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An upper bound for the pseudoisotopy stable range

Oscar Randal-Williams[1]

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Abstract We prove that the pseudoisotopy stable range for manifolds of dimension $2n$ can be no better than $(2n-2)$. In order to do so, we define new characteristic classes for block bundles, extending our earlier work with Ebert, and prove their non-triviality. We also explain how similar methods show that $\mathrm{Top}(2n)/\mathrm{O}(2n)$ is rationally $(4n-5)$-connected.

For a smooth manifold M, possibly with boundary, the space of smooth pseudoisotopies (also known as concordances) is $P(M) := \mathrm{Diff}(M \times [0, 1] \,\mathrm{rel}\, M \times \{0\})$, that is, the space of diffeomorphisms of the cylinder $M \times [0, 1]$ which keep one end fixed. There is a canonical map

$$P(M) \longrightarrow P(M \times I) \tag{0.1}$$

given by crossing with the interval I (and unbending corners), and the (smooth) *pseudoisotopy stable range* is the function

$$\phi(n) := \max\{k \in \mathbb{N} \mid (0.1) \text{ is } k\text{-connected for all manifolds } M \text{ of dimension } \geq n\}.$$

The main theorem concerning this function is due to Igusa [16], and says that

$$\phi(n) \geq \min\left\{\tfrac{n-7}{2}, \tfrac{n-4}{3}\right\}.$$

In this note we establish the following upper bound for this function.

Theorem A $\phi(2n) \leq 2n - 2$ *as long as* $2n \geq 6$.

✉ Oscar Randal-Williams
o.randal-williams@dpmms.cam.ac.uk

1 Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WB, UK

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To explain our approach, let $W_{g,1} := \#^g S^n \times S^n \setminus \mathrm{int}(D^{2n})$ with $2n \geq 6$, and consider the fibration sequence

$$\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})} \longrightarrow B\mathrm{Diff}_\partial(W_{g,1}) \overset{i}{\longrightarrow} B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}) \tag{0.2}$$

from the classifying space of the group of diffeomorphisms of $W_{g,1}$ to the classifying space of the group of block diffeomorphisms of $W_{g,1}$. The rational cohomology of $B\mathrm{Diff}_\partial(W_{g,1})$ has been computed for $g \gg 0$ by Galatius and the author in [12,13]; the rational cohomology of $B\widetilde{\mathrm{Diff}}_\partial(W_{g,1})$ has been computed for $g \gg 0$ by Berglund and Madsen in [1,2] and in a forthcoming revision of [2]. Ebert and the author have shown in [8] that the map i is surjective on rational cohomology in the stable range.

Our approach to Theorem A is motivated by forthcoming work of Berglund and Madsen, in which they show that the map induced by i on rational cohomology is injective in degrees $* < 2n$ and $g \gg 0$, and more importantly for our current purpose they show that this is sharp, in the following sense.

Proposition B (Berglund–Madsen) *For $g \gg 0$,*

$$\mathrm{Ker}(i^* : H^{2n}(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); \mathbb{Q}) \to H^{2n}(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})) \neq 0. \tag{0.3}$$

This has implications for the Serre spectral sequence of (0.2), and it is this that we shall exploit to prove Theorem A. As Proposition B is central to our argument, and its proof is not yet available, in Sects. 2 and 3 we will give an independent proof of it, which works for all $g \geq 1$ and does not require the computation of both groups. It consists of defining Mumford–Morita–Miller classes for block bundles, which extend those that we have already defined with Ebert in [8], and then showing that a certain such class—namely $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n}$, which is easily seen to lie in the kernel (0.3)—is not trivial. The construction of these classes and their non-triviality may be of interest independently of Theorem A.

Finally, in Sect. 4 we show how similar methods can be used to show that the space $\mathrm{Top}(2n)/\mathrm{O}(2n)$ is rationally $(4n-5)$-connected as long as $2n > 4$.

1 Proof of Theorem A

By the work of Weiss–Williams [22, Theorem A], there is a certain map

$$\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})} \longrightarrow \Omega^\infty(S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1})) \tag{1.1}$$

which is $(\phi(2n)+1)$-connected. The $(\mathbb{Z}/2$-)spectrum $\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1})$ is the 1-connected cover of the (smooth) Whitehead spectrum $\mathbf{Wh}^{\mathrm{Diff}}(W_{g,1})$, which in turn is related to Waldhausen's algebraic K-theory of spaces by a (split) cofibre sequence of spectra

$$\Sigma^\infty_+ W_{g,1} \longrightarrow \mathbf{A}(W_{g,1}) \longrightarrow \mathbf{Wh}^{\mathrm{Diff}}(W_{g,1}). \tag{1.2}$$

This identification requires the stable parameterised h-cobordism theorem [20].

Our strategy is then as follows. We use a theorem of Hsiang–Staffeldt to compute the spectrum cohomology $H^*(\mathbf{Wh}^{\mathrm{Diff}}(W_{g,1});\mathbb{Q})$ in degrees $* \leq 2n$. We take care to compute this *as a representation of the mapping class group* $\Gamma_{g,1}$ of $W_{g,1}$, in terms of the standard representation

$$H_g := H_n(W_{g,1};\mathbb{Q})$$

of $\Gamma_{g,1}$. The spectrum cohomology of $S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1})$ is then given by truncating, desuspending, and taking $\mathbb{Z}/2$-invariants, and the cohomology of $\Omega^\infty(S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1}))$ is the free graded-commutative algebra on the result.

