A proof of the uniform boundedness of solutions to the wave equation on slowly rotating Kerr backgrounds

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May 28, 2018

Abstract

We consider Kerr spacetimes with parameters a and M such that $|a| \ll M$, Kerr-Newman spacetimes with parameters $|Q| \ll M$, $|a| \ll M$, and more generally, stationary axisymmetric black hole exterior spacetimes $(\mathcal{M}, g)$ which are sufficiently close to a Schwarzschild metric with parameter $M > 0$, with appropriate geometric assumptions on the plane spanned by the Killing fields. We show uniform boundedness on the exterior for sufficiently regular solutions to the wave equation $\Box_g \psi = 0$, i.e. we show that solutions ψ arising from smooth initial data $(\uppsi, \uppsi')$ prescribed on an arbitrary Cauchy surface Σ satisfy $|\psi| \leq C\,\mathbf{Q}_1(\uppsi, \uppsi')$ in the domain of outer communications. In particular, the bound holds up to and including the event horizon. Here, $\mathbf{Q}_1(\uppsi, \uppsi')$ is a norm on initial data and C depends only on the parameters of the nearby Schwarzschild metric. No unphysical restrictions are imposed on the behaviour of ψ near the bifurcation surface of the event horizon. The norm $\mathbf{Q}_1$ is finite if $\uppsi \in H^2_{\rm loc}(\Sigma)$, $\uppsi' \in H^1_{\rm loc}(\Sigma)$ and $\uppsi$ is well-behaved at spatial infinity, in particular, it is sufficient to assume $\nabla\uppsi$ is supported away from spatial infinity. The pointwise estimate derives in fact from the uniform boundedness of a positive definite energy flux. Note that in view of the very general assumptions, the separability properties of the wave equation on the Kerr background are not used.

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1 Introduction

The Kerr family, discovered in 1963 [23], comprises perhaps the most important family of exact solutions to the Einstein vacuum equations

$$R_{\mu\nu} = 0, \tag{1}$$

the governing equations of general relativity. For parameter values $0 \le |a| < M$ (here M denotes the mass and a angular momentum per unit mass), the Kerr solutions represent black hole spacetimes: i.e. asymptotically flat spacetimes which possess a region which cannot communicate with future null infinity. The celebrated Schwarzschild family sits as the one-parameter subfamily of Kerr corresponding to $a = 0$. Much of current theoretical astrophysics is based on the hypothesis that isolated systems described by Kerr metrics are ubiquitous in the observable universe.

Despite the centrality of the Kerr family to the general relativistic world picture, the most basic questions about the behaviour of linear waves on Kerr backgrounds have remained to this day unanswered. This behaviour is in turn intimately connected to the stability properties of the Kerr metrics themselves as solutions of (1), and thus, with the very physical tenability of the notion of black hole. In particular, even the question of the uniform boundedness (pointwise, or in energy) of solutions ψ to the linear wave equation

$$\Box_g \psi = 0 \tag{2}$$

in the domain of outer communications has not been previously resolved, except for the Schwarzschild subfamily.

The main theorems of this paper give the resolution of the boundedness problem for (2), for the case $|a| \ll M$. Solutions to (2) arising from regular initial data remain uniformly bounded in the domain of outer communications. The bound is quantitative, i.e. it is computable in terms of the initial supremum and initial energy-type quantities on initial data.

In fact, the results of this paper apply to a much more general setting than the specific Kerr metric: Boundedness is proven for solutions of (2) on the exterior region of any stationary axisymmetric spacetime sufficiently close to a Schwarzschild spacetime with mass $M > 0$. Thus, the methods may be of relevance in the ultimate goal of this analysis: understanding the dynamics of the Einstein equations (1) in a neighborhood of a Kerr metric.

We first give a statement of the main results for the special case of Kerr and the related Kerr-Newman family (this is a family of solutions to the coupled Einstein-Maxwell system).

1.1 Statement of the theorem for Kerr and Kerr-Newman

We refer the reader to [7, 21]. Let $(\mathcal{M}, g)$ denote the Kerr solution with parameters

$$0 \leq |a| < M$$

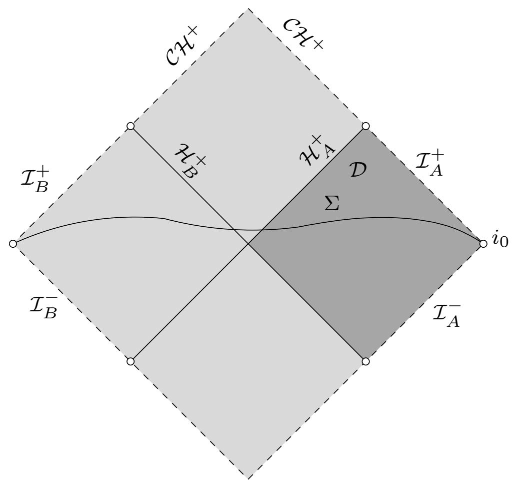
or more generally the Kerr-Newman solution with parameters (a, Q, M), with

$$0 \leq \sqrt{a^2 + Q^2} < M,$$

and let $\mathcal{D}$ denote the closure of a domain of outer communications. (The parameter Q is known as the charge.) Let Σ be a Cauchy hypersurface[1] in $(\mathcal{M}, g)$ crossing the event horizon to the future of the sphere of bifurcation, and such that $\Sigma \cap \mathcal{D}$ coincides with a constant-t hypersurface, for large r, where t and r denote here the standard Boyer-Lindquist coordinates on $\mathrm{int}(\mathcal{D})$. Recall that in such coordinates, the stationary Killing field T is given by $T = \frac{\partial}{\partial t}$. The Kerr-Newman solutions are moreover axisymmetric. The Penrose diagram, say along the axis of symmetry (where the axisymmetric Killing field vanishes), is

[1]For definiteness, our "Kerr solution" or "Kerr-Newman" solution is the Cauchy development of a complete asymptotically flat spacelike hypersurface with two asymptotically flat ends. This is a globally hyperbolic subdomain of the maximal analytic Kerr-Newman described in [21].

depicted below:



Note that $\Sigma \cap \mathcal{D}$ is a past Cauchy hypersurface for $J^+(\Sigma) \cap \mathcal{D}$.[2] We have that $J^+(\Sigma) \cap \mathcal{D}$ is foliated by Σ_τ for $\tau \geq 0$, where Σ_τ is the future translation of $\Sigma \cap \mathcal{D}$ by the flow generated by the stationary Killing field $T = \frac{\partial}{\partial t}$ for time τ. Let n_Σ denote the unit future normal of Σ_τ. Let $n_{\mathcal{H}}$ denote a translation invariant null generator for $\mathcal{H}^+$, and give $\mathcal{H}^+ \cap \mathcal{D}$ the induced volume from g and $n_{\mathcal{H}}$. Let $T_{\mu\nu}(\psi)$ denote the standard energy momentum tensor associated to a solution ψ of the wave equation (2)

$$T_{\mu\nu} = \partial_\mu \psi \partial_\nu \psi - \frac{1}{2} g_{\mu\nu} \nabla^\alpha \psi \nabla_\alpha \psi,$$

define $J_\mu^{n_\Sigma}(\psi)$ by

$$J_\mu^{n_\Sigma}(\psi) \doteq T_{\mu\nu}(\psi) n_\Sigma^\mu$$

and $J_\mu^T(\psi)$ by

$$J_\mu^T(\psi) \doteq T_{\mu\nu}(\psi) T^\mu.$$

Note that the former current is positive definite when contracted with a future-timelike vector field, but is not conserved, whereas the latter current is conserved, but not positive definite when so contracted.

Theorem 1.1. *Let $(\mathcal{M}, g)$, $\mathcal{D}$, Σ_τ be as above. There exists a universal positive constant $\epsilon > 0$, and a constant C depending on M and the choice of Σ_0 such that if*

$$0 \leq a < \epsilon M, \qquad 0 \leq Q < \epsilon M, \tag{3}$$

then the following statement holds. Let ψ be a solution of (2) on $(\mathcal{M}, g)$ such that $\int_{\Sigma_0} J_\mu^{n_\Sigma}(\psi) n_\Sigma^\mu < \infty$. Then

$$\int_{\Sigma_\tau} J_\mu^{n_\Sigma}(\psi) n_\Sigma^\mu \leq C \int_{\Sigma_0} J_\mu^{n_\Sigma}(\psi) n_\Sigma^\mu, \tag{4}$$

[2] Here, J^+ denotes causal future, not to be confused with currents J_μ to be defined later.

$$\left| \int_{\mathcal{H}^+ \cap J^+(\Sigma_0)} J^T_\mu(\psi) n^\mu_{\mathcal{H}} \right| \leq C \int_{\Sigma_0} J^{n_\Sigma}_\mu(\psi) n^\mu_\Sigma, \tag{5}$$

$$\int_{\mathcal{I}^+} J^T_\mu(\psi) n^\mu_{\mathcal{I}} \leq C \int_{\Sigma_0} J^{n_\Sigma}_\mu(\psi) n^\mu_\Sigma. \tag{6}$$

Here the integrals are with respect to the induced volume forms. The integral on the left hand side of (6) can be defined via a limiting procedure.

Theorem 1.2. *Under the assumptions of the previous theorem, the following holds. Let ψ be a solution of the wave equation (2) on $(\mathcal{M}, g)$ such that*

$$\mathbf{Q}_1 \doteq \sqrt{\sup_{\Sigma_0} |\psi|^2 + \int_{\Sigma_0} \left(J^{n_0}_\mu(\psi) + J^{n_0}_\mu(n_\Sigma \psi) \right) n^\mu} < \infty.$$

Then

$$|\psi| \leq C\, \mathbf{Q}_1$$

in $\mathcal{D} \cap J^+(\Sigma_0)$.

The hypothesis of Theorem 1.1 can be re-expressed as the statement that local energy as measured by a local observer be finite, i.e. that $\nabla^{\Sigma_0}\psi|_{\Sigma_0}$, $n_\Sigma \psi|_{\Sigma_0}$ be in L^2_{loc}, together with the global assumption that

$$\int_{\Sigma_0} J^T_\mu(\psi) n^\mu_\Sigma < \infty.$$

The latter in turn is certainly satisfied if $\nabla\psi$ vanishes in a neighborhood of i_0.

Similarly, the hypothesis of Theorem 1.2 is satisfied for $\nabla^{\Sigma_0}\psi$, $n_\Sigma\psi|_{\Sigma_0}$ in H^1_{loc}, if $\nabla\psi$ vanishes in a neighborhood of i_0.

Finally, note that given an arbitrary Cauchy surface $\tilde{\Sigma}$ for Kerr, sufficiently well behaved at i_0, it follows that the right hand side of (4) is bounded by

$$C(\Sigma_0, \tilde{\Sigma}) \int_{\tilde{\Sigma} \cap (J^-(\Sigma_0) \cup J^+(\Sigma_0))} J^{n_{\tilde{\Sigma}}}_\mu(\psi) n^\mu_{\tilde{\Sigma}},$$

thus the above regularity assumptions could be imposed on an arbitrary Cauchy surface. There are no unphysical restrictions on the support of the solution in a neighborhood of $\mathcal{H}^+ \cap \mathcal{H}^-$.

1.2 Statement for general stationary axisymmetric perturbations of Schwarzschild

The results of Theorems 1.1 and 1.2 remain true when the Kerr or Kerr-Newman metric is replaced by an arbitrary stationary axisymmetric black hole exterior metric suitably close to Schwarzschild, and with suitable assumptions on the geometry of the Killing fields. In particular, in addition to smallness, it is required that–as in the Kerr solution–the null generator of the horizon is in the span of the Killing fields. The precise assumptions are outlined in Section 3.

1.3 Dispersion and the redshift vs. superradiance

The elusiveness of the results of Theorems 1.1 and 1.2 stems from the well-known phenomenon of *superradiance*. This is related to the fact that the Killing field T (with respect to which the Kerr solution is stationary) is not everywhere-timelike in the domain of outer communications. In particular, there is a region of spacetime where T is spacelike, the so-called *ergoregion*. The boundary of this region is called the *ergosphere*.

The presence of the ergoregion means that the energy current J^T is not positive definite when integrated over spacelike hypersurfaces. Thus, the conservation of J^T does not imply *a priori* bounds on an L^2-based quantity. In particular, the local energy of the solution can be greater than the initial total energy, even if the energy is initially supported where J^T is positive definite. A test-particle version of this fact, where a particle coming in from infinity splits into one of negative energy entering the black hole and one of greater positive energy returning to infinity, is known as the *Penrose process*. The pioneering study by Christodoulou [8] of the "black hole transformations" obtainable via a Penrose process led to a subject known as "black hole thermodynamics".

In the physics literature, where discussion of these issues is inextricably linked to the separability of (2) and decomposition of ψ into modes, the problem of the ergoregion appears as a formidable and perhaps intractable obstacle. It turns out, however, that there are other physical mechanisms at play which have an important role but are not necessarily well reflected from the point of view of separability. In particular, the tendency of waves to eventually disperse (true in any asymptotically flat spacetime) coupled with the powerful red-shift effect at the horizon. Indeed, these properties, which depend only loosely on the stationarity, tend to make solutions not only stay bounded but decay to a constant in time, even if the local energy increases for a short time.

Unfortunately, the dispersive properties of waves on black hole backgrounds are severely complicated by the presence of trapped null geodesics. (The presence of these can easily be inferred by a continuity argument in view of the fact that there exist both null geodesics crossing the horizon and going to null infinity.) It is only very recently that the role of trapping has been sufficiently well understood in the special case of the Schwarzschild family to allow for the first proofs of decay for general solutions of (2) on such backgrounds. See the results described in Section 1.4.1.

In the case of Kerr, the techniques introduced for controlling trapping on Schwarzschild cannot be readily perturbed. This has to do with the fact that these techniques seem to exploit the special property that the trapping concentrates asymptotically on a set of codimension 1 in physical space, the so-called photon sphere. In contrast, in Kerr the codimensionality of the space of trapped geodesics can only be properly understood in phase space. This indicates that controlling trapping requires a far more delicate analysis.

It would appear from the above that the problem of superradiance could in principle be overcome, but at the expense of a very delicate analysis of trapping. A closer look, however, reveals that the situtation is considerably more favourable. At a heuristic level, the reason for this is the following remark: If one could separate out the "superradiant" part of the solution from the "non-superradiant" part, then one only has to exploit dispersion for the superradiant part. This latter problem turns out to be much easier than understanding dispersion for the whole solution.

To decompose the solution, we must first cut off the solution ψ in the "time"-interval of interest to obtain $\psi_{\text{✂}}$ and then decompose into two pieces

$$\psi_{\text{✂}} = \psi_\flat + \psi_\sharp$$

where $\psi_\flat$ is to be supported in frequency space (real frequencies ω and integer k here defined with respect to coordinates t and ϕ) only in the range $\omega^2 \precsim \omega_0^2\, k^2$, whereas $\psi_\sharp$ is to be supported in frequency space only in the range $\omega^2 \succsim \omega_0^2\, k^2$. For spacetimes sufficiently close to Schwarzschild, for a suitable choice of the parameter ω_0, one can view $\psi_\sharp$ as essentially non-superradiant, and $\psi_\flat$ as the superradiant part. If one can show boundedness for $\psi_\sharp$ and dispersion for $\psi_\flat$, then one will have proven the uniform boundedness of the sum ψ. For spacetimes sufficiently close to Schwarzschild, one can choose ω_0 sufficiently small so that trapping essentially does not occur for $\psi_\flat$, and the dispersive mechanism of Schwarzschild is stable. This relies on the stability of the red-shift effect for considerations close to the horizon. In complete contrast to the standard picture, it is the superradiant part of the solution which would be the better behaved one.

In practice, the analysis is of course not as simple as what has been portrayed above, and here again, the stabilising effect of the red-shift acting near the horizon plays an important role. In view of the cutoffs in time, the equations satisfied by $\psi_\flat$ and $\psi_\sharp$ are coupled. Moreover, the statement that $\psi_\sharp$ is non-superradiant while $\psi_\flat$ is dispersive must also be understood modulo error terms. It turns out that to control these error terms, one of necessity must have at their disposal an energy quantity which does not degenerate on the horizon, that is to say, the L^2-based quantity for which one shows uniform boundedness must be the one of Theorem 1.1, and not a quantity analogous to J^T in Schwarzschild. In particular, one must understand the red-shift mechanism even for the "non-superradiant" part $\psi_\sharp$, for which one does not understand dispersion. Such stable estimates at the horizon (corresponding to the energy measured by local observers) exploiting the red-shift effect were first attained for Schwarzschild in our previous [12]. It is interesting to note, however, that in [12], understanding of the red-shift mechanism was always coupled with understanding dispersion, i.e. controlling the trapping phenomenon. In particular, one had to appeal to an understanding of dispersion even to obtain the result of Theorems 1.1 for Schwarzschild. In this paper, we show how understanding the red-shift can be decoupled from understanding dispersion in the non-superradiant case. In addition, we show that the red-shift effect allows us to commute the wave equation with a vector field transverse to the horizon, yielding a new route to higher order estimates and pointwise estimates. An extra side-benefit of our results is thus a new, simpler and more robust proof of Theorems 1.1 and 1.2 even for the case of Schwarzschild. See Section 14.1.

1.4 Previous results

We review in detail previous work on this and related problems. Results of the type of Theorems 1.1 and 1.2 for static perturbations of Minkowski space pose little difficulty. (Indeed, the analogue of Theorem 1.1 is immediate, and Theorem 1.2 can be proven with the help of Sobolev inequalities after commuting the equation with the static Killing field.) Thus, we shall pass directly to the black hole case.

1.4.1 Schwarzschild

The analogue of Theorem 1.2 for Schwarzschild is a celebrated result of Kay and Wald [22], building on previous work of Wald [27] where the theorem had been proven for the restricted class of data whose support was assumed not to contain the bifurcation sphere $\mathcal{H}^+ \cap \mathcal{H}^-$. In view of the positive definiteness of J^T in the domain of outer communications, the only essential difficulty is obtaining bounds for ψ up to the horizon (where T becomes null), as bounds away from the horizon can be obtained essentially as described immediately above for static perturbations of Minkowski space.

The arguments of Kay and Wald to prove the analogue of Theorem 1.2 relied on the staticity to realize a solution ψ as $\partial_t \tilde{\psi}$ where $\tilde{\psi}$ is again a solution of (2) constructed by inverting an elliptic operator acting on initial data. In addition, a pretty geometric construction exploiting the discrete symmetries of maximal Schwarzschild was used to remove the unphysical restriction on the support near $\mathcal{H}^+ \cap \mathcal{H}^-$ necessary for constructing $\tilde{\psi}$ in the original [27]. Unfortunately, neither of these methods is particularly robust to perturbation. The reason the authors had to resort to such techniques was that Theorem 1.2 was proven *without* proving the analogue of Theorem 1.1, rather, using only the conserved flux J^T whose control degenerates as $\mathcal{H}^+$ is approached. Theorem 1.1 for Schwarzschild was only proven as part of the decay results of [12] to be discussed below.

Turning now to the issue of decay, the first non-quantitative decay result for (2) on Schwarzschild is contained in the thesis of Twainy [26]. The first quantitative decay results for solutions of (2) on Schwarzschild (and more generally, Reissner-Nordström) were proven in [11], but were restricted to spherically symmetric solutions, or alternatively, the 0'th spherical harmonic ψ_0 of a general solution ψ. (In fact, this was a byproduct of the main result of [11], which concerns decay rates for spherically symmetric solutions to the coupled Einstein-(Maxwell)-scalar field system.)

Quantitative decay results for the whole solution ψ, both pointwise and in energy, were proven in [12], in particular, the uniform decay result

$$|\psi| \leq C\,\mathbf{Q} v_+^{-1} \tag{7}$$

in the domain of outer communications. Here v is an Eddington-Finkelstein advanced time coordinate and $\mathbf{Q}$ is an appropriate quantity computable on initial data, and v_+ denotes say $\max\{v, 1\}$. Inequality (7) is sharp as a uniform decay rate in v. The results of [12] exploit both the red-shift effect near the horizon and the dispersive properties. The estimates are derived using a variety of vector field multipliers, in particular, a vector field multiplier Y such that the flux of $T+Y$ gives the local energy at the horizon. The energy identity of Y quantifies the red-shift effect.

Weaker decay results were proven independently by Blue and Sterbenz [6] for initial data vanishing on $\mathcal{H}^+ \cap \mathcal{H}^-$, but with control which degenerates on the horizon. In particular, the estimates of [6] are unstable to perturbation near the horizon. The stability of the estimates of [12] near the horizon will be of critical importance here.

Both [12] and [6] control trapping effects with the help of vector field multipliers which must be carefully chosen for each spherical harmonic separately. These were inspired by a series of papers by Soffer and collaborators, for instance [4]; see, however [5]. The first proof of decay for ψ not relying on spherical harmonic decomposition for the construction of these multipliers is provided by our more recent [13].

1.4.2 Kerr

Since uniform boundedness is the most basic question which can be asked about (2) on Kerr, previous results in this setting are of necessity of a partial nature. In particular, essentially all previous work on (2) is restricted to the projection of ψ to a single azimuthal frequency, or equivalently, to the case where the data are of the form

$$\psi = \psi_k(r,\theta)e^{-ik\phi}, \qquad \psi' = \psi'_k(r,\theta)e^{-ik\phi}. \tag{8}$$

Solutions arising from (8) are then of the form $\psi_k(r,\theta,t)e^{-ik\phi}$. Let us call such solutions azimuthal modes. In principle, one could attempt to deduce properties of general ψ by summing relations deduced for each individual azimuthal mode. As we shall see, however, due to the non-quantitative nature of the results described below, in of themselves they unfortunately yield no information about general ψ. Nonetheless, even the study of such ψ_k without regard to uniform control in k turns out to be a non-trivial problem. Indeed, even for such individual azimuthal modes, the most basic questions had not been previously answered, in particular, the analogue of Theorems 1.1 or 1.2.

This being said, there are interesting partial results concerning (8) that had been previously obtained. In particular, most recently Finster *et al.* [16] had been able to show for smooth ψ_k that for fixed $r > M + \sqrt{M^2 - a^2}$ and θ,

$$\lim_{t\to\infty} \psi_k(t,r,\theta) \to 0, \tag{9}$$

under the assumption that the support of ψ_k does not contain $\mathcal{H}^+ \cap \mathcal{H}^-$. See however [17]. In particular, one can deduce

$$\sup_{-\infty<t<\infty} \psi_k(t,r,\theta) < \infty, \tag{10}$$

for each fixed $r > M + \sqrt{M^2 - a^2}$ and θ, without however a bound on the sup. The results rest on an explicit integral representation of the solution which is derived using the remarkable (but all too fragile) separability properties of the Kerr metric. The arguments contain many pretty applications of contour integral methods of classical complex analysis. Since these techniques are essentially algebraic, no restriction on the size of $|a|$ need be made provided $|a| < M$. In [18], under the same assumption on the initial support, the authors deduce that for each $\delta > 0$,

$$\sup_{\tau\geq 0}\int_{\Sigma_\tau\cap\{r\geq M+\sqrt{M^2-a^2}+\delta\}} J^T_\mu(\psi_k e^{-ik\phi})n^\mu_\Sigma < \infty. \tag{11}$$

Thus, the energy of each mode in the region $r \geq M+\sqrt{M^2-a^2}+\delta$ remains finite but again, no quantitative bound in terms of data can be produced. Moreover, from the results of [16, 18], one cannot deduce that the sup of (10) and (11) commute with taking $\lim_{r\to M+\sqrt{M^2-a^2}}$ or $\lim_{\delta\to 0}$, i.e. (10) is *a priori* compatible with ψ_k blowing up along the horizon:

$$\sup_{\mathcal{H}}|\psi_k| = \infty$$

and (11) is compatible with infinite energy concentration near the horizon:

$$\sup_{\tau\geq 0}\int_{\Sigma_\tau\cap\{M+\sqrt{M^2-a^2}\leq r\leq M+\sqrt{M^2-a^2}+\delta\}} J^T_\mu(\psi_k e^{-ik\phi})n^\mu_\Sigma = \infty.$$

As explained before, no statement could be inferred for the general solution ψ from the above statements on individual azimuthal modes, not even a weak statement like (10) or (11). This is because the lim and sup of (9), (10) and (11) do not *a priori* commute with summation over k.[3] Of course, in view of Theorems 1.1 and 1.2, one can now infer from (9) Corollary 14.1 of Section 14.3.

