
Charged scalar fields on Black Hole space-times

Maxime Claude Robert Van de Moortel

Department of Pure Mathematics and Mathematical Statistics
University of Cambridge



Robinson College

March 2019

This dissertation is submitted
for the degree of *Doctor of Philosophy*.

Summary

Author: Maxime Claude Robert Van de Moortel

Title: Charged scalar fields on Black Hole space-times

The goal of this thesis is to study charged Black Holes in the presence of charged matter. To do so, we investigate the behaviour of spherically symmetric solutions of the Einstein-Maxwell-Klein-Gordon equations, which model the interaction of a charged scalar field with the electromagnetic field originating from the Black Hole charge. The particularity of this model is to putatively admit charged one-ended Black holes with a Cauchy horizon, and thus provides a framework to study simultaneously charged gravitational collapse and the Strong Cosmic Censorship conjecture. The latter problem is related to the question of Determinism of General Relativity, and roughly states that the maximal development of admissible initial data is inextendible. This question is intimately connected to the geometry of the Black Hole interior, which is studied in the first chapter of the present thesis. We prove that perturbed charged Black Holes form a Cauchy horizon which admits generically a singularity. This singularity in turn forms an obstruction to extending the maximal development. To obtain this result, we undertake an asymptotic analysis of the scalar field in the interior of the Black Hole, assuming its exterior region settles towards a stationary solution at a time decay rate that is expected by numeric and heuristic works. In the second chapter of this thesis, we retrieve these time decay rates for weakly charged scalar field on a fixed Reissner–Nordström Black Hole exterior. The result provides a proof of the (gravity-uncoupled) stability of Reissner–Nordström Black Hole exterior against small charged perturbations, which should also be considered as the first step towards the construction of one-ended charged Black Holes with a Cauchy horizon.

This thesis is dedicated to my mother, and
to the memory of my father, in the hope that he is watching.

Declaration

I hereby declare that

1. This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.
2. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text.
3. It does not exceed the prescribed word limit for the relevant Degree Committee.

This thesis is based on research conducted at the Department of Pure Mathematics and Mathematical Statistics between October 2015 and March 2019. None of the work is the outcome of a collaboration.

Chapter 2 is based on my published work:

Stability and instability of the sub-extremal Reissner–Nordström black hole interior for the Einstein–Maxwell–Klein–Gordon equations in spherical symmetry. Comm. Math. Phys., Volume 360, Issue 1, pp 103–168, 2018,

except for sections 2.6 and 2.7 which are unpublished in any form at the time of submission.

Chapter 3 is based on my work: *Decay of weakly charged solutions for the spherically symmetric Maxwell-Charged-Scalar-Field equations on a Reissner–Nordström exterior space-time*, arXiv:1804.04297.

Maxime Van de Moortel
Cambridge
14th March 2019.

Acknowledgements

I am immensely indebted to my Ph.D. advisor Jonathan Luk. Jonathan has been an extraordinary mentor, generously giving his time and sharing his mathematical wisdom on numerous occasions. I am particularly grateful to him for trusting me with interesting problems, for training me to solve them and for transmitting his refined taste in Mathematics. His approach of Science has been and continues to be a source of inspiration for me. As a mentor, Jonathan did not content himself with teaching me Mathematics; he taught me how to become a Mathematician. Being his student was both an honour, and a privilege.

On a more personal level, I would like to thank Jonathan for his patience, his continuous support, his availability, his kindness and his enthusiasm in our interactions and for letting me follow him to Stanford for two years, a decision that had a greater impact on my life trajectory than I ever could have expected.

I want to warmly thank Mihalis Dafermos for his kindness, his invaluable support, his encouragements and his presence, which considerably enlivens the General Relativity group in Cambridge. His philosophical vision of General Relativity that he was kind enough to share with me on numerous occasions, together with the insightful discussions that we had over the years, profoundly influenced my own vision of the field.

I wish to express my gratitude to Igor Rodnianski for his invaluable support, for the interest he showed in my work, and for taking the time to share his inspirational insights with me.

I want to express my gratitude to Sung-Jin Oh, for his kindness, his invaluable support and encouragement, and for the very stimulating discussions that we had on our multiple encounters.

I want to thank Stefanos Aretakis, Dejan Gajic, Georgios Moschidis and Yakov Shlapentokh-Rothman for insightful discussions, for their interest in my work, and for their willingness to explain me their work, and some general concepts in the field in my early years.

I would like to express my warm greetings to my friends belonging, or having belonged to the Mathematical General Relativity group in Cambridge: Christoph, Dejan, Gabriele, Grigorios, Joe, Rita and Tom.

From the bottom of my heart, I express my gratitude to Haydée Pacheco, for being who she is and for her presence by my side and in my soul, everywhere the wind decides to take us.

I am deeply grateful to my family, my grand-mother Jacqueline, my uncle Jacques, my mother Thérèse and my dearly missed father Denis. I am particularly thankful to my mother for igniting my intellectual curiosity, for her understanding and her continuous support in the whole course of my studies, which are now reaching their end. This thesis is dedicated to both my parents.

Chapter 1

Preface

*“Plonger au fond du gouffre,
Enfer ou Ciel, qu’importe ?
Au fond de l’Inconnu pour trouver
du nouveau !”*

Le voyage, Charles Baudelaire,
Les Fleurs du Mal.

When Baudelaire wrote these verses in 1857, how did he imagine the universe ? There is no doubt that his prescient visions encompassed its grandeur and its intricacies, in the words of a man of letters. More than a century and a half after “Les Fleurs du Mal”, the recent development of Physics, from Thermodynamics to Quantum Mechanics and General Relativity, may misguide a novice into believing that most of the cosmos is understood, controlled and that, after all, Physics has reached its end, to paraphrase Kelvin ¹.

Any scientist who reads these lines knows very well that nothing could be further from the truth. Nature is endlessly providing us with wonders, paradoxes, and amazes us always with its beauty, and its complexity. Arguably one of the most beautiful discoveries of modern science is the existence of Black Holes, collapsed stars devouring all the surrounding matter and light into darkness, eternal “black suns of melancholy” ², inaccessible to our senses, that have been beholding the rest of the universe from the early beginnings, long before Science made humanity aware of their very nature.

The existence of Black Holes is encoded into a strikingly simple description of gravitation, resulting from the curvature of space-time, itself caused ³ by the presence of matter or energy. This classical theory can be summed up in a remarkably elegant and concise formula: the tensorial Einstein equation

$$Ric(g) = \tilde{T},$$

where g describes the space-time, $Ric(g)$ its curvature, and \tilde{T} accounts for the distribution of matter, light and energy in the space-time. This equation is deceptively elementary: the movement of galaxies, the genesis of stars and their after-life, the formation of Black Holes, gravitational waves are all encompassed in its apparent simplicity.

“La Mathématique” ⁴, with her quiescent nature, “eternal and silent as matter” ⁵, shares her feature of unity with the Einstein equation. The mission of the mathematician, like the poet, is to unveil the secret of the universe through the intelligible reality, as opposed to the sensible reality, which is bound to the experiments. Black Holes are *par excellence* objects which should mostly be conceptualised, as their direct observation is not possible : literally, one must dive into the abyss, *plonger au fond du gouffre*, to discover the inner arcana of these fascinating astrophysical objects.

The Mathematician interested in General Relativity, in their quest for knowledge, dives in their intelligible self and enlightens the Unknown depths, *le fond de l’Inconnu*. *Son voyage* in a struggle with the difficult concepts of time, space and matter, will lead them to *l’Enfer* disguised as a god-abhorred naked singularity ⁶ or *au Ciel* if they discover that naked singularities are unstable.

¹ “There is nothing new to be discovered in physics now. All that remains is more and more precise measurement”, address to the British Association for the advancement of Science in 1900.

² El Desdichado, Gérard de Nerval.

³ “[The] investigator must feel the need of... knowledge of the immediate connections, say, of the masses of the universe. There will hover before him as an ideal insight into the principles of the whole matter, from which accelerated and inertial motions will result in the same way”, Ernst Mach, The Science of Mechanics; a Critical and Historical Account of its Development, 1906.

⁴ as defined in the “Éléments de Mathématique”, N. Bourbaki, 1939.

⁵ La Beauté, Charles Baudelaire, 1857.

⁶ “God abhors a naked singularity.” Stephen Hawking, A Brief History of Time.

One of the crucial questions that arises in the study of any physical theory, including General Relativity, is of an intelligible nature: are the fundamental laws of the universe deterministic ? This question, in the context of classical mechanics, is answered in the affirmative. Indeed Newton's laws, which play in the non-relativistic setting a unifying role similar to the Einstein equation, are of a deterministic nature, in the sense that the perfect knowledge of the initial conditions, notably position and velocity, predicts entirely the dynamics of a moving body.

Determinism is the soul of all classical theories, and, while it always occupied a special place in the heart of physicists, it experienced upheaval in the 20th century with the advent of Quantum Mechanics. At the centre of the controversy lies a scientific dispute between Niels Bohr and Albert Einstein. While the former, leader of the Copenhagen school, professed that the dynamics of atoms were ruled by probabilistic mechanisms, the latter rejected this interpretation, invoking the existence of "hidden variables". History retained Einstein's apocryphal ⁷ aphorism : "God does not play dice". Science later proved Einstein wrong: Quantum Mechanics indeed has a stochastic nature and there are no hidden variables, as was discovered in the foundational work of Alain Aspect ⁸ who experimentally confirmed Bell's inequalities ⁹. However, one can also argue that the probabilistic nature of Quantum Mechanics does not constitute *per se* a failure of determinism, in the sense that while particles positions are random (in particular, undetermined), their probability of presence is entirely determined by the initial conditions. Concretely, the Schrödinger equation, which governs the evolution in time of the probability density of presence of a quantum particle, is a deterministic equation.

In the context of the relativistic laws of gravitation, determinism can be formulated as follows: is the fate of massive objects, also called observers, entirely determined by the perfect knowledge of their initial condition? As a first attempt one can, therefore, embrace the definition of space-time as the collection of all its possible trajectories, and make sense of its deterministic character of a given space-time as the deterministic character of each trajectory. We must emphasise that a potential failure of determinism in General Relativity, as we discuss further in this preface, would be epistemologically of a much more problematic nature than the one feared by Einstein. This plausible disaster is mostly caused by the hyperbolic nature of the Einstein equation: if determinism fails, then the fate of some observers is unaccounted for. Thus, the theory has lost its predictive character and must be incomplete, even at the classical level. Worse, a negative answer to the great question of determinism, understood in the most radical way, may even shed trouble on our modern cosmogony, in particular, the anthropomorphic idea that the cosmos was born, at an initial time from which it developed into the ordered universe that we know today.

A satisfactory formulation of the respect of determinism of General Relativity was not attained until Penrose formulated ¹⁰ in 1974 a conjecture that now carries his name, and which we state in a modern terminology due to Christodoulou c.f. [14] and Dafermos c.f. [19]:

Conjecture (Strong Cosmic Censorship conjecture, modern version). *The maximal domain of predictability of generic space-times is inextendible as a solution of the Einstein equation.*

We emphasise that this conjecture is, at the time when these lines are written, still an open problem. The conceptual jump between Penrose and the formulation recorded in Dafermos lies in the geometric phrasing of the conjecture, now interpreted as a statement on the uniqueness of the solutions of an initial-value problem for the Einstein equation. This advance was possible notably thanks to the prior work of Choquet-Bruhat and Geroch ¹¹, who geometrically formalised the notion of "maximal domain of predictability". The major subtlety in the formulation of Strong Cosmic Censorship conjecture is the presence of the word "generic", signifying that there do exist space-times with extendible maximal domain of predictability but these space-times are, in a sense, exceptional.

Disturbingly, some of the simplest and most emblematic Black Holes space-times belong to the exceptional problematic category, like the Reissner–Nordström ¹² charged Black Holes, or the rotating Kerr ¹³ Black Holes, which are explicit solutions of the Einstein equation. Crucially, all these solutions are stationary, also called time-independent, meaning that they should be considered as final states of gravitational collapse and do not possess any dynamical components. *A posteriori*, we understand that this is the precise reason why they break

⁷The actual quote is "Quantum theory yields much, but it hardly brings us close to the Old One's secrets. I, in any case, am convinced He does not play dice with the universe", Einstein's letter to Max Born, 1926.

⁸Alain Aspect ; Quelques tests expérimentaux des fondements de la mécanique quantique (en optique), Qu'est-ce que l'Univers ?, Vol. 4 de l'Université de Tous les Savoirs (sous la direction d'Yves Michaux), Odile Jacob (2001).

⁹J. S. Bell, On the Einstein Podolsky Rosen Paradox, Physics 1 (1964), 195.

¹⁰C.f. "Gravitational collapse". In C. Dewitt-Morette, editor, Gravitational Radiation and Gravitational Collapse, Vol.64 of IAU Symposium, pages 82–91.

¹¹Global aspects of the Cauchy problem in general relativity, Communications in Mathematical Physics, December 1969, Volume 14, Issue 4, pp 329–335.

¹²"Über die Eigengravitation des elektrischen Feldes nach der Einsteinschen Theorie", H. Reissner; Annalen der Physik, 1916. "On the Energy of the Gravitational Field in Einstein's Theory", G. Nordström, Verhandl. Koninkl. Ned. Akad. Wetenschap., Afdel. Natuurk., Amsterdam, 1918.

¹³"Gravitational Field of a Spinning Mass as an Example of Algebraically Special Metrics", R. Kerr, Physical Review Letters. 11 (5): 237–238, 1963.

determinism, as it is now expected that any Black Hole formed by the gravitational collapse of a star approaches a Kerr Black Hole, but **does not** retain ¹⁴ its un-deterministic features.

As a consequence, the question of determinism of General Relativity cannot be solved in the sole framework of explicit solutions of the Einstein equation, and the dynamical aspects of Black Holes must be embraced by the mathematician who can rely on the powerful tools of modern analysis, and on the recent advances of the theory of partial differential equations.

We now return to the original formulation of Strong Cosmic Censorship, as stated by Penrose in 1972, in order to clarify the terminology. It was noticed in the 1960s that both Kerr and Reissner–Nordström Black Holes can be extended analytically with an extension featuring so-called time-like singularities, namely singularities that travel through space like observers. Singularities of that type provide a number of conceptual troubles and were deemed to be un-physical by Penrose:

Conjecture (Strong Cosmic Censorship conjecture, old version). *Generically, no time-like singularity ¹⁵ exists.*

While it is difficult to make a precise mathematical sense of the above conjecture, the terminology is clarified: a cosmic censor would prevent the appearance of time-like singularities, so that the theory remains physical. Not so dissimilar is the Weak ¹⁶ cosmic censorship of Penrose, formulated in 1969. This conjecture concerns so-called naked singularities, namely curvature singularities that are not cloaked inside a Black Hole. Those are also deemed to be un-physical, and thus would be concealed by a transcendent cosmic censor:

Conjecture (Weak Cosmic Censorship conjecture, old version). *Naked singularities, if they exist, are unstable.*

The formulation of Weak Cosmic Censorship, like its strong counterpart, is subtle. This is due to the existence of concrete space-times featuring naked singularities, that we must qualify of exceptional, in the same way that Kerr Black Holes are exceptional inside the class of dynamical rotating Black Holes.

A posteriori, and after half a century of mathematical exploration, it is understood that Weak and Strong Cosmic Censorship are two sides of the same coin and must be phrased in the language of well-posedness for hyperbolic systems. More precisely, Strong Cosmic Censorship can be thought of as a geometric formulation of (global) uniqueness of solutions to the Einstein equation, while Weak Cosmic Censorship relates to the existence of global ¹⁷ solutions.

The resolution of both these conjectures is sometimes considered as the holy grail of General Relativity. Still, to these days, a full resolution of the problem in the context of the Einstein vacuum equations, with no symmetry assumptions, seems out of reach. The state of the art is a positive resolution of Weak Cosmic Censorship on the one hand and of Strong Cosmic Censorship on the other hand, for two distinct simplified models, namely the spherically symmetric Einstein equation, coupled with two different types of matter fields. The study of the Einstein equation in the presence of matter fields is motivated by the absence ¹⁸ of dynamics of the vacuum Einstein equation in spherical symmetry and should be considered as an approximation of the real behaviour of the vacuum equation without symmetry.

In the sole presence of a (massless and uncharged) scalar field obeying the wave equation, the non-linear Einstein equation in spherical symmetry has been essentially completely understood in the monumental work of Christodoulou [11], [12], [13], culminating in the proof of the Weak Cosmic Censorship conjecture:

Theorem (Christodoulou, 1999). *For the spherically symmetric Einstein-scalar-field equations, naked singularities exist but are unstable.*

One of the core arguments of Christodoulou is a criterion for Black Hole formation: in the presence of a so-called short pulse, namely a localised and impulsive outgoing gravitational wave, a Black Hole forms and cloaks any naked singularity that might have been present. Another crucial complementary tool to obtain this result is a theory allowing for rough spherically symmetric initial data, belonging to a so-called bounded variation class (functions with integrable derivative), thanks to which Christodoulou formulates genericity (or instability) in a precise manner. It is epistemologically striking that, to obtain a satisfactory existence theory, the method embraced by Christodoulou is to work with rather rough ¹⁹ solutions to construct Black Holes ²⁰, which are in nature singular objects. Yet, paradise is regained as Black Holes immantle the potentially disastrous

¹⁴This is because the interior of the perturbed Black Hole converges only in a weak sense towards the interior of Kerr, a phenomenon that is related to the regularity discrepancy of smooth solutions that we describe further in the preface.

¹⁵While this formulation is historically enlightening and stresses the connection between Weak cosmic censorship and Strong cosmic censorship, the modern mathematical developments have put forth that, to quote Dafermos, “there is no such thing as a “timelike singularity” and we have to learn to talk about cosmic censorship without ever saying those words”.

¹⁶We emphasise that, despite the unfortunate terminology, the Weak cosmic censorship is not implied by the Strong cosmic censorship. These two conjectures are entirely complementary.

¹⁷The formulation of global existence is however delicate, as the presence of Black Holes is associated with the incompleteness of some geodesics.

¹⁸This is due to a rigidity result encoded in the Birkhoff Theorem.

¹⁹By this, we mean **smooth** solutions arising from data which are large in rough norms.

²⁰Those stable Black Holes, constructed as perturbations, bifurcate from the naked singularity.

naked singularities, proven to be dynamically unstable. This state of fact provides physical grounding to study the Einstein equation at a low level of regularity, which is another flourishing branch in the field of General Relativity.

The model of Christodoulou is, however, unsatisfactory with respect to Strong Cosmic Censorship. In particular, the space-times he considers can never converge towards a Reissner–Nordström or a Kerr Black Hole at late times, due to the absence of electric charge or angular momentum in the model at hand. Therefore, while Christodoulou’s space-times provide a suitable simplified setting to study Weak Cosmic Censorship, they fail to capture some major subtleties of Strong Cosmic Censorship and the instability of Reissner–Nordström or Kerr Black Holes, respectively among charged dynamical Black Holes and rotating ones. To study Strong Cosmic Censorship in a simplified setting, one must appeal to the Einstein–Maxwell-scalar-field model, which features an additional electromagnetic field, together with the same uncharged scalar field. This electromagnetic field is, in fact, the very same ²¹ as for the stationary Reissner–Nordström Black Holes and plays an analogous role as the angular momentum in Kerr Black Holes. In particular, a common feature between those two families is the existence of a Cauchy horizon, a future boundary belonging to the Black Hole interior and towards which all in-falling observers converge, but **do not** get destroyed, in the sense that they experience finite tidal deformations. This scenario contrasts with Christodoulou’s generic Black Holes, in which observers get ripped apart by blowing-up tidal deformations. For dynamical Black Holes allowing the presence of electric charge or angular momentum, a Cauchy horizon is still present, thus observers do not experience the same baneful fate as in Christodoulou’s case. However, a new instability mechanism arises under the form of blue-shift, an amplification of high frequencies of radiation, which originates from the application of geometric optics in the Black Hole interior. As it was understood in the pioneering work of Dafermos [19], [20], the first to carry out a mathematical study of the Einstein–Maxwell-scalar-field equations, this blue-shift mechanism gives rise to a mild so-called weak null singularity on the Cauchy horizon which is only present in the dynamical case ²².

Since it so happens that the Cauchy horizon is the boundary of the maximal domain of predictability, this weak null singularity, which purely arises from the dynamical components of the equations, precludes space-time from being extendible and therefore provides evidence in favour of the Strong Cosmic Censorship conjecture, as suggested in [19]. The key insight of Dafermos’ work is to understand the Cauchy horizon singularity as a breakdown in regularity: solutions of the Einstein equation with smooth data **fail** to be continuously differentiable ²³ as measured in a geometric way, i.e. in regular coordinates across the Cauchy horizon, due to the blue-shift effect.

The proof of any such statement, as hinted in [19], must rely on very fine estimates of various components of the non-linear Einstein equation for smooth data, in particular, asymptotic estimates ²⁴ at large time, in the Black Hole interior. Recently, the positive resolution of Strong Cosmic Censorship in the simplified setting of spherical symmetry, for the Einstein–Maxwell-scalar-field equations, was put forth by Luk and Oh in [57], [58], building on the seminal work ²⁵ by Dafermos and Rodnianski [25]:

Theorem (Luk–Oh, 2017). *For the spherically symmetric Einstein–Maxwell-scalar-field equations, the maximal domain of predictability is future inextendible as a classical ²⁶ solution of the Einstein equation.*

In their ground-breaking approach, Luk and Oh identify the specific mechanisms leading to the blue-shift instability and to the singularity of the Cauchy horizon. Their work proceeds with a thorough study of the Black Hole exterior as a necessary preliminary to the interior analysis. The strategy they employ encompasses stability aspects (a Black Hole with a Cauchy horizon must form in the perturbed space-time) and instability aspects (a new singularity arises on this Cauchy horizon in the dynamical ²⁷ case). It is important to emphasise that the stability estimates are crucial preliminaries to proving the instability. This is because the instability manifest itself by the presence of almost conserved quantities which blow up, modulo some error terms that must be controlled via stability estimates. This method is embracing a classical programme ²⁸ in Partial Differential Equations, in which lower bounds are proved by a combination of (approximate) identities, like conservation laws, and robust upper bounds, providing smallness in amplitude or time decay. Methodologically, it is interesting to highlight that Strong Cosmic Censorship, despite its formulation in terms of regularity discrepancy, is not proven via a theory of low regularity solution, unlike its weak counterpart à la Christodoulou, but instead by a precise asymptotic analysis of smooth solutions. While it has been long understood that the question of Black Holes orbital stability cannot be separated from their asymptotic stability ²⁹, we must emphasise that

²¹Due to the uncharged character of the scalar field, it does not interact with the electro-magnetic field which is thus static.

²²We emphasise that it is absent for the Reissner–Nordström Black Holes or Kerr Black Holes.

²³In fact, the best expected regularity is $W_{loc}^{1,1}$, as there has been convincing evidence that the $W_{loc}^{1,p}$ norm blows up for any $p > 1$.

²⁴In contrast with the proof of Christodoulou of Black Holes formations, which possess a more local nature.

²⁵This work [25] pioneered quantitative decay estimates in the Black Hole setting and was one of the first to understand that the structure of the Black Hole interior is strongly correlated to those decay estimates in the exterior.

²⁶The precise statement is “inextendible as a C^2 Lorentzian manifold”.

²⁷A genericity condition is needed for the singularity to form, in order to distinguish the dynamical case from the stationary one.

²⁸See for example the methods to obtain the formation of shocks in the Euler equations, c.f. [15].

²⁹In other words, with no symmetry assumptions, even to prove that small perturbations of a given Black Hole stay close to

the asymptotic estimates required to solve the Strong Cosmic Censorship are much more demanding³⁰ than the ones used to prove the stability of Black Holes.

While the foundational result of Luk and Oh brings hope to solve the full problem of Strong Cosmic Censorship outside of spherical symmetry, the model they consider is still not entirely satisfactory from the epistemological point of view. As was first hinted by Wheeler, electric charge is considered as a poor man’s version of angular momentum, mimicking the repulsive effects due to rotation while allowing the mathematician to remain in the simpler setting of spherical symmetry. Yet, in the model of Dafermos, Luk and Oh, this charge is static – not dynamic – and corresponds to the charge of a Reissner–Norström Black Hole. This is because the equations do not allow for a charged scalar field. Therefore, the uncharged field does not interact with the electromagnetic field, which thus satisfies the spherically symmetric Maxwell equations in vacuum; as a consequence, the electromagnetic field must have a constant charge and also possesses no magnetic component³¹. This rigidity triggers other unpleasant consequences: consistency imposes for instance that the initial data are posed on a two-ended cylindrical topology and that space-time features an un-physical parallel mirror universe. This setting is not suitable to study isolated gravitating systems, like a collapsing star transforming into a Black Hole, unlike Christodoulou’s model which, in contrast, possesses a Euclidean³² topology (one-ended case), and constitutes an ideal simplified setting to study gravitational collapse. Additionally, the model of Dafermos, Luk and Oh does not allow for the existence of naked singularities and thus does not provide a good framework to study Weak Cosmic Censorship as formulated above, which is trivially true in this setting.

To summarise the situation: in spherical symmetry, one can analyse the Einstein equation with an uncharged scalar field in two flavours: either without a Maxwell Field to study Weak Cosmic Censorship and the instability of naked singularities, following Christodoulou’s approach, or with a Maxwell Field to study Strong Cosmic Censorship, the formation of the Cauchy horizon and its generic singular character, à la Dafermos, Luk and Oh. Epistemologically it would be desirable to embrace both aspects, Strong and Weak censorships, in only one model, in the same way that any complete well-posedness theory for a given differential equation addresses simultaneously the existence of solutions and their uniqueness. Evidently, the vacuum Einstein equation without symmetry assumptions encompasses these two considerations, but as we explained, its complete treatment is still too complex to skip over intermediate steps. One possibility is to consider a generalisation of the model of Dafermos, Luk and Oh which allows the scalar field to be **charged**. The resulting Einstein–Maxwell-**charged**-scalar-field model is then freed from the drawbacks we mentioned in the former paragraph, as the charge becomes a dynamical quantity depending on the oscillations of the scalar field. Euclidean data are then allowed and it suddenly becomes possible to study both Weak and Strong Cosmic Censorships in a non-trivial setting, corresponding to the gravitational collapse of a charged, spherically symmetric star into a Black Hole. To study this more complex model, the mathematician must pay a price: most arguments that function well in the uncharged case fail, including all the consequences of some monotonicity properties that do not subsist in the charged setting. They must embrace innovative research paths to address the intricacies of the charged model, which is known to feature physical phenomena that are not present in different settings. Evidently, the hope stands out that their achievements will inspire robust methods that will also shed light on the resolution of the holy grail.

The analytic study of this charged scalar field model on Black Holes space-times is the subject of the present thesis, mostly motivated by Strong Cosmic Censorship. While we do not solve entirely the conjecture in this setting, we identify the important mechanisms behind its positive resolution and we provide conditional results that will hopefully become the building blocks of a future settlement of the question. One of the intermediate and more modest objectives that we would like to pursue, which seems almost in reach in view of the progress stated in this work, is the first construction of Black Holes emerging from gravitational collapse of “a reasonable matter field” and featuring a Cauchy horizon, as enabled by the charged model. While astrophysical Black Holes are expected to satisfy these two properties, no Black Hole which is at the same time one-ended and possesses a Cauchy horizon has ever been dynamically constructed. This gap is mostly due to the limitations of the uncharged field case that we described at length in the above paragraphs. In contrast, a careful understanding of the decay properties of charged scalar fields for large time, in the Black Hole exterior, would in principle allow for such the construction of a large class of Black Holes with an Euclidean topology also featuring a Cauchy horizon, with no further consideration³³. This is why we are hopeful that the analysis of the present manuscript will contribute to bridge this gap in the close future, and pave the way for further studies, notably related to the Weak Cosmic Censorship conjecture.

In addition to the preface, this manuscript is structured into two main chapters. In the first chapter, we the original, we must identify a stationary Black Hole (differing from the original) towards which the space-time converges, and quantify this convergence in terms of time decay rate.

³⁰In particular because they involve lower bounds and a thorough analysis in both the Black Hole interior and exterior.

³¹To be more precise, the “magnetic charges” purely arise from the topology and must be zero in the gravitational collapse case. Otherwise, they can be chosen to be take any constant value with no further incidence on the problem, c.f. [47].

³²By this, we mean that the initial data are posed on \mathbb{R}^3 .

³³In particular, it would not be required to understand the precise behaviour of the scalar field in the vicinity of the centre of symmetry, if we restrict our study to data whose charge is supported away from the horizon.

study the interior of Einstein–Maxwell-charged-scalar-field Black Holes in spherical symmetry, and provide a conditional proof of Strong Cosmic Censorship : providing the exterior is stable at an expected rate, the Cauchy horizon is (locally) singular. Additionally, we allow the scalar field to be massive (or massless) and we also present other results of physical interest, like the absence of a Cauchy horizon for Black Holes converging to the uncharged Schwarzschild background at infinity. In the second chapter, we retrieve the assumption on the exterior stability in the case of weakly charged scalar fields on a fixed Reissner–Nordström background. While the result of the second chapter cannot be strictly speaking combined with the one of the first chapter, which concerns the Einstein equation, one can argue that many technical difficulties in the Black Hole exterior are already addressed in our analysis and it seems reasonable to believe that the additional challenges related to the coupling with the Einstein equation are less severe than the ones encountered in the present manuscript. This is why we profess our optimism concerning the possibility to generalise the results of the second chapter to the Einstein–Maxwell-charged-scalar-field equations in spherical symmetry, thus bringing us closer to the full understanding of this little explored model.

Contents

| | | |
|----------|--|-----------|
| 1 | Preface | 5 |
| 2 | Reissner–Nordström interior stability and instability | 13 |
| 2.1 | Introduction | 13 |
| 2.1.1 | Context of the problem and earlier works | 14 |
| 2.1.2 | A first version of the main results | 17 |
| 2.1.3 | Ideas of proof and methods employed | 21 |
| 2.1.4 | Outline of the chapter | 25 |
| 2.2 | Geometric framework and equations | 25 |
| 2.2.1 | The equations in geometric form | 25 |
| 2.2.2 | Metric in null coordinates, mass, charge and main notations | 26 |
| 2.2.3 | The Reissner–Nordström solution | 27 |
| 2.2.4 | The Einstein–Maxwell–Klein–Gordon equations in null coordinates | 30 |
| 2.3 | Precise statement of the main results | 31 |
| 2.3.1 | Preliminaries on characteristic initial value problem and coordinate choice | 31 |
| 2.3.2 | The stability theorem | 33 |
| 2.3.3 | The instability theorem | 34 |
| 2.3.4 | The zero charge case and the absence of a Cauchy Horizon | 34 |
| 2.3.5 | A sufficient and a necessary condition for continuous extendibility | 35 |
| 2.4 | Proof of the stability Theorem | 37 |
| 2.4.1 | Strategy of the proof | 37 |
| 2.4.2 | A calculus lemma | 38 |
| 2.4.3 | The event horizon | 39 |
| 2.4.4 | The red-shift region | 42 |
| 2.4.5 | The no-shift region | 47 |
| 2.4.6 | The early blue-shift transition region | 49 |
| 2.4.7 | The late blue-shift region | 53 |
| 2.5 | Proof of the instability Theorem | 56 |
| 2.5.1 | Recalling the stability estimates | 56 |
| 2.5.2 | Reduction to the proof of (2.3.7) | 57 |
| 2.5.3 | Strategy to prove (2.3.7) | 58 |
| 2.5.4 | Up to the blue-shift region : the past of γ' | 59 |
| 2.5.5 | Towards the Cauchy horizon : the future of γ' | 60 |
| 2.6 | A criterion for continuous extendibility in the massive or strongly charged case | 62 |
| 2.6.1 | Preliminary estimates for $D_u\psi$ | 62 |
| 2.6.2 | A potential coordinate system (u, V) for a continuous extension | 64 |
| 2.6.3 | The continuous extendibility case | 65 |
| 2.6.4 | The continuous inextendibility case | 67 |
| 2.7 | Absence of the Cauchy Horizon for zero asymptotic charge | 69 |
| 2.8 | Estimates related to the apparent horizon \mathcal{A} | 71 |
| 2.9 | Estimates in \mathcal{N} , cutting the region in small pieces | 72 |
| 2.10 | Local existence for the characteristic initial value problem | 74 |
| 3 | Decay of charged scalar fields on Reissner–Nordström exterior | 77 |
| 3.1 | Introduction | 77 |
| 3.1.1 | Simplified version of the main results | 79 |
| 3.1.2 | Review of previous work and motivation | 81 |
| 3.1.3 | Methods of proof | 88 |
| 3.1.4 | Outline of the chapter | 92 |

| | | |
|--------|---|-----|
| 3.2 | Equations, definitions, foliations and calculus preliminaries | 93 |
| 3.2.1 | Coordinates and vector fields on Reissner–Nordström’s space-time | 93 |
| 3.2.2 | The spherically symmetric Maxwell-Charged-Scalar-Field equations in null coordinates . . | 94 |
| 3.2.3 | Notations for different charges | 96 |
| 3.2.4 | Foliations, domains and curves | 96 |
| 3.2.5 | Energy notations | 97 |
| 3.2.6 | Hardy-type inequalities | 99 |
| 3.3 | Precise statement of the main result | 100 |
| 3.3.1 | The main result | 100 |
| 3.3.2 | Energy boundedness and integrated local energy decay for small charge and scalar field energy data, no point-wise decay assumption | 102 |
| 3.3.3 | Energy time decay without arbitrary charge smallness or point-wise decay assumption, conditional on energy boundedness | 102 |
| 3.4 | Reduction of the Cauchy problem to a characteristic problem and global control of the charge . . | 104 |
| 3.5 | Energy estimates | 107 |
| 3.5.1 | An integrated local energy estimate | 108 |
| 3.5.2 | A Red-shift estimate | 114 |
| 3.5.3 | Boundedness of the energy | 115 |
| 3.6 | Decay of the energy | 120 |
| 3.6.1 | Preliminaries on the decay of the energy | 121 |
| 3.6.2 | Application for the r^p method for $p < 2$ | 124 |
| 3.6.3 | Extension of the r^p method to $p < 3$ | 126 |
| 3.7 | From L^2 bounds to point-wise bounds | 135 |
| 3.7.1 | Point-wise bounds near null infinity, “to the right” of γ | 135 |
| 3.7.2 | Point-wise bounds in the region between γ and γ_R | 137 |
| 3.7.3 | Point-wise bounds in the finite r region $\{r \leq R\}$ | 138 |
| 3.7.4 | Remaining estimates for $D_u\psi$ near \mathcal{I}^+ and $ Q - e $ near \mathcal{H}^+ | 140 |
| 3.8 | A variant of the decay theorem for more decaying initial data | 142 |
| 3.9 | Local boundedness proofs | 144 |
| 3.10 | Useful computations for the vector field method | 155 |
| 3.10.1 | Stress-Energy momentum tensor | 155 |
| 3.10.2 | Deformation tensors | 156 |
| 3.10.3 | Computations of bulk terms in the divergence formula | 156 |
| 3.10.4 | D’Alembertians | 156 |
| 3.10.5 | Volume forms, normals and current fluxes | 157 |

Chapter 2

Stability and instability of the sub-extremal Reissner–Nordström black hole interior for the Einstein-Maxwell-Klein-Gordon equations in spherical symmetry

We show non-linear stability and instability results in spherical symmetry for the interior of a charged black hole -approaching a sub-extremal Reissner–Nordström or Schwarzschild background fast enough- in presence of a massive and charged scalar field, motivated by the strong cosmic censorship conjecture in that setting :

1. *Stability of the Cauchy horizon in the Reissner–Nordström case* : We prove that spherically symmetric characteristic initial data to the Einstein-Maxwell-Klein-Gordon equations approaching a Reissner–Nordström background with a sufficiently decaying polynomial decay rate on the event horizon gives rise to a space-time possessing a Cauchy horizon in a neighbourhood of time-like infinity. Moreover if the decay is even stronger, we prove that the space-time metric admits a continuous extension to the Cauchy horizon. This generalizes the celebrated stability result of Dafermos for Einstein-Maxwell-real-scalar-field in spherical symmetry.
2. *Absence of the Cauchy horizon in the Schwarzschild case* : We prove that spherically symmetric characteristic initial data to the Einstein-Maxwell-Klein-Gordon equations approaching a Schwarzschild background with a sufficiently decaying polynomial decay rate on the event horizon **do not** admit a Cauchy horizon and possess a future boundary on which the area-radius yields 0 uniformly. This result constitutes the first proof of the absence of a Cauchy horizon when the charge is dynamical and converges to 0.
3. *Instability of the Cauchy horizon in the Reissner–Nordström case* : We prove that for the class of space-times considered in the stability part, whose scalar field in addition obeys a polynomial averaged- L^2 (consistent) lower bound on the event horizon, the scalar field obeys an integrated lower bound transversally to the Cauchy horizon. As a consequence we prove that the non-degenerate energy is infinite on any null surface crossing the Cauchy horizon and the Ricci curvature of a geodesic vector field blows up at the Cauchy horizon near time-like infinity. This generalises an instability result due to Luk and Oh for Einstein-Maxwell-real-scalar-field in spherical symmetry.

This instability of the black hole interior can also be viewed as a step towards the resolution of the C^2 strong cosmic censorship conjecture for one-ended asymptotically flat initial data.

2.1 Introduction

In this paper, we study the stability and instability of the Reissner–Nordström Cauchy horizon for the Einstein–Maxwell–Klein–Gordon equations in spherical symmetry :

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{KG}, \quad (2.1.1)$$

$$\mathbb{T}_{\mu\nu}^{EM} = 2 \left(g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \right), \quad (2.1.2)$$

$$\mathbb{T}_{\mu\nu}^{KG} = 2 \left(\Re(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2}(g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi} + m^2 |\phi|^2) g_{\mu\nu} \right), \quad (2.1.3)$$

$$\nabla^\mu F_{\mu\nu} = \frac{q_0}{2} i(\phi \overline{D_\nu \phi} - \overline{\phi} D_\nu \phi), \quad F = dA, \quad (2.1.4)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = m^2 \phi, \quad (2.1.5)$$

where the constants m^2 and q_0 are respectively called the mass and the charge ¹ of the scalar field ϕ .

This problem is motivated by Penrose's strong cosmic censorship conjecture (c.f section 2.1.1.) which claims that general relativity is a deterministic theory. The general strategy to address this question is to exhibit a singularity at the boundary of the maximal domain of predictability, which can be done with instability estimates.

For black hole exteriors that converge towards a Reissner–Nordström at time-like infinity, we prove that assuming an upper and lower bound on the scalar field ϕ on the event horizon of the black hole, the Cauchy horizon exhibits both stability and instability features, namely :

1. Stability : the perturbed black hole still admits a Cauchy horizon -near time-like infinity- like the original unperturbed Reissner–Nordström black hole, and in some cases we can even extend the metric continuously beyond this Cauchy horizon.
2. Instability : the curvature along the Cauchy horizon blows up, which represents an obstruction ² to a C^2 extension, at least near time-like infinity.

In contrast, if the black hole exterior settles towards a Schwarzschild background, then we show the **absence** of the Cauchy horizon and that on any null outgoing curve in the interior, the space-time area-radius converges (towards the future) to 0.

While similar results are known when the black hole exterior converges towards Reissner–Nordström, in the special case $m^2 = q_0 = 0$ see [20] and [57], the result for black hole exterior settling towards a Schwarzschild is entirely new. This is because this absence of the Cauchy horizon (in spherical symmetry) can only be seen through a charged matter model (c.f. section 2.1.2) which has not been as studied as its uncharged counterparts.

Coming back to the Reissner–Nordström case, if the field is massive or charged, the expected decay of the scalar field on the event horizon is much slower than in the $m^2 = q_0 = 0$ case, which makes the stability part more difficult. The previous instability result depends strongly on the special structure of the equation in the absence of mass and charge of the scalar field ³. When $q_0 \neq 0$ but $m^2 = 0$, a previous work of Kommami [47] shows a stability result but his assumed decay on the event horizon is only expected to hold for a sub-range of the charge q_0 that depends on the black hole parameters. In [57], the key argument for the instability is to use an almost conservation law that exists only in the absence of mass and charge. This is the underlying reason why [47] does not contain any instability result.

Our work can also be viewed as a first step towards the understanding of the spherically symmetric charged black holes with Euclidean initial data. This is because when the scalar field is uncharged, the total charge of the space-time arises completely from the topology. On the contrary, the model that we consider allows for a dynamical total charge which makes \mathbb{R}^3 type initial data possible.

The introduction is outlined as follows : in section 2.1.1 we present the strong cosmic censorship conjecture and mention earlier works, then in section 2.1.2 we explain the reasons to study a **charged** and **massive** scalar field and give the results of the present paper. We then sketch the methods of proof in the last section 2.1.3. Finally in section 2.1.4 we outline the rest of the paper.

2.1.1 Context of the problem and earlier works

Strong cosmic censorship conjecture

The study of self-gravitating isolated bodies relies crucially on the vacuum Einstein equation :

$$\text{Ric}_{\mu\nu}(g) - \frac{1}{2} R(g) g_{\mu\nu} = 0. \quad (2.1.6)$$

The simplest non-trivial solution, discovered by Schwarzschild is a spherically symmetric family of black holes, indexed by their mass. These black holes exhibit a very strong singularity, as observers that fall into them experience **infinite tidal deformations**.

¹This charge q_0 is also the constant that couples the electromagnetic and the scalar field tensors.

²Although an appropriate global setting - as opposed to the perturbative one that this paper is concerned with- is necessary to formulate the C^2 inextendibility properly.

³More precisely, in the work of Dafermos [20], it relies on a special mononicity property occurring only in that model.

A more sophisticated family of solutions indexed by mass and angular momentum and which describes rotating black holes has been discovered by Kerr in 1963. Unfortunately, Kerr's black holes have the very undesirable feature that they break determinism : the maximal globally hyperbolic development of their initial data is future extendible as a smooth solution to the Einstein equation (2.1.6) in many **non-unique** ways. In some sense, it represents a failure of global uniqueness of solutions.

One way to restore determinism which has been suggested by numerous heuristic and numerical works is that Kerr black holes feature of non-unique extendibility is **non-generic**, in other words whenever their initial data is slightly perturbed then the maximal globally hyperbolic development is actually future **inextendible** as a suitably regular Lorentzian manifold.

The nature of this singularity was controversial though : it was widely debated in the physics community whether perturbations of Kerr black holes exhibit a Schwarzschild black hole like singularity and observers experience infinite tidal deformations when they get close to it. One convenient way -although not exactly equivalent- to formulate this question geometrically is to study C^0 inextendibility.

The inextendibility question has been formulated by Penrose in the following conjecture :

Conjecture 2.1.1 (Strong Cosmic Censorship). *Maximal globally hyperbolic developments of asymptotically flat initial data are generically future inextendible as a suitably regular Lorentzian manifold .*

In the case of C^0 inextendibility, suitably regular is to be understood as continuous.

Remark 1. Without the word “generically”, the conjecture is false since Kerr black holes would provide counter examples, in the sense that they have a Cauchy horizon over which the metric can be smoothly extended in a non-unique way. Strong cosmic censorship claims that these counter examples are non-generic.

Due to the complexity of the Kerr geometry, early works on this problem studied instead Reissner–Nordström charged black holes. Although they are not solutions to the vacuum Einstein equation (2.1.6), they solve the Einstein-Maxwell equations :

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM}, \quad (2.1.7)$$

$$\mathbb{T}_{\mu\nu}^{EM} = 2 \left(g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \right), \quad (2.1.8)$$

$$\nabla^\mu F_{\mu\nu} = 0, \quad dF = 0. \quad (2.1.9)$$

Reissner–Nordström black holes have the same Penrose diagram as Kerr's but have the simplifying feature that they are spherically symmetric.

In their pioneering numerical work [75], Penrose and Simpson studied linear test fields on Reissner–Nordström black holes and discovered an instability of the Cauchy horizon.

Later Hiscock in [40], Poisson and Israel in [66], [67] and Ori in [65] exhibited - in a spherically symmetric but **non-linear** setting- a so-called weak null singularity with an expected curvature blow-up i.e a C^2 explosion of the metric, but **finite** tidal deformations allowing for a C^0 extension.

They studied the Einstein-null-dust equations which model non self-interacting matter transported on null geodesics : ⁴

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}, \quad (2.1.10)$$

$$\mathbb{T}_{\mu\nu} = f^2 \partial_\mu u \partial_\nu u + h^2 \partial_\mu v \partial_\nu v, \quad (2.1.11)$$

$$g^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad (2.1.12)$$

$$g^{\mu\nu} \partial_\mu v \partial_\nu v = 0, \quad (2.1.13)$$

$$g^{\mu\nu} \partial_\mu u \partial_\nu f + (\square_g u) f = 0, \quad (2.1.14)$$

$$g^{\mu\nu} \partial_\mu v \partial_\nu h + (\square_g v) h = 0. \quad (2.1.15)$$

From characteristic data featuring both outgoing and ingoing dust, they put forth the so-called “mass inflation” scenario, in which the Hawking mass, a quantity involving first derivatives of the metric blows-up, giving the first precise description of the Cauchy horizon instability.

In his seminal work [19], [20], Dafermos studied mathematically the **non-linear** stability of Reissner–Nordström black holes in spherical symmetry for the Einstein-Maxwell-Scalar-Field equations :

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{SF}, \quad (2.1.16)$$

⁴This model can be thought of as a high frequency limit, away from $\{r = 0\}$ of the Einstein-Scalar-Field model.

$$\mathbb{T}_{\mu\nu}^{EM} = 2 \left(g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \right), \quad (2.1.17)$$

$$\mathbb{T}_{\mu\nu}^{SF} = 2 \left(\partial_\mu \phi \partial_\nu \phi - \frac{1}{2} (g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi) g_{\mu\nu} \right), \quad (2.1.18)$$

$$\nabla^\mu F_{\mu\nu} = 0, \quad dF = 0, \quad (2.1.19)$$

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = 0. \quad (2.1.20)$$

Dafermos studied the interior of the black hole and proved conditionally the existence of a Cauchy horizon near time-like infinity with a C^0 extension for the metric, but C^1 inextendibility of the C^0 extension which manifests itself by the blow-up of the (Hawking) mass, which partially confirmed the insights from the work of Poisson–Israel and Ori.

Later Dafermos and Rodnianski in [25] proved a stability result on the black hole exterior (c.f section 2.1.23) which combined with [20] ruled out the C^0 inextendibility scenario :

Theorem 2.1.2 (Dafermos [20], Dafermos-Rodnianski [25]). *For the Einstein-Maxwell-Scalar-Field equations (2.1.16),(2.1.17),(2.1.18), (2.1.19), (2.1.20) in spherical symmetry, the C^0 formulation of strong cosmic censorship is false.*

The question was finally settled in the work of Luk and Oh [57], [58] : they confirmed the weak null singularity scenario, due to a curvature instability :

Theorem 2.1.3 (Luk-Oh [57], [58]). *For the Einstein-Maxwell-Scalar-Field equations (2.1.16),(2.1.17),(2.1.18),(2.1.19),(2.1.20) in spherical symmetry, the C^2 formulation of strong cosmic censorship conjecture is true.*

Earlier works relating to singularities at the Cauchy horizon

As sketched in the previous section, singularities are tightly related to the extendibility question. For the stability of the Cauchy horizon, recent progress have been made in different directions c.f [36], [50] for the linear stability, [56], [59] for the linear instability and [47] for the non-linear problem.

In this section, we review in more details stability and instability results in the black hole interior established in previous works leading to the proof of the C^2 strong cosmic censorship conjecture. These results should be compared to the main theorems of this paper, stated in section 2.3.

In [20], Dafermos proves a C^0 stability and a C^1 instability result of the Reissner–Nordstrom solution for an uncharged massless scalar field perturbation suitably decaying along the event horizon.

The instability essentially relies on a blow-up of the modified mass ϖ over the Cauchy horizon, as a consequence of a lower bound on the scalar field. Hence the metric is not C^1 extendible⁵ in spherical symmetry.

Theorem 2.1.4 (C^0 stability, C^1 instability, Dafermos [20]). *Let (M, g, ϕ, F) be a solution of the Einstein-Maxwell-Scalar-Field equations in spherical symmetry such that for some $s > \frac{1}{2}$, we have on the event horizon parametrized by the coordinate v as defined by gauge (2.3.3) of Theorem 2.3.2 :*

$$|\phi|_{|\mathcal{H}^+}(0, v) + |\partial_v \phi|_{|\mathcal{H}^+}(0, v) \lesssim v^{-s}.$$

Then :

1. **Existence of a Cauchy horizon** : *in a neighbourhood of time-like infinity, the space-time has the Penrose diagram of Figure 1.*
2. **Continuous extension** : *if moreover $s > 1$ then the metric g and the scalar field ϕ extend as continuous functions along the Cauchy horizon \mathcal{CH}^+ . Moreover, the extended metric can be chosen to be spherically symmetric.*
3. **Mass inflation and C^1 inextendibility** : *coming back to general case $s > \frac{1}{2}$, if we assume the following point-wise lower bound ⁶ on the scalar field for some $\epsilon > 0$:*

$$v^{-3s+\epsilon} \lesssim |\partial_v \phi|_{|\mathcal{H}^+} \lesssim v^{-s},$$

*then, the modified mass blows up as one approaches the Cauchy horizon : $\varpi(u, v) \rightarrow_{v \rightarrow +\infty} +\infty$ hence it is impossible to extend the metric g to a **spherically symmetric** C^1 metric across the Cauchy horizon \mathcal{CH}^+ . In particular the constructed C^0 extension is not C^1 .*

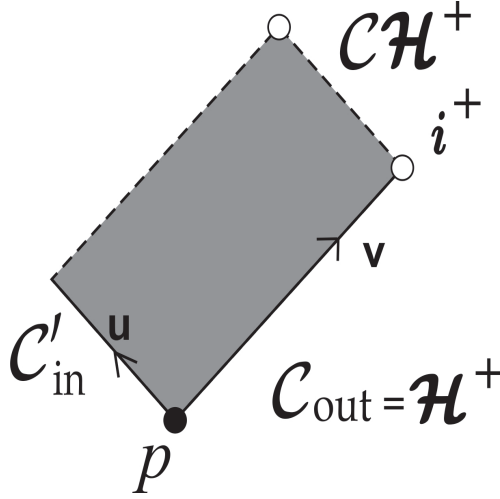


Figure 2.1: Penrose diagram for the characteristic initial value problem appearing in [20] .

In contrast, the C^2 strong cosmic censorship conjecture paper dealing with the black hole interior [57] relies on an averaged polynomial decay, as opposed to point-wise and proves a curvature instability :

Theorem 2.1.5 (C^2 instability Luk-Oh [57]). *Under the same hypothesis as Theorem 2.1.4, we also assume that $s > 2$ and the following lower bound holds for some $2s - 1 \leq p < 4s - 2$ and some $C > 0$:*

$$Cv^{-p} \leq \int_v^{+\infty} |\partial_v \phi|_{\mathcal{H}^+}^2(0, v') dv' \quad (2.1.21)$$

The solution admits a continuous extension \bar{M} across the Cauchy horizon.

*Then a component of the **curvature blows-up** identically along that Cauchy horizon.*

*As a consequence, (M, g, ϕ, F) is C^2 **future-inextendible**.*

Moreover $\phi \notin W_{loc}^{1,2}(\bar{M})$ and the metric is not in C^1 for the constructed continuous extension \bar{M} .

2.1.2 A first version of the main results

In this paper we prove that the expected asymptotic decay of the scalar field on the event horizon -known as generalised Price’s law- ⁷ implies some stability and instability features for a more realistic and richer generalization of the charged space-time model of Dafermos in spherical symmetry.

Instead of studying this problem starting from Cauchy data, we will only consider characteristic initial data on the event horizon with the “expected” behaviour. This should be thought of as an analogue of the previous black hole interior studies [20] and [57].

Motivation to study a massive and charged field and the results of the present paper

The goal of this paper is to generalise the known results for the Einstein-Maxwell-Scalar-Field equations near a Reissner–Nordström background to the case of a massive and charged scalar field model called Einstein-Maxwell-Klein-Gordon. Since the charge and the mass are a priori two different issues, we give motivation for each of them.

1. *A charged scalar field* : The model of Dafermos is a good toy model which gave very good insight on the Kerr case but it suffers from a major disadvantage : the topology of the initial data -i.e the initial time slice which is a Riemannian manifold- is constrained to be that of $\mathbb{S}^2 \times \mathbb{R}$ i.e **two-ended** initial data like for the Reissner–Nordström case. This does not seem so relevant to study isolated collapsing matter : we would like to consider **one-ended** initial data, diffeomorphic to \mathbb{R}^3 , but it is not possible in that model where the radius cannot go to 0 on a fixed time slice.

⁵It can also be proven that the mass blow-up implies also the blow-up of the Kretschmann scalar (c.f [47]) which establishes C^2 inextendibility without spherical symmetry assumptions.

⁶ This lower bound -although supported by numerical evidences- has never been exhibited for any particular solution.

⁷Namely an polynomial decay for an initially compactly supported scalar field on the event horizon of the black hole.

This fact is due to the topological character of the total charge of the space-time. This is better understood by the formula :

$$F = \frac{Q}{2r^2} \Omega^2 du \wedge dv,$$

where (u, v) are null coordinates built from the radius r and the time t , Q is the total charge of the space-time, Ω^2 is the metric coefficient in (u, v) coordinates (c.f section 2.2.2) and F is the electromagnetic field 2-form.

Heuristically we see that, if $Q \equiv e$ is fixed with $e \neq 0$, r is not allowed to tend to 0 without a blow-up of F (if the metric does not degenerate). For more details on these issues, c.f [47].

It turns out that if we impose that the scalar field is uncharged then the charge of the space-time Q is necessary fixed to be some $e \in \mathbb{R}$, as it will be seen in equations (2.2.20) and (2.2.21) of section 2.2.4.

This has two consequences :

- (a) It is not possible to study the presence of a Cauchy horizon dynamically (that we express in Theorem 2.1.8) in the uncharged case. Interestingly, in the uncharged field case, requiring $Q \rightarrow 0$ towards time-like infinity requires that $Q \equiv 0$, which forces the Maxwell form to vanish: therefore, there is no charge in the space-time and we are in the setting of Christodoulou [11], [12], [13].
 - (b) To study \mathbb{R}^3 initial data, which are adapted for self-gravitating systems, we must generalize Dafermos' model and study the Einstein-Maxwell-**Charged**-Scalar-Field equations.
2. *A massive scalar field* : Another variant is to allow for the scalar field to carry a mass, independently of the presence or absence of charge : it now propagates according to the Klein-Gordon equation :

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \phi = m^2 \phi. \quad (2.1.22)$$

One reason to study the Klein-Gordon equation is to understand the effect of a different kind of matter on the results of mathematical general relativity and the strong cosmic censorship in particular.

The Klein-Gordon equation is also fruitful to study **boson stars**. These uncharged objects -already present in the simple framework of spherical symmetry- in addition to being interesting for theoretical physics, give an example of a non-black-hole new "final state" of gravitational collapse.

More importantly, they are soliton-like (even though the metric is static), in particular they are **non-perturbative solutions** which do not converge towards a Schwarzschild or Kerr background ! They even exhibit a new behaviour as the scalar field is time-periodic in contrast to vacuum where periodicity is impossible (all periodic vacuum space-time are actually stationary, c.f [1]). If we let aside the fact that the scalar field is not stationary, boson stars are counter-examples to the generalized **no-hair conjectures** which broadly suggest that the set of stationary and asymptotically flat solutions to the Einstein equations coupled with any reasonable matter should reduce to a finite dimensional family indexed by physical parameters measured at infinity, like Kerr's black hole (indexed by mass and angular momentum) or Reissner-Nordström's (indexed by mass and electric charge). For more developments on boson stars, c.f [5].

Outside of spherical symmetry ⁸, a recent work of Chodosh-Shlapentokh-Rothman [7] constructs a continuous 1-parameter family of periodic space-times between a Kerr black hole and a boson star. Interestingly they exhibit solutions with exponentially growing modes, which is impossible in vacuum as proved (in the linear case) in [30] ! In contrast, LeFloch and Ma prove in [53] that the Minkowski space-time is stable for the Einstein-Klein-Gordon equations.

As a conclusion, the Klein-Gordon model enriches the dynamics of gravitational collapse and generates behaviours that are not present for a simple wave propagation. Despite these rich dynamics, the perturbative regime sometimes behave like the massless case as in [53] or the present paper, and sometimes behaves rather differently as in the work [7].

In this paper, we are going to consider both problems simultaneously by studying a charged and massive field propagating according to the Klein-Gordon equation (2.1.22). The full problem is written in section 2.2.1.

⁸Getting rid of the spherical symmetry assumption allows for a new very important physical phenomenon to arise, namely superradiance. This instability feature results in the presence of exponentially growing modes as discussed in [7] and [73].

3. *Mathematical differences with Dafermos' model* : After dealing with physical aspects, we want to emphasize the technical differences between our new model and the uncharged massless one.

A first remark is that the monotonicity of the modified mass as defined in (2.2.10) and that of the scalar field which is strongly relied on in the instability argument of [20] are no longer available.

More importantly, the expected asymptotics (Price's law (2.1.23)) of the field on the event horizon are different : in particular, the oscillations due to the charge should give only an averaged⁹ polynomial decay -as opposed to point-wise decay- and in many cases, the decay is expected to be always much weaker than for the uncharged and massless case. In particular it should be **non-integrable** in many cases of physical interest.

Moreover, the charge is no longer a topological constraint but a dynamical quantity which obeys an evolutionary P.D.E and that should be controlled like the scalar field or the metric which is what renders **one-ended** asymptotically flat initial data possible.

Price's law conjecture

We now state the expected asymptotics for the scalar field on the event horizon. This was first heuristically discovered by Price in [68] for the Schwarzschild solution, and proven rigorously by Dafermos and Rodnianski in [25] on dynamical spherically symmetric Black Holes and for an uncharged and massless field. The statement that the tail of the scalar field decays polynomially - for all models - is now called generalised Price's law.

This conjecture is still an open problem for the charged and massive model of the present paper and requires a stability study of the black hole exterior. The statement is however supported by numerical studies of the black hole exterior, c.f [6] and [52].

Conjecture 2.1.6 (Price's law decay). *Let (M, g, ϕ, F) be a spherically symmetric solution of the Einstein–Maxwell–Klein–Gordon system which is a perturbation of a Reissner–Nordström background of mass M and charge e satisfying $0 < |e| < M$, with a massive charged field ϕ which is sufficiently regular, of charge q_0 -as appearing in equations (2.2.20), (2.2.21)- and of mass m^2 -as appearing in the Klein-Gordon equation (2.1.22)- where Σ is an asymptotically flat complete Riemannian manifold initial data slice.*

Then on the event horizon of the black hole \mathcal{H}^+ parametrized by the coordinate v as defined by gauge (2.3.3) of Theorem 2.3.2, we have :

$$\phi|_{\mathcal{H}^+}(v) \simeq_1 f(v)v^{-s(e, q_0, m^2)}, \quad (2.1.23)$$

where \simeq_1 denotes the numerical equivalence relation of functions and their first derivatives when $v \rightarrow +\infty$, f is a periodic function and s is defined by :

$$s(e, q_0, m^2) = \begin{cases} \frac{5}{6} & \text{for } m^2 \neq 0, q_0 \neq 0, \\ 1 + \Re(\sqrt{1 - 4e^2 q_0^2}) & \text{for } m^2 = 0, q_0 \neq 0, \\ 3 & \text{for } m^2 = q_0 = 0. \end{cases} \quad (2.1.24)$$

Remark 2. Notice that $s(e, q_0, m^2) > \frac{1}{2}$ always but that the integral decay $s > 1$ holds¹⁰ only for $m^2 = 0$, $|e| < \frac{1}{2|q_0|}$. Since integrability is the crucial point in the C^0 extendibility proof, it explains why we required the field to be massless and not too charged to claim the C^0 extendibility.

Dafermos and Rodnianski in [25] first proved rigorously and in the non-linear setting an upper bound for Price's law in the uncharged and massless case $m^2 = q_0 = 0$.

Later, Luk and Oh proved in [58] the sharpness of this upper bound, still in the non-linear setting, as a consequence of a L^2 averaged¹¹ lower bound.

Statement of the main results

In this section we explain roughly the results of the present work. The stability result is very analogous to Dafermos' in [20] and the instability result is a local near time-like infinity version of Luk and Oh's interior instability of [57].

More precisely, we establish the following result :

⁹Which does not make a difference to prove the C^0 stability because we only need an upper bound but does for the C^1 instability where point-wise estimates are no longer enough.

¹⁰Note that the decay of the massless charged scalar field depends on the dimension-less quantity $q_0 e$ only.

¹¹Note that for the case $q_0 = 0$ it is expected that the function f is constant i.e the oscillations should not arise when the scalar field is uncharged. Nonetheless, no point-wise lower bound has ever been established, even for a particular solution.

Theorem 2.1.7. *We assume Price's law decay of conjecture 2.1.6 on the event horizon for a solution of the Einstein-Maxwell-Klein-Gordon system of section 2.2.1 in spherical symmetry.*

Then, we can define $e \in \mathbb{R}$ to be the asymptotic charge of the space-time measured on the event horizon ¹².

Then, if $e \neq 0$, near time-like infinity, the solution admits ¹³ *a Cauchy horizon emanating from time-like infinity* ¹⁴, *along which a C^2 invariant quantity* ¹⁵ *blows up.*

Moreover, in the massless and weakly charged case i.e. for $m^2 = 0$ and $4q_0^2 e^2 < 1$, the metric is C^0 extendible ¹⁶.

The proof relies on a non-linear stability and instability study of the Reissner–Nordström black hole interior. The C^0 extendibility was first proven by Dafermos in [20] in the uncharged and massless setting but it is really a direct adaptation of the methods of [57] that gives C^0 extendibility in the charged and massless (for $4q_0^2 e^2 < 1$ only) scalar field setting.

Remark 3. One actually needs a much weaker assumption than conjecture 2.1.6 : only a point-wise upper bound on the scalar field and its derivative is needed and an averaged L^2 lower bound on the derivative (c.f section 2.3 for a precise statement).

Remark 4. It is remarkable that the instability part relies only on an (averaged) lower bound on the scalar field but that no lower bound is required for the charge of the space-time.

Remark 5. We do not prove C^0 extendibility in the case $4q_0^2 e^2 \geq 1$ or $m^2 \neq 0$, which remains an open problem. However, we reduce the difficulty to proving that the scalar field is bounded ¹⁷, c.f. Theorem 2.3.5.

Remark 6. Even though we show that a C^2 invariant blows up, we do not show that given characteristic initial data on both event horizon satisfying our assumptions, the maximal globally hyperbolic development is (future) C^2 inextendible. This is because our result only applies in a neighbourhood of time-like infinity, in contrast ¹⁸ with [57], [58]. Nevertheless, it is likely that if one assumes that the data are everywhere close to Reissner–Nordström then one can use the methods of [57] to conclude C^2 inextendibility. We will however not pursue this.

Since the model we consider is charged, we can also consider, for the first time, what happens when the exterior of the Black Hole converges to Schwarzschild, namely $e = 0$. In this case, the result stands in strong contrast with Theorem 2.1.7:

Theorem 2.1.8. *Assume Price's law decay of conjecture 2.1.6 on the event horizon for a solution of the Einstein-Maxwell-Klein-Gordon system of section 2.2.1 in spherical symmetry, and define $e \in \mathbb{R}$ to be the asymptotic charge of the space-time measured on the event horizon as before.*

Then, if $e = 0$, then the solution does not have a Cauchy horizon emanating from time-like infinity, in the sense that no null boundary over which r is bounded away from 0 can be attached to the space-time.

Remark 7. Using the results of [47], we can infer that, for one-ended solutions, space-time admits a future boundary, emanating from i^+ , over which $r \equiv 0$ identically. As a consequence, the Kretschmann scalar blows up on this boundary piece, which provides (at least locally) the C^2 inextendibility of the metric, like in the $e \neq 0$ case.

The causal structure of this boundary, however, remains unknown. In particular, Theorem 2.1.8 does not exclude the case where space-time admits a null boundary over which $r = 0$.

In the last result that we introduce and using the estimates of the present manuscript, we establish a criterion for continuous extendibility:

Theorem 2.1.9. *Under the assumptions of Theorem 2.3.2, we have the following:*

1. *If the scalar field is bounded and possesses a limit towards the Cauchy horizon on the whole Penrose diagram of Figure 2.1, then (M, g) is extendible as a continuous Lorentzian manifold.*
2. *If the scalar field blows up uniformly towards the Cauchy horizon, then (M, g) is (locally) continuously inextendible, in the sense of Definition 3.*

¹²It corresponds to the parameter e of the sub-extremal Reissner–Nordström background (M, e) towards which our space-time converges on the event horizon.

¹³More precisely, the Penrose diagram -locally near timelike infinity- of the resulting black hole solution is the same as Reissner–Nordström's as illustrated by Figure 1.

¹⁴On the other hand in general the metric may not extend even continuously to that Cauchy horizon.

¹⁵Namely $\text{Ric}(V, V)$ where V is a radial null geodesic vector field that is transverse to the Cauchy horizon.

¹⁶In the upcoming [44], we extend the C^0 extendibility statement to the massive and/or strongly charged case, using a different approach.

¹⁷In contrast, if the scalar field blows up on the Cauchy Horizon, we prove that **continuous extendibility fails**.

¹⁸In [57] a special monotonicity property is exploited to propagate the curvature blow-up along the whole Cauchy horizon. Such a property is absent when $q_0 \neq 0$ or $m^2 \neq 0$.

This result is one of the cornerstones of [44], in which we prove that the first case holds under the assumptions of Price's law (conjecture 2.1.6), and also produce examples (not respecting conjecture 2.1.6) satisfying the conditions of the second case. This new physical phenomenon is made possible thanks to the weak decay of massive fields, in contrast to their massless counterparts. In particular, as we establish in Theorem 2.3.2, if the scalar field is massless **and** weakly charged, the second case never arises.

2.1.3 Ideas of proof and methods employed

In this last introductory section, we describe the techniques that we use to prove our main results as stated in section 2.3 later. Some methods are adapted and modified from the work [57] for the stability part and [59] for the instability part.

Methods for the stability part

In the $m^2 = q_0 = 0$ case, stability was first proven by the seminal work of Dafermos [20] in the case $s > \frac{1}{2}$. His work considers geometric quantities (r, ϕ, ϖ) where ϖ is the modified mass defined in (2.2.10), r is the area-radius and ϕ is the scalar field. However, these quantities do not decay - in particular ϖ blows-up. Remarkably, this was overcome using the very special structure of the equation. This structure is not exhibited when the mass or charge of the scalar field are present.

In contrast, the approach of Luk and Oh in [57] controls a non geometric coordinate dependent quantity Ω^2 namely the metric coefficient (c.f section 2.2.2 for a definition). They actually compare (Ω^2, r, ϕ) to their counterpart $(\Omega_{RN}^2, r_{RN}, 0)$ on the Reissner–Nordström background to which the space-time converges.

This has the advantage that the **difference** of these quantities and their degenerate derivatives are bounded and in fact decay towards infinity, allowing for a C^0 stability statement.

They establish this decay using the **non-linear wave structure in a null foliation (u, v)** -as illustrated by Figure 2.2- of the equation. They integrate the difference along the wave characteristics with the help of a **bootstrap** method after splitting the space-time into smaller regions.

The result of Luk and Oh is therefore more quantitative but on the other hand it relies crucially on the hypothesis $s > 1$ giving an initial **integrable decay** of $\Omega^2 - \Omega_{RN}^2$, $r - r_{RN}$ and ϕ .

This is why -although the method can be easily adapted in the presence of a charged and massive field- the proof fails ¹⁹ for $s \leq 1$ which is unfortunately the expectation in many interesting cases as claimed by Price's law of conjecture 2.1.6.

In our proof, we will again control the non-geometric coordinate dependent metric coefficients Ω^2 but since the decay is so weak we cannot consider directly the difference with the background value.

Instead, we consider new natural combinations of these quantities -adapted to the geometry- which obey better estimates, notably those involving the degenerate derivatives ∂_u and ∂_v .

In all previous work²⁰, the proof proceeds in splitting the space-time into a **red-shift region** near the event horizon which is very stable and a **blue-shift region** near the Cauchy horizon where many quantities can blow-up. This is illustrated by Figure 2.2.

In our case, we follow a similar philosophy although we need to further divide the space-time into more regions in view of the slow decay of the scalar field c.f Figure 2.3.

In the red-shift region, decay is proven using that $|\frac{-4\partial_u r}{\Omega^2} - 1|$ and $|\frac{-4\partial_v r}{\Omega^2} - 1|$ decay polynomially ²¹, thanks to the Raychaudhuri equations, which allows us to **replace $\partial_v r$ and $\partial_u r$ by $\Omega^2 \asymp e^{2K+\cdot(u+v)}$** which enjoys an exponential structure. Thus, we do not lose one power when we integrate a polynomial decay on a large region c.f Lemma 2.4.1.

In the blue-shift region, we essentially use the **polynomial decay of $\partial_v r$, $\partial_u r$ and the exponential decay of Ω^2** to propagate the estimates.

Another important point is that we are able to find two **decaying quantities** ²² which capture the red and blue shift effect : $\partial_u \log(\Omega^2) - 2K$ and $\partial_v \log(\Omega^2) - 2K$ -where K is a geometric quantity defined by (2.2.12) - and we control the sign of K : positive in the red-shift region, negative ²³ in the blue-shift region.

In particular the good control of $\partial_v \log(\Omega^2) - 2K$ can be fruitfully integrated to control **the smallness of Ω^2 according to the different regions** but requires a bit of care close to the Cauchy horizon where $\partial_v \log(\Omega^2) - 2K$ is no longer integrable in general.

¹⁹Essentially because $\Omega^2 - \Omega_{RN}^2$, $r - r_{RN}$ and ϕ are no longer integrable.

²⁰ Notably in Dafermos' proof, the gauge derivatives of the scalar field $\frac{\partial_u \phi}{\partial_u r}$ and $\frac{\partial_v \phi}{\partial_v r}$ decay in the red-shift region and grow in the blue-shift region.

²¹Note that on Reissner–Nordström, these quantities are zero.

²² These two quantities are zero on a Reissner–Nordström background so we can expect them to be small in the perturbative setting.

²³Except maybe close to the Cauchy horizon where K may blow-up like the Hawking mass.

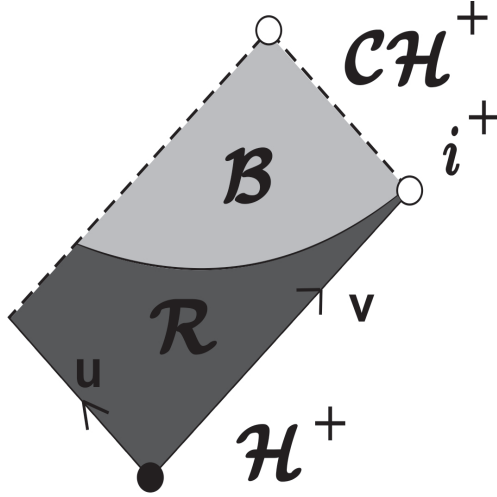


Figure 2.2: Penrose diagram illustrating the division between a red-shift \mathcal{R} and a blue-shift region \mathcal{B} .

To sum up, unlike the strategy of [57] which purely deals with differences whose decay is propagated like a wave, we mainly use propagative arguments for the scalar field only and rely on the geometry of the space-time and on the Raychaudhuri equations (2.2.17), (2.2.18) to prove our estimates.

Methods for the instability part

The first instability result is due to Dafermos in [20]. Like its stability counterpart, it relies crucially on the special structure of the equation and notably a very specific monotonicity property that does not hold in the presence of a massive or charged scalar field.

The work [57] also proves an instability statement. Nevertheless both the presence of the mass or of the charge also destroy the main argument. Indeed the argument makes use of an almost conservation law for the scalar field stress-energy tensor \mathbb{T}^{SF} . With a non-zero mass, a new term appears (c.f (2.1.3)) which has the wrong sign and cannot be easily controlled. If the field is charged, in contrast the two conservation laws -previously independent- coming from \mathbb{T}^{SF} and \mathbb{T}^{EM} are now coupled and therefore Luk and Oh's method does not apply.

Instead, we borrow ideas from a paper of Luk and Sbierski [59] in which the authors prove the linear instability of Kerr's interior. They simplify their methods and adapt them to the Reissner–Nordström case ²⁴ in an introductory section. The point is essentially to prove the blow-up of $\partial_V \phi$ on a constant u hypersurface close to the Cauchy horizon, where (u, V) is a regular coordinate system near the Cauchy horizon thanks to a **polynomial lower bound on $\int_v^{+\infty} |\partial_v \phi|^2(u, v') dv'$** .

For this they use an integrated L^2 stability estimate coupled with a vector field method²⁵ - namely an energy estimate- using the Killing vector field $\partial_t = \partial_v - \partial_u$ -which boils down to the **conservation of the energy**. They manage to control the integral of $\partial_v \phi$ on the event horizon by its values on an intermediate curve γ_σ (which marks the limit between their red-shift and their blue-shift region) **on which Ω^2 decays polynomially** like $v^{-\sigma}$ for a very large power $\sigma > 0$.

After they control this value by the integral of $\partial_v \phi$ on a constant u hypersurface close to the Cauchy horizon using again a vector field method with the vector field ∂_v . They conclude using **the positivity of the energy** which allows for the ∂_v terms to control the $\partial_t = \partial_v - \partial_u$ ones on γ_σ .

Their approach relies on the linearity of the problem and, in particular, the use of a Killing vector field of the Reissner–Nordström background, which does not exist any more in the non-linear setting that we consider.

Another important difference is the existence -in the uncharged field case- of two independent (approximate) conservation laws, namely one for the scalar field \mathbb{T}^{SF} -which the authors of [59] use- and one for the electromagnetic field \mathbb{T}^{EM} - which they ignore. In our case the charged field interacts with the charge of the black hole coupling the Klein-Gordon and the Maxwell equation. This gives a single (approximate) conservation law involving $\mathbb{T} = \mathbb{T}^{KG} + \mathbb{T}^{EM}$.

Moreover, the use of a vector field method in a blue-shift region for a charged and massive scalar field

²⁴For a scalar field that is not necessarily spherically symmetric, unlike in the present paper.

²⁵For an introduction to the vector field method and interesting applications c.f [26].

generates terms which do not decay, in particular, those related to the charge ²⁶ of the black hole Q and which have the inadequate sign.

Fortunately **in the red-shift region the charge terms have a good sign** and the estimates of our stability part are strong enough to prove **decay of the scalar field terms** having the wrong sign.

Moreover, despite Killing vector fields do not exist in general, the Kodama vector field T -which is the non linear analog of ∂_t - induces a conservation law, which renders possible the use of a **vector field method in the red-shift region**.

There is however a difficulty : the coefficients of the Kodama vector field, unlike ∂_t , are expected to blow-up near the Cauchy horizon in general so the limiting curve γ' between the red-shift and the blue-shift region -unlike in [59]- must be **close enough to the Cauchy horizon** so that we enjoy a sufficient decay of Ω^2 in the future to propagate the decay of the wave equations but must also be **close enough to the event horizon** so that the Kodama vector field does not blow-up ! Compared to [59] where the limiting curve was chosen to be as far as possible in the future, this is a completely different strategy.

This challenge is addressed using fine **stability estimates**, notably the quantities $\frac{-4\partial_u r}{\Omega^2}$ and $\frac{-4\partial_v r}{\Omega^2}$ which are precisely the coefficients of T and that are controlled in the vicinity of γ' .

In the blue-shift region, since vector field methods are now hard to use, we simply **propagate point-wise** $\partial_v \phi$ using the wave equation and the sufficient decay of Ω^2 in the future of γ' . We strongly rely on the stability estimates proven in the first part.

Lastly, once this lower bound is proven, we use exactly and without modifications the techniques employed in [57] to prove the blow-up of a C^2 geometric invariant quantity for any $s > \frac{1}{2}$ and the H^1 blow up of the scalar field if $s > 1$, leading to the C^1 inextendibility of the C^0 extension constructed in the stability part.

Methods for the continuous extendibility/inextendibility across the Cauchy horizon for massive or strongly charged fields

When the scalar field enjoys sufficient decay $|\phi|_{\mathcal{H}^+}(v) + |\partial_v \phi|_{\mathcal{H}^+}(v) \lesssim v^{-s}$, $s > 1$, we prove (with Theorem 2.3.2) that the scalar field is bounded up to the Cauchy horizon, in fact the metric (M, g) extends to a continuous Lorentzian metric to which both matter fields ϕ and F extend continuously. According to the massive/charged versions of Price's law (conjecture 2.1.6), an integrable inverse-polynomial tail, as described above, emerges from regular Cauchy data providing the scalar field is massless and uncharged (as it was already proven in [57]), or massless and weakly charged, i.e. the inequality $4(q_0 e)^2 < 1$ holds.

We remind the reader that we obtained this result constructing a "regular" coordinate system (u, V) across the Cauchy horizon. This procedure provides a natural extension: the continuous Lorentzian manifold \tilde{M} , on which (u, V) is (locally) a regular coordinate system. In this language, we prove that we can construct an extension $\phi \in W_{loc}^{1,1}(\tilde{M})$, and, at an even stronger regularity, $\nabla \phi \in L_{loc}^1(\mathbb{R}_V, L_{loc}^\infty(\mathbb{R}_u))$. Since the intersection of these spaces embed in $C^0(\tilde{M})$, we prove that ϕ possesses a continuous extension. To prove that (M, g) is continuously extendible we can exploit a quantified version of the strong regularity $\nabla \phi \in L_{loc}^1(\mathbb{R}_V, L_{loc}^\infty(\mathbb{R}_u))$, with decay rates. The estimate we obtain is so strong that it is sufficient to address the quadratic non-linear terms in the Einstein equations, notably (2.2.16), and to prove eventually that, in the (u, V) coordinate system, the metric coefficient Ω_{CH}^2 obeys $\log(\Omega_{CH}) \in W_{loc}^{1,1}(\tilde{M})$, and $\nabla \log(\Omega_{CH}) \in L_{loc}^1(\mathbb{R}_V, L_{loc}^\infty(\mathbb{R}_u))$, which, a fortiori ²⁷, allows us to prove continuously extendibility.

Taking a step back, it may not be so surprising that we can prove $\phi \in W_{loc}^{1,1}(\tilde{M})$ in the case $s > 1$. Indeed, providing the point-wise bound $|\partial_v \phi| \lesssim v^{-s}$ is propagated (which is in a sense the heart of the analysis of Theorem 2.3.2), then the following estimate holds for a regular coordinate V across the Cauchy horizon ($V(v = +\infty) = 1$):

$$\int_v^{+\infty} |\partial_v \phi|(u, v') dv' = \int_{V(v)}^1 |\partial_V \phi|(u, V') dV' \lesssim \int_v^{+\infty} (v')^{-s} dv' \lesssim 1.$$

The key point is of course the invariance of these integrals by the $v \rightarrow V(v)$ transformation, which provides an estimate in regular coordinates in \tilde{M} , namely a geometric estimate $\phi \in W^{1,1}(\tilde{M})$. This feature is not true $W^{1,p}(\tilde{M})$ for $p > 1$ and we indeed prove that $\phi \notin W^{1,2}(\tilde{M})$ in Theorem 2.3.3. Our C^2 inextendibility result is implied by a blow-up phenomenon for the $W^{1,p}(\tilde{M})$ norms²⁸. This blow-up itself takes its origins in the degenerate character of the coordinate vector field ∂_v , which is a fundamental consequence of the so-called blue-shift instability.

²⁶Which is expected to tend to a constant e so that we cannot hope for decay, unlike for ϕ which is zero on the underlying Reissner-Nordström background.

²⁷Indeed, the continuous extendibility of the function r is always true, for purely geometric reasons, as shown by Kommemi [47].

²⁸In fact, for the uncharged and massless case, it can be proven that for some Reissner-Nordström parameters, the $W^{1,p}(\tilde{M})$ norms blow up for **all** $p > 1$, c.f. [38].

Now, we return to the main latent question of this section: what happens if $s \leq 1$, a case which occurs when the scalar field is massive, or massless and strongly charged, i.e. $4(q_0 e)^2 \geq 1$? In this case, the reasoning that we exposed earlier fails as

$$\int_v^{+\infty} (v')^{-s} dv' = +\infty.$$

For this reason, we expect that in this case, even if regular coordinates (u, V) and a continuous extension \tilde{M} could be constructed, we would have ²⁹, in contrast $\phi \notin W_{loc}^{1,1}(\tilde{M})$. A posteriori, it seems impossible to try to prove so we cannot obtain boundedness or continuous extendibility through that route.

In fact, as we demonstrate in [44], both behaviours are possible in general: there exists data on the event horizon (for the massive case) which give rise to a bounded scalar field on the whole Penrose diagram, and data giving rise to a point-wise blow-up: $\phi \notin L^\infty(M)$. As we show, the determining factors are the oscillation of ϕ on \mathcal{H}^+ , and, in particular, the blow-up of $|\int_v \phi|_{\mathcal{H}^+}(v') dv'|$ giving rise to $\phi \notin L^\infty(M)$. Thus, we capture these oscillations by Fourier methods, for the linear problem $\square_{g_0} \phi_0 = 0$, where g_0 is a fixed Reissner–Nordström interior background. Then, using the method of the present thesis (in physical space), we prove that the difference between the non-linear solution ϕ and the linear solution ϕ_0 **decays** at an ³⁰ integrable rate $s' > 1$, which allows us to conclude that the difference $\phi - \phi_0$ is bounded and continuously extendible, in fact, $\phi - \phi_0 \in W_{loc}^{1,1}(\tilde{M})$, if \tilde{M} exists. Therefore, after some work, one can reduce the problem to the understanding of a linear problem on a fixed Reissner–Nordström interior background.

Coming back to the object of the present section, the remaining task, that is carried out in Theorem 2.3.5, is to connect this behaviour to the geometric extendibility/inextendibility of the metric. Even in the favourable case where $\phi \in L^\infty(M)$, we cannot use the brute force estimate of the $s > 1$ case since ϕ is no longer in $W_{loc}^{1,1}$. Instead, we must realize, from (2.2.16), that $\square_g \log(\Omega^2)$ behaves like $-\square_g(|\phi|^2)$. Therefore, we establish an estimate of the form

$$\square_g(\log(\Omega^2) + |\phi|^2) = \text{errors},$$

and from this, we can produce a coordinate system (u, V) , in which the metric coefficient Ω_{CH}^2 behaves well, in the sense that $\log(\Omega_{CH}^2) + |\phi|^2 \in L^\infty(M)$ and ³¹ is in fact continuously extendible.

After some work, we see the continuous extendibility of (M, g) follows immediately from the continuous extendibility of $|\phi|$. It remains to prove the opposite direction, i.e. proving continuous inextendibility when the scalar field blows up uniformly. Of course, the fact that $\log(\Omega_{CH}^2) \notin L^\infty(M)$, a statement that follows immediately from our analysis, is **not sufficient** to rule out continuous extendibility, as it is merely a failure of the coordinate system (u, V) . However, two arguments indicate that we obtain, in fact, a geometric continuous inextendibility statement:

1. the coordinate system (u, V) is regular across the Cauchy horizon, namely $V < 1$ and $\lim_{v \rightarrow +\infty} V(v) = 1$.
2. The blowing-up factor $|\phi|^2$ is a geometric one, i.e. that cannot be "factored out" by a change of coordinate.

To formalise these intuitions, we must come up with two restrictions: first, inextendibility is only formulated in a neighbourhood of time-like infinity, as this is the only locus in which our estimates are available. Second, we must, at present, only exclude continuous extensions which possess a system of double-null coordinates, to make use of our formalism. However, we do not require these extensions to be spherically symmetric.

This restriction, however, is purely technical and one can hope that a less restrictive formulation could be found in the future. Our construction provides the first continuous inextendibility result that is not due to a $\{r = 0\}$ boundary, in the presence of a Cauchy horizon (in contrast with the C^0 inextendibility proof of Schwarzschild, c.f. [71]).

Methods for the zero charge case

To prove Theorem 2.1.8, we must remark that the whole analysis of Theorem 2.3.2 survives up to the no-shift region \mathcal{N} , even if $e = 0$.

However, it is clear that no blue-shift can be expected to facilitate the estimates in the future of \mathcal{N} . This is because Ω^2 behaves like the first-order polynomial $\frac{2M}{r} - 1$, which only cancels for $r = r_+ = 2M$ but is bounded away from 0 otherwise.

In this absence of our beloved stability mechanism, it is not so clear what to expect in the future of \mathcal{N} . A posteriori, after proving Theorem 2.1.8, it seems that r tends to 0 in the outgoing future direction. Therefore,

²⁹In fact, in [44], we show that in the massive and uncharged setting, a continuous extension \tilde{M} can always be constructed, however generically $\phi \notin W_{loc}^{1,1}(\tilde{M})$.

³⁰To obtain this, we must also estimate $g - g_0$ in an appropriate sense. Such estimates in fact follow from the analysis of Theorem 2.3.2.

³¹In fact, the expression is slightly more complicated, but roughly of the form that we describe, c.f. Proposition 2.6.4 for a precise statement.

to obtain stability estimates, one must understand the full behaviour in r as $r \rightarrow 0$ of the metric components and ϕ . While this analysis has been carried on the fixed Schwarzschild background, see [35], the non-linear case seems more subtle, in particular because, in principle, the future boundary can present some null components on which $r = 0$, c.f. [47]. This scenario could hypothetically lead to a behaviour that differs drastically from the one on a fixed Schwarzschild background.

Therefore, for a direct approach, one would need a completely new set of estimates that are currently not within immediate reach. We will refrain to embrace this route in the present manuscript and will instead work by contradiction.

Thus, we assume that, locally near time-like infinity, our space-time features a Cauchy horizon, like in the $e \neq 0$ case. Essentially, this is equivalent to assuming that r is lower bounded in a neighbourhood of time-like infinity.

Then we can prove that $|\partial_u r|$ is lower bounded in \mathcal{N} , **independently** of the size of \mathcal{N} , quantified by a number N . Making N large enough (which, as a consequence, enlarges Δ' and therefore the size of \mathcal{N}), one can produce values of r that are arbitrarily close to 0, which then contradicts the lower bound.

Therefore, as a direct application of Kommemi extension's principle, c.f. [47], we can infer that the absence of a Cauchy horizon implies the existence of a "right-most" future achronal boundary on which r extends continuously to 0 and thus the Kretschmann scalar must identically blow-up on that boundary. For two-ended Black Holes, our result implies that space-time is inextendible as a C^2 Lorentzian manifold, c.f. [21], [47] and [57].

For the one-ended case, however, the presence of a "left-most" boundary, emanating from the centre of symmetry $\{r = 0\}$ could in principle falsify any reasonable formulation of C^2 Strong Cosmic Censorship conjecture, c.f. the Penrose diagram in [47]. This possibility, which cannot be excluded by pure "extension principle" methods, c.f. [47], is independent of the "late time behaviour" of the scalar field towards time-like infinity. Instead, it is likely that the "left-most" interior structure of one-ended Black Holes is mostly determined by the location of the apparent horizon, in the vicinity of the centre of symmetry, similarly to the uncharged case c.f. [13]. While in the series of papers [11], [12], [13], Christodoulou could exploit monotonicity properties of various quantities to rule out the existence of left boundaries that would contravene Strong Cosmic Censorship, it seems that a more refined analysis would be necessary to address the analogous problem in the charged scalar field context, due to the apparent absence of these monotonicity properties. Such a direction is yet to be explored.

2.1.4 Outline of the chapter

We conclude this introduction by presenting the rest of the chapter.

Section 2.2 is devoted to preliminaries : we notably define the main notations, introduce the equations and express them in the form that we use later. A brief review of the Reissner–Nordström background is also presented.

In section 2.3, we phrase the main results of the paper precisely, namely the stability and the instability theorems. They are preceded by a reminder on the characteristic initial value problem and the coordinate dependency.

In section 2.4, the proof of the stability theorem is carried on. The proof of one minor proposition is deferred to section 2.9 and a simple local existence lemma is proven in section 2.10.

