem 3.2. *Suppose that the regularisation functional \mathcal{J} and the fidelity function \mathcal{H} satisfy assumption 1 and the operators $A, A_n^{lu} : \mathcal{X} \rightarrow \mathcal{Y}$ satisfy assumption 2. Suppose also that the regularisation parameter α_n is chosen such that*

$$\alpha_n \rightarrow 0 \quad \text{and} \quad \frac{\delta_n}{\alpha_n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then any solution u_n of the primal problem (2.3) converges weakly- to a \mathcal{J} -minimising solution of (1.1)*

$$u_n \rightharpoonup^* u_{\mathcal{J}}^{\dagger},$$

and v_n converges weakly- to the exact data in (1.1)*

$$v_n \rightharpoonup^* \bar{f} = Au_{\mathcal{J}}^{\dagger}.$$

Proof. Comparing the value of the objective function at the optimum (u_n, v_n) and $(u_{\mathcal{J}}^{\dagger}, \bar{f})$ (which is a feasible point for all n), we get

$$\frac{1}{\alpha_n} \mathcal{H}(v_n|f_n) + \mathcal{J}(u_n) \leq \frac{1}{\alpha_n} \mathcal{H}(\bar{f}|f_n) + \mathcal{J}(u_{\mathcal{J}}^{\dagger}) \quad (3.2)$$

and

$$\mathcal{J}(u_n) \leq \mathcal{J}(u_{\mathcal{J}}^{\dagger}) + \frac{1}{\alpha_n} \mathcal{H}(\bar{f}|f_n) \leq \mathcal{J}(u_{\mathcal{J}}^{\dagger}) + \frac{\delta_n}{\alpha_n}. \quad (3.3)$$

Since $\frac{\delta_n}{\alpha_n} \rightarrow 0$, the value on the right-hand side is bounded uniformly in n . Hence, since sublevel sets of \mathcal{J} are weakly-* sequentially compact, u_n contains a weakly-* convergent subsequence (that we do not relabel) that converges to some $u_{\infty} \in \mathcal{X}$

$$u_n \rightharpoonup^* u_{\infty}.$$

Since A is weak- * to weak- * continuous by assumption and $\|A_n^{1,u} - A\| \rightarrow 0$, we get that

$$A_n^1 u_n, \rightharpoonup^* A u_\infty \quad \text{and} \quad A_n^u u_n, \rightharpoonup^* A u_\infty.$$

Since (u_n, v_n) is feasible in (2.3) for all n , it holds

$$A_n^1 u_n \leq v_n \leq A_n^u u_n.$$

Using weak- * closedness of order intervals (cf lemma A.4), we infer

$$v_n, \rightharpoonup^* A u_\infty. \tag{3.4}$$

From (3.2) we get that

$$\mathcal{H}(v_n | f_n) \leq \mathcal{H}(\bar{f} | f_n) + \alpha_n \mathcal{J}(u_{\mathcal{J}}^\dagger) \leq \delta_n + \alpha_n \mathcal{J}(u_{\mathcal{J}}^\dagger) \rightarrow 0.$$

Since $\mathcal{H}(\cdot | \cdot)$ is lower semicontinuous jointly in both arguments, we obtain

$$\mathcal{H}(A u_\infty | \bar{f}) \leq \liminf_{n \rightarrow \infty} \mathcal{H}(v_n | f_n) = 0$$

and hence

$$A u_\infty = \bar{f}.$$

Therefore, by (3.4) we have

$$v_n, \rightharpoonup^* \bar{f}.$$

Since \mathcal{J} is lower semicontinuous, (3.3) implies that

$$\mathcal{J}(u_\infty) \leq \liminf_{n \rightarrow \infty} \mathcal{J}(u_n) \leq \mathcal{J}(u_{\mathcal{J}}^\dagger),$$

hence u_∞ is a \mathcal{J} -minimising solution. \square

3.2. Convergence rates

In modern variational regularisation, (generalised) Bregman distances are typically used to study convergence of approximate solutions [32].

Definition 3.3. For a proper convex functional \mathcal{J} the generalised Bregman distance between $u, w \in \mathcal{X}$ corresponding to the subgradient $p \in \partial \mathcal{J}(w)$ is defined as follows

$$D_{\mathcal{J}}^p(u, w) := \mathcal{J}(u) - \mathcal{J}(w) - \langle p, u - w \rangle,$$

where $\partial \mathcal{J}(w)$ denotes the subdifferential of \mathcal{J} at $w \in \mathcal{X}$. The symmetric Bregman distance between u and w corresponding to $q \in \partial \mathcal{J}(u)$ and $p \in \partial \mathcal{J}(w)$ is defined as follows

$$D_{\mathcal{J}}^{\text{symm}}(u, w) := D_{\mathcal{J}}^p(u, w) + D_{\mathcal{J}}^q(w, u) = \langle q - p, u - w \rangle.$$

Bregman distances do not define a metric since they do not satisfy the triangle inequality and $D_{\mathcal{J}}^{\text{symm}}(u, w) = 0$ does not imply $u = w$.

To obtain convergence rates, we will need to make an additional assumption on the regularity of the \mathcal{J} -minimising solution $u_{\mathcal{J}}^\dagger$ called the *source condition*. There are several variants of the

source condition (e.g., [6, 33, 34]); we will use the variant from [6], which in our notation can be written as follows

Assumption 3 (Source condition). There exists $\mu^\dagger \in \mathcal{Y}^* \times \mathcal{Y}^*$, $\mu^\dagger \geq 0$, such that

$$-B^* \mu^\dagger \in \partial \mathcal{J}(u_{\mathcal{J}}^\dagger). \quad (3.5)$$

Remark 3.4. The source condition (3.5) is equivalent to the standard one

$$A^* \omega \in \partial \mathcal{J}(u_{\mathcal{J}}^\dagger), \quad \omega \in \mathcal{Y}^*. \quad (3.6)$$

Indeed, since $B = \begin{pmatrix} A \\ -A \end{pmatrix}$ and $\mu^\dagger = (\mu_1^\dagger, \mu_2^\dagger)$ with $\mu_{1,2}^\dagger \in \mathcal{Y}_+^*$, we get that

$$-B^* \mu^\dagger = A^*(\mu_2^\dagger - \mu_1^\dagger),$$

which implies (3.6) with $\omega := \mu_2^\dagger - \mu_1^\dagger$. For the converse implication we note that since \mathcal{Y}^* is a lattice, we can write an arbitrary $\omega \in \mathcal{Y}^*$ as follows

$$\omega = \omega_+ - \omega_-,$$

where $\omega_\pm \in \mathcal{Y}_+^*$. Hence, (3.6) implies (3.5) with $\mu^\dagger := (\omega_-, \omega_+)$.

3.2.1. Convergence rates in a one-sided Bregman distance. We start with a convergence rate in a one-sided Bregman distance $D_{\mathcal{J}}^{p^\dagger}$, where $p^\dagger := -B^* \mu^\dagger$ is the subgradient from the source condition (3.5).

Theorem 3.5. Let assumptions of theorem 2.1 and assumption 3 be satisfied and (3.1) hold. Then the following estimate holds

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) \leq \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} [\mathcal{H}^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle] + C\eta_n, \quad (3.7)$$

where $p^\dagger = -B^* \mu^\dagger$ is the subgradient from assumption 3.

Proof. We start with the following estimate

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &= \mathcal{J}(u_n) - \mathcal{J}(u_{\mathcal{J}}^\dagger) - \langle -B^* \mu^\dagger, u_n - u_{\mathcal{J}}^\dagger \rangle \\ &= \mathcal{J}(u_n) - \mathcal{J}(u_{\mathcal{J}}^\dagger) + \langle \mu^\dagger, Bu_n \rangle - \langle \mu^\dagger, Bu_{\mathcal{J}}^\dagger \rangle \\ &= \mathcal{J}(u_n) - \mathcal{J}(u_{\mathcal{J}}^\dagger) + \langle \mu^\dagger, B_n u_n \rangle - \langle \mu^\dagger, Bu_{\mathcal{J}}^\dagger \rangle + \langle \mu^\dagger, (B - B_n)u_n \rangle \\ &\leq \mathcal{J}(u_n) - \mathcal{J}(u_{\mathcal{J}}^\dagger) + \langle \mu^\dagger, B_n u_n \rangle - \langle \mu^\dagger, Bu_{\mathcal{J}}^\dagger \rangle + C\eta_n \\ &\leq \mathcal{J}(u_n) - \mathcal{J}(u_{\mathcal{J}}^\dagger) + \langle \mu^\dagger, Ev_n \rangle - \langle \mu^\dagger, Bu_{\mathcal{J}}^\dagger \rangle + C\eta_n, \end{aligned} \quad (3.8)$$

where η_n is as defined in (3.1b) and we used the fact that $B_n u_n \leq Ev_n$. Since (u_n, v_n) is primal optimal and $(u_{\mathcal{J}}^\dagger, \bar{f})$ is feasible, we get that

$$\frac{1}{\alpha_n} \mathcal{H}(v_n | f_n) + \mathcal{J}(u_n) \leq \frac{1}{\alpha_n} \mathcal{H}(\bar{f} | f_n) + \mathcal{J}(u_{\mathcal{J}}^\dagger) \leq \frac{\delta_n}{\alpha_n} + \mathcal{J}(u_{\mathcal{J}}^\dagger)$$

and therefore

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &\leq \frac{\delta_n}{\alpha_n} - \frac{1}{\alpha_n} \mathcal{H}(v_n|f_n) + \langle \mu^\dagger, Ev_n \rangle - \langle \mu^\dagger, Bu_{\mathcal{J}}^\dagger \rangle + C\eta_n \\ &= \frac{\delta_n}{\alpha_n} - \frac{1}{\alpha_n} \mathcal{H}(v_n|f_n) + \langle \mu^\dagger, Ev_n \rangle - \langle \mu^\dagger, E\bar{f} \rangle + C\eta_n \\ &= \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} [\langle \alpha_n E^* \mu^\dagger, v_n \rangle - \mathcal{H}(v_n|f_n)] - \langle E^* \mu^\dagger, \bar{f} \rangle + C\eta_n. \end{aligned}$$

By the Fenchel–Young inequality, the term in the brackets is bounded by $\mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n)$, hence

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &\leq \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} \mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n) - \langle E^* \mu^\dagger, \bar{f} \rangle + C\eta_n \\ &= \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} [\mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle] + C\eta_n. \end{aligned}$$

□

3.2.2. Convergence rates in a symmetric Bregman distance. Under a stronger assumption that \mathcal{Y} is an AM-space (cf theorem 2.6), we can obtain an estimate in a symmetric Bregman distance.