The somewhat unsatisfying nature of the above previous results deduced with the help of separability are indicative of how difficult it is to obtain quantitative statements about solutions of the wave equation (2) even in the algebraically special case where one has explicit representations of the solution. Perhaps this is for the best, however. Remarkable though they are, the separability properties of the Kerr metric are unstable to perturbation. Just as in the case of stability of Minkowski space [10], understanding the stability properties of the Einstein equations near the Kerr solution will undoubtedly require robust methods. We hope that the techniques employed here will have further applications in this direction.

1.4.3 Klein-Gordon

A related problem to the wave equation is that of the Klein-Gordon equation

$$\Box_g\psi = m^2\psi \tag{12}$$

[3]Note that in the abstract of [16], ψ must be understood as the projection ψ_k, to agree with what is proven in the body of the paper.

with $m > 0$. There is a well-developed scattering theory on Schwarzschild for the class of solutions of (12) with finite energy associated to the Killing T. In particular, an asymptotic completeness statement has been proven in [2]. This analysis in of itself, however, when specialised to $H^1_{\rm loc}$ solutions in the geometric sense, only gives very weak information about the solution. In particular, it does not give L^2 control of ψ or its angular derivatives on $\mathcal{H}^+$.

In the case of Kerr, there are again certain partial results for (12) in the direction of scattering for a "non-superradiant" subspace of initial data [19]. These interesting results do not, however, address the characteristic difficulties of superradiance. See also [3].

1.4.4 Dirac on Kerr

Finally, we mention that there has been a series of interesting papers concerning the Dirac equation on Kerr and Kerr-Newman. See [20, 15]. For Dirac, considerations turn out to be much easier as this equation does not exhibit the phenomenon of superradiance. We shall not comment more about this here but refer the reader to the nice article [20].

1.5 Heuristic work

We cannot do justice here to the vast work on this subject in the physics literature. See [24] for a nice survey.

1.6 Acknowledgements

The authors thank G. Holzegel for comments on the manuscript. The authors also thank the Niels Bohr Institute, Copenhagen, for hospitality while this work was being completed. M. D. is support in part by a grant from the European Research Council. I. R. is support in part by NSF grant DMS-0702270.

2 Constants and parameters

Constants will play an important role in this paper and it is imperative to set the conventions early. In the next section we shall fix a Schwarzschild metric with parameter M.

We shall use the notation B and b for general positive constants which only depend on the choice of M. An inequality true with a constant B will be true if B is replaced by a larger constant, and similary, for b if b is replaced by a smaller positive constant.[4] We shall use the notation $f_1 \sim f_2$ to denote

$$bf_1 \leq f_2 \leq Bf_1.$$

[4]In the case of chains of inequalities, e.g. $f_1 \leq Bf_2 \leq Bf_3$ this convention is obviously violated and has to be reinterpreted appropriately.

Since B and b denote general constants, we shall apply without comment the obvious algebraic rules $B^2 = B$, $B^{-1} = b$, $b^2 = b$, etc.

We will also require various particular parameters which can be chosen depending only on M:

$$r^{\pm}_{Y}, r_{\hat{Y}}, \delta, \delta_{\hat{Y}}, q, \lambda, \omega_0, \alpha, R_0, R_1, R, e, \tau_{\mathrm{step}}, \epsilon_{\mathrm{close}}.$$

The above parameters are not explicitly computed but are determined implicitly by various constraints. Before choosing a parameter, say α, we shall use notation like $B(\alpha)$, $b(\alpha)$ to denote constants depending only on M and the as of yet unchosen α. It is to be understood that again here, the notation B indicates that the constant can always be replaced by a bigger one, and b by a smaller one. We shall also use the notation $R_1(\alpha)$ to indicate that the parameter R_1 depends on the still unchosen α. Once α is determined, we may replace the expressions $B(\alpha)$, $R(\alpha)$ etc., with B, R, etc.

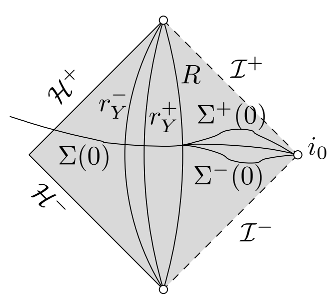
3 The class of spacetimes

In this section we shall describe the general class of metrics for which our results will apply. To set the stage, we must first fix some structures associated to a Schwarzschild metric.

3.1 Schwarzschild

We refer the reader to our previous [12] for a review of the geometry of Schwarzschild. We must first fix a certain subregion of Schwarzschild with parameter $M > 0$, relevant coordinates, and a choice of axisymmetric Killing field. This will provide the underlying manifold with stratified boundary[5] for the class of metrics to be considered later. Let us use the notation g_M to denote the Schwarzschild metric.

Refer to the diagram below:



We will denote by $\mathcal{D}$ the closure of a domain of outer communications in maximal Schwarzschild. We have $\mathcal{D} \setminus \mathrm{int}(\mathcal{D}) = \mathcal{H}^+ \cup \mathcal{H}^-$ where $\mathcal{H}^+$ denotes the *future*

[5]The boundary will be the union of two manifolds with boundary intersecting along their common boundary.

event horizon and $\mathcal{H}^-$ the *past event horizon*. The intersection $\mathcal{H}^+ \cap \mathcal{H}^-$ is known as the *bifurcation sphere*.

Recall the static Killing field T, timelike on $\mathrm{int}(\mathcal{D})$ and null on $\mathcal{H}^+ \cup \mathcal{H}^-$. Flow by integral curves of T defines a one-parameter family of diffeomorphisms $\rho_s : \mathcal{D} \to \mathcal{D}$.

Recall now the area-radius function r. On the horizons $\mathcal{H}^+ \cup \mathcal{H}^-$ we have $r = 2M$. We will use the notation μ for the function defined by $\mu = 2M/r$.

Associated to Schwarzschild will be the constants $2M < r_Y^- < r_Y^+$ determined in Section 8.2. We may assume say that

$$r_Y^- \leq \frac{9M}{4}. \tag{13}$$

Let χ be a cutoff function such that $\chi = 1$ for $r \leq 2M + (r_Y^- - 2M)/2$ and $\chi = 0$ for $r \geq r_Y^-$. Define the hypersurface $\Sigma(0)$ by

$$t = -\chi(r) 2M \log(r - 2M). \tag{14}$$

This can be extended beyond $\mathcal{H}^+$–not by the expression (14), however–to a spacelike hypersurface in maximal Schwarzschild. Let us actually define $\Sigma(0)$ to include its limit points on the horizon $\mathcal{H}^+$. Note of course that in view of the support of χ, it follows that in the region $r \geq r_Y^-$, $\Sigma(0)$ coincides with the constant $t = 0$ hypersurface.

We may define a new coordinate

$$t^* \doteq t + \chi(r) 2M \log(r - 2M)$$

This coordinate is regular on $\mathcal{H}^+ \setminus \mathcal{H}^-$. We have that

$$\Sigma(0) = \{t^* = 0\}.$$

Let us define

$$\Sigma(\tau) = \{t^* = \tau\}.$$

Clearly $\Sigma(\tau) = \rho_\tau(\Sigma(0))$.

We have that

$$B \geq -g_M(\nabla t^*, \nabla t^*) \geq b > 0 \tag{15}$$

for some constants B, b. Recall here the conventions of Section 2.

For technical reasons, we shall require two auxilliary sets of spacelike hypersurfaces. Let $\chi(x)$ be a cutoff function such that $\chi(x) = 1$ for $x \leq 0$ and $\chi(x) = 0$ for $x \geq 1$. Let us define

$$t^+ = t^* - \chi(-r + R)(1 + r - R)^{1/2}$$

and

$$t^- = t^* + \chi(-r + R)(1 + r - R)^{1/2}$$

for an R to be determined later with $R \geq r_Y^- + 1$. Let us define

$$\Sigma^+(\tau) \doteq \{t^+ = \tau\}, \qquad \Sigma^-(\tau) \doteq \{t^- = \tau\}.$$

Independently of the choice of R, we have that $\Sigma^{\pm}$ are spacelike, in fact

$$B \geq -g_M(\nabla t^+, \nabla t^+) \geq b > 0, \qquad B \geq -g_M(\nabla t^-, \nabla t^-) \geq b > 0. \tag{16}$$

In what follows we shall restrict to

$$\mathcal{R} \doteq \mathcal{D} \cap J^+_{g_M}(\Sigma^-(0)).$$

The set $\mathcal{R}$ is again a manifold with stratified boundary (as was $\mathcal{D}$), where the boundary is given by $\Sigma^-(0) \cup (\mathcal{H}^+ \cap J^+_{g_M}(\Sigma^-(0)))$.

Choosing a coordinate atlas consisting of two charts (ξ^A, ξ^B), $(\tilde{\xi}^A, \tilde{\xi}^B)$ on $\mathbb{S}^2$, then setting $x^A = r^{-1}\xi^A$, $\tilde{x}^A = r^{-1}\tilde{\xi}^A$, it follows that

$$(r, t^*, x^A, x^B), \qquad (r, t^*, \tilde{x}^A, \tilde{x}^B) \tag{17}$$

form a coordinate atlas for $\mathcal{R}$. We can ensure moreover that the regions of the sphere covered by the charts are restricted so that the metric functions satisfy

$$(g_M)_{ij} \leq B, \qquad g^{ij}_M \leq B \tag{18}$$

in these coordinates. Note that with respect to both these charts, the vector $\frac{\partial}{\partial t^*}$ is the stationary Killing field T.

We will use the above coordinate atlas (17) in formulating our closeness assumptions. A third set of coordinates will be useful for us, namely the coordinates arising from a choice of standard spherical coordinates[6] (θ, ϕ) on $\mathbb{S}^2$. With respect to

$$(r, t^*, \theta, \phi) \tag{19}$$

coordinates, it follows that $\frac{\partial}{\partial \phi}$ is a Killing field. Let us denote the smooth extension to $\mathcal{D}$ of this Killing field as Φ. Note that Φ vanishes precisely at two points on each sphere of symmetry. This corresponds to the locus of points where the (19) coordinates break down. Because (18) is not satisfied with respect to these coordinates, they will not be as useful in formulating the closeness assumptions.

We will say that Schwarzschild is axisymmetric and Φ is a choice of axisymmetric Killing field.

Finally, we shall also at times refer to so-called Regge-Wheeler coordinates

$$(r^*, t, x^A, x^B).$$

Here t is the standard Schwarzschild time and the coordinate r^* is defined by

$$r^* \doteq r + 2M \log(r - 2M) - 3M - 2M \log M.$$

Note that this coordinate is regular in $\mathrm{int}(\mathcal{R})$, but sends the boundary to $r^* = -\infty$. With respect to these coordinates, we note that ∂_{r^*} extends to a smooth vector field on all of $\mathcal{R}$ (i.e. to the event horizon), and in fact, in the limit $\partial_{r^*} = T$ on $\mathcal{H}^+ \cap \mathcal{R}$.

[6] Here, ϕ denotes an azimuthal coordinate.

This last coordinate system is not useful for formulating closeness assumptions in view of the fact that it breaks down on the horizon. We shall only use Regge-Wheeler coordinates for making calculations with respect to the Schwarzschild metric.

Finally, a word of caution. Since we have several coordinate systems which will be considered, coordinate vectors like ∂_{t^*} will always be referred to in conjunction with a specific coordinate system.

3.2 The general class

We now describe the class of metrics to be allowed.

We consider the manifold with stratified boundary $\mathcal{R}$ defined above. We consider the class of all smooth Lorentzian metrics g such that:

1. For $\epsilon_{\rm close} > 0$ sufficiently small,

$$|g_{ij} - (g_M)_{ij}| \leq \epsilon_{\rm close} r^{-2}, \qquad |g^{ij} - (g_M)^{ij}| \leq \epsilon_{\rm close} r^{-2} \tag{20}$$

$$|\partial_m g_{ij} - \partial_m (g_M)_{ij}| \leq \epsilon_{\rm close} r^{-2} \tag{21}$$

   with respect to the atlas (17).[7]

2. The vector fields $T = \partial_{t^*}$ and $\Phi = \partial_\phi$ with respect to (r, t^*, θ, ϕ) coordinates are again Killing with respect to g.

3. There is a function γ defined on $\mathcal{H}^+$ such that $T + \gamma\Phi$ is null on the horizon, and

$$|\gamma| < \epsilon_{\rm close}. \tag{22}$$

In particular, Assumption 3 above implies that $\mathcal{H}^+$ is null with respect to g and its null generator lies in the span of T and Φ. We may define the *ergoregion* to be the region where T itself is *not* timelike.

For sufficiently small $\epsilon_{\rm close}$, assumptions (20) and (15) imply that $\Sigma(0)$ is spacelike with respect to g, in fact, with our conventions on constants,

$$B \geq -g(\nabla t^*, \nabla t^*) \geq b. \tag{23}$$

Similarly, we have from (16) that for $\epsilon_{\rm close}$ sufficiently small, $\Sigma^+(\tau)$ and $\Sigma^-(\tau)$ are spacelike, in fact

$$B \geq -g(\nabla t^\pm, \nabla t^\pm) \geq b, \tag{24}$$

independently of the choice of R.

Note that $\Sigma(\tau)$ is again isometric to $\Sigma(0)$ with respect to g, and similarly $\Sigma^\pm(\tau)$ is isometric to $\Sigma^\pm(0)$. We will denote by n_Σ the future normal of $\Sigma(\tau)$:

$$n^\mu_\Sigma \doteq (-g(\nabla t^*, \nabla t^*))^{-1/2} \nabla^\mu t^*.$$

[7] When specialized to the case of Kerr-Newman, this clearly will *not* be the Boyer-Lindquist r referred to previously. For the relation to Kerr-Newman, see Section 3.3.

This defines a translation invariant smooth timelike unit vector field on $\mathcal{R}$. Similarly, we define

$$n^\mu_{\Sigma^\pm} \doteq (-g(\nabla t^\pm, \nabla t^\pm))^{-1/2}\nabla^\mu t^\pm.$$

We will use the notations

$$\mathcal{R}(\tau', \tau'') \doteq \bigcup_{\tau' \le \bar\tau \le \tau''} \Sigma(\bar\tau),$$

$$\mathcal{R}^+(\tau', \tau'') \doteq \bigcup_{\tau' \le \bar\tau \le \tau''} \Sigma^+(\bar\tau),$$

$$\mathcal{R}^-(\tau', \tau'') \doteq \bigcup_{\tau' \le \bar\tau \le \tau''} \Sigma^-(\bar\tau),$$

$$\mathcal{H}(\tau', \tau'') \doteq \mathcal{H}^+ \cap \mathcal{R}(\tau', \tau'').$$

All integrals without an explicit measure of integration are to be taken with respect to the volume form in the case of a region of spacetime or a spacelike hypersurface, and an induced volume form connected to the choice of a ρ_t-invariant tangential vector field $n^\mu_{\mathcal{H}}$, in the case of $\mathcal{H}(\tau', \tau'')$.

Note the following property of the volume integral with respect to the (almost) global (t, r, ϕ, θ) coordinate system: There exist smooth $\nu(\theta, r) \ge 0$, $\tilde\nu(\theta) \ge 0$ such that for all continuous f:

$$\int_{\mathcal{R}(\tau',\tau'')} f \quad = \quad \int_{2M}^{\infty}\int_0^\pi \nu(\theta, r)\left(\int_{\tau'}^{\tau''}\left(\int_0^{2\pi} f\, d\phi\right) dt^*\right) d\theta\, dr,$$

$$\int_{\mathcal{H}(\tau',\tau'')} f \quad = \quad \int_0^\pi \tilde\nu(\theta)\left(\int_{\tau'}^{\tau''}\left(\int_0^{2\pi} f\, d\phi\right) dt^*\right) d\theta.$$

Also let us note that

$$\int_{\mathcal{R}(\tau',\tau'')} f = \int_{\tau'}^{\tau''}\left(\int_{\Sigma(\bar\tau)} (-g(\nabla t^*, \nabla t^*))^{-1/2}\, f\right) d\bar\tau.$$

By (23), it follows that if $f_1 \sim f_2$ in the sense $0 < bf_1 \le f_2 \le Bf_1$, it follows that

$$\int_{\mathcal{R}(\tau',\tau'')} f_1 \sim \int_{\tau'}^{\tau''}\left(\int_{\Sigma(\bar\tau)} f_2\right) d\bar\tau.$$

A similar relation holds with $\mathcal{R}^\pm$ and $\Sigma^\pm$.

3.3 The Kerr and Kerr-Newman metrics

Proposition 3.1. *Let $Q \ll M$, $a \ll M$. Then the Kerr-Newman metric with parameters Q, a satisfies the assumptions of Section 3.2.*

Let us sketch how one can implicitly define a Kerr-Newman metric on $\mathcal{R}$ in our (r, t^*, θ, ϕ) coordinate system.

For convenience, let us do this by defining a new set of coordinates on $\mathrm{int}(\mathcal{R})$ which are to represent Boyer-Lindquist coordinates $(\hat{r}, \hat{t}, \hat{\theta}, \hat{\phi})$. For this define $\hat{r}$ by

$$r^2 - 2Mr = \hat{r}^2 - 2M\hat{r} + Q^2 + a^2$$

$\hat{t}$ by

$$\hat{t} = t^* - h(\hat{r})$$

where h is defined by $\frac{dh}{d\hat{r}} = \frac{2M\hat{r}-Q^2}{\hat{r}^2-2M\hat{r}+Q^2+a^2}$, and $\hat{\phi}$ by

$$\hat{\phi} = \phi - P(\hat{r})$$

where $\frac{dP}{d\hat{r}} = \frac{a}{\hat{r}^2-2M\hat{r}+Q^2+a^2}$, and $\hat{\theta}$ by

$$\hat{\theta} = \theta.$$

Now consider the metric on $\mathrm{int}(\mathcal{R})$ defined in these new coordinates by

$$\begin{aligned}
-\left(1 - \frac{2M - \frac{Q^2}{\hat{r}}}{\hat{r}\left(1+\frac{a^2\cos^2\hat{\theta}}{\hat{r}^2}\right)}\right) d\hat{t}^2 + \frac{1+\frac{a^2\cos^2\hat{\theta}}{\hat{r}^2}}{1-\frac{2M}{\hat{r}}+\frac{Q^2}{\hat{r}^2}+\frac{a^2}{\hat{r}^2}} d\hat{r}^2 + \hat{r}^2\left(1+\frac{a^2\cos^2\hat{\theta}}{\hat{r}^2}\right) d\hat{\theta}^2 \\
+\hat{r}^2\left(1+\frac{a^2}{\hat{r}^2}+\left(\frac{2M}{\hat{r}}-\frac{Q^2}{\hat{r}^2}\right)\frac{a^2\sin^2\hat{\theta}}{\hat{r}^2\left(1+\frac{a^2\cos^2\hat{\theta}}{\hat{r}^2}\right)}\right)\sin^2\hat{\theta}\, d\hat{\phi}^2 \\
-2\left(2M-\frac{Q^2}{\hat{r}}\right)\frac{a\sin^2\hat{\theta}}{\hat{r}\left(1+\frac{a^2\cos^2\hat{\theta}}{\hat{r}^2}\right)} d\hat{t}\, d\hat{\phi}.
\end{aligned}$$

Writing the metric in (r, t^*, θ, ϕ) coordinates, and then relating this form in turn to the coordinates of (17) one sees immediately that

$$r^2(g_{ij} - (g_M)_{ij}) \to 0, \qquad r^2(g^{ij} - g_M^{ij}) \to 0$$

uniformly as $a \to 0$, and

$$r^3(\partial_k g_{ij} - \partial_k (g_M)_{ij}) \to 0$$

uniformly as $a \to 0$, where i, j, k denote coordinates of (17). It follows that given $\epsilon_{\mathrm{close}}$, the assumptions (20) and (21) hold. The remaining assumptions are well-known properties of Kerr which are manifest from the Boyer-Lindquist form.

4 The class of solutions ψ

Let $(\mathcal{R}, g)$, $\Sigma(\tau)$ be as in Section 3.2, and let $\uppsi$ be an H^1_{loc} function on $\Sigma(0)$, and let $\uppsi'$ be an L^2_{loc} function on $\Sigma(0)$. Here the L^2 norm is defined naturally with respect to the induced Riemannian metric on $\Sigma(0)$. By standard theory, there exists a unique solution ψ of the initial value problem

$$\Box_g \psi = 0, \qquad \psi|_{\Sigma(0)} = \uppsi, \qquad n_\Sigma \psi|_{\Sigma(0)} = \uppsi', \tag{25}$$

with the property that

$$\psi \in C^1(H^1_{\mathrm{loc}}(\Sigma(\tau)), \qquad n_\Sigma \psi \in C^0(L^2_{\mathrm{loc}}(\Sigma(\tau)).$$

We will in fact require that

$$\nabla^\Sigma \uppsi \in L^2(\Sigma(0)), \qquad \uppsi' \in L^2(\Sigma(0)). \tag{26}$$

By density arguments, the main results of this paper would follow if they were proven under the additional restriction that $\uppsi$, $\uppsi'$ are in fact smooth, and thus, that ψ is smooth. Moreover, we can safely assume that $\nabla^\Sigma \uppsi$ and $\uppsi'$ are supported away from infinity. Let us assume this in what follows so as not to have to comment on regularity issues or the *a priori* finiteness of certain quantities. It follows in particular from this assumption that

$$\nabla \psi \in L^2(\Sigma(\tau)), \qquad \nabla \psi \in L^2(\Sigma^\pm(\tau)), \tag{27}$$

moreover, that $\nabla \psi$ is supported away from spatial infinity.

5 The main theorem

For a sufficiently regular function Ψ, let us define

$$T_{\mu\nu}(\Psi) \doteq \partial_\mu \Psi \partial_\nu \Psi - \frac{1}{2} g_{\mu\nu} g^{\alpha\beta} \partial_\alpha \Psi \partial_\beta \Psi \tag{28}$$

and for V^μ a vector field,

$$J^V_\mu(\Psi) \doteq T_{\mu\nu}(\Psi) V^\nu. \tag{29}$$

In addition, let us define the quantity

$$\mathbf{q}(\Psi) \doteq J^{n_\Sigma}_\mu(\Psi) n^\mu_\Sigma.$$

Note that this is non-negative. Moreover, in the coordinate charts of the atlas (17), we have

$$\mathbf{q}(\Psi) \sim \sum_i (\partial_i \Psi)^2. \tag{30}$$

By (26) and (27), we have that,

$$\int_{\Sigma(0)} \mathbf{q}(\psi) \leq B \left(\|\uppsi'\|^2_{L^2} + \|\nabla^\Sigma \uppsi\|^2_{L^2} \right)$$

and for all $\tau \geq 0$,

$$\int_{\Sigma(\tau)} \mathbf{q}(\psi) < \infty, \qquad \int_{\Sigma^{\pm}(\tau)} \mathbf{q}(\psi) < \infty.$$

Key to our results will be the uniform boundedness of this quantity.

Theorem 5.1. *There exist positive constants $\epsilon_{\mathrm{close}}$, C depending only on $M > 0$ such that the following holds. Let g, $\Sigma(\tau)$ be as in Section 3.2 and let Ψ, Ψ', ψ be as in Section 4 where ψ satisfies (2). Then, for $\tau \geq 0$,*

$$\int_{\Sigma(\tau)} \mathbf{q}(\psi) \leq C \int_{\Sigma(0)} \mathbf{q}(\psi). \tag{31}$$

Inequality (4) of Theorem 1.1 follows from Theorem 5.1. (The universality of the constant ϵ in the statement of that theorem follows *a posteriori* from a simple scaling argument.)