We now suppose for a contradiction that $\phi(2n) \geq 2n-1$, so the map (1.1) is $2n$-connected and hence we have a computation of the rational cohomology of $\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})}$ in degrees $* \leq 2n-1$, as a $\Gamma_{g,1}$-module. We then study the Serre spectral sequence for (0.2), and derive a contradiction.

1.1 Rational homology of the Whitehead spectrum

We shall use Corollary 1.2 of Hsiang–Staffeldt [15], which shows that

$$H_*(\mathbf{A}(W_{g,1});\mathbb{Q}) = \pi_*(\mathbf{A}(W_{g,1})) \otimes \mathbb{Q} \cong (K_*(\mathbb{Z}) \otimes \mathbb{Q}) \oplus (\Sigma \bar{K}_{ab})$$

where K is a minimal model for the dga $C_*(\Omega W_{g,1};\mathbb{Q})$, $\bar{K}$ denotes the augmentation ideal, which inherits the structure of a graded Lie algebra with bracket given by $[x,y] := x \cdot y - (-1)^{|x|\cdot|y|} y \cdot x$, and $\bar{K}_{ab} = \bar{K}/[\bar{K},\bar{K}]$ is the abelianisation of this graded Lie algebra.

As $W_{g,1}$ is a suspension, the homology of $\Omega W_{g,1}$ is the tensor algebra on the vector space $H_g[n-1]$. In particular it is a free (non-commutative) algebra, so is quasi-isomorphic to $C_*(\Omega W_{g,1};\mathbb{Q})$, and we may take $K = H_*(\Omega W_{g,1};\mathbb{Q})$ with trivial differential. It follows that $\bar{K}_{ab}$ is the augmentation ideal of the free graded commutative algebra on $H_g[n-1]$, that is

$$\begin{aligned}\bar{K}_{ab} = (H_g[n-1]) \oplus &\begin{pmatrix} \mathrm{Sym}^2(H_g)[2n-2] & \text{if } n \text{ is odd} \\ \wedge^2(H_g)[2n-2] & \text{if } n \text{ is even} \end{pmatrix} \\ &\oplus \text{(terms of degree} \geq 3n-3).\end{aligned}$$

Let us write

$$U := \begin{cases} \mathrm{Sym}^2(H_g) & \text{if } n \text{ is odd} \\ \wedge^2(H_g) & \text{if } n \text{ is even.} \end{cases}$$

Then we have

$$H_*(\mathbf{A}(W_{g,1});\mathbb{Q}) \cong (K_*(\mathbb{Z})\otimes\mathbb{Q}) \oplus (H_g[n]) \oplus (U[2n-1])$$

in degrees $* \leq 2n$. Applying the cofibre sequence (1.2), we obtain

$$H_*(\mathbf{Wh}^{\mathrm{Diff}}(W_{g,1});\mathbb{Q}) \cong (\tilde{K}_*(\mathbb{Z})\otimes\mathbb{Q}) \oplus (U[2n-1])$$

in degrees $* \leq 2n$. The rational homology of $\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1})$ is therefore the same, as it is already 1-connected. Thus, dualising, we have

$$H^*(S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1});\mathbb{Q}) \cong ((\tilde{K}_{*-1}(\mathbb{Z})\otimes\mathbb{Q}) \oplus (U[2n-2]))^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$, for some involution. It follows from Farrell–Hsiang [10] (which considers the case $g=0$) that this involution acts as -1 on $\tilde{K}_{*-1}(\mathbb{Z})\otimes\mathbb{Q}$, so this summand does not contribute to the invariants. Thus

$$H^*(S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1});\mathbb{Q}) \cong (U[2n-2])^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$, for some involution on U. Taking the free graded-commutative algebra on this, it follows that

$$H^*(\Omega^\infty(S^\infty_+ \wedge_{\mathbb{Z}/2} \Omega\mathbf{Wh}^{\mathrm{Diff}}_s(W_{g,1}));\mathbb{Q}) \cong \mathbb{Q}[0] \oplus (U[2n-2])^{\mathbb{Z}/2}$$

in degrees $* \leq 2n-1$.

1.2 The Serre spectral sequence argument

The Serre spectral sequence for the fibration (0.2) takes the form

$$E_1^{p,q} = H^p\left(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); H^q\left(\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})};\mathbb{Q}\right)\right) \Longrightarrow H^{p+q}(B\mathrm{Diff}_\partial(W_{g,1});\mathbb{Q}).$$

Under the assumption that $\phi(2n) \geq 2n-1$ we have identified the coefficients in degrees $q \leq 2n-1$, to be $\mathbb{Q}$ for $q=0$ and to be $V := U^{\mathbb{Z}/2}$ for $q = 2n-2$. In order for (0.3) to be possible, we must therefore have a non-trivial differential

$$d^{2n-1} : H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); V) \longrightarrow H^{2n}(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1});\mathbb{Q}).$$

In particular, the source must be non-trivial. Note that $H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); V)$ is a summand of $H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); U)$, so the following will give a contradiction.