In section 2.5, the proof of the instability theorem is carried on.

In section 2.6, we prove Theorem 2.3.5 and provide the "dichotomy" between boundedness of the scalar field and continuous extendibility of the metric on one side, or blow-up of the scalar field and continuous inextendibility on the other side.

In section 2.7, we prove Theorem 2.1.8: if the charge tends to 0 on the event horizon, then no Cauchy horizon forms and the future boundary yields $\{r = 0\}$.

Finally, in the section 2.8, we use our stability framework to "localise" in coordinates the part of the apparent horizon that is close to time-like infinity.

2.2 Geometric framework and equations

2.2.1 The equations in geometric form

We look for solutions to the Einstein-Maxwell equations coupled with a charged and massive scalar field ϕ of constant mass ³² $m^2 \geq 0$ and constant charge $q_0 \neq 0$ propagating according to the Klein-Gordon equation (2.2.5) in curved space-time ³³ :

³² $m^2 \geq 0$ ensures that the dominant energy condition is satisfied. It does not play a role for the proof of the stability estimates but is crucial for the instability part.

³³One important difference compared to real scalar field models is that the Maxwell and the wave equations are now coupled because the field is charged.

A solution is described by a quadruplet (M, g, ϕ, F) - where (M, g) is a Lorentzian manifold of dimension $3+1$, ϕ is a complex-valued ³⁴ function on M and F is a real-valued 2-form on M - which satisfies the following equations :

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{KG}, \quad (2.2.1)$$

$$\mathbb{T}_{\mu\nu}^{EM} = 2 \left(g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu} \right), \quad (2.2.2)$$

$$\mathbb{T}_{\mu\nu}^{KG} = 2 \left(\Re(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2} (g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi} + m^2 |\phi|^2) g_{\mu\nu} \right), \quad (2.2.3)$$

$$\nabla^\mu F_{\mu\nu} = \frac{q_0}{2} i (\phi \overline{D_\nu \phi} - \overline{\phi} D_\nu \phi), \quad F = dA, \quad (2.2.4)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = m^2 \phi, \quad (2.2.5)$$

where $D := \nabla + iq_0 A$ is the gauge derivative, ∇ is the Levi-Civita connection of g and A is the potential one-form ³⁵. $\mathbb{T}_{\mu\nu}^{EM}$ and $\mathbb{T}_{\mu\nu}^{KG}$ are the electromagnetic and the Klein-Gordon stress-energy tensor respectively.

(2.2.1) is the Einstein equation, (2.2.4) is the Maxwell equation and (2.2.5) is the Klein-Gordon equation. Note that they are all coupled one to another.

2.2.2 Metric in null coordinates, mass, charge and main notations

Let (M, g, ϕ, F) be a spherically symmetric solution of the Einstein-Maxwell-Klein-Gordon equations. By this we mean that $SO(3)$ acts on (M, g) by isometry with spacelike orbits and for all $R_0 \in SO(3)$, the pull-back of F and ϕ by R_0 coincides with itself.

We define $\mathcal{Q} = M/SO(3)$, the quotient 2-dimensional manifold induced by the action of $SO(3)$.

$\Pi : M \rightarrow \mathcal{Q}$ is the canonical projection taking a point of M into its spherical orbit.

The metric on M is then given by $g = g_{\mathcal{Q}} + r^2 d\sigma_{\mathbb{S}^2}$ where $g_{\mathcal{Q}}$ is the push-forward of g by Π and $d\sigma_{\mathbb{S}^2}$ the standard metric on the sphere.

$g_{\mathcal{Q}}$ as a general Lorentzian metric over a 2-dimensional manifold, can be written in null coordinates (u, v) as a conformally flat metric :

$$g_{\mathcal{Q}} := -\frac{\Omega^2}{2} (du \otimes dv + dv \otimes du).$$

We define the area-radius function r over \mathcal{Q} by $r(p) = \sqrt{\frac{\text{Area}(\Pi^{-1}(p))}{4\pi}}$.

We can then define κ and ι as :

$$\kappa = \frac{-\Omega^2}{4\partial_u r} \in \mathbb{R} \cup \{\pm\infty\}, \quad (2.2.6)$$

$$\iota = \frac{-\Omega^2}{4\partial_v r} \in \mathbb{R} \cup \{\pm\infty\}. \quad (2.2.7)$$

Remark 8. Notice that κ is invariant under u -coordinate change : if $du' = f(u)du$, then in the new coordinate system (u', v) , $\kappa(u', v) = \kappa(u, v)$. Similarly , ι is invariant under v -coordinate change. ³⁶

We can also define the Hawking mass and mass ratio as geometric quantities, at least in spherical symmetry

$$\rho := \frac{r}{2} (1 - g_{\mathcal{Q}}(\nabla r, \nabla r)),$$

$$\mu := \frac{2\rho}{r}.$$

In what follows, we will abuse notation and denote by F the 2-form over \mathcal{Q} that is the push-forward by Π of the electromagnetic 2-form originally on M , and same for ϕ .

It turns out that spherical symmetry allows us to set :

$$F = \frac{Q}{2r^2} \Omega^2 du \wedge dv,$$

where Q is a scalar function that we call the electric charge.

³⁴The second important difference with the uncharged case is that it is not no longer possible to take a real scalar field : ϕ must be complex-valued.

³⁵ $F = dA$ is to be interpreted as “ there exists real-valued a one-form A such that $F = dA$ ”. This determines A up to a closed form only. It means that there is a gauge freedom, c.f section 2.2.2.

³⁶Note however that rescaling v also rescales κ and rescaling u rescales ι .

Remark 9. It should be noted that in the Einstein-Maxwell-Scalar-Field of [20] and [57], $Q \equiv e$ was forced to be a constant because it was coupled with vacuum Maxwell's equation $\text{div } F = 0$.

$F = dA$ also allows us to chose a spherically symmetric potential A written as :

$$A = A_u du + A_v dv.$$

The equations of section 2.2.1 are invariant under the following gauge transformation :

$$\begin{aligned}\phi &\rightarrow e^{-iq_0 f} \phi, \\ A &\rightarrow A + df.\end{aligned}$$

where f is a smooth real-valued function.

Therefore we can choose the following gauge for some constant v_0 and for all (u, v) :

$$A_v(u, v) \equiv 0, \tag{2.2.8}$$

$$A_u(u, v_0) = 0. \tag{2.2.9}$$

Remark 10. Notice that this gauge depends only on the null foliation and therefore is invariant if u or v is re-parametrized.

This gauge will be used in the rest of the paper, for v_0 to be specified in the statement of Theorem 2.3.2. For a more justified and complete discussion of the Einstein-Maxwell-Klein-Gordon setting, c.f [47].

Now we introduce the modified mass ϖ that takes the charge Q into account :

$$\varpi := \rho + \frac{Q^2}{2r} = \frac{\mu r}{2} + \frac{Q^2}{2r}. \tag{2.2.10}$$

An elementary computation relates coordinate-dependent quantities to geometric ³⁷ ones :

$$1 - \mu = \frac{-4\partial_u r \partial_v r}{\Omega^2} = \frac{-\Omega^2}{4\iota\kappa} = 1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2}. \tag{2.2.11}$$

We then define the geometric quantity³⁸ $2K$:

$$2K = \frac{2}{r^2}(\varpi - \frac{Q^2}{r}). \tag{2.2.12}$$

We will also denote, for fixed constants M and e :

$$2K_{M,e}(r) = \frac{2}{r^2}(M - \frac{e^2}{r}).$$

Finally we introduce the following notation, first used by Christodoulou :

$$\begin{aligned}\lambda &= \partial_v r, \\ \nu &= \partial_u r.\end{aligned}$$

2.2.3 The Reissner–Nordström solution

In this section we present the sub-extremal Reissner–Nordström solution. Because the space-time that we consider converges at late time towards a member of the Reissner–Nordström family and that we aim at proving stability estimates, it is important to recall their main qualitative features to see which are conserved in the presence of a perturbation.

³⁷Notice that $1 - \mu$ and K do not depend on the coordinate choice (u, v) .

³⁸On Reissner–Nordström, $2K = \partial_u \log |1 - \mu| = \partial_v \log |1 - \mu|$.

The Reissner–Nordström interior metric

The Reissner–Nordström black hole is a 2-parameter family of spherically symmetric and static space-times indexed by the charge and the mass (e, M) , which satisfy the Einstein-Maxwell equations i.e the system of section 2.2.1 with $\phi \equiv 0$ with $\mathbb{R}_+^* \times \mathbb{S}^2$ initial data.

We are interested in sub-extremal Reissner–Nordström black holes, which is expressed by the condition $0 < |e| < M$.

Define then for such (e, M) :

$$r_+(M, e) = M + \sqrt{M^2 - e^2} > 0,$$

$$r_-(M, e) = M - \sqrt{M^2 - e^2} > 0.$$

The metric in the interior of the black hole can be written in coordinates as :

$$g_{RN} = \frac{\Omega_{RN}^2}{4} dt^2 - 4\Omega_{RN}^{-2} dr^2 + r^2[d\theta^2 + \sin(\theta)^2 d\psi^2], \quad (2.2.13)$$

$$\Omega_{RN}^2(r) := -4\left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right), \quad (2.2.14)$$

Where $(r, t, \theta, \psi) \in (r_-, r_+) \times \mathbb{R} \times [0, \pi] \times [0, 2\pi]$.

(u, v) coordinate system on Reissner–Nordström background

We have seen in Section 2.2.2 how to build any null coordinate (u, v) . Now that the metric is explicit, we would like to find such a (u, v) system that is related to the variables (r, t) appearing in equation (2.2.13).

Define

$$r^* = r + \frac{1}{2K_+} \log(r_+ - r) + \frac{1}{2K_-} \log(r - r_-),$$

where $2K_+(M, e)$ and $2K_-(M, e)$, respectively called the surface gravity³⁹ of the event horizon and the surface gravity of the Cauchy horizon, are defined by⁴⁰ :

$$K_+(M, e) = \frac{1}{r_+^2} \left(M - \frac{e^2}{r_+}\right) = \frac{r_+ - r_-}{2r_+^2} > 0,$$

$$K_-(M, e) = \frac{1}{r_-^2} \left(M - \frac{e^2}{r_-}\right) = \frac{r_- - r_+}{2r_-^2} < 0.$$

Remark 11. Note that if $\varpi = M$ and $Q = e$ then $K(r_+) = K_+(M, e) > 0$ and $K(r_-) = K_-(M, e) < 0$, where K is defined in equation (2.2.12).

We then set $(u, v) \in \mathbb{R} \times \mathbb{R}$ as :

$$v = \frac{1}{2}(r^* + t), \quad u = \frac{1}{2}(r^* - t),$$

and claim that equation (2.2.13) can then be rewritten as :

$$g_{RN} = -\frac{\Omega_{RN}^2}{2} (du \otimes dv + dv \otimes du) + r^2[d\theta^2 + \sin(\theta)^2 d\psi^2].$$

Behaviour of Ω_{RN}^2

We define⁴¹ the event horizon $\mathcal{H}^+ = \{u \equiv -\infty, v \in \mathbb{R}\}$, and the Cauchy horizon $\mathcal{CH}^+ = \{v \equiv +\infty, u \in \mathbb{R}\}$. Ω_{RN}^2 cancels on both \mathcal{H}^+ and \mathcal{CH}^+ . A computation shows that :

$$\Omega_{RN}^2 \sim_{r \rightarrow r_+} C_{e,M} e^{2K_+ r^*} = C_{e,M} e^{2K_+ \cdot (u+v)},$$

and similarly that :

$$\Omega_{RN}^2 \sim_{r \rightarrow r_-} C'_{e,M} e^{2K_- r^*} = C'_{e,M} e^{2K_- \cdot (u+v)},$$

for some $C_{e,M} > 0$, $C'_{e,M} > 0$.

³⁹For an physical explanation of the terminology, c.f [69].

⁴⁰Note that $K_- < 0$ like in [56] but unlike in [57].

⁴¹We could have defined in more generality the event horizon to be the past boundary of the black hole region and the Cauchy horizon the future boundary of the maximal globally hyperbolic development. Strictly speaking the Cauchy horizon is not part of the space-time but can be attached as a double null boundary and we then consider the space-time as a manifold with corners.

Remark 12. Notice that Ω_{RN}^2 exhibits an exponential behaviour in $(u+v)$, exponentially increasing from 0 near the event horizon and exponentially decreasing to 0 near the Cauchy horizon.

Notice also that for r bounded away from r_+ and r_- , Ω_{RN}^2 is upper and lower bounded.

Kruskal coordinates (U, V) and Eddington–Finkelstein coordinates (U, v) , (u, V)

From the previous section, one could fear that the metric could be singular across the horizons \mathcal{H}^+ and \mathcal{CH}^+ . Actually it is not : like for the Schwarzschild's event horizon, it suffices to define Kruskal-like coordinates (U, V) from the (u, v) coordinates as :

$$U := \frac{1}{2K_+} e^{2K_+ u},$$

and

$$V := 1 - \frac{1}{2|K_-|} e^{2K_- v}.$$

Note that U and V now range in $(U, V) \in [0, +\infty) \times (-\infty, 1]$ and that $\mathcal{H}^+ = \{U \equiv 0\}$; $\mathcal{CH}^+ = \{V \equiv 1\}$.

We then write the metric in the Eddington–Finkelstein-type mixed (U, v) coordinates as :

$$g_{RN} := -\frac{\Omega_{RN,H}^2}{2} (dU \otimes dv + dv \otimes dU) + r^2 [d\theta^2 + \sin(\theta)^2 d\psi^2].$$

We find that (U, v) is a regular coordinate system near the event horizon \mathcal{H}^+ :

$$\Omega_{RN,H}^2(U, v) = -\frac{1}{2K_+ U} \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) \rightarrow_{U \rightarrow 0} C_{e,M} e^{2K_+ v}.$$

In (u, V) coordinates we write now write the metric as :

$$g_{RN} := -\frac{\Omega_{RN,CH}^2}{2} (du \otimes dV + dV \otimes du) + r^2 [d\theta^2 + \sin(\theta)^2 d\psi^2].$$

We then see that (u, V) is a regular ⁴² coordinate system near the Cauchy horizon \mathcal{CH}^+ :

$$\Omega_{RN,CH}^2(u, V) = \frac{1}{2K_- (1-V)} \left(1 - \frac{2M}{r} + \frac{e^2}{r^2}\right) \rightarrow_{V \rightarrow 1} C'_{e,M} e^{-2K_- u}.$$

Constant quantities on Reissner–Nordström

Since we consider the stability of a Reissner–Nordström background under perturbation, it is useful to identify which quantities are zero on this fixed background : these are the ones that we can hope decay for in the non-linear perturbative setting with the Klein-Gordon field.

Reissner–Nordström has four main qualitative features which distinguishes it from general dynamical solutions :

1. Both the charge and the modified mass are fixed :

$$\varpi \equiv M,$$

$$Q \equiv e.$$

$$\text{Hence } 1 - \mu = 1 - \frac{2M}{r} + \frac{e^2}{r^2}.$$

2. The metric is symmetric ⁴³ in u and v and in particular :

$$\partial_u r = \partial_v r,$$

$$\partial_u \log(\Omega_{RN}^2) = \partial_v \log(\Omega_{RN}^2) = \frac{2}{r^2} \left(M - \frac{e^2}{r}\right) = 2K_{M,e}(r).$$

⁴²Moreover, as mentioned in remark 1 the metric is actually smoothly extendible beyond \mathcal{CH}^+ , which would pose a problem for the strong cosmic censorship conjecture but does not because Reissner–Nordström is expected to be non generic.

⁴³Which is essentially equivalent to the fact that ∂_t is a Killing vector field or that $\Omega_{RN}^2(r)$ is a sole function of r .

3. The horizons are constant r null hypersurfaces :

$$\mathcal{H}^+ = \{u \equiv -\infty, v \in \mathbb{R}\} = \{r \equiv r_+\},$$

$$\mathcal{CH}^+ = \{v \equiv +\infty, u \in \mathbb{R}\} = \{r \equiv r_-\}.$$

Hence $\partial_v r|_{\mathcal{H}^+} \equiv 0$ and $\partial_u r|_{\mathcal{CH}^+} \equiv 0$ which is consistent with the following relation :

$$\partial_u r = \partial_v r = 1 - \frac{2M}{r} + \frac{e^2}{r^2}.$$

4. The event horizon \mathcal{H}^+ coincides with the apparent horizon $\mathcal{A} := \{\partial_v r = 0\}$ so all the 2-spheres inside the black hole are trapped.

This does not hold for dynamical space-times where \mathcal{A} is in the future of \mathcal{H}^+ in general.

However, in the perturbative regime, we can expect that \mathcal{A} is not too far ⁴⁴ from \mathcal{H}^+ , c.f section 2.8.

In the end, we can sum up all the relations by :

$$\partial_u r = \partial_v r = 1 - \frac{2M}{r} + \frac{e^2}{r^2} = -\frac{\Omega_{RN}^2}{4} \leq 0,$$

which also means that :

$$\kappa_{RN} = \iota_{RN} \equiv 1.$$

2.2.4 The Einstein–Maxwell–Klein–Gordon equations in null coordinates

Finally, we express the Einstein–Maxwell–Klein–Gordon system in spherical symmetry in any (u, v) coordinates as in section 2.2.2 and under the gauge choice for the potential (2.2.8), (2.2.9).

We start by the wave part of the Einstein equation :

$$\partial_u \partial_v r = \frac{-\Omega^2}{4r} - \frac{\partial_u r \partial_v r}{r} + \frac{\Omega^2}{4r^3} Q^2 + \frac{m^2 r}{4} \Omega^2 |\phi|^2 = -\frac{\Omega^2}{4} .2K + \frac{m^2 r}{4} \Omega^2 |\phi|^2, \quad (2.2.15)$$

$$\partial_u \partial_v \log(\Omega^2) = -2\Re(D_u \phi \partial_v \bar{\phi}) + \frac{\Omega^2}{2r^2} + \frac{2\partial_u r \partial_v r}{r^2} - \frac{\Omega^2}{r^4} Q^2, \quad (2.2.16)$$

the Raychaudhuri equations :

$$\partial_u \left(\frac{\partial_u r}{\Omega^2} \right) = \frac{-r}{\Omega^2} |D_u \phi|^2, \quad (2.2.17)$$

$$\partial_v \left(\frac{\partial_v r}{\Omega^2} \right) = \frac{-r}{\Omega^2} |\partial_v \phi|^2, \quad (2.2.18)$$

the Klein–Gordon wave equation :

$$\partial_u \partial_v \phi = -\frac{\partial_u \phi \partial_v r}{r} - \frac{\partial_u r \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q \phi - \frac{m^2 \Omega^2}{4} \phi - i q_0 A_u \frac{\phi \partial_v r}{r} - i q_0 A_v \partial_u \phi, \quad (2.2.19)$$

and the propagative part of Maxwell's equation :

$$\partial_u Q = -q_0 r^2 \Im(\phi \overline{D_u \phi}). \quad (2.2.20)$$

$$\partial_v Q = q_0 r^2 \Im(\phi \overline{\partial_v \phi}). \quad (2.2.21)$$

Also the existence of an electro-magnetic potential A implies that :

$$\partial_v A_u = \frac{-Q \Omega^2}{2r^2}. \quad (2.2.22)$$

Now we can reformulate the equations to put them in a form that is more convenient to use.

It is interesting to use (2.2.15), (2.2.17), (2.2.18), (2.2.20), (2.2.21) to derive an equation for the modified mass :

⁴⁴We can prove that $0 \leq \partial_v r|_{\mathcal{H}^+} \lesssim v^{-2s}$ if $|\partial_v \phi| \lesssim v^{-s}$ and that this estimate is sharp under appropriate lower bounds.

$$\partial_u \varpi = \frac{r^2}{2\iota} |D_u \phi|^2 + \frac{m^2}{2} r \partial_u r |\phi|^2 - i \frac{q_0}{2} Q r \Im(\bar{\phi} D_u \phi), \quad (2.2.23)$$

$$\partial_v \varpi = \frac{r^2}{2\kappa} |\partial_v \phi|^2 + \frac{m^2}{2} r \partial_v r |\phi|^2 + i \frac{q_0}{2} Q r \Im(\bar{\phi} \partial_v \phi). \quad (2.2.24)$$

Moreover, the following reformulation of (2.2.15) will be useful :

$$\partial_v \log(|\partial_u r|) = \kappa(2K - r m^2 |\phi|^2). \quad (2.2.25)$$

Remark 13. Note that the left-hand-side, like κ is invariant under u-coordinate changes.

We also reformulate (2.2.16) as :

$$\partial_u \partial_v \log(\Omega^2) = \kappa \partial_u(2K) - 2\Re(D_u \phi \partial_v \bar{\phi}) - \frac{2\kappa}{r^2} (\partial_u \varpi - \frac{\partial_u Q^2}{r}) = \iota \partial_v(2K) - 2\Re(D_u \phi \partial_v \bar{\phi}) - \frac{2\iota}{r^2} (\partial_v \varpi - \frac{\partial_v Q^2}{r}). \quad (2.2.26)$$

We can also rewrite (2.2.19) to control $|\partial_v \phi|$ more easily :

$$e^{-iq_0 \int_{u_0}^u A_u} \partial_u (e^{iq_0 \int_{u_0}^u A_u} \partial_v \phi) = -\frac{\partial_v r D_u \phi}{r} - \frac{\partial_u r \partial_v \phi}{r} + \frac{q_0 i \Omega^2}{4r^2} Q \phi - \frac{m^2 \Omega^2}{4} \phi, \quad (2.2.27)$$

or to control $|D_u \phi|$ more easily :

$$\partial_v(D_u \phi) = -\frac{\partial_u r \partial_v \phi}{r} - \frac{\partial_v r D_u \phi}{r} - \frac{m^2 \Omega^2}{4} \phi - \frac{q_0 i \Omega^2}{4r^2} Q \phi. \quad (2.2.28)$$

Finally we can also write the Raychaudhuri equations as :

$$\partial_u(\kappa^{-1}) = \frac{4r}{\Omega^2} |D_u \phi|^2, \quad (2.2.29)$$

$$\partial_v(\iota^{-1}) = \frac{4r}{\Omega^2} |\partial_v \phi|^2. \quad (2.2.30)$$

2.3 Precise statement of the main results

2.3.1 Preliminaries on characteristic initial value problem and coordinate choice

Before stating the theorem, we want to demystify a little the framework used to define the gauges and the coordinate dependent objects. The context is the same as for [20] and [57] , the only difference is the presence of the (dynamical) charge of the space-time Q .

We want to phrase the characteristic initial value problem for the Einstein-Maxwell-Klein-Gordon system of section 2.2.1. The reader familiar with the framework can skip this section.

We first consider two connected and oriented smooth, 1-dimensional manifolds \mathcal{C}_{in} and \mathcal{C}_{out} -each with a boundary point (c.f Figure 2.1.)

We can identify the surfaces at their boundary point to get $\mathcal{C}_{in} \cup_{\{p\}} \mathcal{C}_{out}$, on which we now want to build a (U, v) **null** regular coordinate system. For this, we have four choices to make :

1. Choosing an increasing ⁴⁵ parametrization U of \mathcal{C}_{in} .
2. Choosing an increasing parametrization v of \mathcal{C}_{out} .
3. Choosing the U -coordinate $U_0 \in \mathbb{R} \cup \{\pm\infty\}$ of the intersection point p .
4. Choosing the v -coordinate $v_0 \in \mathbb{R} \cup \{\pm\infty\}$ of the intersection point p .

In this coordinate system, \mathcal{C}_{in} and \mathcal{C}_{out} can be written as :

$$\mathcal{C}_{in} = \{(U, v_0), U \in [U_0, U_{max}]\},$$

$$\mathcal{C}_{out} = \{(U_0, v), v \in [v_0, v_{max}]\},$$

⁴⁵By increasing, we mean parallel to the orientation of the 1-dimensional surface.

with $U_{max} \in \mathbb{R} \cup \{\pm\infty\}$, $v_{max} \in \mathbb{R} \cup \{\pm\infty\}$.

As our initial data we shall consider (r, Ω_H^2, ϕ, A) as follows :

(r, ϕ) are C^1 scalar ⁴⁶ functions and A a C^1 1-form on $\mathcal{C}_{in} \cup_{\{p\}} \mathcal{C}_{out}$.

r and ϕ induce in the (U, v) coordinate system- some functions on $[v_0, v_{max}) \times \{u_0\} \cup \{v_0\} \times [U_0, U_{max})$ that we still call r and ϕ by notation abuse.

A induces a function A_v on $[v_0, v_{max}) \times \{u_0\}$ by $A|_{\mathcal{C}_{out}} = A_v dv$ and another function A_U on $\{v_0\} \times [U_0, U_{max})$ by $A|_{\mathcal{C}_{in}} = A_U dU$.

The remaining part of the data will be a C^1 function $\Omega_H^2 : [v_0, v_{max}) \times \{u_0\} \cup \{v_0\} \times [U_0, U_{max}) \rightarrow \mathbb{R}_+^*$. We will use this later to build a metric of the form $g = -\frac{\Omega_H^2}{2}(dU \otimes dv + dv \otimes dU) + r^2[d\theta^2 + \sin(\theta)^2 d\psi^2]$.

The prescription of Ω_H^2 as above will be coordinate dependent.

This coordinate dependent framework allows us to define the Raychaudhuri equations on the initial surfaces, seen as constraints for the characteristic initial value problem.

However, they are still valid under any re-parametrization of U or v :

Definition 1 (Raychaudhuri equations). We say that the data (r, Ω_H^2, ϕ, A) satisfy the Raychaudhuri equations if on $\{v_0\} \times [U_0, U_{max})$:

$$\partial_U \left(\frac{\partial_U r}{\Omega_H^2} \right) = \frac{-r}{\Omega_H^2} |D_U \phi|^2. \quad (2.3.1)$$

And on $[v_0, v_{max}) \times \{U_0\}$:

$$\partial_v \left(\frac{\partial_v r}{\Omega_H^2} \right) = \frac{-r}{\Omega_H^2} |D_v \phi|^2, \quad (2.3.2)$$

where D depends on A by $D = \partial + iq_0 A$ as an operator on scalar functions.

We now want to talk of “the solution” - up to gauge transforms- of the Einstein-Maxwell-Klein-Gordon equations. To do so, we solve the partial differential equation system of section 2.2.4 “abstractly” for some data (r, Ω^2, ϕ, A) . Since it is standard that the Raychaudhuri equations -once satisfied on the initial surfaces- are propagated, we see the solution actually satisfies the Einstein-Maxwell-Klein-Gordon equations in their geometric form of section 2.2.1.

Theorem 2.3.1 (Characteristic initial value problem). *Let $\mathcal{C}_{in}, \mathcal{C}_{out}$ be as before.*

We assume moreover that the data (r, Ω_H^2, ϕ, A) are as before and satisfy the Raychaudhuri equations (2.3.1) and (2.3.2). Moreover we suppose ⁴⁷ that $r > 0$.

Then there exists a unique C^1 maximal globally hyperbolic development (M, g, ϕ, F) , spherically symmetric solution of Einstein-Maxwell-Klein-Gordon equations of section 2.2.1 such that

1. \mathcal{C}_{out} and \mathcal{C}_{in} embed into $\mathcal{Q} = \mathcal{M}/SO(3)$ as null boundaries with respect to the metric g .
2. $D^+(\mathcal{C}_{in} \cup_{\{p\}} \mathcal{C}_{out}) \cap \mathcal{Q} = J^+(\{p\}) \cap \mathcal{Q}$
where D^+ denotes the future domain of dependence and J^+ the causal future ⁴⁸.
3. (M, g, ϕ, F) satisfy :

$$g = -\frac{\Omega_H^2}{2}(dU \otimes dv + dv \otimes dU) + r^2[d\theta^2 + \sin(\theta)^2 d\psi^2],$$

$$F = dA.$$

And (r, Ω_H^2, ϕ, A) restrict on the initial surfaces to the value prescribed

by the initial data $(r, \Omega_H^2, \phi, A)|_{\mathcal{C}_{in} \cup_{\{p\}} \mathcal{C}_{out}}$.

4. *The equations in null coordinates of section 2.2.4 are satisfied.*

For a more thorough discussion of the uniqueness problem in that framework, c.f [18].

⁴⁶It should be emphasized that r and ϕ -like the metric g will be later- are geometric quantities, namely they do not depend on the coordinate choice. However Ω_H^2 **does** depend on the coordinate choice.

⁴⁷However, $\inf_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} r = 0$ is allowed.

⁴⁸For a definition c.f [69].

2.3.2 The stability theorem

We can now formulate the main stability theorem. The main point is the presence of a Cauchy horizon, reflected by the form of the Penrose diagram, instead of a space-like Schwarzschild-type singularity.

Theorem 2.3.2 (Non-linear stability theorem). *Let \mathcal{C}_{in} , \mathcal{C}_{out} and (r, ϕ, Ω_H^2, A) satisfy the assumptions of Theorem 2.3.1.*

Moreover, we will make the following geometric assumptions :

Assumption 1. \mathcal{C}_{out} is future affine complete ⁴⁹.

Assumption 2. $r > 0$ is a strictly decreasing function on \mathcal{C}_{in} with respect to any increasing parametrization.

From now on we will denote $\mathcal{H}^+ := \mathcal{C}_{out}$ and call \mathcal{H}^+ the event horizon.

For some constant $v_0 > 0$, we parametrize $\mathcal{H}^+ := \mathcal{C}_{out} = \{U \equiv 0, v \geq v_0\}$ with a coordinate v defined ⁵⁰ by

$$\kappa|_{\mathcal{H}^+} = \left(\frac{-\Omega_H^2(0, v)}{4\partial_U r(0, v)} \right)|_{\mathcal{H}^+} \equiv 1, \quad (2.3.3)$$

and for some $U_{max} > 0$, we parametrize $\mathcal{C}_{in} = \{v \equiv v_0, 0 \leq U \leq U_{max}\}$ with a coordinate U defined by

$$(\partial_U r)|_{\mathcal{C}_{in}}(U, v_0) \equiv -1. \quad (2.3.4)$$

We also make the following no-anti-trapped surfaces ⁵¹ assumption :

Assumption 3. $\partial_U r(0, v)|_{\mathcal{H}^+} < 0$

We assume the following decay on the field in (U, v) coordinates : there exists $C > 0$ and $s > \frac{1}{2}$ such that

Assumption 4.

$$|\phi(0, v)|_{\mathcal{H}^+} + |\partial_v \phi(0, v)|_{\mathcal{H}^+} \leq C v^{-s},$$

Assumption 5. ⁵²

$$|D_U \phi|(U, v_0) \leq C.$$

We also ask the following convergences towards infinity on the event horizon :

Assumption 6.

$$r|_{\mathcal{H}^+}(0, v) \rightarrow r_\infty$$

as $v \rightarrow +\infty$,

where $r_\infty > 0$ is a constant.

Assumption 7.

$$0 < Q_+ < r_\infty,$$

where $Q_+ := \limsup_{v \rightarrow +\infty} |Q|_{\mathcal{H}^+}$

We consider the unique C^1 maximal globally hyperbolic development (M, g, ϕ, F) of Theorem 2.3.1,

Then, after restriction to a small enough connected subset $p \in \mathcal{C}'_{in} \subset \mathcal{C}_{in}$, i.e $\mathcal{C}'_{in} = \{v \equiv v_0, 0 \leq U \leq U_s\}$ for $0 < U_s$ small enough, $D^+(\mathcal{C}'_{in} \cup_{\{p\}} \mathcal{C}_{out}) \cap \mathcal{Q}$ has the Penrose diagram of Figure 2.1.

Moreover, if $s > 1$, (M, g, ϕ, F) admits a continuous extension to the Cauchy horizon.

More precisely, we can attach a future null boundary $\mathcal{CH}^+ := \{v \equiv +\infty, 0 \leq U \leq U_s\}$ to the space-time (M, g) such that (g, ϕ, F) each admits a continuous extension to the new space-time $\bar{M} := M \cup \mathcal{CH}^+$ seen as a manifold with boundary.

Remark 14. Because of (2.2.11), the relation (2.3.3) is exactly equivalent⁵³ to :

$$\partial_v r|_{\mathcal{H}^+} = 1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2} = 1 - \mu. \quad (2.3.5)$$

⁴⁹We define affine completeness by the relation $\int_{v_0}^{v_{max}} \Omega_H^2(U_0, v) dv = +\infty$. This is a **coordinate-independent** statement.

⁵⁰It is then easy to see that (2.2.15) and assumption 4 together with the affine completeness prove that $v_{max} = +\infty$.

⁵¹This assumption combined with (2.2.17) proves that $\partial_U r < 0$ everywhere, an assumption first made by Christodoulou.

⁵²Notice that in the gauge (2.2.9), this is equivalent to saying $|\partial_U \phi|(U, v_0) \leq C$.

⁵³ Notice that the gauge (2.3.3) is the same as [20] but slightly different from [57], although it actually only differs from a multiplicative function of v bounded above and below.

Remark 15. The present paper introduces the first stability result dealing with **all** the possible values of m^2 and q_0 . However the continuous extension statement when $s > 1$ was already established in the work [20] and [57] although stated in the chargeless case $q_0 = 0$ only. Some continuous extension results for the charged case have also been proved in [47]. Notice (c.f section 2.1.2) that the case $s > 1$ should be relevant in our context only if the scalar field is massless and not too charged⁵⁴ compared to the black hole.

Remark 16. Notice also that the assumptions are (almost) the same as those of [57], except for the strength of the decay rate, which was integrable unlike in the present paper.

In the rest of the paper, we will write $A \lesssim B$ if there exists a constant $\tilde{C} = \tilde{C}(C, Q_+, q_0, m^2, r_\infty, s, v_0)$ such that $A \leq \tilde{C}B$.

If we need to specify this constant, we shall call it consistently \tilde{C} when there are no ambiguities.

We denote also $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2.3.3 The instability theorem

We can now phrase our instability theorem that relies very much on the non-linear stability claimed in the preceding section.

Theorem 2.3.3 (Non-linear instability theorem). *Let \mathcal{C}_{in} , \mathcal{C}_{out} and (r, Ω_H^2, ϕ, A) satisfying all the assumptions of Theorem 2.3.2 and in particular assumption 4 with $s > \frac{1}{2}$.*

We assume, using the same gauges as for Theorem 2.3.2, that the field in addition satisfies the following L^2 averaged polynomial lower-bound on the event horizon $\mathcal{C}_{out} = \mathcal{H}^+$:

Assumption 8.

$$v^{-p} \lesssim \int_v^{+\infty} |\partial_v \phi|_{\mathcal{H}^+}^2(0, v') dv', \quad (2.3.6)$$

for $2s - 1 \leq p < \min\{2s, 6s - 3\}$.

Then for any $u \in \mathbb{R}$ negative enough, and for all v large enough (depending on u),

$$\int_v^{+\infty} |\partial_v \phi|^2(u, v') dv' \gtrsim v^{-p}. \quad (2.3.7)$$

In particular the following component of the curvature blows-up on the Cauchy horizon :

$$\limsup_{v \rightarrow +\infty} Ric(\Omega^{-2}\partial_v, \Omega^{-2}\partial_v)(u, v) = +\infty.$$

Moreover for $s > 1$, $\phi \notin W_{loc}^{1,2}$ and the metric is not C^1 for the continuous extension constructed in Theorem 2.3.2.

Remark 17. This theorem is the very first instability result outside the uncharged and massless case. As explained in section 2.1.3, the methods of previous instability works do not apply here.

Remark 18. In view of the result of [57], one can very reasonably hope that this curvature blow up leads to a **C^2 inextendibility of the metric** in an appropriate global setting⁵⁵. The reason for this is that $Ric(\Omega^{-2}\partial_v, \Omega^{-2}\partial_v)$ is a geometric quantity since $\Omega^{-2}\partial_v$ is a geodesic vector field. The only remaining argument is to extend the blow-up far from time-like infinity namely to get a global statement as opposed to perturbative.

2.3.4 The zero charge case and the absence of a Cauchy Horizon

In this section, we show that when the exterior of the space-time converges towards a Schwarzschild Black Hole, which is a co-dimension one family inside the Reissner–Nordstrom family of charged Black Holes, then space-time does **not** feature a Cauchy horizon emanating from time-like infinity. Instead, we know that the future boundary consist of achronal pieces on which r yields the zero function. This knowledge is in principle sufficient to establish C^2 inextendibility across the future “right-most” boundary of the space-time, as the Kretschmann scalar K blows up, c.f. [47]. We will however not pursue this direction.

Theorem 2.3.4. *We work under the same hypothesis as Theorem 2.3.2, except for assumption 7, replaced by*

Assumption 9. $\lim_{v \rightarrow +\infty} Q|_{\mathcal{H}^+}(v) = 0$.

⁵⁴Namely $m^2 = 0$ and $|q_0 e| < \frac{1}{2}$ with the notation of section 2.1.2.

⁵⁵At least for two-ended black holes.

Then, the conclusion of Theorem 2.3.2 does **not** hold. In particular, a Cauchy horizon emanating from time-like infinity **does not** form, in the following sense: for any $U_s \geq 0$,

$$\inf_{U \leq U_s, v \geq v_0} r(U, v) = 0$$

Moreover, one can attach a future boundary S to the space-time, having the property that S lies to the (time-like) future of the incoming ⁵⁶ initial characteristic hyper-surface. More precisely, recalling that we set the initial data on a characteristic double null surface $\mathcal{C}_{in} \cup_p \mathcal{C}_{out}$,

$$S \subset I^+(\mathcal{C}_{in}),$$

and moreover r extends continuously as the zero function on S : $r|_S \equiv 0$ and the Kretschmann scalar K blows up uniformly on S : $K|_S \equiv +\infty$.

2.3.5 A sufficient and a necessary condition for continuous extendibility

As discussed before, the presence of a Cauchy horizon suggests the failure of Strong Cosmic Censorship, formulated in C^0 regularity (although it still holds in C^2 regularity, like in the uncharged and massive case). While we proved this scenario for the case of weakly charged and massless fields in Theorem 2.3.2 using standard techniques, the scenario is more complex in the massive or massless and strongly charged case, where physical space methods fail to provide a satisfactory answer.

In this section, we provide a criterion for continuous extendibility of the metric. As we are going to see, continuous extendibility of the metric g is roughly equivalent to the boundedness of the scalar field ϕ . As established in Theorem 2.3.2, if the scalar field ϕ decays on the event horizon at a rate $s > 1$, then one can extend continuously the solution through a future null boundary — the Cauchy horizon. In fact, denoting the extension \tilde{M} , it can be shown that the extension of the scalar field ⁵⁷ satisfies $\nabla\phi \in L^1_{loc}(\mathbb{R}_V, L^\infty_{loc}(\mathbb{R}_u))$, where (u, V) are regular ⁵⁸ coordinates across the Cauchy horizon, hence $\phi \in W^1_{loc}(\tilde{M}) \cap C^0(\tilde{M})$ like in the uncharged and massless case c.f. [57]. In particular, since for weakly charged scalar fields, it is expected that $s > 1$ (c.f. chapter 3), then the metric is continuously extendible.

However, for massive or strongly charged scalar fields, the rate is expected to be $\frac{5}{6} \leq s \leq 1$, hence Theorem 2.3.2 does not apply. Based on the recent work [44], it seems that the hypothesis of Theorem 2.3.2, when $\frac{1}{2} < s \leq 1$ are **not sufficient** to determine whether the metric is C^0 extendible. In particular, for such a weak decay, continuous extendibility depends on the oscillations of the scalar field on the event horizon. We make this statement precise in [44], using a combination of linear Fourier methods and the analysis of this chapter 2.

The major obstruction is the (potential) failure of the boundedness of the scalar field. Indeed, when $s < 1$, it is expected that $\nabla\phi \notin L^1_{loc}(\mathbb{R}_V, L^\infty_{loc}(\mathbb{R}_u))$ and even $\phi \notin W^1_{loc}(\tilde{M})$, unlike in the case $s > 1$. Nevertheless, it is still possible, in principle, that the scalar field could be bounded up to the Cauchy horizon, and even continuously extendible. In [44], we demonstrate that this happens for the case of physically realistic oscillating (as prescribed by (2.1.23)) data on the event horizon, for massive (charged or uncharged) fields. We also produce (putatively non-generic) data which **do not** oscillate and for which the scalar field blows up uniformly on the Cauchy horizon.

The point of this section is to relate this blow-up (respectively boundedness) for a field obeying the wave equation, to a geometric inextendibility result (respectively extendibility). While it is easy to prove that a blowing up scalar field does not extend continuously across the Cauchy Horizon, we obtain a more ambitious claim : the space-time (M, g) does not extend as a continuous Lorentzian manifold, at least locally in a sense that will be made precise.

Definition 2 (Double null coordinate system). A double null coordinate system is a coordinate patch $(u, v, \theta_1, \theta_2)$ on an open set of the Lorentzian manifold (M, g) , with u and v solving the eikonal equation :

$$g^{\mu\nu} \partial_\mu u \partial_\nu u = 0, \quad g^{\mu\nu} \partial_\mu v \partial_\nu v = 0,$$

and for $A = 1, 2$:

$$\mathcal{L}\theta^A = 0,$$

where \mathcal{L} is the restriction of the Lie derivative to $TS_{u,v}$ and $S_{u,v} := \{u' = u, v' = v\}$. Then, there ⁵⁹ exist functions Ω^2 , b^A and γ_{AB} such that

$$g = -\frac{\Omega^2}{2} \cdot (du \otimes dv + dv \otimes du) + \gamma_{AB} \cdot (d\theta^A - b^A du) \otimes (d\theta^B - b^B dv). \quad (2.3.8)$$

⁵⁶Informally, we can say that S is the "right-most" future boundary of the space-time, c.f. the Penrose diagram in [47].

⁵⁷On the other hand, it can be shown that $\phi \notin H^1_{loc}(\tilde{M})$, as we see in Theorem 2.3.3. This is the manifestation of the blue-shift instability.

⁵⁸Meaning coordinates in which the metric components extend continuously across the Cauchy horizon.

⁵⁹If L and \underline{L} are the geodesic vector fields associated to the eikonal functions u and v respectively, then Ω^2 is defined by $2g(L, \underline{L}) = -\Omega^{-2}$.

Definition 3. Assume that we can attach ⁶⁰ a null boundary $CH^+(u_1) := [-\infty, u_1] \times \{V(v) = 1\}$, indexed in null coordinates (u, v) , for some $u_1 \in \mathbb{R}$ and $V(v) \rightarrow 1$ as $v \rightarrow +\infty$. We will still denote M the manifold with boundary and $\text{int}(M) := M - \partial M = M - CH^+(u_1)$.

Then we say that (M, g) is continuously extendible across the null boundary $CH^+(u_1)$ if there exists a differentiable manifold \tilde{M} equipped with a continuous Lorentzian metric \tilde{g} and a differentiable isometric embedding $i : \text{int}(M) \rightarrow \tilde{M}$, such that $i(\text{int}(M))$ is a proper subset of \tilde{M} and moreover the following conditions hold true:

1. There exists **two** curves in \tilde{M} that cross the Cauchy horizon (at two distinct points): more precisely, for $k = 1, 2$, there exists points $q_k \in \tilde{M} - i(\text{int}(M))$, $p_k \in i(\text{int}(M))$, and two continuous curve $\sigma_k : [0, 1] \rightarrow \tilde{M}$ such that for some $0 < s_k < 1$, $\sigma_k(s_k) \in i(CH^+(u_1))$, $\sigma_1(s_1) \neq \sigma_2(s_2)$ and $\sigma_k(0) = p_k$ and $\sigma_k(1) = q_k$.
2. \tilde{M} can be (locally) equipped with a double null coordinate system in the sense of definition 2.

If no such extension exists, we say that (M, g) continuously **in**-extendible across the null boundary $CH^+(u_1)$.

Remark 19. Although references are made to double null coordinates in Definition 3 of inextendibility, this property should still be understood as a geometric one. Indeed, we emphasize that, if (M, g) a continuously inextendible Lorentzian metric, **no** double-null coordinate system can be found on the extension (even locally).

Theorem 2.3.5. *We work under the hypothesis of Theorem 2.3.2 and we assume moreover ⁶¹ that $s > \frac{3}{4}$. We recall that by Theorem 2.3.2, we obtain estimates on the rectangle $[-\infty, u_s] \times [v_0, +\infty]$ in (u, v) coordinates and that the Penrose diagram corresponds to Figure 2.1.*

1. *Sufficient condition for continuous extendibility: suppose that there exists $u_1 \leq u_s$ and $l \geq 0$ such that*

$$\lim_{v \rightarrow +\infty} |\phi|(u_1, v) = l.$$

We will denote $l := |\phi|_{CH}(u_1)$. Then, (M, g) is continuously extendible across the Cauchy horizon and $(|\phi|, F)$ extend also continuously to the extension.

2. *"Necessary" condition for continuous extendibility: suppose that there exists $u_1 \leq u_s$ such that*

$$\limsup_{v \rightarrow +\infty} |\phi|(u_1, v) = +\infty$$

*Then, (M, g) is continuously **inextendible** across the Cauchy horizon $CH^+(u_s) := \{(u, v) \in [-\infty, u_s] \times \{+\infty\}\}$, in the sense of definition 3.*

Remark 20. As artificial as these conditions may seem at first glance, we construct in [44] generic examples for which the "sufficient condition" holds for massive fields, and "fine-tuned" ones for which the blow-up scenario occurs ("necessary condition" part of the theorem).

Remark 21. We claim extendibility (respectively inextendibility) properties only locally, i.e. on a portion $[-\infty, u_s]$ of the Cauchy horizon: $CH^+(u_s)$. This is because we rely crucially on the stability estimates of Theorem 2.3.2, which are only valid in the rectangle $[-\infty, u_s] \times [v_0, +\infty]$ of Figure 2.1. Part of the proof is to prove that ϕ is bounded towards one point of the Cauchy horizon $\{u_1\}$, $u_1 \leq u_s$, of the Cauchy horizon if and only if ϕ is bounded towards the whole of $CH^+(u_s)$. This is because we prove that $|D_u(r\phi)|$ is locally bounded (although it is not integrable as $u \rightarrow -\infty$).

Evidently, restricting the study to a local portion of the Cauchy horizon near time-like infinity $CH^+(u_s)$ is unimportant to the "sufficient condition" part of the Theorem, since constructing an extension, even locally, is sufficient to falsify any "continuous inextendibility statement", in particular is sufficient to falsify the C^0 version of Strong Cosmic Censorship conjecture.

For inextendibility properties, the question is more delicate : proving inextendibility across the local portion $CH^+(u_s)$ does not preclude space-time (M, g) to be continuously extendible on the whole of the Cauchy horizon, for instance on some $[u_s, u_0]$ later portion.

Remark 22. We only claim continuous extendibility of $|\phi|$ and not ϕ . This is because, in the charged case, the complex value field ϕ is gauge dependent: a different gauge choice for the potential A will change ϕ by a phase term. However, $|\phi|$ is a gauge invariant quantity. In particular, it is (trivially) **not true** that ϕ can extend continuously across the Cauchy horizon in all gauges. Consistently, the extendibility property of (M, g) (which must be independent of the electro-magnetic gauge choice) depend only $|\phi|^2$, which is indeed also a gauge-invariant quantity.

⁶⁰This is equivalent to saying that the Penrose diagram is that of Figure 2.1. In particular it does not involve any extendibility properties.

⁶¹Note that this is a realistic assumption, as it is expected that the weakest decay rate for our model is $s = \frac{5}{6} > \frac{3}{4}$, c.f. 2.1.23.

2.4 Proof of the stability Theorem

We recall that we write $A \lesssim B$ if there exists a constant ⁶² $\tilde{C} = \tilde{C}(C, Q_+, q_0, m^2, r_\infty, s, v_0)$ such that $A \leq \tilde{C}B$.

If we need to specify this constant, we shall call it consistently \tilde{C} when they are no ambiguities.

We denote also $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

When we write “with respect to the parameters”, we actually mean “with respect to $C, Q_+, q_0, m^2, r_\infty$ and s ”.

We shall use repeatedly the following technique : if we are in a region where $|u| \leq Dv$ where D is a constant, then we can take $|u_s|$ large enough (equivalently U_s small enough) so that for any $u \leq u_s$ and any function of v , $\epsilon(u, \cdot) = o(1)$ where $v \rightarrow +\infty$ and any positive number η then $|\epsilon(u, v)| \leq \eta$ for all $|u| \leq Dv$. When we do so, we write “for $|u_s|$ large enough” or equivalently in (U, v) coordinates “for U_s small enough”.

2.4.1 Strategy of the proof

The main idea of the proof is to split the space-time into smaller regions where the red-shift and blue-shift effect manifest themselves as already done in [20] and [57] and to integrate along characteristics for the wave equations.

The main novelty is to deal with a non-integrable field decaying on \mathcal{H}^+ like v^{-s} with $s > \frac{1}{2}$ only. The reason why stability estimates still proceed is that the Raychaudhuri equation on \mathcal{H}^+ involve the square of the field of the order v^{-2s} which is integrable.

We will use five different regions, which form a partition of the rectangle $[-\infty, u_s] \times [v_0, +\infty]$:

1. The event horizon $\mathcal{H}^+ := \{U = 0, v \geq v_0\}$ where we use crucially the Raychaudhuri equation and exhibit the correct Reissner–Nordström space-time to which our dynamical space-time is expected to converge at infinity. We find that Ω^2 behaves likes $e^{2K_+ \cdot (u+v+h(v))} = 2K_+ U e^{2K_+ \cdot (v+h(v))}$ where $h(v) = o(v)$.
2. The red-shift region $\mathcal{R} = \{u + v + h(v) \leq -\Delta\}$: this is a large region where Ω^2 is small enough and $|D_u \phi| \lesssim \Omega^2 v^{-s}$. This strong stability feature is the key to prove the estimates. Another important feature is that Ω^2 can almost be written as a product $f(u) \cdot g(v)$ which simplifies most of the calculations. This comes from the fact that $\Omega_H^2(U, v)$ is almost $\Omega_H^2(0, v)$, up to a arbitrary small constant $e^{-\tilde{C}\Delta}$.
3. The no-shift region $\mathcal{N} := \{-\Delta \leq u + v + h(v) \leq \Delta_N\}$: the function of this small region is to allow r to vary from its event horizon limit value r_+ to its Cauchy horizon limit value r_- , up to arbitrarily small constants. The smallness of the region, in which both u and v differences are bounded, allows us to conserve the estimates of its past region \mathcal{R} while initiating the blue-shift effect in its future.
4. The early blue-shift transition region ⁶³ $\mathcal{EB} := \{\Delta_N \leq u + v + h(v) \leq -\Delta' + \frac{2s}{2[K_-]} \log(v)\}$: this small region is the first where the blue-shift happens and as a consequence the metric coefficients $\Omega^2(u, v)$ start to be small enough to facilitate the decay of propagating waves but do not decay too much so that we can still treat the problem as almost linear : in particular ⁶⁴ κ^{-1} and ι^{-1} stay bounded.
5. The late blue-shift region $\mathcal{LB} := \{-\Delta' + \frac{2s}{2[K_-]} \log(v) \leq u + v + h(v)\}$: this very large region exhibits the strongest blue-shift : the metric coefficients $\Omega^2(u, v)$ start from inverse polynomial decay but decrease exponentially in v near the Cauchy horizon. We use this smallness to prove decay for the propagation problem. However, we do not prove enough decay to get a continuous extension of the space-time in the case $s \leq 1$.

The core of the proof is to control $\partial_v \log(\Omega^2)$ and $\partial_u \log(\Omega^2)$ and use a calculus Lemma (Lemma 2.4.1) :

In \mathcal{H}^+ and \mathcal{R} , as a consequence of the red-shift effect, they are lower bounded by a strictly positive constant, which allows us to consider Ω^2 as an increasing exponential in u and as an increasing exponential in v , avoiding the loss of one power when we integrate a polynomial decay.

In \mathcal{N} , $\partial_v \log(\Omega^2)$ and $\partial_u \log(\Omega^2)$ change sign and can be close to 0, but it does not matter for the decay of the scalar field because the region is small enough ⁶⁵.

⁶²This is equivalent to saying that \tilde{C} will depend only on q_0, m^2, v_0 , the initial data and on (e, M) as defined in section 2.4.3.

⁶³The idea to have a curve at a logarithmic distance from the no-shift region comes back -in a different form- to the early papers of Dafermos [19], [20].

⁶⁴Recall that κ and ι were defined in (2.2.6) and (2.2.7).

⁶⁵More precisely the u difference is bounded.

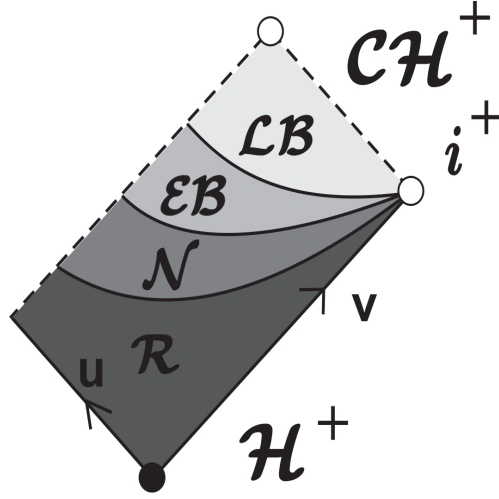


Figure 2.3: Penrose diagram of the space-time $\mathcal{M} = \mathcal{R} \cup \mathcal{N} \cup \mathcal{EB} \cup \mathcal{LB}$

In \mathcal{EB} and \mathcal{LB} , as a consequence of the blue-shift effect, they are upper bounded⁶⁶ by a strictly negative constant, which allows us to consider Ω^2 as a decreasing exponential in u and as a decreasing exponential in v , which also avoids the loss of power when we integrate a polynomial decay.

2.4.2 A calculus lemma

We begin this proof section by a calculus lemma, which broadly says that integrating a polynomial decay -as expected for ϕ^- with a Ω^2 or Ω^{-2} weight avoids to lose one power as we would otherwise.

Lemma 2.4.1. *Let $q \geq 0$, $a = a(e, M, q_0, m^2, s) > 0$ and γ_1 be a one-dimensional curve on which $|u| \approx v$ with $u_1(v)$ being the only u such that $(u, v) \in \gamma_1$ and $v_1(u)$ being the only v such that $(u, v) \in \gamma_1$.*

Then for any positive C^1 function Ω^2 , the following hold true :

1. **Red-shift bounds in $|u|$:** *assume that for all $u' \in [u_1(v), u]$, $\partial_u \log(\Omega^2)(u', v) > a$. Then :*

$$\int_{u_1(v)}^u \Omega^2(u', v) |u'|^{-q} du' \lesssim \Omega^2(u, v) |u|^{-q},$$

$$\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du' \lesssim \Omega^{-2}(u_1(v), v) v^{-q}.$$

2. **Red-shift bounds in v :** *assume that for all $v' \in [v_1(u), v]$, $\partial_v \log(\Omega^2)(u', v) > a$. Then :*

$$\int_{v_1(u)}^v \Omega^2(u, v') v'^{-q} dv' \lesssim \Omega^2(u, v) v^{-q}.$$

$$\int_{v_1(u)}^v \Omega^{-2}(u, v') v'^{-q} dv' \lesssim \Omega^{-2}(u, v_1(u)) |u|^{-q}.$$

3. **Blue-shift bounds in $|u|$:** *assume that for all $u' \in [u_1(v), u]$, $\partial_u \log(\Omega^2)(u', v) < -a$. Then :*

$$\int_{u_1(v)}^u \Omega^2(u', v) |u'|^{-q} du' \lesssim \Omega^2(u_1(v), v) v^{-q}.$$

$$\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du' \lesssim \Omega^{-2}(u, v) |u|^{-q}.$$

⁶⁶Strictly speaking, we do not prove however that $\partial_u \log(\Omega^2)$ is upper bounded in \mathcal{LB} if $s \leq 1$.

4. **Blue-shift bounds in v** : assume that for all $v' \in [v_1(u), v]$, $\partial_v \log(\Omega^2)(u', v) < -a$. Then :

$$\int_{v_1(u)}^v \Omega^2(u, v') v'^{-q} dv' \lesssim \Omega^2(u, v_1(u)) |u|^{-q},$$

$$\int_{v_1(u)}^v \Omega^{-2}(u, v') v'^{-q} dv' \lesssim \Omega^{-2}(u, v) v^{-q}.$$

Proof. We will only prove one case when $\partial_u \log(\Omega^2) > a$, the others being similar. For $u \geq u_1(v)$:

$$\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du' \leq \frac{1}{a} \int_{u_1(v)}^u \Omega^{-2}(u', v) \partial_u \log(\Omega^2)(u', v) |u'|^{-q} du' = -\frac{1}{a} \int_{u_1(v)}^u \partial_u (\Omega^{-2})(u', v) |u'|^{-q} du'.$$

Then we integrate by parts to write :

$$\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du' \leq \frac{q}{a} \int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q-1} du' + \frac{1}{a} \Omega^{-2}(u_1(v), v) |u_1(v)|^{-q} - \frac{1}{a} \Omega^{-2}(u, v) |u|^{-q}.$$

Then clearly $\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q-1} du' = o(\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du')$ so the dominant term is the second, and a depends on the parameters only, giving :

$$\int_{u_1(v)}^u \Omega^{-2}(u', v) |u'|^{-q} du' \lesssim \Omega^{-2}(u_1(v), v) |u_1(v)|^{-q}.$$

□

2.4.3 The event horizon

Convergence at large advanced time towards a Reissner–Nordström background

Proposition 2.4.2. *There exists constants $0 < |e| < M$ such that on the event horizon $\mathcal{H}^+ = \{U = 0, v \geq v_0\}$*

$$|\varpi(0, v) - M| \lesssim v^{1-2s}, \quad (2.4.1)$$

$$|Q(0, v) - e| \lesssim v^{1-2s}. \quad (2.4.2)$$

Moreover $r_\infty = r_+(M, e)$ where r_∞ is as in hypothesis 6 and

$$K(0, v) \rightarrow K_+(M, e) > 0,$$

as $v \rightarrow +\infty$.

Proof. First we use (2.2.21) together with the decay of assumption 4 and the boundedness of r to get the existence of $e \in \mathbb{R}$ such that (2.4.2) holds. In particular Q is bounded. Moreover, due to assumption 7, $e \neq 0$.

For the mass, notice that by integration by parts and the decay of assumption 4 :

$$\left| \int_v^{+\infty} r \partial_v r |\phi|^2 dv' \right| = \left| - \int_v^{+\infty} r^2 \Re(\bar{\phi} \partial_v \phi) dv' - \frac{r^2}{2} |\phi|^2(0, v) \right| \lesssim v^{1-2s}.$$

Therefore - the other terms being easier in (2.2.24)- by using the gauge (2.3.3) and assumption 4, together with the boundedness of r , we prove that there exists $M \in \mathbb{R}$ such that (2.4.1) holds.

Gauge (2.3.3) then gives the following convergence when v tends to $+\infty$ on \mathcal{H}^+ :

$$\partial_v r = 1 - \mu = 1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2} \rightarrow 1 - \frac{2M}{r_\infty} + \frac{e^2}{r_\infty^2} := l.$$

Since r admits a limit at infinity, $l = 0$ so r_∞ is a strictly positive root of the polynomial $x^2 - 2Mx + e^2$ hence :

$$r_\infty = M \pm \sqrt{M^2 - e^2},$$

$$|e| \leq |M|,$$

$$0 < M.$$

We then use assumption 7 to rule out the case $r_\infty = M - \sqrt{M^2 - e^2}$ since $r_-(M, e) \leq |e|$ for all $0 < |e| \leq M$. Assumption 7 also gives the sub-extremality condition $|e| < M$.

The last claim follows from the definition of K and the fact that for all $0 < |e| < |M|$, $M - \frac{e}{r_+(M, e)} > 0$. \square

Now that M and e are known, we shall denote K_+ instead of $K_+(M, e)$ and K_- instead of $K_-(M, e)$.

We know the Reissner–Nordström background -indexed by (M, e) - towards which our space-time converges at infinity and we can define the null coordinates u and V in the spirit of section 2.2.3 - given that the (U, v) coordinates are already defined by the statement of Theorem 2.3.2 - :

Definition 4. Recalling that $(U, v) \in [0, U_s] \times [v_0, +\infty]$, we define $u \in [-\infty, u_s]$ by the relation :

$$U := \frac{1}{2K_+} e^{2K_+ u},$$

and $V \in [V_0, 1]$ by :

$$V := 1 - \frac{1}{2|K_-|} e^{2K_- v}.$$

We write the metric ⁶⁷ on \mathcal{Q} in these different coordinates systems as :

$$g_{\mathcal{Q}} = -\frac{\Omega^2}{2}(du \otimes dv + dv \otimes du) = -\frac{\Omega_H^2}{2}(dU \otimes dv + dv \otimes dU) = -\frac{\Omega_{CH}^2}{2}(du \otimes dV + dV \otimes du).$$

Notice that :

$$2K_+ U \Omega_H^2(U, v) = \Omega^2(u, v) = 2|K_-|(1 - V) \Omega_{CH}^2(u, V).$$

We will also define $\nu_H := \partial_U r$. Notice that $\nu_H < 0$ everywhere on the space-time. This is because it is strictly negative on \mathcal{H}^+ -due to the no anti-trapped surface assumption- therefore so is $\frac{\nu_H}{\Omega_H^2}$ and this quantity is decreasing in U due to (2.2.17).

Now that the parameters (M, e) are determined, we translate the notation \lesssim : $A \lesssim B$ means that there exists a constant $\tilde{C} = \tilde{C}(C, e, q_0, m^2, M, s, v_0)$ such that $A \leq \tilde{C}B$.

Reduction to the case where K is lower bounded on the event horizon.

In order to use the red-shift effect in all its strength near the event horizon, we have to prove that K is close enough to its limit value -the surface gravity K_+ - and in particular is lower bounded by a strictly positive constant on the event horizon.

To do so, we need to be far away in the future, i.e to consider large v .

We are going to prove that for $v'_0 = v'_0(C, e, M, q_0, m^2, s)$ large enough -with the assumptions of Theorem 2.3.2- bounds of the following form are still true :

$$|D_U \phi(U, v'_0)| \lesssim D(v'_0),$$

$$|\partial_U r(U, v'_0)| \gtrsim 1.$$

In the second step, we restart our problem, replacing v_0 by v'_0 in the hypothesis of Theorem 2.3.2 - in particular C_{in} is redefined to be $C_{in} = \{v \equiv v'_0, 0 \leq U \leq U_s\}$ and (2.2.9), (2.3.4) are true on $v \equiv v'_0$ instead.

This can be done introducing a new coordinate system (U', v) with $\partial_{U'} r(U', v'_0) = -1$. This can only multiply the bound for $D_{U'} \phi(U', v'_0)$ by a constant. Notice that $|D_{U'} \phi(U, v'_0)|$ is not modified by any gauge transform on A . After this section, we will abuse notation and still call (U, v) this new coordinate system (U', v) .

We now take $v'_0 = v'_0(C, e, M, q_0, m^2, s)$ to be large enough so that $2K - 2K_+ + rm^2|\phi|^2$ is arbitrarily close to 0.

To be able to do it, we must use ⁶⁸ the Einstein-Maxwell-Klein-Gordon equations on the space-time rectangle $[0, U_s] \times [v_0, v'_0]$ which is the object of the following lemma :

⁶⁷C.f section 2.2.2 for a definition.

⁶⁸This essentially boils down to an easy local existence theorem.

Lemma 2.4.3. *Under the same hypothesis than before and for $v'_0 > v_0$, if U_s is sufficiently small there exists a constant $D > 0$ depending on $C, e, M, q_0, m^2, s, v_0$ and v'_0 such that*

$$|\partial_U r(U, v'_0)|^{-1} + |D_U \phi(U, v'_0)| \leq D. \quad (2.4.3)$$

Therefore, for any $\eta > 0$ independent⁶⁹ of any parameter, there exists a $v'_0 > 0$ such that

$$|D_U \phi(U, v'_0)| \lesssim C,$$

and for all $v \geq v'_0$:

$$|2K(0, v) - 2K_+| \leq \eta K_+,$$

$$rm^2|\phi|^2(0, v) \leq \eta K_+.$$

The proof, which is not difficult, is deferred to section 2.10.

In what follows, we will not refer to v'_0 any longer, and when we will write v_0 in the rest of the paper, we actually mean v'_0 .

Main bounds on the event horizon

Proposition 2.4.4. *The following bounds hold on the event horizon :*

$$0 \leq \lambda = 1 - \mu \lesssim v^{-2s}, \quad (2.4.4)$$

$$0 \leq r_+ - r(0, v) \lesssim v^{1-2s}, \quad (2.4.5)$$

$$|\partial_v \log(\Omega_H^2)(0, v) - 2K(0, v)| \lesssim v^{-2s}, \quad (2.4.6)$$

$$|\partial_U \log(\Omega_H^2)|(0, v) \lesssim \Omega_H^2(0, v), \quad (2.4.7)$$

$$|\partial_U \phi|(0, v) \lesssim \Omega_H^2(0, v)v^{-s} \quad (2.4.8)$$

$$|A_U|(0, v) \lesssim \Omega_H^2(0, v). \quad (2.4.9)$$

Moreover there exists a fixed function $h(v)$ such that :

$$\Omega_H^2(0, v) = -4\nu_H(0, v) = e^{2K_+(v+h(v))}, \quad (2.4.10)$$

with

$$|\partial_v h(v)| \lesssim v^{1-2s}. \quad (2.4.11)$$

Proof. We use (2.2.25) and gauge (2.3.3) to write :

$$\partial_v \log(\Omega_H^2) = \partial_v \log(-\nu_H) = 2K - rm^2|\phi|^2. \quad (2.4.12)$$

(2.4.6) then follows directly from assumption 4.

We first prove that

$$\frac{\lambda}{\Omega_H^2}(v = +\infty) = 0.$$

Let $0 < \delta_0 < 1$ suitably small enough to be chosen later, independently of all the parameters.

Then, by section 2.4.3 we are allowed to assume that :

$$|2K - rm^2|\phi|^2 - 2K_+| \leq 2\delta_0 K_+.$$

Then, we integrate (2.4.12) on $[v_0, v]$ to get :

$$e^{2K_+(1-\delta_0)v} \lesssim \Omega_H^2(0, v) \lesssim e^{2K_+(1+\delta_0)v}.$$

Using (2.2.18) written as $\partial_v(\frac{\lambda}{\Omega_H^2}) = \frac{-r}{\Omega_H^2}|\partial_v \phi|^2$, we get that

$$|\partial_v(\frac{\lambda}{\Omega_H^2})| \lesssim e^{-2K_+(1-\delta_0)v} v^{-2s},$$

⁶⁹We insist that η must be a numerical constant that do **not** depend on **any** of the C, e, M, q_0, m^2, v_0 or v'_0 .

which is integrable. Therefore $\frac{\lambda}{\Omega_H^2}$ admits a limit $l \in \mathbb{R}$ when $v \rightarrow +\infty$. Integrating ⁷⁰ on $[v, +\infty]$, we get after multiplication by $\Omega_H^2(0, v)$:

$$|\lambda - l\Omega_H^2| \lesssim e^{4K_+\delta_0 v} v^{-2s}.$$

Integrating again and using the boundedness of r , we get after absorbing the r difference in $e^{4K_+\delta_0 v} v^{-2s}$

$$|l \int_{v_0}^v \Omega_H^2| \lesssim e^{4K_+\delta_0 v} v^{-2s}.$$

Hence, using the lower bound for Ω_H^2 :

$$|l|e^{2K_+(1-\delta_0)v} \lesssim e^{4K_+\delta_0 v} v^{-2s}.$$

If $\delta_0 < \frac{1}{3}$, it proves that $l = 0$. Since $\partial_v(\frac{\lambda}{\Omega_H^2}) \leq 0$, we have that

$$\lambda \geq 0.$$

Using (2.4.12) and the earlier section 2.4.3, we are allowed to assume that :

$$\partial_v \log(\Omega_H^2) \geq K_+ > 0.$$

Therefore using a variant of Lemma 2.4.1 on $[v, +\infty]$:

$$0 \leq \lambda(0, v) = \Omega_H^2(0, v) \int_v^{+\infty} \frac{r|\partial_v \phi|^2}{\Omega_H^2(0, v')} dv' \lesssim \Omega_H^2(0, v) \int_v^{+\infty} \frac{|v'|^{-2s}}{\Omega_H^2(0, v')} dv' \lesssim v^{-2s}.$$

Therefore we proved (2.4.4) and (2.4.5). It also gives -using (2.4.1) and (2.4.2)- :

$$|2K(U, v) - 2K_+(M, e)| \lesssim v^{1-2s},$$

and therefore giving (2.4.11) from (2.4.6).

(2.4.9) follows from (2.2.9) and (2.2.22) written as $\partial_v A_U = -\frac{Q\Omega_H^2(0, v)}{2r^2}$, using Lemma 2.4.1 with $q = 0$. From then it is easy to use (2.2.28), the gauge (2.3.3) and the decay of ϕ and $\partial_v \phi$ to establish (2.4.8). Now writing (2.2.16) as

$$|\partial_v \partial_U \log(\Omega_H^2)| = |-2\Re(D_U \phi \partial_v \phi) + \frac{\Omega_H^2}{2r^2} + \frac{2\partial_U r \partial_v r}{r^2} - \frac{\Omega_H^2}{r^4} Q^2| \lesssim \Omega_H^2(0, v)$$

gives immediately (2.4.7) after integration. □

2.4.4 The red-shift region

We define for $\delta > 0$ suitably small to be chosen later, the red-shift region as :

$$\mathcal{R} := \{U\Omega_H^2(0, v) \leq \delta\} = \{u + v + h(v) \leq \frac{\log(2K_+\delta)}{2K_+} := -\Delta\}.$$

In this region, we expect that Ω^2 will be exponentially growing in $u + v$ while still remaining very small as it is the case for Reissner–Nordström, which is a manifestation of the red-shift effect.

However already on the event horizon $\frac{\Omega_H^2(0, v)}{e^{2K_+v}}$ may be unbounded ⁷¹ so we decide to set $e^{2K_+(u+v+h(v))} = 2K_+ U \Omega_H^2(0, v)$ to be small instead of $e^{2K_+(u+v)}$.

The most emblematic consequence of the red-shift effect - and the main difficulty- is the bound for the field $|D_u \phi| \lesssim \Omega^2 v^{-s}$ from which we derive the others.

⁷⁰Recall that $\int_v^{+\infty} e^{-2K_+(1-\delta_0)v'} v'^{-2s} dv' \lesssim e^{-2K_+(1-\delta_0)v} v^{-2s}$. Similarly, $\int_{v_0}^v e^{4K_+\delta_0 v'} v'^{-2s} dv' \lesssim e^{4K_+\delta_0 v} v^{-2s}$.

⁷¹This quantity may grow like v^{2-2s} . If $s > 1$ like in [57], this problem does not exist so \mathcal{R} can be defined using $e^{2K_+(u+v)}$ directly.

Main bounds on the red-shift region

Proposition 2.4.5. *We have the following control ⁷² on the field and the potential on \mathcal{R} :*

$$|\phi| + |\partial_v \phi| \lesssim v^{-s}, \quad (2.4.13)$$

$$|\partial_U \phi| \lesssim \Omega_H^2(0, v) v^{-s}, \quad (2.4.14)$$

$$|A_U| \lesssim \Omega_H^2(0, v). \quad (2.4.15)$$

We also have :

$$|\log(\Omega^2(u, v)) - 2K_+ \cdot (u + v + h(v))| = |\log(\frac{\Omega_H^2(U, v)}{\Omega_H^2(0, v)})| \lesssim U \Omega_H^2(0, v), \quad (2.4.16)$$

$$0 \leq 1 - \kappa(U, v) \lesssim \Omega^2(U, v) v^{-2s}, \quad (2.4.17)$$

$$|\partial_U \log(\Omega_H^2)(U, v)| \lesssim \Omega_H^2(0, v), \quad (2.4.18)$$

$$|\partial_v \log(\Omega^2)(U, v) - 2K(U, v)| \lesssim v^{-2s}, \quad (2.4.19)$$

$$0 \leq r_+ - r(U, v) \lesssim \Omega^2 + v^{1-2s}, \quad (2.4.20)$$

$$|Q(U, v) - e| \lesssim v^{1-2s}, \quad (2.4.21)$$

$$|\varpi(U, v) - M| \lesssim v^{1-2s}, \quad (2.4.22)$$

$$|2K(U, v) - 2K_+| \lesssim \Omega^2 + v^{1-2s}. \quad (2.4.23)$$

Proof. We bootstrap ⁷³ the following estimates ⁷⁴ in \mathcal{R} :

$$|\phi| + |\partial_v \phi| \leq 4C v^{-s}, \quad (2.4.24)$$

$$|D_U \phi| \leq D \Omega_H^2(0, v) v^{-s}, \quad (2.4.25)$$

$$-\nu_H(U, v) \leq \Omega_H^2(0, v), \quad (2.4.26)$$

$$\frac{1}{2} \leq \kappa \leq 1, \quad (2.4.27)$$

$$|Q - e| \leq 4\bar{C} v^{1-2s}. \quad (2.4.28)$$

Where \bar{C} is the constant of estimate (2.4.2) and D is a large enough constant -independent of δ - to be chosen later. Recall also that C is defined in the statement of Theorem 2.3.2.

We can first write (2.2.23) using bootstraps (2.4.24), (2.4.25), (2.4.26), (2.4.27) as :

$$|\partial_U \varpi| \lesssim (D^2 |\lambda| + 1) \Omega_H^2(0, v) v^{-2s}.$$

Using (2.2.15), it is not difficult to prove that $|\lambda|$ is bounded hence after integrating in U :

$$|\varpi(U, v) - \varpi(0, v)| \lesssim D^2 \delta v^{-2s}. \quad (2.4.29)$$

Then it gives (2.4.22) , using the bound on the event horizon with δ small enough with respect to D notably. Similarly we get :

$$|Q(U, v) - Q(0, v)| \lesssim D \delta v^{-2s}, \quad (2.4.30)$$

which proves (2.4.21) and closes bootstrap (2.4.28) for δ small enough.

We now write (2.2.22) as :

$$\partial_v A_U = -\frac{2Q\kappa}{r} \nu_H(U, v).$$

Then bootstraps (2.4.26), (2.4.27) and (2.4.28) give

⁷²Note that (2.4.13), (2.4.14) and (2.4.15) also give $|D_U \phi(U, v)| \lesssim U \Omega_H^2(0, v)$.

⁷³For an introduction to bootstrap methods, c.f chapter 1 of [76].

⁷⁴Notice that bootstrap (2.4.26) and (2.4.27) combined give $\Omega_H^2(U, v) \leq 4\Omega_H^2(0, v)$.

$$|\partial_v A_U| \lesssim \Omega_H^2(0, v).$$

Hence with gauge (2.2.9) and the bound on the event horizon (2.4.10), (2.4.11), we use Lemma 2.4.1 with $q = 0$ to get (2.4.15) :

$$|A_U| \lesssim \Omega_H^2(0, v).$$

Now using the last equation we get with bootstrap (2.4.24) and (2.4.25) :

$$|\partial_U \phi| \lesssim D \Omega_H^2(0, v) v^{-s}.$$

We can then integrate to get :

$$|\phi(U, v) - \phi(0, v)| \lesssim D \delta v^{-s}. \quad (2.4.31)$$

which implies that for δ small enough :

$$|\phi| \leq 2C v^{-s}.$$

Let $0 < a$ be a constant suitably chosen later. We can rewrite (2.2.19) together with (2.2.15) as :

$$\partial_v(e^{av} r \frac{D_U \phi}{\nu_H}) = (a - \kappa(2K - rm^2|\phi|^2)) e^{av} r \frac{D_U \phi}{\nu_H} - e^{av} \partial_v \phi + \kappa e^{av} rm^2 \phi. \quad (2.4.32)$$

We first need to prove that K is lower bounded in \mathcal{R} . The bootstrap (2.4.26) gives :

$$0 \leq r(0, v) - r(U, v) \leq \delta.$$

Then, making use of (2.4.29) and (2.4.30), we write :

$$|K(U, v) - K_+| \leq |K(U, v) - K(0, v)| + |K(0, v) - K_+| \lesssim (1 + D + D^2)\delta + |K(0, v) - K_+|.$$

We then recall that the discussion of section 2.4.3 allows us to consider that $|K(0, v) - K_+| \leq \eta K_+$ and also that $rm^2|\phi|^2(0, v) < \eta K_+$ for any η not depending on the parameters. Hence for δ small enough, we can assume that

$$2K(U, v) - rm^2|\phi|^2(U, v) > K_+.$$

Choosing say $0 < a < \frac{K_+}{4}$ gives with bootstrap (2.4.27) that $a - \kappa(2K - rm^2|\phi|^2) \leq -\frac{K_+}{4}$

We then use the Grönwall Lemma combined with the boundedness of bootstrap (2.4.27), the lower boundedness of r , the decay of bootstrap (2.4.24) and assumption (5) with gauge (2.3.4) for the initial condition to get :

$$|r \frac{D_U \phi}{\nu_H}| \lesssim v^{-s} + e^{a(v_0 - v)} \lesssim v^{-s}.$$

It also closes⁷⁵ bootstrap (2.4.25) if D is large enough compared⁷⁶ to the constant that arises which depends on $C, e, M, q_0, m^2, s, v_0$ only and proves :

$$|D_U \phi|(U, v) \lesssim \Omega_H^2(0, v) v^{-s}. \quad (2.4.33)$$

$$|\partial_U \varpi|(U, v) + |\partial_U Q^2|(U, v) \lesssim \Omega_H^2(0, v) v^{-2s}. \quad (2.4.34)$$

Using the preceding bounds on ϕ and A_U , we get (2.4.14) :

$$|\partial_U \phi| \lesssim |\nu_H| v^{-s}.$$

Now using (2.4.15), we can write (2.2.19) as :

$$|\partial_U(\partial_v \phi)| \lesssim |\partial_U \phi| + \Omega_H^2(|\phi| + |\partial_v \phi|) \lesssim -\nu_H(U, v) v^{-s}.$$

Hence by (2.4.26), bootstrap (2.4.24) is validated for δ small enough.

⁷⁵We used that r is bounded below by a constant depending of v_0 and the parameters for δ small enough.

⁷⁶In particular, D is taken large enough independently of δ , hence taking δ small enough compared to D was licit and boiled down to taking δ small enough compared to the parameters.

Recall from section 2.4.3 that we established that everywhere on the space-time :

$$0 \leq \kappa \leq 1.$$

Writing (2.2.17) in (U, v) coordinates, we get -using (2.4.33)- :

$$|\partial_U \log(\kappa)| = \frac{r}{-\nu_H} |D_U \phi|^2 \lesssim |\nu_H| v^{-2s}.$$

Using bootstrap (2.4.26) we get the amelioration :

$$0 \leq 1 - \kappa \lesssim U \Omega_H^2(0, v) v^{-2s}. \quad (2.4.35)$$

Hence bootstrap (2.4.27) is validated for δ small enough.

Now we write (2.2.16) as :

$$|\partial_v \partial_U \log(\Omega_H^2)| \lesssim |\nu_H| \leq \Omega_H^2(0, v) = e^{2K_+(v+h(v))}.$$

Hence we establish (2.4.18) using Lemma 2.4.1 and (2.4.11) :

$$|\partial_U \log(\Omega_H^2)(U, v)| \lesssim |\partial_U \log(\Omega_H^2)|(U, v_0) + \int_{v_0}^v e^{2K_+(v'+h(v'))} dv' \lesssim \frac{1}{K_+} e^{2K_+(v+h(v))} \lesssim \Omega_H^2(0, v),$$

where we used that on C_{v_0} and due to (2.2.17) , assumption 5 and gauge (2.3.4) :

$$|\partial_U \log(\Omega_H^2)|(U, v_0) = r |D_U \phi|^2(U, v_0) \lesssim 1.$$

Hence we establish (2.4.16), that we write with a constant $\tilde{C} > 0$ as :

$$e^{-\tilde{C} U \Omega_H^2(0, v)} \leq \frac{\Omega_H^2(U, v)}{\Omega_H^2(0, v)} \leq e^{\tilde{C} U \Omega_H^2(0, v)},$$

and in particular :

$$e^{-\tilde{C} \delta} \leq \frac{\Omega_H^2(U, v)}{\Omega_H^2(0, v)} \leq e^{\tilde{C} \delta},$$

which together with (2.4.35) closes bootstrap (2.4.26) for δ small enough . It gives⁷⁷ also (2.4.17). Moreover we have the more precise estimate :

$$e^{-\tilde{C} \delta} \leq \frac{-4\nu_H(U, v)}{\Omega_H^2(0, v)} \leq \frac{e^{\tilde{C} \delta}}{1 - \tilde{C}' \delta v^{-2s}}. \quad (2.4.36)$$

We get the more refined bound (2.4.20) on r , using (2.4.5) :

$$0 \leq r_+ - r(U, v) \leq \frac{1}{4} e^{\tilde{C} \delta} U \Omega_H^2(0, v) + \tilde{C} v^{1-2s} \lesssim \Omega^2(U, v) + v^{1-2s}.$$

As a consequence of (2.4.20), (2.4.21) and (2.4.22) we get (2.4.23).

Finally we can rewrite (2.2.26) in (U, v) coordinates and using our estimates we get :

$$|\partial_U (\partial_v \log(\Omega^2) - 2K)| \lesssim |\kappa - 1| |\partial_U (2K)| + |D_U \phi| |\partial_v \phi| + |\partial_U \varpi| + |\partial_U Q| \lesssim \Omega_H^2(0, v) v^{-2s}.$$

Hence with (2.4.6), we prove (2.4.19). □

⁷⁷Notice that δ small enough is to be understood as $\delta \leq \epsilon(C, e, M, q_0, m^2, s, v_0)$ with ϵ small enough.

Control of ι in the late red-shift transition region

Notice that in Proposition 2.4.5, we have an estimate for $1 - \kappa$ but nothing for the v -analogue $1 - \iota$. This is because ι^{-1} blows-up in general near the event horizon where $1 - \iota^{-1}(0, v) = +\infty$.

It is important to get a bound for $1 - \iota$ as it will give control of $\partial_u \log(\Omega^2) - 2K$, in the same manner $1 - \kappa$ bounds in \mathcal{R} gave control of $\partial_v \log(\Omega^2) - 2K$.

Still we will show that we can control $1 - \iota$ on a subset⁷⁸ of \mathcal{R} defined as

$$\mathcal{LR} := \{C_0 v^{-q(s)} \leq U\Omega_H^2(0, v) \leq \delta\},$$

where $q(s) = 1_{\{s \leq 1\}} + s1_{\{s > 1\}}$ and we call this subset the late red-shift transition region.

The name transition simply comes from the fact we aim at bounding $\partial_u \log(\Omega^2) - 2K$ instead of $\partial_u \log(\Omega^2) - 2K_+ = \partial_u \log(\Omega_H^2)$ so there is a transition from $2K_+$ to $2K$.

Notice that in this region $|u| \sim v$.

Proposition 2.4.6. *In $\mathcal{LR} := \{C_0 v^{-q(s)} \leq U\Omega_H^2(0, v) \leq \delta\}$, we have the following estimates :*

$$|1 - \iota^{-1}| \lesssim v^{-p(s)}, \quad (2.4.37)$$

$$|\partial_u \log(\Omega^2) - 2K| \lesssim v^{-p(s)}, \quad (2.4.38)$$

where ⁷⁹ $p(s) = (2s - 1)1_{\{s \leq 1\}} + s1_{\{s > 1\}}$.

Proof. Use (2.2.15) to write :

$$\partial_u(\Omega^2 + 4\lambda) = \Omega^2(\partial_u \log(\Omega^2) - 2K + rm^2|\phi|^2).$$

We can integrate from the event horizon for $u' \in (-\infty, u]$ to get :

$$|\Omega^2 + 4\lambda|(u, v) \lesssim |\lambda|(-\infty, v)|_{\mathcal{H}^+} + \int_{-\infty}^u \Omega^2(u', v) |rm^2|\phi|^2 + \partial_u \log(\Omega^2) - 2K du'.$$

Notice that (2.4.18)- thanks to (2.4.16)- can be alternatively written as

$$|\partial_u \log(\Omega^2)(u, v) - 2K_+| \lesssim U\Omega_H^2(0, v) \sim \Omega^2(u, v).$$

In particular if δ is chosen to be small enough, $\partial_u \log(\Omega^2) > K_+$.

Moreover, (2.4.13) and (2.4.23) give :

$$|\Omega^2 + 4\lambda| \lesssim |\lambda|_{\mathcal{H}^+} + \int_{-\infty}^u \Omega^2(u', v)(\Omega^2(u', v) + v^{1-2s}) du' = |\lambda|_{\mathcal{H}^+} + \int_{-\infty}^u \Omega^4(u', v) du' + v^{1-2s} \int_{-\infty}^u \Omega^2(u', v) du'.$$

We then divide by $\partial_u \log(\Omega^2)$ which is lower bounded to use Lemma 2.4.1 and with (2.4.4) we get ⁸⁰

$$|\Omega^2 + 4\lambda|(u, v) \lesssim \Omega^2(u, v)(\Omega^2(u, v) + v^{1-2s}) + v^{-2s}.$$

Therefore -dividing by Ω^2 - on the past boundary of \mathcal{LR} defined as $\gamma_{\mathcal{LR}} := \{U\Omega_H^2(0, v) = Cv^{-q(s)}\}$ we get

$$|1 - \iota^{-1}|_{\gamma_{\mathcal{LR}}} \lesssim v^{-q(s)} + v^{1-2s} + v^{q(s)-2s} \lesssim v^{-p(s)}.$$

We then integrate (2.2.18) from $\gamma_{\mathcal{LR}}$ i.e on $[v_{\gamma_{\mathcal{LR}}}(u), v]$, using (2.4.13) :

$$|1 - \iota^{-1}|(u, v) \leq |1 - \iota^{-1}|(u, v_{\gamma_{\mathcal{LR}}}(u)) + \int_{v_{\gamma_{\mathcal{LR}}}(u)}^v \Omega^{-2}(u, v') v'^{-2s} dv'.$$

Thanks to (2.4.19) and for $|u_s|$ large enough, $\partial_v \log(\Omega^2) > K_+$ hence using Lemma 2.4.1 :

$$\int_{v_{\gamma_{\mathcal{LR}}}(u)}^v \Omega^{-2}(u, v') v'^{-2s} dv' \lesssim \Omega^{-2}(u, v_{\gamma_{\mathcal{LR}}}(u)) v_{\gamma_{\mathcal{LR}}}(u)^{-2s} \lesssim v^{q(s)-2s},$$

where we have used in the last inequality that in this region $v_{\gamma_{\mathcal{LR}}}(u) \sim |u| \sim v$.

⁷⁸ C_0 is chosen such that $C_0 v_0^{-q(s)} < \delta$.

⁷⁹The behaviour is different for $s > 1$ but still gives integrability when $s > 1$ and non-integrability if $s \leq 1$.

⁸⁰Recall that $\Omega^2(u, v) = 2K_+ U\Omega_H^2(U, v)$.

Hence (2.4.37) is proved :

$$|1 - \iota^{-1}| \lesssim v^{-p(s)} + v^{q(s)-2s} \lesssim v^{-p(s)}.$$

Notice that because of (2.4.13) and the boundedness⁸¹ of ι^{-1} we have :

$$\begin{aligned} |\partial_v \varpi| + |\partial_v Q^2| &\lesssim v^{-2s}, \\ |\partial_v(2K)| &\lesssim \Omega^2 + v^{-2s}. \end{aligned}$$

Hence using (2.2.26) and the red-shift region main bounds we get :

$$|\partial_v(\partial_u \log(\Omega^2) - 2K)| \lesssim (\Omega^2 + v^{-2s})v^{-p(s)} + \Omega^2 v^{-2s} + \iota v^{-2s} \lesssim v^{-2s} + \Omega^2 v^{-p(s)}.$$

Integrating using that $\partial_v \log(\Omega^2) > K_+$ and Lemma 2.4.1 gives (2.4.38), after noticing that :

$$|\partial_u \log(\Omega^2)(u, v_{\gamma_{\mathcal{LR}}}(u)) - 2K(u, v_{\gamma_{\mathcal{LR}}}(u))| \lesssim \Omega^2(u, v_{\gamma_{\mathcal{LR}}}(u)) + |2K - 2K_+|(u, v_{\gamma_{\mathcal{LR}}}(u)) \lesssim v^{-q(s)} + v^{1-2s}.$$

□

2.4.5 The no-shift region

We now define the no-shift region as :

$$\mathcal{N} := \bigcup_{k=1}^N \mathcal{N}_k,$$

where

$$\mathcal{N}_k := \{\Delta_{k-1} := -\Delta + (k-1)\epsilon \leq u + v + h(v) \leq \Delta_k := -\Delta + k\epsilon\},$$

$\epsilon > 0$ small enough and $N \in \mathbb{N}$ large enough are to be chosen⁸² later.

