Quasinormal Modes of Nearly Extremal Black Holes

Jason Joykutty

Supervisor: Prof. Claude Warnick

Department of Applied Mathematics and Theoretical Physics
University of Cambridge

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To my parents, Joykutty and Glancy.

Declaration

I declare that except where specific reference is made to the work of others, the content of this dissertation is original work which has not been submitted in whole or in part for consideration for any other degree or qualification in this or any other university. Furthermore, it does not contain work done in collaboration with others except as specified in the text and the acknowledgements: Chapters 1 and 2 review the literature surrounding quasinormal modes while Chapter 3 contains calculations which were completed before and independently of [75]. Chapters 4 and 5 contain original results from [92] and 6 contains original work which has not been published in a journal or appeared in a preprint. This dissertation does not exceed the prescribed word limit for the relevant Degree Committee.

Jason Joykutty
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Quasinormal modes are the gravitational wave analogue to the overtones heard after striking a bell; like many physical systems, black holes emit radiation as a response to perturbations. After a dynamical event, for example a black hole merger, the system is expected to relax to a stationary black hole solution. After sufficient time, the system can be treated as a perturbation to this stationary solution in what is called the ringdown phase. The observed gravitational wave signal is dominated by the ringing associated with these solutions to the linear perturbation equations in this period of the evolution. Each quasinormal mode is characterised by a complex frequency which encodes its behaviour in time: the imaginary part determines its oscillation and the real part its exponential decay.

In light of the observation of gravitational wave signals in the past few years, quasinormal modes are important from an astronomical perspective. By comparing the observed gravitational wave signal from some dynamical event with the predictions provided by computing quasinormal frequencies, one can compare the fit given by general relativity against some modified theory of gravity and test which is a better model for these phenomena. This black hole spectroscopy could also be used to deduce the parameters of an astrophysical object from the gravitational wave signal.

As horizons become extremal, various computations (from a range of authors including Detweiler, Hod and Zimmerman) have shown that in many cases, there exists a sequence of frequencies which become purely oscillatory in the limit and which cluster on a line in the complex plane. These zero-damped modes are typically the most slowly decaying resonances of the equation and hence are key to understanding stability. In the case of a positive cosmological constant, they are closely tied to the Strong Cosmic Censorship Conjecture: if the spectral gap is too large, the modes don't decay slowly enough to destabilise the Cauchy horizon.

From the large variety of examples in the literature of nearly extremal black holes with zero-damped modes, it is natural to conjecture that this phenomenon is generic. This thesis explores mathematically rigorous results that can be obtained toward resolving this question. In particular, we shall review the literature on quasinormal modes (focussing on zero-damped modes), discuss the mathematical definition of these objects and the idea of co-modes or dual resonant states: solutions to the adjoint problem which can make identifying the frequencies easier. Finally, we shall use this framework and Gohberg-Sigal theory to prove existence results for zero-damped modes: firstly in the case of wave equations with potentials which decay sufficiently rapidly, then for a large class of static, spherically symmetric black hole spacetimes. There are also partial results toward resolving the question for the Kerr-de Sitter spacetime.

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Finally, but certainly not least, I would like to thank my family for their love and support throughout my life, especially my parents to whom I dedicate this thesis.

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Contents

Chapter 1

Introduction

Quasinormal modes are the gravitational wave equivalent of the overtones heard when striking a bell. After a dynamical event involving black holes (for example, a binary merger), there is a short initial burst of gravitational radiation where the signal is dominated by highly non-linear, strong-field effects. As the system relaxes, the non-linearities die down and one can use linear perturbation theory around some stationary black hole solution to study the gravitational field. Just as systems where energy is conserved in a compact region have normal modes with imaginary frequencies associated with pure oscillation, a dispersive system where energy leaks out (for example through a black hole horizon) has *quasinormal* modes with *complex* frequencies which encode decay as well.

These mode solutions have been observed to dominate the evolution of the system for a period of time called the ringdown phase, in analogy with a bell (see [1, 2]). The late time behaviour is determined by the asymptotics of the spacetime and hence the sign of the cosmological constant [38]. For $\Lambda > 0$, the system continues to ring down to the stationary black hole solution (see [23, 73, 102, 116]). If the spacetime is asymptotically flat ($\Lambda = 0$), the ringdown is followed by a Price's Law tail where the signal decays inverse polynomially in time (see [119, 120, 9, 10, 72, 127]). For spacetimes with $\Lambda < 0$, the decay is even slower than inverse polynomial, see [84].

Despite initially being described implicitly in the pioneering work of Regge and Wheeler in black hole perturbation theory [121], the phenomenon was first observed by the numerical simulations of Vishveshwara in [138]. Incident wavepackets of gravitational radiation were scattered after intermediate times into solutions with damped sinusoidal time-dependence whose frequencies depended only on the parameters of the black hole. This was supported by further numerics performed by Press in [118], where the term quasinormal mode was coined.

With the advent of gravitational wave detectors, it has been discovered that this is a phenomenon typical in black hole dynamics: detected gravitational wave signals have exhibited ringdown where they are dominated by a superposition of decaying sinusoids [1, 2, 128]. Furthermore, it has opened up the prospect of black hole spectroscopy: attempting to extract the ringing quasinormal frequencies from the observed data [52, 85]. An isolated, astrophysical black hole is expected to be characterised completely by its mass and angular momentum by the no-hair theorem, since its electric charge will be small and can be taken to be zero. The black holes in modified theories of gravity may possess additional 'hair' (parameters required to fully characterise the solution) which will affect their quasinormal spectra [133, 96, 58, 104]. Identifying black holes with such 'hair' through observation would identify violations of general relativity so gravitational wave data can be used to compare proposed models extending it [3, 24, 4, 25, 86, 66].

While the considerations above are promising, stability of the predicted quasinormal spectrum is a key issue [45]. Naturally the black hole solutions used to provide predictions are idealised, isolated models and the astrophysical black holes which are observed will have gravitational fields which differ (at least slightly) due to the other matter content of the universe. If quasinormal spectra are very sensitive to perturbations (they exhibit spectral instability), then the predictions of these idealisations will not be of much utility when compared to observation.

The first evidence of such spectral instability was given by Nollert in [108]. This was followed up in [90, 46], where the authors use pseudospectral methods to analyse the sensitivity of the spectrum to perturbing potentials. The cause of these instabilities can be made apparent in the framework of regularity quasinormal modes (see Chapter 2). The pseudospectral techniques make use of the energy norm (which involves only one derivative) to measure the size of the perturbing potential, however we shall see that for higher overtones it is more appropriate to use Sobolev norms including more derivatives. These higher regularity norms can grow quite rapidly as more derivatives are included, so perturbations that are small in the energy norm can turn out to be large when considered from the regularity point of view: the concept of spectral stability naturally depends on the notion of size used to measure the perturbation.

Although the pseudospectral methods in [90, 46] implied the possibility of an instability in the fundamental mode, none was observed in the examples considered. This lines up with the regularity quasinormal mode picture since this frequency would not require high regularity norms to control it. However perturbations which reasonably model physical phenomena have been constructed which destabilise even this mode [36]. Although it has subsequently been pointed out that this instability is not detectable in the currently observable phase of the ringdown waveform [21], this is still of mathematical interest and warrants further investigation.

In spacetimes of positive cosmological constant, quasinormal modes play an even greater role in the late time behaviour since there is no tail. When considering linear fields on these backgrounds, one can obtain precise asymptotics: the field can be expressed as a sum of quasinormal modes and an exponentially decaying remainder at late times. Understanding the asymptotics of linear fields is a key first step in proving non-linear stability results such as the proof of the stability of slowly rotating Kerr-de Sitter black holes [74], though it is often sufficient to have mode stability and detailed knowledge of the spectrum is not necessary. The overall decay of a perturbation is determined by the frequency of the fundamental mode and is exponential. If this rate of decay is too quick, this could lead to a violation of the strong cosmic censorship conjecture since there isn't sufficient time for the perturbation to be blueshifted and destabilise the Cauchy horizon. For discussions of this idea see [28, 29, 44, 47] for applications to Reissner-Nordström-de Sitter black holes and [50, 42] for the rotating case.

1.1 Black hole perturbation theory

Quasinormal modes have a history spanning nearly 70 years in the physics literature, beginning with the work of Regge and Wheeler on perturbations to the Schwarzschild black hole [121]. Since this seminal work, the Einstein field equations have been linearised around several other black hole spacetimes and the theory of quasinormal modes has developed around these perturbative settings [93, 109, 20]. In this section, we shall review their origins in black hole perturbation theory.

1.1.1 Spherically symmetric black holes

In [121], the authors considered perturbations to the Schwarzschild metric

$$g_0 = -\left(1 - \frac{2\,\mathfrak{m}}{r}\right) dt^2 + \left(1 - \frac{2\,\mathfrak{m}}{r}\right)^{-1} dr^2 + r^2 \not{g},$$

where $\not{g}$ is the usual round metric on the unit sphere. Writing the perturbed metric as $g = g_0 + h$, this is a vacuum spacetime if it solves the equation

$$\mathrm{Ric}[g] = 0,$$

where $\mathrm{Ric}[g]$ is the Ricci tensor of the Lorentzian metric g. Since we take h to be small, we can linearise the above equation around g_0 to obtain a linear partial differential equation for h:

$$\mathcal{R}_{g_0} h = 0.$$

The spherical symmetry of g_0 implies that it should be possible to separate the angular dependence from the equation using spherical harmonics. The naïve approach of decomposing each component function does not achieve this since it doesn't take into account how h as a whole transforms under rotations. Regge and Wheeler noted that under these transformations, h is composed of three scalar functions, two vectors and a tensor:

$$\begin{pmatrix} h_{00} & h_{01} & \mathbf{v_1} \\ * & h_{11} & \mathbf{v_2} \\ & * & T \end{pmatrix},$$

where $*$ denotes components determined by symmetry. Each of these objects can now be decomposed into a suitable generalisation of spherical harmonics and it suffices to consider radial wave equations in each angular momentum sector (l, m). We can simplify the problem further by considering the behaviour of the tensor spherical harmonics under inversion (the map which sends a point on the sphere to its antipode): some components gain a factor of $(-1)^{l+1}$ (called 'odd' or axial perturbations), while even or polar perturbations change by a factor of $(-1)^l$. The terminology odd and even can cause confusion, so we shall use the axial/polar terms used in [34].

The spherical symmetry of the background spacetime implies that the axial and polar perturbations cannot mix and thus can be considered separately. By choosing the Regge-Wheeler gauge, the equations for the axial perturbations simplify immensely: all the information required to construct the perturbation can be found by solving a single radial wave equation with an effective potential:

$$-\partial_t^2 Q + \partial_{r_*}^2 Q + V_{RW} Q = 0, \tag{1.1.1}$$

where $r_* = r - 2\,\mathfrak{m} + \log(r - 2\,\mathfrak{m})$ is a tortoise coordinate and (taking r as a function of r_*)

$$V_{RW} = \left(1 - \frac{2\,\mathfrak{m}}{r}\right)\left(\frac{6\,\mathfrak{m}}{r^3} - \frac{l(l+1)}{r^2}\right)$$

is called the Regge-Wheeler potential [121]. The polar perturbation equations are more complicated and don't simplify as readily, however by choosing a suitable gauge and auxiliary functions, one can also reduce these to a single wave equation in one-dimension:

$$-\partial_t^2 Q + \partial_{r_*}^2 Q + V_Z Q = 0, \tag{1.1.2}$$

where (setting $k = (l-1)(l+2)/2 \in \mathbb{Z}$)

$$V_Z = -\left(1 - \frac{2\,\mathfrak{m}}{r}\right)\left(\frac{2k^2(k+1)}{(kr+3\,\mathfrak{m})^2} + \frac{6k^2\,\mathfrak{m}}{r(kr+3\,\mathfrak{m})^2} + \frac{18k\,\mathfrak{m}^2}{r^2(kr+3\,\mathfrak{m})^2} + \frac{18\,\mathfrak{m}^3}{r^3(kr+3\,\mathfrak{m})^2}\right)$$

is called the Zerilli potential [144].

Chandrasekhar [33, 34] took an alternative but equivalent approach to the above which makes the axial/polar terminology clearer. Noting that the background metric is spherically symmetric, the first order perturbation h can be decomposed using spherical harmonics and each (l, m) sector can be analysed separately. Since each harmonic is symmetric about some axis, it suffices to insert an axisymmetric ansatz

$$g = -e^{2\nu}dt^2 + e^{2\psi}\left(d\phi - \omega dt - q^r dr - q^\theta d\theta\right)^2 + e^{2\lambda}dr^2 + e^{2\mu}d\theta^2 \tag{1.1.3}$$

into the Einstein equations. The Schwarzschild solution satisfies

$$\omega_0 = q_0^r = q_0^\theta = 0, \qquad e^{2\nu_0} = e^{-2\lambda_0} = 1 - \frac{2\,\mathfrak{m}}{r}, \qquad e^{\mu_0} = r,$$

so in the Einstein equations for Equation (1.1.3), ω, q^r, q^θ are small parameters (these are the axial perturbations since one can see from the metric that they introduce rotation about the axis of symmetry) and $\lambda - \lambda_0, \mu - \mu_0, \nu - \nu_0$ are taken to be small (these are called polar perturbations since they do not correspond to rotation). Taking the full Einstein equations to first order in these parameters and performing various simplifications also leads to the equations above.

If we now seek solutions of the above equations with the e^{st} t-dependence we expect for resonant states[1]), we obtain elliptic equations of the form

$$\partial_{r_*}^2 Q + \left(V - s^2\right) Q = 0,$$

for which we require boundary conditions to impose. A natural choice motivated by the physics of the situation is that there is no radiation coming in from infinity and none escaping the black hole horizon i.e. the solution looks like a purely outgoing wave near infinity and a purely ingoing wave at the horizon. Taking these boundary conditions in a suitable sense (for a full discussion making this notion more precise, see the next chapter) we may pick out a discrete set of frequencies s (the quasinormal spectrum) and solutions corresponding to them (the quasinormal modes). Chandrasekhar [33, 35] showed that through suitable differential transforms one can go from the Regge-Wheeler equation to the Zerilli equation and back again. Through this relationship, he furthermore showed that the two equations have the same quasinormal spectrum.

One can similarly linearise the Einstein-Maxwell system around the Reissner-Nordström solution (which models a spherically symmetric charged black hole) and find that the problem similarly reduces to the analysis of a one-dimensional wave equation with a

[1] Note that this is notation following [93, 110, 139], which is more convenient when using the Laplace transform. More frequently in the literature, the Fourier transform is used with frequency $\omega = is$.

potential [145, 105, 106]. Much like the Schwarzschild case, it can also be shown that the quasinormal spectra of the axial and of the polar perturbations can be deduced from each other [32].

1.1.2 Rotating black holes

The Kerr metric modelling an isolated rotating black hole can be expressed in Boyer-Lindquist coordinates as

$$g = -\frac{\mu}{\rho^2}\left(dt - \mathfrak{a}\sin^2 d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+\mathfrak{a}^2)d\phi - \mathfrak{a}\,dt\right)^2 + \frac{\rho^2}{\mu}dr^2 + \rho^2 d\theta^2$$

where $\mathfrak{m}$ is the mass of the black hole, $\mathfrak{a}$ is the Kerr parameter (its angular momentum per unit mass) and we have

$$\mu = r^2 + 2\,\mathfrak{m}\,r + \mathfrak{a}^2, \qquad \rho^2 = r^2 + \mathfrak{a}^2\cos^2\theta.$$

The perturbation equations for this spacetime metric are more complicated than the static, spherically symmetric cases considered above, however it is of Petrov type D and hence the Newman-Penrose formalism is especially effective in its analysis. Indeed, in [135, 134], Teukolsky used this formalism to obtain perturbation equations for two of the Weyl scalars:

$$\begin{aligned}
&\mu^{-\sigma}\partial_r\left(\mu^{\sigma+1}\partial_r\Psi\right) + \frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta\Psi\right) + \left[\frac{1}{\sin^2\theta} - \frac{\mathfrak{a}^2}{\mu}\right]\partial_\phi^2\Psi \\
&\quad + 2\sigma\left[\frac{\mathfrak{a}(r-\mathfrak{m})}{\mu} + \frac{i\cot\theta}{\sin\theta}\right]\partial_\phi\Psi - \frac{4\,\mathfrak{m}\,\mathfrak{a}\,r}{\mu}\partial_t\partial_\phi\Psi - \left[\frac{(r^2+\mathfrak{a}^2)^2}{\mu} - \mathfrak{a}^2\sin^2\theta\right]\partial_t^2\Psi \\
&\quad + 2\sigma\left[\frac{\mathfrak{m}(r^2-\mathfrak{a}^2)}{\mu} - r - i\,\mathfrak{a}\cos\theta\right]\partial_t\Psi - \sigma(\sigma\cot^2\theta - 1)\Psi = 0
\end{aligned}$$

where $\sigma = \pm 2$ is the spin weight of the field Ψ. These are also the perturbation equations for other linearised fields on the Kerr background: $\sigma = \pm 1$ corresponds to Maxwell fields and $\sigma = 0$ to scalar fields. We again seek stationary solutions and separate into angular modes i.e. set $\Psi = e^{st}e^{im\phi}\psi$. The resultant equation for $\psi = R(r)S(\theta)$ separates further into a pair of coupled ordinary differential equations

$$\begin{aligned}
\mu^{-\sigma}\partial_r\left(\mu^{\sigma+1}\partial_r R\right) + V_r R = 0 \\
\frac{1}{\sin\theta}\partial_\theta\left(\sin\theta\partial_\theta S\right) + V_\theta S = 0
\end{aligned}$$

where, setting the separation constant to be A and $K = (r^2 + \mathfrak{a}^2)is - \mathfrak{a}\, m$,

$$V_r = \frac{K^2 - 2i\sigma(r - \mathfrak{m})K}{\mu} - 4s\sigma r - A + s^2\, \mathfrak{a}^2 + 2ism\, \mathfrak{a},$$
$$V_\theta = -s^2\, \mathfrak{a}^2 \cos^2\theta - \frac{m^2}{\sin^2\theta} - 2i\sigma s\, \mathfrak{a} \cos\theta - \frac{2m\sigma \cot\theta}{\sin\theta} - \sigma^2 \cot^2\theta + \sigma + A.$$

Similarly to the spherically symmetric cases, one can define boundary conditions at the black hole event horizon and asymptotic infinity to pick out a discrete set of quasinormal frequencies and modes.

1.2 Finding the quasinormal spectrum

Much like computing the eigenvalues of an elliptic operator (like the Laplace-Beltrami operator on a manifold), there are few examples where one can find exactly the quasinormal spectrum of a spacetime. One example of note is the radial wave equation with a Pöschl-Teller potential,

$$V = \frac{V_0}{\cosh^2 \kappa(r_* - \bar{r}_*)},$$

where V_0 and $\bar{r}_*$ are parameters characterising the peak of the potential and κ a parameter characterising its curvature at the peak. The quasinormal spectrum of such a system can be found exactly to be

$$s = -\left(n + \frac{1}{2}\right)\kappa \pm i\sqrt{V_0 - \frac{\kappa^2}{4}}$$

for $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. This is closely related to the quasinormal modes of Klein-Gordon equation on the de Sitter spacetime (see Chapter 3) since similar techniques are used to find the frequencies in that case. To tackle more complicated examples such as Schwarzschild, an early technique [57] was to approximate the true Regge-Wheeler or Zerilli potential with a Pöschl-Teller potential of suitable parameters. This provides good agreement with the numerically computed frequencies which improves in the eikonal limit $l \to \infty$ using WKB methods (which we shall discuss below).

The intuition for this approximation is that qualitatively the Pöschl-Teller potential has a similar shape to the actual potentials in consideration for Schwarzschild: $V \to 0$ as $r_* \to \pm\infty$ and V has a unique maximum. However, there is some subtlety with the asymptotic behaviour: while all the potentials decay exponentially as $r_* \to -\infty$, the Schwarzschild potentials decay much more slowly as $r_* \to \infty$ (inverse polynomially). It is for this reason that the Schwarzschild-de Sitter spacetime is a natural candidate for using this approximation since the potential in this case has exponential decay towards both asymptotic regions, taking $r_* \in (-\infty, \infty)$ to be the region between the event and cosmologial horizons. Indeed, this method is particularly effective for Schwarzschild-de Sitter black holes in the Nariai extremal limit as the event and cosmological horizons coalesce [107, 26].

The WKB (or JWKB) method was initially developed to approximate solutions for a large class of differential equations by Jeffreys [91] before the independent work of Wentzel, Kramers and Brillouin applying it to the Schrödinger equation. The technique was adapted to the problem of identifying quasinormal modes and frequencies for the Schwarzschild black hole by Schutz and Will [126]. The idea is to relate the two WKB solutions corresponding to each asymptotic end across a matching region whose end-points are characterised by $V - s^2 = 0$. Assuming these end-points are close to each other (i.e. $|V_0 - s^2| \ll |s^2|$ where V_0 is the value of V at its peak), the potential in this region can be approximated by a quadratic:

$$V - s^2 = V_0 - s^2 + \frac{1}{2}V_0'' x^2 + O(x^3)$$

where $x = r_* - \bar{r}_*$. The resulting equation can be solved in terms of special functions and imposition of appropriate boundary conditions for quasinormal modes yields the Bohr-Sommerfeld type condition

$$\frac{V_0 - s^2}{\sqrt{2V_0''}} = i\left(n + \frac{1}{2}\right). \tag{1.2.1}$$

The above expression was improved by Iyer and Will [88, 87] through considering a sixth order Taylor polynomial approximation for the potential and taking the WKB expansion to third order to obtain corrections (see also more recent work by Konoplya going to sixth order in WKB [94]).

This semianalytic approach is best suited to picking up slowly decaying modes in the eikonal limit $l \to \infty$ since the quantity $l(l+1)$ appearing in the potential is a natural large parameter to use in the WKB expansion. Furthermore, the frequencies picked up by this method have a nice physical interpretation: they can be thought of as waves trapped on the light ring (the unstable circular null geodesic) which slowly leak out, either escaping to infinity or into the black hole region. This idea was first proposed by Goebel [67] and subsequently worked out in more detail by Mashhoon [101]. The more recent work [27] goes into even more depth and shows that WKB frequencies can be written in the form

$$s = -|\lambda|\left(n + \frac{1}{2}\right) - il\Omega$$

where l is the angular momentum of the geodesic, Ω is its angular velocity and $|\lambda|$ is its Lyapunov exponent, a quantity that measures the instability of the orbit. This was extended to Kerr spacetimes in [142]: using the geometric-optics approximation, one can construct approximate solutions to the wave equation in the eikonal limit using null geodesics and relate these to quasinormal modes. This idea has also appeared in the mathematical literature, see for example [125, 117] studying the problem in spherical symmetry and [54] in axisymmetry. The authors of these papers obtain lattices approximating quasinormal frequencies with $l \gg 1$.

Many numerical approaches have been developed since the initial methods of Vishveshwara [138] observed the phenomenon and identified frequencies from analysing the resulting waveform. These are typically applied to the stationary problem: one of the first examples is the shooting approach of Chandrasekhar and Detweiler [35]. The idea of this method is to solve the equations with appropriate boundary conditions at both asymptotic ends and identify frequencies with a matching condition (see the next chapter for more details). A more effective numerical method is the continued fraction method of Leaver [97], where conditions on the Frobenius expansion of the solution and the resulting recurrence relations are leveraged to provide a continued fraction expression whose roots are precisely quasinormal frequencies. More recently, Galerkin or pseudospectral methods have been used due to their robustness near extremality.

1.3 Zero-damped modes

In the examples we have considered so far, we have seen multiple approximate expressions for the quasinormal frequencies that take the form

$$s = -\lambda\left(n + \frac{1}{2}\right) + i\Omega \tag{1.3.1}$$

for $n \in \mathbb{N}_0$. In many of these examples, the parameter λ is proportional to the surface gravity κ of a Killing horizon present in the background spacetime and in the limit that $\kappa \to 0$, s becomes purely imaginary (i.e. the mode becomes purely oscillatory). This zero surface gravity limit of a horizon can be thought of as an extremal limit since this is precisely what occurs as the Cauchy and event horizons of Kerr or Reissner-Nordström black holes coalesce.

In the case of a spacetime with cosmological constant $\Lambda > 0$, we see that the surface gravity associated to the cosmological horizon $\kappa = O(\sqrt{\Lambda})$. Thus the limit $\kappa \to 0$ corresponds to the singular limit where the cosmological horizon goes to infinity and the spacetime becomes asymptotically flat. Using a suitable compactification, we see that the asymptotically flat end of the limiting spacetime is conformal to an extremal horizon and we can interpret asymptotically flat ends as 'extremal horizons at infinity'. Since the structure of the equations is very similar in these different scenarios, we can investigate a nearly extremal event horizon using a conformal transformation and considering the resulting 'nearly extremal' cosmological horizon (see Section 5.3 in Chapter 5).

When studying the quasinormal spectrum of spacetimes with extremal horizons (or asymptotically flat ends), we find that the resolvent operator associated with this problem (see Chapter 2) develops a branch cut emanating from a purely imaginary value which contributes the Price's law tail observed at late times. Returning to Equation (1.3.1) in the extremal limit $\kappa \to 0$, we also see that the family of frequencies cluster closer together on the line $\operatorname{Im} s = \Omega$. Since there are infinitely many frequencies on this line which are almost stationary in the extremal limit, we can informally think of these as forming the branch cut described above. For this thesis, quasinormal modes whose frequencies exhibit this behaviour are called zero-damped modes, although other authors have used different terminology.

This phenomenon has been observed in several spacetimes with horizons approaching extremality. For example, in [140, 141], the authors study the quasinormal modes of the Teukolsky equation for nearly extremal Kerr. After decomposing the equation using the symmetries of Kerr, they use both a WKB analysis in the eikonal limit and the method of matched asymptotic expansions to find zero-damped modes. The latter approach to estimating quasinormal modes uses the fact that near extremality, one can approximate the equation far away from the black hole to one which can be solved exactly using confluent hypergeometric functions and outgoing boundary conditions can be imposed. A similar approximation can be made in the near horizon region and conditions for quasinormal frequencies can be obtained through matching these asymptotic solutions in an intermediary region. The slowly decaying modes were first identified using this method by Detweiler [48], however with a mistake: this was corrected by Hod [82].

The above works found that in nearly extremal Kerr spacetimes, there are sequences of frequencies which cluster on the lines $\operatorname{Im} s = -m\Omega_H$ for each m, where m is the azimuthal mode number of the quasinormal mode and Ω_H is the horizon frequency. The key observations in [140, 141] were the existence of these modes and the fact that the spectrum bifurcates: there is a critical value of $m/(l + 1/2)$ below which there are both zero-damped modes and what the authors called damped modes (quasinormal modes which still exhibit decay in the extremal limit). Above this critical value, only the zero-damped modes persist: this is related to the fact that with respect to the usual radial coordinates, the zero-damped modes are concentrated near the turning point in the potential associated with the horizon (in contrast with the WKB damped modes associated with the peak of the potential).

The more general case of nearly extremal Kerr-Newman black holes was discussed in [80] for the slowly rotating case and more recently by Zimmerman and Mark for tractable fields in [146] without that assumption. In the latter paper, the authors tackle the

Dudley-Finley equation for general nearly extremal Kerr-Newman spacetimes (this is a toy model for the full gravitational perturbation equations) and find approximate expressions for both zero-damped and damped modes for any value of $\mathfrak{a}$. Existence for general perturbations is supported by the numerical work provided in [51]. Zimmerman and Mark also consider gravito-electromagnetic perturbations of near-extremal Reissner-Nordström in [146] and demonstrate the existence of zero-damped modes in this case.

The case of a charged scalar field on a Reissner-Nordström background was discussed in [78, 81] and later in [148] using similar techniques to [82, 140, 141]. Since both situations contain a $U(1)$ symmetry (Kerr is axisymmetric, while the charged scalar field has a gauge symmetry), many results carry over from one of the cases to the other by simply swapping q for m where q is the charge of the scalar field and m is the azimuthal mode number of a Kerr perturbation. As such, the phenomenon of zero-damped modes arises in this situation as well.

This phenomenon has also been observed on a Reissner-Nordström-de Sitter background for both scalar [28, 29] and fermionic [44] fields. These results have been generalised to higher dimensions [99].

As we can see from the above, the importance of quasinormal modes in linearised gravity and their application to observation has naturally led to a great deal of theoretical and numerical study. Many authors have computed the quasinormal spectra of a variety of black hole spacetimes (see Table 1.1).

Background spacetime	Field	Extremal limit	References
Kerr	Teukolsky	rapidly rotating	[82, 77, 140, 141]
Kerr	Klein-Gordon	rapidly rotating	[79]
Kerr-Newman (KN)	Dudley-Finley	rapidly rotating	[146]
KN (slowly rotating)	Teukolsky	extremal charge	[80, 146]
Reissner-Nordström (RN)	GEM[2]	extremal charge	[146]
RN	charged scalar	extremal charge	[81, 78, 83, 148]
RN-de Sitter (RNdS)	massless scalar	extremal charge	[28]
RNdS	charged scalar	extremal charge	[29]
RNdS	charged fermion	extremal charge	[44]
De Sitter	Klein-Gordon	$\Lambda \to 0$	[75]
Schwarzschild-de Sitter	Klein-Gordon	$\Lambda \to 0$[3]	[76]
Kerr-de Sitter	Klein-Gordon	$\Lambda\,\mathfrak{m}^2 \to 0$	[73]

Table 1.1: A list of fields on $3+1$-dimensional spacetime backgrounds that exhibit zero-damped modes.

In light of these observations, it is natural to ask if this phenomenon is generic. Sup-

[2] gravito-electromagnetic perturbations

[3] The limit considered explicitly in the paper is $\mathfrak{m} \to 0$, however by this is in some sense equivalent to $\Lambda \to 0$ via scaling transformations (see Chapter 5).

posing that it is, establishing the link between these zero-damped modes and horizons becoming extremal could provide an understanding of the polynomial decay observed in extremal or asymptotically flat spacetimes in terms of the limit of a mode sum. Furthermore, finding generic features of quasinormal spectra will be of use when studying gravitational wave signals and is especially of interest due to astrophysical evidence of the existence of extremal or nearly extremal black holes. In this thesis, we shall present some results proving that a range of spacetimes with suitable assumptions exhibit this phenomenon.

An outline of the thesis. In Chapter 2, we lay out the setting in which we study the problem. First we review the two more traditional approaches to mathematically characterising quasinormal modes in the literature: the first using ingoing/outgoing boundary conditions and the second defining quasinormal modes as resonances of the equation. We settle on a third, more recent definition referred to as 'regularity' quasinormal modes, which has two key advantages above the former approaches: it allows for initial data supported on horizons and one can show any quasinormal mode characterised by the other two definitions is also a regularity quasinormal mode. The rest of the chapter sets up the tools we use from Gohberg-Sigal theory and provides the definition of quasinormal co-modes (or dual resonant states in [75]), essentially distributional solutions to the original quasinormal mode problem.
In Chapter 3, we apply the definitions and tools described in the previous chapter to a concrete example: the static patch of de Sitter space. This is one of the simplest examples of a spacetime with a horizon suitable for this treatment and provides a nice model problem which will be used in the later chapters. After performing the well-known calculation to find the quasinormal frequencies and corresponding modes for a Klein-Gordon field on this spacetime, we construct the co-modes as distributions.

In Chapter 4, we present results from [92] proving continuity of the quasinormal spectrum for a Klein-Gordon field in de Sitter when the equation is altered by adding a smooth potential (Theorem 4.3.1). Assuming the potential is spherically symmetric, we can even obtain a series expansion for the quasinormal frequencies of the new equation (Theorem 4.4.1). The main result of the chapter establishes the existence of zero-damped modes for suitably decaying potentials:

Theorem 4.5.1 (rough version). *Let $V \in C^\infty(\mathbb{R})$ such that V and its derivatives decay faster than an inverse square and its derivatives respectively. Let g be the metric on the static patch of de Sitter described in Section 3.1.2 with cosmological constant $3\kappa^2$. Then the equation*

$$-\Box_g \psi + 2\kappa^2 \psi + V\psi = 0$$

exhibits the phenomenon of zero-damped modes converging to 0 as $\kappa \to 0$.

In Chapter 5, we study the more complicated problem concerning zero-damped modes when the metric differs from de Sitter, also appearing in [92]. Beyond assuming the metric is still static and spherically symmetric, we place additional technical conditions on the form it takes which still allow a fairly generic class of spacetimes including Schwarzschild-de Sitter and Reissner-Nordström-de Sitter. The main result of this chapter is

Theorem 5.2.1 (rough version). *Let $(\mathcal{M}, g)$ be a spacetime with a metric satisfying the conditions outlined at the start of Chapter 5: in particular with an event horizon and a cosmological horizon and let $R[g]$ be its Ricci scalar. Then in the limit as the cosmological horizon becomes an asymptotically flat end, the equation*

$$-\Box_g \psi + \frac{R[g]}{6}\psi = 0$$

exhibits the phenomenon of zero-damped modes converging to 0.

A straightforward applications of this result obtains:

Corollary 5.3.1 (rough version). *Consider the conformal Klein-Gordon equation on the Reissner-Nordström-de Sitter spacetime:*

$$-\Box_g \psi + \frac{R[g]}{6}\psi = 0.$$

This equation exhibits the phenomenon of zero-damped modes converging to 0 in the limit the Cauchy and event horizons coalesce.

Finally, in Chapter 6, we study the problem on the Kerr-de Sitter spacetime which models a rotating black hole. Using similar arguments to Chapter 5, one can prove existence of zero-damped modes in the asymptotically flat limit, so it is more interesting to consider the limit the black hole event horizon spins up to extremality. The main result of this chapter is

Theorem 6.3.1 (rough version). *Pick $N \in \mathbb{N}$. Then there exists a region R of the Kerr-de Sitter parameter space including points arbitrarily close to the rapidly-rotating extremal boundary such that any spacetime with parameters in R has N frequencies well-approximated by zero-damped modes associated with the event horizon.*

This is a weaker result than the previous chapter where we had stronger symmetry assumptions: it falls short of proving existence of the full family of zero-damped modes in the extremal limit. However, if we simultaneously take the cosmological constant to zero as we spin the black hole up, we recover existence of an infinite family of frequencies:

Corollary 6.3.1. *Consider the wave equation in a subextremal Kerr-de Sitter spacetime. Then in any extremal Kerr limit, it exhibits the phenomenon of zero-damped axisymmetric modes converging to 0.*

Throughout we shall use geometric units $G = c = 1$ and the mostly positive metric signature $(-,+,+,+)$.

Chapter 2

Mathematical Framework

In this chapter, we shall introduce and review the mathematial tools we shall make use of in this thesis. We begin by discussing the three main characterisations of quasinormal modes. The first approach to characterising quasinormal modes (which we discuss in more detail in Section 2.1.1) uses ingoing and outgoing boundary conditions to pick out a special set of solutions to the equation in question. In Section 2.1.2, we outline the second definition which considers quasinormal modes as resonances of the equation: the frequencies are identified as poles of the resolvent operator and quasinormal modes are related to the residues of these poles.

While these are two very natural approaches to the problem, it is not clear how they are related to each other: it is entirely possible that there is a situation where the two definitions will pick out different sets of quasinormal modes and frequencies. For this reason (and a few others, see Section 2.1.3), we settle on the notion of regularity quasinormal modes: one can prove that any quasinormal mode or frequency identified by either of the other two methods will also be picked out by this approach. The key idea is to change time coordinate so that the constant-time slices incorporate the ingoing/outgoing boundary conditions into their geometry. In this setting, the equation takes a suitable form to define a time-translation operator whose eigenvalues are quasinormal frequencies and whose eigenvectors are quasinormal modes: this definition is discussed in Section 2.1.3 and can be understood more explicitly through application in Section 3.1.2.

After providing a mathematical definition of zero-damped modes for use in the statement of results later, we review the theory of meromorphic families of Fredholm operators in Section 2.2. Besides the basic definitions and properties, we will also outline various useful results including those of Gohberg-Sigal theory (generalisations of standard complex analysis to these objects). Finally in Section 2.3, we discuss the notion of quasinormal co-modes, essentially distributions which satisfy analogous conditions to regularity quasinormal modes and are in a sense dual to them.

2.1 Defining quasinormal modes

Recalling from the previous chapter that the linear perturbation equations for black hole spacetimes can often be reduced to scalar wave equations (for example the Regge-Wheeler, Zerilli and Teukolsky equations), we shall assume that we are studying the quasinormal modes of an equation of the form:

$$(-\Box_g + V)\psi = 0, \tag{2.1.1}$$

where V is a suitable potential and $\Box_g$ is a wave operator which is expressed in a gauge we shall choose later. We shall discuss various definitions of quasinormal modes that have been proposed and some of the relations between them.

2.1.1 Ingoing and outgoing boundary conditions

The first definition of quasinormal modes to be given in the literature was to characterise them using ingoing and outgoing boundary conditions. For simplicity, let us consider the spherically symmetric case first. Once we have decomposed the perturbation equations using spherical harmonics, we are left with a radial equation of the form

$$-\partial_{r_*}^2 \hat{\psi} + (\tilde{V}(r_*) + s^2)\hat{\psi} = 0 \tag{2.1.2}$$

where r_* is a tortoise coordinate so $r_* \to -\infty$ as we approach the black hole horizon and $r_* \to \infty$ toward the cosmological horizon or asymptotically flat end. If $\tilde{V}$ is compactly supported, we see that for large $|r_*|$ the equation becomes

$$\partial_{r_*}^2 \hat{\psi} - s^2 \hat{\psi} = 0.$$

Thus, we have solutions $\hat{\psi} = e^{\pm s r_*}$ as we approach the horizon or infinity. Reintroducing the time coordinate and momentarily setting $s = -i\omega$, we have

$$\psi(t, r_*) = e^{-i\omega(t \pm r_*)},$$

i.e. ψ is asymptotically a linear combination of ingoing or outgoing waves. A natural condition to impose at these boundaries is that the wave is purely 'ingoing' at the event horizon $r_* \to -\infty$ (i.e. $\hat{\psi} = e^{s r_*}$) and purely 'outgoing' for $r_* \to \infty$ (i.e. $\hat{\psi} = e^{-s r_*}$) since we expect nothing to exit the event horizon and we impose the condition that no more radiation enters from infinity to perturb the black hole. Some authors (for example the author of [109]) refer to both conditions as outgoing, since a similar analysis can be done outside the black hole scenario, which we shall use from now onwards. We can define quasinormal frequencies as the values s with $\operatorname{Re} s < 0$ such that Equation (2.1.2) admits a solution (the quasinormal mode) which is purely outgoing at both asymptotic ends.

For practical purposes, we may identify frequencies by first finding two families of solutions which exhibit this purely outgoing behaviour at each end. We then compute the Wronskian $W(s)$, which for Equation (2.1.2) is independent of r_*. Whenever $W(s) = 0$, the two families of solutions described above are linearly dependent i.e. there exists a solution which is purely outgoing at both asymptotic ends. Thus we can identify quasinormal frequencies s as the zeroes of W with $\operatorname{Re} s < 0$ and the quasinormal modes as the corresponding solutions. This was the approach (though applied to a larger class of potentials) of Chandrasekhar and Detweiler in [35].

The potentials $\tilde{V}$ that arise in the situations we are interested in (and those considered in [35]) are not compactly supported, so the above idea needs some refinements. Since $\tilde{V}$ no longer vanishes identically for large r_*, we can no longer specify the precise behaviour of the solutions in this region. However, $\tilde{V} \to 0$ so we expect solutions to be exponentials asymptotically:

$$\hat{\psi} \sim e^{\pm s r_*}.$$

However, the above condition is not precise: we cannot expect to pick out just one solution by imposing growing exponential asymptotics as arbitrary combinations of the growing mode and decaying mode will have this behaviour (see Appendix A in [110] for a more detailed account). One method of overcoming this in the spherically symmetric case is the approach in [35], where the transformation

$$\hat{\psi} = \exp\left(\int^{r_*} \phi \, dx\right)$$

yields a Riccati equation:

$$-\partial_{r_*}\phi - \phi^2 + s^2 + \tilde{V} = 0, \tag{2.1.3}$$

and we impose the boundary conditions

$$\phi \to s \quad \text{as} \quad r_* \to -\infty, \qquad \phi \to -s \quad \text{as} \quad r_* \to \infty.$$

For $\tilde{V}$ corresponding to the Schwarzschild black hole, solutions of this problem exist only for a discrete set of values of s [34] and we can define quasinormal frequencies and modes using this fact. The frequencies can be computed using shooting methods: at each asymptotic end, we impose the appropriate boundary condition and integrate the equation numerically to some common intermediate point. The Wronskian of the two solutions we obtain is then computed and if it is zero, we have identified a quasinormal frequency. In fact, integrating Equation (2.1.3) is more numerically stable than the original linear equation: the natural error in the numerical scheme will introduce small quantities of the ingoing mode which, when integrated from infinity, will grow exponentially. The numerical methods outlined in [35] work well for determining frequencies with $|\operatorname{Re} s| < |\operatorname{Im}(s)|$. This notion of outgoing boundary conditions can be extended to the Kerr case where the equations can only be reduced to a pair of coupled ordinary differential equations as well (for a detailed exposition of this, we refer the reader to Appendix B of [17]).

To make the above precise, we first make the set up a little clearer. Let r be the areal coordinate of the spacetime, so $dr_* = f(r)dr$ for a suitable f such that the coordinate transformation $r_* \mapsto r$ sends $\mathbb{R} \to (r_-, r_+)$. The event horizon is located at $r = r_- < \infty$ and we take r_+ as a cosmological horizon if it is finite or an asymptotically flat end otherwise. Near the event horizon, we can write solutions to Equation (2.1.2) as a sum of two linearly independent parts:

$$\hat{\psi} = A_+ e^{sr_*} \psi_+^{-\infty} + A_- e^{-sr_*} \psi_-^{-\infty}$$

where $\psi_\pm^{-\infty}$ are smooth in the areal coordinate r near r_- and we take $\operatorname{Re} s < 0$. Assuming the horizon is subextremal, the potential in Equation (2.1.2) decays exponentially so this decomposition is unique[1]. We can identify purely outgoing solutions as those for which $A_- = 0$. If the horizon is extremal, the analysis is similar to an asymptotically flat end: assuming the latter is the case as $r_* \to \infty$, we take $x = 1/r$ and suppose there exist $\hat{\psi}_\pm^\infty$ smooth in x near $x = 0$ such that solutions to Equation (2.1.2) can be written

$$\hat{\psi} = B_+ e^{sr_*} \psi_+^\infty + B_- e^{-sr_*} \psi_-^\infty .$$

Naïvely, we may expect that the outgoing solutions are those for which $B_+ = 0$ in analogy with the subextremal case. However, the potential decays more slowly near $x = 0$ in this extremal case, so for $\operatorname{Re} s < 0$, e^{2sr_*} is smooth in x in this region. This means $e^{2sr_*} \psi_+^\infty$ is smooth near $x = 0$ so $e^{-sr_*} \varphi_-^\infty = e^{sr_*} \psi_+^\infty$ could also be taken to be outgoing.