Proposition 1.1 $H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); U) = 0$ *for* $g \gg 0$.

Proof The action of $\Gamma_{g,1}$ on $H_n(W_{g,1};\mathbb{Z})$ preserves the intersection form, determining a homomorphism

$$\Gamma_{g,1} \longrightarrow \begin{cases} O_{g,g}(\mathbb{Z}) & \text{if } n \text{ is even} \\ Sp_{2g}(\mathbb{Z}) & \text{if } n \text{ is odd.} \end{cases}$$

This is onto if n is even or $n = 1, 3, 7$, but for the remaining odd n its image is the finite-index subgroup—often denoted $\Gamma_g(1,2) \leq Sp_{2g}(\mathbb{Z})$ in the theory of theta functions—of those symplectic matrices which preserve the standard quadratic form, cf. [2, Example 4.2]. Let us write G for the algebraic group $O_{g,g}$ or Sp_{2g}, depending on the parity of n, and $A_g \leq G(\mathbb{Z})$ for the image of this homomorphism. As Sp_{2g} and $SO_{g,g}$ are connected semisimple algebraic groups defined over $\mathbb{Q}$, it follows from a theorem of Borel–Harish-Chandra [5, Theorem 7.8] that A_g is a lattice in $G(\mathbb{R})$, and hence by the Borel Density Theorem [3] that A_g is Zariski dense in $G(\mathbb{R})$, so also in $G(\mathbb{C})$.

Consider the fibration sequence

$$B\widetilde{\mathfrak{Tor}}_{g,1} \longrightarrow B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}) \longrightarrow BA_g,$$

where $B\widetilde{\mathfrak{Tor}}_{g,1}$ is defined to be the homotopy fibre. By [2, Proposition 4.1] we have

$$H^1(B\widetilde{\mathfrak{Tor}}_{g,1};\mathbb{Q}) \cong \begin{cases} H_g & n \equiv 3 \mod 4 \\ 0 & \text{else} \end{cases}$$

so if $n \not\equiv 3 \mod 4$ then $H^1(A_g;U) \to H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1});U)$ is an isomorphism, and if $n \equiv 3 \mod 4$ then we have an exact sequence

$$0 \longrightarrow H^1(A_g;U) \longrightarrow H^1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1});U) \longrightarrow (H_g \otimes U)^{A_g}.$$

In the case $n \equiv 3 \mod 4$, n is odd and Zariski density of $A_g \leq Sp_{2g}(\mathbb{C})$ implies that the complexification of $(H_g \otimes \mathrm{Sym}^2(H_g))^{A_g}$ is $(H_g \otimes \mathrm{Sym}^2(H_g) \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$, which is contained in $(H_g^{\otimes 3} \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$ and so vanishes by standard invariant theory (for which we refer to [11, §F.2]).

It remains to show that $H^1(A_g;U) = 0$. The representation U is arithmetic, so a theorem of Borel [4, Theorem 1] can be used to identify this with $H^1(A_g;\mathbb{Q}) \otimes U^{A_g}$ as long as $g \gg 0$; see [9, Proposition 3.9] for a statement of this result adapted to our situation. Hence it is enough to show the vanishing of U^{A_g}.

If n is odd then U^{A_g} is $\mathrm{Sym}^2(H_g)^{A_g}$, whose complexification is the same as $\mathrm{Sym}^2(H_g \otimes \mathbb{C})^{Sp_{2g}(\mathbb{C})}$ by Zariski density, and this vanishes by standard invariant theory. If n is even then U^{A_g} is $\wedge^2(H_g)^{A_g}$, whose complexification is $\wedge^2(H_g \otimes \mathbb{C})^{O_{g,g}(\mathbb{C})}$, which also vanishes by standard invariant theory (noting that $O_{g,g}(\mathbb{C}) \cong O_{2g}(\mathbb{C})$). $\square$

2 Characteristic classes of block bundles

We should like to give a proof of Proposition B, as it does not require the entire corpus [1,2,12,13] and beyond to see that the kernel (0.3) is non-trivial. We shall show that this kernel is non-trivial by producing an explicit element in it, which will be described in terms of generalised Mumford–Morita–Miller classes. If $(\pi : E \to |K|, \mathcal{A})$ is a smooth oriented block bundle with fibre a closed d-manifold M (we refer to [8, Section 2] for this notation), in [8, Section 3] Ebert and the author have associated to it

(i) a Leray–Serre spectral sequence $H^p(|K|, \mathcal{H}^q(M)) \Rightarrow H^{p+q}(E)$, and hence a fibre-integration map $\pi_!(-) : H^{k+d}(E) \to H^k(|K|)$,
(ii) a transfer map $\mathrm{trf}_\pi^* : H^*(E) \to H^*(|K|)$ of Becker–Gottlieb type,
(iii) a *stable* vertical tangent bundle $T_\pi^s E \to E$,

such that if $(\pi : E \to |K|, \mathcal{A})$ arises from a smooth fibre bundle then these data reduce to those coming from the bundle structure. In the case $d = 2n$, we then employed the following ruse: If π came from a smooth fibre bundle with $2n$-dimensional fibres, so there was an *unstable* vertical tangent bundle $T_\pi E$, then we would have $e(T_\pi E)^2 = p_n(T_\pi E)$, and $\pi_!(e(T_\pi E) \cdot -) = \mathrm{trf}_\pi^*(-) : H^*(E) \to H^*(|K|)$. Therefore, for a monomial p_I in Pontrjagin classes, if we define

$$\tilde{\kappa}_{p_I}(\pi) := \pi_!(p_I(T_\pi^s E)) \qquad \tilde{\kappa}_{ep_I} := \mathrm{trf}_\pi^*(p_I(T_\pi^s E))$$

then these classes restrict to the usual κ_{p_I} and κ_{ep_I} on fibre bundles, and these give all generalised Mumford–Morita–Miller classes on fibre bundles.