Theorem 3.6. *Let assumptions of theorem 2.6 and assumption 3 be satisfied and (3.1) hold. Then the following estimate holds*

$$D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) \leq \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} [\mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle] + C\eta_n, \quad (3.9)$$

where the symmetric Bregman distance corresponds to the subgradients $p^\dagger := -B^* \mu^\dagger \in \partial \mathcal{J}(u_{\mathcal{J}}^\dagger)$ from assumption 3 and $p_n := -B_n^* \mu_n \in \partial \mathcal{J}(u_n)$.

Proof. The symmetric Bregman distance between u_n and $u_{\mathcal{J}}^\dagger$ is given by

$$\begin{aligned} D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) &= \langle -B^* \mu^\dagger + B_n^* \mu_n, u_{\mathcal{J}}^\dagger - u_n \rangle \\ &= \langle \mu_n, B_n u_{\mathcal{J}}^\dagger - B_n u_n \rangle - \langle \mu^\dagger, B u_{\mathcal{J}}^\dagger - B u_n \rangle. \end{aligned}$$

Since the pair $(u_{\mathcal{J}}^\dagger, \bar{f})$ is feasible for all n , we get that $B_n u_{\mathcal{J}}^\dagger \leq E\bar{f}$. It is also evident that $B u_{\mathcal{J}}^\dagger = E\bar{f}$. Combining this with the complementarity condition (2.9), we obtain

$$\begin{aligned} D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) &= \langle \mu_n, E\bar{f} - Ev_n \rangle + \langle \mu^\dagger, B u_n - E\bar{f} \rangle \\ &= \langle \mu_n, E\bar{f} - Ev_n \rangle + \langle \mu^\dagger, B_n u_n - E\bar{f} \rangle + \langle \mu^\dagger, (B - B_n) u_n \rangle. \end{aligned}$$

Since the pair (u_n, v_n) is also feasible, we get that $B_n u_n \leq Ev_n$ and hence

$$\begin{aligned} D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) &\leq \langle \mu_n, E\bar{f} - Ev_n \rangle + \langle \mu^\dagger, Ev_n - E\bar{f} \rangle + \langle \mu^\dagger, (B - B_n) u_n \rangle \\ &\leq \langle E^* \mu_n, \bar{f} - v_n \rangle - \langle E^* \mu^\dagger, \bar{f} - v_n \rangle + \|\mu^\dagger\| \|u_n\| \|B - B_n\| \\ &\leq \langle E^* \mu_n, \bar{f} - v_n \rangle - \langle E^* \mu^\dagger, \bar{f} - v_n \rangle + C\eta_n, \end{aligned}$$

where $\|u_n\|$ is bounded due to theorem 3.2. From the Fenchel–Young inequality and theorem 2.8 we get that

$$\langle \alpha_n E^* \mu_n, \bar{f} \rangle \leq \mathcal{H}(\bar{f}|f_n) + \mathcal{H}^*(\alpha_n E^* \mu_n|f_n), \quad (3.10a)$$

$$\langle \alpha_n E^* \mu_n, v_n \rangle = \mathcal{H}(v_n|f_n) + \mathcal{H}^*(\alpha_n E^* \mu_n|f_n), \quad (3.10b)$$

$$\langle \alpha_n E^* \mu^\dagger, v_n \rangle \leq \mathcal{H}(v_n|f_n) + \mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n), \quad (3.10c)$$

hence

$$\begin{aligned} \alpha_n D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) &\leq \mathcal{H}(\bar{f}|f_n) - \mathcal{H}(v_n|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} - v_n \rangle + \alpha_n C \eta_n \\ &\leq \delta_n - \mathcal{H}(v_n|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} - v_n \rangle + \alpha_n C \eta_n \\ &\leq \delta_n + \mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle + \alpha_n C \eta_n, \end{aligned}$$

which yields the desired estimate upon dividing by α_n . \square

3.3. Applications to different fidelity terms

To apply theorems 3.5 or 3.6, we need to study the term $\mathcal{H}^*(\alpha_n E^* \mu^\dagger|f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle$ separately for each fidelity term.

3.3.1. φ -divergences. Let $\varphi : (0, \infty) \rightarrow \mathbb{R}$ be a convex function. For two probability measures ρ, ν on Ω with $\rho \ll \nu$ the φ -divergence (often called f -divergence) is defined as follows

$$d_\varphi(\rho|\nu) := \int_\Omega \varphi \left(\frac{d\rho}{d\nu} \right) d\nu, \quad (3.11)$$

where $\varphi(1) = 0$. We refer to [35] for many examples and fundamental properties of φ -divergences. Since ρ and ν have unit mass, function φ is only determined up to the additive term $c(x - 1)$ for $c \in \mathbb{R}$. In particular, since φ is convex and meets $\varphi(1) = 0$, it is straightforward to see that one can always find a suitable $c \in \mathbb{R}$ such that $\varphi(x) + c(x - 1) \geq 0$ for all $x > 0$. Hence, we will without loss of generality assume that $\varphi \geq 0$.

We take $\mathcal{Y} = \mathcal{M}(\Omega)$ to be space of Radon measures on Ω equipped with the total variation norm and consider

$$\mathcal{H}(v|f) := \begin{cases} d_\varphi(v|f), & \text{if } v \in \mathcal{P}(\Omega), v \ll f, \\ \infty, & \text{else,} \end{cases} \quad (3.12)$$

where $\mathcal{P}(\Omega) \subset \mathcal{M}(\Omega)$ is the set of probability measures and $f \in \mathcal{P}(\Omega)$.

We estimate the convex conjugate of $\mathcal{H}(\rho|\nu)$ as follows:

$$\begin{aligned} \mathcal{H}^*(h|\nu) &= \sup_{\rho \ll \nu} \langle h, \rho \rangle - \mathcal{H}(\rho|\nu) \\ &= \sup_{\rho \ll \nu} \int_\Omega \left(h \frac{d\rho}{d\nu} - \varphi \left(\frac{d\rho}{d\nu} \right) \right) d\nu \\ &= \sup_{f \in L_+^1(\Omega)} \int_\Omega (h(x)f(x) - \varphi(f(x))) d\nu(x) \\ &\leq \int_\Omega \sup_{y > 0} [h(x)y - \varphi(y)] d\nu(x) \end{aligned}$$

$$\begin{aligned}
&= \int_{\Omega} \varphi^*(h) \, d\nu \\
&= \langle \varphi^*(h), \nu \rangle,
\end{aligned} \tag{3.13}$$

for any $h \in \mathcal{C}(\Omega)$.

Since $\varphi(1) = 0$ and $\varphi \geq 0$, we know that $\varphi^*(0) = 0$ and $\varphi^*(x) \geq x$. Indeed, we have $\varphi^*(0) = \sup_x -\varphi(x) = -\inf_x \varphi(x) = 0$ and, by the Fenchel–Young inequality, $\varphi^*(x) \geq x - \varphi(1) = x$. This motivates us to assume

$$\varphi^*(x) = x + r(x), \tag{3.14}$$

where $r(x)/x \rightarrow 0$ as $x \rightarrow 0$. This is satisfied in many cases (examples will be provided later on).

Theorem 3.7. *Let $\mathcal{H}(\cdot)$ be as defined in (3.12) and let the assumptions of theorem 3.5 be satisfied. Suppose that $E^* \mu^\dagger \in \mathcal{C}(\Omega)$, where μ^\dagger is the source element from assumption 3, and that (3.14) holds. Then the following convergence rate holds*

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\frac{\delta_n}{\alpha_n} + \frac{r(\alpha_n)}{\alpha_n} + \eta_n\right), \tag{3.15}$$

where $p^\dagger = -B^* \mu^\dagger$ is the subgradient from assumption 3.

Under the additional assumption that $A, A^{1,u}$ are bounded from as operators $\mathcal{X} \rightarrow L^\infty(\Omega) \subset \mathcal{M}(\Omega)$, we get the same rate for the symmetric Bregman distance $D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger)$ (cf theorem 3.6).