The issue of this ambiguity can be (at least partially) remedied by imposing a stronger condition than smoothness. For the examples of most interest, the symmetry reductions lead to ordinary differential equations in the radial coordinate with meromorphic coefficients. Transforming back to the original areal coordinate, the equation would have regular singular points at the origin and the (subextremal) black hole horizon (say $r_- = 1$), while the asymptotically flat end at infinity would be an irregular singular point. The indices of the regular singular point at $r = 1$ are $\pm s$, so a solution to the

[1]This only applies away from $s = -n\kappa_-$ for $n \in \mathbb{N}_0$, where $\kappa_- > 0$ is the surface gravity of the horizon. This is because $r_* \sim \frac{1}{2\kappa_-} \log(r - r_-)$ as $r_* \to -\infty$ so e^{-2sr_*} behaves like $(r - r_-)^n$ near $r = r_-$. Thus e^{-2r_*} is smooth in r near the horizon at r_- and we have a similar ambiguity as described in the extremal case.

equation can be expressed in an expansion:

$$\hat{\psi} = (r-1)^s r^{-2s} e^{-sr} \sum_{n=0} a_n \left(1 - \frac{1}{r}\right)^n$$

where we have also imposed the outgoing boundary condition $\hat{\psi} \to r^{-s} e^{-sr}$ as $r \to \infty$ and taken the example of the Schwarzschild spacetime with unit mass. The coefficients a_n are determined by recurrence relations using the usual Frobenius analysis near a regular singular point. The outgoing boundary condition is satisfied when the series above converges absolutely as $r \to \infty$: namely when $\sum_{n=0}^{\infty} |a_n| < \infty$. Asymptotic analysis of the behaviour of this sequence yields a continued fraction depending on s whose roots are quasinormal frequencies. Numerically finding these roots is quite efficient (much more than the shooting methods that preceded it) and constitute the continued fraction method of finding quasinormal frequencies.

This approach was first proposed by Leaver in [97], where the author noticed that the Teukolsky equations were similar in structure to the Schrödinger equation for the hydrogen molecule ion studied in an earlier work by Jaffé [89]. By using further generalisations to find the spectrum developed by Baber and Hassé in [11], the author could accurately compute quasinormal frequencies for the Kerr black hole. This approach works well for most of the parameter space for Kerr black holes, however it breaks down in the case of an extremal black hole horizon since the associated regular singular point becomes irregular. This can be overcome by modifying the method slightly and expanding around an ordinary point. This was done for extremal Reissner-Nordström in [113] and for Kerr in [123].

2.1.2 Quasinormal modes as resonant states

Another way to bypass the ambiguities outlined above is to consider where the resolvent operator (or Green's function) of Equation (2.1.2) breaks down. First, observe that for $\operatorname{Re} s$ sufficiently large, the resolvent is well-defined and holomorphic[2]: given initial data, we can find a unique solution to the equation. Restricting to spherical symmetry for the moment, we find that Equation (2.1.2) is

$$-\partial_{r_*}^2 \hat{\psi} + \tilde{V} \hat{\psi} = -s^2 \hat{\psi}, \tag{2.1.4}$$

which is a stationary Schrödinger equation with eigenvalue $-s^2$. Provided $\tilde{V}$ is of short-range (which is the case for the potentials considered in black hole perturbation theory), there exist Jost solutions $J_\pm(r_*, s)$ for $\operatorname{Re} s > 0$ (see Chapter 7 of [129]). These are

[2] See Section 2.2 for a precise definition.

solutions to Equation (2.1.4) which satisfy the conditions:

$$\lim_{r_* \to \pm\infty} e^{\pm s r_*} J_\pm(r_*, s) = 1.$$

Thus (for $\operatorname{Re} s > 0$) we can construct a Green's function $G_s(r_*; r_*')$

$$G_s(r_*; r_*') = \frac{1}{W(s)} \begin{cases} J_+(r_*, s) J_-(r_*', s) & r_* > r_*' \\ J_+(r_*', s) J_-(r_*, s) & r_*' > r_* \end{cases}, \tag{2.1.5}$$

where $W(s) := J_+(r_*, s)\partial_{r_*} J_-(r_*, s) - J_-(r_*, s)\partial_{r_*} J_+(r_*, s)$ is the Wronskian of J_+ and J_-. Since the dependence of the equation is polynomial in s, the solutions $J_\pm$ have holomorphic dependence on s and thus W is holomorphic for $\operatorname{Re} s > 0$. This allows us to invert the original hyperbolic problem (given suitable initial data) using a Bromwich inversion contour.

Provided the Jost solutions can be analytically continued in s to the left of the complex plane (and ignoring the subtleties caused by the branch cuts that usually appear), we can analytically continue the function G_s pointwise in (r_*, r_*') with the caveat that singularities may be introduced by $W(s)$. It is important to note here that this continuation means G_s will grow exponentially in r_* at both asymptotic ends for $\operatorname{Re} s < 0$, so the operator defined by convolution with the Green's function is only well-defined when acting on functions of compact support in the left-half plane in contrast to the space of square integrable functions it can act on for $\operatorname{Re} s > 0$.

With this set-up, there is a pole in G_s wherever $W(s)$ has a zero i.e. when J_+ and J_- are linearly dependent. Moreover, a pole at s_i contributes a term of the form $e^{s_i t} u_i$ where u_i is a function of the spatial variables by the residue theorem (see Section 3.1.1 in the next chapter for more details). Thus we can define quasinormal frequencies s_i as poles of the meromorphic extension of the Green's function and the corresponding u_i as quasinormal modes. Further considerations on the function u_i show that the contribution from this pole is itself a solution to the hyperbolic problem and one can show that u_i satisfies the outgoing boundary conditions outlined in the previous section.

It turns out in the Schwarzschild case that this analytic continuation can be done up to a branch point singularity in $J_+(r_*, s)$ (see [30] and the references therein). This branch cut arises due to the slower than exponential decay of the potential toward the asymptotically flat end [39, 37] and is associated with a zero surface gravity 'horizon'. This is not present for asymptotically (anti-)de Sitter spacetimes precisely because of the faster decay in $\tilde{V}$ as $r_* \to \pm\infty$.

This approach first appeared in the physics literature (see for example [49, 98, 110]). In

[110], Nollert and Schmidt propose a refinement to this technique for the Schwarzschild case by appealing to a theorem of Weyl to show that for each asymptotic end, there is (up to rescaling) a unique solution which is square integrable near the corresponding end. This provides a more firm foundation for selecting the $J_{\pm}$ solutions to construct the Green's function over the (ambiguous) asymptotic behaviour conditions given above.

The first mathematically rigorous results regarding quasinormal modes (in particular their definition and the distribution of frequencies) were obtained by Bachelot and Motet-Bachelot in [14]. The authors perform a careful analysis on the radial wave equations of the form obtained from the perturbation equations for the Schwarzschild spacetime:

$$\partial_t^2\psi - \partial_{r_*}^2\psi + \tilde{V}\psi = 0, \tag{2.1.6}$$

where $\tilde{V}$ satisfies suitable decay conditions. First, a scattering theory for Equation (2.1.6) is developed by treating it as a perturbation of the free problem using similar techniques to the earlier works [12, 13]. Spectral representations for the propagator for the hyperbolic problem are constructed and the issue of analytic continuation of the solutions $J_{\pm}$ to $\mathbb{C} \setminus (-\infty, 0]$ is treated rigorously through the use of complex scaling. Known in the numerical analysis literature as the perfectly matched layer method [19], this was initially developed mathematically for studying the Schrödinger equation in [5, 16]. It provides several practical advantages for computations of spectra and as such has been used extensively in atomic physics and quantum chemistry [122] and has been extended to a very general setting in works by Sjöstrand and Zworski [130, 131].

The principal idea of the technique is to use the analyticity of the metric coefficients in our chosen coordinates to observe that $\tilde{V}$ is real analytic and satisfies suitable decay conditions in a cone containing $\mathbb{R}$ for sufficiently large $|r_*|$[3]. Once this is established, we can continue r_* into a complex variable and consider a deformed operator acting on a curve in $\mathbb{C}$. By selecting a favourable direction with which to approach infinity in the complex plane, we can side-step the troublesome exponential growth that arises due to the outgoing boundary conditions.

Once we have the analytic continuation of $J_{\pm}$, we can extend the definition of the Green's function in Equation (2.1.5) for a larger range of values of s, however it can now only serve as the kernel for an operator acting on functions of compact support since the $J_{\pm}$ diverge exponentially at each end for $\operatorname{Re} s < 0$. In [14] the poles of this meromorphic continuation of the Green's function were identified with quasinormal modes and it was further shown that there are infinitely many in the Schwarzschild case.

[3] In fact, this technique has since been refined to include lower regularity potentials $\tilde{V}$ with even weaker decay assumptions [61].

This approach to defining quasinormal modes can be extended to more general spacetimes through the use of techniques in geometric scattering theory developed by Mazzeo and Melrose [103]. Consider a compact Riemannian manifold with boundary X (corresponding to the spatial slices of a spacetime with Killing horizons) whose metric is g and let ρ be a boundary defining function. We further assume that

$$g = \frac{h}{\rho^2}$$

i.e. near ∂X, the metric is similar to the one for hyperbolic space near infinity. In this set-up, one can show that the resolvent of the Laplace-Beltrami operator of g extends to a meromorphic family of operators $C_0^\infty(X) \to C^\infty(X)$. This holds for other differential operators Q on the manifold with this given differential structure, provided some technical conditions are satisfied: Q is an elliptic polynomial in vector fields which vanish at the boundary (a polynomial which only vanishes at 0) and at each point on the boundary, the normal operator of Q (intuitively, this is obtained by freezing coefficients of Q at the point on the boundary to get a differential operator on the tangent space at this point) takes a certain form. For a precise statement of this result, we refer the reader to Proposition 2.2 of [125], where the authors apply this theory to the Schwarzschild-de Sitter black hole and obtain the asymptotic distribution of quasinormal frequencies. The advantage of this more general setting is that it does not rely on the spherical symmetry of the spacetime and can thus be applied to rotating black hole spacetimes: see for example work by Dyatlov on the Kerr-de Sitter spacetime [55].

2.1.3 Regularity quasinormal modes

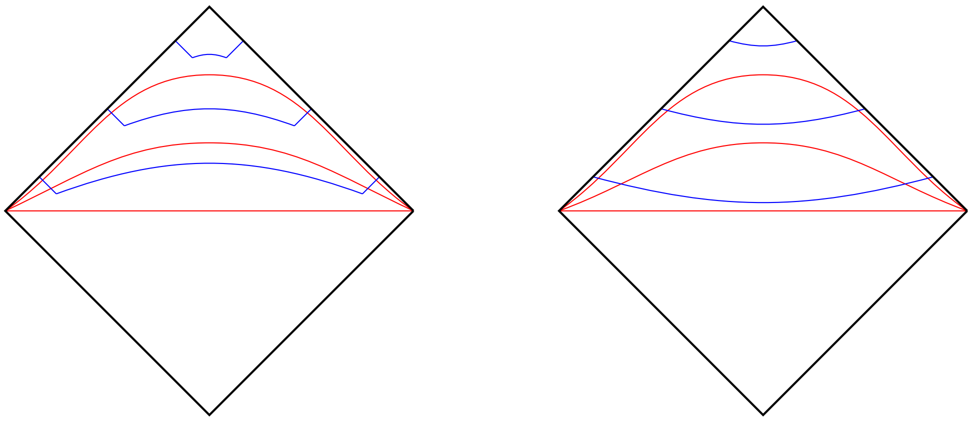
The two approaches above give very natural settings for studying quasinormal modes, however one can raise some objections:

1. While the restriction to compactly supported data is a physically reasonable assumption for an asymptotically flat end, $r_* \to -\infty$ corresponds to a black hole horizon and there is no physical reason to assume initial data is supported away from it.

2. The quasinormal modes defined this way do not belong to any natural Hilbert space due to the exponential growth in r_* for $\operatorname{Re} s < 0$.

3. This exponential growth of quasinormal modes near horizons also means that the asymptotics obtained by picking up quasinormal modes can never be uniform in the spatial variables.

The primary cause of these issues is the fact that quasinormal modes are best described as dispersive phenomena. They exhibit their characteristic decaying behaviour since



(a) A τ-slicing which is null at the horizons (b) A τ-slicing which is hyperboloidal at the horizons

Figure 2.1: Penrose diagrams depicting a static slicing in red and choices of regular slicings in blue.

energy leaks out of the system either through the horizons or to infinity. In the usual static coordinates, it would take infinite time for waves to escape, which is intuitively why the modes grow exponentially as $r_* \to \pm\infty$. To overcome this issue, we simply need to change to a time slicing which is regular at the horizon i.e. either null at the horizon or hyperboloidal (see Figure 2.1). This coordinate transformation typically takes the from

$$\tau = t + h(r)$$

where h is a suitably chosen height function [143, 100]. For the Schwarzschild spacetime, one possible choice for h is a function which is precisely the tortoise coordinate r_* from earlier near the horizons, which yields a foliation where the spatial slices are null at the horizons. With respect to these horizon penetrating coordinates, we consider the wave equation as it serves as a useful model for the the perturbation equations:

$$\Box_g \psi = 0.$$

After proving existence and boundedness results for the hyperbolic problem above, we can consider the Laplace-transformed wave operator

$$P(s)\hat{\psi} = e^{-s\tau}\Box_g(e^{s\tau}\hat{\psi}),$$

which acts on functions of the spatial variables only. If we consider the equation

$$P(s)\hat{\psi} = f,$$

we see that in the region we are interested (outside the event horizon of a black hole), the operator is elliptic away from the boundary where the principal symbol degenerates. The classical theory of elliptic partial differential equations [56] implies that if we select appropriate function spaces $\{X^k\}_{k=0}^{\infty}$ (where k is some measure of regularity), we may be able to get a set up where $P(s)$ is a family of Fredholm operators between them (see Definition 2.2.3). This was the approach taken by Vasy in the seminal paper [137], where methods from microlocal analysis were applied to spacetimes with positive cosmological constant: in particular the de Sitter space and sufficiently slowly rotating Kerr-de Sitter. Vasy used microlocal methods and the b-analysis developed by Melrose to create a Fredholm framework for understanding the resolvent for $P(s)$ above and its poles. One of the key ingredients was the use of the microlocal radial point structure of the phase space at subextremal horizons: the Hamiltonian vector field of the geodesic flow points in the same direction as the generator of dilations on the cotangent bundle. Thus the geodesic flow infinitesimally dilates the cotangent vectors of null geodesics when they arrive at the horizon: this is related to the redshift effect.

Supposing we work in a region of spacetime bounded by two subextremal horizons of surface gravities $\kappa_\pm$ and suppose furthermore that $0 < \kappa_- < \kappa_+$. Then we may roughly state Vasy's results as follows:

Proposition 2.1.1. *With X^k as above, suppose $f \in X^{k-1}$. For each s such that $\operatorname{Re} s > (1/2 - k)\kappa_-$, we have either*

(i) there exists unique $u \in X^k$ such that $P(s)u = f$

(ii) there exists a finite dimensional space of solutions $u \in \cap_{k=0}^{\infty} X^k$ to $P(s)v = 0$. Moreover this can only occur for isolated values of s.

The upshot of this result is that we can define quasinormal frequencies as values of s such that we are in case (ii) and the associated quasinormal modes as the solutions u. We saw before that given a frequency s_i and corresponding mode solution u_i, we can construct a solution to the wave equation $e^{s_i\tau}u_i$. Converting this back to the original coordinates, we see that the solution we obtained is

$$e^{s_i\tau}u_i = e^{s_i t} \cdot e^{s_i h(r)}u_i.$$

Recalling that for Schwarzschild r_* is a suitable choice of height function, we see that our change of time coordinate has moved the outgoing boundary conditions into the time dependence of our mode solution.

The results in [137] are quite general for spacetimes with positive cosmological constant, however the full subextremal range of parameters for the Kerr-de Sitter spacetime was not covered by the techniques of that paper. This was due to the fact that for sufficiently rapidly spinning black holes, the ergoregions associated with the event and cosmological horizons intersect, which causes issues with the analysis. This can be remedied by choosing a slightly different transformation for the time coordinate [116, 115]. These results can also be extended to the asymptotically flat Kerr case (see [136, 114]) and the asymptotically anti-de Sitter case [64, 62, 63].

Much of Vasy's microlocal approach to quasinormal modes can be reproduced using physical space methods, as was done by Warnick in [139]. Although the results in this paper are focussed on the asymptotically anti-de Sitter (negative cosmological constant) case, the method of proof also works for the positive cosmological constant spacetimes considered above. Here, the radial point estimates of [137] are replaced by the celebrated redshift estimates of Dafermos and Rodnianski [41] which rely on the fact that the surface gravities of any horizons are non-zero. Furthermore, the results of [139] allows the interpretation of the quasinormal frequencies as eigenvalues of the time translation generator of the wave equation and quasinormal modes as eigenfunctions. For a worked example in the static patch of the de Sitter spacetime of this approach to defining quasinormal modes, see Section 3.1.2.

The next natural step is to extend these methods to cases where there is an extremal horizon or an asymptotically flat end. Since the surface gravity of an extremal horizon is zero, the redshift effect can no longer be used. If we consider the problem with $\kappa_- > 0$ and attempt to take the limit $\kappa_- \to 0$, we see that given finite regularity k, the region where quasinormal modes are defined $\operatorname{Re} s > (1/2 - k)\kappa_-$ gets smaller and vanishes in the limit. To retain any sensible notion of invertibility of $P(s)$ in the left half plane, we must restrict the function spaces we consider. It turns out that even smoothness is not restrictive enough and real analyticity is too restrictive (see the counterexamples constructed in [59]).

Recall that a smooth function f is real analytic on U if and only if for every compact $K \subset U$, there exists C_K such that for all multi-indices α,

$$\sup_{x \in K} |\partial^\alpha f(x)| < C_K^{|\alpha|+1} \alpha!.$$

We can relax this condition to define spaces which 'lie inbetween' smoothness and real analyticity:

Definition 2.1.1. A function $f \in C^\infty(U)$ is (σ, k)-*Gevrey regular* with $k, \sigma > 0$ if for each K there exists C_K such that for all multi-indices α,

$$\sup_{x \in K} |\partial^\alpha f(x)| < C_K^{|\alpha|+1} \sigma^{-|\alpha|} (\alpha!)^k.$$

In [60], the authors use this as the basis to define Hilbert spaces and (in lieu of the redshift estimates) make use of the r^p-weighted energy method of Dafermos and Rodnianski [40] to establish the necessary estimates for a Fredholm theory and analogous definition of quasinormal modes for extremal Reissner-Nordström black holes outside a conic sector of the negative real axis in $\mathbb{C}$. In particular, this means that the damped modes of subextremal Reissner-Nordström are stable in the extremal limit. In [59], the companion to the above paper, a model problem which captures the key features of an extremal horizon is considered in detail and the relation between the various definitions of quasinormal modes is proved in this situation (in particular that quasinormal modes arising from the continued fraction method are indeed regularity quasinormal modes).

2.1.4 Zero-damped modes

We wish to prove results about the zero-damped modes discussed in the previous chapter within the framework of regularity quasinormal modes outlined above. In order to make mathematically rigorous statements, we need a definition of this phenomenon. All the examples of zero-damped modes we have discussed satisfy two key features which we shall try and pick out: namely that there is an infinite family of them with increasing rates of damping and that each mode individually has a rate of decay converging to zero in some extremal limit.

Definition 2.1.2. Let $(\mathcal{M}, g)_\kappa$ be a family of spacetimes with non-degenerate Killing horizons of surface gravity $0 < \kappa \leq \kappa_0$. Consider a geometric partial differential equation on this background for which the notion of regularity quasinormal modes as described in the previous subsection is well-defined. We say that the equation exhibits the phenomenon of *zero-damped quasinormal frequencies* if there exists a sequence of functions $\{s_n : (0, \kappa_0] \to \mathbb{C}\}_{n=1}^{\infty}$ such that:

1. $s_n(\kappa)$ is a quasinormal frequency for each $n \in \mathbb{N}, \kappa \in (0, \kappa_0]$,
2. $\operatorname{Re} s_n(\kappa) \to -\infty$ as $n \to \infty$ for each $\kappa \in (0, \kappa_0]$,
3. there exists $\alpha \in \mathbb{R}$ such that $s_n(\kappa) \to i\alpha$ as $\kappa \to 0$ for each $n \in \mathbb{N}$.

The last two components of the above definition capture the fact that for nearly extremal spacetimes, zero-damped modes are almost stationary and can be thought of as the components of the quasinormal spectrum corresponding to the branch cut responsible for the tail present in the extremal case.

2.2 Meromorphic families of operators

In the previous section we saw that quasinormal frequencies can be defined in terms of the poles of a meromorphic family of Fredholm operators. To further study the properties of the spectrum of quasinormal modes, we will need several tools from the theory meromorphic families of operators which we shall outline in this section. We begin with the following definition:

Definition 2.2.1. Let X, Y be Banach spaces so $\mathcal{L}(X, Y)$ is the Banach space of bounded linear operators $X \to Y$. Furthermore, let $\Omega \subset \mathbb{C}$ be connected and open. We say $A : \Omega \to \mathcal{L}(X, Y)$ is a *holomorphic family of operators* on Ω if for all $z \in \Omega$,

$$\lim_{h \to 0} \frac{A(z+h) - A(z)}{h}$$

exists in the operator norm topology.

In other words, we can take complex derivatives of the family of operators. We can similarly generalise the notion of meromorphic function from complex analysis to this case:

Definition 2.2.2. In the same set up as above, A defines a *meromorphic family of operators* on Ω if for all $z_0 \in \Omega$, there exist finite rank operators $\{A_k\}_{k=1}^K$ and a family of operators $A_0(z)$ holomorphic near z_0 such that

$$A(z) = A_0(z) + \sum_{k=1}^{K} \frac{A_k}{(z - z_0)^k}$$

If $K \neq 0$, we say z_0 is a pole of order K of $A(z)$. Otherwise, $A(z) = A_0(z)$ is holomorphic at z_0.

In fact, many results from complex analysis carry over to analytic families of operators. For example, one can make sense of contour integrals for holomorphic families of operators (for a discussion of vector-valued integration, see Chapter 3 of [124]) and we have Cauchy's integral theorem: for a positively oriented closed contour γ enclosing a point z_0,

$$A(z_0) = \frac{1}{2\pi i} \oint_\gamma \frac{A(z)}{z - z_0} dz.$$

Recall from the discussion of regularity quasinormal modes that the families of operators we wish to work with are Fredholm: for completeness we include a definition

Definition 2.2.3. We say an operator $P \in \mathcal{L}(X, Y)$ is *Fredholm* if both its kernel

$$\ker P := \{x \in X \mid Px = 0\},$$

and its cokernel

$$\operatorname{coker} P := Y / \operatorname{im} P$$

are finite-dimensional. The *index* of P is

$$\operatorname{ind} P := \dim \ker P - \dim \operatorname{coker} P$$

If $A : \Omega \to \mathcal{L}(X, Y)$ is a meromorphic family of operators such that $A_0(z)$ from Definition 2.2.2 is Fredholm for each $z \in \Omega$, we say A is a meromorphic family of Fredholm operators. Fredholm operators are useful in the sense that despite generally acting between infinite-dimensional spaces, they can be thought of as behaving like matrices in a sense which will be made clear in the next subsection.

2.2.1 Grushin problems

When studying spectral problems, a simple and useful technique to aid the analysis is to enlarge to a system of equations which is invertible. Suppose we study the holomorphic family of operators $A(s) : \Omega \to \mathcal{L}(X, Y)$. We introduce auxiliary Banach spaces $X_\pm$ and operators $R_+ : X \to X_+$, $R_- : X_- \to Y$ so that the enlarged system

$$P(s) = \begin{pmatrix} A(s) & R_- \\ R_+ & 0 \end{pmatrix} \tag{2.2.1}$$

is invertible. The key observation for this approach is a generalisation of Schur's complement formula for matrices: supposing we have

$$P(s)^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}, \tag{2.2.2}$$

then $A(s)$ is invertible if and only if E_{-+} is. This technique appeared in the context of linear partial differential equations in work on hypoelliptic operators by Grushin [70] and for this reason problems of the form

$$P(s)u = v, \tag{2.2.3}$$

have been called Grushin problems [132]. The problem is said to be well-posed if the system above is invertible and we write the inverse as in Equation (2.2.2). This technique has appeared many times in the mathematics and physics literature with a several names: in the physics literature it is usually called the Feshbach method and formula

$$A(s)^{-1} = E - E_+ E_{-+}^{-1} E_-$$

(arising from Schur's complement formula) is called the Feshbach formula. It typically arises from splitting a Hilbert space into a direct sum so the quantum Hamiltonian acting on it decomposes in a favourable way. For precise details on how this splitting is used in the study of spectra, we refer the reader to [43]. We shall focus on using this technique when considering analytic families of Fredholm operators.

Fredholm operators have a particularly nice characterisation in terms of Grushin problems. Dropping the s dependence momentarily, we consider the Fredholm operator

$A : X \to Y$ and shall construct a well-posed Grushin problem for it. Since the kernel and cokernel are of finite dimension (say n_+ and n_- repectively), we can set the auxiliary spaces $X_\pm$ above to be $\mathbb{C}^{n_\pm}$. We pick a basis $\{x_i\}_{i=1}^{n_+} \subset \ker A(s)$ and note that we can find $x_i^* \in X^*$ (the dual of X) such that $x_i^*(x_j) = \delta_{ij}$. Then we define the map $R_+ : X \to \mathbb{C}^{n_+}$:

$$R_+(x) = \begin{pmatrix} x_1^*(x) \\ x_2^*(x) \\ \vdots \\ x_{n_+}^*(x) \end{pmatrix}.$$

Similarly, we pick a basis of $\operatorname{coker} A(s)$ and for each basis element, a representative $y_i \in Y$, $i = 1, 2, \dots n_-$. We define the map $R_- : \mathbb{C}^{n_-} \to Y$ to be

$$R_-(v_1, v_2, \dots v_{n_-}) = \sum_{i=1}^{n_-} v_i y_i$$

Then it is not hard to see from the constructions above that for

$$\begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix}$$

mapping $X \oplus \mathbb{C}^{n_-} \to Y \oplus \mathbb{C}^{n_-}$ has trivial kernel and is surjective. Hence the Grushin problem is well-posed and has inverse

$$\begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix}.$$

In particular A is invertible if and only if the matrix $E_{-+} : \mathbb{C}^{n_-} \to \mathbb{C}^{n_+}$ is. Furthermore, given an operator B sufficiently close to A in operator norm, one can show that the Grushin problem

$$\begin{pmatrix} B & R_- \\ R_+ & 0 \end{pmatrix}$$

is also well-posed by using a Neumann series argument. Since E_{-+} in this perturbed case is still a map $\mathbb{C}^{n_-} \to \mathbb{C}^{n_+}$, we see that B is Fredholm and has the same index as A. Hence the set of Fredholm operators of a given index is open in $\mathcal{L}(X, Y)$.

2.2.2 Analytic Fredholm theory

A useful standard result [53] that can be proved using these Grushin problems is the following:

Theorem 2.2.1 (Analytic Fredholm theory)**.** *Suppose $\Omega \subset \mathbb{C}$ is open and connected. Let $A(z) : \Omega \to \mathcal{L}(X, Y)$ be a meromorphic family of Fredholm operators. If there exists $z_0 \in \Omega$ such that $A(z_0)$ is invertible, then the family of operators $z \mapsto A(z)^{-1}$ is a meromorphic family of operators with poles of finite rank.*

Proof. First we consider the case when $A(z)$ is a holomorphic family of Fredholm operators.

1. Since $A(z_0)$ is invertible, it is of index 0 so by stability of perturbations of index, $\operatorname{ind} A(z) = 0$ for all $z \in \Omega$. Hence for each $z \in \Omega$, we can construct a Grushin problem as done above with E_{-+} an $n \times n$ matrix for some $n \in \mathbb{N}$. For a sufficiently small neighbourhood U_z of z, we can define the holomorphic function $f_z(w) = \det E_{-+}(w)$ which vanishes if and only if $A(z)$ is not invertible.

2. Since Ω is connected and there exists z_0 such that $A(z_0)$ is invertible, none of the f_z can vanish identically: if this were the case for some w, we could connect z_0 and w by a path and consider a finite set of neighbourhoods $\{U_i\}_{i=0}^N$ covering it where $U_0 = U_w$ and $U_N = U_{z_0}$. Without loss of generality, we assume that these sets are ordered so $U_i \cap U_{i+1} \neq \varnothing$. Since $f_w \equiv 0$ on $U_0 \cap U_1$, $A(z)$ is not invertible in this set so $f_1 \equiv 0$ on U_1. This continues until we have $A(z_0)$ not invertible, a contradiction.

3. Since $f_z(w)$ is not identically zero in U_z, $E_{-+}(w)^{-1}$ is a meromorphic family of matrices and therefore

$$A(w)^{-1} = E(w) - E_+(w)E_{-+}(w)^{-1}E_-(w)$$

   is a meromorphic family of operators on U_z. Not that the poles are of finite rank since the singular behaviour comes from the second term. Since z was arbitrary, we have the result.

The case when $A(z)$ is a meromorphic family of operators follows from Theorem 2.2.3 in Section 2.2.3. $\square$

2.2.3 Gohberg-Sigal theory

It is natural to ask given the above set up whether other classical theorems from complex analysis generalise for meromorphic families of operators. In particular, since quasinormal frequencies can be thought of as 'zeroes' of the family of operators we are working with, we seek a generalisation of a continuity result like Rouché's theorem:

Theorem 2.2.2 (Rouché)**.** *Let $U \subset \mathbb{C}$ be connected, open and bounded so ∂U is a closed contour. Suppose $f, g : U \to \mathbb{C}$ are holomorphic and that*

$$|g(z)| < |f(z)| \quad \forall z \in \partial U.$$

Then f and $f+g$ have the same number of zeroes (counted with multiplicity) in U.

The proof of this theorem makes use of the argument principle i.e. the fact that

$$\oint_{\partial U} \frac{f'(z)}{f(z)} dz = n_0 - n_p$$

where n_0 is the number of zeroes of f contained in U and n_p is the number of poles. Both of these results were generalised to families of operators by Gohberg and Sigal in [68] (an alternative proof with a discussion on which classes of Banach algebras such results hold on is presented in [18]). A key difference from the classical case is the fact that near a 'zero', the function can vanish at different rates in different directions since the operators no longer act on one-dimensional spaces. This is more clearly seen in light of the following theorem:

Theorem 2.2.3. *Suppose X is a Banach space and $A(s)$ is a meromorphic family of Fredholm operators $X \to X$ in Ω where Ω is a connected, open subset of $\mathbb{C}$. Given $s_0 \in \Omega$, there exists some neighbourhood of s_0 such that*

$$A(s) = A_0(s) + \sum_{k=1}^{K} \frac{A_k}{(s-s_0)^k},$$

where $A_0(s)$ is holomorphic. If $A_0(s_0)$ has index zero, then there exist families of operators U_1, U_2 which are holomorphic and invertible in a neighbourhood of s_0 and operators $\{P_m\}_{m=0}^M$ such that

$$A(s) = U_1(s) \left(P_0 + \sum_{m=1}^{M} (s-s_0)^{k_m} P_m \right) U_2(s)$$

in a neighbourhood of s_0. The k_m are non-zero integers and the P_m obey

$$P_m P_n = \delta_{mn} P_m,$$

with $\{P_m\}_{m=1}^M$ being of rank 1 and $I - P_0$ being of finite rank. The inverse $A(s)^{-1}$ exists as a meromorphic family of operators near s_0 if and only if $\sum_{m=0}^M P_m = I$, in which case it takes the form:

$$A(s)^{-1} = U_2(s)^{-1} \left(P_0 + \sum_{m=1}^{M} (s-z_0)^{-k_m} P_m \right) U_1(s)^{-1}.$$

The proof of this result is quite involved and requires several lemmas from the theory of Banach spaces. The intuition behind the finite rank piece follows from Gaussian elimination for meromorphic families of operators: it is not hard to see that in the finite dimensional case, a meromorphic family of matrices can be locally diagonalised. To be more precise: suppose $M(s)$ is a meromorphic family of matrices near some $s_0 \in \mathbb{C}$.

Then there exist holomorphic families of matrices invertible in a neighbourhood of s_0 $E(s)$ and $F(s)$ such that

$$M(s) = E(s)\Lambda(s)F(s)$$

where $\Lambda(s)$ is a diagonal matrix where the diagonal entries either take the form $(s-s_0)^{k_i}$ or are 0. For a proof of the full result, we direct the reader to Theorem C.10 in [53] or Theorem 3.1 in [68]. This allows us to generalise the notion of the order of a zero via the notion of null multiplicity:

Definition 2.2.4. The *null multiplicity* at s_0 of a meromorphic family of operators $A(s)$ is:

$$N_{s_0}(A) := \begin{cases} \sum_{k_m>0} k_m & M = \operatorname{rank}(I - P_0) \\ \infty & M < \operatorname{rank}(I - P_0) \end{cases}.$$

When $N_{s_0}(A) < \infty$, $A(s)^{-1}$ is meromorphic and we can compute its null multiplicity:

$$N_{s_0}(A^{-1}) = -\sum_{k_m<0} k_m.$$

Now we can state a generalisation of the argument principle for these families of operators:

Theorem 2.2.4. *Let H be a Hilbert space and suppose $A(s)$ and $A(s)^{-1}$ are meromorphic families of operators $H \to H$ in Ω. Let*

$$\Pi_{s_0} = \frac{1}{2\pi i}\oint_{\Gamma_{s_0,\delta}} \partial_s A(s)A(s)^{-1}ds,$$

where $\Gamma_{s_0,\delta}$ is a positively oriented circle of radius δ containing s_0 and no other pole of the integrand. Then Π_{s_0} has finite rank and

$$\operatorname{tr}\Pi_{s_0} = N_{s_0}(A) - N_{s_0}(A^{-1}).$$

The proof of this result relies heavily on Theorem 2.2.3 since the factorisation splits the right hand side to holomorphic terms (which vanish after the integral) and finite rank singular terms. Thus we can use cyclicity of the trace to simplify the calculations and obtain the result. For the full details of the proof, the reader is directed to Theorem C.11 in [53] or Theorem 2.1 in [68].

Theorem 2.2.5. *Suppose $A(s)$ and $B(s)$ for $s \in \Omega$ are meromorphic families of Fredholm operators as in Theorem 2.2.4. Suppose further that $U \Subset \Omega$ is a simply connected open subset with C^1 boundary ∂U such that A and B are invertible on ∂U and*

$$\|A(s)^{-1}(A(s) - B(s))\|_{H\to H} < 1, \quad s \in \partial U.$$

Then

$$\frac{1}{2\pi i}\operatorname{tr}\oint_{\partial U}\partial_s A(s)A(s)^{-1}ds = \frac{1}{2\pi i}\operatorname{tr}\oint_{\partial U}\partial_s B(s)B(s)^{-1}ds.$$

The idea of the proof of this is very similar to that of the classical Rouché theorem: using the generalised argument principle, it suffices to show that a suitable contour integral vanishes. However proving the result in its full generality requires a few technical lemmas (see Theorem 2.2 in [68]). A weaker version of the result where the operator $A(s)^{-1}(A(s) - B(s))$ is assumed to be trace class uses the same idea, however the fact that the difference is trace class simplifies the argument: see Theorem C.12 in [53]. These results form the basis of Gohberg-Sigal theory which has been used to give rigorous justification for eigenvalue expansions in the analysis of spectral problems for wave propagation in various media with complicated domains [6, 7, 8].

2.3 Quasinormal Co-modes

Recall from Section 2.2.1 that for a family of Fredholm operators, one can construct a Grushin problem so that (at least locally) invertibility of the operator of interest is equivalent to invertibility of a matrix. Supposing A is a Fredholm operator between Hilbert spaces with well-posed Grushin problem

$$\begin{pmatrix} A & R_- \\ R_+ & 0 \end{pmatrix}^{-1} = \begin{pmatrix} E & E_+ \\ E_- & E_{-+} \end{pmatrix},$$

we can construct a Grushin problem for $A^\dagger$, the adjoint of A:

$$\begin{pmatrix} A^\dagger & R_+^\dagger \\ R_-^\dagger & 0 \end{pmatrix}$$

where $R_\pm^\dagger$ are the adjoints of $R_\pm$. It is not hard to construct these adjoints explicitly and see that this Grushin problem is also well-posed with inverse

$$\begin{pmatrix} E^\dagger & E_-^\dagger \\ E_+^\dagger & E_{-+}^\dagger \end{pmatrix}.$$

Thus $A^\dagger$ is invertible if and only if $E_{-+}^\dagger$, the adjoint of E_{-+}, is invertible. Thus A is invertible if and only if $A^\dagger$ is since E_{-+} is a matrix. In particular, their kernels have the same dimension. This motivates the idea of seeking quasinormal frequencies by considering the adjoint problem. This idea was first discussed in [75], where the authors defined solutions to the adjoint problem as *dual resonant states*.

The notion of co-modes (or dual resonant states) is also what occurs in the background of [69], where the authors construct a bilinear form with respect to which the quasinormal modes are orthogonal. One can think of this bilinear form as a mapping from the quasinormal modes to the co-modes in a way which introduces orthogonality. For a finite dimensional example, consider the matrix

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}.$$

It is not hard to see that

$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \qquad \begin{pmatrix} 1, -2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 0, 0 \end{pmatrix},$$
$$\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4 \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \qquad \begin{pmatrix} 1, 2 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} = 4 \begin{pmatrix} 1, 2 \end{pmatrix}.$$

Note that the right and left-eigenvectors are not identified via the usual map $v \mapsto v^\top$: the map which sends a right-eigenvector to its corresponding left-eigenvector can be represented by the matrix

$$\begin{pmatrix} 1/2 & 0 \\ 0 & 2 \end{pmatrix},$$

which induces a bilinear form on $\mathbb{R}^2$ with respect to which the eigenvectors are orthogonal.

We shall formulate an equivalent definition to the one presented in [75] after making a few more considerations. From the results leading to the definition of quasinormal modes discussed in Section 2.1.3, we see that $P(s)$ is a holomorphic family of Fredholm operators on a suitable domain $\Omega \subset \mathbb{C}$. In light of Theorem 2.2.1, we can deduce that $P(s)^{-1}$ is a meromorphic family of operators and, near some $s_0 \in \Omega$, can be written

$$P(s)^{-1} = A_0(s) + \sum_{j=1}^{J} \frac{A_{-j}}{(s-s_0)^j},$$

where $A_0(s)$ is holomorphic near s_0 and the A_{-j} are finite rank operators. Assuming for the moment that the poles of $P(s)^{-1}$ are simple (the more complicated higher order case gives similar results), we use the fact that

$$(s-s_0)P(s)^{-1}P(s)u = (s-s_0)P(s)P(s)^{-1}u = (s-s_0)u \to 0 \quad \text{as} \quad s \to s_0$$

to deduce that

$$\operatorname{im} A_{-1} = \ker P(s_0), \qquad \ker A_{-1} = \operatorname{im} P(s_0).$$

So the residue projects onto the space of solutions to $P(s_0)u = 0$, which we know is finite dimensional. Hence if the solutions to $P(s_0)u = 0$ are spanned by $\{w_j\}_{j=1}^N$, we can write

$$A_{-1} = \sum_{j=1}^{N} w_j \otimes \theta_j \tag{2.3.1}$$

where the θ_i are continuous linear functionals which vanish on $\operatorname{im} P(s_0)$. Since we are working in a Hilbert space, they can be identified (via the Riesz representation theorem) with kernel of $P(s_0)^\dagger$. As we will see for the spacetimes we work with in later sections, the Hilbert spaces we use are Sobolev spaces on some manifold with boundary $\mathcal{M}$. Since $\partial\mathcal{M}$ corresponds to the intersections of the horizons with the spatial slices and the coordinates we use penetrate these, we can define $\mathcal{M}_\epsilon$ to be the extension of $\mathcal{M}$ by a fixed small amount outside its boundary. Thus we can define the space

$$\mathfrak{C} = \left\{\theta \in H^{-k}(\mathcal{M}_\epsilon) \,\middle|\, \operatorname{supp}\theta \subset \mathcal{M}\right\},$$

where we say $\operatorname{supp}\theta \subset \mathcal{M}$ if for any $u \in H^k(\mathcal{M}_\epsilon)$ with $\operatorname{supp} u \subset \mathcal{M}_\epsilon \setminus \mathcal{M}$, $\theta(u) = 0$. This space can be identified with $H^k(\mathcal{M})'$, the continuous dual of $H^k(\mathcal{M})$, so since $\mathfrak{C} \subset H^{-k}(\mathcal{M}_\epsilon) \subset \mathcal{D}'(\mathcal{M}_\epsilon)$, we can seek distributional solutions to the adjoint problem $P(s)^\dagger\theta = 0$. Taking κ_- to be the smallest surface gravity in our problem, we make the following definition:

Definition 2.3.1. We say $\theta \in \mathcal{D}'(\mathcal{M}_\epsilon)$ is a *quasinormal co-mode* if it satisfies for some $\operatorname{Re} s > (1/2 - k)\,\kappa_-$:

(i) there exists $C > 0$ such that $|\theta(u)| < C\|u\|_{H^k(\mathcal{M})}$ for all $u \in C_0^\infty(\mathcal{M}_\epsilon)$ ($\Rightarrow \operatorname{supp}\theta \subset \mathcal{M}$)

(ii) $P(s)^\dagger\theta = 0$ in the sense of distributions, i.e. $\theta(P(s)u) = 0$ for any $u \in C_0^\infty(\mathcal{M}_\epsilon)$.

By noting that $P(s)^{-1}P(s) = I$, we see that the co-modes defined above are precisely the θ_i appearing in Equation (2.3.1). While this definition is equivalent to considering these co-modes as elements of some Sobolev space, in some situations (for example de Sitter space considered in the next chapter) thinking of them as distributions simplifies matters considerably.

We see here as well a link to [69], since the co-modes as defined above are used to pick out the coefficient in front of a corresponding quasinormal mode in the residue of $P(s)^{-1}u$. The terms of the form $w_j \otimes \theta_j$ serve as projections into the space of quasinormal frequencies with θ_j picking out the excitation coefficient: one can think of the map $w_j \mapsto \theta_j$ as the bilinear form relating the two orthogonal systems.

