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FULL LENGTH PAPER

On representing the positive semidefinite cone using the second-order cone

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Abstract The second-order cone plays an important role in convex optimization and has strong expressive abilities despite its apparent simplicity. Second-order cone formulations can also be solved more efficiently than semidefinite programming problems in general. We consider the following question, posed by Lewis and Glineur, Parrilo, Saunderson: is it possible to express the general positive semidefinite cone using second-order cones? We provide a negative answer to this question and show that the 3×3 positive semidefinite cone does not admit any second-order cone representation. In fact we show that the slice consisting of 3×3 positive semidefinite Hankel matrices does not admit a second-order cone representation. Our proof relies on exhibiting a sequence of submatrices of the slack matrix of the 3×3 positive semidefinite cone whose "second-order cone rank" grows to infinity.

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1 Introduction

Let $\mathcal{Q} \subset \mathbb{R}^3$ denote the three-dimensional *second-order* cone (also known as the "ice-cream" cone or the Lorentz cone):

$$\mathcal{Q} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : \|x\| \leq t\}.$$

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It is known that $\mathcal{Q}$ is linearly isomorphic to the cone of 2×2 real symmetric positive semidefinite matrices. Indeed we have:

$$(x_1, x_2, t) \in \mathcal{Q} \iff \begin{bmatrix} t - x_1 & x_2 \\ x_2 & t + x_1 \end{bmatrix} \succeq 0. \tag{1}$$

Despite its apparent simplicity the second-order cone $\mathcal{Q}$ has strong expressive abilities and allows us to represent various convex constraints that go beyond "simple quadratic constraints". For example it can be used to express geometric means ($x \mapsto \prod_{i=1}^n x_i^{p_i}$ where $p_i \geq 0$, rational, and $\sum_{i=1}^n p_i = 1$), ℓ_p-norm constraints, multifocal ellipses (see e.g., [11, Equation (3.5)]), robust counterparts of linear programs, etc. We refer the reader to [4, Section 3.3] for more details.

Given this strong expressive ability one may wonder whether the general positive semidefinite cone can be represented using $\mathcal{Q}$. This question was posed in particular by Adrian Lewis (personal communication) and Glineur, Parrilo and Saunderson [7]. In this paper we show that this is not possible, even for the 3×3 positive semidefinite cone. To make things precise we use the language of lifts (or extended formulations), see [8]. We denote by $\mathcal{Q}^k$ the Cartesian product of k copies of $\mathcal{Q}$:

$$\mathcal{Q}^k = \mathcal{Q} \times \cdots \times \mathcal{Q} \quad (k \text{ copies}).$$

A *linear slice* of $\mathcal{Q}^k$ is an intersection of $\mathcal{Q}^k$ with a linear subspace. We say that a convex cone $K \subset \mathbb{R}^m$ has a *second-order cone lift of size* k (or simply $\mathcal{Q}^k$-lift) if it can be written as the projection of a slice of $\mathcal{Q}^k$, i.e.:

$$K = \pi\left(\mathcal{Q}^k \cap L\right) \tag{2}$$

where $\pi : \mathbb{R}^{3k} \to \mathbb{R}^m$ is a linear map and L is a linear subspace of $\mathbb{R}^{3k}$. Let $\mathbf{S}^n_+$ be the cone of $n \times n$ real symmetric positive semidefinite matrices. In this paper we prove:

Theorem 1 *The cone $\mathbf{S}^3_+$ does not admit any $\mathcal{Q}^k$-lift for any finite k.*

Actually our proof allows us to show that the slice of $\mathbf{S}^3_+$ consisting of Hankel matrices does not admit any second-order representation (see Sect. 4 for details). Note that higher-dimensional second order cones of the form

$$\{(x, t) \in \mathbb{R}^n \times t : \|x\| \leq t\}$$

where $n \geq 3$ can be represented using the three-dimensional cone $\mathcal{Q}$, see e.g., [5, Section 2]. Thus Theorem 1 also rules out any representation of $\mathbf{S}^3_+$ using the higher-dimensional second-order cones. Moreover since $\mathbf{S}^3_+$ appears as a slice of higher-order positive semidefinite cones Theorem 1 also shows that one cannot represent $\mathbf{S}^n_+$, for $n \geq 3$ using second-order cones.

2 Preliminaries

The paper [8] introduced a general methodology to prove existence or nonexistence of lifts in terms of the *slack matrix* of a cone. In this section we review some of the definitions and results from this paper, and introduce the notion of a *second-order cone factorization* and the *second-order cone rank*.