By way of apology for this ruse, we add to the list above

(iv) an Euler class $e(T_\pi E) \in H^d(E; \mathbb{Z})$.

(Of course $e(T_\pi E)$ is merely notation: there is no d-dimensional bundle $T_\pi E$ of which it is the Euler class.) Using this Euler class, we may then define

$$\tilde{\kappa}_{e^i p_I}(\pi) := \pi_!(e(T_\pi E)^i \cdot p_I(T_\pi^s E)) \in H^*(|K|; \mathbb{Z}).$$

The symbol $\tilde{\kappa}_{ep_I}$ has the same meaning as before, by Lemma 2.2 (iv) below.

The existence of this Euler class is a consequence of the Fibre Inclusion Theorem of [7] (or rather its proof, which constructs a canonical such class), and the fact that the homotopy fibre of π is homotopy equivalent to a Poincaré duality space of dimension d, namely M [8, Proposition 2.8]. As the construction is quite pretty, let us describe it.

Construction 2.1 Embed $|K|$ into $\mathbb{R}^k$ for some $k \gg 0$, and let B' be a closed regular neighbourhood, so that there is a retraction $r : B' \to |K|$. Let $B = D(B')$ be the double of B', a closed smooth manifold. This has a retraction $s : D(B') \to B'$, and let $p : X \to B$ be the Hurewicz fibration obtained by turning π into a fibration $\pi^f : E^f \to |K|$ and pulling it back along rs. As B and the fibre of p are Poincaré duality spaces, of dimensions k and d respectively, X is too [14], of dimension $(d+k)$. But $X \times_B X = p^*(X) \to X$ is also a fibration over a Poincaré duality space with

Poincaré duality fibre, so is again a Poincaré duality space, of dimension $(2d + k)$. Writing $\Delta : X \to X \times_B X$ for the fibrewise diagonal map, which admits an umkehr map $\Delta_!$ as source and target are both Poincaré, we define

$$e(T_p X) := \Delta^* \Delta_!(1) \in H^d(X; \mathbb{Z}).$$

We then define $e(T_\pi E)$ by restriction along $E \subset E^f \subset X|_{B'} \subset X$.

It is easy to see that the class so obtained is independent of all choices, and it is shown in [7, §4] that it restricts to the Euler class on the fibre M. The definition given in [7, §4] seems to differ by a sign, but it does not, by Lemma 2.2 (i) below.

Lemma 2.2 *The Euler class defined enjoys the following properties:*

(i) *If d is odd then $2e(T_\pi E) = 0 \in H^*(E; \mathbb{Z})$,*
(ii) *if $(\pi : E \to |K|, \mathcal{A})$ arises from a smooth fibre bundle with vertical tangent bundle $T_\pi E$, then $e(T_\pi E)$ agrees with the Euler class of the vertical tangent bundle,*
(iii) *if there is a map $r : E \to M$ such that $\pi \times r : E \to |K| \times M$ is a homotopy equivalence, then $e(T_\pi E) = r^*(e(TM))$,*
(iv) *the equation $\pi_!(e(T_\pi E) \cdot -) = \mathrm{trf}_\pi^*(-) : H^*(E; \mathbb{Z}) \to H^*(|K|; \mathbb{Z})$ is satisfied.*

Proof For (i), consider the involution τ of $X \times_B X$ which interchanges the two factors. When d is odd, this has degree -1, and so $\tau^* \Delta_! = -\Delta_!$. On the other hand $\Delta^* \tau^* = \Delta^*$, so $e(T_\pi E) = -e(T_\pi E)$.

For (ii), note that if $(\pi : E \to |K|, \mathcal{A})$ arises from a smooth fibre bundle then in Construction 2.1 we do not need to replace it by a fibration. The resulting $p : X \to B$ is a smooth fibre bundle with vertical tangent bundle $T_p X$, and the map $\Delta : X \to X \times_B X$ is a smooth embedding with normal bundle $T_p X$. Hence $\Delta^* \Delta_!(1)$ is the Euler class of $T_p X$, which restricts to the Euler class of $T_\pi E$.

For (iii), if such an r exists then the fibration $p : X \to B$ admits a similar fibre homotopy trivialisation, $p \times \rho : X \xrightarrow{\sim} B \times M$. Then $X \times_B X \simeq B \times M \times M$ and the map Δ is given by the identity map on B and the diagonal map on M. Hence $\Delta^* \Delta_!(1) = 1 \otimes e(TM)$.