We take the convention that $\mathcal{N}_0 = \gamma_{-\Delta}$ is the past boundary of \mathcal{N} .

This is the region where the transition between the red-shift effect and the blue-shift effect occurs : $2K$ goes from positive values for r close to r_+ towards negative values for r close to r_- .

Since the derivatives of $\log(\Omega^2)$ are broadly $2K$ which changes sign hence cancels, we cannot use the technique arising from Lemma 2.4.1 as before.

Moreover, we cannot hope for any decay of Ω^2 that is small on the past and future boundary but is only bounded in between.

However, this region is easy because the $u + v + h(v)$ difference is finite so that essentially, we do not lose the bounds proved in the red-shift region.

There are two difficulties : the first is to prove decay for the wave equations. We do it by splitting \mathcal{N} into small enough pieces which allows us to close the bootstrapped bounds.

The second and main difficulty is to prove that the blue-shift indeed appears, i.e that r is decreasing enough so that it reaches $M - \frac{\epsilon^2}{r} < 0$ i.e $K_{M,\epsilon}(r) < 0$, giving also $K < 0$.

Note that in \mathcal{N} : $|u| \sim v$, due to (2.4.11) which gives $h(v) = o(v)$.

We will denote for $0 \leq k \leq N$: $\gamma_k := \{u + v + h(v) = \Delta_k\}$. We also denote $u_k(v)$ the unique u such that $(u_k(v), v) \in \gamma_k$. We define similarly $v_k(u)$.

The main estimates in the no-shift region

This is the first part where we address the propagation of the bounds established in the past sections.

Since Δ is now fixed definitively, we define the new notation : $A \lesssim B$ if there exists a constant $\bar{C} = \bar{C}(\Delta)$ such that $A \lesssim \bar{C}B$.

If we need to specify this constant, we shall call it consistently \bar{C} when there are no ambiguities.

⁸¹Since $v \sim |u|$ in this region we can take $|u_s|$ to be large enough so that -say- $|1 - \iota^{-1}| \leq 0.01$.

⁸²Later, we will first choose ϵ small compared to C, e, M, q_0, m^2 and δ in this section. Once ϵ is chosen and small enough, we will choose $N\epsilon$ large enough compared to C, e, M, q_0, m and δ in the next section.

Proposition 2.4.7. *For small enough $\epsilon > 0$, we have :
the following control on the field and the potential on \mathcal{N} :*

$$|\phi| + |\partial_v \phi| \lesssim 2^N v^{-s}, \quad (2.4.39)$$

$$|D_u \phi| \lesssim 2^N |u|^{-s} \sim 2^N v^{-s}, \quad (2.4.40)$$

$$|A_u| \lesssim (N+1)\delta. \quad (2.4.41)$$

and we also have ⁸³ :

$$|\log \Omega^2(u, v) - \log(-4(1 - \frac{2M}{r} + \frac{e^2}{r^2}))| \lesssim 4^N v^{1-2s}, \quad (2.4.42)$$

$$0 \leq 1 - \kappa \lesssim 5^N v^{-2s}, \quad (2.4.43)$$

$$|1 - \iota| \lesssim 5^N v^{-p(s)}, \quad (2.4.44)$$

$$|\partial_u \log(\Omega^2) - 2K| \lesssim 5^N v^{-p(s)}, \quad (2.4.45)$$

$$|\partial_v \log(\Omega^2) - 2K| \lesssim 5^N v^{-2s}, \quad (2.4.46)$$

$$|Q(u, v) - e| \lesssim 4^N v^{1-2s}. \quad (2.4.47)$$

$$|\varpi(u, v) - M| \lesssim 4^N v^{1-2s}. \quad (2.4.48)$$

The proof essentially relies on a partition of \mathcal{N} into sub-regions with small $u + v + h(v)$ difference, in the style of the methods of [20] and [57]. Since the proof does not present so many original ideas, we put it in section 2.9 for the sake of completeness.

Estimates on the future boundary of the no-shift region

We now address the second difficulty : we need to have $K < 0$ at some point to initiate the blue-shift effect, get Ω^2 small on the future boundary and therefore r close to r_- . To do that, we use a simple contradiction argument.

Proposition 2.4.8. *There exists a constant $K_* > 0$, independent of N and ϵ such that, for $u \leq u_s$:*

$$\Omega_{|\gamma_N}^2 \lesssim e^{-K_* N \epsilon}, \quad (2.4.49)$$

$$|r_{|\gamma_N} - r_-| \lesssim e^{-K_* N \epsilon}, \quad (2.4.50)$$

$$|2K_{|\gamma_N} - 2K_-| \lesssim e^{-K_* N \epsilon}. \quad (2.4.51)$$

Proof. We will start by the following lemma, proved by contradiction :

Lemma 2.4.9. *For all $\delta_* > 0$, there exists $0 < \Delta_*$ large enough so that $r < r_-(e, M) + \delta_*$ on $\gamma_{\Delta_*} \cap \{u \leq u_s\}$.*

Proof. By contradiction, take a $\delta_* > 0$ such that for all $0 < \Delta_*$, there exists $u \leq u_s$ such that on γ_{Δ_*} ,

$$r(u, v_{\Delta_*}(u)) \geq r_- + \delta_*$$

Then because $\lambda, \nu < 0$, for all $u_0(v_{\Delta_*}(u)) \leq u' \leq u$ we have :

$$r_- + \delta_* \leq r(u', v_{\Delta_*}(u)) \leq r_+ - \delta. \quad (2.4.52)$$

Using (2.4.42) and (2.4.43), we see that for $|u_s|$ large enough, there exists a constant $\bar{C} > 0$ depending on Δ only such that for all $u_0(v_{\Delta_*}(u)) \leq u' \leq u$

$$-\nu(u', v_{\Delta_*}(u)) \gtrsim \frac{\bar{C}}{\delta_*}.$$

Then we can integrate in u' from γ_0 to γ_{Δ_*} :

$$r(u_0(v_{\Delta_*}(u)), v_{\Delta_*}(u)) - r(u, v_{\Delta_*}(u)) \geq \frac{\bar{C}}{\delta_*}(u - u_0(v_{\Delta_*}(u))) = \frac{\bar{C}}{\delta_*} \Delta_*.$$

Hence, using (2.4.52) :

⁸³Being a bit more careful, we can prove an improved version of (2.4.47) and (2.4.48) without the 4^N factor.

$$r_+ - \delta \geq r_- + \delta_* + \frac{\bar{C}}{\delta_*} \Delta_*.$$

So at fixed δ_* , we can take Δ_* large enough so that the inequality is absurd. Therefore the lemma is proved. \square

Now, since $r_-(e, M) < \frac{e^2}{M}$, we choose a δ_* such that $0 < \delta_* < \frac{e^2}{M} - r_-(e, M)$ and pick a Δ_* such that $r < r_- + \delta_*$ on γ_{Δ_*} .

Then, because $\nu, \lambda < 0$, $r < r_- + \delta_*$ as well in the future of γ_{Δ_*} .

Therefore there exists ⁸⁴ $K_* > 0$ depending on (e, M) only, such that on $\Delta_* \leq u + v + h(v) \leq \Delta_N$:

$$K(u, v) < -K_*,$$

-where we used again (2.4.47), (2.4.48) with $|u_s|$ large enough.

So from (2.4.46), we see that (2.4.49) is true :

$$\Omega_{|\gamma_N}^2 \lesssim \Omega_{|\gamma_{\Delta_*}}^2 e^{K_* \Delta_*} e^{-K_*(-\Delta + N\epsilon)} \lesssim e^{-K_* N \epsilon}.$$

Then recalling from (2.2.11) that

$$\frac{1}{r^2} \left(r - (\varpi + \sqrt{\varpi^2 - Q^2}) \right) \left(r - (\varpi - \sqrt{\varpi^2 - Q^2}) \right) = 1 - \mu = \frac{-4\lambda\nu}{\Omega^2} = \frac{-\Omega^2}{4\epsilon\kappa},$$

we prove that, thanks to (2.4.43) , (2.4.44) and (2.4.47) , (2.4.48) :

$$|r_{|\gamma_{\Delta'}} - (\varpi - \sqrt{\varpi^2 - Q^2})| \lesssim \frac{e^{-K_* N \epsilon}}{|r_{|\gamma_{\Delta'}} - (\varpi + \sqrt{\varpi^2 - Q^2})|} \lesssim \frac{e^{-K_* N \epsilon}}{|r_{|\gamma_{\Delta'}} - r_+| - \tilde{C}v^{1-2s}}.$$

Then , since the monotonicity of r ensures that r is uniformly bounded away from r_+ on $\gamma_{\Delta'}$ and using (2.4.47) and (2.4.48) again on the left-hand-side, we get (2.4.50) and (2.4.51) for $|u_s|$ large enough. \square

2.4.6 The early blue-shift transition region

We define the early blue-shift transition region :

$$\mathcal{EB} := \{ \Delta_N \leq u + v + h(v) \leq -\Delta' + \frac{2s}{2|K_-|} \log(v) \},$$

where $|u_s|$ is large enough so that $v_0 + h(v_0) - \frac{2s}{2|K_-|} \log(v_0) < |u_s| - \Delta'$ and Δ' is a large ⁸⁵ constant to be chosen later.

We will denote⁸⁶ $\gamma := \{u + v + h(v) = -\Delta' + \frac{2s}{2|K_-|} \log(v)\}$, the future boundary of \mathcal{EB} .

Similarly to the region of section (2.4.4), the goal in \mathcal{EB} is to obtain bounds for $\partial_v \log(\Omega^2) - 2K_-$ and $\partial_u \log(\Omega^2) - 2K_-$ on the future boundary instead of $\partial_v \log(\Omega^2) - 2K$ and $\partial_u \log(\Omega^2) - 2K$. For this to be true, we need to prove that the blue-shift in this region is strong enough, in particular we need $|r - r_-| \lesssim |u|^{1-2s} \sim v^{1-2s}$ close enough to the future boundary ⁸⁷.

This region exhibits enough blue-shift so that there is a good decay of the interesting quantities, but not too much so that κ^{-1} and ι^{-1} are still under control. Moreover, the size of the region is small enough -of the order of $\log(v)$ - so that we do not lose too much the control proved in the previous sections- but the decay of the metric coefficients has started and will be strong enough in the future to make the wave propagation decay easier to prove.

Note that in \mathcal{EB} again : $|u| \sim v$.

We define the new notation : $A \lesssim B$ if there exists a constant $\hat{C} = \hat{C}(N, \epsilon)$ such that $A \lesssim \hat{C}B$. We denote $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

If we need to specify this constant, we will call it consistently \hat{C} when there are no ambiguities.

⁸⁴Notice that if $r < \frac{e^2}{M}$ then $K_{M,e}(r) < 0$.

⁸⁵Compared to N , ϵ , Δ and the initial data.

⁸⁶A similar curve has been first introduced by Dafermos in [20].

⁸⁷Actually this bound is already attained in the future of the curve $u + v + h(v) = \frac{2s-1}{2|K_-|} \log(v)$ and in fact, one cannot get better in general. Note that this last curve is very close to γ' exhibited in the instability section.

Proposition 2.4.10. *For $N\epsilon$ large enough, we have the following control on the field on \mathcal{EB} :*

$$|\phi| \lesssim v^{-s} \log(v), \quad (2.4.53)$$

$$|\partial_v \phi| \lesssim v^{-s}, \quad (2.4.54)$$

$$|D_u \phi| \lesssim |u|^{-s} \approx v^{-s}. \quad (2.4.55)$$

and we also have :

$$|\log \Omega^2(u, v) - 2K_- \cdot (u + v + h(v))| \lesssim \Delta e^{-2K_+ \Delta} \sim \delta |\log(\delta)|, \quad (2.4.56)$$

$$0 \leq 1 - \kappa \leq \frac{1}{3}, \quad (2.4.57)$$

$$|1 - \iota| \leq \frac{1}{3}, \quad (2.4.58)$$

$$|\partial_u \log(\Omega^2) - 2K| \lesssim v^{-p(s)} \log(v)^3, \quad (2.4.59)$$

$$|\partial_v \log(\Omega^2) - 2K| \lesssim v^{-2s} \log(v)^3, \quad (2.4.60)$$

$$|Q(u, v) - e| \lesssim v^{1-2s}, \quad (2.4.61)$$

$$|\varpi(u, v) - M| \lesssim v^{1-2s}. \quad (2.4.62)$$

Moreover, on the future boundary γ we have :

$$|\lambda(u_\gamma(v), v)| \lesssim e^{2|K_-| \Delta'} v^{-2s}, \quad (2.4.63)$$

$$|\nu(u, v_\gamma(u))| \lesssim e^{2|K_-| \Delta'} |u|^{-2s}, \quad (2.4.64)$$

$$|r(u_\gamma(v), v) - r_-(M, e)| \lesssim e^{2|K_-| \Delta'} v^{1-2s}, \quad (2.4.65)$$

$$|\partial_v \log(\Omega_{CH}^2)(u_\gamma(v), v)| \lesssim v^{1-2s}, \quad (2.4.66)$$

$$\Omega^2(u_\gamma(v), v) \lesssim e^{2|K_-| \Delta'} v^{-2s}. \quad (2.4.67)$$

Proof. First we take ⁸⁸ $v_0 \geq 2$ so that $1 \lesssim |\log(v)| = \log(v)$.

We make the following bootstrap assumptions :

$$|\partial_v \phi| \leq 4C_\Delta 2^N v^{-s}, \quad (2.4.68)$$

$$|D_u \phi| \leq 4C_\Delta 2^N v^{-s}, \quad (2.4.69)$$

$$|1 - \kappa| \leq \frac{1}{2}, \quad (2.4.70)$$

$$|1 - \iota| \leq \frac{1}{2}, \quad (2.4.71)$$

$$\partial_u \log(\Omega^2) \leq K_- < 0, \quad (2.4.72)$$

$$\partial_v \log(\Omega^2) \leq K_- < 0. \quad (2.4.73)$$

For a constant C_Δ such that $|\partial_v \phi| \leq C_\Delta 2^N v^{-s}$ and $|D_u \phi| \leq C_\Delta 2^N v^{-s}$ are true initially on the past boundary $\gamma_N := \{u + v + h(v) = \Delta_N\}$, using the estimates of \mathcal{N} .

An immediate consequence of bootstrap (2.4.72), (2.4.73) and the boundedness of Ω^2 in \mathcal{N} (c.f section 2.9) is the existence of a constant $\Omega_{max}^2(M, e) > 0$ such that

$$\Omega^2 \leq \Omega_{max}^2(M, e).$$

We now want to prove a decay on $\Omega^\eta \phi$ for η arbitrarily small.

Let $\eta > 0$. We write :

⁸⁸This can be assumed by section (2.4.3) but is really not a restriction, we simply write $|\log(2 + |v|)|$ instead of $\log(v)$.

$$\partial_v(\Omega^{2\eta}\phi) = \eta \cdot \partial_v \log(\Omega^2) \cdot \Omega^{2\eta}\phi + \Omega^{2\eta}\partial_v\phi.$$

Then, because of bootstraps (2.4.68), (2.4.73) we have

$$\partial_v(\Omega^{4\eta}|\phi|^2) = 2\eta \cdot \partial_v \log(\Omega^2) \cdot \Omega^{4\eta}|\phi|^2 + 2\Omega^{4\eta}\Re(\partial_v\phi\bar{\phi}) \leq 8C_\Delta 2^N v^{-s}\Omega^{4\eta}|\phi|,$$

which implies :

$$\partial_v(\Omega^{2\eta}|\phi|) \leq 4C_\Delta 2^N v^{-s}\Omega^{2\eta}v^{-s}.$$

Then it is enough to integrate using (2.4.73) and Lemma 2.4.1, the bound on the previous region and the fact that $|\Omega^{2\eta}(u, v_\gamma(u))\phi(u, v_\gamma(u))| \lesssim |u|^{-s}$ to get :

$$\Omega^{2\eta}|\phi| \lesssim C_\eta |u|^{-s} \sim C_\eta v^{-s}. \quad (2.4.74)$$

Using (2.2.28) together with bootstraps (2.4.68), (2.4.69), (2.4.70), (2.4.71) and (2.4.74) we show that for all $0 < \eta < 1$:

$$|\partial_v(D_u\phi)| \lesssim (1 + C_\eta)C_\Delta 2^N \Omega^{2-2\eta}v^{-s}.$$

We can take $\eta = \frac{1}{2}$.

Integrating using (2.4.73) with Lemma 2.4.1 and $|u| \sim v$ gives :

$$|D_u\phi| \leq C_\Delta 2^N v^{-s} + \tilde{C} 2^N \Omega_{|\gamma_N}(u, v_N(u))v^{-s} \leq C_\Delta 2^N v^{-s} + \tilde{C}\tilde{C} 2^N e^{-\frac{K_*}{2}N\epsilon} v^{-s}.$$

Therefore, we can choose $N\epsilon$ large enough compared to Δ and parameters so that $\tilde{C}\tilde{C} 2^N e^{-K_0 N\epsilon} \leq C_\Delta 2^N$ which closes bootstrap (2.4.69).

Bootstrap (2.4.68) is validated similarly, using (2.4.72), (2.2.27) and the boundedness of Q .

Notice that bootstrap (2.4.69) and (2.4.72) used together with Lemma 2.4.1 give :

$$\int_{u_N(v)}^u \frac{|D_u\phi|^2}{\Omega^2}(u', v) du' \lesssim 4^N \frac{v^{-2s}}{\Omega^2(u, v)}.$$

We integrate (2.2.17) on $[u_N(v), u]$ and multiply by Ω^2 to get, using the bounds from the past :

$$|4\nu + \Omega^2|(u, v) \lesssim v^{-2s}. \quad (2.4.75)$$

Similarly with (2.2.18) :

$$|4\lambda + \Omega^2|(u, v) \lesssim v^{-2s} + \Omega^2 v^{-p(s)} \lesssim v^{-p(s)}. \quad (2.4.76)$$

Integrating bootstrap (2.4.68) over $[v_\gamma(u), v]$ whose size is at most $\tilde{C}\log(v)$, we get (2.4.53) :

$$|\phi| \lesssim C_{\Delta, N} v^{-s} \log(v).$$

From this, we get :

$$|\partial_u Q| + |\partial_v Q| \lesssim C_{\Delta, N}^2 v^{-2s} \cdot \log(v) \quad (2.4.77)$$

And we can integrate to get (2.4.61). The main contribution comes from the past since $v^{-2s}\log(v) = o(v^{1-2s})$ so for $|u_s|$ large enough :

$$|Q - e| \lesssim 4^N v^{1-2s}.$$

Using this together with bootstrap (2.4.70), (2.4.71) and equations (2.2.23), (2.2.24) we get :

$$|\partial_u \varpi| + |\partial_v \varpi| \lesssim C_{\Delta, N}^2 v^{-2s} \log(v)^2. \quad (2.4.78)$$

We also integrate to get (2.4.62) :

$$|\varpi - M| \lesssim 4^N v^{1-2s}.$$

Notice that under our bootstrap assumptions we have -using (2.4.77) and (2.4.78)- :

$$|(\kappa - 1)\partial_u(2K)| \lesssim |4\nu + \Omega^2| + C_{\Delta, N}^2 v^{-2s} \log(v)^2.$$

Now integrating (2.2.26) in u and remembering that $|u - u_N(v)| + |v - v_N(u)| \lesssim \log(v)$, we get (2.4.60) as

$$|\partial_v \log(\Omega^2) - 2K| \lesssim C_{\Delta,N}^2 v^{-2s} \log(v)^3.$$

Where we used (2.4.75). Similarly, using (2.4.76) we prove (2.4.59) :

$$|\partial_u \log(\Omega^2) - 2K| \lesssim C_{\Delta,N}^2 v^{-p(s)} \log(v)^3.$$

Notice that with (2.4.61), (2.4.62) and bootstrap (2.4.70), (2.4.71) used with (2.2.11) we have, for $|u_s|$ large enough ⁸⁹ and using the precedent section :

$$|2K_- - 2K| \lesssim \Omega^2 + C_{\Delta} 4^N v^{1-2s} \lesssim \Omega_{|\gamma_N}^2 \lesssim e^{-K_* N \epsilon}.$$

Hence if $N\epsilon$ is large enough and $|u_s|$ is large enough, bootstrap (2.4.72) and (2.4.73) are validated.

Notice that since $\log(v)v^{1-2s} = o(1)$, we still have :

$$v - v_N(u) = u + v + h(v) - \Delta_N + o(1).$$

From what precedes, we know that :

$$|\partial_v \log(\Omega_{CH}^2)| = |\partial_v \log(\Omega^2) - 2K_-| \lesssim \Omega^2 + C_{\Delta,N}^2 (v^{-2s} \log(v)^3 + v^{1-2s}). \quad (2.4.79)$$

Hence we can integrate from $v_N(u)$ to v , using the upper bound (2.4.73) with Lemma 2.4.1 and the bounds from the past :

$$|\log(\Omega^2) - 2K_- \cdot (u + v + h(v))| \lesssim (\log(\Omega^2(u, v_N(u))) - 2K_- \Delta_N) + \Omega^2(u, v_N(u)) + C_{\Delta,N}^2 (v^{-2s} \log(v)^3 + v^{1-2s}) \cdot \log(v).$$

With what precedes, we see that

$$\Omega^2(u, v_N(u)) + C_{\Delta,N}^2 (v^{-2s} \log(v)^3 + v^{1-2s}) \log(v) \lesssim e^{-K_* N \epsilon}.$$

Hence to get (2.4.56), we choose $N\epsilon$ large enough compared to δ and the initial data.

Now we have proved that

$$\Omega^2 \approx e^{2K_-(u+v+h(v))}.$$

It proves (2.4.67). Using (2.4.79), we get (2.4.66).

Notice that it also proves (2.4.63), (2.4.64) using (2.4.75) and (2.4.76).

Then, dividing (2.4.75) by Ω^2 we get :

$$|\kappa^{-1} - 1| \lesssim e^{2|K_-|(u+v+h(v))} v^{-2s} \lesssim e^{-2|K_-|\Delta'}.$$

Hence for Δ' large enough compared to N, ϵ, Δ and the initial data, we close bootstrap (2.4.70) and prove (2.4.57) with

$$|\kappa^{-1} - 1| \leq \frac{1}{4}.$$

Similarly using (2.4.76), we get :

$$|\iota^{-1} - 1| \lesssim e^{2|K_-|(u+v+h(v))} v^{-2s} + v^{-p(s)},$$

which closes (2.4.71) and proves (2.4.58), for $|u_s|$ large enough.

Finally (2.4.57), (2.4.58) and (2.4.67) give -using (2.2.11)- that

$$|\left(r(u_\gamma(v), v) - (\varpi + \sqrt{\varpi^2 - Q^2})\right) \left(r(u_\gamma(v), v) - (\varpi - \sqrt{\varpi^2 - Q^2})\right)| \leq \tilde{C} e^{-2|K_-|\Delta'} v^{-2s}.$$

Then using (2.4.61) and (2.4.62) with the same type of argument as in section 2.4.5 -notably that r is far away from $r_+(M, e) = M + \sqrt{M^2 - e^2}$ -, we get (2.4.65) :

$$|r(u_\gamma(v), v) - r_-(M, e)| \lesssim e^{2|K_-|\Delta'} v^{1-2s}.$$

□

⁸⁹ $|u_s|$ is taken large enough to annihilate the dependence in N and Δ of $C_{\Delta,N}^2 v^{1-2s}$.

2.4.7 The late blue-shift region

We then define the late blue-shift region :

$$\mathcal{LB} := \{-\Delta' + \frac{2s}{2|K_-|} \log(v) \leq u + v + h(v)\}.$$

This large region is where the essential of the blue-shift occurs : Ω^2 goes from a polynomial decay in v on the past boundary to an exponential decay in v .

In this region, κ^{-1} and ι^{-1} are expected to blow-up⁹⁰ exponentially near the Cauchy horizon if the initial bound on the field is sharp so we cannot trade λ and ν -which decay no better than what (2.4.63) and (2.4.64) suggest- for Ω^2 which decays exponentially.

However, there is enough decay of Ω^2 , ν and λ on the past boundary γ so that we can prove decay for the scalar field with (2.2.19) using a bootstrap method.

In \mathcal{LB} , we will not prove decay for ϕ and $D_u\phi$ -due to $|u| \lesssim v$ only- and we do not know if $-\partial_u \log(\Omega^2)$ is lower bounded like before if $s \leq 1$.

Nevertheless, we can still prove that $-\partial_v \log(\Omega^2)$ is lower bounded which will allow us to prove most of the estimates.

We now recapitulate the constants choice : we have chosen Δ large enough depending on C, e, M, q_0, m^2, v_0 in 2.4.4, then ϵ small enough depending on Δ and C, e, M, q_0, m^2, v_0 in 2.4.5, then $N\epsilon$ large enough depending on Δ and C, e, M, q_0, m^2, v_0 in 2.4.6 and finally Δ' large enough depending on N, ϵ, Δ and C, e, M, q_0, m^2, v_0 also in 2.4.6.

This been said, we can consider that all the constants mentioned above depend on C, e, M, q_0, m^2, v_0 so we are going to write again $A \lesssim B$ if there exists a \tilde{D} depending on these constants such that $A \leq \tilde{D}B$.

Proposition 2.4.11. *We have the following estimates in \mathcal{LB} :*

For all $\eta > 0$, there exists $C_\eta > 0$:

$$\Omega^{2\eta}|\phi| \lesssim C_\eta v^{-s}, \quad (2.4.80)$$

$$\Omega^{2\eta}|Q - e| \lesssim C_\eta v^{1-2s}. \quad (2.4.81)$$

And

$$|\partial_v \phi| \lesssim v^{-s}, \quad (2.4.82)$$

$$|\partial_v \log(\Omega_{CH}^2)| \lesssim |u|^{1-s} v^{-s} 1_{\{s>1\}} + v^{1-2s} 1_{\{s<1\}} + 1_{\{s=1\}} \log(v) v^{-1}, \quad (2.4.83)$$

$$0 < -\lambda \lesssim v^{-2s}, \quad (2.4.84)$$

$$0 < -\nu \lesssim |u|^{-2s}. \quad (2.4.85)$$

Moreover if $s > 1$ we have :

$$|D_u \phi| \lesssim |u|^{-s}, \quad (2.4.86)$$

$$|\partial_u \log(\Omega_{CH}^2) - 2K_-| \lesssim |u|^{1-2s}, \quad (2.4.87)$$

$$|\partial_v Q| \lesssim |u|^{1-s} v^{-s}, \quad (2.4.88)$$

$$|\partial_u Q| \lesssim |u|^{1-2s}. \quad (2.4.89)$$

Proof. We make the following bootstrap assumptions :

$$|r\partial_v \phi| \leq 2\check{C}v^{-s}, \quad (2.4.90)$$

$$|\lambda| \leq 2\check{D}v^{-2s}, \quad (2.4.91)$$

$$\partial_v \log(\Omega^2) \leq K_-, \quad (2.4.92)$$

for $\check{C} > 0$ chosen so that on the past boundary γ we have : $|r\partial_v \phi| \leq \check{C}v^{-s}$ and \check{D} is a large enough constant to be chosen later such that $|\lambda| \leq \check{D}v^{-2s}$ on γ .

Notice that because of (2.2.18), ι decreases in v so by the previous bound on γ we can write :

$$\Omega^2 \leq -6\lambda \leq 12\check{D}v^{-2s}. \quad (2.4.93)$$

For the proof, we introduce a curve $\gamma_{\mathcal{V}} := \{u + v + h(v) = \frac{v}{2}\}$ whose future domain $\mathcal{V} = \{u + v + h(v) \geq \frac{v}{2}\}$ is called the vicinity of the Cauchy horizon.

⁹⁰Indeed, we prove in the instability part that ι^{-1} blows up identically on the Cauchy horizon, for $u \leq u_s$.

We start to integrate (2.4.92) to get, using the bounds on the previous region and choosing $|u_s|$ large enough so that $\log(\Omega^2) \leq 0$:

$$\log(\Omega^2)(u, v) \leq \log(\Omega^2)(u, v_\gamma(u)) + K_-(v - v_\gamma(u)) \leq K_-(v - v_\gamma(u)),$$

and since $v_\gamma(u) \leq -\frac{3}{2}u$ for u_s negative enough, we get :

$$\Omega^2 \leq e^{K_-(v - \frac{3}{2}|u|)}.$$

Notice that on \mathcal{V} , $|u| \leq \frac{v}{2} + h(v)$ hence $v - \frac{3|u|}{2} \geq \frac{v}{4} + o(v)$ so in \mathcal{V} for $|u_s|$ large enough ⁹¹ since $K_- < 0$:

$$\Omega^2 \lesssim e^{\frac{K_-}{5}v}. \quad (2.4.94)$$

The following lemma will prove (2.4.80) and (2.4.81) :

Lemma 2.4.12. *Assuming the bootstraps stated above, we have the following estimates in \mathcal{LB} : for all $\eta > 0$, there exists $C_\eta > 0$ such that :*

$$\Omega^{2\eta}|\phi| \lesssim C_\eta v^{-s}, \quad (2.4.95)$$

$$\Omega^{2\eta}|Q - e| \lesssim C_\eta v^{1-2s}. \quad (2.4.96)$$

Proof. Let $\eta > 0$. We write :

$$\partial_v(\Omega^{2\eta}\phi) = \eta \cdot \partial_v \log(\Omega^2) \cdot \Omega^{2\eta}\phi + \Omega^{2\eta}\partial_v\phi.$$

Then, because of bootstraps (2.4.90), (2.4.92) we have

$$\partial_v(\Omega^{4\eta}|\phi|^2) = 2\eta \cdot \partial_v \log(\Omega^2) \cdot \Omega^{4\eta}|\phi|^2 + 2\Omega^{4\eta}\Re(\partial_v\phi\bar{\phi}) \leq \frac{4\check{C}}{r}v^{-s}\Omega^{4\eta}|\phi|,$$

which implies :

$$\partial_v(\Omega^{2\eta}|\phi|) \leq \frac{2\check{C}}{r}\Omega^{2\eta}v^{-s}.$$

Then it is enough to integrate using (2.4.92) and Lemma 2.4.1, the bound on the previous region and the fact that

$$|\Omega^{2\eta}(u, v_\gamma(u))\phi(u, v_\gamma(u))| \lesssim |u|^{-s}$$

to get :

$$\Omega^{2\eta}|\phi| \lesssim C_\eta |u|^{-s}.$$

Now in the past of $\gamma_\mathcal{V}$, $|u| \sim v$ so (2.4.95) is true.

In \mathcal{V} , we can integrate (2.4.90) to get $|\phi| \lesssim |u|^{1-s}1_{\{s>1\}} + v^{1-s}1_{\{s<1\}} + \log(v)1_{\{s=1\}}$ but the exponential decay of Ω^2 in v from (2.4.94) is stronger than this potential growth for $|u_s|$ large enough, so that (2.4.95) is true also.

We use the same technique to get (2.4.96), using (2.4.95), bootstrap (2.4.90) and (2.2.21).

□

Now we can use (2.2.15) and what precedes to write :

$$|\partial_v(r\nu)| \lesssim \Omega^2 + C_\eta \Omega^{2-2\eta}v^{1-2s}.$$

Integrating, choosing η small enough and using (2.4.92) with Lemma 2.4.1 and the bounds on the former region we prove (2.4.85) :

$$|\nu| \lesssim |u|^{-2s}.$$

Then we can use (2.2.28), (2.4.85) and (2.4.90) to get :

$$|\partial_v(rD_u\phi)| \lesssim |u|^{-2s}v^{-s} + C_\eta \Omega^{2-2\eta}v^{-s}.$$

⁹¹Of course this bound is far from sharp : actually for all $\epsilon_0 > 0$, there exists a region sufficiently close to the Cauchy horizon so that $\Omega^2 \lesssim e^{(2K_- + \epsilon_0)v}$. We will not need such a sharp bound.

Integrating, choosing η small enough and using (2.4.92) with Lemma 2.4.1 to absorb of the $C_\eta \Omega^{2-2\eta} v^{-s}$ term in $|u|^{-s}$, we get :

$$|D_u \phi| \lesssim |u|^{-s} + |u|^{-2s} b(u, v), \quad (2.4.97)$$

with $b(u, v) := v^{1-s} 1_{\{s < 1\}} + |u|^{1-s} 1_{\{s > 1\}} + \log(v) 1_{\{s=1\}}$.

We can then use (2.2.27) and bootstrap (2.4.91) to get :

$$|\partial_u(e^{iq_0 \int_{u_0}^u A_u r \partial_v \phi})| \lesssim \check{D} v^{-2s} |u|^{-s} + \check{D} v^{-2s} |u|^{-2s} b(u, v) + C_\eta \Omega^{2-2\eta} v^{-s}.$$

Integrating on $[u_\gamma(v), u]$ and taking the absolute value we get :

$$|r \partial_v \phi| \leq \check{C} v^{-s} + \check{C} (\check{D} v^{-s} b(u, v) + \check{D} |u|^{1-2s} v^{-s} b(u, v) + v^{1-\frac{2s}{1-\eta}}) v^{-s},$$

where we used that $\Omega^{2-2\eta} \lesssim v^{-\frac{2s}{1-\eta}}$ because of (2.4.93) and $|u - u_\gamma(v)| \lesssim v$.

Noticing that $v^{-s} b(u, v) = o(1)$ when $v \rightarrow +\infty$, uniformly in u and $v^{1-\frac{2s}{1-\eta}} = o(1)$ for η small enough, we can close bootstrap (2.4.90) for $|u_s|$ large enough ⁹².

Now in the past of γ_γ , we can prove, using $v \sim |u|$, the bounds proved before, (2.2.16) and arguments similar to those of section (2.4.6) that :

$$|\partial_u \log(\Omega^2) - 2K_-| \lesssim v^{1-2s}.$$

Hence $\partial_u \Omega^2 \leq 0$ for $|u_s|$ large enough so -denoting C_γ the constant appearing in estimate (2.4.67)- we have :

$$\Omega^2(u, v) \leq \Omega^2(u_\gamma(v), v) \leq C_\gamma v^{-2s}.$$

Moreover the exponential decay of (2.4.94) makes $\Omega^2(u, v) \leq C_\gamma v^{-2s}$ also true for $|u_s|$ large enough in \mathcal{V} . Now we integrate (2.2.18), using (2.4.92) and the bound (2.4.58) to get :

$$4|\lambda| \leq \frac{3}{2} \Omega^2 + \check{C} v^{-2s}.$$

So for $4\check{D} > \frac{3}{2} C_\gamma + \check{C}$, bootstrap (2.4.91) is validated.

Now using the preceding bounds, we get ⁹³ :

$$|\partial_u \partial_v \log(\Omega_{CH}^2)| \lesssim |u|^{-s} v^{-s} + |u|^{-2s} v^{-s} b(u, v) + v^{-2s} + \Omega^{2-2\eta} v^{1-2s}.$$

We can integrate and -using similar methods than before- for η small enough we get (2.4.83), which also closes bootstrap (2.4.92) for $|u_s|$ large enough :

$$|\partial_v \log(\Omega_{CH}^2)| \lesssim b(u, v) v^{-s}.$$

Where we used that $v^{1-2s} = O(v^{-s} b(u, v))$. To prove (2.4.86), (2.4.87), (2.4.88), (2.4.89), it is enough to use the equations, (2.4.97) and the fact that $b(u, v) = |u|^{1-s}$ when $s > 1$, similarly to what was done in the past regions. □

Then we finish the proof of Theorem 2.3.2 : from (2.4.84) and (2.4.85), it is clear the r admits a continuous limit $r_{CH}(u)$ when v tends to $+\infty$ and that $r_{CH}(u) \rightarrow r_-(M, e)$ when $|u|$ tend to $+\infty$.

This is because we can integrate from γ as :

$$r_{CH}(u) = r(u, v_\gamma(u)) + \int_{v_\gamma(u)}^{+\infty} \lambda(u, v') dv' = r(u, v_\gamma(u)) + O(|u|^{1-2s}).$$

Where we used (2.4.84) and $v_\gamma(u) \sim |u|$. Then (2.4.65) proves the claim.

Moreover, we see that ⁹⁴ $|\nu_{CH^+}(u)| \lesssim |u|^{-2s}$ is integrable, therefore $r_{CH}(u)$ is lower bounded for $|u_s|$ large enough. Hence the space-time admits the claimed Penrose diagram for $|u_s|$ large enough.

Moreover if $s > 1$, v^{1-2s} and v^{-s} are integrable in v so we can use the estimates of the last proposition and the argument from Proposition 8.14 of [57] to get a continuous extension of the space-time.

⁹²Notice that \check{D} is absorbed by the decay and does not play any role.

⁹³Recall that $\partial_v \log(\Omega_{CH}^2) = \partial_v \log(\Omega^2) - 2K_-$.

⁹⁴The fact the ν admits a continuous limit when v tends to $+\infty$ follows easily from the estimates.

2.5 Proof of the instability Theorem

2.5.1 Recalling the stability estimates

Before starting the proof of Theorem 2.3.3, we recall the stability estimates -established in the proof of Theorem 2.3.2- that are necessary to prove the instability argument. Notice that they are valid in this framework because all the hypothesis of Theorem 2.3.2 are present in the hypothesis of Theorem 2.3.3.

First we recall the different regions :

1. The event horizon $\mathcal{H}^+ = \{u \equiv -\infty, v \geq v_0\}$
2. The red-shift region $\mathcal{R} = \{u + v + h(v) \leq -\Delta\}$.
3. The no-shift region $\mathcal{N} := \{-\Delta \leq u + v + h(v) \leq \Delta_N\}$
4. The early blue-shift transition region $\mathcal{EB} := \{\Delta_N \leq u + v + h(v) \leq -\Delta' + \frac{2s}{2|K_-|} \log(v)\}$
5. The late blue-shift region $\mathcal{LB} := \{-\Delta' + \frac{2s}{2|K_-|} \log(v) \leq u + v + h(v)\}$ composed of the past of $\gamma_{\mathcal{V}} := \{u + v + h(v) = \frac{v}{2}\}$ and its future called $\mathcal{V} = \{u + v + h(v) \geq \frac{v}{2}\}$.

Then we gather the different bounds from section 2.4 that we will use in this section :

1. on \mathcal{H}^+ , we know that :

$$\lambda \geq 0. \quad (2.5.1)$$

2. We have the following estimates : in \mathcal{R} ,

$$|D_u \phi|(u, v) \lesssim \Omega^2(u, v) v^{-s}. \quad (2.5.2)$$

3. In $\mathcal{N} \cup \mathcal{EB}$:

$$|D_u \phi|(u, v) \lesssim v^{-s}, \quad (2.5.3)$$

$$0 < \iota^{-1} \sim 1, \quad (2.5.4)$$

$$0 < \kappa^{-1} \sim 1. \quad (2.5.5)$$

4. In $\mathcal{R} \cup \mathcal{N} \cup \mathcal{EB}$:

$$|\varpi - M| + |Q - e| \lesssim v^{1-2s}. \quad (2.5.6)$$

5. In \mathcal{EB} :

$$\Omega^2 \sim e^{2K_-(u+v+h(v))}, \quad (2.5.7)$$

$$|r - r_+| \gtrsim 1. \quad (2.5.8)$$

6. In $\mathcal{EB} \cup \mathcal{LB}$:

$$\partial_v \log(\Omega^2) < K_- < 0, \quad (2.5.9)$$

$$|D_u \phi|(u, v) \lesssim |u|^{-s} + |u|^{-2s} b(u, v), \quad (2.5.10)$$

$$|\partial_v \log(\Omega_{CH}^2)|(u, v) \lesssim v^{-s} b(u, v), \quad (2.5.11)$$

with $b(u, v) := 1_{\{s>1\}} |u|^{1-s} + v^{1-s} 1_{\{s<1\}} + \log(v) 1_{\{s=1\}}$.

For all $\epsilon_0 > 0$, there exists a constant $C_{\epsilon_0} > 0$ such that :

$$\Omega^{2\epsilon_0} |\phi| \lesssim C_{\epsilon_0} v^{-s}, \quad (2.5.12)$$

$$\Omega^{2\epsilon_0} |Q - e| \lesssim C_{\epsilon_0} v^{1-2s}, \quad (2.5.13)$$

$$|\lambda| \lesssim \Omega^2 + v^{-2s}. \quad (2.5.14)$$

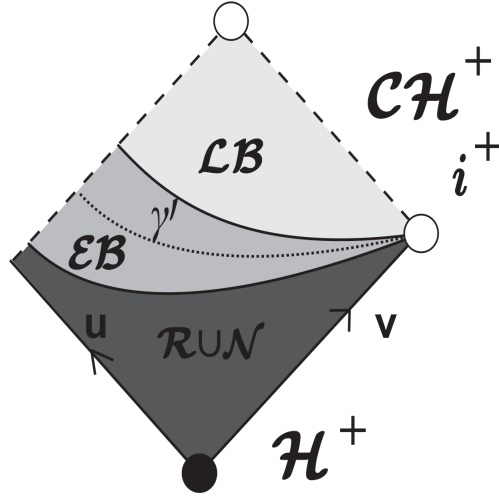


Figure 2.4: Penrose diagram of the space-time $\mathcal{M} = \mathcal{R} \cup \mathcal{N} \cup \mathcal{EB} \cup \mathcal{LB}$ with the inclusion of γ' .

7. In $\mathcal{EB} \cup \mathcal{LB} - \mathcal{V}$:

$$\partial_u \log(\Omega^2) < K_- < 0. \quad (2.5.15)$$

8. In \mathcal{LB} :

$$\Omega^2 \lesssim v^{-2s}, \quad (2.5.16)$$

$$|\lambda| \lesssim v^{-2s}. \quad (2.5.17)$$

9. In \mathcal{V} :

$$\Omega^2 \lesssim e^{\frac{\kappa_-}{5}v}. \quad (2.5.18)$$

2.5.2 Reduction to the proof of (2.3.7)

In this section, we want to highlight that the polynomial lower bound (2.3.7) for the derivative of ϕ transversally to the Cauchy horizon is enough to establish all the other claims of Theorem 2.3.3

The blow-up of the curvature follows directly from 2.3.7 as first highlighted in the pioneering work⁹⁵ [57] : indeed we can consider

$$Ric(\Omega^{-2}\partial_v, \Omega^{-2}\partial_v) = \Omega^{-4}|\partial_v\phi|^2.$$

(2.3.7) then gives that :

$$\limsup_{v \rightarrow +\infty} Ric(\Omega^{-2}\partial_v, \Omega^{-2}\partial_v)(u, v) = +\infty,$$

using for instance the exponential lower bound for Ω^{-4} given by (2.5.18) in \mathcal{V} .

If $s > 1$, we consider the continuous extension \bar{M} and the future boundary null $\mathcal{CH}^+ := \{V \equiv 1, 0 \leq U \leq U_0\}$ mentioned in the statement of Theorem 2.3.2.

Notice that (2.5.11) proves in that case that $\partial_v \log(\Omega_{CH}^2)(u, \cdot)$ is integrable in v hence (u, V) is a regular coordinate system across the extension : in particular $\Omega_{CH}^2 > 0$ on \mathcal{CH}^+ .

If \mathcal{U} is a neighbourhood in \bar{M} with compact closure -in particular with a finite range of u - of a point $p \in \mathcal{CH}^+$, and ϕ is a spherically symmetric function, its $W_{\mathcal{U}}^{1,2}$ norm can be expressed in (u, V) and (u, v) coordinates - as developed in [56]- as :

$$\|\phi\|_{W^{1,2}(\mathcal{U})}^2 = \int_{\mathcal{U}} (|\partial_V\phi|^2 + |\partial_u\phi|^2 + |\phi|^2) dudV \sim \int_{\mathcal{U}} (\Omega^{-2}|\partial_v\phi|^2 + \Omega^2(|\partial_u\phi|^2 + |\phi|^2)) dudv. \quad (2.5.19)$$

Since \mathcal{U} is a neighbourhood of p , consider the smaller neighbourhood $\mathcal{U}' := \mathcal{U} \cap \mathcal{V}$.

Then, using the fact from (2.5.9) that $\partial_v\Omega^2 \leq 0$:

⁹⁵Note that [57] proves more : in a appropriate global setting, they manage to prove the C^2 inextendibility of the metric.

$$\|\phi\|_{W^{1,2}(\mathcal{U})}^2 \gtrsim \int_{\mathcal{U}'} \Omega^{-2} |\partial_v \phi|^2 du' dv'.$$

We can then use (2.5.18) -valid in \mathcal{U}' - with (2.3.7) to get

$$\|\phi\|_{W^{1,2}(\mathcal{U})} = +\infty,$$

i.e

$$\phi \notin W_{loc}^{1,2}.$$

Now we want to prove that the continuous extension to \mathcal{CH}^+ of Theorem 2.3.2 is not C^1 .

We integrate (2.2.18) on $[v_{\gamma_V}(u), v]$. Using that $\iota^{-1} \geq 0$ we get :

$$\iota^{-1}(u, v) \geq \iota^{-1}(u, v_{\gamma_V}(u)) + \int_{v_{\gamma_V}(u)}^v \frac{4r}{\Omega^2} |\partial_v \phi|^2(u, v') dv' \gtrsim \int_{v_{\gamma_V}(u)}^v \frac{|\partial_v \phi|^2(u', v')}{\Omega^2} dv'.$$

Which means using the same argument as a few lines above that for all $u \leq u_s$ and when $v \rightarrow +\infty$:

$$\iota^{-1}(u, v) = \frac{-4\lambda}{\Omega^2} \rightarrow +\infty.$$

And since ι^{-1} is unchanged for the coordinate system (u, V) that is regular near the Cauchy horizon, i.e the system allowing for the continuous extension, it proves that the metric is not⁹⁶ C^1 in the continuous extension of Theorem 2.3.2 for $s > 1$.

2.5.3 Strategy to prove (2.3.7)

This time we split the space-time into two sub-regions only, namely the past and the future of the curve $\gamma' := \{r - r_- = v^{1-2s+\eta}\}$ for a well-chosen $0 < \eta < 2s - 1$ small enough. This curve is similar to γ introduced in section 2.4.6 -although it has a different power-, we will see that it is comparable near infinity to $\{u + v + h(v) = \Delta_N + \frac{2s-1-\eta}{2|K_-|} \log(v)\}$.

For the sake of comparison, as we will see γ' lies entirely in the early blue-shift transition region \mathcal{EB} for $|u_s|$ large enough c.f Figure 2.4. The key use of this property is that κ^{-1} and ι^{-1} are still bounded in \mathcal{EB} .

Since only the averaged - opposed to pointwise- lower bound of hypothesis 8 is available, we use a vector field method in the past of γ' with the Kodama vector field $T := \kappa^{-1} \partial_v - \iota^{-1} \partial_u$ which is the geometric analog of the Killing vector field ∂_t on Reissner-Nordström. However notice that unlike ∂_t on Reissner-Nordström, T is not a Killing vector field in general i.e $\Pi^{(T)} \neq 0$.

The study of T is particularly relevant for two reasons : first there is no bulk term when we contract the deformation tensor $\Pi_{\mu\nu}^{(T)} := \nabla_{(\mu} T_{\nu)}$ of T with the stress-energy tensor $\mathbb{T} = \mathbb{T}_{EM} + \mathbb{T}_{KG} : \Pi_{\mu\nu}^{(T)} \mathbb{T}^{\mu\nu} = 0$.

Despite $\Pi^{(T)} \neq 0$, this is remarkable that we still get an **exact conservation law**⁹⁷, that we want to integrate.

Second, the good control of κ^{-1} and ι^{-1} allows us to capture $|\partial_v \phi|$ appropriately. In particular on the event horizon \mathcal{H}^+ , we see $\int_{\mathcal{H}^+} |\partial_v \phi|^2$ in gauge (2.3.3) which is exactly the term for which we have a lower bound that we want to propagate. The other terms, notably crossed terms, either enjoy a stronger decay or have a favourable sign.

In the future of γ' , we simply use the propagation equation (2.2.27) and integrate along the u characteristic taking advantage on the upper bound⁹⁸ $\Omega^2 \lesssim v^{-2s}$ on γ' , using similar techniques to that of section 2.4.7. The key point is that the energy flux on γ' is controlled by the integral of $|\partial_v \phi|^2$ on γ' . This is due to the fact that κ^{-1} and ι^{-1} are bounded on γ' and also that γ' is rather symmetric in u and v apart from the term $v^{1-2s+\eta}$ which decays sufficiently⁹⁹. This symmetry avoids to consider terms of the form $\kappa^{-1} - \iota^{-1}$ which are bounded but do not a priori decay.

⁹⁶More precisely, $|\partial_V r|$ blows up identically on \mathcal{CH}^+ because $\Omega_{CH}^2 > 0$ and ι^{-1} blows up.

⁹⁷This can be interpreted as the conservation of the Hawking mass.

⁹⁸This is actually where the remainder term $O(v^{3-6s+4\eta})$ of Lemma 2.5.6 comes from.

⁹⁹This is actually where the remainder term $O(v^{-2s})$ of Lemma 2.5.5 comes from.

2.5.4 Up to the blue-shift region : the past of γ'

We will use the same notations as in the stability part.

Moreover, for $v \geq v_0$, we introduce $\gamma'_v := \{(u', v') \in \gamma', v' \geq v\}$ and denote $u_{\gamma'}(v)$ the unique u such that $(u, v) \in \gamma'$ and $\mathcal{H}_v^+ := \mathcal{H}^+ \cap \{v' \geq v\}$. n' denotes the future directed unit normal of γ' .

vol is the standard volume form induced by the metric, and is written in (u, v) coordinates as

$$vol = \Omega^2 r^2 \sin(\theta) du dv d\theta d\psi,$$

where (θ, ψ) are the standard coordinates on \mathbb{S}^2 .

We also define the Kodama vector field $T := \kappa^{-1} \partial_v - \iota^{-1} \partial_u$.

Proposition 2.5.1. *Under the hypothesis of Theorem 2.3.3 and for v large enough, we have :*

$$\int_{\gamma'_v} \mathbb{T}(T, n') vol(n', \cdot) \gtrsim v^{-p}. \quad (2.5.20)$$

Proof. We state the following lemma, which is proven using elementary calculus only :

Lemma 2.5.2.

$$\mathbb{T}_{uu} = 2|D_u \phi|^2, \quad (2.5.21)$$

$$\mathbb{T}_{vv} = 2|\partial_v \phi|^2, \quad (2.5.22)$$

$$\mathbb{T}_{uv} = \frac{\Omega^2}{2} (m^2 |\phi|^2 + \frac{Q^2}{r^4}), \quad (2.5.23)$$

$$\Pi_{uu}^{(T)} = -2r|D_u \phi|^2, \quad (2.5.24)$$

$$\Pi_{vv}^{(T)} = 2r|\partial_v \phi|^2, \quad (2.5.25)$$

$$\Pi_{uv}^{(T)} = \Pi_{\theta\theta}^{(T)} = \Pi_{\psi\psi}^{(T)} = 0. \quad (2.5.26)$$

As a consequence, we see that $\Pi_{\mu\nu}^{(T)} \mathbb{T}^{\mu\nu} = 0$.

Using the precedent lemma and applying the divergence theorem¹⁰⁰ to the region delimited by \mathcal{H}_v^+ , γ'_v and $\{v' = v, u \leq u_{\gamma'}(v)\}$ we get :

$$\int_{\gamma'_v} \mathbb{T}(T, n') vol(n', \cdot) \gtrsim \int_{\mathcal{H}_v^+} |\partial_v \phi|^2 + 4\lambda Q^2 + 4\lambda |\phi|^2 + \int_{v'=v, u \leq u_{\gamma'}(v)} -\iota^{-1} |D_u \phi|^2 - 4\nu Q^2 - 4\nu |\phi|^2. \quad (2.5.27)$$

Now notice that $\lambda|_{\mathcal{H}^+} \geq 0$ as proved in 2.4.3, and $\nu \leq 0$ so all the terms in the right hand side are non-negative, except $-\iota^{-1} |D_u \phi|^2$. For this one, we write :

$$\int_{-\infty}^{u_{\gamma'}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du' = \int_{-\infty}^{u_{\mathcal{R}}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du' + \int_{u_{\mathcal{R}}(v)}^{u_{\gamma'}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du',$$

where $u_{\mathcal{R}}(v)$ is the unique u such that $u + v + h(v) = -\Delta$ i.e (u, v) belongs to the future boundary of \mathcal{R} .

The first term can be bounded using (2.5.2) :

$$\left| \int_{-\infty}^{u_{\mathcal{R}}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du' \right| \leq v^{-2s} \left| \int_{-\infty}^{u_{\mathcal{R}}(v)} 4\lambda \Omega^2 du' \right| \lesssim v^{-2s}.$$

The second term using (2.5.3), (2.5.4) and $|u_{\gamma'}(v) - u_{\mathcal{R}}(v)| \lesssim \log(v)$:

$$\left| \int_{u_{\mathcal{R}}(v)}^{u_{\gamma'}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du' \right| \lesssim v^{-2s} \log(v).$$

To sum up since $p < 2s$, it proves that $\int_{-\infty}^{u_{\gamma'}(v)} \iota^{-1}(u', v) |D_u \phi|^2(u', v) du' = o(v^{-p})$ hence, as claimed

$$\int_{\gamma'_v} \mathbb{T}(T, n') vol(n', \cdot) \gtrsim v^{-p}.$$

□

¹⁰⁰This is the classical use of the vector field method : the key point being that $\mathbb{T}(\phi, F)$ is divergence-free because (ϕ, F) is a solution to the Maxwell-Klein-Gordon equations.

Before moving to the next section, we will need to localise γ' with respect to the regions of the stability part to be able to use the stability estimates. This is done by the following lemma :

Lemma 2.5.3. *For $|u_s|$ large enough and $\eta > 0$ small enough, $\gamma' := \{r - r_- = v^{1-2s+\eta}\} \subset \mathcal{EB}$.
Moreover we have :*

$$\Omega^2(u_{\gamma'}(v), v) \sim v^{1-2s+\eta}. \quad (2.5.28)$$

Proof. Using (2.2.11), we can write :

$$(r - r_-)(r - r_+) = \frac{r^2 \Omega^2}{4\iota\kappa} - 2r(\varpi - M) + Q^2 - e^2.$$

As a consequence of this equation and (2.5.4), (2.5.5), (2.5.6), (2.5.7) and (2.5.8) - all valid in \mathcal{EB} - we get that $|r - r_-| \lesssim v^{1-2s}$ on $\gamma_{2s-1} := \{u + v + h(v) = \frac{2s-1}{2|K_-|} \log(v)\}$ so , since $\nu \leq 0$, γ' lies in the past of γ_{2s-1} for $|u_s|$ large enough.

Using the same equation as above, we prove easily, still using (2.5.6) that on $\gamma_N = \{u + v + h(v) = \Delta_N\}$ and for $|u_s|$ large enough,

$$r - r_- \gtrsim 1.$$

Hence, because $\nu \leq 0$, it is clear that γ' lies in the future of γ_N , providing $2s - 1 - \eta > 0$.

We conclude by noticing that the intersection of the future of γ_N and the past of γ_{2s-1} is included in \mathcal{EB} for $|u_s|$ large enough.

The last claim (2.5.28) follows from using the above equality in the other way around : there exists $\tilde{C} > 0$ such that :

$$\Omega^2 = \tilde{C}|r - r_-| + O(v^{1-2s}),$$

where we used the remarks mentioned earlier in the proof. □

2.5.5 Towards the Cauchy horizon : the future of γ'

We now want to propagate our lower bounds to the future of γ' . To circumvent the lack of decay of Q and ϖ near the Cauchy horizon, we do not use a vector field method any more but a more classical integration along the constant v characteristic, as it was done in the stability part.

Given the bound of Proposition 2.5.1, and since $p < \min\{2s, 6s - 3\}$, it will be enough to prove the following

Proposition 2.5.4. *The following lower bound for $\partial_v \phi$ near the Cauchy horizon is true :*

$$\int_{\gamma'_v} \mathbb{T}(T, n') \text{vol}(n', \cdot) \lesssim \int_v^{+\infty} |\partial_v \phi|^2(u_0, v') dv' + O(v^{-2s}) + O(v^{3-6s+4\eta}). \quad (2.5.29)$$

Proof. The proof will be decomposed into two steps : the first one is expressed by the following lemma : we identify $\mathbb{T}(T, n')$ in terms of the scalar field using the decay of Ω^2 and the control of κ^{-1} :

Lemma 2.5.5. *The following estimate is true :*

$$\int_{\gamma'_v} \mathbb{T}(T, n') \text{vol}(n', \cdot) \lesssim \int_{\gamma'_v} |\partial_v \phi|^2(u_{\gamma'}(v'), v') dv' + O(v^{-2s}). \quad (2.5.30)$$

Proof. We now write $\gamma' = f^{-1}\{0\}$ where $f(u, v) := r(u, v) - r_- - v^{1-2s+\eta}$.

Using $g^{uv} = -2\Omega^{-2}$, we can write for $0 < \eta < 2s - 1$:

$$df^\# = \left(\frac{\iota^{-1}}{2} - 2\Omega^{-2}(2s - 1 - \eta)v^{-2s+\eta}\right)\partial_u + \frac{\kappa^{-1}}{2}\partial_v.$$

Using the definition of T , we can derive :

$$\mathbb{T}(T, df^\#) = \frac{\kappa^{-2}}{2}T_{vv} - \frac{\iota^{-2}}{2}T_{uu} - \frac{2(2s - 1 - \eta)\kappa^{-1}v^{-2s+\eta}}{\Omega^2}T_{uv} + \frac{2(2s - 1 - \eta)\iota^{-1}v^{-2s+\eta}}{\Omega^2}T_{uu}.$$

Now notice that the second and third term are negative if $2s - 1 - \eta > 0$, which can be arranged for η sufficiently small.

For the fourth, notice that on γ' :

$$\frac{\iota^{-1}v^{-2s+\eta}}{\Omega^2}T_{uu} = \frac{\iota^{-1}v^{-2s+\eta}}{\Omega^2}|D_u\phi|^2 = O(v^{-2s-1}),$$

where we used (2.5.3), (2.5.28) and the fact that ι^{-1} is bounded on γ' by (2.5.4). This gives, recalling that $T_{vv} = 2|\partial_v\phi|^2$ and that κ^{-2} is bounded on γ' by (2.5.5) :

$$\mathbb{T}(T, df^\#) \lesssim |\partial_v\phi|^2 + O(v^{-2s-1}).$$

Now, an elementary computation gives that there exists a bounded function w such that :

$$\frac{1}{\sqrt{-g(df^\#, df^\#)}} \text{vol}(n', \cdot) = w(u, v) dv d\sigma_{S^2}.$$

Noticing that $n' = \frac{df^\#}{\sqrt{-g(df^\#, df^\#)}}$, we integrate $\mathbb{T}(T, n') \text{vol}(n', \cdot)$ on γ'_v which gives the claimed lemma. \square

Now we want to propagate point-wise using (2.2.27) and then integrate :

Lemma 2.5.6. *The following estimate is true for all $u \leq u_s$:*

$$\int_{\gamma'_v} |\partial_v\phi|^2(u_{\gamma'}(v'), v') dv' \lesssim \int_v^{+\infty} |\partial_v\phi|^2(u, v') dv' + O(v^{3-6s+4\eta}) + o(v^{-2s}). \quad (2.5.31)$$

Proof. We now place ourselves in the future of γ'_v , a region that lies in $\mathcal{EB} \cup \mathcal{LB}$.

We use (2.2.27) to get -after adding and subtracting a $\Omega^2 e^2 |\phi|$ term- :

$$|\partial_u(e^{iq_0 \int_{u_0}^u A_u(u', v) du'} r \partial_v \phi)| \lesssim |\lambda| |D_u \phi| + \Omega^2((m^2 + e^2)|\phi| + |Q^2 - e^2||\phi|).$$

We now deal with each term separately. To the future of γ'_v , included in $\mathcal{EB} \cup \mathcal{LB}$ we use (2.5.14) :

$$|\lambda| \lesssim \Omega^2 + v^{-2s}.$$

We can also use in the same region the estimate (2.5.10) :

$$|D_u \phi| \lesssim |u|^{-s} + |u|^{-2s} b(u, v),$$

with $b(u, v) := |u|^{1-s} 1_{\{s>1\}} + v^{1-s} 1_{\{s<1\}} + \log(v) 1_{\{s=1\}}$.

All put together, we get :

$$|\lambda| |D_u \phi|(u, v) \lesssim \Omega^2 |u|^{-s} + \Omega^2 |u|^{-2s} b(u, v) + v^{-2s} |u|^{-s} + v^{-2s} |u|^{-2s} b(u, v).$$

We start by the third and fourth terms :

$$\int_{u_{\gamma'}(v)}^u (v^{-2s} |u'|^{-s} + v^{-2s} |u'|^{-2s} b(u', v)) du' \lesssim v^{-2s} b(u, v).$$

The first and second terms are more complicated : at fixed v we have to split between the part of $[u_{\gamma'}(v), u]$ that is in \mathcal{EB} : $[u_{\gamma'}(v), u_\gamma(v)]$ and the one that is in \mathcal{LB} : $[u_\gamma(v), u]$.

For $[u_\gamma(v), u]$, we use (2.5.16) :

$$\int_{u_\gamma(v)}^u (\Omega^2(u', v) |u'|^{-s} + \Omega^2(u', v) |u'|^{-2s} b(u', v)) du' \lesssim \int_{u_\gamma(v)}^u (v^{-2s} |u'|^{-s} + v^{-2s} |u'|^{-2s} b(u', v)) du' \lesssim v^{-2s} b(u, v).$$

On $[u_{\gamma'}(v), u_\gamma(v)]$, we use (2.5.15) the strictly negative lower bound on $\partial_u \log(\Omega^2)$ with Lemma 2.4.1 to get that :

$$\begin{aligned} & \int_{u_{\gamma'}(v)}^{u_\gamma(v)} (\Omega^2(u', v) |u'|^{-s} + \Omega^2(u', v) |u'|^{-2s} b(u', v)) du' \lesssim \\ & \Omega^2(u_{\gamma'}(v), v) |u_{\gamma'}(v)|^{-s} + \Omega^2(u_{\gamma'}(v), v) |u_{\gamma'}(v)|^{-2s} b(u_{\gamma'}(v), v) \lesssim v^{1-3s+\eta}, \end{aligned}$$

where we used in the last inequality that $v^{1-4s+\eta} b(u_{\gamma'}(v), v) = o(v^{1-3s+\eta})$.

To estimate $\Omega^2((m^2+e^2)|\phi|+|\phi||Q^2-e^2|)$, we use a similar technique, splitting $[u_{\gamma'(v)}, u]$ into $[u_{\gamma'(v)}, u_{\gamma v}(v)] \cup [u_{\gamma v}(v), u]$ where γv is defined in section 2.4.7 as the past boundary of \mathcal{V} .

Using estimates¹⁰¹ (2.5.12), (2.5.13) in \mathcal{EB} together with calculus Lemma 2.4.1 and (2.5.15), we prove that -choosing $\epsilon_0 = \frac{\eta}{2s-1-\eta}$:

$$\int_{u_{\gamma'(v)}}^{u_{\gamma v}(v)} \Omega^2(u', v)((m^2+e^2)|\phi|(u', v) + |\phi|(u', v)|Q^2 - e^2|(u', v))du' \lesssim v^{1-3s+2\eta}.$$

Using (2.5.18) in \mathcal{V} and $|\phi| + |Q - e| \lesssim b(u, v)$ gives a negligible contribution on $[u_{\gamma v}(v), u]$, because Ω^2 is exponentially decreasing, which proves :

$$\int_{u_{\gamma'(v)}}^u \Omega^2(u', v)(|\phi|(u', v) + |\phi|(u', v)|Q^2 - e^2|(u', v))du' \lesssim v^{1-3s+2\eta}.$$

Now we can use that $v^{-2s}b(u, v) = v^{-2s}|u|^{1-s}1_{\{s>1\}} + o(v^{1-3s+2\eta})$ if η is small enough, combine all the estimates and integrate the first equation :

$$|e^{iq_0 \int_{u_0}^u A_u(u', v)du'} r \partial_v \phi(u, v) - e^{iq_0 \int_{u_0}^{u_{\gamma'(v)}} A_u(u', v)du'} r \partial_v \phi(u_{\gamma'(v)}, v)| \lesssim v^{1-3s+2\eta} + v^{-2s}|u|^{1-s}1_{\{s>1\}}.$$

Making the difference, using upper and lower bounds for r and squaring, we get :

$$|\partial_v \phi(u_{\gamma'(v)}, v)|^2 \lesssim |\partial_v \phi(u, v)|^2 + v^{2-6s+4\eta} + v^{-4s}|u|^{2-2s}1_{\{s>1\}}.$$

To conclude, it is enough to integrate the last estimate on $[v, +\infty]$ and noticing that $v^{1-4s}|u|^{2-2s}1_{\{s>1\}} = o(v^{-2s})$. □

The combination of the two lemmas proves the proposition after choosing η small enough so that $p < 6s - 3 - 4\eta$. □

2.6 A criterion for continuous extendibility in the massive or strongly charged case

In this section, we provide the proof of Theorem 2.3.5. The key estimate, proved in the Proposition 2.6.4, proceeds factoring out a term which is roughly proportional to $|\phi|^2$, and can either blow-up or be bounded according to the cases.

Before this, we must prove that essentially, the blow-up of $|\log(\Omega^2)|$ (in a regular coordinate system) and of $|\phi|^2$ are the strongest potential instabilities: the other quantities are well-behaved. This is the object of section 2.6.1, in which we derive preliminary estimates.

Thanks to the refined analysis of section 2.6.2, we will later able to show estimates continuity estimates for $|\log(\Omega^2)|$ if the scalar field is bounded, or a blow-up of $|\log(\Omega^2)|$ if the scalar field is not bounded.

2.6.1 Preliminary estimates for $D_u \psi$

To reach the goals of this section, we must first prove preliminary estimates on $D_u \psi$, where $\psi := r\phi$ is (what is called in the black hole exterior) the radiation field. Since r is upper and lower bounded in our region of interest, it may be very surprising to consider this quantity in the black hole *interior*. However, as it turns out, $D_u \psi$ is always bounded, while $D_u \phi$ is bounded if and only if ϕ is (providing $\liminf_{v \rightarrow +\infty} |\nu|(u, v) > 0$, which is putatively a generic condition, related to the blow-up of the Hawking mass).

Proposition 2.6.1. *We have the following estimate in \mathcal{LB} :*

$$|D_u \psi| \lesssim |u|^{2-3s} + |u|^{-s}. \quad (2.6.1)$$

Moreover, if $s > \frac{2}{3}$ then $D_u \psi$ admits a continuous and bounded extension to the Cauchy horizon, denoted $(D_u \psi)_{CH}$ where we recall that $\psi := r\phi$.

¹⁰¹These bounds are not strictly speaking stated in \mathcal{EB} in the stability part but they are an easy consequence of the estimates.

Proof. Using (2.2.19) it is not hard to see that

$$\partial_v(D_u\psi) = \partial_v\partial_ur \cdot \phi - \left(m^2\frac{\Omega^2}{4} + iq_0Q\frac{\Omega^2}{4r^2}\right) \cdot r\phi.$$

Using (2.2.15) and (2.4.93) and the boundedness of ν we get

$$|\partial_v(D_u\psi)| \lesssim |\lambda||\phi|.$$

Finally with (2.4.84) and (2.4.80) we get

$$|\partial_v(D_u\psi)| \lesssim v^{1-3s}.$$

Now the left hand side is integrable in v so $D_u\psi$ admits a continuous extension and the estimate is true. \square

Remark 23. Notice that, at this stage, we only obtain the continuous extendibility of $D_u\psi$ but we ignore whether ψ extends or not to the Cauchy horizon (there exist cases in which it does not). This is why we use the notation $(D_u\psi)_{CH}$ for the extension (not to be confused with $D_u(\psi_{CH})$, which does not always exist).

We also derive an additional estimate, which is useful in some degenerate cases ($\nu_{CH} \equiv 0$) that we are unable to exclude (in particular, because those cases occur if $\phi \equiv 0$, i.e. on the exact Reissner–Nordström space-time).

Lemma 2.6.2. *If $s > \frac{2}{3}$, there exists $0 < \alpha < \frac{1}{2}$ small enough so that we have the estimate*

$$|D_u\psi|(u, v) \lesssim |\nu|^\alpha |u|^{-s \cdot (1-2\alpha)} \quad (2.6.2)$$

Proof. Using (2.2.27) and (2.2.15) we prove that

$$\partial_v \left(\frac{D_u\psi}{(-r\nu)^\alpha} \right) = \frac{\phi}{|r\nu|^\alpha} \left(\partial_u\partial_v r - \frac{m^2 r \Omega^2}{4} - \frac{i q_0 Q \Omega^2}{4r^2} \right) + \alpha \kappa \frac{D_u\psi}{(-r\nu)^\alpha} \left(\frac{-1}{r} + \frac{Q^2}{r^3} - \frac{m^2 r \Omega^2}{4} |\phi|^2 \right)$$

Now the idea is to bootstrap $\frac{|D_u\psi|}{(-r\nu)^\alpha} \leq D|u|^{-s(1-2\alpha)}$ to the future of the curve γ_N and to retrieve the bootstrap by taking α small enough. Note that there exists a D , depending on ϵ for which this estimate is true on γ_N , using the estimates of section 2.4.5, and in particular $\Omega^2 \sim |\nu| \gtrsim 1$.

We estimate the integral of the first term, using the estimates of section 2.4.5 and section 2.4.7 and the "gauge estimate" $\kappa \leq 1$:

$$\int_{v_{\gamma_N}(u)}^v \left| \frac{\phi}{|r\nu|^\alpha} \left(\partial_u\partial_v r - \frac{m^2 r \Omega^2}{4} - \frac{i q_0 Q \Omega^2}{4r^2} \right) \right| \lesssim \int_{v_{\gamma_N}(u)}^v |\nu|^{1-\alpha} |\lambda| \cdot |\phi| + \kappa^\alpha \cdot \Omega^{2(1-\alpha-\eta)} v'^{-s} dv' \lesssim |u|^{2-(5-2\alpha)s}$$

and clearly, $5s - 1 > s$ so $|u|^{2-s \cdot (5-2\alpha)} = o(|u|^{-s \cdot (1-2\alpha)})$.

Now, the second estimate in \mathcal{LB} that we need is:

$$\int_{v_{\gamma_N}(u)}^v \frac{\Omega^2}{-4\nu} (1 + Q^2 + |\phi|^2) dv' \lesssim 1$$

which can be proven using that $\partial_v \log(\Omega^2) \leq K_-$ with $K_- < 0$ and integration by parts. Details are left to the reader.

Now, using a standard Gronwall type argument, we can close the bootstrap estimate and obtain the estimate required for the lemma. \square

Proposition 2.6.3. *If there exists $u_1 < u_2 \leq u_s$ such that $r_{CH}(u) = r_0 > 0$ for all $u_1 \leq u \leq u_2$, then for all $u \leq u_s$:*

$$\limsup_{V \rightarrow 1} |\phi|(u, V) < +\infty.$$

Proof. Using (2.2.15), we can integrate $\partial_v(r\nu)(u, v)$ at fixed $u_1 \leq u \leq u_2$ on $[v, +\infty]$ and using the estimates of section 2.4.7, especially $\partial_v \log(\Omega^2) < 0$ and bounded away from 0, we get, for all $0 < \eta < 1$:

$$|\nu|(u, v) \lesssim \Omega^2(u, v) \cdot (1 + Q^2(u, v) + |\phi|^2(u, v)) \lesssim \Omega^{2-\eta}(u, v).$$

Now, notice that by the Raychaudhuri equation (2.2.29), $\partial_u \log(\kappa^{-1}) \geq 0$, hence, by monotonicity and the bound we proved for $u = u_1$, for all $u \leq u_1$:

$$|\nu|(u, v) \lesssim \Omega^2(u, v) \cdot (1 + Q^2(u_1, v) + |\phi|^2(u_1, v)),$$

thus, $\nu_{CH}(u) \equiv 0$ for all $u \leq u_2$, hence $r_{CH}(u) \equiv r_-$ for all $u \leq u_2$.

Additionally, with the former Lemma, there exists $0 < \alpha' < \frac{1}{2}$ such that for all $u_1 \leq u \leq u_2$:

$$|D_u \psi|(u, v) \lesssim \Omega^{2\alpha'}(u, v) \cdot |u|^{-s \cdot (1-2\alpha')}.$$

Thus, integrating on $[u_\gamma(v), u]$ for fixed v , we clearly establish that for all $u \leq u_2$:

$$|\phi|(u, v) \lesssim v^{-s} |\log(v)|.$$

This also implies (using (2.2.21)) that Q possesses a continuous extension Q_{CH} at least for $u \leq u_2$, and a similar conclusion for ϖ and for all $u \leq u_2$:

$$Q_{CH}(u) = e, \quad |\phi|_{CH}(u) = 0, \quad \varpi_{CH}(u) = M, \quad r_{CH} = r_-(e, M).$$

Now, on $[u_2, u_s]$, $|D_u \psi| \lesssim 1$ thanks to the estimates of this section. Thus, we can integrate and easily conclude that $|\phi|(u, V)$ is bounded uniformly in $u \in (-\infty, u_s]$ as $V \rightarrow 1$. \square

2.6.2 A potential coordinate system (u, V) for a continuous extension

In this section, we construct a "good" coordinate system (u, V) , in which the boundedness of the metric coefficient Ω_{CH}^2 related to (u, V) is essentially equivalent to the boundedness of the scalar field ϕ .

Proposition 2.6.4. *Under the hypothesis of Theorem 2.3.2, there exists a coordinate system (u, V) for which $V(v) < 1$, and $\lim_{v \rightarrow +\infty} V(v) = 1$ and for which, defining the metric coefficient $\Omega_{CH}^2 du dv = \Omega^2 du dv$, we have*

$$|\partial_v \left(\log(\Omega_{CH}^2)(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2(u', v) du' \right)| \lesssim v^{2-4s} + v^{-2s} |\log(v)|^3, \quad (2.6.3)$$

$$|\partial_v \partial_u \left(\log(\Omega_{CH}^2)(u, v) + |\phi|^2(u, v) + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2(u', v) du' \right)| \lesssim |u|^{-2s} \cdot (v^{2-4s} + v^{-2s} |\log(v)|^3) + (|u|^{-s} + |u|^{2-3s}) \cdot v^{1-3s}, \quad (2.6.4)$$

in \mathcal{LB} . Since both right-hand-side are integrable for $s > \frac{3}{4}$, under this condition, both $\log(\Omega_{CH}^2) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2 du'$ and $\partial_u \left(\log(\Omega_{CH}^2) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2 du' \right)$ are bounded as $v \rightarrow +\infty$ and in fact continuously extendible¹⁰² across the Cauchy horizon.

Proof. We first use (2.2.27) to establish the following formulae :

$$\begin{aligned} -2\Re(D_u \phi \overline{\partial_v \phi}) &= \frac{-\partial_u \partial_v (r|\phi|^2)}{r} + \left(\frac{\partial_u \partial_v r}{r} - \frac{m^2 \Omega^2}{2} \right) |\phi|^2 \\ \frac{\partial_u \partial_v (r|\phi|^2)}{r} &= \partial_u \partial_v (|\phi|^2) + \frac{\nu}{r} \partial_v (|\phi|^2) + \frac{1}{r} \partial_u (\lambda |\phi|^2) \end{aligned}$$

Now we define $2K_\gamma(v) := 2K(u_{\gamma(v)}, v)$ and we write (2.2.16) as, using the two last formulae

$$|\partial_u (\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v (|\phi|^2)) + \frac{\nu}{r} \partial_v (|\phi|^2) + \frac{1}{r} \partial_u (\lambda |\phi|^2)| \lesssim |\lambda \nu| (1 + |\phi|^2) + \Omega^2 (1 + Q^2 + m^2 |\phi|^2)$$

First note that the right hand side is $O(|u|^{-2s} \cdot v^{-2s} + |u|^{-2s} \cdot v^{2-4s})$, using the estimates of Proposition 2.4.11. Using (2.2.15), (2.6.1) and the other estimates of section 2.4.7 we get

$$|\partial_u (\lambda |\phi|^2)| = |\partial_u (r^{-2} \lambda |\psi|^2)| \lesssim |u|^{-2s} v^{2-4s} + (|u|^{-s} + |u|^{2-3s}) \cdot v^{1-3s}.$$

This gives:

$$|\partial_u (\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v (|\phi|^2)) + \frac{\nu}{r} \partial_v (|\phi|^2)| \lesssim |u|^{-2s} \cdot v^{-2s} + |u|^{-2s} v^{2-4s} + (|u|^{-s} + |u|^{2-3s}) \cdot v^{1-3s}. \quad (2.6.5)$$

Now we want to integrate both sides on $[u_{\gamma(v)}, u]$. Now recall that on γ , $|\partial_v \log(\Omega^2)(u_{\gamma(v)}, v) - 2K_\gamma(v)| \lesssim v^{-2s} |\log(v)|^3$ and $|\partial_v (\phi^2)| \lesssim v^{-2s} |\log(v)|$, as established in Proposition 2.4.10.

Thus, we obtain

$$|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \partial_v (|\phi|^2) + \int_{u_{\gamma(v)}}^u \frac{\nu}{r} \partial_v (|\phi|^2) du'| \lesssim v^{2-4s} + v^{-2s} |\log(v)|^3. \quad (2.6.6)$$

¹⁰²(2.6.4) even proves that the extension of $\log(\Omega_{CH}^2) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2 du'$ is integrable as $u \rightarrow -\infty$ on the Cauchy horizon.

Now we write

$$\int_{u_{\gamma(v)}}^u \frac{\nu}{r} \partial_v(|\phi|^2) du' = \int_{u_{\gamma(v)}}^{u_s} \frac{\nu}{r} \partial_v(|\phi|^2) du' - \partial_v \left(\int_u^{u_s} \frac{\nu}{r} |\phi|^2 du' \right) + \int_u^{u_s} \partial_v \left(\frac{\nu}{r} \right) |\phi|^2 du'.$$

Now using (2.2.15) we can see, using the estimates of Proposition 2.4.10 again, that

$$\left| \int_u^{u_s} \partial_v \left(\frac{\nu}{r} \right) |\phi|^2 du' \right| \lesssim \int_u^{u_s} (|\nu||\lambda| + \Omega^2(1 + Q^2 + |\phi|^2)) |\phi|^2 du' \lesssim v^{2-4s}.$$

Therefore we actually showed that

$$|\partial_v \log(\Omega^2) - 2K_\gamma(v) + \int_{u_{\gamma(v)}}^{u_s} \frac{\nu}{r} \partial_v(|\phi|^2) du' + \partial_v(|\phi|^2) - \partial_v \left(\int_u^{u_s} \frac{\nu}{r} |\phi|^2 du' \right)| \lesssim v^{2-4s} + v^{-2s} |\log(v)|^3. \quad (2.6.7)$$

Note that the second and the third term of the left-hand-side only depend on v and not on u .

We define a new coordinate system (u, V) with the following equations :

$$\frac{dV}{dv} = e^{f(v)}, \quad (2.6.8)$$

$$f'(v) = 2K_\gamma(v) + \int_{u_{\gamma(v)}}^{u_s} \frac{|\nu|}{r} \partial_v(|\phi|^2)(u', v) du'. \quad (2.6.9)$$

By the estimates of Proposition 2.4.10, note that $|f'(v) - 2K_-| \lesssim v^{1-2s}$ and we recall that $K_- < 0$; thus it is clear that $V'(v)$ is integrable as $v \rightarrow +\infty$, and $V(v)$ increases towards a limit V_∞ that we can choose to be 1 without loss of generality. Therefore, we also have, as $v \rightarrow +\infty$:

$$1 - V(v) \approx e^{f(v)} \approx e^{2K_- v}.$$

We also denote Ω_{CH}^2 the metric coefficient in this system defined by

$$\Omega_{CH}^2 dudV = \Omega^2 dudv.$$

We then have the claimed estimate (2.6.3) :

$$|\partial_v \left(\log(\Omega_{CH}^2) + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2 du' \right)| \lesssim v^{2-4s} + v^{-2s} |\log(v)|^3.$$

Clearly, (2.6.5) is a reformulation of (2.6.4).

Since the right hand sides of (2.6.3) and (2.6.4) are integrable for $s > \frac{3}{4}$, a limit exist and we can extend by continuity: thus the proposition is proved. \square

Now, we have obtained a coordinate system (u, V) in which the metric is "almost continuously extendible", up to the factor involving $|\phi|^2$, which is a geometric one, i.e. that does not depend on the coordinate system. Now, we will experience handle two cases¹⁰³ either the field is continuously extendible or it blows up uniformly on the Cauchy horizon. These cases are treated separately in the last two sub-sections.

2.6.3 The continuous extendibility case

Now, we work under the hypothesis of the "sufficient condition" part of Theorem 2.3.5: we assume that for some $u_1 \leq u_s$ and $l \geq 0$ such that

$$\lim_{v \rightarrow +\infty} |\phi|(u_1, v) = l.$$

We will denote $l := |\phi|_{CH}(u_1)$. First, using the results of last section, we are going to show that this condition is sufficient to obtain a bounded scalar field ϕ whose module extends continuously across the Cauchy horizon :

Lemma 2.6.5. *Under the hypothesis of Theorem 2.3.2, we also assume that $s > \frac{2}{3}$ and the "sufficient condition": for some $u_1 \leq u_s$ and $l \geq 0$ such that*

$$\lim_{v \rightarrow +\infty} |\phi|(u_1, v) = l.$$

Then ϕ is bounded on the whole space-time $[-\infty, u_s] \times [v_0, +\infty]$ in (u, v) coordinates and moreover, $|\phi|$ admits a continuous extension across the Cauchy horizon, denoted by $|\phi|_{CH}$.

¹⁰³That do not cover all possibilities: in principle it is still possible to have a bounded field $\phi(u, v)$ as $v \rightarrow +\infty$ with multiple limit values. We will not consider this case in the present manuscript.

Proof. First, we want to prove that, in our gauge condition $A_v = 0$, $\int_u^{u_s} A_u(u', v) du'$ is bounded and extends continuously towards the Cauchy horizon. We recall the estimate (2.4.41). Using the estimates of section 2.4.6, it is not hard to see that on γ , the future boundary of \mathcal{EB} :

$$|A_u|(u, v_\gamma(u)) \lesssim 1.$$

Then using (2.2.22) and the estimates of section 2.4.7, it is not hard to see that

$$|\partial_v A_u|(u, v) \lesssim v^{-2s},$$

which is integrable. Hence A_u is bounded on the whole of $[-\infty, u_s] \times [v_0, +\infty]$ in (u, v) coordinates and moreover there exists $A_{u, CH}(u)$ (that need not be continuous) such that for all $u \leq u_s$:

$$\lim_{v \rightarrow +\infty} A_u(u, v) = A_{u, CH}(u).$$

Now, denote $B(u, v) := \int_u^{u_s} A_u(u', v) du'$. Recall the (u, V) coordinate system, defined by (2.6.8), (2.6.9), in which $V(v = \infty) = 1$ and $1 - V(v) \approx e^{2K-v}$. Now take $u_\infty < u_s$ and two sequences $u_i \rightarrow u_\infty$, $V_i \rightarrow 1$, $V_i < 1$. With the estimates that precede, it is clear that

$$|\partial_u B(u, V)| \lesssim 1,$$

$$|\partial_V B(u, V)| \lesssim \log((1 - V)^{-1})^{-2s},$$

which are both integrable (in their respective directions). Hence $B(u_i, V_i)$ is Cauchy, and in fact converges towards $\int_u^{u_s} A_{CH}(u') du'$. Therefore, extending B to $\{V = 1\}$ by this value (denoted $B_{CH}(u)$), we produced a continuous extension, at least on $[-\infty, u_s] \times [v_0, +\infty]$.

Now notice that $e^{iq_0 \cdot B(u, v)} \partial_u (e^{-iq_0 \cdot B(u, v)} \psi(u, v)) = D_u \psi(u, v)$, thus for all $u \leq u_s$:

$$\psi(u, v) = e^{iq_0 \cdot (B(u, v) - B(u_1, v))} \psi(u_1, v) + \int_{u_1}^u e^{-iq_0 \cdot B(u', v)} D_u \psi(u', v) du'.$$

We write the complex number $\phi(u_1, v) = e^{i\theta_{u_1}(v)} \cdot |\phi|(u_1, v)$ where $\theta_{u_1}(v)$ is a real valued function (not necessarily continuous or bounded). Evidently, the hypothesis means that $\lim_{v \rightarrow +\infty} e^{-i\theta_{u_1}(v)} \phi(u_1, v) = l$. Therefore, using (easy) continuity theorems under the integral, we get (since r extends as a continuous function r_{CH} as it has been first noticed by Kommemi [47]):

$$\lim_{v \rightarrow +\infty} e^{-i\theta_{u_1}(v)} \psi(u, v) = e^{iq_0 \cdot (B_{CH}(u) - B_{CH}(u_1))} \cdot r_{CH}(u_1) \cdot l + \int_{u_1}^u e^{-iq_0 \cdot B_{CH}(u')} (D_u \psi)_{CH}(u') du'.$$

Hence, if one denotes the absolute value of the right-hand-side by $|\psi|_{CH}(u)$, we proved that

$$\lim_{v \rightarrow +\infty} |\psi|(u, v) = |\psi|_{CH}(u).$$

Moreover, by what was done earlier, it is also clear that for any $u_\infty < u_s$, and sequences $u_k \rightarrow u_\infty$, $V_k \rightarrow 1$:

$$\lim_{k \rightarrow +\infty} e^{iq_0 \cdot (B(u_k, V_k) - B(u_1, V_k))} \cdot r(u_k, V_k) \cdot |\psi|(u_1, V_k) + \int_{u_1}^{u_k} e^{-iq_0 \cdot B(u', V_k)} D_u \psi(u', V_k) du' = |\psi|_{CH}(u_\infty).$$

Therefore, extending $|\psi|(u, V)$ by $|\psi|_{CH}(u)$ as $V \rightarrow 1$, it is clear that have created a continuous extension for ϕ . Since r^{-1} also extends continuously (the key point is that r is lower bounded), then we can also extend continuously $|\phi|$ by $r_{CH}^{-1}(u) \cdot |\psi|_{CH}(u)$, which concludes the proof. \square

To complete the proof of the "sufficient condition" part of Theorem 2.3.5, we also need to extend continuously $\int_u^{u_s} \frac{\nu}{r} |\phi|^2(u', V) du'$ towards $\{V = 1\}$. This step is, at this stage, much easier and is carried out in the next lemma:

Lemma 2.6.6. $\int_u^{u_s} \frac{\nu}{r} |\phi|^2(u', V) du'$ extends as a continuous function across the boundary $\{V = 1\}$. In fact, for all $u \leq u_s$, there exists $\nu_{CH}(u) := \lim_{v \rightarrow +\infty} \nu(u, v)$ and $\int_u^{u_s} \frac{\nu}{r} |\phi|^2(u', V) du'$ extends continuously as $\int_u^{u_s} \frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u') du'$.

Remark 24. In fact, we do not prove directly that ν extends as continuous function across the Cauchy horizon, as we miss a control of $\partial_u \nu$. However, it is clear that, even though ν_{CH} might not be continuous in u , it is clearly in L_{loc}^1 (and even in $L^1(CH^+(u_s))$, as $|\nu_{CH}| \lesssim |u|^{-2s}$) which is sufficient for our purpose.