6 The auxiliary positive definite quantities $\mathbf{q}_e$ and $\mathbf{q}_e^\bigstar$

We note that given $e > 0$, for small enough $\epsilon_{\mathrm{close}} \ll e$, the vector field $T + en_\Sigma$ is timelike. For sufficiently regular Ψ, let us define

$$\mathbf{q}_e(\Psi) = J_\mu^{T+en_\Sigma}(\Psi) n_\Sigma^\mu.$$

Note that

$$eb\,\mathbf{q}(\Psi) \leq \mathbf{q}_e(\Psi) \leq B\,\mathbf{q}(\Psi).$$

Thus, to prove Theorem 5.1, it is sufficient to prove (31) with $\mathbf{q}_e$ replacing $\mathbf{q}$. The significance of the parameter e will become clear in the context of the proof.

We shall need also a weaker positive definite quantity defined as follows. Let χ denote a cutoff function such that $\chi = 1$ for $r \geq r_Y^-$, and $\chi = 0$ for say $r \leq r_Y^- - (r_Y^- - 2M)/2$. For a sufficiently regular function Ψ, define

$$\mathbf{q}_e^\bigstar(\Psi) = r^{-2} J_\mu^{\chi T + en_\Sigma}(\Psi) n_\Sigma^\mu.$$

Note that we have

$$eb\,r^{-2}\mathbf{q}(\Psi) \leq \mathbf{q}_e^\bigstar(\Psi) \leq B\,r^{-2}\mathbf{q}(\Psi).$$

Note also that for $r \geq r_Y^-$, we have

$$\mathbf{q}_e(\Psi) \sim \mathbf{q}(\Psi).$$

and

$$\mathbf{q}_e^\bigstar(\Psi) \sim r^{-2}\,\mathbf{q}(\Psi) \sim r^{-2}\,\mathbf{q}_e(\Psi). \tag{32}$$

For all $r \geq 2M$, we have

$$\mathbf{q}_e(\Psi) \leq Be^{-1} r^2 \mathbf{q}_e^\bigstar(\Psi), \tag{33}$$

$$\mathbf{q}(\Psi) \leq Be^{-1} r^2 \mathbf{q}_e^\bigstar(\Psi). \tag{34}$$

7 The basic identity for currents

For an arbitrary suitably regular function Ψ such that $\nabla\Psi$ is supported away from spatial infinity, recall from (28) and (29) the definitions of $T_{\mu\nu}$ and J_μ. Define also

$$K^V(\Psi) \doteq T_{\mu\nu}(\Psi)\nabla^\mu V^\nu.$$

We have

$$\nabla^\mu J_\mu(\Psi) = K^V(\Psi) + F\,V^\nu\Psi_\nu$$

where

$$F \doteq \Box_g\Psi.$$

Thus, setting

$$\mathcal{E}^V(\Psi) = -F\,V^\nu\Psi_\nu, \tag{35}$$

we have the identity

$$\begin{aligned}&\int_{\mathcal{H}(\tau',\tau'')} J^V_\mu(\Psi)n^\mu_{\mathcal{H}} + \int_{\Sigma(\tau'')} J^V_\mu(\Psi)n^\mu_\Sigma + \int_{\mathcal{R}(\tau'.\tau'')} K^V(\Psi)\\ &= \int_{\Sigma(\tau')} J^V_\mu(\Psi)n^\mu_\Sigma + \int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}^V(\Psi).\end{aligned} \tag{36}$$

We will also consider currents modified as follows. Given a function w, define $J^{V,w}_\mu$ by

$$J^{V,w}_\mu(\Psi) = J^V_\mu(\Psi) + \frac{1}{8}w\partial_\mu(\Psi^2) - \frac{1}{8}(\partial_\mu w)\Psi^2, \tag{37}$$

$$K^{V,w}(\Psi) = K^V(\Psi) - \frac{1}{8}\Box_g w(\Psi^2) + \frac{1}{4}w\nabla^\alpha\Psi\nabla_\alpha\Psi,$$

$$\mathcal{E}^{V,w}(\psi) = \mathcal{E}^V(\psi) - \frac{1}{4}w\Psi F. \tag{38}$$

Identity (36) also holds for $J^{V,w}$ as long as appropriate assumptions are made in a neighborhood of spatial infinity. We will always apply $J^{V,w}$ to Ψ with $\Psi_0 = 0$, and thus, by our assumptions on $\nabla\Psi$, such Ψ will in fact be supported away from spatial infinity.

It will be useful to have a separate notation for currents as defined with respect to the Schwarzschild metric. For these we use the notation $(J^V_{g_M})_\mu$, $K^V_{g_M}$, $(J^{V,w}_{g_M})_\mu$, etc.

Suppose that V is a vector field such that its components V^i are bounded in the atlas (17). It follows from (20) that

$$\left|(J^{V,w}_{g_M})_\mu(\Psi)n^\mu - J^{V,w}_\mu(\Psi)n^\mu\right| \leq B\epsilon_{\rm close}r^{-2}\max_i|V_i|\sum(\partial_i\Psi)^2. \tag{39}$$

The above applies in particular if $w = 0$, i.e. for the case $J^V_{g_M}$. (In fact, the w term disappears from the difference above.) Note that if the components of $n_\mu - \tilde{n}_\mu$ are less than $B\epsilon_{\rm close}r^{-2}$ we have by the triangle inequality

$$\begin{aligned}\left|(J^{V,w}_{g_M})_\mu(\Psi)n^\mu - J^{V,w}_\mu(\Psi)\tilde{n}^\mu\right| &\leq B\epsilon_{\rm close}r^{-2}(|w| + \max_i(|V_i| + |\partial_i w|))\\ &\quad\cdot\sum(\partial_i\Psi)^2.\end{aligned} \tag{40}$$

Note also that if V^j, $\partial_i V^j$, w, $\partial_i w$ and $\partial_i\partial_j w$ are bounded with respect to (17), where then from (20), (21), we obtain

$$\begin{aligned} \left|K^{V,w}_{g_M}(\Psi) - K^{V,w}(\Psi)\right| &\leq B\epsilon_{\mathrm{close}} r^{-2}\left(\max_{ij}\max_{p_k=0,1}|\partial_j^{p_j}V^i| + |\partial_i^{p_i}\partial_j^{p_j}w|\right)\\ &\quad\cdot\sum(\partial_i\Psi)^2. \end{aligned} \tag{41}$$

If F above vanishes, then $J^{V,w}_\mu$ are examples of *compatible currents* in the sense of [9]. This is a unifying principle for understanding the structure behind much of the analysis for Lagrangian equations like (2).

8 The vector fields and their currents

8.1 The vector field T

Since T is Killing we have

$$K^T(\Psi) = 0.$$

In $r \geq r_Y^-$, T is timelike and moreover we have

$$J^T_\mu(\Psi)n^\mu_\Sigma \sim J^{n_\Sigma}_\mu(\Psi)n^\mu_\Sigma$$

in that region. In all regions we have

$$|J^T_\mu(\Psi)n^\mu_\Sigma| \leq B\, J^{n_\Sigma}_\mu(\Psi)n^\mu_\Sigma, \qquad |J^T_\mu(\Psi)n^\mu_{\mathcal{H}}| \leq B\, J^{n_\Sigma}_\mu(\Psi)n^\mu_{\mathcal{H}}.$$

For $\epsilon_{\mathrm{close}} \ll e$ we have

$$|J^T_\mu(\Psi)n^\mu_\Sigma| \leq B\,\mathbf{q}_e(\Psi). \tag{42}$$

8.2 The vector fields Y and $N_e = T + eY$

Let (u,v) denote Eddington-Finkelstein null coordinates[8] on $\mathrm{int}(\mathcal{D})$ and let r^* denote the Regge-Wheeler coordinate. In the paragraph that follows, coordinate derivatives are with respect to say (u, v, x^A, x^B) coordinates, whereas y_1', y_2' denote $\frac{dy_1}{dr^*}$, etc.

Recall from [12] that for a vector field Y of the form:

$$Y = y_1(r^*)\frac{1}{1-\mu}\frac{\partial}{\partial u} + y_2(r^*)\frac{\partial}{\partial v},$$

we have

$$\begin{aligned} K_{g_M}(\Psi) &= \frac{(\partial_u\Psi)^2}{2(1-\mu)^2}\left(\frac{y_1\mu}{r} - y_1'\right) + (\partial_v\Psi)^2\frac{y_2'}{2(1-\mu)}\\ &\quad + \frac{1}{2}|\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{y_1'}{1-\mu} - \frac{(y_2(1-\mu))'}{1-\mu}\right)\\ &\quad - \frac{1}{r}\left(\frac{y_1}{1-\mu} - y_2\right)\partial_u\Psi\partial_v\Psi. \end{aligned}$$

[8]See [13, 14]. Our use of this terminology is somewhat non-standard. Here $v \doteq (t + r^*)/2$, $u \doteq (t - r^*)/2$.

Let us define $y_1 = \xi(r^*)(1+(1-\mu))$, $y_2 = \xi(r^*)\delta^{-1}(1-\mu)$ where ξ is a cutoff function such that $\xi = 1$ for $r \le r_Y^-$, and $\xi = 0$ for $r \ge r_Y^+$, for two parameters $2M < r_Y^- < r_Y^+$, and a small constant δ. One sees easily that there exist such parameters such that for $r \le r_Y^-$,

$$\left(\frac{y_1\mu}{r} - y_1'\right) \ge b, \qquad \frac{y_2'}{2(1-\mu)} \ge b,$$

$$\frac{y_1'}{1-\mu} - \frac{(y_2(1-\mu))'}{1-\mu} \ge b$$

$$\left|-\frac{1}{r}\left(\frac{y_1}{1-\mu} - y_2\right)\partial_u\Psi\partial_v\Psi\right| \le \frac{1}{2}\left(\frac{(\partial_u\Psi)^2}{2(1-\mu)^2}\left(\frac{y_1\mu}{r} - y_1'\right) + (\partial_v\Psi)^2\frac{y_2'}{2(1-\mu)}\right).$$

Let us return now to the coordinate charts of our (17). We see from the above that the vector field Y has the property that in $r \le r_Y^-$,

$$B\sum_i(\partial_i\Psi)^2 \ge K^Y_{g_M}(\Psi) \ge b\sum_i(\partial_i\Psi)^2 \tag{43}$$

where i, j refer to the coordinate charts of (17) whereas we easily see also that in $r_Y^- \le r \le r_Y^+$

$$|K^Y_{g_M}| \le B\sum_i(\partial_i\Psi)^2. \tag{44}$$

Finally, for $r \ge r_Y^+$, $Y = 0$.

Moreover, we note that Y is a regular vector field, in particular, when expressed with respect to the coordinates of (17), we have $\max|Y^i| \le B$, $\max|\partial_i Y^j| \le B$.

Because all derivatives appear on the right hand sides of (43) and (44), these inequalities are stable, i.e. it follows from (41) that for $\epsilon_{\rm close}$ sufficiently small,

$$K^Y(\Psi) \sim \sum_i(\partial_i\Psi)^2 \sim J^{n_\Sigma}_\mu(\Psi)n^\mu_\Sigma \tag{45}$$

in $r \le r_Y^-$, and

$$\left|K^Y\right| \le B\sum_i(\partial_i\Psi)^2 \le B\,J^{n_\Sigma}_\mu(\Psi)n^\mu_\Sigma \tag{46}$$

in $r_Y^- \le r \le r_Y^+$, while certainly $K^Y = 0$ for $r \ge r_Y^+$.

Define

$$N_e = T + eY.$$

Note that

$$K^{N_e} = K^T + e\,K^Y = e\,K^Y.$$

In the region $r_Y^- < r < r_Y^+$, we have by (46)

$$|K^{N_e}(\Psi)| \le Be\,J^{n_\Sigma}_\mu(\Psi)n^\mu_\Sigma \le eB\,\mathbf{q}_e^\star(\Psi).$$

Note the factor of e. In the region $r \leq r_Y^-$, we certainly have by (45)

$$K^{N_e}(\Psi) \geq b\, \mathbf{q}_e^\star(\Psi). \tag{47}$$

For $r \geq r_Y^+$, we have of course

$$K^{N_e} = e\, K^Y = 0.$$

In particular, the bound

$$-K^{N_e}(\Psi) \leq eB\, \mathbf{q}_e^\star(\Psi) \tag{48}$$

holds in all regions.

With the help of (45) and (48), we obtain easily that

$$\mathbf{q}_e(\Psi) \leq B\left(K^{N_e}(\Psi) + J^T_\mu(\Psi) n^\mu_\Sigma\right) \tag{49}$$

holds everywhere, if e is sufficiently small.

By similar considerations to the above, we see that given e, by requiring $\epsilon_{\mathrm{close}} \ll e$ sufficiently small, we have that N_e is timelike everwhere up to the boundary, and in fact

$$J^{N_e}_\mu(\Psi) n^\mu_\Sigma \sim \mathbf{q}_e(\Psi). \tag{50}$$

On the other hand, since by Assumption 3, $\mathcal{H}^+$ is null, $J^{N_e}_\mu(\Psi) n^\mu_{\mathcal{H}}$ controls all tangential derivatives. More precisely, we have

$$(\partial_t\psi)^2 \leq (B + B\epsilon_{\mathrm{close}} e^{-1}) J^{N_e}_\mu(\Psi) n^\mu_{\mathcal{H}} \leq B\, J^{N_e}_\mu(\Psi) n^\mu_{\mathcal{H}}, \tag{51}$$

$$(\partial_\phi\psi)^2 \leq B e^{-1} J^{N_e}_\mu(\Psi) n^\mu_{\mathcal{H}}, \tag{52}$$

on $\mathcal{H}^+$. For the above we have used the full content of Assumption 3, as well as the translation invariance of $n_{\mathcal{H}}$, n_Σ, ∂_ϕ, ∂_t and N_e, which allows us to choose uniform constants B.

8.3 The vector fields X^a and X^b

In this section we shall often use Regge-Wheeler coordinates as many of the computations refer to the Schwarzschild metric g_M.

In particular, we will consider vector fields of the form $V = f(r^*)\partial_{r^*}$. In what follows f' will denote $\frac{df}{dr^*}$.

In (t, r^*, x^A, x^B) coordinates[9] we have

$$\begin{aligned} K^V_{g_M} \quad = \quad & \frac{f'}{1-\mu}(\partial_{r^*}\Psi)^2 + \frac{1}{2}|\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{2-3\mu}{r}\right) f \\ & - \frac{1}{4}\left(2f' + 4\frac{1-\mu}{r} f\right) g_M^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \end{aligned} \tag{53}$$

[9] Careful, t not the t^* of our chart! Of course, t^* coincides with t for $r \geq r_Y^-$.

where $|\nabla\!\!\!/\Psi|^2_{g_M}$ denotes the induced metric from g_M on the spheres. We may rewrite the above as

$$
\begin{aligned}
K^V_{g_M} &= \left(\frac{f'}{2(1-\mu)} - \frac{f}{r}\right)(\partial_{r^*}\Psi)^2 + |\nabla\!\!\!/\Psi|^2_{g_M}\left(-\frac{\mu}{2r}f - \frac{1}{2}f'\right) \\
&\quad + \left(\frac{f'}{2(1-\mu)} + \frac{f}{r}\right)(\partial_t\Psi)^2. \qquad (54)
\end{aligned}
$$

Let α, $R_1(\alpha) \gg M$ be parameters to be chosen in what follows. Let $R(\alpha) = \exp(4)R_1(\alpha)$. Given these, we define a function f_a such that

$$
\begin{aligned}
f_a &= -r^{-4}(r_Y^-)^4, \qquad \text{for } r \le r_Y^- \\
f_a &= -1, \qquad \text{for } r_Y^- \le r \le R_1(\alpha), \\
f_a &= -1 + \int_{R_1(\alpha)}^{r} \frac{d\tilde{r}}{4\tilde{r}} \qquad \text{for } R_1(\alpha) \le r \le R(\alpha), \\
f_a &= 0 \text{ for } r \ge R(\alpha).
\end{aligned}
$$

(One can smooth this function, although this is irrelevant.) We call the resulting vector field X_a.

We obtain that in $R_1(\alpha) \ge r > r_Y^-$

$$
K^{X_a}_{g_M}(\Psi) = |\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{\mu}{2r}\right) + r^{-1}|\partial_{r^*}\Psi|^2 - r^{-1}|\partial_t\Psi|^2. \qquad (55)
$$

Since $t = t^*$ for $r \ge r_Y^-$, we can rewrite this as

$$
K^{X_a}_{g_M}(\Psi) = |\nabla\!\!\!/\Psi|^2_{g_M}\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 - r^{-1}|\partial_{t^*}\Psi|^2, \qquad (56)
$$

where the coordinate derivatives in the last line can now be understood with respect to the atlas (17). For $\epsilon_{\mathrm{close}}$ sufficiently small we obtain from (41)

$$
\begin{aligned}
K^{X_a}(\Psi) &\ge |\nabla\!\!\!/\Psi|^2\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 \\
&\quad - r^{-1}|\partial_{t^*}\Psi|^2 - \epsilon_{\mathrm{close}}B\,\mathbf{q}_e^{\bigstar}(\Psi) \qquad (57)
\end{aligned}
$$

in this region, where we have used (32).

By (48), it follows that in $r_Y^- \le r \le R_1(\alpha)$.

$$
\begin{aligned}
K^{X_a} + K^{N_e}(\Psi) &\ge |\nabla\!\!\!/\Psi|^2\left(\frac{\mu}{2r}\right) + r^{-1}(1-\mu)^2|\partial_r\Psi|^2 \\
&\quad - r^{-1}|\partial_{t^*}\Psi|^2 - eB\,\mathbf{q}_e^{\bigstar}(\Psi) \qquad (58)
\end{aligned}
$$

for small enough $\epsilon_{\mathrm{close}} \ll e$.

Consider now the region $2M \le r \le r_Y^-$. We have

$$
f' = 4r^{-5}(r_Y^-)^4(1-\mu),
$$

and thus

$$
\left(\frac{f'}{2(1-\mu)} + \frac{f}{r}\right) = (r_Y^-)^4 r^{-5},
$$

$$\left(\frac{f'}{2(1-\mu)} - \frac{f}{r}\right) = 3(r_Y^-)^4 r^{-5},$$
$$\left(-\frac{\mu}{2r}f - \frac{1}{2}f'\right) = (r_Y^-)^4\left(\frac{5\mu-4}{2}r^{-5}\right).$$

We have thus
$$K_{g_M}^{X_a}(\Psi) \geq 0$$
in this region.

Thus, by (41), (30) and (34) we have
$$K_g^{X_a}(\Psi) \geq -\epsilon_{\rm close} e^{-1} B\, \mathbf{q}_e^{\bigstar}(\Psi)$$
in this region. It follows now from (47) that
$$K^{X_a}(\Psi) + K^{N_e}(\Psi) \geq b\, \mathbf{q}_e^{\bigstar}(\Psi) \tag{59}$$
in this region, for small enough $\epsilon_{\rm close} \ll e$.

In view of (55), K^{X_a} will "have[10] a sign" when applied to $\psi_\flat^\tau$ (see Section 10.1) except for very large values of r, namely $r \geq R_1(\alpha)$. To control the behaviour there we will need an additional current. First, let us notice that for the X_a we have selected, the coefficient of $(\partial_r\Psi)^2$ is always nonnegative. Finally we notice that for $r \geq R_1(\alpha)$, the coefficient of $|\nabla\!\!\!/\Psi|^2$ satisfies
$$-\frac{\mu}{2r}f_a - \frac{1}{2}f_a' \geq -\frac{1}{8r}. \tag{60}$$

To choose an additional vector field, let us define
$$f_b \doteq \chi(r^*)\pi^{-1}\int_0^{r^*}\frac{\alpha\, dx}{x^2+\alpha^2},$$
where χ is a smooth cutoff with $\chi = 0$ for $r^* \leq 0$ and $\chi = 1$ for $r^* \geq 1$, and let X_b be the vector
$$X_b = f_b\partial_{r^*}.$$
Finally, define the function
$$w_b \doteq f_b' + 2\frac{1-\mu}{r}f_b - \frac{2M(1-\mu)f_b}{r^2}$$
and consider the modified current $J_\mu^{X_b,w_b}$ defined by (37), as well as the associated K^{X_b,w_b} and $\mathcal{E}^{X_b,w_b}$.

Note that for general f, we can rewrite
$$\begin{aligned} K_{g_M}^V \;=\;& \left(\frac{f'}{1-\mu}\right)(\partial_{r^*}\Psi)^2 + \frac{1}{2}\left(\frac{2-3\mu}{r}\right)f|\nabla\!\!\!/\Psi|_{g_M}^2 \\ &- \frac{M(1-\mu)f}{r^2}g_M^{\mu\nu}\partial_\mu\Psi\partial_\nu\Psi \\ &- \frac{1}{8}\left(2f' + 4\frac{1-\mu}{r}f - \frac{4M(1-\mu)f}{r^2}\right)(\Box_{g_M}\Psi^2 - 2\Psi F) \end{aligned} \tag{61}$$

[10] After integration over appropriate domains and modulo error terms

from which we see

$$\begin{aligned}
K_{g_M}^{X_b,w_b}(\Psi) &= \left(\frac{f_b'}{1-\mu}-\frac{Mf_b}{r^2}\right)(\partial_{r^*}\Psi)^2+\frac{Mf_b}{r^2}(\partial_t\Psi)^2 \\
&+\left(\frac{2-3\mu}{2r}-\frac{M(1-\mu)}{r^2}\right)f_b|\nabla\!\!\!/\Psi|^2_{g_M} \\
&-\frac{1}{8}\Box_{g_M}\left(2f_b'+4\frac{1-\mu}{r}f_b-\frac{4M(1-\mu)}{r^2}f_b\right)\Psi^2.
\end{aligned}$$

Note also the modified error term

$$\mathcal{E}^{X_b,w_b}(\Psi)=\mathcal{E}^{X_b}(\Psi)-\frac{1}{4}\left(2f_b'+4\frac{1-\mu}{r}f_b-\frac{4M(1-\mu)f_b}{r^2}\right)\Psi F.$$

Finally, let us define the currents

$$\begin{aligned}
J_\mu^{\mathbf{X}} &= J_\mu^{X_a}+J_\mu^{X_b,w_b}, \\
K^{\mathbf{X}} &= K^{X_a}+K^{X_b,w_b}, \\
\mathcal{E}^{\mathbf{X}} &= \mathcal{E}^{X_a}+\mathcal{E}^{X_b,w_b}.
\end{aligned}$$

By our previous remarks, (36) holds for $J^{\mathbf{X}}$. Also, in view of the definition of w, identities (39), (40) and (41) hold for $J^{\mathbf{X}}$, $K^{\mathbf{X}}$.