Chapter 3

The de Sitter spacetime

In this chapter, we shall apply the ideas discussed in the previous one to a concrete example: the static patch of the de Sitter spacetime. In particular, we shall explicitly construct the resolvent described in the resonance approach to quasinormal modes (Section 2.1.2) before we apply the regularity approach (Section 2.1.3) which identifies frequencies as eigenvalues (Proposition 3.1.1) and allows us to prove discreteness of the spectrum (Proposition 3.1.2). Following this, we can obtain the main results of this section: finding the quasinormal spectrum and the corresponding modes for Klein-Gordon fields in this spacetime explicitly (this is possible due to the simplicity and symmetry of the spacetime and is a well-known result), and constructing the co-modes for the spacetime (this is Equation (3.2.2)), a calculation first appearing in [75] but was done independently before its publication.

Consider the vacuum Einstein field equations with cosmological constant Λ:

$$\mathrm{Ric}[g] - \frac{1}{2}R[g]g + \Lambda g = 0,$$

where g is a Lorentzian metric, $\mathrm{Ric}[g]$ is its Ricci tensor and $R[g]$ is its Ricci scalar. After contraction with the metric, the equation reduces to

$$\mathrm{Ric}[g] = \Lambda g$$

in $1+3$ dimensions which we restrict to for the remainder of this chapter. Seeking constant curvature solutions of the above leaves us three cases, each corresponding to maximally symmetric spacetimes. If $\Lambda = 0$, we have the familiar Minkowski spacetime which we will sidestep for simplicity of exposition, since null infinity behaves like an extremal horizon in the set-up of the previous chapter. If $\Lambda < 0$, the solution is an anti-de Sitter spacetime where the Fredholm set-up of [139] applies, however the conformal boundary is timelike. This means that we must consider initial boundary value problems to get well-posed equations and the complications of these boundary conditions are added to the function spaces we work with. This leaves the case $\Lambda > 0$, the de Sitter spacetime, which will be the focus of this chapter. This spacetime is a good model to play with since it possesses a lot of symmetry and, crucially from the perspective of quasinormal modes, the static patch of de Sitter admits subextremal horizons due to the spacelike conformal boundary.

The isometries of the de Sitter spacetime are most easily seen when it is constructed as a Lorentzian submanifold of a Minkowski spacetime of one higher dimension. With respect to the usual Cartesian coordinates $X = (X^0, X^i)$ on $\mathbb{R}^{1,4}$, consider the level sets of the quadratic form

$$\eta(X, X) = -(X^0)^2 + \sum_{i=0}^{4} (X^i)^2,$$

which (except $\eta(X, X) = 0$) are naturally codimension one smooth submanifolds of $\mathbb{R}^{1,4}$. Fixing some $\Lambda > 0$, we consider in particular sets of the form:

$$\eta(X, X) = \frac{3}{\Lambda},$$

which are connected (in fact homeomorphic to $\mathbb{R} \times S^3$) and admit a Lorentzian-signature metric induced from the usual Minkowski one on $\mathbb{R}^{1,4}$. This manifold and pulled-back metric pair is the de Sitter spacetime of radius $\sqrt{3/\Lambda}$ and it is clear from this construction that the isometry group is $O(4, 1)$, the indefinite orthogonal group. This is the ten dimensional Lie group of real matrices which preserve the quadratic form η and has four connected components. The subgroup of matrices with determinant one, $SO(4, 1)$, has two: the component containing the identity, $SO^+(4, 1)$, preserves orientation of both time and space while the other piece consists of transformations which reverse both subspaces and thus preserves the overall orientation in $\mathbb{R}^{1,4}$.

Following Hawking and Ellis [71], we can introduce global intrinsic coordinates (except for the usual polar coordinate singularities arising from parametrising S^3) using the following transformation:

$$\begin{aligned}
X^0 &= \sqrt{\frac{3}{\Lambda}} \sinh\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right), \\
X^1 &= \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right) \cos\chi, \\
X^2 &= \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right) \sin\chi\cos\theta, \\
X^3 &= \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right) \sin\chi\sin\theta\cos\phi, \\
X^4 &= \sqrt{\frac{3}{\Lambda}} \cosh\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right) \sin\chi\sin\theta\sin\phi,
\end{aligned}$$

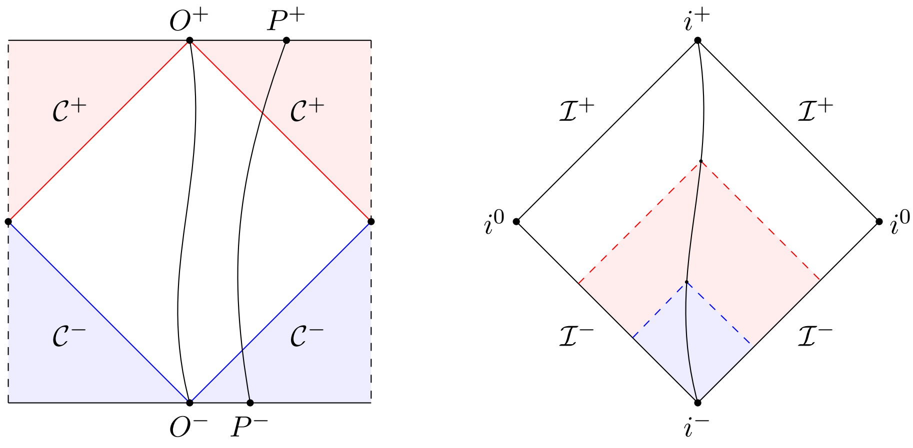
where $(\hat{t}, \chi, \theta, \phi) \in \mathbb{R}_{\hat{t}} \times (0,\pi)_\chi \times (0,\pi)_\theta \times (0, 2\pi)_\phi$. This yields the metric

$$g_{\text{dS}} = -d\hat{t}^2 + \frac{3}{\Lambda}\cosh^2\left(\sqrt{\frac{\Lambda}{3}}\hat{t}\right) g_{S^3},$$

where g_{S^3} is the usual metric on the unit round 3-sphere. The constant time spatial surfaces are 3-spheres of radius at least $\sqrt{3/\Lambda}$: the radius exponentially decreases in time as we approach $\hat{t} = 0$ from $\hat{t} = -\infty$ and exponentially increases for positive $\hat{t}$.

We can conformally compactify the spacetime and study infinity in the usual way by introducing coordinates where de Sitter is conformal to a patch of the Einstein static universe. This process yields a spacelike infinity which leads to the existence of cosmological horizons for a free-falling, timelike observer: consider such an observer O following a worldline starting at O^- on past infinity and finishing at O^+ on future infinity (see Figure 3.1). Drawing the past null cone from O^+, we shade in red the region which cannot be observed by O at all. Similarly, we shade in blue the region which cannot be influenced by O. The boundary of the unshaded region acts like the event horizon of a black hole for the observer: the red segments like a future event horizon and the blue ones like a past event horizon.

To see this more explicitly, suppose some particle P crosses the future horizon at p. From the diagram, we can see that O can only observe this event at O^+ i.e. it will take infinite time. This is similar to the infinite redshift effect observed at the event horizon of a black hole. For a more detailed explanation of this, we direct to reader to the fifth chapter of [71]. To work in the frame of reference of an observer which experiences this redshift effect, we can introduce static coordinates on a patch of this manifold via the



(a) de Sitter

(b) Minkowski

Figure 3.1: Penrose diagrams for de Sitter space and Minkowski. From the Penrose diagram of Minkowski (right), it is clear that an observer traveling from i^- to i^+ will be able to observe all of the spacetime after sufficient time. This is not the case in de Sitter (left).

transformation

$$
\begin{aligned}
X^0 &= \left(\frac{3}{\Lambda} - r^2\right)^{1/2} \sinh\left(\sqrt{\frac{\Lambda}{3}}t\right), \\
X^1 &= \left(\frac{3}{\Lambda} - r^2\right)^{1/2} \cosh\left(\sqrt{\frac{\Lambda}{3}}t\right), \\
X^2 &= r\sin\theta\cos\phi, \\
X^3 &= r\sin\theta\sin\phi, \\
X^4 &= r\cos\theta,
\end{aligned}
$$

where $(t, r, \theta, \phi) \in \mathbb{R}_t \times (0, \sqrt{3/\Lambda})_r \times (0, \pi)_\theta \times (0, 2\pi)_\phi$. This can naturally be extended to $\mathbb{R}_t \times B_{\sqrt{3/\Lambda}} = \mathbb{R}_t \times \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid \sum_{i=1}^3 x_i^2 < 3/\Lambda\}$ by the usual methods of overcoming the coordinate singularity brought on by polar coordinates. Note that this parametrisation has also introduced a coordinate singularity at $r = \sqrt{3/\Lambda}$. This is the cosmological horizon for an observer at the origin $r = 0$ as described above. The metric in these coordinates takes the form

$$
g_{\mathrm{dS}} = -(1 - \kappa^2 r^2)dt^2 + \frac{dr^2}{1 - \kappa^2 r^2} + r^2 \not{g}, \tag{3.0.1}
$$

where $\kappa = \sqrt{\Lambda/3}$ is the surface gravity of the cosmological horizon and $\not{g}$ is the usual metric on the unit round 2-sphere. Note that ∂_t is now a Killing vector for this spacetime, which is why this patch of de Sitter the coordinates cover is called the static patch. We see that as $\kappa \to 0$, the cosmological horizon goes to infinity and a comparison of the Penrose diagrams in Figure 3.1 motivates the interpretation of the conformal boundary in Minkowski as an extremal cosmological horizon.

3.1 Quasinormal modes

Now we have set up the spacetime we wish to work in and fixed suitable coordinates on it, we shall specialise the definitions given in the Chapter 2 to the de Sitter case and demonstrate the techniques discussed previously. We shall shall focus on the following initial value problem:

$$-\Box_g\psi + V\psi = 0, \quad \psi|_{t=0} = \psi_0, \quad \partial_t\psi|_{t=0} = \psi_1 \tag{3.1.1}$$

where V is a bounded, t-independent potential satisfying various further conditions we specify later, g is the metric for the de Sitter spacetimes in suitable coordinates and $\Box_g$ is the associated wave operator i.e.

$$\Box_g\psi = \frac{1}{\sqrt{-\det g}}\partial_a\left(\sqrt{-\det g}\cdot g^{ab}\partial_b\psi\right).$$

3.1.1 The traditional approach

We begin with the method to defining quasinormal modes initially outlined by Bachelot in [14]: we shall find the Green's function and analytically continue this to $\operatorname{Re} s < 0$. Following this approach, we consider the tortoise coordinate $\kappa r_* = \operatorname{artanh}(\kappa r)$ which transforms the metric to

$$g = \operatorname{sech}^2(\kappa r_*)(-dt^2 + dr_*^2) + \frac{\tanh^2(\kappa r_*)}{\kappa^2}\not{g} \tag{3.1.2}$$

which can be extended to $\mathbb{R}_t \times \mathbb{R}^3$. The equation in these coordinates becomes

$$\cosh^2(\kappa r_*)\partial_t^2\psi - \frac{\cosh^2(\kappa r_*)}{\tanh^2(\kappa r_*)}\partial_{r_*}\left(\tanh^2(\kappa r_*)\partial_{r_*}\psi\right) - \frac{\kappa^2\not{\Delta}\psi}{\tanh^2(\kappa r_*)} + V\psi = 0,$$

where $\not{\Delta}$ is the Laplace-Beltrami operator on the unit round sphere. Dividing by $\cosh^2(\kappa r_*)$, the equation becomes

$$\partial_t^2\psi - \frac{\partial_{r_*}\left(\tanh^2(\kappa r_*)\partial_{r_*}\psi\right)}{\tanh^2(\kappa r_*)} - \frac{\kappa^2\not{\Delta}\psi}{\sinh^2(\kappa r_*)} + \frac{V}{\cosh^2(\kappa r_*)}\psi = 0, \tag{3.1.3}$$

We shall further assume that $V \in C^\infty(\mathbb{R}^3)$ is real-valued since we wish to write Equation (3.1.3) in the form

$$(\partial_t^2 + H)\psi = 0 \tag{3.1.4}$$

where H is a densely defined self-adjoint operator acting on L^2 with respect to the measure $d\nu = \tanh^2(\kappa r_*)dr_*d\sigma$, with $d\sigma$ the usual measure on the unit round sphere.

We Laplace transform this equation in time to obtain

$$(H + s^2)\hat{\psi} = s\psi_0 + \psi_1. \tag{3.1.5}$$

Since H is self-adjoint, we have

$$\mathrm{Re} \int_{\mathbb{R}^3} (su)^*(H + s^2)u d\nu = \mathrm{Re}(s) \int_{\mathbb{R}^3} \left(u^* H u + |s|^2|u|^2\right) d\nu, \tag{3.1.6}$$

and from Equation (3.1.3),

$$\int_{\mathbb{R}^3} u^* H u d\nu \geq - \int_{\mathbb{R}^3} \frac{V}{\cosh^2(\kappa r_*)} |u|^2 d\nu \geq -V_0 \int_{\mathbb{R}^3} |u|^2 d\nu, \tag{3.1.7}$$

where V_0 is a positive constant depending only on V. Combining Equation (3.1.6) and Equation (3.1.7), we get the estimate

$$\frac{\mathrm{Re}(s)}{|s|}(|s|^2 - V_0) \left(\int_{\mathbb{R}^3} |u|^2 d\nu\right)^{1/2} \leq \left(\int_{\mathbb{R}^3} |(H + s^2)u|^2 d\nu\right)^{1/2}. \tag{3.1.8}$$

This allows us (using standard arguments from the theory of self-adjoint operators) to deduce the existence and analyticity of the resolvent operator $R(s) : L^2(\mathbb{R}^3, d\nu) \to L^2(\mathbb{R}^3, d\nu)$ for $H + s^2$ provided

$$\mathrm{Re}(s) > 0 \quad \text{and} \quad |s|^2 > V_0.$$

It is not too hard to see that $\mathrm{Re}(s) > c_0 = \sqrt{V_0}$ is a region contained in the set defined by the inequalities above. Hence we can use a Bromwich inversion contour to solve the initial value problem:

$$\psi = \frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} e^{st} R(s)(s\psi_0 + \psi_1) ds \tag{3.1.9}$$

taking any $c > c_0$. One can then apply the results of Mazzeo and Melrose in [103] to obtain a meromorphic extension of:

$$R(s) : C_0^\infty(\mathbb{R}^3) \to C^\infty(\mathbb{R}^3),$$

to $\mathbb{C}$ with poles of finite rank. This could also be achieved by using the spherical symmetry of the spacetime to separate variables and doing an appropriate analysis of the ordinary differential equations in a similar vein to [14], however the former approach applies more readily to situations where such symmetry is not present. For more details on applying the results of Mazzeo and Melrose from geometric scattering theory to de Sitter, the reader is directed to [125], where the authors do this explicitly for the very similar case of Schwarzschild-de Sitter. We can now define the quasinormal frequencies as the poles of this continuation and the quasinormal modes as the corresponding solutions to Equation (3.1.5).

Returning to Equation (3.1.9), one can consider deforming the contour in Equation (3.1.9) in order to compute ψ using the residue theorem since the extension of $R(s)$ only has poles. For example, assuming suitable resolvent estimates hold for large $\operatorname{Im} s$, one can push the ends of the contour deeper into the left half plane (see Figure 3.2). This deformed contour can be expressed as the sum of a vertical line of $\operatorname{Re}(s) = -C$ and a closed contour enclosing any quasinormal frequencies s_i such that $\operatorname{Re} s_i > -C$. Writing $\hat{\psi}(s) = R(s)(s\psi_0 + \psi_1)$ and assuming there are (counting with multiplicity) N such frequencies, this process yields:

$$\begin{aligned}\psi(x) - &\sum_{i=1}^{N} \Psi_i w_i(x) e^{s_i t} \\ &= \lim_{\varepsilon \to 0} e^{-Ct} \left(\int_{-\infty}^{-\varepsilon} e^{i\omega t} \hat{\psi}(-C + i\omega) d\omega + \int_{\varepsilon}^{\infty} e^{i\omega t} \hat{\psi}(-C + i\omega) d\omega \right),\end{aligned}$$

where w_i are quasinormal modes and Ψ_i are constants depending only on the initial data. Assuming the integral is bounded by some $C_0 > 0$, this gives

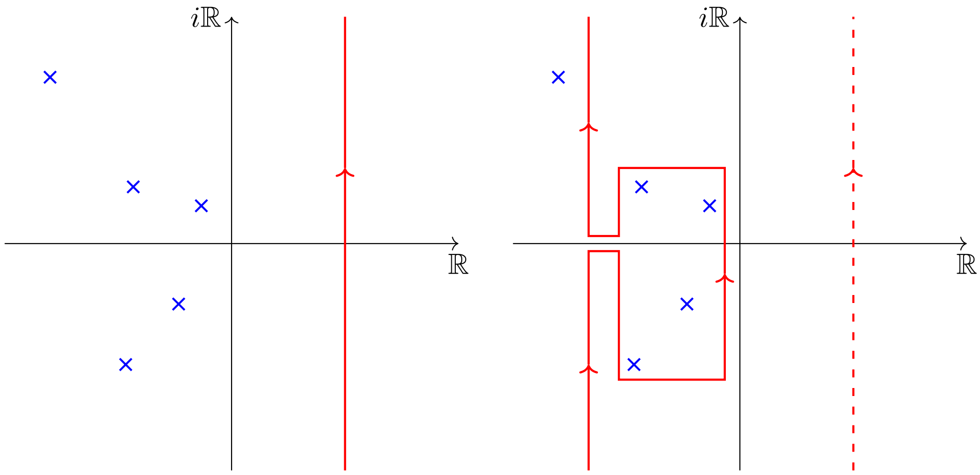
$$\left| \psi(x) - \sum_{i=1}^{N} \Psi_i w_i(x) e^{s_i t} \right| \leq C_0 e^{-Ct}.$$

These are typical asymptotics for positive cosmological constant spacetimes, as the only singularities in the extension of $R(s)$ are poles, so there is only ringdown and no polynomial tail. In the Schwarzschild case, the complications at the zero frequency mentioned in [125] can be thought of as a branch point singularity in $R(s)$ and generates a Price's law tail (see [72] for more details).

The above discussion enables us to define the quasinormal frequencies and express solutions to the equation as a sum of quasinormal modes with an exponentially decaying remainder, however we are no closer to identifying which values these frequencies take. To do so, we must first construct $R(s)$ to study its poles. This is challenging for the full three-dimensional problem, so we shall further assume the potential V is spherically symmetric and decompose Equation (3.1.5) into a family of uncoupled ordinary differential equations in r_*. For each angular momentum sector (l, m) we have the equation:

$$-\frac{\partial_{r_*} \left(\tanh^2(\kappa r_*) \partial_{r_*} \hat{\psi}_{lm} \right)}{\tanh^2(\kappa r_*)} + \left(\frac{l(l+1)\kappa^2}{\sinh^2(\kappa r_*)} + s^2 + \frac{V}{\cosh^2(\kappa r_*)} \right) \hat{\psi}_{lm} = s\psi_{0,lm} + \psi_{1,lm}, \tag{3.1.10}$$

where $\hat{\psi}_{lm}(r_*)$, $\psi_{0,lm}$ and $\psi_{1,lm}$ are the projections of $\hat{\psi}$, ψ_0 and ψ_1 respectively into the corresponding angular momentum subspaces. The assumption that V is spherically



(a) The original Bromwich contour (b) The deformed contour

Figure 3.2: By deforming the Bromwich inversion contour, we can use the residue theorem to express our solution as a sum of quasinormal modes with a more rapidly decaying remainder term.

symmetric isn't strictly necessary here, since we can solve for any given $\hat{\psi}_{lm}$ by considering a system of $l+1$ coupled equations, however for the sake of exposition we shall restrict to this simpler case. Noting that Equation (3.1.10) has the same structure as the full problem, the resolvent $R_{lm}(s)$ will be holomorphic in a region of the form $\operatorname{Re} s > c_0$ for some $c_0 \geq 0$. Dropping the l, m subscripts from now on to reduce clutter, we find a Green's function $G_s(r_*; r_*')$ which solves

$$-\frac{\partial_{r_*}\left(\tanh^2(\kappa r_*)\partial_{r_*} G_s\right)}{\tanh^2(\kappa r_*)} + \left(\frac{l(l+1)\kappa^2}{\sinh^2(\kappa r_*)} + s^2 + \frac{V}{\cosh^2(\kappa r_*)}\right) G_s = \delta(r_* - r_*'), \quad (3.1.11)$$

so that

$$\left(R(s)u\right)(r_*) = \int_0^\infty G_s(r_*; r_*')u(r_*')dr_*',$$

subject to suitable behaviour (i.e. sufficiently rapid decay at 0 and ∞). Returning to Equation (3.1.11), the exponential decay of the lower order terms imply that for large r_*, we will have smooth solutions which obey

$$\hat{\psi}^\pm \sim e^{\pm s r_*} \quad \text{as} \quad r_* \to \infty.$$

Since we are interested in finding $R(s)$ where it is holomorphic, we shall assume $\operatorname{Re} s > 0$ and the solution we seek is $\hat{\psi}^-$. Furthermore, by considering a Taylor series of the equation around $r_* = 0$, we see that the solutions should be either $O(r_*^l)$ or $O(r_*^{-l-1})$ as $r_* \to 0$. Let $\hat{\psi}^0$ be the regular solution. Using similar methods that we used to obtain Equation (3.1.8), we see that $\hat{\psi}^0$ and $\hat{\psi}^-$ are linearly independent for $\operatorname{Re}(s) > c_0$ for some constant. Therefore, for these values of s, we can write the Green's function

$$G_s(r_*;r'_*) = \frac{1}{W(s,r'_*)}\begin{cases} \hat{\psi}^0(r_*)\hat{\psi}^-(r'_*) & 0 \le r_* < r'_* < \infty \\ \hat{\psi}^0(r'_*)\hat{\psi}^-(r_*) & r'_* < r_* < \infty \end{cases}, \tag{3.1.12}$$

where $W(s,r_*) = \hat{\psi}^0\partial_{r_*}\hat{\psi}^- - \hat{\psi}^-\partial_{r_*}\hat{\psi}^0$ is the Wronskian of the above two solutions. Despite the general nature of the equation, the r_* dependence of $W(s,r_*)$ is not hard to fix. Since the equation is of the form:

$$\partial^2_{r_*}\hat{\psi} + \partial_r(\log(\tanh^2(\kappa r_*))\partial_{r_*}\hat{\psi} + (q-s^2)\hat{\psi} = 0, \tag{3.1.13}$$

and a simple computation gives

$$\begin{aligned} \partial_{r_*}W &= -\partial_r(\log(\tanh^2(\kappa r_*))W \\ \Rightarrow W(s,r_*) &= \frac{\tilde{W}(s)}{\tanh^2(\kappa r_*)}. \end{aligned}$$

Thus we can rewrite Equation (3.1.12):

$$G_s(r_*;r'_*) = \frac{\tanh^2(\kappa r'_*)}{\tilde{W}(s)}\begin{cases} \hat{\psi}^0(r_*)\hat{\psi}^-(r'_*) & 0 \le r_* < r'_* < \infty \\ \hat{\psi}^0(r'_*)\hat{\psi}^-(r_*) & r'_* < r_* < \infty \end{cases}. \tag{3.1.14}$$

Since G_s is analytic in a suitable region of the plane, we can try to analytically continue the above expression to the full complex plane. The only obstruction to doing so are the zeroes of $\tilde{W}$, which correspond to poles of $R(s)$. Thus we can find the quasinormal frequencies by seeking the poles of this meromorphic function and the modes by seeking solutions to Equation (3.1.13) which obey the boundary conditions $\hat{\psi}(0) = 0$ and $\hat{\psi}(r_*) \sim e^{-sr_*}$ as $r_* \to \infty$. Given a frequency s and a solution $\hat{\psi}$ satisfying the boundary conditions, we can find a solution to Equation (3.1.3):

$$\psi = e^{st}\hat{\psi}(r_*)Y_{lm}(\theta,\phi) \sim e^{s(t-r_*)} \quad \text{as} \quad r_* \to \infty.$$

As expected, this is asymptotic to a right-moving wave, so the boundary conditions we have imposed are the 'outgoing' boundary conditions of [35].

As discussed in the previous chapter, the asymptotic boundary condition is ambiguous for $\mathrm{Re}(s) < 0$ since we seek the growing mode solution, which is hard to pick out from solutions which contain both e^{-sr_*} and e^{sr_*} terms. This can be often be overcome through the method of complex scaling. Since the metric components of the de Sitter spacetime are analytic in a suitable set of coordinates, we can continue r_* into the complex plane and consider a deformed operator acting on a curve in $\mathbb{C}$ whose discrete spectrum is the same as the spectrum of quasinormal frequencies. By choosing a suitable deformation in $\mathbb{C}$, the outgoing mode can be made to be decaying and thus the problem is converted to a more standard one: finding the L^2 spectrum of some operator (albeit no longer a self-adjoint one). As a result, resonances can now be computed numerically through the usual discretisation techniques applied to the deformed operator. For a more detailed exposition of the technique of complex scaling and its extension to higher dimensions, we direct the reader to the book [53].

It is important to note at this stage that this does not necessarily mean that we have solved the full three-dimensional problem: we cannot rule out accumulation of poles as $l \to \infty$. Nevertheless, considering each of these subspaces individually can give some insight into the quasinormal spectrum.

3.1.2 Regularity quasinormal modes

In the last chapter, we discussed the advantages of the regularity approach of defining quasinormal modes first developed by Vasy in [137] using microlocal methods with many of the results reproduced using physical space methods by Warnick in [139]. We shall apply these physical space methods to Equation (3.1.1). In order to do so, we need to use coordinates regular at the horizon. The following change of coordinates gives a hyperboloidal foliation of the spacetime (see Figure 3.3):

$$\begin{aligned} \tau &= t + \frac{1}{2\kappa}\log\left(1-\kappa^2 r^2\right), \\ \rho &= \kappa r. \end{aligned}$$

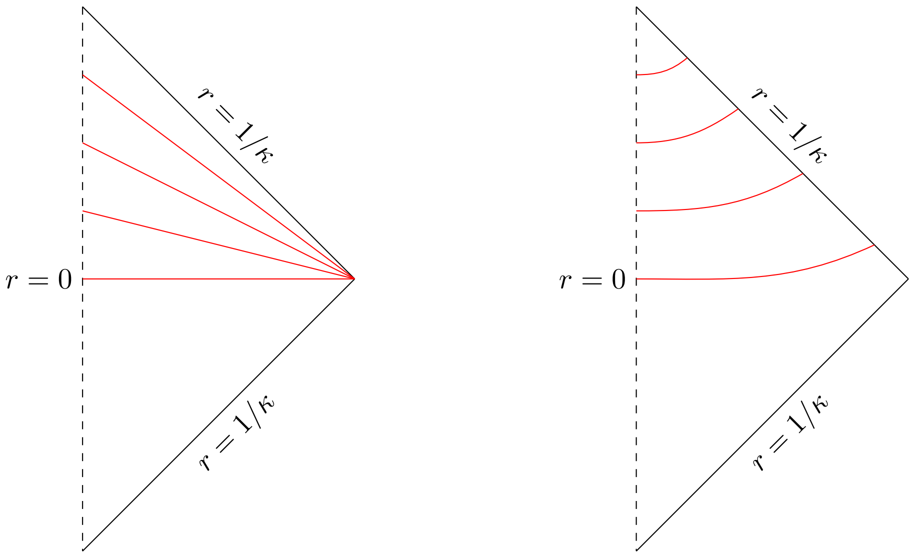
Using the coordinate τ gives a foliation where the leaves intersect the horizon and are regular there (see Figure 3.3) After this change of coordinates, Equation (3.0.1) becomes

$$g_\kappa = -(1-\rho^2)d\tau^2 - \frac{2\rho}{\kappa}d\tau d\rho + \frac{1}{\kappa^2}\left(d\rho^2 + \rho^2 g_{S^2}\right) \tag{3.1.15}$$

and can be extended to the boundary of the open ball so it is defined on $[0,\infty) \times \overline{B_1}$. We further change from polar coordinates on $\overline{B_1}$ to Cartesian ones $\{x_i\}_{i=3}^{3}$ such that $\sum_{i=1}^{3}(x_i)^2 = \rho^2$. We can write Equation (3.1.1) as:

$$-\kappa^2\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial_i\partial_j\psi + 4\kappa^2\sum_{i=1}^{3} x_i\partial_i\psi + 2\kappa\sum_{i=1}^{3} x_i\partial_i\partial_\tau\psi + 3\kappa\partial_\tau\psi + \partial_\tau^2\psi + V\psi = 0, \tag{3.1.16}$$

where $a_{ij} = \delta_{ij} - x_i x_j$. We impose initial conditions at $\tau = 0$ now instead: $\psi(0,\mathbf{x}) = \psi_0(\mathbf{x})$, $\partial_\tau\psi(0,\mathbf{x}) = \psi_1(\mathbf{x})$. Setting $(u,v) = (\psi, \partial_\tau\psi)$, we can recast the problem to the



(a) Constant t-slicing (b) Constant τ-slicing

Figure 3.3: Penrose diagrams of the static patch of de Sitter depicting (a) the static slicing and (b) the hyperboloidal slicing.

following form:

$$\frac{\partial}{\partial \tau}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\kappa^2 P_0 - V & -2\kappa P_1 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}, \quad \begin{pmatrix} u \\ v \end{pmatrix}\Bigg|_{\tau=0} = \begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix}, \tag{3.1.17}$$

where

$$P_0 u := -\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial_i\partial_j u + 4\sum_{i=1}^{3} x_i \partial_i u, \tag{3.1.18}$$

$$= -\sum_{i=1}^{3}\sum_{j=1}^{3} \partial_i \left(a_{ij}\partial_j u\right) \tag{3.1.19}$$

$$P_1 u := \sum_{i=1}^{3} x_i \partial_i u + \frac{3}{2}u. \tag{3.1.20}$$

By the theory developed in [139] (assuming V is $C^\infty(B_1)$), we have the following result:

Proposition 3.1.1. *For each $\tau \geq 0$, define the operator $\mathcal{S}(\tau) : H^k(B_1) \times H^{k-1}(B_1) \to H^k(B_1) \times H^{k-1}(B_1)$ such that*

$$\mathcal{S}(\tau)\begin{pmatrix} \psi_0 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}(\tau),$$

where (u, v) is the unique solution to (3.1.17) *and $H^n(B_1)$ denote the usual Sobolev spaces with the case $n = 0$ denoting $L^2(B_1)$. Then the family of operators $\{\mathcal{S}(\tau)\}_{\tau\geq 0}$ forms a C^0-semigroup acting on $H^k(B_1) \times H^{k-1}(B_1)$. The infinitesimal generator of $\mathcal{S}$*

is the closed, densely defined operator $\mathcal{A} : D^k(\mathcal{A}) \to H^k(B_1) \times H^{k-1}(B_1)$ given by

$$\mathcal{A} = \begin{pmatrix} 0 & 1 \\ -\kappa^2 P_0 - V & -2\kappa P_1 \end{pmatrix},$$

where $D^k(\mathcal{A}) = \{\psi \in H^k(B_1) \times H^{k-1}(B_1) \mid \mathcal{A}\psi \in H^k(B_1) \times H^{k-1}(B_1)\}$ is the domain of the unbounded operator $\mathcal{A}$. The resolvent $(\mathcal{A} - s)^{-1} : H^k(B_1) \times H^{k-1}(B_1) \to H^k(B_1) \times H^{k-1}(B_1)$ is well-defined and holomorphic on $\mathrm{Re}(s) > C$ for some real constant C.

Proof. For shorthand, we shall use the notation $H^k = H^k(B_1)$ and $\mathbf{H}^k = H^k \times H^{k-1}$. We also note that for $u, v \in C^\infty(B_1)$,

$$\begin{aligned}(u, P_0 v)_{L^2} &= -\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} u^* \partial_i(a_{ij}\partial_j v)dx \\ &= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} a_{ij}\partial_i u^* \partial_j v dx\end{aligned}$$

where the boundary term vanishes since $\sum_{j=1}^{3} a_{ij} n_j = 0$ on the boundary with n_j the outward pointing normal. By continuity, the above equality holds for $u, v \in H^1$ and so we define the sesquilinear form on H^1

$$\langle u, v\rangle = \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} a_{ij}\partial_i u^* \partial_j v dx + 2\int_{B_1} u^* v dx.$$

We shall begin with the case $V = 2\kappa^2$ and write $\mathcal{A}_0$ as the corresponding densely defined operator on $\mathbf{H}^k$.

1. Consider the map $B_0 : \mathbf{H}^1 \times \mathbf{H}^1 \to \mathbb{C}$ defined by

   $$B_0[u, v; f, g] := \kappa^2\langle f, u\rangle + \kappa^2(P_1 u, P_1 f)_{L^2} + (v + \kappa P_1 u, g + \kappa P_1 f)_{L^2}.$$

   Note that it is clearly $\mathbb{R}$-linear and that $B_0[u, v; f, g] = B_0[f, g; u, v]^*$. It is also non-degenerate: $B_0[u, v; u, v] = 0$ implies each term must vanish individually since they are non-negative. From there, it is not hard to see that $\langle u, u\rangle = 0$ if and only if u is constant and we can deduce from the $\|P_1 u\|_{L^2}$ term that $u \equiv 0$. It then follows from the remaining term that $v \equiv 0$. We can use these properties to deduce a Cauchy-Schwarz-type inequality:

   $$(\mathrm{Re}\, B_0[u, v; f, g])^2 \leq B_0[u, v; u, v] B_0[f, g; f, g].$$

   It is clear that there exists a constant $C > 0$ such that

   $$B_0[u, v; u, v] \leq C\left(\kappa^2\|u\|_{H^1}^2 + \|v\|_{L^2}^2\right).$$

We also note that

$$\begin{aligned}
\|u\|_{H^1}^2 &= \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1}(a_{ij}+x_ix_j)\partial_i u^*\partial_j u dx + \int_{B_1}|u|^2 dx \\
&= \langle u,u\rangle - \|u\|_{L^2}^2 + \|(P_1-3/2)u\|_{L^2}^2 \\
&= \langle u,u\rangle + \frac{5}{4}\|u\|_{L^2}^2 + \|P_1u\|_{L^2}^2 - 3\operatorname{Re}(u,P_1u)_{L^2}
\end{aligned}$$

Since

$$\begin{aligned}
\operatorname{Re}(u,P_1u)_{L^2} &= \frac{1}{2}\sum_{i=1}^{3}\int_{B_1} x_i\partial_i(|u|^2)dx + \frac{3}{2}\int_{B_1}|u|^2 dx \\
&= \int_{S^2}|u|^2 d\sigma \geq 0,
\end{aligned}$$

we have the inequality

$$\|u\|_{H^1}^2 \leq \langle u,u\rangle + \frac{5}{4}\|u\|_{L^2}^2 + \|P_1u\|_{L^2}^2.$$

Furthermore, we have

$$\begin{aligned}
\|v\|_{L^2}^2 &= \|v+\kappa P_1u - \kappa P_1u\|_{L^2}^2 \\
&= \|v+\kappa P_1u\|_{L^2}^2 - \kappa^2\|P_1u\|_{L^2}^2 - 2\kappa\operatorname{Re}(v,P_1u)_{L^2} \\
&\leq \|v+\kappa P_1u\|_{L^2}^2 + \kappa^2\|P_1u\|_{L^2}^2 - \frac{1}{2}\|v\|_{L^2}^2,
\end{aligned}$$

where we have used Young's inequality to get the last line. Combining these estimates shows that $\sqrt{B_0[u,v;u,v]}$ defines a norm on $\mathbf{H}^1$ with the same topology as the usual norm.

2. It will follow from arguments in the proof of Proposition 3.1.2 that $\mathcal{A}_0$ is closed and for $s>0$, given $(f,g)\in\mathbf{H}^1$ there exists a unique solution to

$$(\mathcal{A}_0 - s)\begin{pmatrix} u \\ v\end{pmatrix} = \begin{pmatrix} f \\ g\end{pmatrix}$$

and hence the resolvent $(\mathcal{A}_0-s)^{-1}$ exists. It will also follow from arguments in the proof of Proposition 3.1.2 that if $f,g\in C^\infty(B_1)$, $u,v\in C^\infty(B_1)$ where

$$\begin{pmatrix} u \\ v\end{pmatrix} = (\mathcal{A}_0-s)^{-1}\begin{pmatrix} f \\ g\end{pmatrix}.$$

Assuming the above equation holds, this means we have smooth functions u,v,f,g

such that

$$\begin{aligned} -su + v &= f \\ -\kappa^2(P_0+2)u - 2\kappa P_1 v - sv &= g. \end{aligned}$$

Rearranging these equations, we are left with

$$\kappa^2(P_0+2)u + s\left(v + 2\kappa P_1 u\right) = -g - 2\kappa P_1 f. \tag{3.1.21}$$

Taking the L^2-inner product of the above with $v + \kappa P_1 u = su + f + \kappa P_1 u$, we have

$$\begin{aligned} s\left(\kappa^2\langle u,u\rangle \right. &\left. + \kappa^2\|P_1 u\|_{L^2}^2 + \|v+\kappa P_1 u\|_{L^2}^2\right) + s(v,\kappa P_1 u)_{L^2} + \kappa^3\langle P_1 u, u\rangle \\ &= -\kappa^2\langle f,u\rangle - \kappa^2(P_1 f, P_1 u)_{L^2} - (v+\kappa P_1 u, g + \kappa P_1 f) - (v, \kappa P_1 f)_{L^2} \end{aligned}$$

which can be rewritten

$$\begin{aligned} sB_0[u,v;u,v] + \kappa^3\langle P_1 u, u\rangle &= -B_0[u,v;f,g] - (v, P_1(su+f))_{L^2} \\ &= -B_0[u,v;f,g] - (v, P_1 v)_{L^2}. \end{aligned}$$

From a simple calculation, it follows that

$$\mathrm{Re}\langle P_1 u, u\rangle = \langle u,u\rangle - \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} \rho\partial_\rho a_{ij}\partial_i u^* \partial_j u \geq 0.$$

The last inequality follows from the fact that $\partial_\rho a_{ij}$ is negative definite up to the boundary of B_1 (since the surface gravity of the horizon is non-zero) and is related to the celebrated redshift estimate [41]. Thus we have

$$\begin{aligned} sB_0[u,v;u,v] &\leq -\mathrm{Re}\, B_0[u,v;f,g] \\ &\leq \sqrt{B_0[u,v;u,v]B_0[f,g;f,g]}. \end{aligned}$$

By continuity, we have (for $s > 0$)

$$\sqrt{B_0[u,v;u,v]} \leq \frac{1}{s}\sqrt{B_0[f,g;f,g]}$$

for all $(f,g) \in \mathbf{H}^1$ and $(u,v) = (\mathcal{A}_0 - s)^{-1}(f,g)$. The result follows by the Hille-Yosida theorem.

3. Let α be a multi-index of order k and note that by differentiating Equation (3.1.21), we have

$$\kappa^2(P_0 + 2 + 2kP_1 + k^2)\partial^\alpha u + s(\partial^\alpha v + 2\kappa(P_1+k)\partial^\alpha u) = -(\partial^\alpha g + 2\kappa(P_1+k)\partial^\alpha f).$$

We construct $B_k[u, v; f, g]$ similarly to before:

$$\begin{aligned} B_k[u, v; f, g] := &\kappa^2 \left(\langle f, u \rangle + k^2 (f, u)_{L^2} + ((P_1 + k)u, (P_1 + k)f)_{L^2} \right) \\ &+ (v + \kappa(P_1 + k)u, g + \kappa(P_1 + k)f)_{L^2} \end{aligned}$$

and note that we can define a norm on $\mathbf{H}^k$ with the same topology of the usual one by writing

$$\|(u, v)\|^2 = \sum_{j=0}^{k-1} B_k[u, v; u, v].$$

Then by a similar procedure as before, we obtain the resolvent estimate necessary to apply the Hille-Yosida theorem.

4. Now take V a sufficiently smooth and set

$$\mathcal{V} = \begin{pmatrix} 0 & 0 \\ 2\kappa^2 - V & 0 \end{pmatrix}. \tag{3.1.22}$$

Furthermore, let $V_0 > 0$ be the operator norm of $\mathcal{V}$ on $\mathbf{H}^k$. Then we can write $\mathcal{A} = \mathcal{A}_0 + \mathcal{V}$ and we have

$$\begin{aligned} (\mathcal{A} - s)^{-1} &= (\mathcal{A}_0 - s)^{-1} \left(I_{\mathbf{H}^k} + \mathcal{V}(\mathcal{A}_0 - s)^{-1} \right) \\ &= \sum_{j=0}^{\infty} (\mathcal{A}_0 - s)^{-1} \left(\mathcal{V}(\mathcal{A}_0 - s)^{-1} \right)^j, \end{aligned}$$

provided $V_0/s > 1$. Therefore

$$\begin{aligned} \|(\mathcal{A} - s)^{-1}\|_{\mathbf{H}^k \to \mathbf{H}^k} &\leq \frac{1}{s} \sum_{k=0}^{\infty} \|\mathcal{V}(\mathcal{A}_0 - s)^{-1}\|^k \\ &\leq \frac{1}{s} \sum_{k=0}^{\infty} \left(\frac{V_0}{s} \right)^k \\ &= \frac{1}{s - V_0} \end{aligned}$$

so the result holds.

□

From the definition in the previous chapter, the quasinormal frequencies are eigenvalues of $\mathcal{A}$ and the modes are the corresponding eigenvectors. We see that eigenvectors of $\mathcal{A}$

must have components obeying:

$$\begin{aligned} su &= v, \\ sv &= -\kappa^2 P_0 u - Vu - 2\kappa P_1 v, \end{aligned}$$

which means they are of the form

$$\mathbf{u} = \begin{pmatrix} u \\ su \end{pmatrix}, \tag{3.1.23}$$

where $\kappa^2 P_0 u + 2s\kappa P_1 u + s^2 u + Vu = 0$. We define the densely defined operator:

$$P(s) := P_0 + 2sP_1 + s^2$$

and set $\hat{V}(\mathbf{x}) = V(\mathbf{x}/\kappa)/\kappa^2$. Note that there is a one-to-one correspondence between eigenvectors of $\mathcal{A}$ and solutions to $(P(s/\kappa) + \hat{V})u = 0$. While the dependence on s of this problem is quadratic (versus the linear dependence of the eigenvalue problem for the semigroup generator), it is an elliptic problem (away from the cosmological horizon where ellipticity degenerates) and we can apply Fredholm theory to it.