Proof. Using the estimates of Proposition 2.4.11, it is easy to see that

$$|\partial_v \nu| \lesssim v^{-2s},$$

which shows, by integrability, that for all $u \leq u_s$ there exists $\nu_{CH}(u)$ such that $\lim_{v \rightarrow +\infty} \nu(u, v) = \nu_{CH}(u)$, and, as we knew already, $|\nu|$ is uniformly bounded. Now take again $u_\infty < u_s$ and two sequences $u_i \rightarrow u_\infty$, $V_i \rightarrow 1$, $V_i < 1$ and write

$$\begin{aligned} \left| \int_{u_i}^{u_s} \frac{\nu}{r} |\phi|^2(u', V_i) du' - \int_{u_i}^{u_s} \frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u') du' \right| &\leq \left| \int_{u_i}^{u_\infty} \frac{\nu}{r} |\phi|^2(u', V_i) du' \right| + \left| \int_{u_i}^{u_\infty} \frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u') du' \right| \\ &\quad + \left| \int_{u_\infty}^{u_s} \left(\frac{\nu}{r} |\phi|^2(u', V_i) - \frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u') \right) du' \right|. \end{aligned}$$

Now, clearly, both functions $\frac{\nu}{r} |\phi|^2(u, V)$ and $\frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u')$ are uniformly bounded in u and v on a set of form $(u, V) \in [u_\infty - \epsilon, u_s] \times [1 - \epsilon, 1]$ and $\lim_{i \rightarrow +\infty} \frac{\nu}{r} |\phi|^2(u', V_i) = \frac{\nu_{CH}(u')}{r_{CH}(u')} |\phi|_{CH}^2(u')$ so by the dominated convergence theorem, the last term tends to 0 as i tends to $+\infty$.

Moreover, the integrands of the first two terms being uniformly bounded, it is also very easy to see that these two terms tend to 0. This concludes the proof of the lemma. \square

Now, combining the results of this section to those of section 2.6.2, it is clear that in the (u, V) coordinate system, Ω_{CH}^2 extends continuously to $\{V = 1\}$. As we know already that r extends continuously to $\{V = 1\}$, this fact achieves the proof of the "sufficient condition" of Theorem 2.3.5.

2.6.4 The continuous inextendibility case

Now, we treat the case where the scalar field blows up as $V \rightarrow 1$. First, we prove that if the scalar field blows up towards one point on the Cauchy horizon, then it blows up towards every point of $CH^+(u_s)$:

Lemma 2.6.7. *We work in the hypothesis of the "necessary condition" of Theorem 2.3.5, namely that there exists $u_1 \leq u_s$ such that*

$$\limsup_{v \rightarrow +\infty} |\phi|(u_1, v) = +\infty.$$

Then, for every $u \leq u_s$,

$$\limsup_{v \rightarrow +\infty} |\phi|(u, v) = +\infty.$$

Proof. This is an easy consequence of the fact that $D_u \psi$ extends as a continuous, hence $L_{loc,u}^1$ function, as seen in section 2.6.1. As this is even easier than the bounded case of the former section, we do not repeat the proof and leave the details to the reader. \square

Recall that we attached a null boundary $CH^+(u_s)$ to M , making M a manifold-with-boundaries. In the rest of the section, we are going to denote $int(M) := M - \partial M = M - CH^+(u_s)$.

Now, to prove continuous inextendibility, we argue by contradiction: we make the following contradiction hypothesis, under which we are going to work for the rest of this section :

Assumption 10. Assume that (M, g) is continuously extendible across $CH^+(u_s)$, in the sense of Definition 3. Therefore, there exists C^0 Lorentzian manifold (\tilde{M}, \tilde{g}) , a differential isometric embedding $i : M \rightarrow \tilde{M}$ and two continuous curve $\sigma_a : [0, 1] \rightarrow \tilde{M}$ with $\sigma_a(0) \in i(int(M))$, $\sigma_a(1) \in \tilde{M} - i(int(M))$ and $\sigma_a(s_a) \in i(CH^+(u_s))$, for some $0 < s_a < 1$, $\sigma_1(s_1) \neq \sigma_2(s_2)$.

Under the contradiction hypothesis (assumption 10), we are going to prove the following Lemma, which is purely a matter of Lorentzian geometry :

Lemma 2.6.8. *There exists a double-null spherically symmetric coordinate system (u', v') on M for which, denoting $(\Omega')^2 du' dv' = \Omega^2 du dv = \Omega_{CH}^2 du dV$, we can show that there exists two sequences of points $(u_{a,i}, V_i)$, $a = 1, 2$, $u_{a,i} \rightarrow u_{a,\infty}$, $u_{1,\infty} \neq u_{2,\infty}$ with $V_i \rightarrow 1$, $V_i < 1$ as $i \rightarrow +\infty$ such that $u_{a,\infty} < u_s$ and for all $u \leq u_s$:*

$$\lim_{i \rightarrow +\infty} |\phi|(u, V_i) = +\infty. \quad (2.6.10)$$

$$\sup_{i \rightarrow +\infty} |\log(\Omega')(u_{a,i}, V_i)| < +\infty. \quad (2.6.11)$$

Proof. Since $\text{int}(M)$ is an open set in the topology of M , one can assume without loss of generality that for all $s < s_a$, $\sigma_a(s) \in i(\text{int}(M))$.

Without loss of generality (restricting the range of σ_a if need be), one can assume that the range of σ_a lies in an open set W of \tilde{M} on which we have a double null coordinate system $(\tilde{u}', \tilde{v}', \tilde{\theta}'_1, \tilde{\theta}'_2)$, in the sense of Definition 2. We define the pull-back of these four functions $(u', v', \theta'_1, \theta'_2)$. This quadruplet defines also (locally) a coordinate system on $M \cap i^{-1}(W)$.

It is also easy to see that $(u', v', \theta'_1, \theta'_2)$ is a double-null coordinate system in the sense of Definition 2: indeed, as i is an isometry, du' and dv' satisfy the eikonal equation $g^{-1}(du', du') = 0$ and $g^{-1}(dv', dv') = 0$, because that are the pull-backs by i of $d\tilde{u}'$ and $d\tilde{v}'$ (and g coincides with the pull-back of \tilde{g}).

Thus, for some scalar functions $\tilde{\Omega}'$, \tilde{b}^A and $\gamma_{\tilde{A}\tilde{B}}$, with their pull-backs by i : Ω' , b^A and γ_{AB} respectively, we have, as g is the pull-back of \tilde{g} :

$$\tilde{g} = -\frac{(\tilde{\Omega}')^2}{2} \cdot (d\tilde{u}' \otimes d\tilde{v}' + d\tilde{v}' \otimes d\tilde{u}') + \tilde{\gamma}_{\tilde{A}\tilde{B}} \cdot (d\tilde{\theta}^{\tilde{A}} - \tilde{b}^{\tilde{A}} d\tilde{u}) \otimes (d\tilde{\theta}^{\tilde{B}} - \tilde{b}^{\tilde{B}} d\tilde{v}),$$

$$g = -\frac{(\Omega')^2}{2} \cdot (du' \otimes dv' + dv' \otimes du') + \gamma_{AB} \cdot (d\theta^A - b^A du') \otimes (d\theta^B - b^B dv').$$

Now, clearly, since g is spherically symmetric in $\text{int}(M)$, then $\gamma_{AB} \cdot (d\theta^A - b^A du') \otimes (d\theta^B - b^B dv') = r^2 d\sigma_{S^2}$ and moreover, $(\Omega')^2 du' dv' = \Omega_{CH}^2 du dV$.

Now, $\tilde{\Omega}$ is a continuous function on W , by continuity of the metric. Thus, we have the following estimate

$$\sup_{0 \leq s < s_a} |\log(\tilde{\Omega}')(\sigma_a(s))| \leq \sup_{0 \leq s \leq 1} |\log(\tilde{\Omega}')(\sigma_a(s))| < +\infty. \quad (2.6.12)$$

Now, since $\sigma_a(s) \in \text{int}(M)$ for $s < s_a$, we can find two sequences of points $(u_{a,i}, V_i) \in \sigma_a([0, s_a])$ with $u_{a,i} \rightarrow u_{a,\infty}$ and $V_i \rightarrow 1$ as $i \rightarrow +\infty$ such that $u_{a,\infty} := u(\sigma_a(s_a)) < u_s$ and (2.6.10) is satisfied, as $\limsup_{V \rightarrow 1} |\phi|(u, V) = \infty$ for¹⁰⁴ all u . Since Ω' coincides with the pull-back of $\tilde{\Omega}'$, we also get downstairs

$$\sup_{i \rightarrow +\infty} |\log(\Omega')|(u_{a,i}, V_i) < +\infty,$$

which is (2.6.11). □

We are now going to derive a contradiction to (2.6.11). Denote $\Delta^2 := \Omega_{CH}^2 + |\phi|^2 + \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2 du'$. As we have seen, $\log(\Delta^2)$ is bounded and in fact continuously extendible as $V \rightarrow 1$. We are going to exploit this fact in the following lemma :

Lemma 2.6.9. *Since $(\Omega')^2 du' dv' = \Omega_{CH}^2 du dV$, there exists functions $\chi = \chi(u)$ and $\underline{\chi} = \underline{\chi}(v)$ such that $(\Omega')^2(u, v) = e^{\chi(u)} \cdot e^{\underline{\chi}(v)} \cdot \Omega_{CH}^2(u, v)$. Then, we have the following boundedness result:*

$$\sup_{i \in \mathbb{N}} |\underline{\chi}(V_i) - |\phi|^2(u_{a,i}, V_i) - \int_{u_{a,i}}^{u_s} \frac{|\nu|}{r} |\phi|^2(u', V_i) du'| < +\infty, \quad (2.6.13)$$

for the sequences $(u_{a,i}, V_i)$, $a = 1, 2$ of the former lemma, satisfying 2.6.10, where we recall $u_{1,\infty} < u_{2,\infty}$.

Proof. First, we write the simple identity

$$\log(\Omega')(u, v) = \frac{1}{2} \cdot \chi(u) + \frac{1}{2} \cdot \underline{\chi}(v) + \log(\Delta)(u, v) - \frac{1}{2} \cdot |\phi|^2(u, v) - \frac{1}{2} \cdot \int_u^{u_s} \frac{|\nu|}{r} |\phi|^2(u', v) du'. \quad (2.6.14)$$

Now take the ∂_u derivative of (2.6.14) and fix $v_0 < v_1 < \infty$:

$$\frac{1}{2} \cdot \chi'(u) = \partial_u \log(\Omega')(u, v_1) + \partial_u \log(\Delta)(u, v_1) - \frac{1}{2} \cdot \partial_u (|\phi|^2)(u, v_1) + \frac{1}{2} \cdot \frac{|\nu|}{r} |\phi|^2(u, v_1).$$

As Ω' can be chosen C^1 away from the Cauchy horizon, it is clear that the right-hand-side is a continuous function of u . Therefore, $\chi(u)$ is a C^1 function of u , with bounded derivative. Therefore, coming back to (2.6.14), since $\log(\Delta)$ is uniformly bounded as $V \rightarrow 1$, we obtain immediately the result, combining with the former lemma. □

Now, we are going to prove that the bound (2.6.13) can only occur in a very degenerate situation :

¹⁰⁴To do this, we can use the same ideas as those leading to the proof of Lemma 2.6.7, since $D_u \psi$ is locally integrable. The key point is that we can choose a sequence $V_i \rightarrow 1$ such that for all $u \leq u_s$, $|\phi|(u, V_i) \rightarrow +\infty$.

Lemma 2.6.10. *Under assumption 10, there exists $r_0 > 0$ such that $r_{CH}(u) = r_0$ for all $u_{1,\infty} \leq u \leq u_{2,\infty}$.*

Proof. Recall that $B(u, V) := \int_u^{u_s} A_u(u', V) du'$ and $e^{iq_0 B(u, V)} \partial_u (e^{-iq_0 B(u, v)} \psi) = D_u \psi$. Then for all $u_1, u_2 \leq u_s$, we write

$$|\phi|^2(u_2, V) = r^{-2}(u_2, V) |\psi(u_2, V)|^2 = r^{-2}(u_2, V) |\psi(u_1, V) + e^{iq_0(B(u_1, v) - B(u_2, v))} \int_{u_1}^{u_2} e^{-iq_0 B(u', V)} D_u \psi(u', V) du'|^2.$$

Now since $e^{iq_0(B(u_1, v) - B(u_2, v))} \int_{u_1}^{u_2} e^{-iq_0 B(u', V)} D_u \psi(u', V) du'$ is bounded as $V \rightarrow 1$ and $|\psi|(u_1, V_i) \rightarrow +\infty$ as $i \rightarrow +\infty$ for all $u_1 \leq u_s$, it is pretty clear that for $u_1, u_2 \leq u_s$:

$$\begin{aligned} \lim_{i \rightarrow +\infty} \frac{|\phi|^2(u_{a,i}, V_i)}{|\psi|^2(u_1, V_i)} &= r_{CH}^{-2}(u_{a,\infty}), \\ \lim_{i \rightarrow +\infty} \frac{\int_{u_{a,i}}^{u_s} \frac{|\nu|}{r} |\phi|^2(u', V_i) du'}{|\psi|^2(u_1, V_i)} &= \int_{u_{a,\infty}}^{u_s} \frac{|\nu|_{CH}}{r_{CH}^3}(u') du', \end{aligned}$$

therefore, summing and writing $\frac{|\nu|_{CH}}{r_{CH}^3} = \frac{1}{2} \partial_u (\frac{1}{r^2})$, we get that for all $u_1 \leq u_s$:

$$\lim_{i \rightarrow +\infty} \frac{|\phi|^2(u_{a,i}, V_i) + \int_{u_{a,i}}^{u_s} \frac{|\nu|}{r} |\phi|^2(u', V_i) du'}{|\psi|^2(u_1, V_i)} = \frac{r_{CH}^{-2}(u_{a,\infty}) + r_{CH}^{-2}(u_s)}{2}.$$

Therefore, using (2.6.13), it is clear that we proved

$$\lim_{i \rightarrow +\infty} \frac{\chi(V_i)}{|\psi|^2(u_1, V_i)} = \frac{r_{CH}^{-2}(u_{1,\infty}) + r_{CH}^{-2}(u_s)}{2} = \frac{r_{CH}^{-2}(u_{2,\infty}) + r_{CH}^{-2}(u_s)}{2},$$

by uniqueness of the limit. This is only possible if $r_{CH}(u_{2,\infty}) = r_{CH}(u_{1,\infty})$, and since r_{CH} is a non-increasing function of u , $r_{CH}(u) = r_0 > 0$ for all $u_{1,\infty} \leq u \leq u_{2,\infty}$. \square

Now, we finish the proof of Theorem 2.3.5 deriving a contradiction. Clearly, the result of the last lemma contradicts the blow-up of Assumption 10, using Proposition 2.6.3. Thus, the "necessary part" of Theorem 2.3.5 is proved.

2.7 Absence of the Cauchy Horizon for zero asymptotic charge

In this section, we are going to provide a proof of Theorem 2.1.8. Recall that in the earlier sections, we proved that, in the case $e \neq 0$, the solution was regular in a whole rectangle $(U, v) \in [0, U_s] \times [v_0, +\infty]$, up to the Cauchy horizon. In the case $e = 0$, we employ a contradiction argument: we assume that space-time features a Cauchy horizon, more precisely

Assumption 11 (Contradiction hypothesis). Consider our characteristic initial value problem, assuming additionally that $\lim_{v \rightarrow +\infty} Q|_{H^+}(v) = 0$ (convergence towards Schwarzschild), i.e. $e = 0$. Then, we make the hypothesis that the solution is regular in a region $[0, U_0] \times [v_0, +\infty]$ and that a null boundary $CH^+ := [0, U_0] \times \{+\infty\}$ can be attached to our space-time. Moreover, we assume that r can be extended as a continuous function $r_{CH}(u)$ on $[0, U_0]$ and there exists $r_0 > 0$ such that $\inf_{U \in [0, U_0]} r_{CH}(u) \geq r_0$.

Using the monotonicity properties of r , it is clear from Assumption 11 that, for U_0 small enough, there exists $r'_0 > 0$, such that $r(U, v) \geq r'_0$ for all $(U, v) \in [0, U_s] \times [v_0, +\infty]$. Thus we have the following Proposition

Proposition 2.7.1. *In the framework of section 2.4, the estimates of Propositions 2.4.2, 2.4.4, 2.4.5, 2.4.6, 2.4.7 hold true for $e = 0$, for implicit constants that now depend also on r'_0 . More specifically, $\lambda < 0$ and $\nu < 0$ in ¹⁰⁵ \mathcal{N} and there exists $C_0 = C_0(M, e, q_0, m^2, \Delta, r'_0, N) > 0$ such that in \mathcal{N}*

$$|\log \Omega^2(u, v) - \log(-4(1 - \frac{2M}{r}))| \leq C_0 \cdot v^{1-2s}, \quad (2.7.1)$$

$$0 \leq 1 - \kappa \leq C_0 \cdot v^{-2s}, \quad (2.7.2)$$

where we recall that \mathcal{N} is defined as

$$\mathcal{N} := \bigcup_{k=1}^N \mathcal{N}_k,$$

¹⁰⁵This statement mostly means that the apparent horizon is located strictly to the past of \mathcal{N} , namely the result concerns Proposition 2.4.6.

for

$$\mathcal{N}_k := \{\Delta_{k-1} := -\Delta + (k-1)\epsilon \leq u + v + h(v) \leq \Delta_k := -\Delta + k\epsilon\},$$

and $0 < \epsilon \leq \epsilon_0$ for $\epsilon_0 = \epsilon_0(M, e, q_0, m^2, \Delta, r'_0)$ small enough and $\Delta \geq \Delta_0$ for $\Delta_0 = \Delta_0(M, e, q_0, m^2)$ large enough.

Additionally, we will need the following refinement of one estimate on γ_Δ , the past boundary of \mathcal{N} : for $U_s > 0$ small enough, and Δ large enough, we have

$$\frac{\delta}{8} \leq r_+ - r(u_{\gamma_\Delta}(v), v) \leq \delta, \quad (2.7.3)$$

where we recall that $\delta := (2K_+)^{-1} \cdot e^{-2K_+ \cdot \Delta}$.

Proof. We revisit the proofs of Propositions 2.4.2, 2.4.4, 2.4.5, 2.4.6, 2.4.7. Now, we still have the $|\log(r)|$ is bounded like in the case $e \neq 0$ case, but $|\log(r)| \leq D_0$ for $D_0 = D_0(r'_0)$, making the assumption 11.

The crucial point is that, up to Proposition 2.4.7, we have not used anywhere the fact that $|Q|$ was lower bounded. Thus, the proof proceeds as before. For the refined estimate (2.7.3), we recall (2.4.36):

$$e^{-\tilde{C}\delta} \leq \frac{-4\nu_H(U, v)}{\Omega_H^2(0, v)} \leq \frac{e^{\tilde{C}\delta}}{1 - \tilde{C}'\delta v^{-2s}},$$

for some $\tilde{C} = \tilde{C}(M, e, q_0, m^2) > 0$ and $\tilde{C}' = \tilde{C}'(M, e, q_0, m^2) > 0$. Thus, integrating in U on $[0, U_{\gamma_\Delta}(v)]$:

$$\frac{\delta \cdot e^{-\tilde{C}\delta}}{4} \leq r_+ - r(U_{\gamma_\Delta}(v), v) \leq \frac{\delta \cdot e^{\tilde{C}\delta}}{4 \cdot (1 - \tilde{C}'\delta v^{-2s})},$$

where we also make use of the fact that $U_{\gamma_\Delta}(v)\Omega_H^2(0, v) = \delta$ by definition. Now, since on the curve γ_Δ , we have $u + v + h(v) = -\Delta$, it is also clear that $v + h(v) \geq -\Delta + |u_s|$. Hence, for $U_s = U_s(M, e, q_0, m^2, \Delta) > 0$ small enough, one can assume that $\tilde{C}'\delta v^{-2s} < \frac{1}{2}$. Now, for $\Delta \geq \Delta_0(M, e, q_0, m^2)$ large enough, one can assume $e^{\tilde{C}\delta} \leq 2$ and $e^{\tilde{C}'\delta} \leq 2$. This finally gives (2.7.3). \square

Remark 25. Notice that to prove (2.7.3), we do not need Assumption 11, because everything is done in the Red-shift region, where we already know that r is lower bounded. This is why the constants involved in the proof of (2.7.3) did not depend on r'_0 .

Remark 26. Of course, we cannot go beyond Proposition 2.4.7. In particular, Proposition 2.4.8 is **not true** in the case $e = 0$, as it is concerned with initiating the blue-shift effect, i.e. obtaining values of r for which $K(r) := r^{-2} \cdot (M - \frac{e^2}{r}) < 0$. This is also why we will not use (nor need) the estimates of sections 2.4.6 and 2.4.7, as they do not hold true in the case $e = 0$.

From now on, we will take $\Delta = \Delta_0(M, e, q_0, m^2)$ and $\epsilon = \epsilon_0(M, e, q_0, m^2, r'_0)$ so that the former proposition is satisfied. We also denote, correspondingly, $\delta_0 := (2K_+)^{-1} \cdot e^{-2K_+ \cdot \Delta_0}$.

Now, we establish our contradiction :

Corollary 2.7.2. *There exists $v^* > v_0$ such that for all $v \geq v^*$,*

$$r(u_{\gamma_{\Delta_N}}(v), v) \leq \frac{r'_0}{2},$$

where we recall that γ_{Δ_N} , the future boundary of \mathcal{N} , defined by $\{u + v + h(v) = \Delta_N = -\Delta_0 + \epsilon_0 N\}$. Thus, Assumption 11 is contradicted, which proves Theorem 2.1.8.

Proof. Since $\lambda < 0$ in \mathcal{N} , we get, combining with (2.7.3) that on the whole of \mathcal{N} :

$$r(u, v) \leq 2M - \frac{\delta_0}{8},$$

where we also used the fact that $e = 0$ so $r_+ := M + \sqrt{M^2 - e^2} = 2M$. Now using (2.7.2) as $\kappa - 1 \geq 1$ and (2.7.1), we find that

$$|\partial_u r| = \frac{\kappa^{-1}\Omega^2}{4} \geq \left(\frac{2M}{r} - 1\right) \cdot e^{-C_0 \cdot v^{1-2s}} \geq \frac{\delta_0}{16M - \delta_0} \cdot e^{-C_0 \cdot v^{1-2s}},$$

where for the last lower bound we used $r \leq 2M - \frac{\delta_0}{8}$. Then, we integrate in u on $[u_{\gamma_\Delta}(v), u_{\gamma_{\Delta_N}}(v)]$ to obtain

$$r(u_{\gamma_{\Delta_N}}(v), v) \leq r(u_{\gamma_\Delta}(v), v) - \epsilon_0 N \cdot \frac{\delta_0}{16M - \delta_0} \cdot e^{-C_0 \cdot v^{1-2s}} \leq 2M - \frac{\delta_0}{8} - \epsilon_0 N \cdot \frac{\delta_0}{16M - \delta_0} \cdot e^{-C_0 \cdot v^{1-2s}},$$

where we have used the fact that $u_{\gamma_{\Delta_N}}(v) - u_{\gamma_{\Delta}}(v) = \epsilon_0 N$ and the bound $r \leq 2M - \frac{\delta_0}{8}$ again. Now, recall that ϵ_0 and δ_0 only depend on M, e, q_0 and m^2 . Thus, it is clear that one can choose $N \geq N_0$ for an integer $N_0 = N_0(M, e, q_0, m^2) \geq 1$ such that

$$\epsilon_0 N \cdot \frac{\delta_0}{16M - \delta_0} > 2 \cdot (2M - \frac{\delta_0}{8}).$$

In particular, we also obtain the following (much weaker) estimate

$$\limsup_{v \rightarrow +\infty} r(u_{\gamma_{\Delta_N}}(v), v) \leq \frac{r'_0}{4},$$

which easily proves the corollary. \square

This proves the first and most important statement of Theorem 2.1.8. The other statements of Theorem 2.1.8 follow directly from the corollary, c.f. [47] (in particular Theorem 1.1 and its Penrose diagram).

2.8 Estimates related to the apparent horizon \mathcal{A}

As a straightforward by-product of our framework, we prove that in a non-linear setting, the apparent horizon¹⁰⁶ $\mathcal{A} := \{\partial_v r = 0\}$ cannot be too far or too close of the event horizon if the decay of the perturbation is upper and lower bounded.

Proposition 2.8.1. *We keep the same hypothesis as for Theorem 2.3.2. h is defined in equation (2.4.10).*

We assume the following on the event horizon \mathcal{H}^+ :

Assumption 12.

$$C' v^{-p-1} \leq \Omega_H^2(0, v) \int_v^{+\infty} \frac{|\partial_v \phi|^2(0, v')}{\Omega_H^2(0, v')} dv', \quad (2.8.1)$$

$$|\partial_v \phi|(0, v') \leq C v^{-s}, \quad (2.8.2)$$

for $2s - 1 \leq p$ and $C, C' > 0$.

Then there exists constants $C_+ > 0, C_- > 0$ such that

$$\mathcal{A} \subset \{C_- v^{-p-1} \leq \Omega^2(u, v) \leq C_+ v^{-2s}\} = \{-\tilde{C} - \frac{(p+1)}{2K_+} \log(v) \leq u + v + h(v) \leq -\frac{2s}{2K_+} \log(v) + \tilde{C}\}.$$

Remark 27. Notice that because of the exponential growth of Ω_H^2 established in section 2.4.3, assumption 12 is consistent with the conjectured tail of the *field* as formulated in Price's law of conjecture 2.1.6.

Remark 28. Notice that if ϕ does not become constant near infinity on the event horizon, \mathcal{A} is strictly to the future of \mathcal{H}^+ . This is in particular true¹⁰⁷ if one assumes a lower bound on $\partial_v \phi$, like (2.3.6) or (2.8.1). Coupled with (2.8.2), it proves that \mathcal{A} must asymptotically approach time-like infinity.

Proof. Using assumption (12) and (2.2.18) on the event horizon and recalling (2.4.4), we get :

$$v^{-p-1} \lesssim \lambda \lesssim v^{-2s}.$$

We can rewrite (2.2.15) in (U, v) coordinates as :

$$\partial_U \lambda = \frac{-\Omega_H^2}{4} (2K - m^2 r |\phi|^2).$$

Using (2.4.13), 2.4.23 and section 2.4.3, there exists $\delta > 0$ small enough so that $K_+ < 2K - m^2 r |\phi|^2 < 3K_+$ in \mathcal{R} .

We can then integrate between the event horizon and the apparent horizon for $U \in [0, U_{\mathcal{A}}]$ to get :

$$v^{-p-1} \lesssim \frac{4}{3K_+} \lambda(0, v) < \int_0^{U_{\mathcal{A}}} \Omega_H^2(U', v) dU' < \frac{4}{K_+} \lambda(0, v) \lesssim v^{-2s}.$$

¹⁰⁶Indeed \mathcal{A} coincides with $\{\lambda = 0\}$ on the whole space-time in our coordinate choice. This is because λ becomes strictly negative while $\kappa^{-1} \approx 1$.

¹⁰⁷Notice that an upper bound that assures that ϕ tends to 0, like that of hypothesis 4, is enough to reduce the problem to either the trivial case $\phi \equiv 0$ where $\mathcal{A} = \mathcal{H}^+$ or the case where \mathcal{A} asymptotically approaches time-like infinity.

Then we can use (2.4.16) to prove that :

$$\int_0^{U_A} \Omega_H^2(U', v) dU' \sim \Omega^2(U_A, v).$$

Which gives the result. \square

2.9 Estimates in \mathcal{N} , cutting the region in small pieces

In this section, we provide a proof of Proposition 2.4.7 of section 2.4.5. We first recall proposition 2.4.7 for convenience :

Proposition. *For small enough $\epsilon > 0$, we have :*

$$|\phi| + |\partial_v \phi| + |D_u \phi| \lesssim 2^N v^{-s}, \quad (2.9.1)$$

$$|A_u| \lesssim (N+1)\delta, \quad (2.9.2)$$

we also have :

$$|\log \Omega^2(u, v) - \log(-4(1 - \frac{2M}{r} + \frac{e^2}{r^2}))| \lesssim 4^N v^{1-2s}, \quad (2.9.3)$$

$$0 \leq 1 - \kappa \lesssim 5^N v^{-2s}, \quad (2.9.4)$$

$$|1 - \iota| \lesssim 5^N v^{-p(s)}, \quad (2.9.5)$$

$$|\partial_u \log(\Omega^2) - 2K| \lesssim 5^N v^{-p(s)}, \quad (2.9.6)$$

$$|\partial_v \log(\Omega^2) - 2K| \lesssim 5^N v^{-2s}, \quad (2.9.7)$$

$$|Q(u, v) - e| \lesssim 4^N v^{1-2s}, \quad (2.9.8)$$

$$|\varpi(u, v) - M| \lesssim 4^N v^{1-2s}. \quad (2.9.9)$$

Proof. We want to prove by induction on k the following estimates on $\mathcal{N}_k := \{u + v + h(v) = -\Delta + k\epsilon\}$:

$$|\phi| + |\partial_v \phi| \leq D_k v^{-s}, \quad (2.9.10)$$

$$|r D_u \phi| \leq D_k v^{-s}, \quad (2.9.11)$$

$$|A_u| \lesssim A_k, \quad (2.9.12)$$

$$|\log \Omega^2(u, v)| \leq C_k, \quad (2.9.13)$$

$$\Omega^2 \leq \frac{3}{2} \Omega_{max}^2(M, e), \quad (2.9.14)$$

$$0 \leq 1 - \kappa \lesssim E_k v^{-2s}, \quad (2.9.15)$$

$$|1 - \iota| \lesssim E_k v^{-p(s)}, \quad (2.9.16)$$

$$|\partial_u \log(\Omega^2) - 2K| \lesssim E_k v^{-p(s)}, \quad (2.9.17)$$

$$|\partial_v \log(\Omega^2) - 2K| \lesssim E_k v^{-2s}, \quad (2.9.18)$$

$$|Q(u, v) - e| \lesssim D_k^2 v^{1-2s}, \quad (2.9.19)$$

$$|\varpi(u, v) - M| \lesssim D_k^2 v^{1-2s}, \quad (2.9.20)$$

with $D_k = 2D_{k-1}$, $E_k = 5E_{k-1}$, $C_k = C_{k-1} + K_{max}\epsilon$, $A_k = (k+1)\delta$ and K_{max} depending on (e, M) only. $\Omega_{max}^2(M, e)$ is defined as :

$$\Omega_{max}^2(M, e) := 4\left(\frac{M^2}{e^2} - 1\right) = \sup_{r \in [r_-(M, e), r_+(M, e)]} 4\left|1 - \frac{2M}{r} + \frac{e^2}{r^2}\right|.$$

The initialization of the induction comes directly from the bounds of proposition (2.4.5), after choosing D_0 , E_0 and A_0 consistently. Notice that $A_0 \lesssim \delta$.

Supposing the bounds are established for \mathcal{N}_{k-1} , we bootstrap the following on \mathcal{N}_k :

$$\Omega^2 \leq 2\Omega_{max}^2(M, e), \quad (2.9.21)$$

$$|\phi| + |\partial_v \phi| \leq 2D_k v^{-s}, \quad (2.9.22)$$

$$|1 - \kappa| \leq 2E_k v^{-2s}, \quad (2.9.23)$$

$$|1 - \iota| \leq 2E_k v^{-2s}. \quad (2.9.24)$$

Notice that because $\nu < 0$ and $\lambda|_{\mathcal{H}^+} \geq 0$, we have $r \leq r_+$ everywhere.

We first use (2.9.22) to prove (2.9.19) with (2.2.21) :

$$|Q(u, v) - e| \lesssim D_k^2 v^{1-2s}.$$

Then we can use (2.9.21) with (2.9.24) to prove that $|\lambda|$ is bounded, (2.9.22), (2.9.23) to prove (2.9.20) with (2.2.24) for $|u_s|$ large enough :

$$|\varpi(u, v) - M| \lesssim D_k^2 v^{1-2s}.$$

Notice that since $\Omega^2 = -4\kappa(1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2})$ -as seen in equation (2.2.11)- we have -forming the differences $\varpi - M$ and $Q^2 - e^2$ and using (2.9.23), (2.9.24) for $|u_s|$ large enough :

$$0 \leq -(1 - \frac{2\varpi}{r} + \frac{Q^2}{r^2}) = -(1 - \frac{2M}{r} + \frac{e^2}{r^2}) + O(D_k^2 v^{1-2s}).$$

So -since r_- cancels $(1 - \frac{2M}{r} + \frac{e^2}{r^2})$ - for all $\eta > 0$, there exists $|u_s|(\eta)$ large enough so that $r_- - \eta < r$. For $\eta > 0$ small enough, it can be easily shown that the supremum on $[r_- - \eta, r_+]$ is attained on $[r_-, r_+]$:

$$\Omega_{max}^2(M, e) = \sup_{r \in [r_-(M, e) - \eta, r_+(M, e)]} 4|1 - \frac{2M}{r} + \frac{e^2}{r^2}|.$$

Since $|\Omega^2| \leq 4|1 - \frac{2M}{r} + \frac{e^2}{r^2}| + \tilde{C}v^{1-2s}$, bootstrap (2.9.21) is validated for $|u_s|$ large enough and proves (2.9.14).

Moreover, with the same technique using (2.2.11), (2.9.19), (2.9.20) and bootstrap (2.9.23), (2.9.24) we can prove (2.9.3), choosing $|u_s|$ large enough.

(2.9.19), (2.9.20) also prove that :

$$|2K(u, v) - 2K_{M, e}(r(u, v))| \lesssim D_k^2 v^{1-2s}. \quad (2.9.25)$$

Using the same argument as in the red-shift region and (2.9.24) we get -for $|u_s|$ large enough- :

$$|\partial_v(2K)| \lesssim \Omega^2 \leq 2\Omega_{max}^2. \quad (2.9.26)$$

We denote $v_i = v_i(u)$ the unique v such that $u + v + h(v) = \Delta_i$.

Notice that from (2.4.11) :

$$|h(v_{k-1}) - h(v)| \lesssim v_{k-1}^{1-2s} |v - v_{k-1}| \approx |u|^{1-2s} |v - v_{k-1}| \approx v^{1-2s} |v - v_{k-1}|.$$

Hence , because $u + v + h(v) - \Delta_{k-1} \leq \epsilon$ is bounded :

$$v - v_{k-1} = \frac{u + v + h(v) - \Delta_{k-1}}{1 + O(v^{1-2s})} = u + v + h(v) - \Delta_{k-1} + O(v^{1-2s}) \quad (2.9.27)$$

We use (2.2.22) and (2.9.21) to get :

$$|\partial_v A_u| \lesssim 1.$$

Hence by induction, we get (2.9.12) with

$$|A_u| \leq A_{k-1} + \tilde{C}\epsilon \leq A_k,$$

after choosing ϵ small enough compared to δ .

Then we use (2.2.28) with (2.9.21), (2.9.22), (2.9.23) to get :

$$|rD_u\phi| \leq D_{k-1}|u|^{-s} + \tilde{C}D_k\epsilon v^{-s}.$$

Hence for ϵ small enough compared to (C, e, M, q_0, m) , we get (2.9.11).

Using (2.2.27) and the same type of argument, we close bootstrap (2.9.22) and get (2.9.10) after integrating $\partial_v\phi$ on a ϵ -small region.

We can then use bootstrap (2.9.21), (2.9.22) and (2.9.23) and notice that $|\partial_u\varpi| + |\partial_u Q^2| \lesssim D_k^2 v^{-2s}$ to get from (2.2.26):

$$|\partial_v \log(\Omega^2) - 2K| \lesssim (E_k + D_k^2)v^{-2s} \lesssim E_k v^{-2s}. \quad (2.9.28)$$

Then, because of the discussion above, $\frac{r_-(e, M)}{2} < r < r_+(e, M)$ therefore there exists $K_{max} = K_{max}(e, M) > 0$ such that $|K| < K_{max}$.

Using (2.9.28) and the induction hypothesis, we prove (2.9.13) and get -choosing $|u_s|$ large enough- :

$$\Omega^{-2} \gtrsim e^{2K_{max}k\epsilon}.$$

Hence from (2.2.29) and (2.9.11) we get :

$$|\kappa - 1| \leq E_{k-1} + \bar{C}D_k^2 e^{2K_{max}k\epsilon} \epsilon = E_{k-1} + \bar{C}D_0^2 e^{(\log(4) + 2K_{max}\epsilon)k} \epsilon.$$

We proceed in two times : first with choose ϵ small enough so that $\log(4) + 2K_{max}\epsilon \leq \log(5)$. We get :

$$|\kappa - 1| \leq E_{k-1} + \bar{C}D_k^2 e^{2K_{max}k\epsilon} \epsilon = E_0 5^{k-1} + \bar{C}D_0^2 5^k \epsilon.$$

Then we can choose ϵ even smaller so that bootstrap (2.9.23) is validated. (2.9.15), and (2.9.18) are proved simultaneously, using (2.2.26) for (2.9.18) similarly to what was done before.

Symmetrically in v we use the same methods to close bootstrap (2.9.24) and to prove (2.9.16), (2.9.17).

The induction is then proved and the estimates of the proposition follow directly. \square

2.10 Local existence for the characteristic initial value problem

In this section, we provide a proof of Lemma 2.4.3 of section 2.4.3. First, we recall Lemma 2.4.3 :

Lemma. *Under the same hypothesis than before and for $v'_0 > v_0$, if U_s is sufficiently small there exists a constant $D > 0$ depending on $C, e, M, q_0, m^2, s, v_0$ and v'_0 such that*

$$|D_U\phi(U, v'_0)| \leq D. \quad (2.10.1)$$

$$|\partial_U r(U, v'_0)|^{-1} \leq D. \quad (2.10.2)$$

Therefore, for any $\eta > 0$ independent ¹⁰⁸ of any parameter, there exists a $v'_0 > 0$ such that

$$|D_U\phi(U, v'_0)| \lesssim C,$$

and for all $v \geq v'_0$:

$$\begin{aligned} |2K(0, v) - 2K_+| &\leq \eta K_+, \\ rm^2|\phi|^2(0, v) &\leq \eta K_+. \end{aligned}$$

Proof. We will make the following bootstrap assumptions :

$$|\nu_H| \leq B_1 \quad (2.10.3)$$

$$|Q| + |\phi| + |\partial_v\phi| \leq B_2 \quad (2.10.4)$$

The set of points such that the bootstraps are valid is non empty because of the hypothesis of Theorem 2.3.2, for B_2 large enough with respect to C, e and v_0 and $B_1 > 1$.

¹⁰⁸We insist that η must be a numerical constant that do **not** depend on **any** of the C, e, M, q_0, m^2, v_0 or v'_0 .

Notice that with our hypothesis $r(0, v) > 0$ and since $[v_0, v'_0]$ is a compact, it is clear that $r(0, v)$ is upper and lower bounded by strictly positive constants that depend on v_0 and v'_0 .

If we integrate (2.10.3) for U_s small enough compared to B_1 , we see that the same conclusion holds true for $r(U, v)$ on the whole rectangle $[0, U_s] \times [v_0, v'_0]$. We write $0 < r_{min} < r < r_{max}$.

Then, notice that $\kappa(0, v) \equiv 1$ and the positive right hand side of (2.2.29) give that $0 \leq \kappa \leq 1$ everywhere on the space-time namely $\Omega_H^2 \leq -4\nu_H$.

Then we write (2.2.15) as

$$|\partial_v(\log(-r\nu_H))| \leq \frac{1}{r} \left(1 + \frac{Q^2}{r^2}\right) + m^2 r |\phi|^2 \leq \frac{1}{r_{min}} + \frac{B_2^2}{r_{min}^3} + m^2 r_{max} B_2^2.$$

We can then integrate in v and use gauge (2.3.4) to get :

$$|\log(-\nu_H)| \leq |\log(\frac{r_{max}}{r_{min}})| + \left(\frac{1}{r_{min}} + \frac{B_2^2}{r_{min}^3} + m^2 r_{max} B_2^2\right)(v'_0 - v_0).$$

This closes bootstrap (2.10.3) for B_1 large enough with respect to B_2 , v_0 , v'_0 and the parameters and proves (2.10.2).

Now we want to bound λ : to do so we write (2.2.15) as :

$$|\partial_U(r\lambda)| \leq -\nu_H \left(1 + \frac{Q^2}{r^2} + m^2 r^2 |\phi|^2\right) \leq B_1 \left(1 + \frac{B_2^2}{r_{min}^2} + m^2 r_{max}^2 B_2^2\right).$$

Now notice that on the compact $[v_0, v'_0]$, $|\lambda|(0, v) \leq \lambda_{max}$ where λ_{max} depends on v_0 , v'_0 and the parameters. Then we can integrate the previous equation and take U_s small enough to get everywhere :

$$|\lambda|(U, v) \leq 2\lambda_{max}.$$

Now we write (2.2.28) as

$$|\partial_v(rD_U\phi)| \leq -\nu_H(m^2|\phi| + |\partial_v\phi|) \leq (m^2 + 1)B_1B_2.$$

Then we integrate and use assumption 5 and the bounds on r to get :

$$|D_U\phi| \leq \frac{r_{max}}{r_{min}}C + \frac{(v'_0 - v_0)(m^2 + 1)B_1B_2}{r_{min}}. \quad (2.10.5)$$

Now we use gauge ¹⁰⁹ (2.2.9) to integrate (2.2.22) :

$$|A_U| \leq \frac{2B_1B_2}{r_{min}^2}(v'_0 - v_0).$$

This, with bootstrap (2.10.4) and (2.10.5) gives :

$$|\partial_U\phi| \leq \frac{r_{max}}{r_{min}}C + \frac{(v'_0 - v_0)B_1B_2}{r_{min}}[(m^2 + 1) + \frac{2q_0B_2}{r_{min}}].$$

It now suffices to integrate for U_s small enough to close the ϕ part of bootstrap (2.10.4).

The Q part of bootstrap (2.10.4) is validated when we integrate (2.2.20) using (2.10.5).

For the $\partial_v\phi$ part, we write (2.2.27) as :

$$|\partial_U(e^{iq_0 \int_{U_0}^U A_U} r \partial_v \phi)| \leq |\lambda| |D_U\phi| + |\nu_H| (rm^2|\phi| + \frac{q_0|Q||\phi|}{r})$$

Then, from all the bounds that precedes it is clear that we can integrate on $[0, U]$ and close the $\partial_v\phi$ part of bootstrap (2.10.4) if we chose U_s small enough.

Notice that B_2 can be chosen to depend on C , v_0 and e only. Hence B_1 can be chosen to depend on v_0 , v'_0 and the parameters only.

In the end both bootstraps are validated.

Notice that (2.10.5) gives actually (2.10.1) now that the bootstraps assumptions are proved.

¹⁰⁹Note that it does not matter whether the gauge is on $v \equiv v_0$ or $v \equiv v'_0$: we simply integrate from the curve where $A_u = 0$.

From the last section 2.4.3, $2K(0, v) - 2K_+ \rightarrow 0$ when $v \rightarrow +\infty$ and from the hypothesis 4 and the boundedness¹¹⁰ of r we know that $rm^2|\phi|^2 \rightarrow 0$ when $v \rightarrow +\infty$. We can write $\max\{|2K(0, v) - 2K_+|, rm^2|\phi|^2\} = K_+\epsilon(v)$ and $\epsilon(v) \rightarrow 0$ when $v \rightarrow +\infty$.

Therefore for all $\eta > 0$ -independent of all the other constants- there exists v'_0 -depending only on the parameters and v_0 such that for all $v' \geq v'_0$, $|\epsilon(v')| \leq \eta$.

Therefore, combining with (2.10.1), the lemma is proven. □

¹¹⁰Indeed r converges to r_∞ when v tends to $+\infty$ and the interval $[v_0, +\infty]$ is lower bounded.

Chapter 3

Decay of weakly charged solutions for the spherically symmetric Maxwell-Charged-Scalar-Field equations on a Reissner–Nordström exterior space-time

We consider the Cauchy problem for the (non-linear) Maxwell-Charged-Scalar-Field equations with spherically symmetric initial data, on a fixed sub-extremal ¹ Reissner–Nordström exterior space-time. We prove that the solutions are bounded and decay at an inverse polynomial rate towards time-like infinity and along the black hole event horizon, provided the charge of the Maxwell equation is sufficiently small.

This condition is in particular satisfied for small data in energy space that enjoy a sufficient decay towards the asymptotically flat end.

Some of the decay estimates we prove are arbitrarily close to the conjectured optimal rate in the limit where the charge tends to 0, following the heuristics of [41].

Our result can also be interpreted as a step towards the stability of Reissner–Nordström black holes for the **gravity coupled** Einstein–Maxwell-Charged-Scalar-Field model. This problem is closely connected to the understanding of strong cosmic censorship and charged gravitational collapse in this setting.

3.1 Introduction

The model In this paper, we study the asymptotic behaviour of solutions to the Maxwell-Charged-Scalar-Field equations, sometimes referred to as massless Maxwell–Klein–Gordon, arising from spherically symmetric and asymptotically decaying initial data on a fixed sub-extremal Reissner–Nordström exterior space-time :

$$\nabla^\mu F_{\mu\nu} = iq_0 \frac{(\phi \overline{D_\nu \phi} - \overline{\phi} D_\nu \phi)}{2}, \quad F = dA, \quad (3.1.1)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0, \quad (3.1.2)$$

$$g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 [d\theta^2 + \sin(\theta)^2 d\varphi^2], \quad (3.1.3)$$

$$\Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2}, \quad (3.1.4)$$

where $q_0 \geq 0$ is a constant called the charge ² of the scalar field ϕ , M , ρ are respectively the mass and the charge of the Reissner–Nordström black hole ³ with $0 \leq |\rho| < M$, ∇_μ is the Levi–Civita connection and $D_\mu = \nabla_\mu + iq_0 A_\mu$ is the gauge derivative. Note that — due to the interaction between the Maxwell field F and

¹Our convention is that it includes Schwarzschild black holes as well.

²This charge q_0 is also the coupling constant between the Maxwell and the scalar field equations. This is not to be confused with the charge of the Maxwell equation or the parameter ρ . For a precise definition of all “charges”, c.f. section 3.2.3.

³Notice the slight notation difference with [78] where Ω^2 was defined to be $-4(1 - \frac{2M}{r} + \frac{\rho^2}{r^2})$ on the Reissner–Nordström interior.

the charged scalar field ϕ — this system of equations is **non-linear** when $q_0 \neq 0$, the case of interest for this paper. This is in contrast to the uncharged case $q_0 = 0$, where (3.1.2) is then the linear wave equation.

Main results Since global regularity is known for this system ⁴, we focus on the asymptotic behaviour of solutions.

The case of a charged scalar field on a black hole space-time that we hereby consider is considerably different from the analogous problem on Minkowski space-time. While on the flat space-time, the charge of the Maxwell equation tends to 0 towards time-like infinity, this is not expected to be the case on black hole space-times. This fact constitutes a major difference and renders the proof of decay harder, already in the spherically symmetric case and when the charge is small.

We show that if the charge in the Maxwell equation and the scalar field energies are initially smaller than a constant depending on the black hole parameters M and ρ , then

1. the charge in the Maxwell equation is bounded and stays small on the whole Reissner–Nordström exterior space-time.
2. Boundedness of the scalar field energy holds.
3. A local integrated energy decay estimate holds for the scalar field.

If now we relax the smallness hypothesis on the charge, requiring only that the initial charge is smaller than a numerical constant ⁵ and assume 2 and 3, then

4. the energy ⁶ of the scalar field decays at an inverse polynomial rate, depending on the charge.
5. The scalar field enjoys point-wise decay estimates at an inverse polynomial rate consistent with 4.

These results are stated in a simplified version in Theorem 3.1.1 and later in a more precise way in Theorem 3.3.1, Theorem 3.3.2 and Theorem 3.3.3.

The decay rate of the energy — and of some point-wise estimates that we derive — has been conjectured to be optimal in [41], in the limit when the asymptotic charge tends to zero, c.f. section 3.1.2. Other point-wise estimates, notably along the black hole event horizon, are however not sharp in that sense.

In the case of an uncharged scalar field $q_0 = 0$ (namely the wave equation on sub-extremal black holes), it is well-known that the long term asymptotics are governed by the so-called Price’s law, first put forth heuristically by Price in [68] and later proved in [2], [3], [25], [32], [61], [77]. Generic solutions then decay in time at an universal inverse polynomial decay rate, in the sense that this rate does not depend on any physical parameter or on the initial data. This is in contrast to our charged case $q_0 \neq 0$ where the optimal decay rate is conjectured to be **slower** and moreover depending on the charge in the Maxwell equation, itself determined by the data.

Roughly, this can be explained by the fact that in the uncharged case $q_0 = 0$, the equation effectively looks like the wave equation on Minkowski space in the presence of a potential decaying like r^{-3} .

This decay of the potential-like term is somehow more “forgiving” than in the charged case $q_0 \neq 0$. In the latter case, the equation becomes similar to the wave equation in the presence of a potential decaying like r^{-2} . As explained beautifully in pages 7 and 8 of [54], while the system is sub-critical with respect to the conserved energy in dimensions (3+1), the new term coming from the charge induces a form of criticality with respect to decay at space-like infinity, i.e. a criticality with respect to r weights.

One important consequence — in the black hole case — is that the sharp decay rate is expected to depend on the charge, c.f. section 3.1.2. Interestingly in our paper, in order to deal with this criticality, we need to use the full non-linear structure of the system. Also, the criticality with respect to decay implies the absence of any “extra convergence factor” that facilitates the proof of bounds on long time intervals, in the language of [54]. This is in contrast to the uncharged case and requires a certain sharpness in the estimates, as explained in [54].

Motivation Our result can be viewed as a first step towards the understanding of the analogous Einstein–Maxwell–Charged–Scalar–Field model, where the Maxwell and Scalar Field equations are now coupled with gravity, c.f. equations (3.1.16), (3.1.17), (3.1.18), (3.1.19), (3.1.20) when $m^2 = 0$. In this setting, the asymptotic behaviour of the scalar field is important, in particular because it determines the black hole interior structure, c.f. section 3.1.2. This is also closely related to the so-called Strong Cosmic Censorship Conjecture, c.f. section 3.1.2.

⁴It follows essentially from the global regularity of Yang–Mills equations on globally hyperbolic (3+1) Lorentzian manifolds, established in [17]. In spherical symmetry, this can also be deduced from the methods of [47].

⁵More precisely, the maximal value is $q_0|e_0| = \frac{1}{4}$ for the weakest claimed decay and $q_0|e_0| = 0.8267$ for the improved one, where e_0 is the initial asymptotic charge, c.f. section 3.2.3.

⁶By this, we mean all the energies transverse or parallel to the event horizon or null infinity, or L^2 flux on any constant r curve.

Before discussing the relevance of our result for the interior of black holes, let us mention that the Einstein–Maxwell–Charged–Scalar–Field system possesses a number of new features compared to its uncharged analogue $q_0 = 0$. One of them is the existence of one-ended ⁷ charged black holes solutions, which are of great interest to study the formation of a Cauchy horizon during gravitational collapse. Indeed one-ended initial data are required to study gravitational collapse. Also, unlike their uncharged analogue in spherical symmetry, charged black holes admit a Cauchy horizon, a feature which is also expected when no symmetry assumption is made, c.f. [24]. While previously studied models in spherical symmetry admit either uncharged one-ended black holes (c.f. [11], [12], [13]) or charged two-ended black holes (c.f. [19], [20], [25], [57], [58]), the Einstein–Maxwell–Charged–Scalar–Field model admits **charged one-ended** black holes. This is permitted by the coupling between the Maxwell–Field and the charged scalar field, which allows for a non constant charge.

We now return to the interior structure of black holes. Various previous works for different models have highlighted that the interior possesses both stability and instability ⁸ features that depend strongly on the asymptotic behaviour of the scalar field on the event horizon c.f. [19], [20], [24], [56], [57], [58], [59]. For the Einstein–Maxwell–Charged–Scalar–Field model, stability and instability results in the interior have been established in [78]. Concerning stability, it is proven in [78] that a scalar field decaying point-wise at a **strictly integrable** inverse-polynomial rate on the event horizon gives rise to a C^0 stable Cauchy horizon over which the metric is continuously extendible. In the present paper, we establish such strictly integrable point-wise decay for the scalar field on a fixed black hole. If those bounds can be extrapolated to the Einstein–Maxwell–Charged–Scalar–Field case, then the hypotheses of [78] are verified and the stability result applies. In particular, the continuous extendibility result of [78] would then disprove the continuous formulation of the so-called Strong Cosmic Censorship Conjecture as discussed in more details in section 3.1.2.

Another important aspect is the singularity structure of the black hole interior, which is related to the instability feature we stated earlier. This is relevant to a (weaker) C^2 formulation of the Strong Cosmic Censorship Conjecture we mentioned before, c.f. section 3.1.2. More specifically, it is proven in [78] that lower bounds for the energy on the event horizon imply the formation of a singular Cauchy horizon. It was proven in [57] for the uncharged analogue model that the positive resolution of the C^2 strong cosmic censorship conjecture actually follows from this singular nature of the Cauchy horizon. For this, lower bounds on the event horizon must be proven for solutions of a Cauchy problem, c.f. [56], [58] for the uncharged model. Even though in the present paper, we only focus on proving upper bounds for the Cauchy problem, the energy estimates that we carry out are conjectured to be **sharp** in the limit when the charge tends to zero, c.f. section 3.1.2. Therefore, our work can also be seen as a first step towards the understanding of energy lower bounds, and consequently towards the resolution of strong cosmic censorship in spherical symmetry, for the Einstein–Maxwell–Charged–Scalar–Field model. While this philosophy has been successfully applied to uncharged fields on two-ended black holes [57], [58], the analogous question in the charged case, both for one-ended and two-ended black holes remains open.

Outline The introduction is outlined as followed : first in section 3.1.1 we describe a rough version of the main results, namely asymptotic decay estimates for small decaying data. Then in section 3.1.2, we review previous works on related topics. More specifically in section 3.1.2 we state the conjectured asymptotic behaviour of charged scalar fields on black holes. Then in section 3.1.2 we sum up the known results in the black hole interior for this model, as a motivation for the present study. Then in section 3.1.2 we review previous results for the Maxwell–Charged–Scalar–Field equations on Minkowski space-time and for small energy data. Then section 3.1.2 deals with previous works on the wave equations on black holes space-times and strategies to reach Price’s law optimal decay. After in section 3.1.3 we explain the main ideas of the proof. This includes mostly the strategy to prove energy boundedness, integrated energy estimates and energy decay. Finally in section 3.1.4, we outline the rest of the chapter, section by section.

3.1.1 Simplified version of the main results

We now give a first version of the main results. The formulation of this section is extremely simplified and a more precise version is available in section 3.3. See also remark 29 for important precisions.

Theorem 3.1.1. *Consider spherically symmetric regular data (ϕ_0, Q_0) , for the Maxwell–Charged–Scalar–Field equations on a sub-extremal Reissner–Nordström exterior space-time of mass M and charge ρ . Assume that ϕ_0 and its derivatives decay sufficiently towards spatial infinity.*

*If Q_0 and ϕ_0 are small enough in appropriate norms then **energy boundedness** (3.3.1) is true and an **integrated local energy** estimate (3.3.2) holds.*

⁷In spherical symmetry, the initial data is one-ended if it is diffeomorphic to \mathbb{R}^3 and two-ended if it is diffeomorphic to $\mathbb{R} \times \mathbb{S}^2$.

⁸See also the recent [35] that investigates a very different kind of scalar field instability on Schwarzschild black hole, which is somehow stronger but specific to uncharged and non-rotating black holes.

Also we can define the future asymptotic ⁹ charge e such that on any curve constant r curve $\gamma_{R_0} := \{r = R_0\}$ for $r_+ \leq R_0 \leq +\infty$, we have $Q|_{\gamma_{R_0}}(t) \rightarrow e$ as $t \rightarrow +\infty$.

Then we have **energy decay** : there exists $2 < p(e) < 3$, with $p(e) \rightarrow 3$ as $e \rightarrow 0$ and such that for all $R_0 > r_+$, for all $u > 0$:

$$E(u) + \int_{\gamma_{R_0} \cap [u, +\infty]} |\phi|^2 + |D_v \phi|^2 + \int_{\mathcal{H}^+ \cap [v_R(u), +\infty]} |\phi|^2 + |D_v \phi|^2 \lesssim u^{-p(e)}, \quad (3.1.5)$$

where $\gamma_{R_0} := \{r = R_0\}$, (u, v) are defined in section 3.2.1, $v_R(u) = u + R^*$ is defined in section 3.2.4, and E is defined in section 3.2.5.

We also have the following ¹⁰ **point-wise decay**, for any $R_0 > r_+$, for $u > 0$, $v > 0$:

$$r^{\frac{1}{2}} |\phi|(u, v) + r^{\frac{3}{2}} |D_v \phi|(u, v) \lesssim (\min\{u, v\})^{-\frac{p(e)}{2}}, \quad (3.1.6)$$

$$|\psi|_{\mathcal{I}^+}(u) \lesssim u^{\frac{1-p(e)}{2}}, \quad (3.1.7)$$

$$|D_u \psi| \lesssim \Omega^2 \cdot u^{-\frac{p(e)}{2}}, \quad (3.1.8)$$

$$|D_v \psi|_{\{v \geq 2u + R^*\}}(u, v) \lesssim v^{\frac{1-p(e)}{2}}, \quad (3.1.9)$$

$$|Q - e|(u, v) \lesssim u^{1-p(e)} 1_{\{r \geq R_0\}} + v^{-p(e)} 1_{\{r \leq R_0\}}, \quad (3.1.10)$$

where $\psi := r\phi$ denotes the radiation field and Q is the Maxwell charge ¹¹ defined by $F_{uv} = \frac{2Q\Omega^2}{r^2}$.

Remark 29. In reality, this theorem —which is a broad version of Theorem 3.3.1 — contains two different intermediate results.

The first one, Theorem 3.3.2, proves simultaneously energy boundedness and the integrated local energy estimate, on condition that the charge Q is **everywhere** smaller than a constant depending on the black hole parameters. In particular, this is the case if the r weighted energies of the scalar field data and the initial charge are sufficiently small.

The second one, Theorem 3.3.3, proves the decay of the energy at a polynomial rate and subsequent point-wise estimates. The decay rate depends only on the dimensionless quantity $q_0 e$ and is conjectured to be almost optimal when $q_0 e \rightarrow 0$. This theorem only requires that the boundedness of the energy and the integrated local energy estimate are verified, together with the bound $q_0 |e| < 0.8267$. In particular, this is the case if the limit of the initial charge e_0 verifies $q_0 |e_0| < 0.8267$ and if the r weighted energies of the scalar field data are sufficiently small. In a sense, the charge smallness —which determines the decay rate— is more precise in these theorems and independent of the black hole mass. This relates to the physical expectation that the decay rate depends only on the asymptotic charge, c.f. section 3.1.2.

Remark 30. Estimates (3.1.5), (3.1.7) are expected to be **sharp** ¹² when $q_0 e$ tends to 0, c.f. section 3.1.2. Estimate (3.1.6) is also expected to be sharp in a region of the form $\{v \geq 2u + R^*\}$.

Remark 31. It is also possible to prove an alternative to (3.1.9), c.f. Theorem 3.8.1 in section 3.8. Essentially, if we require more point-wise decay of the initial data, we can prove that $r^2 \partial_v \psi$, $r^2 \partial_v (r^2 \partial_v \psi)$ etc. admit a finite limit on \mathcal{I}^+ , say in the gauge $A_v = 0$.

Broadly speaking, the present paper contains three different new ingredients : a non-degenerate energy boundedness statement, an integrated local energy decay Morawetz estimate and a hierarchy of r^p weighted estimates.

The presence of an interaction between the Maxwell charge and the scalar field renders the problem non-linear, which makes the estimates very coupled. In particular, we need to prove energy boundedness and the Morawetz estimate together, and the r^p estimates hierarchy also depends on them.

This coupling represents a major difficulty that we overcome proving that the charge is small. Even then however, the absorption of the interaction term requires great care. This is essentially due to the presence of a non-decaying quantity, the charge Q , that is not present for the uncharged problem or on Minkowski, c.f. sections 3.1.2, 3.1.2 and to the criticality of the equations, c.f. section 3.1.2 and the introduction.

⁹Note that $e \neq 0$ in general.

¹⁰Note that estimate (3.1.6) implies a limiting point-wise decay rate of $\frac{3}{2}$ as $q_0 |e| \rightarrow 0$ for ϕ and $D_v \phi$ on the event horizon.

¹¹The charge Q from the Maxwell equation, should not be confused with the charge of the black hole ρ . Because the problem is not coupled with gravity, these are different objects, in contrast to the model of [78]. For definitions, c.f. also section 3.2.3.

¹²At least for the 0th order term of the Taylor expansion in $q_0 e$.

Remark 32. Following the conjecture of [41], one expects that outside spherical symmetry, the spherically symmetric mode dominates the late time behaviour like in the uncharged case, c.f. Remark 36. One can hope that some of the main ideas of the present paper can be adapted to the case where no symmetry assumption is made. In particular — as it can be seen in the statement of Theorem 3.3.3 — the decay of the energy is **totally independent** of point-wise bounds which do not propagate easily without any symmetry assumption.

Remark 33. One of the novelty of the present work is to give an asymptotic expansion¹³ of the decay rate in terms of $q_0 e$, as $e \rightarrow 0$, c.f. the Taylor expansion (3.6.21). This was not present in previous work [4], [54], [80] precisely because on Minkowski space-time $e = 0$ so the long time effect of the charge is not as determinant.

Remark 34. It should be noted that everything said in the present paper also works for the case of a spherically symmetric charged scalar field on a Schwarzschild black hole, i.e. when $\rho = 0$.

3.1.2 Review of previous work and motivation

The motivations to study the Maxwell-Charged-Scalar-Field model are multiple.

First charged scalar fields trigger a lot of interest in the Physics community, sometimes in connection with problems of Mathematical General Relativity, as an example see the recent [31], which discusses strong cosmic censorship for cosmological space-times in the presence of a charged scalar field.

Second, the difficulty of this problem, due to its criticality with respect to decay at space-like infinity, c.f. section 3.1.2 demands a certain robustness in the estimates. Therefore, the methods of proof may be used in different, potentially more complicated situations where traditional strategies are insufficient.

Finally, if the estimates of the paper can be transposed to the problem where the Maxwell and scalar fields are coupled with gravity, this proves the stability of Reissner–Nordström black holes against charged perturbations. Additionally studying this model on black holes space-time can be considered as a first step towards understanding strong cosmic censorship and gravitational collapse for charged scalar fields in spherical symmetry, which may have different geometric characteristics from its uncharged analogue, see the introduction and section 3.1.2.

The goal of this section is to review previous works related to the problem of the present manuscript. This allows us to motivate the problem and to compare our results to what already exists in the literature.

The latter is the sole object of section 3.1.2 where we express the conjectured asymptotic behaviour of charged scalar fields, obtained by heuristic considerations in [41]. The former is the object of section 3.1.2, which summarizes the conditions to apply the results of [78], as one of the main motivation for the study of the exterior we perform.

The previous known results for this model are essentially all proved on Minkowski space-time, although outside spherical symmetry¹⁴. They either count crucially on conformal symmetries, a method that cannot be generalized easily to black hole space-times, or on the fact that the charge in the Maxwell equations has to tend to 0 towards time-like infinity. This fact greatly simplifies the analysis on Minkowski but is not true on black hole space-times because the charge asymptotes a finite generically non-zero value e . These works are discussed in section 3.1.2.

Finally in section 3.1.2, we review some of the numerous works on wave equations on black holes space-time. This is the uncharged analogue $q_0 = 0$ of the equations we study. This is the occasion to review the r^p method which is central to our argument and its application to proving exact Price’s law tail in [3].

Conjectured asymptotic behaviour of weakly charged scalar fields on black hole space-times

We now state the expected asymptotics for a charged scalar field on the event horizon of asymptotic flat black hole space-times. The Physics literature is surprisingly scarce. We base this section on [41], which is a heuristics-based work and the subsequent papers of the same authors. They state inverse polynomial decay estimates for the charged scalar field on Reissner–Nordström space-time when the asymptotic charge of the Maxwell equation is arbitrarily close to 0.

It is argued that the limiting decay rate 2 is a consequence of multiple scattering already present in flat space-time. Therefore, they suggest that the limit decay rate on black holes space-time —when the charge tends to zero— is the same as on Minkowski, which is consistent with the best decay rate on a constant r curves found in [54], c.f. section 3.1.2.

This slow decay stands in contrast to the faster rate prescribed by Price’s law for uncharged perturbations, c.f. Theorem 3.1.3. This is because for charged perturbations, the curvature term coming directly from the black hole metric decays faster than the term proportional to the charge of the scalar field. The work [41] was one of the first to notice, in the language of Physics, that charged scalar hairs decay slower than neutral ones.

¹³Which is probably not sharp.

¹⁴Outside spherical symmetry, dynamics of the Maxwell equation notably are much richer and the Maxwell field cannot be reduced to the charge, unlike in the present paper.

From an heuristic argument, one can understand from the wave equation 3.2.6 why the limit decay rate is 2 : we state the charged scalar field equations in spherically symmetry on a (M, ρ) Reissner–Nordström background :

$$D_u(D_v\psi) = \frac{\Omega^2}{r^2}\psi \left(iq_0Q - \frac{2M}{r} + \frac{2\rho^2}{r^2} \right),$$

while its uncharged analogue when $q_0 = 0$ is

$$\partial_u(\partial_v\psi) = \frac{\Omega^2}{r^3}\psi \left(-2M + \frac{2\rho^2}{r} \right).$$

If we expect that the radiation field ψ and the charge Q are bounded, we can infer that the charged equations may give a t^{-2} decay of ϕ on a constant r curve for compactly supported data, while the uncharged analogue gives t^{-3} decay, because of the different r power. This is because for asymptotically flat hyperbolic problems, we expect the r decay at spatial infinity to be translated into t decay towards time-like infinity.

It should also be noted that, since higher angular modes are expected to decay **faster** than the spherically symmetric average, the decay of scalar fields without any symmetry assumption is expected to be the same.

Another interesting discovery made in [41] is the oscillatory behaviour of the scalar field on the event horizon. This is connected to the fact that the scalar field is complex, otherwise the dynamics of the Maxwell equation is trivial as it can be seen in equation 3.1.1. This makes the proof of decay much more delicate than in the uncharged case, c.f. section 3.1.2 for a discussion.

The result of [41] can be summed up as follows :

Conjecture 3.1.2 (Asymptotic behaviour of weakly charged scalar fields, [41]). *Let (ϕ, F) be a solution of the Maxwell-Charged-Scalar field system **with no particular symmetry assumption** on a Reissner–Nordström space-time, and define the Maxwell charge Q with the relationship $F_{uv} = \frac{2Q\Omega^2}{r^2}$.*

Suppose that the data for $|\phi|$ is sufficiently decaying towards spatial infinity.

Denote the asymptotic charge $e = \lim_{v \rightarrow +\infty} Q|_{\mathcal{H}^+}(v)$.

Let $\epsilon > 0$. Then there exists $\delta > 0$, such if $q_0|e| < \delta$ we have ¹⁵ on the event horizon of the black hole, parametrized by an advance time coordinate v , as defined in section 3.2.1 :

$$\phi|_{\mathcal{H}^+}(v) \sim \Gamma_0 \cdot e^{iq_0e\frac{r^*}{r_+}} \cdot v^{-2+\eta(q_0e)}, \quad (3.1.11)$$

$$\psi|_{\mathcal{I}^+}(u) \sim \Gamma'_0 \cdot \left(\frac{u}{v}\right)^{iq_0e} \cdot u^{-1+\eta(q_0e)}, \quad (3.1.12)$$

$$\phi|_{\gamma}(t) \sim \Gamma''_0 \cdot t^{iq_0e} \cdot v^{-2+\eta(q_0e)}, \quad (3.1.13)$$

as $v \rightarrow +\infty$ and where $\Gamma_0, \Gamma'_0, \Gamma''_0$ are constants, $0 < \eta(q_0e) < \epsilon$ and γ is a far-away curve on which $\{t \sim u \sim v \sim r\}$, defined in section 3.2.

This also implies ¹⁶ that the energy on say the V foliation of section 3.2.4, behaves like

$$E(u) \sim E_0 \cdot u^{-3+2\eta(q_0e)}, \quad (3.1.14)$$

where E_0 is a constant.

Remark 35. Note that the decay of the charged scalar field depends on the dimension-less quantity q_0e only.

Remark 36. Notice also that the conjecture of [41] does not imply any spherical symmetry assumption. It is actually argued that — due to a better decay of the higher angular modes — generic solutions decay as the same rate as spherically symmetric ones.

The present paper mostly solves this conjecture : in particular we are able to prove the upper bounds corresponding to (3.1.12), (3.1.13) and (3.1.14). Moreover, we give a Taylor expansion ¹⁷ $\eta(q_0e) = O(\sqrt{q_0|e|})$.

The upper bound corresponding to (3.1.11) is more difficult to prove due to the degeneration of r weights in the bounded r region. We indeed prove

$$r^{\frac{1}{2}}|\phi|(u, v) \lesssim (E(u))^{\frac{1}{2}} \lesssim u^{-\frac{3}{2}+\eta(q_0e)},$$

which gives the optimal $t^{-2+\eta(q_0e)}$ decay on γ but only $v^{-\frac{3}{2}+\eta(q_0e)}$ on \mathcal{H}^+ . While this issue can be circumvented for the uncharged scalar field using the better decay of the derivative, such a strategy is out of reach

¹⁵(2.1.23) has to be understood as $\lim_{u \rightarrow +\infty} \phi(u, v) e^{-iq_0e\frac{r^*}{r_+}} \sim_{v \rightarrow +\infty} \Gamma_0 \cdot v^{-2+\eta}$ and (3.1.12) has to be understood as $\lim_{v \rightarrow +\infty} \psi(u, v) v^{iq_0e} \sim_{u \rightarrow +\infty} \Gamma'_0 \cdot u^{-1+\eta+iq_0e}$.

¹⁶Although this is not explicitly stated in [41].

¹⁷A more precise version is found in (3.6.21).

here, c.f. section 3.1.2. Nonetheless, the strictly integral point-wise estimates on the event horizon are sufficient for future work on the interior of black holes, c.f. section 3.1.2.

The decay of charged scalar fields should be compared to its uncharged analogue prescribed by Price's law, which is faster :

Theorem 3.1.3 (Price's law, [3], [25], [58], [61], [68], [77]). *Let ϕ be a finite energy solution of the wave equation on a Reissner–Nordström space-time.*

*Then ϕ is bounded everywhere on the space-time and **generically** decays in time on the event horizon parametrized by the coordinate v of section 3.2.1 :*

$$\phi|_{\mathcal{H}^+}(v) \sim_{\infty} \Gamma_0 \cdot v^{-3}, \quad (3.1.15)$$

where \sim_{∞} denotes the numerical equivalence relation as $v \rightarrow +\infty$ for two functions and their derivatives at any order and $\Gamma_0 \neq 0$ is a constant.

Towards the gravity coupled model in spherical symmetry

In this section, we mention some of the consequences of a future adaptation of this work to the case of the Einstein–Maxwell-Charged-Scalar-Field equations in spherical symmetry.

If all carries through, this proves in particular the **non-linear stability of Reissner–Nordström** space-time against small charged perturbations.

Additionally, one can understand the **interior structure** of black holes for this model. To that effect, we briefly summarize the results of [78] on the black hole interior for the Einstein–Maxwell-Charged-Scalar-Field equations in spherical symmetry. In particular, if the results of the present paper can be extrapolated to the gravity coupled case, it would give interesting information on the structure of **one-ended black holes**, not modelled by the uncharged analogous equations.

The Einstein–Maxwell-Klein-Gordon equations, whose Einstein–Maxwell-Charged-Scalar-Field is a particular case where the constant $m^2 = 0$, can be formulated as

$$Ric_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{KG}, \quad (3.1.16)$$

$$\mathbb{T}_{\mu\nu}^{EM} = g^{\alpha\beta}F_{\alpha\nu}F_{\beta\mu} - \frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g_{\mu\nu}, \quad (3.1.17)$$

$$\mathbb{T}_{\mu\nu}^{KG} = \Re(D_{\mu}\phi\overline{D_{\nu}\phi}) - \frac{1}{2}(g^{\alpha\beta}D_{\alpha}\phi\overline{D_{\beta}\phi} + m^2|\phi|^2)g_{\mu\nu}, \quad (3.1.18)$$

$$\nabla^{\mu}F_{\mu\nu} = iq_0 \frac{(\phi\overline{D_{\nu}\phi} - \overline{\phi}D_{\nu}\phi)}{2}, \quad F = dA, \quad (3.1.19)$$

$$g^{\mu\nu}D_{\mu}D_{\nu}\phi = m^2\phi. \quad (3.1.20)$$

We then have the following result :

Theorem 3.1.4 ([78]). *Let (M, g, F, ϕ) a spherically symmetric solution of the Einstein–Maxwell-Klein-Gordon system. Suppose that for some $s > \frac{1}{2}$ the following bounds hold, for some advanced time¹⁸ coordinate v on the event horizon :*

$$|\phi(0, v)|_{|\mathcal{H}^+} + |D_v\phi(0, v)|_{|\mathcal{H}^+} \lesssim v^{-s}. \quad (3.1.21)$$

Then near time-like infinity, the solution remains regular¹⁹ up to its Cauchy horizon.

If in addition $s > 1$ then the metric extends continuously across that Cauchy horizon.

If moreover the following lower bound on the energy holds , for a p such that $2s - 1 \leq p < \min\{2s, 6s - 3\}$

$$v^{-p} \lesssim \int_v^{+\infty} |D_v\phi|_{|\mathcal{H}^+}^2(0, v')dv', \quad (3.1.22)$$

then the Cauchy horizon is C^2 singular²⁰.

Hence the metric is (locally) C^2 inextendible across the Cauchy horizon.

¹⁸This v is the v coordinate defined in section 3.2.1, although in the uncoupled case a gauge choice is necessary, c.f. [78].

¹⁹More precisely, the Penrose diagram -locally near timelike infinity- of the resulting black hole solution is the same as Reissner–Nordström's .

²⁰A C^2 invariant quantity blows up, namely $Ric(V, V)$ where V is a radial null geodesic vector field that is transverse to the Cauchy horizon.

Therefore, if the results of the present manuscript can be extended to the gravity coupled problem, it would imply the continuous extendibility of the metric, at least for small enough data (in particular small initial charge).

Moreover, if we assume that an energy boundedness statement and an integrated local energy estimate ²¹ hold, then we can prove the continuous extendibility result for a larger class of initial data, namely for an initial asymptotic charge ²² in the range $q_0|e_0| \in [0, 0.8267)$.

This is relevant to the so-called Strong Cosmic Censorship Conjecture. Its C^k formulation states that for generic admissible ²³ initial data, the maximal globally hyperbolic development is inextendible as a C^k Lorentzian manifold. While its continuous formulation — for $k = 0$ — is often conjectured, it has been disproved in the context of the Einstein–Maxwell–**Uncharged**-Scalar-Field black holes in spherical symmetry, c.f. [19], [20], [25] and more recently for the Vacuum Einstein equation with no symmetry assumption in ²⁴ the seminal work [24]. Roughly, the continuous formulation of Strong Cosmic Censorship is false in these contexts because one expects dynamic rotating or charged black hole interiors to admit a null boundary — the Cauchy horizon — over which tidal deformations are finite. Therefore, provided that the decay rate assumed in (3.1.21) can be proved for Cauchy data, the continuous extendibility result of Theorem 3.1.4 disproves the continuous version of Strong Cosmic Censorship for the Einstein–Maxwell–**Charged**-Scalar-Field black holes in spherical symmetry.

While the (strongest) continuous version of the conjecture is false, the (weaker) C^2 formulation of Strong Cosmic Censorship has been proven for Einstein–Maxwell–**Uncharged**-Scalar-Field spherically symmetric black holes in the seminal works [57], [58]. For the analogous Einstein–Maxwell–**Charged**-Scalar-Field model, provided one can prove that (3.1.22) holds for generic data, the result of Theorem 3.1.4 proves the C^2 inextendibility of the metric along a part of the Cauchy horizon, near time-like infinity. This should be thought of as a first step towards the proof of the C^2 formulation of Strong Cosmic Censorship in the charged case, c.f. [78] for a more extended discussion.

Remark 37. The case $k = 2$, i.e. the C^2 inextendibility property of the metric is of particular physical interest. Indeed, classical solutions of the Einstein equations are considered in the C^2 class : therefore, a C^2 inextendibility property implies the impossibility to extend the metric as a classical solution of the Einstein equation. On the other hand, in principle the solution can still be extended as a weak solution of the Einstein equation, for which we only require the Christoffel symbols to be in L^2 . An interesting but unexplored direction would be to prove the equivalent of Strong Cosmic Censorship in the Sobolev H^1 regularity class, that would also exclude extensions as weak solutions of the Einstein equation, in that sense.

For a discussion on other motivations to study the Einstein–Maxwell–Charged-Scalar-Field model in spherical symmetry, we refer to section 1.2.1 of [78] and the pages 23-24, section 1.42 of [24].

Previous works on the Maxwell-Charged-Scalar-Field

We now turn to the study of the Maxwell-Charged-Scalar-Field model. It should be noted that, even in spherical symmetry, the equations are **non-linear**.

Although this problem on Minkowski space-time has received a lot of attention in the last decade, it should be noted that the only quantitative and rigorous result for that model ²⁵ on black hole space-times was derived in [78], for the black hole interior (c.f. section 3.1.2). We also mention that [47] contains many interesting preliminary results and geometric arguments for the Einstein–Maxwell–Klein–Gordon model in spherical symmetry, although no quantitative study is carried out, either in the interior or the exterior of the black hole.

In the rest of this section, we review previous works on flat space-time.

The first step is to study the global existence problem for variously regular data. This question has been extensively studied, starting with the global existence for smooth data first established by Eardley and Moncrief [33], [34]. Since then, various works have made substantial progress in different directions [17], [43], [45], [48], [49], [60], [64], [70].

After global existence, the next natural question is to study the asymptotic behaviour of solutions. While this problem was pioneered by Shu [74], one of the first modern result is due to Lindblad and Sterbenz [54] who establish point-wise inverse polynomial bounds for the scalar and the Maxwell Field, provided the data is small enough. More precisely they prove the following :

Theorem 3.1.5 (Lindblad-Sterbenz, [54]). *Consider asymptotically flat energy Cauchy data, namely a scalar field/Maxwell form couple (ϕ_0, F_0) such that for some $\alpha > 0$ and $s > 0$,*

²¹Which we prove in the present paper for small enough data.

²²We remind the reader that a definition of the initial asymptotic charge is present in section 3.2.3.

²³For a precise formulation of Strong Cosmic Censorship, where the notion of admissible data is explained, we refer to [57], [58].

²⁴The authors of [24] assume the widely-believed stability of Kerr black holes and prove that it implies the continuous extendibility of the metric.

²⁵More exactly, for the **Einstein**–Maxwell–Charged-Scalar-Field and in the context of spherical symmetry.

$$\mathcal{E}^{EM+SF} := \|r^\alpha \phi_0\|_{H^s(\mathbb{R}^3)} + \|r^\alpha D_t \phi_0\|_{H^s(\mathbb{R}^3)} + \|r^\alpha F_0\|_{H^s(\mathbb{R}^3)} < \infty.$$

We also denote the asymptotic initial charge $e_0 := \lim_{r \rightarrow +\infty} \frac{r^2 F_0(\partial_u, \partial_v)}{2}$.
For every $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\mathcal{E}^{EM+SF} < \delta$$

then we have the following estimates, for u and v large enough

$$|\phi|(u, v) \lesssim v^{-1} u^{-1+\epsilon}, \quad (3.1.23)$$

$$|\psi|(u, v) \lesssim u^{-1+\epsilon}, \quad (3.1.24)$$

$$|D_v \psi|_{\{v \geq 2u+R^*\}}(u, v) \lesssim v^{-2+\epsilon}, \quad (3.1.25)$$

$$|Q|(u, v) \lesssim |e_0| \cdot 1_{\{u \leq u_0(R)\}} + u^{-1+\epsilon}. \quad (3.1.26)$$

As explained in pages 9 and 10 of [54], the Maxwell-Charged-Scalar-Field equations are **critical** with respect to decay at $r \rightarrow +\infty$. This makes the estimates very tight, as opposed for instance to the Einstein equations, which allow for more room. This is due to the presence of a non-linear term that scales exactly like the dominant terms in the energy while its sign cannot be controlled. This very fact relates to the dependence of the decay rate on the asymptotic charge e , as we explain in section 3.1.1.

To overcome this difficult, the authors of [54] make use of a conformal energy, a fractional Morawetz estimate and some L^2/L^∞ Strichartz-type estimates. These arguments rely on the specific form of the Minkowski metric and are difficult to transpose to any black hole space-time. We also mention the work [4] where similar results are derived but with a simpler proof and the very recent [42] that treats the more general case of Minkowski perturbations, still for the gravity uncoupled case.

Recently, this problem has been revisited by Yang [80] using the modern r^p method invented in [28] to establish decay estimates. While the proven decay was weaker than that of [54], it made the decay of energy more explicit. More importantly, the proof requires weaker hypothesis. In particular, while the scalar field initial data need to be small, the Maxwell field is allowed to be large. This is summed up by :

Theorem 3.1.6 (Yang, [80]). *Consider asymptotically flat energy Cauchy data, namely a scalar field/Maxwell form couple (ϕ_0, F_0) such that for some $\alpha > 0$ and $s > 0$,*

$$\mathcal{E}^{SF} := \|r^\alpha \phi_0\|_{H^s(\mathbb{R}^3)} + \|r^\alpha D_t \phi_0\|_{H^s(\mathbb{R}^3)},$$

$$\mathcal{E}^{EM} := \|r^\alpha F_0\|_{H^s(\mathbb{R}^3)} < \infty.$$

We also denote the asymptotic initial charge $e_0 := \lim_{r \rightarrow +\infty} \frac{r^2 F_0(\partial_u, \partial_v)}{2}$.
For every $\epsilon > 0$, there exists $\delta > 0$ such that if

$$\mathcal{E}^{SF} < \delta$$

then we have the following estimates, for u and v large enough

$$r^{\frac{1}{2}} |\phi|(u, v) \lesssim u^{-1+\epsilon}, \quad (3.1.27)$$

$$|\psi|(u) \lesssim u^{-\frac{1}{2}+\epsilon}, \quad (3.1.28)$$

$$|D_v \psi|_{\{v \geq 2u+R^*\}}(u, v) \lesssim v^{-1+\epsilon}, \quad (3.1.29)$$

$$|Q - e_0 \cdot 1_{\{u \leq u_0(R)\}}|(u, v) \lesssim u^{-1+\epsilon}, \quad (3.1.30)$$

$$E(u) \lesssim u^{-2+2\epsilon}, \quad (3.1.31)$$

where $E(u)$ is the energy of the scalar field, similar to the one defined in section 3.2.5, and $u_0(R)$ is defined in section 3.2.4.

While the r^p method utilized by Yang is very robust, his work cannot be generalized easily to the case of black hole space-times. This is because on Minkowski space-time, the long range effect of the charge manifests itself only in the exterior of a fixed forward light cone, as it can be seen in estimates (3.1.26), (3.1.30). In contrast to the charge on black hole space-times that always admits a ²⁶ limit e , the charge on Minkowski space-time

²⁶The future asymptotic charge e , defined as the limit of Q towards infinity on \mathcal{H}^+ , \mathcal{I}^+ or any constant r curve.

tends to 0 towards time-like infinity. As a result ²⁷, studying compactly supported initial data on a black hole space-time is not a priori more difficult than studying data that decay sufficiently towards spatial infinity. The strategy employed in [80] is to count on the u decay of the Maxwell term to absorb the interaction terms in the r^p weighted estimate for the scalar field. However, because of the existence of a **non-zero** asymptotic charge, this fails on any black hole space-time and instead we need to rely on the smallness of the charge in the present paper.

It should be noted that these works [4], [54], [74], [80] are treating the Maxwell-Charged-Scalar-Field outside spherical symmetry, which makes the dynamics of the Maxwell field much richer than for the symmetric case considered in the present paper where the Maxwell form is reduced to the charge.

We also mention the extremely recent [81] extending the results of [80], with better decay rates and remarkably for large Maxwell field and large scalar field.

Previous works on wave equations on black hole space-times and r^p method

The wave equation on black hole space-times has been an extremely active field of research over the last fifteen years, c.f. [2], [3] [25], [27], [28], [29], [30], [56], [58], [63], [77] to cite a few. This is the uncharged analogue — when $q_0 = 0$ — of the problem we study in the present paper.

It is related to one of the main open problems of General Relativity, the question of black holes stability for the Einstein equations without symmetries, c.f. [23], [51], [46] for some recent remarkable advances in various directions.

Subsequently, the black holes interior structure could be inferred from the resolution of this problem, c.f. [19], [20], [24], [57], [78]. In addition to the analyst curiosity to understand the wave equation in different contexts, these works also aim at exploring toy models. This may give valuable insight on the mentioned problems coming from Physics. This is also one of the goals of the present paper.

In this section, we review some results that are related to the decay of scalar fields on spherically symmetric space-times, which is the uncharged version of the model considered in the present manuscript. We are going to mention in particular the different uses of the new r^p method, pioneered in [28].

After the broader discussion of section 3.1.2, we would like to emphasize how the quantitative late time behaviour of scalar fields impacts the geometry of black holes.

This was first understood in [19], [20] in the context of Einstein–Maxwell–Uncharged–Scalar–Field in spherical symmetry. It is proved that Price’s law of Theorem 3.1.3 implies that generic black holes for this model possess a Cauchy horizon over which the metric is continuously extendible. Therefore, from [25], the continuous formulation of Strong Cosmic Censorship conjecture is false for this model.

This insight, provided by the toy model, gave a good indication about black holes that satisfy the Einstein equation without symmetry assumptions. This is best illustrated by the remarkable and recent work of Dafermos and Luk [24], where it is proven that the decay of energy-like quantities on the event horizon implies the formation of a C^0 regular Cauchy horizon, **with no symmetry assumption**.

Once the first step — namely understanding (almost) sharp upper bounds — has been carried out, the next step is to understand lower bounds. This is the object of the work [58] for the Einstein–Maxwell–Uncharged–Scalar–Field in spherical symmetry, in which L^2 lower bounds are proved on the event horizon. Then, in [57], it is shown that these lower bounds propagate to the interior of the black hole. The result implies that generic black holes possess a C^2 singular ²⁸ Cauchy horizon. This means that the C^2 formulation of Strong Cosmic Censorship conjecture is true for this model. Note that the upper bounds of [25] are extremely useful in the instability proof of [58].

The main intake of this short review is the idea that the fine geometry of the black hole is determined by the decaying quantities on the event horizon, which makes the study of black hole exteriors all the more important. The problem gathers both ²⁹ stability and instability features (c.f. also [78]) : stability of scalar fields implies the formation of a C^0 regular Cauchy horizon while instability ensures its C^2 singular nature.

Now we start a short review of the r^p method from [28]. This should be thought of as a new vector field method, which makes use of r weights as opposed to conformal vector fields that were used more traditionally, like in [54]. The objective is to prove that energy decays in time. Nevertheless, it is a well-known fact that the energy on constant time slice is constant and **does not** decay. The idea is to consider instead the energy on a J -shaped foliation $\Sigma_\tau = \{r \leq R, \tau' = \tau\} \cup \{r \geq R, u = u_R(\tau)\}$ as depicted in [28] or [80]. We can then establish a hierarchy of r^p weighted energy from which time decay can be obtained, using the pigeon-hole principle.

²⁷In [54] and [80], it has been argued that it is not sufficient to study compactly supported data to understand how decaying data behave. This is because the main charge term cancels for the former and not for the latter. This fact is not any longer true on a black hole space-time.

²⁸Precisely, there exists a geodesic vector field ∂_V that is transverse to the Cauchy horizon and regular, so that $Ric(\partial_V, \partial_V) = \infty$.

²⁹This apparent paradox is resolved once one realizes that stability estimates are proven for a very weak norm, whereas instability estimates originate from a blow-up of stronger norms. c.f. [57] and [78].

In this article we are going to consider a V -shaped foliation instead $\mathcal{V}_u = \{r \leq R, v = v_R(u)\} \cup \{r \geq R, u' = u\}$, c.f. Figure 3.1 and section 3.2.4. This is purely for the sake of simplicity and does not change anything.