Let us expand

$$K_{g_M}^{\mathbf{X}}=H_1(\partial_{r^*}\Psi)^2+H_2(\partial_t\Psi)^2+H_3|\nabla\!\!\!/\Psi|^2_{g_M}+H_4\Psi^2$$

where

$$\begin{aligned}
H_1 &= \frac{f_a'}{2(1-\mu)}-\frac{f_a}{r}+\frac{f_b'}{1-\mu}-\frac{Mf_b}{r^2}, \\
H_2 &= \frac{f_a'}{2(1-\mu)}+\frac{f_a}{r}+\frac{Mf_b}{r^2}, \\
H_3 &= -\frac{\mu}{2r}f_a-\frac{1}{2}f_a'+\left(\frac{2-3\mu}{2r}-\frac{M(1-\mu)}{r^2}\right)f_b, \\
H_4 &= -\frac{1}{8}\Box_{g_M}\left(2f_b'+4\frac{1-\mu}{r}f_b-\frac{4M(1-\mu)}{r^2}f_b\right).
\end{aligned}$$

Note that for $r^*\geq 1$, we have

$$f_b'=\frac{1}{\pi}\frac{\alpha}{(r^*)^2+\alpha^2}.$$

In particular, for $r\geq R_1(\alpha)$ for sufficiently large $R_1(\alpha)$ we have that

$$H_1=\frac{f_a'}{2(1-\mu)}-\frac{f_a}{r}+\frac{f_b'}{1-\mu}-\frac{Mf_b}{r^2}\geq\frac{\alpha}{2\pi r^2}$$

while in $r_Y^- \leq r \leq R_1(\alpha)$, we have

$$H_1 = \frac{1}{r} + \frac{f_b'}{1-\mu} - \frac{Mf_b}{r^2} \geq \frac{1}{2r}.$$

For H_2, let us simply remark that for $r \geq R(\alpha)$, we have

$$H_2 = \frac{Mf_b}{r^2} \geq b(\alpha)r^{-2}.$$

For H_3, we note first that we have the following asymptotic formula

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \sim \frac{1}{r},$$

i.e. for $r \geq R_1(\alpha)$ for sufficiently big $R_1(\alpha)$, we have

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{7}{8r}$$

and thus by (60)

$$H_3 = -\frac{\mu}{2r} f_a - \frac{1}{2} f_a' + \left(\frac{2-3\mu}{r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{3}{4r}.$$

To consider the behaviour for $r \leq R_1(\alpha)$, let us first note that there exists an R_0 depending only on M–i.e. independent of α if we require α to be sufficiently large–such that for $r > R_0$ we have

$$\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq 0$$

and thus, in $R_0 \leq r \leq R_1(\alpha)$ we have

$$H_3 = \frac{\mu}{2r} + \left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f_b \geq \frac{M}{r^2}.$$

For $r_Y^- \leq r \leq R_0$ we have

$$\left|\left(\frac{2-3\mu}{2r} - \frac{M(1-\mu)}{r^2}\right) f\right| \leq B\alpha^{-1}$$

and thus, say

$$H_3 \geq \frac{M}{2r^2}$$

for α sufficiently large.

Turning to H_4, we note first

$$\begin{aligned}
-\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)}{r^2}f_b\right) &= -\frac{1}{4}\frac{1}{1-\mu}f_b''' - \frac{1}{r}f'' + \frac{\mu'}{r(1-\mu)}f_b' \\
&\quad - \frac{1}{2(1-\mu)r}\left(\frac{\mu'(1-\mu)}{r} - \mu''\right)f_b \\
&\quad + \frac{1}{2}\Box_{g_M}\left(\frac{M(1-\mu)}{r^2}f_b\right) \\
&\sim \frac{7\alpha}{2\pi r^4}
\end{aligned}$$

for large r, i.e. we have

$$H_4 = -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)}{r^2}f_b\right) \geq \frac{7\alpha}{4\pi r^4}$$

for $r \geq R_1(\alpha)$ for $R_1(\alpha)$ suitably chosen. On the other hand, one sees easily that R_0 before could have been chosen such that for all α we have

$$H_4 = -\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)}{r^2}f_b\right) \geq 0$$

for $r \geq R_0$. For $r_Y^- \leq r \leq R_0$, we have

$$\left|-\frac{1}{8}\Box_{g_M}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)}{r^2}f_b\right)\right| \leq B\alpha^{-1}.$$

We may thus choose α large enough so that in this region

$$|H_4| \leq \frac{M}{8r^4} \leq \frac{1}{4r^2}H_3.$$

Let α be now chosen. It follows that $R_1 = R_1(\alpha)$ and $R = R(\alpha)$ can be chosen. These choices thus can be made to depend only on M.

Let us assume in what follows in this section that $\Psi_0 = 0$. We thus have

$$\int_0^{2\pi} \Psi^2\, d\phi \leq \int_0^{2\pi} (\partial_\phi \Psi)^2 d\phi. \tag{62}$$

It follows that

$$\int_0^{2\pi} \Psi^2\, d\phi \leq \int_0^{2\pi} (\partial_\phi \Psi)^2\, d\phi \leq r^2 \int_0^{2\pi} |\nabla\!\!\!\!/\,\Psi|^2_{g_M}\, d\phi.$$

Thus, in the region $r_Y^- \leq r \leq R$, we have

$$\int_0^{2\pi} (H_3|\nabla\!\!\!\!/\,\Psi|^2_{g_M} + H_4\Psi^2)\, d\phi \geq \frac{1}{2}\int_0^{2\pi} H_3|\nabla\!\!\!\!/\,\Psi|^2_{g_M}\, d\phi.$$

Note that, in the support of f_b, we have

$$(\partial_r^*\Psi)^2 \sim (\partial_r\Psi)^2, \qquad |\nabla\!\!\!/\Psi|^2 \sim |\nabla\!\!\!/\Psi|^2_{g_M}.$$

We have then by the above bounds and (41), (30) and (32) that for $r \geq R$,

$$\begin{aligned}
\int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\Psi)\, d\phi \quad &\geq \quad \int_0^{2\pi} (K^{\mathbf{X}}_{g_M} + K^{N_e}_{g_M})(\Psi)\, d\phi \\
&\quad - \int_0^{2\pi} \epsilon_{\text{close}} B\, \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi \\
&= \quad \int_0^{2\pi} \big(H_1(\partial_{r^*}\Psi)^2 + H_2(\partial_t\Psi)^2 + H_3|\nabla\!\!\!/\Psi|^2_{g_M} \\
&\quad + H_4\Psi^2\big)\; d\phi \\
&\quad - \int_0^{2\pi} \epsilon_{\text{close}} B\, \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi \\
&\geq \quad \int_0^{2\pi} \big(b\, \mathbf{q}_e^{\bigstar}(\Psi) - \epsilon_{\text{close}} B\, \mathbf{q}_e^{\bigstar}(\Psi)\big)\; d\phi \\
&\geq \quad \int_0^{2\pi} b\, \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi
\end{aligned} \tag{63}$$

for ϵ_{close} suitably small, whereas for $r_Y^- \leq r \leq R$ we may write

$$\begin{aligned}
\int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\Psi)\, d\phi \quad &\geq \quad b\int_0^{2\pi} \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi \\
&\quad + \int_0^{2\pi} (b|\nabla\!\!\!/\Psi|^2 - B(\partial_t\Psi)^2)\, d\phi \\
&\quad - b\epsilon_{\text{close}} \int_0^{2\pi} \mathbf{q}_e^{\bigstar}(\Psi) \\
&\geq \quad b\int_0^{2\pi} \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi \\
&\quad + \int_0^{2\pi} (b|\nabla\!\!\!/\Psi|^2 - B(\partial_t\Psi)^2)\, d\phi
\end{aligned} \tag{64}$$

where for the second inequality we require that ϵ_{close} be sufficiently small. From (59) and the fact that f_b vanishes identically in $r \leq r_Y^-$, we have

$$\begin{aligned}
\int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\Psi)\, d\phi \quad &= \quad \int_0^{2\pi} (K^{X_a} + K^{N_e})(\Psi)\, d\phi \\
&\geq \quad b\int_0^{2\pi} \mathbf{q}_e^{\bigstar}(\Psi)\, d\phi,
\end{aligned} \tag{65}$$

in the region $r \leq r_Y^-$.

To give bounds for the boundary terms, note first that $X_a = -\left(\frac{r_Y^-}{2M}\right)^4 T$ on $\mathcal{H}^+$. It follows that on the horizon, we have

$$J_\mu^{X_a} n_{\mathcal{H}}^\mu = -\left(\frac{r_Y^-}{2M}\right)^4 J_\mu^T n_{\mathcal{H}}^\mu.$$

One sees easily that for $\mathcal{H}^+$ or $\Sigma(\tau)$ where $n^\mu = n_{\mathcal{H}}^\mu$ or $n^\mu = n_\Sigma^\mu$, we have

$$\left|J_\mu^{X_a} n^\mu\right| \leq B\left|J_\mu^T n^\mu\right| + B\epsilon_{\mathrm{close}} e^{-1} J_\mu^{N_e} n^\mu \leq B\, J_\mu^{N_e} n^\mu$$

for $\epsilon_{\mathrm{close}}$ sufficiently small. In view of the fact that we also have

$$|J_\mu^T n^\mu| \leq J_\mu^T n^\mu + B\epsilon_{\mathrm{close}} e^{-1} J_\mu^{N_e} n^\mu \leq B\, J_\mu^{N_e} n^\mu,$$

it follows that

$$\left|J_\mu^{X_a} n^\mu\right| \leq B\left|J_\mu^T n^\mu\right| + B\epsilon_{\mathrm{close}} e^{-1} J_\mu^{N_e} n^\mu \leq B\, J_\mu^{N_e} n^\mu$$

on $\mathcal{H}^+$ or $\Sigma(\tau)$. On the other hand, in view of the the assumption $\Psi_0 = 0$, we have similarly

$$\begin{aligned}\left|\int_0^{2\pi} J_\mu^{X_b} n^\mu\, d\phi\right| &\leq B\left|\int_0^{2\pi} J_\mu^T n^\mu d\phi\right| + B\epsilon_{\mathrm{close}} e^{-1}\int_0^{2\pi} J_\mu^{N_e} n^\mu\, d\phi\\ &\leq B\int_0^{2\pi} J_\mu^{N_e} n^\mu\, d\phi.\end{aligned}$$

It follows from the above inequalities that

$$\left|\int_0^{2\pi} J_\mu^{\mathbf{X}} n^\mu\, d\phi\right| \leq B\int_0^{2\pi} J_\mu^{N_e} n^\mu d\phi \tag{66}$$

on both $\Sigma(\tau)$ and $\mathcal{H}^+$.

9 The high-low frequency decomposition

As explained in the introduction, the arguments of this paper hinge on separating the "superradiant" part of the solution from the non-"superradiant" part, and then exploiting dispersion for the former and positive definiteness for the J^T flux through the event horizon for the latter. These two parts will be characterized by their support in frequency space. As we certainly do not know, however, *a priori* that ψ is in $L^2(t^*)$, we will first need to cut off ψ in t^*. This construction, together with propositions which control the errors that arise, are given in this section.

9.1 ψ cut off: the definition of $\psi^\tau_{\scissors}$

Let $\chi(x)$ be a cutoff function such that $\chi(x) = 1$ for $x \le 0$ and $\chi(x) = 0$ for $x \ge 1$. Given $\tau \ge 2$, define

$$\psi^\tau_{\scissors} \quad = \quad \chi(t^+ + 1 - \tau)\chi(-t^- + 1)\psi.$$

We may express this as

$$\psi^\tau_{\scissors} = \chi^\tau_{\scissors}\psi = ({}^+\chi^\tau_{\scissors} + {}^-\chi^\tau_{\scissors})\psi,$$

where ${}^+\chi^\tau_{\scissors}$ and ${}^-\chi^\tau_{\scissors}$ are smooth functions on $\mathcal{R}$ with

$$\mathrm{supp}({}^+\chi^\tau_{\scissors}) \subset \mathcal{R}^+(\tau - 1, \tau),$$

$$\mathrm{supp}({}^-\chi^\tau_{\scissors}) \subset \mathcal{R}^-(0, 1),$$

$$0 \le {}^-\chi^\tau_{\scissors} \le 1, \qquad 0 \le {}^+\chi^\tau_{\scissors} \le 1$$

and

$$|\partial_{(i)}{}^+\chi^\tau_{\scissors}| \le B_q, \qquad |\partial_{(i)}{}^-\chi^\tau_{\scissors}| \le B_q,$$

with respect to the charts of (17), for any multi-index (i) of order q. Moreover,

$$\partial_\theta{}^+\chi^\tau_{\scissors} = 0, \qquad \partial_\theta{}^-\chi^\tau_{\scissors} = 0. \tag{67}$$

The reader may wonder why the cutoff region is related to $\Sigma^\pm$, indeed, why $\Sigma^\pm$ have been introduced in the first place. Essentially, this is necessary to achieve the propositions of Section 9.4–9.6. We would like to express all errors in terms of the positive definite quantity $\mathbf{q}_e(\psi)$. This quantity does not contain ψ itself but only derivatives. Of course, in view of the fact that, as we shall see, the spherical average ψ_0 does not give rise to errors, this does not generate problems for the region $r \le R$ for $(\psi - \psi_0)^2$ can be controlled by $\mathbf{q}_e(\psi)$ via a Poincaré inequality. As $r \to \infty$, one needs extra negative powers of r. Our cutoff region diverges from $\mathcal{R}(0, \tau)$ as $r \to \infty$ and this allows us to "gain" powers of r necessary to control 0'th order terms via a Poincaré inequality in $\mathcal{R}(0, \tau)$. One can then retrieve estimates all the way to the boundary of the cutoff region using the positive definiteness of J^T for large r.

9.2 Definition of $\Psi_\flat$ and $\Psi_\sharp$

Let ζ be a smooth cutoff supported in $[-2, 2]$ with the property that $\zeta = 1$ in $[-1, 1]$, and let $\omega_0 > 0$ be a parameter to be determined later.

For a smooth function $\Psi(t^*, \cdot)$ of compact support in t^*, let Ψ_k denote its k'th azimuthal mode. Let $\hat{\Psi}$ denote the Fourier transform of Ψ in t^*. Note that $\widehat{\Psi_k} = \hat{\Psi}_k$.

Define

$$\Psi_\flat(t^*, \cdot) \doteq \sum_{k \neq 0} e^{-ik\theta} \int_{-\infty}^{\infty} \zeta((\omega_0 k)^{-1}\omega)\, \hat{\Psi}_k(\omega, \cdot)\, e^{i\omega t^*} d\omega,$$

$$\Psi_\sharp(t^*,\cdot) \doteq \Psi_0 + \sum_{k\neq 0} e^{-ik\theta} \int_{-\infty}^{\infty} \big(1-\zeta((\omega_0 k)^{-1}\omega)\big)\ \hat{\Psi}_k(\omega,\cdot)\, e^{i\omega t^*}\, d\omega.$$

Note of course

$$\Psi_\flat + \Psi_\sharp = \Psi. \tag{68}$$

Note in addition that

$$(\Psi_\flat)_0 = 0 \tag{69}$$

whereas

$$(\Psi_\sharp)_0 = \Psi_0. \tag{70}$$

In the application to $\Psi = \psi^{\tau}_{\text{✂}}$, we shall write simply $\psi^{\tau}_{\sharp}$ and $\psi^{\tau}_{\flat}$. Note finally, that in view of (67), $(\psi_k)^{\tau}_{\text{✂}} = (\psi^{\tau}_{\text{✂}})_k$.

Note that for $k \neq 0$,

$$(\Psi_\flat)_k(t^*) = \int_{-\infty}^{\infty} \zeta((\omega_0 k)^{-1}\omega)\, \hat{\Psi}_k(\omega)\, e^{i\omega t^*} d\omega = \int_{-\infty}^{\infty} P^{<}_{k}(t^*-s^*)\Psi_k(s^*)\, ds^*,$$

where

$$P^{<}_{k}(t^*) = \omega_0 k \int_{-\infty}^{\infty} \zeta(\omega) e^{-i\omega(\omega_0 k t^*)}\, d\omega.$$

The kernel $P^{<}_{k}(t^*)$ is a rescaled copy of a Schwarz function of t^*. As a consequence, for any $m, q \geq 0$,

$$|\partial^m_{t^*} P^{<}_{k}(t^*)| \leq B_{mq}(\omega_0|k|)^{m+1}\, (1+|\omega_0 k t^*|)^{-q}\,. \tag{71}$$

On the other hand, let $\tilde{\zeta}$ be a smooth cut-off function supported in $(-3,3)$ such that $\tilde{\zeta} = 1$ on $[-2,2]$. Then, since $\tilde{\zeta}\,\zeta = \zeta$, we have the reproducing formula

$$(\Psi_\flat)_k(t^*) = \int_{-\infty}^{\infty} \tilde{\zeta}((\omega_0 k)^{-1}\omega)(\hat{\Psi}_\flat)_k(\omega)\, e^{i\omega t^*}\, d\omega = \int_{-\infty}^{\infty} \tilde{P}^{<}_{k}(t^*-s^*)(\Psi_\flat)_k(s^*)\, ds^*,$$

where the kernel $\tilde{P}^{<}_{k}$ also satisfies (71).

Finally, let $\xi(\omega)$ be a function smooth away from $\omega = 0$ and with the property that $\xi(\omega) = \omega^{-1}$ for $|\omega| \leq 1/2$ and $\xi(\omega) = 1$ for $|\omega| \geq 1$. In particular, the function $\tilde{\xi}(\omega) = \omega\xi(\omega)$ is smooth and $\tilde{\xi}(\omega) = 1$ for $|\omega| \leq 1/2$ and $\tilde{\xi}(\omega) = \omega$ for $|\omega| \geq 1$. Since $\xi(1-\zeta) = 1-\zeta$, we can write for $k \neq 0$,

$$(\Psi_\sharp)_k(t^*) = \int_{-\infty}^{\infty} Q^{>}_{k}(t^*-s^*)(\Psi_\sharp)_k(s^*)\, ds^*,$$

where

$$Q^{>}_{k}(t^*) = \omega_0 k \int_{-\infty}^{\infty} \xi(\omega)\, e^{i\omega(\omega_0 k t^*)}\, d\omega.$$

Furthermore,

$$\partial_{t^*}(\Psi_\sharp)_k(t^*) = \omega_0 k \int_{-\infty}^{\infty} \tilde{Q}^{>}_{k}(t^*-s^*)(\Psi_\sharp)_k(s^*)\, ds^*,$$

where

$$\tilde{Q}_k^{>}(t^*) = \omega_0 k \int_{-\infty}^{\infty} \tilde{\xi}(\omega)\, e^{i\omega(\omega_0 k t^*)}\, d\omega.$$

and

$$(\Psi_\sharp)_k(t^*) = (\omega_0 k)^{-1} \int_{-\infty}^{\infty} R_k^{>}(t^* - s^*)\partial_{s^*}(\Psi_\sharp)_k(s^*)\, ds^*,$$

where

$$R_k^{>}(t^*) = \omega_0 k \int_{-\infty}^{\infty} \left(\tilde{\xi}(\omega)\right)^{-1} e^{i\omega(\omega_0 k t^*)}\, d\omega.$$

The function $a(\omega) = (\tilde{\xi}(\omega))^{-1}$ is equal to one on $(-1/2, 1/2)$ and ω^{-1} for $|\omega| \geq 1$. The kernel $R_k^{>}(t^*)$ satisfies

$$|R_k^{>}(t^*)| \leq B_q(\omega_0|k|)^{1-q}(t^*)^{-q}$$

for any $q > 0$. In addition, we have a uniform bound (coming from $1/\omega$ decay)

$$|R_k^{>}(t^*)| \leq B\omega_0|k|\,|\log(\omega_0|k|t^*)|.$$

Combining we obtain

$$|R_k^{>}(t^*)| \leq B_q\omega_0|k|\,|\log(\omega_0|k|t^*)|\,(1 + |\omega_0 k t^*|)^{-q}.$$

9.3 Comparing $\partial_{t^*}\Psi$ and $\partial_\phi\Psi$

The decomposition of Ψ into $\Psi_\flat$ and $\Psi_\sharp$ is motivated by the desire to compare various L^2-type norms of the ∂_ϕ and ∂_{t^*} derivatives. Since this is required at a localised level, however, error terms arise. The precise relations one can make are recorded in this section. The estimates of this section employ standard techniques of elementary Fourier analysis. We must be careful, however, to express all "error terms" in a form which can be related to our bootstrap assumptions which will be introduced later on.

9.3.1 Comparisons for $\Psi_\flat$

First a lemma.

Lemma 9.1. *Let $\tau'' \geq \tau'$ and let Ψ be smooth and of compact support in t^*. Then*

$$\begin{aligned}\int_{\mathcal{R}(\tau',\tau'')\cap\{r_Y^- \leq r \leq R\}} (\partial_{t^*}\Psi_\flat)^2 \quad \leq \quad & B{\omega_0}^2 \int_{\mathcal{R}(\tau',\tau'')\cap\{r_Y^- \leq r \leq R\}} (\partial_\phi\Psi_\flat)^2 \\ & + B\omega_0 \sup_{-\infty \leq \tilde{\tau} \leq \infty} \int_{\tilde{\tau}}^{\tilde{\tau}+1} \left(\int_{\Sigma(\tilde{\tau})\cap\{r_Y^- \leq r \leq R\}} (\partial_\phi\Psi_\flat)^2 \right) d\tilde{\tau}.\end{aligned}$$

Proof. Recall (69). Note first that by the relations of Section 9.2, it follows that for any $q > 0$, we have

$$|\partial_{t^*}(\Psi_\flat)_k(t^*,\cdot)| \leq B_q(\omega_0 k)^2 \int_{-\infty}^{\infty} (1+|\omega_0 k(t^*-s^*)|)^{-q}\,|(\Psi_\flat)_k|\,(s^*,\cdot)ds^*.$$

We have thus

$$\begin{aligned}
|\partial_{t^*}(\Psi_\flat)_k(t^*,\cdot)| &\leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (1+|\ell|)^{-q}\,|(\Psi_\flat)_k|\,(s^*,\cdot)ds^* \\
&\leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\Psi_\flat)_k|\,(s^*,\cdot)ds^* \\
&\leq B_q(\omega_0|k|)^{3/2} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)ds^*\right)^{1/2}.
\end{aligned}$$

It follows that, for $q > 1$,

$$\begin{aligned}
&\int_{\tau'}^{\tau''} (\partial_{t^*}(\Psi_\flat)_k)^2(t^*,\cdot)\,dt^* \\
\leq\ & B_q(\omega_0|k|)^3 \int_{\tau'}^{\tau''} \left(\sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)\,ds^*\right)^{1/2}\right)^2 dt^* \\
\leq\ & B_q(\omega_0|k|)^3 \int_{\tau'}^{\tau''} \left(\sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q}\right) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)\,ds^*\,dt^* \\
\leq\ & B_q(\omega_0|k|)^3 \int_{\tau'}^{\tau''} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)\,ds^*\,dt^* \\
=\ & B_q(\omega_0|k|)^3 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tau'}^{\tau''} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)\,ds^*\,dt^* \\
=\ & B_q(\omega_0|k|)^3 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\frac{\ell}{\omega_0|k|}}^{\frac{\ell+1}{\omega_0|k|}} \int_{\tau'}^{\tau''} (\Psi_\flat)_k^2(s^*+t^*,\cdot)\,dt^*\,ds^* \\
=\ & B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(t^*,\cdot)\,dt^*.
\end{aligned}$$

Thus,

$$
\begin{aligned}
& \int_{\tau'}^{\tau''} (\partial_{t^*}(\Psi_\flat)_k)^2(t^*,\cdot)\,dt^* \\
\leq\ & B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}} \chi_{\tau',\tau''}(s^*)(\Psi_\flat)_k^2(s^*,\cdot)\,ds^* \\
& + B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau''+\frac{\ell+1}{\omega_0|k|}} (1-\chi_{\tau',\tau''}(s^*))(\Psi_\flat)_k^2(s^*,\cdot)\,ds^* \\
\doteq\ & T_{1,k}+T_{2,k}
\end{aligned}
\tag{72}
$$

where $\chi_{\tau',\tau''}(s^*)=1$ if $s^*\in[\tau',\tau'']$ and 0 otherwise.

To prove the lemma, in view of the comments in Section 3.2 on the volume form and Plancherel, it would suffice to show that

$$
\begin{aligned}
& \sum_{|k|\geq 1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi} T_{1,k}\,d\phi\,d\theta\,dr \\
& \leq B{\omega_0}^2\int_{\tau'}^{\tau''}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}(\partial_\phi\Psi_\flat)^2\,d\phi\,d\theta\,dr\,dt^*,
\end{aligned}
\tag{73}
$$

$$
\begin{aligned}
& \sum_{|k|\geq 1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi} T_{2,k}\,d\phi\,d\theta\,dr \\
& \leq B\omega_0 \sup_{-\infty\leq\bar\tau\leq\infty}\int_{\bar\tau}^{\bar\tau+1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}(\partial_\phi\Psi_\flat)^2\,d\phi\,d\theta\,dr d\tilde\tau.
\end{aligned}
\tag{74}
$$

The first term on the right hand side of (72) is bounded by

$$
\begin{aligned}
T_{1,k} &\leq B_q(\omega_0 k)^2\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'}^{\tau''}(\Psi_\flat)_k^2(s^*,\cdot)\,ds^* \\
&\leq B_q(\omega_0 k)^2\int_{\tau'}^{\tau''}(\Psi_\flat)_k^2(s^*,\cdot)\,ds^*.
\end{aligned}
$$

Thus, it follows that

$$
\begin{aligned}
\sum_{|k|\geq 1}\int_0^{2\pi} T_{1,k}d\phi &\leq \sum_{|k|\geq 1} B_q\omega_0^2 k^2\int_0^{2\pi}\int_{\tau'}^{\tau''}(\Psi_\flat)_k^2(s^*,\cdot)\,ds^* d\phi \\
&\leq \sum_{|k|\geq 1} B_q\omega_0^2\int_0^{2\pi}\int_{\tau'}^{\tau''}(\partial_\phi\Psi_\flat)^2(s^*,\cdot)\,ds^* d\phi.
\end{aligned}
$$

We have established (73).