For (iv), we must involve ourselves in the details of the construction of the transfer in [7], with which we assume the reader is familiar. We begin by constructing a commutative diagram

$$\begin{tikzcd}
 & F \arrow[r, bend right] \arrow[d] & W \times T^k \arrow[l, bend right] \arrow[r, bend right] \arrow[d] & D(W) \times T^k \arrow[l, bend right] \arrow[d] \\
E \arrow[r, bend right] \arrow[d, "\pi"] & X \arrow[l, bend right, "r"'] \arrow[r, bend right, "u"'] \arrow[d, "p"] & X' \arrow[l, bend right, "t"'] \arrow[r, bend right, "v"'] \arrow[d, "p'"] & X'' \arrow[l, bend right, "s"'] \arrow[d, "p''"] \\
{|K|} \arrow[r, bend right] & B \arrow[l, bend right] \arrow[r, equal] & B \arrow[r, equal] & B.
\end{tikzcd}$$

In this diagram, B is a Poincaré duality space and p is a Hurewicz fibration with fibre $F \simeq M^d$ (obtained as in Construction 2.1). W is a smooth oriented manifold of dimension $(d+\ell)$ with boundary, which is homotopy equivalent to M, and p' is a smooth fibre bundle (obtained from the Closed Fibre Smoothing Theorem of [7]). The map p'' is obtained as the fibrewise double of p', and is a smooth oriented fibre bundle with closed fibres. Finally, the horizontal arrows express each left-hand space as a (fibrewise) retract of the right-hand space.

For a fibration $p : S \to T$ with fibre homotopy equivalent to a finite CW complex, and a fibrewise map $f : S \to S$, let us write $\mathrm{trf}^*_{p,f} : H^*(S) \to H^*(T)$ for the associated transfer map. This is the map denoted τ^f in [7]. When $f = \mathrm{Id}_S$, we shorten this to trf^*_p.

By the definition of the transfer in [7, §6], we have $\mathrm{trf}^*_p = \mathrm{trf}^*_{p'',vuts} s^* t^*$. By the construction of the transfer for smooth fibre bundles in [7, §5], if we write

$$\begin{aligned} \delta &= (\mathrm{Id}_{X''}, vuts) : X'' \longrightarrow X'' \times_B X'' \\ d &= (\mathrm{Id}_{X''}, \mathrm{Id}_{X''}) : X'' \longrightarrow X'' \times_B X'' \end{aligned}$$

then we have $\mathrm{trf}^*_{p'',vuts}(-) = p''_!(\delta^*(d_!(1))\cdot -)$. Thus the map $\mathrm{trf}^*_p(-)$ is $p''_!(\delta^*(d_!(1))\cdot s^*t^*(-)) = (pts)_!(\delta^*(d_!(1)) \cdot s^*t^*(-))$, which we may write as $p_!((ts)_!(\delta^*(d_!(1)) \cdot -)$, so we will be done if $(ts)_!(\delta^*(d_!(1)))$ is equal to the class $e(T_pX)$ defined by Construction 2.1. Consider the homotopy cartesian squares

$$\begin{tikzcd} X'' \arrow[r, "ts"] \arrow[d, "\delta"] & X \arrow[d, "(\mathrm{Id}\times vu)\circ\Delta"] \\ X'' \times_B X'' \arrow[r, "ts\times\mathrm{Id}"] & X \times_B X'' \end{tikzcd} \qquad \begin{tikzcd} X'' \arrow[r, "(ts\times\mathrm{Id})\circ d"] \arrow[d, "ts"] & X \times_B X'' \arrow[d, "\mathrm{Id}\times ts"] \\ X \arrow[r, "\Delta"] & X \times_B X \end{tikzcd}$$

of Poincaré duality spaces, to which Lemma 2.3 below applies and shows that

$$(ts)_!\delta^* = \Delta^*(\mathrm{Id}\times vu)^*(ts\times\mathrm{Id})_! \qquad (ts\times\mathrm{Id})_! d_!(ts)^* = (\mathrm{Id}\times ts)^*\Delta_!.$$

(The signs can be determined by restricting each square to a single fibre over B.) Thus, writing $1 = (ts)^*(1)$, we have

$$\begin{aligned} (ts)_!\delta^* d_!(ts)^*(1) &= \Delta^*(\mathrm{Id}\times vu)^*(ts\times\mathrm{Id})_! d_!(ts)^*(1) \\ &= \Delta^*(\mathrm{Id}\times vu)^*(\mathrm{Id}\times ts)^*\Delta_!(1) = \Delta^*\Delta_!(1) \end{aligned}$$

which is $e(T_pX)$, as required. □

Lemma 2.3 *Consider a homotopy cartesian square*

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"] & C \arrow[d, "v"] \\ B \arrow[r, "f"] & D \end{tikzcd}$$

of oriented Poincaré duality spaces. Then $g_!u^* = \pm v^* f_!$.

The sign ambiguity is unavoidable under the given hypotheses: changing the orientation of B, say, does not change $g_! u^*$, but changes $v^* f_!$ by a sign.