Theorem 3.1.7 (r^p method for $0 \leq p \leq 2$, Dafermos-Rodnianski, [28]). *Let ϕ be a finite energy solution of*

$$\square_g \phi = 0,$$

where g is a Schwarzschild exterior metric. The following hierarchy of r^p weighted estimates is true : for all $u_0(R) \leq u_1 < u_2$:

$$\begin{aligned} \int_{u_1}^{u_2} E_1[\psi](u) du + E_2[\psi](u_2) &\lesssim E_2[\psi](u_1), \\ \int_{u_1}^{u_2} E_0[\psi](u) du + E_1[\psi](u_2) &\lesssim E_1[\psi](u_1), \end{aligned}$$

where $E_q[\psi]$ is defined in section 3.2.5. Therefore³⁰ the following estimates are true :

$$r^{\frac{1}{2}} |\phi|_{|\{r \geq R\}}(u, v) \lesssim u^{-1}, \quad (3.1.32)$$

$$|\psi|(u, v) \lesssim u^{-\frac{1}{2}}, \quad (3.1.33)$$

$$E(u)[\phi] \lesssim u^{-2}, \quad (3.1.34)$$

where $E(u) = E(u)[\phi]$ is defined in section 3.2.5.

The method was then subsequently extended to the case of n -dimensional Schwarzschild black holes for $n \geq 3$ by Schlue in [72]. The main novelty is the existence of a better energy decay estimate for $\partial_t \phi$ which proves a better point-wise for ϕ as well : for all $\epsilon > 0$

$$r^{\frac{1}{2}} |\phi|_{|\{r \geq R\}}(u, v) \lesssim u^{-\frac{3}{2} + \epsilon}, \quad (3.1.35)$$

$$|\psi|(u, v) \lesssim u^{-1 + \epsilon}, \quad (3.1.36)$$

$$E(u)[\partial_t \phi] \lesssim u^{-4 + 2\epsilon}. \quad (3.1.37)$$

The work of Moschidis [63], which generalizes the r^p method and point-wise decay estimates to a very general class of space-times, should also be mentioned.

The r^p hierarchy has been subsequently extended to $p < 5$ in [2]. This has led to the proof of the (almost) optimal decay for the scalar field and its derivatives on Reissner–Nordström space-time.

Theorem 3.1.8 (r^p method for $0 \leq p < 5$, Angelopoulos-Aretakis-Gajic, [2]). *Let ϕ be a compactly supported³¹ solution of*

$$\square_g \phi = 0,$$

where g is a sub-extremal Reissner–Nordström exterior metric.

For all $0 \leq p < 5$, the following hierarchy of r^p weighted estimates is true : for all $u_0(R) \leq u_1 < u_2$:

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + E_p[\psi](u_2) \lesssim E_p[\psi](u_1).$$

For all $2 \leq q < 6$, the following hierarchy of r^q weighted estimates is true : for all $u_0(R) \leq u_1 < u_2$:

$$\int_{u_1}^{u_2} E_{q-1}[\partial_t \psi](u) du \lesssim E_q[\partial_r \psi](u) + E_{q-2}[\psi](u) + E(u)[\phi] + E(u)[\partial_t \phi].$$

For all $6 \leq s < 7$, the following hierarchy of r^s weighted estimates is true : for all $u_0(R) \leq u_1 < u_2$:

$$\int_{u_1}^{u_2} E_{s-1}[\partial_r \psi](u) du + E_s[\partial_r \psi](u_2) \lesssim E_s[\partial_r \psi](u) + E_{s-2}[\psi](u) + E(u)[\phi] + E(u)[\partial_t \phi].$$

Therefore for all $\epsilon > 0$, we have the following energy decay

$$E(u)[\phi] \lesssim v^{-5 + \epsilon}, \quad (3.1.38)$$

$$E(u)[\partial_t \phi] \lesssim v^{-7 + \epsilon}. \quad (3.1.39)$$

³⁰We must then crucially make use of a Morawetz estimate and of the energy boundedness on Schwarzschild space-time.

³¹More precisely, with sufficient initial decay that the Newman–Penrose constant vanishes.

Remark 38. This gives an alternative proof of estimate 3.1.15 for the linear problem, i.e. the wave equation on a Reissner–Nordström background. Actually, after commuting twice with ∂_t , one can also obtain the estimate $|\partial_v \phi|_{\mathcal{H}^+} \lesssim v^{-4+\epsilon}$. This is therefore a better estimate than in [25], although obtained only for the linear problem.

This strategy to prove the almost optimal energy decay is the first step towards understanding lower bounds. The second step, carried out in [3], is to identify a conservation law that allows for precise estimates.

Corollary 3.1.9 (Angelopoulos-Aretakis-Gajic, [3]). *With the same hypothesis as for Theorem 3.1.8, for every $r_0 > r_+$, where r_+ is the radius of the black hole, there exists a constant $C > 0$ and $\epsilon > 0$ such that on a $\{r = r_0\}$ curve :*

$$\phi(r_0, t) = \frac{C}{t^3} + O(t^{-3-\epsilon}).$$

In the present paper and although we do not explicitly use any of the techniques of [2] and [3], we intend to pursue the same program for the **non-linear** Maxwell-charged scalar field equations. In our case the decay mechanism is more complicated, in particular the decay rate is not universal and depends on the asymptotic charge e .

We find that the maximum p for which we can derive a hierarchy of r^p weighted energies (with no loss) is $2 < p(e) < 2 + \sqrt{1 - 4q_0|e|}$, described in (3.6.21). Since $p(e) \rightarrow 3$ as $e \rightarrow 0$, we reach the optimal energy decay rate predicted by [41] as $q_0 e$ tends to 0, at least for the energy. However, we cannot retrieve the optimal point-wise bound on the event horizon from this : our method only proves $|\phi(v)| \lesssim v^{-s}$, $s \rightarrow \frac{3}{2}$ as $q_0 e$ tends to 0. This is because it is not clear whether $E(u)[D_t \phi]$ enjoys a better u decay than $E(u)[\phi]$, in contrast to the uncharged problem as it can be seen in equations (3.1.37), (3.1.39).

The physical explanation behind this phenomenon is the presence of an oscillatory term, because the scalar field is complex, as explained in [41]. This makes the analysis more difficult. Therefore, while in the uncharged problem ϕ and $\partial_t \phi$ decay like v^{-3} and v^{-4} respectively, it is not certain that an analogous property is true in the charged case.

Notice however that the bounds we prove on the charge —depending only on the energy— and the bounds towards null infinity are expected to be almost sharp. Moreover, the point-wise bounds on the event horizon are always strictly integrable, which is crucial to study the interior of the black hole, see section 3.1.2.

3.1.3 Methods of proof

We now briefly discuss some of the main ideas involved in the proofs.

Smallness of the charge

During the whole paper, we require $q_0 Q$ to be smaller than some constant. This smallness originates from that of the initial data. More precisely, we prove that provided the asymptotic initial charge e_0 and of the r weighted initial scalar field energies are small, then so is the charge, everywhere.

We explain heuristically why this is the case. Schematically, the Maxwell equation looks like

$|\partial Q| \lesssim r^2 |\phi| |D\phi|$. Then by using Cauchy-Schwarz with some Hardy inequality to control the zero order term in $|\phi|$, we see that the charge difference is roughly controlled by the r weighted energy \tilde{E}_1 (c.f. section 3.2.5 for a precise definition), which is itself bounded by its initial value $\tilde{\mathcal{E}}_1$. Therefore, broadly $|Q - e_0| \lesssim \tilde{\mathcal{E}}_1$ so Q is small if both $|e_0|$ and $\tilde{\mathcal{E}}_1$ are small.

This issue relying essentially on the boundedness of various energy-like quantity by the initial data, an estimate of the form $\tilde{E}_1(u) \lesssim \tilde{\mathcal{E}}_1$ suffices.

This is sensibly easier to prove than an estimate of the form $\tilde{E}_1(u_2) \lesssim \tilde{E}_1(u_1)$ for all $u_1 < u_2$, which we prove later and is required to prove decay, with the use of a pigeon-hole like argument.

This is why this charge smallness step is carried out first, as a preliminary estimate, before the much more precise versions later required to prove decay. This first step carried out in section 3.4 should be thought of as the analogue of a boundedness proof.

Doing so, we reduce the Cauchy problem to a characteristic initial value problem on the double null surface $\{u = u_0(R)\} \cup \{v = v_0(R)\}$. Therefore, section 3.4 is also the only part of the paper where estimates cover the whole space-time, including the region $\{u \leq u_0(R)\} \cup \{v \leq v_0(R)\}$. In later sections dealing with decay, we use the results of this first part and only consider a characteristic initial value problem on the domain $\mathcal{D}(u_0(R), +\infty)$, the complement of $\{u < u_0(R)\} \cup \{v < v_0(R)\}$.

Overview of decay estimates

Now we turn to the core of the present paper : decay estimates. As explained earlier, they rely on

1. Degenerate energy boundedness, c.f. section 3.1.3.

2. An integrated local energy decay, also called a Morawetz estimate, see section 3.1.3.
3. Non-degenerate energy boundedness, using the red-shift effect c.f. section 3.1.3.
4. A hierarchy of r^p weighted estimates, see section 3.1.3 .
5. A pigeon-hole principle like argument from which time decay can be retrieved for the un-weighted energy, using the last three estimates.
6. Point-wise decay estimates, using crucially the energy decay, c.f. section 3.1.3.

Steps 1, 2 and 3 are inter-connected and must be carried out all simultaneously, in contrast to the uncharged case (the wave equation) c.f. [26]. Step 4 and 5 are also connected and moreover rely crucially on the results of steps 2 and 3. The last step 6 requires the results of 4 and 5 together with some additional point-wise decay hypothesis of the scalar field Cauchy data.

The distinction between the degenerate and non-degenerate energy is due to the causal character of ∂_t , the Killing vector field which allows for energy conservation. While on Minkowski space-time ∂_t is everywhere time-like, this is not true on black hole space-times since ∂_t then becomes null on the event horizon. For this reason, the energy conserved by ∂_t is called degenerate. To obtain the so-called non-degenerate energy, which is the most natural to consider, the use of red-shift estimates is required, c.f. section 3.1.3 for a more precise description.

To prove time decay of the energy — one of the main objectives of this paper — we use the r^p method : from the boundedness of the r^p weighted energies, one can roughly retrieve time decay t^{-p} of the un-weighted energy.

For steps 1 to 4, we mainly use the vector field method, which is a robust technique to establish L^2 estimates with the use of geometry-inspired vector fields and the divergence theorem, c.f. section 3.10. The principal difficulty, when we apply the energy identity to a vector field X , is to absorb an interaction ³² term between the scalar field and the electromagnetic part of the form

$$\frac{q_0 Q}{r^2} \Im (\phi(X^v \overline{D_v \phi} - X^u \overline{D_u \phi})) .$$

It comes from the second term of the identity $\nabla^\mu (\mathbb{T}_{\mu\nu}^{SF} X^\nu) = \mathbb{T}_{\mu\nu}^{SF} \Pi_X^{\mu\nu} + F_{\mu\nu} X^\mu \mathcal{J}^\nu(\phi)$, c.f. (3.10.11) and section 3.10.

This term must be absorbed by a controlled quantity to close the energy identities of steps 1 to 4. However it has the same r weight as the positive main term controlled by the energy. The strategy is then to apply Cauchy-Schwarz inequality to turn this interaction term into a product of L^2 norms. Thereafter we use the smallness of $q_0 Q$ to absorb it. For a more precise description, c.f. for instance section 3.1.3.

Because the main term controlled by the energy is proportional to $|D\phi|^2$, we also need to use Hardy-type inequalities throughout the paper, to absorb any term proportional to $|\phi|^2$. These estimates are proven in section 3.2.

More details on each step are provided in the subsequent sub-sections.

Degenerate energy boundedness

We define the degenerate energy of the scalar field on our \mathcal{V}_u foliation by

$$E_{deg}(u) = \int_u^{+\infty} r^2 |D_u \phi|^2(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv,$$

c.f. section 3.2.4 and section 3.2.5.

We want to prove an estimate of the form $E_{deg}(u_2) \lesssim E_{deg}(u_1)$ for any $u_1 < u_2$. For this, we make use of the Killing vector field $T = \partial_t$ and notice that $\nabla^\mu (\mathbb{T}_{\mu\nu} T^\nu) = 0$ where $\mathbb{T} = \mathbb{T}^{SF} + \mathbb{T}^{EM}$, c.f. section 3.10.1 and section 3.10.2.

While the analogous boundedness estimate in the uncharged case of the wave equation is trivial, even on Reissner–Nordström space-time, this is in the charged case one of the technical hearts of the paper.

Indeed, the method we described above now gives rise to an equation of the form

$$\begin{aligned} E_{deg}(u_2) &+ \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + \int_{u_2}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_2)) du \\ &+ \int_{v_R(u_2)}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u_2, v) dv = E_{deg}(u_1) + \int_{u_1}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u_1, v) dv. \end{aligned} \quad (3.1.40)$$

³²This term is the source of the criticality with respect to r decay that we describe in the introduction and in section 3.1.2.

This is problematic, since the quadratic terms involving the charge do not decay, Q tending to a finite limit at infinity in the black hole case. This is of course in contrast to the Minkowski case where Q tends to 0 towards time-like infinity. However, there is a hope that the difference of such terms, e.g. a term like $\int_{v_R(u_2)}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u_2, v) dv - \int_{v_R(u_1)}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u_1, v) dv$ can be absorbed into the energy of the scalar field.

To prove this, we require the charge to be small and we have to estimate the difference carefully. More precisely, we need to transport some charge differences towards constant r curves and then to use the Morawetz estimate of step 3.

Integrated local energy decay

It has been known since [62] that an integrated local energy decay estimate — now called Morawetz estimate — is useful to prove time decay of the energy. This is classically an estimate roughly of the form $\int_{space-time} r^{-1-\delta} (|\phi|^2 + r^2 |D\phi|^2) \lesssim E(u)$ where $E(u)$ is the energy coming from ∂_t and $\delta > 0$ can be taken arbitrarily small. Note that this is a global estimate with sub-optimal r weights but involving all derivatives. We prove such an estimate in the step 3 but for $\delta > 0$ that can be actually large. This does not make the later proof of decay harder, since our argument — carried out in the r bounded region $\{r \leq R\}$ — is unaffected by the value of δ .

The Morawetz estimate is probably — together with the degenerate energy boundedness of step 1 — one of the most delicate point of the present paper. This is because in the charged case, the customary use of the vector field method with $f(r)\partial_{r^*}$ now involves a supplementary term of the form $error = \int_{space-time} q_0 Q \cdot f(r) \cdot \Im(\bar{\phi} D\phi)$ that was not present in the uncharged case. This creates additional difficulties :

1. the zero order term $A_0 = \int_{space-time} r^{-1-\delta} |\phi|^2$ cannot be controlled independently. This is in contrast to the uncharged case where one can first control $\int_{space-time} r^{-1-\delta'} \cdot r^2 |D\phi|^2$ for some $\delta' > \delta$ and then use this preliminary bound to finally control $\int_{space-time} r^{-1-\delta} |\phi|^2$, c.f. [58].
2. The integrand of $error$ decays in r at the same rate as the main controlled term, for any reasonable choice of $f(r)$. To absorb $error$ in the large r region, we require $r \cdot f'(r) \gtrsim |f(r)|$ as $r \rightarrow +\infty$. This is because, unlike on Minkowski space-time where Q tends to 0 towards time-like infinity, we can only rely on $|Q| \lesssim e$, where $e > 0$ is a (small) constant. This roughly gives an estimate of the form

$$\int_{space-time} f'(r) \cdot r^2 |D\phi|^2 \lesssim A_0 + q_0 e \int_{space-time} |f(r)| \cdot |\phi| \cdot |D\phi| \lesssim A_0 + q_0 e \int_{space-time} r |f(r)| \cdot |D\phi|^2, \quad (3.1.41)$$

where for the last inequality, we used Cauchy-Schwarz and the Hardy inequality roughly under the form $\int_{space-time} r^{-1} |f(r)| \cdot |\phi|^2 \lesssim \int_{space-time} r |f(r)| \cdot |D\phi|^2$. The condition $r \cdot f'(r) \gtrsim |f(r)|$ together with the smallness of $q_0 e$ then allows us to close the estimate, up to the zero order term A_0 . This line of thought suggests that $f(r) \approx -r^{-\delta}$, $\delta > 0$ is an appropriate choice.

3. On a black hole space-time, the zero order term A_0 is harder to control than on Minkowski space. This is because (3.1.41) can actually be written in a more precise manner as

$$\int_{space-time} f'(r) \cdot r^2 |D\phi|^2 + r^2 \square_g (\Omega^2 \cdot r^{-1} f(r)) \cdot |\phi|^2 \lesssim q_0 e \int_{space-time} r |f(r)| \cdot |D\phi|^2 + E(u). \quad (3.1.42)$$

For $f(r) = -r^{-\delta}$, we compute $r^2 \square_g (\Omega^2 \cdot r^{-1} f(r)) = r^{-\delta-1} P_{M,\rho}(r)$ where $P_{M,\rho}(r)$ is a second order polynomial ³³ in r^{-1} that is positive on $[r_+, r(\delta)] \cup [R(\delta), +\infty]$ and negative on $(r(\delta), R(\delta))$ for some $r_+ < r(\delta) < R(\delta)$. An analogous computation on Minkowski gives a strictly positive constant polynomial $P_{0,0}(r) = (\delta + 1)(\delta + 4)$.

To deal with these difficulties, we first need to prove an estimate for A_0 in a region $\{r_+ \leq r \leq R_0\}$ for R_0 close enough from r_+ , using the vector field $-\partial_{r^*}$ and the smallness of $q_0 e$. We then rely on the crucial but elementary fact that $R(\delta) \rightarrow r_+$ as $\delta \rightarrow +\infty$. Therefore, for δ **large** enough, we get a positive control of A_0 on $[R_0, +\infty]$ using (3.1.42). We conclude combining this with the estimate on $[r_+, R_0]$.

³³whose coefficients involve M and ρ , the parameters of the Reissner–Nordström black hole.

Non-degenerate energy boundedness and red-shift

We now define the non-degenerate energy of the scalar field on our \mathcal{V}_u foliation by

$$E(u) = \int_u^{+\infty} r^2 \frac{|D_u \phi|^2}{\Omega^2}(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv,$$

c.f. section 3.2.4 and section 3.2.5.

This is called non-degenerate precisely because on a fixed v line, $\Omega^{-2} \partial_u$ is a non-degenerate vector field across the event horizon $\{u = +\infty\}$. Therefore, we expect (and prove in step 6) a bound of the form $|D_u \phi| \lesssim \Omega^2$, consistent with the boundedness of the quantity $E(u)$. This also means that $D_u \phi = 0$ on the event horizon, which explains why ∂_u is degenerate and needs to be renormalized to obtain a finite limit.

Our goal is then to prove an estimate of the form $E(u_2) \lesssim E(u_1)$ for any $u_1 < u_2$.

For this, we use a so-called red-shift estimate, pioneered in [25], [26], [27]. In our context, it boils down to using the vector field method with $X = \Omega^{-2} \partial_u$. While this is not the hardest part of the paper, we still need to use the Morawetz estimate of step 3 to conclude, unlike in the uncharged case. This is because in our case, we must absorb a bulk term coming from the charge (c.f. section 3.1.3) into a controlled scalar field bulk term, while for the wave equation, no control of the bulk term is needed at this stage, c.f. [26].

Energy decay and r^p method

To prove time decay of the energy, we use the r^p method, pioneered in [28].

The idea is to prove the boundedness of r^p weighted energies

$$E_p(u) := \int_{v_R(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv,$$

for a certain range of p , ultimately responsible for time decay t^{-p} . Actually, we prove a hierarchical estimate of the form

$$\int_{u_1}^{u_2} E_{p-1}(u) du + E_p(u_2) \lesssim E_p(u_1) \lesssim 1.$$

This kind of estimate is obtained by applying the vector field method in a region $\{r \geq R\}$ with the vector field $r^p \partial_v$.

Thereafter applying the mean-value theorem or a pigeon-hole like argument, we can retrieve an estimate of the form $E_{p-1}(u) \lesssim u^{-1}$ and eventually $E(u) \lesssim u^{-p}$.

Now in the case of the Maxwell-Charged-Scalar-Field model, we also have to control an interaction term coming from the charge c.f. section 3.1.3. We now have an estimate of the form

$$\int_{u_1}^{u_2} E_{p-1}(u) du + E_p(u_2) \lesssim E_p(u_1) + \text{error},$$

where $\text{error} \approx q_0 e \int \int_{\{r \geq R\}} r^{p-2} \Im(\bar{\psi} D_v \psi)$ is the interaction term we mentioned earlier.

Due to the shape of this term, as we explained already, we need a Hardy inequality to absorb to $|\phi|$ into the energy term.

More explicitly we roughly estimate error using the following type of bounds :

$$\left| \int \int_{\{r \geq R\}} r^{p-2} \Im(\bar{\psi} D_v \psi) \right| \lesssim \left(\int \int_{\{r \geq R\}} r^{p_1} |\psi|^2 \right)^{\frac{1}{2}} \left(\int \int_{\{r \geq R\}} r^{p_2} |D_v \psi|^2 \right)^{\frac{1}{2}},$$

where $p_1 + p_2 = 2p - 4$, simply using Cauchy-Schwarz. We then apply a Hardy inequality to roughly find if $p_1 < -1$:

$$\left| \int \int_{\{r \geq R\}} r^{p-2} \Im(\bar{\psi} D_v \psi) \right| \lesssim |1 + p_1|^{-1} \left(\int \int_{\{r \geq R\}} r^{p_1+2} |D_v \psi|^2 \right)^{\frac{1}{2}} \left(\int \int_{\{r \geq R\}} r^{p_2} |D_v \psi|^2 \right)^{\frac{1}{2}}.$$

Now because $\int \int_{\{r \geq R\}} r^{p_2} |D_v \psi|^2 = \int_{u_1}^{u_2} E_{p_2}(u) du$, which is already controlled for $p_2 = p - 1$, the choice $(p_1, p_2) = (p - 3, p - 1)$ seems natural, c.f. section 3.6.2.

We then roughly need to absorb a term $q_0 |e| \cdot |2 - p|^{-1} \int_{u_1}^{u_2} E_{p-1}(u) du$ into $\int_{u_1}^{u_2} E_{p-1}(u) du$ which is essentially doable if $q_0 |e|$ is small but requires $p \in [0, 2 - \epsilon(q_0 e)]$ in particular $p < 2$. Calling p_0 the maximal p for which we can do this, we then essentially prove for $u > 0$ large :

$$E_{p_0-1}(u) \lesssim \frac{E_{p_0}(u)}{u}. \quad (3.1.43)$$

We employ this strategy in section 3.6.2. While the estimates we get are necessary to “start” the argument, they are insufficient to reach the best possible decay advertised in the theorems.

This is why in section 3.6.3, we adopt a completely different strategy. This time we chose $(p_1, p_2) = (p-4, p)$ for $p = p_0 + 1$. Then the error term we need to absorb is roughly ³⁴ of the form

$$q_0|e| \cdot |2 - p_0|^{-1} \left(\int_{u_1}^{u_2} E_{p-2}(u) du \right)^{\frac{1}{2}} \left(\int_{u_1}^{u_2} E_p(u) du \right)^{\frac{1}{2}} \lesssim q_0|e| \cdot |3 - p|^{-1} \left(\frac{E_{p-1}(u)}{u} \right)^{\frac{1}{2}} \left(\int_{u_1}^{u_2} E_p(u) du \right)^{\frac{1}{2}},$$

where we used the equation (3.1.43) and the fact that $p_0 = p - 1$.

Now we need to absorb the right-hand side into $\int_{u_1}^{u_2} E_{p-1}(u) du + E_p(u_2)$. This requires a Grönwall-like method, in which the r^p weighted energy experiences a controlled u growth : $E_p(u) \lesssim u^{2\epsilon}$. Eventually, we get $E(u) \lesssim u^{-p+2\epsilon}$ for some small ϵ .

The most delicate part of the proof is to chose $2 < p = p(e) < 3$ and $0 < \epsilon(e)$ so as to close the estimates on the one hand, and to maximise the decay rate on the other hand. This requires an optimisation procedure which is explained in more details at the beginning of section 3.6.3.

With this last argument, a r^p weighted hierarchy is proven for all $0 \leq p \leq p(e)$, where $p(e) > 2$ and $p(e) \rightarrow 3$ as $e \rightarrow 0$, which is a significant improvement with respect to the first method and allows us to claim a stronger time decay of the energy.

Point-wise bounds

To prove point-wise bounds on the scalar field and the charge, we need two essential ingredients : the weighted energy decay of step 5 and a point-wise estimate on a fixed light cone of the form $|D_v \psi|(u_0(R), v) \lesssim v^{-\omega}$ for some $\omega > 0$.

The latter comes from the point-wise decay of the scalar field Cauchy data, that implies consistent point-wise bounds in the past of a fixed forward light cone, c.f. section 3.4. We then use the former to “initiate” some decay estimate for ϕ . For this we essentially use Cauchy-Schwarz under the form $r^{\frac{1}{2}}|\phi| \lesssim E(u) \lesssim u^{-p}$, for the maximum p in the r^p hierarchy.

Point-wise bounds are then established in the rest of the space-time integrating (3.2.6) and (3.2.7) along constant u and constant v lines, after carefully splitting the space-time into regions where the scalar field behaves differently.

These regions are roughly :

1. A far away region — including $\mathcal{I}^+ = \{r = +\infty\}$ — where $r \sim v$, which is somehow the easiest.
2. An intermediate region $\{R \leq r \lesssim v\}$ where $R > r_+$ is a large constant.
3. The bounded r region $\{r_+ \leq r \leq R\}$, which includes $\mathcal{H}^+ = \{r = r_+\}$.

The far away region is the one where r weights are strong so the “conversion” between the L^2 and the L^∞ occurs “with no loss”.

The bounded r region is also not so difficult due to a point-wise version of the red-shift effect : there is again no loss between the estimates on the curve $r = R$ and the event horizon $r = r_+$.

However, in the intermediate region, the r weights, strong on $\{r \sim v\}$ degenerate to a mere constant near $\{r = R\}$. For this reason, the estimates imply a loss of $v^{\frac{1}{2}}$, which explains why the point-wise decay rate obtained on the event horizon is not expected to be optimal, while the rates on the energy and in the far away region are, at least in the limit $q_0|e| \rightarrow 0$.

In every section and subsection, the precise strategy of the proof is discussed. We refer the reader to these paragraphs for more details.

3.1.4 Outline of the chapter

The paper is outlined as follows : after introducing the equations, some notations and our foliation in section 3.2, we announce in section 3.3 a more precise version of our results.

³⁴In fact, the right-hand-side also has a multiplicative factor of the form $(p - 1 - \frac{4q_0|e|}{3-p})^{-\frac{1}{2}}$, that we do not write here for the sake of exposition. The real factor is denoted $f(p, e)$ in the proof.

Because our techniques (with the V shaped foliation of section 3.2) only deal with characteristic initial value data, we explain in section 3.4 how the Cauchy problem can be reduced to a characteristic initial value problem, with the correct assumptions. This is also the occasion to derive a priori smallness estimates on the charge, useful to close the harder estimates of the following sections. To avoid repetition and because the estimates are easier than the ones in the next sections, we postpone the proof to section 3.9.

Then in section 3.5, we prove Theorem 3.3.2. This section is divided as follows : first the subsection 3.5.1 where the integrated local energy decay is established, modulo boundary terms. Then the subsection 3.5.2 where the red-shift effect is used to establish a non-degenerate energy boundedness statement, modulo degenerate energy boundary terms. Finally we close together the energy boundedness and the Morawetz estimate in subsection 3.5.3, using crucially the results of the former two subsections.

Then, we turn to the proof of the second main result of the present paper, Theorem 3.3.3. Establishing energy decay is the object of section 3.6. It is divided as follows : first in subsection 3.6.1, we carry out preliminary computations. We also re-prove explicitly the version of the r^p method needed for our purpose, with the notations of the paper. Then a first energy decay estimate is proven, for $p < 2$, in section 3.6.2, using the smallness of the future asymptotic charge under the form $q_0|e| < \frac{1}{4}$.

Finally, in section 3.6.3, we prove a hierarchy of r^p estimates for $2 < p < 3$, requiring now that $q_0|e| < 0.8267$. The employed strategy differs radically from the one of section 3.6.2 but we use crucially the r^p hierarchy proven in the previous section. This proves Theorem 3.3.3.

Section 3.6.3 is one of the technical hearts of the paper, and the part that allows eventually for strictly integrable bounds on the event horizon.

The intake of section 3.6.2 and 3.6.3 is that energy decay at a rate $p(e) = 3 + O((q_0|e|)^{\frac{1}{2}})$ holds, provided energy boundedness is true. This rate $p(e)$ tends to 3 as q_0e tends to 0, which is the expected optimal limit. It also approaches to 2 as $q_0|e|$ gets closer to its maximal³⁵ value 0.8267.

We can then retrieve point-wise estimates from the energy decay in section 3.7. We use methods that only work in spherical symmetry and which are optimal in a region near null infinity. However, the presence of r weights that degenerate in the bounded r region creates a $u^{\frac{1}{2}}$ loss in the decay that cannot be compensated otherwise, like it was³⁶ in the uncharged case, c.f. Section 3.1.2. This is why the optimal energy decay does not give rise to optimal point-wise bounds on the event horizon. However, since the decay rate of the energy is always superior to 2, we retrieve **strictly integrable** point-wise decay rate for ϕ on the event horizon, for the full range of $q_0|e|$ we consider.

Finally, we prove a slight variant³⁷ of Theorem 3.3.3 in section 3.8, the proofs of Section 3.4 are carried out in section 3.9 and a few computations are made explicit in section 3.10.

3.2 Equations, definitions, foliations and calculus preliminaries

3.2.1 Coordinates and vector fields on Reissner–Nordström’s space-time

The sub-extremal Reissner–Nordström exterior metric can be written in coordinates (t, r, θ, φ)

$$g = -\Omega^2 dt^2 + \Omega^{-2} dr^2 + r^2 [d\theta^2 + \sin(\theta)^2 d\psi^2],$$

$$\Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2},$$

for some $0 \leq |\rho| < M$ and where $(r, t, \theta, \varphi) \in (r_+, +\infty) \times \mathbb{R}^+ \times [0, \pi) \times [0, 2\pi]$.

$r_+(M, \rho) := M + \sqrt{M^2 - \rho^2}$ is one of the two positives roots of $\Omega^2(r)$ and represents the radius of the event horizon, defined to be

$$\mathcal{H}^+ := \{r = r_+, (t, \theta, \varphi) \in \mathbb{R}^+ \times [0, \pi) \times [0, 2\pi]\}.$$

Symmetrically we define future null infinity to be

$$\mathcal{I}^+ := \{r = +\infty, (t, \theta, \varphi) \in \mathbb{R}^+ \times [0, \pi) \times [0, 2\pi]\}.$$

\mathcal{H}^+ and \mathcal{I}^+ are then null and complete hyper-surfaces for the Reissner–Nordström metric.

The other root $r_-(M, \rho) := M - \sqrt{M^2 - \rho^2}$ corresponds to the locus of the Cauchy horizon, inside the black hole, c.f. [78]. Therefore r_- does not play any role in the exterior.

We want to build a double null coordinate system (u, v) on spheres that can replace (t, r) .

³⁵In fact, there exists a maximal value $0.8267 < e_{max} < 0.8269$, which is not explicit, and $p(e) \rightarrow 2$ as $|e| \rightarrow e_{max}$, c.f. the proof of Lemma 3.6.10.

³⁶The reason is that in the uncharged case, $\partial_t \phi$ decays better than ϕ but in this charged case, $D\phi$ decays like ϕ , due to oscillations.

³⁷The main intake is that, for slightly more decaying data than required in Theorem 3.3.3, one can prove that $v^2 D_v \psi$ admits a bounded limit when $v \rightarrow +\infty$, for fixed u .

One possibility is to define a function $r^*(r)$, sometimes called tortoise radial coordinate, such that

$$\frac{dr^*}{dr} = \Omega^{-2}(r),$$

$$\lim_{r \rightarrow +\infty} \frac{r^*}{r} = 1.$$

Notice that the new coordinate $r^*(r)$ is an increasing function of r that takes its values in $(-\infty, +\infty)$.

We denote by ∂_t and ∂_{r^*} the corresponding vector fields in the $(r^*, t, \theta, \varphi)$ coordinate system. Notice that ∂_t is a time-like **Killing vector field** for this metric.

We can then define the functions $u(r^*, t)$ and $v(r^*, t)$ as

$$v = \frac{t + r^*}{2}, \quad u = \frac{t - r^*}{2}.$$

Notice that u takes its values in $(-\infty, +\infty)$ and $\{u = +\infty\} = \mathcal{H}^+$.

v takes its value in $(-\infty, +\infty)$ and $\{v = +\infty\} = \mathcal{I}^+$.

Spatial infinity i^0 is $\{u = -\infty, v = +\infty\}$ and the bifurcation sphere is $\{u = +\infty, v = -\infty\}$.

Defining ∂_u and ∂_v , the corresponding vector fields in the (u, v, θ, φ) coordinate system, we can check that $(\partial_u, \partial_v, \partial_\theta, \partial_\varphi)$ is a null frame for the Reissner–Nordström metric.

Notice that we have

$$\partial_v = \partial_t + \partial_{r^*}, \quad \partial_u = \partial_t - \partial_{r^*}.$$

The Reissner–Nordström's metric can then be re-written as

$$g = -2\Omega^2(du \otimes dv + dv \otimes du) + r^2[d\theta^2 + \sin(\theta)^2 d\varphi^2].$$

Note that in this coordinate system we also have

$$\Omega^2(r) = \partial_v r = -\partial_u r.$$

Then we define the quantity $2K$, the log derivative of Ω^2 that plays a role in the present paper :

$$2K(r) := \frac{2}{r^2}(M - \frac{\rho^2}{r}) = \partial_v \log(\Omega^2) = -\partial_u \log(\Omega^2). \quad (3.2.1)$$

We also define the surface gravity of the event horizon $2K_+ := 2K(r_+)$ and that of the Cauchy horizon $2K_- := 2K(r_-)$. These definitions allow for an explicit and simple expression of r^* as

$$r^* = r + \frac{\log(r - r_+)}{2K_+} + \frac{\log(r - r_-)}{2K_-}.$$

This implies that when $r \rightarrow r_+$:

$$\Omega^2(r) \sim C_+^2 \cdot e^{2K_+ \cdot (v-u)}, \quad (3.2.2)$$

where $C_+ = C_+(M, \rho) > 0$ is defined by $C_+^2 = e^{-2K_+ r_+} \frac{(r_+ - r_-)^{1 - \frac{K_+}{K_-}}}{r_+^2}$.

3.2.2 The spherically symmetric Maxwell-Charged-Scalar-Field equations in null coordinates

We now want to express Maxwell-Charged-Scalar-Field system

$$\nabla^\mu F_{\mu\nu} = iq_0 \frac{(\phi \overline{D_\nu \phi} - \overline{\phi} D_\nu \phi)}{2}, \quad F = dA, \quad (3.2.3)$$

$$g^{\mu\nu} D_\mu D_\nu \phi = 0 \quad (3.2.4)$$

in the (u, v) coordinates of section 3.2.1 and for spherically symmetric solutions.

Here ϕ represents a complex valued function while F is a real-valued 2-form.

In what follows, for a spherically symmetric solution (ϕ, F) , we are going to consider the projection of (ϕ, F) —that we still denote (ϕ, F) —on the 2-dimensional manifold indexed by the null coordinate system (u, v) and on which every point represents a sphere of radius $r(u, v)$ on Reissner–Nordström space-time. Much more details can be found on the procedure in [78], section 2.2.

It can be shown (c.f. [47]) that in spherical symmetry, we can express the two-form F as

$$F = \frac{2Q\Omega^2}{r^2} du \wedge dv, \quad (3.2.5)$$

where Q is a scalar function that we call the charge of the Maxwell equation.

Remark 39. Notice that this formula, that defines Q , differs from a multiplicative factor 4 from the formula of [37], [47] and [78]. This is because, in the present paper, Ω^2 is defined to be $\Omega^2 = 1 - \frac{2M}{r} + \frac{\rho^2}{r^2}$. In the previous papers, Ω^2 was actually defined as $\Omega^2 = 4(1 - \frac{2M}{r} + \frac{\rho^2}{r^2})$, hence the difference.

$F = dA$ also allows us to chose a spherically symmetric potential A 1-form written as :

$$A = A_u du + A_v dv.$$

We then define the gauge derivative D as $D_\mu = \partial_\mu + iq_0 A_\mu$.

The equations (3.2.3), (3.2.4) are invariant under the following gauge transformation :

$$\begin{aligned} \phi &\rightarrow \tilde{\phi} = e^{-iq_0 f} \phi \\ A &\rightarrow \tilde{A} = A + df, \end{aligned}$$

where f is a smooth real-valued function.

There is therefore a gauge freedom. However, all the estimates derived in this paper are essentially gauge invariant, so **we do not need to choose a gauge**.

Notice that for the gauge derivative $\tilde{D} := \nabla + \tilde{A}$, we have

$$\tilde{D}\tilde{\phi} = e^{-iq_0 f} D\phi.$$

Therefore, as explained in [37], the quantities $|\phi|$ and $|D\phi|$ are gauge invariant and so are the energies defined in section 3.2.5.

It should be noted that gauge derivatives do not compute : indeed one can show that

$$[D_u, D_v] = iq_0 F_{uv} Id = \frac{2iq_0 Q \Omega^2}{r^2} Id.$$

We now express equation (3.2.4) in (u, v) coordinates. For this, it is convenient to define the radiation field $\psi := r\phi$. We then find :

$$D_u(D_v \psi) = \frac{\Omega^2}{r} \phi \left(iq_0 Q - \frac{2M}{r} + \frac{2\rho^2}{r^2} \right) \quad (3.2.6)$$

$$D_v(D_u \psi) = \frac{\Omega^2}{r} \phi \left(-iq_0 Q - \frac{2M}{r} + \frac{2\rho^2}{r^2} \right) \quad (3.2.7)$$

Maxwell's equation (3.2.3) can also be written in (u, v) coordinates as

$$\partial_u Q = -q_0 r^2 \Im(\bar{\phi} D_u \phi) \quad (3.2.8)$$

$$\partial_v Q = q_0 r^2 \Im(\bar{\phi} D_v \phi). \quad (3.2.9)$$

Notice that in the spherically symmetric case, the Maxwell form is reduced to the charge Q .

Then finally the existence of an electro-magnetic potential A can be written as:

$$\partial_u A_v - \partial_v A_u = F_{uv} = \frac{2Q\Omega^2}{r^2}.$$

Finally we would like to finish this section by a Lemma stated in [37] (Lemma 2.1) that says that “gauge derivatives can be integrated normally”. More precisely :

Lemma 3.2.1. *For every $u_1 \in (-\infty, +\infty)$ and $v_1 \in (-\infty, +\infty)$ and every function f ,*

$$|f(u, v)| \leq |f(u_1, v)| + \int_{u_1}^u |D_u f|(u', v) du',$$

$$|f(u, v)| \leq |f(u, v_1)| + \int_{v_1}^v |D_v f|(u, v') dv'.$$

We refer to [37] for a proof, which is identical in the exterior case.

This lemma will be used implicitly throughout the paper.

3.2.3 Notations for different charges

In this paper, we make use of several quantities that we call “charge” although they may not be related. This section is present to clarify the notations and the relationships between these different quantities.

First we work on a Reissner–Nordström space-time of parameters (M, ρ) .

The Reissner–Nordström charge ρ is defined by the expression of the metric (3.1.3), (3.1.4).

We assume the sub-extremality condition $0 \leq |\rho| < M$, where M is the Reissner–Nordström mass.

Note that ρ is here **constant**, as a parameter of the black hole that does not vary dynamically because we study the **gravity uncoupled** problem.

Note also that the value of ρ **plays no role** in the paper, provided $|\rho| < M$.

In this paper we consider the Maxwell-**Charged**-Scalar-Field model. This means that the **charged** scalar field interacts with the Maxwell field.

The interaction constant $q_0 \geq 0$ is called the charge of the scalar field, as appearing explicitly in the system (3.2.3), (3.2.4).

We require the condition $q_0 \geq 0$ uniquely for aesthetic reasons and this is purely conventional. Everything said in this paper also works if $q_0 < 0$, after replacing $q_0|e|$ by $|q_0e|$ and $q_0|e_0|$ by $|q_0e_0|$.

Note that we consider q_0 as a universal constant, not as a parameter. Note also that q_0 has the dimension of the inverse of a charge : q_0Q is a dimensionless quantity.

Note also that the uncharged case $q_0 = 0$ corresponds to the well-known linear wave equation.

Now we define the charge of the Maxwell equation Q , defined explicitly from the Maxwell Field form F by (3.2.5), itself appearing in (3.2.3).

Q is a scalar **dynamic** function on the space-time —in contrast to the uncharged case $q_0 = 0$ where Q is forced to be a constant — that determines completely the Maxwell Field in spherical symmetry.

Note also that the Maxwell equations can be completely written in terms of Q , c.f. equations (3.2.8), (3.2.9).

Because we consider a Cauchy initial value problem, we also consider the initial charge of the Maxwell equation Q_0 defined as $Q_0 = Q|_{\Sigma_0}$ where Σ_0 is the initial space-like Cauchy surface.

Q_0 is then one part of the initial data (ϕ_0, Q_0) .

Then we define the initial asymptotic charge e_0 as

$$e_0 = \lim_{r \rightarrow +\infty} Q_0(r),$$

when it exists. This is the limit value of the Maxwell charge at spatial infinity.

Finally we define the future asymptotic charge e as

$$e = \lim_{t \rightarrow +\infty} Q(t, r),$$

for all $r \in [r_+, +\infty]$, when it exists. It can be proven³⁸ that the limit e does not depend on r . This is the limit value of the Maxwell charge at time-like and null infinity.

3.2.4 Foliations, domains and curves

In this section we define the foliation over which we control the energy, c.f. section 3.2.5. This is a V -shaped foliation, similar to the one of [58] but different from the J -shaped foliation of [28] and subsequent works.

For any $r_+ \leq R_1$ we define the curve $\gamma_{R_1} = \{r = R_1\}$.

For any u , we denote $v_{R_1}(u)$ the only v such that $(u, v_{R_1}(u)) \in \gamma_{R_1}$. Such a v is given explicitly by the formula $v - u = R_1^*$. Similarly for every v , we introduce $u_{R_1}(v)$.

Then for some $R > r_+$ large enough to be chosen in course of the proof, we considered the curve γ_R .

We also denote $(u_0(R), v_0(R))$, the coordinates of the intersection of γ_R and $\{t = 0\}$:

$$v_0(R) = -u_0(R) = \frac{R^*}{2}.$$

We then define the foliation \mathcal{V} : for every $u \geq u_0(R)$

$$\mathcal{V}_u = (\{v_R(u)\} \times [u, +\infty]) \cup ([v_R(u), +\infty] \times \{u\}).$$

³⁸The proof is made in section 3.7.4.

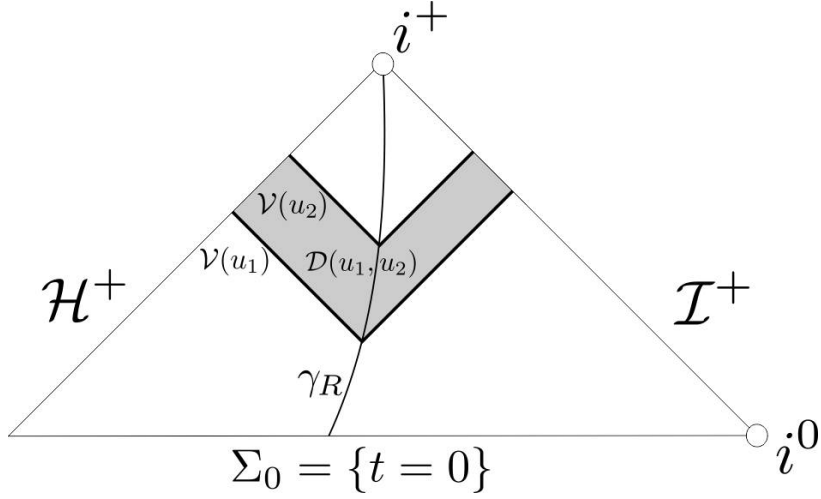


Figure 3.1: Illustration of the foliation \mathcal{V}_u , $u \geq u_0(R)$

This foliation is composed from a constant v segment in the region $\{r \leq R\}$ joining a constant u segment in the region $\{r \geq R\}$. It is illustrated in Figure 3.1.

Notice that the foliation does not cover the regions $\{u \leq u_0(R)\}$ and $\{v \leq v_0(R)\}$ as it can be seen in Figure 3.3. This is why we need section 3.4 to connect the energy on this foliation, and in particular on $\mathcal{V}_{u_0(R)}$ to the energy on the initial space-like hyper-surface \mathcal{E} .

We now defined the domain $\mathcal{D}(u_1, u_2)$, for all $u_0(R) \leq u_1 < u_2$ as

$$\mathcal{D}(u_1, u_2) := \cup_{u_1 \leq u \leq u_2} \mathcal{V}_u. \quad (3.2.10)$$

Numerous L^2 identities of this paper are going to be integrated either on $\mathcal{D}(u_1, u_2)$, or $\mathcal{D}(u_1, u_2) \cap \{r \leq R\}$ or $\mathcal{D}(u_1, u_2) \cap \{r \geq R\}$.

We also define the initial space-like hyper-surface $\Sigma_0 = \{t = 0\} = \{v = -u\}$ on which we set the Cauchy data (ϕ_0, Q_0) c.f. section 3.4.

We are going to make use of the following notations $u_0(v) = -v$, $v_0(u) = -u$, $r_0(u)$ the radius corresponding to $(r_0(u))^* = -2u$ and $r_0(v)$ the radius corresponding to $(r_0(v))^* = 2v$, when there is no ambiguity between u and v .

Finally we will also need a curve close enough to null infinity, in particular to retrieve point-wise bounds.

We define $\gamma = \{r^* = \frac{v}{2} + R^*\}$. Notice that on this curve, $r \sim u \sim \frac{v}{2}$ as $v \rightarrow +\infty$.

For any u , we then define $v_\gamma(u)$, the only v such that $(u, v_\gamma(u)) \in \gamma$. Similarly, we introduce $u_\gamma(v)$.

Remark 40. Note that because γ and γ_{R_1} are time-like curves, talking of their future domain is not very interesting. Instead, notably in section 3.7, we are going to make use of the domain “to the right” of γ : $\{r^* \geq \frac{v}{2} + R^*\}$ and the domain “to the right” of γ_{R_1} : $\{r \geq R_1\}$.

3.2.5 Energy notations

In this section, we gather the definitions of all the energies used in the paper. Those definitions are however repeated just before being used for the first time.

The main definition concerns the non-degenerate energy, for all $u \geq u_0(R)$:

$$E(u) = E_R(u) = \int_u^{+\infty} r^2 \frac{|D_u \phi|^2}{\Omega^2}(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv.$$

By default, $E(u)$ is the energy related to the γ_R -based foliation \mathcal{V} we mention in section 3.2.1. Sometimes however, we may talk of $E_{R_1}(u)$ for a R_1 that is different from R , in section 3.4.

We also define the degenerate energy :

$$E_{deg}(u) = \int_u^{+\infty} r^2 |D_u \phi|^2(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv.$$

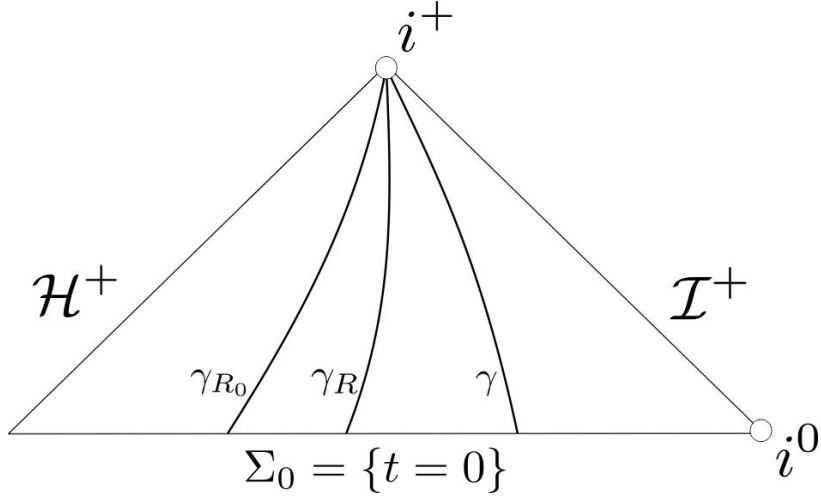


Figure 3.2: Curves γ_{R_0} for $r_+ < R_0 = R_0(M, \rho) < R$, γ_R and γ

It will be convenient to use for all $u_0(R) \leq u_1 < u_2$:

$$E^+(u_1, u_2) := E(u_2) + E(u_1) + \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du,$$

$$E_{deg}^+(u_1, u_2) := E_{deg}(u_2) + E_{deg}(u_1) + \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du.$$

For the r^p hierarchy we are also going to use for all $u_0(R) \leq u_1 < u_2$:

$$E_p[\psi](u) := \int_{v_R(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv,$$

$$\tilde{E}_p(u) := E_p[\psi](u) + E(u).$$

Finally we define the initial non-degenerate energy on the initial slice $\Sigma_0 := \{t = 0\}$

$$\mathcal{E} = \int_{\Sigma_0} \frac{r^2 |D_u \phi_0|^2 + r^2 |D_v \phi_0|^2}{\Omega^2} dr^*. \quad (3.2.11)$$

Remark 41. Notice that this energy is proportional to the regular H^1 norm for ϕ on Σ_0 . Indeed the vector field $\Omega^{-1} \partial_t = \frac{\partial_u + \partial_v}{2\Omega}$ — time-like and unitary — is regular across the bifurcation sphere. But from (3.2.2), we see that towards the future or the past event horizon $\{r = r_+\}$, $\Omega^2 = (C_+)^2 \cdot e^{2K_+v} e^{-2K_+u}$. Therefore, for on any constant v slice transverse to the future event horizon, $(C_+)^{-1} e^{2K_+u} \partial_u$ is regular. Symmetrically on any constant u slice transverse to the past event horizon, $(C_+)^{-1} e^{-2K_+v} \partial_v$ is regular. We then see that $(2C_+)^{-1} (e^{2K_+u} \partial_u + e^{-2K_+v} \partial_v)$ is then also a regular vector field across the bifurcation sphere, actually proportional to $\Omega^{-1} \partial_t$ because on Σ_0 , $v \equiv -u$.

Notice that consistently, to prove point-wise bounds, we are going assume $|D_u \phi_0|(r) \lesssim e^{-2K_+u} \sim \Omega(u, v_0(u))$ — c.f. Hypothesis 4 — as opposed to the too strong but somehow naively intuitive hypothesis $|D_u \phi_0|(r) \lesssim \Omega^2$.

Actually, the L^2 bound is “morally” slightly stronger than the point-wise one. Indeed $\Omega^{-2} |D_u \phi_0|^2 dr^* = \Omega^{-4} |D_u \phi_0|^2 dr$ is integrable if $\Omega^{-4} |D_u \phi_0|^2 \lesssim (r - r_+)^{-1+\epsilon}$ for any $\epsilon > 0$, i.e $|D_u \phi_0|(u, v_0(u)) \lesssim e^{-2K_+(1+\epsilon)u}$.

We also define the initial r^p weighted energy for $\psi_0 := r \phi_0$:

$$\mathcal{E}_p := \int_{\Sigma_0} (r^p |D_v \psi_0|^2 + r^p |D_u \psi_0|^2 + \Omega^2 r^p |\phi_0|^2) dr^*. \quad (3.2.12)$$

Note that \mathcal{E}_p includes a 0 order term $r^p |\phi_0|^2$, of the “same homogeneity” as $r^p |D_v \psi_0|^2$. Notice also that

$$\int_{\Sigma_0} r^p |D_{r^*} \psi_0|^2 dr^* \leq \frac{\mathcal{E}_p}{2}.$$

Finally we regroup the two former definitions into

$$\tilde{\mathcal{E}}_p = \mathcal{E}_p + \mathcal{E}.$$

We also want to recall that Stress-Energy momentum tensors are defined as :

$$\mathbb{T}_{\mu\nu}^{SF} = \Re(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2}(g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi}) g_{\mu\nu}, \quad (3.2.13)$$

$$\mathbb{T}_{\mu\nu}^{EM} = g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu}. \quad (3.2.14)$$

For an expression in (u, v) coordinates and explicit computations, c.f. section 3.10.

3.2.6 Hardy-type inequalities

In this paper, we use Hardy-type inequalities numerous times. As explained in section 3.1.4, this is mainly due to the necessity to absorb the 0 order term that appears in the interaction term with the Maxwell field. In this section, we are going to state and prove the different Hardy-type inequalities that we use throughout the paper. The only objective is to prove the estimates exactly the way they are later used in the paper.

Lemma 3.2.2. *For $R > 0$ as in the definition of the foliation and all $u_0(R) \leq u_1 < u_2$, the following estimates are true, for any $q \in [0, 2)$, and any $r_+ < R_1 < R$:*

$$\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du \leq 4 \int_{u_i}^{+\infty} \Omega^2 \frac{r^2 |D_u \phi|^2(u, v_R(u_i))}{r^2 \cdot 2K} du + 2\Omega^2(R) \cdot |\phi|^2(u_i, v_R(u_i)). \quad (3.2.15)$$

$$R|\phi|^2(u_i, v_R(u_i)) \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv \quad (3.2.16)$$

$$\int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq \frac{4}{\Omega^2(R_1)} \int_{u_i}^{+\infty} r^2 |D_u \phi|^2(u, v_R(u_i)) du + 2R|\phi|^2(u_i, v_R(u_i)). \quad (3.2.17)$$

$$\int_{v_R(u_i)}^{+\infty} |\phi|^2(u_i, v) dv \leq \frac{4}{\Omega^4(R)} \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv. \quad (3.2.18)$$

$$\left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-3} \Omega^2 |\psi|^2 dudv \right)^{\frac{1}{2}} \leq \frac{2}{(2-q)\Omega(R)} \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-1} |D_v \psi|^2 dudv \right)^{\frac{1}{2}} + \left(\frac{R^{q-2}}{2-q} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du \right)^{\frac{1}{2}} \quad (3.2.19)$$

Proof. We start by (3.2.15) : after noticing that $\Omega^2 \cdot 2K = (-\partial_u \Omega^2)$ we integrate by parts and get :

$$\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du = \int_{u_i}^{+\infty} (-\partial_u \Omega^2) |\phi|^2(u, v_R(u_i)) du \leq 2 \int_{u_i}^{+\infty} \Omega^2 \Re(\bar{\phi} D_u \phi)(u, v_R(u_i)) du + \Omega^2(R) \cdot |\phi|^2(u_i, v_R(u_i)).$$

Then using Cauchy-Schwarz, we end up having an inequality of the form $A^2 \leq 2AB + C^2$, where

$$A^2 = \int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du, \quad B^2 = \int_{u_i}^{+\infty} \Omega^2 \frac{|D_u \phi|^2(u, v_R(u_i))}{2K} du, \quad C^2 = \Omega^2(R) \cdot |\phi|^2(u_i, v_R(u_i)).$$

Then using what we know on roots of second order polynomials we get (3.2.15) under the form :

$$A^2 \leq \left(B + \sqrt{B^2 + C^2} \right)^2 \leq 4B^2 + 2C^2.$$

Now we prove (3.2.16). Since $\lim_{v \rightarrow +\infty} \phi(u, v) = 0$, we³⁹ can write

$$\phi(u, v) = - \int_v^{+\infty} e^{\int_v^{v'} i q_0 A_v} D_v \phi(u, v') dv',$$

where we used the fact that $\partial_v (e^{\int_{v_0(R)}^v i q_0 A_v} \phi) = e^{\int_{v_0(R)}^v i q_0 A_v} D_v \phi$.

³⁹This comes from the finiteness of \mathcal{E} , c.f. the proof of Proposition (3.4.1).

Now using Cauchy-Schwarz we have

$$|\phi|(u, v) \leq \left(\int_v^{+\infty} \Omega^2 r^{-2}(u, v') dv' \right)^{\frac{1}{2}} \left(\int_v^{+\infty} \Omega^{-2} r^2 |D_v \phi|^2(u, v') dv' \right)^{\frac{1}{2}}.$$

After squaring, this directly implies (3.2.16) under the form

$$r|\phi|^2(u, v) \leq \Omega^{-2}(u, v) \int_v^{+\infty} r^2 |D_v \phi|^2(u, v') dv'.$$

Then we turn to (3.2.17) : after an integration by parts we get

$$\int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq 2 \int_{u_i}^{+\infty} r \Re(\bar{\phi} D_u \phi)(u, v_R(u_i)) du + R |\phi|^2(u_i, v_R(u_i)).$$

Then using Cauchy-Schwarz, we find an inequality of the form $A^2 \leq 2AB + C^2$, where

$$A^2 = \int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du, \quad B^2 = \int_{u_i}^{u_{R_1}(v_R(u_i))} r^2 \frac{|D_u \phi|^2}{\Omega^2}(u, v_R(u_i)) du, \quad C^2 = R |\phi|^2(u_i, v_R(u_i)).$$

Similarly to (3.2.15), this concludes the proof of (3.2.17), after we notice that $\Omega^{-2} \leq \Omega^{-2}(R_1)$.

We now turn to (3.2.18) : after notice that $\Omega^2(R) \leq \partial_v r$ we integrate by parts and get

$$\int_{v_R(u_i)}^{+\infty} |\phi|^2(u_i, v) dv \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} \partial_v r |\phi|^2(u_i, v) dv \leq -2\Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r \Re(\bar{\phi} D_v \phi)(u_i, v) dv.$$

Notice that the sum of the boundary terms is negative because $\lim_{r \rightarrow +\infty} r |\phi|^2 = 0$. Then Cauchy-Schwarz inequality directly gives (3.2.18).

Finally we prove (3.2.19) : after noticing that $r^{q-3}\Omega^2 = -(2-q)^{-1}\partial_v(r^{q-2})$ and an integration by parts in v we get, using that $q < 2$

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-3} \Omega^2 |\psi|^2 dudv \leq \frac{2}{(2-q)} \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-2} \Re(\bar{\psi} D_v \psi) dudv + \frac{R^{q-2}}{2-q} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du.$$

Then using Cauchy-Schwarz, we end up having an inequality of the form $A^2 \leq 2AB + C^2$, where

$$A^2 = \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-3} \Omega^2 |\psi|^2 dudv, \quad B^2 = \frac{1}{(2-q)^2} \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{q-1} |D_v \psi|^2 dudv, \quad C^2 = \frac{R^{q-2}}{2-q} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du.$$

Then, as seen for the proof of (3.2.15), we have $A \leq B + \sqrt{B^2 + C^2}$. This implies that $A \leq 2B + C$, which is exactly (3.2.19). We used the fact that for all $a, b \geq 0$, $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$. This concludes the proof. \square

3.3 Precise statement of the main result

3.3.1 The main result

We now state the results of the present paper, with precise hypothesis. The main result is an almost-optimal energy and point-wise decay statement, for **small** and point-wise decaying initial data.

It should be noted that all the required hypotheses are only needed to prove all the claims.

For example, to prove only **energy decay**, **point-wise decay rates are not necessary**, and one can assume weaker r^p weighted initial energy boundedness, c.f. the theorems of next sub-sections.

Theorem 3.3.1 (Energy and point-wise decay for small and decaying data). *Consider spherically symmetric Cauchy data (ϕ_0, Q_0) on the surface $\Sigma_0 = \{t = 0\}$ — on a Reissner–Nordström exterior space-time of mass M and charge ρ with $0 \leq |\rho| < M$ — that satisfy the constraint equation (3.4.1).*

Assume that the data satisfy the following regularity hypotheses :

1. *The initial energy is finite : $\mathcal{E} < \infty$, where \mathcal{E} is defined in section 3.2.5.*

⁴⁰This comes from the finiteness of $\tilde{\mathcal{E}}_{1+\eta}$, c.f. the proof of Proposition (3.4.1).

2. $Q_0 \in L^\infty(\Sigma_0)$ and admits a limit $e_0 \in \mathbb{R}$ as $r \rightarrow +\infty$.
3. We have the following data smallness assumption, for some $\delta > 0$ and $\eta > 0$:

$$\|Q_0\|_{L^\infty(\Sigma_0)} + \tilde{\mathcal{E}}_{1+\eta} < \delta.$$

4. Defining $\psi_0 := r\phi_0$, assume $\psi_0 \in C^1(\Sigma_0)$ enjoys⁴¹ point-wise decay :
for all $\epsilon_0 > 0$, there exist $C_0 = C_0(\epsilon_0) > 0$, such that

$$\begin{aligned} r|D_v\psi_0|(r) + |\psi_0|(r) &\leq C_0 \cdot r^{\frac{-1-\sqrt{1-4q_0|\epsilon_0|}}{2} + \epsilon_0}. \\ |D_u\phi_0|(r) &\leq C_0 \cdot \Omega. \end{aligned}$$

Then, there exists $\delta_0 = \delta_0(M, \rho) > 0$, $C = C(M, \rho) > 0$ such that if $\delta < \delta_0$, the following are true :

1. The charge Q admits a future asymptotic charge : there exists $e \in \mathbb{R}$ such that for every $R_1 > r_+$:

$$\lim_{t \rightarrow +\infty} Q|_{\mathcal{H}^+}(t) = \lim_{t \rightarrow +\infty} Q|_{\mathcal{I}^+}(t) = \lim_{t \rightarrow +\infty} Q|_{\gamma_{R_1}}(t) = e.$$

Moreover **the charge is small** : on the whole-space-time

$$|Q(u, v) - e_0| \leq C \cdot \left(\|Q_0\|_{L^\infty(\Sigma_0)} + \tilde{\mathcal{E}}_{1+\eta} \right),$$

in particular $|e - e_0| \leq C \cdot \delta$ and $|e| \leq (C + 1) \cdot \delta$.

2. **Non-degenerate energy boundedness**⁴² holds : for all $u_0(R) \leq u_1 < u_2$:

$$\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v\phi|_{|\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u\phi|_{|\mathcal{I}^+}^2(u) du + E(u_2) \leq C \cdot E(u_1) \leq C^2 \cdot \mathcal{E}, \quad (3.3.1)$$

where E and \mathcal{E} are defined in section 3.2.5.

3. **Integrated local decay estimate** holds : there exists $\sigma = \sigma(M, \rho) > 1$ —potentially large— such that for all $u_0(R) \leq u_1 < u_2$:

$$\int_{\mathcal{D}(u_1, u_2)} \left(\frac{r^2 |D_v\phi|^2 + r^2 \Omega^{-2} |D_u\phi|^2 + |\phi|^2}{r^\sigma} \right) dudv \leq C \cdot E(u_1) \leq C^2 \cdot \mathcal{E}. \quad (3.3.2)$$

4. **Weighted r^p energies are bounded and decay** : There exists $2 < p(e) < 3$, $p(e) \rightarrow 3$ as $e \rightarrow 0$ obeying the asymptotics of (3.6.21), such that for all $0 \leq p \leq p(e)$, there exists $\delta_p = \delta_p(M, \rho, p, e) > 0$, such that if $\delta < \delta_p$, then there exists $R_0 = R_0(e, p, M, \rho) > r_+$ such that if $R > R_0$, there exists $C'_p = C'_p(p, M, \rho, R, C_0) > 0$ such that for all $u \geq u_0(R)$:

$$\tilde{E}_p(u) \leq C'_p \cdot |u|^{p-p(e)},$$

where \tilde{E}_p is defined in section 3.2.5.

5. In particular **the non-degenerate energy decays in time** : for all $u \geq u_0(R)$:

$$E(u) \leq C'_0 \cdot |u|^{-p(e)}. \quad (3.3.3)$$

We also have the decay of the scalar field L^2 flux along constant r curves and the event horizon : for all $R_0 > r_+$, there exists $\tilde{C} = \tilde{C}(R_0, M, \rho, R) > 0$ such that for all $v \geq v_0(R)$:

$$\int_v^{+\infty} \left[|\phi|^2(u_{R_0}(v'), v') + |D_v\phi|^2(u_{R_0}(v'), v') \right] du' \leq \tilde{C} \cdot v^{-p(e)}, \quad (3.3.4)$$

$$\int_v^{+\infty} \left[|\phi|_{|\mathcal{H}^+}^2(v') + |D_v\phi|_{|\mathcal{H}^+}^2(v') \right] dv' \leq C'_0 \cdot v^{-p(e)}. \quad (3.3.5)$$

⁴¹By Hypothesis 3, one can in particular assume $q_0|\epsilon_0| < \frac{1}{4}$, without loss of generality.

⁴²Here energy boundedness is stated for all $u \geq u_0(R)$ for convenience, because we have a specific foliation but the argument can be slightly modified to prove the boundedness of the analogous quantity when $u < u_0(R)$.

6. Finally, we obtain **point-wise decay estimates** : there exists $R_0 = R_0(e, M, \rho) > r_+$ such that if $R > R_0$ then there exists $C' = C'(e, R, M, \rho) > 0$ such that for all $u > 0, v > 0$:

$$r^{\frac{1}{2}} |\phi|(u, v) + r^{\frac{3}{2}} |D_v \phi|(u, v) \leq C' \cdot (\min\{u, v\})^{-\frac{p(e)}{2}}, \quad (3.3.6)$$

$$|D_u \psi| \leq C' \cdot \Omega^2 \cdot (\min\{u, v\})^{-\frac{p(e)}{2}}, \quad (3.3.7)$$

$$|\psi|_{\mathcal{I}^+}(u) \leq C' \cdot u^{\frac{1-p(e)}{2}}, \quad (3.3.8)$$

$$|D_v \psi|_{\{v \geq 2u + R^*\}}(u, v) \leq C' \cdot v^{\frac{1-p(e)}{2}} |\log(u)|, \quad (3.3.9)$$

$$|Q - e|(u, v) \leq C' \cdot \left(u^{1-p(e)} |\log(u)| 1_{\{r \geq 2r_+\}} + v^{-p(e)} 1_{\{r \leq 2r_+\}} \right), \quad (3.3.10)$$

Remark 42. As explained in sections 3.1.1 and 3.1.2, most of the estimates are expected to be (almost) optimal⁴³, in the limit $e \rightarrow 0$, including (3.3.4), (3.3.5), (3.3.8), (3.3.10). While (3.3.6) is expected to be optimal in a region near null infinity $\{v \geq 2u + R^*\}$, it is not optimal on any constant r curve or on the event horizon for reasons explained in sections 3.1.2 and 3.1.2. However, the L^2 bound on the event horizon from (3.3.4) gives point-wise bounds $|\phi|(v_n) + |D_v \phi|(v_n) \lesssim v_n^{-\frac{1+p(e)}{2}}$ along a **dyadic sequence** (v_n) that are expected to be **almost optimal** in the limit $e \rightarrow 0$, c.f. section 3.1.2.

Remark 43. Notice that we stated (3.3.7) for $v > 0$, in particular far away from $-\infty$. What happens near the bifurcation sphere is more subtle, as explained in Remark 41 and Remark 57. Indeed we can prove that for any $V_0 \in \mathbb{R}$, $|D_u \psi|(u, v) \lesssim \Omega^2$ in $\{v \geq V_0\}$. The constant in the inequality blows up as $V_0 \rightarrow -\infty$ however, remarkably one can still prove that $|D_u \psi|(u, v) \lesssim e^{-2K+u}$ on the whole space-time, in conformity with our hypothesis 4 that the regular derivative⁴⁴ of the scalar field $\Omega^{-1} D_t \phi_0 \sim e^{2K+u} D_t \phi_0$ is initially bounded. Notice that at fixed $v = V_0$, $\Omega^{-2} \partial_u$ is a regular vector field transverse to the event horizon and it degenerates when approaching the bifurcation sphere $v = -\infty$.

This unconditional result is issued as a combination of several other statements, that we believe to be of independent interest. These are the object of the following sub-sections.

3.3.2 Energy boundedness and integrated local energy decay for small charge and scalar field energy data, no point-wise decay assumption

First we want to emphasize that energy boundedness and integrated local energy decay have nothing to do with point-wise property of the data. This is the content of the following short boundedness theorem :

Theorem 3.3.2. *[Boundedness of the energy and integrated local energy decay for small data] In the context of Theorem 3.3.1, assume hypothesis 1, 2 and 3 for some $\eta > 0$ and $\delta < \delta_0(M, \rho)$ sufficiently small.*

Suppose also that $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.

Then statements 1, 2 and 3 are true : the charge stays small, the energy is bounded and local integrated energy decay holds.

Remark 44. Notice that the energy boundedness statement (3.3.1) actually consists of two estimates : the first inequality is more subtle⁴⁵ to prove than the second, but **necessary** to later prove decay. The only difficulty of the second inequality is to prove the boundedness of the **non-degenerate** energy⁴⁶ using the red-shift effect, which is more difficult in this context than for the uncharged case, c.f. section 3.4.

Then there are a few conditions we would like to relax in Theorem 3.3.1, at the cost of other assumptions. This is the object of the next section.

3.3.3 Energy time decay without arbitrary charge smallness or point-wise decay assumption, conditional on energy boundedness

In this section, we try to relax some of the assumptions made in Theorem 3.3.1.

The most important is that energy decay should not rely on the point-wise decay of the initial data. This is because we aim at providing estimates with no symmetry assumptions, where point-wise bounds are harder to propagate. We can prove that this is indeed the case, c.f. Theorem 3.3.3.

⁴³(3.3.3) is also expected to be optimal in the limit $e \rightarrow 0$, because of its component in the large r region that decays slower than the bounded r term, c.f. section 3.2.5 to see the precise terms.

⁴⁴Thus, the "regular vector field across the bifurcation sphere" is $\Omega^{-1} \partial_t$ **on the initial surface** Σ_0 .

⁴⁵This is because we want to have a right-hand-side that depends only on the scalar field and **not on the charge**.

⁴⁶The boundedness of the degenerate energy is easy, using the vector field ∂_t , if we do not care of initial charge terms.

Another important physical fact is that the decay rate should only depend on the future asymptotic charge of the Maxwell equation e , and in particular not on the mass of the black hole ⁴⁷. The first theorem that we state does not do justice to this fact, for the charge is required to be smaller than a constant **depending on the parameters of the black holes**, in particular on the mass. This is however due to the difficulty to prove energy boundedness estimates in the presence of a large charge. We overcome it using the red-shift effect and the argument requires such a smallness assumption on the initial charge.

In the following result, assuming the boundedness of the energy and an integrated local energy decay estimate, we can retrieve energy decay for $q_0|e|$ smaller than a **universal** numeric constant ⁴⁸.

We now state the theorem, in which we assume energy boundedness (3.3.1) and an integrated local energy decay estimate (3.3.2), but **no point-wise decay** of the data. Smallness of the scalar field initial energy is still required but the initial charge e_0 is only required to satisfy $q_0|e_0| < \frac{1}{4}$, which makes in principle the class of admissible initial data larger.

The following theorem includes two statements: first, if $q_0|e_0| \leq 0.08267$, we obtain a decay rate for the energy $2 < p = p(e) < 3$, a decay which we call "almost optimal" as $p(e) \rightarrow 3$, the (putatively) largest possible for charged scalar fields. If $0.08267 < q_0|e_0| < \frac{1}{4}$, we still obtain some weaker decay at a rate $1 < p < 1 + \sqrt{1 - 4q_0|e_0|}$. To encompass these two aspects, we are going to denote $\check{p}(e) := p(e) \in (2, 3]$ if $q_0|e_0| \leq 0.08267$ and $\check{p}(e) := 1 + \sqrt{1 - 4q_0|e|} \in (1, 2]$ if $0.08267 < q_0|e_0| < \frac{1}{4}$.

Theorem 3.3.3 (Almost Optimal decay for a small scalar field energy and larger $q_0|e|$). *In the same context as for Theorem 3.3.1, we make the different following assumptions :*

1. *The initial energy is finite : $\mathcal{E} < \infty$, where \mathcal{E} is defined in section 3.2.5.*
2. *$Q_0(r)$ admits a limit $e_0 \in \mathbb{R}$ as $r \rightarrow +\infty$.*
3. *The initial asymptotic charge is smaller than an universal constant :*

$$q_0|e_0| < \frac{1}{4}.$$

4. *We exclude constant solutions by the condition $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.*
5. *We have the finiteness condition : for every $0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|}$,*

$$\tilde{\mathcal{E}}_p < +\infty.$$

6. *We have the smallness condition : for some $\eta > 0$ and some $\delta > 0$,*

$$\tilde{\mathcal{E}}_{1+\eta} < \delta.$$

7. *Energy boundedness (3.3.1) and integrated local decay (3.3.2) hold.*

Then there exists $C = C(M, \rho) > 0$, $\delta_0 = \delta_0(e_0, M, \rho) > 0$ such that if $\delta < \delta_0$ we have the following

1. *The charge is bounded and there exists a future asymptotic charge $e \in \mathbb{R}$ such that for every $R_1 > r_+$*

$$\lim_{t \rightarrow +\infty} Q|_{\mathcal{H}^+}(t) = \lim_{t \rightarrow +\infty} Q|_{\mathcal{I}^+}(t) = \lim_{t \rightarrow +\infty} Q|_{\gamma_{R_1}}(t) = e.$$

Moreover the charge is close to its initial asymptotic value : on the whole-space-time

$$|Q(u, v) - e_0| \leq C \cdot \tilde{\mathcal{E}}_{1+\eta},$$

in particular $|e - e_0| \leq C \cdot \delta$ and therefore if δ is small enough, $q_0|e| < \frac{1}{4}$.

2. *Moreover, **boundedness of r^p weighted energies and energy decay** hold :*

for every $0 \leq p < \check{p}(e)$, if $\delta < \delta_p$ then statements 4 and 5 of Theorem 3.3.1 are true, where we recall that $\check{p}(e) > 2$ if $q_0|e| \leq 0.8267$ and $\check{p}(e) \rightarrow 3$ as $e \rightarrow 0$.

⁴⁷Because we study the gravity uncoupled problem, it should not depend on the charge of the black hole ρ either. However, in the gravity coupled problem $\rho = e$ so the decay should depend on the charge of the black hole this time.

⁴⁸depending on the strength of the decay, this constant can be $\frac{1}{4}$ or 0.8267

3. If additionally, one assumes initial point-wise decay under the form :
for all $\epsilon_0 > 0$, there exist $C_0 = C_0(\epsilon_0) > 0$, such that

$$r|D_v\psi_0|(r) + |\psi_0|(r) \leq C_0 \cdot r^{\frac{-1-\sqrt{1-4q_0|\epsilon_0|}}{2} + \epsilon_0},$$

$$|D_u\phi_0|(r) \leq C_0 \cdot \Omega,$$

then one can retrieve **point-wise decay estimates** : statement 6 of Theorem 3.3.1 is true, with $\check{p}(e)$ replacing $p(e)$ in the range $0.08267 < q_0|e| < \frac{1}{4}$.

Remark 45. Finally, note that the requirements $q_0|e_0| < \frac{1}{4}$ or $q_0|e_0| < 0.8267$ should be understood — together with the initial scalar field energy smallness — as $q_0|Q| < \frac{1}{4}$, respectively $q_0|Q| < 0.8267$ everywhere.

Therefore, it is also equivalent to state it as $q_0|e| < \frac{1}{4}$, respectively $q_0|e| < 0.8267$, which is what we do in section 3.6.

Remark 46. Note that in both theorems, the only smallness initial energy condition is on $\tilde{E}_{1+\eta}$, to control the variations of Q . In particular, **no smallness condition** is imposed on the initial r^p energies for $p > 1 + \eta$, even though we require them to be finite. This is related to the fact that the only smallness needed for this problem is that of the charge Q but not of the scalar field.

Remark 47. Another version of Theorem 3.3.3 is proven in section 3.8. While this different version requires more point-wise decay of the initial data, it also proves more v decay for $D_v\psi$, at the expense of growing u weights. In particular, we can prove that in ⁴⁹ the gauge $A_v = 0$, denoting $X = r^2\partial_v$, $X^n\psi(u, v)$ admits a finite limit when $v \rightarrow +\infty$, for fixed u .

3.4 Reduction of the Cauchy problem to a characteristic problem and global control of the charge

In this section, we explain how the Cauchy problem can be reduced to a characteristic problem, with suitable hypothesis. This is the step that we described earlier as “boundedness in terms of initial data”, which is simpler than boundedness with respect to be past values that we prove in later sections.

The main object is to show how the smallness of the initial charge propagates to the whole space-time, providing the initial energy of the scalar field is also small.

The results split into four parts : in the first Proposition 3.4.1, we show how the smallness of the scalar field energy **and the smallness of the initial charge** imply energy boundedness and integrated local energy decay, because the charge stays small everywhere. This is the boundedness part of Theorem 3.3.2.

Then in Proposition 3.4.2, we show that if the two latter hold, then, if the initial energy is small enough, the charge stays close to its limit value at spatial infinity e_0 on the whole space-time, provided $q_0|e_0| < \frac{1}{4}$, a bound which is **independent of the black holes parameters**. This part is related to Theorems 3.3.3, more precisely to statement 1.

In Proposition 3.4.3, we show how certain initial data point-wise decay rates can be extended towards a fixed forward light cone. This is useful for point-wise decay rates in Theorem 3.3.3 or as an alternative the finiteness assumption of higher order r^p weighted initial energy, like in the statement of Theorem 3.1.1, although this is technically a stronger statement and that **initial point-wise decay is not needed for energy decay**.

Finally in Proposition 3.4.4, we prove that — provided the initial r^p weighted energies of large order are finite — they are still finite on a constant u hyper-surface transverse to null infinity.

To do this, we make use of a $r^p u^s$ weighted energy estimate, for $s > 0$.

In the next sections, whose goal is to prove energy u decay, the strategy is in contrast to what we do here : we will aim at estimating the energy in terms of its past values. This is sensibly more difficult than to bound the energy in terms of the data.

This is why in this section, we only state the results, while their proofs are postponed to section 3.9. This allows us to focus on the main difficulty of the paper — the energy decay — and to avoid repeating very similar arguments.

We denote $\Sigma_0 = \{t = 0\}$ the 3-dimensional Riemannian manifold on which we set the Cauchy data $(\phi_0, D_t\phi_0, Q_0)$ satisfying the following constraint equation

$$\partial_{r^*} Q_0 = q_0 r^2 \Im(\bar{\phi}_0 D_t \phi_0). \quad (3.4.1)$$

We first show energy boundedness and global smallness of the charge, on condition that the data is small.

⁴⁹The result that we prove is actually gauge invariant but we state it here in the gauge $A_v = 0$ for simplicity.

Proposition 3.4.1. *Suppose that there exists $p > 1$ such that $\tilde{\mathcal{E}}_p < \infty$ and that $Q_0 \in L^\infty(\Sigma_0)$.*

Assume also that $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.

We denote $Q_0^\infty = \|Q_0\|_{L^\infty(\Sigma_0)}$.

There exists $r_+ < R_0 = R_0(M, \rho)$ large enough, $\delta = \delta(M, \rho) > 0$ small enough and $C = C(M, \rho) > 0$ so that for all $R_1 > R_0$, and if

$$Q_0^\infty + \tilde{\mathcal{E}}_p < \delta,$$

then for all $u \geq u_0(R_1)$:

$$E_{R_1}(u) \leq C \cdot \mathcal{E}. \quad (3.4.2)$$

Also for all $v \leq v_{R_1}(u)$:

$$\int_{\tilde{u}(v)}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v) du' \leq C \cdot \mathcal{E}, \quad (3.4.3)$$

where $\tilde{u}(v) = u_0(v)$ if $v \leq v_0(R_1)$ and $\tilde{u}(v) = u_{R_1}(v)$ if $v \geq v_0(R_1)$.

Moreover for all (u, v) in the space-time :

$$|Q|(u, v) \leq C \cdot (Q_0^\infty + \tilde{\mathcal{E}}_p). \quad (3.4.4)$$

Finally there exists $C_1 = C_1(R_1, M, \rho) > 0$ such that for all $u \geq u_0(R_1)$:

$$\int_{\{v \leq v_{R_1}(u)\} \cap \{r \leq R_1\}} (|D_t \phi|^2(u', v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du dv \leq C_1 \cdot \mathcal{E}, \quad (3.4.5)$$

Remark 48. As a by product of our analysis , one can show that

1. Q_0 admits a limit $e_0 \in \mathbb{R}$ when $r \rightarrow +\infty$. This just comes from $\tilde{\mathcal{E}}_p < \infty$, for $p > 1$.
2. The future asymptotic charge exists : there exists $e \in \mathbb{R}$ such that for every $R_1 > r_+$

$$\lim_{t \rightarrow +\infty} Q|_{\gamma_{R_1}}(t) = e.$$

3. The asymptotic charges are small : $\max\{|e|, |e_0|\} \leq C \cdot (Q_0^\infty + \tilde{\mathcal{E}}_p) < C \cdot \delta$.

Remark 49. Notice also that no qualitative strong decay is required on the data for the statement ⁵⁰. The condition $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$ is present simply to exclude constant solutions that do not decay.

Remark 50. Notice that (3.4.3) is stated for **all** $v \leq v_R(u)$, in particular v can be arbitrarily close to $-\infty$. This is consistent with $e^{2K+u} D_u \phi \in L^\infty$, as we request initially in our hypothesis 4 and prove everywhere on the space-time in Lemma 3.7.5, c.f. also Remark 57. This is because — by (3.2.2) — if $e^{2K+u} |D_u \phi| \lesssim 1$:

$$\int_{u_0(v)}^{+\infty} \frac{|D_u \phi|^2}{\Omega^2} du \sim e^{-2K+v} \int_{u_0(v)}^{+\infty} e^{-2K+u} |e^{2K+u} D_u \phi|^2 du \lesssim e^{-2K+v} \int_{u_0(v)}^{+\infty} e^{-2K+u} du \lesssim 1.$$

This subtlety is related to the degenerescence of the vector field $\Omega^{-2} \partial_u$ as v tends to $-\infty$, c.f. Remark 41.

Now, we want to prove the following fact : in the far away region, the only smallness condition required on the charge is $q_0 |Q| < \frac{1}{4}$. Provided energy boundedness and the integrated local energy decay hold, one can prove that decay of the energy follows, in the spirit of Theorem 3.3.3.

For this, we want to show that, if $q_0 |e_0| < \frac{1}{4}$, then for small enough initial energies, $|Q - e_0|$ is small :

Proposition 3.4.2. *Suppose that there exists $1 < p < 2$ such that $\tilde{\mathcal{E}}_p < \infty$.*

It follows that there exists $e_0 \in \mathbb{R}$ such that

$$\lim_{r \rightarrow +\infty} Q_0(r) = e_0.$$

Without loss of generality one can assume that $1 < p < 1 + \sqrt{1 - 4q_0 |e_0|}$.

Assume also that $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.

Now assume (3.4.2), (3.4.3) for $R_1 = R$: there exists $\bar{C} = \bar{C}(M, \rho) > 0$ such that for all $u \geq u_0(R)$ and for all $v \leq v_R(u)$:

⁵⁰Even though the finiteness of the energy gives already some mild decay towards spatial infinity.

$$E(u) = E_R(u) \leq \bar{C} \cdot \mathcal{E}. \quad (3.4.6)$$

$$\int_{\tilde{u}(v)}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v) du' \leq \bar{C} \cdot \mathcal{E}, \quad (3.4.7)$$

where $\tilde{u}(v) = u_0(v)$ if $v \leq v_0(R)$ and $\tilde{u}(v) = u_R(v)$ if $v_0(R) \leq v \leq v_R(u)$.

Assume also (3.4.5) : there exists $\bar{R}_0 = \bar{R}_0(M, \rho) > r_+$ such that for all $\bar{R}_1 > \bar{R}_0$,

there exists $\bar{C}_1 = \bar{C}_1(\bar{R}_1, M, \rho) > 0$ such that

$$\int_{\{u' \leq u\} \cap \{r \leq \bar{R}_1\}} (|D_t \phi|^2(u', v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du dv \leq \bar{C}_1 \cdot \mathcal{E}. \quad (3.4.8)$$

Make also the following smallness hypothesis : for some $\delta > 0$:

$$\tilde{\mathcal{E}}_p < \delta,$$

$$q_0 |e_0| < \frac{1}{4}.$$

There exists $\delta_0 = \delta_0(\mathbf{e}_0, \mathbf{M}, \rho) > 0$ and $C = C(M, \rho) > 0$ such that if $\delta < \delta_0$ then for all (u, v) in the space-time :

$$|Q(u, v) - e_0| \leq C \cdot \tilde{\mathcal{E}}_p, \quad (3.4.9)$$

$$q_0 |Q|(u, v) < \frac{1}{4}. \quad (3.4.10)$$

Moreover, there ⁵¹ exists $\delta_p = \delta_p(\mathbf{e}_0, \mathbf{p}, \mathbf{M}, \rho) > 0$ and $C' = C'(e_0, p, M, \rho) > 0$ such that if $\delta < \delta_p$ then for all $u \geq u_0(R)$:

$$E_p[\psi](u) \leq C' \cdot \tilde{\mathcal{E}}_p. \quad (3.4.11)$$

Remark 51. Note that in this context, we also have

$$|e - e_0| \leq C \cdot \tilde{\mathcal{E}}_p, \quad (3.4.12)$$

so the difference between the initial charge and the asymptotic charge is arbitrarily small, when the initial energies are small. This fact will be used extensively, in particular in the statement of the theorems.

In spherical symmetry, some point-wise decay rates propagate, at least in the past of a forward light cone. This is important to derive point-wise decay estimate, because in the proofs of section 3.7, an initial decay estimate is needed for $D_v \psi$ on the forward light cone $\{u = u_0(R)\}$.

This is also useful to obtain the **finiteness** ⁵² of the r^p weighted energies, for larger p .

We now prove point-wise bounds in the past of a forward light cone in the following proposition :

Proposition 3.4.3. *In the conditions of Proposition 3.4.2, assume moreover that there exists $\omega \geq 0$ and $C_0 > 0$ such that*

$$r |D_v \psi_0| + |\psi_0| \leq C_0 \cdot r^{-\omega}.$$

Then in the following cases

1. $\omega = 1 + \theta$ with $\theta > \frac{q_0 |e_0|}{4}$
2. $\omega = \frac{1}{2} + \beta$ with $\beta \in (-\frac{\sqrt{1-4q_0|e_0|}}{2}, \frac{\sqrt{1-4q_0|e_0|}}{2})$, if $q_0 |e_0| < \frac{1}{4}$,

there exists $\delta = \delta(e_0, \omega, M, \rho) > 0$ and $R_0 = R_0(\omega, e_0, M, \rho) > r_+$ such that if $\tilde{\mathcal{E}}_p < \delta$ and $R > R_0$ then the decay is propagated : there exists $C'_0 = C'_0(C_0, \omega, R, M, \rho, e_0) > 0$ such that for all $u \leq u_0(R)$:

$$|D_v \psi|(u, v) \leq C'_0 \cdot r^{-1-\omega'}, \quad (3.4.13)$$

$$|\psi|(u, v) \leq C'_0 \cdot |u|^{-\omega}, \quad (3.4.14)$$

⁵¹The dependence of δ_p on p only exists as p approaches $1 + \sqrt{1 - 4q_0|e_0|}$.

⁵²Notice that we do not prove or require the **boundedness** of such energy by the initial data. Indeed, for the decay, only the finiteness of these energies is required but not their smallness, unlike for the smaller p energies which ensure that the charge Q is small.

where $\omega' = \min\{\omega, 1\}$.

In that case, for every $0 < p < 2\omega' + 1$, we have the finiteness of the r^p weighted energy on $\mathcal{V}_{u_0(R)}$

$$E_p[\psi](u_0(R)) < \infty.$$

Remark 52. Notice that, even for very decaying data, one cannot obtain a better r decay for $D_v\psi$ than r^{-2} . This is in contrast to the uncharged case $q_0 = 0$ where decay rate was r^{-3} in spherical symmetry, due to the sub-criticality with respect to r decay of the uncharged wave equation, c.f. the discussion in section 3.1.2.