Proof. Taking $h = \alpha_n E^* \mu^\dagger$ and $\nu = f_n$ in (3.13), and using (3.14), we get

$$\begin{aligned}
\mathcal{H}(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle &= \langle \varphi^*(\alpha_n E^* \mu^\dagger), f_n \rangle - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\
&= \langle \varphi^*(\alpha_n E^* \mu^\dagger) + \varphi^*(-\alpha_n E^* \mu^\dagger), f_n \rangle \\
&\quad + \langle -\alpha_n E^* \mu^\dagger, \bar{f} \rangle - \langle \varphi^*(-\alpha_n E^* \mu^\dagger), f_n \rangle \\
&\leq \langle \varphi^*(\alpha_n E^* \mu^\dagger) + \varphi^*(-\alpha_n E^* \mu^\dagger), f_n \rangle \\
&\quad + \mathbf{d}_\varphi(\bar{f}, f_n) \\
&\leq \langle \varphi^*(\alpha_n E^* \mu^\dagger) + \varphi^*(-\alpha_n E^* \mu^\dagger), f_n \rangle + \delta_n \\
&= \langle r(\alpha_n E^* \mu^\dagger) + r(-\alpha_n E^* \mu^\dagger), f_n \rangle + \delta_n,
\end{aligned}$$

and in combination with (3.7) this yields the assertion. \square

KL-divergence. Here $\varphi(x) = x, \log(x) - (x - 1)$, $\varphi^*(x) = e^x - 1 = x + r(x)$ with $r(x) = x^2/2 + x^3/6 \dots$ and we get that

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\frac{\delta_n}{\alpha_n} + \alpha_n + \eta_n\right). \tag{3.16}$$

which coincides with [4] in the case of an exact operator. For $\alpha_n \sim (\delta_n)^{\frac{1}{2}}$ we get the optimal rate

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left((\delta_n)^{\frac{1}{2}} + \eta_n\right). \tag{3.17}$$

χ^2 -divergence. Here $\varphi(x) = (x - 1)^2$ and $\varphi^*(x) = x + \frac{x^2}{4}$. Again,

$$D_{\mathcal{J}}^{p\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\frac{\delta_n}{\alpha_n} + \alpha_n + \eta_n\right) \quad (3.18)$$

and the optimal rate coincides with (3.17).

Squared Hellinger distance. Here $\varphi(x) = (\sqrt{x} - 1)^2$, $\varphi^*(x) = \frac{x}{x-1} \approx x + x^2 + \dots$ and we get

$$D_{\mathcal{J}}^{p\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\frac{\delta_n}{\alpha_n} + \alpha_n + \eta_n\right) \quad (3.19)$$

and the optimal rate coincides with (3.17).

Total variation. For the total variation (of measures) we have $\varphi(x) = \frac{1}{2}|x - 1|$ and

$$\varphi^*(x) = \begin{cases} x, & |x| \leq \frac{1}{2}, \\ \infty, & \text{otherwise.} \end{cases}$$

Then for any $\alpha_n = \text{const}$ such that $\|\alpha_n E^* \mu^\dagger\|_\infty \leq \frac{1}{2}$ we get that

$$D_{\mathcal{J}}^{p\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O(\delta_n + \eta_n). \quad (3.20)$$

Remark 3.8 (Poisson noise). The main motivation for the use of the Kullback–Leibler divergence as a fidelity term is the modelling of Poisson noise [1]. If t denotes the exposure time, the measured data can be assumed to be generated by a Poisson process with intensity $t\bar{f}$. In this case, the upper bound on the error in the Kullback–Leibler divergence is given by [36]

$$\mathcal{H}(\bar{f}|f_n) \leq \frac{1}{\sqrt{t_n}}.$$

While in the deterministic setting, this estimate is sufficient to obtain convergence rates, the statistical setting requires further assumptions, in particular some concentration inequalities [2, 36, 37].

3.3.2. Strongly coercive fidelity terms.

Theorem 3.9. Suppose that the fidelity function \mathcal{H} is coercive in the following sense

$$\frac{C}{\lambda} \|v - f\|_{\mathcal{Y}}^\lambda \leq \mathcal{H}(v|f) \quad (3.21)$$

for all $v, f \in \mathcal{Y}$, where $\lambda \geq 1$ and $C > 0$ are constants (we will assume with loss of generality that $C = 1$). Then under the assumptions of theorem 3.5 the following convergence rates hold

$$D_{\mathcal{J}}^{p\dagger}(u_n, u_{\mathcal{J}}^\dagger) = \begin{cases} O\left(\frac{\delta_n}{\alpha_n} + \alpha_n^{\frac{1}{\lambda-1}} + \delta_n^{\frac{1}{\lambda}} + \eta_n\right), & \lambda > 1, \\ O\left(\frac{\delta_n}{\alpha_n} + \delta_n + \eta_n\right), & \lambda = 1, \end{cases}$$

where $p^\dagger = -B^* \mu^\dagger$ is the subgradient from assumption 3. If α_n is chosen such that $\alpha_n \sim (\delta_n)^{\frac{\lambda-1}{\lambda}}$ if $\lambda > 1$ and $\alpha_n = \text{const} \leq \frac{1}{\|E^* \mu^\dagger\|}$ if $\lambda = 1$, we get the optimal rate

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\delta_n^{\frac{1}{\lambda}} + \eta_n\right).$$

If \mathcal{Y} is an AM-space (cf theorem 3.6), the same rate holds for the symmetric Bregman distance $D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger)$.

Proof. Since convex conjugation is order-reversing, from (3.21) we obtain that for any $q \in \mathcal{Y}^*$ (we will drop the subscripts \mathcal{Y} and \mathcal{Y}^* after the norms to simplify notation)

$$\begin{aligned} \mathcal{H}^*(q|f) &\leq \left(\frac{1}{\lambda} \|\cdot - f\|^\lambda\right)^*(q) = \left(\frac{1}{\lambda} \|\cdot\|^\lambda\right)^*(q) + \langle f, q \rangle \\ &= \begin{cases} \frac{1}{\lambda^*} \|q\|^{\lambda^*} + \langle f, q \rangle & \lambda > 1, \\ \chi_{\|\cdot\| \leq 1}(q) + \langle f, q \rangle & \lambda = 1, \end{cases} \end{aligned}$$

where $\lambda^* = \frac{\lambda}{\lambda-1}$. We will consider the cases $\lambda > 1$ and $\lambda = 1$ separately.

Let $\lambda > 1$. Then from theorem 3.5 we obtain

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &\leq \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} \left(\frac{1}{\lambda^*} \|\alpha_n E^* \mu^\dagger\|^{\lambda^*} + \langle \alpha_n E^* \mu^\dagger, f_n \rangle - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \right) + C\eta_n \\ &= \frac{\delta_n}{\alpha_n} + \frac{1}{\alpha_n} \left(\frac{\alpha_n^{\lambda^*}}{\lambda^*} \|E^* \mu^\dagger\|^{\lambda^*} + \alpha_n \langle E^* \mu^\dagger, f_n - \bar{f} \rangle \right) + C\eta_n \\ &= \frac{\delta_n}{\alpha_n} + \frac{\alpha_n^{\lambda^*-1}}{\lambda^*} \|E^* \mu^\dagger\|^{\lambda^*} + \langle E^* \mu^\dagger, f_n - \bar{f} \rangle + C\eta_n. \end{aligned}$$

Condition (3.21) implies that $\|f_n - \bar{f}\| \leq C\delta_n^{\frac{1}{\lambda}}$. Hence, using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &\leq \frac{\delta_n}{\alpha_n} + C \frac{\alpha_n^{\lambda^*-1}}{\lambda^*} \|E^* \mu^\dagger\|^{\lambda^*} + \|E^* \mu^\dagger\| \|f_n - \bar{f}\| + C\eta_n \\ &\leq \frac{\delta_n}{\alpha_n} + C \frac{\alpha_n^{\lambda^*-1}}{\lambda^*} \|E^* \mu^\dagger\|^{\lambda^*} + \|E^* \mu^\dagger\| \delta_n^{\frac{1}{\lambda}} + C\eta_n \\ &= O\left(\frac{\delta_n}{\alpha_n} + \alpha_n^{\frac{1}{\lambda-1}} + \delta_n^{\frac{1}{\lambda}} + \eta_n\right). \end{aligned}$$

Let now $\lambda = 1$. Then for sufficiently small $\alpha_n \leq \frac{1}{\|E^* \mu^\dagger\|}$ we obtain from theorem 3.5

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) &\leq \frac{\delta_n}{\alpha_n} + \langle E^* \mu^\dagger, f_n - \bar{f} \rangle + C\eta_n \\ &\leq \frac{\delta_n}{\alpha_n} + \|E^* \mu^\dagger\| \|f_n - \bar{f}\| + C\eta_n \\ &\leq \frac{\delta_n}{\alpha_n} + C\delta_n + C\eta_n. \end{aligned}$$

For a sufficiently small but fixed α_n we get that

$$D_{\mathcal{J}}^{p\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O(\delta_n + \eta_n).$$

□

Remark 3.10. The value $\frac{1}{\|E^* \mu^\dagger\|}$ matches the exact penalisation parameter in regularisation with one-homogeneous fidelity terms (e.g. [4, 6, 38]). Exact penalisation means that the regularisation parameters α_n do not have to be sent to zero in order to obtain convergence in the Bregman distance. It is observed if the subdifferential $\partial\mathcal{H}(\cdot|_{\bar{f}})|_{\bar{f}}$ is no singleton.