Proof Let us write a for the formal dimension of A, and so on. We assume some familiarity with the notion of Poincaré embeddings, for which we refer to [17] for details. It is enough to prove the identity for the larger square

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "u"] & C \arrow[r, "\mathrm{Id}_C \times \{*\}"] \arrow[d, "v"] & C \times S^N \arrow[d, "v \times \mathrm{Id}_{S^N}"] \\ B \arrow[r, "f"] & D \arrow[r, "\mathrm{Id}_D \times \{*\}"] & D \times S^N. \end{tikzcd}$$

By this device, we may suppose [17, Lemma 3.1] that f admits the structure of a Poincaré embedding, with complement K and normal spherical fibration ξ of dimension $(d - b - 1)$. Let $u^*\xi \to A$ denote the pulled back spherical fibration, and $v^*K \to C$ denote the homotopy pullback of the map $K \to D$ along v. There is then a homotopy commutative cube

$$\begin{tikzcd}[row sep=small, column sep=small] u^*\xi \arrow[rr] \arrow[dr] \arrow[dd] & & v^*K \arrow[dr] \arrow[dd] & \\ & A \arrow[rr] \arrow[dd] & & C \arrow[dd] \\ \xi \arrow[rr] \arrow[dr] & & K \arrow[dr] & \\ & B \arrow[rr] & & D \end{tikzcd}$$

in which the bottom face is homotopy cocartesian, and the vertical faces are all homotopy cartesian. It follows by Mather's Second Cube Theorem [18, Theorem 25] that the top face is also homotopy cocartesian. We therefore have a map

$$C \simeq A \cup_{u^*\xi} v^*K \longrightarrow A/u^*\xi = \mathrm{Th}(u^*\xi)$$

by collapsing v^*K, and similarly for K. This gives a homotopy commutative diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d, "v"] & \mathrm{Th}(u^*\xi \to A) \arrow[d, "\mathrm{Th}(u)"] \\ D \arrow[r] & \mathrm{Th}(\xi \to B) \end{tikzcd}$$

which in cohomology yields the required equation. From this point of view, the sign ambiguity arises from the two possible choices of Thom class for $u^*\xi$: the one compatible with the fundamental classes $[C]$ and $[A]$, or the pullback of the one compatible with $[D]$ and $[B]$. □

3 Proof of Proposition B

We can extend the definition of the classes $\tilde{\kappa}_{e^i p_I}$ to block bundles having fibres $W_{g,1}$ by filling in a disc in each fibre, giving a new block bundle with fibre $W_g := W_{g,1} \cup_\partial D^{2n} = \#^g S^n \times S^n$. There are therefore defined universal characteristic classes $\tilde{\kappa}_{e^i p_I} \in H^*(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); \mathbb{Q})$, by the proof of [8, Theorem 3.4].

In particular, we have a class $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n} \in H^{2n}(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); \mathbb{Q})$ which vanishes in $H^{2n}(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})$, because $e^2 = p_n$ on the total space of a smooth fibre bundle. Proposition B is an immediate consequence of the following.

Proposition 3.1 *For each $g \geq 1$ and each $n \geq 3$ there is a block bundle $(\pi : E \to |K|, \mathcal{A})$ with fibre $W_{g,1}$, such that*

(i) $\tilde{\kappa}_{e^2}(\pi) = 0 \in H^{2n}(|K|; \mathbb{Q})$,
(ii) $\tilde{\kappa}_{p_n}(\pi) \neq 0 \in H^{2n}(|K|; \mathbb{Q})$.

Therefore $\tilde{\kappa}_{e^2} - \tilde{\kappa}_{p_n} \neq 0 \in H^{2n}(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); \mathbb{Q})$.

Proof From Lemma 2.2 (iii) it follows that the $\tilde{\kappa}_{e^i}$ vanish for all $i > 0$ on all fibre homotopically trivial block bundles. We will therefore construct π to be fibre homotopically trivial, guaranteeing that $\tilde{\kappa}_{e^2}(\pi) = 0$.

We will use the (space-level) surgery fibration of Quinn [19], which following the discussion in [1, Section 3.2], in particular equation (43), may be put in the form

$$\left(\frac{\mathrm{hAut}_\partial(W_{g,1})}{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}\right)_{(1)} \longrightarrow \mathrm{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)} \xrightarrow{\sigma} \mathbb{L}_{2n}(\mathbb{Z})_{(1)}.$$

Thus to construct a fibre homotopically trivial block bundle over B (with some triangulation) it is enough to give a map $f : B \to \mathrm{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)}$ and a nullhomotopy of $\sigma \circ f$.

For simplicity of exposition we restrict to the case $n = 2k$. We let $B = S^n \times S^n$, write $a, b \in H^n(B; \mathbb{Q})$ for a hyperbolic basis, and write $e_1, f_1, \ldots, e_g, f_g \in H^n(W_{g,1}, \partial W_{g,1}; \mathbb{Q})$ for a hyperbolic basis. Write the nth Hirzebruch L-polynomial as $\mathcal{L}_n = Ap_n + Bp_{n/2}^2$ modulo other Pontrjagin classes, for some constants A and B. It is well-known that $A \neq 0$, and less well-known but true [21, Lemma A.1] that $B \neq 0$.