For $\operatorname{Re} s > 1/2-k$, the $P(s)$ form a holomorphic family of Fredholm operators $D^k(P(s)+\hat{V}) \to H^{k-1}(B_1)$ where we define $D^k(P(s)+\hat{V})$ to be the domain of $P(s)$ i.e. the closure with respect to the graph norm of

$$\left\{ u \in C^\infty(B_1) \,\middle|\, \|u\|_{D^k} := \|u\|_{H^{k-1}} + \|(P(s) + \hat{V})u\|_{H^{k-1}} < \infty \right\}. \tag{3.1.24}$$

Note that for the range of values of s we are considering, this set is independent of the value of s and is in fact a subset of $H^k(B_1)$. Since the space does not depend on s, we shall write it as D^k for short from now on.

Proposition 3.1.2. *Let $f \in H^{k-1}(B_1)$, $\hat{V} \in C^\infty(B_1)$ and $\Re(s) > 1/2 - k$. If $(P(s) + \hat{V})u = f$, we have either:*

(i) there exists a unique solution to $(P(s) + \hat{V})u = f$ where $u \in H^k(B_1)$

(ii) there exists a finite dimensional space of solutions $v \in C^\infty(\overline{B_1})$ to $(P(s)+\hat{V})v = 0$. Moreover this can only occur at isolated values of s.

Proof. This result would follow from standard elliptic theory, however the degeneration of the principal symbol at the cosmological horizon prevents us from using it near the horizon. To get around this, we must use the redshift effect [41] at the horizon. We begin with the case $k = 1$.

1. Taking $\gamma > 0$ to be chosen later, we have

$$
\begin{aligned}
&\operatorname{Re}\int_{B_1} s^* u^* \left(P(s)+\hat{V}+\gamma\right) u dx \\
&\qquad = \operatorname{Re}(s)\int_{B_1}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial_i u^* \partial_j u + (\gamma+|s|^2)|u|^2\right) dx \\
&\qquad\quad + \int_{B_1}\operatorname{Re}(s^*\hat{V})|u|^2 dx + |s|^2\int_{S^2}|u|^2 d\sigma \\
&\qquad \geq \operatorname{Re}(s)\int_{B_1}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial_i u^* \partial_j u + (\gamma+|s|^2-|\hat{V}|)|u|^2\right) dx
\end{aligned}
$$

where we have used the divergence theorem and the fact that $\sum_{i=1}^{3} a_{ij}x_i = 0$ for $\mathbf{x}\in S^2$. Provided $\operatorname{Re}(s) > \|\hat{V}\|_{L^\infty}^{1/2}$, this means for any $\epsilon > 0$ we have

$$
\begin{aligned}
E(u) :=& \int_{B_1}\left(\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\partial_i u^* \partial_j u + \gamma|u|^2\right) dx \\
\leq& \frac{1}{\operatorname{Re}(s)}\operatorname{Re}\int_{B_1} s^* u^*\left(P(s)+\hat{V}+\gamma\right) u dx \\
\leq& \frac{|s|}{\operatorname{Re}(s)}\|u\|_{L^2}\|(P(s)+\hat{V}+\gamma)u\|_{L^2} \\
\leq& \frac{\epsilon}{2}\|(P(s)+\hat{V}+\gamma)u\|_{L^2}^2 + \frac{|s|^2}{2\epsilon\operatorname{Re}(s)^2}\|u\|_{L^2}^2.
\end{aligned} \tag{3.1.25}
$$

This is related to the Killing energy estimate for the hyperbolic problem.

2. Next, we consider the expression

$$
\begin{aligned}
\operatorname{Re}\int_{B_1} P_1 u^*\left(P(s)+\hat{V}+\gamma\right) u dx =& \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} a_{ij}\partial_i u^* \partial_j u dx \\
&- \frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1}\rho\partial_\rho a_{ij}\partial_i u^*\partial_j u dx \\
&+ 2\operatorname{Re}(s)\int_{B_1}|P_1 u|^2 dx + \frac{\gamma}{2}\int_{S^2}|u|^2 d\sigma \\
&+ \operatorname{Re}\int_{B_1}\left(s^2+\hat{V}\right) u P_1 u^* dx.
\end{aligned}
$$

Adding a multiple of $E(u)$ to the above, we have for any $\delta > 0$

$$\begin{aligned}
\operatorname{Re}\int_{B_1} P_1 u^* \left(P(s) + \hat{V} + \gamma\right) u dx + E(u) \\
\geq \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} \left(2\delta_{ij} + (2\operatorname{Re}(s) - 1)x_i x_j\right) \partial_i u^* \partial_j u dx \\
- \delta\|P_1 u\|_{L^2}^2 - \frac{\|s^2 + \hat{V}\|_{L^\infty}^2}{4\delta}\|u\|_{L^2}^2 + \gamma\|u\|_{L^2}^2.
\end{aligned}$$

This is analogous to the redshift estimate we made in the proof of Proposition 3.1.1: again, it will be crucial in our argument that $\rho\partial_\rho a_{ij}|_{\rho=1}$ is non-degenerate. Since the matrix $2\delta_{ij} + (2\operatorname{Re}(s) - 1)x_i x_j$ has eigenvalues 2 and $2 + (2\operatorname{Re}(s) - 1)|\mathbf{x}|^2$ and orthogonal eigenvectors, we in fact have

$$\begin{aligned}
\operatorname{Re}\int_{B_1} P_1 u^* \left(P(s) + \hat{V} + \gamma\right) u dx + E(u) \\
\geq C_0(s)\|\nabla u\|_{L^2}^2 - \delta\|P_1 u\|_{L^2}^2 + \left(\gamma - \frac{\|s^2 + \hat{V}\|_{L^\infty}^2}{4\delta}\right)\|u\|_{L^2}^2,
\end{aligned}$$

where $C_0(s) = \min\{2, 2\operatorname{Re}(s) + 1\}$. This means that this estimate in fact holds for $\operatorname{Re}(s) > -1/2$. Combining this with Equation (3.1.25) and taking ϵ, δ sufficiently small and γ sufficiently large, we have for $\operatorname{Re}(s) > \|\hat{V}\|_{L^\infty}^{1/2}$,

$$\|u\|_{H^1} \leq C_1(s)\|(P(s) + \hat{V} + \gamma)u\|_{L^2}$$

where $C_1(s)$ is a constant depending only on s.

3. Let $s_0 = \|\hat{V}\|_{L^\infty}^{1/2} + 1/2$. Then for $\operatorname{Re} s > -1/2$, we have

$$E(u) \leq \frac{\epsilon}{2}\|(P(s + s_0) + \hat{V} + \gamma)u\|_{L^2}^2 + \frac{|s + s_0|^2}{2\epsilon\operatorname{Re}(s + s_0)^2}\|u\|_{L^2}^2$$

Since $P(s + s_0) = P(s) + 2s_0 P_1 + s_0(2s + s_0)$, it follows that

$$\|(P(s + s_0) + \hat{V} + \gamma)u\|_{L^2}^2 \leq \|(P(s) + \hat{V} + \gamma)u\|_{L^2}^2 + C_2(s)\|u\|_{H^1}^2.$$

Substituting and running through the arguments as before, we have for $\operatorname{Re} s > -1/2$,

$$\|u\|_{H^1} \leq C_3(s)\|(P(s) + \hat{V} + \gamma)u\|_{L^2}. \tag{3.1.26}$$

4. The estimate above establishes that $P(s)+\hat{V}+\gamma$ defines an injective map $H^1(B_1) \to L^2(B_1)$ for $\operatorname{Re}(s)$ sufficiently large. By considering an adjoint problem and performing similar estimates to above, one can show that it is surjective for these values of s as well. Hence the inverse exists and is a compact operator $L^2(B_1) \to$

$L^2(B_1)$. The result follows from an application of the Fredholm alternative and noting that $P(s) + \hat{V} + \gamma$ is a polynomial in s.

5. Finally, we consider the cases $k > 1$. We first note that

$$\partial_i P(s)u = P(s+1)\partial_i u,$$

which means that differentiating the equation yields a system of equations with a similar structure: for example in the case $k = 2$, we have

$$\begin{aligned}\left(P(s+1) + \hat{V} + \gamma\right)\begin{pmatrix} u \\ \nabla u \end{pmatrix} - \begin{pmatrix} 2s+1 & \mathbf{x}^\top \\ \mathbf{0} & 0 \end{pmatrix}\begin{pmatrix} u \\ \nabla u \end{pmatrix} \\ + \begin{pmatrix} 0 & \mathbf{0}^\top \\ \nabla \hat{V} & 0 \end{pmatrix}\begin{pmatrix} u \\ \nabla u \end{pmatrix} = \begin{pmatrix} f \\ \nabla f \end{pmatrix}.\end{aligned}$$

After absorbing the extra terms into a potential term $\hat{V_1}$, we have the system of equations

$$\left(P(s+1) + \hat{V_1} + \gamma\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix},$$

which is of the same structure as the $k = 1$ case but acts on vectors of functions. We can get similar estimates to the $k = 1$ case except with s replaced with $s + 1$ and with respect to higher regularity norms. We can repeat this process to get the result for each $k \in \mathbb{N}$.

□

The upshot of this result is that given $f \in H^{k-1}(B_1)$, we have a family of solutions $u(s) \in H^k(B_1)$ which are meromorphic in s for $\mathrm{Re}(s) > 1/2 - k$ and that the locations of the poles of $u(s) = (P(s) + \hat{V})^{-1} f$ are quasinormal frequencies. It is important to note here that for any s to the left of the region defined (i.e. with $\mathrm{Re}(s) < 1/2 - k$), there exists $v \in H^k(B_1)$ such that $(P(s) + \hat{V})v = 0$, so we must restrict to this subset of $\mathbb{C}$[1].

We shall now compute the quasinormal frequencies for a Klein-Gordon field of mass $\kappa\,\mathfrak{m} \geq 0$ propagating on a de Sitter background ($V = \kappa^2\,\mathfrak{m}^2$). Note that to determine the corresponding modes, it suffices to determine the scalar function u in (3.1.23) since the second component is just a multiple of it. To aid our computation, we use the

[1]This goes some way to explaining the spectral instability observed for higher overtones: see the discussion in Section 4.3.1

spherical symmetry of de Sitter to decompose u into spherical harmonics:

$$u = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} u_{lm}(\rho) Y_{lm}(\theta, \phi),$$

where we have switched back to spherical polar coordinates. Using the orthogonality of the Y_{lm}, we can separate the equations to reduce the problem to a set of ordinary differential equations on $[0, 1]$:

$$(1-\rho^2)\partial_\rho^2 u_{lm} + \left(\frac{2}{\rho} - 4\rho - 2\tilde{s}\rho\right)\partial_\rho u_{lm} - \left(\frac{l(l+1)}{\rho^2} + \tilde{s}^2 + 3\tilde{s} + \mathfrak{m}^2\right) u_{lm} = 0,$$

where $\tilde{s} := s/\kappa$. This is a second order Fuchsian equation with four regular singular points: 0, ± 1 and ∞. Considering the indicial equation at $\rho = 0$, we see that the roots are l and $-l-1$. The second exponent will clearly lead to solutions which are not smooth at the origin and so can be discarded. We seek a solution of the form $u_{lm}(\rho) = \rho^l v_{lm}(\rho)$ which leads to the equation

$$(1-\rho^2)\partial_\rho^2 v_{lm} + \left(\frac{2l+2}{\rho} - (2\tilde{s} + 2l + 4)\rho\right)\partial_\rho v_{lm} - (\tilde{s} + l + \mathfrak{m}_-)(\tilde{s} + l + \mathfrak{m}_+) v_{lm} = 0,$$

where

$$\mathfrak{m}_\pm = \frac{3}{2} \pm \sqrt{\frac{9}{4} - \mathfrak{m}^2}.$$

Changing variable to $z = \rho^2$, we have:

$$z(1-z)\partial_z^2 v_{lm} + \left(l + \frac{3}{2} - \left(\tilde{s} + l + \frac{5}{2}\right) z\right)\partial_z v_{lm} - \frac{\tilde{s} + l + \mathfrak{m}_-}{2} \cdot \frac{\tilde{s} + l + \mathfrak{m}_+}{2} v_{lm} = 0.$$

This is the hypergeometric equation, which gives us a unique solution that is smooth at $\rho = 0$:

$$u_{lm}(\rho) = \rho^l {}_2F_1\left[\frac{\tilde{s} + l + \mathfrak{m}_-}{2}, \frac{\tilde{s} + l + \mathfrak{m}_+}{2}; \frac{3}{2} + l; \rho^2\right],$$

where ${}_2F_1$ is the hypergeometric function as defined in [112]. Thus we have found candidates for quasinormal modes: vectors constructed from functions of the form

$$u(\rho, \theta, \phi) = \rho^l Y_{lm}(\theta, \phi) \cdot {}_2F_1\left[\frac{\tilde{s} + l + \mathfrak{m}_-}{2}, \frac{\tilde{s} + l + \mathfrak{m}_+}{2}; \frac{3}{2} + l; \rho^2\right]$$

satisfy the appropriate equation, we simply need to check that they are smooth at the horizon. Since $\rho^l Y_{lm}(\theta, \phi)$ is smooth in $\overline{B_1}$, it suffices to check smoothness of the hypergeometric function given above on $[0, 1]$.

From standard results for Fuchsian equations and the Taylor series about $\rho = 0$ of v_{lm}, the solution is analytic on the unit disc, so we only need to check behaviour at $\rho = 1$. First, we define the sets

$$Q_l^{\pm} = \{-\mathfrak{m}_{\pm} - l, -\mathfrak{m}_{\pm} - l - 2, -\mathfrak{m}_{\pm} - l - 4, \dots\}$$

and $Q_l = Q_l^- \cup Q_l^+$. Provided $\tilde{s} \notin Q_l$, the radius of convergence of the Taylor series about $\rho = 0$ for this hypergeometric function is 1 and hence there must be a singularity of the function on the unit circle in $\mathbb{C}$. This can only occur at a regular singular point of the equation, namely $0, \pm 1, \infty$. Hence we have a singularity at either $\rho = 1$ or $\rho = -1$, but since $v_{lm}(\rho)$ is even, there must be one at both. Whether this singularity arises from a pole or a branch point, after sufficiently many derivatives, $\partial_\rho^k u_{lm}$ will not be continuous on $[0, 1]$ and hence we cannot have quasinormal frequencies for $\tilde{s} \notin Q_l$. Now it suffices to check that elements of this set are indeed quasinormal frequencies: for these values of $\tilde{s}$, the Taylor series for v_{lm} about $\rho = 0$ terminates. This gives us the following expressions for u:

$$\begin{aligned}
\tilde{s} = -l - 2n - \mathfrak{m}_-, \quad u &= \rho^l Y_{lm}(\theta, \phi) \sum_{k=0}^{n} \frac{n(n - \mathfrak{m}_\Delta) \dots (n - k + 1)(n - k + 1 - \mathfrak{m}_\Delta)}{\left(\frac{3}{2} + l\right)_k k!} \rho^{2k}, \\
\tilde{s} = -l - 2n - \mathfrak{m}_+, \quad u &= \rho^l Y_{lm}(\theta, \phi) \sum_{k=0}^{n} \frac{n(n + \mathfrak{m}_\Delta) \dots (n - k + 1)(n - k + 1 + \mathfrak{m}\,\Delta)}{\left(\frac{3}{2} + l\right)_k k!} \rho^{2k},
\end{aligned} \tag{3.1.27}$$

where $\mathfrak{m}_\Delta = (\mathfrak{m}_+ - \mathfrak{m}_-)/2$ and $n \in \mathbb{N}_0$. We have used the notation $(a)_k$ for the Pochhammer symbol i.e. $(a)_0 = 1$ and $(a)_k = a(a + 1) \dots (a + k - 1)$ for $a \in \mathbb{C}$ and $k \in \mathbb{N}$. These are polynomials and thus clearly smooth, so the quasinormal frequencies for a given angular momentum sector l are Q_l with the corresponding modes determined by the above functions. Returning to Cartesian coordinates, we see that for each frequency $\tilde{s} \in Q = \cup_l Q_l$ there is a finite dimensional subspace of analytic solutions (spanned by polynomials in $\{x_1, x_2, x_3\}$) as expected from the results in [114].

If we consider the quasinormal spectrum of a Klein-Gordon field of mass $\kappa\,\mathfrak{m}$ propagating on a de Sitter background with cosmological constant $3\kappa^2$, we see that the spectrum of frequencies is given by κQ above. For $0 \leq \mathfrak{m}^2 \leq 9/4$, the frequencies all lie on the real line: in the special cases $\mathfrak{m}^2 = 0$ (wave equation) and $\mathfrak{m}^2 = 2$ (conformal Klein-Gordon equation), the set of frequencies are $-\kappa\mathbb{N}_0$ and $-\kappa\mathbb{N}$ respectively. For $\mathfrak{m}^2 > 9/4$, they all lie on one of the lines $\operatorname{Im} s = \pm\kappa\sqrt{\mathfrak{m}^2 - 9/4}$. This existence of a critical mass above which the decay is fixed and only rate of oscillation varies is analogous to the situation in [79], where a massive field on a near-extremal Kerr background was studied. In either case, when we take the (singular) limit $\kappa \to 0$, the frequencies all converge to 0 individually, bunch up and get closer to the real line (for the heavy field case, the frequencies move on straight lines to the origin so they bunch up near the real line - see Figure 3.4). Thus the frequencies for this family of potentials fit the definition of zero-damped modes given in Chapter 2.

In the above calculations, we have assumed that the mass of the Klein-Gordon field varies with the cosmological constant: that it goes to 0 as $\Lambda \to 0$. One can ask if zero-damped modes are present if we take a fixed mass, $V = \mu^2 > 0$. In this case, our calculations above are the same except with $\mathfrak{m}$ replaced with μ/κ. Thus the quasinormal spectrum of such a field consists of frequencies of the form

$$s = -\kappa\left(n + \frac{1}{2}\right) \pm \sqrt{\frac{9}{4}\kappa^2 - \mu^2}$$

with $n \in \mathbb{N}$. Taking the extremal limit ($\kappa \to 0$), we see that for κ sufficiently small, we have

$$s = \pm i\mu - \kappa\left(n + \frac{1}{2}\right) + O(\kappa^2). \tag{3.1.28}$$

Thus we still have zero-damped modes in this case, however this time they cluster on the lines $\operatorname{Im} s = \pm\mu$. The intuition behind the zeroeth order part of this expansion is straightforward: in the limit $\kappa \to 0$, we expect the equation to look like the Klein-Gordon equation on Minkowski. The late-time asymptotics for a Klein-Gordon field ψ of mass μ in Minkowski are [95]

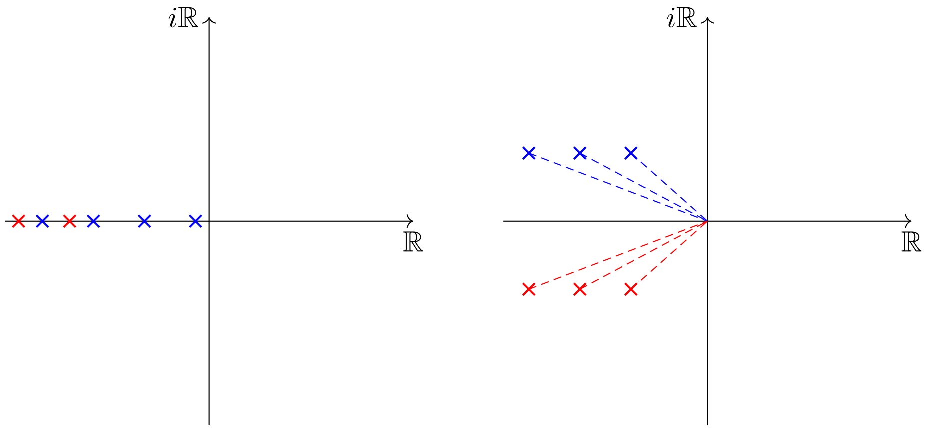
$$\psi \sim t^{-3/2}\sin(\mu t) = \frac{t^{-3/2}}{2i}\left(e^{i\mu t} - e^{-i\mu t}\right),$$

where the $t^{-3/2}$ factor arises from integrating along the branch cuts introduced by the asymptotically flat end. Expressions like Equation (3.1.28) are fairly typical in calculations of near-extremal quasinormal modes for rotating black holes (see [82, 111] for just a couple of examples). This invites the possibility of using the de Sitter spacetime as a model problem for studying zero-damped modes in spacetimes with Killing horizons of surface gravity κ. This idea will be explored further in later chapters.

The approach we used to identify the frequencies and modes above is similar to Leaver's method. However, the hyperboloidal slicing was crucial to being able to do this: a quasinormal mode generates a solution to the wave equation of the form

$$e^{-(\mathfrak{m}_\pm + l + 2n)\kappa\tau}\rho^l Y_{lm}(\theta,\phi)q(\rho^2) = e^{-(\mathfrak{m}_\pm + l + 2n)\kappa t}(1-\kappa^2 r^2)^{-(\mathfrak{m}_\pm + l + 2n)/2}\,(\kappa r)^l\, Y_{lm}(\theta,\phi)q\left(\kappa^2 r^2\right)$$

where q is a polynomial, n is some positive integer and we have converted back to static coordinates. This is (as one would expect from the slicing) clearly divergent at $r = 1/\kappa$, so using regularity as a boundary condition at the horizon would not be possible.



(a) $\mathfrak{m}^2 \leq 9/4$ (b) $\mathfrak{m}^2 > 9/4$

Figure 3.4: The quasinormal spectrum of the Klein-Gordon equation on de Sitter. The set Q^+ is marked in red and the set Q^- in blue.

3.2 Quasinormal co-modes

In this section we shall compute the quasinormal co-modes for the Klein-Gordon field in de Sitter that we considered above. This recreates the calculations initially found in [75] using the equivalent definition from the last chapter. We begin by noticing that $P_0^\dagger = P_0$ since we have

$$-\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} v^*\partial_i\left(a_{ij}\partial_j u\right)dx = -\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} u\partial_i\left(a_{ij}\partial_j v^*\right)dx,$$

using the divergence theorem twice and noting that the boundary terms vanish due to the nature of our test functions. Furthermore, we have

$$\sum_{i=1}^{3}\int_{B_1} v^* x_i\partial_i u dx + \frac{3}{2}\int_{B_1} v^* u dx = -\sum_{i=1}^{3}\int_{B_1} u x_i \partial_i v^* dx - 3\int_{B_1} v^* u dx + \frac{3}{2}\int_{B_1} v^* u dx.$$

Hence $P(s)^\dagger = P_0 - 2s^* P_1 + (s^*)^2 = P(-s^*)$. Recalling that $\tilde{s} = s/\kappa$, we already know the solutions which are regular at $\rho = 0$ in the case $V = -\kappa^2\,\mathfrak{m}^2$:

$$\begin{aligned} u(\rho,\theta,\phi) &= \rho^l Y_{lm}(\theta,\phi)\cdot {}_2F_1\left[\frac{-\tilde{s}^* + l + \mathfrak{m}_-}{2}, \frac{-\tilde{s}^* + l + \mathfrak{m}_+}{2}; \frac{3}{2} + l; \rho^2\right] \\ &= \rho^l Y_{lm}(\theta,\phi)\cdot (1-\rho^2)^{\tilde{s}^*}\, {}_2F_1\left[\frac{\tilde{s}^* + l + \mathfrak{m}_+}{2}, \frac{\tilde{s}^* + l + \mathfrak{m}_-}{2}; \frac{3}{2} + l; \rho^2\right] \end{aligned}$$

where we have used a standard identity of the hypergeometric function [112] and the fact that $\mathfrak{m}_+ + \mathfrak{m}_- = 3$. The hypergeometric function ${}_2F_1$ is holomorphic in the unit disc around 0, so the only place it can diverge is at $\rho = 1$. By Gauss' Hypergeometric

Theorem [15] we have for $\operatorname{Re} s < 0$,

$$ {}_2F_1\left[\frac{\tilde{s}^*+l+\mathfrak{m}_+}{2}, \frac{\tilde{s}^*+l+\mathfrak{m}_-}{2}; \frac{3}{2}+l; \rho^2\right] = \frac{\Gamma\left(\frac{3}{2}+l\right)\Gamma\left(-\tilde{s}\right)}{\Gamma\left(\frac{l-\tilde{s}+\mathfrak{m}_-}{2}\right)\Gamma\left(\frac{l-\tilde{s}+\mathfrak{m}_+}{2}\right)}. $$

The right hand side of the above has singularities for $\tilde{s} \in \mathbb{N}_0$, so by analytic continuation, the hypergeometric part of u is well-behaved for away from non-negative integers. Now the only issues with convergence arise from the $(1-\rho)^{\tilde{s}^*}$ part. Thus, for $\operatorname{Re} \tilde{s} > -1/2$ and away from the poles mentioned above, this defines a distribution on $L^2(B_1)$ by simply taking the L^2 inner product with the above function rescaled for convenience:

$$ T(u) = \frac{1}{\Gamma(\tilde{s}+1)} \int_{B_1} \rho^l Y_{lm}(\theta,\phi) \cdot (1-\rho^2)^{\tilde{s}} {}_2F_1\left[\frac{\tilde{s}+l+\mathfrak{m}_-}{2}, \frac{\tilde{s}+l+\mathfrak{m}_+}{2}; \frac{3}{2}+l; \rho^2\right] u dV, $$

where $dV = \rho^2 d\rho d\sigma$. We shall now work in Cartesian coordinates: first note that $\rho^l Y_l m(\theta,\phi) = S_{lm}(\mathbf{x})$ for some symmetric polynomial S_{lm} in the x_i and define

$$ F(\mathbf{x}) = {}_2F_1\left[\frac{\tilde{s}+l+\mathfrak{m}_-}{2}, \frac{\tilde{s}+l+\mathfrak{m}_+}{2}; \frac{3}{2}+l; |\mathbf{x}|^2\right]. $$

Note that

$$ \begin{aligned} \mathbf{x}\cdot\nabla\left(\frac{(1-|\mathbf{x}|^2)^{\tilde{s}+1}}{\Gamma(\tilde{s}+2)}\right) &= -2|\mathbf{x}|^2\frac{(1-|\mathbf{x}|^2)^{\tilde{s}}}{\Gamma(\tilde{s}+1)}, \\ &= \frac{2}{\Gamma(\tilde{s}+1)}\left((1-|\mathbf{x}|^2)^{\tilde{s}+1} - (1-|\mathbf{x}|^2)^{\tilde{s}}\right) \end{aligned} $$

so that

$$ \frac{(1-|\mathbf{x}|^2)^{\tilde{s}}}{\Gamma(\tilde{s}+1)} = \left(-\frac{1}{2}(\mathbf{x}\cdot\nabla) + \tilde{s} + 1\right)\frac{(1-|\mathbf{x}|^2)^{\tilde{s}+1}}{\Gamma(\tilde{s}+2)}. $$

We define the family of operators

$$ \hat{D}(s)u := \frac{1}{2}\nabla\cdot(\mathbf{x}u) + (\tilde{s}+1)u, \tag{3.2.1} $$

which, by the divergence theorem, gives

$$ T(u) = \frac{1}{\Gamma(\tilde{s}+2)}\int_{B_1}(1-|\mathbf{x}|^2)^{\tilde{s}+1}\hat{D}(\tilde{s})\left(S_{lm}Fu\right)dx. $$

Note that the above integral converges for $u \in H^1(B_1)$ and $\operatorname{Re}\tilde{s} > -3/2$. We repeat the above process k times so that for $u \in H^k(B_1)$,

$$ T(u) = \frac{1}{\Gamma(\tilde{s}+k+1)}\int_{B_1}(1-|\mathbf{x}|^2)^{\tilde{s}+k}\hat{D}(\tilde{s}+k-1)\ldots\hat{D}(\tilde{s}+1)\hat{D}(\tilde{s})\left(S_{lm}Fu\right)dx, $$

which is a distribution satisfying (i) in Definition 2.3.1 for $\operatorname{Re}\tilde{s} > -1/2 - k$. For shorthand, we shall write $\hat{D}_k(\tilde{s}) = \hat{D}(\tilde{s}+k-1)\ldots\hat{D}(\tilde{s}+1)\hat{D}(\tilde{s})$. At the quasinormal frequency $\tilde{s} = -k + 3 - \mathfrak{m}_\pm$, we see that $\operatorname{Re}\tilde{s} \geq -k$ since $\operatorname{Re}\mathfrak{m}_- \leq \operatorname{Re}\mathfrak{m}_+ \leq 3$. We integrate by parts k times to get

$$T(u) = \frac{1}{\Gamma(4-\mathfrak{m}_\pm)} \int_{B_1} (1-|\mathbf{x}|^2)^{3-\mathfrak{m}_\pm} \hat{D}_{k+3}(\tilde{s}) \ldots (S_{lm}Fu)\, dx. \tag{3.2.2}$$

The above distribution is well-defined provided u has sufficient regularity (this is built in to the definition of the frequency as we can only extend into the left that far if u is regular enough) and defines a co-mode associated with that frequency. Similar expressions can be obtained when we consider the mass to be independent of κ but with $\mathfrak{m}$ replaced with μ/κ as before.

Returning to the special cases of the wave and conformal Klein-Gordon equations where $\mathfrak{m}_\pm \in \mathbb{Z}$, the distribution Equation (3.2.2) is special: suppose now that we are considering the frequency at $\tilde{s} = -k$ and note that $2\hat{D}(\tilde{s}+k-1)u = \nabla \cdot (\mathbf{x}u)$, so

$$\begin{aligned} T(u) &= \frac{1}{2}\int_{B_1} \nabla \cdot \Big(x\hat{D}_{k-1}(-k)\,(S_{lm}Fu)\Big)\, dx \\ &= \frac{1}{2}\int_{S^2} \hat{D}_{k-1}(-k)\,(S_{lm}Fu)\,(\sigma) d\sigma \end{aligned}$$

where σ is a coordinate on the unit 2-sphere and, as before, $d\sigma$ is the usual measure on it. Note that in polar coordinates, the above distribution can be expressed as a sum over derivatives of a delta function. This concentration of the co-modes on the cosmological horizon will be crucial to proving the results in the next couple of chapters. Examining the argument above closely also reveals that the fact that the frequencies $\tilde{s}$ are negative integers is crucial to the fact that this concentration occurs: we can see from Equation (3.2.2) that the value $T(u)$ takes depends on u in the whole interval.

Chapter 4

Potentials in the de Sitter spacetime

At the end of the previous chapter, we discussed in some detail the case of a constant potential $V \geq 0$ (i.e. the Klein-Gordon equation on de Sitter). In this chapter, we shall focus in more detail on the effect of various classes of potential on the quasinormal spectrum of the wave operator in de Sitter. This time, we consider the initial value problem:

$$-\Box_g \psi + 2\kappa^2 \psi + V\psi = 0, \quad \psi|_{\tau=0} = \psi_0, \quad \partial_\tau \psi|_{\tau=0} = \psi_1, \tag{4.0.1}$$

using the hyperboloidal coordinates described in the previous chapter. We have chosen to include a conformal mass term here for convenience. Through the same reasoning as before, we see that the quasinormal frequencies depend on the invertibility of the analytic family of operators $L(\tilde{s}) + \hat{V} = P(\tilde{s}) + 2 + \hat{V}$ where, as before, $\tilde{s} = s/\kappa$ and $\hat{V}(x) = V(x/\kappa)/\kappa^2$.

We begin by considering a few special cases: the inverse square potential (motivated by a model problem for the Kerr-de Sitter problem considered in Chapter 6) and compactly supported potentials where co-modes make the problem trivial. We then prove Theorem 4.3.1, where continuity of the quasinormal spectrum up to some finite order is established. We can extend this result further for spherically symmetric potentials, in which case we can project into each angular momentum sector and obtain a series expansion for the new quasinormal frequency (see Theorem 4.4.1). Finally we establish the main result of this chapter, Theorem 4.5.1, which establishes the existence of zero-damped modes for the conformal Klein-Gordon equation in de Sitter with a sufficiently rapidly decaying smooth potential in the limit the cosmological constant is taken to zero.

4.1 The inverse square potential

First we shall discuss in detail an inverse square potential, where the equation is similar in structure to those for non-axisymmetric perturbations in the Kerr-de Sitter problem described in Section 6.2 and can be used as a toy model for this more complicated example. It stands out from the rest of the potentials we consider in this chapter due its singular nature at the origin, however one can find the quasinormal modes and frequencies explicitly in this case. The equation Equation (4.0.1) becomes

$$-\Box_g\psi + 2\kappa^2 + \frac{V_0}{r^2}\psi = 0,$$

where $V_0 > 0$. We can obtain similar results to Proposition 3.1.1 and Proposition 3.1.2 for this equation, however the singular behaviour of the potential at the origin results in modifications to our approach. We define a new energy

$$E_1(u) := \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{B_1} a_{ij}\partial_i\overline{u}\partial_j u dx + \gamma\int_{B_1}|u|^2 dx + \int_{B_1}\frac{V_0|u|^2}{\rho^2}dx,$$

which is well-defined for $u \in H^1(B_1)$ by virtue of Hardy's inequality. We can use this and the redshift effect at the horizon to obtain a result analogous to Proposition 3.1.2, however we lose the ability to differentiate the equation at the origin due to the singularity in the potential. The best we can expect is a solution in $H^1(B_1)$. Since the origin is the only obstacle to obtaining higher regularity (and we are only really interested in behaviour near the horizon, see [60]), we can remedy this by introducing a cut-off χ and splitting the problem into the two regions. We now consider a coupled system of partial differential equations with solutions in two different spaces ($H^1(\operatorname{supp}\chi)$ for the equation near the origin and $H^k(\operatorname{supp}(1-\chi))$ for the equation near the horizon). Since we are only really interested in the behaviour near the horizon (see [60]), we can simply work in $H^1(B_1) \cap H^k(B_1 \setminus B_{2/3})$:

Proposition 4.1.1. *Let $f \in H^{k-1}(B_1)$ and $\mathrm{Re}(\tilde{s}) > 1/2 - k$. If $(L(\tilde{s}) + V_0/\rho^2)u = f$, we have either:*

(i) there exists a unique solution to $(L(\tilde{s}) + V_0/\rho^2)u = f$ where $u \in H^1(B_1) \cap H^k(B_1 \setminus B_{2/3})$

(ii) there exists $v \in H^1(B_1) \cap C^\infty(\overline{B_1 \setminus B_{2/3}})$ and which obeys $(L(\tilde{s}) + V_0/\rho^2)v = 0$. Moreover this can only occur at isolated values of s.

We can use this to establish estimates to get a result similar to Proposition 3.1.1 and define quasinormal modes and frequencies. To actually compute these, we may again use spherical symmetry to decompose into spherical harmonics and reduce the problem to an ordinary differential equation. The resulting equation is a second order Fuchsian equation with the same ordinary points:

$$(1-\rho^2)\partial_\rho^2 u_{lm} + \left(\frac{2}{\rho} - 4\rho - 2\tilde{s}\rho\right)\partial_\rho u_{lm} - \left(\frac{l(l+1)+V_0}{\rho^2} + \tilde{s}^2 + 3\tilde{s} + 2\right)u_{lm} = 0.$$

We can use precisely the same transformations as before, however the potential has changed the exponents at $\rho = 0$ to

$$\sigma_l^\pm = -\frac{1}{2} \pm \sqrt{\left(l+\frac{1}{2}\right)^2 + V_0}.$$

The solution with exponent σ_l^- at the origin gives a singular solution and thus through exactly the same methods as before, we see that solutions are:

$$u_{lm}(\rho) = \rho^{\sigma_l^+}\,{}_2F_1\left[\frac{\tilde{s}+\sigma_l^+ + 1}{2}, \frac{\tilde{s}+\sigma_l^+ + 2}{2}; \frac{3}{2} + \sigma_l^+; \rho^2\right].$$

Since we require solutions which are smooth at the horizon, we see that quasinormal frequencies are of the form $\tilde{s} = -\sigma_l^+ - k$ for $k \in \mathbb{N}$ and the corresponding modes are polynomials multiplied by $\rho^{\sigma_l^+}$. We observe that we have zero-damped modes from the relation $s = \kappa\tilde{s}$.

We can find the co-modes using precisely the arguments we used for Klein-Gordon fields, since $(L(s)+\hat{V})^\dagger = L(-s^*) + \hat{V}$ and the quasinormal modes are constructed from similar hypergeometric functions. Let n_l be the smallest integer such that $n_l - \sigma_l^+ > -1/2$. Then the co-modes corresponding to frequency $\tilde{s} = -\sigma_l^+ - k$ take the form:

$$T(u) = \frac{1}{\Gamma(n_l - \sigma_l^+ + 1)} \int_{B_1} (1-|\mathbf{x}|^2)^{n_l - \sigma_l^+}\,\hat{D}_{n_l+k}(\tilde{s})\ldots(S_{lm}Fu)\,dx, \tag{4.1.1}$$

where S_{lm} and $D_k(\tilde{s})$ are defined exactly as they were in Section 3.2, however we now instead have

$$F(\mathbf{x}) = |x|^{\sigma_l^+ - l}\,{}_2F_1\left[\frac{1-k}{2}, \frac{2-k}{2}; \frac{3}{2} + \sigma_l^+; |\mathbf{x}|^2\right],$$

i.e. F is a polynomial in $|\mathbf{x}|^2$ multiplied by a positive, non-integer power of $|\mathbf{x}|$. It should be noted that for l sufficiently large,

$$\begin{aligned} n_l - \sigma_l^+ &= n_l + \frac{1}{2} - \left(l+\frac{1}{2}\right)\left(1 + \frac{V_0}{(l+\frac{1}{2})^2}\right)^{1/2} \\ &= n_l - l - \frac{V_0}{2l+1} + O\left(\frac{1}{l^2}\right). \end{aligned}$$

In particular, if l is large enough, we have $\alpha = l - \sigma_l^+ > -1/2$ (in fact $\alpha = l - \sigma_l^+ =$

$O(1/l)$) so the co-mode takes the form:

$$T_\alpha(u) = \frac{1}{\Gamma(\alpha+1)} \int_{B_1} (1-|\mathbf{x}|^2)^\alpha \hat{D}_{l+k}(\tilde{s}) \dots (S_{lm} F u) \, dx,$$

recalling that $\hat{D}_k(\tilde{s}) = \hat{D}(\tilde{s}+k-1)\dots\hat{D}(\tilde{s}+1)\hat{D}(\tilde{s})$ where

$$\hat{D}(s)u := \frac{1}{2}\nabla \cdot (\mathbf{x}u) + (\tilde{s}+1)u.$$

Thus when $\alpha = 0$, this becomes an integral over the boundary and this distribution concentrates on the horizon. So consider the following:

$$\begin{aligned} |T_\alpha(u) - T_0(u)| &= \left| \int_{B_1} \left(\frac{(1-|\mathbf{x}|^2)^\alpha}{\Gamma(\alpha+1)} - 1 \right) \hat{D}_{l+k}(\tilde{s}) \dots (S_{lm} F u) \, dx \right|, \\ &< C \left[\int_{B_1} \left(\frac{(1-|\mathbf{x}|^2)^\alpha}{\Gamma(\alpha+1)} - 1 \right)^2 dx \right]^{1/2} \|u\|_{H^{l+k}} \\ &< C \left[\frac{1}{3} - \frac{\sqrt{\pi}}{2} \frac{1}{\Gamma(\alpha+5/2)} + \frac{\sqrt{\pi}}{4} \frac{\Gamma(2\alpha+1)}{\Gamma(\alpha+1)^2 \Gamma(2\alpha+5/2)} \right] \|u\|_{H^{l+k}} \end{aligned}$$

The expression in the square brackets above is real analytic in α for $\alpha > -1/2$ and takes the value 0 when $\alpha = 0$, hence there exists a constant C such that

$$|T_\alpha(u) - T_0(u)| < C|\alpha| \|u\|_{H^{l+k}} = O\left(\frac{1}{l}\right).$$

So for l sufficiently large, the co-modes are increasingly well-approximated by ones which concentrate on the horizons.

4.2 Compactly supported potentials

The fact that the co-modes are concentrated on the horizon for suitable values of $\hat{V}$ (namely 0 and -2) implies that the behaviour of the potential near the horizon is all that matters when considering these zero-damped mode frequencies.

Proposition 4.2.1. *Let $\hat{V} \in C^k(\overline{B_1})$ such that*

$$\partial^\alpha \hat{V}|_{\partial B_1} = 0$$

for each multi-index α such that $|\alpha| \le k$. Take $n < k$. Then $\tilde{s} = -1, -2, \dots, -(n-1)$ are quasinormal frequencies of $L(\tilde{s}) + \hat{V} : D^{n+1} \to H^n(B_1)$.

Proof. It suffices to apply the co-modes from the previous chapter to $(L(\tilde{s}) + \hat{V})u$ and note that $\theta((L(\tilde{s}) + \hat{V})u) = \theta(L(\tilde{s})u)$ when the trace of these derivatives of $\hat{V}$ is zero, so a quasinormal frequency of $L(\tilde{s})$ is a frequency of $L(\tilde{s}) + \hat{V}$. $\square$

We see in particular that if $\hat{V} \in C_0^\infty(B_1)$ that $-\kappa\mathbb{N}$ is contained in the quasinormal spectrum of $L(\tilde{s}) + \hat{V}$. It is important to note at this point that this may not be the whole spectrum - additional frequencies could arise in $\operatorname{Re}\tilde{s} < -1/2$ due to the potential. Since $s = \kappa\tilde{s}$, we have established the existence of zero-damped modes in this case. In fact the above argument also applies for $P(\tilde{s}) + \hat{V}$, in which case the quasinormal spectrum would contain $-\kappa\mathbb{N}_0$.