The second term on the right hand side of (72) is bounded by

$$
\begin{aligned}
T_{2,k} &\leq B_q(\omega_0 k)^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau'} (\Psi_\flat)_k^2(s^*,\cdot)\, ds^* \\
&\quad + B_q(\omega_0 k)^2 \sum_{\ell=0}^{\infty} (1+|\ell|)^{-q} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0|k|}} (\Psi_\flat)_k^2(s^*,\cdot)\, ds^* \\
&\doteq T_{21,k} + T_{22,k}. \qquad (75)
\end{aligned}
$$

We have

$$
\begin{aligned}
&\sum_k \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} T_{21,k}\, d\phi\, d\theta\, dr \\
\leq\ & B_q\omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0|k|}}^{\tau'} k^2(\Psi_\flat)_k^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q\omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} k^2(\Psi_\flat)_k^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \sum_{|k|\geq 1} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} k^2(\Psi_\flat)_k^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\tau'+\frac{\ell}{\omega_0}}^{\tau'} (\partial_\phi\Psi_\flat)^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q\omega_0^2 \sum_{\ell=-\infty}^{-1} (1+|\ell|)^{-q}\, |\ell|\omega_0^{-1} \sup_{-\infty\leq\bar\tau\leq\infty} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi\Psi_\flat)^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B\omega_0 \sup_{-\infty\leq\bar\tau\leq\infty} \int_{r_Y^-}^{R} \int_0^\pi \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi\Psi_\flat)_k^2(s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr, \qquad (76)
\end{aligned}
$$

for q chosen sufficiently large.

As for $T_{22,k}$, we have

$$
\begin{aligned}
& \sum_k \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} T_{22,k} d\phi\, d\theta\, dr \\
\leq\ & B_q \omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=0}^{\infty} (1+\ell)^{-q} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0|k|}} k^2 (\Psi_\flat)_k^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q \omega_0^2 \sum_{|k|\geq 1} \sum_{\ell=0}^{\infty} (1+\ell)^{-q} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0}} k^2 (\Psi_\flat)_k^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q \omega_0^2 \sum_{\ell=0}^{\infty} (1+\ell)^{-q} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \sum_{|k|\geq 1} \int_0^{2\pi} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0}} k^2 (\Psi_\flat)_k^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
=\ & B_q \omega_0^2 \sum_{\ell=0}^{\infty} (1+\ell)^{-q} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\tau''}^{\tau''+\frac{\ell+1}{\omega_0}} (\partial_\phi \Psi_\flat)^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q \omega_0^2 \sum_{\ell=0}^{\infty} (1+\ell)^{-q} \frac{1+\ell}{\omega_0} \sup_{-\infty\leq\bar\tau\leq\infty} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi \Psi_\flat)^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr \\
\leq\ & B_q \omega_0 \sup_{-\infty\leq\bar\tau\leq\infty} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau}^{\bar\tau+1} (\partial_\phi \Psi_\flat)^2 (s^*,\cdot)\, ds^*\, d\phi\, d\theta\, dr.
\end{aligned}
$$

The above and (76) give (74). $\square$

Lemma 9.2. *Under the assumptions of the previous lemma, if $\omega_0 \leq 1$, then*

$$
\begin{aligned}
& \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\bar\tau-1}^{\bar\tau} (\partial_\phi \Psi_\flat)^2 dt^*\, d\phi \\
& \qquad \leq B\omega_0^{-1} \sup_{-\infty\leq\tilde\tau\leq\infty} \int_{r_Y^-}^{R} \int_0^{\pi} \nu(\theta,r) \int_0^{2\pi} \int_{\tilde\tau-1}^{\tilde\tau} (\partial_\phi \Psi)^2 dt^*\, d\phi.
\end{aligned}
$$

Proof. For any $q > 0$, we have

$$
\begin{aligned}
|(\Psi_\flat)_k(t^*,\cdot)| &\leq B_q(\omega_0|k|) \int_{-\infty}^{\infty} (1+|\omega_0 k(t^*-s^*)|)^{-q} |\Psi_k|(s^*,\cdot)\, ds^* \\
&\leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|(s^*,\cdot)\, ds^* \\
&\leq B_q(\omega_0|k|)^{\frac12} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\, ds^* \right)^{\frac12}.
\end{aligned}
$$

It follows, with the help of Cauchy-Schwarz, that for $q > 1$,

$$
|(\Psi_\flat)_k(t^*,\cdot)|^2 \leq B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |\Psi_k|^2(s^*,\cdot)\, ds^*,
$$

and thus,

$$
\begin{aligned}
&\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1}^{\bar\tau}|(\Psi_\flat)_k(t^*,\cdot)|^2dt^*\\
&\quad\le B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1}^{\bar\tau}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*\,dt^*\\
&\quad\le B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}\int_{s^*-\frac{\ell+1}{\omega_0|k|}}^{s^*-\frac{\ell}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,dt^*\,ds^*\\
&\quad\le B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*.
\end{aligned}
$$

We then obtain

$$
\begin{aligned}
&\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1}^{\bar\tau}(\partial_\phi\Psi_\flat)^2dt^*\,d\phi\\
&\quad=\sum_{|k|\ge1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1}^{\bar\tau}k^2|(\Psi_\flat)_k(t^*,\cdot)|^2dt^*\\
&\quad\le B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\sum_{|k|\ge1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1+\frac{\ell}{\omega_0|k|}}^{\bar\tau+\frac{\ell+1}{\omega_0|k|}}k^2|\Psi_k|^2(s^*,\cdot)\,ds^*\\
&\quad\le B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\sum_{|k|\ge1}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1-\frac{\ell}{\omega_0}}^{\bar\tau+\frac{\ell+1}{\omega_0}}k^2|\Psi_k|^2(s^*,\cdot)\,ds^*\\
&\quad=B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\bar\tau-1-\frac{\ell}{\omega_0}}^{\bar\tau+\frac{\ell+1}{\omega_0}}(\partial_\phi\Psi)^2\,ds^*\\
&\quad\le B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}(2\ell+2){\omega_0}^{-1}\sup_{-\infty\le\tilde\tau\le\infty}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\tilde\tau-1}^{\tilde\tau}(\partial_\phi\Psi)^2\,ds^*\\
&\quad\le B_q\omega_0^{-1}\sup_{-\infty\le\tilde\tau\le\infty}\int_{r_Y^-}^{R}\int_0^{\pi}\nu(\theta,r)\int_0^{2\pi}\int_{\tilde\tau-1}^{\tilde\tau}(\partial_\phi\Psi)^2\,dt^*,
\end{aligned}
$$

where we have assumed q sufficiently large, and have used ${\omega_0}^{-1}\ge1$. The lemma follows after fixing q. □

9.3.2 Application to $\psi_\flat^\tau$

From the above lemmas, we easily obtain the following statement, which is the form we shall use later in this paper:

Proposition 9.1. *Let $\tau'' \geq \tau'$ and $\omega_0 \leq 1$. Then*

$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r_Y^- \leq r \leq R\}} (\partial_{t^*}\psi^\tau_\flat)^2 \quad \leq \quad B{\omega_0}^2 \int_{\mathcal{R}(\tau',\tau'')\cap\{r_Y^- \leq r \leq R\}} (\partial_\phi \psi^\tau_\flat)^2 + B \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. In view of (67), it follows that

$$(\partial_\phi \psi^\tau_{\text{✂}})^2 = (\chi^\tau_{\text{✂}})^2 (\partial_\phi\psi)^2 \leq B\,\mathbf{q}_e(\psi)$$

in the region $r_Y^- \leq r \leq R$. In view also of the support of $\psi^\tau_{\text{✂}}$, we may thus bound the right hand side of the statement of Lemma 9.2 applied to $\psi^\tau_{\text{✂}}$ by

$$B\omega_0^{-1} \sup_{1\leq\bar\tau\leq\tau} \int_{\bar\tau-1}^{\bar\tau} \left(\int_{\Sigma(\tilde\tau)} \mathbf{q}_e(\psi) \right) d\tilde\tau.$$

The proposition now follows from Lemmas 9.1 and 9.2. $\square$

9.3.3 Comparisons for $\Psi_\sharp$

First a lemma:

Lemma 9.3. *Let $\tau' \leq \tau''$ let Ψ be smooth and of compact support in t^*. Then*

$$\int_{\mathcal{H}(\tau',\tau'')} (\partial_{t^*}\Psi_\sharp)^2 \geq B{\omega_0}^2 \int_{\mathcal{H}(\tau',\tau'')} (\partial_\phi \Psi_\sharp)^2 - B{\omega_0}^{-1} \sup_{-\infty\leq\bar\tau\leq\infty} \int_{\mathcal{H}(\bar\tau,\bar\tau+1)} (\partial_{t^*}\Psi_\sharp)^2,$$

Proof. Note first that the lemma holds trivially for $(\Psi_\sharp)_0$. We may thus assume that $(\Psi_\sharp)_0 = 0$. For $|k| \geq 1$, we note first that from Section 9.2 we obtain

$$|(\Psi_\sharp)_k(t^*,\cdot)| \leq B_q\omega_0^{-1}|k|^{-1} \int_{-\infty}^{\infty} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \, |\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)| \, ds^*.$$

Thus,

$$\begin{aligned}
& \int_{\tau'}^{\tau''} k^2 (\Psi_\sharp)_k^2(t^*,\cdot)\, dt^* \\
\leq\ & B_q\omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{-\infty}^{\infty} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \, |\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)| \, ds^* \right)^2 dt^* \\
\leq\ & B_q\omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{\tau'}^{\tau''} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \, |\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)| \, ds^* \right)^2 dt^* \\
& + B_q\omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{-\infty}^{\tau'} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \, |\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)| \, ds^* \right)^2 dt^* \\
& + B_q\omega_0^{-2} \int_{\tau'}^{\tau''} \left(\int_{\tau''}^{\infty} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \, |\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)| \, ds^* \right)^2 dt^* \\
\doteq\ & T_{1,k} + T_{2,k} + T_{3,k}.
\end{aligned}$$

We obtain immediately that for sufficiently large q, since

$$\int_{\tau'}^{\tau''} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q}\, ds^* \leq B_q$$

then

$$\begin{aligned}
&\int_{\tau'}^{\tau''} \left(\int_{\tau'}^{\tau''} \omega_0|k| \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| \, ds^* \right)^2 dt^* \\
&\leq B_q \int_{\tau'}^{\tau''} (\partial_t \Psi_\sharp)_k^2(t^*,\cdot)\, dt^*
\end{aligned}$$

and thus

$$T_{1,k} \leq B_q \omega_0^{-2} \int_{\tau'}^{\tau''} (\partial_{t^*}(\Psi_\sharp)_k)^2(t^*,\cdot)\, dt^*.$$

On the other hand,

$$\begin{aligned}
&\sum_{|k|\geq 1} \int_0^\pi \tilde{\nu}(\theta) \int_0^{2\pi} T_{2,k}\, d\phi\, d\theta \\
=\ & B_q \omega_0^{-2} \sum_{|k|\geq 1} \int_0^\pi \tilde{\nu}(\theta) \\
& \int_0^{2\pi} \int_{\tau'}^{\tau''} \left(\int_{-\infty}^{\tau'} \omega_0 k \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| \, ds^* \right)^2 dt^*\, d\phi\, d\theta \\
=\ & B_q \omega_0^{-2} \sum_{|k|\geq 1} \int_0^\pi \tilde{\nu}(\theta) \\
& \int_0^{2\pi} \int_{\tau'}^{\tau''} \left(\sum_{\ell=0}^{\infty} \int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}} \omega_0 k \frac{|\log|\omega_0 k(t^*-s^*)||}{(1+|\omega_0 k(t^*-s^*)|)^q} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right| \, ds^* \right)^2 dt^*\, d\phi\, d\theta \\
\leq\ & B_q \omega_0^{-2} \sum_{|k|\geq 1} \int_0^\pi \tilde{\nu}(\theta) \int_0^{2\pi} \int_{\tau'} \left(\sum_{\ell=0}^{\infty} \left(\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}} \omega_0^2 k^2 \frac{\left|\log^2|\omega_0 k(t^*-s^*)|\right|}{(1+|\omega_0 k(t^*-s^*)|)^{2q}} ds^* \right)^{1/2} \right. \\
& \left. \cdot \left(\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2 \, ds^* \right)^{1/2} \right)^2 dt^*\, d\phi\, d\theta \\
\leq\ & B_q \omega_0^{-2} \sum_{|k|\geq 1} \int_0^\pi \tilde{\nu}(\theta) \\
& \int_0^{2\pi} \int_{\tau'}^{\tau''} \omega_0|k| \left(\sum_{\ell=0}^{\infty} \frac{|\log|\omega_0 k(t^*-\tau')+\ell||}{(1+|\omega_0 k(t^*-\tau')+\ell|)^q} \left(\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}} \left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2 \, ds^* \right)^{1/2} \right)^2 dt^*\, d\phi\, d\theta
\end{aligned}$$

$$
\begin{aligned}
&\leq B_q\omega_0^{-2}\sum_{|k|\geq 1}\int_0^\pi \tilde{\nu}(\theta) \\
&\quad \int_0^{2\pi}\int_0^\infty\left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell)||}{(1+|t^*+\ell|)^q}\left(\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}}\left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2\,ds^*\right)^{1/2}\right)^2 dt^*\,d\phi\,d\theta \\
&\leq B_q\omega_0^{-2}\sum_{|k|\geq 1}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_0^\infty\left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell)||}{(1+|t^*+\ell|)^q}\right) \\
&\quad \left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell)||}{(1+|t^*+\ell|)^q}\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}}\left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2\,ds^*\right)dt^*\,d\phi\,d\theta \\
&\leq B_q\omega_0^{-2}\int_0^\infty\left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell)||}{(1+|t^*+\ell|)^q}\right) \\
&\quad \left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell)||}{(1+|t^*+\ell|)^q}\sum_{|k|\geq 1}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_{\tau'-\frac{\ell+1}{\omega_0|k|}}^{\tau'-\frac{\ell}{\omega_0|k|}}\left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2\,ds^*\,d\phi\,d\theta^*\right)dt^* \\
&\leq B_q\omega_0^{-2}\int_0^\infty \frac{|\log|t^*||}{(1+|t^*|)^q} \\
&\quad \left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell||}{(1+|t^*+\ell|)^q}\sum_{|k|\geq 1}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_{\tau'-\frac{\ell+1}{\omega_0}}^{\tau'}\left|\partial_{s^*}(\Psi_\sharp)_k(s^*,\cdot)\right|^2\,ds^*\,d\phi\,d\theta^*\right)dt^* \\
&\leq B_q\omega_0^{-2}\int_0^\infty \frac{|\log|t^*||}{(1+|t^*|)^q} \\
&\quad \left(\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell||}{(1+|t^*+\ell|)^q}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_{\tau'-\frac{\ell+1}{\omega_0}}^{\tau'}\left|\partial_{s^*}(\Psi_\sharp)(s^*,\cdot)\right|^2\,ds^*\,d\phi\,d\theta^*\right)dt^* \\
&\leq B_q\omega_0^{-2}\int_0^\infty \frac{|\log|t^*||}{(1+|t^*|)^q}\sum_{\ell=0}^\infty \frac{|\log|t^*+\ell||}{(1+|t^*+\ell|)^q}\frac{\ell+1}{\omega_0} \\
&\quad \left(\sup_{-\infty\leq\bar{\tau}\leq\infty}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_{\bar{\tau}+1}^{\bar{\tau}}\left|\partial_{s^*}(\Psi_\sharp)(s^*,\cdot)\right|^2\,ds^*\,d\phi\,d\theta^*\right)dt^* \\
&\leq B_q\omega_0^{-3}\sup_{-\infty\leq\bar{\tau}\leq\infty}\int_0^\pi \tilde{\nu}(\theta)\int_0^{2\pi}\int_{\bar{\tau}+1}^{\bar{\tau}}\left|\partial_{s^*}(\Psi_\sharp)(s^*,\cdot)\right|^2\,ds^*\,d\phi\,d\theta^*.
\end{aligned}
$$

A similar bound holds for $T_{3,k}$. We obtain the lemma after appropriate fixing of q. $\square$

Lemma 9.4. *Under the assumptions of the previous lemma, if $\omega_0 \leq 1$,*

$$
\sup_{-\infty\leq\bar{\tau}\leq\infty}\int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)}(\partial_{t^*}\Psi_\sharp)^2 \leq B\omega_0^{-1}\sup_{-\infty\leq\bar{\tau}\leq\infty}\int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)}(\partial_{t^*}\Psi)^2
$$

Proof. Since

$$
\partial_{t^*}\Psi_\sharp = \partial_{t^*}\Psi - \partial_{t^*}\Psi_\flat,
$$

and $\omega_0{}^{-1} \geq 1$, it suffices in fact to prove

$$\sup_{-\infty \leq \bar{\tau} \leq \infty} \int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} (\partial_{t^*}\Psi_\flat)^2 \leq B\omega_0^{-1} \sup_{-\infty \leq \bar{\tau} \leq \infty} \int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} (\partial_{t^*}\Psi)^2.$$

Recall from Section 9.2 we have

$$|(\partial_{t^*}\Psi_\flat)_k| \leq B_q \int_{-\infty}^{\infty} \omega_0|k|(1+\omega_0|k||s^*-t^*|)^{-q}|(\partial_{t^*}\Psi)_k|\, ds^*.$$

We obtain

$$\begin{aligned}
& \int_{\tilde{\tau}}^{\tilde{\tau}+1} (\partial_{t^*}\Psi_\flat)_k^2(t^*,\cdot)dt^* \\
\leq \quad & B_q \int_{\tilde{\tau}}^{\tilde{\tau}+1} \left(\int_{-\infty}^{\infty} \frac{\omega_0|k|}{(1+\omega_0|k||s^*-t^*|)^q} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|\, ds^* \right)^2 dt^* \\
\leq \quad & B_q \int_{\tilde{\tau}}^{\tilde{\tau}+1} \left(\sum_{\ell=-\infty}^{\infty} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} \frac{\omega_0|k|}{(1+\omega_0|k||s^*-t^*|)^q} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|\, ds^* \right)^2 dt^* \\
\leq \quad & B_q(\omega_0 k)^2 \int_{\tilde{\tau}}^{\tilde{\tau}+1} \left(\sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|\, ds^* \right)^2 dt^* \\
\leq \quad & B_q(\omega_0 k)^2 \int_{\tilde{\tau}}^{\tilde{\tau}+1} \left(\sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \right) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q}(\omega_0|k|)^{-1} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, ds^*\, dt^* \\
\leq \quad & B_q(\omega_0|k|) \int_{\tilde{\tau}}^{\tilde{\tau}+1} \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, ds^*\, dt^* \\
\leq \quad & B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tilde{\tau}}^{\tilde{\tau}+1} \int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, ds^*\, dt^* \\
\leq \quad & B_q(\omega_0|k|) \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tilde{\tau}+\frac{\ell}{\omega_0|k|}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0|k|}} \int_{s^*-\frac{\ell+1}{\omega_0|k|}}^{s^*-\frac{\ell}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, dt^*\, ds^* \\
\leq \quad & B_q \sum_{\ell=-\infty}^{\infty} (1+|\ell|)^{-q} \int_{\tilde{\tau}+\frac{\ell}{\omega_0|k|}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0|k|}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, ds^* \\
\leq \quad & B_q \sum_{\ell=0}^{\infty} (1+|\ell|)^{-q} \int_{\tilde{\tau}-\frac{\ell}{\omega_0}}^{\tilde{\tau}+1+\frac{\ell+1}{\omega_0}} |(\partial_{t^*}\Psi)_k(s^*,\cdot)|^2\, ds^*
\end{aligned}$$

for $q > 1$.