Example 3.11 (Powers of norms). Theorem 3.9 obviously applies if the fidelity function is given by a power of the norm, i.e.

$$\mathcal{H}(v|f) = \frac{1}{\lambda} \|v - f\|^\lambda, \quad \lambda \geq 1.$$

This covers important cases such as the squared L^2 norm fidelity which is used to model Gaussian noise and the L^1 norm fidelity which is often used to model salt-and-pepper noise [39].

Example 3.12 (Wasserstein distances). For any $p \geq 1$, the p -Wasserstein distance between two probability measures $\rho, \nu \in \mathcal{P}(\Omega)$ is defined as follows (cf [40])

$$W_p(\rho, \nu) := \left(\inf_{\gamma \in \Pi(\rho, \nu)} \int_{\Omega} \int_{\Omega} |x - y|^p \, d\gamma(x, y) \right)^{\frac{1}{p}},$$

where $\Pi(\rho, \nu)$ is the space of probability measures on $\Omega \times \Omega$ with marginals ρ and ν .

Let the data space $\mathcal{Y} = \text{KR}(\Omega)$ be the closure of the space of Radon measures $\mathcal{M}(\Omega)$ with respect to the Kantorovich–Rubinstein norm

$$\|\mu\|_{\text{KR}} := \sup \left\{ \int g \, d\mu : \text{Lip}(g) \leq 1, \|g\|_{\infty} \leq 1 \right\},$$

where Lip denotes the Lipschitz constant [41]. Obviously it holds $\|\mu\|_{\text{KR}} \leq |\mu|(\Omega)$ for all $\mu \in \mathcal{M}(\Omega)$ and $\|\mu\|_{\text{KR}} \geq |\mu|(\Omega)$ if $\mu \geq 0$ by choosing $g \equiv 1$ (it is known that the positive cone $\mathcal{M}_+(\Omega)$, and hence also the set of probability measures $\mathcal{P}(\Omega)$, is closed in the KR norm [41, theorem 8.9.4]). For any $v \in \mathcal{Y}$ and a probability measure $f \in \mathcal{P}(\Omega)$ we let

$$\mathcal{H}(v|f) := \begin{cases} W_p^p(v|f), & \text{if } v \in \mathcal{P}(\Omega), \\ \infty, & \text{else.} \end{cases} \quad (3.22)$$

It is well known that for any two probability measures $\rho, \nu \in \mathcal{P}(\Omega)$

$$W_1(\rho, \nu) = \|\rho - \nu\|_{\text{KR}}.$$

It is also known that for any $q \leq p$ and any two probability measures $\rho, \nu \in \mathcal{P}(\Omega)$, the following relation holds [40]

$$W_q(\rho, \nu) \leq W_p(\rho, \nu).$$

Hence, the data term defined in (3.22) satisfies

$$\|v - f\|_{\text{KR}}^p \leq \mathcal{H}(v|f),$$

i.e. it is strongly coercive on $\text{KR}(\Omega)$. Note that it is not strongly coercive on $\mathcal{M}(\Omega)$ equipped with the total variation norm.

Hence, using theorem 3.9 we get the following optimal rate

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O\left(\delta_n^{\frac{1}{p}} + \eta_n\right).$$

3.3.3. Characteristic function of a norm ball. Let the fidelity function be as follows

$$\mathcal{H}(v|f) = \chi_{\|\cdot\| \leq \delta}(v - f). \quad (3.23)$$

This type of fidelity functions corresponds to the so-called residual method [15, 42] and allows one to explicitly use the noise level δ in the reconstruction (another way of doing so is the discrepancy principle, see section 4). It is clear that

$$\mathcal{H}(v|f) \leq \delta \iff \|v - f\| \leq \delta.$$

With this particular fidelity function the parameter α does not have any effect on the solutions of (2.3), hence with no loss of generality we will assume $\alpha_n = \text{const}$ for all n .

The coercivity assumption (3.21) is not satisfied for this fidelity function (it is only *weakly* coercive, i.e. $\|v - f\| \rightarrow \infty$ implies $\mathcal{H}(v|f) \rightarrow \infty$) and theorem 3.9 does not apply.

Theorem 3.13. *Let the fidelity function be as defined in (3.23). Then under the assumptions of theorem 3.5 the following convergence rate holds*

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O(\delta_n + \eta_n), \quad (3.24)$$

where $p^\dagger = -B^* \mu^\dagger$ is the subgradient from assumption 3.

If \mathcal{Y} is an AM-space (cf theorem 3.6), the same rate holds for the symmetric Bregman distance $D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger)$.

Proof. Taking the convex conjugate of $\mathcal{H}(\cdot|f)$ defined in (3.23), we get

$$\begin{aligned} \mathcal{H}^*(q|f) &= \sup_{v: \|v-f\| \leq \delta} \langle q, v \rangle = \sup_{v: \|v-f\| \leq \delta} \langle q, v-f \rangle + \langle q, f \rangle \\ &\leq \sup_{v: \|v-f\| \leq \delta} \|q\| \|v-f\| + \langle q, f \rangle \leq \delta \|q\| + \langle q, f \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \mathcal{H}^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle &\leq \delta_n \alpha_n \|E^* \mu^\dagger\| + \langle \alpha_n E^* \mu^\dagger, f_n - \bar{f} \rangle \\ &\leq 2\delta_n \alpha_n \|E^* \mu^\dagger\| \end{aligned}$$

since $\|f_n - \bar{f}\| \leq \delta_n$. Plugging this into the estimate in theorem 3.5 (resp. theorem 3.6) and remembering that $\alpha_n = \text{const}$ for all n , we get the assertion. \square

3.3.4. Sum of fidelities. Having studied a plethora of explicit examples of fidelity functions, we now turn to combinations of several fidelities, each of which can be studied as above. Let us assume that \mathcal{H} is the sum of two other fidelity functions \mathcal{H}_1 and \mathcal{H}_2 , i.e.,

$$\mathcal{H}(v|f) = \mathcal{H}_1(v|f) + \mathcal{H}_2(v|f). \quad (3.25)$$

Such fidelities were studied e.g. in [43] and allow to simultaneously handle data from different modalities. Furthermore, in [44–46] fidelities of $L^1 + L^2$ -type were analysed and used for image restoration in the presence of mixed Gaussian and impulse noise.

If \mathcal{H}_1 and \mathcal{H}_2 are proper, it holds

$$\mathcal{H}^*(q|f) \leq \inf_{r \in \mathcal{Y}^*} \{ \mathcal{H}_1^*(r|f) + \mathcal{H}_2^*(q-r|f) \} =: (\mathcal{H}_1^*(\cdot|f) \square \mathcal{H}_2^*(\cdot|f))(q), \quad (3.26)$$

where the term on the right-hand side is the so-called infimal convolution of \mathcal{H}_1 and \mathcal{H}_2 . Let us assume that we have estimates of the form

$$\mathcal{H}_i^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \leq R_i(\alpha_n, \mathcal{H}_i(\bar{f} | f_n)), \quad i = 1, 2, \quad (3.27)$$

for each of the fidelities. The functions R_i are assumed to be non-decreasing in both arguments and we set $R_i(\alpha, \cdot) = \infty$ for $\alpha < 0$. Combining (3.26) and (3.27) we obtain

$$\begin{aligned} & \mathcal{H}^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &= \inf_{w \in \mathcal{Y}} \{ \mathcal{H}_1^*(w | f_n) + \mathcal{H}_2^*(\alpha_n E^* \mu^\dagger - w | f_n) \} - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &\leq \inf_{\lambda \in [0,1]} \mathcal{H}_1^*(\lambda \alpha_n E^* \mu^\dagger | f_n) - \langle \lambda \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &\quad + \mathcal{H}_2^*((1-\lambda)\alpha_n E^* \mu^\dagger | f_n) - \langle (1-\lambda)\alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &\leq \inf_{\lambda \in [0,1]} R_1(\lambda \alpha_n, \mathcal{H}_1(\bar{f} | f_n)) + R_2((1-\lambda)\alpha_n, \mathcal{H}_2(\bar{f} | f_n)) \\ &\leq \inf_{\lambda \in [0,1]} R_1(\lambda \alpha_n, \delta_n) + R_2((1-\lambda)\alpha_n, \delta_n) \\ &= (R_1(\cdot, \delta_n) \square R_2(\cdot, \delta_n))(\alpha_n), \end{aligned}$$

where we used the monotonicity properties of R_i in the last two steps. This shows that the convergence rate for \mathcal{H} can be estimated by the infimal convolution of the rates associated to \mathcal{H}_1 and \mathcal{H}_2 , i.e.

$$D_{\mathcal{J}}^{\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O[(R_1(\cdot, \delta_n) \square R_2(\cdot, \delta_n))(\alpha_n) + \eta_n]. \quad (3.28)$$

If \mathcal{Y} is an AM-space (cf theorem 3.6), the same rate holds for the symmetric Bregman distance $D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger)$.