Remark 53. Making an hypothesis on point-wise decay can be thought of as an alternative to circumvent the assumption that $\tilde{\mathcal{E}}_{3-\epsilon} < \infty$ to prove that $E_{3-\epsilon}[\psi](u_0(R)) < \infty$, like we do in the next proposition. This is why we do not need to assume any higher order initial r^p weighted energy boundedness for Theorem 3.3.1 : the result we need is already included in the point-wise assumption, that is stronger.

Finally, we prove that higher order r^p weighted energies boundedness holds on a characteristic constant u surface transverse to null infinity, provided it holds on the initial surface. This proves the boundedness of higher r^p weighted energies, for p close to 3, in order to close the argument of section 3.6.6.

This also allows us to avoid making initial point-wise decay assumptions on the data, starting with “weaker” weighted L^2 boundedness hypotheses.

The proof mainly makes use of a $r^p|u|^s$ weighted estimate in the past of a forward light cone.

Proposition 3.4.4. *Suppose that for all $0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|}$, $\tilde{\mathcal{E}}_p < \infty$.*

We also assume the other hypotheses of Theorem 3.3.3.

Then for all $0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|}$, there ⁵³ exists $\delta_p = \delta_p(\mathbf{e}_0, \mathbf{p}, \mathbf{M}, \rho) > 0$, such that if $\delta < \delta_p$ then for all $u \leq u_0(R)$:

$$E_p[\psi](u) < \infty.$$

3.5 Energy estimates

In the former section, we explained how the global smallness of the charge can be monitored from assumptions on the initial data, and how various energies on characteristic surfaces or domains are bounded by the data on a space-like initial surface Σ_0 .

Taking this for granted, we turn to the proof of energy decay. We are going to assume that the charge is suitably small — according to the needs — and that the energies on the initial characteristic surface $\mathcal{V}_{u_0(R)}$ are finite.

The goal of this section 3.5 and the next section 3.6 is to prove u decay for a characteristic initial value problem on the domain $\mathcal{D}(u_0(R), +\infty)$ — c.f. Figure 3.3 — with data on $\mathcal{V}_{u_0(R)}$, **assuming that the charge is sufficiently small everywhere**.

As explained in the introduction, the proof of decay requires, similarly to the wave equation, three different estimates.

The first is a robust energy boundedness statement, which takes the red-shift effect into account. This allows us in principle to prove point-wise boundedness of the scalar field, following the philosophy of [26].

The second is an integrated local energy estimate, also called Morawetz estimate in reference to the seminal work [62], which is a global estimate on all derivatives but with a sub-optimal r weight at infinity.

Proving and closing these two estimates in the goal of the present section.

Finally the last ingredient which is going to be developed in section 3.6 is a r^p weighted estimate, which gives inverse-polynomial time decay of the energy over the new foliation.

The main difficulty, compared to the wave equation, is that these estimates are all very coupled. In particular the energy boundedness statement is not established independently and requires the use of the Morawetz estimate ⁵⁴, because of the charge terms that cannot be absorbed easily otherwise. These terms are moreover “critical” with respect to r decay, more precisely they possess the same r weight as the positive terms controlled by the energy while their sign is not controllable.

Moreover, whenever an estimate is proven, a term of the form $q_0 Q \Im(\bar{\phi} D\phi)$ arises and these terms necessitate a control of both the zero order term and the derivative at the same time to be absorbed, even with a very small $q_0 Q$. This is already very demanding in the large r region where the estimates are very tight while in the bounded r region, we need to use the smallness of $q_0 Q$ crucially to absorb the r weights.

⁵³The dependence of δ_p on p only exists as p approaches $2 + \sqrt{1 - 4q_0|e_0|}$.

⁵⁴The converse is always true even for the wave equation : the Morawetz estimate cannot close without the boundedness of the energy. However, for the wave equation the boundedness of the energy is already a closed independent estimate.

We start by proving a Morawetz estimate, which bounds bulk terms with sub-optimal weights but **does not control** the boundary terms, proportional to the degenerate energy.

Then we prove a Red-Shift estimate, which gives a good control of the regular derivative of the scalar field $\Omega^{-2}D_u\phi$.

Then we control the boundary terms using the Killing vector field ∂_t .

The difficulty in this last estimate is the presence of the electromagnetic terms which need to be absorbed in the energy of the scalar field. We require the full strength of the Morawetz and the Red-Shift estimate, together with the smallness of q_0Q , to overcome this issue.

Before starting, we are going to recall a few notations from section 3.2.5. We have defined the non-degenerate energy $E(u)$ as

$$E(u) = \int_u^{+\infty} r^2 \frac{|D_u\phi|^2}{\Omega^2}(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v\phi|^2(u, v) dv.$$

It will also be convenient to have a short notation $E^+(u_1, u_2)$ for the sum of all the boundary terms having a role in the energy identity : for all $u_0(R) \leq u_1 \leq u_2$:

$$E^+(u_1, u_2) := E(u_2) + E(u_1) + \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v\phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u\phi|_{\mathcal{I}^+}^2(u) du.$$

It will also be useful for the Red-Shift estimate to have an equivalent notation for the degenerate energy

$$E_{deg}(u) = \int_u^{+\infty} r^2 |D_u\phi|^2(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v\phi|^2(u, v) dv,$$

and for the sum of all scalar field terms appearing when one contracts $\mathbb{T}_{\mu\nu}^{SF}$ with the vector field ∂_t :

$$E_{deg}^+(u_1, u_2) := E_{deg}(u_2) + E_{deg}(u_1) + \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v\phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u\phi|_{\mathcal{I}^+}^2(u) du.$$

3.5.1 An integrated local energy estimate

The goal of this upcoming Morawetz estimate is to control the derivative of ϕ but also $|\phi|^2$ the term of order 0. To do so we need to proceed in two times, using the vector field $X_\alpha = -r^{-\alpha}\partial_{r^*}$.

First we bound the zero order bulk term on a region $r \leq R_0$ for some $R_0(M, \rho) > r_+$, at the cost of some boundary terms and using and the modified current $\tilde{J}_\mu^{X_0}(\phi)$, without obtaining any control of the boundary terms or of the electromagnetic bulk term. It turns out that the boundary term on the time-like boundary $\{r = R_0\}$ coming from $\mathbb{T}_{\mu\nu}^{SF}$ has the right sign. Other boundary terms appear due to the use of χ in the modified current $\tilde{J}_\mu^{X_0}(\phi)$ but they can be absorbed using the red-shift effect, more precisely the smallness of Ω^2 for R_0 close enough to r_+ .

In a second time, we use X_α — for some large $\alpha > 1$ — and the modified current $\tilde{J}_\mu^{X_\alpha}(\phi)$ with an appropriate χ_α on the whole domain $\mathcal{D}(u_1, u_2)$. While in the bulk term we control the derivatives of ϕ everywhere, we only control the zero order term $|\phi|^2$ near infinity, i.e. on a region $[R_\alpha, +\infty[$. The remarkable key feature is that R_α tends to r_+ as α tends to infinity.

Therefore it is enough to take $\alpha > 1$ large enough to have $R_\alpha < R_0$ and we can take a linear combination of the two identities to obtain the control of the zero and first order terms.

The other electromagnetic bulk term are then absorbed using the smallness of the charge.

In more details, we are going to prove the following :

Proposition 3.5.1. *There exists $\alpha = \alpha(M, \rho) > 1$, $\delta = \delta(M, \rho) > 0$ and $C = C(M, \rho) > 0$ so that if $\|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$, then for all $u_0(R) \leq u_1 < u_2$ we have :*

$$\int_{\mathcal{D}(u_1, u_2)} \left(\frac{|D_u\phi|^2 + |D_v\phi|^2}{r^{\alpha-1}} + \frac{|\phi|^2}{r^{\alpha+1}} \right) \Omega^2 du dv \leq C \cdot E_{deg}^+(u_1, u_2) \quad (3.5.1)$$

Proof. We start by a computation, based on the identities of section 3.10.1 and 3.10.2 and on (3.10.11).

In the identities we use the vector field $X_\alpha = -r^{-\alpha}\partial_{r^*} = \frac{r^{-\alpha}}{2}(\partial_u - \partial_v)$ and the function $\chi(r) = \frac{-r^{-\alpha-1}}{2}\Omega^2$. We get, for all $\alpha \in \mathbb{R}$

$$\nabla^\mu \tilde{J}_\mu^{X_\alpha}(\phi) = \frac{\alpha}{r^{\alpha+1}} (|D_u\phi|^2 + |D_v\phi|^2) + \frac{\square(\Omega^2 r^{-\alpha-1})}{4} |\phi|^2 + q_0 Q r^{-\alpha-2} \Im(\bar{\phi} D_t \phi), \quad (3.5.2)$$

where we also used (3.10.12) for the last term.

We first take care of the region $\{r_+ \leq r \leq R_0\}$ where we only aim at controlling the 0 order term $|\phi|^2$. For this we are going to prove the following lemma :

Lemma 3.5.2. *There exists $C = C(M, \rho) > 0$ and $r_+ < R'_0 < R$, $R'_0 = R'_0(M, \rho)$ such that*

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} \Omega^2 r^2 |\phi|^2 dudv \leq C \cdot \left(E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} \Omega^2 q_0 |Q| |\phi| |D_t \phi| dudv \right). \quad (3.5.3)$$

Proof. Using (3.10.13) at $r = r_+$, i.e. where $\Omega^2(r) = 0$, we can prove that for all $\beta \in \mathbb{R}$

$$\square(\Omega^2 r^{-\beta})(r = r_+) = 4(K_+)^2 r_+^{-\beta}. \quad (3.5.4)$$

We take $\beta = 1$. Now since $K_+ > 0$, the coefficients of $\square(\Omega^2 r^{-1})$ only depend on ρ and M and by continuity, there exists $\tilde{R}_0 = \tilde{R}_0(M, \rho)$ such that for all $r_+ \leq r \leq \tilde{R}_0$:

$$\square(\Omega^2 r^{-1}) > 2(K_+)^2 r_+^{-1}.$$

Then, we take $r_+ < R_0 < \tilde{R}_0$ with R_0 to be chosen later and we ⁵⁵ integrate (3.5.2) $\times dvol$ for $\alpha = 0$ on $\mathcal{D}(u_1, u_2) \cap \{r \geq R_0\}$. Notice that the exterior unitary normal on the time-like boundary $\{r = R_0\}$ is $\frac{\partial_u - \partial_v}{2\Omega}$.

$$\begin{aligned} \frac{(K_+)^2}{r_+} \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R_0\}} \Omega^2 r^2 |\phi|^2 dudv + \frac{1}{2} \int_{u_{R_0}(v_R(u_2))}^{+\infty} \left(r^2 |D_u \phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2K) |\phi|^2 - \frac{\Omega^2 r}{2} \partial_u(|\phi|^2) \right) (u', v_R(u_2)) du' \\ + \hat{E}_{R_0}(u_1, u_2) \leq \frac{1}{2} \int_{u_{R_0}(v_R(u_1))}^{+\infty} \left(r^2 |D_u \phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2K) |\phi|^2 - \frac{\Omega^2 r}{2} \partial_u(|\phi|^2) \right) (u', v_R(u_1)) du' \\ + \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R_0\}} 2q_0 |Q| \Omega^2 |\phi| |D_t \phi| dudv + \int_{v_R(u_1)}^{v_R(u_2)} \frac{r_+^2 |D_v \phi|_{\mathcal{H}^+}^2 dv'}{2} dv' \end{aligned} \quad (3.5.5)$$

where $\hat{E}_{R_0}(u_1, u_2)$, the L^2 flux through $\{r = R_0\}$ is defined by

$$\hat{E}_{R_0}(u_1, u_2) = \int_{r=R_0} R_0^2 \left(\frac{|D_u \phi|^2 + |D_v \phi|^2}{4} \right) - \frac{R_0 \Omega^2(R_0)}{8} (\partial_u |\phi|^2 - \partial_v |\phi|^2) + \frac{\Omega^2(R_0) |\phi|^2}{4} (\Omega^2(R_0) - R_0 \cdot 2K(R_0)) dt.$$

First we want to absorb all the boundary terms except the \hat{E}_{R_0} term into a $C'(\rho, M) \cdot E_{deg}^+(u_1, u_2)$ term.

We start to write, using that $\Omega^2 r |\partial_u(|\phi|^2)| \leq \Omega^4 |\phi|^2 + r^2 |D_u \phi|^2$, we see that

$$\frac{r^2 |D_u \phi|^2}{2} - \Omega^2 r \cdot K \cdot |\phi|^2 \leq r^2 |D_u \phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2K) |\phi|^2 - \frac{\Omega^2 r}{2} \partial_u(|\phi|^2).$$

Now we want to control the 0 order term on this constant v boundaries.

For this we use a version of Hardy's inequality (3.2.15) in u of the form

$$\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du \leq 4 \int_{u_i}^{+\infty} \Omega^2 \frac{r^2 |D_u \phi|^2(u, v_R(u_i))}{r^2 \cdot 2K} du + 2\Omega^2(R) \cdot |\phi|^2(u_i, v_R(u_i)).$$

The last term of the right-hand side can be bounded, using Hardy's inequality (3.2.16) in v :

$$R |\phi|^2(u_i, v_R(u_i)) \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Notice also that $r^2 \cdot 2K(r) = 2(M - \frac{\rho^2}{r}) \geq 2K_+ r_+^2$. Therefore, we have

$$\int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du \leq \frac{2}{K_+ r_+^2} \int_{u_i}^{+\infty} \Omega^2 r^2 |D_u \phi|^2(u, v_R(u_i)) du + \frac{2}{R} \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Then, taking R large enough so that $\frac{2}{R} < \frac{2}{K_+ r_+^2}$, we get

$$\int_{u_{R_0}(v_R(u_i))}^{+\infty} \Omega^2 r \cdot 2K |\phi|^2(u, v_R(u_i)) du \leq R_0 \int_{u_i}^{+\infty} \Omega^2 \cdot 2K \cdot |\phi|^2(u, v_R(u_i)) du \leq \frac{2R_0}{K_+ r_+^2} E_{deg}(u_i) \quad (3.5.6)$$

⁵⁵ $dvol$ is defined section 3.10.

Therefore, combining with (3.5.5), it is clear that there exists a constant $C' = C'(M, \rho, R_0) > 0$ such that

$$\frac{(K_+)^2}{r_+} \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R_0\}} \Omega^2 r^2 |\phi|^2 dudv + \hat{E}_{R_0}(u_1, u_2) \leq C' \cdot E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R_0\}} 2q_0 |Q| \Omega^2 |\phi| |D_t \phi| dudv. \quad (3.5.7)$$

Then we take care of the \hat{E}_{R_0} term. First we want to absorb the $\frac{R_0 \Omega^2(R_0)}{8} (\partial_u |\phi|^2 - \partial_v |\phi|^2)$ into the first and the third term of \hat{E}_{R_0} . For this, we simply notice that $\Omega^2 |\partial_\mu \phi|^2 \leq \Omega^4 \frac{|\phi|^2}{R_0} + R_0 |D_\mu \phi|^2$. We then get

$$\hat{E}_{R_0}(u_1, u_2) \geq \int_{r=R_0} R_0^2 \left(\frac{|D_u \phi|^2 + |D_v \phi|^2}{8} \right) - \Omega^2(R_0) \frac{R_0 \cdot K(R_0) |\phi|^2}{2} dt.$$

Then for some small $\epsilon > 0$ to be chosen later, we choose R_0 sufficiently close from r_+ so that

$$\sup_{r_+ \leq r \leq R_0} \Omega^2(r) \frac{K(r)}{2} < \epsilon^2.$$

Then, applying the mean-value theorem in r on $[(1-\epsilon)R_0, R_0]$ for ϵ sufficiently small so that $r_+ < (1-\epsilon)R_0$, we see that there exists $(1-\epsilon)R_0 < R'_0 < R_0$ so that

$$\int_{r=R'_0} |\phi|^2 dt = \frac{\int_{\mathcal{D}(u_1, u_2) \cap \{(1-\epsilon)R_0 \leq r \leq R_0\}} \Omega^2 |\phi|^2 dudv}{\epsilon R_0}.$$

The presence of Ω^2 in the integral is due to the integration in r : indeed $dt dr = \Omega^2 dudv$.

Thus

$$\int_{r=R'_0} \Omega^2(R'_0) \frac{R'_0 K(R'_0) |\phi|^2}{2} dt \leq \epsilon \int_{\mathcal{D}(u_1, u_2) \cap \{(1-\epsilon)R_0 \leq r \leq R_0\}} \Omega^2 |\phi|^2 dudv \leq \epsilon \int_{\mathcal{D}(u_1, u_2) \cap \{r_+ \leq r \leq R_0\}} \Omega^2 |\phi|^2 dudv.$$

Hence if $0 < \epsilon < \frac{(K_+)^2}{2r_+}$, applying the former identities for R'_0 , there exists a constant $C = C(M, \rho) > 0$ and a $R'_0 = R'_0(M, \rho) > r_+$ such that

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} \Omega^2 r^2 |\phi|^2 dudv \leq C \cdot \left(E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} q_0 Q \Omega^2 |\phi| |D_t \phi| dudv \right),$$

which proves the lemma. \square

Now that the 0 order bulk term is controlled on a region $\{r_+ \leq r \leq R_0\}$ **near the event horizon**, we would like a global estimate that can also control the derivative of the scalar field **everywhere**, and the 0 order term **near infinity**.

We use the vector field X_α with α sufficiently large to get a $r^{-\alpha}$ weighted control of $|D\phi|^2$ on the whole region $\mathcal{D}(u_1, u_2)$. However, this identity alone necessarily comes with a loss of control of the 0 order term in a bounded region $[r_+, R(\alpha)]$.

The key point is to notice that $R(\alpha) \rightarrow r_+$ as $\alpha \rightarrow +\infty$. Therefore, at the cost ⁵⁶ of a worse $r^{-\alpha}$ weight, we can take α large enough so that $R(\alpha) < R'_0$. As a result, the loss of control of the X_α estimate can be compensated by the 0 order term estimate obtained prior in $\{r_+ \leq r \leq R'_0\}$.

The proof of this key fact is the object of the following lemma :

Lemma 3.5.3. *For all $\Delta > 0$ and for all $R_1 > r_+$, there exists $\tilde{\alpha}(R_1) > 1$ sufficiently large such that for all $\alpha \geq \tilde{\alpha}(R_1)$ and for all $r \geq R_1$,*

$$\square(\Omega^2 r^{-\alpha})(r) > \Delta \cdot r^{-\alpha-2}. \quad (3.5.8)$$

Proof. For all $\alpha \in \mathbb{R}$, we use (3.10.13) to compute :

$$\square(r^{-\alpha}) = \alpha r^{-\alpha-2} ((\alpha-1)\Omega^2 - r \cdot 2K) = \alpha r^{-\alpha-2} \left((\alpha-1) - \alpha \frac{2M}{r} + (\alpha+1) \frac{\rho^2}{r^2} \right). \quad (3.5.9)$$

Then, using the same identity for α and $\alpha-1$ we get

⁵⁶Which does not matter because we actually want to apply the Morawetz estimate in section 3.6 to a bounded r region $\{r_+ \leq r \leq R\}$.

$$\square((1 - \frac{2M}{r})r^{-\alpha}) = \alpha(\alpha-1)r^{-\alpha-2} - (2\alpha^2 - 3\alpha + 2)2Mr^{-\alpha-3} + (4(\alpha-1)^2M^2 + \alpha(\alpha+1)\rho^2)r^{-\alpha-4} - \alpha(\alpha-1)(2M\rho^2)r^{-\alpha-5}.$$

Then using again the identity for $\alpha - 2$ we get

$$\square(\Omega^2 r^{-\alpha}) = \alpha(\alpha-1)r^{-\alpha-2} - (2\alpha^2 - 3\alpha + 2)2Mr^{-\alpha-3} + (4(\alpha-1)^2M^2 + 2(\alpha^2 - 2\alpha + 3)\rho^2)r^{-\alpha-4} - (2\alpha^2 - 5\alpha + 4)(2M\rho^2)r^{-\alpha-5} + (\alpha-2)(\alpha-1)\rho^4 r^{-\alpha-6}. \quad (3.5.10)$$

Now we want to take α very large. Notice first that

$$\square(\Omega^2 r^{-\alpha}) = \alpha^2 r^{-\alpha-2} \left(1 - \frac{4M}{r} + \frac{4M^2 + 2\rho^2}{r^2} - \frac{4M\rho^2}{r^3} + \frac{\rho^4}{r^4} + p_\alpha(r) \right),$$

where $p_\alpha(r)$ is a degree four polynomial in r^{-1} whose coefficients are all $O(\alpha^{-1})$ as α tends to $+\infty$.

Now, notice that

$$1 - \frac{4M}{r} + \frac{4M^2 + 2\rho^2}{r^2} - \frac{4M\rho^2}{r^3} + \frac{\rho^4}{r^4} = \Omega^4.$$

Hence

$$\square(\Omega^2 r^{-\alpha}) = \alpha^2 r^{-\alpha-2} (\Omega^4 + p_\alpha(r)).$$

We now denote $b_\alpha(r) := \square(\Omega^2 r^{-\alpha})r^{\alpha+2} = \alpha^2 (\Omega^4 + p_\alpha(r))$.

Then we have $\lim_{r \rightarrow +\infty} b_\alpha(r) = \alpha(\alpha-1)$.

Then define $R(\alpha)$ as the maximum r such that $b_\alpha(r) > 0$ on $(R(\alpha), +\infty]$, i.e.

$$R(\alpha) = \sup\{r_+ \leq r \mid \forall r' \geq r, b_\alpha(r') > 0\}.$$

Because of what precedes, it is clear that for all $\alpha > 1$, $R(\alpha) < +\infty$.

Moreover, because of the continuity of b_α , we also have $b_\alpha(R(\alpha)) = 0$.

We want to prove that $R(\alpha)$ is bounded. Since for all $\alpha > 1$, $R(\alpha) < +\infty$, it is actually enough to prove that $R(\alpha)$ is bounded for α in any neighbourhood of $+\infty$.

Suppose not : then there exists a sequence α_n such that

$$\lim_{n \rightarrow +\infty} \alpha_n = +\infty,$$

$$\lim_{n \rightarrow +\infty} R(\alpha_n) = +\infty.$$

Therefore we have

$$b_{\alpha_n}(R(\alpha_n)) = 0 = \alpha_n^2 \cdot \Omega^4(R(\alpha_n)) + p_{\alpha_n}(R(\alpha_n)) \sim \alpha_n^2 \rightarrow +\infty,$$

which is a contradiction. To obtain the infinite limit, we used the fact that $p_{\alpha_n}(R(\alpha_n)) \rightarrow 0$ and $\Omega^4(R(\alpha_n)) \rightarrow 1$ when $n \rightarrow +\infty$.

Now that $R(\alpha)$ is bounded, it admits at least one limit value when $\alpha \rightarrow +\infty$. We call $R_1 \geq r_+$ such a limit value and we take a sequence α_n such that

$$\lim_{n \rightarrow +\infty} \alpha_n = +\infty,$$

$$\lim_{n \rightarrow +\infty} R(\alpha_n) = R_1.$$

then we see that because $p_{\alpha_n}(R(\alpha_n)) \rightarrow 0$:

$$\lim_{n \rightarrow +\infty} \alpha_n^2 \cdot \Omega^2(R_1) = 0,$$

which automatically implies that $\Omega^2(R_1) = 0$ hence $R_1 = r_+$. Since r_+ is the only admissible limit value for $R(\alpha)$ when $\alpha \rightarrow +\infty$, we actually proved that

$$\lim_{\alpha \rightarrow +\infty} R(\alpha) = r_+.$$

More precisely, we have $R(\alpha) = r_+ + o(\alpha^{-2})$ when $\alpha \rightarrow +\infty$.

This implies that for all $\Delta > 0$ and for all $R_1 > r_+$, there exists $\alpha(R_1) > 1$ sufficiently large so that for all $r \geq R_1$,

$$b_\alpha(r) > \Delta,$$

which proves the lemma. \square

Now we come back to the main proof : the next step is to establish the global estimate using X_α . We integrate ⁵⁷ identity (3.5.2) $\times d\text{vol}$ on $\mathcal{D}(u_1, u_2)$. We get

$$\int_{\mathcal{D}(u_1, u_2)} \left[\frac{2\alpha}{r^{\alpha-1}} \Omega^2 (|D_u \phi|^2 + |D_v \phi|^2) + \frac{\square(\Omega^2 r^{-\alpha-1})}{2} \Omega^2 r^2 |\phi|^2 \right] dudv \leq \tilde{E} + \int_{\mathcal{D}(u_1, u_2)} \frac{2q_0|Q|}{r^\alpha} \Omega^2 |\phi| |D_t \phi| dudv,$$

where \tilde{E} accounts for the boundary terms in the identity.

We now need to prove that for some $C' = C'(M, \rho) > 0$,

$$\tilde{E} \leq C' \cdot E_{deg}^+(u_1, u_2).$$

The terms appearing on \mathcal{H}^+ is one component of $E_{deg}^+(u_1, u_2)$, since the terms involving χ are 0.

The term on \mathcal{I}^+ is proportional to

$$\int_{u_1}^{u_2} \left(r^{2-\alpha} |D_u \phi|^2 + \frac{\Omega^2}{2r^\alpha} (\Omega^2 - r \cdot 2K) |\phi|^2 - \frac{\Omega^2 r^{\alpha-1}}{2} \partial_u (|\phi|^2) \right) (u') du'.$$

The first term appears in the X_α identity with a positive sign on the left-hand-side. The third term can be absorbed into the first and second term, using that $\Omega^2 r^{1-\alpha} |\partial_u (|\phi|^2)| \leq \Omega^4 r^{-\alpha} |\phi|^2 + r^{2-\alpha} |D_u \phi|^2$. Only remains a $-\frac{\Omega^2}{2r^\alpha} (r \cdot 2K) |\phi|^2$ term. Now notice that since ⁵⁸ $\lim_{r \rightarrow +\infty} \phi = 0$, this term is actually 0 on \mathcal{I}^+ , for any $\alpha \geq 0$.

The terms on $\{u_i \leq u \leq +\infty, v = v_R(u_i)\}$, $i = 1, 2$ can be written as :

$$\int_{u_i}^{+\infty} \left(r^{2-\alpha} |D_u \phi|^2 + \frac{\Omega^2}{2} (\Omega^2 - r \cdot 2K) r^{-\alpha} |\phi|^2 - \frac{\Omega^2}{2} r^{1-\alpha} \partial_u (|\phi|^2) \right) (u', v_R(u_i)) du'.$$

Similarly to what was written for \mathcal{I}^+ , we can absorb the third term so that we only need to control

$$\int_{u_i}^{+\infty} (r^{2-\alpha} |D_u \phi|^2 - \Omega^2 \cdot 2K \cdot r^{1-\alpha} |\phi|^2) (u', v_R(u_i)) du'.$$

Since $\alpha > 0$, the first term can obviously be controlled by a term proportional to $E_{deg}(u_i)$. This only leaves the $\Omega^2 \cdot 2K \cdot r^{1-\alpha} |\phi|^2$ term to control.

Now we proceed in two times : let $r_+ < R_1 < R$. On $\{r_+ \leq r \leq R_1\}$, we use a similar method to the one leading to (3.5.6). We find that there exists a constant $C_1 = C_1(M, \rho, R_1) > 0$ such that

$$\int_{u_{R_1}(v_R(u_i))}^{+\infty} \Omega^2 r^{1-\alpha} \cdot 2K |\phi|^2(u, v_R(u_i)) du \leq C_1 \cdot E_{deg}(u_i). \quad (3.5.11)$$

Now we can take care of the region $\{R_1 \leq r \leq R\}$. Using Hardy inequality (3.2.17) in u we get :

$$\int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq \frac{4}{\Omega^2(R_1)} \int_{u_i}^{+\infty} r^2 |D_u \phi|^2(u, v_R(u_i)) du + 2R |\phi|^2(u_i, v_R(u_i)).$$

The last term of the right-hand side can be bounded, using Hardy's inequality (3.2.16) in v :

$$R |\phi|^2(u_i, v_R(u_i)) \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Hence there exists a constant $C'_1 = C'_1(M, \rho, R_1) > 0$ such that

$$\int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq C'_1 \cdot E_{deg}(u_i).$$

Now notice that $r^{1-\alpha} \cdot 2K \leq r_+^{-1-\alpha} \cdot 2K_+ r_+^2 \leq r^{1-\alpha} \cdot 2K_+$. Therefore there exists a constant

⁵⁷ $d\text{vol}$ is defined in section 3.10.

⁵⁸This comes from the finiteness of \mathcal{E} , c.f. the proof of Proposition 3.4.1.

$C_1'' = C_1''(M, \rho, R_1) > 0$ such that

$$\int_{u_i}^{u_{R_1}(v_R(u_i))} \Omega^2 r^{1-\alpha} \cdot 2K |\phi|^2(u, v_R(u_i)) du \leq C_1'' \cdot E_{deg}(u_i). \quad (3.5.12)$$

Combining with (3.5.11) and after choosing $R_1 = R_1(M, \rho)$, for instance $R_1 = 2r_+$ we see that the boundary terms on $\{u_i \leq u \leq +\infty, v = v_R(u_i)\}$, $i = 1, 2$ are controlled by $\tilde{C}(M, \rho) \cdot E_{deg}^+(u_1, u_2)$.

Now we take care of the terms on $\{v_R(u_i) \leq v \leq +\infty, u = u_i\}$ for $i = 1, 2$.

We can see that it is enough to control $\int_{v_R(u_i)}^\infty (r^{2-\alpha} |D_v \phi|^2(u_i, v') + r^{-\alpha} |\phi|^2(u_i, v')) dv'$.

We can use Hardy's inequality (3.2.18) under the form

$$\int_{v_R(u_i)}^{+\infty} |\phi|^2(u_i, v) dv \leq \frac{4}{\Omega^4(R)} \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Therefore,

$$\int_{v_R(u_i)}^\infty (r^{2-\alpha} |D_v \phi|^2(u_i, v') + r^{-\alpha} |\phi|^2(u_i, v')) dv' \leq R^{-\alpha} [1 + 4\Omega^{-4}(R)] E_{deg}(u_i) \leq E_{deg}^+(u_1, u_2),$$

after taking R large enough.

Therefore we proved that for some $C' = C'(M, \rho) > 0$,

$$\int_{\mathcal{D}(u_1, u_2)} \left[\frac{2\alpha}{r^{\alpha-1}} \Omega^2 (|D_u \phi|^2 + |D_v \phi|^2) + \frac{\square(\Omega^2 r^{-\alpha-1})}{2} \Omega^2 r^2 |\phi|^2 \right] dudv \leq C' \cdot E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2)} \frac{2q_0 |Q|}{r^\alpha} \Omega^2 |\phi| |D_t \phi| dudv. \quad (3.5.13)$$

Now take $\Delta = 2$ and $R_1 = R'_0$ in Lemma 3.5.3, where $r_+ < R'_0(M, \rho)$ is the radius of Lemma 3.5.2. Then following Lemma 3.5.3 we can take $\alpha = \alpha(M, \rho) > 1$ large enough so that

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} \frac{\square(\Omega^2 r^{-\alpha-1})}{2} \Omega^2 r^2 |\phi|^2 dudv + \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R'_0\}} \Omega^2 r^{-\alpha-1} |\phi|^2 dudv < \int_{\mathcal{D}(u_1, u_2)} \frac{\square(\Omega^2 r^{-\alpha-1})}{2} \Omega^2 r^2 |\phi|^2 dudv.$$

Then, since on $[r_+, R'_0]$, $\square(\Omega^2 r^{-\alpha-1})$ is bounded by a constant only depending on M and ρ , we find using Lemma 3.5.2 that there exists $\tilde{C} = \tilde{C}(M, \rho) > 0$ such that

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R'_0\}} \Omega^2 \left(r^{-\alpha-1} + \frac{|\square(\Omega^2 r^{-\alpha-1})|}{2} r^2 \right) |\phi|^2 dudv \leq \tilde{C} \cdot \left(E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R'_0\}} q_0 |Q| \Omega^2 |\phi| |D_t \phi| dudv \right).$$

Therefore, combining with (3.5.13) we get that there exists $\bar{C} = \bar{C}(M, \rho) > 0$ such that

$$\int_{\mathcal{D}(u_1, u_2)} \left[\frac{\alpha}{r^{\alpha-1}} \Omega^2 (|D_u \phi|^2 + |D_v \phi|^2) + \Omega^2 \frac{|\phi|^2}{r^{\alpha+1}} \right] dudv \leq \bar{C} \cdot \left(E_{deg}^+(u_1, u_2) + \int_{\mathcal{D}(u_1, u_2)} \frac{q_0 |Q|}{r^\alpha} \Omega^2 |\phi| |D_t \phi| dudv \right). \quad (3.5.14)$$

Now notice that

$$\frac{2|\phi| |D_t \phi|}{r^\alpha} \leq \frac{|\phi|^2}{r^{\alpha+1}} + \frac{|D_v \phi|^2 + |D_u \phi|^2}{2r^{\alpha-1}}.$$

Then, if $|Q|$ is small enough so that

$$\bar{C} \cdot q_0 \sup_{\mathcal{D}(u_1, u_2)} |Q| < \min\{1, 2\alpha\},$$

then the interaction term can be absorbed into the left-hand-side of (3.5.14), which proves Proposition 3.5.1. \square

3.5.2 A Red-shift estimate

In this section, we are going to prove that the non-degenerate energy near the event horizon is bounded by the degenerate energy. This echoes with the red-shift estimates pioneered in [26] and [27]. The main difference here is that we cannot obtain non-degenerate energy boundedness as an independent statement, due to the presence of the charge term. Indeed we need to use the control of the 0 order term near the event horizon, obtained priorly by the Morawetz estimate (3.5.1).

This red-shift estimate is proved using the vector field $\Omega^{-2}\partial_u$, which is regular across the event horizon.

We integrate the resulting vector field identity on $\{r_+ \leq r \leq R\}$. Although a term appears on the time-like boundary $\{r = R_0\}$, we can control it by the degenerate energy, in the same way we did for the Morawetz estimate. We are borrowing a few important arguments from section 3.5.1. We are going to prove the following proposition :

Proposition 3.5.4. *For all $r_+ < R_0 < R$ and for all $0 < \epsilon < 1$ sufficiently small, there exists $C = C(R_0, \epsilon, M, \rho) > 0$, $\delta = \delta(R_0, \epsilon, M, \rho) > 0$ and $(1 - \epsilon)R_0 < \tilde{R}_0 < (1 + \epsilon)R_0$ such that if $\|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$ then for all $u_0(R) \leq u_1 < u_2$:*

$$\begin{aligned} & \int_{u_{\tilde{R}_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du + \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq \tilde{R}_0\}} \frac{|D_u \phi|^2}{\Omega^2} dudv \\ & \leq C \cdot \left[E_{deg}^+(u_1, u_2) + \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_1)) du \right] \end{aligned} \quad (3.5.15)$$

Proof. We are going to consider the vector field $X_{RS} = \frac{\partial_u}{\Omega^2}$. We also choose $\chi = -\frac{1}{r}$. Then we get

$$\nabla^\mu \tilde{J}_\mu^{X_{RS}}(\phi) = \frac{4K(r)|D_u \phi|^2}{\Omega^4} - \frac{K(r)|\phi|^2}{r^2} + \frac{q_0 Q}{r^2 \Omega^2} \Im(\bar{\phi} D_u \phi),$$

We now integrate this identity, multiplied⁵⁹ by $dvol$, on $\{r_+ \leq r \leq R_0\}$:

$$\begin{aligned} & \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R_0\}} \left(\frac{8K(r)r^2 |D_u \phi|^2}{\Omega^2} - 2K(r)\Omega^2 |\phi|^2 + 2q_0 Q \Im(\bar{\phi} D_u \phi) \right) dudv \\ & + \int_{u_{R_0}(v_R(u_2))}^{+\infty} \left(\frac{r^2 |D_u \phi|^2}{\Omega^2} + \frac{\Omega^2}{2} |\phi|^2 - \frac{r}{2} \partial_u(|\phi|^2) \right) (u', v_R(u_2)) du' \\ & + \hat{E}_{R_0}^{RS}(u_1, u_2) = \int_{u_{R_0}(v_R(u_1))}^{+\infty} \left(\frac{r^2 |D_u \phi|^2}{\Omega^2} + \frac{\Omega^2}{2} |\phi|^2 - \frac{r}{2} \partial_u(|\phi|^2) \right) (u', v_R(u_1)) du', \end{aligned} \quad (3.5.16)$$

where $\hat{E}_{R_0}^{RS}(u_1, u_2)$, the L^2 flux through $\{r = R_0\}$ is defined by

$$\hat{E}_{R_0}^{RS}(u_1, u_2) = \int_{r=R_0} R_0^2 \left(\frac{|D_u \phi|^2}{2\Omega^2(R_0)} \right) - \frac{R_0}{4} (\partial_u |\phi|^2 - \partial_v |\phi|^2) + \frac{\Omega^2(R_0)}{2R_0^2} |\phi|^2 dt.$$

For the constant v boundary term, similarly to what was done in section 3.5.1, we can absorb the second term into the others using the fact that $r|\partial_u |\phi|^2| \leq \Omega^2 |\phi|^2 + \frac{r^2 |D_u \phi|^2}{\Omega^2}$.

This only leaves the $\frac{\Omega^2}{2} |\phi|^2$ term to control on $[u_{R_0}(v_R(u_1)), +\infty) \times \{v_R(u_1)\}$. Using (3.5.6), we see that there exists $\tilde{C} = \tilde{C}(M, \rho, R_0) > 0$ such that

$$\int_{u_{R_0}(v_R(u_1))}^{+\infty} \Omega^2 |\phi|^2(u', v_R(u_1)) du' \leq \tilde{C} \cdot E_{deg}(u_1).$$

Therefore we proved that for some $\bar{C} = \bar{C}(M, \rho, R_0) > 0$

$$\begin{aligned} & \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R_0\}} \left(\frac{8K(r)r^2 |D_u \phi|^2}{\Omega^2} - 2K(r)\Omega^2 |\phi|^2 + 2q_0 Q \Im(\bar{\phi} D_u \phi) \right) dudv \\ & + \int_{u_{R_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v_R(u_2)) du' \\ & + \hat{E}_{R_0}^{RS}(u_1, u_2) \leq \int_{u_{R_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v_R(u_1)) du' + \bar{C} E_{deg}^+(u_1, u_2). \end{aligned}$$

⁵⁹ $dvol$ is defined in section 3.10.

Now we want to control $\hat{E}_{R_0}^{RS}$ term. Notice that its first term has the right sign. For the other two terms, we use the same technique as for the proof of Lemma (3.5.2). Therefore, for any $0 < \epsilon < 1$ and after an application of the mean-value theorem on $[(1 - \epsilon)R_0, (1 + \epsilon)R_0]$ with the Morawetz estimate proved in former section, we see that there exists $C' = C'(M, \rho, R_0, \epsilon) > 0$ and $(1 - \epsilon)R_0 < \tilde{R}_0 < (1 + \epsilon)R_0$ such that

$$\left| \int_{r=\tilde{R}_0} -\frac{\tilde{R}_0}{4} (\partial_u |\phi|^2 - \partial_v |\phi|^2) + \frac{\Omega^2(\tilde{R}_0)}{\tilde{R}_0^2} |\phi|^2 dt \right| \leq C' \cdot E_{deg}^+(u_1, u_2).$$

Therefore we have

$$\begin{aligned} & \left(\frac{8K(r)r^2|D_u\phi|^2}{\Omega^2} - 2K(r)\Omega^2|\phi|^2 + 2q_0Q\Im(\bar{\phi}D_u\phi) \right) dudv + \int_{u_{\tilde{R}_0}(v_R(u_2))}^{+\infty} \frac{r^2|D_u\phi|^2}{\Omega^2}(u', v_R(u_2))du' \\ & \leq \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2|D_u\phi|^2}{\Omega^2}(u', v_R(u_1))du' + (\bar{C} + C') \cdot E_{deg}^+(u_1, u_2). \end{aligned} \quad (3.5.17)$$

Then we use the Morawetz estimate (3.5.1) to control the 0 order term : there exists $C_0 = C_0(M, \rho, R_0) > 0$ such that

$$\left| \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq \tilde{R}_0\}} K(r)\Omega^2|\phi|^2 dudv \right| \leq C_0 \cdot E_{deg}^+(u_1, u_2). \quad (3.5.18)$$

Then, combining with (3.5.17) and noticing that $2K \cdot r^2 \geq 2K_+ r_+^2$, we see that there exists $C'_0 = C'_0(M, \rho, R_0, \epsilon) > 0$ such that

$$\begin{aligned} & \int_{\mathcal{D}(u_1, u_2) \cap \{r \leq \tilde{R}_0\}} \frac{r^2|D_u\phi|^2}{\Omega^2} dudv + \int_{u_{\tilde{R}_0}(v_R(u_2))}^{+\infty} \frac{r^2|D_u\phi|^2}{\Omega^2}(u', v_R(u_2))du' \\ & \leq C'_0 \cdot \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq \tilde{R}_0\}} \frac{q_0|Q|}{4} |\bar{\phi}| |D_u\phi| dudv + \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2|D_u\phi|^2}{\Omega^2}(u', v_R(u_1))du' + E_{deg}^+(u_1, u_2) \right). \end{aligned} \quad (3.5.19)$$

Then for $|Q|$ small enough and using (3.5.18), we can absorb the interaction term into $\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq \tilde{R}_0\}} \frac{r^2|D_u\phi|^2}{\Omega^2} dudv$ and $E_{deg}^+(u_1, u_2)$. The idea is the same as the one that was used to conclude the proof of Proposition 3.5.1.

This concludes the proof of Proposition 3.5.4. \square

3.5.3 Boundedness of the energy

This estimate is by far the most delicate in this section. Even though for a charged scalar, a positive conserved quantity —arising from the coupling of the scalar field and the electromagnetic tensors— is still available via the vector field ∂_t , it is of little direct use because one of its component, involving the charge, does not decay in time since the charge approaches a constant value at infinity.

This problem does not exist on Minkowski space-time —where the charge is constrained to tend to 0 towards time-infinity— or in the uncharged case $q_0 = 0$ where the conservation laws are uncoupled and the use of ∂_t gives an estimate only in terms of scalar fields quantities.

Our strategy to deal with this charge term is to absorb the charge difference into a term involving the scalar field. More precisely, the key point is to control the **fluctuations of the charge** by the energy of the scalar field. This idea already appeared in [37], in the context of the black hole interior.

In the exterior, we do this in four different steps.

In the first one, we take care of the domain $\mathcal{D}(u_1, u_2) \cap \{r \geq R\}$ where we integrate Q^2 towards γ_R . This estimate leaves a term $R^{-1} \cdot (Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1)))$ on γ_R to be controlled.

In the second step, we transport in u the $R^{-1} \cdot (Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1)))$ term towards the curve γ_{R_0} for some fixed $r_+ < R_0 < R$, $R_0 = R_0(M, \rho)$. This term can be controlled by the $r^2|D_u\phi|^2$ part of the energy on $\{R_0 \leq r \leq R\}$ and by the $r^2|D_v\phi|^2$ part of the energy on $\{R \leq r\}$, using a Hardy-type inequality in each region. Every-time a small term $q_0|Q|$ multiplies the terms controlled by the energy, which is why they can be absorbed.

This estimate leaves this time a term ⁶⁰ $R^{-1} \cdot (Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)))$.

In the third step, we take care of the domain $\mathcal{D}(u_1, u_2) \cap \{r \leq R\}$ where we integrate Q^2 towards γ_{R_0} . This leaves a charge difference term on γ_{R_0} : $(\frac{1}{r_+} - \frac{1}{R}) (Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)))$.

⁶⁰The first multiplicative term is not a typo : it is indeed R and not R_0 .

In the fourth step, we control the two terms $(Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)))$ using the Morawetz estimate of section 3.5.1. The key point is that R_0 is chosen “in between” r_+ and R , so that it is not too close to the event horizon and not too close either from infinity. Since the argument loses a lot of Ω^{-2} and r weights, it is fortunate that these losses are bounded (they only depend on ρ and M) and can therefore be absorbed using the smallness of $q_0|Q|$. The argument does not depend on the smallness of R , which can still be taken arbitrarily large for the following sections, in particular section 3.6.

The different curves are summed up by the Penrose diagram of Figure 3.2.

The use of the vector field ∂_t on the domain $\mathcal{D}(u_1, u_2)$ gives rise to an energy identity, summed up by the following lemma :

Lemma 3.5.5. *For all $u_0(R) \leq u_1 < u_2$ we have the following energy identity :*

$$\begin{aligned} & \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_2}^{+\infty} \left(r^2 |D_u \phi|^2(u, v_R(u_2)) + \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_2)) \right) du \\ & \quad + \int_{v_R(u_2)}^{+\infty} \left(r^2 |D_v \phi|^2(u_2, v) + \frac{2\Omega^2 Q^2}{r^2}(u_2, v) \right) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \\ & = \int_{u_1}^{+\infty} \left(r^2 |D_u \phi|^2(u, v_R(u_1)) + \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_1)) \right) du + \int_{v_R(u_1)}^{+\infty} \left(r^2 |D_v \phi|^2(u_1, v) + \frac{2\Omega^2 Q^2}{r^2}(u_1, v) \right) dv. \end{aligned} \quad (3.5.20)$$

With the notations of this section, it can also be written as

$$\begin{aligned} & \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_2}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_2)) du + \int_{v_R(u_2)}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u_2, v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + E_{deg}(u_2) \\ & = \int_{u_1}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} \frac{2\Omega^2 Q^2}{r^2}(u_1, v) dv + E_{deg}(u_1). \end{aligned} \quad (3.5.21)$$

Proof. The proof is an elementary computation based on the fact that $\nabla^\mu (\mathbb{T}_{\mu\nu}^{EM} + \mathbb{T}_{\mu\nu}^{SF}) = 0$ and that ∂_t is a Killing vector field of the Reissner–Nordström metric, c.f. section 3.10 for more details. We omit the (easy) argument in which one writes the estimate on a truncated domain $\mathcal{D}(u_1, u_2) \cap \{v \leq v_0\}$ and sends v_0 toward $+\infty$ to obtain the claimed boundary terms on \mathcal{I}^+ . We are going to repeat this omission in what follows. \square

The goal of this section is to prove the following :

Proposition 3.5.6. *There exists $C = C(M, \rho) > 0$, $\delta = \delta(M, \rho) > 0$ such that*

if $\|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$, we have for all $u_0(R) \leq u_1 < u_2$:

$$\begin{aligned} & \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_2}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du + \int_{v_R(u_2)}^{+\infty} r^2 |D_v \phi|^2(u_2, v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \\ & \leq C \cdot \left(\int_{u_1}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} r^2 |D_v \phi|^2(u, v_R(u_1)) dv \right). \end{aligned} \quad (3.5.22)$$

With the notations of this section, it can also be written as

$$\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + E(u_2) \leq C \cdot E(u_1). \quad (3.5.23)$$

In short, the strategy is to bound all the terms involving Q^2 by $C'(M, \rho) \cdot q_0|Q| \cdot E^+(u_1, u_2)$ for a constant $C'(M, \rho) > 0$ that only depends on the black hole parameters. Then we take $|Q|$ to be small enough so that $C'(M, \rho) \cdot q_0|Q| < 1$, in order to absorb those terms into the others.

Step 1 : the region $\mathcal{D}(u_1, u_2) \cap \{r \geq R\}$

Lemma 3.5.7.

$$\begin{aligned} & \int_{v_R(u_2)}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u_2, v) dv - \int_{v_R(u_1)}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u_1, v) dv \leq \frac{Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1))}{R} \\ & + 4q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \Omega^{-2}(R) \left[\int_{v_R(u_2)}^{+\infty} r^2 |D_v \phi|^2(u_2, v) dv + \int_{v_R(u_1)}^{+\infty} r^2 |D_v \phi|^2(u_1, v) dv \right] \end{aligned} \quad (3.5.24)$$

Proof. We start to prove the identity :

$$\int_{v_R(u_i)}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u_i, v) dv = \frac{Q^2(u_i, v_R(u_i))}{R} + 2 \int_{v_R(u_i)}^{+\infty} q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v) dv \quad (3.5.25)$$

For this we write using (3.2.9), for all $v_R(u_i) \leq v$:

$$Q^2(u_i, v) = Q^2(u_i, v_R(u_i)) + 2 \int_{v_R(u_i)}^v q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v') dv'.$$

Then for the first term, we just need to notice that because $\Omega^2 = \partial_v r$:

$$\int_{v_R(u_i)}^{+\infty} \frac{\Omega^2}{r^2}(u_i, v) dv = \frac{1}{R}.$$

For the second term we integrate by parts as :

$$\int_{v_R(u_i)}^{+\infty} \frac{\Omega^2}{r^2}(u_i, v) \left(\int_{v_R(u_i)}^v q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v') dv' \right) dv = \int_{v_R(u_i)}^{+\infty} q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v) dv.$$

The boundary terms cancelled because Q^2 is bounded :

$$\lim_{v \rightarrow +\infty} \frac{1}{r(u_i, v)} \int_{v_R(u_i)}^v 2q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v') dv' = \frac{Q^2(u_i, v) - Q^2(u_i, v_R(u_i))}{r(u_i, v)} = 0.$$

Therefore (3.5.25) is proven.

Thereafter we use Hardy's inequality (3.2.18) under the form

$$\int_{v_R(u_i)}^{+\infty} |\phi|^2(u_i, v) dv \leq \frac{4}{\Omega^4(R)} \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

We can then use Cauchy-Schwarz to get that

$$\int_{v_R(u_i)}^{+\infty} q_0 Q r^2 \Im(\phi \overline{D_v \phi})(u_i, v) dv \leq 2q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv,$$

which concludes the proof of Lemma 3.5.7. □

Step 2 : transport in u of $Q^2_{|\gamma_R}$ to $Q^2_{|\gamma_{R_0}}$

Lemma 3.5.8. For all $r_+ < R_0 < R$ we have

$$\begin{aligned} \frac{Q^2(u_2, v_R(u_2)) - Q^2(u_1, v_R(u_1))}{R} & \leq \frac{Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1))}{R} \\ & + 5q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2). \end{aligned} \quad (3.5.26)$$

Proof. We start by an identity coming directly from (3.2.8) :

$$Q^2(u_i, v_R(u_i)) = Q^2(u_{R_0}(v_R(u_i)), v_R(u_i)) + 2 \int_{u_i}^{u_{R_0}(v_R(u_i))} q_0 Q r^2 \Im(\phi \overline{D_u \phi})(u, v_R(u_i)) du.$$

Using Hardy inequality (3.2.17) in u we get :

$$\int_{u_i}^{u_{R_0}(v_R(u_i))} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq 4 \int_{u_i}^{u_{R_0}(v_R(u_i))} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_i)) du + 2R |\phi|^2(u_i, v_R(u_i)).$$

The last term of the right-hand side can be bounded, using Hardy's inequality (3.2.16) in v :

$$R |\phi|^2(u_i, v_R(u_i)) \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

We then use Cauchy-Schwarz, taking advantage of the fact that $r \leq R$:

$$2 \left| \int_{u_i}^{u_{R_0}(v_R(u_i))} q_0 Q r^2 \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du \right| \leq R \cdot q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \int_{u_i}^{u_{R_0}(v_R(u_i))} \left(\frac{r^2 |D_u \phi|^2}{\Omega^2} + \Omega^2 |\phi|^2 \right)(u, v_R(u_i)) du.$$

Then we combine with the estimates proven formerly to get

$$\begin{aligned} & 2 \left| \int_{u_i}^{u_{R_0}(v_R(u_i))} q_0 Q r^2 \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du \right| \\ & \leq R \cdot q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \left(5 \int_{u_i}^{u_{R_0}(v_R(u_i))} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_i)) du + 2\Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv \right). \end{aligned} \quad (3.5.27)$$

Then if we take R large enough so that $2\Omega^{-2}(R) < 5$, this last estimate proves Lemma 3.5.8. \square

Step 3 : the region $\mathcal{D}(u_1, u_2) \cap \{r \leq R\}$

Lemma 3.5.9. *For all $r_+ < R_0 < R$ we have*

$$\begin{aligned} & \int_{u_2}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u, v_R(u_2)) du - \int_{u_1}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u, v_R(u_1)) du \leq \\ & \frac{Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1))}{r_+} \left(1 - \frac{r_+}{R}\right) + \frac{5}{2} q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2). \end{aligned} \quad (3.5.28)$$

Proof. We start to prove an identity for all $r_+ < R_0 < R$:

$$\begin{aligned} & \int_{u_i}^{+\infty} \frac{\Omega^2 Q^2}{r^2}(u, v_R(u_i)) du = \frac{Q^2(u_{R_0}(v_R(u_i)), v_R(u_i))}{r_+} \left(1 - \frac{r_+}{R}\right) \\ & + 2 \int_{u_{R_0}(v_R(u_i))}^{+\infty} q_0 Q r \left(1 - \frac{r}{r_+}\right) \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du + 2 \int_{u_i}^{u_{R_0}(v_R(u_i))} q_0 Q r \left(1 - \frac{r}{R}\right) \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du \end{aligned} \quad (3.5.29)$$

After dividing $[u_i, +\infty]$ into $[u_i, u_{R_0}(v_R(u_i))]$ and $[u_{R_0}(v_R(u_i)), +\infty]$, the method of proof is similar to that of the estimate 3.5.25, except that we integrate Q^2 in u towards γ_{R_0} this time.

Taking $R_0 < 2r_+$ and R large enough so that $\frac{R_0}{r_+} - 1 < 1 - \frac{r_+}{R}$, we can write that

$$\begin{aligned} & \left| \int_{u_{R_0}(v_R(u_i))}^{+\infty} q_0 Q r \left(1 - \frac{r}{r_+}\right) \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du + \int_{u_i}^{u_{R_0}(v_R(u_i))} q_0 Q r \left(1 - \frac{r}{R}\right) \Im(\bar{\phi} D_u \phi)(u, v_R(u_i)) du \right| \\ & \leq \left(1 - \frac{r_+}{R}\right) q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \int_{u_i}^{+\infty} r |\Im(\bar{\phi} D_u \phi)|(u, v_R(u_i)) du \end{aligned} \quad (3.5.30)$$

Using Hardy inequality (3.2.17) in u we get :

$$\int_{u_i}^{+\infty} \Omega^2 |\phi|^2(u, v_R(u_i)) du \leq 4 \int_{u_i}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_i)) du + 2R |\phi|^2(u_i, v_R(u_i)).$$

The last term of the right-hand side can be bounded, using Hardy's inequality (3.2.16) in v :

$$R|\phi|^2(u_i, v_R(u_i)) \leq \Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Combining all the former estimates and using Cauchy-Schwarz, we finally get :

$$\int_{u_i}^{+\infty} r |\Im(\bar{\phi} D_u \phi)| (u, v_R(u_i)) du \leq \frac{5}{2} \int_{u_i}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2} (u, v_R(u_i)) du + 2\Omega^{-2}(R) \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv \leq \frac{5}{2} E(u_i), \quad (3.5.31)$$

where we took R large enough so that $2\Omega^{-2}(R) < \frac{5}{2}$.

We then combine with (3.5.30), which proves Lemma 3.5.9. \square

Step 4 : control of the $Q^2_{|\gamma_{R_0}}$ terms by the Morawetz estimate

Lemma 3.5.10. *For all $r_+ < R'_0 < R$ there exists $r_+ < R_0 < R'_0$ and $C_4 = C_4(R'_0, M, \rho) > 0$ such that*

$$Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)) \leq C_4 q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) \cdot E^+(u_1, u_2). \quad (3.5.32)$$

Proof. For this proof, we work in (t, r^*) coordinates. We define $t_i = 2v_R(u_i) - R_0^*$ such that the space-time point (t_i, R_0^*) corresponds to the space-time point $(u_{R_0}(v_R(u_i)), v_R(u_i))$.

Using (3.2.8), (3.2.9) we see that on γ_{R_0} :

$$Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)) = Q^2(t_2, R_0^*) - Q^2(t_1, R_0^*) = 2R_0^2 \int_{t_1}^{t_2} q_0 Q \Im(\bar{\phi} D_{r^*} \phi)(t, R_0^*) dt.$$

This estimate is valid for any $r_+ < R_0 < R$.

Now fix some $r_+ < R'_0 < R$. Using the mean-value theorem in r , we see that there exists a $r_+ < \tilde{R}_0 < R'_0$ such that :

$$\int_{r_+}^{R'_0} \left[\int_{t_1}^{t_2} q_0 Q \Im(\bar{\phi} D_{r^*} \phi)(t, r^*) dt \right] dr = (R'_0 - r_+) \int_{t_1}^{t_2} q_0 Q \Im(\bar{\phi} D_{r^*} \phi)(t, (\tilde{R}_0)^*) dt.$$

From now on, we choose R_0 to be exactly that \tilde{R}_0 and we take $R'_0 < 2r_+$.

Therefore, since $R_0 < R'_0$ and $R'_0 < 2r_+$ we have

$$Q^2(u_{R_0}(v_R(u_2)), v_R(u_2)) - Q^2(u_{R_0}(v_R(u_1)), v_R(u_1)) \leq 4R'_0 \int_{r_+}^{R'_0} \left[\int_{t_1}^{t_2} q_0 Q \Im(\bar{\phi} D_{r^*} \phi)(t, r^*) dt \right] dr.$$

Then Lemma 3.5.10 follows from a direct application of Cauchy-Schwarz and the Morawetz estimate (3.5.1). \square

Completion of the proof of Proposition 3.5.6

Proof. Now we choose $R'_0 = 2r_+$ and take the $R_0 > r_+$ of Lemma 3.5.10. We now apply Lemma 3.5.8 and Lemma 3.5.9 with that R_0 . Coupled with Lemma 3.5.7 and identity (3.5.20) we find that there exists $C_5 = C_5(M, \rho) > 0$ such that

$$\begin{aligned} & \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_2}^{+\infty} r^2 |D_u \phi|^2(u, v_R(u_2)) du + \int_{v_R(u_2)}^{+\infty} r^2 |D_v \phi|^2(u_2, v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \\ & \leq \int_{u_1}^{+\infty} r^2 |D_u \phi|^2(u, v_R(u_1)) du + \int_{v_R(u_1)}^{+\infty} r^2 |D_v \phi|^2(u_1, v) dv + C_5 q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2). \end{aligned} \quad (3.5.33)$$

Now we apply the Red-shift estimate (3.5.15) for say $R_0 = 2r_+$ and $\epsilon = \frac{1}{4}$: there exists $C_6 = C_6(M, \rho) > 0$ such that

$$C_6^{-1} \int_{u_{\tilde{R}_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du \leq \frac{1}{2} E_{deg}^+(u_1, u_2) + \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_1)) du. \quad (3.5.34)$$

Therefore, adding (3.5.33) and (3.5.34) and absorbing the $\frac{1}{2} E_{deg}^+(u_1, u_2)$, both in the left and right-hand side we get

$$\begin{aligned} & \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_2}^{+\infty} r^2 |D_u \phi|^2(u, v_R(u_2)) du + 2C_6^{-1} \int_{u_2}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du \\ & \quad + \int_{v_R(u_2)}^{+\infty} r^2 |D_v \phi|^2(u_2, v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \leq \quad (3.5.35) \\ & 3E_{deg}(u_1) + 2 \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_1)) du + 2C_5 q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2), \end{aligned}$$

Without loss of generality⁶¹, we can take $C_6 > 2\Omega^{-2}(\tilde{R}_0)$ so that

$$2C_6^{-1} \int_{u_2}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du \leq \int_{u_2}^{u_{\tilde{R}_0}(v_R(u_2))} r^2 |D_u \phi|^2(u, v_R(u_2)) du + 2C_6^{-1} \int_{u_{\tilde{R}_0}(v_R(u_2))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_2)) du.$$

Therefore, also using the fact that $3E_{deg}(u_1) + 2 \int_{u_{\tilde{R}_0}(v_R(u_1))}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u, v_R(u_1)) du \leq 5E(u_1)$ we get

$$\int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + 4C_6^{-1} E(u_2) + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \leq 5E(u_1) + 2C_5 q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) E^+(u_1, u_2). \quad (3.5.36)$$

Now choosing $|Q|$ small enough so that $C_5 C_6 q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right) < 1$, we can absorb the $E^+(u_1, u_2)$ term on the left-hand-side, which proves Proposition 3.5.6. \square

As a by-product of the boundedness of the energy, we finally close the Morawetz estimate of section 3.5.1. We indeed proved that :

Proposition 3.5.11. *There exists $\alpha(M, \rho) > 1$, $C = C(M, \rho) > 0$, $\delta = \delta(M, \rho) > 0$ such that if $\|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$, we have for all $u_0(R) \leq u_1 < u_2$:*

$$\int_{\mathcal{D}(u_1, u_2)} \left(\frac{|D_u \phi|^2 + |D_v \phi|^2}{r^{\alpha-1}} + \frac{|\phi|^2}{r^{\alpha+1}} \right) \Omega^2 dudv \leq C \cdot E(u_1). \quad (3.5.37)$$

As a consequence — also combining with the red-shift estimate — we have the following

Corollary 3.5.12. *There exists $C = C(M, \rho, R) > 0$, $\delta = \delta(M, \rho) > 0$ such that if $\|Q\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$, we have for all $u_0(R) \leq u_1 < u_2$:*

$$\int_{\mathcal{D}(u_1, u_2) \cap \{r \leq R\}} \left(r^2 |D_v \phi|^2 + \frac{r^2 |D_u \phi|^2}{\Omega^4} + |\phi|^2 \right) \Omega^2 dudv \leq C \cdot E(u_1). \quad (3.5.38)$$

3.6 Decay of the energy

We now establish the time decay of the energy. We use the r^p method developed in [28]. The main idea is that the boundedness of r^p weighted energies \tilde{E}_p can be converted into time decay for the standard energy E on the V -shaped foliation \mathcal{V}_u we introduced in section 3.2.4.

The papers [2], [28], [63] and [72] all treat the linear wave equation on black hole space-times, in different contexts. Compared to these works, the main difference and difficulty for the charged scalar field model is the presence of a **non-linear** term, coming from the interaction with the Maxwell field.

⁶¹This only makes the former estimate worse, and C_6 can still be taken to depend on M and ρ only.

The whole objective of this section is to find a way absorb this interaction term, for various values of p and to prove the boundedness of the r^p weighted energy.

We are going to assume the energy boundedness and the Morawetz estimates of the former section. As before, to treat the interaction term, we rely on the smallness of the charge Q , however it is not as drastic as in the former section : indeed, $q_0|Q|$ must be smaller than a **universal** constant, independent on the black holes parameters. This is one of the key ingredients of the proof of Theorem 3.3.3.

Notice also that, thanks to the charge a priori estimates from section 3.4, provided that the initial energy of the scalar field is sufficiently small, it is equivalent to talk about the smallness of Q , of e or of e_0 , c.f. Remark 45. This remark will be used implicitly throughout this section.

The first step is establish a r^p hierarchy for $p < 2$. For this, we use a Hardy type inequality and we absorb the interaction term into the $\int_{u_1}^{u_2} E_{p-1}[\psi](u)du$ term ⁶²of the left-hand-side. The smallness of the charges Q and e is essential and limits the maximal p to $p_{max} < 1 + \sqrt{1 - 4q_0|e|}$, which tends to 2 as e tends to 0.

This method cannot be extended for $p > 2$ because for a scalar field ϕ , $r\phi$ admits a finite generically non-zero limit ψ — the radiation field — towards \mathcal{I}^+ . This very fact imposes that the r^{p-3} weight in the Hardy estimate is strictly inferior to r^{-1} , hence $p < 2$ is necessary to apply the same argument, even when $e \rightarrow 0$.

Still, this first step already gives some decay of the energy that will be crucial for the second step.

The second step uses a different strategy : this time, while we still use the Hardy inequality, we apply it to a weaker r weight. This is because the Hardy inequality proceeds with a maximal r^{-1} weight. This allows us to take p larger, up to almost 3 as e tends to 0. On the other hand, we now have to absorb the interaction term into $E_p[\psi](u)$, in contrast to what was done in the first step.

This is done in three steps : first we prove that a differential inequation involving $E_p[\psi](u)$ holds, where the error terms are multiplied by a constant $\nu(e)$ depending *essentially* only on the charge e .

The next step is to prove that $\nu(e)$ is small indeed: for this, we mostly require the smallness of $q_0|e|$, after various optimisation procedures.

Then, we integrate the differential inequation à la Grönwall, making use of the smallness of $\nu(e)$ to prove that $E_p(u)$ enjoys a small control growth in u , of the order $u^{2\epsilon(e)}$.

Finally we use the pigeon-hole principle to get u decay of $\tilde{E}_{p-1}(u)$, of the order $u^{-1+2\epsilon(e)}$.

It is interesting to notice that the procedure does **not** close the boundedness of the r^p weighted energy for the largest p that we consider but allows for a small u growth, due to the use of a Grönwall-type argument.

More details on this crucial and delicate step can be found in the introduction of section 3.6.3.

This procedure imposes to take $p < 3 - \epsilon(q_0e)$, with $\epsilon(q_0e) = O((q_0|e|)^{\frac{1}{2}})$ as $e \rightarrow 0$ and gives the boundedness of r^p weighted energies for such a p range.

This finally proves that the standard energy decays at a rate that tends to 3 when e tends to 0.

More precisely, the goal of this section is to prove the following result :

Proposition 3.6.1. *Suppose that the energy boundedness (3.5.23) and the Morawetz estimates (3.5.38) are valid. Assume that $q_0|e| \leq 0.08267$. Assume also that for all $0 \leq p' < 2 + \sqrt{1 - 4q_0|e|}$, $E_{p'}(u_0(R)) < \infty$.*

Then there exists $2 < p(e) < 2 + \sqrt{1 - 4q_0|e|}$, with $p(e) \rightarrow 3$ as $e \rightarrow 0$ such that,

For all $0 \leq p \leq p(e)$, there exist $\delta = \delta(p, e, M, \rho) > 0$ such that

if $\|Q - e\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$ then there exists $R_0 = R_0(p, M, \rho, e) > r_+$ such that if $R > R_0$, there exists $D = D(M, \rho, R, p, e) > 0$ such that for all $u > 1$:

$$\tilde{E}_p(u) \leq D_p \cdot u^{p-p(e)}, \quad (3.6.1)$$

in particular,

$$E(u) \leq D_0 \cdot u^{-p(e)}. \quad (3.6.2)$$

3.6.1 Preliminaries on the decay of the energy

In this section, we are going to establish a few preliminary results on the energy decay. The main goal is to understand how the boundedness of r^p weighted energies implies the time decay of the un-weighted energy.

The estimates are so tight that they can be closed only with the smallness of the charge. Therefore it is very important to monitor the constants as carefully as possible.

As we will see, the lowest order term, which should be controlled together with the r^p weighted energy, is bounded by a large constant and the smallness of the charge cannot make it smaller. The key point is that this term actually enjoys additional u decay so that the large constant can be absorbed for large u .

This motivates the following definitions, some of which have already been encountered by the reader :

⁶²For a definition of $E_p[\psi](u)$, c.f. section 3.2.5 or section 3.6.1.

$$\begin{aligned}
E(u) &= \int_u^{+\infty} r^2 \frac{|D_u \phi|^2}{\Omega^2}(u', v_R(u)) du' + \int_{v_R(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv, \\
E^+(u_1, u_2) &:= E(u_2) + E(u_1) + \int_{v_R(u_1)}^{v_R(u_2)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{u_1}^{u_2} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du, \\
E_p[\psi](u) &:= \int_{v_R(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv, \\
\tilde{E}_p(u) &:= E_p[\psi](u) + E(u).
\end{aligned}$$

It should be noted that $\tilde{E}_0(u)$ and $E(u)$ are comparable.

In what follows, we make a repetitive implicit use of the trivial inequality $E_p[\psi](u) \leq \tilde{E}_p(u)$.

We now establish a preliminary lemma, that reduces the problem to understanding the interaction term.

Lemma 3.6.2. *There exists $C_1 = C_1(M, \rho, R) > 0$ such that for every $0 < p < 3$ and for all $u_0(R) \leq u_1 < u_2$*

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq [1 + P_0(R)] \left(E_p[\psi](u_1) + \left| \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv \right| \right) + C_1 \cdot E(u_1), \quad (3.6.3)$$

where $P_0(r)$ is a polynomial in r that behaves like $O(r^{-1})$ as r tends to $+\infty$ and with coefficients depending only on M and ρ .

Proof. We multiply (3.2.6) by $2r^p \overline{D_v \psi}$ and take the real part. We get :

$$r^p \partial_u (|D_v \psi|^2) = \Omega^2 r^{p-3} \left(-2M + \frac{2\rho^2}{r} \right) \partial_v (|\psi|^2) - 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}).$$

We integrate this identity on $\mathcal{D}(u_1, u_2) \cap \{r \geq R\}$ and after a few integrations by parts we get

$$\begin{aligned}
& \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} \left(pr^{p-1} |D_v \psi|^2 + \left(2M(3-p) - (4-p) \frac{2\rho^2}{r} \right) r^{p-4} |\psi|^2 \right) [1 + P_1(r)] dudv \\
& + \int_{v_R(u_2)}^{+\infty} r^p |D_v \psi|^2(u_2, v) dv + \int_{u_1}^{u_2} \Omega^2 \left(2M - \frac{2\rho^2}{r} \right) r^{p-3} |\psi|_{\mathcal{I}^+}^2(u) du \\
& = \int_{u_1}^{u_2} \Omega^2 \left(2M - \frac{2\rho^2}{R} \right) R^{p-3} |\psi|^2(u, v_R(u)) du + \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv + \int_{v_R(u_1)}^{+\infty} r^p |D_v \psi|^2(u_1, v) dv, \quad (3.6.4)
\end{aligned}$$

where $P_1(r)$ is a polynomial in r that behaves like $O(r^{-1})$ as r tends to $+\infty$ and with coefficients depending only on M and ρ .

Since $p < 3$ we can take R large enough so that $|P_1(r)| < 1$ and $2M(3-p) - (4-p) \frac{2\rho^2}{r} > 0$.

Now, we can use the Morawetz estimate and an argument similar to the one employed to prove (3.7.15) to establish that there exists $\tilde{C} = \tilde{C}(M, \rho, R) > 0$ such that

$$\int_{u_1}^{u_2} \Omega^2 \left(2M - \frac{2\rho^2}{R} \right) R^{p-3} |\psi|^2(u, v_R(u)) du \leq \tilde{C} \cdot E(u_1).$$

We then have established :

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + E_p[\psi](u_2) \leq [1 + P_0(R)] \left(E_p[\psi](u_1) + \left| \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv \right| \right) + \tilde{C} \cdot E(u_1), \quad (3.6.5)$$

where $P_0(r)$ is a polynomial in r that behaves like $O(r^{-1})$ as r tends to $+\infty$ and with coefficients depending only on M and ρ .

Finally, to obtain the claimed estimate (3.6.3), it is enough to add (3.5.22) to the previous inequality. This concludes the proof of the lemma. \square

We are now going to explain how the boundedness of the r^p weighted energies $\tilde{E}_p(u)$ implies the time decay of the standard energy $E(u)$. This is the core of the new method invented in [28]. We write a **refinement** of the classical argument with the notations of the paper. This will be important in section 3.6.6.

Lemma 3.6.3. Suppose that there exists $1 < p < 2$ and $C > 0$ such that for all $u_0(R) \leq u_1 < u_2$

$$\int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq C \cdot \tilde{E}_p(u_1), \quad (3.6.6)$$

and

$$\int_{u_1}^{u_2} E_0[\psi](u) du + \tilde{E}_1(u_2) \leq C \cdot \tilde{E}_1(u_1). \quad (3.6.7)$$

Then for all $0 \leq s \leq p$ and for all $k \in \mathbb{N}$, there exists $C' = C'(M, \rho, C, R, s, E(u_0(R)), k) > 0$ such that for all $u > 1$:

$$\tilde{E}_s(u) \leq C' \cdot \left(u^{-k(1-\frac{s}{p})} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_p(u')}{u^{p-s}} \right). \quad (3.6.8)$$

Remark 54. In the case $s = 0$, this lemma broadly says that, up to an arbitrarily fast decaying polynomial term and a *sup*, the energy decays like $u^{-p} \cdot \tilde{E}_p$. This formulation that was not present in the original article [28] will be useful to obtain the almost optimal energy decay in section 3.6.6.

Remark 55. The lemma is stated for $1 < p < 2$ because it is going to be applied in this paper to

$p < 1 + \sqrt{1 - 4q_0|e|}$. Of course, a similar result still holds without that restriction, using a similar method. In the present paper, we will need more than this lemma in section 3.6.6, where a major improvement of the method will become necessary.

Proof. During this proof, we make use of the following notation : $A \lesssim B$ if there exists a constant

$C_0 = C_0(M, \rho, C, R, s, \tilde{E}_p(u_0(R)), k) > 0$ such that $A \leq C_0 B$.

We take $(u_n)_{n \in \mathbb{N}}$ to be a dyadic sequence, i.e. $u_{n+1} = 2u_n$ and $u_0 > 1$.

We first apply the mean-value theorem on $[u_n, u_{n+1}]$: there exists $u_n < \bar{u}_n < u_{n+1}$ such that

$$E_{p-1}[\psi](\bar{u}_n) = \frac{\int_{u_n}^{u_{n+1}} E_{p-1}[\psi](u) du}{u_n},$$

where we used the fact that $u_{n+1} - u_n = u_n$.

Applying (3.6.6) gives

$$E_{p-1}[\psi](\bar{u}_n) \lesssim \frac{\tilde{E}_p(u_n)}{u_n}. \quad (3.6.9)$$

Now notice that because $1 < p < 2$, $(p-1) \in (0, 1)$ and $(2-p) \in (0, 1)$. We then apply Hölder's inequality under the form :

$$E_1[\psi](u) \leq (E_{p-1}[\psi](u))^{p-1} (E_p[\psi](u))^{2-p},$$

where we used the fact that $1 = (p-1)(p-1) + (2-p)p$.

Then we apply (3.6.9) and the precedent inequality for $u = \bar{u}_n$:

$$E_1[\psi](\bar{u}_n) \lesssim \frac{\sup_{u_n \leq u \leq u_{n+1}} \tilde{E}_p(u)}{(u_n)^{p-1}}. \quad (3.6.10)$$

We now use (3.6.7) and (3.6.10) to get :

$$\int_{u_n}^{u_{n+1}} E_0[\psi](u) du \lesssim \tilde{E}_1(u_n) \lesssim \tilde{E}_1(\bar{u}_{n-1}) \lesssim E(u_{n-1}) + \frac{\sup_{u_{n-1} \leq u \leq u_n} \tilde{E}_p(u)}{(u_n)^{p-1}}.$$

We used that $u_n \sim u_{n-1} \leq \bar{u}_{n-1} \leq u_n$ and for the last inequality, we also used the energy boundedness (3.5.22) under the form $E(\bar{u}_{n-1}) \lesssim E(u_{n-1})$.

Then, following the method developed in [28], we add a multiple of the Morawetz estimate (3.5.38) to cover for the $\{r \leq R\}$ energy bulk terms. After a standard computation, identical to that done in [28] we get

$$\int_{u_n}^{u_{n+1}} E(u) du \lesssim E(u_{n-1}) + \frac{\sup_{u_{n-1} \leq u \leq u_n} \tilde{E}_p(u)}{(u_n)^{p-1}}.$$

Now using the mean-value theorem again on $[u_{n-1}, u_n]$: there exists $\tilde{u}_{n-1} \in (u_{n-1}, u_n)$ such that

$$E(\tilde{u}_{n-1}) = \frac{\int_{u_{n-1}}^{u_n} E(u) du}{u_n - u_{n-1}} \lesssim \frac{E(u_{n-2})}{u_n} + \frac{\sup_{u_{n-2} \leq u \leq u_{n-1}} \tilde{E}_p(u)}{(u_n)^p},$$

Making use of the energy boundedness from (3.5.22) finally gives

$$E(u_n) \lesssim \frac{E(u_{n-2})}{u_n} + \frac{\sup_{u_{n-2} \leq u \leq u_{n-1}} \tilde{E}_p(u)}{(u_n)^p}. \quad (3.6.11)$$

Now take an integer $k \geq 1$ and iterate (3.6.11) k times :

$$E(u_n) \lesssim \frac{E(u_{n-2k})}{(u_n)^k} + \sum_{i=1}^k \frac{\sup_{u_{n-2i} \leq u \leq u_{n-2i+1}} \tilde{E}_p(u)}{(u_n)^{p+i-1}}.$$

Now we can use the energy boundedness to bound $E(u_{n-2k})$: there exists a constant $\tilde{C} = \tilde{C}(M, \rho) > 0$ such that

$$E(u_{n-2k}) \leq C \cdot E(u_0(R)).$$

We then have

$$E(u_n) \lesssim (u_n)^{-k} + \frac{\sup_{u_{n-2k} \leq u \leq u_{n-1}} \tilde{E}_p(u)}{(u_n)^p}.$$

For $u \in (u_n, u_{n+1})$, we get

$$E(u) \lesssim u^{-k} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_p(u')}{u^p}, \quad (3.6.12)$$

where we used the fact that $\frac{u}{2} \leq u_n \leq u$.

This already proves the lemma for $s = 0$.

For the general case $0 < s \leq p$, we use the Hölder inequality under the form

$$E_s[\psi](u) \leq (E_0[\psi](u))^{1-\frac{s}{p}} (E_p[\psi](u))^{\frac{s}{p}} \lesssim (E(u))^{1-\frac{s}{p}} \left(\tilde{E}_p(u) \right)^{\frac{s}{p}},$$

where we used the fact that $s = (1 - \frac{s}{p}) \cdot 0 + \frac{s}{p} \cdot p$. For the last inequality, we also used the fact that for R large enough we have

$$E_0[\psi] \leq 12E(u).$$

This is because $|D_v \psi|^2 \leq 2(r^2 |D_v \phi|^2 + \Omega^4 |\phi|^2) \leq 2(r^2 |D_v \phi|^2 + |\phi|^2)$ and the use of the Hardy inequality (3.2.18) under the form

$$\int_{v_R(u_i)}^{+\infty} |\phi|^2(u_i, v) dv \leq \frac{4}{\Omega^4(R)} \int_{v_R(u_i)}^{+\infty} r^2 |D_v \phi|^2(u_i, v) dv.$$

Then using (3.6.12) we get

$$\tilde{E}_s(u) \lesssim u^{-k} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_p(u')}{u^p} + u^{-k(1-\frac{s}{p})} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_p(u')}{u^{p-s}},$$

where we used the boundedness of $\tilde{E}_p(u)$ for the third term and the inequality $(a+b)^\theta \leq C(\theta) \cdot (a^\theta + b^\theta)$, for $\theta = \frac{s}{p}$ or $\theta = 1 - \frac{s}{p}$.

Putting everything together and taking u large we get

$$\tilde{E}_s(u) \lesssim u^{-k(1-\frac{s}{p})} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_p(u')}{u^{p-s}},$$

which concludes the proof of the lemma. □

3.6.2 Application for the r^p method for $p < 2$.

In this section, we establish the boundedness of the r^p weighted energy for p slightly inferior to 2. The argument relies on the absorption of the interaction term into the $\int_{u_1}^{u_2} E_{p-1}[\psi](u) du$ term. To do this, we make use of a Hardy-type inequality, coupled with the Morawetz estimate to treat the boundary terms on the curve $\{r = R\}$.

As a corollary, we establish the u^{-p} decay of the un-weighted energy for the range of p considered. This step is crucial to establish the almost-optimal decay claimed in section 3.6.3.

The maximal value of $q_0|Q|$ authorized in this section is $\frac{1}{4}$.

We start by the main result of this section, which is the computation that illustrates how the interaction term can be absorbed into the left-hand-side.

Proposition 3.6.4 (r^p weighted energy boundedness). *Assume that $q_0|e| < \frac{1}{4}$.*

For all $\eta_0 > 0$, there exists $R_0 = R_0(M, \rho, \eta_0, e) > r_+$, $\delta = \delta(M, \rho, \eta_0, e) > 0$ such that if $\|Q - e\|_{L^\infty(\mathcal{D}(u_0(R), +\infty))} < \delta$, then for all $1 - \sqrt{1 - 4q_0|e|} < p < 1 + \sqrt{1 - 4q_0|e|}$ and $R > R_0$, there exists $C_2 = C_2(M, \rho, R, \eta_0, p, e) > 0$ such that for all $u_0(R) \leq u_1 < u_2$, we have

$$\left(p - \frac{4q_0|e|}{2-p} - \eta_0\right) \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + \eta_0) \cdot E_p[\psi](u_1) + C_2 \cdot E(u_1). \quad (3.6.13)$$

Proof. We apply Cauchy-Schwarz to control the charge term as :

$$\left| \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv \right| \leq \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-1} |D_v \psi|^2 dudv \right)^{\frac{1}{2}} \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-3} \Omega^4 |\psi|^2 dudv \right)^{\frac{1}{2}}.$$

Using a version of Hardy's inequality (3.2.19), we then establish that

$$\left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-3} \Omega^2 |\psi|^2 dudv \right)^{\frac{1}{2}} \leq \frac{2}{(2-p)\Omega(R)} \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-1} |D_v \psi|^2 dudv \right)^{\frac{1}{2}} + \left(\frac{R^{p-2}}{2-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du \right)^{\frac{1}{2}}$$

Then, using an averaging procedure in r , similar to the one in Lemma 3.5.10, we find that for all $R > 0$, there exists $\frac{R}{2} < \tilde{R} < R$.

$$\int_{u_1}^{u_2} |\psi|^2(u, v_{\tilde{R}}(u)) du \leq \frac{2}{\tilde{R}} \int_{\mathcal{D}(u_1, u_2) \cap \{\frac{R}{2} \leq r \leq R\}} |\psi|^2 dudv.$$

We are going to abuse notations and still call R this \tilde{R} . Using the Morawetz estimate (3.5.1), we see that there exists $C = C(M, \rho, R) > 0$ such that

$$\frac{R^{p-2}}{2-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du \leq \frac{C}{2-p} \cdot E(u_1).$$

Then we apply Cauchy-Schwarz to the charge term and we combine the result with the inequality of Lemma 3.6.2 to get that for all $\eta_0 > 0$:

$$\left(p - \frac{4q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right)}{(2-p)\Omega(R)} - \frac{\eta_0}{2} \right) \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + P(R)) \cdot E_p[\psi](u_1) + \frac{C}{2\eta_0 \cdot (2-p)} \cdot E(u_1). \quad (3.6.14)$$

Then, we can taking R_0 large enough and δ small enough so that :

$$\left| \frac{4q_0 \left(\sup_{\mathcal{D}(u_1, u_2)} |Q| \right)}{(2-p)\Omega(R)} - \frac{4q_0|e|}{(2-p)} \right| \leq \frac{\eta_0}{2},$$

$$1 + P(R) \leq 1 + \eta_0.$$

More precisely, it suffices to have $R_0 > \frac{D(M, \rho) \cdot q_0|e|}{(1 - \sqrt{1 - 4q_0|e|}) \cdot \eta_0}$ and $\delta < 2(1 - \sqrt{1 - 4q_0|e|}) \cdot \eta_0$, for some $D(M, \rho) > 0$.

This concludes the proof of Proposition 3.6.4. \square

To finish this section, we establish the best decay of the energy that we can attain so far. The methods employed are similar to that of [28]. In particular the constants do not need to be monitored as sharply as in section 3.6.3.

Corollary 3.6.5. *Suppose that $q_0|e| < \frac{1}{4}$.*

Then for all $1 < p' < 1 + \sqrt{1 - 4q_0|e|}$, for all $0 \leq s \leq p'$ and for all $k \in \mathbb{N}$ large enough, there exists $C_0 = C_0(M, \rho, e, R, p', s, k, E(u_0(R))) > 0$ and $C'_0 = C'_0(M, \rho, e, R, p', s, k, \tilde{E}_{p'}(u_0(R))) > 0$ such that for all $u > 1$, we have the following energy decay

$$\tilde{E}_s(u) \leq C_0 \cdot \left(u^{-k(1 - \frac{s}{p'})} + \frac{\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_{p'}(u')}{u^{p'-s}} \right) \leq \frac{C'_0}{u^{p'-s}}. \quad (3.6.15)$$

Proof. We make use of the result of Proposition 3.6.4 and take η_0 sufficiently small so that there exists $D_0 = D_0(M, \rho, p', e, R)$ so that for all $u_0(R) \leq u_1 < u_2$:

$$\int_{u_1}^{u_2} E_{p'-1}[\psi](u) du + \tilde{E}_{p'}(u_2) \leq D_0 \cdot \tilde{E}_{p'}(u_1), \quad (3.6.16)$$

and

$$\int_{u_1}^{u_2} E_0[\psi](u) du + \tilde{E}_1(u_2) \leq D_0 \cdot \tilde{E}_1(u_1). \quad (3.6.17)$$

Thus the hypothesis of Lemma 3.6.3 are satisfied for the chosen p' range.

This proves the first inequality of (3.6.15).

The second one solely relies on the boundedness of the r^p weighted energy :

$$\sup_{2^{-2k-1}u \leq u' \leq u} \tilde{E}_{p'}(u') \leq D_0 \tilde{E}_p(u_0(R)).$$

We also need to take $k > p'$ so that $k(1 - \frac{s}{p'}) > p' - s$.

This concludes the proof of Corollary 3.6.5. □

3.6.3 Extension of the r^p method to $p < 3$.

In this section, we establish the improved decay of the energy, at a rate $u^{-p(e)}$, for $2 < p(e) < 2 + \sqrt{1 - 4q_0|e|} < 3$ and provided that $0 \leq q_0|e| < 0.08267$. We prove that $p(e) \rightarrow 3$ as $e \rightarrow 0$, which is the (limit) optimal rate as $e \rightarrow 0$, and we also produce an asymptotic expansion in $q_0|e|$. Unfortunately, $p(e)$ cannot be made explicit easily, for it is obtained as a solution of an optimisation problem.

Since that $p(e) > 2$, we obtain an integrable point-wise decay on the event horizon, crucial for the interior study, c.f. section 3.1.2.

The strategy employed to absorb the interaction term differs radically from that of section 3.6.2, where the maximal p was strictly inferior to 2.

Indeed, we now aim at absorbing this term inside the $\tilde{E}_p(u_2)$ term. To do this, we have to solve an ordinary differential inequation and, making use of the relevant smallness, prove a small growth of $\tilde{E}_{p(e)+2\epsilon}(u) \lesssim u^{2\epsilon}$. This is sufficient to prove the claimed decay.

An optimisation problem arises, as the estimates close more easily when $0 < \epsilon$ is close to $\frac{1}{2}$. On the other hand, taking ϵ too large deteriorates the decay rate so ϵ must also be small enough: therefore a compromise must be found.

The core of the proof is to use a Grönwall like argument (although it involves a square-root, which is not standard) to handle an estimate of the form $E_p(u) \lesssim \nu(e) \cdot (E_{p-1}(u))^{\frac{1}{2}} \cdot (\int_{u_1}^u E_p(u))^{\frac{1}{2}}$, where $\nu(e) > 0$ is a small constant, related the ϵ we mentioned. Ultimately, we will obtain a small growth of the r^p weighted energy by this method : $\tilde{E}_p(u) \lesssim u^{2\epsilon}$, for u large.

The first part of the argument is to prove that $\nu(e)$ is indeed small: to do this, we must impose that $q_0|e|$ is small enough, hence the restriction $q_0|e| < 0.08267$. This is the part which involves the optimisation: c.f. Lemma 3.6.10.

The second part is to establish an approximate version of the estimate $E_p(u_2) \lesssim \nu(e) \cdot (E_{p-1}(u_1))^{\frac{1}{2}} \cdot (\int_{u_1}^{u_2} E_p(u) du)^{\frac{1}{2}}$. The most crucial argument is of a non-linear nature, and is best summed up by the alternative of Lemma 3.6.8. In short, according to how $(\int_{u_1}^{u_2} E_p[\psi](u) du)^{\frac{1}{2}}$ compares with some initial energies, we either have (3.6.28) or (3.6.29).