Integrating and summing over k, we obtain

$$\int_0^\pi \tilde\nu(\theta)\int_0^{2\pi}\int_{\tilde\tau}^{\tilde\tau+1}(\partial_{t^*}\Psi_\flat)^2(t^*,\cdot)dt^*\,d\phi\,d\theta$$
$$\leq B_q\sum_{\ell=0}^{\infty}(1+\ell)^{-q}\int_0^\pi\tilde\nu(\theta)\int_0^{2\pi}\int_{\tilde\tau-\frac{\ell}{\omega_0}}^{\tilde\tau+1+\frac{\ell+1}{\omega_0}}|(\partial_{t^*}\Psi)(s^*,\cdot)|^2\,ds^*$$
$$\leq B_q\omega_0^{-1}\sum_{\ell=0}^{\infty}(1+\ell)^{-q}(2\ell+2)\sup_{-\infty<\bar\tau<\infty}\int_0^\pi\tilde\nu(\theta)\int_0^{2\pi}\int_{\bar\tau}^{\bar\tau+1}|(\partial_{t^*}\Psi)(s^*,\cdot)|^2\,ds^*,$$

where we have used in the last line that $\omega_0\leq 1$. The lemma follows after fixing $q>2$. $\square$

9.3.4 Application to $\psi_\sharp^\tau$

We may now easily prove

Proposition 9.2. *Let $0\leq\tau'<\tau''\leq\tau$ and let ψ be as in Theorem 5.1. We have*

$$\int_{\mathcal{H}(\tau',\tau'')}(\partial_{t^*}\psi_\sharp^\tau)^2\geq B{\omega_0}^2\int_{\mathcal{H}(\tau',\tau'')}(\partial_\phi\psi_\sharp^\tau)^2-B{\omega_0}^{-2}e^{-1}\sup_{0\leq\bar\tau\leq\tau-1}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)}J_\mu^{N_e}(\psi)n^\mu.$$

Proof. To prove the proposition from the above lemmas, we first remark that it suffices to prove the inequality under the assumption

$$(\psi_\sharp^\tau)_0=\int_0^{2\pi}\psi_\sharp^\tau\,d\phi=0,$$

for the inequality is trivially true for $(\psi_\sharp^\tau)_0$. By (70), this is equivalent to assuming

$$\int_0^{2\pi}\psi^\tau_{\text{✂}}d\phi=0,$$

and by (67)

$$\int_0^{2\pi}\psi\,d\phi=0$$

in the support of $\psi_{\text{✂}}$. Of course, under this assumption it follows that this holds in all of $\mathcal{R}$. Thus we may assume

$$\int_0^{2\pi}\psi^2\,d\phi\leq\int_0^{2\pi}(\partial_\phi\psi)^2\,d\phi \tag{77}$$

in the relevant region. From the above lemma, we just notice that on $\mathcal{H}(0,\tau)$

$$
\begin{aligned}
\int_0^{2\pi} (\partial_{t^*}\psi^\tau_{\text{✂}})^2\, d\phi &\leq \int_0^{2\pi} ((\partial_{t^*}\chi^\tau_{\text{✂}})\psi + \chi^\tau_{\text{✂}}\partial_{t^*}\psi)^2\, d\phi \\
&\leq \int_0^{2\pi} (B\psi^2 + B(\partial_t\psi)^2)\, d\phi \\
&\leq \int_0^{2\pi} (B(\partial_\phi\psi)^2 + B(\partial_t\psi)^2) d\phi \\
&\leq \int_0^{2\pi} Be^{-1} J^{N_e}_\mu n^\mu_{\mathcal{H}},
\end{aligned}
$$

where we have used (77), (51) and (52). The proposition follows. $\square$

9.4 Comparing $\mathbf{q}_e(\psi^\tau_\flat)$, $\mathbf{q}_e(\psi^\tau_\sharp)$ and $\mathbf{q}_e(\psi)$

In view of (68), we clearly have the pointwise relation

$$
\mathbf{q}_e(\psi) \leq 2\left(\mathbf{q}_e(\psi^\tau_\flat) + \mathbf{q}_e(\psi^\tau_\sharp)\right) \tag{78}
$$

in $\mathcal{R}(1, \tau-1)$. It will be necessary, however, to compare also in the opposite direction. We have

Proposition 9.3. *Let $\omega_0 \leq 1 \leq \tau_{\text{step}} \leq \tau' \leq \tau - \tau_{\text{step}}$. Then*

$$
\begin{aligned}
\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e(\psi^\tau_\flat) &\leq B{\omega_0}^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar\tau \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\quad + B{\omega_0}^{-7} e^{-1} \tau_{\text{step}}^{-2} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi),
\end{aligned}
$$

$$
\begin{aligned}
\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e(\psi^\tau_\sharp) &\leq B\omega_0^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar\tau \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\quad + B{\omega_0}^{-7} e^{-1} \tau_{\text{step}}^{-2} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).
\end{aligned}
$$

Proof. Since $\psi^\tau_\sharp = \psi^\tau_{\text{✂}} - \psi^\tau_\flat$ and $\mathbf{q}_e(\psi^\tau_\sharp) \leq 2(\mathbf{q}_e(\psi^\tau) + \mathbf{q}_e(\psi^\tau_\flat))$ it will be sufficient to prove the first statement of the proposition. We begin with the following

Lemma 9.5. *Let Ψ be smooth of compact support in t^* and $\omega_0 \leq 1$. Then*

$$
\begin{aligned}
\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \Psi^2_\flat &\leq B\omega_0^{-1} \sup_{\tau'-\tau_{\text{step}} \leq \bar\tau \leq \tau'+\tau_{\text{step}}} \int_{\Sigma(\bar\tau)} \Psi^2 \\
&\quad + B \sup_{-\infty \leq \bar\tau \leq \tau'-\tau_{\text{step}} \cup \tau'+\tau_{\text{step}} \leq \bar\tau \leq \infty} \omega_0^{-7} |\bar\tau - \tau_{\text{step}}|^{-6} \int_{\Sigma(\bar\tau)} \Psi^2.
\end{aligned}
$$

Proof. For any $q > 0$, we have

$$\begin{aligned}
|(\Psi_\flat)_k(t^*,\cdot)| &\leq B_q(\omega_0|k|)\int_{-\infty}^{\infty}(1+|\omega_0 k(t^*-s^*)|)^{-q}|\Psi_k|(s^*,\cdot)\,ds^*\\
&\leq B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|(s^*,\cdot)\,ds^*\\
&\leq B_q(\omega_0|k|)^{\frac{1}{2}}\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\left(\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*\right)^{\frac{1}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\tau'-1}^{\tau'}|(\Psi_\flat)_k(t^*,\cdot)|^2dt^* &\leq B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1}^{\tau'}dt^*\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*\\
&\leq B_q(\omega_0|k|)\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1}^{\tau'}dt^*\int_{t^*+\frac{\ell}{\omega_0|k|}}^{t^*+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*\\
&\leq B_q\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1+\frac{\ell}{\omega_0|k|}}^{\tau'+\frac{\ell+1}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,ds^*.
\end{aligned}$$

As a consequence,

$$\begin{aligned}
&\int_{\tau'-1}^{\tau'}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)\int_0^{2\pi}|(\Psi_\flat)(t^*,\cdot)|^2d\phi\,d\theta\,dr\,dt^*\\
&\quad=\sum_{|k|\geq 1}\int_{\tau'-1}^{\tau'}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)\int_0^{2\pi}|(\Psi_\flat)_k(t^*,\cdot)|^2dt^*\\
&\quad\leq B_q\sum_{|k|\geq 1}\sum_{\ell=-\infty}^{\infty}(1+|\ell|)^{-q}\int_{\tau'-1+\frac{\ell}{\omega_0|k|}}^{\tau'+\frac{\ell+1}{\omega_0|k|}}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)\int_0^{2\pi}\int_{s^*-\frac{\ell+1}{\omega_0|k|}}^{s^*-\frac{\ell}{\omega_0|k|}}|\Psi_k|^2(s^*,\cdot)\,dt^*\,ds^*\\
&\quad\leq B_q\sum_{\ell\geq 0}(1+|\ell|)^{-q}\int_{2M}^{\infty}\int_0^{\pi}\nu(r,\theta)\int_0^{2\pi}\int_{\tau'-1-\frac{\ell}{\omega_0}}^{\tau'+\frac{\ell+1}{\omega_0}}|\Psi|^2(s^*,\cdot)\,ds^*\\
&\quad\leq B\omega_0^{-1}\sup_{\tau'-\tau_{\mathrm{step}}\leq\bar\tau\leq\tau'+\tau_{\mathrm{step}}}\int_{\Sigma(\bar\tau)}\Psi^2\\
&\qquad+B\sup_{-\infty\leq\bar\tau\leq\tau'-\tau_{\mathrm{step}}\cup\tau'+\tau_{\mathrm{step}}\leq\bar\tau\leq\infty}\omega_0^{-7}\left(\tau_{\mathrm{step}}+|\bar\tau-\tau_{\mathrm{step}}|\right)^{-6}\int_{\Sigma(\bar\tau)}\Psi^2
\end{aligned}$$

for q chosen sufficiently large. $\square$

Note that

$$\left((\partial_v\Psi)^2+(\partial_u\Psi)^2+|\nabla\!\!\!/\Psi|^2+e\frac{(\partial_u\Psi)^2}{(1-\mu)^2}\right)\sim\mathbf{q}_e(\Psi),$$

and as a consequence,

$$(\partial_v \Psi)^2_\flat + (\partial_u \Psi)^2_\flat + (\partial_{z^A} \Psi)^2_\flat + (\partial_{z^B} \Psi)^2_\flat + e \left(\frac{\partial_u \Psi}{1-\mu} \right)^2_\flat \sim \mathbf{q}_e(\Psi_\flat),$$

where z^A denote alternative coordinates x^A or $\tilde{x}^A$ of our atlas (17). Thus, we obtain from the above lemma applied to $\Psi = \partial_v \psi^\tau_{\scissors}$, $\Psi = \partial_u \psi^\tau_{\scissors}$, $\Psi = \partial_{z^A} \psi^\tau_{\scissors}$,[11] $\Psi = \sqrt{e}\frac{\partial_u \psi^\tau_{\scissors}}{1-\mu}$, the statement

$$\int_{\tau'-1}^{\tau'} dt^* \int_{\Sigma(t^*)} \mathbf{q}_e(\psi^\tau_\flat) \leq B\omega_0^{-1} \sup_{\tau'-\tau_{\rm step} \leq \bar{\tau} \leq \tau' + \tau_{\rm step}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi^\tau_{\scissors}) \tag{79}$$
$$+ B \sup_{-\infty \leq \bar{\tau} \leq \tau' - \tau_{\rm step} \cup \tau' + \tau_{\rm step} \leq \bar{\tau} \leq \infty} {\omega_0}^{-7} \left(\tau_{\rm step} + |\bar{\tau} - \tau_{\rm step}|\right)^{-6} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi^\tau_{\scissors}).$$

Note that it is sufficient to prove the inequality under the assumption $\psi_0 = 0$, and thus we may assume (77). Note the inequality

$$\mathbf{q}_e(\psi^\tau_{\scissors})(t,r,\cdot) \leq \mathbf{q}_e(\psi)(t,r,\cdot) + B\chi(t^+ + 1 - \tau)\chi(-t^-)\psi^2. \tag{80}$$

Now, B can be chosen such that in the support of the first term on the right hand side of (79) either $r \leq B$ or $\psi^\tau_{\scissors} = \psi$. In view of (77), it follows thus that we may there replace $\mathbf{q}_e(\psi^\tau_{\scissors})$ with $\mathbf{q}_e(\psi)$.

Turning to the second supremum term of (79) and applying

$$|t^* - \tau_{\rm step}|^{-4}({}^+\chi^\tau_{\scissors} +^- \chi^\tau_{\scissors}) \leq Br^{-2},$$

the statement of the proposition follows immediately in view of the restriction on $\tau_{\rm step}$ and (77). $\square$

9.5 Estimates for $\mathcal{E}$

In view of the cutoffs, $\psi^\tau_\flat$ and $\psi^\tau_\sharp$ no longer satisfy (2).

Define

$$F^\tau_{\scissors} = \psi \, \Box_g \chi^\tau_{\scissors} + g^{\mu\nu} \partial_\mu(\chi^\tau_{\scissors}) \partial_\nu \psi. \tag{81}$$

Note that $F^\tau_{\scissors}$ is supported in $\mathcal{R}^+(\tau-1,\tau) \cup \mathcal{R}^-(0,1)$.

We may write

$$\Box_g \psi^\tau_\flat = F^\tau_\flat, \tag{82}$$

$$\Box_g \psi^\tau_\sharp = F^\tau_\sharp, \tag{83}$$

where $F^\tau_\flat$ and $F^\tau_\sharp$ are defined from $F^\tau_{\scissors}$ as in Section 9.2.

The right hand sides of (82) and (83) generate error terms in applying (36) with our various currents. We have the following

[11] Of course, one needs to multiply this by a cutoff on the sphere to make it a well defined smooth function.

Proposition 9.4. *Let $\omega_0 \le 1 \le \tau_{\rm step} \le \tau' \le \tau'' \le \tau - \tau_{\rm step}$ and consider $V = \mathbf{X}$, N_e, or T. Then the following holds*

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}^V(\psi^\tau_\flat) \quad \le \quad B\omega_0^{-8}\tau_{\rm step}^{-2}e^{-1} \sup_{0\le\bar\tau\le\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi), \tag{84}$$

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}^V(\psi^\tau_\sharp) \quad \le \quad B\omega_0^{-8}\tau_{\rm step}^{-2}e^{-1} \sup_{0\le\bar\tau\le\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi). \tag{85}$$

Proof. Decompose

$$F^\tau_{\scissors} = {}^1F^\tau_{\scissors} + {}^2F^\tau_{\scissors}$$

where

$${}^1F^\tau_{\scissors} = {}^-\chi^\tau_{\scissors} F^\tau_{\scissors}, \qquad {}^2F^\tau_{\scissors} = {}^+\chi^\tau_{\scissors} F^\tau_{\scissors},$$

and consider ${}^jF^\tau_\flat$, ${}^jF^\tau_\sharp$, defined in Section 9.2, for $j = 1, 2$.

Recall the definitions (35), (38) of $\mathcal{E}^V$ and $\mathcal{E}^{V,w}$. Since $(F^\tau_\flat)_0 = 0$ and $(F^\tau_\sharp)_0 = \int_0^{2\pi} F^\tau_{\scissors}\, d\phi = 0$ in $\mathcal{R}(\tau',\tau'')$, it follows that $\mathcal{E}^V((\psi^\tau_\flat)_0) = \mathcal{E}^V((\psi^\tau_\sharp)_0) = 0$ in $\mathcal{R}(\tau',\tau'')$ and thus equations (84) and (85) are trivially satisfied. By subtraction, we may thus assume in what follows that

$$\psi_0 = \int_0^{2\pi} \psi \, d\phi = 0,$$

and thus

$$r^{-2}\int_0^{2\pi} \psi^2 d\phi \le r^{-2}\int_0^{2\pi} (\partial_\phi\psi)^2\, d\phi \le Be^{-1}\int_0^{2\pi} J^{N_e}_\mu(\psi) n^\mu_{\Sigma^+}\, d\phi \tag{86}$$

and similarly with $n^\mu_{\Sigma^-}$.

Lemma 9.6. *For any $q \ge 0$, $\tau_0 \le \tau - 1$, there exists a B_q such that*

$$|({}^2F^\tau_\flat)_k|(t^+ = \tau_0, \cdot) \le B_q {\omega_0}^{1-q}(\tau - \tau_0)^{-q}k^{1-q}\int_{\tau-1}^{\tau} |({}^2F^\tau_{\scissors})_k(t^+,\cdot)|\, dt^+,$$

$$|({}^2F^\tau_\sharp)_k|(t^+ = \tau_0, \cdot) \le B_q {\omega_0}^{1-q}(\tau - \tau_0)^{-q}k^{1-q}\int_{\tau-1}^{\tau} |({}^2F^\tau_{\scissors})_k(t^+,\cdot)|\, dt^+.$$

Proof. This is standard. $\square$

It follows from the above lemma applied to $q = 6$, the restriction on τ', τ'', and the relation between t^* and t^+ that

$$\begin{aligned}\int_{\tau'}^{\tau''} (1 + (\tau - t^*))^3({}^2F^\tau_\flat)^2_k\, dt^* \quad &\le \quad B\tau_{\rm step}^{-5}{\omega_0}^{-8}\int_{\tau-1}^{\tau} r^{-2}({}^2F^\tau_{\scissors})^2_k\, dt^+ \\ &\le \quad B\tau_{\rm step}^{-5}{\omega_0}^{-8}\int_{\tau-1}^{\tau} r^{-2}(\psi_k^2 + e^{-1}J^{N_e}_\mu(\psi_k)n^\mu_{\Sigma^+})\, dt^+.\end{aligned}$$

We remark that the powers of $\tau_{\rm step}^{-1}$ and r^{-1} can be chosen arbitrarily above, at the expense of the constant B and powers of ${\omega_0}^{-1}$, but this would give no advantage in what follows. Thus,

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')} &(1+(\tau-t^*))^3({}^2F^\tau_\flat)^2 \\
&\le \sum_k B\tau_{\rm step}^{-5}{\omega_0}^{-8}\int_{\mathcal{R}^+(\tau-1,\tau)}(r^{-2}\psi_k^2+r^{-2}e^{-1}\,J^{N_e}_\mu(\psi_k)n^\mu_{\Sigma^+})\\
&\le B\tau_{\rm step}^{-5}{\omega_0}^{-8}\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma^+(\bar\tau)} r^{-2}(\partial_\phi\psi)^2+r^{-2}e^{-1}\,J^{N_e}_\mu(\psi)n^\mu_{\Sigma^+}\\
&\le B\tau_{\rm step}^{-5}e^{-1}{\omega_0}^{-8}\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma^+(\bar\tau)} J^{N_e}_\mu(\psi)n^\mu_{\Sigma^+},
\end{aligned}
$$

where we have used (86). On the other hand, by conservation of energy we have that

$$
\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma^+(\bar\tau)} J^{N_e}_\mu(\psi)n^\mu_{\Sigma^+}\le 2\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)} J^{N_e}_\mu(\psi)n^\mu_{\Sigma},
$$

and thus,

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')}(1+(\tau-t^*))^3({}^2F^\tau_\flat)^2\,dt^* &\le B\tau_{\rm step}^{-5}e^{-1}{\omega_0}^{-8}\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)} J^{N_e}_\mu(\psi)n^\mu_{\Sigma}\\
&\le B\tau_{\rm step}^{-5}e^{-1}{\omega_0}^{-8}\sup_{\tau-1\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e(\psi)\\
&\le B\tau_{\rm step}^{-5}e^{-1}{\omega_0}^{-8}\sup_{0\le\bar\tau\le\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e(\psi). \quad (87)
\end{aligned}
$$

Clearly, an identical bound holds for

$$
\int_{\mathcal{R}(\tau',\tau'')}(1+t^*)^3({}^1F^\tau_\flat)^2\,dt^*.
$$

Let us consider first the cases where $V \neq \mathbf{X}$. For $V = T, N_e$ we have

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}^V(\psi_\flat^\tau) &= \int_{\mathcal{R}(\tau',\tau'')} {}^1F_\flat^\tau V^\nu(\psi_\flat^\tau)_\nu + {}^2F_\flat^\tau V^\nu(\psi_\flat^\tau)_\nu \\
&\leq \int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^3\,({}^1F_\flat^\tau)^2 + \int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^{-3}(V^\nu(\psi_\flat^\tau)_\nu)^2 \\
&\quad + \int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^3({}^2F_\flat^\tau)^2 \\
&\quad + \int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^{-3}(V^\nu(\psi_\flat^\tau)_\nu)^2 \\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}{\omega_0}^{-8} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) + B\int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^{-3}\,\mathbf{q}_e(\psi_\flat^\tau) \\
&\quad + B\int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^{-3}\,\mathbf{q}_e(\psi_\flat^\tau) \\
&\leq B\tau_{\mathrm{step}}^{-5}e^{-1}{\omega_0}^{-8} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\quad + B\tau_{\mathrm{step}}^{-2} \sup_{\tau_{\mathrm{step}}\leq\bar\tau\leq\tau-\tau_{\mathrm{step}}} \int_{\bar\tau-1}^{\bar\tau} \left(\int_{\Sigma(\tilde\tau)} \mathbf{q}_e(\psi_\flat^\tau) \right) d\tilde\tau \\
&\leq (B\tau_{\mathrm{step}}^{-5}e^{-1}{\omega_0}^{-8} + \tau_{\mathrm{step}}^{-2}B(\omega_0^{-1} + \omega_0^{-7}e^{-1}\tau_{\mathrm{step}}^{-2})) \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi)
\end{aligned}
$$

where for the last inequality we have used Proposition 9.3. We argue similarly for $\mathcal{E}^V(\psi_\sharp^\tau)$.

For the case of $\mathcal{E}^V$ where $V = \mathbf{X}$, we have an additional error term

$$
\tilde{\mathcal{E}}^X(\psi_\flat^\tau) = -\frac{1}{4}\left(2f_b' + 4\frac{1-\mu}{r}f_b - \frac{4M(1-\mu)f_b}{r^2}\right)\psi_\flat^\tau F_\flat^\tau.
$$

Recall that $|f_b| \leq B\chi$, and $|f_b'| \leq Br^{-2}\chi$, where χ is a cutoff function such that

$\chi = 0$ in $r^* \leq 0$. Arguing as in the previous bound we obtain

$$\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')} \tilde{\mathcal{E}}^X(\psi_\flat^\tau) &\leq \int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^3\,({}^1F_\flat^\tau)^2 + B\int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^{-3}\chi^2 r^{-2}(\psi_\flat^\tau)^2 \\
&\quad + \int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^3({}^2F_\flat^\tau)^2 \\
&\quad + B\int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^{-3}\chi^2 r^{-2}(\psi_\flat^\tau)^2 \\
&\leq B\tau_{\text{step}}^{-5} e^{-1}\omega_0{}^{-8} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) + \int_{\mathcal{R}(\tau',\tau'')} (1+t^*)^{-3}\,\mathbf{q}_e(\psi_\flat^\tau) \\
&\quad + \int_{\mathcal{R}(\tau',\tau'')} (1+(\tau-t^*))^{-3}\,\mathbf{q}_e(\psi_\flat^\tau) \\
&\leq B\tau_{\text{step}}^{-5} e^{-1}\omega_0{}^{-8} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \\
&\quad + B\tau_{\text{step}}^{-2} \sup_{\tau_{\text{step}}\leq\bar{\tau}\leq\tau-\tau_{\text{step}}} \int_{\bar{\tau}-1}^{\bar{\tau}} \left(\int_{\Sigma(\tilde{\tau})} \mathbf{q}_e(\psi_\flat^\tau)\right) d\tilde{\tau} \\
&\leq (B\tau_{\text{step}}^{-5} e^{-1}\omega_0{}^{-8} + \tau_{\text{step}}^{-2} B(\omega_0^{-1} + \omega_0^{-7} e^{-1}\tau_{\text{step}}^{-2})) \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi).
\end{aligned}$$

In the above, we have used again Proposition 9.3 as well as the inequality

$$r^{-2}\int_0^{2\pi} (\psi_\flat^\tau)^2 d\phi \leq r^{-2}\int_0^{2\pi} (\partial_\phi\psi_\flat^\tau)^2 d\phi \leq \mathbf{q}_e(\psi_\flat^\tau)$$

in the support of χ. The other terms of $\mathcal{E}^{\mathbf{X}}$ can be handled as in the argument for $\mathcal{E}^T$, $\mathcal{E}^{N_e}$. Again, the argument for $\psi_\sharp^\tau$ is identical. $\square$

9.6 Revisiting the relation between $\mathbf{q}_e(\psi_\flat^\tau)$, $\mathbf{q}_e(\psi_\sharp^\tau)$ and $\mathbf{q}_e(\psi)$

With the Proposition of the previous section, we may now refine Proposition 9.3 to a pointwise-in-time bound:

Proposition 9.5. *Let* $\omega_0 \leq 1 \leq \tau_{\text{step}} \leq \tau' \leq \tau - \tau_{\text{step}}$. *Then*

$$\begin{aligned}
\int_{\Sigma(\tau')} \mathbf{q}_e(\psi_\flat^\tau) &\leq B \sup_{\tau'-\tau_{\text{step}}\leq\bar{\tau}\leq\tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \\
&\quad + B\omega_0{}^{-8} e^{-1}\tau_{\text{step}}^{-2} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi),
\end{aligned}$$

$$\begin{aligned}
\int_{\Sigma(\tau')} \mathbf{q}_e(\psi_\sharp^\tau) &\leq B \sup_{\tau'-\tau_{\text{step}}\leq\bar{\tau}\leq\tau'+\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \\
&\quad + B\omega_0{}^{-8} e^{-1}\tau_{\text{step}}^{-2} \sup_{0\leq\bar{\tau}\leq\tau} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi).
\end{aligned}$$

Proof. Once again it is sufficient to establish this for $\psi_\flat^\tau$.

We write the energy identity (36) for the vector field N_e to obtain

$$\int_{\mathcal{H}(\tau_0,\tau')} J_\mu^{N_e}(\psi_\flat^\tau) n_{\mathcal{H}}^\mu + \int_{\Sigma(\tau')} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu + \int_{\mathcal{R}(\tau_0,\tau')} K^{N_e}(\psi_\flat^\tau)$$
$$= \int_{\mathcal{R}(\tau_0,\tau')} \mathcal{E}^{N_e}(\psi_\flat^\tau) + \int_{\Sigma(\tau_0)} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu.$$

By (48), (50), and the nonnegativity of the first term on the left hand side above, we obtain

$$\int_{\Sigma(\tau')} \mathbf{q}_e(\psi_\flat^\tau) \leq eB \int_{\tau_0}^{\tau'} \int_{\Sigma(t^*)} \mathbf{q}_e(\psi_\flat^\tau) dt^* + B \left| \int_{\mathcal{R}(\tau_0,\tau')} \mathcal{E}^{N_e}(\psi_\flat^\tau) \right| + B \int_{\Sigma(\tau_0)} \mathbf{q}_e(\psi_\flat^\tau).$$

We integrate the above inequality with respect to τ_0 between $\tau' - 1$ and τ' and use Propositions 9.3, 9.4 to obtain the desired estimate. □

10 The main estimates

10.1 Estimates for $\psi_\flat^\tau$

Let us assume always

$$\tau_{\text{step}} \leq \tau' \leq \tau'' \leq \tau - \tau_{\text{step}}. \tag{88}$$

Proposition 10.1. *For $\psi_\flat^\tau$ we have*

$$\int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}_e^\star(\psi_\flat^\tau) \right) d\bar\tau \quad \leq \quad B \int_{\mathcal{R}(\tau'.\tau'')} \left(K^{\mathbf{X}} + K^{N_e} \right)(\psi_\flat^\tau)$$
$$+ B \sup_{0 \leq \bar\tau \leq \tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. Recall that $\int_0^{2\pi} \psi_\flat^\tau d\phi = 0$.