3.3.5. Infimal convolution of fidelities. Let us consider the case that \mathcal{H} is given by the infimal convolution of two other fidelities \mathcal{H}_1 and \mathcal{H}_2

$$\mathcal{H}(v|f) = \inf_{w \in \mathcal{Y}} \mathcal{H}_1(w|0) + \mathcal{H}_2(v-w|f) = (\mathcal{H}_1(\cdot|0) \square \mathcal{H}_2(\cdot|f))(v) \quad (3.29)$$

Such fidelities are also chosen for the removal of mixed noise in image restoration (see e.g. [47] for an application to hyperspectral unmixing and [48] and the references therein for image denoising with mixtures of Gaussian, impulse, and Poisson noise). Since the infimal convolution *optimally* decomposes v into a noise part w , which is small in \mathcal{H}_1 , and a residual $v-w$, which is close to the data f in \mathcal{H}_2 , such fidelities are more suitable for this purpose than the plain sum of fidelities, studied in the previous section. By standard calculus for infimal convolutions, if \mathcal{H}_1 and \mathcal{H}_2 are proper, it holds

$$\mathcal{H}^*(q|f) = \mathcal{H}_1^*(q|0) + \mathcal{H}_2^*(q|f). \quad (3.30)$$

Furthermore, under the hypothesis that \mathcal{H}_1 is coercive, \mathcal{H}_2 is bounded from below, and both are weakly- $*$ lower semicontinuous convex functions, it holds that \mathcal{H} is weakly- $*$ lower semicontinuous, proper, and exact (see [49] for the statement and [50] for a proof on Hilbert spaces which generalises to Banach spaces). The latter means that the infimum in the definition of \mathcal{H} is attained. In particular, there are $\bar{g}, \bar{h} \in \mathcal{Y}$ such that $\bar{f} = \bar{g} + \bar{h}$ and

$$\delta_n = \mathcal{H}(\bar{f}, f_n) = \mathcal{H}_1(\bar{g}|0) + \mathcal{H}_2(\bar{h}|f_n). \quad (3.31)$$

Furthermore, from (3.30) we get

$$\begin{aligned} & \mathcal{H}^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &= \mathcal{H}_1^*(\alpha_n E^* \mu^\dagger | 0) + \mathcal{H}_2^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle \\ &= (\mathcal{H}_1^*(\alpha_n E^* \mu^\dagger | 0) - \langle \alpha_n E^* \mu^\dagger, \bar{g} \rangle) + (\mathcal{H}_2^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{h} \rangle). \end{aligned}$$

Consequently, we have to estimate the two terms in brackets which only depend on the individual fidelities \mathcal{H}_1 and \mathcal{H}_2 . In all the examples studied above, such estimates are available. Using the functions R_i defined in (3.27) above together with (3.31), we can estimate

$$\begin{aligned} \mathcal{H}^*(\alpha_n E^* \mu^\dagger | f_n) - \langle \alpha_n E^* \mu^\dagger, \bar{f} \rangle &\leq R_1(\alpha_n, \mathcal{H}_1(\bar{g}|0)) + R_2(\alpha_n, \mathcal{H}_2(\bar{h}|f_n)) \\ &\leq R_1(\alpha_n, \delta_n) + R_2(\alpha_n, \delta_n). \end{aligned}$$

Hence, we get the statement that the rate of convergence of a infimal convolution of fidelities can be estimated by the sum of the individual rates associated to \mathcal{H}_1 and \mathcal{H}_2 , i.e.

$$D_{\mathcal{J}}^{p^\dagger}(u_n, u_{\mathcal{J}}^\dagger) = O(R_1(\alpha_n, \delta_n) + R_2(\alpha_n, \delta_n) + \eta_n). \quad (3.32)$$

This is in contrast to the rate of a sum of fidelities being given by the infimal convolution of the rates, as shown in the previous section.

If \mathcal{Y} is an AM-space (cf theorem 3.6), the same rate holds for the symmetric Bregman distance $D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger)$.

4. Discrepancy principle

When the operator is known exactly, Morozov's discrepancy principle [10, 33] can be used to select the regularisation parameter α_n . In the case of a squared norm fidelity $\mathcal{H}(v|f) = \|v - f\|^2$ this amounts to selecting α_n such that

$$\alpha_n = \sup \{ \alpha > 0 : \|A u_n^{\alpha_n} - f_n\|^2 \leq \tau \delta_n \}, \quad (4.1)$$

where $u_n^{\alpha_n}$ is the regularised solution corresponding the regularisation parameter α_n and $\tau > 1$ is a parameter. Here we assume that $\|\bar{f} - f_n\|^2 \leq \delta_n$ (and not $\|\bar{f} - f_n\|^2 \leq \delta_n^2$) to be consistent with our earlier notation. Convergence rates for this choice of α_n in the case of an exact operator and an arbitrary convex regularisation functional were obtained in [11]. For the data fidelity given by the Kullback–Leibler divergence, the discrepancy principle is studied in [13].

In the case of an imperfect operator, the discrepancy principle needs to be modified. When the operator error is measured using the operator norm, i.e. one assumes that an approximate operator A_h is available such that

$$\|A - A_h\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y})} \leq h_n,$$

one can choose α_n as follows [15] (in the case of a squared norm fidelity in the Hilbert space setting)

$$\alpha_n = \sup \{ \alpha > 0 : \|Au_n^{\alpha_n} - f_n\|^2 = (\sqrt{\delta_n} + h_n \|u_n^{\alpha_n}\|)^2 \}. \quad (4.2)$$

If the fidelity term is not based on a norm and does not satisfy the triangle inequality, such a generalisation is not available.

Since in our case the operator error is explicitly accounted for through the constraints in (2.3), we can use the discrepancy principle in its original form (4.1) with an *arbitrary* fidelity term. We will choose α_n such that

$$\alpha_n = \sup \{ \alpha > 0 : \mathcal{H}(v_n^\alpha | f_n) \leq \tau \delta_n \}, \quad (4.3)$$

where $v_n^{\alpha_n}$ solves (2.3) with the regularisation parameter α_n and $\tau > 1$ is a parameter.

Remark 4.1. If the solution $v_n^{\alpha_n}$ is unique, then we have

$$\mathcal{H}(v_n^{\alpha_n} | f_n) = \tau \delta_n. \quad (4.4)$$

In case of non-uniqueness, we can always choose a solution $v_n^{\alpha_n}$ such that (4.4) is satisfied, following the argument in [12, proposition 3.5–remark 3.8] and using convexity of the objective function in (2.3).

4.1. Existence

In this section we study well-posedness of the discrepancy principle, meaning that there is a regularisation parameter α_n which meets (4.3). Let (u^α, v^α) be a solution of (2.3) corresponding to the parameter $\alpha > 0$. Define the following functions:

$$h(\alpha) := \mathcal{H}(v^\alpha | f_n), \quad j(\alpha) := \mathcal{J}(u^\alpha). \quad (4.5)$$

Lemma 4.2. *The function $j(\alpha)$ is monotone non-increasing and $h(\alpha)$ is monotone non-decreasing in α .*

Proof. The proof is similar to [51]. □

Remark 4.3. If either $\mathcal{H}(\cdot | f_n)$ or $\mathcal{J}(\cdot)$ is strictly convex, then $h(\alpha)$ and $j(\alpha)$ are indeed uniquely defined (the argument is similar to [38]). Otherwise the lemma applies to $\mathcal{H}(v^\alpha | f_n)$ and $\mathcal{J}(u^\alpha)$ for any solution (u^α, v^α) of (2.3).

Remark 4.4. Since j and h are monotone functions, they are in particular continuous for almost all values of $\alpha > 0$.

Lemma 4.5. *Functions h and j defined in (4.5) are lower semicontinuous.*

Proof. We just sketch the proof. Letting $\alpha_k \rightarrow \alpha$, one can easily see that the corresponding solutions (v_k, u_k) converge (up to a subsequence) weakly- $*$ to (v, u) which solve the problem for α . Hence, by the lower semicontinuity of \mathcal{H} and \mathcal{J} the assertion follows. □

Theorem 4.6. *Suppose that for all n*

$$C\delta_n \leq \liminf_{\alpha \rightarrow \infty} \mathcal{H}(v^\alpha | f_n)$$

for some constant $C > 1$, which does not depend on n .

Then the discrepancy principle (4.3) is well-posed for all $\tau \in (1, C)$, i.e. there exists $\alpha_n > 0$ and a solution $(u^{\alpha_n}, v^{\alpha_n})$ of (2.3) corresponding to $\alpha = \alpha_n$ and $f = f_n$ such that (4.3) is satisfied.

Proof. For every $\alpha > 0$ because of the feasibility of $(u_{\mathcal{J}}^\dagger, \bar{f})$ we get

$$\mathcal{H}(v^\alpha | f_n) + \alpha \mathcal{J}(u^\alpha) \leq \mathcal{H}(\bar{f} | f_n) + \alpha \mathcal{J}(u_{\mathcal{J}}^\dagger)$$

and in particular

$$h(\alpha) = \mathcal{H}(v^\alpha | f_n) \leq \delta_n + \alpha \mathcal{J}(u_{\mathcal{J}}^\dagger),$$

for almost all $\alpha > 0$. Letting $\alpha \downarrow 0$ we obtain using the monotonicity of h that

$$h(0+) \leq \delta_n. \quad (4.6)$$

On the other hand, by assumption it holds

$$C\delta_n \leq \liminf_{\alpha \rightarrow \infty} \mathcal{H}(v^\alpha | f_n). \quad (4.7)$$

Hence, in light of (4.6) and (4.7), and the monotonicity of h , there exists $\alpha_n > 0$ such that

$$h(\alpha) \leq \tau \delta_n, \quad \forall 0 < \alpha < \alpha_n,$$

and τ can be chosen in $(1, C)$. Since h is lower semicontinuous according to lemma 4.5, we get that

$$\sup_{\alpha < \alpha_n} h(\alpha) \leq \tau \delta_n$$

which proves the assertion. \square

Remark 4.7. The assumption of theorem 4.6 is rather weak. For instance, if $\mathcal{H}(0 | f_n) < \infty$, one can show that $v^\alpha \rightharpoonup^* 0$ as $\alpha \rightarrow \infty$. Hence, one can relax the assumption to $C\delta_n \leq \mathcal{H}(0 | f_n)$ which, for δ_n sufficiently small, is fulfilled in many applications.