4.3 More general potentials

The first important thing to note is that the results of Gohberg-Sigal theory outlined in Chapter 2 require a family of operators which maps some Hilbert space to itself, so we need to modify the operators we are considering to apply them to our problem. To reduce clutter, we set $\kappa = 1$ for the remainder of this section and define

$$\begin{aligned} A(s) &= L(s)L_0^{-1} = I_{H^k} + 2sPL_0^{-1} + s^2 L_0^{-1}, \\ B(s) &= A(s) + \epsilon\hat{V}L_0^{-1}, \end{aligned}$$

which are holomorphic families of Fredholm operators $H^k(B_1) \to H^k(B_1)$ for $\hat{V} \in C^k(\overline{B_1})$, since $L_0 = L(0) : D^{k+1} \to H^k(B_1)$ is invertible. We first note that by decomposing into spherical harmonics, we can find the eigenvalues and eigenvectors of $L(s)$ in D^{k+1}: this is equivalent to solving for quasinormal modes of the Klein-Gordon equation for a different mass as we did in the previous chapter. We see that the eigenvectors (i.e. solutions of $L(s)u = \lambda u$) are

$$u_{n,l,m}(\mathbf{x}, s) = \rho^l Y_{lm}(\theta, \phi)\, {}_1F_2\left[-n, s + l + \frac{3}{2} + n; \frac{3}{2} + l; \rho^2\right]$$

for $n \in \mathbb{N}_0$ with corresponding eigenvalues

$$\lambda_{n,l,m}(s) = (s + l + 1 + 2n)(s + l + 2 + 2n).$$

This means that for the factorisation of $A(s)$ given by Theorem 2.2.3, all the positive k_m are 1 and none are negative, since $A(s)$ is holomorphic in the region of interest. Hence we have

$$\frac{1}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,1/2}} 2(P_1 + z)L(z)^{-1}dz = n^2$$

where $\Gamma_{-n,1/2}$ is the circle of radius $1/2$ around $-n$ in $\mathbb{C}$. Noting that we can write

$$L(s)^{-1} = \frac{A_{-1}}{s + n} + A_0(s),$$

we see that

$$A(s)^{-1}(A(s)-B(s)) = \frac{\epsilon L_0 A_{-1}\hat{V}L_0^{-1}}{s+n} + \epsilon L_0 A_0(s)\hat{V}L_0^{-1}.$$

We define the constants

$$C_{k,n} = \sup_{s\in\overline{D(-n,1/2)}} \left\{ \|L_0 A_0(s)\hat{V}L_0^{-1}\|_{H^k\to H^k} \right\},$$

where $\overline{D(-n,1/2)}$ is the closed disc of radius $1/2$ around $-n$. We can use Theorem 2.2.5 to obtain the following result:

Proposition 4.3.1. *Suppose $\hat{V}\in C^k(\overline{B_1})$, pick $0<\delta<1/2$ and $n\in\mathbb{N}$ such that $n<k$. Let*

$$\tilde{C}_{k,n} = \min\left\{ \frac{1}{2\|L_0 A_{-1}\hat{V}L_0^{-1}\|_{H^k\to H^k}}, \frac{1}{C_{k,n}} \right\}.$$

Then for $0<\epsilon<\tilde{C}_{k,n}\delta$, we have

$$\frac{1}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2(P_1+z)(L(z)+\epsilon\hat{V})^{-1}dz = \frac{1}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2(P_1+z)L(z)^{-1}dz.$$

Proof. We see that for $\epsilon<\tilde{C}_{k,n}\delta$, we have on a circle of radius δ around $-n$,

$$\begin{aligned}\|A(s)^{-1}(A(s)-B(s))\|_{H^k\to H^k} &\le \frac{\epsilon}{\delta}\|L_0 A_{-1}\hat{V}L_0^{-1}\|_{H^k\to H^k} + \epsilon C_{k,n}\\ &< \tilde{C}_{k,n}\|L_0 A_{-1}\hat{V}L_0^{-1}\|_{H^k\to H^k} + \tilde{C}_{k,n}C_{k,n}\delta\\ &\le \frac{1}{2}+\delta<1.\end{aligned}$$

Thus the conditions in Theorem 2.2.5 are met and we have

$$\frac{1}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2(P_1+z)(L(z)+\epsilon\hat{V})^{-1}dz = \frac{1}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2(P_1+z)L(z)^{-1}dz = n^2.$$

□

The upshot of this result is that we can control the distance, δ, which a quasinormal frequency arising from a zero-damped mode can move from its unperturbed value provided the perturbation is of size $O(\delta)$.

Theorem 4.3.1. *Let $\hat{V}\in C^\infty(\overline{B_1};\mathbb{R})$ and fix $n\in\mathbb{N}$ and $0<\delta<1/2$. We assume further that $\|\hat{V}\|_{C^k}=1$. Then there exists $M_k>0$ such that for $\epsilon<M_k\delta$, there exist n^2 quasinormal frequencies (counted with multiplicity) of $L(s)+\epsilon\hat{V}$ inside $D(-n,\delta)$ for $n\in\{1,2,\ldots,k\}$.*

Proof. This is simply an application of Proposition 4.3.1 above. □

4.3.1 Relation to pseudospectral instability

The result we have just proved shows a sort of spectral stability up to some given finite number of frequencies, which may at first glance seem to contradict the pseudospectral work identifying instabilities associated to non-self-adjoint operators [90, 46, 65]. However, observe that to obtain the stability of these frequencies, we need to measure the size of the perturbation $\epsilon\hat{V}$ in the C^k norm. The pseudospectral methods typically use an energy norm involving at most one derivative to measure the size of perturbations and thus naturally higher overtone regularity quasinormal modes may appear to be unstable since they are only well-defined with high regularity norms: two potentials can have very similar C^1 norms, yet vastly different C^k norms for k sufficiently large (e.g. $2\sin x$ and $\sin x + \sin \alpha x/\alpha$ for α a large constant).

Furthermore note that $1/M_k$ can be thought of as the maximum of a set of operator norms on H^k and thus is expected to grow rapidly with k. As a result, we will likely require ϵ to get very small as k increases for this notion of stability of frequencies. Outside this range of ϵ, we can make no statements about the spectrum.

4.4 Computing the perturbed spectrum

The above result tells us the maximum size of a potential given how much control we want on the frequencies. It does not give an explicit expression for what the perturbed frequencies are and how they depend on the potential we add to the equation. In the case of a spherically symmetric potential, we can achieve this in principle: we can obtain a series expansion for the perturbed frequencies.

The $SO(3)$ symmetry of the system means that at frequency $-n$, there are n^2 modes corresponding to different spherical harmonics. Each of these will be affected by $\hat{V}$ in a different way in general and thus it is difficult to get precise information on how they are perturbed in this situation. When $\hat{V}$ is spherically symmetric, multiplication by $\hat{V}$ commutes with projections onto any angular momentum sector, so we can separate the equation and lift this degeneracy. If $\hat{V}$ is a non-trivial superposition of angular modes, this doesn't work since there is mixing between angular momentum sectors and the equations are coupled. At this stage, the poles are still simple and the residues are now rank one operators, which enables us to use a generalisation of Theorem 2.2.4 (this is Theorem 4.1 in [68]) to get an expression for perturbed frequencies.

First we define the projections. For $u \in C^\infty(\overline{B_1})$, let us define

$$(\Pi_{lm}u)(r,\theta,\phi) = Y_{lm}(\theta,\phi)\int_{S^2} u(r,\theta',\phi')Y_{lm}(\theta',\phi')\sin^2\theta' d\theta' d\phi'.$$

One can show that this is in fact bounded on $C^\infty(\overline{B_1})$ with respect to the $H^k(B_1)$ norm for each $k \geq 1$ and so extends to a continuous linear map $H^k(B_1) \to H^k(B_1)$.

We assume that $\hat{V} \in C^k(\overline{B_1})$ is spherically symmetric and apply the above methods on $\Pi_{lm}(L(s) + \hat{V})$ where Π_{lm} is the projection onto the angular momentum sector associated with the spherical harmonic $Y_{lm}(\theta, \phi)$. We shall omit explicitly writing the projection Π_{lm} when considering operators like $(\Pi_{lm} L(s))^{-1}$ in the following calculations to reduce clutter. Then for $n < k$, consider the functional:

$$s_n[\hat{V}] := \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} z\partial_z(L(z) + \hat{V}) \left(L(z) + \hat{V}\right)^{-1} dz \tag{4.4.1}$$

$$= \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z) \left(L(z) + \hat{V}\right)^{-1} dz, \tag{4.4.2}$$

where $\delta < 1$. We note that

$$\begin{aligned} s_n[0] &= \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} z\partial_z(L(z))L(z)^{-1} dz \\ &= \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} (z+n)\partial_z(L(z))L(z)^{-1} dz - \frac{n}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} \partial_z(L(z))L(z)^{-1} dz \end{aligned}$$

and since $L(z)^{-1}$ has a simple pole at $z = -n$, the integrand in the first term is holomorphic and hence the integral evaluates to 0. Since the residue is a one-dimensional projection, it follows that $s_n[0] = -n$. Now, take $0 < \epsilon < 1$ sufficiently small so that

$$\frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2(P_1 + z)L(z)^{-1} dz = \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2(P_1 + z)(L(z) + \epsilon\hat{V})^{-1} dz.$$

This ensures that we are enclosing just one pole of $(L(s) + \epsilon\hat{V})^{-1}$ and that it is also simple. We observe that

$$\begin{aligned} s_n[\epsilon\hat{V}] + n &= \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z) \left(\left(L(z) + \epsilon\hat{V}\right)^{-1} - L(z)^{-1} \right) dz \\ &= \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1} \left(\left(I + \epsilon\hat{V}L(z)^{-1}\right)^{-1} - I \right) dz. \end{aligned}$$

Since $L(z)$ is invertible for $z \in \Gamma_{-n,\delta}$, $\hat{V}L(z)^{-1}$ is bounded in the operator norm topology along the contour and hence its operator norm will take a maximum value on $\Gamma_{-n,\delta}$ by continuity. So if we take ϵ sufficiently small, we have $\|\epsilon\hat{V}L(z)^{-1}\| < 1$ along the contour and we may use a von Neumann series to expand it:

$$s_n[\epsilon\hat{V}] + n = \frac{1}{2\pi i} \operatorname{tr} \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1} \sum_{m=1}^{\infty} (-\epsilon)^m (\hat{V}L(z)^{-1})^m dz.$$

By uniform convergence in the operator norm topology of the sum, we may exchange the sum and the integral and consider for each m the following:

$$\frac{1}{2\pi i}\oint_{\Gamma_{-n,\delta}} 2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz.$$

We write

$$L(z)^{-1} = \frac{A_{-1}}{z+n} + A_0(z)$$

and consider its poles to see that the radius of convergence of the Taylor series of $A_0(z)$ about $-n$ is 1, so we can write in this disc

$$A_0(z) = A_0(-n) + \sum_{m=1}^{\infty} \frac{A_0^{(m)}(-n)}{m!}(z+n)^m,$$

where $A_0^{(m)}(z)$ is the mth derivative of A_0. Let us write $A_0 = A_0(-n)$ and, for $m \in \mathbb{N}$,

$$A_m := \frac{A_0^{(m)}(-n)}{m!}.$$

This gives us the expansion

$$L(z)^{-1}(\hat{V}L(z)^{-1})^m = \sum_{i_0=-1}^{\infty} \cdots \sum_{i_m=-1}^{\infty} A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})(z+n)^{i_0+\dots+i_m},$$

and hence (noting that $2z(P_1+z) = 2(z+n)^2 + 2(P_1-2n)(z+n) - 2n(P_1-n)$), we see that

$$\begin{aligned}
&2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m \\
&\quad= \sum_{i_0=-1}^{\infty} \cdots \sum_{i_m=-1}^{\infty} 2A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})(z+n)^{i_0+\dots+i_m+2} \\
&\qquad + 2(P_1-2n) \sum_{i_0=-1}^{\infty} \cdots \sum_{i_m=-1}^{\infty} A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})(z+n)^{i_0+\dots+i_m+1} \\
&\qquad - 2n(P_1-n) \sum_{i_0=-1}^{\infty} \cdots \sum_{i_m=-1}^{\infty} A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})(z+n)^{i_0+\dots+i_m}.
\end{aligned}$$

Many of these terms will not contribute to the integral by the residue theorem: the only ones that will survive are those which correspond to $(z+n)^{-1}$ i.e. those for which

$i_0 + \dots i_m = -2$ in the first sum and $i_0 + \dots i_m = -1$ in the second. Thus, we have

$$\begin{aligned}
&\frac{1}{2\pi i} \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz \\
&\qquad\qquad = \sum_{i_0+\dots+i_m=-3} 2A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m}) \\
&\qquad\qquad\quad + 2(P - 2n) \sum_{i_0+\dots+i_m=-2} A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m}) \\
&\qquad\qquad\quad - 2n(P - n) \sum_{i_0+\dots+i_m=-1} A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m}).
\end{aligned}$$

Since $A_0(s)$ is holomorphic at $-n$ with radius of convergence 1, $\|A_m\|_{H^k \to H^k} < 1$. Let $B_k = \max\{1, \|A_{-1}\|_{H^k \to H^k}\}$.

Lemma 4.4.1. *For $\epsilon < 1/(16B_k\|\hat{V}\|_{C^k})$,*

$$\sum_{m=1}^{\infty} (-\epsilon)^m \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz$$

converges in the trace norm topology.

Proof. We note that since, $i_0 + \dots + i_m < 0$, each of the $A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})$ is a rank 1 operator and hence the integral gives a finite rank operator with rank bounded by the number of $(i_0, i_1, \dots, i_m)$ which obey the conditions given. By setting $j_p = i_p + 1$ for $p = 1, 2, \dots m$, we see that each of the conditions are equivalent to one of the following:

$$\begin{aligned}
j_0 + j_1 + \dots + j_m &= m, \\
j_0 + j_1 + \dots + j_m &= m - 1, \\
j_0 + j_1 + \dots + j_m &= m - 2,
\end{aligned}$$

for j_p non-negative integers. By standard arguments, the rank of the operator is at most

$$\binom{2m}{m} + \binom{2m-1}{m} + \binom{2m-2}{m} < 3 \cdot 2^{2m}.$$

We can bound the trace norm $\|\cdot\|_1$ of a finite rank operator on a Hilbert space by its operator norm and its rank, so we have:

$$\begin{aligned}
&\frac{1}{2\pi} \left\| \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz \right\|_1 \\
&\quad < \frac{3 \cdot 4^m}{2\pi} \left\| \oint_{\Gamma_{-n,\delta}} 2z(P_1 + z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz \right\|_{H^k \to H^k}.
\end{aligned}$$

We also have the following bound:

$$\|A_{i_0}(\hat{V}A_{i_1})\dots(\hat{V}A_{i_m})\|_{H^k\to H^k} \leq B_k^{m+1}\|\hat{V}\|_{C^k}^m$$

and hence

$$\left\|\frac{1}{2\pi i}\oint_{\Gamma_{-n,\delta}} 2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz\right\|_1 < 18M16^m B_k^{m+1}\|\hat{V}\|_{C^k}^m,$$

where M depends on n and the operator norm of P. This gives the result. $\square$

Theorem 4.4.1. *Let $\hat{V} \in C^\infty(\overline{B_1};\mathbb{R})$ be a spherically symmetric potential. Then for ϵ sufficiently small, there exist quasinormal frequencies s_n of $L(s)+\epsilon\hat{V}$ and $S_m \in \mathbb{C}$ such that*

$$s_n = -n + \sum_{m=1}^{\infty} S_m \epsilon^m.$$

Furthermore, we can compute the S_m explicitly:

$$S_m = \frac{(-1)^m}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz.$$

Proof. We begin by projecting into any given angular momentum sector (l,m) as above and performing the same steps to obtain a von Neumann series within a trace. Since the von Neumann series given above converges absolutely in the trace norm topology for small enough ϵ by Lemma 4.4.1, we can use continuity of trace to exchange the summation and trace operations to get

$$\begin{aligned} s_n[\epsilon\hat{V}] + n &= \frac{1}{2\pi i}\sum_{m=1}^{\infty}(-1)^m\epsilon^m \operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz \\ &= \sum_{m=1}^{\infty} S_m\epsilon^m, \end{aligned}$$

where

$$S_m = \frac{(-1)^m}{2\pi i}\operatorname{tr}\oint_{\Gamma_{-n,\delta}} 2z(P_1+z)L(z)^{-1}(\hat{V}L(z)^{-1})^m dz.$$

$\square$

The first order change to the frequency, δs_n, can be expressed as:

$$\delta s_n = -2\operatorname{tr}\left[(P_1-2n)A_{-1}\hat{V}A_{-1} - n(P_1-n)(A_0\hat{V}A_{-1} + A_{-1}\hat{V}A_0)\right]. \tag{4.4.3}$$

We can write $A_{-1} = u_n \otimes \theta_n$ where u_n is the quasinormal mode corresponding to this angular momentum sector and frequency normalized so that $\|u_n\|_{H^k} = 1$. Note that due to Theorem 2.2.4, the operator $2(P_1 - n)A_{-1}$ is a projection, so the normalization of θ_n is fixed i.e. $2\theta_n(P_1 u_n) - 2n\theta_n(u_n) = 1$. This gives us

$$\begin{aligned}\delta s_n =& -2\theta_n(\hat{V}u_n)\theta_n((P-2n)u_n) + 2n\theta_n(P_1 A_0 \hat{V}u_n + \hat{V}A_0 P_1 u_n)\\ &- 2n^2\theta_n(A_0\hat{V}u_n + \hat{V}A_0 u_n).\end{aligned}$$

In general, this is quite difficult to compute explicitly since it involves finding the holomorphic part of the inverse operator in a neighbourhood of the pole. To check the result, we shall tackle the long yet tractable calculations for δs_1 in the case of a constant potential $\hat{V}$ and compare it to what we already found for the Klein-Gordon equation this corresponds to in Chapter 3. This frequency corresponds to exactly one mode: the $l = 0$ constant solution. This means $(P_1 - 3/2)u_1 = 0$ which, combined with the condition on the normalisation of θ_1, gives $\theta_1(u_1) = 1$. Thus Equation (4.4.3) becomes:

$$\begin{aligned}\delta s_1 &= [-2\theta_1(u_1)\theta_1((P_1-2)u_1) + 2\theta_1(P_1 A_0 u_1 + A_0 P_1 u_1) - 4\theta_1(A_0 u_1)]\,\hat{V}\\ &= [1 + 2\theta_1(P_1 A_0 u_1) - \theta_1(A_0 u_1)]\,\hat{V} \qquad (4.4.4)\end{aligned}$$

To finish this calculation, we must find the holomorphic part of the operator. This can be done by noting that we are looking at one of the modes in the $l = 0$ angular momentum sector and using spherical polar coordinates. This reduces our problem to a one dimensional radial equation and we can construct the inverse operator in this sector through Green's functions. Two linearly independent solutions to this second order ordinary differential equation are

$$\begin{aligned}g_1(\rho, s) &= {}_2F_1\left[\frac{s+1}{2}, \frac{s+2}{2}; \frac{3}{2}; \rho^2\right]\\ g_2(\rho.s) &= {}_2F_1\left[\frac{s+1}{2}, \frac{s+2}{2}; s+1; 1-\rho^2\right],\end{aligned}$$

where we have selected g_1 to be regular at $\rho = 0$ and g_2 to be regular at $\rho = 1$ for all s. We can compute the Wronskian $W(\rho) := g_1(\rho)g_2'(\rho) - g_1'(\rho)g_2(\rho)$ using the differential equation relatively straightforwardly:

$$\begin{aligned}W(\rho) &\propto \exp\left(-\int \frac{2-4\rho^2-2s\rho^2}{\rho(1-\rho^2)}d\rho\right)\\ &\propto \frac{(1-\rho^2)^{-s-1}}{\rho^2}.\end{aligned}$$

We can fix the constant of proportionality by considering $\lim_{\rho\to 0}\rho^2 W(\rho)$. Since g_1 and its derivatives are regular at $\rho = 1$ (in fact $g_1(0) = 1$), the constant of proportionality is in fact $\lim_{\rho\to 0}\rho^2 g_2'(\rho)$. A property of the hypergeometric function when $\mathrm{Re}(c-a-b) < 0$

(see 15.4 in [112]) is that

$$\lim_{z\to 1_-} \frac{{}_2F_1(a,b;c;z)}{(1-z)^{c-a-b}} = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)},$$

so it follows that

$$\begin{aligned}\lim_{\rho\to 0} \rho^2 g_2'(\rho) &= \frac{\Gamma(s+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s+2}{2}\right)} \\ &= \frac{\sqrt{\pi}\Gamma(s+1)}{2^{1-s-1}\sqrt{\pi}\Gamma(s+1)} \\ &= 2^s\end{aligned}$$

So the Green's function for this equation is:

$$G(\rho,\xi) := \begin{cases} 2^{-s}\xi^2(1-\xi^2)^s g_1(\rho)g_2(\xi) & 0\le\rho<\xi\le 1 \\ 2^{-s}\xi^2(1-\xi^2)^s g_1(\xi)g_2(\rho) & 0\le\xi<\rho\le 1 \end{cases} \tag{4.4.5}$$

From more properties of the hypergeometric function, we have

$$\begin{aligned} g_1(\rho,s) = (1-\rho^2)^{-s} &\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(s)}{\Gamma\left(\frac{1+s}{2}\right)\Gamma\left(\frac{2+s}{2}\right)} {}_2F_1\left[\frac{1-s}{2},\frac{2-s}{2};1-s;1-\rho^2\right] \\ &+\frac{\Gamma\left(\frac{3}{2}\right)\Gamma(-s)}{\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(\frac{2-s}{2}\right)} {}_2F_1\left[\frac{1+s}{2},\frac{2+s}{2};1+s;1-\rho^2\right] \\ = (1-\rho^2)^{-s} &\frac{2^s}{2s} g_2(\rho,-s) - \frac{2^{-s}}{2s} g_2(\rho,s), \end{aligned}$$

so the inverse operator on the $l=0$ angular momentum subspace can be written

$$\begin{aligned} \left(L(s)^{-1}u_1\right)(\rho) = &\frac{1}{2s} g_2(\rho,s)\int_0^\rho g_2(\xi,-s)u_1\xi^2 d\xi - \frac{4^{-s}}{2s} g_2(\rho,s)\int_0^1 (1-\xi^2)^s g_2(\xi,s)u_1\xi^2 d\xi \\ &+ \frac{(1-\rho^2)^{-s}}{2s} g_2(\rho,-s)\int_\rho^1 (1-\xi^2)^s g_2(\xi,s)u_1\xi^2 d\xi. \end{aligned} \tag{4.4.6}$$

The above expression is well-defined provided $\operatorname{Re} s > -1/2$ and to extend it to $\operatorname{Re} s > -3/2$, we must integrate by parts. We extend into the left of the complex plane in exchange for a derivative of u_1, as we would expect in light of Proposition 3.1.2. To perform this, we first repeat the observation made at the end of Chapter 3:

$$(1-\xi^2)^s = \left(-\frac{\xi}{2(s+1)}\partial_\xi + 1\right)(1-\xi^2)^{s+1},$$

which implies that

$$\int_a^b (1-\xi^2)^s f(\xi)d\xi = \int_a^b (1-\xi^2)^{s+1}\left(f(\xi)\right)d\xi$$
$$+\frac{1}{2(s+1)}\left(\int_a^b (1-\xi^2)^{s+1}\partial_\xi\left(\xi f(\xi)\right)d\xi - \left[\xi f(\xi)(1-\xi^2)^{s+1}\right]_a^b\right).$$

Applying this to Equation (4.4.6) yields

$$\begin{aligned}
L(s)^{-1}u_1 =& \frac{1}{2s}g_2(\rho,s)\int_0^\rho g_2(\xi,-s)u_1\xi^2 d\xi - \frac{4^{-s}}{2s}g_2(\rho,s)\int_0^1 (1-\xi^2)^{s+1}g_2(\xi,s)u_1\xi^2 d\xi \\
&+\frac{(1-\rho^2)^{-s}}{2s}g_2(\rho,-s)\int_\rho^1 (1-\xi^2)^{s+1}g_2(\xi,s)u_1\xi^2 d\xi \\
&-\frac{4^{-s}}{4s(s+1)}g_2(\rho,s)\int_0^1 (1-\xi^2)^{s+1}\partial_\xi\left(u_1\xi^3 g_2(\xi,s)\right)d\xi \\
&+\frac{(1-\rho^2)^{-s}}{4s(s+1)}g_2(\rho,-s)\int_\rho^1 (1-\xi^2)^{s+1}\partial_\xi\left(u_1\xi^3 g_2(\xi,s)\right)d\xi \\
&+\frac{(1-\rho^2)}{4s(s+1)}\rho^3 u_1 g_s(\rho,-s)g_2(\rho,s).
\end{aligned} \tag{4.4.7}$$

From the expression for $g_2(\rho,s)$, one may be concerned about the behaviour of the hypergeometric function near $s=-1$, however we see that in fact g_2 is holomorphic in s near $s=-1$ by considering its Taylor series:

$$\begin{aligned}
g_2(\rho,s) &= 1+\sum_{k=1}^\infty \frac{\left(\frac{s+1}{2}\right)_k\left(\frac{s+2}{2}\right)_k}{(s+1)_k k!}(1-\rho^2)^k \\
&= 1+\frac{s+2}{4}(1-\rho^2)+\sum_{k=2}^\infty \frac{(s+k+1)\dots(s+2k)}{4^k k!}(1-\rho^2)^k.
\end{aligned}$$

We know from the normalisation of u_1 and θ_1, we have

$$\theta_1(L(s)^{-1}u_1) = \frac{1}{s+1}+\theta_1(A_0(s)u_1),$$

so by continuity of θ_1, that

$$\theta_1(A_0 u_1) = \lim_{s\to -1}\left(\theta_1(L(s)^{-1}u_1) - \frac{1}{s+1}\right).$$

Furthermore, we have

$$\theta_1(P_1 A_0 u_1) = \lim_{s\to -1}\left(\theta_1(P_1 L(s)^{-1}u_1) - \frac{3}{2(s+1)}\right).$$

Using Equation (3.2.2), we see that θ_1 is (up to a constant) simply evaluating the $l=0$ component of the function at $\rho=1$. Since Equation (4.4.7) is quite an unwieldy expression, it will be easier to use Equation (4.4.6) to find $\theta_1(L(s)^{-1}u_1)$ for $\operatorname{Re} s > -1/2$

and use uniqueness of analytic continuation to find the appropriate limit. Noting that $\theta_1(u_1) = 1$ and $g_2(1, s) = 1$ for all s, we have for $\operatorname{Re} s > -1/2$

$$\theta_1(L(s)^{-1}u_1) = \frac{1}{2s}\int_0^1 g_2(\xi, -s)\xi^2 d\xi - \frac{4^{-s}}{2s}\int_0^1 (1-\xi^2)^s g_2(\xi, s)\xi^2 d\xi$$

We can evaluate the above using the uniform convergence of Taylor series of $g_2(\rho, s)$ in $1-\rho^2$ to swap limits. To begin, we note that for $\operatorname{Re} z > -1$,

$$\begin{aligned}
\int_0^1 \xi^2(1-\xi^2)^z d\xi &= \frac{1}{2}\int_0^1 y^{1/2}(1-y)^z dy \\
&= \frac{1}{2}B\left(z+1, \frac{3}{2}\right) \\
&= \frac{\Gamma(z+1)\Gamma\left(\frac{3}{2}\right)}{2\Gamma\left(z+\frac{5}{2}\right)} \\
&= \frac{\sqrt{\pi}}{4}\frac{\Gamma(z+1)}{\Gamma\left(z+\frac{5}{2}\right)},
\end{aligned}$$

where we have used the substitution $y = \xi^2$ and the a couple of properties of the Beta function. Thus we have

$$\begin{aligned}
\int_0^1 g_2(\xi, -s)\xi^2 d\xi &= \sum_{k=0}^{\infty} \frac{\Gamma(-s+2k+1)}{\Gamma(-s+k+1)}\frac{4^{-k}}{k!}\int_0^1 \xi^2(1-\xi^2)^k d\xi \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(-s+2k+1)}{\Gamma(-s+k+1)}\frac{4^{-k}}{k!}\frac{\sqrt{\pi}}{4}\frac{\Gamma(k+1)}{\Gamma\left(k+2+\frac{1}{2}\right)},
\end{aligned}$$

which is in fact holomorphic at $s = -1$:

$$\int_0^1 g_2(\xi, 1)\xi^2 d\xi = \frac{\sqrt{\pi}}{4}\sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{\Gamma(k+2)}\frac{4^{-k}}{\Gamma\left(k+2+\frac{1}{2}\right)}.$$

Using the duplication formula, we can simplify this expression to

$$\begin{aligned}
\int_0^1 g_2(\xi, 1)\xi^2 d\xi &= 2\sum_{k=0}^{\infty} \frac{\Gamma(2k+2)}{\Gamma(2k+4)} \\
&= 2\sum_{k=0}^{\infty} \frac{1}{(2k+2)(2k+3)} \\
&= 2(1-\log 2)
\end{aligned}$$

where we have used the Taylor series of $\log(1+x)$ about $x = 0$ to evaluate the sum. We can similarly evaluate

$$
\begin{aligned}
\int_0^1 (1-\xi^2)^s g_2(\xi,s)\xi^2 d\xi &= \sum_{k=0}^{\infty} \frac{\Gamma(s+2k+1)}{\Gamma(s+k+1)} \frac{4^{-k}}{k!} \int_0^1 \xi^2 (1-\xi^2)^{s+k} d\xi \\
&= \sum_{k=0}^{\infty} \frac{\Gamma(s+2k+1)}{\Gamma(s+k+1)} \frac{4^{-k}}{k!} \frac{\sqrt{\pi}}{4} \frac{\Gamma\left(s+k+1\right)}{\Gamma\left(s+k+\frac{5}{2}\right)} \\
&= \frac{\sqrt{\pi}}{4} \sum_{k=0}^{\infty} \frac{\Gamma(s+2k+1)}{\Gamma\left(s+k+\frac{5}{2}\right)} \frac{4^{-k}}{k!} \\
&= \frac{\sqrt{\pi}}{4} \frac{\Gamma(s+1)}{\Gamma\left(s+\frac{5}{2}\right)} + \frac{\sqrt{\pi}}{4} \sum_{k=1}^{\infty} \frac{\Gamma(s+2k+1)}{\Gamma\left(s+k+\frac{5}{2}\right)} \frac{4^{-k}}{k!}.
\end{aligned}
$$

This second term in the above also holomorphic at $s=-1$, so we can substitute this in as before:

$$
\begin{aligned}
\frac{\sqrt{\pi}}{4} \sum_{k=1}^{\infty} \frac{\Gamma(2k)}{\Gamma\left(k+\frac{3}{2}\right)} \frac{4^{-k}}{\Gamma(k+1)} &= \frac{1}{2} \sum_{k=1}^{\infty} \frac{\Gamma(2k)}{\Gamma\left(2k+2\right)} \\
&= \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{(2k)(2k+1)} \\
&= \frac{1}{2} \left(1-\log 2\right).
\end{aligned}
$$

Finally, we focus on

$$
\begin{aligned}
(s+1) \cdot \frac{\sqrt{\pi}}{4} \frac{\Gamma(s+1)}{\Gamma\left(s+\frac{5}{2}\right)} &= \frac{\sqrt{\pi}}{4} \frac{\Gamma(s+2)}{\Gamma\left(s+\frac{5}{2}\right)} \\
&= \frac{\Gamma\left(\frac{3}{2}\right)}{2} \frac{1+\Gamma'(1)(s+1)+O((s+1)^2)}{\Gamma\left(\frac{3}{2}\right)+\Gamma'\left(\frac{3}{2}\right)(s+1)+O((s+1)^2)} \\
&= \frac{1}{2} + \frac{\psi(1)-\psi\left(\frac{3}{2}\right)}{2}(s+1) + O((s+1)^2),
\end{aligned}
$$

where $\psi(z) = \Gamma'(z)/\Gamma(z)$ is the digamma function. Thus we can use these expressions in terms of gamma functions to analytically continue the meromorphic function $\theta_1(L(s)^{-1}u_1)$ further into the left: in particular, we can find a Laurent series for this function near $s=-1$. Using the functional relation of the gamma function and the duplication formula, we have the following identities

$$
\begin{aligned}
\psi(z+1) &= \psi(z) + \frac{1}{z} \\
\psi\left(z+\frac{1}{2}\right) &= 2\psi(2z) - \psi(z) - 2\log 2,
\end{aligned}
$$

from which we can conclude that

$$
\psi\left(\frac{3}{2}\right) = \psi(1) + 2 - 2\log 2.
$$

Putting all this together, we have

$$\begin{aligned}
\theta_1(L(s)^{-1}u_1) =& -(1-\log 2) - \frac{4^{-(s+1)}}{s}\left(1-\log 2+\frac{1}{s+1}+2(\log 2-1)\right)+O(s+1)\\
=& -1+\log 2+\left(\log 2-1+\frac{1}{s+1}\right)(1+(s+1))(1-2(s+1)\log 2)\\
&+O(s+1)\\
=& -1+\log 2+\left(\log 2-1+\frac{1}{s+1}\right)(1+(1-2\log 2)(s+1))+O(s+1)\\
=& \frac{1}{s+1}-2+2\log 2+1-2\log 2+O(s+1)\\
=& \frac{1}{s+1}-1+O(s+1)
\end{aligned}$$

Therefore, $\theta_1(A_0u_1) = -1$. The next step is to work out $\theta_1(P_1A_0u_1)$. First, we work out $\theta_1(\rho\partial_\rho L(s)^{-1}u_1)$ using Equation (4.4.7):

$$\begin{aligned}
\theta_1(\rho\partial_\rho L(s)^{-1}u_1) =& \frac{g_2'(1,s)}{2s}\int_0^1 g_2(\xi,-s)\xi^2 d\xi - \frac{4^{-s}}{2s}g_2'(1,s)\int_0^1(1-\xi^2)^{s+1}g_2(\xi,s)\xi^2 d\xi\\
&-\frac{4^{-s}}{4s(s+1)}g_2'(1,s)\int_0^1(1-\xi^2)^{s+1}\partial_\xi\left(\xi^3 g_2(\xi,s)\right)d\xi\\
&+\frac{1}{2(s+1)}\\
=& -\frac{s+2}{2}\theta_1(L(s)^{-1}u_1)+\frac{1}{2(s+1)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\theta_1(P_1A_0u_1) &= \lim_{s\to-1}\left(\frac{1-s}{2}\theta_1(L(s)^{-1}u_1)-\frac{1}{s+1}\right)\\
&= \lim_{s\to-1}\left(\theta_1(L(s)^{-1}u_1)-\frac{s+1}{2}\theta_1(L(s)^{-1}u_1)-\frac{1}{s+1}\right)\\
&= \theta_1(A_0u_1)-\frac{1}{2}\\
&= -\frac{3}{2}.
\end{aligned}$$

Plugging this into Equation (4.4.4) yields

$$\begin{aligned}
\delta s_1 &= (1-3+1)\hat{V}\\
&= -\hat{V}.
\end{aligned}$$

This agrees with our exact calculations in the previous chapter: for $|\hat{V}| < 1/4$, the corresponding frequency is:

$$\begin{aligned}
s &= -\frac{3}{2} + \frac{1}{2}\sqrt{1 - 4\hat{V}} \\
&= -\frac{3}{2} + \frac{1}{2}\left(1 - \frac{4\hat{V}}{2} + O(\hat{V}^2)\right) \\
&= -1 - \hat{V} + O(\hat{V}^2).
\end{aligned}$$

4.5 The extremal limit

Let us now reintroduce a varying κ and recall that $\kappa \to 0$ corresponds to a cosmological horizon going to infinity. To capture this limit, we need to take $V \in C^\infty(\mathbb{R}^3; \mathbb{R})$ and we impose the conditions

$$|\mathbf{x}|^{|\alpha|+2}\partial^\alpha V(\mathbf{x}) \to 0 \quad \text{as} \quad |\mathbf{x}| \to \infty$$

for all multi-indices α so that V and its derivatives are small near the cosmological horizon $|\mathbf{x}| = 1/\kappa$ in the extremal limit. In this section, we consider Equation (4.0.1) with this potential:

$$-\Box_g\psi + 2\kappa^2\psi + V\psi = 0.$$

We can perform the same manipulations as before, so we just need to study the family of operators $L(\tilde{s}) + \hat{V}$. Let $\chi \in C^\infty(\overline{B_1})$ be such that $\chi \equiv 0$ on $B_{1/3}$ and $\chi \equiv 1$ on $B_1 \setminus B_{2/3}$ and note that our problem is now considering the invertibility of

$$L(s/\kappa) + (1 - \chi)\hat{V} + \chi\hat{V}.$$

Since $1 - \chi$ vanishes to all orders on the boundary, we see that $-\kappa\mathbb{N}$ is contained in the quasinormal spectrum of $L(s/\kappa) + (1 - \chi)\hat{V}$ from previous results. Again, we must note that there may be additional frequencies that result due to the addition of this potential, which will be important when we try to apply Theorem 2.2.5 to this operator while treating $\chi\hat{V}$ as the perturbation. The following lemma establishes that the perturbation due to $\chi\hat{V}$ gets smaller as $\kappa \to 0$ from the decay properties of the potential. Note that we require decay of the derivatives to be faster than that of the original function: this is to be expected since we need the increased regularity to define the quasinormal modes further to the left of the complex plane and these higher order frequencies are more sensitive to perturbations than the more slowly decaying ones.

Lemma 4.5.1. *Let $\chi\hat{V}$ be defined as above. Then given $\epsilon > 0$, there exists a constant R depending on ϵ and $\hat{V}$ and a constant C depending only on k and χ such that for all*

$0 < \kappa < 1/(3R)$,

$$\|\chi \hat{V}\|_{C^k} < C\epsilon.$$

Proof. We note that the condition on V means that given $\epsilon > 0$, there exists $R > 1/\epsilon$ such that for $|\mathbf{x}| > R$,

$$|\mathbf{x}|^{|\alpha|+2} |\partial^\alpha V(\mathbf{x})| < \epsilon$$

for each multi-index α such that $|\alpha| \leq k$. Thus if $0 < \kappa < 1/(3R)$, we have for $|\mathbf{x}| > 1/3$,

$$\begin{aligned}
|\partial^\alpha \hat{V}(\mathbf{x})| &= \frac{1}{\kappa^{|\alpha|+2}} |(\partial^\alpha V)(\mathbf{x}/\kappa)| \\
&= \frac{1}{|\mathbf{x}|^{|\alpha|+2}} \left| \frac{|\mathbf{x}|^{|\alpha|+2}}{\kappa^{|\alpha|+2}} (\partial^\alpha V)(\mathbf{x}/\kappa) \right| \\
&< 3^{|\alpha|+2} \epsilon \\
&< 3^{k+2} \epsilon.
\end{aligned}$$

This gives us

$$\begin{aligned}
|\partial^\alpha (\chi \hat{V})(\mathbf{x})| &\leq \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} |\partial^\beta \hat{V}(\mathbf{x})| |\partial^{\alpha-\beta} \chi(\mathbf{x})| \\
&< \|\chi\|_{C^k} \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} 3^{|\beta|+2} \epsilon \\
&\leq 9\epsilon \|\chi\|_{C^k} 3^k \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \\
&= 9 \cdot 6^k \epsilon \|\chi\|_{C^k},
\end{aligned}$$

which gives the estimate. □

With this in mind, we define

$$\tilde{V} := \frac{\chi \hat{V}}{\|\chi \hat{V}\|_{C^k}}$$

and note the operator norm of this as a map $H^k(B_1) \to H^k(B_1)$ is bounded independently of κ. We write

$$\begin{aligned}
A^V(s) &= (L(s) + (1-\chi)\hat{V}) L_0^{-1}, \\
B^V(s) &= A^V(s) + \chi \hat{V} L_0^{-1},
\end{aligned}$$

and note that as before we can decompose

$$(L(s)+(1-\chi)\hat{V})^{-1}=\frac{A^V_{-1}}{s+n}+A^V_0(s),$$

where $A^V_0(s)$ is bounded in an open neighbourhood of $-n$. The fact that this is a simple pole follows from noting that $L'(s)L(s)^{-1}$ has simple poles and that the kernel of $L'(s)$ is trivial at each of these. Thus there exists $0<r<1/2$ such that $A_0(s)$ is bounded on the disc of radius r around $-n$. Note that we have suppressed the dependence of r on k and n - this will in general depend on how many extra frequencies are introduced by the potential. Let us define

$$C^V_{k,n}=\sup_{s\in D(-n,r)}\left\{\|L_0A^V_0(s)\tilde{V}L_0^{-1}\|_{H^k\to H^k}\right\}$$

and

$$\tilde{C}^V_{k,n}=\min\left\{\frac{1}{2\|L_0A^V_{-1}\widehat{\chi V}L_0^{-1}\|_{H^k\to H^k}},\frac{1}{C^V_{k,n}}\right\}.$$

We can obtain analogous results to before.

Proposition 4.5.1. *For each $n\in\mathbb{N}$ and $0<\delta<r(n,k)$, there exists $R>0$ such that for all $0<\kappa<1/(3R)$, there exists at least one quasinormal frequency s inside $D(-n\kappa,\delta\kappa)$ i.e.*

$$|s+n\kappa|<\delta\kappa.$$

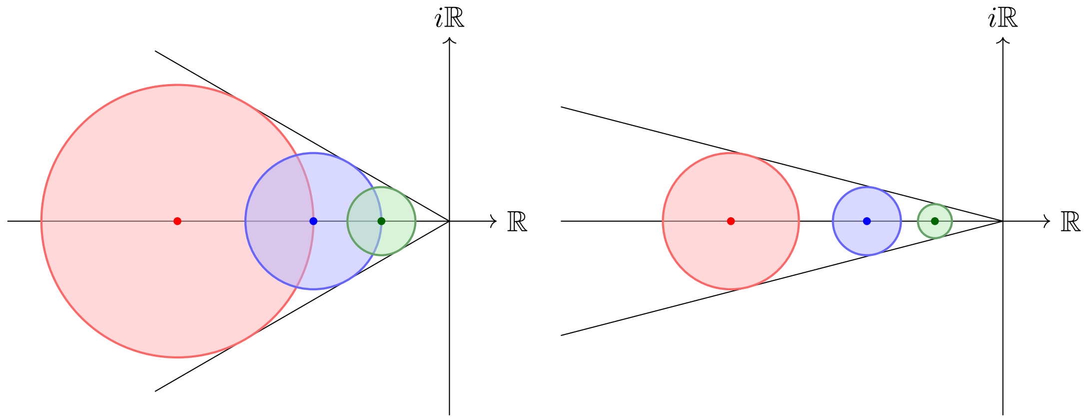
Proof. This is analogous to the proofs of the results from before. We see again that on the circle of radius δ around $-n$ that

$$\|A^V(\tilde{s})^{-1}(A^V(\tilde{s})-B^V(\tilde{s}))\|_{H^k\to H^k}\leq\frac{\|\chi\hat{V}\|_{C^k}}{\delta}\|L_0A^V_{-1}\tilde{V}L_0^{-1}\|_{H^k\to H^k}+\|\chi\hat{V}\|_{C^k}C^V_{k,n},$$

where again $\tilde{s}=s/\kappa$. We note that by Lemma 4.5.1, there exists R such that for all $0<\kappa<1/(3R)$ $\|\chi\hat{V}\|_{C^k}<\tilde{C}^V_{k,n}\delta$. Thus we have

$$\begin{aligned}\|A^V(\tilde{s})^{-1}(A^V(\tilde{s})-B^V(\tilde{s}))\|_{H^k\to H^k}&<\tilde{C}^V_{k,n}\|L_0A^V_{-1}\tilde{V}L_0^{-1}\|_{H^k\to H^k}+\tilde{C}^V_{k,n}C^V_{k,n}\delta\\&\leq\frac{1}{2}+\delta<1,\end{aligned}$$

since $\delta<r\leq 1/2$. The conditions in Theorem 2.2.5 are met and so the number of quasinormal frequencies (counted with multiplicity) are unchanged. So there exists at



(a) $s = -\kappa$ (b) $s = -2\kappa$

Figure 4.1: The shaded regions correspond to the possible location of quasinormal frequencies for differing values of κ. We see that as κ gets smaller, the frequency approaches 0 in a sector with angle $\arcsin(1/2n)$.

least one s such that

$$\begin{aligned} &\left|\frac{s}{\kappa} + n\right| < \delta \\ \Rightarrow &\ |s + n\kappa| < \delta\kappa < \frac{\kappa}{2}. \end{aligned}$$

$\square$

Theorem 4.5.1. *Let $V \in C^\infty(\mathbb{R}^3; \mathbb{R})$ be such that*

$$|\mathbf{x}|^{|\alpha|+2}\partial^\alpha V(\mathbf{x}) \to 0 \quad \textit{as} \quad |\mathbf{x}| \to \infty$$

for each multi-index α and g be the metric on de Sitter outlined in Section 3.1.2. Then

$$-\Box_g \psi + 2\kappa^2\psi + V\psi = 0$$

exhibits the phenomenon of zero-damped quasinormal frequencies converging to 0.