As the composition

$$p : G/O \xrightarrow{i} BO \xrightarrow{\prod p_i} \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i)$$

has homotopy fibre with finite homotopy groups, we claim that may find a map f whose adjoint $\hat{f} : (B \times W_{g,1}, B \times \partial W_{g,1}) \to (G/O, *)$ composed with i gives a class

$$\xi \in KO^0(B \times W_{g,1}, B \times \partial W_{g,1})$$

which has $p_{n/2}(\xi) = C \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi) = -\frac{2BC^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other rational Pontrjagin classes zero, for some constant $C \neq 0$. To establish this claim, let the map

$$\varphi : (B \times W_{g,1}, B \times \partial W_{g,1}) \longrightarrow \left(\prod_{i=1}^{\infty} K(\mathbb{Z}, 4i), * \right)$$

classify the pair of relative cohomology classes $L \cdot (a \otimes e_1 + b \otimes f_1)$ and $-\frac{2BL^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, for some integer $L \neq 0$ large enough that these classes are integral. For each $N > 0$ consider the map $\phi_N : \prod_i K(\mathbb{Z}, 4i) \to \prod_i K(\mathbb{Z}, 4i)$ which multiplies by N^i on $K(\mathbb{Z}, 4i)$. The diagram

$$\begin{tikzcd}
B \times \partial W_{g,1} \arrow[r, "\varphi|_{B \times \partial W_{g,1}}"] \arrow[d, hook] & * \arrow[r] \arrow[d] & G/O \arrow[d, "p"] \\
B \times W_{g,1} \arrow[r, "\varphi"] \arrow[urr, dashed, "\hat{f}"] & \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i) \arrow[r, "\phi_N"] & \prod_{i=1}^{\infty} K(\mathbb{Z}, 4i)
\end{tikzcd}$$

then admits a dotted lift $\hat{f}$ for N large enough, as the universal obstructions to finding such a lift lie in the cohomology of $\prod_i K(\mathbb{Z}, 4i)$ with finite coefficients, and are therefore annihilated (on each skeleton) by some ϕ_N. The resulting map $\hat{f}$ gives $p_{n/2}(\xi) = L \cdot N^{n/2} \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi) = -\frac{2BL^2}{A} \cdot N^n \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other Pontrjagin classes zero, as required (with $C = L \cdot N^{n/2}$).

We must show that the composition

$$B = S^n \times S^n \xrightarrow{f} \mathrm{map}_*(W_{g,1}/\partial W_{g,1}, G/O)_{(1)} \xrightarrow{\sigma} \mathbb{L}_{2n}(\mathbb{Z})_{(1)}$$

is nullhomotopic, but we shall allow ourselves to precompose f with self-maps $k_N : S^n \times S^n \to S^n \times S^n$ having degree $N \neq 0$ on both factors (such a precomposition preserves the form of Pontrjagin classes which has been arranged above). With this in mind, it is enough to show that

$$\sigma \circ f = 0 \in [B, \mathbb{L}_{2n}(\mathbb{Z})] \otimes \mathbb{Q}.$$

This group may be identified with $H^{4*}(B; \mathbb{Q})$. If $n \equiv 0 \mod 4$ then the component of degree $n = 2k = 4\ell$ is identified with the Künneth factor of

$$\tfrac{1}{8}\mathcal{L}_{3\ell}(\xi) \in H^{12\ell}(B \times W_{g,1}, B \times \partial W_{g,1}; \mathbb{Q})$$

in $H^{4\ell}(B;\mathbb{Q}) \cong H^{4\ell}(B;\mathbb{Q}) \otimes H^{8\ell}(W_{g,1}, \partial W_{g,1}; \mathbb{Q})$. But $\mathcal{L}_{3\ell}(\xi) = 0$ by observation, as only $p_{4\ell}(\xi)$ and $p_{2\ell}(\xi)$ are non-zero. Whatever the class of n modulo 4, the component of degree $2n = 4k$ is identified with the Künneth factor of

$$\tfrac{1}{8}\mathcal{L}_{2k}(\xi) \in H^{8k}(B \times W_{g,1}, B \times \partial W_{g,1}; \mathbb{Q})$$

in $H^{4k}(B;\mathbb{Q})$. But by construction

$$\mathcal{L}_{2k}(\xi) = A \cdot \left(-\tfrac{2BC^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1\right) + B \cdot (C \cdot (a \otimes e_1 + b \otimes f_1))^2 = 0.$$

We therefore obtain a map f, with $\sigma \circ f$ nullhomotopic and $i \circ \hat{f}$ classifying a vector bundle ξ' having $p_{n/2}(\xi') = D \cdot (a \otimes e_1 + b \otimes f_1)$, $p_n(\xi') = -\frac{2BD^2}{A} \cdot a \cdot b \otimes e_1 \cdot f_1$, and all other Pontrjagin classes zero, for some constant $D \neq 0$. (The constant will have changed when we precomposed the original choice of f with the maps k_N.) The associated block bundle $\pi : E \to |K| \approx B$ has $T_v^s E \simeq_s TE - \pi^* TB = \epsilon^{2n} + \xi'$ (see [8, Lemma 3.3]) and so

$$\tilde{\kappa}_{p_n}(\pi) = \pi_!(p_n(T_v^s E)) = \pi_!(p_n(\xi')) = -\tfrac{2BD^2}{A} \cdot a \cdot b \neq 0$$

as required.