Let E be a Euclidean space with inner product $\langle \cdot, \cdot \rangle$ and let $K \subseteq E$ be a cone. The dual cone K^* is defined as:

$$K^* = \{x \in E : \langle x, y \rangle \geq 0 \quad \forall y \in K\}.$$

We also denote by $\operatorname{ext}(K)$ the extreme rays of a cone K. The notion of *slack matrix* plays a fundamental role in the study of lifts.

Definition 1 (*Slack matrix*) The slack matrix of a cone K, denoted S_K, is a (potentially infinite) matrix where columns are indexed by extreme rays of K, and rows are indexed by extreme rays of K^* (the dual of K) and where the (x, y) entry is given by:

$$S_K[x, y] = \langle x, y \rangle \quad \forall (x, y) \in \operatorname{ext}(K^*) \times \operatorname{ext}(K). \tag{3}$$

Note that, by definition of dual cone, all the entries of S_K are nonnegative. Also note that an element $x \in \operatorname{ext}(K^*)$ (and similarly $y \in \operatorname{ext}(K)$) is only defined up to a positive multiple. Any choice of scaling gives a valid slack matrix of K and the properties of S_K that we are interested in will be independent of the scaling chosen.

The existence/nonexistence of a second-order cone lift for a convex cone K will depend on whether S_K admits a certain *second-order cone factorization* which we now define.

Definition 2 (*$\mathcal{Q}^k$-factorization and second-order cone rank*) Let $S \in \mathbb{R}^{|I|\times|J|}$ be a matrix with nonnegative entries. We say that S has a $\mathcal{Q}^k$-factorization if there exist vectors $a_i \in \mathcal{Q}^k$ for $i \in I$ and $b_j \in \mathcal{Q}^k$ for $j \in J$ such that $S[i, j] = \langle a_i, b_j \rangle$ for all $i \in I$ and $j \in J$. The smallest k for which such a factorization exists will be denoted $\operatorname{rank}_{\mathrm{soc}}(S)$.

Remark 1 Recall that for any $a, b \in \mathcal{Q}$ we have $\langle a, b \rangle \geq 0$. This means that any matrix with a second-order cone factorization is elementwise nonnegative.

Remark 2 It is important to note that the second-order cone rank of any matrix S can be equivalently expressed as the smallest k such that S admits a decomposition

$$S = M_1 + \cdots + M_k \tag{4}$$

where $\operatorname{rank}_{\mathrm{soc}}(M_l) = 1$ for each $l = 1, \ldots, k$ (i.e., each M_l has a factorization $M_l[i, j] = \langle a_i, b_j \rangle$ where $a_i, b_j \in \mathcal{Q}$). This simply follows from the fact that $\mathcal{Q}^k$ is the Cartesian product of k copies of $\mathcal{Q}$.

We now state the result from [8] that we will need.

Theorem 2 (Existence of a lift, special case of [8]) *Let K be a convex cone. If K has a $\mathcal{Q}^k$-lift then its slack matrix S_K has a $\mathcal{Q}^k$-factorization.*

This theorem can actually be turned into an if and only if condition under mild conditions on K (e.g., K is proper), see [8], but we have only stated here the direction that we will need.

The cone $\mathbf{S}_+^3$ In this paper we are interested in the cone $K = \mathbf{S}_+^3$ of real symmetric 3×3 positive semidefinite matrices. The extreme rays of $\mathbf{S}_+^3$ are rank-one matrices of the form xx^T where $x \in \mathbb{R}^3$. Also $\mathbf{S}_+^3$ is self-dual, i.e., $(\mathbf{S}_+^3)^* = \mathbf{S}_+^3$. The slack matrix of $\mathbf{S}_+^3$ thus has its rows and columns indexed by three-dimensional vectors and

$$S_{\mathbf{S}_+^3}[x, y] = \langle xx^T, yy^T \rangle = \left(x^T y\right)^2 \quad \forall (x, y) \in \mathbb{R}^3 \times \mathbb{R}^3. \tag{5}$$

In order to prove that $\mathbf{S}_+^3$ does not admit a second-order representation, we will show that its slack matrix does not admit any $\mathcal{Q}^k$-factorization for any finite k. In fact we will exhibit a sequence (A_n) of submatrices of $S_{\mathbf{S}_+^3}$ where $\text{rank}_{\text{soc}}(A_n)$ grows to $+\infty$ as $n \to +\infty$.

Before introducing this sequence of matrices we record the following simple (known) proposition concerning orthogonal vectors in the cone $\mathcal{Q}$ which will be useful later.