If we worked directly on the interval $[1, u]$ for large $u > 1$ and apply Lemma 3.6.8, we would obtain the boundedness of \tilde{E}_p immediately **only if** alternative 1 applies, i.e. (3.6.28) is true. On the other hand, in the event when (3.6.29) holds (alternative 2), we obtain a linear growth in u , which is disastrous.

This is why we work with a λ -adic sequence, $\lambda > 1$, to obtain the decay of the energy by a pigeon-hole argument. In this case, in the event that alternative 2 holds on $[u_0 \cdot \lambda^n, u_0 \cdot \lambda^{n+1}]$, we do obtain (almost ⁶³) boundedness of \tilde{E}_p . The traditional use of the r^p method makes use of $\lambda = 2$, namely dyadic sequence. However, the presence of the two alternatives (3.6.28) or (3.6.29) renders necessary to chose λ appropriately. This is because, if (3.6.28) (alternative 1) applies on every λ -adic intervals between 1 and u , then we obtain some $u^{2\epsilon_0}$ growth, where ϵ_0 depends on λ . Since this possibility cannot be excluded, λ has to be chosen in accordance with the ϵ resulting from the optimisation we mentioned earlier.

Because the argument is of a non-linear nature, we proceed by induction, as the induction hypothesis can be fed into Lemma 3.6.8 to obtain an estimate which is sufficient to close the induction step. Upon completion

⁶³The reason why (3.6.29) gives us a $u^{2\epsilon}$ growing weight is because in the induction hypothesis, we already introduced a growth, as seen in (3.6.27). It seems that this growth, however, cannot be avoided.

of the induction, we show that for some $2 < \tilde{p}(e)$ and $0 < \epsilon(e)$, $\tilde{E}_{\tilde{p}(e)} \lesssim u^{-1+2\epsilon(e)}$ and we eventually obtain the energy decay $E(u) \lesssim u^{-\tilde{p}(e)+2\epsilon(e)}$, with Corollary 3.6.5.

Proposition 3.6.6. *Assume that $q_0|e| < 0.08267$.*

Assume also that for all $0 \leq p' < 2 + \sqrt{1 - 4q_0|e|}$, $E_{p'}(u_0(R)) < \infty$ and that the energy boundedness (3.5.23) and the Morawetz estimate (3.5.38) hold.

Then there exists $2 < p(e) < 2 + \sqrt{1 - 4q_0|e|}$ such that, if , then there exists $D = D(M, \rho, R, e) > 0$ such that for all $u > 1$:

$$E(u) \leq \frac{D}{u^{p(e)}}, \quad (3.6.18)$$

$$\tilde{E}_{p(e)-1}(u) \leq \frac{D}{u}, \quad (3.6.19)$$

$$\tilde{E}_{p(e)}(u) \leq D. \quad (3.6.20)$$

Moreover , $p(e) \rightarrow 3$ as $e \rightarrow 0$, and $p(e)$ has the following Taylor expansion when $e \rightarrow 0$:

$$p(e) = 3 - 2\sqrt{\frac{\sqrt{6}}{3}} \cdot (q_0|e|)^{\frac{1}{2}} + O(q_0|e|). \quad (3.6.21)$$

Remark 56. The point of this proposition is two-fold : first, we want to find the maximal $q_0|e|$ such that there exists a $s > 2$ with, eventually, $E(u) \lesssim u^{-s}$, implying an integrable decay for the scalar field on the event horizon $|\phi_{\mathcal{H}^+}|(v) \lesssim v^{-\frac{s}{2}}$. As it turns out, this cannot be manage on the full range $q_0|e| \in (0, \frac{1}{4})$, as our techniques use the smallness of $q_0|e|$: this is why we impose $q_0|e| < 0.08267$. We also want to prove the conjectured limit rate $s = 3$ when the charge is small predicted by [41], i.e. that there exists $0 < \zeta(e) = o(1)$ when $|e| \rightarrow 0$ such that $E(u) \lesssim u^{-3+\zeta(e)}$. This is provided by (3.6.21) and $\zeta(e) = 2\sqrt{\frac{\sqrt{6}}{3}} \cdot (q_0|e|)^{\frac{1}{2}} + O(q_0|e|)$. To achieve this result, the smallness of $q_0|e|$ is exploited again. In the former section, the r^p weighted energy was bounded for a maximal p which was $p_{max} = 1 + \sqrt{1 - 4q_0|e|}$. In this section, we try to increase the maximal p of 1. However, a (necessary) loss is occurred in the process, which is why $p(e) < p_{max} + 1 = 2 + \sqrt{1 - 4q_0|e|}$. For the sake of comparison, observe that the asymptotic expansion of $2 + \sqrt{1 - 4q_0|e|}$ as $e \rightarrow 0$ is $2 + \sqrt{1 - 4q_0|e|} = 3 - 2 \cdot q_0|e| + O((q_0|e|)^2)$, to be compared with (3.6.21), where we "lost" a $(q_0|e|)^{\frac{1}{2}}$ power in the expansion.

Proof. Like in section 3.6.2, we apply Cauchy-Schwarz to control the charge term but we distribute the r weights differently as :

$$|\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv| \leq \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^p |D_v \psi|^2 dudv \right)^{\frac{1}{2}} \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-4} \Omega^4 |\psi|^2 dudv \right)^{\frac{1}{2}}.$$

Using a version of Hardy's inequality (3.2.19), we then establish that

$$\left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-4} \Omega^2 |\psi|^2 dudv \right)^{\frac{1}{2}} \leq \frac{2}{(3-p)\Omega(R)} \left(\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} r^{p-2} |D_v \psi|^2 dudv \right)^{\frac{1}{2}} + \left(\frac{R^{p-3}}{3-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du \right)^{\frac{1}{2}}$$

Combining both inequalities we see that

$$\begin{aligned} |\int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv| &\leq \frac{2}{(3-p)\Omega(R)} \left(\int_{u_1}^{u_2} E_{p-2}[\psi](u) du \right)^{\frac{1}{2}} \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \\ &\quad + \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \left(\frac{R^{p-3}}{3-p} \int_{u_1}^{u_2} |\psi|^2(u, v_R(u)) du \right)^{\frac{1}{2}}. \end{aligned} \quad (3.6.22)$$

The main contribution of the right-hand-side is the first term.

Using the results of section 3.6.2, we see that there exists $C' = C'(M, \rho, R, p, e) > 0$ such that ⁶⁴ for all $u_0(R) \leq u_1 < u_2$:

⁶⁴It is very important that C' , which depends on the large constant R , is multiplying $(E(u_1))^{\frac{1}{2}}$, which enjoys a faster decay in u_1 , so that, for large u_1 , the large constant C' can be absorbed.

$$\left(\int_{u_1}^{u_2} E_{p-2}[\psi](u) du \right)^{\frac{1}{2}} \leq (1 + \eta_0) \cdot (p - 1 - \frac{4q_0|e|}{3-p})^{-\frac{1}{2}} \cdot \left[(E_{p-1}[\psi](u_1))^{\frac{1}{2}} + C' \cdot (E(u_1))^{\frac{1}{2}} \right],$$

where $\eta_0 > 0$ is arbitrarily small. We used the fact that for all $\alpha, \beta > 0$, $\sqrt{\alpha + \beta} \leq \sqrt{\alpha} + \sqrt{\beta}$.

Similarly to the method employed in section 3.6.2, we deal with the second term in the right-hand-side of (3.6.22) using the Morawetz estimate of section 3.5.1. Therefore there exists $D = D(M, \rho, p, e, R) > 0$ such that

$$\begin{aligned} & \left| \int_{\mathcal{D}(u_1, u_2) \cap \{r \geq R\}} \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) du dv \right| \\ & \leq \left[(1 + \eta_0) \cdot \frac{2}{3-p} \cdot (p - 1 - \frac{4q_0|e|}{3-p})^{-\frac{1}{2}} \cdot (E_{p-1}[\psi](u_1))^{\frac{1}{2}} + D \cdot (E(u_1))^{\frac{1}{2}} \right] \cdot \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}}. \end{aligned} \quad (3.6.23)$$

Again, the main contribution in the right-hand-side is the first term. Actually we will see that the $E(u_1)$ term enjoys a better decay in u than $E_{p-1}[\psi](u_1)$ term because in this section $p > 1$. This will later allow us to absorb⁶⁵ the second term in the right-hand-side into the first term, for u_1 large enough.

We then combine the bound from Lemma 3.6.2 with the bound on the charge term (3.6.23) to get for any $\eta > 0$, $\eta_0 > 0$ small enough:

$$\begin{aligned} p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) & \leq (1 + \eta) \cdot E_p[\psi](u_1) + \tilde{f}(p, e, \eta_0) \cdot (E_{p-1}[\psi](u_1))^{\frac{1}{2}} \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \\ & \quad + \tilde{D} \cdot \left[(E(u_1))^{\frac{1}{2}} \cdot \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} + E(u_1) \right], \end{aligned} \quad (3.6.24)$$

where $\tilde{f}(p, e, \eta_0) := (1 + \eta_0) \cdot f(p, e) = (1 + \eta) \cdot 4q_0|e| \cdot (3-p)^{-1} \cdot (p - 1 - \frac{4q_0|e|}{3-p})^{-\frac{1}{2}}$ and we took $R > R'_0(M, \rho, \eta)$ so $P_0(r)$ — defined in the statement of Lemma (3.6.2) — satisfies $|1 + P_0(r)| < (1 + \eta)$ and $\tilde{D} = \tilde{D}(M, \rho, p, e, R) > 0$.

In contrast to the strategy adopted in section 3.6.2, we now aim at absorbing the error term into the $E_p[\psi](u_2)$ term of the left-hand-side.

Therefore we are going to drop temporarily the first term of the left-hand-side and consider the differential functional inequality, with u_2 being the variable, u_1 the constant and $\int_{u_1}^{u_2} E_p[\psi](u) du$ the unknown function

$$E_p[\psi](u_2) \leq \left[\tilde{f}(p, e, \eta_0) \cdot (E_{p-1}[\psi](u_1))^{\frac{1}{2}} + \tilde{D} \cdot (E(u_1))^{\frac{1}{2}} \right] \cdot \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} + (1 + \eta) \cdot E_p[\psi](u_1) + \tilde{D} \cdot E(u_1). \quad (3.6.25)$$

As usual, the main contribution of the right-hand-side is on the first term, the others being treated as errors.

To include also the error terms, we will require a small technical lemma dealing with integration of certain square-root functions :

Lemma 3.6.7.

$$\int_{u_1}^{u_2} \frac{du}{a\sqrt{u} + b} = \frac{2}{a} \left[\sqrt{u} - \frac{b}{a} \log(\sqrt{u} + \frac{b}{a}) \right]_{u_1}^{u_2}, \quad (3.6.26)$$

where for every function f we define $[f(u)]_{u_1}^{u_2} := f(u_2) - f(u_1)$.

Proof. The proof is elementary, using the change of variable $x = \sqrt{u}$. We leave the details to the reader. \square

Now, we are going to integrate the differential equation we obtained earlier. Two behaviours are possible: integrated decay or boundedness. We show that only one of those behaviour can occur on any given interval, which is the object of the following "lemma of two alternatives" :

Lemma 3.6.8. Assume that for some $1 \leq u_1$, $\Delta > 0$, $\epsilon > 0$, $2 < p < 2 + \sqrt{1 - 4q_0|e|}$, we have

$$E_{p-1}[\psi](u_1) \leq \frac{\Delta}{u_1^{1-2\epsilon}}. \quad (3.6.27)$$

Then for any $\beta > 0$, $u_2 \geq u_1$ we either have (what we later call the alternative 1: boundedness)

⁶⁵Even though the constant D can be arbitrarily large.

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + \beta) \cdot \left((1 + \eta) \cdot E_p[\psi](u_1) + \frac{D_0}{(u_1)^{p-1}} \right), \quad (3.6.28)$$

or we have (what we later call the alternative 2: integrated decay)

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq \frac{\left(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta_0) + D'_0 \cdot (u_1)^{\frac{2-p}{2}-\epsilon} \right)^2}{2} \cdot \left(\frac{1 + \beta}{\beta - \log(1 + \beta)} \right) \cdot \left(\frac{u_2 - u_1}{u_1^{1-2\epsilon}} \right), \quad (3.6.29)$$

where $D_0 = \tilde{D} \cdot C'_0$, $D'_0 := \tilde{D} \cdot \sqrt{C'_0}$ and C'_0 is the constant appearing in Corollary 3.6.5.

Proof. We now combine (3.6.27) and the u^{1-p} decay of the energy from Corollary 3.6.5 with (3.6.25) to get:

$$E_p[\psi](u_2) \leq p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + E_p[\psi](u_2) \leq a \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} + b, \quad (3.6.30)$$

with $a = \left(\tilde{f}(p, e, \eta_0) \cdot \sqrt{\Delta} \cdot (u_1)^{-\frac{1}{2}+\epsilon} + D'_0 \cdot (u_1)^{-\frac{p-1}{2}} \right) > 0$ and $b = (1 + \eta) \cdot E_p[\psi](u_1) + \frac{D_0}{(u_1)^{p-1}} > 0$.

Now, using Lemma 3.6.7 we see that for all $0 < u_1 < u_2$:

$$\left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \leq \frac{a}{2} (u_2 - u_1) + \frac{b}{a} \log \left(1 + \frac{a}{b} \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \right) \quad (3.6.31)$$

Now for all $\beta > 0$ we have the following alternative:

Either

$$\left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \leq \beta \cdot \frac{b}{a},$$

in which case, combining with (3.6.30) we find

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq (1 + \beta) \cdot b,$$

which is (3.6.28).

Or

$$\left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \geq \beta \frac{b}{a}.$$

In that case (3.6.30) and (3.6.24) give

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq \left(1 + \frac{1}{\beta} \right) \cdot a \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}}. \quad (3.6.32)$$

Now because the function $\frac{\log(1+x)}{x}$ is decreasing, we also have

$$\frac{b}{a} \log \left(1 + \frac{a}{b} \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \right) \leq \frac{\log(1 + \beta)}{\beta} \left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}}.$$

Hence from (3.6.31) we have

$$\left(\int_{u_1}^{u_2} E_p[\psi](u) du \right)^{\frac{1}{2}} \leq \frac{a}{2(1 - \frac{\log(1+\beta)}{\beta})} (u_2 - u_1).$$

The combination with (3.6.32) gives

$$p \int_{u_1}^{u_2} E_{p-1}[\psi](u) du + \tilde{E}_p(u_2) \leq \frac{a^2 \cdot (1 + \beta)}{2(\beta - \log(1 + \beta))} \cdot (u_2 - u_1).$$

which is exactly (3.6.29), after replacing a by its definition.

□

The function $z(\beta_0) = \frac{(1+\beta_0)^{\frac{1}{2\epsilon}}}{\beta_0 - \log(1+\beta_0)}$ will play a major role, in particular when alternative (3.6.29) holds, as we are going to see later. Assuming $0 < \epsilon < \frac{1}{2}$, it can be shown that function z admits a unique minimum on $(0, +\infty)$ that we denote $\beta(\epsilon)$: this is the β value to which we will later apply Lemma 3.6.8.

Using Taylor expansions, it can be shown that $\beta(\epsilon) \rightarrow 0$, $z(\beta(\epsilon)) \rightarrow +\infty$ as $\epsilon \rightarrow 0$ and more precisely

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \frac{\beta(\epsilon)}{\epsilon} &= 4, \\ \lim_{\epsilon \rightarrow 0} z(\beta(\epsilon)) \cdot \epsilon^2 &= \frac{1}{8}. \end{aligned} \tag{3.6.33}$$

Moreover, it can be proven that $\epsilon \rightarrow z(\beta(\epsilon))$ is strictly decreasing and, on the other end of the interval $(0, \frac{1}{2})$

$$\begin{aligned} \lim_{\epsilon \rightarrow (\frac{1}{2})^-} \beta(\epsilon) &= +\infty, \\ \lim_{\epsilon \rightarrow (\frac{1}{2})^-} z(\beta(\epsilon)) &= 1. \end{aligned} \tag{3.6.34}$$

We are now going to apply the pigeon-hole principle. Because the constants now matter for the decay, we need to use a different version from that developed in [28] or the other subsequent papers. In particular the difference is that we actually use the mean-value theorem instead of the pigeon-hole principle and moreover, we abandon dyadic sequence to use λ -adic sequences⁶⁶ that provide more flexibility.

We take (\tilde{u}_n) to be a λ -adic sequence, i.e. $\tilde{u}_{n+1} = \lambda \cdot \tilde{u}_n$ and $\tilde{u}_0 = U_0 > 1$ and we define $\lambda = \lambda(\epsilon, \eta) > 1$ as

$$\lambda = [(1 + \beta(\epsilon)) \cdot (1 + \eta)]^{\frac{1}{2\epsilon}}, \tag{3.6.35}$$

where $\eta > 0$ is the (arbitrarily small) constant appearing on the right-hand-side of (3.6.28).

To prove the proposition, we are going to proceed by induction, ultimately taking $u_2 = \tilde{u}_{n+1}$, $u_1 = \tilde{u}_n$, $2 < p = \tilde{p}(\epsilon)$, $0 < \epsilon = \epsilon(e) < \frac{1}{2}$, $0 < \beta = \beta(\epsilon(e))$ and apply Lemma 3.6.8.

Let $\Delta > 1$ to be determined later. We make the following induction⁶⁷ hypothesis, for $k \in \mathbb{N}$:

$$\tilde{E}_{p-1}(\tilde{u}_k) \leq \frac{\Delta}{(\tilde{u}_k)^{1-2\epsilon}}, \tag{3.6.36}$$

$$\tilde{E}_p(\tilde{u}_k) \leq \frac{p \cdot \lambda}{\lambda - 1} \cdot \Delta \cdot (\tilde{u}_k)^{2\epsilon}. \tag{3.6.37}$$

First, it is clear that the induction hypothesis is true at $k = 0$ if the following two conditions are satisfied:

$$U_0^{1-2\epsilon} \cdot \tilde{E}_{p-1}(U_0) < \Delta, \tag{3.6.38}$$

$$U_0^{-2\epsilon} \cdot \frac{\lambda - 1}{p \cdot \lambda} \cdot \tilde{E}_p(U_0) < \Delta. \tag{3.6.39}$$

We will check at the end of the induction that these conditions, together with the others we will encounter on the way, can be satisfied for a licit choice of parameters.

Once the induction is closed, we will simply use the boundedness of the \tilde{E}_{p-1} energy proven in former sections to retrieve the claimed decay of the present proposition.

Before we start applying Lemma 3.6.8, we will prove a small technical lemma:

Lemma 3.6.9. *There exists $C'_2 = C'_2(M, \rho, R, p, e) > 0$ such that for all $\eta_0 > 0$ and $n \in \mathbb{N}$, we have :*

$$\tilde{E}_{p-1}(\tilde{u}_{n+1}) \leq (1 + \eta_0) \cdot \frac{\lambda}{\lambda - 1} \cdot \frac{\int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du'}{\tilde{u}_{n+1}} + \frac{C'_2}{(\tilde{u}_n)^{p-1}}, \tag{3.6.40}$$

Proof. Using the mean-value theorem on $[\tilde{u}_n, \tilde{u}_{n+1}]$, we see that there exists $\tilde{u}_n < u < \tilde{u}_{n+1}$ so that

$$E_{p-1}[\psi](u) = \frac{\int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du'}{\tilde{u}_{n+1} - \tilde{u}_n} = \frac{\lambda}{\lambda - 1} \cdot \frac{\int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du'}{\tilde{u}_{n+1}}.$$

Then we use the result of Proposition 3.6.4 of section 3.6.2 to obtain that for some $C_2 = C_2(M, \rho, R, p, e) > 0$, some $C = C(M, \rho) > 0$ and for every $\eta_0 > 0$ small enough :

$$\tilde{E}_{p-1}(\tilde{u}_{n+1}) \leq (1 + \eta_0) \cdot E_{p-1}[\psi](u) + C_2 \cdot E(u) \leq (1 + \eta_0) \cdot E_{p-1}[\psi](u) + C \cdot C_2 \cdot E(\tilde{u}_n),$$

⁶⁶It may seem paradoxical to abandon dyadic sequences for λ -adic ones as in most cases $\lambda > 2$. We make this choice to compensate for the "logarithmic loss" incurred in the event that alternative 1 applies on every (or most) intervals $[\tilde{u}_n, \tilde{u}_{n+1}]$, as we will see.

⁶⁷Notice that we expect $E_p(u)$ to grow slightly in u , at a rate $u^{2\epsilon}$.

where we also used the boundedness of the energy (3.5.22) in the last inequality.

Since $p-1 < 1 + \sqrt{1-4q_0|e|}$, we use the decay of the energy of Corollary 3.6.5 and⁶⁸ setting $C'_2 := C \cdot C_2 \cdot C'_0$, the lemma is proven. \square

Now, we turn to the induction step. We assume that the induction hypothesis (3.6.36), (3.6.37) hold for all $k \in [[0, n]]$ and we want to prove it for $k = n+1$.

As advertised earlier, we apply Lemma 3.6.8 successively to $u_2 = \tilde{u}_{k+1}$, $u_1 = \tilde{u}_k$, $2 < p$, $0 < \epsilon < \frac{1}{2}$, $0 < \beta = \beta(\epsilon)$ for all $k \in [[0, n]]$. Notice that (3.6.27) is always satisfied by (strong) induction.

We are now going to make a case disjunction. The idea is that, if alternative 2 holds for $k = n$, then the claimed decay holds immediately, with a smallness constant that allows us to close the bootstrap. If this is not the case, then we descend in the λ -adic interval until we find a $k(n)$ for which alternative 2 holds. It may not exist, but in any case we can close the estimates provided we are willing to abandon the boundedness of the r^p weighted energy, allowing for an arbitrary small u growth, using⁶⁹ crucially (3.6.35).

To prove the induction step, we are going to show the following stronger estimates, for some $0 < \nu < 1$ (which is authorised to depend on n , although this detail is of no importance):

$$\tilde{E}_{p-1}(\tilde{u}_{n+1}) \leq (1-\nu) \cdot \frac{\Delta}{(\tilde{u}_{n+1})^{1-2\epsilon}}, \quad (3.6.41)$$

$$\tilde{E}_p(\tilde{u}_{n+1}) \leq (1-\nu) \cdot \frac{p \cdot \lambda}{\lambda-1} \cdot \Delta \cdot (\tilde{u}_{n+1})^{2\epsilon}. \quad (3.6.42)$$

The first case of our disjunction is when alternative 2 holds for $u_2 = \tilde{u}_{n+1}$ and $u_1 = \tilde{u}_n$. Then, we see immediately that

$$p \int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u) du \leq \frac{(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta_0) + D'_0 \cdot (\tilde{u}_n)^{\frac{2-p}{2}-\epsilon})^2}{2} \cdot \left(\frac{1+\beta}{\beta - \log(1+\beta)} \right) \cdot (\lambda-1) \cdot (\tilde{u}_n)^{2\epsilon}.$$

Then combining this estimate with (3.6.40) we see that

$$\begin{aligned} \tilde{E}_{p-1}(\tilde{u}_{n+1}) &\leq (1+\eta_0) \cdot \frac{(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta_0) + D'_0 \cdot (\tilde{u}_n)^{\frac{2-p}{2}-\epsilon})^2}{2p} \cdot \left(\frac{1+\beta}{\beta - \log(1+\beta)} \right) \cdot \frac{\lambda \cdot (\tilde{u}_n)^{2\epsilon}}{\tilde{u}_{n+1}} + \frac{C'_2}{(\tilde{u}_n)^{p-1}} \\ &\leq (1+\eta_0)^{\frac{1}{2\epsilon}} \cdot \frac{(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta_0) + D'_0 \cdot (\tilde{u}_n)^{\frac{2-p}{2}-\epsilon})^2}{2p} \cdot \frac{z(\beta(\epsilon))}{\tilde{u}_{n+1}^{1-2\epsilon}} + \frac{C'_2}{(\tilde{u}_n)^{p-1}}, \end{aligned} \quad (3.6.43)$$

where we recall $z(\beta) = \frac{(1+\beta)^{\frac{1}{2\epsilon}}}{\beta - \log(1+\beta)}$, we took $\eta \leq \eta_0$ and we used (3.6.35) for the second inequality. To prove (3.6.36), we first need to insure that

$$\frac{f(p, e)^2}{2p} \cdot z(\beta(\epsilon)) < 1. \quad (3.6.44)$$

For this, we will⁷⁰ chose $2 < p = \tilde{p}(e)$ and $0 < \epsilon = \epsilon(e) < \frac{1}{2}$, after the following (small) lemma :

Lemma 3.6.10. *For all $q_0|e| < 0.08267$, there exists $2 < \tilde{p}(e) < 2 + \sqrt{1-4q_0|e|}$ and $0 < \epsilon(e) < \frac{1}{2}$ such that (3.6.44) holds for $p = \tilde{p}(e)$, $\epsilon = \epsilon(e)$ and*

$$2 < \tilde{p}(e) - 2\epsilon(e) < 2 + \sqrt{1-4q_0|e|}. \quad (3.6.45)$$

Moreover, $\tilde{p}(e) \rightarrow 3$, $\epsilon(e) \rightarrow 0$ as $e \rightarrow 0$ and more precisely, we have the following Taylor expansions

$$\tilde{p}(e) = 3 - \sqrt{\frac{\sqrt{6}}{3}} \cdot (q_0|e|)^{\frac{1}{2}} + O(q_0|e|), \quad (3.6.46)$$

$$\epsilon(e) = \frac{1}{2} \cdot \sqrt{\frac{\sqrt{6}}{3}} \cdot (q_0|e|)^{\frac{1}{2}} + O(q_0|e|), \quad (3.6.47)$$

⁶⁸Notice that C'_2 effectively only depends on M, ρ and ϵ , using (3.6.35) (there is no actual dependence on η , as $\eta \leq 1$).

⁶⁹This is the only reason why we require (3.6.35), imposing that λ grows when ϵ becomes small. Having a large λ allows to compensate for the "logarithmic loss" occurred by the repeated use of alternative 1 on λ -adic intervals, at the cost of a larger constant when alternative 2 occurs (which, in turn, demands a smaller $|e|$ or a smaller $p(e)$ to close the bootstrap).

⁷⁰The $p(e)$ appearing in the statement of the proposition will end up being $p(e) = \tilde{p}(e) - \epsilon(e)$, as $E_{\tilde{p}(e)}$ grows like $u^{2\epsilon(e)}$.

Proof. We start to handle the case when $e \rightarrow 0$ and the asymptotics of (3.6.46), (3.6.47).

First denote $g(p, e) := \frac{f(p, e)^2}{2p} = \frac{8(q_0 e)^2}{p \cdot (p - p_-(e)) \cdot (p_+(e) - p) \cdot (3 - p)}$, where $p_{\pm}(e) := 2 \pm \sqrt{1 - 4q_0|e|}$.

Define, for some $1 - \sqrt{1 - 4q_0|e|} < \alpha(e) < 1$, with $\alpha(e) \rightarrow 0$ as $e \rightarrow 0$: $q_{\alpha}(e) := 3 - \alpha(e)$; notice that $2 < q_{\alpha}(e) < p_+(e)$. Denoting $g_{\alpha}(e) := g(q_{\alpha}(e), e)$ and $\zeta(e) = 1 - \sqrt{1 - 4q_0|e|}$ we see that, we have

$$g_{\alpha}(e) = \frac{4 \cdot (q_0 e)^2}{(3 - \alpha(e)) \cdot \alpha(e) \cdot (\alpha(e) - \zeta(e)) \cdot (1 - \frac{\alpha(e) + \zeta(e)}{2})},$$

hence, as $e \rightarrow 0$, $g_{\alpha}(e) \sim \frac{4 \cdot (q_0 e)^2}{3 \cdot \alpha(e) \cdot (\alpha(e) - \zeta(e))}$. Now, take $\epsilon(e) \rightarrow 0$ as $e \rightarrow 0$. We will try to find the right $\epsilon(e)$ such that (3.6.44) is satisfied, and that maximizes ⁷¹ $\tilde{p}(e) - 2\epsilon(e)$.

With what we said earlier, in particular (3.6.33), it is clear that $z(\beta(\epsilon(e))) \sim (8 \cdot \epsilon^2(e))^{-1}$ as $e \rightarrow 0$, thus

$$g_{\alpha}(e) \cdot z(\beta(\epsilon(e))) \sim \frac{(q_0 e)^2}{6 \cdot \alpha(e) \cdot (\alpha(e) - \zeta(e)) \cdot \epsilon^2(e)}.$$

To satisfy (3.6.44), it is equivalent to require, solving a second order polynomial equation:

$$\alpha(e) > \alpha_-(e) := \frac{\zeta(e) + \sqrt{\zeta(e)^2 + \frac{2(q_0 e)^2}{3\epsilon^2}}}{2}.$$

Using a Taylor expansion, as $\zeta(e) \sim 2q_0|e|$, it is also easy to see as that $e \rightarrow 0$:

$$\alpha_-(e) \sim \frac{q_0|e|}{\epsilon(e) \cdot \sqrt{6}}.$$

We want to find $\epsilon(e)$ so as to maximise $q_{\alpha}(e) - 2\epsilon(e)$ for $\alpha = \alpha_-(e)$ or equivalently minimise $\alpha_-(e) + 2\epsilon(e)$: we find that the function $\epsilon \rightarrow \frac{q_0|e|}{\epsilon \cdot \sqrt{6}} + 2\epsilon$ possess a minimum at $\epsilon = \sqrt{\frac{q_0|e|}{2\sqrt{6}}}$, whose value is $2\sqrt{\frac{\sqrt{6}}{3}} \cdot \sqrt{q_0|e|}$. This gives (3.6.47), noticing that $\sqrt{\frac{1}{2\sqrt{6}}} = \frac{1}{2} \cdot \sqrt{\frac{\sqrt{6}}{3}}$. Noticing that $2\sqrt{\frac{\sqrt{6}}{3}} - 2\sqrt{\frac{1}{2\sqrt{6}}} = \sqrt{\frac{\sqrt{6}}{3}}$, we also obtain (3.6.46). Finally, denoting $p(e) = q_{\alpha}(e) - 2\epsilon(e) = \tilde{p}(e) - 2\epsilon(e)$, we obtain the claimed (3.6.21).

Now, we want to find the largest number $r < \frac{1}{4}$ such that for all $q_0|e| < r$, there exists $0 < \epsilon(e)$, $2 + 2\epsilon(e) < \tilde{p}(e)$ such that (3.6.44) holds. By what we did earlier in the small $|e|$ case, such a r exists. We introduce $p_{\nu} := 2 + \nu$ for some $2\epsilon < \nu < \delta(e)$, to be determined, where we also denoted $\delta = \delta(e) = \sqrt{1 - 4q_0|e|}$. Then we compute

$$g(p_{\nu}, e) = \frac{8(q_0 e)^2}{(2 + \nu) \cdot (1 - \nu) \cdot (\delta^2(e) - \nu^2)}.$$

If we can prove that $g(p_{\nu=2\epsilon}, e) \cdot z(\beta(\epsilon)) < 1$, then, since this is an open condition, it will imply that there exists some $2\epsilon < \nu$ such that $g(p_{\nu}, e) \cdot z(\beta(\epsilon)) < 1$ is true. The earlier condition can be written as

$$\frac{4(q_0 e)^2}{(1 + \epsilon) \cdot (1 - 2\epsilon) \cdot (\delta^2(e) - 4\epsilon^2)} \cdot z(\beta(\epsilon)) < 1. \quad (3.6.48)$$

First, denote $v(\epsilon) := \frac{z(\beta(\epsilon))}{(1 + \epsilon) \cdot (1 - 2\epsilon)}$. The condition (3.6.48) is equivalent, in terms of e to :

$$q_0|e| < \frac{-1 + \sqrt{1 + (1 - 4\epsilon^2) \cdot v(\epsilon)}}{2 \cdot v(\epsilon)}.$$

Now denote $w(\epsilon) := \frac{-1 + \sqrt{1 + (1 - 4\epsilon^2) \cdot v(\epsilon)}}{2 \cdot v(\epsilon)}$: we want to **maximise** $w(\epsilon)$ for $0 < \epsilon < \frac{1}{2}$. This computation is not explicit but can be done numerically to obtain a range of values. To do so, we can notice that $\beta(\epsilon)$ can be expressed "explicitly" (thus plotted easily) with the (-1) branch of the Lambert ⁷² function W_{-1} as

$$\beta(\epsilon) = -\frac{W_{-1}(-(1 - 2\epsilon) \cdot e^{2\epsilon-1}) + 1 - 2\epsilon}{1 - 2\epsilon}.$$

This is because $\beta(\epsilon)$ solves the equation of a critical point $z'(\beta)|_{\beta=\beta(\epsilon)} = 0$, which can also be written as

$$\left(\frac{1}{2\epsilon} - 1\right) \cdot \beta(\epsilon) = \log(1 + \beta(\epsilon)).$$

⁷¹As we will end-up proving $E(u) \lesssim u^{-p(e)}$, for $p(e) := \tilde{p}(e) - 2\epsilon(e)$ for $\tilde{p}(e) = q_{\alpha}(e)$.

⁷² $W_{-1}(x)$, taking values on $[-\exp(-1), 0]$, is defined as the unique solution $y \in (-\infty, -1]$ of $y \exp(y) = x$.

Using a calculus software, we find numerically that $w(\epsilon)$ has a global maximum on $(0, \frac{1}{2})$ at $0.2728 < \epsilon_M < 0.2729$ and moreover $0.08267414 < w(\epsilon_M) < 0.08267415$. Thus the condition $q_0|e| \leq 0.08267$ is sufficient ⁷³ to obtain our $\tilde{p}(e)$ and $\epsilon(e)$, as required by the Lemma. \square

Thus, there exists $0 < \tilde{\nu}(e) < 1$ such that

$$\frac{f(\tilde{p}(e), e)^2}{2\tilde{p}(e)} \cdot z(\beta(\epsilon(e))) \leq 1 - \tilde{\nu}(e). \quad (3.6.49)$$

From now on, we will take $p = \tilde{p}(e)$ and $\epsilon = \epsilon(e)$, and we will omit to write the e dependence.

Therefore, with (3.6.43), and since $\tilde{p}(e) > 2$, it is clear that there exists $\tilde{U}_0(e) > 1$ large enough and $\tilde{\eta}(e) > 0$ small enough such that, if $0 < \eta_0 < \tilde{\eta}(e)$ and

$$U_0 > \tilde{U}_0(e), \quad (3.6.50)$$

then (3.6.41) is satisfied ⁷⁴ for $p = \tilde{p}(e)$, with $\nu = \tilde{\nu}(e)^2$.

Notice that (3.6.29) also gives the following for $\beta = \beta(\epsilon(e))$:

$$\begin{aligned} \tilde{E}_p(\tilde{u}_{n+1}) &\leq \frac{\left(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta) + D'_0 \cdot (\tilde{u}_n)^{\frac{2-p}{2}-\epsilon}\right)^2}{2} \cdot \left(\frac{1+\beta}{\beta - \log(1+\beta)}\right) \cdot (\lambda - 1) \cdot (\tilde{u}_n)^{2\epsilon} \\ &\leq \frac{p \cdot (\lambda - 1)}{\lambda^{1-2\epsilon}} \cdot (1 - \nu) \cdot \Delta \cdot (\tilde{u}_n)^{2\epsilon} \leq \frac{p \cdot (\lambda - 1)}{\lambda} \cdot (1 - \nu) \cdot \Delta \cdot (\tilde{u}_{n+1})^{2\epsilon}. \end{aligned} \quad (3.6.51)$$

Thus, (3.6.37) is true with $\nu = \tilde{\nu}(e)^2$.

Now we treat the other case when alternative 1 of Lemma 3.6.8 holds for $u_2 = \tilde{u}_{n+1}$ and $u_1 = \tilde{u}_n$ (and the same $\beta(\epsilon(e))$ as before). (3.6.28) can then be written as

$$p \int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du' + \tilde{E}_p(\tilde{u}_{n+1}) \leq (1 + \beta) \cdot \left((1 + \eta) \cdot E_p[\psi](\tilde{u}_n) + \frac{D_0}{(\tilde{u}_n)^{p-1}} \right). \quad (3.6.52)$$

Now we can define the integer $k(n)$ as the minimum of $k \in [[0, n]]$ such that for all above integers $k \leq k' \leq n$, alternative 1 holds on for $u_2 = \tilde{u}_{k'+1}$ and $u_1 = \tilde{u}_{k'}$. We are in the case where alternative 1 holds for $u_2 = \tilde{u}_{n+1}$ and $u_1 = \tilde{u}_n$, thus $k(n)$ is well-defined and $k(n) \leq n$.

Using (3.6.52) repetitively ⁷⁵ we see that :

$$\begin{aligned} p \int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du' + \tilde{E}_p(\tilde{u}_{n+1}) &\leq (1 + \beta)^{n-k(n)+1} (1 + \eta)^{n-k(n)+1} \cdot \tilde{E}_p(u_{k(n)}) \\ &\quad + D_0 \sum_{i=0}^{n-k(n)} \frac{(1 + \beta)^{i+1}}{(\tilde{u}_{n-i})^{p-1}} \leq (1 + \beta)^{n-k(n)+1} (1 + \eta)^{n-k(n)+1} \cdot \tilde{E}_p(u_{k(n)}) \\ &\quad + \frac{D_0}{(\tilde{u}_n)^{p-1}} \cdot \frac{(1 + \beta)}{(1 + \beta) \cdot \lambda^{p-1} - 1} \cdot \left([(1 + \beta) \cdot \lambda^{p-1}]^{n-k(n)+1} - 1 \right) \\ &\leq (1 + \beta)^{n-k(n)+1} (1 + \eta)^{n-k(n)+1} \cdot \tilde{E}_p(u_{k(n)}) + \frac{D_0 \cdot (\tilde{u}_{n+1})^{2\epsilon}}{U_0^{p-1+2\epsilon}} \cdot \frac{1}{\lambda^{p-1+2\epsilon} - (1 + \eta)} \cdot \frac{[\frac{\lambda^{p-1+2\epsilon}}{1+\eta}]^{n-k(n)+1} - 1}{\lambda^{(p-1+2\epsilon) \cdot n}}, \end{aligned} \quad (3.6.53)$$

where we used the fact that $\tilde{u}_{n-i} = \tilde{u}_n \cdot \lambda^{-i}$, (3.6.35) and geometric series.

Now there are two cases : either $k(n) = 0$ or $k(n) \geq 1$, in which case alternative 2 of Lemma 3.6.8 holds for $u_2 = \tilde{u}_{k(n)}$, $u_1 = \tilde{u}_{k(n)-1}$. We treat these two sub-cases separately again.

Suppose that $k(n) = 0$ thus alternative 1 applies on all intervals $[\tilde{u}_k, \tilde{u}_{k+1}]$ for $0 \leq k \leq n$. Thus, with (3.6.53) and (3.6.40) we get ⁷⁶

⁷³We give this value arbitrarily, as we could give a slightly higher maximal value of $q_0|e|$.

⁷⁴The point being that $1 - \tilde{\nu}(e) < 1 - \tilde{\nu}(e)^2$, so there is a bit of room in this estimate.

⁷⁵We drop the integral term whenever we apply (3.6.28), except for the first iteration.

⁷⁶Now that we fixed $\epsilon = \epsilon(e)$, C'_2 depends only on M , ρ and e .

$$\begin{aligned}
& \tilde{E}_{p-1}(\tilde{u}_{n+1}) \leq \frac{(1+\eta_0) \cdot \lambda \cdot (1+\beta)^{n+1} (1+\eta)^{n+1} \cdot \tilde{E}_p(U_0)}{p \cdot (\lambda-1) \cdot \tilde{u}_{n+1}} \\
& + \frac{(1+\eta_0) \cdot D_0 \cdot (\tilde{u}_{n+1})^{2\epsilon-1}}{p \cdot U_0^{p-1+2\epsilon}} \cdot \frac{\lambda}{(\lambda-1) \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} \cdot \frac{[\frac{\lambda^{p-1+2\epsilon}}{1+\eta}]^{n+1} - 1}{\lambda^{(p-1+2\epsilon) \cdot n}} + \frac{C'_2}{(\tilde{u}_n)^{p-1}} \leq \frac{(1+\eta_0) \cdot \lambda \cdot U_0^{-2\epsilon} \tilde{E}_p(U_0)}{p \cdot (\lambda-1) \cdot (\tilde{u}_{n+1})^{1-2\epsilon}} \\
& + \frac{(1+\eta_0) \cdot D_0}{p \cdot U_0^{p-1+2\epsilon} \cdot (\tilde{u}_{n+1})^{1-2\epsilon}} \cdot \frac{\lambda^{p+2\epsilon}}{(\lambda-1) \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} \cdot [(1+\eta)^{-(n+1)} - \lambda^{-(p-1+2\epsilon) \cdot (n+1)}] + \frac{C'_2}{(\tilde{u}_n)^{p-1}} \\
& \leq (\tilde{u}_{n+1})^{-1+2\epsilon} \cdot \left(\frac{(1+\eta_0) \cdot \lambda \cdot U_0^{-2\epsilon} \tilde{E}_p(U_0)}{p \cdot (\lambda-1)} + \frac{(1+\eta_0) \cdot D_0}{p \cdot U_0^{p-1+2\epsilon}} \cdot \frac{\lambda^{p+2\epsilon}}{(\lambda-1) \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} + \frac{C'_2 \cdot \lambda^{1-2\epsilon}}{U_0^{p-2+2\epsilon}} \right) \\
& \leq (\tilde{u}_{n+1})^{-1+2\epsilon} \cdot \left(\frac{2\lambda \cdot U_0^{-2\epsilon} \tilde{E}_p(U_0)}{p \cdot (\lambda-1)} + \frac{2D_0}{p \cdot U_0^{p-1+2\epsilon}} \cdot \frac{\lambda^{p+2\epsilon}}{(\lambda-1) \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} + \frac{C'_2 \cdot \lambda^{1-2\epsilon}}{U_0^{p-2+2\epsilon}} \right)
\end{aligned} \tag{3.6.54}$$

where we took $\eta_0 \leq 1$, used (3.6.35) as $(1+\beta)^{n+1} (1+\eta)^{n+1} \cdot U_0^{2\epsilon} = (\tilde{u}_{n+1})^{2\epsilon}$, $\tilde{u}_n^{-p+2-2\epsilon} \leq U_0^{-p+2-2\epsilon}$ and the estimate $\frac{[\frac{\lambda^{p-1+2\epsilon}}{1+\eta}]^{n+1} - 1}{\lambda^{(p-1+2\epsilon) \cdot n}} = \lambda^{p-1+2\epsilon} \cdot [(1+\eta)^{-(n+1)} - \lambda^{-(p-1+2\epsilon) \cdot (n+1)}] \leq \lambda^{p-1+2\epsilon}$.

Thus (3.6.41) holds with $\nu = \frac{1}{4}$ provided that the following conditions are true

$$\frac{8\lambda \cdot U_0^{-2\epsilon} \tilde{E}_p(U_0)}{p \cdot (\lambda-1)} < \Delta, \tag{3.6.55}$$

$$\left(\frac{8D_0}{\tilde{p}(e)} \cdot \frac{(\lambda(e, \eta))^{\tilde{p}(e)+2\epsilon(e)}}{(\lambda(e, \eta) - 1) \cdot ((\lambda(e, \eta))^{\tilde{p}(e)-1+2\epsilon(e)} - (1+\eta))} \right)^{(\tilde{p}(e)-1+2\epsilon(e))^{-1}} < U_0, \tag{3.6.56}$$

$$\left(4C'_2 \cdot (\lambda(e, \eta))^{1-2\epsilon(e)} \right)^{(\tilde{p}(e)-2+2\epsilon(e))^{-1}} < U_0, \tag{3.6.57}$$

where we recall that $\lambda > 1$ depends only on e and η .

Coming back to (3.6.53), we see that

$$\tilde{E}_p(\tilde{u}_{n+1}) \leq \left(\tilde{E}_p(U_0) \cdot U_0^{-2\epsilon} + \frac{D_0 \cdot \lambda^{p-1+2\epsilon}}{U_0^{p-1+2\epsilon} \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} \right) \cdot (\tilde{u}_{n+1})^{2\epsilon}.$$

Thus (3.6.42) is true for $\nu = \frac{1}{4}$ if conditions similar to (3.6.55) and (3.6.56), (3.6.57) are satisfied.

Now we treat the case where $k(n) \geq 1$, thus alternative 2 of Lemma 3.6.8 holds on $[\tilde{u}_{k(n)-1}, \tilde{u}_{k(n)}]$, i.e. for $u_2 = \tilde{u}_{k(n)}$, $u_1 = \tilde{u}_{k(n)-1}$ and $\beta(\epsilon(e))$. We repeat ⁷⁷ the argument leading to (3.6.51) and get

$$\tilde{E}_p(\tilde{u}_{k(n)}) \leq \frac{(\sqrt{\Delta} \cdot \tilde{f}(p, e, \eta) + D'_0 \cdot (\tilde{u}_{k(n)-1})^{\frac{2-p}{2}-\epsilon})^2}{2} \cdot \left(\frac{1+\beta}{\beta - \log(1+\beta)} \right) \cdot (\lambda-1) \cdot (\tilde{u}_{k(n)-1})^{2\epsilon} \leq \frac{p \cdot (\lambda-1)}{\lambda} \cdot (1-\tilde{\nu}(e)^2) \cdot \Delta \cdot (\tilde{u}_{k(n)})^{2\epsilon}$$

Thus, combining with (3.6.53), we get, using that $\lambda^{n-k(n)} \cdot \tilde{u}_{k(n)} = \tilde{u}_n$ and (3.6.35):

$$p \int_{\tilde{u}_n}^{\tilde{u}_{n+1}} E_{p-1}[\psi](u') du' + \tilde{E}_p(\tilde{u}_{n+1}) \leq \left(\frac{p \cdot (\lambda-1)}{\lambda} \cdot (1-\tilde{\nu}(e)^2) \cdot \Delta + \frac{D_0 \cdot \lambda^{p-1+2\epsilon}}{U_0^{p-1+2\epsilon} \cdot (\lambda^{p-1+2\epsilon} - (1+\eta))} \right) \cdot (\tilde{u}_{n+1})^{2\epsilon}. \tag{3.6.58}$$

Thus, combining with (3.6.40) we get

$$\tilde{E}_{p-1}(\tilde{u}_{n+1}) \leq (\tilde{u}_{n+1})^{-1+2\epsilon} \cdot \left((1+\eta_0) \cdot (1-\tilde{\nu}(e)^2) \cdot \Delta + \frac{C'_2 \cdot \lambda^{1-2\epsilon}}{U_0^{p-2+2\epsilon}} \right).$$

Therefore, we obtain (3.6.41) for $\nu = \tilde{\nu}(e)^4$ providing the following two conditions hold (since $1 < \Delta$):

$$1 + \eta_0 < \frac{1 - \tilde{\nu}(e)^3}{1 - \tilde{\nu}(e)^2} \tag{3.6.59}$$

⁷⁷Note that we cannot directly use the induction hypothesis for $k = k(n)$, as we need the $(1-\nu)$ factor.

$$\left(\frac{C'_2 \cdot \lambda(e, \eta)^{1-2\epsilon}}{\tilde{\nu}(e)^3 \cdot (1 - \tilde{\nu}(e))} \right)^{(\tilde{p}(e) - 2 + 2\epsilon(e))^{-1}} < U_0 \quad (3.6.60)$$

(3.6.42) is proven under similar conditions, for $\nu = \tilde{\nu}(e)^4$.

Thus, if all conditions (3.6.38), (3.6.39), (3.6.55) (3.6.50), (3.6.56), (3.6.57), (3.6.60), , (3.6.59) can be satisfied, then the induction hypothesis, i.e. (3.6.36) and (3.6.37), is proved.

Recall that we chose already $\epsilon = \epsilon(e)$, $2 < 2 + \epsilon(e) < p = \tilde{p}(e) < 2 + \sqrt{1 - 4q_0|e|}$ according to Lemma 3.6.10 and $\lambda = \lambda(e, \eta)$ according to (3.6.35), where $\eta > 0$ is a number that can still be taken arbitrarily small without restriction. We are going to take $\eta = \eta_0$ in all that follows.

First, it is clear that there exists $\tilde{\eta}_0 = \tilde{\eta}_0(e) > 0$ such that if $\eta_0 \leq \tilde{\eta}_0(e)$, then (3.6.59) is satisfied. We now take $\eta_0 = \tilde{\eta}_0$. Thus, λ now only depends on e .

Then, there exists $\tilde{U}_0 = \tilde{U}_0(M, \rho, e, R) > 1$ large enough so that, if $U_0 \geq \tilde{U}_0$, then (3.6.50), (3.6.56), (3.6.57), (3.6.60) are satisfied. We fix $U_0 = \tilde{U}_0(M, \rho, e, R)$.

Now, U_0 being fixed, there exists $\tilde{\Delta}_0 := \tilde{\Delta}_0(M, \rho, e, R) > 1$ such that, if $\Delta \geq \tilde{\Delta}_0$, then conditions (3.6.38), (3.6.39), (3.6.55) are satisfied. Thus, we chose $\Delta = \tilde{\Delta}_0(M, \rho, e, R)$.

Thus, by induction, we proved that there exists a constant $\tilde{D} = \tilde{D}(M, \rho, e, R) > 0$ such that for all $n \in \mathbb{N}$:

$$\tilde{E}_{\tilde{p}(e)}(\tilde{u}_n) \leq \tilde{D} \cdot (\tilde{u}_n)^{2\epsilon(e)},$$

$$\tilde{E}_{\tilde{p}(e)-1}(\tilde{u}_n) \leq \frac{\tilde{D}}{(\tilde{u}_n)^{1-2\epsilon(e)}}.$$

Now for all $u \geq U_0$, there exists $\tilde{u}_n \leq u \leq \tilde{u}_{n+1}$ and thus, using Lemma (3.6.8) for $u_1 = \tilde{u}_n$, $u_2 = u$ and say $\beta = 1$, it is not hard to see that there exists $\tilde{D} = \tilde{D}(M, \rho, e, R) > 0$ such that

$$\tilde{E}_{\tilde{p}(e)}(u) \leq \tilde{D} \cdot u^{2\epsilon(e)}, \quad (3.6.61)$$

$$\tilde{E}_{\tilde{p}(e)-1}(u) \leq \frac{\tilde{D}}{u^{1-2\epsilon(e)}}, \quad (3.6.62)$$

where we also used the boundedness of the \tilde{E}_{p-1} energy of section 3.6.2. Using the Holder inequality, together with Corollary 3.6.5 we obtain, for $p(e) = \tilde{p}(e) - 2\epsilon(e)$:

$$\tilde{E}_{p(e)}(u) \leq \tilde{D}, \quad (3.6.63)$$

$$E(u) \leq \frac{\hat{D}}{u^{p(e)}}, \quad (3.6.64)$$

which is the object of the proposition.

The only remaining thing to show is that $\tilde{E}_p(u) < +\infty$ for any $u > u_0(R)$, in particular $\tilde{E}_p(U_0) < +\infty$. To do this, one can use (3.6.25) together with a very soft Grönwall type argument (making use of Lemma (3.6.7) with $u_1 = u_0(R)$ and the fact that $\tilde{E}_p(u_0(R)) < +\infty$). Details are left to the reader.

This concludes the proof of Proposition 3.6.6. □

3.7 From L^2 bounds to point-wise bounds

In this section we indicate how the energy decay can be translated into point-wise decay, provided initial point-wise decay assumptions for $D_v \psi_0$ are available.

It should be noted that this decay is to be understood as $u \rightarrow +\infty$ or $v \rightarrow +\infty$, namely near time-like or null infinity.

All the bounds that we prove in the form of a decay estimate e.g. $|\phi| \leq v^{-s}$ for $v > 0$ actually also contain a point-wise boundedness statement for v close to 0 or negative. Because we take more interest in decay for large v , we do not state those explicitly but the reader should keep in mind that these estimates are simultaneously derived in the proofs and do not present any supplementary difficulty.

Note also that later in this section, we assume the energy decay result of section 3.6 under the form

$$\tilde{E}_p(u) \leq D_p \cdot u^{p-(3-\alpha(e))},$$

where we defined $\alpha(e) := 3 - p(e) \in [0, 1]$, see Proposition 3.6.6 for the first occurrence of $p(e)$, to make the notations lighter.

3.7.1 Point-wise bounds near null infinity, “to the right” of γ

In this section, we indicate how the decay of the energy implies some point-wise decay and the bounds proved are **sharp** near null infinity ⁷⁸, according to heuristics c.f. section 3.1.2.

⁷⁸In the sense that the energy bounds are almost sharp when e tends to 0 and the method does not waste any decay.

This strategy to prove quantitative decay rates has been initiated in [55] although the argument we use here varies slightly.

The main idea is to start by the energy decay and to use a Hardy type inequality to get a point-wise bound of $r|\phi|^2$. After, we can integrate the equation to establish L^∞ estimates, using the decay of the initial data.

Lemma 3.7.1. *Suppose that the energy boundedness (3.5.23) and the Morawetz estimates (3.5.38) are valid and that the charge is sufficiently small in $\mathcal{D}(u_0(R), +\infty)$ so that the result of Proposition 3.6.1 applies.*

Then for all $0 \leq \beta < \frac{1}{2}$, there exists $C = C(\beta, R, M, \rho) > 0$ such that for all $(u, v) \in \mathcal{D}(u_0(R), +\infty) \cap \{r \geq R\}$

$$r^{\frac{1}{2}+\beta}|\phi|(u, v) \leq C \cdot \left(\tilde{E}_{2\beta}(u) \right)^{\frac{1}{2}}. \quad (3.7.1)$$

Proof. Since $\lim_{v \rightarrow +\infty} \phi(u, v) = 0$ — this is a consequence of the finiteness of \mathcal{E} , c.f. the proof of Proposition 3.4.1 — we can write

$$\phi(u, v) = - \int_v^{+\infty} e^{\int_v^{v'} i q_0 A_v} D_v \phi(u, v') dv',$$

after noticing that $\partial_v(e^{\int_{v_0(R)}^v i q_0 A_v} \phi) = e^{\int_{v_0(R)}^v i q_0 A_v} D_v \phi$.

Now using Cauchy-Schwarz we can write that for every $0 < \beta < \frac{1}{2}$

$$|\phi|(u, v) \leq \left(\int_v^{+\infty} \Omega^2 r^{-2-2\beta}(u, v') dv' \right)^{\frac{1}{2}} \left(\int_v^{+\infty} \Omega^{-2} r^{2+2\beta} |D_v \phi|^2(u, v') dv' \right)^{\frac{1}{2}}.$$

This gives

$$r^{\frac{1}{2}+\beta}|\phi|(u, v) \leq (1+2\beta)^{-\frac{1}{2}} \left(\int_v^{+\infty} \Omega^{-2} r^{2+2\beta} |D_v \phi|^2(u, v') dv' \right)^{\frac{1}{2}}.$$

Now we square this inequality and write :

$$r^{1+2\beta}|\phi|^2(u, v) \leq (1+2\beta)^{-1} \cdot \int_v^{+\infty} \Omega^{-2} r^{2+2\beta} |D_v \phi|^2(u, v') dv'. \quad (3.7.2)$$

Now, based on the fact that $D_v \psi = r D_v \phi + \Omega^2 \phi$, we write the identity

$$\Omega^{-2} r^{2+2\beta} |D_v \phi|^2 = \Omega^{-2} r^{2\beta} |D_v \psi|^2 - \Omega^2 r^{2\beta} |\phi|^2 - r^{1+2\beta} \partial_v(|\phi|^2).$$

We now integrate on $\{u\} \times [v_R(u), +\infty]$. After one integration by parts ⁷⁹ we get

$$\int_{v_R(u)}^{+\infty} \Omega^{-2} r^{2+2\beta} |D_v \phi|^2(u, v') dv' = \int_{v_R(u)}^{+\infty} \Omega^{-2} r^{2\beta} |D_v \psi|^2(u, v') dv' + 2\beta \int_{v_R(u)}^{+\infty} \Omega^2 r^{2\beta} |\phi|^2(u, v') dv' + R^{1+2\beta} |\phi|^2(u, v_R(u)).$$

Now we use Hardy's inequality (3.2.19) coupled with the Morawetz ⁸⁰ estimate (3.5.37) : there exists $D = D(M, \rho, R) > 0$ such that

$$R^{1+2\beta} |\phi|^2(u, v_R(u)) + \int_{v_R(u)}^{+\infty} \Omega^2 r^{2\beta} |\phi|^2(u, v') dv' \leq \left(\frac{9}{1-2\beta} \right) \cdot E_{2\beta}[\psi](u) + D \cdot E(u). \quad (3.7.3)$$

Combining with (3.7.2), we see that there exists $C = C(R, M, \rho)$ so that

$$r^{1+2\beta}|\phi|^2(u, v) \leq \frac{C}{1-2\beta} \cdot \tilde{E}_{2\beta}(u).$$

This concludes the proof of Lemma 3.7.1

□

From now on , we are going to assume the energy decay result of section 3.6 with $p(e) = 3 - \alpha(e)$.

We can now establish the decay of $D_v \psi$ in $\{r^* \geq \frac{v}{2} + R^*\}$ — which is the “right” of γ , c.f. Figure 3.3 — together with estimates for the radiation field ψ , in particular on \mathcal{I}^+ :

⁷⁹Using the fact that $r^{1+2\beta} \phi$ tends to 0 when v tends to $+\infty$, guaranteed by the finiteness of $\mathcal{E}_{1+\epsilon}$, c.f. the proofs of Section 3.4.

⁸⁰The precise way to use it, averaging on R , has been carefully explained in section 3.5.10. We do not repeat the argument.

Corollary 3.7.2. *Make the same assumptions as for Lemma 3.7.1.*

Suppose also that there exists $D_0 > 0$ such that for all $v_0(R) \leq v$:

$$|D_v \psi|(u_0(R), v) \leq D_0 \cdot v^{-2+\frac{\alpha}{2}}.$$

Then there exists $C = C(M, \rho, R, e, D_0) > 0$ so that for all $(u, v) \in \mathcal{D}(u_0(R), +\infty) \cap \{r^ \geq \frac{v}{2} + R^*\}$ we have for $u > 0$:*

$$|D_v \psi| \leq C \cdot v^{-2+\frac{\alpha}{2}} \cdot |\log(u)|, \quad (3.7.4)$$

$$|\psi| \leq C \cdot u^{-1+\frac{\alpha}{2}} \cdot |\log(u)|, \quad (3.7.5)$$

$$|Q - e| \leq C \cdot u^{-2+\alpha} \cdot |\log(u)|, \quad (3.7.6)$$

where $\alpha(e) := 3 - p(e) \in [0, 1)$, as introduced in the beginning of the section.

Proof. We take $\beta = \frac{1}{2} - \epsilon$ for $0 < \epsilon < \frac{1}{2}$.

We are working to the “right” of γ where $r \sim v \gtrsim u$. Therefore using 3.2.6 and the result of Lemma 3.7.1, there exists $C = C(R, \epsilon, e, M, \rho) > 0$ such that

$$|D_u(D_v \psi)| \leq C \cdot r^{-2+\epsilon} u^{-1-\epsilon+\frac{\alpha}{2}}.$$

Then we choose $\epsilon = \frac{\alpha}{2}$ and we integrate on $[u_0(R), u]$. For R large enough we get

$$|D_v \psi|(u, v) \leq |D_v \psi|(u_0(R), v) + 2C \cdot v^{-2+\frac{\alpha}{2}} |\log(u)|.$$

Making use of the decay of the initial data gives (3.7.4).

Now we notice that from Lemma 3.7.1, we have that there exists $C' = C'(R, M, \rho, e) > 0$ such that

$$|\psi|(u, v_\gamma(u)) \leq C' u^{-1+\frac{\alpha}{2}}.$$

Therefore, integrating (3.7.4) in v to the right of γ , we prove (3.7.5).

Now we turn to the charge : integrating (3.2.9) towards null infinity and using Cauchy-Schwarz we get that to the right of γ and for all $u \geq u_0(R)$ and for all $\epsilon > 0$:

$$|Q(u, v) - Q|_{\mathcal{I}^+}(u)| \leq q_0 \left(\int_v^{+\infty} r^{1+\epsilon} |D_v \psi|^2(u, v) dv \right)^{\frac{1}{2}} \left(\int_v^{+\infty} \frac{|\psi|^2(u, v)}{r^{1+\epsilon}} dv \right)^{\frac{1}{2}} \leq \frac{D}{\sqrt{\epsilon}} \cdot (\tilde{\mathcal{E}}_{1+\epsilon})^{\frac{1}{2}} \cdot (\tilde{\mathcal{E}}_{1-\epsilon})^{\frac{1}{2}} \leq D^2 \cdot u^{-2+\alpha},$$

where $D = D(R, M, \rho, e) > 0$ and we used (3.7.3) and the energy decay of Section 3.6, for $\epsilon = \frac{1}{2}$.

Now using (3.2.8) and Cauchy-Schwarz again we get that for all $u \geq u_0(R)$:

$$|Q|_{\mathcal{I}^+}(u) - e| \leq q_0 \left(\int_u^{+\infty} r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du \right)^{\frac{1}{2}} \left(\int_u^{+\infty} |\psi|_{\mathcal{I}^+}^2(u) du \right)^{\frac{1}{2}} \leq q_0 (E(u))^{\frac{1}{2}} \left(\int_u^{+\infty} (C \cdot |\log(u)| \cdot u^{-1+\frac{\alpha}{2}})^2 du \right)^{\frac{1}{2}},$$

where we used (3.7.5) in the last estimate. Combining the two estimates for Q , it proves (3.7.6).

This concludes the proof of Corollary 3.7.2. □

Notice that these two results prove estimates (3.3.8) and (3.3.9) of Theorem 3.3.1, together with (3.3.6), (3.3.10) to the right of γ .

3.7.2 Point-wise bounds in the region between γ and γ_R

We now propagate the bounds obtained “to the right” of γ towards the right of γ_R . Since we already have an estimate from Lemma (3.7.1), the argument is very soft.

Proposition 3.7.3. *Make the same assumptions as for Corollary 3.7.2.*

Then there exists $C = C(M, \rho, R, e) > 0$ so that for all $(u, v) \in \mathcal{D}(u_0(R), +\infty) \cap \{R^ \leq r^* \leq \frac{v}{2} + R^*\}$ we have for $u > 0$:*

$$r^{\frac{1}{2}} |\phi| + r^{\frac{3}{2}} |D_v \phi| \leq C \cdot u^{-\frac{3+\alpha}{2}}, \quad (3.7.7)$$

$$|Q - e| \leq C \cdot u^{-2+\alpha} \cdot |\log(u)|, \quad (3.7.8)$$

where $\alpha(e) := 3 - p(e) \in [0, 1)$, as introduced in the beginning of the section.

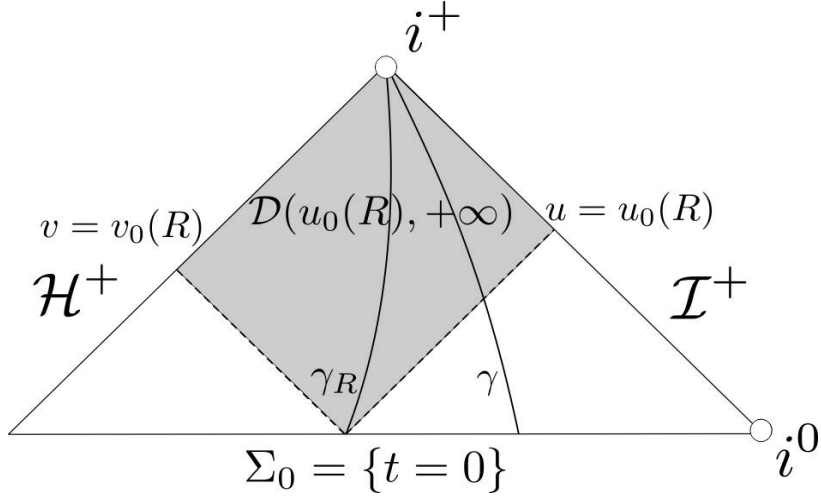


Figure 3.3: Illustration of $\mathcal{D}(u_0(R), +\infty)$ and γ

Proof. With the work that we have already done, this proof is easy.

First notice that in this region $u \sim v \sim 2u$ when v tends to $+\infty$.

Then, making use of the boundedness of Q it is enough to integrate (3.2.6), using the estimate of Lemma 3.7.1 for $\beta = 0$ and the bound on $|D_v \psi|_{|\gamma}$ of Corollary 3.7.2.

We get for all $(u, v) \in \mathcal{D}(u_0(R), +\infty) \cap \{R^* \leq r^* \leq \frac{v}{2} + R^*\}$

$$|D_v \psi|(u, v) \leq C' \cdot r^{-\frac{1}{2}}(u, v) \cdot u^{-\frac{3+\alpha}{2}},$$

for some $C' = C'(M, \rho, R, e) > 0$ since $E(u)$ decays like $u^{-3+\alpha}$. Combining with the estimate of Lemma 3.7.1 for $\beta = 0$ gives directly (3.7.7), noting that $D_v \psi = \Omega^2 \phi + r D_v \phi$.

Then using (3.2.9), we see that, by (3.7.7)

$$|\partial_v Q| \leq C \cdot u^{-3+\alpha},$$

which gives (3.7.8) easily using the former estimate for $|Q - e|$, after integrating in this region where $u \sim v \sim 2u$. \square

3.7.3 Point-wise bounds in the finite r region $\{r \leq R\}$

Now, notice that by construction of the domain $\mathcal{D}(u_0(R), +\infty)$, bounds have been proven on the whole curve $\gamma_R = \{r = R\}$. In what follows, we can completely forget about the foliation \mathcal{V}_u and \mathcal{D} and we consider the full region $\{r \leq R\}$, bounded by \mathcal{H}^+ , Σ_0 and γ_R , c.f. Figure 3.3.

We are going to prove the following proposition, that also includes a red-shift estimate.

Proposition 3.7.4. *Suppose that Q is bounded in $\{r_+ \leq r \leq R\}$ and define $Q^+ = \|Q\|_\infty$.*

By section 3.4, Q^+ can be taken arbitrarily close to $|e|$ or $|e_0|$. We also have assumed naturally that $\mathcal{E} < \infty$.

Suppose that $\frac{D_u \phi_0}{\Omega} \in L^\infty(\Sigma_0)$ and ϕ_0 in $L^\infty(\Sigma_0)$. We denote $N_\infty := \|\frac{D_u \phi_0}{\Omega}\|_{L^\infty(\Sigma_0)} + \|\phi_0\|_{L^\infty(\Sigma_0)}$.

Then there exists $C = C(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ such that

for all $(u, v) \in \{r_+ \leq r \leq R\}$, if $v > 0$:

$$|\phi|(u, v) + |D_v \phi|(u, v) + \frac{|D_u \phi|(u, v)}{\Omega^2} \leq C \cdot v^{-\frac{3+\alpha}{2}}, \quad (3.7.9)$$

where $\alpha(e) := 3 - p(e) \in [0, 1)$, as introduced in the beginning of the section.

Proof. The first step in the proof is to prove a red-shift estimate with no time decay in a region

$\{V_0 \leq v \leq v_0(R)\}$. This is the object of the following lemma :

Lemma 3.7.5. *Under the same assumptions as in Proposition 3.7.4, then for any $V_0 < v_0(R)$, there exists $D_0 = D_0(V_0, M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ such that for all $(u, v) \in \{V_0 \leq v \leq v_0(R)\}$,*

$$\frac{|D_u \psi|(u, v)}{\Omega^2} \leq D_0. \quad (3.7.10)$$

Proof. Using the red-shift estimate (3.4.5) from Proposition 3.4.1 with the help of Cauchy-Schwarz, one finds $C = C(M, \rho) > 0$ such that in $\{v \leq v_0(R)\}$

$$|\phi| \leq \|\phi_0\|_\infty + C \cdot \sqrt{\mathcal{E}} \leq C \cdot (N_\infty + \sqrt{\mathcal{E}}). \quad (3.7.11)$$

Now we write (3.2.7) as

$$D_v\left(\frac{D_u\psi}{\Omega^2}\right) = \frac{-2K}{\Omega^2}D_u\psi + \frac{\phi}{r}(iq_0Q - r \cdot 2K),$$

which can also be expressed as

$$D_v\left(\exp\left(\int_{v_0(u)}^v 2K(u, v')dv'\right) \cdot \frac{D_u\psi}{\Omega^2}\right) = \exp\left(\int_{v_0(u)}^v 2K(u, v')dv'\right) \cdot \frac{\phi}{r}(iq_0Q - r \cdot 2K).$$

We can then integrate such an estimate on $\{u\} \times [v_0(u), v]$ and obtain

$$\frac{|D_u\psi(u, v)|}{\Omega^2} \leq \exp\left(-\int_{v_0(u)}^v 2K(u, v')dv'\right) \cdot \Omega^{-1}(u, v_0(u)) \cdot N_\infty + C' \cdot (N_\infty + \sqrt{\mathcal{E}}) \cdot \int_{v_0(u)}^v \exp\left(-\int_{v'}^v 2K(u, v'')dv''\right)dv',$$

for some $C' = C'(M, \rho, R) > 0$ where we used the fact that $\frac{1}{r}(iq_0Q - r \cdot 2K)$ is bounded and (3.7.11).

Now remember from (3.2.2) that $\Omega(u, v_0(u)) \sim C_+ \cdot e^{-2K_+u}$ as $u \rightarrow +\infty$.

Also it can be proven easily that there exists $D_+ = D_+(M, \rho) > 0$ such that in this region, since $(r - r_+)e^{-2K_+(v-u)}$ is bounded,

$$|2K(u, v) - 2K_+| \leq D^+ e^{2K_+(v-u)}.$$

If we integrate this as a lower bound, we get that there exists $D'_+ = D'_+(M, \rho) > 0$ such that

$$\exp\left(-\int_{v_0(u)}^v 2K(u, v')dv'\right) \leq D'_+ \cdot e^{-2K_+(v+u)}.$$

Then we get that for some $D''_+ = D''_+(M, \rho) > 0$:

$$\frac{|D_u\psi(u, v)|}{\Omega^2} \leq D''_+ \cdot e^{-2K_+v} \cdot N_\infty + C' \cdot (N_\infty + \sqrt{\mathcal{E}}) \cdot \int_{v_0(u)}^v \exp\left(-\int_{v'}^v 2K(u, v'')dv''\right)dv',$$

Now, because we stand in a region where $r \leq R$, one can find a constant $K_0 = K_0(M, \rho, R) > 0$ such that $K(r) > K_0$. Therefore we can write

$$\int_{v_0(u)}^v \exp\left(-\int_{v'}^v 2K(u, v'')dv''\right)dv' \leq \int_{v_0(u)}^v \exp(-2K_0 \cdot (v - v'))dv' \leq \frac{1}{2K_0}.$$

This concludes the proof of the lemma, after controlling e^{-2K_+v} by $e^{-2K_+V_0}$.

□

Then we use the time decay of the non-degenerate energy under the form :

for all $(u, v) \in \{v \geq v_0(R)\} \cap \{r \leq R\}$

$$|\phi(u, v) - \phi(u_R(v), v)| \leq \left(\int_{u_R(v)}^u \Omega^2 du'\right)^{\frac{1}{2}} \cdot \left(\int_{u_R(v)}^u \frac{|D_u\phi|^2(u', v)}{\Omega^2} du'\right)^{\frac{1}{2}} \leq \tilde{C} \cdot v^{\frac{-3+\alpha}{2}},$$

where $\tilde{C} = \tilde{C}(M, \rho, R, e) > 0$ and we took advantage of the fact that $|u_R(v)| \sim v$.

Therefore, using the bounds of the former section we see that there exists $C' = C'(M, \rho, R, e) > 0$ such that for all $(u, v) \in \{v \geq v_0(R)\} \cap \{r \leq R\}$

$$|\phi(u, v)| \leq C' \cdot v^{\frac{-3+\alpha}{2}}. \quad (3.7.12)$$

Now integrating (3.2.6) we can, allowing C' to be larger, derive the same estimate for $D_v\psi$:

$$|D_v\psi(u, v)| \leq C' \cdot v^{\frac{-3+\alpha}{2}}. \quad (3.7.13)$$

Now using a reasoning similar to one of the lemma, we get that for some $C'' = C''(M, \rho, R, e) > 0$:

$$\frac{|D_u\psi(u, v)|}{\Omega^2} \leq \exp\left(-\int_{v_0(R)}^v 2K(u, v')dv'\right) \cdot D_0 + C'' \int_{v_0(R)}^v (v')^{\frac{-3+\alpha}{2}} \exp\left(-\int_{v'}^v 2K(u, v'')dv''\right)dv'.$$

Now observe that for all $\beta > 0$, $s > 0$ —using an integration by parts— when $v \rightarrow +\infty$:

$$e^{-\beta v} \int_{v_0(R)}^v e^{\beta v'} (v')^{-s} dv' \leq \frac{v^{-s}}{\beta} + O(v^{-s-1}).$$

Now, because there exists $K_0 = K_0(M, \rho, R) > 0$ such that $K > K_0$, this concludes the proof of the proposition, after noticing that $\exp(-\int_{v_0(R)}^v 2K(u, v') dv') = o(v^{-\frac{3+\alpha}{2}})$. \square

Remark 57. Notice that (3.7.9) was stated for $v > 0$ only. This is related to a degenerescence of the Ω^2 weight towards the bifurcation sphere c.f. Remark 43. Actually, from Lemma 3.7.5 it is not hard to infer that $e^{2K+u}|D_u\phi|$ is bounded **on the whole space-time**, in conformity with Hypothesis 4. For a discussion of the L^2 analogue, c.f. also Remark 41.

3.7.4 Remaining estimates for $D_u\psi$ near \mathcal{I}^+ and $|Q - e|$ near \mathcal{H}^+

For this last sub-section of section 3.7, we make a little summary of what we did.

First, we derived an estimate for $r^{\frac{1}{2}}\phi$ on the whole region $\{r \geq R\}$ using energy decay.

Then, with the help of the point-wise decay of $D_v\psi$ on $u = u_0(R)$ —itself coming from the point-wise decay of ψ_0 and $D_v\psi_0$ on Σ_0 , c.f. Proposition 3.4.3— we derived v decay of $D_v\psi$, still on $\{r \geq R\}$.

This gives directly almost optimal point-wise estimate for ψ and $|Q - e|$ on null infinity and nearby.

Then the v decay of ϕ can be propagated from γ_R to the event horizon using again the decay of the energy. As a consequence v decay can also be retrieved for $D_v\phi$ on $\{r \leq R\}$. After we use a red-shift estimate to prove the same v decay for the regular derivative $\Omega^{-2}D_u\phi$.

Now compared to the statement of our theorems, we are missing four estimates.

The first is an estimate for $D_u\psi$ in the large r region $\{r \geq R\}$. It is the object of Proposition 3.7.6.

The second is the almost optimal L^2 flux bounds on ϕ and $D_v\phi$ on any constant r curve. We prove them in Proposition 3.7.7.

The third is the existence of the future asymptotic charge e . We proved that Q admits a future limit when $t \rightarrow +\infty$ on constant r curves, the event horizon and null infinity but we never proved that they were all the same.

The fourth is related to the third : it is the v decay of $|Q - e|$ in the bounded r region. These two are the object of Proposition 3.7.8

Proposition 3.7.6. *We make the same assumptions as for the former propositions. Then there exists a constant $C = C(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ such that in $\{r \geq R\} \cap \{u \geq u_0(R)\}$, for $u > 0$:*

$$|D_u\psi| \leq C \cdot u^{-\frac{3+\alpha}{2}},$$

where $\alpha(e) := 3 - p(e) \in [0, 1)$, as introduced in the beginning of the section.

Proof. Using the estimates for ϕ in $\{r \geq R\} \cap \{u \geq u_0(R)\}$ and (3.2.7) we get that for some $D = D(M, \rho, R) > 0$

$$|D_v D_u\psi| \leq D \cdot r^{-\frac{3}{2}} \cdot u^{-\frac{3+\alpha}{2}}.$$

It is enough to integrate this bound — given that $|\partial_u r| \geq \Omega^2(R)$ and make use of the estimate in the past of Proposition 3.7.4. \square

Proposition 3.7.7. *For every $r_+ \leq R_0 \leq R$, there exists a constant $C_0 = C_0(R_0, M, \rho, R, e) > 0$ such that*

$$\int_v^{+\infty} [|\phi|^2(u_{R_0}(v'), v') + |D_v\phi|^2(u_{R_0}(v'), v')] dv' \leq C_0 \cdot v^{-3+\alpha}. \quad (3.7.14)$$

Proof. Take any $r_+ \leq R_0 < R_1 \leq R$.

First, using Cauchy-Schwarz, we find that there exists $D = D(R_0, R_1, M, \rho) > 0$ such that

$$|\phi|^2(u_{R_1}(v), v) \leq 2|\phi|^2(u_{R_0}(v), v) + D \cdot \int_{u_{R_0}(v)}^{u_{R_1}(v)} \frac{|D_u\phi|^2(u, v)}{\Omega^2} du,$$

where we squared the Cauchy-Schwarz estimate and bounded $2 \int_{u_{R_0}(v)}^{u_{R_1}(v)} \Omega^2 du \leq D$. This implies that

$$\int_v^{+\infty} |\phi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_v^{+\infty} |\phi|^2(u_{R_0}(v'), v') dv' + D \cdot \int_{\{v' \geq v\} \cap \{R_0 \leq r \leq R_1\}} \frac{|D_u \phi|^2(u, v')}{\Omega^2} du dv'. \quad (3.7.15)$$

Now since $\{R_0 \leq r \leq R_1\}$ is included inside $\{r \leq R\}$, one can use the Morawetz estimate (3.5.38) : there exists $C = C(M, \rho, e, R) > 0$ such that

$$\int_v^{+\infty} |\phi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_v^{+\infty} |\phi|^2(u_{R_0}(v'), v') dv' + C \cdot E(u_R(v)). \quad (3.7.16)$$

Similarly, using Cauchy-Schwarz and (3.2.6), we find that there exists $D' = D'(R_0, R_1, M, \rho) > 0$ such that

$$|D_v \psi|^2(u_{R_1}(v), v) \leq 2|D_v \psi|^2(u_{R_0}(v), v) + D' \cdot \int_{u_{R_0}(v)}^{u_{R_1}(v)} \Omega^2 |\phi|^2(u, v) du.$$

Therefore one can use again the non-degenerate Morawetz estimate (3.5.38) : there exists $C' = C'(M, \rho, e, R) > 0$ such that

$$\int_v^{+\infty} |D_v \psi|^2(u_{R_1}(v'), v') dv' \leq 2 \int_v^{+\infty} |D_v \psi|^2(u_{R_0}(v'), v') dv' + C' \cdot E(u_R(v)). \quad (3.7.17)$$

Now we use the mean-value theorem with the Morawetz estimate (3.5.38), like we did several times in section 3.5 : there exists $r_+ < \tilde{R} < R$ such that

$$\int_v^{+\infty} [|\phi|^2(u_{\tilde{R}}(v'), v') + |D_v \phi|^2(u_{\tilde{R}}(v'), v')] dv' \leq C'' \int_{\mathcal{D}(u_R(v), +\infty) \cap \{r \leq R\}} (r^2 |D_v \phi|^2 + |\phi|^2) \Omega^2 du dv \leq (C'')^2 \cdot E(u_R(v)),$$

where $C'' = C''(M, \rho, e, R) > 0$.

Therefore, taking $R_0 = \tilde{R}$, it means that for all $r_+ \leq R_1 \leq R$,

$$\int_v^{+\infty} [|\phi|^2(u_{R_1}(v'), v') + |D_v \psi|^2(u_{R_1}(v'), v')] dv' \leq C_1 \cdot E(u_R(v)) \leq C_1^2 \cdot v^{-3+\alpha}, \quad (3.7.18)$$

for some $C_1 = C_1(M, \rho, e, R) > 0$.

Notice that since $D_v \psi = r D_v \phi + \Omega^2 \phi$, this estimate is equivalent to the claimed (3.7.14). This concludes the proof of the proposition. \square

Proposition 3.7.8. *The future asymptotic charge e exists — in the sense explained above — and the following estimate is true : for some $C' = C'(M, \rho, R, e) > 0$*

$$|Q - e|(u, v) \leq C' \cdot (u^{-2+\alpha} |\log(u)| 1_{\{r \geq R\}} + v^{-3+\alpha} 1_{\{r \leq R\}}). \quad (3.7.19)$$

Proof. For now on, e is defined to be the asymptotic limit of $Q|_{\mathcal{I}^+}(t)$ when $t \rightarrow +\infty$.

Temporarily we also define $e_{\mathcal{H}^+}$ as the asymptotic limit of $Q|_{\mathcal{H}^+}(t)$ when $t \rightarrow +\infty$, and $e(R_0)$ the asymptotic limit of $Q|_{\gamma_{R_0}}(t)$ when $t \rightarrow +\infty$. One of the goals of the proposition is to prove $e = e_{\mathcal{H}^+} = e(R_0)$.

We apply the estimate (3.7.9) of former section to get some $\tilde{C} = \tilde{C}(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ such that

$$|\partial_u Q| \leq \tilde{C} \cdot \Omega^2 \cdot v^{-3+\alpha}.$$

Integrating this gives that for some $\bar{C} = \bar{C}(M, \rho, R, e, \mathcal{E}, N_\infty) > 0$ and for all $r_+ \leq R_0 \leq R_1 \leq R$

$$|Q(u_{R_0}(v), v) - Q(u_{R_1}(v), v)| \leq \bar{C} \cdot v^{-3+\alpha}. \quad (3.7.20)$$

In particular, taking v to $+\infty$, it proves that for all $r_+ \leq R_0 \leq R_1 \leq R$, $e(R_0) = e(R_1) = e_{\mathcal{H}^+}$.

But (3.7.8) also gives $e(R) = e$. Hence $e(R_0) = e(R_1) = e_{\mathcal{H}^+} = e$ as requested.

Now apply (3.7.14) on the event horizon, for $R_0 = r_+$ and integrate (3.2.9) : we get that for some $C'_0 = C'_0(R_0, M, \rho, R, e) > 0$

$$|Q|_{\mathcal{H}^+}(v) - e| \leq C'_0 \cdot v^{-3+\alpha}.$$

Combining with estimate with (3.7.20) taken in $R_0 = r_+$ we get that for all $v \geq v_0(R)$ and $u \geq u_R(v)$

$$|Q(u, v) - e| \leq (C'_0 + \bar{C}) \cdot v^{-3+\alpha}, \quad (3.7.21)$$

which is the claimed estimate on $\{r \leq R\}$. The second part in $\{r \geq R\}$ was already proven in section 3.7.3. This concludes the proof of the proposition. \square

3.8 A variant of the decay theorem for more decaying initial data

While the point-wise results stated in Theorem 3.3.1 and Theorem 3.3.3 are established assuming the weakest decay of the initial data that made the proofs work, they do not give the same r decay rate for $D_v\psi$ towards null infinity as we would expect for say compactly supported data, which is $|D_v\psi| \lesssim r^{-2}$.

Notice that by what was done in the proof of Theorem 3.3.3, we already know that $|D_v\psi| \lesssim v^{-1-\epsilon}$ for some $\epsilon > 0$. Therefore $\partial_v(e^{iq_0 \int_0^v A_v(u,v')dv'}\psi(u,v))$ is integrable in v so $e^{iq_0 \int_0^v A_v(u,v')dv'}\psi(u,v)$ admits a finite limit $\tilde{\varphi}_0(u)$ when $v \rightarrow +\infty$.

In this section, we want to prove that for data decaying slightly more than in the assumption of Theorem 3.3.3, $e^{iq_0 \int_0^v A_v(u,v')dv'}r^2D_v\psi$ admits a bounded limit towards null infinity, like for the wave equation without any symmetry assumption.

It is equally interesting to notice that — similarly to the wave equation c.f. [3] — $e^{iq_0 \int_0^v A_v(u,v')dv'}r^2D_v(r^2D_v\psi)$ also admits a bounded limit towards null infinity, providing the data decay even more. In fact, one can apply r^2D_v to the radiation field infinitely many times and still find a finite limit towards null infinity⁸¹, providing the data decay rapidly.

This feature is reminiscent of what happens for the uncharged wave equation, although the analogous of the Newman-Penrose quantity is no longer conserved in the charged case.

Theorem 3.8.1. *We make the same assumptions as for Theorem 3.3.3 and we also make the following additional assumption : There exists $\epsilon_0 > 0$ and $C_0 > 0$, such that*

$$|D_v(r^2D_v\psi_0)| + r|D_v\psi_0|(r) + |\psi_0|(r) \leq C_0 \cdot r^{-1-\frac{q_0|\epsilon_0|}{4}-\epsilon_0}, \quad (3.8.1)$$

then — in addition to all the conclusions of Theorem 3.3.3 being true — we can also conclude that for all $u \in \mathbb{R}$, $e^{iq_0 \int_0^v A_v(u,v')dv'}v^2D_v\psi(u,v)$ admits a finite limit $\tilde{\varphi}_1(u)$ as $v \rightarrow +\infty$. Moreover in the whole region $\mathcal{D}(u_0(R), +\infty) \cap \{r^ \geq \frac{v}{2} + R^*\}$, there exists $C > 0$ such that for all $u > 0$:*

$$r^2|D_v\psi| \leq C \cdot u^{\frac{3-p(e)}{2}}. \quad (3.8.2)$$

In particular $|\tilde{\varphi}_1(u)| \leq C \cdot u^{\frac{3-p(e)}{2}}$.

If now we assume that we have rapidly decaying data, i.e. that for all $\omega > 0$, there exists $C_0 = C_0(\omega) > 0$ such that on Σ_0 :

$$r|D_v\psi_0|(r) + |\psi_0|(r) \leq C_0 \cdot r^{-\omega}, \quad (3.8.3)$$

and for all $n \in \mathbb{N}$, on Σ_0 :

$$|\varphi_n|(r) \leq C_0, \quad (3.8.4)$$

where we defined $\varphi_0 := \psi$ and $\varphi_{n+1} = r^2D_v\varphi_n$.

Then one can prove that for all $n \in \mathbb{N}$, for all $u \in \mathbb{R}$, $e^{iq_0 \int_0^v A_v(u,v')dv'}\varphi_n(u,v)$ admits a finite limit $\tilde{\varphi}_n(u)$ as $v \rightarrow +\infty$.

Proof. First we only assume (3.8.1). We start by the proof of (3.8.2), the easiest claim. The hypothesis of Proposition 3.4.3 are satisfied so (3.4.13) and (3.4.14) are true : in particular there exists $D_0 > 0$ such that for all $v \geq v_0(R)$,

$$|D_v\psi(u_0(R), v)| \leq D_0 \cdot v^{-2}. \quad (3.8.5)$$

We write (3.2.6) as :

$$D_u(r^2D_v\psi) = -2r\Omega^2D_v\psi + \Omega^2\psi \cdot (iq_0Q - \frac{2M}{r} + \frac{2\rho^2}{r^2}).$$

We are going to apply Corollary 3.7.2 : we recall that we defined $\alpha(e) := 3 - p(e)$. Applying (3.7.4) and (3.7.5) and given that in this region $v \gtrsim |u|$, there exists $D > 0$ such that for $u > 0$:

$$|D_u(r^2D_v\psi)| \leq D \cdot u^{-1+\frac{\alpha}{2}}.$$

Integrating this in u and using (3.8.5) gives (3.8.2) in this region.