Proof. This is simply an application of Proposition 4.5.1, noting that for each κ sufficiently small, there is a subset of the quasinormal spectrum $\{s_n\}_{n=1}^\infty$ such that

$$|s_n + n\kappa| < \frac{\kappa}{2}.$$

$\square$

This is sufficient to establish the existence of these frequencies, however we don't have much control on the size of the error - we just have $O(\kappa)$ circles where we may find them (see Figure 4.1). If we assume faster decay of V, we can get a better idea of how far away this subset of the spectrum can stray from $-\kappa\mathbb{N}$. For example, if

$$|\mathbf{x}|^{|\alpha|+3}|\partial^\alpha V(\mathbf{x})| < M_1$$

for some constant $M_1 > 0$ and for each multi-index α, we have existence of quasinormal frequencies $s_n(\kappa)$ such that for κ small enough,

$$|s_n(\kappa) + \kappa n| < M_2\kappa^2$$

for some $M_2 > 0$ independent of κ and thus $s_n(\kappa) = -n\kappa + O(\kappa^2)$.

Chapter 5

Static, spherically symmetric spacetimes

In this chapter, we turn our focus to the effect of changing the metric on the quasinormal spectrum. We maintain certain symmetries of the spacetime (namely that it is static and spherically symmetric), but do consider a generic class of metrics which include important examples like Schwarzschild and Reissner-Nordström-de Sitter black holes.

The main result of this chapter is Theorem 5.2.1, which establishes existence of zero-damped modes for the conformal Klein-Gordon equation as a cosmological horizon becomes an asymptotically flat end for these spacetimes. Indeed, examining the proof and using similar arguments as found in Chapter 4, this result establishes the existence of these modes for the Regge-Wheeler equation for Schwarzschild and Reissner-Nordström-de Sitter black holes. We conclude the chapter by applying Theorem 5.2.1 to a special case: after performing a transformation to flip the event and cosmological horizons, the Reissner-Nordström-de Sitter metric is conformal to one considered in our generic class of spacetimes. Using the conformal symmetry of the equation

$$\left(-\Box_g + \frac{R[g]}{6}\right)\psi = 0,$$

and applying Theorem 5.2.1, we obtain Corollary 5.3.1: existence of zero-damped modes for the above equation in the limit the event and Cauchy horizons coalesce. Again this can be extended to the Regge-Wheeler equation for this spacetime, however we must additionally take $\Lambda \to 0$ (i.e. the extremal Reissner-Nordström limit) to obtain existence of zero-damped modes.

In the mathematical literature, most of the results obtained concerning the quasinormal spectrum have been focussed on either proving mode stability (showing there are no unstable modes) or approximating the high frequency parts of the quasinormal spectrum typically probed in computation using the WKB method. The first result of the latter type was obtained for Schwarzschild-de Sitter by Sá Barreto and Zworski in [125] and this was extended by Dyatlov in [54] to the slowly rotating Kerr-de Sitter space.

Studying zero-damped modes rigorously in black hole spacetimes was first done in the mathematical literature by Hintz and Xie [76], where the Klein-Gordon equation was considered on small Schwarzschild-de Sitter black holes. The authors showed that in the limit that the black hole mass goes to zero, the set of quasinormal frequencies converge to the values one would expect from the pure de Sitter spacetime. While this limit is not obviously extremal, one can see that for special values of the Klein-Gordon mass that this limit is in some sense equivalent to taking the cosmological constant to 0. Using the notation of [76], we define the set $\mathrm{QNM}(\mathfrak{m}, \Lambda, \nu)$ as the set of quasinormal frequencies of a Klein-Gordon field of mass $\sqrt{\nu}$ propagating on a Schwarzschild-de Sitter background with black hole mass $\mathfrak{m}$ and cosmological constant Λ. Then by considering the equation (see [76] for details) one can show that for any $h > 0$,

$$\mathrm{QNM}(\mathfrak{m}, \Lambda, \nu) = \frac{1}{h}\mathrm{QNM}\left(\frac{\mathfrak{m}}{h}, h^2\Lambda, h^2\nu\right).$$

In particular, if we take $\mathfrak{m} = \kappa\mu$ where $\kappa \to 0$, we see that

$$\mathrm{QNM}(\mathfrak{m}, 3, \hat{\nu}) = \mathrm{QNM}(\kappa\mu, 3, \hat{\nu}) = \frac{1}{\kappa}\mathrm{QNM}\left(\mu, 3\kappa^2, \kappa^2\hat{\nu}\right).$$

Setting $\kappa = \sqrt{\Lambda/3}$, we see that the set on the left hand side converges pointwise to $\mathrm{QNM}(0, 3, \hat{\nu})$ as $\mathfrak{m} \to 0$, while the set on the right hand side (a rescaled set of quasinormal frequencies) converges pointwise to this set as $\Lambda \to 0$ i.e. after zooming into the complex plane, the limits pointwise give the same spectra. It is important to note here that there is no guarantee of uniform convergence of the sets here. The authors use the spherical symmetry of the spacetime and careful analysis of the resulting ordinary differential equations near each horizon with respect to suitable weighted spaces to obtain the convergence of quasinormal modes to this set.

In this chapter, we seek to extend the results of [76] to a more general class of spherically symmetric spacetimes with a cosmological horizon. The arguments of the previous chapter are robust in the sense that they can be applied to this more complicated situation, albeit with some technicalities. We shall again consider the conformally coupled Klein-Gordon equation but now with metrics of the form:

$$g = -f_\Lambda(r)dt^2 + \frac{dr^2}{f_\Lambda(r)} + r^2\not{g}$$

where

$$f_\Lambda(r) = 1 + w_\Lambda(r) + \frac{\Lambda}{3}\alpha_\Lambda r - \frac{\Lambda}{3}r^2$$

for some $\alpha_\Lambda \in \mathbb{R}$ bounded for $\Lambda \in [0, \Lambda_0]$ such that $\alpha_\Lambda \to \alpha_0$ as $\Lambda \to 0$ and with $w_\Lambda(r) \in C^\infty((0,\infty);\mathbb{R})$ for each $\Lambda \in [0,\Lambda_0]$ obeying the following additional conditions:

1. for each $k \in \mathbb{N}_0, \Lambda \in [0, \Lambda_0]$, $r^k \partial_r^k w_\Lambda(r) \to 0$ as $r \to \infty$,
2. $1 + w_0(r)$ has at least one root and finitely many others. We further assume the largest root, $r_e(0)$, is simple and $w_0(r) < 0$ for $r > r_e(0)$,
3. there exists $0 < t < r_e(0)$ such that for each $k \in \mathbb{N}_0$, there exists $\beta_k > 0$ so the following holds

$$\sup_{r \geq t} \left| r^k \partial_r^k w_\Lambda(r) - r^k \partial_r^k w_0(r) \right| < \frac{\Lambda}{3}\beta_k.$$

These conditions are sufficient by standard arguments (for sufficiently small Λ) to establish the following properties:

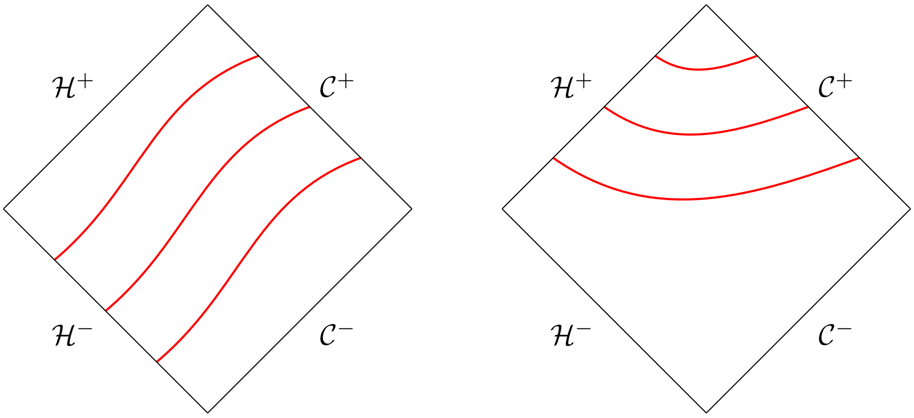
1. $f_\Lambda(r)$ has a simple root at $r = r_c(\Lambda)$ (cosmological horizon) such that $r_c(\Lambda) \to \infty$ and $\Lambda r_c^2/3 \to 1$ as $\Lambda \to 0$,
2. $f_\Lambda(r)$ has a simple root at $r = r_e(\Lambda)$ (event horizon) such that $r_e(\Lambda) \to r_e(0)$ as $\Lambda \to 0$,
3. $f_\Lambda(r)$ has no roots in the interval $(r_e(\Lambda), r_c(\Lambda))$ for Λ sufficiently small,
4. $1 - f_\Lambda(r) > 0$ for $r_e(\Lambda) < r < r_c(\Lambda)$ provided Λ is sufficiently small and furthermore, there exists $\epsilon > 0$ such that $1 - f_\Lambda(r) > \epsilon$ for any $r_c(\Lambda)/3 < r < r_c(\Lambda)$.

Important examples of w_Λ which obey these conditions are $w_\Lambda(r) = -2\,\mathfrak{m}/r$ (the Schwarzschild-de Sitter solution) and $w_\Lambda(r) = -2\,\mathfrak{m}/r + \mathfrak{q}^2/r^2$ (Reissner-Nordström-de Sitter). For an example of $w_\Lambda(r)$ which depends on Λ, see Section 5.3. From now on, we shall suppress the dependence on Λ of some quantities. The equation we shall consider is the conformal Klein-Gordon equation for this spacetime:

$$-\Box_g \psi + \frac{R[g]}{6}\psi = 0,$$

where $R[g]$ is the Ricci scalar of the spacetime. In this case, we see that

$$\begin{aligned} R[g] &= 4\Lambda - \frac{2\Lambda\alpha_\Lambda}{r} - w_\Lambda''(r) - \frac{4w_\Lambda'(r)}{r} - \frac{2w_\Lambda(r)}{r^2} \\ &= 4\Lambda + \frac{\Lambda}{3}V^w(r). \end{aligned}$$



(a) The slicing when $F \equiv 1$.

(b) The slicing when F interpolates between -1 and 1.

Figure 5.1: Penrose diagrams depicting the coordinate transformation with and without the smooth interpolation F.

We use a coordinate transformation analogous to the pure de Sitter case to get a hyperboloidal slicing:

$$\tau = t - \int_{r_0}^{r} \frac{\sqrt{1 - f_\Lambda(\xi)}}{f_\Lambda(\xi)} F(\xi) d\xi,$$
$$\rho = \frac{r}{r_c}.$$

where r_0 is fixed so that it lies in $(r_e(\Lambda), r_c(\Lambda))$ for all $\Lambda \in [0, \Lambda_0]$ and $F \in C^\infty(\mathbb{R})$ is chosen so that

$$F(r) = \begin{cases} -1 & r < 2r_e(0) \\ 1 & r > 3r_e(0) \end{cases}.$$

For Λ_0 sufficiently small, this means that $F \equiv -1$ in a neighbourhood of the event horizon and $F \equiv 1$ near the cosmological horizon. We must introduce this factor to 'bend' the spatial slices toward the future event horizon (see Figure 5.1). Note that the fourth condition given above is necessary to make this transformation well-defined. As before (since the horizons we consider have non-zero surface gravity) the results in [139] allow us to define the quasinormal frequencies and similar reasoning to Chapter 2 and Chapter 3 means it suffices to consider the following equation on $\overline{B_1 \setminus B_{\rho_e}}$

$$-e^{-s\tau} \Box_g \left(e^{s\tau} u\right) + \frac{R[g]}{6} u = \Psi_0, \tag{5.0.1}$$

where $\rho_e = r_e/r_c$ and Ψ_0 is constructed from the initial data. Near the cosmological horizon, Equation (5.0.1) takes the form:

$$
\begin{aligned}
-\frac{f_\Lambda(r_c\rho)}{r_c^2}\partial_\rho^2 u - \left(\frac{f'_\Lambda(r_c\rho)}{r_c} + \frac{2}{r_c^2\rho}f_\Lambda(r_c\rho) - \frac{2s}{r_c}\sqrt{1-f_\Lambda(r_c\rho)}\right)\partial_\rho u \\
+\left(s^2 + \frac{2\Lambda}{3} + \frac{2s}{r_c\rho}\sqrt{1-f_\Lambda(r_c\rho)} - \frac{s}{2}\frac{f'_\Lambda(r_c\rho)}{\sqrt{1-f_\Lambda(r_c\rho)}} + \frac{\Lambda}{3}V^w(r_c\rho)\right)u \\
-\frac{\not{\Delta} u}{r_c^2\rho^2} = \Psi_0,
\end{aligned}
\tag{5.0.2}
$$

It will be useful to introduce the notation $U_r := B_1 \setminus B_r$ from this point onwards. Dividing through by $3/\Lambda$ and setting $\tilde{s} = s\sqrt{3/\Lambda}$ we obtain the family of operators

$$
L^w(\tilde{s})u := -\, e^{-s\tau}\frac{3}{\Lambda}\Box_g\left(e^{s\tau}u\right) + \frac{R[g]u}{2\Lambda}, \tag{5.0.3}
$$

which (near the cosmological horizon where $F \equiv 1$) takes the form:

$$
\begin{aligned}
L^w(\tilde{s})u = &-\frac{3}{\Lambda r_c^2}f_\Lambda(r_c\rho)\partial_\rho^2 u - \frac{3}{\Lambda r_c^2}\frac{\not{\Delta} u}{\rho^2} \\
&-\left(\frac{3}{\Lambda r_c^2}\left(r_c f'_\Lambda(r_c\rho) + \frac{2f_\Lambda(r_c\rho)}{\rho}\right) - 2\tilde{s}\sqrt{\frac{3}{\Lambda r_c^2}\left(1-f_\Lambda(r_c\rho)\right)}\right)\partial_\rho u \\
&+\left(\tilde{s}^2 + 2 + \frac{2\tilde{s}}{\rho}\sqrt{\frac{3}{\Lambda r_c^2}\left(1-f_\Lambda(r_c\rho)\right)} - \frac{\tilde{s}}{2}\sqrt{\frac{3}{\Lambda r_c^2}}\frac{r_c f'_\Lambda(r_c\rho)}{\sqrt{1-f_\Lambda(r_c\rho)}} + V^w(r_c\rho)\right)u.
\end{aligned}
\tag{5.0.4}
$$

5.1 Comparing families of operators

In order to use the results outlined in Section 2.2.3, we need to compare the $L^w(s)$ as defined in Equation (5.0.3) with $L(s)$. These operators are, however, defined on different domains. To get around this, we define a smooth, spherically symmetric cut-off $\chi \in C^\infty(\overline{B_1})$ such that

$$
\chi(\mathbf{x}) = \begin{cases} 0 & \rho < \frac{1}{3} \\ 1 & \rho > \frac{2}{3} \end{cases}
$$

and note that for Λ sufficiently small, this vanishes to all orders away from the set where $F \equiv 1$: in particular it does so on the event horizon $\rho = \rho_e$. We now define the family of operators

$$
K(s) := L^w(s) \circ \chi - L(s) \circ E_\chi,
$$

where E_χ multiplies a function by χ and extends it to B_1 by setting it to zero outside its domain of definition. Note also that we have omitted explicitly writing the map which restricts functions on B_1 to functions on U_{ρ_e}. This gives us the following decomposition of $L^w(s)$:

$$L^w(s) = L^w(s) \circ (1-\chi) + L(s) \circ E_\chi + K(s).$$

The unbounded operator $L^w(s)\circ(1-\chi)+L(s)\circ E_\chi$ is a degenerate elliptic operator with the degeneracy occurring precisely at the event and cosmological horizons. Furthermore the surface gravity at these horizons is non-zero, so from the results of [139], this defines a family of Fredholm operators from its domain to $H^k(U_{\rho_e})$. We also have the following estimates:

Lemma 5.1.1. *There exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$, there exists a constant $C > 0$ depending only on k such that*

$$\begin{aligned}\|L^w(0)(1-\chi)u + L_0 \circ E_\chi u)\|_{H^k} &\leq C\left(\|L^w(0)u\|_{H^k} + \|u\|_{H^k}\right),\\ \|L^w(0)u\|_{H^k} &\leq C\left(\|L^w(0)(1-\chi)u + L_0\circ E_\chi u)\|_{H^k} + \|u\|_{H^k}\right).\end{aligned}$$

Proof. Throughout this section the constant C may change value from line to line to reduce clutter. The key point is that it depends only on k. We begin by proving the intermediary estimate

$$\|(\Delta - \partial_\rho^2)u\|_{H^k} \leq C\|u\|_{D^{k+1}(L^w(s))}.$$

Observe that

$$\begin{aligned}\|L^w(0)(\chi u)\|^2_{H^k} - \left(\frac{3}{\Lambda r_c^2}\right)^2 \|\left(\Delta - \partial_\rho^2\right)(\chi u)\|^2_{H^k}&\\ \geq -\frac{6}{\Lambda r_c^2}\left(L^w(0)(\chi u) + \frac{3}{\Lambda r_c^2}\left(\Delta - \partial_\rho^2\right)(\chi u), \left(\Delta-\partial_\rho^2\right)(\chi u)\right)_{H^k}&,\end{aligned}$$

where Δ is the Laplacian on U_{ρ_e}. We aim to show that the right-hand side can be controlled by a suitable multiple of $\|u\|_{D^{k+1}(L^w(s))}$. A useful observation from the theory developed in [139] is the fact that there exists a constant C depending only on k such that $\|u\|_{H^{k+1}} \leq C\|L^w(0)u\|_{H^k}$, so it suffices to show that the terms we are interested in can be dominated by a multiple of $\|u\|_{H^{k+1}}$. We switch to polar coordinates so $\Delta - \partial_\rho^2 = (2/\rho)\partial_\rho + (1/\rho^2)\not{\Delta}$ and consider each of these separately. Since the operators we are interested in commute with angular derivatives, it suffices to prove this for radial ones only.

$$\partial_\rho^k\left(\frac{2}{\rho}\partial_\rho(\chi u)\right) = \frac{2}{\rho}\partial_\rho^{k+1}(\chi u) + \sum_{j=1}^k p_j(\rho)\partial_\rho^j(\chi u)$$

for some functions p_j which are bounded on $\operatorname{supp}\chi$. We combine this with

$$\begin{aligned}\partial_\rho^k\left(L^w(0)(\chi u)+\frac{3}{\Lambda r_c^2}\left(\Delta-\partial_\rho^2\right)(\chi u)\right)=&-\frac{3f_\Lambda}{\Lambda r_c^2}\partial_\rho^{k+2}(\chi u)+\sum_{j=1}^k q_j(\rho)\partial_\rho^j(\chi u)\\&+\frac{3}{\Lambda r_c^2}\left(\frac{2}{\rho}(1-f_\Lambda)-(k+1)\partial_\rho f_\Lambda\right)\partial_\rho^{k+1}(\chi u),\end{aligned}\tag{5.1.1}$$

where the q_j are some functions constructed from $3f_\Lambda(r_c\rho)/(\Lambda r_c^2)$ and observe that

$$\frac{3}{\Lambda r_c^2}f_\Lambda(r_c\rho)-(1-\rho^2)=\frac{3}{\Lambda r_c^2}-1+\frac{3}{\Lambda r_c^2}w_\Lambda(r_c\rho)+\frac{\alpha_\Lambda\rho}{r_c}.$$

Restricting to the support of χ, we see that the right hand side and its derivatives converge uniformly to 0 as $\Lambda\to 0$ by observing that

$$\begin{aligned}\left|\partial_\rho^k(w_\Lambda(r_c\rho))\right|&\le\frac{\left|r_c^k\rho^k(w_\Lambda^{(k)}(r_c\rho)-w_0^{(k)}(r_c\rho))\right|}{\rho^k}+\frac{\left|r_c^k\rho^k w_0^{(k)}(r_c\rho)\right|}{\rho^k}\\&<3^k\left(\frac{\Lambda}{3}\beta_k+|r_c^k\rho^k w_0^{(k)}(r_c\rho)|\right)\end{aligned}$$

and that the quantity in the brackets above goes to 0 as $\Lambda\to 0$. Hence $3f_\Lambda(r_c\rho)/(\Lambda r_c^2)\to 1-\rho^2$ in $C^l(\operatorname{supp}\chi)$ as $\Lambda\to 0$ for each $l\in\mathbb{N}_0$. Thus, there exists $\Lambda_0>0$ such that for all $\Lambda<\Lambda_0$, we can control these functions by some constant. Taking the L^2-inner product of Equation (5.1.1) with $\partial_\rho^k((\Delta-\partial_\rho^2)(\chi u))$, we see that the only pairing that cannot clearly be controlled by a suitable multiple of $\|u\|_{H^{k+1}}$ is

$$\begin{aligned}-\frac{3}{\Lambda r_c^2}\int_{U_{\rho_e}}\frac{2}{\rho}\partial_\rho^{k+1}(\chi\bar u)\cdot f_\Lambda\partial_\rho^{k+2}(\chi u)dx&=-\frac{3}{\Lambda r_c^2}\int_{U_{\rho_e}}\frac{f_\Lambda}{\rho}\partial_\rho\left(\left|\partial_\rho^{k+1}(\chi u)\right|^2\right)dx\\&=\frac{3}{\Lambda r_c^2}\int_{U_{\rho_e}}\partial_\rho\left(\frac{f_\Lambda}{\rho}\right)\left|\partial_\rho^{k+1}(\chi u)\right|^2dx,\end{aligned}$$

where the boundary term vanishes due to f_Λ. Thus this term can also be made positive by adding a multiple of $\|u\|_{H^{k+1}}$ where the constant is independent of Λ for Λ small enough. To deal with the other term, we write $\chi u=\rho^2\cdot\chi u/\rho^2=\rho^2 v$ and consider

$$\partial_\rho^k\left(L^w(0)(\chi u)+\frac{3}{\Lambda r_c^2}\left(\Delta-\partial_\rho^2\right)(\chi u)\right)=\partial_\rho^k\left(L^w(0)(\rho^2 v)+\frac{3}{\Lambda r_c^2}\left(\Delta-\partial_\rho^2\right)(\rho^2 v)\right).$$

We only need to consider the two highest order derivative terms since after multiplying by $\not{\Delta}\partial_\rho^k v$ and integrating by parts on the sphere, we are left with terms of the form $t(\rho)\partial_\rho^l\not{\nabla}v\cdot\partial_\rho^k\not{\nabla}v$ for some function t and $0\le l\le k+2$. These can be controlled by $\|u\|_{H^{k+1}}$ for $l\le k$ so we only have to consider the terms with $l=k+1$ and $l=k+2$.

These are

$$-\frac{3}{\Lambda r_c^2}\left(\rho^2 f_\Lambda \partial_\rho^{k+2} v + 2(k+2)\rho f_\Lambda \partial_\rho^{k+1} v + k\rho^2 \partial_\rho f_\Lambda \partial_\rho^{k+1} v - \partial_\rho(\rho^2(1-f_\Lambda))\partial_\rho^{k+1} v\right). \tag{5.1.2}$$

We start with the highest order term:

$$\begin{aligned}
\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} \rho^2 f_\Lambda \partial_\rho^{k+2} \bar{v} \not\Delta \partial_\rho^k v dx = &-\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} \rho^2 f_\Lambda \partial_\rho^{k+2} \not\nabla \bar{v} \cdot \partial_\rho^k \not\nabla v dx \\
=&\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} \rho^2 f_\Lambda |\partial_\rho^{k+1} \not\nabla v|^2 dx \\
&+\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} (\rho^2 \partial_\rho f_\Lambda + 4\rho f_\Lambda)\partial_\rho^{k+1} \not\nabla \bar{v} \cdot \partial_\rho^k \not\nabla v dx \\
\geq& \frac{1}{2}\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} (\rho^2 \partial_\rho f_\Lambda + 4\rho f_\Lambda)\partial_\rho |\partial_\rho^k \not\nabla v|^2 dx.
\end{aligned}$$

Since f_Λ vanishes on the boundary, we can integrate by parts to deduce that the term involving f_Λ can be dealt with using $\|u\|_{H^{k+1}}$. Combining this with (5.1.2), we are left with

$$\begin{aligned}
&\frac{1}{2}\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} (2\rho - k\rho^2 \partial_\rho f_\Lambda)\partial_\rho |\partial_\rho^k \not\nabla v|^2 dx \\
&\qquad \geq -\frac{1}{2}\left(\frac{3}{\Lambda r_c^2}\right)^2 \int_{U_{\rho e}} (6 - 4k\rho \partial_\rho f_\Lambda - k\rho^2 \partial_\rho^2 f_\Lambda)|\partial_\rho^k \not\nabla v|^2 dx.
\end{aligned}$$

The inequality follows from the fact that the cut-off allows us to ignore the inner boundary and that $\partial_\rho f_\Lambda < 0$ at the outer boundary, so $2 - k\partial_\rho f_\Lambda > 0$. Thus this is also dominated by a suitable multiple of $\|u\|_{H^{k+1}}$ and we have the following result: there exists $C, \Lambda_0 > 0$ depending only on k such that for all $\Lambda < \Lambda_0$,

$$\|\left(\Delta - \partial_\rho^2\right)(\chi u)\|_{H^k} \leq C\left(\|L^w(0)(\chi u)\|_{H^k} + \|u\|_{H^{k+1}}\right) \leq C\|u\|_{D^{k+1}(L^w(s))}.$$

We now use this to prove the lemma. From the product rule and the usual estimates,

$$\|L^w(0)(1-\chi)u + L_0 \circ E_\chi u)\|_{H^k} \leq C\left(\|L^w(0)u\|_{H^k} + \|u\|_{H^{k+1}}\right) + \|L_0 \circ E_\chi u\|_{H^k}.$$

Since f_Λ vanishes only at the horizons and both these zeroes are simple, $(1-\rho^2)/f_\Lambda(r_c\rho)$ is smooth and bounded on $\operatorname{supp}\chi$ and thus has a finite C^k norm. Thus there exists a constant C such that for all Λ small enough,

$$\begin{aligned}
\|L_0 \circ E_\chi u\|_{H^k} &\leq \|\frac{\Lambda r_c^2}{3}\frac{1-\rho^2}{f_\Lambda(r_c\rho)}\left(L^w(0)(\chi u) + \frac{3}{\Lambda r_c^2}(\Delta - \partial_\rho^2)(\chi u)\right)\| + C\|u\|_{D^{k+1}(L^w(s))} \\
&\leq C\left(\|L^w(0)u\|_{H^k} + \|u\|_{H^{k+1}}\right) \\
&\leq C\|u\|_{D^{k+1}(L^w(s))}.
\end{aligned}$$

Next we observe that

$$\begin{aligned}
\|L^w(0)u\|_{H^k} &= \|L^w(0)(1-\chi)u + L^w(0)(\chi u)\|_{H^k} \\
&\leq \|L^w(0)(1-\chi)u + L_0 \circ E_\chi u\|_{H^k} + \|L^w(0)(\chi u) - L_0 \circ E_\chi u\|_{H^k}.
\end{aligned}$$

Focussing on the second term, we have the following inequality

$$\begin{aligned}
\|L^w(0)(\chi u) - L_0 \circ E_\chi u\|_{H^k} \leq &\|\left((1-\rho^2) - 3f_\Lambda/(\Lambda r_c^2)\right)\partial_\rho^2(\chi u)\|_{H^k} \\
&+ \|\left(1 - 3/(\Lambda r_c^2)\right)(\Delta - \partial_\rho^2)(\chi u)\|_{H^k} + \|V^w \chi u\|_{H^k} \\
&+ \|\left(\frac{3}{\Lambda r_c^2\rho^2}\partial_\rho(\rho^2(1 - f_\Lambda(r_c\rho)) - \frac{1}{\rho^2}\partial_\rho(\rho^2\cdot\rho^2)\right)\partial_\rho(\chi u)\|_{H^k}.
\end{aligned}$$

We can deal with the term involving V^w since

$$V^w(r_c\rho) = -\frac{6\alpha_\Lambda}{r_c\rho} - \frac{3}{\Lambda r_c^2}\left(\frac{r_c^2\rho^2 w_\Lambda''(r_c\rho)}{\rho^2} + \frac{4r_c\rho w_\Lambda'(r_c\rho)}{\rho^2} + \frac{2w_\Lambda(r_c\rho)}{\rho^2}\right)$$

and in $\operatorname{supp}\chi$ that $V^w(r_c\rho)$ and finitely many of its derivatives can be made arbitrarily small by similar reasoning to before. Thus we have both $V^w(r_c\rho) \to 0$ and $3f_\Lambda(r_c\rho)/(\Lambda r_c^2) \to 1-\rho^2$ in $C^{k+1}(\operatorname{supp}\chi)$ as $\Lambda \to 0$, so there exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$ we have all the results above and furthermore

$$\|L^w(0)(\chi u) - L_0 \circ E_\chi u\|_{H^k} \leq \frac{1}{2}\|L^w(0)u\|_{H^k}.$$

Therefore

$$\frac{1}{2}\|L^w(0)u\|_{H^k} \leq \|L^w(0)(1-\chi)u + L_0 \circ E_\chi u\|_{H^k} + C\|u\|_{H^{k+1}}$$

which yields the result. □

The lemma above establishes that $L^w(s)\circ(1-\chi) + L(s)\circ E_\chi$ is a holomorphic family of Fredholm operators $D^{k+1}(L^w(s)) \to H^k(U_{\rho_e})$. Furthermore, for $\rho \in [1/3, 1]$, the cut-off allows the distributions defined in Section 3.2 to satisfy the appropriate conditions to be co-modes of this family of operators at the usual frequencies i.e. $-\mathbb{N}$ is contained in the quasinormal spectrum.

5.2 The proof of the main result

We shall treat $K(s)$ as a perturbation to this operator and apply the results of Section 2.2.3. Using similar notation, we set

$$A^w(s) = (L^w(s) \circ (1-\chi) + L(s) \circ E_\chi) L^w(0)^{-1},$$
$$B^w(s) = A^w(s) + K(s) L^w(0)^{-1},$$

noting that $L^w(0)$ is invertible since its kernel is trivial and it is Fredholm of index 0. By Theorem 2.2.5, the number of frequencies contained within a contour (counted with multiplicity) is the same when

$$\|L^w(0)\left(L^w(s)\circ(1-\chi) + L_0 \circ E_\chi\right)^{-1} K(s) L^w(0)^{-1}\|_{H^k \to H^k} < 1$$

holds for all $s \in \Gamma_{-n,\delta}$. We have

$$\begin{aligned}
&\|L^w(0)\left(L^w(s)\circ(1-\chi) + L_0 \circ E_\chi\right)^{-1} K(s) L^w(0)^{-1}\|_{H^k \to H^k} \\
&\leq \|L^w(0)\left(L^w(s)\circ(1-\chi) + L_0 \circ E_\chi\right)^{-1}\|_{H^k \to H^k} \|K(s)\|_{D^{k+1}\to H^k} \|L^w(0)^{-1}\|_{H^k \to D^{k+1}} \\
&\leq \|L^w(0)\left(L^w(s)\circ(1-\chi) + L_0 \circ E_\chi\right)^{-1}\|_{H^k \to H^k} \|K(s)\|_{D^{k+1}\to H^k}.
\end{aligned}$$

To finish the argument, we just need to show that $K(s)$ can be made arbitrarily small as $\Lambda \to 0$.

Lemma 5.2.1. *Fix $s \in \mathbb{C}$. Given $\epsilon > 0$, there exists $\Lambda_0 > 0$ and depending only on s and k such that for all $0 < \Lambda < \Lambda_0$,*

$$\|K(s)\|_{D^{k+1}\to H^k} < \epsilon.$$

Proof. We first fix $k \in \mathbb{N}$ and $0 < \epsilon < 1$. We also assume Λ is sufficiently small so that $1/2 < 3/(\Lambda r_c^2) < 3/2$ and to give the results derived above ($K(s)$ is a holomorphic family of Fredholm operators, $F \equiv 1$ on $\operatorname{supp}\chi$ and $\rho_e < 1/3$). For $u \in D^{k+1}(L^w(s))$, we can write

$$\begin{aligned}
K(s)u = &\left((1-\rho^2) - \frac{3f_\Lambda(r_c\rho)}{\Lambda r_c^2}\right)\partial_\rho^2(\chi u) + \left(1 - \frac{3}{\Lambda r_c^2}\right)\left(\Delta - \partial_\rho^2\right)(\chi u) \\
&- \frac{3}{\Lambda r_c^2}\left(r_c w_\Lambda'(r_c\rho) + \frac{2}{\rho} w_\Lambda(r_c\rho) + \frac{3\alpha_\Lambda}{r_c}\right)\partial_\rho(\chi u) \\
&+ 2s\left(\sqrt{\frac{3}{\Lambda r_c^2}(1 - f_\Lambda(r_c\rho))} - \rho\right)\partial_\rho(\chi u) + V^w(r_c\rho)\chi u \\
&+ s\left(\frac{2}{\rho}\sqrt{\frac{3}{\Lambda r_c^2}\left(1 - f_\Lambda(r_c\rho)\right)} - \frac{1}{2}\sqrt{\frac{3}{\Lambda r_c^2}}\frac{r_c f_\Lambda'(r_c\rho)}{\sqrt{1 - f_\Lambda(r_c\rho)}} - 3\right)\chi u.
\end{aligned}$$

We deal with each term separately. We have already observed that $V^w(r_c\rho) \to 0$ in

$C^l(\operatorname{supp}\chi)$ as $\Lambda \to 0$ for each $l \in \mathbb{N}$, so the $V^w\chi u$ term can be made small using that. Since Λ is small enough, we have

$$\begin{aligned}\frac{3}{\Lambda r_c^2}&\|(r_c w_\Lambda'(r_c\rho) + 2w_\Lambda(r_c\rho)/\rho + 3\alpha_\Lambda/r_c)\partial_\rho(\chi u)\|_{H^k}\\ &\leq \frac{3}{2}\left(\|r_c w_\Lambda'(r_c\rho) + 2w_\Lambda(r_c\rho)/\rho\|_{C^k} + \frac{3\alpha_\Lambda}{r_c}\right)\|u\|_{H^{k+1}},\end{aligned}$$

where $H^l = H^l(U_{\rho_e})$ and $C^l = C^l(\overline{U_{1/3}})$ for each $l \in \mathbb{N}$. From the properties of w_Λ, we see that there exists $\Lambda_0 > 0$ such that for $\Lambda < \Lambda_0$,

$$\frac{3}{\Lambda r_c^2}\|(r_c w_\Lambda'(r_c\rho) + 2w_\Lambda(r_c\rho)/\rho + 3\alpha_\Lambda/r_c)\partial_\rho(\chi u)\|_{H^k} < \epsilon\|u\|_{D^{k+1}},$$

where $D^{k+1} = D^{k+1}(L^w(s))$. By taking Λ_0 smaller if necessary, we can also ensure that

$$\frac{3}{\Lambda r_c^2}(1 - f_\Lambda(r_c\rho)) = \rho^2 - \frac{3}{\Lambda r_c^2}w(r_c\rho) + \frac{\alpha_\Lambda}{r_c}\rho \in \left[\frac{1}{16}, \frac{151}{144}\right] := I,$$

since $3f_\Lambda(r_c\rho)/(\Lambda r_c^2) \to 1 - \rho^2$ in C^l for each $l \in \mathbb{N}$. We note that $\sqrt{x}$ belongs to $C^\infty(I)$ and thus all its derivatives are Lipschitz on I. Hence we have

$$\sqrt{\frac{3}{\Lambda r_c^2}(1 - f_\Lambda(r_c\rho))} \to \rho \text{ in } C^k \text{ as } \Lambda \to 0.$$

Thus (taking Λ_0 smaller if necessary) we have

$$\begin{aligned}\|2s\left(\sqrt{\frac{3}{\Lambda r_c^2}(1 - f_\Lambda(r_c\rho))} - \rho\right)\partial_\rho(\chi u)\|_{H^k} &< \epsilon\|u\|_{D^{k+1}},\\ \|s\left(\frac{2}{\rho}\sqrt{\frac{3}{\Lambda r_c^2}(1 - f_\Lambda(r_c\rho))} - \frac{1}{2}\sqrt{\frac{3}{\Lambda r_c^2}}\frac{r_c f_\Lambda'(r_c\rho)}{\sqrt{1 - f_\Lambda(r_c\rho)}} - 3\right)\chi u\|_{H^k} &< \epsilon\|u\|_{D^{k+1}}.\end{aligned}$$

We shall now focus on the second order terms. We have already proved that

$$\|(\Delta - \partial_\rho^2)(\chi u)\|_{H^k} \leq C\|u\|_{D^{k+1}},$$

where C depends only on k from the discussion in the previous lemma. Thus for $\Lambda < \Lambda_0$

$$\|\left(1 - \frac{3}{\Lambda r_c^2}\right)\left(\Delta - \partial_\rho^2\right)(\chi u)\|_{H^k} < \epsilon\|u\|_{D^{k+1}}.$$

The radial second order term is

$$\left((1 - \rho^2) - \frac{3f_\Lambda(r_c\rho)}{\Lambda r_c^2}\right)\partial_\rho^2(\chi u) = \left(\frac{\Lambda r_c^2}{3}\frac{1 - \rho^2}{f_\Lambda} - 1\right)\frac{3f_\Lambda(r_c\rho)}{\Lambda r_c^2}\partial_\rho^2(\chi u).$$

For $\rho \in [1/3, 1]$, we see that both $f_\Lambda(r_c\rho)$ and $1 - \rho^2$ vanish only at $\rho = 1$ with simple zeroes there, so the fraction in the above expression extends to a smooth function on this interval. We also know that since it obeys that condition, it will converge uniformly to 1 in that interval if and only if its reciprocal does. We write

$$\begin{aligned} f_\Lambda(r) &= 1 + w_\Lambda(r) + \frac{\Lambda}{3}\alpha_\Lambda r - \frac{\Lambda}{3}r^2 \\ &= w_\Lambda(r) - w_\Lambda(r_c) + \frac{\Lambda}{3}\alpha_\Lambda(r - r_c) - \frac{\Lambda}{3}(r^2 - r_c^2) \\ &= \int_{r_c}^{r} w'_\Lambda(x)dx + \frac{\Lambda}{3}\alpha_\Lambda(r - r_c) - \frac{\Lambda}{3}(r^2 - r_c^2), \end{aligned}$$

using the fact that f_Λ vanishes at $r = r_c$. Writing $r = r_c\rho$ and focussing on $\rho \in [1/3, 1]$, we see that we have

$$f_\Lambda(r_c\rho) = (\rho - 1)\left(\int_0^1 r_c w'_\Lambda(r_c\rho t + r_c(1-t))dt + \frac{\Lambda}{3}\alpha_\Lambda r_c - \frac{\Lambda}{3}r_c^2(1+\rho)\right).$$

From this we see that

$$\frac{3}{\Lambda r_c^2}\frac{f_\Lambda(r_c\rho)}{1-\rho^2} = 1 - \frac{1}{1+\rho}\left(\frac{\alpha_\Lambda}{r_c} + \int_0^1 r_c w'_\Lambda(r_c\rho t + r_c(1-t))dt\right)$$

and hence that

$$\|\frac{3}{\Lambda r_c^2}\frac{f_\Lambda(r_c\rho)}{1-\rho^2} - 1\|_{C^k} \leq C\left(\frac{\alpha_\Lambda}{r_c} + \|\int_0^1 r_c w'_\Lambda(r_c\rho t + r_c(1-t))dt\|_{C^k}\right),$$

where C is a constant depending on only on k. The α_Λ/r_c term can be made arbitrarily small by taking Λ sufficiently small, so we focus on the other term

$$\partial_\rho^j\left(\int_0^1 r_c w'_\Lambda(r_c\rho t + r_c(1-t))dt\right) = \int_0^1 r_c^{j+1}t^j w_\Lambda^{(j+1)}(r_c\rho t + r_c(1-t))dt.$$

We know that

$$\begin{aligned} |w_\Lambda^{(j+1)}(r)| &= |w_\Lambda^{(j+1)}(r) - w_0^{(j+1)}(r) + w_0^{(j+1)}(r)| \\ &\leq \frac{\Lambda}{3}\frac{\beta_{j+1}}{r^{j+1}} + \frac{\epsilon}{r^{j+1}} \end{aligned}$$

for any $\epsilon > 0$ provided $r = r_c\rho$ is sufficiently large. Hence we have

$$\begin{aligned} \left|\int_0^1 r_c^{j+1}t^j w_\Lambda^{(j+1)}(r_c\rho t + r_c(1-t))dt\right| &\leq \left(\frac{\Lambda}{3}\beta_{j+1} + \epsilon\right) r_c^{j+1}\left|\int_0^1 \frac{t^j}{(r_c(\rho-1)t + r_c)^{j+1}}dt\right| \\ &\leq \left(\frac{\Lambda}{3}\beta_{j+1} + \epsilon\right)\frac{1}{(j+1)\rho^{j+1}}. \end{aligned}$$

The right hand side can be made arbitrarily small by taking Λ sufficiently small and hence

$$\frac{3}{\Lambda r_c^2}\frac{f_\Lambda(r_c\rho)}{1-\rho^2} \to 1 \quad \text{in} \quad C^k([1/3,1]) \quad \text{as} \quad \Lambda \to 0$$
$$\Rightarrow \frac{\Lambda r_c^2}{3}\frac{1-\rho^2}{f_\Lambda(r_c\rho)} \to 1 \quad \text{in} \quad C^k([1/3,1]) \quad \text{as} \quad \Lambda \to 0.$$

To reduce clutter, we shall set

$$g_\Lambda(\rho) = 1 - \frac{\Lambda r_c^2}{3}\frac{1-\rho^2}{f_\Lambda(r_c\rho)}$$

and observe that

$$\left((1-\rho^2) - \frac{3f_\Lambda(r_c\rho)}{\Lambda r_c^2}\right)\partial_\rho^2(\chi u)$$
$$= g_\Lambda(\rho)\left(L^w(0)(\chi u) + \frac{3}{\Lambda r_c^2}(\Delta - \partial_\rho^2)(\chi u) - \frac{3}{\Lambda r_c^2\rho^2}\partial_\rho(\rho^2(1-f_\Lambda))\partial_\rho(\chi u) - (V^w+2)\chi u\right),$$

which yields the estimate

$$\|\left((1-\rho^2) - \frac{3f_\Lambda}{\Lambda r_c^2}\right)\partial_\rho^2(\chi u)\|_{H^k} \le C\|g_\Lambda\|_{C^k}\|u\|_{D^{k+1}}.$$

Note that similarly to before, this C depends only on k provided we already assume Λ is small enough. Since $g_\Lambda \to 0$ in C^k, there exists $\Lambda_0 > 0$ depending only on k such that for $\Lambda < \Lambda_0$,

$$\|\left((1-\rho^2) - \frac{3f_\Lambda}{\Lambda r_c^2}\right)\partial_\rho^2(\chi u)\|_{H^k} < \epsilon\|u\|_{D^{k+1}}.$$

Combining all these estimates, we have

$$\|K(s)u\|_{D^{k+1}\to H^k} < \epsilon.$$

□

We know that $(L^w(s)\circ(1-\chi) + L(s)\circ E_\chi)^{-1}$ has a pole of finite order at $-n$ for $n = 1, 2, \ldots k$ which is enough to establish the result, however we can do better and show that these poles are in fact simple.