In the region $r \leq r_Y^-$, we have immediately from (65) that

$$\int_0^{2\pi} \mathbf{q}_e^\star(\psi_\flat^\tau)\, d\phi \leq B \int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\psi_\flat^\tau) d\phi.$$

Similarly, in the region $r \geq R$, we have from (63) that

$$\int_0^{2\pi} \mathbf{q}_e^\star(\psi_\flat^\tau) d\phi \leq B \int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\psi_\flat^\tau)\, d\phi.$$

For $r_Y^- \leq r \leq R$, we have from (64) that

$$\int_0^{2\pi} \mathbf{q}_e^\star(\psi_\flat^\tau)\, d\phi \quad \leq \quad B \int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\psi_\flat^\tau)\, d\phi$$
$$- \int_0^{2\pi} \left(b|\nabla\!\!\!/ \psi_\flat^\tau|^2 - B(\partial_t \psi_\flat^\tau)^2 \right)\, d\phi$$

Note also that

$$\int_0^{2\pi} |\nabla\!\!\!/ \psi_\flat^\tau|^2 \, d\phi \geq b \int_0^{2\pi} |\partial_\phi \psi_\flat^\tau|^2 \, d\phi$$

for constant (r, θ, t) curves in the region $r_Y^- \leq r \leq R$. We have thus

$$\begin{aligned}
\int_0^{2\pi} \mathbf{q}_e^\bigstar(\psi_\flat^\tau) \, d\phi \quad &\leq \quad B \int_0^{2\pi} (K^{\mathbf{X}} + K^{N_e})(\psi_\flat^\tau) \, d\phi \\
&\quad - \int_0^{2\pi} \left(b(\partial_\phi \psi_\flat^\tau)^2 - B(\partial_t \psi_\flat^\tau)^2 \right) \, d\phi
\end{aligned}$$

The Proposition follows now from Proposition 9.1 for ω_0 chosen appropriately, in view also of our remarks on the measure of integration. □

In what follows we shall consider ω_0 to have been chosen and absorb such factors into the constants B.

Proposition 10.2. *For $\psi_\flat^\tau$, we have*

$$\begin{aligned}
\int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}_e^\bigstar(\psi_\flat^\tau) \right) d\bar\tau \quad &\leq \quad B \left(\int_{\Sigma(\tau')} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu + \int_{\Sigma(\tau'')} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu \right. \\
&\qquad \left. + \int_{\mathcal{H}(\tau', \tau'')} J_\mu^{N_e}(\psi_\flat^\tau) n_{\mathcal{H}}^\mu \right) \\
&\qquad + B \sup_{0 \leq \bar\tau \leq \tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).
\end{aligned}$$

Proof. To prove Proposition 10.2 from Proposition 10.1, note that from (36) applied to the current $J^{\mathbf{X}} + J^{N_e}$ we have

$$\begin{aligned}
\int_{\mathcal{R}(\tau', \tau'')} (K^{\mathbf{X}} + K^{N_e})(\psi_\flat^\tau) \quad &\leq \quad \left| \int_{\Sigma(\tau')} (J_\mu^{\mathbf{X}} + J_\mu^{N_e})(\psi_\flat^\tau) n_\Sigma^\mu \right| \\
&\qquad + \left| \int_{\Sigma(\tau'')} (J_\mu^{\mathbf{X}} + J_\mu^{N_e})(\psi_\flat^\tau) n_\Sigma^\mu \right. \\
&\qquad \left. + \int_{\mathcal{H}(\tau', \tau'')} (J_\mu^{\mathbf{X}} + J_\mu^{N_e})(\psi_\flat^\tau) n_{\mathcal{H}}^\mu \right| \\
&\qquad + \left| \int_{\mathcal{R}(\tau', \tau'')} \mathcal{E}^{\mathbf{X} + N_e}(\psi_\flat^\tau) \right| \\
&\leq \quad B \left(\int_{\Sigma(\tau')} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu + \int_{\Sigma(\tau'')} J_\mu^{N_e}(\psi_\flat^\tau) n_\Sigma^\mu \right. \\
&\qquad \left. + \int_{\mathcal{H}(\tau', \tau'')} J_\mu^{N_e}(\psi_\flat^\tau) n_{\mathcal{H}}^\mu \right) \\
&\qquad + B \tau_{\mathrm{step}}^{-2} e^{-1} \sup_{0 \leq \bar\tau \leq \tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).
\end{aligned}$$

Above we have used (66) and Proposition 9.4. It will be important for later that $eB \ll 1$. The Proposition now follows immediately. □

Proposition 10.3.

$$\int_{\mathcal{H}(\tau',\tau'')} J^T_\mu(\psi^\tau_\flat) n^\mu_{\mathcal{H}} + \int_{\Sigma(\tau'')} J^T_\mu(\psi^\tau_\flat) n^\mu_\Sigma$$
$$\leq B \int_{\Sigma(\tau')} \mathbf{q}_e(\psi_\flat) + B\tau_{\rm step}^{-2} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. This follows from the divergence identity (36) for the current $J^T(\psi^\tau_\flat)$ and the fact that $K^T = 0$ and the inequality

$$\int_{\mathcal{R}(\tau',\tau'')} \mathcal{E}^T(\psi^\tau_\flat) \leq B\tau_{\rm step}^{-2} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi)$$

of Proposition 9.4. □

Proposition 10.4.

$$\begin{aligned}\int_{\mathcal{H}(\tau',\tau'')} J^{N_e}_\mu(\psi^\tau_\flat) n^\mu_{\mathcal{H}} + \int_{\Sigma(\tau'')} J^{N_e}_\mu(\psi^\tau_\flat) n^\mu_\Sigma \quad \leq \quad & \int_{\Sigma(\tau')} J^{N_e}_\mu(\psi^\tau_\flat) n^\mu_\Sigma \\ & + Be \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}^\star_e(\psi^\tau_\flat) \right) d\bar\tau \\ & + B\tau_{\rm step}^{-2} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).\end{aligned}$$

Proof. This follows just from the divergence identity (36) for J^{N_e} together with the bounds (48) and (84). □

Proposition 10.5.

$$\int_{\Sigma(\tau'')} \mathbf{q}_e(\psi^\tau_\flat) + \int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}^\star_e(\psi^\tau_\flat) \right) d\bar\tau$$
$$\leq B \int_{\Sigma(\tau')} \mathbf{q}_e(\psi^\tau_\flat) + B \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. This follows immediately from Propositions 10.2 and 10.4 in view of (50) and the fact that for e small we have $Be \ll 1$. □

Proposition 10.6.

$$\int_{\Sigma(\tau'')} J^T_\mu(\psi^\tau_\flat) n^\mu_\Sigma$$
$$\leq B \int_{\Sigma(\tau')} \mathbf{q}_e(\psi^\tau_\flat) + (B\tau_{\rm step}^{-2} e^{-1} + B\epsilon_{\rm close} e^{-1}) \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. This follows from Propositions 10.3, 10.4 and 10.5 together with the one-sided bound

$$-\int_{\mathcal{H}(\tau',\tau'')} J^T_\mu(\psi^\tau_\flat)n^\mu_{\mathcal{H}} \leq B\epsilon_{\text{close}}e^{-1}\int_{\mathcal{H}(\tau',\tau'')} J^{N_e}_\mu(\psi^\tau_\flat)n^\mu_{\mathcal{H}}.$$

□

10.2 Estimates for $\psi^\tau_\sharp$

We assume always (88).

Proposition 10.7. *For* $\psi^\tau_\sharp$,

$$\frac{1}{2}\int_{\mathcal{H}(\tau',\tau'')} \left|J^T_\mu(\psi^\tau_\sharp)n^\mu_{\mathcal{H}}\right| + \int_{\Sigma(\tau'')} J^T_\mu(\psi^\tau_\sharp)n^\mu \leq \int_{\Sigma(\tau')} J^T_\mu(\psi^\tau_\sharp)n^\mu$$
$$+B\tau_{\text{step}}^{-2}\epsilon^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\Sigma(\bar\tau)}\mathbf{q}_e(\psi) + B\epsilon_{\text{close}}e^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)} J^{N_e}_\mu(\psi)n^\mu_{\mathcal{H}}.$$

Proof. From (36) applied to $\psi^\tau_\sharp$ with $V = T$ we have

$$\begin{aligned}&\int_{\Sigma(\tau'')} J^T_\mu(\psi^\tau_\sharp)n^\mu_\Sigma\\ =&\int_{\Sigma(\tau')} J^T_\mu(\psi^\tau_\sharp)n^\mu_\Sigma - \int_{\mathcal{H}(\tau',\tau'')} J^T_\mu(\psi^\tau_\sharp)n^\mu_{\mathcal{H}}\\ &+\int_{\mathcal{R}(\tau',\tau'')}\mathcal{E}^T(\psi^\tau_\sharp).\end{aligned}$$

On the other hand, by (22), we have the one-sided bound

$$\begin{aligned}-\int_{\mathcal{H}(\tau',\tau'')} J^T_\mu(\psi^\tau_\sharp)n^\mu_{\mathcal{H}} \quad\leq\quad & B\epsilon_{\text{close}}\int_{\mathcal{H}(\tau',\tau'')}\partial_t\psi^\tau_\sharp\partial_\phi\psi^\tau_\sharp\\ &-b\int_{\mathcal{H}(\tau',\tau'')}(\partial_t\psi^\tau_\sharp)^2\\ \leq\quad & B\epsilon_{\text{close}}\int_{\mathcal{H}(\tau',\tau'')}(\partial_\phi\psi^\tau_\sharp)^2\\ &-b\int_{\mathcal{H}(\tau',\tau'')}(\partial_t\psi^\tau_\sharp)^2\end{aligned}$$

and thus by Proposition 9.2 we have

$$-\int_{\mathcal{H}(\tau',\tau'')} J^T_\mu(\psi^\tau_\sharp)n^\mu_{\mathcal{H}} \leq B\epsilon_{\text{close}}e^{-1}\sup_{0\leq\bar\tau\leq\tau}\int_{\mathcal{H}(\bar\tau,\bar\tau+1)} J^{N_e}_\mu(\psi)n^\mu_{\mathcal{H}}.$$

The desired result now follows from Proposition 9.4. □

Proposition 10.8.

$$\int_{\Sigma(\tau'')} J^{N_e}_\mu(\psi^\tau_\sharp) n^\mu_\Sigma + \int_{\mathcal{R}(\tau',\tau'')} K^{N_e}(\psi^\tau_\sharp)$$
$$\leq \int_{\Sigma(\tau')} J^{N_e}_\mu(\psi^\tau_\sharp) n^\mu_\Sigma + B\tau_{\mathrm{step}}^{-2} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi).$$

Proof. This is the energy identity (36) for N_e in view of the nonnegativity of the flux on the horizon and the estimate (85). □

Proposition 10.9.

$$b\int_{\Sigma(\tau'')} \mathbf{q}_e(\psi^\tau_\sharp) + b\int_{\tau'}^{\tau''} \left(\int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi^\tau_\sharp)\right) d\bar\tau$$
$$\leq (\tau''-\tau') \int_{\Sigma(\tau_{\mathrm{step}})} J^T_\mu(\psi^\tau_\sharp) n^\mu_\Sigma + B\int_{\Sigma(\tau')} \mathbf{q}_e(\psi^\tau_\sharp)$$
$$+ (\tau''-\tau'+1)\left(B\tau_{\mathrm{step}}^{-2} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) + B\epsilon_{\mathrm{close}} e^{-1} \sup_{0\leq\bar\tau\leq\tau} \int_{\mathcal{H}(\bar\tau,\bar\tau+1)} J^{N_e}_\mu(\psi) n^\mu_{\mathcal{H}}\right).$$

Proof. The proof follows from Propositions 10.7 (applied with $\tau' = \tau_{\mathrm{step}}$ and $\tau'' = \bar\tau$), Proposition 10.8 applied to the given τ' and τ'', (49) and (50). □

11 The bootstrap

Let C be given, and consider the set $\mathcal{T} \subset [0,\infty)$ of all τ such that for $0 \leq \bar\tau \leq \tau$, we have

$$\int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \leq C\int_{\Sigma(0)} \mathbf{q}_e(\psi). \tag{89}$$

Theorem 5.1 would follow from

Proposition 11.1. *For suitable choice of C, then $\mathcal{T} = [0,\infty)$, i.e. (89) holds for all $\tau \geq 0$.*

For this it suffices to show that $\mathcal{T}$ is non-empty, open and closed. The non-emptyness is clear for sufficiently large C. It thus suffices to show that C can be chosen such that for all $\tau \in \mathcal{T}$, then

$$\int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \leq \frac{C}{2}\int_{\Sigma(0)} \mathbf{q}_e(\psi) \tag{90}$$

for $0 \leq \bar\tau \leq \tau$.

11.1 Evolution for time $\tau_{\rm step}$

We will need the following proposition

Proposition 11.2. *Let $\tau_{\rm step}$ be given. For small enough e depending on $\tau_{\rm step}$, $\epsilon_{\rm close} \ll e$, it follows that for all τ_0 and $\bar{\tau} \in [\tau_0, \tau_0 + \tau_{\rm step}]$,*

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \leq 2 \int_{\Sigma(\tau_0)} \mathbf{q}_e(\psi). \tag{91}$$

Proof. We write the energy identity (36) for the vector field N_e to obtain

$$\int_{\mathcal{H}(\tau_0,\bar{\tau})} J^{N_e}_\mu(\psi) n^\mu_{\mathcal{H}} + \int_{\Sigma(\bar{\tau})} J^{N_e}_\mu(\psi) n^\mu_\Sigma + \int_{\mathcal{R}(\tau_0,\bar{\tau})} K^{N_e}(\psi)$$
$$= \int_{\Sigma(\tau_0)} J^{N_e}_\mu(\psi) n^\mu_\Sigma.$$

By (48), (50), and the nonnegativity of the first term on the left hand side above, we obtain

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \leq eB \int_{\tau_0}^{\bar{\tau}} \int_{\Sigma(\hat{\tau})} \mathbf{q}_e(\psi) d\hat{\tau} + \int_{\Sigma(\tau_0)} \mathbf{q}_e(\psi)$$

and thus

$$\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \leq \exp(eB(\bar{\tau} - \tau_0)) \int_{\Sigma(\tau_0)} \mathbf{q}_e(\psi).$$

The result follows thus if e is chosen so that

$$\exp(eB\tau_{\rm step}) \leq 2.$$

□

11.2 Estimate for the local horizon flux of $J^{N_e}_\mu(\psi)$

A corollary of the proof of the previous Proposition is the following

Proposition 11.3.

$$\int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} J^{N_e}_\mu(\psi) n^\mu \leq B \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi).$$

Of course, if we choose e to be sufficiently small as in the previous Proposition, we may replace B with 2.

11.3 Bounds for $\psi_\flat^\tau$

From Proposition 10.5 applied for $\tau' = n\tau_{\rm step}$, $\tau'' = (n+1)\tau_{\rm step}$, $n = 1, 2, \ldots, n_f$ where n_f is the largest integer such that $(n_f+1)\tau_{\rm step} \le \tau - \tau_{\rm step}$, Proposition 9.5 and the bootstrap assumption (89), it follows that in each interval $[n\tau_{\rm step}, (n+1)\tau_{\rm step}]$, we can find a τ_n such that

$$\begin{aligned}
\int_{\Sigma(\tau_n)} \mathbf{q}_e^\star(\psi_\flat^\tau) &\le \frac{B}{\tau_{\rm step}} \sup_{0\le\bar\tau\le\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) + \frac{B}{\tau_{\rm step}} \int_{\Sigma(n\tau_{\rm step})} \mathbf{q}_e(\psi_\flat^\tau) \\
&\le B\tau_{\rm step}^{-1} C \int_{\Sigma(0)} \mathbf{q}_e(\psi)
\end{aligned}$$

for appropriate choice of $\tau_{\rm step}$. On the other hand, by Proposition 10.6 applied with $\tau' = \tau_{\rm step}$, $\tau'' = \tau_n$, Proposition 9.5, (91) applied to $\tau_0 = 0$ and again to $\tau_0 = \tau_{\rm step}$, and the bootstrap assumption (89), we have

$$\begin{aligned}
\int_{\Sigma(\tau_n)} J_\mu^T(\psi_\flat^\tau) n_\Sigma^\mu &\le B \int_{\Sigma(\tau_{\rm step})} \mathbf{q}_e(\psi_\flat^\tau) \\
&\quad + (B\tau_{\rm step}^{-2} e^{-1} + B\epsilon_{\rm close} e^{-1}) \sup_{0\le\bar\tau\le\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\le B \sup_{0\le\bar\tau\le 2\tau_{\rm step}} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\quad + (B\tau_{\rm step}^{-2} e^{-1} + B\epsilon_{\rm close} e^{-1}) \sup_{0\le\bar\tau\le\tau} \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi) \\
&\le (B + B\tau_{\rm step}^{-2} e^{-1} C + B\epsilon_{\rm close} e^{-1} C) \\
&\quad \cdot \int_{\Sigma(0)} \mathbf{q}_e(\psi).
\end{aligned} \tag{92}$$

It follows that

$$\begin{aligned}
\int_{\Sigma(\tau_n)} \mathbf{q}_e(\psi_\flat^\tau) &\le B \int_{\Sigma(\tau_n)} \mathbf{q}_e^\star(\psi_\flat^\tau) + \int_{\Sigma(\tau_n)} J_\mu^T(\psi_\flat^\tau) n_\Sigma^\mu \\
&\le (B + B\epsilon_{\rm close} e^{-1} C + B\tau_{\rm step}^{-2} e^{-1} C + B\tau_{\rm step}^{-1} C) \\
&\quad \cdot \int_{\Sigma(0)} \mathbf{q}_e(\psi).
\end{aligned} \tag{93}$$

11.4 Bounds for $\psi_\sharp^\tau$

Since

$$\int_{\Sigma(\bar\tau)} J_\mu^T(\psi_\sharp^\tau) \le B \int_{\Sigma(\bar\tau)} \mathbf{q}_e(\psi_\sharp^\tau),$$

it follows from Proposition 10.9 applied to $\tau' = n\tau_{\rm step}$, $\tau'' = (n+1)\tau_{\rm step}$, $n = 1, 2, \ldots, n_f$, Proposition 11.3 and (91) applied (twice) with $\tau_0 = 0$ and

$\tau_0 = \tau_{\text{step}}$, and Proposition 9.5 that in each interval $[n\tau_{\text{step}}, (n+1)\tau_{\text{step}}]$, we can find an τ_n such that

$$
\begin{aligned}
b\int_{\Sigma(\tau_n)} \mathbf{q}_e(\psi_\sharp^\tau) &\leq \int_{\Sigma(\tau_{\text{step}})} J_\mu^T(\psi_\sharp^\tau) n_\Sigma^\mu + \tau_{\text{step}}^{-1} B \int_{\Sigma(n\tau_{\text{step}})} \mathbf{q}_e(\psi_\sharp^\tau) \\
&\quad + B\tau_{\text{step}}^{-2} e^{-1} C \int_{\Sigma(0)} \mathbf{q}_e(\psi) \\
&\quad + B\epsilon_{\text{close}} \sup_{0\leq\bar{\tau}\leq\tau-1} \int_{\mathcal{H}(\bar{\tau},\bar{\tau}+1)} J_\mu^{N_e}(\psi) n_{\mathcal{H}}^\mu \\
&\leq B\int_{\Sigma(\tau_{\text{step}})} \mathbf{q}_e(\psi_\sharp^\tau) + \tau_{\text{step}}^{-1} B \int_{\Sigma(n\tau_{\text{step}})} \mathbf{q}_e(\psi_\sharp^\tau) \\
&\quad + (B\tau_{\text{step}}^{-2} e^{-1} C + B\epsilon_{\text{close}} C) \int_{\Sigma(0)} \mathbf{q}_e(\psi) \\
&\leq B \sup_{0\leq\bar{\tau}\leq 2\tau_{\text{step}}} \int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) \\
&\quad + (\tau_{\text{step}}^{-1} BC + \tau_{\text{step}}^{-2} e^{-1} BC + B\epsilon_{\text{close}} C) \int_{\Sigma(0)} \mathbf{q}_e(\psi) \\
&\leq (B + B\tau_{\text{step}}^{-1} C + B\tau_{\text{step}}^{-2} e^{-1} C + B\epsilon_{\text{close}} C) \\
&\quad \cdot \int_{\Sigma(0)} \mathbf{q}_e(\psi). \qquad (94)
\end{aligned}
$$

11.5 Bounds for ψ

Choosing C sufficiently large, τ_{step} sufficiently large, e sufficiently small so that Proposition 11.2 holds, and $\epsilon_{\text{close}} \ll e$ sufficiently small, from (78), (93), and (94) it follows that

$$\int_{\Sigma(\tau_n)} \mathbf{q}_e(\psi) \leq \frac{C}{8} \int_{\Sigma(0)} \mathbf{q}_e(\psi).$$

From Proposition 11.2 it follows that for $\bar{\tau} \in [\tau_n, \tau_{n+1}]$

$$
\begin{aligned}
\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) &\leq 2\int_{\Sigma(\tau_n)} \mathbf{q}_e(\psi) \\
&\leq \frac{C}{4} \int_{\Sigma(0)} \mathbf{q}_e(\psi).
\end{aligned}
$$

For $\bar{\tau} \in [0, \tau_1]$ we apply twice Proposition 11.2

$$
\begin{aligned}
\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) &\leq 2\int_{\Sigma(\tau_{\text{step}})} \mathbf{q}_e(\psi) \\
&\leq 4\int_{\Sigma(0)} \mathbf{q}_e(\psi) \\
&\leq \frac{C}{2} \int_{\Sigma(0)} \mathbf{q}_e(\psi),
\end{aligned}
$$

as long as we assume $C \geq 8$. Similarly, for $\bar{\tau} \in [\tau_{n_f}, \tau]$, we apply Proposition 11.2 twice to obtain

$$\begin{aligned}\int_{\Sigma(\bar{\tau})} \mathbf{q}_e(\psi) &\leq 2\int_{\Sigma(\tau-\tau_{\text{step}})} \mathbf{q}_e(\psi) \\ &\leq 4\int_{\Sigma(\tau_{n_f})} \mathbf{q}_e(\psi) \\ &\leq \frac{C}{2}\int_{\Sigma(0)} \mathbf{q}_e(\psi).\end{aligned}$$

We have shown (90).

12 Estimate for the total horizon and null-infinity flux of $J^T_\mu(\psi)$

For the rest of this paper, all small quantities can be considered fixed. We will use in the following C as a general constant depending on the constant of Proposition 11.1.

Proposition 12.1. *For all $\tau \geq 0$ we have*

$$\int_{\mathcal{H}(0,\tau)} \left|J^T_\mu(\psi)n^\mu_{\mathcal{H}}\right| \leq C\int_{\Sigma(0)} \mathbf{q}_e(\psi) \tag{95}$$

Proof. For (95), in view of the relations

$$J^T_\mu(\psi) \leq B\left(\left|J^T_\mu(\psi^\tau_\sharp)\right| + \left|J^T_\mu(\psi^\tau_\flat)\right|\right)$$

valid in $\mathcal{R}(1, \tau-1)$, and

$$\left|J^T_\mu(\Psi)n^\mu_{\mathcal{H}}\right| \leq C\, J^{N_e}_\mu(\Psi)n^\mu_{\mathcal{H}} \tag{96}$$

on $\mathcal{H}^+$, it follows from Proposition 11.3 and Proposition 11.1, that it suffices to show

$$\int_{\mathcal{H}(\tau_{\text{step}},\tau-\tau_{\text{step}})} \left|J^T_\mu(\psi^\tau_\flat)n^\mu\right| \leq C\int_{\Sigma(0)} \mathbf{q}_e(\psi), \tag{97}$$

$$\int_{\mathcal{H}(\tau_{\text{step}},\tau-\tau_{\text{step}})} \left|J^T_\mu(\psi^\tau_\sharp)n^\mu\right| \leq C\int_{\Sigma(0)} \mathbf{q}_e(\psi). \tag{98}$$

Inequality (97) follows immediately from (96) applied to $\psi^\tau_\flat$, and Propositions 10.4, 10.5, 9.5 and 11.1.