4.2. Convergence rates

Our goal in this section is to obtain convergence rates similar to those in theorem 3.5 (respectively theorem 3.6) for the parameter choice rule (4.3).

Lemma 4.8. *Let α_n be chosen according to (4.3). Then the following inequality holds for all n*

$$\mathcal{J}(u_n^{\alpha_n}) \leq \mathcal{J}(u_{\mathcal{J}}^\dagger). \quad (4.8)$$

If conditions of theorem 2.6 are satisfied, then also the following inequality holds

$$\langle E^* \mu_n^{\alpha_n}, \bar{f} - v_n^{\alpha_n} \rangle \leq 0. \quad (4.9)$$

Proof. Comparing the value of the objective function in (2.3) at the optimal solution $(u_n^{\alpha_n}, v_n^{\alpha_n})$ and $(u_{\mathcal{J}}^\dagger, \bar{f})$ and using (4.3), we get that

$$\begin{aligned} \tau\delta_n + \alpha_n \mathcal{J}(u_n^{\alpha_n}) &= \mathcal{H}(v_n^{\alpha_n}|f_n) + \alpha_n \mathcal{J}(u_n^{\alpha_n}) \\ &\leq \mathcal{H}(\bar{f}|f_n) + \alpha_n \mathcal{J}(u_{\mathcal{J}}^\dagger) \leq \delta_n + \alpha_n \mathcal{J}(u_{\mathcal{J}}^\dagger). \end{aligned}$$

Since $\tau > 1$, this yields the first inequality.

For the second one we use the Fenchel–Young inequality. Subtracting (3.10b) from (3.10a) we obtain

$$\begin{aligned} \langle \alpha_n E^* \mu_n^{\alpha_n}, \bar{f} - v_n^{\alpha_n} \rangle &\leq \mathcal{H}(\bar{f}|f_n) + \mathcal{H}^*(\alpha_n E^* \mu_n^{\alpha_n}|f_n) - \mathcal{H}(v_n^{\alpha_n}|f_n) \\ &\quad - \mathcal{H}^*(\alpha_n E^* \mu_n^{\alpha_n}|f_n) \\ &\leq \delta_n - \tau\delta_n \leq 0, \end{aligned}$$

which completes the proof. \square

Theorem 4.9. Under assumptions of theorem 3.2 and with α_n chosen according to (4.3), $u_n^{\alpha_n}$ converges weakly-* to a \mathcal{J} -minimising solution of (1.1), i.e.

$$u_n^{\alpha_n} \rightharpoonup^* u_{\mathcal{J}}^\dagger.$$

Proof. Since $\mathcal{J}(u_n^{\alpha_n})$ is bounded uniformly in n and $\mathcal{H}(v_n^{\alpha_n}|f_n) = \tau\delta_n \rightarrow 0$, we immediately get the desired result following the proof of theorem 3.2. \square

Theorem 4.10. Let α_n be chosen according to (4.3). Then, under the assumptions of theorem 3.5, the following estimate holds for the one-sided Bregman distance between $u_n^{\alpha_n}$ and $u_{\mathcal{J}}^\dagger$

$$D_{\mathcal{J}}^{p^\dagger}(u_n^{\alpha_n}, u_{\mathcal{J}}^\dagger) \leq \langle E^* \mu^\dagger, v_n^{\alpha_n} - \bar{f} \rangle + C\eta_n,$$

where $p^\dagger = -B^* \mu^\dagger$ is the subgradient from assumption 3. Under the assumptions of theorem 3.6 the same estimate holds for the symmetric Bregman distance.

Proof. We start with the estimate (3.8). Using lemma 4.8, we obtain

$$\begin{aligned} D_{\mathcal{J}}^{p^\dagger}(u_n^{\alpha_n}, u_{\mathcal{J}}^\dagger) &\leq \mathcal{J}(u_{\mathcal{J}}^\dagger) + \langle E^* \mu^\dagger, v_n^{\alpha_n} \rangle + C\eta_n \\ &= \langle -B^* \mu^\dagger, u_{\mathcal{J}}^\dagger \rangle + \langle E^* \mu^\dagger, v_n^{\alpha_n} \rangle + C\eta_n \\ &= \langle -\mu^\dagger, E\bar{f} \rangle + \langle E^* \mu^\dagger, v_n^{\alpha_n} \rangle + C\eta_n \\ &= \langle E^* \mu^\dagger, v_n^{\alpha_n} - \bar{f} \rangle + C\eta_n, \end{aligned}$$

which yields the first assertion. For the second assertion, we use (3.10) and lemma 4.8 and obtain

$$\begin{aligned} D_{\mathcal{J}}^{\text{symm}}(u_n, u_{\mathcal{J}}^\dagger) &\leq \langle E^* \mu_n, \bar{f} - v_n^{\alpha_n} \rangle - \langle E^* \mu^\dagger, \bar{f} - v_n^{\alpha_n} \rangle + C\eta_n \\ &\leq \langle E^* \mu^\dagger, v_n^{\alpha_n} - \bar{f} \rangle + C\eta_n. \end{aligned}$$

\square

Table 1. Summary of convergence rates for different fidelities in terms of the data error δ , the operator error η and the regularisation parameter α . Whenever α is absent in the *a priori* rate, exact penalisation occurs and the rate is independent of α as long as it is smaller than a fixed constant. Optimal rates correspond to an optimal choice of α in the *a priori* rate.

Fidelity	<i>A priori</i> rate	Optimal rate	Discr. principle
KL- and χ^2 -divergences, sq. Hellinger distance	$O\left(\frac{\delta}{\alpha} + \alpha + \eta\right)$	$O(\sqrt{\delta} + \eta)$	$O(\sqrt{\delta} + \eta)$
Total variation	$O(\delta + \eta)$	$O(\delta + \eta)$	$O(\delta + \eta)$
Wasserstein- p distance	$O\left(\frac{\delta}{\alpha} + \alpha^{\frac{1}{p-1}} + \delta^{\frac{1}{p}} + \eta\right), p > 1$ $O(\delta + \eta), p = 1$	$O(\delta^{\frac{1}{p}} + \eta)$	$O(\delta^{\frac{1}{p}} + \eta)$
Characteristic function of a norm ball	$O(\delta + \eta)$	$O(\delta + \eta)$	$O(\delta + \eta)$
λ -strongly coercive fidelities	$O\left(\frac{\delta}{\alpha} + \alpha^{\frac{1}{\lambda-1}} + \delta^{\frac{1}{\lambda}} + \eta\right), \lambda > 1$ $O(\delta + \eta), \lambda = 1$	$O(\delta^{\frac{1}{\lambda}} + \eta)$	$O(\delta^{\frac{1}{\lambda}} + \eta)$

Strongly coercive fidelities. For a strongly coercive fidelity terms such that (3.21) holds, we immediately get, using the Cauchy-Schwarz inequality, that

$$\begin{aligned} D_{\mathcal{J}}^{p\dagger}(u_n^{\alpha_n}, u_{\mathcal{J}}^{\dagger}) &\leq \|E^* \mu^{\dagger}\| \|v_n^{\alpha_n} - \bar{f}\| + C\eta_n \\ &\leq \|E^* \mu^{\dagger}\| (\|v_n^{\alpha_n} - f_n\| + \|f_n - \bar{f}\|) + C\eta_n \\ &\leq \|E^* \mu^{\dagger}\| (\mathcal{H}(v_n^{\alpha_n}|f_n) + \mathcal{H}(\bar{f}|f_n))^{\frac{1}{\lambda}} + C\eta_n \end{aligned}$$

and therefore we get the following rate

$$D_{\mathcal{J}}^{p\dagger}(u_n^{\alpha_n}, u_{\mathcal{J}}^{\dagger}) = O\left(\delta_n^{\frac{1}{\lambda}} + \eta_n\right),$$

which coincides with the optimal rate in theorem 3.9.

φ -divergences. For any φ -divergence that satisfies Pinsker's inequality [52] with exponent λ

$$\|v - f\|^{\lambda} \leq C\mathcal{H}(v|f),$$

where $v, f \in \mathcal{P}(\Omega)$, we have the same situation as above. In particular, for the Kullback-Leibler divergence, the χ^2 -divergence and the squared Hellinger distance $\lambda = 2$ and

$$D_{\mathcal{J}}^{p\dagger}(u_n^{\alpha_n}, u_{\mathcal{J}}^{\dagger}) = O(\sqrt{\delta_n} + \eta_n),$$

which coincides with the optimal rate (3.17).

We summarise all convergence rates for obtained in this paper in table 1.

5. Conclusions

In this work we have proven convergence rates in Bregman distances for variational regularisation in Banach lattices for problems with imperfect forward operators and general fidelity

functions. Our results apply to many classes of fidelity functions and recover known convergence rates for norm-type fidelities and the Kullback–Leibler divergence in the case of exact operators. In addition, we have derived convergence rates for sums and infimal convolutions of fidelity functions, as used for mixed-noise removal. Furthermore, we have analysed an extension of Morozov’s discrepancy principle to problems with operator errors in the Banach lattice setting, which does not rely on the triangle inequality and hence applies to a broader class of fidelity functions.