Proposition 5.2.1. *Fix $k \in \mathbb{N}$. Then for each $n \in \{1, 2, \ldots k\}$, there exists $\Lambda_0 > 0$, $A_{-1}^w : H^k \to D^{k+1}$ a finite rank operator and $A_0^w(s)$ is a holomorphic family of Fredholm operators $H^k \to D^{k+1}$ such that for $\Lambda < \Lambda_0$,*

$$(L^w(s)\circ(1-\chi) + L(s)\circ E_\chi)^{-1} = \frac{A_{-1}^w}{s+n} + A_0^w(s)$$

in a suitable neighbourhood of $-n$.

Proof. First let us note that since the quasinormal modes of $A(s) := L^w(s) \circ (1-\chi) + L(s) \circ E_\chi$ are smooth, for a quasinormal mode u in the region $2/3 < \rho \leq 1$

$$A(s)u = 0 \Rightarrow L(s)u = 0.$$

Thus for $2/3 < \rho \leq 1$, the solutions match the modes we calculated for $L(s)$ previously. It also means that the dimension of the kernel of $A(-n)$ is n^2 and that the n^2 corresponding co-modes $\{\theta_i\}_{i=0}^{n^2}$ are the same distributions from earlier. If we set $\{u_i\}_{i=1}^{n^2}$ to be a basis for the kernel of $A(-n)$ and $\{v_i\}_{i=1}^{n^2}$ a corresponding basis for the kernel of $L(-n)$ i.e. such that $u_i|_{U_{2/3}} = v_i$, we note that

$$\theta_i(A'(-n)u_j) = \theta_i(A'(-n)v_j) = \theta_i(L'(-n)v_j)$$

since the θ_i are concentrated on the horizon where $A(s) = L(s)$. We have already seen that $L(s)^{-1}$ has simple poles and in a neighbourhood of $-n$ is of the form

$$L(s)^{-1} = \frac{\Pi_1}{s+n} + B(s)$$

where Π_1 is a linear combination of terms of the form $v_i\theta_j$ and $B(s)$ is holomorphic in a neighbourhood of $-n$. For $v \in \ker L(-n)$ and $s \neq -n$, we see

$$\begin{aligned}
v &= L(s)^{-1}L(s)v \\
&= \frac{\Pi_1 L(-n)v}{s+n} + \Pi_1 L'(-n)v + (s+n)\Pi_1 v + (s+n)B(s)(L'(-n) + s + n)v \\
&= \Pi_1 L'(-n)v + (s+n)\left(\Pi_1 v + B(s)(L'(-n) + s + n)\right) v
\end{aligned}$$

Taking the limit as $s \to -n$ gives that for all $v \in \ker L(-n)$,

$$v = \Pi_1 L'(-n)v$$

Hence it is possible to arrange the θ_i and v_i such that $\theta_i(L'(-n)v_j) = \delta_{ij}$. We relabel the bases of modes $\{u_i\}_{i=1}^{n^2}$ and co-modes $\{\theta_i\}_{i=1}^{n^2}$ of $A(s)$ such that $\theta_i(A'(-n)u_j) = \delta_{ij}$. Now let us suppose the pole is of order $N > 1$. In this case, we have

$$A(s)^{-1} = \sum_{j=1}^{N} \frac{A^w_{-j}}{(s+n)^j} + A^w_0(s).$$

where the A^w_{-j} are of finite rank, $A^w_{-N} \neq 0$ and $A^w_0(s)$ is holomorphic in a neighbourhood of $-n$. Recalling that $A(s)u = (s+n)^2 u + (s+n)A'(-n)u + A(-n)u$, we see that

$$\begin{aligned}
I_{D^{k+1}} &= A(s)^{-1}A(s) \\
&= \sum_{j=1}^{N} \frac{A^w_{-j}}{(s+n)^{j-2}} + \sum_{j=1}^{N} \frac{A^w_{-j}A'(-n)}{(s+n)^{j-1}} + \sum_{j=1}^{N} \frac{A^w_{-j}A(-n)}{(s+n)^{j}} + A^w_0(s)A(s), \\
I_{H^k} &= A(s)^{-1}A(s) \\
&= \sum_{j=1}^{N} \frac{A^w_{-j}}{(s+n)^{j-2}} + \sum_{j=1}^{N} \frac{A'(-n)A^w_{-j}}{(s+n)^{j-1}} + \sum_{j=1}^{N} \frac{A(-n)A^w_{-j}}{(s+n)^{j}} + A(s)A^w_0(s).
\end{aligned}$$

Since the above holds for all $s \neq -n$ in some neighbourhood of $-n$, it follows from equating the terms corresponding to $(s+n)^{-N}$ that the following equations hold:

$$A^w_{-N}A(-n) = A(-n)A^w_{-N} = 0$$

This implies that there exist constants c_{ij} such that

$$A^w_{-N} = \sum_{i,j=1}^{n^2} c_{ij}u_i\theta_j.$$

We also have

$$A^w_{-N}A'(-n) + A^w_{-N+1}A(-n) = 0$$

so for any $u \in \ker A(-n)$ (in particular u_m for $m = 1, 2, \dots n^2$),

$$A^w_{-N}A'(-n)u_m = \sum_{i,j=0}^{n^2} c_{ij}u_i\theta_j(A'(-n)u_m) = \sum_{i=1}^{n^2} c_{im}u_i = 0.$$

From the linear independence of the u_i, the above implies that $c_{ij} = 0$ for each $i, j = 1, 2, \dots n^2$ i.e. $A^w_{-N} = 0$, a contradiction. Hence the pole is simple. $\square$

With this in mind, we can proceed similarly to the previous chapter in the case of potentials. For each $n \in \mathbb{N}$, there exists $0 < r < 1/2$ such that $A^w_0(s)$ is bounded on the disc of radius r around $-n$ and we define the following constants

$$C^w_{k,n} = \sup_{s \in \overline{D(-n,r)}} \{\|L^w(0)A^w_0(s)\|_{H^k \to H^k}\}$$

and

$$\tilde{C}^w_{k,n} = \min\left\{\frac{1}{2\|L^w(0)A_{-1}\|_{H^k \to H^k}}, \frac{1}{C^w_{k,n}}\right\}.$$

Proposition 5.2.2. *For each $n \in \mathbb{N}$ and $0 < \delta < r(n,k)$, there exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$, there exists at least one quasinormal frequency s inside $D(-n\sqrt{\Lambda/3}, \delta\sqrt{\Lambda/3})$ i.e.*

$$\left| s + n\sqrt{\frac{\Lambda}{3}} \right| < \delta\sqrt{\frac{\Lambda}{3}}.$$

Proof. The proof is exactly analogous to the previous results. We see again that on the circle of radius δ around $-n$ that

$$\|A^w(\tilde{s})^{-1}(A^w(\tilde{s}) - B^w(\tilde{s}))\|_{H^k \to H^k} \leq \frac{\|K(\tilde{s})\|_{D^{k+1} \to H^k}}{\delta} \|L_0 A^w_{-1}\|_{H^k \to H^k} + C^w_{k,n}\|K(\tilde{s})\|_{D^{k+1} \to H^k},$$

where again $\tilde{s} = s\sqrt{3/\Lambda}$. We note that by Lemma 5.2.1, there exists $\Lambda_0 > 0$ such that for all $\Lambda < \Lambda_0$, $\|K(\tilde{s})\|_{D^{k+1} \to H^k} < \tilde{C}^w_{k,n}\delta$. Thus we have

$$\begin{aligned} \|A^w(\tilde{s})^{-1}(A^w(\tilde{s}) - B^w(\tilde{s}))\|_{H^k \to H^k} &< \tilde{C}^w_{k,n}\|L_0 A^w_{-1}\|_{H^k \to H^k} + \tilde{C}^w_{k,n} C^w_{k,n}\delta \\ &\leq \frac{1}{2} + \delta < 1, \end{aligned}$$

since $\delta < r \leq 1/2$. The conditions in Theorem 2.2.5 are met and so the number of quasinormal frequencies (counted with multiplicity) are unchanged. So there exists at least one s such that

$$\begin{aligned} &\left| s\sqrt{\frac{3}{\Lambda}} + n \right| < \delta \\ \Rightarrow &\left| s + n\sqrt{\frac{\Lambda}{3}} \right| < \delta\sqrt{\frac{\Lambda}{3}} < \frac{1}{2}\sqrt{\frac{\Lambda}{3}}. \end{aligned}$$

□

Theorem 5.2.1. *Let $w_\Lambda \in C^\infty(0,\infty)$ be a family of functions for $\Lambda \in [0, \Lambda_0]$ for some $\Lambda_0 > 0$ such that $1 + w_0$ has finitely many roots, the largest of which, r_e, is simple and such that $w_0 < 0$ for $r > r_e$. Suppose further that for each $k \in \mathbb{N}_0$ and $\Lambda \in [0, \Lambda_0]$,*

$$(r^k \partial_r^k w_\Lambda)(r) \to 0 \text{ as } r \to \infty$$

and let us suppose that there exists $t < r_e$ and β_k such that

$$\sup_{r \geq t} \left| r^k \partial_r^k (w_\Lambda - w_0)(r) \right| < \frac{\Lambda}{3}\beta_k$$

for all $k \in \mathbb{N}_0$. Let $\alpha : [0, \Lambda_0] \to \mathbb{R}$ be a continuous function and write $\alpha(\Lambda) = \alpha_\Lambda$. Let g be the metric defined using the above functions in the same way as at the start of this *section and $R[g]$ be its Ricci scalar. Then the equation*

$$-\Box_g \psi + \frac{R[g]}{6}\psi = 0$$

exhibits the phenomenon of zero-damped quasinormal frequencies converging to 0.

Proof. This is simply an application of Proposition 5.2.2, noting that for each Λ sufficiently small, there is a subset of the quasinormal spectrum $\{s_n\}_{n=1}^{\infty}$ such that

$$\left| s_n + n\sqrt{\frac{\Lambda}{3}} \right| < \frac{1}{2}\sqrt{\frac{\Lambda}{3}}.$$

□

It should be noted that the arguments above (combined with the results of the previous chapter) apply also to equations of the form $-\Box_g \psi + V_\Lambda \psi = 0$ provided V_Λ satisfies suitable conditions so it is can be made arbitrarily small near the cosmological horizon as $\Lambda \to 0$: in particular, the proof follows straightforwardly for the wave equation. Furthermore, the Regge-Wheeler equation for a Reissner-Nordström-de Sitter black hole is equivalent to

$$\left(-\Box_g + \frac{R[g]}{6} - \frac{8\,\mathfrak{m}}{r^3} + \frac{8\,\mathfrak{q}^2}{r^4} \right) \psi = 0, \tag{5.2.1}$$

where the potential does satisfy the appropriate decay conditions. Thus the quasinormal spectrum of the Reissner-Nordström-de Sitter black hole exhibits zero-damped modes in the Reissner-Nordström limit and we get a similar result for Schwarzschild simply by setting $\mathfrak{q} = 0$.

This method differs from that used in [76] since we keep $\mathfrak{m} > 0$ fixed: in the $\mathfrak{m} \to 0$ setting, the family of operators associated with the problem converge on the whole space to that of pure de Sitter (albeit with some singular behaviour near the origin) so convergence of the spectrum to $-n\sqrt{\Lambda/3}$ is possible. Our methods can only show the existence of this family of zero-damped modes in the spectrum of Schwarzschild-de Sitter since when $\mathfrak{m}$ is kept bounded away from zero, the family of operators diverges from the model problem away from the cosmological horizon. Another consequence of this is that we require the co-modes of the model problems we use to concentrate on the horizon in question.

5.3 Nearly extremal Reissner-Nordström-de Sitter

We expect to be able to prove that zero-damped modes exist for scalar fields minimally coupled to Reissner-Nordström-de Sitter from the results of [28] (in this paper, these are called near-extremal modes), so we shall use this example to present an application of the results obtained above. We consider the Reissner-Nordström-de Sitter metric

$$g_1 = -f_1(r)dt^2 + \frac{dr^2}{f_1(r)} + r^2 \not{g},$$

where

$$f_1(r) = 1 - \frac{2\,\mathfrak{m}}{r} + \frac{\mathfrak{q}^2}{r^2} - \frac{\Lambda}{3}r^2 = -\frac{\Lambda}{3}\frac{(r-r_0)(r-r_-)(r-r_+)(r_c-r)}{r^2}.$$

We can write f_1 in the second form provided $\mathfrak{m}, \mathfrak{q}$ and Λ obey suitable conditions so that f_1 has four real roots $r_0 < 0 < r_- < r_+ < r_c$, so we assume this is the case. This metric is defined on the manifold $\mathcal{M}_1 = [0,\infty) \times [r_+, r_c] \times S^2$. By Vieta's formulae, we have

$$r_0 + r_- + r_+ + r_c = 0, \tag{5.3.1}$$

$$-\frac{\Lambda}{3}(r_0 r_c + r_+ r_- + (r_0 + r_c)(r_+ + r_-)) = 1, \tag{5.3.2}$$

$$-\frac{\Lambda}{3}(r_0 r_c (r_+ + r_-) + r_+ r_- (r_c + r_0)) = 2\,\mathfrak{m}, \tag{5.3.3}$$

$$-\frac{\Lambda}{3} r_0 r_- r_+ r_c = \mathfrak{q}^2, \tag{5.3.4}$$

Note that these can be inverted, which allows us to write the roots as continuous functions of $\mathfrak{m}, \mathfrak{q}$ and Λ. Thus we can parametrise the spacetime using three of the roots (the fourth is determined by (5.3.1)) and consider the extremal limit using this picture. We define the quantities

$$\begin{aligned} \bar{r} &= \frac{r_+ + r_-}{2} = -\frac{r_0 + r_c}{2}, \\ h &= \frac{r_+ - r_-}{2}, \end{aligned}$$

and consider the limit where we keep $\bar{r}$ and r_c fixed and take $h \to 0$. Consider the coordinate transformation inspired by the one used in [22]

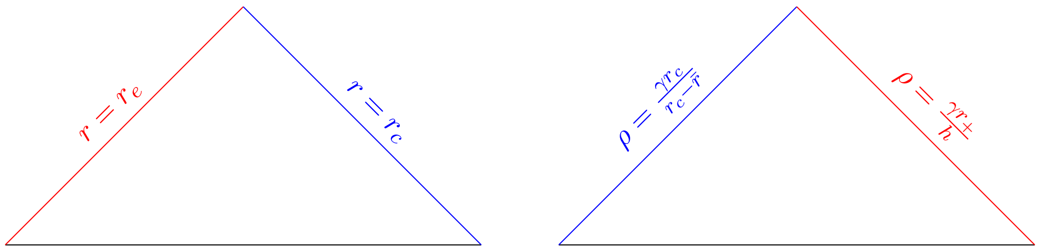
$$\rho = \frac{\gamma r}{r - \bar{r}}, \quad r = \frac{\bar{r}\rho}{\rho - \gamma}$$

for some $\gamma(h) > 0$ to be determined later. Under this transformation, we see that

$$r^2 = \frac{\bar{r}^2 \rho^2}{(\rho - \gamma)^2},$$

which motivates the introduction of a conformal factor

$$\Omega = \frac{\bar{r}}{\rho - \gamma}.$$



(a) The spacetime $(\mathcal{M}_1, g_1)$ (b) The spacetime $(\mathcal{M}_2, g_2)$

Figure 5.2: The original Reissner-Nordström-de Sitter spacetime is conformal to another spacetime where the horizons are swapped.

Noting that

$$f_1(\rho) = \frac{\Lambda}{3} \frac{\left((\bar{r} - r_0)\rho + \gamma r_0\right)\left(h\rho + \gamma r_-\right)\left(\gamma r_+ - h\rho\right)\left((r_c - \bar{r})\rho - \gamma r_c\right)}{\bar{r}^2 \rho^2 (\rho - \bar{r})^2},$$

we define

$$f_2(\rho) = \Omega^{-2} f_1(\rho).$$

A quick calculation yields

$$\frac{dr^2}{f_1(r)} = \frac{\Omega^2 d\rho^2}{f_2(\rho)}$$

and hence

$$g_1 = \Omega^2 \left(-f_2 dt^2 + \frac{d\rho^2}{f_2} + \rho^2 \not{g} \right) = \Omega^2 g_2.$$

The manifold is mapped to $\mathcal{M}_2 = [0, \infty) \times [\gamma r_c/(r_c - \bar{r}), \gamma r_+/h] \times S^2$ and we see that g_1 is conformal to the metric g_2. We stress here that $(\mathcal{M}_2, g_2)$ is not another Reissner-Nordström-de Sitter spacetime (in fact there is no guarantee g_2 obeys the vacuum Einstein equations with positive cosmological constant): the purpose of this transformation is to map the event horizon in $(\mathcal{M}_1, g_1)$ to the cosmological horizon that is present in $(\mathcal{M}_2, g_2)$ so we can apply the results we just obtained (see Figure 5.2). As before, we study the conformal Klein-Gordon equation on $(\mathcal{M}_1, g_1)$:

$$\left(-\Box_{g_1} + \frac{R[g_1]}{6} \right) \psi = 0.$$

We use a standard result to see that for a smooth function ψ, we have

$$\left(-\Box_{g_1} + \frac{R[g_1]}{6} \right) \psi = \Omega^{-3} \left(-\Box_{g_2} + \frac{R[g_2]}{6} \right) (\Omega \psi)$$

and hence it suffices to consider the quasinormal spectrum of the equation

$$\left(-\Box_{g_2} + \frac{R[g_2]}{6}\right)\varphi = 0$$

where $\varphi = \Omega\psi$.

Corollary 5.3.1. *Consider the conformal Klein-Gordon equation on $(\mathcal{M}_1, g_1)$ as defined above with $\mathfrak{m}, \mathfrak{q}, \Lambda$ chosen appropriately. This equation exhibits the phenomenon of zero-damped quasinormal frequencies in the limit as the event horizon becomes extremal.*

Proof. This is simply an application of Theorem 5.2.1 to this particular example. Bearing in mind the discussion above, it suffices to check that f_2 satisfies the conditions required to apply the result.

$$f_2(\rho) = -\frac{\Lambda}{3\bar{r}^4}\frac{h^2(r_c-\bar{r})(\bar{r}-r_0)}{\rho^2}\left(\rho + \frac{\gamma r_0}{\bar{r}-r_0}\right)\left(\rho - \frac{\gamma r_c}{r_c-\bar{r}}\right)\left(\rho + \frac{\gamma r_-}{h}\right)\left(\rho - \frac{\gamma r_+}{h}\right).$$

We define a new 'cosmological constant' λ by

$$\lambda = \frac{\Lambda}{\bar{r}^4}(r_c-\bar{r})(\bar{r}-r_0)h^2$$

and we find that f_2 takes the form:

$$f_2(\rho) = -\frac{\lambda}{3}\rho^2 + F_1\rho + F_0 + \frac{F_{-1}}{\rho} + \frac{F_{-2}}{\rho^2}.$$

We again use Vieta's formulae and the relations between the roots, $\bar{r}$ and h to see

$$\begin{aligned}
F_1 &= \frac{4\lambda}{3}\gamma\left(r_c^2 + 2r_c\bar{r} - 2\bar{r}^2\right),\\
F_0 &= \gamma^2\left(\frac{\Lambda}{3}\frac{(r_c-\bar{r})(r_c+3\bar{r})}{\bar{r}^2} + \frac{\lambda}{3}\frac{7\bar{r}^2 - 6\bar{r}r_c - 6r_c^2}{(r_c-\bar{r})(r_c+3\bar{r})}\right) = \gamma^2(a_0 + \lambda b_0),\\
F_{-1} &= 2\gamma^3\left(\frac{\Lambda}{3}\frac{(\bar{r}^2 - 2\bar{r}r_c - r_c^2)}{\bar{r}^2} - \frac{\lambda}{3}\frac{\bar{r}^2 - 4\bar{r}r_c - 2r_c^2}{(r_c-\bar{r})(r_c+3\bar{r})}\right) = 2\gamma^3(a_1 + \lambda b_1),\\
F_{-2} &= \gamma^4\left(\frac{\Lambda}{3}\frac{r_c(r_c+2\bar{r})}{\bar{r}^2} - \frac{\lambda}{3}\frac{r_c(r_c+2\bar{r})}{(r_c-\bar{r})(\bar{r}-r_0)}\right) = \gamma^4(a_2 + \lambda b_2).
\end{aligned}$$

At this stage we are in a position to make a selection for γ

$$\gamma = \frac{1}{\sqrt{a_0 + \lambda b_0}}$$

for $\lambda \in [0, \lambda_0]$ where $\lambda_0 < -b_0/a_0$ is taken sufficiently small. We define

$$\alpha_\lambda = \frac{4(r_c^2 + 2r_c\bar{r} - 2\bar{r}^2)}{\sqrt{a_0 + \lambda b_0}}$$

and note that it is a continuous function of λ and converges to a well-defined, finite limit as $\lambda \to 0$. Thus we have

$$\begin{aligned} f_2(\rho) &= 1 + \frac{\lambda}{3}\alpha_\lambda \rho - \frac{\lambda}{3}\rho^2 + \frac{2a_1 + 2\lambda b_1}{(a_0+\lambda b_0)^{3/2}}\frac{1}{\rho} + \frac{a_2+\lambda b_2}{(a_0+\lambda b_0)^2}\frac{1}{\rho^2} \\ &= 1 + \frac{\lambda}{3}\alpha_\lambda \rho - \frac{\lambda}{3}\rho^2 + w_\lambda(\rho). \end{aligned}$$

Now we check that w_λ satisfies the conditions defined above. Some of these properties are clear: since w_λ is a polynomial in $1/\rho$, naturally $w_\Lambda \in C^\infty(0,\infty)$ and there exists $\lambda_0 > 0$ such that for each $k \in \mathbb{N}_0$ and $\lambda \in [0, \lambda_0]$,

$$\rho^k(\partial_\rho^k w_\lambda)(\rho) \to 0 \quad \text{as} \quad \rho \to \infty.$$

Furthermore it is clear that $1 + w_0$ has finitely many zeroes in $(0,\infty)$ (it is a quadratic so it will have 0, 1 or 2), but we need to check that it has at least one. Since $r_c > \bar{r}$, we see that $a_1 < 0$ and $a_2 > 0$ and hence any real roots of

$$\begin{aligned} 1 + w_0(\rho) &= 1 + \frac{2a_1/a_0^{3/2}}{\rho} + \frac{a_2/a_0^2}{\rho^2} \\ &= \frac{\rho^2 + 2a_1/a_0^{3/2}\rho + a_2/a_0^2}{\rho^2} \\ &= \frac{\rho^2 - 2\mu\rho + \nu}{\rho^2} \end{aligned}$$

will be positive. So it suffices to consider the discriminant of the quadratic,

$$\text{Disc} = \frac{4}{a_0^3}(a_1^2 - a_0 a_2).$$

Since we are only interested in the sign of this quantity, we can multiply out positive factors (such as $\Lambda/(9\bar{r}^4)$) and consider

$$\begin{aligned} \text{Disc}' &= (\bar{r}^2 - 2\bar{r}r_c - r_c^2)^2 - r_c(r_c + 2\bar{r})(r_c - \bar{r})(r_c + 3\bar{r}) \\ &= \bar{r}^4 + \bar{r}^2 r_c^2 + 2\bar{r}^3 r_c \\ &= \bar{r}^2(\bar{r} + r_c)^2 > 0. \end{aligned}$$

Hence $1 + w_0$ has two positive roots in $(0,\infty)$ and furthermore these are simple. The largest root is $\rho_e = \mu + \sqrt{\mu^2 - \nu}$. Plugging this in to our expression for w_0, we see

$$\begin{aligned} w_0(\rho) &= -\frac{1}{\rho^2}(2\mu\rho - \nu) \\ &= -\frac{1}{\rho^2}\left(2\mu^2 + 2\mu\sqrt{\mu^2 - \nu} - \nu + 2\mu(\rho - \rho_e)\right) \\ &= -\frac{1}{\rho^2}\left(\rho_e^2 + 2\mu(\rho - \rho_e)\right), \end{aligned}$$

which is negative for $\rho > \rho_e$. Now we check the final condition:

$$\left|\rho^k \partial_\rho^k (w_\lambda - w_0)(\rho)\right| \leq \frac{k!\,|F_{-1}(\lambda) - F_{-1}(0)|}{\rho^{k+1}} + \frac{(k+1)!\,|F_{-2}(\lambda) - F_{-2}(0)|}{\rho^{k+2}}.$$

In a sufficiently small neighbourhood of $\lambda = 0$, the F_i are continuously differentiable with bounded derivatives and hence Lipschitz. Thus, there exists a constant $C > 0$ such that

$$|F_j(\lambda) - F_j(0)| \leq \frac{\lambda}{3} C$$

for $j = -1, -2$. After fixing some $0 < t < \rho_e$ defined above, we see that

$$\begin{aligned}\sup_{\rho \geq t} \left|\rho^k \partial_\rho^k (w_\lambda - w_0)(\rho)\right| &\leq \frac{k!\,|F_{-1}(\lambda) - F_{-1}(0)|}{t^{k+1}} + \frac{(k+1)!\,|F_{-2}(\lambda) - F_{-2}(0)|}{t^{k+2}} \\ &\leq \frac{\lambda}{3}\beta_k\end{aligned}$$

for some $\beta_k > 0$. $\square$

Returning to the Regge-Wheeler equation (Equation (5.2.1)), our manipulations leave us with:

$$\left(-\Box_{g_2} + \frac{R[g_2]}{6}\right)\varphi + \frac{8}{\bar{r}^2\rho^2}\left[\mathfrak{q}^2\left(1 - \frac{\gamma}{\rho}\right)^2 - \mathfrak{m}\,\bar{r}\left(1 - \frac{\gamma}{\rho}\right)\right]\varphi = 0.$$

To apply the extension of Theorem 5.2.1 to include potentials, it suffices to check if the potential above satisfies the necessary decay condition: in fact it does up to the term leading in $1/\rho$:

$$\frac{8}{\bar{r}^2\rho^2}(\mathfrak{q}^2 - \mathfrak{m}\,\bar{r}).$$

However, in the extremal Reissner-Nordström limit, we see that $\bar{r} \to \mathfrak{m}$ which we can use to make this term small and obtain existence of zero-damped modes associated with the event horizon in this limit. We will see in the next chapter that a similar limit must be considered for event horizon modes in Kerr-de Sitter.

Chapter 6

The Kerr-de Sitter spacetime

In the previous chapter, we proved the existence of zero-damped modes for a class of static, spherically symmetric spacetimes. The natural next step is to relax some of the symmetry assumptions we have placed on the spacetime, in particular spherical symmetry. In this chapter, we shall consider the Kerr-de Sitter spacetime, a stationary solution to the positive cosmological constant vacuum Einstein equations which models a rotating black hole. As a consequence, spherical symmetry is broken, however we still have some additional symmetries to exploit (namely axisymmetry is still and an additional Killing tensor symmetry which enables us to separate variables using Carter's constant) and make the analysis more tractable.

The bulk of this chapter is devoted to providing the set-up for the proof of its main result, Theorem 6.3.1, which establishes that one can find finitely many quasinormal frequencies approximating zero-damped modes provided the black hole parameters are sufficiently close to extremal Kerr. This falls short of establishing the existence of zero-damped modes since we do not obtain an infinite family of frequencies as the event and Cauchy horizons coalesce: as we introduce more frequencies, the black hole parameters need to be closer to extremal Kerr for the result to hold. Thus to get a family of zero-damped modes, we are forced to take the extremal Kerr limit which is the content of Corollary 6.3.1.

The mathematical study of the quasinormal modes of the Kerr-de Sitter spacetime was initiated by Dyatlov in [55]. In this paper, Dyatlov gives a definition of quasinormal modes for slowly rotating spacetimes as poles of the resolvent operator (the resonance approach in Chapter 2). He constructs the full resolvent from the separated one-dimensional resolvent operators for radial and angular equations and uses discreteness of the set of frequencies to prove decay of linear waves orthogonal to the zero resonance. These ideas were later applied in [54] to obtain a generalisation of [125] for Kerr-de Sitter spacetimes and compared to the regularity approach of Vasy in Appendix A of [137].

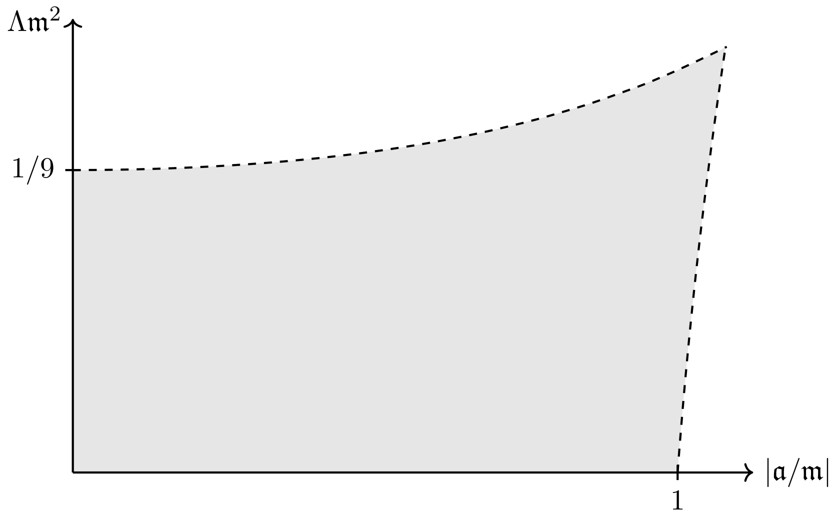


Figure 6.1: The range of parameters for subextremal Kerr-de Sitter spacetimes.

More recently, the low frequency spectrum has been studied by Hintz in [73], where the mode stability of the Kerr-de Sitter spacetime for a subset of the subextremal parameter range was established, complementing other work [31]. The proof uses microlocal methods and an adaptation of Melrose's b-analysis to this situation (q- and Q-analysis) to prove that in the limit $\Lambda\,\mathfrak{m}^2 \to 0$, the frequencies converge (in a suitable sense) to the set $-n\sqrt{\Lambda/3}$ i.e. they are zero-damped modes. In this chapter, we shall use different methods to [73] to prove the existence of zero-damped modes. In particular, we approach extremal Kerr from rapidly rotating Kerr-de Sitter spacetimes and as a result pick up the frequencies associated with the event horizon (as opposed to the cosmological horizon modes of [73]).

6.1 The Kerr-de Sitter spacetime

Consider the quartic polynomial

$$\mu(r) = (r^2 + \mathfrak{a}^2)\left(1 - \frac{\Lambda}{3}r^2\right) - 2\mathfrak{m}r \tag{6.1.1}$$

with real parameters $\Lambda, \mathfrak{m} > 0$ and $\mathfrak{a}$. If we impose the condition

$$-\left(1 + \frac{\Lambda}{3}\,\mathfrak{a}^2\right)^4 \frac{\mathfrak{a}^2}{\mathfrak{m}^2} + 12\Lambda\mathfrak{a}^2\left(1 - \frac{\Lambda}{3}\,\mathfrak{a}^2\right) + \left(1 - \frac{\Lambda}{3}\,\mathfrak{a}^2\right)^3 - 9\Lambda\mathfrak{m}^2 > 0,$$

the quartic has four distinct real roots: $r_0 < 0 < r_- < r_+ < r_c$. The parameter space obeying this condition is depicted by the shaded region in Figure 6.1. Given μ, we can construct a solution to the Einstein equations with cosmological constant Λ by endowing the manifold

$$\mathbb{R} \times (r_+, r_c) \times S^2$$

with the Lorentzian metric (defined up to coordinate singularities due to the sphere)

$$g = \varrho^2 \left(\frac{dr^2}{\mu} + \frac{d\theta^2}{c} \right) + \frac{c \sin^2 \theta}{b^2 \varrho^2} (\mathfrak{a}\, dt - (r^2 + \mathfrak{a}^2) d\varphi)^2 - \frac{\mu}{b^2 \varrho^2} (dt - \mathfrak{a} \sin^2 \theta d\varphi)^2 \tag{6.1.2}$$

where

$$\begin{aligned} \varrho^2 &= r^2 + \mathfrak{a}^2 \cos^2 \theta \\ b &= 1 + \frac{\Lambda}{3} \mathfrak{a}^2 \\ c &= 1 + \frac{\Lambda}{3} \mathfrak{a}^2 \cos^2 \theta. \end{aligned}$$

This is the domain of outer communication for a subextremal Kerr-de Sitter spacetime in Boyer-Lindquist coordinates. The metric has singularities at the surfaces corresponding to the roots of μ; r_- corresponds to a Cauchy horizon, r_+ to an event horizon and r_c to a cosmological horizon. These are coordinate singularities and can be removed using the transformation (see for example [114, 116])

$$\begin{aligned} t_* &= t - T(r), \\ \varphi_* &= \varphi - \Phi(r), \end{aligned}$$

where

$$\begin{aligned} T'(r) &= \frac{bF(r)}{\mu} (r^2 + \mathfrak{a}^2), \\ \Phi'(r) &= \frac{bF(r)}{\mu} \mathfrak{a}, \end{aligned}$$

for some $F \in C^\infty(\mathbb{R}_{\geq 0})$ such that $F(r_c) = -F(r_+) = 1$. We additionally need to impose the condition

$$F(r)^2 \leq 1 - \frac{a^2 \mu(r)}{(r^2 + \mathfrak{a}^2)^2}$$

to ensure that the constant t_* hypersurfaces are spacelike or null. To this end, we choose some $\eta \in C^\infty(\mathbb{R}_{\geq 0})$ such that

$$\eta(r) = \begin{cases} -1 & r < 2r_+ - \bar{r}, \\ 1 & r > r_c - r_+ + \bar{r}. \end{cases}$$

and set

$$F(r) = \eta(r)\sqrt{1 - \frac{a^2\mu(r)}{(r^2 + \mathfrak{a}^2)^2}}$$

This transforms the metric to

$$\begin{aligned} g = (1 - F^2)\frac{\varrho^2}{\mu}dr^2 + \frac{\varrho^2}{c}d\theta^2 + \frac{c\sin^2\theta}{b^2\varrho^2}\left(\mathfrak{a}\,dt_* - (r^2 + \mathfrak{a}^2)d\varphi_*\right)^2 \\ - \frac{2F}{b}\left(dt_* - \mathfrak{a}\sin^2\theta d\varphi_*\right)dr - \frac{\mu}{b^2\varrho^2}\left(dt_* - \mathfrak{a}\sin^2\theta d\varphi_*\right)^2 \end{aligned}$$

and the dual metric to

$$\begin{aligned} \varrho^2 G = \mu\partial_r^2 + c\partial_\theta^2 - 2bF\left((r^2 + a^2)\partial_{t_*} + \mathfrak{a}\,\partial_{\varphi_*}\right)\partial_r + \frac{b^2}{c\sin^2\theta}\left(\mathfrak{a}\sin^2\theta\partial_{t_*} + \partial_{\varphi_*}\right)^2 \\ + \frac{b^2}{\mu}(F^2 - 1)\left((r^2 + \mathfrak{a}^2)\partial_{t_*} + \mathfrak{a}\,\partial_{\varphi_*}\right)^2. \end{aligned}$$

These can now be smoothly extended (again, up to the spherical singularities) to the spacetime manifold

$$\mathbb{R} \times (r_-, \infty) \times S^2.$$

This solution now covers the black hole interior (up to the Cauchy horizon) and outside the cosmological horizon. Recalling the parameter space depicted in Figure 6.1, approaching the edges of the shaded region (barring the vertical axis) corresponds to some 'extremal' limit where the surface gravity of some horizon or pair of horizons goes to zero: the upper line represents the rotating Nariai limit where the cosmological and event horizons coalesce (whose frequencies are studied in [111]), the horizontal axis represents the Kerr limit where the cosmological horizon becomes an asymptotically flat end (studied in [73]) and the right-most line corresponds to a rapidly rotating black hole where the Cauchy and event horizons coalesce.

To get a suitable Fredholm set-up to serve as a framework for studying the quasinormal frequencies on this spacetime, we restrict to the subextremal range of parameters and make use of a further coordinate transformation so we can define quasinormal modes with respect to a particular stationary Killing vector field. We make use the results of [115], a generalisation of [116], and change coordinates so that $\partial_\tau = b\partial_{t_*} + \Omega_+\partial_{\phi_*}$ where $\Omega_+ = \mathfrak{a}\,b/(r_+^2 + \mathfrak{a}^2)$, i.e.

$$\begin{aligned} \tau &= \frac{t_*}{b}, \\ \phi &= \varphi_* - \frac{\mathfrak{a}}{r_+^2 + \mathfrak{a}^2}t_*. \end{aligned}$$

Note that our definition of Ω_+ differs from the usual definition by a factor of b.