It is not difficult to adapt the above argument to work for $n = 2k+1$. The essential point is that if we write $\mathcal{L}_n = Ap_n + Bp_{\frac{n-1}{2}} p_{\frac{n+1}{2}}$ modulo all other Pontrjagin classes, then $A \neq 0$ and again by [21, Lemma A.1] $B \neq 0$. We then take $B = S^{2k-1} \times S^{2k+3}$ and proceed as above. □

4 Rational connectivity of $\mathrm{Top}(2n)/\mathrm{O}(2n)$

Our goal in this section is to show how similar techniques to those we have been using imply the following.

Theorem 4.1 $\pi_*(B\mathrm{Diff}_\partial(D^{2n})) \otimes \mathbb{Q} = 0$ *for* $1 \leq * \leq 2n-5$.

This extends the analogous calculation of Farrell–Hsiang [10], which established the same result in degrees $1 \leq * \leq \phi(2n)$. By smoothing theory we have a homotopy equivalence $B\mathrm{Diff}_\partial(D^{2n}) \simeq \Omega_0^{2n}(\mathrm{Top}(2n)/\mathrm{O}(2n))$ as long as $2n > 4$, from which we deduce that

Corollary 4.2 $\mathrm{Top}(2n)/\mathrm{O}(2n)$ *is rationally* $(4n-5)$*-connected as long as* $n > 2$.

On the other hand, it has been shown by Weiss [21] that

$$H^{4n}(B\mathrm{Top}(2n);\mathbb{Q}) \longrightarrow H^{4n}(B\mathrm{O}(2n);\mathbb{Q})$$

has nontrivial kernel for $n \gg 0$ (namely, the class $e^2 - p_n$), so $\mathrm{Top}(2n)/\mathrm{O}(2n)$ is *not* rationally $(4n-1)$-connected.

Proof of Theorem 4.1 Let $W_g = \#^g S^n \times S^n \setminus \mathrm{int}(D^{2n})$, with $2n \geq 6$, and choose a collar $[0,1) \times \partial W_{g,1} \subset W_{g,1}$ and a disc $D^{2n} \subset (0,1) \times \partial W_{g,1}$. The map

$$\frac{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}{\mathrm{Diff}_\partial(D^{2n})} \longrightarrow \frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})} \tag{4.1}$$

is $(2n-4)$-connected, by Morlet's lemma of disjunction [6, Corollary 3.2]. Consider the fibration

$$\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})} \longrightarrow B\mathrm{Diff}_\partial(W_{g,1}) \xrightarrow{i} B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}).$$

Up to isotopy any diffeomorphism φ representing an element of $\pi_1(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}))$ may be supposed to be equal to the identity on the collar: the map (4.1) is then preserved by that induced by φ, and it then follows that φ acts trivially on $H^*\left(\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})}; \mathbb{Q}\right)$ in the range of degrees $* \leq 2n-4$ where the map (4.1) is a cohomology injection. Hence the Serre spectral sequence

$$H^p\left(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); H^q\left(\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})}; \mathbb{Q}\right)\right) \Longrightarrow H^{p+q}(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})$$

associated to this fibration has a product structure in this range of degrees. But Berglund–Madsen have shown that the map

$$i^* : H^*(B\widetilde{\mathrm{Diff}}_\partial(W_{g,1}); \mathbb{Q}) \longrightarrow H^*(B\mathrm{Diff}_\partial(W_{g,1}); \mathbb{Q})$$

is an isomorphism in degrees $* \leq 2n-1$ as long as $g \gg 0$. This result will appear soon in a revision of [2]. It follows that $H^q\left(\frac{\widetilde{\mathrm{Diff}}_\partial(W_{g,1})}{\mathrm{Diff}_\partial(W_{g,1})}; \mathbb{Q}\right) = 0$ for $1 \leq q \leq 2n-4$, and hence by the map (4.1) that $H^q\left(\frac{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}{\mathrm{Diff}_\partial(D^{2n})}; \mathbb{Q}\right) = 0$ for $1 \leq q \leq 2n-5$.

On the other hand, the surgery fibration sequence shows that $\frac{\mathrm{hAut}_\partial(D^{2n})}{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}$ is rationally acyclic, and $\mathrm{hAut}_\partial(D^{2n}) \simeq *$ by the Alexander trick, so $B\widetilde{\mathrm{Diff}}_\partial(D^{2n})$ is rationally acyclic. Boundary connect-sum makes this into an H-space, so it has trivial rational homotopy groups. Thus $\widetilde{\mathrm{Diff}}_\partial(D^{2n})$ has finitely-many components, and each one is rationally acyclic. The group $\mathrm{Diff}_\partial(D^{2n})$ has the same components, and so the quotient $\frac{\widetilde{\mathrm{Diff}}_\partial(D^{2n})}{\mathrm{Diff}_\partial(D^{2n})}$ is rationally homotopy equivalent to $B\mathrm{Diff}_\partial(D^{2n})_0$, the classifying space of the component of the identity in $B\mathrm{Diff}_\partial(D^{2n})$. It follows from the above that its rational cohomology, and hence rational homotopy, vanishes in degrees $1 \leq * \leq 2n-5$. $\square$

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