Acknowledgments

This work was supported by the European Union’s Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 777826 (NoMADS). MB acknowledges support by the Bundesministerium für Bildung und Forschung under the project id 05M16PMB (MED4D). YK acknowledges the support of the Royal Society (Newton International Fellowship NF170045), the EPSRC (Fellowship EP/V003615/1), the Cantab Capital Institute for the Mathematics of Information and the National Physical Laboratory. CBS acknowledges support from the Leverhulme Trust project on ‘Breaking the non-convexity barrier’, the Philip Leverhulme Prize, the EPSRC grants EP/S026045/1 and EP/T003553/1, the EPSRC Centre Nr. EP/N014588/1, the Wellcome Innovator Award RG98755, European Union Horizon 2020 research and innovation programmes under the Marie Skłodowska-Curie grant agreement No. 777826 NoMADS and No. 691070 CHiPS, the Cantab Capital Institute for the Mathematics of Information and the Alan Turing Institute.

Appendix. Banach lattices and duality

The following definitions and results can be found, e.g., in [24].

Let \mathcal{U} be a vector space and ‘ \leq ’ be a partial order relation on \mathcal{U} (i.e. a reflexive, antisymmetric and transitive binary relation). For all $x, y \in \mathcal{U}$ we write $x \geq y$ if $y \leq x$. The pair (\mathcal{U}, \leq) is called an ordered vector space if the following conditions hold

$$\begin{aligned}x \leq y &\implies x + z \leq y + z \quad \forall z \in \mathcal{U}, \\x \leq y &\implies ax \leq ay \quad \forall a \in \mathbb{R}_+.\end{aligned}$$

An ordered vector space (\mathcal{U}, \leq) is called a vector lattice (or a Riesz space) if any two elements $x, y \in \mathcal{U}$ have a unique supremum $x \vee y$ and infimum $x \wedge y$. For any $x \in \mathcal{U}$ we define

$$x_+ := x \vee 0, \quad x_- := (-x)_+, \quad |x| := x_+ + x_-.$$

For any $x \in \mathcal{U}$ it holds that

$$x = x_+ - x_-.$$

Let $\|\cdot\|$ be a norm on \mathcal{U} . The triple $(\mathcal{U}, \leq, \|\cdot\|)$ is called a Banach lattice if (\mathcal{U}, \leq) is a vector lattice, $(\mathcal{U}, \|\cdot\|)$ is a Banach space (i.e. it is norm complete) and for all $x, y \in \mathcal{U}$

$$|x| \geq |y| \implies \|x\| \geq \|y\|,$$

or equivalently that $\|x\| \geq \|y\|$ for any $x \geq y \geq 0$.

A linear operator T acting between two vector lattices $\mathcal{U}_1, \mathcal{U}_2$ is called positive, and we write $T \geq 0$, if $u \geq 0$ implies $Tu \geq 0$ (the inequalities are understood in the sense of partial orders in \mathcal{U}_1 and \mathcal{U}_2 , respectively). A linear operator T is called *regular* if it can be written as a difference of two positive operators, $T = T_1 - T_2$ with $T_{1,2} \geq 0$. The space of all regular operators $\mathcal{U}_1 \rightarrow \mathcal{U}_2$ is itself an ordered vector space with the following partial order

$$T_1 \geq T_2 \iff T_1 - T_2 \geq 0.$$

Proposition A.1 ([24, proposition 1.3.5]). *Let $\mathcal{U}_1, \mathcal{U}_2$ be Banach lattices. Then every regular operator $\mathcal{U}_1 \rightarrow \mathcal{U}_2$ is (norm) continuous.*

The converse is in general false, i.e. not every continuous operator is regular. However, in some settings this is true. We repeat definition 2.5 for readers' convenience.

Definition A.2. A Banach lattice \mathcal{Y} with norm $\|\cdot\|$ is called an AM-space (abstract maximum space) if

$$\|x \vee y\| = \|x\| \vee \|y\|, \quad \forall x, y \geq 0.$$

An element $\mathbb{1} \in \mathcal{Y}$ which meets

$$\mathbb{1} \geq 0, \quad \|\mathbb{1}\| = 1, \quad \|x\| \leq 1 \implies |x| \leq \mathbb{1},$$

is called *unit* of \mathcal{Y} .

Definition A.3. A Banach lattice \mathcal{Y} with norm $\|\cdot\|$ is called an AL-space (abstract Lebesgue space) if

$$\|x \vee y\| = \|x\| + \|y\|, \quad \forall x, y \geq 0.$$

If either \mathcal{Y} is an AM-space with an order unit or \mathcal{X} is an AL-space, then every linear bounded operator is regular (under some additional conditions, see [24, theorem 1.5.11] for a precise statement).

We need the following result.

Lemma A.4 (Partial order on the dual). *Let \mathcal{U} be a Banach space and \mathcal{U}^* be its dual. If $(\mathcal{U}, \leq, \|\cdot\|)$ is a Banach lattice, then so is \mathcal{U}^* , equipped with the dual norm and the following partial order*

$$\varphi \geq 0: \iff \varphi(x) \geq 0, \quad \forall x \in \mathcal{U}, x \geq 0, \tag{A.1a}$$

$$\varphi \geq \psi: \iff \varphi - \psi \geq 0. \tag{A.1b}$$

Furthermore, order intervals in \mathcal{U}^* are weakly-* closed.

Proof. We need to check that $\varphi \geq \psi \geq 0$ implies $\|\varphi\|_{\mathcal{U}^*} \geq \|\psi\|_{\mathcal{U}^*}$. Splitting $x \in \mathcal{U}$ into positive and negative part as $x = x_+ - x_-$ with $x_{\pm} \geq 0$, we obtain by linearity and non-negativity that

$$\chi(x) = \chi(x_+) - \chi(x_-) \leq \chi(x_+), \quad \chi \in \{\varphi, \psi\}.$$

This implies

$$\|\chi\|_{\mathcal{U}^*} = \sup_{\|x\|_{\mathcal{U}}=1} \chi(x) = \sup_{\substack{\|x\|_{\mathcal{U}}=1 \\ x \geq 0}} \chi(x), \quad \chi \in \{\varphi, \psi\}.$$

Hence, we obtain

$$\|\varphi\|_{\mathcal{U}^*} = \sup_{\substack{\|x\|_{\mathcal{U}}=1 \\ x \geq 0}} \varphi(x) \geq \sup_{\substack{\|x\|_{\mathcal{U}}=1 \\ x \geq 0}} \psi(x) = \|\psi\|_{\mathcal{U}^*},$$

which proves that \mathcal{U}^* is a Banach lattice. Now we prove weak-* closedness of order intervals. Here it is sufficient to show that whenever $(\varphi_k) \subset \mathcal{U}^*$ converges weakly* to some $\varphi \in \mathcal{U}^*$ and meets $\varphi_k \geq 0$ for all $k \in \mathbb{N}$ it holds $\varphi \geq 0$. Using the assumptions we get

$$0 \leq \lim_{k \rightarrow \infty} \varphi_k(x) = \varphi(x), \quad \forall x \in \mathcal{U}, x \geq 0,$$

which according to (A.1) means $\varphi \geq 0$. \square

We also need the following result unrelated to Banach lattices.

Lemma A.5. *Let $A : \mathcal{U}^* \rightarrow \mathcal{V}^*$ be a bounded linear operator mapping between the duals of two Banach spaces \mathcal{U} and \mathcal{V} , and let $J_{\mathcal{U}}$ and $J_{\mathcal{V}}$ be the canonical embeddings of \mathcal{U} and \mathcal{V} into \mathcal{U}^{**} and \mathcal{V}^{**} . If $A^*J_{\mathcal{V}}(\mathcal{V}) \subset J_{\mathcal{U}}(\mathcal{U})$, then A is weak-* to weak-* continuous.*

Proof. Let $(\eta_k) \subset \mathcal{U}^*$ converge weakly-* to $\eta \in \mathcal{U}^*$. Using that for any $y \in \mathcal{V}$ it holds $A^*J_{\mathcal{V}}(y) = J_{\mathcal{U}}(x)$ for some $x \in \mathcal{U}$, we obtain

$$\begin{aligned} \langle A\eta, y \rangle_{\mathcal{V}^*, \mathcal{V}} &= \langle J_{\mathcal{V}}(y), A\eta \rangle_{\mathcal{V}^{**}, \mathcal{V}^*} = \langle A^*J_{\mathcal{V}}(y), \eta \rangle_{\mathcal{U}^{**}, \mathcal{U}^*} = \langle J_{\mathcal{U}}(x), \eta \rangle_{\mathcal{U}^{**}, \mathcal{U}^*} \\ &= \langle \eta, x \rangle_{\mathcal{U}^*, \mathcal{U}} = \lim_{k \rightarrow \infty} \langle \eta_k, x \rangle_{\mathcal{U}^*, \mathcal{U}} = \lim_{k \rightarrow \infty} \langle J_{\mathcal{U}}(x), \eta_k \rangle_{\mathcal{U}^{**}, \mathcal{U}^*} \\ &= \lim_{k \rightarrow \infty} \langle A^*J_{\mathcal{V}}(y), \eta_k \rangle_{\mathcal{U}^{**}, \mathcal{U}^*} = \lim_{k \rightarrow \infty} \langle J_{\mathcal{V}}(y), A\eta_k \rangle_{\mathcal{V}^{**}, \mathcal{V}^*} \\ &= \lim_{k \rightarrow \infty} \langle A\eta_k, y \rangle_{\mathcal{V}^*, \mathcal{V}}, \end{aligned}$$

which means that $(A\eta_k)$ converges weakly-* to $A\eta$. \square

Remark A.6. A sufficient condition for $A^*J_{\mathcal{V}}(\mathcal{V}) \subset J_{\mathcal{U}}(\mathcal{U})$ in lemma A.5 is that $A = B^*$ for a bounded linear operator $B : \mathcal{V} \rightarrow \mathcal{U}$. In this case $A^* = B^{**} : \mathcal{V}^{**} \rightarrow \mathcal{U}^{**}$ and it is easy to see that $B^{**}J_{\mathcal{V}}(y) = J_{\mathcal{U}}(By)$ for every $y \in \mathcal{V}$ which means $A^*J_{\mathcal{V}}(\mathcal{V}) = B^{**}J_{\mathcal{V}}(\mathcal{V}) = J_{\mathcal{U}}(B\mathcal{V}) \subset J_{\mathcal{U}}(\mathcal{U})$.