Now, to prove that $e^{iq_0 \int_0^v A_v(u,v')dv'}r^2D_v\psi(u,v)$ admits a limit when $v \rightarrow +\infty$, we prove that

$\partial_v(e^{iq_0 \int_0^v A_v(u,v')dv'}r^2D_v\psi) = e^{iq_0 \int_0^v A_v(u,v')dv'}D_v(r^2D_v\psi)$ is integrable. From now on, we denote $r^2D_v\psi = \varphi_1$. Applying D_v on (3.2.6) it is possible to prove that

⁸¹After multiplying the last term by $e^{iq_0 \int_0^v A_v(u,v')dv'}$.

$$D_u D_v \varphi_1 = \frac{-2\Omega^2}{r} D_v \varphi_1 + \frac{\Omega^2}{r^2} \varphi_1 \cdot \left[2 + 3iq_0 Q - \frac{10M}{r} - \frac{8\rho^2}{r^2} \right] + \frac{\Omega^2 \psi}{r^2} \left[(iq_0 Q - r \cdot 2K) \cdot (2K \cdot r^2) + iq_0^2 \Im(\varphi_1 \bar{\psi}) + (2M - \frac{4\rho^2}{r}) \cdot \Omega \right]$$

where we recall that $2K(r) := \frac{2M}{r^2} - \frac{2\rho^2}{r^3}$: hence $r^2 \cdot 2K$ is bounded. This can also be written as :

$$D_u(r^{-2} D_v \varphi_1) = \frac{\Omega^2}{r^4} \varphi_1 \cdot \left[2 + 3iq_0 Q - \frac{10M}{r} - \frac{8\rho^2}{r^2} \right] + \frac{\Omega^2 \varphi_0}{r^4} \left[(iq_0 Q - r \cdot 2K) \cdot (2K \cdot r^2) + iq_0^2 \Im(\varphi_1 \bar{\varphi}_0) + (2M - \frac{4\rho^2}{r}) \cdot \Omega^2 \right]. \quad (3.8.6)$$

First, in the region $\{u \leq u_0(R)\}$, we come back to the proof of Proposition 3.4.3. If we examine more closely the estimate (3.9.39), we see that we actually proved that for some $D' > 0$ and for all $u \leq u_0(R)$:

$$|\varphi_1| = |r^2 D_v \psi| \leq D' \cdot |u|^{1-\omega}, \quad (3.8.7)$$

where $\omega = 1 + \frac{q_0 |e_0|}{4} + \epsilon_0$. From Proposition 3.4.3, we also have the estimate

$$|\psi| \leq D' \cdot |u|^{-\omega}.$$

Using (3.8.6) in the region $\{u \leq u_0(R)\}$ and the boundedness of relevant quantities, we see that there exists $D'' > 0$ such that

$$|D_u(r^{-2} D_v \varphi_1)| \leq D'' \cdot r^{-4} \cdot |u|^{1-\omega}.$$

Integrating in u towards Σ_0 , since $1 < \omega < 2$, we get that there exists $C'' > 0$ such that

$$|r^{-2} D_v \varphi_1| \leq D'' \cdot r^{-2-\omega},$$

where we also used the first term of the (3.8.1). This means in particular that for all $v \geq v_0(R)$:

$$|D_v \varphi_1|(u_0(R), v) \leq D'' \cdot r^{-\omega}. \quad (3.8.8)$$

Now we can use the bound on φ_1 (3.8.2) with (3.7.5) and integrate (3.8.6) in u towards $\{u = u_0(R)\}$ to get that there exists $C' > 0$ such that for all $(u, v) \in \mathcal{D}(u_0(R), +\infty) \cap \{r^* \geq \frac{v}{2} + R^*\}$

$$|D_v \varphi_1|(u, v) \leq D'' \cdot r^{-\omega}, \quad (3.8.9)$$

where we used the fact that $0 < \alpha < 1$ and (3.8.8). Now because $\omega > 1$, it is clear that $\partial_v(e^{iq_0 \int_0^v A_v(u, v') dv'} r^2 D_v \psi) = e^{iq_0 \int_0^v A_v(u, v') dv'} D_v(r^2 D_v \psi)$ is integrable. Hence $e^{iq_0 \int_0^v A_v(u, v') dv'} r^2 D_v \psi(u, v)$ admits a limit when $v \rightarrow +\infty$, as claimed.

We now assume (3.8.3), (3.8.4) and we want to prove that for all $n \geq 2$, $e^{iq_0 \int_0^v A_v(u, v') dv'} \varphi_n(u, v)$ admits a limit when $v \rightarrow +\infty$.

For this, we need to commute the equation (3.2.6) with the operator $X = r^2 D_v$ n times. While the precise formula is very complicated one can write the following :

$$D_u D_v \varphi_n = -\frac{2n\Omega^2}{r} D_v \varphi_n + \sum_{k=0}^n \left[(B_k + \sum_{q=k}^n iC_{kq} X^{n-q} Q) \cdot \frac{\varphi_k}{r^2} \right], \quad (3.8.10)$$

where $B_k(r)$, C_{kq} are real valued entire functions of r^{-1} (therefore bounded on the space-time) whose coefficients depend only on M , ρ and q_0 .

We check this by a quick induction : by what we derived earlier, the formula is obviously true for $n = 0$ and $n = 1$, since $X(Q) = q_0 \Im(\varphi_1 \bar{\varphi}_0)$. Now if (3.8.10) is true, we can rewrite it using $\varphi_{n+1} = r^2 D_v \varphi_n$ as

$$D_u(\varphi_{n+1}) = -\frac{2(n+1)\Omega^2}{r} \varphi_{n+1} + \sum_{k=0}^n \left[(B_k + \sum_{q=k}^n C_{kq} X^{n-q} Q) \cdot \varphi_k \right].$$

Then we apply D_v to get

$$D_v D_u(\varphi_{n+1}) = -\frac{2(n+1)\Omega^2}{r} D_v \varphi_{n+1} - 2(n+1) \partial_v \left(\frac{\Omega^2}{r} \right) \varphi_{n+1} + \sum_{k=0}^n \left[(B_k + \sum_{q=k}^n C_{kq} X^{n-q} Q) \cdot \frac{\varphi_{k+1}}{r^2} \right] + \sum_{k=0}^n \sum_{q=k}^n C_{kq} \partial_v(X^{n-q} Q) \cdot \varphi_k$$

Now, because $r^2 \partial_v(\frac{\Omega^2}{r})$ is an entire function in r^{-1} , $\partial_v(X^{n-q} Q) = r^{-2} X^{n+1-q} Q$ and

$D_v D_u(\varphi_{n+1}) = D_u D_v(\varphi_{n+1}) - \frac{2\Omega^2}{r^2} \varphi_{n+1}$, it is clear that (3.8.10) is true at the level $(n+1)$, therefore for all n .

Now that the equation is written, let us assume first we have in the region $\{u \leq u_0(R)\}$ that for all n , there exists $D_n > 0$ such that for all $v \geq v_0(R)$

$$|D_v \psi_n|(u_0(R), v) \leq D_n \cdot r^{-2}. \quad (3.8.11)$$

Then we can prove by induction that $|\varphi_n|(u, v) \lesssim u^{n-1+\frac{\alpha}{2}}$ for $u > 0$. This is indeed the case for $n = 0$ and $n = 1$. Now if we assume the result for all integer $0 \leq k \leq n$ we want to prove it at the level $(n+1)$. For this we first write (3.8.10) as

$$D_u(r^{-2n} D_v \varphi_n) = r^{-2n-2} \sum_{k=0}^n \left[(B_k + \sum_{q=k}^n C_{kq} X^{n-q} Q) \cdot \varphi_k \right]. \quad (3.8.12)$$

Now notice that $X^q Q$ can be written as a linear combination of the form $X^q Q = \sum_{i=0}^q a_i \mathfrak{F}(\varphi_{q-i} \bar{\varphi}_i)$, with $a_i \in \mathbb{R}$. Using the induction hypothesis we get that there exists $C_n > 0$ such that for $u > 0$

$$|D_u(r^{-2n} D_v \varphi_n)|(u, v) \leq C_n \cdot r^{-2n-2} u^{n-1+\frac{\alpha}{2}}.$$

Then integrating in u towards $\{u = u_0(R)\}$ and using (3.8.11) we get that for some $C'_n > 0$

$$|\varphi_{n+1}| = |r^2 D_v \varphi_n| \leq C'_n \cdot u^{n+\frac{\alpha}{2}},$$

which finishes the induction. Now, this proves that for all n , $|D_v \varphi_n| \lesssim r^{-2} u^{n+\frac{\alpha}{2}}$ hence $\partial_v(e^{iq_0 \int_0^v A_v(u, v') dv'} \varphi_n(u, v))$ is integrable in v so $e^{iq_0 \int_0^v A_v(u, v') dv'} \psi(u, v)$ admits a finite limit $\tilde{\psi}_0(u)$ when $v \rightarrow +\infty$, as claimed.

Now, the last part of the proof is to establish (3.8.11). As noticed in (3.8.7), we can prove, from Proposition 3.4.3 that for all $\omega > 1$, there exists $\tilde{D} > 0$ such that in $\{u \leq u_0(R)\}$:

$$\begin{aligned} |\psi| &= |\varphi_0| \leq \tilde{D} \cdot |u|^{-\omega}, \\ |\varphi_1| &\leq \tilde{D} \cdot |u|^{1-\omega}. \end{aligned}$$

Then this time we want to prove by induction that $|\varphi_n| \lesssim |u|^{n-\omega}$ for all $0 < \omega < n$. It is enough to use (3.8.12) with an argument similar to the one developed in the region $\{u \geq u_0(R)\}$. We also need the hypothesis (3.8.4) to deal with the term on Σ_0 when integrating in u .

This finally gives (3.8.11) and concludes the proof. □

3.9 Local boundedness proofs

In this section, we carry out the proofs of the results claimed in section 3.4 . We are going to state once more the propositions, for the convenience of the reader.

We start by the proof of Proposition 3.4.1.

Proposition 3.9.1. *Suppose that there exists $p > 1$ such that $\tilde{\mathcal{E}}_p < \infty$ and that $Q_0 \in L^\infty(\Sigma_0)$.*

Assume also that $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.

We denote $Q_0^\infty = \|Q_0\|_{L^\infty(\Sigma_0)}$.

There exists $r_+ < R_0 = R_0(M, \rho)$ large enough, $\delta = \delta(M, \rho) > 0$ small enough and $C = C(M, \rho) > 0$ so that for all $R_1 > R_0$, and if

$$Q_0^\infty + \tilde{\mathcal{E}}_p < \delta,$$

then for all $u \geq u_0(R_1)$:

$$E_{R_1}(u) \leq C \cdot \mathcal{E}. \quad (3.9.1)$$

Also for all $v \leq v_{R_1}(u)$:

$$\int_{\tilde{u}(v)}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v) du' \leq C \cdot \mathcal{E}, \quad (3.9.2)$$

where $\tilde{u}(v) = u_0(v)$ if $v \leq v_0(R_1)$ and $\tilde{u}(v) = u_{R_1}(v)$ if $v \geq v_0(R_1)$.

Moreover for all (u, v) in the space-time :

$$|Q|(u, v) \leq C \cdot (Q_0^\infty + \tilde{\mathcal{E}}_p). \quad (3.9.3)$$

Finally there exists $C_1 = C_1(R_1, M, \rho)$ such that for all $u \geq u_0(R_1)$:

$$\int_{\{v \leq v_{R_1}(u)\} \cap \{r \leq R_1\}} (|D_t \phi|^2(u', v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du' dv \leq C_1 \cdot \mathcal{E}, \quad (3.9.4)$$

Proof. Let $R_0 > r_+$ large to be chosen later and $R_1 \geq R_0$.

We are going to apply various identities to the domain

$\Theta_u(R_1) := \{v \leq v_{R_1}(u), r \leq R_1\} \cup \{u' \leq u, r \geq R_1\}$, defined for all $u \geq u_0(R_1)$, having boundaries $\mathcal{H}^+ \cap \{v \leq v_{R_1}(u)\}$, $[u, +\infty] \times \{v_{R_1}(u)\}$, $\{u\} \times [v_{R_1}(u), +\infty]$, $\mathcal{I}^+ \cap \{u' \leq u\}$ and Σ_0 , c.f. Figure 3.4

Notice that for all $u \geq u_0(R_1)$, $\Theta_u(R_1)$ is the complement of the interior of $\mathcal{D}_{R_1}(u, +\infty)$.

We apply the divergence identity in $\Theta_u(R_1)$ for the Killing vector field ∂_t , c.f. section 3.10 for more details. We get that for all $u \geq u_0(R_1)$:

$$\begin{aligned} & \int_{-\infty}^{v_{R_1}(u)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{-\infty}^u r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + \int_u^{+\infty} r^2 |D_u \phi|^2(u', v_{R_1}(u)) du' + \int_{v_{R_1}(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv \\ & + \int_u^{+\infty} \frac{2\Omega^2 Q^2(u', v_{R_1}(u))}{r^2} du' + \int_{v_{R_1}(u)}^{+\infty} \frac{2\Omega^2 Q^2(u, v)}{r^2} dv = \int_{\Sigma_0} r^2 \left(\frac{|D_u \phi|^2 + |D_v \phi|^2}{2} + \frac{2\Omega^2 Q^2}{r^4} \right) dr^*, \end{aligned} \quad (3.9.5)$$

Therefore it implies that

$$E_{deg, R_1}^+(u) \leq \frac{\mathcal{E}}{2} + C \cdot (Q_0^\infty)^2, \quad (3.9.6)$$

where $C = C(M, \rho) > 0$ is defined as $C = \int_{\Sigma_0} \frac{2\Omega^2}{r^4} dr^*$ and $E_{deg, R_1}^+(u)$ is defined as

$$E_{deg, R_1}^+(u) := \int_{-\infty}^{v_{R_1}(u)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{-\infty}^u r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + \int_u^{+\infty} r^2 |D_u \phi|^2(u', v_{R_1}(u)) du' + \int_{v_{R_1}(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv. \quad (3.9.7)$$

Now we want to prove a similar estimate for the non-degenerate energy $E_{R_1}(u)$. We are going to use a slight variation of the red-shift effect as demonstrated in section 3.5.2.

We start to prove a Morawetz type estimate in the region Θ_u .

The method of proof is the same as in section 3.5.1 :

Lemma 3.9.2. *There exists $\bar{e} = \bar{e}(M, \rho) > 0$, $\bar{C} = \bar{C}(M, \rho) > 0$, $\bar{\sigma} = \bar{\sigma}(M, \rho) > 1$ such that for all $u \geq u_0(R_1)$, if $\sup_{\Theta_u(R_1)} |Q| < \bar{e}$ then*

$$\int_{\Theta_u(R_1)} \left(\frac{|D_t \phi|^2(u', v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)}{r^{\bar{\sigma}}(u', v)} \right) \Omega^2 du' dv \leq \bar{C} \cdot E_{deg, R_1}^+(u) + \mathcal{E} \leq (\bar{C})^2 \cdot (\mathcal{E} + (Q_0^\infty)^2), \quad (3.9.8)$$

where $E_{deg, R_1}^+(u)$ is defined in (3.9.7).

Proof. The proof is exactly almost the same as the one in the interior \mathcal{D} . The bulk term is identical, the only difference is the boundary term on $\{t = 0\}$, which has already been controlled : we first get that for some $C' = C'(M, \rho) > 0$

$$\begin{aligned} & \int_{\Theta_u(R_1)} \left(\frac{|D_t \phi|^2(u, v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)}{r^{\bar{\sigma}}(u', v)} \right) \Omega^2 du' dv \\ & \leq C' \cdot \left(\int_{-\infty}^{v_{R_1}(u)} r^2 |D_v \phi|_{\mathcal{H}^+}^2(v) dv + \int_{-\infty}^u r^2 |D_u \phi|_{\mathcal{I}^+}^2(u) du + \int_u^{+\infty} r^2 |D_u \phi|^2(u', v_{R_1}(u)) du' + \int_{v_{R_1}(u)}^{+\infty} r^2 |D_v \phi|^2(u, v) dv \right), \end{aligned}$$

and then we apply (3.9.6) to conclude. \square

Now we are going to prove a Red-shift type estimate in the region $\{v \leq v_{R_1}(u)\}$. The method of proof is the same as in section 3.5.2 :

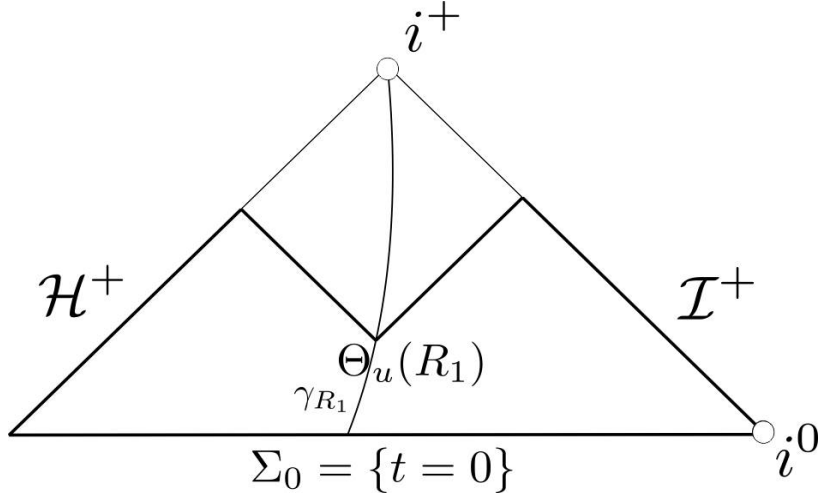


Figure 3.4: Illustration of the domain $\Theta_u(R_1)$, $u \geq u_0(R_1)$, $R_0(M, \rho) \leq R_1 \leq R$

Lemma 3.9.3. *There exists $\bar{R}_0 = \bar{R}_0(M, \rho)$, sufficiently close to r_+ such that all $r_+ < \tilde{R}_0 < \bar{R}_0$, there exists $\bar{e} = \bar{e}(M, \rho, \tilde{R}_0) > 0$, $\tilde{C} = \tilde{C}(M, \rho, \tilde{R}_0) > 0$ such that for all $u \geq u_0(R_1)$, if $\sup_{\Theta_u \cap \{r \leq \tilde{R}_0\}} |Q| < \bar{e}$ then for all $v \leq v_{R_1}(u)$*

$$\int_{u_{\tilde{R}_0}(v)}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v) du' + \int_{\{r_+ \leq r \leq \tilde{R}_0\} \cap \Theta_u(R_1)} \frac{r^2 |D_u \phi|^2}{\Omega^2} dudv \leq \tilde{C} \cdot E_{deg, R_1}^+(u) + \mathcal{E} \leq (\tilde{C})^2 \cdot (\mathcal{E} + (Q_0^\infty)^2), \quad (3.9.9)$$

where $E_{deg, R_1}^+(u)$ is defined in (3.9.7).

Proof. The proof is exactly almost the same as the one in the interior \mathcal{D} . Similarly, we make use crucially of estimate (3.9.8) to control the 0 order term, which is the exterior analogue of the Morawetz estimate, c.f. section 3.5.2 for more details. \square

Now take such a $\tilde{R}_0 = \tilde{R}_0(M, \rho)$ and assume $\tilde{R}_0 < R_0$. We also define $e^+ = e^+(M, \rho) < \max\{\bar{e}, \tilde{e}\}$, with the notations of the former lemmata. For fixed $u \geq u_0(R_1)$, we bootstrap in Θ_u :

$$|Q| \leq e^+. \quad (3.9.10)$$

If we assume $Q_0^\infty < e^+$, then the set of points which verify the bootstrap is non-empty. We now want to establish preliminary “weak” estimates. First we write, on the constant $v = V$ surface

$$|e^{\int_{-V}^u i q_0 A_u} \phi(u, V) - \phi_0(-V, V)| \leq \left| \int_{-V}^u e^{\int_{-V}^{u'} i q_0 A_u} D_u \phi(u', V) du' \right| \leq \left(\int_{-V}^u \Omega^2 r^{-2}(u', V) du' \right)^{\frac{1}{2}} \left(\int_{-V}^u \Omega^{-2} r^2 |D_u \phi|^2(u', V) du' \right)^{\frac{1}{2}}$$

where we used the fact that $\partial_u(e^{\int_{-V}^u i q_0 A_u} \phi) = e^{\int_{-V}^u i q_0 A_u} D_u \phi$.

Then we take the limit $V \rightarrow +\infty$, we use the fact from the hypothesis that ϕ_0 tends to 0 towards spatial infinity, together with (3.9.6) to get, for some $C' = C'(M, \rho) > 0$:

$$r^{\frac{1}{2}} |\phi|_{\mathcal{I}^+}(u) \leq \left(\int_{-\infty}^u r^2 |D_u \phi|^2(u', V) du' \right)^{\frac{1}{2}} \leq \sqrt{\frac{\mathcal{E}}{2} + C \cdot (Q_0^\infty)^2} \leq C' \cdot [\sqrt{\mathcal{E}} + Q_0^\infty].$$

Hence we get that $\lim_{v \rightarrow +\infty} \phi(u, v) = 0$.

Now because we know that $\lim_{v \rightarrow +\infty} \phi(u, v) = 0$, one can use Hardy’s inequality under the form (3.2.16) (the proof is the same although the statement differs slightly) and (3.9.6) to get that for all $(u, v) \in \{r \geq R_0\}$

$$r |\phi|^2(u, v) \leq \int_v^{+\infty} \frac{r^2 |D_v \phi|^2(u, v')}{\Omega^2} dv' \leq \frac{\mathcal{E}}{\Omega^2}. \quad (3.9.11)$$

We were able to claim the estimate in the whole region $\{r \geq R_0\}$ because it is true in $\{r \geq R_1\}$ and R_1 is allowed to vary in the range $[R_0, +\infty]$

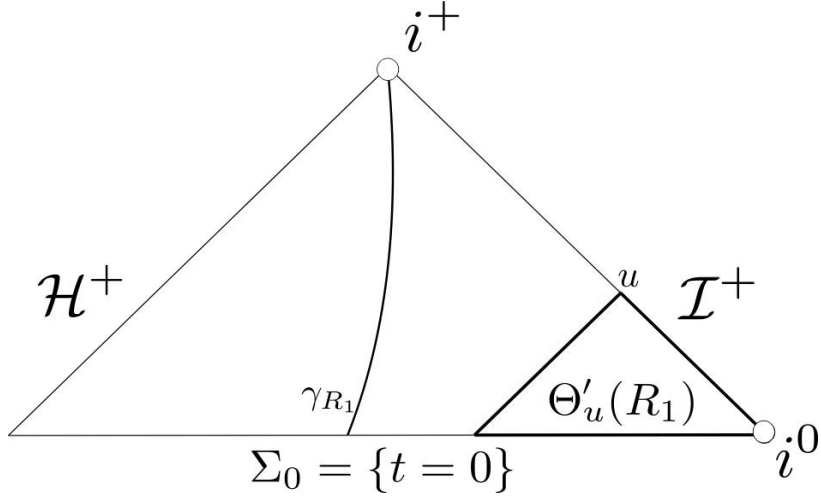


Figure 3.5: Illustration of the domain $\Theta'_u(R_1)$ for $u < u_0(R_1)$, $R_0(M, \rho) \leq R_1 \leq R$

Now using a method that has been made explicit already several times in section 3.5, we use the mean-value theorem with the Morawetz estimate (3.9.8) : we find that there exists $R'_0 \in (0.8R_0, 0.9R_0)$ and $\bar{C}_0 = \bar{C}_0(M, \rho)$ such that

$$\int_0^{t_{R_1, R'_0}(u)} (|\phi|^2(R'_0, t) + |D_{r^*}\phi|^2(R'_0, t)) dt \leq \bar{C}_0 \cdot (\mathcal{E} + (Q_0^\infty)^2),$$

this provided $r_+ < 0.8R_0$ and where $t_{R_1, R'_0}(u) := 2v_{R_1}(u) - (R'_0)^* = 2u + 2R_1^* - (R'_0)^*$ is defined such that $(t_{R_1, R'_0}(u), (R'_0)^*) = \gamma_{R'_0} \cap \{v = v_{R_1}(u)\}$.

Therefore, using (3.2.8), (3.2.9) as $|\partial_t Q| \leq q_0 r^2 |\phi| |D_{r^*}\phi|$ we integrate and find that there exists $C'_0 = C'_0(M, \rho) > 0$ so that for all $0 \leq t \leq t_{R_1, R'_0}(u)$

$$|Q(t, R'_0)| \leq Q_0^\infty + C'_0 \cdot (\mathcal{E} + (Q_0^\infty)^2)$$

Then we integrate $\partial_u Q$ in u towards $\gamma_{R'_0}$, using the red-shift estimate (3.9.9). Tedious details of the integration are left to the reader.

We find that in $\{v_0(R'_0) \leq v \leq v_{R_1}(u)\} \cap \{r \leq R'_0\}$, there exists $C''_0 = C''_0(M, \rho) > 0$ such that

$$|Q(u, v)| \leq Q_0^\infty + C''_0 \cdot (\mathcal{E} + (Q_0^\infty)^2).$$

Using the same technique with (3.9.9) in $\{v \leq v_0(R'_0)\}$ and then in the bounded region $\{R'_0 \leq r \leq R_0\}$ we then get that there exists $C = C(M, \rho) > 0$ such that in the whole region $\{v \leq v_{R_1}(u)\} \cap \{r \leq R_0\}$

$$|Q(u, v)| \leq Q_0^\infty + C \cdot (\mathcal{E} + (Q_0^\infty)^2), \quad (3.9.12)$$

where we used the fact that R_0 depends only on M and ρ .

This should be thought of as a local in space smallness propagation of the charge, for potentially large times.

We now want to “globalise” this result in space. For this, we need to control higher r^p weighted energies, for any $p > 1$. We will need to use the r^p method, as developed in section 3.6.

We establish the r^p estimate, in the very same way as in section 3.6 but this time on the domain $\Theta_u \cap \{r \geq R_0\}$.

We are going to write two different r^p estimates, according to $u < u_0(R_0)$ or $u > u_0(R_0)$. The proof is however the same.

For this we define $\Theta'_u(R_0) = \Theta_u(R_0) \cap \{r \geq R_0\}$ if $u \geq u_0(R_0)$ and $\Theta'_u(R_0) = \{u' \leq u\}$ if $u \leq u_0(R_0)$, c.f. Figure 3.5. We also define $\tilde{v}(u) = v_{R_0}(u)$ if $u \geq u_0(R_0)$ and $\tilde{v}(u) = v_0(u)$ if $u \leq u_0(R_0)$. We can then write

$$\begin{aligned} & \int_{\Theta'_u(R_0)} \left(pr^{p-1} \Omega^2 |D_v \psi|^2 + [1 + P_1(r)] \left(2M(3-p) - (4-p) \frac{2\rho^2}{r} \right) r^{p-4} |\psi|^2 \right) dudv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \\ & + \int_{-\infty}^u \Omega^2 \left(2M - \frac{2\rho^2}{r} \right) r^{p-3} |\psi|_{\mathcal{I}^+}^2(u) du = \int_{\Theta'_u(R_0)} 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv + \int_{\Sigma_0 \cap \{v \geq \tilde{v}(u)\}} r^p |D_v \psi_0|^2(r^*) dr^*, \end{aligned}$$

where $P_1(r)$ is a polynomial in r that behaves like $O(r^{-1})$ as r tends to $+\infty$ and with coefficients depending only on M and ρ .

Since $p < 3$ we can take R_0 large enough (depending on M and ρ) so that $|P_1(r)| < 1$ and $2M(3-p) - (4-p)\frac{2\rho^2}{r} > 0$. We get

$$\int_{\Theta'_u(R_0)} pr^{p-1}\Omega^2|D_v\psi|^2 dudv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v\psi|^2(u, v)dv \leq \int_{\Theta'_u(R_0)} 2q_0Q\Omega^2r^{p-2}\Im(\psi\overline{D_v\psi})dudv + \mathcal{E}_p. \quad (3.9.13)$$

Now we use a variant of Hardy's inequality (3.2.19) for any $0 \leq q < 2$ under the form

$$\left(\int_{\Theta'_u(R_0)} r^{q-3}\Omega^2|\psi|^2 dudv \right)^{\frac{1}{2}} \leq \frac{2}{(2-q)\Omega(R_0)} \left(\int_{\Theta'_u(R_0)} r^{q-1}|D_v\psi|^2 dudv \right)^{\frac{1}{2}} + \left(\frac{1}{2-q} \int_{-\infty}^u r^{q-2}|\psi|^2(u, \tilde{v}(u))du \right)^{\frac{1}{2}}.$$

Now we are free to choose $1 < p < 2$ without loss of generality, and with p as close to 1 as needed. Taking $q = p$, this implies, by the hypothesis and using the Morawetz estimate (3.9.8) that there exists a constant $C_0 = C_0(M, \rho) > 0$ such that

$$\frac{1}{2-p} \int_{-\infty}^u r^{p-2}|\psi|^2(u, \tilde{v}(u))du \leq C_0 \cdot \tilde{\mathcal{E}}_p.$$

Combining this with the Cauchy-Schwarz inequality and the bootstrap assumption (3.9.10) we see —like in section 3.6.2 — that for all $\eta > 0$ small enough we have

$$\left(p - \frac{4q_0e^+}{(2-p)\Omega(R_0)} - \frac{\eta}{2} \right) \int_{\Theta'_u(R_0)} r^{p-1}\Omega^2|D_v\psi|^2 dudv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v\psi|^2(u, v)dv \leq \left[1 + \frac{C_0}{2\eta} \right] \cdot \tilde{\mathcal{E}}_p, \quad (3.9.14)$$

so if say $4q_0e^+ < \frac{(2-p)p\Omega(R_0)}{2}$ — which can be assumed, taking δ small enough — we proved that for some $C'_0 = C'_0(M, \rho) > 0$ and for all $u \in \mathbb{R}$:

$$\int_{\tilde{v}(u)}^{+\infty} r^p |D_v\psi|^2(u, v)dv \leq C'_0 \cdot \tilde{\mathcal{E}}_p. \quad (3.9.15)$$

Now we re-write (3.2.9) as $|\partial_v Q| \leq q_0|\psi||D_v\psi|$ and we integrate towards γ_{R_0} if $u \geq u_0(R_0)$ and towards Σ_0 if $u \leq u_0(R_0)$, using Cauchy-Schwarz :

$$|Q(u, v) - Q(u, \tilde{v}(u))| \leq q_0 \left(\int_{\tilde{v}(u)}^{+\infty} r^p |D_v\psi|^2(u, v)dv \right)^{\frac{1}{2}} \left(\int_{\tilde{v}(u)}^{+\infty} r^{-p} |\psi|^2(u, v)dv \right)^{\frac{1}{2}}. \quad (3.9.16)$$

We now use the “boundary term” version of Hardy's inequality (3.2.19) and we get that there exists $C_1 = C_1(M, \rho) > 0$ such that

$$\left(\int_{\tilde{v}(u)}^{+\infty} r^{-p} |\psi|^2(u, v)dv \right)^{\frac{1}{2}} \leq C_1 \cdot \left[\left(\int_{\tilde{v}(u)}^{+\infty} r^p |D_v\psi|^2(u, v)dv \right)^{\frac{1}{2}} + [r(u, \tilde{v}(u))]^{\frac{1-p}{2}} |\psi|(u, \tilde{v}(u)) \right], \quad (3.9.17)$$

where we used the fact that $2-p < p$ because $p > 1$.

Now, to estimate ψ_0 we can use the Cauchy-Schwarz under the following form, taking advantage of the fact that $p > 1$: for all $r \geq R_0$

$$|\psi_0(r) - \psi_0(R_0)| \leq \int_{(R_0)^*}^{r^*} |D_{r^*}\psi_0|((r')^*)d(r')^* \leq \sqrt{2} \left(\int_{(R_0)^*}^{r^*} (r')^{-p}d(r')^* \right)^{\frac{1}{2}} (\mathcal{E}_p)^{\frac{1}{2}},$$

where we used the fact that $\int_{\Sigma_0} r^p |D_{r^*}\psi_0|^2 dr^* \leq 2\mathcal{E}_p$.

Using also (3.9.11) and remembering that $R_0 = R_0(M, \rho)$ depends only on M and ρ we can then write, for some $\tilde{C} = \tilde{C}(M, \rho) > 0$:

$$|\psi_0(r)| \leq \tilde{C} \cdot (\tilde{\mathcal{E}}_p)^{\frac{1}{2}}.$$

Coupled with (3.9.11) applied on γ_{R_0} , there exists also $\bar{C} = \bar{C}(M, \rho) > 0$ such that for all $u \in \mathbb{R}$:

$$|\psi|(u, \tilde{v}(u)) \leq \bar{C} \cdot (\tilde{\mathcal{E}}_p)^{\frac{1}{2}}. \quad (3.9.18)$$

Now combining (3.9.15), (3.9.16), (3.9.17) and (3.9.18) we get that there exists $C_2 = C_2(M, \rho) > 0$ such that

$$|Q(u, v) - Q(u, \tilde{v}(u))| \leq C_2 \cdot \tilde{\mathcal{E}}_p.$$

Now combining with (3.9.12), we see that for all $u \in \mathbb{R}$:

$$|Q(u, v)| \leq Q_0^\infty + C \cdot (Q_0^\infty)^2 + C'_2 \cdot \tilde{\mathcal{E}}_p < \delta \cdot (1 + C'_2 + C \cdot \delta), \quad (3.9.19)$$

where we defined $C'_2 = C_2 + C$.

Therefore the bootstrap (3.9.10) is retrieved provided if $\delta = \delta(M, \rho)$ is small enough so that

$$\delta \cdot (1 + C'_2 + C \cdot \delta) < e^+.$$

This proves the charge bound claimed in the statement of the Proposition.

The last step we need to carry out is to prove the claimed boundedness of the energy. Indeed we only proved in (3.9.6)

$$E_{R_1}(u) \leq \frac{\mathcal{E}}{2} + C \cdot (Q_0^\infty)^2,$$

and we would like a right-hand-side that only depends on \mathcal{E} .

For this, we have to revisit the proof of section 3.5.3, to absorb the charge terms properly in the energy identity (3.9.5).

The term on the future boundary of Θ_u are treated in the very same way so we do not repeat the argument, c.f. sections 3.5.3, 3.5.3, 3.5.3, 3.5.3. We only take care of the charge term on Σ_0 .

First, using the same strategy as in section 3.5.3, one can prove that

$$\left| \int_{(R_1)^*}^{+\infty} \frac{\Omega^2 Q_0^2(r)}{r^2} dr^* - \frac{Q_0^2(R_1)}{R_1} \right| \leq \frac{2q_0 Q_0^\infty \mathcal{E}}{\Omega^2(R_1)}. \quad (3.9.20)$$

Then, like in section 3.5.3 one can prove

$$\left| \frac{Q_0^2(R_1)}{R_1} - \frac{Q_0^2(R_0)}{R_1} \right| \leq \frac{2q_0 Q_0^\infty \mathcal{E}}{\Omega^2(R_0)}. \quad (3.9.21)$$

Now for the analogue of the proof in section 3.5.3, we need to prove a few preliminary estimates.

First, using an argument similar to the one used in the proof of Hardy inequality (3.2.15) and the fact that $\phi_0(r) \rightarrow 0$ when $r \rightarrow +\infty$ one can prove that there exists $C_+ = C_+(M, \rho) > 0$ such that

$$\int_{-\infty}^{+\infty} \Omega^2 |\phi_0|^2 dr^* \leq C_+ \cdot \mathcal{E}.$$

The rough idea is to apply an Hardy argument to the integral on $[-\infty, (R_0)^*]$ then we pick a term on γ_{R_0} that can be controlled (3.9.11). The integral on $[(R_0)^*, +\infty]$ can be treated similarly. Every-time, we lose a weight on γ_{R_0} but that weight depends only on M and ρ .

Consequently with this estimate, like in section 3.5.3 one can prove

$$\left| \int_{-\infty}^{(R_1)^*} \frac{\Omega^2 Q_0^2(r)}{r^2} dr^* - \left(1 - \frac{r_+}{R_1}\right) \frac{Q_0^2(R_0)}{r_+} \right| \leq \left(\sqrt{2} \left| \frac{R_0}{r_+} - 1 \right| + \frac{2 \left| 1 - \frac{R_0}{R_1} \right|}{\Omega^2(R_0)} \right) \cdot q_0 Q_0^\infty \mathcal{E}, \quad (3.9.22)$$

We can then use the Morawetz estimate (3.9.8) to deal with the charge difference on γ_{R_0} , recalling the dependence $R_0 = R_0(M, \rho)$, everything like in section 3.5.10. We conclude that — provided δ is small enough — there exists $C' = C'(M, \rho) > 0$ such that for all $u \geq u_0(R_1)$

$$E_{R_1}(u) \leq C' \cdot \mathcal{E}.$$

This concludes the proof of the proposition. □

We now turn to the proof of Proposition 3.4.2 : this time we assume already energy boundedness and the Morawetz estimate but **not arbitrary charge smallness**. The method of proof is very similar to that of Proposition 3.4.1.

Proposition 3.9.4. *Suppose that there exists $1 < p < 2$ such that $\tilde{\mathcal{E}}_p < \infty$.
It follows that there exists $e_0 \in \mathbb{R}$ such that*

$$\lim_{r \rightarrow +\infty} Q_0(r) = e_0.$$

Without loss of generality one can assume that $1 < p < 1 + \sqrt{1 - 4q_0|e_0|}$.

Assume also that $\lim_{r \rightarrow +\infty} \phi_0(r) = 0$.

Now assume (3.4.2), (3.4.3) for $R_1 = R$: there exists $\bar{C} = \bar{C}(M, \rho) > 0$ such that for all $u \geq u_0(R)$ and for all $v \leq v_R(u)$:

$$E(u) = E_R(u) \leq \bar{C} \cdot \mathcal{E}. \quad (3.9.23)$$

$$\int_{\tilde{u}(v)}^{+\infty} \frac{r^2 |D_u \phi|^2}{\Omega^2}(u', v) du' \leq \bar{C} \cdot \mathcal{E}, \quad (3.9.24)$$

where $\tilde{u}(v) = u_0(v)$ if $v \leq v_0(R)$ and $\tilde{u}(v) = u_R(v)$ if $v_0(R) \leq v \leq v_R(u)$.

Assume also (3.4.5) : there exists $\bar{R}_0 = \bar{R}_0(M, \rho) > r_+$ such that for all $\bar{R}_1 > \bar{R}_0$, there exists $\bar{C}_1 = \bar{C}_1(\bar{R}_1, M, \rho) > 0$ such that

$$\int_{\{u' \leq u\} \cap \{r \leq \bar{R}_1\}} (|D_t \phi|^2(u', v) + |D_{r^*} \phi|^2(u', v) + |\phi|^2(u', v)) \Omega^2 du dv \leq \bar{C}_1 \cdot \mathcal{E}. \quad (3.9.25)$$

Make also the following smallness hypothesis : for some $\delta > 0$:

$$\begin{aligned} \tilde{\mathcal{E}}_p &< \delta, \\ q_0 |e_0| &< \frac{1}{4}. \end{aligned}$$

There exists $\delta_0 = \delta_0(\mathbf{e}_0, \mathbf{M}, \rho) > 0$ and $C = C(M, \rho) > 0$ such that if $\delta < \delta_0$ then for all (u, v) in the space-time :

$$|Q(u, v) - e_0| \leq C \cdot \tilde{\mathcal{E}}_p, \quad (3.9.26)$$

$$q_0 |Q|(u, v) < \frac{1}{4}. \quad (3.9.27)$$

Moreover, there ⁸² exists $\delta_p = \delta_p(\mathbf{e}_0, \mathbf{p}, \mathbf{M}, \rho) > 0$ and $C' = C'(e_0, p, M, \rho) > 0$ such that if $\delta < \delta_p$ then for all $u \geq u_0(R)$:

$$E_p[\psi](u) \leq C' \cdot \tilde{\mathcal{E}}_p. \quad (3.9.28)$$

Proof. First take $R_0 = R_0(M, \rho)$. Using Cauchy-Schwarz and the Maxwell equation under the form $|\partial_{r^*} Q| \leq q_0 |\psi| |D_t \psi|$, we can show that there exists $\bar{C}_0 = \bar{C}_0(M, \rho) > 0$ such that for all $r \geq R_0$,

$$|Q_0(r) - e_0| \leq \bar{C}_0 \cdot \mathcal{E}_p.$$

Similarly on $\{r_+ \leq r \leq R_0\}$ one can prove that there exists $\tilde{C}_0 = \tilde{C}_0(M, \rho) > 0$ such that for all $r \leq R_0$,

$$|Q_0(r) - Q_0(R_0)| \leq \tilde{C}_0 \cdot \mathcal{E},$$

where we used Cauchy-Schwarz and the Maxwell equation under the form $|\partial_{r^*} Q| \leq \frac{q_0 R_0^2}{2} (|\phi| |D_u \phi| + |\phi| |D_v \phi|)$.

This gives, on the whole Σ_0 :

$$|Q_0(r) - e_0| \leq (\tilde{C}_0 + \bar{C}_0) \cdot \tilde{\mathcal{E}}_p. \quad (3.9.29)$$

Now assume that $R_0 > \bar{R}_0(M, \rho)$ so that estimate (3.9.25) is valid.

Now, using the Morawetz estimate (3.9.25) in a way that was explained numerous times in this paper, one can find $r_+ < \bar{R}_0 = \bar{R}_0(M, \rho) < \bar{R}_0$ and $D_0 = D_0(M, \rho) > 0$ such that for all $t \geq 0$

$$|Q(t, \bar{R}_0) - Q_0(\bar{R}_0)| \leq D_0 \cdot \mathcal{E}, \quad (3.9.30)$$

where we used Cauchy-Schwarz and the Maxwell equation under the form $|\partial_t Q| \leq q_0 R_0^2 |\phi| |D_{r^*} \phi|$.

Then using (3.9.24) and (3.2.8), one can prove that there exists $D'_0 = D'_0(M, \rho) > 0$ such that for all $(u, v) \in \{r \leq R\}$,

⁸²The dependence of δ_p on p only exists as p approaches $1 + \sqrt{1 - 4q_0|e_0|}$.

$$|Q(u, v) - Q(\hat{u}(v), v)| \leq D'_0 \cdot \tilde{\mathcal{E}}_p,$$

where $\hat{u}(v) = u_0(v)$ if $v \leq v_0(R)$ and $\hat{u}(v) = u_{\tilde{R}_0}(v)$ if $v \geq v_0(R)$.

This combined with 3.9.29 and 3.9.30 proves that there exists $C_0 = C_0(M, \rho) > 0$ such that for all $(u, v) \in \{r \leq R\}$,

$$|Q(u, v) - e_0| \leq C_0 \cdot \tilde{\mathcal{E}}_p. \quad (3.9.31)$$

In the same way as in Proposition 3.4.1, we can derive the estimate for $R_0 = R_0(M, \rho)$ large enough, for all $R_1 \geq R_0$ and for all $u \in \mathbb{R}$:

$$\int_{\Theta'_u(R_1)} pr^{p-1} \Omega^2 |D_v \psi|^2 dudv + \int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \leq \int_{\Theta'_u(R_1)} 2q_0 Q \Omega^2 r^{p-2} \Im(\psi \overline{D_v \psi}) dudv + \mathcal{E}_p, \quad (3.9.32)$$

where all the notations are introduced in the proof of Proposition 3.4.1. Note that this time we consider $\Theta'_u(R_1)$ for any $R_1 \geq R_0$ to be chosen later, in contrast to the proof of Proposition 3.4.1 where only $\Theta'_u(R_0)$ was considered.

Then we bootstrap for some $\bar{e} = |e_0| + 2\epsilon$, $\epsilon > 0$ and in $\Theta'_u(R_1)$:

$$|Q| < \bar{e}. \quad (3.9.33)$$

Because $Q_0 \rightarrow e_0$ towards spatial infinity, it is clear that the set of points for which this bootstrap is verified is non-empty.

For ϵ small enough, one can assume that $q_0 \cdot (|e_0| + 2\epsilon) < \frac{1}{4}$.

By assumption, one can assume that there exists $1 < p < \sqrt{1 - 4q_0 \bar{e}}$ such that $\tilde{\mathcal{E}}_p < \infty$.

Using this, we find constants $C_{R_1} = C_{R_1}(R_1, M, \rho) > 0$ and $D = D(M, \rho)$ such that for all $\eta > 0$ small enough and for all $u \in \mathbb{R}$:

$$\int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \leq \frac{D \cdot \tilde{\mathcal{E}}_p}{\eta \cdot \left(p - \frac{4q_0 \bar{e}}{(2-p)\Omega(R_1)} - \eta \right)}, \quad (3.9.34)$$

$$\int_{\tilde{v}(u)}^{+\infty} r^{p-2} |\psi|^2(u, v) \leq \frac{C_{R_1} \cdot \tilde{\mathcal{E}}_p}{\eta \cdot \left(p - \frac{4q_0 \bar{e}}{(2-p)\Omega(R_1)} - \eta \right)}, \quad (3.9.35)$$

where for the second estimate, we used a Hardy inequality coupled with the Morawetz estimate (3.9.25).

Now we take $R_1 = R_1(e_0, M, \rho)$ large enough so that $4q_0 \bar{e} < \Omega(R_1)$. Then we can take temporarily $p > 1$ sufficiently close to 1 and η small enough so that $p - \frac{4q_0 \bar{e}}{(2-p)\Omega(R_1)} - \eta > 0$.

We then find using (3.2.9) that there exists $C = C(e_0, M, \rho) > 0$ such that on $\{r \geq R_1\}$

$$|Q(u, v) - Q(u, \bar{v}(u))| \leq C \cdot \tilde{\mathcal{E}}_p, \quad (3.9.36)$$

where $\bar{v}(u) = v_{R_1}(u)$ if $u \geq u_0(R_1)$ and $\bar{v}(u) = v_0(u)$ if $u \leq u_0(R_1)$. Then with (3.9.31) and provided that $R_1 < R$ — we actually proved that for some $C' = C'(e_0, M, \rho) > 0$ and on the whole $\Theta'_u(R_1)$:

$$|Q(u, v) - e_0| \leq C' \cdot \tilde{\mathcal{E}}_p < C' \cdot \delta. \quad (3.9.37)$$

Then it suffices to take $C' \cdot \delta < \epsilon$ to retrieve bootstrap (3.9.33). This evidently gives the first two claims on the whole space-time.

Now come back to general $1 < p < \sqrt{1 - 4q_0 |e_0|}$ and notice that (3.9.34) can be written as, for all $\eta > 0$:

$$\int_{\tilde{v}(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \leq \frac{D \cdot \tilde{\mathcal{E}}_p}{\eta \cdot \left(p - \frac{4q_0 |e_0| + \eta}{(2-p)} - \eta \right)}, \quad (3.9.38)$$

for δ small enough and for the choice $R_1 = R$, for R large enough. This gives directly (3.9.28) and concludes the proof of the proposition. \square

Now we turn to the proof of Proposition 3.4.3.

Proposition 3.9.5. *In the conditions of Proposition 3.4.2, assume moreover that there exists $\omega > 0$ and $C_0 > 0$ such that*

$$r|D_v\psi_0| + |\psi_0| \leq C_0 \cdot r^{-\omega}.$$

Then in the following cases

1. $\omega = 1 + \theta$ with $\theta > \frac{q_0|e_0|}{4}$
2. $\omega = \frac{1}{2} + \beta$ with $\beta \in (-\frac{\sqrt{1-4q_0|e_0|}}{2}, \frac{\sqrt{1-4q_0|e_0|}}{2})$, if $q_0|e_0| < \frac{1}{4}$,

there exists $\delta = \delta(e_0, \omega, M, \rho) > 0$ and $R_0 = R_0(\omega, e_0, M, \rho) > r_+$ such that if $\tilde{\mathcal{E}}_p < \delta$ and $R > R_0$ then the decay is propagated : there exists $C'_0 = C'_0(C_0, \omega, R, M, \rho, e_0) > 0$ such that for all $u \leq u_0(R)$:

$$\begin{aligned} |D_v\psi|(u, v) &\leq C'_0 \cdot r^{-1-\omega'}, \\ |\psi|(u, v) &\leq C'_0 \cdot |u|^{-\omega}, \end{aligned}$$

where $\omega' = \min\{\omega, 1\}$.

In that case, for every $0 < p < 2\omega' + 1$, we have the finiteness of the r^p weighted energy on $\mathcal{V}_{u_0(R)}$:

$$E_p[\psi](u_0(R)) < \infty.$$

Proof. We start the proof in a region $\{u \leq U_0\}$ for $|U_0|$ large enough to be chosen later.

We are going to use the notations $u_0(v), v_0(u), r_0(u), r_0(v)$, c.f. section 3.2.4 for a definition.

For some $B > 0$ large enough to be chosen appropriately later, we bootstrap the following in $\{u \leq U_0\}$

$$|\psi| \leq B|u|^{-\omega}.$$

Notice that with the assumptions and the fact that $r_0(u) \sim 2|u|$ when $u \rightarrow -\infty$, the set of points for which the bootstrap is verified is non-empty, for B large enough.

We also denote $Q^+ = \sup_{\{u \leq U_0\}} |Q|$. By Proposition 3.4.2, Q^+ can be taken arbitrarily close to $|e_0|$ or $|e|$ for δ appropriately small.

Then we use (3.2.6) under the form

$$|D_u D_v \psi| \leq \frac{B|u|^{-\omega}}{r^2} (q_0 Q^+ + |2r \cdot K(r)|).$$

Now take $\epsilon > 0$. $\{u \leq U_0\} \subset \{r \geq r_0(U_0)\}$ so by taking $|U_0|$ large enough, one can assume that $|2r \cdot K(r)| < \epsilon$ since this quantity tends to 0 as r tends to $+\infty$.

There are two cases : either $\omega > 1$ or $\omega \leq 1$. We start by $\omega > 1$: we can integrate in u the inequality above and get

$$|D_v\psi|(u, v) \leq C_0 \cdot (r_0(v))^{-1-\omega} + \frac{B|u|^{1-\omega}}{(\omega-1)r^2} (q_0 Q^+ + \epsilon). \quad (3.9.39)$$

Now we want to integrate this in v : first notice that $\frac{d(r_0(v))}{dv} = 2\Omega^2(-v, v) \geq 2\Omega^2(r_0(U_0)) \geq \frac{2}{(1+\epsilon)}$ if $|U_0|$ is large enough. Therefore

$$\begin{aligned} \int_{v_0(u)}^v (r_0(v'))^{-1-\omega} dv' &\leq \frac{(1+\epsilon)}{2(-\omega)} \int_{v_0(u)}^v \frac{d[(r_0(v'))^{-\omega}]}{dv'} dv' \leq \frac{(1+\epsilon)}{2\omega} (r_0(u))^{-\omega}, \\ \int_{v_0(u)}^v (r_0(v'))^{-2} dv' &\leq \frac{(1+\epsilon)}{2} (r_0(u))^{-1}, \end{aligned}$$

after noticing that $r_0(v_0(u)) = r_0(u)$. Hence we have

$$|\psi(u, v)| \leq C_0 \cdot (1 + \frac{(1+\epsilon)}{2\omega}) \cdot (r_0(u))^{-\omega} + \frac{B(q_0 Q^+ + \epsilon)(1+\epsilon)}{2(\omega-1)} (r_0(u))^{-1} |u|^{1-\omega}.$$

Since $r_0(u) \sim 2|u|$ when $u \rightarrow -\infty$, it is clear that for $|U_0|$ large enough, ϵ small enough and B large enough, the bootstrap is retrieved if

$$\omega > 1 + \frac{q_0 Q^+}{4},$$

or equivalently if δ is small enough,

$$\omega > 1 + \frac{q_0|e_0|}{4}.$$

Now we turn to $\omega \leq 1$. We ignore the case $\omega = 1$ and consider $\omega < 1$. Integrating in u we get this time

$$|D_v \psi|(u, v) \leq C_0 \cdot (r_0(v))^{-1-\omega} + \frac{B \cdot v^{1-\omega}}{(1-\omega)r^2} (q_0 Q^+ + \epsilon),$$

where we used that $|u_0(v)| = v$.

Now, taking $|U_0|$ large enough, one can assume that everywhere on $\{u \leq U_0\} : r^{-2} \leq \frac{(r^*)^{-2}}{1-\epsilon} \leq \frac{v^{-2}}{1-\epsilon}$, since $u \leq 0$. Using this, we get

$$\int_{v_0(u)}^v \frac{(v')^{1-\omega}}{r^2(u, v')} dv' \leq \frac{|u|^{-\omega}}{(1-\epsilon)\omega},$$

hence we have

$$|\psi_0(u, v)| \leq C_0 \cdot \left(1 + \frac{(1+\epsilon)}{2\omega}\right) \cdot (r_0(u))^{-\omega} + \frac{B(q_0 Q^+ + \epsilon)}{(1-\omega)\omega(1-\epsilon)} |u|^{-\omega}.$$

We now see that the bootstrap is retrieved on the condition :

$$q_0 Q^+ < (1-\omega)\omega,$$

Otherwise said

$$\frac{1 - \sqrt{1 - 4q_0 Q^+}}{2} < \omega < \frac{1 + \sqrt{1 - 4q_0 Q^+}}{2},$$

or equivalently if δ is small enough :

$$\frac{1 - \sqrt{1 - 4q_0|e_0|}}{2} < \omega < \frac{1 + \sqrt{1 - 4q_0|e_0|}}{2}.$$

This proves the proposition in the region $\{u \leq U_0\}$, for $|U_0|$ large enough with respect to ω and e_0 and **independently of R** .

Now it is enough to take R large enough so that $|u_0(R)| > |U_0|$ and the result is proven, if we accept that the constants now depend on R . □

Then we turn to our last step, the proof of Proposition 3.4.4.

Proposition 3.9.6. *Suppose that for all $0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|}$, $\tilde{\mathcal{E}}_p < \infty$.*

We also assume the other hypotheses of Theorem 3.3.3.

Then for all $0 \leq p < 2 + \sqrt{1 - 4q_0|e_0|}$, there ⁸³ exists $\delta_p = \delta_p(\mathbf{e}_0, \mathbf{p}, \mathbf{M}, \rho) > 0$, such that if $\delta < \delta_p$ then for all $u \leq u_0(R)$:

$$E_p[\psi](u) < \infty.$$

Proof. The proof relies on the generalization of the r^p weighted estimate, namely a $r^p u^s$ weighted estimate, for $s > 0$.

We are going to state this identity on the neighbourhood of spatial infinity $\{u \leq u_0(R)\}$ where R is large enough so that $u_0(R) < 0$. As a consequence, $|u| = -u$ in this region.

Using (3.2.6) that we multiply by $(-u)^s r^p \overline{D_v \psi}$, we take the real part, integrate by parts and get, for all $u \leq u_0(R)$

$$\begin{aligned} & \int_{\{u' \leq u\}} [p r^{p-1} |u'|^s + s r^p |u'|^{s-1}] |D_v \psi|^2 du' dv + |u|^s \int_{v_0(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \\ & \leq [1 + P_0(r)] \left[\int_{\Sigma_0 \cap \{v \geq v_0(u)\}} |u_0(v)|^s (r_0(v))^p |D_v \psi|^2(u_0(v), v) dv + \int_{\{u' \leq u\}} 2q_0 Q r^{p-2} |u'|^s \Im(\psi \overline{D_v \psi}) du' dv \right], \end{aligned}$$

where $P_0(r)$ is a polynomial in r that behaves like $O(r^{-1})$ as r tends to $+\infty$ and with coefficients depending only on M and ρ .

⁸³The dependence of δ_p on p only exists as p approaches $2 + \sqrt{1 - 4q_0|e_0|}$.

Now because $|u_0(v)| \sim \frac{r_0(v)}{2}$ as $v \rightarrow +\infty$, for R large enough we can say that

$$\int_{\Sigma_0 \cap \{v \geq v_0(u)\}} |u_0(v)|^s (r_0(v))^p |D_v \psi|^2(u_0(v), v) dv \leq \mathcal{E}_{p+s}.$$

Denoting $Q^+ = \sup_{u' \leq u_0(R)} |Q|$: note that $|Q^+ - |e_0|| \lesssim \delta$ by the last proposition.

With this, we find that for all $\eta > 0$, taking R large enough so that $|1 + P_0(r)| < (1 + \eta)$:

$$\begin{aligned} \int_{\{u' \leq u\}} [pr^{p-1}|u'|^s + sr^p|u'|^{s-1}] |D_v \psi|^2 du' dv + |u|^s \int_{v_0(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \\ \leq (1 + \eta) \cdot \left[\mathcal{E}_{p+s} + 2q_0 Q^+ \int_{\{u' \leq u\}} r^{p-2} |u'|^s |\psi| |D_v \psi| du' dv \right]. \end{aligned} \quad (3.9.40)$$

As it was seen in section 3.6.2, if $1 - \sqrt{1 - 4q_0 Q^+} < p' < 1 + \sqrt{1 - 4q_0 Q^+}$ the interaction term can be absorbed inside the bulk term using Hardy's inequality because the presence of the u weight does not change anything. Therefore for all $s' \geq 0$, there exists a ⁸⁴ constant $D > 0$ such that

$$\int_{\{u' \leq u\}} \left[r^{p'-1} |u'|^{s'} + r^{p'} |u'|^{s'-1} \right] |D_v \psi|^2 du' dv + |u|^{s'} \int_{v_0(u)}^{+\infty} r^{p'} |D_v \psi|^2(u, v) dv \leq D \cdot \tilde{\mathcal{E}}_{p'+s'}. \quad (3.9.41)$$

Now we take $2 - \sqrt{1 - 4q_0 |e_0|} < p < 2 + \sqrt{1 - 4q_0 |e_0|}$ and come back to (3.9.40). If δ is small enough —depending on p — one can assume that actually $2 - \sqrt{1 - 4q_0 Q^+} < p < 2 + \sqrt{1 - 4q_0 Q^+}$.

Now applying Cauchy-Schwarz we find that

$$\int_{\{u' \leq u\}} r^{p-2} |u'|^s |\psi| |D_v \psi| du' dv \leq \left(\int_{\{u' \leq u\}} r^{p-4} |u'|^{s+1} |\psi|^2 du' dv \right)^{\frac{1}{2}} \left(\int_{\{u' \leq u\}} r^p |u'|^{s-1} |D_v \psi|^2 du' dv \right)^{\frac{1}{2}}.$$

□

Then we prove a Hardy inequality, very similar to (3.2.19) under the form : for all $u' \leq u$:

$$\int_{v_0(u')}^{+\infty} r^{p-4} |\psi|^2(u', v) dv \leq \frac{2}{(3-p)\Omega^2(R)} (r_0(u'))^{p-3} |\psi_0|^2(u', v_0(u')) + \frac{4}{(3-p)^2 \Omega^4(R)} \int_{v_0(u')}^{+\infty} r^{p-2} |D_v \psi|^2(u', v) dv.$$

Using Fubini's theorem, one can now prove that for ⁸⁵ some $D' > 0$:

$$\int_{\{u' \leq u\}} r^{p-4} |u'|^{s+1} |\psi|^2 du' dv \leq D' \cdot \left[\mathcal{E}_{p+s} + \int_{\{u' \leq u\}} r^{p-2} |u'|^{s+1} |D_v \psi|^2 du' dv \right],$$

where we controlled the integral of $|u|^{s+1} (r_0(u'))^{p-3} |\psi_0|^2(u', v_0(u'))$ by the zero order term of \mathcal{E}_{p+s} .

Combining with (3.9.41) for $p' = p - 1$ and $s' = s + 1$ we finally get

$$\int_{\{u' \leq u\}} r^{p-4} |u'|^{s+1} |\psi|^2 du' dv \leq D' \cdot (1 + D) \cdot \tilde{\mathcal{E}}_{p+s}. \quad (3.9.42)$$

Now denoting $D'' = 2q_0 Q^+ \cdot (1 + \eta) \cdot \sqrt{D' \cdot (1 + D)}$ and coming back to (3.9.40) we get :

$$\begin{aligned} \int_{\{u' \leq u\}} [pr^{p-1}|u'|^s + sr^p|u'|^{s-1}] |D_v \psi|^2 du' dv + |u|^s \int_{v_0(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv \\ \leq D'' \cdot \tilde{\mathcal{E}}_{p+s} \cdot \left(\int_{\{u' \leq u\}} r^p |u'|^{s-1} |D_v \psi|^2 du' dv \right)^{\frac{1}{2}}. \end{aligned} \quad (3.9.43)$$

Once we reach this step, the proof is over if $s > 0$: it suffices to absorb the last term on the right-hand-side into the first term of the left-hand-side, using the inequality $|ab| \leq \epsilon a + \frac{b}{4\epsilon}$ for small enough $\epsilon > 0$.

This proves that $E_p[\psi](u) = \int_{v_0(u)}^{+\infty} r^p |D_v \psi|^2(u, v) dv < +\infty$ if $\tilde{\mathcal{E}}_{p+s} < \infty$ which concludes the proof of the proposition.

⁸⁴Since we just need a finiteness statement, the parameters on which D depends do not matter so we do not specify them.

⁸⁵Since we just need a finiteness statement, the parameters on which D' depends do not matter so we do not specify them.

3.10 Useful computations for the vector field method

This section carries out explicitly a few computations to apply the vector field method to the case of Maxwell-Charged-Scalar-Field.

For the linear wave equation, the traditional vector field method proceeds as follows : we can construct a quadratic quantity $\mathbb{T}_{\mu\nu}^{Wave}(\phi) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}(g^{\alpha\beta} \partial_\alpha \phi \partial_\beta \phi) g_{\mu\nu}$. We then see that the wave equation corresponds to a conservation law :

$$\nabla^\mu \mathbb{T}_{\mu\nu}^{Wave} = 0 \iff \square \phi = 0.$$

Then for any solution of the wave equation ϕ and for a well chosen vector field X we construct the current $J_\mu^X = \mathbb{T}_{\mu\nu}^{Wave}(\phi) X^\nu$ and we integrate $\nabla^\mu J_\mu^X = \mathbb{T}_{\mu\nu}^{Wave} \nabla^{(\mu} X^{\nu)}$ on a space-time domain.

Making use of the divergence theorem, we see that a bulk term, namely the integral on a space-time domain of $\mathbb{T}_{\mu\nu}^{Wave} \nabla^{(\mu} X^{\nu)}$, equals some boundary terms involving the current J_μ^X .

Notice that if X is a Killing vector field, then $\nabla^{(\mu} X^{\nu)} = 0$ and the identity only includes boundary terms.

Compared to the classical vector field method, a major difference — in the case we consider in this paper — is the presence of a Maxwell stress-energy tensor that is coupled to the scalar field's. This still gives rise of a conservation law that couples the scalar field and the charge. While compactly supported scalar fields on asymptotically flat space-time decay, this is not the case for charges on black hole space-times.

Therefore, in many cases we do not apply this conservation law directly, except in subsection 3.5.3. Instead, we treat the Maxwell term as an error term, that can be controlled by the energy of the scalar field. To put it otherwise, instead of looking at charges —that do not decay— we look at the fluctuation of these charges, that do enjoy time decay estimates. To see the main related computations, c.f. subsection 3.10.3.

3.10.1 Stress-Energy momentum tensor

For a spherically symmetric scalar field ϕ and 2-form F we define the stress energy momentum tensor of the scalar field $\mathbb{T}_{\mu\nu}^{SF}(\phi)$ and the one of the Maxwell field $\mathbb{T}_{\mu\nu}^{EM}(F)$. Notice that the Maxwell-Charged-Scalar-Field equations (3.2.3), (3.2.4) imply the following conservation law :

$$\nabla^\mu (\mathbb{T}_{\mu\nu}^{SF} + \mathbb{T}_{\mu\nu}^{EM}) = 0,$$

where $\mathbb{T}_{\mu\nu}^{SF}(\phi)$ and $\mathbb{T}_{\mu\nu}^{EM}(F)$ are defined as :

$$\mathbb{T}_{\mu\nu}^{SF} = \Re(D_\mu \phi \overline{D_\nu \phi}) - \frac{1}{2}(g^{\alpha\beta} D_\alpha \phi \overline{D_\beta \phi}) g_{\mu\nu},$$

$$\mathbb{T}_{\mu\nu}^{EM} = g^{\alpha\beta} F_{\alpha\nu} F_{\beta\mu} - \frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} g_{\mu\nu}.$$

In the (u, v, θ, φ) coordinate system of section 3.2.1 this gives :

$$\mathbb{T}_{vv}^{SF} = |D_v \phi|^2, \tag{3.10.1}$$

$$\mathbb{T}_{uu}^{SF} = |D_u \phi|^2, \tag{3.10.2}$$

$$\mathbb{T}_{\theta\theta}^{SF} = \mathbb{T}_{\varphi\varphi}^{SF} \sin^{-2}(\theta) = \frac{r^2 \Re(D_u \phi \overline{D_v \phi})}{2\Omega^2}. \tag{3.10.3}$$

$$\mathbb{T}_{uv}^{EM} = \frac{2\Omega^2 Q^2}{r^4}, \tag{3.10.4}$$

$$\mathbb{T}_{\theta\theta}^{EM} = \mathbb{T}_{\varphi\varphi}^{EM} \sin^{-2}(\theta) = \frac{Q^2}{r^2}. \tag{3.10.5}$$

$$\mathbb{T}_{uv}^{SF} = \mathbb{T}_{vu}^{SF} = \mathbb{T}_{vv}^{EM} = \mathbb{T}_{uu}^{EM} = 0. \tag{3.10.6}$$

While (3.10.1), (3.10.2), (3.10.3) are used everywhere in the paper, particularly in subsection 3.5.1 and 3.5.2, (3.10.4) is only useful in subsection 3.5.3 while (3.10.5) is not used.

Notice that in section 3.6, we do not make use of the divergence theorem but directly of equations (3.2.6), (3.2.7). Hence this section is mainly useful for section 3.5.

3.10.2 Deformation tensors

As we saw in the beginning of this section, to use the divergence theorem we need to compute the derivative of vector fields.

More precisely for any vector field X we define the deformation tensor as

$$\Pi_X^{\mu\nu} := \nabla^{(\mu} X^{\nu)}.$$

In the (u, v, θ, φ) coordinate system of section 3.2.1 we compute :

$$\Pi_X^{vv} = \frac{-2}{\Omega^2} \partial_u X^v, \quad (3.10.7)$$

$$\Pi_X^{uu} = \frac{-2}{\Omega^2} \partial_v X^u, \quad (3.10.8)$$

$$\Pi_X^{uv} = \frac{-1}{\Omega^2} (\partial_v X^v + \partial_v \log(\Omega^2) X^v + \partial_u X^u + \partial_u \log(\Omega^2) X^u), \quad (3.10.9)$$

$$\Pi_X^{\theta\theta} = \Pi_X^{\phi\phi} \sin^2(\theta) = \frac{1}{r^2} \left(\frac{\partial_v r}{r} X^v + \frac{\partial_u r}{r} X^u \right). \quad (3.10.10)$$

Notice that for $\partial_t = \frac{\partial_v + \partial_u}{2}$,

$$\Pi_{\partial_t}^{\mu\nu} = 0.$$

This is because ∂_t is a Killing vector field, corresponding to the t invariance of the Reissner–Nordström metric.

3.10.3 Computations of bulk terms in the divergence formula

Let (ϕ, F) be a solution to the Maxwell-Charged-Scalar-Field equations (3.2.3), (3.2.4), and we consider a vector field X .

Even though the traditional method considers the current $J_\mu^X = \mathbb{T}_{\mu\nu}^{SF}(\phi) X^\nu$, it is sometimes useful to create a modified current, c.f. [26], [27].

For this we introduce a **real-valued** scalar function χ and define the modified energy current:

$$\tilde{J}_\mu^X(\phi, \chi) := \mathbb{T}_{\mu\nu}^{SF} X^\nu + \frac{\chi \cdot \partial_\mu |\phi|^2 - \partial_\mu \chi \cdot |\phi|^2}{2}.$$

Thus we can compute its divergence :

$$\nabla^\mu \tilde{J}_\mu^X := \mathbb{T}_{\mu\nu}^{SF} \Pi_X^{\mu\nu} + F_{\mu\nu} X^\mu \mathcal{J}^\nu(\phi) + \chi D^\mu \phi \overline{D_\mu \phi} - \frac{\square \chi}{2} |\phi|^2, \quad (3.10.11)$$

where the particular current $\mathcal{J}_\mu(\phi)$ is defined by

$$\mathcal{J}_\mu(\phi) = q_0 \Im(\phi \cdot \overline{D_\mu \phi}).$$

For this computation, we used the fact that $\nabla^\mu \mathbb{T}_{\mu\nu}^{SF} = -\nabla^\mu \mathbb{T}_{\mu\nu}^{EM} = F_{\nu\mu} \mathcal{J}^\mu(\phi)$, where the last identity makes use of the Maxwell equation (3.2.3).

In order to compute this term in (u, v) coordinates, recall that $F_{uv} = \frac{2\Omega^2 Q}{r^2}$. Then we can establish the following expression for the interaction term :

$$F_{\mu\nu} X^\mu \mathcal{J}^\nu(\phi) = \frac{Q}{r^2} [X^v \mathcal{J}_v(\phi) - X^u \mathcal{J}_u(\phi)]. \quad (3.10.12)$$

3.10.4 D'Alembertians

As seen in the former section, the use of a modified current involves a $\square \chi$ term, multiplying the 0 order term $|\phi|^2$.

Notice that the control of this 0 order term is one of the difficulties of this paper.

We compute the expression of the \square operator in (u, v) coordinates, for a spherically symmetric χ :

$$\square(\chi) = \frac{-1}{\Omega^2} \left(\partial_u \partial_v \chi + \frac{\partial_v r}{r} \partial_u \chi + \frac{\partial_u r}{r} \partial_v \chi \right) = -\frac{\partial_u \partial_v \chi}{\Omega^2} + \frac{\partial_v \chi}{r} - \frac{\partial_u \chi}{r}. \quad (3.10.13)$$

3.10.5 Volume forms, normals and current fluxes

To conclude this section, we include a few computations of space-time volume forms and volume forms induced on the curves we use in this paper, together with exterior unit normals.

Note that as far as null surfaces are concerned, there **no canonical** notion of exterior unit normals or induced volume forms. However, the contraction of an induced volume form with its unit normal **does not depend** on the normalization choice.

The volume form corresponding to Reissner–Nordström metric is defined by

$$dvol = 2\Omega^2 r^2 dudvd\sigma_{\mathbb{S}^2},$$

where $d\sigma_{\mathbb{S}^2}$ is the standard volume form on the unit sphere.

We now give consecutively the normals, the induced volumes forms and the current flux for each of the following : constant u hyper-surfaces, constant v hyper-surfaces, constant r hyper-surfaces γ_{R_1} and $\Sigma_0 = \{t = 0\}$.

$$\begin{aligned} n_{u=cst}^\mu &= \frac{1}{\Omega^2} \partial_v, \\ dvol_{u=cst} &= \Omega^2 r^2 dv d\sigma_{\mathbb{S}^2}, \\ J_\mu n_{u=cst}^\mu dvol_{u=cst} &= J_v r^2 dv d\sigma_{\mathbb{S}^2}. \end{aligned}$$

$$\begin{aligned} n_{v=cst}^\mu &= \frac{1}{\Omega^2} \partial_u, \\ dvol_{v=cst} &= \Omega^2 r^2 du d\sigma_{\mathbb{S}^2}, \\ J_\mu n_{v=cst}^\mu dvol_{v=cst} &= J_u r^2 du d\sigma_{\mathbb{S}^2}. \end{aligned}$$

For these constant u and constant v hyper-surfaces, we expressed the **future-directed** normal.

Notice that if such a null hyper-surface appears as the **future** boundary of a space-time domain, than the **future-directed** is also the **exterior** normal.

Symmetrically, if such a null hyper-surface appears as the **past** boundary of a space-time domain, than the **exterior** normal is the **past-directed** normal, i.e. the opposite of the future-directed one that we wrote.

Then we carry out the same computations for $\gamma_{R_1} = \{r = R_1\}$, for any $r_+ < R_1$:

$$\begin{aligned} n_{\gamma_{R_1}}^\mu &= \frac{1}{2\Omega(R_1)} (\partial_u - \partial_v), \\ dvol_{\gamma_{R_1}} &= \Omega(R_1) (dv + du) r^2 d\sigma_{\mathbb{S}^2}, \\ J_\mu n_{\gamma_{R_1}}^\mu dvol_{\gamma_{R_1}} &= \frac{J_u - J_v}{2} (dv + du) r^2 d\sigma_{\mathbb{S}^2}. \end{aligned}$$

Notice that if γ_{R_1} appears as the **right-most** boundary of a space-time domain, e.g. for $\{r \leq R_1\}$, than the **exterior** normal is the one we wrote.

Symmetrically, if γ_{R_1} appears as the **left-most** boundary of a space-time domain, e.g. for $\{r \geq R_1\}$, than the **exterior** normal is the **opposite** of the one we wrote.

Finally we write the same computations for $\Sigma_0 = \{t = 0\}$:

$$\begin{aligned} n_{\Sigma_0}^\mu &= \frac{1}{2\Omega} (\partial_v + \partial_u), \\ dvol_{\Sigma_0} &= \Omega \cdot (dv - du) r^2 d\sigma_{\mathbb{S}^2}, \\ J_\mu n_{\Sigma_0}^\mu dvol_{\Sigma_0} &= \frac{J_u + J_v}{2} (dv - du) r^2 d\sigma_{\mathbb{S}^2}. \end{aligned}$$

We wrote the future directed normal, which is in fact the opposite of the exterior unit normal, because the space-time is included inside $\{t \geq 0\}$.

Bibliography

- [1] Spyros Alexakis, Volker Schlue *Non-existence of time periodic vacuum space-times*. Journal of Differential Geometry, to appear. arXiv:1504.04592.
- [2] Yannis Angelopoulos, Stefanos Aretakis, Dejan Gajic *A vector field approach to almost-sharp decay for the wave equation on spherically symmetric, stationary spacetimes*. Preprint, 2016. arXiv:1612.01565
- [3] Yannis Angelopoulos, Stefanos Aretakis, Dejan Gajic *Late-time asymptotics for the wave equation on spherically symmetric, stationary spacetimes*. Advances in Mathematics, Vol. 323, 529–621, 2018.
- [4] Lydia Bieri, Shuang Miao, and Sohrab Shahshahani *Asymptotic properties of solutions of the Maxwell Klein Gordon equation with small data*. Communications in Analysis and Geometry 25(1), 2014.
- [5] Piotr Bizoń, Arthur Wasserman *On existence of mini-boson stars*. Comm.Math.Phys.215:357-373 , 2000.
- [6] Lior Burko, Gaurav Khanna *Universality of massive scalar field late-time tails in black-hole spacetimes*. Phys.Rev. D70, 044018, 2004.
- [7] Otis Chodosh, Yakov Shlapentokh-Rothman, *Time-periodic Einstein-Klein-Gordon bifurcations of Kerr*. Preprint, 2016. arXiv:1510.08025.
- [8] Yvonne Choquet-Bruhat, *Theoreme d’existence pour certains systemes d’equations aux derivees partielles non linaires*. Acta Math. 88, 141–225, 1952.
- [9] Yvonne Choquet-Bruhat, Robert Geroch, *Global Aspects of the Cauchy Problem in General Relativity*. Comm. Math. Phys. 14, 329–335, 1969.
- [10] Demetrios Christodoulou, *Violation of cosmic censorship in the gravitational collapse of a dust cloud*. Comm. Math. Phys. 93, 171–195, 1984.
- [11] Demetrios Christodoulou, *The Formation of Black Holes and Singularities in Spherically Symmetric Gravitational Collapse*. Comm. Pure Appl. Math. 44, no. 3, 339–373, 1991.
- [12] Demetrios Christodoulou, *Bounded Variation Solutions of the Spherically Symmetric Einstein-Scalar Field Equations*. Comm. Pure Appl. Math. 46, 1131–1220, 1993.
- [13] Demetrios Christodoulou, *The instability of naked singularities in the gravitational collapse of a scalar field*. Ann. of Math. (2) 149, no. 1, 183–217, 1999.
- [14] Demetrios Christodoulou, *The global initial value problem and the issue of singularities*. Classical Quantum Gravity 16, no. 12A,A23–A35, 1999.
- [15] Demetrios Christodoulou, *The Formation of Shocks in 3-Dimensional Fluids*. European Mathematical Society, 2007.
- [16] Demetrios Christodoulou, *The formation of black holes in general relativity*. European Mathematical Society, 2009.
- [17] Piotr T. Chruściel, Jalal Shatah *Global existence of solutions of the Yang-Mills equations on globally hyperbolic four dimensional Lorentzian manifolds*. Asian journal of Mathematics, Vol.1, No.3,pp. 530-548, 1997.
- [18] João L Costa, Pedro M Girão, José Natário, Jorge Drumond Silva, *On the global uniqueness for the Einstein-Maxwell-scalar field system with a cosmological constant: I. Well posedness and breakdown criterion*. Classical and Quantum Gravity, Volume 32, Number 1, 2014.

- [19] Mihalis Dafermos, *Stability and instability of the Cauchy horizon for the spherically symmetric Einstein–Maxwell-scalar field equations*. Ann. of Math. 158 , 875–928, 2003.
- [20] Mihalis Dafermos, *The Interior of Charged Black Holes and the Problem of Uniqueness in General Relativity*. Comm. Pure Appl. Math. 58, 0445–0504, 2005.
- [21] Mihalis Dafermos, *Spherically symmetric spacetimes with a trapped surface*. Classical and Quantum Gravity, Volume 22, Number 11, 2005.
- [22] Mihalis Dafermos, *Black holes without spacelike singularities*. Commun. Math. Phys. 332, 729–757, 2014.
- [23] Mihalis Dafermos, Gustav Holzegel, Igor Rodnianski *The linear stability of the Schwarzschild solution to gravitational perturbations*. Preprint, 2017. arXiv:1601.06467
- [24] Mihalis Dafermos, Jonathan Luk, *The interior of dynamical vacuum black holes I: The C^0 -stability of the Kerr Cauchy horizon*. Preprint, 2017. arXiv:1710.01772
- [25] Mihalis Dafermos, Igor Rodnianski *A proof of Price’s law for the collapse of a self-gravitating scalar field*. Invent. math. 162, 381–457, 2005.
- [26] Mihalis Dafermos, Igor Rodnianski , *Lectures on black holes and linear waves*. 2008.
- [27] Mihalis Dafermos, Igor Rodnianski *The red-shift effect and radiation decay on black hole spacetimes*. Comm. Pure Appl. Math., 62(7):859:919, 2009.
- [28] Mihalis Dafermos, Igor Rodnianski , *A new physical-space approach to decay for the wave equation with applications to black hole spacetimes*. pp. 421–432 in XVIth International Congress on Mathematical Physics, edited by P. Exner, World Sci. Publ., Hackensack, NJ, 2010.
- [29] Mihalis Dafermos, Igor Rodnianski *A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds*. Invent. Math., 185, 2011.
- [30] Mihalis Dafermos, Igor Rodnianski, Yakov Shlapentokh-Rothman, *Decay for solutions to the wave equation on Kerr exterior spacetimes III : the full sub-extremal case $|a| < M$* . Ann. of Math., 183, 787–913, 2016.
- [31] Oscar J. C. Dias, Harvey S. Reall, Jorge E. Santos *Strong cosmic censorship for charged de Sitter black holes with a charged scalar field* . Preprint, 2018. arXiv:1808.04832.
- [32] Roland Donniger, Wilhelm Schlag, Avy Soffer *A proof of Price’s Law on Schwarzschild black hole manifolds for all angular momenta* . Advances in Mathematics Volume 226, Issue 1, Pages 484-540, 2011.
- [33] Douglas M. Eardley, Vincent Moncrief, *The global existence of Yang–Mills–Higgs fields in 4- dimensional Minkowski space, I: Local existence and smoothness properties* . Comm. Math. Phys. , 83:2, 171–191, 1982.
- [34] Douglas M. Eardley, Vincent Moncrief, *The global existence of Yang–Mills–Higgs fields in 4- dimensional Minkowski space, II: Completion of proof* . Comm. Math. Phys. , 83:2, 193–212, 1982.
- [35] Grigorios Fournodavlos, Jan Sbierski, *Generic blow-up results for the wave equation in the interior of a Schwarzschild black hole*. Preprint, 2018.
- [36] Anne Franzen *Boundedness of massless scalar waves on Reissner-Nordström interior backgrounds*. Commun. Math. Phys. 343: 601. doi:10.1007/s00220-015-2440-7, 2016.
- [37] Dejan Gajic, Jonathan Luk *The interior of dynamical extremal black holes in spherical symmetry*. Preprint, 2017. arXiv:1709.09137 .
- [38] Eeavan Gleeson *Linear Instability of the Reissner-Nordström Cauchy Horizon*. Preprint, 2017. arXiv:1701.06668 .
- [39] Stephen Hawking, George Ellis, *The large scale structure of space-time*. Cambridge Monographs on Mathematical Physics, 1973.
- [40] William Hiscock, *Evolution of the interior of a charged black hole*. Physics Letters A Volume 83, Issue 3, Pages 110–112, 1981.
- [41] Shahar Hod, Tsvi Piran *Late-Time Evolution of Charged Gravitational Collapse and Decay of Charged Scalar Hair- I*. Phys.Rev D58 024017, 1998.

- [42] Christopher Kauffman *Global Stability for Charged Scalar Fields in an Asymptotically Flat Metric in Harmonic Gauge*. Preprint, 2018. arXiv:1801.09648 .
- [43] Markus Keel, Tristan Roy, Terence Tao *Global well-posedness of the Maxwell–Klein–Gordon equation below the energy norm*. Discrete Contin. Dyn. Syst. 30:3, 573–621, 2011.
- [44] Christoph Kehle, Maxime Van de Moortel *Continuous (in)-extendibility across the Cauchy horizon of Einstein–Maxwell–Klein–Gordon Black Holes in spherical symmetry*, in preparation.
- [45] Sergiu Klainerman, Matei Machedon *On the Maxwell–Klein–Gordon equation with finite energy*. Duke Math. J. 74:1 , 19–44, 1994.
- [46] Sergiu Klainerman, Jérémie Szeftel *Global Nonlinear Stability of Schwarzschild Spacetime under Polarized Perturbations*. Preprint, 2017. arXiv:1711.07597
- [47] Jonathan Kommemi, *The Global Structure of Spherically Symmetric Charged Scalar Field Spacetimes*. Comm. Math. Phys., 323: 35. doi:10.1007/s00220-013-1759-1, 2013.
- [48] Joachim Krieger, Jonas Lührmann, *Concentration compactness for the critical Maxwell–Klein–Gordon equation*. Annals of PDE 1:1, Art. 5, 208, 2015.
- [49] Joachim Krieger, Jacob Sterbenz, Daniel Tataru, *Global well-posedness for the Maxwell–Klein–Gordon equation in $(4+1)$ dimensions: small energy*. Duke Math. J. 164:6, 973–1040, 2015.
- [50] Peter Hintz, *Boundedness and decay of scalar waves at the Cauchy horizon of the Kerr spacetime*. Commentarii Mathematici Helvetici, to appear. arXiv:1512.08003.
- [51] Peter Hintz, András Vasy *The global non-linear stability of the Kerr-de Sitter family of black holes*. Preprint, 2016. arXiv:1606.04014 .
- [52] Roman Konoplya, Alexander Zhidenko; *A massive charged scalar field in the Kerr–Newman background I: quasi-normal modes, late-time tails and stability*. Phys.Rev. D88 024054, 2013.
- [53] Philippe G. LeFloch, Yue Ma, *The global nonlinear stability of Minkowski space for self-gravitating massive fields. The wave-Klein-Gordon model*. Preprint, 2015. arXiv:1507.01143.
- [54] Hans Lindblad, Jacob Sterbenz *Global stability for charged-scalar fields on Minkowski space*. IMRP Int. Math. Res. Pap., Art. ID 52976, 109, 2006.
- [55] Jonathan Luk, Sung-Jin Oh, *Quantitative decay rates for dispersive solutions to the Einstein-scalar field system in spherical symmetry*. Analysis and PDE, 8(7):1603:1674, 2015.
- [56] Jonathan Luk, Sung-Jin Oh *Proof of Linear Instability of the Reissner–Nordström Cauchy Horizon under Scalar Perturbations*. Duke Math. J. Volume 166, Number 3, 437-493, 2017.
- [57] Jonathan Luk, Sung-Jin Oh, *Strong Cosmic Censorship in Spherical Symmetry for two-ended Asymptotically Flat Initial Data I. The Interior of the Black Hole Region*. Preprint, 2017.
- [58] Jonathan Luk, Sung-Jin Oh, *Strong Cosmic Censorship in Spherical Symmetry for two-ended Asymptotically Flat Initial Data II. The Exterior of the Black Hole Region*. Preprint, 2017.
- [59] Jonathan Luk, Jan Sbierski *Instability results for the wave equation in the interior of Kerr black holes*. Journal of Functional Analysis, Volume 271, Issue 7, Pages 1948-1995, 2016.
- [60] Matei Machedon, Jacob Sterbenz *Almost optimal local well-posedness for the $(3+1)$ -dimensional Maxwell–Klein–Gordon equations*. . Journal of the American Mathematical Society 17:2, 297–359, 2004.
- [61] Jason Metcalfe, Daniel Tataru, Mihai Tohaneanu *Price’s Law on Nonstationary Spacetimes*. Adv. Math. 230, no. 3, 995-1028, 2012.
- [62] Cathleen Synge Morawetz *Time decay for the nonlinear Klein-Gordon equations*. Proc. Roy. Soc. Ser. A, 306:291:296, 1968.
- [63] Georgios Moschidis *The r^p -weighted energy method of Dafermos and Rodnianski in general asymptotically flat spacetimes and applications*. Annals of PDE 2:6 , 1-194, 2016.
- [64] Sung-Jin Oh, Daniel Tataru *Global well-posedness and scattering of the $(4+1)$ -dimensional Maxwell–Klein–Gordon equation*. Invent. Math. 205:3 , 781–877, 2016.

- [65] Amos Ori *Inner structure of a charged black hole: An exact mass-inflation solution*. Phys. Rev. Lett. 67, 789 – P, 1990.
- [66] Eric Poisson and Werner Israel *Internal structure of black holes*. Phys. Rev. D, 63:1663-1666, 1989.
- [67] Eric Poisson and Werner Israel *Inner-horizon instability and mass inflation in black holes*. Phys. Rev. Lett., 67(7):789-792, 1991.
- [68] Richard Price *Nonspherical perturbations of relativistic gravitational collapse. I. Scalar and gravitational perturbations* Phys. Rev. D (3) 5, 2419-2438, 1972.
- [69] Harvey Reall, *Black Holes Lecture notes*. Part III lecture notes, University of Cambridge, 2015.
- [70] Igor Rodnianski, Terence Tao *Global regularity for the Maxwell–Klein–Gordon equation with small critical Sobolev norm in high dimensions* . Comm. Math. Phys., 251:2, 377–426, 2004.
- [71] Jan Sbierski, *The C^0 -inextendibility of the Schwarzschild spacetime and the spacelike diameter in Lorentzian geometry*. Preprint, 2016. arXiv:1507.00601.
- [72] Volker Schlue *Decay of linear waves on higher dimensional Schwarzschild black holes* . Analysis and PDE 6-3, 515–600, 2013.
- [73] Yakov Shlapentokh-Rothman, *Exponentially growing finite energy solutions for the Klein-Gordon equation on sub-extremal Kerr spacetimes*. Comm. Math. Phys. 309 no.3 859-891, 2014.
- [74] Wei-Tong Shu *Asymptotic properties of the solutions of linear and nonlinear spin field equations in Minkowski space* . Comm. Math. Phys. 140:3, 449–480, 1991.
- [75] Michael Simpson, Roger Penrose *Internal instability in a Reissner–Nordström black hole*. Int J Theor Phys, 7: 183. doi:10.1007/BF00792069, 1973.
- [76] Terence Tao, *Nonlinear dispersive equations : local and global analysis*. CBMS, 2006.
- [77] Daniel Tataru *Local decay of waves on asymptotically flat stationary space-times* . Amer. J. Math. 130, no. 3, 571–634, 2008.
- [78] Maxime Van de Moortel *Stability and instability of the sub-extremal Reissner–Nordström black hole interior for the Einstein–Maxwell-Klein-Gordon equations in spherical symmetry*. Comm. Math. Phys., Volume 360, Issue 1, pp 103–168, 2018.
- [79] Maxime Van de Moortel *Decay of weakly charged solutions for the spherically symmetric Maxwell-Charged-Scalar-Field equations on a Reissner–Nordström exterior space-time.*, preprint, 2018. arXiv:1804.04297.
- [80] Shiwu Yang *Decay of solutions of Maxwell-Klein-Gordon equations with arbitrary Maxwell fields*. Analysis and PDE, Vol. 9, No. 8, 2016.
- [81] Shiwu Yang, Pin Yu *On global dynamics of the Maxwell-Klein-Gordon equations*. Preprint, 2018. arXiv 1804.00078.