For (98), in view of the bound

$$\int_{\Sigma(\tau')} J^T_\mu(\psi^\tau_\sharp)n^\mu_\Sigma \leq C\int_{\Sigma(\tau')} \mathbf{q}_e(\psi^\tau_\sharp)$$

we need only apply Propositions 9.5, 10.7, 11.3, and 11.1. □

Proposition 12.2.

$$\int_{\mathcal{I}^+} J^T_\mu(\psi)n^\mu \leq C \int_{\Sigma(0)} \mathbf{q}_e(\psi). \tag{99}$$

Proof. This follows now by the previous and the statement $K^T = 0$. We omit the details concerning the definition of the left hand side of the inequality. $\square$

The propositions of this section prove in particular (5) and (6). The complete statement of Theorem 1.1 is thus proven.

13 Higher order energies and pointwise bounds

A deficiency of previous understanding of boundedness, even in the Schwarzschild case, is that it relied on commuting the equation with a full basis of angular momentum operators Ω_i, $i = 1 \ldots 3$. In view of the loss of symmetry when passing from Schwarzschild to Kerr, this approach is no longer tenable. A much more robust approach to boundedness is via commutation with n_Σ, or equivalently, the vector field $\hat{Y}$ to be discussed below. It turns out that the dangerous extra terms arising have a good sign. This can be viewed of as yet another manifestation of the redshift effect.

In Section 13.1 below, we will first derive L^2 estimates for higher order energies. These will rely on certain elliptic estimates derived in Section 13.2. Pointwise estimates will then follow in Section 13.3 from standard Sobolev inequalities.

13.1 Higher order energies

Let us consider now the quantity

$$\mathbf{q}^j(\Psi) \doteq \sum_{i=0}^{j} J^{n_\Sigma}_\mu(n^i_\Sigma(\Psi))n^\mu_\Sigma$$

where $n^j\Psi$ denotes $n(n(n \ldots \psi))$ where j n's appear. Under our smoothness assumptions, coupled with our assumptions about the support, we have that

$$\int_{\Sigma(\tau)} \mathbf{q}^j(\psi) < \infty.$$

We have the following

Theorem 13.1. *For all $j \geq 0$, there exist constants C_j depending only on j and M such that under the assumptions of Theorem 5.1, then for all $\tau \geq 0$,*

$$\int_{\Sigma(\tau)} \mathbf{q}^j(\psi) \leq C_j \int_{\Sigma(0)} \mathbf{q}^j(\psi).$$

Proof. We shall give the proof only for the case $j = 1$, as this will be sufficient for deriving pointwise bounds for ψ.

Commute (2) with T. One obtains from (31) that for $\tau \geq 0$

$$\int_{\Sigma(\tau)} \mathbf{q}(T\psi) \leq C \int_{\Sigma(0)} \mathbf{q}(T\psi). \tag{100}$$

Note that from (31) we have for $\tau'' > \tau' \geq 0$,

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(\psi) \leq C(\tau'' - \tau') \int_{\Sigma(0)} \mathbf{q}(\psi), \tag{101}$$

and from (100),

$$\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi) \leq C(\tau'' - \tau') \int_{\Sigma(0)} \mathbf{q}(T\psi). \tag{102}$$

Now commute (2) with the vector field

$$\hat{Y} \doteq \frac{1}{1-\mu}\partial_u$$

where ∂_u refers to the coordinate system described in Section 8.2. We obtain

Lemma 13.1. *Let ψ satisfy (2). Then we may write*

$$\Box_g(\hat{Y}\psi) = \frac{2}{r}\hat{Y}(\hat{Y}(\psi)) - \frac{4}{r}(\hat{Y}(T\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{Y}, P_2]\psi$$

where P_1 is the first order operator $P_1 \doteq \frac{2}{r^2}(T\psi - \hat{Y}\psi)$, and P_2 is the second order operator $P_2 = \Box_{g_M} - \Box_g$.

Now apply the basic identity (36) to $\hat{Y}$ with Y. We have that for $r \leq r_Y^-$,

$$K^Y(\hat{Y}\psi) \geq b\,\mathbf{q}(\hat{Y}\psi)$$

while for $r \geq r_Y^-$,

$$K^Y(\hat{Y}\psi) \leq B\,\mathbf{q}(\hat{Y}\psi).$$

On the other hand,

$$\begin{aligned} \mathcal{E}^Y(\hat{Y}\psi) &= -\left(\frac{2}{r}\hat{Y}(\hat{Y}(\psi)) - \frac{4}{r}(\hat{Y}(T\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{Y}, P_2]\psi\right) Y(\hat{Y}(\psi)) \\ &\leq -\left(\frac{2}{r}(\hat{Y} - Y)\hat{Y}\psi - \frac{4}{r}(\hat{Y}(T\psi)) + P_1\psi - \frac{2}{r}P_2\psi + [\hat{Y}, P_2]\psi\right) Y(\hat{Y}(\psi)). \end{aligned}$$

The following lemmas are immediate:

Lemma 13.2.

$$\int_{\mathcal{R}(\tau',\tau'')} \frac{16}{r^2}(Y(T\psi))^2 \leq B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi),$$

$$\int_{\mathcal{R}(\tau',\tau'')} (P_1\psi)^2 \leq B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(\psi).$$

Lemma 13.3.

$$\int_{\mathcal{R}(\tau',\tau'')} 4\frac{(P_2\psi)^2}{r^2} + ([P_2,\hat{Y}]\psi)^2 \leq B\epsilon_{\mathrm{close}}^2 \int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}^1(\psi).$$

Lemma 13.4. *Given $r_{\hat{Y}} > 2M$, we may choose a $\delta_{\hat{Y}}$ (with $\delta_{\hat{Y}} \to 0$ as $r_{\hat{Y}} \to 2M$) such that*

$$\int_{\mathcal{R}(\tau',\tau'')} \frac{4}{r^2}((\hat{Y}-Y)\hat{Y}\psi)^2 \leq B\delta_{\hat{Y}} \int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) + B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_{\hat{Y}}\}} \mathbf{q}^1(\psi).$$

For convenience, let us require in what follows that $\delta_{\hat{Y}} \leq \delta_Y^-$.

It follows from (36), the above lemmas and Cauchy-Schwarz (applied with a small parameter λ) that

$$\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \quad \leq \quad & B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \\
& + B\lambda^{-1}\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi) + \mathbf{q}(\psi) \\
& + B\lambda^{-1}\epsilon_{\mathrm{close}}^2 \int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}^1(\psi) \\
& + B\lambda^{-1}\delta_{\hat{Y}} \int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& + B\lambda^{-1}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& + B\lambda \int_{\mathcal{R}(\tau',\tau'')} (Y(\hat{Y}\psi))^2 \\
& + B\int_{\Sigma(\tau')} \mathbf{q}(\hat{Y}\psi).
\end{aligned}$$

Since $(Y(\hat{Y}\psi))^2 \leq B\,\mathbf{q}(\hat{Y}\psi)$, it follows that λ can be chosen so that

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \quad \leq \quad & B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \\
& + B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi) \\
& + B\epsilon_{\mathrm{close}}^2\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}^1(\psi) \\
& + B\delta_{\hat{Y}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& + B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& +B\int_{\Sigma(\tau')} \mathbf{q}(\hat{Y}\psi) \\
\leq \quad & B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \\
& + B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi) \\
& + B\epsilon_{\mathrm{close}}^2\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_Y^-\}} \mathbf{q}^1(\psi) \\
& + B\delta_{\hat{Y}}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& + B\epsilon_{\mathrm{close}}^2\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_Y^-\}} \mathbf{q}^1(\psi) \\
& + B\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_{\hat{Y}}\}} \mathbf{q}^1(\psi) \\
& +B\int_{\Sigma(\tau')} \mathbf{q}(\hat{Y}\psi).
\end{aligned}
$$

From the Propositions of Section 13.2 below, we obtain

$$
\begin{aligned}
\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_Y^-\}} \mathbf{q}(\hat{Y}\psi) \quad \leq \quad & B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi) \\
& + B(\epsilon_{\mathrm{close}}^2+\delta_{\hat{Y}})\int_{\mathcal{R}(\tau',\tau'')\cap\{r\leq r_Y^-\}} \mathbf{q}(\hat{Y}\psi)+\mathbf{q}(T\psi)+\mathbf{q}(\psi) \\
& + B(r_{\hat{Y}})\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi) \\
& +B\int_{\Sigma(\tau')} \mathbf{q}(\hat{Y}\psi),
\end{aligned}
$$

and thus, for small enough $\epsilon_{\mathrm{close}}$, and choosing $r_{\hat{Y}}$ close enough to $2M$ (and thus small enough $\delta_{\hat{Y}}$), we obtain

$$\begin{aligned}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_Y^-\}} \mathbf{q}(\hat{Y}\psi) &\le B\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi)\\ &\quad +B\int_{\Sigma(\tau')}\mathbf{q}(\hat{Y}\psi).\end{aligned}$$

(The choice of $r_{\hat{Y}}$ having been made, we have written above $B(r_{\hat{Y}})$ as B following our convention.) From (101) and (102), it now follows that

$$\begin{aligned}\int_{\mathcal{R}(\tau',\tau'')\cap\{r\le r_Y^-\}} \mathbf{q}(\hat{Y}\psi) &\le B(\tau''-\tau')\int_{\Sigma(0)} \mathbf{q}(T\psi)+\mathbf{q}(\psi)\\ &\quad +B\int_{\Sigma(\tau')}\mathbf{q}(\hat{Y}\psi).\end{aligned}$$

It follows immediately that there exists a sequence $0=\tau_0<\tau_i<\tau_{i+1}$ such that

$$|\tau_i-\tau_j|\le B, \qquad \tau_i\to\infty \tag{103}$$

with

$$\begin{aligned}\int_{\Sigma(\tau_i)\cap\{r\le r_Y^-\}} \mathbf{q}(\hat{Y}\psi) &\le B\int_{\Sigma(0)} \mathbf{q}(T\psi)+\mathbf{q}(\psi)\\ &\quad +B\int_{\Sigma(0)}\mathbf{q}(\hat{Y}\psi).\end{aligned}$$

From (100), we have on the other hand

$$\int_{\Sigma(\tau_i)}\mathbf{q}(T\psi)\le B\int_{\Sigma(0)}\mathbf{q}(T\psi)+\mathbf{q}(\psi),$$

and from (31)

$$\int_{\Sigma(\tau_i)}\mathbf{q}(\psi)\le B\int_{\Sigma(0)}\mathbf{q}(\psi).$$

From Proposition 13.1 it follows that

$$\int_{\Sigma(\tau_i)\cap\{r\le r_Y^-\}}\mathbf{q}^1(\psi)\le B\int_{\Sigma(0)}\mathbf{q}(\hat{Y}\psi)+\mathbf{q}(T\psi)+\mathbf{q}(\psi),$$

while from Proposition 13.2, it follows that

$$\int_{\Sigma(\tau_i)\cap\{r\ge r_Y^-\}}\mathbf{q}^1(\psi)\le B\int_{\Sigma(0)}\mathbf{q}(T\psi)+\mathbf{q}(\psi).$$

Thus in fact,

$$\int_{\Sigma(\tau_i)}\mathbf{q}^1(\psi)\le B\int_{\Sigma(0)}\mathbf{q}(\hat{Y}\psi)+\mathbf{q}(T\psi)+\mathbf{q}(\psi).$$

In view of (103), we obtain now easily

$$\begin{aligned}\int_{\Sigma(\tau)} \mathbf{q}^1(\psi) &\leq B\int_{\Sigma(0)} \mathbf{q}(\hat{Y}\psi) + \mathbf{q}(T\psi) + \mathbf{q}(\psi)\\ &\leq B\int_{\Sigma(0)} \mathbf{q}^1(\psi).\end{aligned}$$

$\square$

13.2 Elliptic estimates

We have the following elliptic estimate on spheres:

Proposition 13.1. *Let S_r denote a set of constant r in a t^*, r, x^A, x^B coordinate system. For ψ a solution of $\Box_g\psi = 0$, we have*

$$\int_{S_r} \mathbf{q}^1(\psi) \leq B\int_{S_r} \mathbf{q}(T\psi) + \mathbf{q}(\hat{Y}\psi) + \mathbf{q}(\psi).$$

Proof. Note first that

$$\begin{aligned}\mathbf{q}^1(\psi) &\leq B\Big(|\nabla\!\!\!/^2\psi|^2 + |\nabla\!\!\!/(T\psi)|^2 + |\nabla\!\!\!/(\hat{Y}\psi)|^2 + |TT\psi|^2\\ &\quad + |\hat{Y}\hat{Y}\psi|^2 + |T\hat{Y}\psi|^2 + \mathbf{q}(\psi)\Big)\\ &\leq B\left(|\nabla\!\!\!/^2\psi|^2 + \mathbf{q}(T\psi) + \mathbf{q}(\hat{Y}\psi) + \mathbf{q}(\psi)\right). \qquad (104)\end{aligned}$$

Let $\triangle_{\mathbb{S}^2}$ denote the standard Laplacian on the unit sphere. In the coordinates of the first paragraph of Section 8.2, we may write

$$\frac{1}{r^2}\triangle_{\mathbb{S}^2}\psi = \partial_v(\hat{Y}\psi) - \frac{2}{r}(T\psi - \hat{Y}\psi) - P_2\psi.$$

Integrating over S_r endowed with metric of the standard unit sphere, we obtain the elliptic estimate

$$\frac{1}{r^2}\int_{S_r} |\nabla^2_{\mathbb{S}^2}\psi|^2\, dA_{\mathbb{S}^2} \leq B\int_{S_r} \mathbf{q}(T\psi) + \mathbf{q}(\hat{Y}\psi) + \mathbf{q}(\psi) + (P_2\psi)^2\, dA_{\mathbb{S}^2},$$

i.e., in view of the assumptions (20) on the metric,

$$\int_{S_r} |\nabla\!\!\!/^2\psi|^2 \leq B\int_{S_r} \mathbf{q}(T\psi) + \mathbf{q}(\hat{Y}\psi) + \mathbf{q}(\psi) + (P_2\psi)^2. \qquad (105)$$

On the other hand, from (20), (21) we have

$$(P_2\psi)^2 \leq B\epsilon_{\mathrm{close}}\mathbf{q}^1(\psi).$$

The above, (105) and (104) yield the proposition, for $\epsilon_{\mathrm{close}}$ sufficiently small. $\square$

In addition, we have the following elliptic estimates away from the event horizon.

Proposition 13.2. *For ψ a solution of $\Box_g\psi = 0$, and r_0 a parameter with $r_0 > 2M$, then, for $\epsilon_{\mathrm{close}}$ sufficiently small, we have*

$$\int_{\Sigma(\tau')\cap\{r\geq r_0\}} \mathbf{q}^1(\psi) \leq B(r_0)\int_{\Sigma(\tau')} \mathbf{q}(T\psi)+\mathbf{q}(\psi),$$

$$\int_{\mathcal{R}(\tau',\tau'')\cap\{r\geq r_0\}} \mathbf{q}^1(\psi) \leq B(r_0)\int_{\mathcal{R}(\tau',\tau'')} \mathbf{q}(T\psi)+\mathbf{q}(\psi).$$

Proof. The proof of this straightforward elliptic estimate is left to the reader. $\square$

13.3 Pointwise bounds

We have the following Sobolev-type estimate on Schwarzschild

Proposition 13.3. *Let Ψ be a smooth function on Σ_τ of compact support. Then*

$$\sup_{\Sigma_\tau}\Psi^2 \leq B\int_{\Sigma_\tau} |\nabla^2_{\Sigma_\tau}\Psi|^2_{(g_M)_{\Sigma_\tau}} + |\nabla_{\Sigma_\tau}\Psi|^2_{(g_M)_{\Sigma_\tau}}.$$

This in turn follows from the following Euclidean space estimate:

Proposition 13.4. *There exists a constant K such that the following holds. Let Ψ be a smooth function on $\mathbb{R}^3\cap\{r\geq 1\}$ of compact support. Then*

$$\sup_{r\geq 1}\Psi^2 \leq K\int_{\{r\geq 1\}} (|\nabla^2\Psi|^2+|\nabla\Psi|^2)\,dx^1dx^2dx^3.$$

Proof. Omitted. $\square$

We obtain

Theorem 13.2. *Let $n\geq 0$. There exists a positive constant $\epsilon_{\mathrm{close}}$, depending only on M, and a positive constant C_n, depending on M and n, such that the following holds.Let g, $\Sigma(\tau)$ be as in Section 3.2 and let $\uppsi$, $\uppsi'$, ψ be as in Section 4 where ψ satisfies (2). Then, for $\tau\geq 0$,*

$$|\nabla^{(n)}\psi|^2_{g_{\Sigma_\tau}} \leq \lim_{r\to\infty}\uppsi^2 + C_n\int_{\Sigma(0)}\mathbf{q}^{n+1}(\psi)$$

where $\nabla^{(n)}\psi$ denotes the n'th order spacetime covariant derivative tensor and $|\cdot|_{g_{\Sigma_\tau}}$ denotes the induced Riemannian norm.

Theorem 1.2 in particular follows from the above.

14 Further notes

14.1 The Schwarzschild case

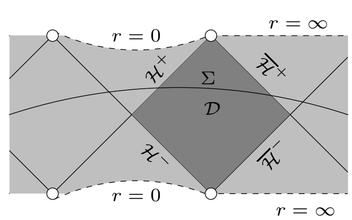
In the Schwarzschild case, we may apply the estimates proven here for $\psi_\sharp$ in Sections 10.2 and 11.4 directly to the whole ψ. Since no frequency decomposition need be made, no associated error terms arise and the whole argument can be reduced to a few pages. The resulting energy estimate, coupled with the higher order and pointwise estimates of Section 13, yield a new proof for uniform boundedness of solutions to the wave equation on Schwarzschild which is in some sense the simplest one yet–using neither the discrete isometry exploited by Kay-Wald [22], nor the vector field X of our [12] or [13], nor commuting with angular momentum operators. Moreover, one shows the uniform boundedness of all derivatives on the event horizon up to all order, whereas previous results could control only tangential derivatives.

In fact, one can obtain a much more general statement applying to all static spherically symmetric non-extremal black holes. We have

Theorem 14.1. *Let $(\mathcal{D}, g)$ be a static spherically symmetric asymptotically flat exterior black hole spacetime bounded by a non-extremal event horizon $\mathcal{H}^+$. Then the estimates of Theorems 1.1 and 1.2 hold.*

14.2 Kerr-de Sitter

Our argument is easily adapted to spacetimes which are small perturbations of non-extremal Schwarzschild-de Sitter, in particular to slowly rotating non-extremal Kerr-de Sitter, or Kerr-Newman-de Sitter. See [14] for the setting. One fixes the manifold structure on a subregion $\mathcal{D} \cap J^+(\Sigma_0)$ where $\mathcal{D}$ is here the region between a set of black/white hole and cosmological horizons and Σ_0 is a Cauchy surface crossing both horizons to the future of the bifurcate spheres.



One continues as in the Schwarzschild case. The argument is in fact easier at several points. Because r is bounded in $\mathcal{D}$, the zero-order terms pose no difficulty. In particular, one need not introduce the Σ^+ and Σ^- surfaces, nor must one modify J^{X_a} by the addition of J^{X_b,w_b}. We leave the details for a subsequent paper.

14.3 Non-quantitative decay

As a final application, we note that uniform boundedness is sufficient to translate non-quantitative results for fixed angular frequency into non-quantitative results for ψ itself. For instance

Corollary 14.1. *Suppose for each k we have $\psi_k(\cdot, t) \to 0$ where ψ_k denotes the projection to the k'th azimuthal mode. Then $\psi(\cdot, t) \to 0$.*

The assumption of the above corollary is obtained in [16] away from the event horizon for Kerr solutions for the very special case where the initial data is supported away from the horizon.

References

[1] S. Alinhac *Energy multipliers for perturbations of Schwarzschild metrics*, preprint, 2008

[2] A. Bachelot *Asymptotic completeness for the Klein-Gordon equation on the Schwarzschild metric*, Ann. Inst. H. Poincaré Phys. Théor. **16** (1994), no. 4, 411–441

[3] H. Beyer *On the Stability of the Kerr Metric* Comm. Math. Phys. **221** (2001), 659–676

[4] P. Blue and A. Soffer *Semilinear wave equations on the Schwarzschild manifold. I. Local decay estimates*, Adv. Differential Equations **8** (2003), no. 5, 595–614

[5] P. Blue and A. Soffer *Errata for "Global existence and scattering for the nonlinear Schrodinger equation on Schwarzschild manifolds", "Semilinear wave equations on the Schwarzschild manifold I: Local Decay Estimates", and "The wave equation on the Schwarzschild metric II: Local Decay for the spin 2 Regge Wheeler equation"*, gr-qc/0608073, 6 pages

[6] P. Blue and J. Sterbenz *Uniform decay of local energy and the semi-linear wave equation on Schwarzschild space* Comm. Math. Phys. **268** (2006), no. 2, 481–504

[7] B. Carter *Black hole equilibrium sates* In: Black holes/Les astres occlus (École d'Été Phys. Théor., Les Houches, 1972), pp. 57–214. Gordon and Breach, New York, 1973.

[8] D. Christodoulou *Reversible and irreversible transformations in black-hole physics* Phys. Rev. Lett. **25** (1970) 1596–1597

[9] D. Christodoulou *The action principle and partial differential equations*, Ann. Math. Studies No. 146, 1999

[10] D. Christodoulou and S. Klainerman *The global nonlinear stability of the Minkowski space* Princeton University Press, 1993

[11] M. Dafermos and I. Rodnianski *A proof of Price's law for the collapse of a self-gravitating scalar field*, Invent. Math. **162** (2005), 381–457

[12] M. Dafermos and I. Rodnianski *The redshift effect and radiation decay on black hole spacetimes*, gr-qc/0512119

[13] M. Dafermos and I. Rodnianski *A note on energy currents and decay for the wave equation on a Schwarzschild background*, arXiv:0710.0171

[14] M. Dafermos and I. Rodnianski *The wave equation on Schwarzschild-de Sitter spacetimes* arXiv:0709.2766

[15] F. Finster, N. Kamran, J. Smoller, S.-T. Yau *The long-time dynamics of Dirac particles in the Kerr-Newman black hole geometry* Adv. Theor. Math. Phys. **7** (2003), 25–52

[16] F. Finster, N. Kamran, J. Smoller, S. T. Yau *Decay of solutions of the wave equation in Kerr geometry* Comm. Math. Phys. **264** (2006), 465–503

[17] F. Finster, N. Kamran, J. Smoller, S.-T. Yau *Erratum: Decay of solutions of the wave equation in Kerr geometry* Comm. Math. Phys., online first

[18] F. Finster and J. Smoller *A Time Independent Energy Estimate for Outgoing Scalar Waves in the Kerr Geometry*, preprint

[19] D. Häfner *Sur la théorie de la diffusion pour l'équation de Klein-Gordon dans la métrique de Kerr* Dissertationes Mathematicae **421** (2003)

[20] D. Häfner and J.-P. Nicolas *Scattering of massless Dirac fields by a Kerr black hole* Rev. Math. Phys. **16**(1) (2004), 29–123

[21] S. W. Hawking and G. F. R. Ellis *The large scale structure of space-time* Cambridge Monographs on Mathematical Physics, No. 1. Cambridge University Press, London-New York, 1973

[22] B. Kay and R. Wald *Linear stability of Schwarzschild under perturbations which are nonvanishing on the bifurcation* 2*-sphere* Classical Quantum Gravity **4** (1987), no. 4, 893–898

[23] R. Kerr *Gravitational field of a spinning mass as an example of algebraically special metrics* Phys. Rev. Lett. **11** (1963) 237–238

[24] K. Kokkotas and B. Schmidt *Quasi-normal modes of stars and black holes* Living Rev. Relativity **2** (1999)

[25] J.-P. Nicolas *A non linear Klein-Gordon equation on Kerr metrics* J. Math. Pures. Appl. **81** (2002), 885–914

[26] F. Twainy *The Time Decay of Solutions to the Scalar Wave Equation in Schwarzschild Background*, Thesis, University of California, San Diego, 1989

[27] R. M. Wald *Note on the stability of the Schwarzschild metric* J. Math. Phys. **20** (1979), 1056–1058