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References

- [1] Le T, Chartrand R and Asaki T J 2007 A variational approach to reconstructing images corrupted by Poisson noise *J. Math. Imaging Vis.* **27** 257–63
- [2] Hohage T and Werner F 2016 Inverse problems with Poisson data: statistical regularization theory, applications and algorithms *Inverse Problems* **32** 093001
- [3] Pöschl C 2008 Tikhonov regularization with general residual term *PhD Thesis* (Department of Mathematics, Computer Science, and Physics, Leopold-Franzens-Universität Innsbruck)

- [4] Benning M and Burger M 2011 Error estimates for general fidelities *Electron. Trans. Numer. Anal.* **38** –68
- [5] Engl H W, Kunisch K and Neubauer A 1989 Convergence rates for Tikhonov regularisation of non-linear ill-posed problems *Inverse Problems* **5** 523
- [6] Burger M and Osher S 2004 Convergence rates of convex variational regularization *Inverse Problems* **20** 1411
- [7] Resmerita E 2005 Regularization of ill-posed problems in Banach spaces: convergence rates *Inverse Problems* **21** 1303
- [8] Hofmann B, Kaltenbacher B, Pöschl C and Scherzer O 2007 A convergence rates result for Tikhonov regularization in Banach spaces with non-smooth operators *Inverse Problems* **23** 987
- [9] Grasmair M 2011 Linear convergence rates for Tikhonov regularization with positively homogeneous functionals *Inverse Problems* **27** 075014
- [10] Morozov V A 1966 On the solution of functional equations by the method of regularisation *Dokl. Akad. Nauk SSSR* **167** –
- [11] Bonesky T 2008 Morozov’s discrepancy principle and Tikhonov-type functionals *Inverse Problems* **25** 015015
- [12] Anzengruber S W and Ramlau R 2009 Morozov’s discrepancy principle for Tikhonov-type functionals with nonlinear operators *Inverse Problems* **26** 025001
- [13] Sixou B, Hohweiller T and Ducros N 2018 Morozov principle for Kullback–Leibler residual term and Poisson noise *Inverse Problems Imaging* **12** 607–34
- [14] Neubauer A and Scherzer O 1990 Finite-dimensional approximation of Tikhonov regularized solutions of nonlinear ill-posed problems *Inverse Problems* **11** 85–99
- [15] Tikhonov A N, Goncharsky A V, Stepanov V V and Yagola A G 1995 *Numerical Methods for the Solution of Ill-Posed Problems* (Dordrecht: Kluwer)
- [16] Pöschl C, Resmerita E and Scherzer O 2010 Discretization of variational regularization in Banach spaces *Inverse Problems* **26** 105017
- [17] Bleyer I R and Ramlau R 2013 A double regularization approach for inverse problems with noisy data and inexact operator *Inverse Problems* **29** 025004
- [18] Goncharskii A V, Leonov A S and Yagola A G 1973 A generalized discrepancy principle *USSR Comput. Math. Math. Phys.* **13** 25–37
- [19] Hofmann B 1986 Optimization aspects of the generalized discrepancy principle in regularization *Optimization* **17** 305–16
- [20] Lu S, Pereverzev S V, Shao Y and Tautenhahn U 2010 On the generalized discrepancy principle for Tikhonov regularization in Hilbert scales *J. Integr. Equ. Appl.* **22** 483–517
- [21] Korolev Y and Yagola A 2013 Making use of a partial order in solving inverse problems *Inverse Problems* **29** 095012
- [22] Korolev Y 2014 Making use of a partial order in solving inverse problems: II *Inverse Problems* **30** 085003
- [23] Burger M, Korolev Y and Rasch J 2019 Convergence rates and structure of solutions of inverse problems with imperfect forward models *Inverse Problems* **35** 024006
- [24] Meyer-Nieberg P 1991 *Banach Lattices* (Berlin: Springer)
- [25] Gorokh A, Korolev Y and Valkonen T 2016 Diffusion tensor imaging with deterministic error bounds *J. Math. Imaging Vis.* **56** 137–57
- [26] Schachermayer W 1981 Integral operators on L^p spaces, part I *Indiana Univ. Math. J.* **30** 123–40
- [27] Diestel J and Uhl J J 1977 *Vector Measures* (Providence, RI: American Mathematical Society)
- [28] Bonnans J and Shapiro A 2000 *Perturbation Analysis of Optimization Problems* (Berlin: Springer)
- [29] Chambolle A, Duval V, Peyré G and Poon C 2017 Geometric properties of solutions to the total variation denoising problem *Inverse Problems* **33** 015002
- [30] Robinson S M 1975 Stability theory for systems of inequalities. Part I: linear systems *SIAM J. Numer. Anal.* **12** 754–69
- [31] Burnham K P and Anderson D R 2002 *Model Selection and Multimodel Inference* (Berlin: Springer)
- [32] Benning M and Burger M 2018 Modern regularization methods for inverse problems *Acta Numer.* **27** 1–111
- [33] Engl H, Hanke M and Neubauer A 1996 *Regularization of Inverse Problems* (Berlin: Springer)
- [34] Hofmann B and Yamamoto M 2010 On the interplay of source conditions and variational inequalities for nonlinear ill-posed problems *Appl. Anal.* **89** 1705–27
- [35] Liese F and Vajda I 2006 On divergences and informations in statistics and information theory *IEEE Trans. Inf. Theory* **52** 4394–412

- [36] Hohage T and Werner F 2013 Iteratively regularized Newton-type methods for general data misfit functionals and applications to Poisson data *Numer. Math.* **123** 745–79
- [37] Werner F and Hohage T 2012 Convergence rates in expectation for Tikhonov-type regularization of inverse problems with Poisson data *Inverse Problems* **28** 104004
- [38] Bungert L and Burger M 2019 Solution paths of variational regularization methods for inverse problems *Inverse Problems* **35** 105012
- [39] Chan R H, Chung-Wa C-W and Nikolova M 2005 Salt-and-pepper noise removal by median-type noise detectors and detail-preserving regularization *IEEE Trans. Image Process.* **14** 1479–85
- [40] Santambrogio F 2015 *Optimal Transport for Applied Mathematicians* (NY: Birkäuser) vol 55
- [41] Bogachev V 2007 *Measure Theory* vol 2 (Berlin: Springer)
- [42] Grasmair M, Haltmeier M and Scherzer O 2011 The residual method for regularizing ill-posed problems *Appl. Math. Comput.* **218** 2693–710
- [43] Holler M, Huber R and Knoll F 2018 Coupled regularization with multiple data discrepancies *Inverse Problems* **34** 084003
- [44] Hintermüller M and Langer A 2013 Subspace correction methods for a class of nonsmooth and nonadditive convex variational problems with mixed $L1/L2$ data-fidelity in image processing *SIAM J. Imag. Sci.* **6** 2134–73
- [45] Langer A 2017 Automated parameter selection in the-TV model for removing Gaussian plus impulse noise *Inverse Problems* **33** 074002
- [46] Yue L, Shen H, Yuan Q and Zhang L 2014 A locally adaptive $L1-L2$ norm for multi-frame super-resolution of images with mixed noise and outliers *Signal Process.* **105** 156–74
- [47] Aggarwal H K and Majumdar A 2016 Hyperspectral unmixing in the presence of mixed noise using joint-sparsity and total variation *IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing* **9** 4257–66
- [48] Calatroni L, De Los Reyes J C and Schönlieb C-B 2017 Infimal convolution of data discrepancies for mixed noise removal *SIAM J. Imag. Sci.* **10** 1196–233
- [49] Strömberg T 1994 A study of the operation of infimal convolution *PhD Thesis* (Luleå Tekniska Universitet)
- [50] Bauschke H H and Combettes P L 2011 *Convex Analysis and Monotone Operator Theory in Hilbert Spaces* (Berlin: Springer)
- [51] Burger M and Osher S 2013 A guide to the TV zoo *Level-Set and PDE-Based Reconstruction Methods* ed M Burger and S Osher (Berlin: Springer)
- [52] Sason I and Verdú S 2016 f -divergence inequalities *IEEE Trans. Inf. Theory* **62** 5973–6006