6.1.1 The extremal limit

First, we fix values $\Lambda, \mathfrak{m}, \mathfrak{a}_0$ to yield a rapidly-rotating, extremal Kerr-de Sitter spacetime. In our choice of coordinates, the extremal horizon is located at some $r = \bar{r}$ and hence the polynomial μ_0 can be written

$$\mu_0(r) = \frac{1}{2}V^2(r-\bar{r})^2 - \frac{4\Lambda\bar{r}}{3}(r-\bar{r})^3 - \frac{\Lambda}{3}(r-\bar{r})^4,$$

where $V^2 = \mu_0''(\bar{r}) > 0$ and $V > 0$. In fact, using the conditions $\mu_0(\bar{r}) = \mu_0'(\bar{r}) = 0$, we can find expressions for $\bar{r}$ and V^2 in terms of Λ and $\mathfrak{a}_0$:

$$\bar{r}^2 = \frac{1}{2\Lambda}\left(1 - \frac{\Lambda}{3}\mathfrak{a}_0^2 - \sqrt{\left(1-\frac{\Lambda}{3}\mathfrak{a}_0^2\right)^2 - 4\Lambda\,\mathfrak{a}_0^2}\right)$$
$$V^2 = 2\sqrt{\left(1-\frac{\Lambda}{3}\mathfrak{a}_0^2\right)^2 - 4\Lambda\,\mathfrak{a}_0^2}$$

From the allowed range of parameters for our spacetime, we note that $\epsilon_0 = \Lambda\,\mathfrak{a}_0^2/3 < 7-4\sqrt{3} < 1/13$ is a dimensionless small parameter. To further work with dimensionless parameters, we define the following constants:

$$\lambda = \Lambda\,\mathfrak{m}^2, \quad \hat{\mathfrak{a}}_0 = \frac{\mathfrak{a}_0}{\mathfrak{m}}, \quad \bar{\rho} = \frac{\bar{r}}{\mathfrak{m}}, \quad \rho_\bullet = \frac{r_\bullet}{\mathfrak{m}}, \qquad \omega_+ = \frac{\Omega_+}{\mathfrak{m}},$$

where $\bullet \in \{0, -, +, c\}$. We also make use of the coordinate transformation $\rho = r/\,\mathfrak{m}$. With respect to these parameters we see that

$$\begin{aligned}\mu_0(r) &= \mathfrak{m}^2\left(\frac{1}{2}V^2(\rho-\bar{\rho})^2 - \frac{4\lambda\bar{\rho}}{3}(\rho-\bar{\rho})^3 - \frac{\lambda}{3}(\rho-\bar{\rho})^4\right)\\ &= \mathfrak{m}^2\,\hat{\mu}_0(\rho),\end{aligned}$$

which we use to define the polynomial $\hat{\mu}_0(\rho)$. We consider the family of spacetimes corresponding to

$$\hat{\mu}_{\hat{\kappa}}(\rho) = -\frac{2\left(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2\right)^2}{V^2}\hat{\kappa}^2 + \frac{1}{2}V^2(\rho-\bar{\rho})^2 - \frac{4\lambda\bar{\rho}}{3}(\rho-\bar{\rho})^3 - \frac{\lambda}{3}(\rho-\bar{\rho})^4,$$

which are subextremal when $0 < \hat{\kappa} < \hat{\kappa}_{\max}$ for some $\hat{\kappa}_{\max}$ sufficiently small. By comparing coefficients with μ_0, we see that λ is unchanged and the rescaled rotation parameter $\hat{\mathfrak{a}} = \mathfrak{a}\,/\,\mathfrak{m}$ satisfies

$$\hat{\mathfrak{a}}^2 = \hat{\mathfrak{a}}_0^2 - \frac{2\left(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2\right)^2}{V^2}\hat{\kappa}^2,$$

where we take the sign of $\hat{\mathfrak{a}}$ to be the same as that of $\hat{\mathfrak{a}}_0$. The effect of this with respect to the diagram of the parameter space in Figure 6.1 is to shift a point on the rightmost boundary to the left. Therefore it is straightforward to see that taking the limit as $\hat{\kappa} \to 0$ can be interpreted spinning the black hole up to extremality. The name of the parameter $\hat{\kappa}$ is suggestive: in fact, it is a first-order approximation of the surface gravity of the event horizon κ scaled up by $\mathfrak{m}$ to make it dimensionless. This can be seen by considering the exact expression for this rescaled surface gravity:

$$\mathfrak{m}\,\kappa = \frac{1}{2}\frac{\hat{\mu}'_{\hat{\kappa}}(\rho_+)}{\rho_+^2 + \hat{\mathfrak{a}}^2}. \tag{6.1.3}$$

To proceed we must approximate $x_+ = \rho_+ - \bar{\rho}$, which is the smallest positive root of the equation

$$\frac{1}{2}V^2 x^2 - \frac{4\lambda\bar{\rho}}{3}x^3 - \frac{\lambda}{3}x^4 = \frac{2\left(\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2\right)^2}{V^2}\hat{\kappa}^2.$$

For $\hat{\kappa}$ sufficiently small, we have:

$$\rho_+ = \bar{\rho} + \frac{2(\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2)}{V^2}\hat{\kappa} + O(\hat{\kappa}^2) \tag{6.1.4}$$

and plugging this in to Equation (6.1.3), we see

$$\begin{aligned}\mathfrak{m}\,\kappa &= \frac{1}{2}\frac{V^2}{\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2 + O(\hat{\kappa})}\frac{2\left(\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2\right)}{V^2}\hat{\kappa} + O(\hat{\kappa}^2) \\ &= \hat{\kappa} + O(\hat{\kappa}^2).\end{aligned}$$

6.1.2 The wave equation

We shall study the wave equation on the family of backgrounds with the set-up above:

$$\Box_g \Psi = 0. \tag{6.1.5}$$

In a similar manner to [92], we see that to find the quasinormal frequencies of this equation, it suffices to consider the following family of operators on the spatial slices:

$$P(s)\hat{\Psi} = e^{-s\tau}\varrho^2\Box_g(e^{s\tau}\hat{\Psi})$$

which is the Laplace transform in τ of $\varrho^2\Box_g$ (with zero initial conditions). We shall exploit the symmetries of the system by separating variables, beginning with axisymmetry:

we restrict to solutions of the form $\hat{\Psi} = e^{im\phi}\psi$. We are left with the equation

$$\begin{aligned}
P_m(s)\psi &= e^{-im\phi}P(s)(e^{im\phi}\psi) \\
&= \partial_\rho(\hat{\mu}\partial_\rho\psi) - 2F\left(\hat{\kappa}s'(\rho^2+\hat{\mathfrak{a}}^2) + im\omega_+(\rho_+^2-\rho^2)\right)\partial_\rho\psi \\
&\quad - \partial_\rho(\rho^2F)(\hat{\kappa}s' - im\omega_+)\psi - (\hat{\mathfrak{a}}^2\hat{\kappa}s' + im\omega_+\rho_+^2)\partial_\rho F\psi \\
&\quad + \frac{F^2-1}{\hat{\mu}}\left[\hat{\kappa}s'(\rho^2+\hat{\mathfrak{a}}^2) + im\omega_+(\rho_+^2-\rho^2)\right]^2\psi \\
&\quad + \frac{\partial_\theta\left(c\sin\theta\partial_\theta\psi\right)}{\sin\theta} + \frac{1}{c\sin^2\theta}\left[(\hat{\kappa}s' - im\omega_+)\hat{\mathfrak{a}}\sin^2\theta + imb\right]^2\psi,
\end{aligned}$$

where we have written $\hat{\mu} = \hat{\mu}_{\hat{\kappa}}(\rho)$ for shorthand and $s' = s\,\mathfrak{m}\,/\hat{\kappa}$ is the rescaled frequency. Now we can separate the ρ and θ variables by writing

$$P_m(s)\psi = P_m^\rho(s')\psi + P_m^\theta(s')\psi$$

where

$$\begin{aligned}
P_m^\rho(s')\psi &= \partial_\rho(\hat{\mu}\partial_\rho\psi) - 2F\left(\hat{\kappa}s'(\rho^2+\hat{\mathfrak{a}}^2) + im\omega_+(\rho_+^2-\rho^2)\right)\partial_\rho\psi \\
&\quad - \partial_\rho(\rho^2F)(\hat{\kappa}s' - im\omega_+)\psi - (\hat{\mathfrak{a}}^2\hat{\kappa}s' + im\omega_+\rho_+^2)\partial_\rho F\psi \\
&\quad + \frac{F^2-1}{\hat{\mu}}\left[\hat{\kappa}s'(\rho^2+\hat{\mathfrak{a}}^2) + im\omega_+(\rho_+^2-\rho^2)\right]^2\psi \\
P_m^\theta(s')\psi &= \frac{\partial_\theta\left(c\sin\theta\partial_\theta\psi\right)}{\sin\theta} + \frac{1}{c\sin^2\theta}\left[(\hat{\kappa}s' - im\omega_+)\hat{\mathfrak{a}}\sin^2\theta + imb\right]^2\psi
\end{aligned}$$

Since we are interested in the behaviour near the event horizon, we consider the behaviour of F for $\bar{\rho} < \rho < 2\rho_+ - \bar{\rho}$:

$$\begin{aligned}
F(\rho) &= -\sqrt{1 - \frac{\hat{\mathfrak{a}}^2\hat{\mu}(\rho)}{(\rho^2+\hat{\mathfrak{a}}^2)^2}} \\
&= -1 + \frac{\hat{\mathfrak{a}}^2\hat{\mu}(\rho)}{(\rho^2+\hat{\mathfrak{a}}^2)^2} + \sum_{k=2}^{\infty} F_k\left(\frac{\hat{\mathfrak{a}}^2\hat{\mu}(\rho)}{(\rho^2+\hat{\mathfrak{a}}^2)^2}\right)^k \qquad (6.1.6) \\
\Rightarrow \partial_\rho F(\rho) &= \frac{\hat{\mathfrak{a}}^2}{(\rho^2+\hat{\mathfrak{a}}^2)^2}\left[\partial_\rho\hat{\mu}(\rho) - \frac{4\rho\hat{\mu}(\rho)}{\rho^2+\hat{\mathfrak{a}}^2}\right]\left[1 + \sum_{k=2}^{\infty} kF_k\left(\frac{\hat{\mathfrak{a}}^2\hat{\mu}(\rho)}{(\rho^2+\hat{\mathfrak{a}}^2)^2}\right)^{k-1}\right]. \qquad (6.1.7)
\end{aligned}$$

For the range of ρ the above hold, we have $\rho = \bar{\rho} + O(\hat{\kappa})$. In particular, this means that $\hat{\mu}(\rho) = O(\hat{\kappa}^2)$ and $\partial_\rho\hat{\mu}(\rho) = O(\hat{\kappa})$, so $F(\rho) = -1 + O(\hat{\kappa}^2)$ and $\partial_\rho F(\rho) = O(\hat{\kappa})$. To be more precise, we write $\rho = \bar{\rho} + ux_+$ where $u \in (0,2)$ and substitute into Equation (6.1.7):

$$\begin{aligned}
\partial_\rho F(\rho) &= \frac{\hat{\mathfrak{a}}_0^2}{(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2}(1+O(\hat{\kappa}))\left[V^2ux_+ + O(\hat{\kappa}^2)\right]\left[1+O(\hat{\kappa})\right] \\
&= \frac{\hat{\mathfrak{a}}_0^2}{\bar{\rho}^2+\hat{\mathfrak{a}}_0^2}\hat{\kappa}u + O(\kappa^2),
\end{aligned}$$

recalling that

$$\begin{aligned}x_+ &= \rho_+ - \bar{\rho}\\ &= \frac{2(\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2)}{V^2}\hat{\kappa} + O(\hat{\kappa}^2).\end{aligned}$$

Hence, in this neighbourhood of $\rho = \rho_+$, the radial operator can be written:

$$\begin{aligned}P_m^\rho(s')\psi =&\partial_\rho(\hat{\mu}\partial_\rho\psi) + 2\left(\hat{\kappa}s'(\rho^2 + \hat{\mathfrak{a}}^2) + im\omega_+(\rho_+^2 - \rho^2)\right)\partial_\rho\psi\\ &+ 2\bar{\rho}(\hat{\kappa}s' - im\omega_+)\psi + \hat{\kappa}(\hat{\mathfrak{a}}s')^2\psi + O(\hat{\kappa}).\end{aligned} \tag{6.1.8}$$

We further make a conformal coordinate change:

$$R = \frac{\rho_+ - \bar{\rho}}{\rho - \bar{\rho}},$$

which fixes the event horizon to $R = 1$ and sends the Cauchy horizon to

$$R_- = \frac{\rho_+ - \bar{\rho}}{\rho_- - \bar{\rho}} = -1 + O(k\hat{appa}).$$

In fact, we see that Equations (6.1.6) and (6.1.7) hold for $R > 1/2$, so we can make use of the simpler form of Equation (6.1.8) when studying the equation in this region. The cosmological horizon is mapped to $R = R_c$ where $R_c = O(\hat{\kappa})$ and thus gets closer to $R = 0$ in the extremal limit $\hat{\kappa} \to 0$. The radial derivative operators are now defined on $I_{\hat{\kappa}} := [R_c, 1]$ and take the following form in these coordinates:

$$\begin{aligned}\partial_\rho &= -\frac{R^2}{x_+}\partial_R\\ \partial_\rho^2 &= \frac{1}{x_+^2}\left(R^4\partial_R^2 + 2R^3\partial_R\right)\end{aligned}$$

We also have

$$\begin{aligned}\hat{\mu} &= -\frac{2(\bar{\rho}^2 + \hat{\mathfrak{a}}_0^2)^2}{V^2}\hat{\kappa}^2 + \frac{1}{2}V^2\frac{x_+^2}{R^2} - \frac{4\lambda\bar{\rho}}{3}\frac{x_+^3}{R^3} - \frac{\lambda}{3}\frac{x_+^4}{R^4},\\ \partial_\rho\hat{\mu} &= \frac{x_+V^2}{R} - \frac{4\lambda\bar{\rho}x_+^2}{R^2} - \frac{4\lambda}{3}\frac{x_+^3}{R^3},\end{aligned}$$

which implies (when restricting to $R > 1/2$)

$$\begin{aligned}
\partial_\rho\left(\hat{\mu}\partial_\rho u\right) =& \left(-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}R^4+\frac{1}{2}V^2R^2-\frac{4\lambda\bar{\rho}x_+}{3}R-\frac{\lambda}{3}x_+^2\right)\partial_R^2 u\\
&+\left(-\frac{4(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}R^3+V^2R-\frac{8\lambda\bar{\rho}x_+}{3}-\frac{2\lambda}{3}\frac{x_+^2}{R}\right)\partial_R u\\
&+\left(-V^2R+4\lambda\bar{\rho}x_++\frac{4\lambda}{3}\frac{x_+^2}{R}\right)\partial_R u\\
=&\left(-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}R^4+\frac{1}{2}V^2R^2-\frac{4\lambda\bar{\rho}}{3}x_+R-\frac{\lambda}{3}x_+^2\right)\partial_R^2 u\\
&+\left(-V^2R^3+\frac{4\lambda\bar{\rho}}{3}x_++\frac{2\lambda}{3}\frac{x_+^2}{R}\right)\partial_R u+O(\hat{\kappa})\\
=&\frac{1}{2}V^2R^2\partial_R\left((1-R^2)\partial_R u\right)+O(\hat{\kappa}),
\end{aligned}$$

We shall continue to restrict to the region $R > 1/2$ and note that the remaining first order term takes the form:

$$\begin{aligned}
\left((\rho^2+\hat{\mathfrak{a}}^2)\hat{\kappa}s'+(\rho_+^2-\rho^2)im\omega_+\right)\partial_\rho u =& -\frac{\hat{\kappa}}{x_+}\left((\bar{\rho}^2+\hat{\mathfrak{a}}^2)R^2+2x_+\bar{\rho}R+x_+^2\right)s'\partial_R u\\
&+im\omega_+(1-R)\left((2\bar{\rho}+x_+)R+x_+\right)\partial_R u.
\end{aligned}$$

Using Equation (6.1.4), we can simplify this expression to

$$\left((\rho^2+\hat{\mathfrak{a}}^2)\hat{\kappa}s'+(\rho_+^2-\rho^2)im\omega_+\right)\partial_\rho u = -\left(\frac{1}{2}V^2R^2s'-2im\bar{\omega}\bar{\rho}R(1-R)\right)\partial_R u+O(\hat{\kappa}),$$

where $\bar{\omega} = \hat{\mathfrak{a}}_0 b/(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)$. Hence for $R > 1/2$, the pointwise limit of the radial operator as $\hat{\kappa} \to 0$ is

$$\frac{1}{2}V^2R^2\partial_R((1-R^2)\partial_R u)-2\left(\frac{1}{2}V^2R^2s'-2im\bar{\omega}\bar{\rho}R(1-R)\right)\partial_R u-2im\bar{\omega}\bar{\rho}u,$$

which implies that a good model equation for our problem involves the radial operator

$$(1-R^2)\partial_R^2 u-2\left(s'+R-\frac{4im\bar{\omega}\bar{\rho}}{V^2}\frac{1-R}{R}\right)\partial_R u-\frac{4im\bar{\omega}\bar{\rho}}{V^2R^2}u. \tag{6.1.9}$$

Noting that $\epsilon = \Lambda\,\mathfrak{a}^2/3$ is a dimensionless small parameter we shall take to zero in the extremal Kerr limit, the angular operator is close to the usual Laplace-Beltrami operator on the sphere restricted to $\mathcal{K}_m$, the kernel of $\partial_\phi - im$, which we denote $\not{\Delta}_m$.

6.2 A model problem

Given the motivation from the previous section, we now construct a model problem which exhibits the phenomenon of zero-damped modes which we can use to compare with the full Kerr-de Sitter operator and apply similar methods to [92]. To simplify the problem, we shall model the angular dependence by the Laplacian on the sphere, giving the following equation on the unit ball:

$$(1-R^2)\partial_R^2 u - 2\left(s' + R - \frac{4im\bar{\omega}\bar{\rho}}{V^2}\frac{1-R}{R}\right)\partial_R u - \frac{4im\bar{\omega}\bar{\rho}}{V^2R^2}u + \frac{\not{\Delta}_m u}{R^2} = 0. \tag{6.2.1}$$

Note that for the case $l=0$, $u(R) = \text{const.}$ is a solution to the above equation for any s': to impose any sensible notion of invertibility for this family of operators, we must quotient this out by imposing $u(0)=0$. Since this is a degenerate elliptic problem which degenerates at the horizon, we can define quasinormal modes here in much the same way as in the previous sections. Furthermore, since the dependence on s' is linear, we have that any poles of the resolvent operator will be simple. Decomposing the angular dependence into spherical harmonics, we are left with the following ordinary differential equation:

$$(1-R^2)\partial_R^2 u - 2\left(s' + R - imQ\frac{1-R}{R}\right)\partial_R u - \frac{l(l+1)+imQ}{R^2}u = 0$$

where we have set $Q = 4\bar{\omega}\bar{\rho}/V^2$ and noting that we must take $l \geq |m|$ since we have already restricted to modes with azimuthal eigenvalue m. This equation is Fuchsian: it is clear from the above that $0, \pm 1$ are regular singular points and a suitable change of variables shows that the singularity at ∞ is removable and thus ∞ is an ordinary point. Hence the solutions to this equation can be found using the theory of hypergeometric functions by simply considering the indices at $0, \pm 1$. The indicial equations are:

$$\begin{cases} R = 0 \quad : \quad \left(\sigma - \frac{1}{2} + imQ\right)^2 = \left(l + \frac{1}{2}\right)^2 - m^2Q^2, \\ R = 1 \quad : \quad \sigma(\sigma + s') = 0, \\ R = -1 : \quad \sigma(\sigma - s' - 2imQ) = 0. \end{cases} \tag{6.2.2}$$

We write the roots of the indicial equation at zero as:

$$\sigma_\pm = \frac{1}{2} - imQ \pm \sqrt{\left(l + \frac{1}{2}\right)^2 - m^2Q^2}$$

and note that the information from the equations can be represented using the Papperitz symbol (also known as a Riemann P-symbol)

$$P\begin{Bmatrix} 0 & 1 & -1 & \\ \sigma_+ & 0 & 0 & R \\ \sigma_- & -s' & s'+2imQ & \end{Bmatrix}$$

For square integrability of derivatives of the solution near $R = 0$, we need to take the solution which behaves like R^{σ_+} (much like the inverse square potential in [92]). Focussing on a solution which behaves like this, we write

$$u(R) = \left(\frac{2R}{1+R}\right)^{\sigma_+} v\left(\frac{2R}{1+R}\right)$$

From the exponent shifting properties of P-symbols and their behaviour under Möbius maps (see Chapter 15 of [112]), we see that $v(z)$ solves the equation corresponding to the following P-symbol:

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & \sigma_+ & z \\ \sigma_- - \sigma_+ & -s'+imQ & s'+\sigma_+ + imQ & \end{Bmatrix}.$$

The solution that is regular at $R = 0$ to the above is the hypergeometric function, yielding the solution

$$u(R) = \left(\frac{2R}{1+R}\right)^{\sigma_+} {}_2F_1\left[\sigma_+, s'+\sigma_+ + imQ; 1+\sigma_+ - \sigma_-; \frac{2R}{1+R}\right].$$

Imposing the regularity at the horizon $R = 1$ required of quasinormal modes restricts s' to the values

$$s' = -n - \frac{1}{2} - \sqrt{\left(l+\frac{1}{2}\right)^2 - m^2Q^2}, \qquad n = 0, 1, 2, \ldots$$

Since this model is only a good approximation of the full problem in a neighbourhood of the horizon $R = 1$ and behaves very differently elsewhere, we require the co-modes to be suitably localised near the horizon and preferably zero outside the range of validity of the model. This can only occur if the quasinormal frequencies take integer values and as such, we restrict to axisymmetric ($m = 0$) modes from now onwards.

6.2.1 Co-modes of the model problem

We shall now find the co-modes for the operator pencil

$$L(s)u = R^2(1-R^2)\partial_R^2 u - 2R^2(s+R)\partial_R u + \not{\Delta}_0 u.$$

This is very similar to the de Sitter case outlined in Section 3.2 since a simple computation shows that the formal adjoint $L(s)^\dagger = L(-s^*)$. Thus we see that solutions to $L(s)^\dagger u = 0$ take the form

$$\begin{aligned} u(R) &= \left(\frac{2R}{1+R}\right)^{l+1} {}_2F_1\left[l+1, -s^*+l+1; 2l+2; \frac{2R}{1+R}\right] Y_{l0}(\theta) \\ &= \left(\frac{2R}{1+R}\right)^{l+1} \left(\frac{1-R}{1+R}\right)^{s^*} {}_2F_1\left[l+1, s^*+l+1; 2l+2; \frac{2R}{1+R}\right] Y_{l0}(\theta), \end{aligned}$$

and we can use them to define distributions as candidates for co-modes. To reduce clutter, we define the function

$$F(l,s;R,\theta) := (2R)^{l+1}(1+R)^{-s-l-1} {}_2F_1\left[l+1, s+l+1; 2l+2; \frac{2R}{1+R}\right] Y_{l0}(\theta),$$

and (for $\mathrm{Re}(s) > -1/2$) we set

$$T(u) := \int_{B_1} \frac{(1-R)^s}{\Gamma(s+1)} F(l,s;R,\theta) u dx.$$

In a similar manner to Section 3.2, we extend the definition of this to $\mathrm{Re}(s) > 1/2 - k$ by integrating by parts (assuming u is sufficiently regular). In particular, we see that since the quasinormal frequencies are integers, the co-modes concentrate on the horizons.

6.3 Existence of zero-damped modes

We now return to the original equation restricted to axisymmetric solutions. Let $\mathring{P}(s')$ denote $P_0(s)$ as defined in Section 6.1.2 but with respect to the R, θ coordinates where $s = \kappa s'/\mathfrak{m}$. We wish to relate this to the model problem operator $L(s)$ from the previous section, however these operator pencils are defined on different domains. To overcome this, we interpolate between them using a cut-off function and extension operators. Again, we choose some $\chi \in C^\infty(\mathbb{R})$ such that

$$\chi(R) = \begin{cases} 0 & R < 1/2 \\ 1 & R > 2/3 \end{cases}$$

and take κ sufficiently small so $R_c < 1/3$. Let E_χ be the operator that extends $f : I_\kappa \times S^2 \to \mathbb{C}$ to $\overline{B_1}$ by setting $f = 0$ outside the support of χ. We now decompose the operator as

$$\begin{aligned} \mathring{P}(s') &= \mathring{P}(s') \circ (1-\chi) + \mathring{P}(s') \circ \chi \\ &= \mathring{P}(s') \circ (1-\chi) + \frac{1}{2} V^2 L(s') \circ E_\chi + \mathring{P}(s') \circ \chi - \frac{1}{2} V^2 L(s') \circ E_\chi. \end{aligned}$$

We define the interpolated operator

$$\mathring{L}(s') := \mathring{P}(s') \circ (1-\chi) + \frac{1}{2}V^2 L(s') \circ E_\chi$$

and the error term

$$K(s') := \mathring{P}(s') \circ \chi - \frac{1}{2}V^2 L(s') \circ E_\chi.$$

The interpolated operator is an unbounded, degenerate elliptic operator where ellipticity fails precisely at the horizons. Since the surface gravities of these horizons are non-zero, we can use the theory developed in [139] and [116] to establish the fact that this defines a family of Fredholm operators from their domain to $\mathcal{H}^k_0 := H^k(I_\kappa \times S^2) \cap \mathcal{K}_0$. As in the spherically symmetric case, we also have the following estimates:

Lemma 6.3.1. *There exists $\hat{\kappa}_0, \epsilon_0 > 0$ such that for all $\hat{\kappa} < \hat{\kappa}_0$ and $\epsilon < \epsilon_0$, there exists C depending only on k such that the following estimates hold:*

$$\|\mathring{L}(s')u\|_{\mathcal{H}^k_0} < C\left(\|\mathring{P}(s')u\|_{\mathcal{H}^k_0} + \|u\|_{\mathcal{H}^{k+1}_0}\right)$$
$$\|\mathring{P}(s')u\|_{\mathcal{H}^k_0} < C\left(\|\mathring{L}(s')u\|_{\mathcal{H}^k_0} + \|u\|_{\mathcal{H}^{k+1}_0}\right)$$

Proof. In this proof, the constant C may change value from line to line in inequalities: the point is that for $\hat{\kappa}$ and ϵ sufficiently small, it depends only on k. We begin with the observation that

$$\|\mathring{L}(s')u\|_{\mathcal{H}^k_0} \leq \|\mathring{P}(s')u\|_{\mathcal{H}^k_0} + \|K(s')u\|_{\mathcal{H}^k_0}$$
$$\|\mathring{P}(s)u\|_{\mathcal{H}^k_0} \leq \|\mathring{L}(s')u\|_{\mathcal{H}^k_0} + \|K(s')u\|_{\mathcal{H}^k_0}$$

so it suffices to establish the appropriate estimates for $K(s')$. We first consider the highest order terms for each variable. For the operator $\mathring{P}(s')$, we see that this is

$$\mathring{\mu}(R)\partial_R^2 u + c\partial_\theta^2 u$$

where

$$\mathring{\mu}(R) = \frac{\lambda}{3}(1-R)\left(1 + \frac{\bar{\rho}-\rho_-}{x_+}R\right)(x_+ + (\bar{\rho}-\rho_0)R)\left((\rho_c - \bar{\rho})R - x_+\right).$$

For $\mathring{L}$, it is

$$R^2(1-R^2)\partial_R^2 u + \partial_\theta^2 u$$

Letting K_h be the highest order part of $K(s')$, we have

$$K_h u = \left(\mathring{\mu}(R) - \frac{1}{2}V^2R^2(1-R^2)\right)\partial_R^2 u + \left(1 + \epsilon\cos^2\theta - \frac{1}{2}V^2\right)\partial_\theta^2 u. \tag{6.3.1}$$

Since $\mathring{\mu}$ and $R^2(1-R^2)$ both have simple zeroes at $R=1$, we can define a smooth function $f_1 : \operatorname{supp}\chi \to \mathbb{R}$ such that

$$f_1(R) = \frac{2\mathring{\mu}(R)}{V^2R^2(1-R^2)},$$

and $f_2 = 1/f_1$ is smooth. Furthermore, there is a constant depending on k (and can be made independent of κ and ϵ provided they are taken to be sufficiently small) such that

$$\|f_i - 1\|_{C^k} < C. \tag{6.3.2}$$

We also note that $g_1 = 2c/V^2$ and $g_2 = V^2/2c$ are smooth on $\operatorname{supp}\chi$ and satisfy

$$\|g_i - 1\|_{C^k} < C. \tag{6.3.3}$$

Thus we have

$$\begin{aligned}\|K(s')u\|_{\mathcal{H}_0^k} &< \|(1-f_2)\partial_R^2(\chi u) + (1-g_2)\partial_\theta^2(\chi u)\|_{\mathcal{H}_0^k} + C\|u\|_{\mathcal{H}_0^{k+1}} \\ &< C\left(\|\mathring{P}(s')u\|_{\mathcal{H}_0^k} + \|u\|_{\mathcal{H}_0^{k+1}}\right)\end{aligned}$$

The proof of the other inequality is similar. □

To obtain a set-up which satisfies the hypotheses of Theorem 2.2.5, we pick some s_0 with positive real part such that $\mathring{P}(s_0)^{-1}$ exists and define the following families of operators:

$$\begin{aligned}A(s') &:= \mathring{L}(s')\mathring{P}(s_0)^{-1} \\ B(s') &:= A(s') + K(s')\mathring{P}(s_0)^{-1}.\end{aligned}$$

All that remains is to establish that the operator norm of $K(s')$ from the domain of $\mathring{P}$ to $\mathcal{H}_0^k$ can be controlled by $\hat{\kappa}$ and ϵ.

Lemma 6.3.2. *Let $\mathring{D}^{k+1}$ be the domain of $\mathring{P}(s')$ in $\mathcal{H}_0^k$ i.e.*

$$\mathring{D}^{k+1} := \left\{u \in C^\infty(I_\kappa \times S^2) \,\middle|\, \|u\|_{\mathring{D}^k} := \|u\|_{\mathcal{H}_0^k} + \|\mathring{P}(s_0)u\|_{\mathcal{H}_0^k} < \infty\right\},$$

and take $\hat{\kappa}, \epsilon$ sufficiently small as before. Then there exist constants $C_1, C_2 > 0$ depending on k and s' such that

$$\|K(s')\|_{\mathring{D}^{k+1} \to \mathcal{H}_0^k} < C_1\hat{\kappa} + C_2\epsilon.$$

Proof. The proof is similar to Lemma 6.3.1.

1. Recall from the previous sections that

$$\begin{aligned}
\mathring{\mu}(R) =& -\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}+\frac{1}{2}V^2R^2-\frac{4\lambda\bar{\rho}}{3}x_+R-\frac{\lambda}{3}x_+^2\\
=&\frac{1}{2}V^2R^2(1-R^2)+\left(\frac{1}{2}V^2-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}\right)(1+R-1)^4\\
&-\frac{4\lambda\bar{\rho}}{3}x_+(R-1)-\frac{4\lambda\bar{\rho}}{3}x_+-\frac{\lambda}{3}x_+^2\\
=&\frac{1}{2}V^2R^2(1-R^2)+\left(\frac{1}{2}V^2-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}\right)(R^2-1)(R^2+1)\\
&-\frac{4\lambda\bar{\rho}}{3}x_+(R-1)-\frac{4\lambda\bar{\rho}}{3}x_+-\frac{\lambda}{3}x_+^2+\frac{1}{2}V^2-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}
\end{aligned}$$

From the definition of x_+, we have

$$-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}+\frac{1}{2}V^2x_+^2-\frac{4\lambda\bar{\rho}x_+^3}{3}-\frac{\lambda}{3}x_+^4=0,$$

so, defining

$$\nu(R)=\left(\frac{1}{2}V^2-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}\right)(R+1)(R^2+1)-\frac{4\lambda\bar{\rho}}{3}x_+$$

we have

$$\mathring{\mu}(R)=\frac{1}{2}V^2R^2(1-R^2)+(R-1)\nu(R).$$

It will be key to understand the function

$$f(R):=\frac{\mathring{\mu}(R)-\frac{1}{2}V^2R^2(1-R^2)}{\mathring{\mu}(R)}=\frac{\nu(R)}{\frac{1}{2}V^2R^2(1+R)+\nu(R)}$$

so let us first consider ν. Firstly we note that since $x_+>0$, we have

$$\frac{1}{2}V^2-\frac{2(\bar{\rho}^2+\hat{\mathfrak{a}}_0^2)^2\hat{\kappa}^2}{x_+^2V^2}=\frac{4\lambda\bar{\rho}}{3}x_++\frac{\lambda}{3}x_+^2>\frac{4\lambda\bar{r}}{3}x_+>0,$$

so on $\operatorname{supp}\chi$, we have

$$\frac{7}{8}\frac{4\lambda\bar{\rho}}{3}x_++\frac{15}{8}\frac{\lambda}{3}x_+^2<\nu(R)<\frac{4\lambda}{3}x_+\left(3\bar{\rho}+x_+\right).$$

Since we have already seen that $x_+ = O(\hat{\kappa})$, we have

$$\|\nu\|_{C^k(\operatorname{supp}\chi)} < C_1\hat{\kappa}$$

where C_1 is a constant bounded in ϵ so can be take to depend only on k as $\hat{\kappa} \to 0$. Furthermore, for ϵ sufficiently small, we have

$$\frac{1}{2} < \frac{1}{2}V^2 \leq 1,$$

so for $\epsilon, \hat{\kappa}$ sufficiently small,

$$\|V^2R^2(1-R^2)/2 + \nu\|_{C^k(\operatorname{supp}\chi)} > \frac{1}{C_1}$$

which gives us

$$\|f\|_{C^k(\operatorname{supp}\chi)} < C_1\hat{\kappa}.$$

2. It will also be key to understand the function

$$g(\theta) = \frac{1 + \epsilon\cos^2\theta - \frac{1}{2}V^2}{1 + \cos^2\theta}.$$

For ϵ sufficiently small (i.e. bounded away from its maximum value where $V^2 = 0$), we see that there exists a constant C_2 such that

$$\left|1 - \frac{1}{2}V^2\right| < C_2\,\epsilon\,.$$

We can obtain similar estimates for the denominator as in the radial case, so we have

$$\|g\|_{C^k(\operatorname{supp}\chi)} < C_2\epsilon.$$

3. Combining these estimates, we have

$$\begin{aligned}\|K(s')u\|_{\mathcal{H}_0^k} <& \|f\mathring{\mu}\partial_R^2(\chi u) + gc\partial_\theta^2(\chi u)\|_{\mathcal{H}_0^k} + (C_1\hat{\kappa} + C_2\epsilon)\|u\|_{\mathcal{H}_0^{k+1}}\\ <& \|f\|_{C^k(\operatorname{supp}\chi)}\|\mathring{P}(s_0)(\chi u)\|_{\mathcal{H}_0^k} + \|g\|_{C^k(\operatorname{supp}\chi)}\|\mathring{P}(s_0)(\chi u)\|_{\mathcal{H}_0^k}\\ &+ (C_1\hat{\kappa} + C_2\epsilon)\|u\|_{\mathcal{H}_0^{k+1}}\end{aligned}$$

It follows from the theory developed in [139] that there exists a constant C such that $\|u\|_{\mathcal{H}_0^{k+1}} < C\|u\|_{\mathring{D}^{k+1}}$. Thus we have

$$\|K(s')u\|_{\mathcal{H}_0^k} < (C_1\hat{\kappa} + C_2\,\epsilon)\|u\|_{\mathring{D}^{k+1}}$$

and the result follows.

□

We also have the following result:

Proposition 6.3.1. *Fix $k \in \mathbb{N}$. Then for each $n \in \{1, 2, \dots k\}$, there exists $\Lambda_0 > 0$, $L_{-1} : \mathcal{H}_0^k \to \mathring{D}^{k+1}$ a finite rank operator and $L_0(s)$ is a holomorphic family of Fredholm operators $\mathcal{H}_0^k \to \mathring{D}^{k+1}$ such that for κ sufficiently small (so that $R_c \notin \operatorname{supp} \chi$),*

$$\mathring{L}(s')^{-1} = \frac{L_{-1}}{s' + n} + L_0(s')$$

in a suitable neighbourhood of $-n$.

Proof. This utilises the concentration of the co-modes calculated in Section 6.2.1 to deduce that $-1, -2, \cdots - k$ are indeed quasinormal frequencies and analogous reasoning to the proof of Proposition 5.2.1 to establish that these are simple poles. □

Hence for each $n \in \mathbb{N}$, there exists $0 < r < 1/2$ such that $L_0(s')$ as defined above is bounded on the closed disc of radius r around $-n \in \mathbb{C}$. We then set

$$C'_{k,n} := \sup_{s' \in D(-n,r)} \left\{ \|\mathring{P}(s_0) L_0(s')\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k} \right\}$$

and

$$\mathring{C}_{k,n} := \min \left\{ \frac{1}{2\|\mathring{P}(s_0) L_{-1}\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k}}, \frac{1}{C'_{k,n}} \right\}$$

Proposition 6.3.2. *For each $n \in \mathbb{N}$ and $0 < \delta < r$, there exists $\hat{\kappa}_0, \epsilon_0$ such that for all $\hat{\kappa} < \hat{\kappa}_0$ and $\epsilon < \epsilon_0$, there exists a quasinormal frequency s such that*

$$\left| s + \frac{\hat{\kappa}}{\mathfrak{m}} n \right| < \frac{\hat{\kappa}}{\mathfrak{m}} \delta$$

where $\mathfrak{m}$ is the mass parameter of the Kerr-de Sitter black hole.

Proof. Again we use Theorem 2.2.5. On the circle of radius δ around $s' = -n$, we have

$$\|A(s')^{-1}(A(s') - B(s'))\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k} \leq \left(\frac{1}{\delta} \|\mathring{P}(s_0) L_-1\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k} + C'_{k,n} \right) \|K(s')\|_{\mathring{D}^{k+1} \to \mathcal{H}_0^k}.$$

Since $\|K(s')\|_{\mathring{D}^{k+1} \to \mathcal{H}_0^k} < C_1 \hat{\kappa} + C_2 \epsilon$, provided we take κ and ϵ sufficiently small, we can arrange to have $\|K(s')\|_{\mathring{D}^{k+1} \to \mathcal{H}_0^k} < \mathring{C}_{k,n} \delta$. Thus

$$\begin{aligned} \|A(s')^{-1}(A(s') - B(s'))\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k} &\leq \mathring{C}_{k,n} \|\mathring{P}(s_0) L_-1\|_{\mathcal{H}_0^k \to \mathcal{H}_0^k} + C'_{k,n} \mathring{C}_{k,n} \delta \\ &< \frac{1}{2} + \delta < 1. \end{aligned}$$

$\square$

Theorem 6.3.1. *Pick $N \in \mathbb{N}$ and $\delta > 0$ sufficiently small. Then there exist $\lambda_0, \hat{\mathfrak{a}}_- > 0$ such that for any subextremal Kerr-de Sitter spacetime with dimensionless parameters $\lambda < \lambda_0$ and $|\hat{\mathfrak{a}}| > \hat{\mathfrak{a}}_-$, the wave equation has quasinormal frequencies s_n satisfying*

$$\left| s_n + \frac{\hat{\kappa}}{\mathfrak{m}} n \right| < \frac{\hat{\kappa}}{\mathfrak{m}} \delta.$$

for each $n = 1, 2, \dots N$ where $\mathfrak{m}$ is the mass parameter of the black hole.

Proof. To prove this result, we simply make use of Proposition 6.3.2:

1. From the proof of Proposition 6.3.2, we know that given N, δ we can find C, M such that the result holds provided $\epsilon + C\hat{\kappa} < M$. So we select some $\epsilon_0 < M$ (which also fixes some $\kappa_0 < (M - \epsilon_0)/C$) and note that the curve $\lambda \hat{\mathfrak{a}}^2 = 3\epsilon_0$ intersects the boundary of the parameter space exactly twice (see Figure 6.2). The rightmost point defines a rapidly rotating extremal spacetime and we can solve exactly for λ_0 and $\hat{\mathfrak{a}}_0$ using a suitable quadratic. Finally, we define

$$\hat{\mathfrak{a}}_- = \sqrt{\hat{\mathfrak{a}}_0^2 - \hat{\kappa}_0^2}$$

2. Now we simply observe that for $\lambda < \lambda_0$ we necessarily have $\hat{\mathfrak{a}} < \hat{\mathfrak{a}}_0$ so for any spacetime in this region,

$$\epsilon = \frac{\lambda \hat{\mathfrak{a}}^2}{3} < \frac{\lambda_0 \hat{\mathfrak{a}}_0^2}{3} = \epsilon_0$$

Furthermore, given a spacetime with dimensionless parameters $(\lambda, \hat{\mathfrak{a}})$, we have a unique rapidly rotating extremal spacetime with parameters $(\lambda, \hat{\mathfrak{a}}_+)$ such that $\lambda \hat{\mathfrak{a}}_+^2 = 3\,\epsilon_+ < 3\,\epsilon_0$. With this in mind, and assuming $\hat{\mathfrak{a}}_- < |\hat{\mathfrak{a}}| < \hat{\mathfrak{a}}_+$, we have for this spacetime

$$\begin{aligned}
\hat{\kappa}^2 =& \frac{1}{2} V^2(\epsilon_+) \cdot \frac{9\,\epsilon_+^2}{(\lambda \bar{\rho}^2 + 3\,\epsilon_+)^2} \cdot \frac{\hat{\mathfrak{a}}_+^2 - \hat{\mathfrak{a}}^2}{\hat{\mathfrak{a}}_+^4} \\
<& \hat{\mathfrak{a}}_0^2 - \hat{\mathfrak{a}}_-^2 \\
<& \hat{\kappa}_0^2
\end{aligned}$$

Thus for any subextremal Kerr-de Sitter spacetime with $\Lambda\, \mathfrak{m}^2 = \lambda < \lambda_0$ and $|\, \mathfrak{a} \,/\, \mathfrak{m} \,| > \hat{\mathfrak{a}}_-$, we have $\epsilon + C\hat{\kappa} < M$ and the result holds.

$\square$

This result falls short of proving the existence of zero-damped modes in the rapidly rotating limit for some fixed $\Lambda > 0$ since we do not have an infinite family of them. In order to establish a result like that, we need to be able to control the constants associated

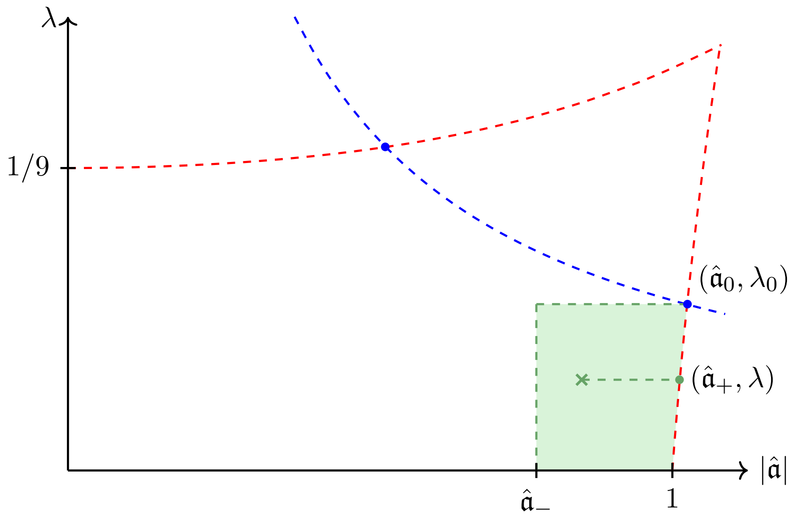


Figure 6.2: Any subextremal Kerr-de Sitter spacetime with parameters in the shaded region has quasinormal frequencies approximated by $-n\hat{\kappa}/\,\mathfrak{m}$ for $n = 1, 2, \dots N$.

with higher Sobolev norms appearing in Proposition 6.3.2 or take ϵ arbitrarily small to counteract their growth. This latter can be made to occur in the extremal Kerr limit:

Corollary 6.3.1. *Consider the wave equation in a subextremal Kerr-de Sitter spacetime. Then in any extremal Kerr limit, it exhibits the phenomenon of axisymmetric zero-damped modes converging to 0.*

In fact the above result also be holds for the conformal Klein-Gordon equation (or any Klein-Gordon mass proportional to Λ) provided we use the same model problem as for the wave equation: since we take $\epsilon \to 0$, it is straightforward to see that we can obtain analogous estimates to those found in Lemma 6.3.2.

With the process described in the proof of the theorem, we can approach extremal Kerr arbitrarily close to the right-hand component of the boundary (see Figure 6.2): the black holes considered actually rotate faster than the limiting spacetime. This offers a different perspective from the results of [73], where the existence of zero-damped modes is established in the limit as $\Lambda\,\mathfrak{m}^2 \to 0$ for a range of parameters which doesn't quite go all the way to the edge.

6.4 Future directions

While the result above probes a small part of the rapidly rotating Kerr-de Sitter parameter space, we do not have an understanding of the full spectrum in the extremal limit for some fixed Λ_0 bounded away from 0. Considerations from the model problem imply that we should expect zero-damped modes to occur and we have a candidate lattice of points, however a proof using our methods remains elusive. The main obstacle is the deformation of the sphere caused by $\epsilon = \Lambda\, \mathfrak{a}^2 /3$: the angular operator is deformed so its eigenvalues are no longer of the form $-l(l+1)$. There are two approaches to constructing a model problem we can take here: either use the deformed angular operator as is or try and approximate it with the Laplace-Beltrami operator on the round sphere.

In the first case, the model problem is unlikely to have integer quasinormal frequencies which means the co-modes are no longer localised. In order to make progress, one would need to develop a thorough understanding of the co-modes in this case and in particular if they can be approximated by localised ones. This is unlikely to be the case if the corresponding frequencies are not close to integers: this is especially a concern if they have non-trivial imaginary part as is the case for $m \neq 0$ modes in rotating black hole spacetimes. Furthermore, the error in these co-modes from the localised ones will be of zeroeth order in κ so cannot be taken to be increasingly small in the extremal limit. A better understanding of oscillating co-modes (such as those for the fixed mass Klein-Gordon example in Chapter 3 or the inverse square potential in Chapter 4) may shed further light on the matter.

In the second case, we would need to have deep quantitative knowledge of the interpolated operator $\mathring{L}(s')^{-1}$ near each pole. In particular we would need to be able to bound the holomorphic part of the operator on circles around $s' = -n$. Provided the constants defined in the bounds above do not grow without bound as we go deeper into the left half plane, one could hope that there is some fixed ϵ_0 such that for all $\epsilon < \epsilon_0$ the bounds required to use Theorem 2.2.5 hold provided we take $\kappa \to 0$. This is, however, unlikely to be the case since it is well known that highly damped quasinormal frequencies are more unstable to perturbations than the more slowly decaying ones [90, 65]. This is because to probe these frequencies, we require higher regularity norms which are more sensitive: the constants in the operator norms given above are likely to grow with k so we would need to take ϵ to be arbitrarily small.

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