Proposition 1 *Let $a, b_1, b_2 \in \mathcal{Q}$ nonzero and assume that $\langle a, b_1 \rangle = \langle a, b_2 \rangle = 0$. Then b_1 and b_2 are collinear.*

Proof This is easy to see geometrically by visualizing the "ice cream" cone. We include a proof for completeness: let $a = (a', t) \in \mathbb{R}^2 \times \mathbb{R}$ and $b_i = (b_i', s_i) \in \mathbb{R}^2 \times \mathbb{R}$ where $\|a'\| \le t$ and $\|b_i'\| \le s_i$. Note that for $i = 1, 2$ we have $0 = \langle a, b_i \rangle = \langle a', b_i' \rangle + ts_i \ge -\|a'\|\|b_i'\| + ts_i \ge 0$ where in the first inequality we used Cauchy-Schwarz and in the second inequality we used the definition of the second-order cone. It thus follows that all the inequalities must be equalities: by the equality case in Cauchy-Schwarz we must have that $b_i' = \alpha_i a'$ for some constant $\alpha_i < 0$ and we must also have $t = \|a'\|$ and $s_i = \|b_i'\|$. Thus we get that $b_i = (\alpha_i a', |\alpha_i|\|a'\|) = |\alpha_i|(-a', \|a'\|)$. This shows that b_1 and b_2 are both collinear to the same vector $(-a', \|a'\|)$ and thus completes the proof. □

3 Proof of Theorem 1

A sequence of matrices We now define our sequence A_n of submatrices of the slack matrix of $\mathbf{S}_+^3$. For any integer i define the vector

$$v_i = (1, i, i^2) \in \mathbb{R}^3. \tag{6}$$

Note that this sequence of vectors satisfies the following:

$$\text{For all distinct integers } i_1, i_2, i_3 \;\; \det(v_{i_1}, v_{i_2}, v_{i_3}) \neq 0. \tag{7}$$

Our matrix A_n has size $\binom{n}{2} \times n$ and is defined as follows (rows are indexed by 2-subsets of $[n]$ and columns are indexed by $[n]$):

$$\begin{aligned} A_n[\{i_1, i_2\}, j] :&= \left((v_{i_1} \times v_{i_2})^T v_j\right)^2 \\ &= \det(v_{i_1}, v_{i_2}, v_j)^2 \quad \forall \{i_1, i_2\} \in \binom{[n]}{2}, \ \forall j \in [n] \end{aligned} \tag{8}$$

where $\times$ denotes the cross-product of three-dimensional vectors. It is clear from the definition of A_n that it is a submatrix of the slack matrix of $\mathbf{S}^3_+$. Note that the sparsity pattern of A_n satisfies the following:

$$\begin{array}{ll} A_n[e, j] = 0 & \text{if } j \in e \\ A_n[e, j] > 0 & \text{otherwise} \end{array} \qquad e \in \binom{[n]}{2}, \ j \in [n]. \tag{9}$$

Also note that A_n satisfies the following important recursive property: for any subset C of $[n]$ of size n_0 the submatrix $A_n[\binom{C}{2}, C]$ has the same sparsity pattern as A_{n_0} (up to relabeling of rows and columns). In our main theorem we will show that the second-order cone rank of A_n grows to infinity with n.

Remark 3 (Geometric interpretation of (9)) The property (9) of the matrices A_n will be the key to prove a lower bound on their second-order cone rank. Geometrically, the property (9) reflects a certain *2-neighborliness* property of the extreme rays[1] $\operatorname{ext}(\mathbf{S}^3_+)$ of $\mathbf{S}^3_+$: for any two distinct extreme rays xx^T and yy^T of $\mathbf{S}^3_+$, there is a supporting hyperplane H to $\mathbf{S}^3_+$ that touches $\operatorname{ext}(\mathbf{S}^3_+)$ precisely at xx^T and yy^T. This 2-neighborliness property turns out to be the key geometric obstruction for the existence of second-order cone lifts for $\mathbf{S}^3_+$.

Covering numbers Our analysis of the matrix A_n will only rely on its sparsity pattern. Given two matrices A and B of the same size we write $A \overset{supp}{=} B$ if A and B have the same support (i.e., $A_{ij} = 0$ if and only if $B_{ij} = 0$ for all i, j). We now define a combinatorial analogue of the *second-order cone rank*:

Definition 3 Given a nonnegative matrix A, we define the *soc*-covering number of A, denoted $\operatorname{cov}_{\text{soc}}(A)$ to be the smallest number k of matrices $M_1, \dots, M_k$ with $\operatorname{rank}_{\text{soc}}(M_l) = 1$ for $l = 1, \dots, k$ that are needed to cover the nonzero entries of A, i.e., such that

$$A \overset{supp}{=} M_1 + \cdots + M_k. \tag{10}$$

Proposition 2 *For any nonnegative matrix A we have* $\operatorname{rank}_{\text{soc}}(A) \geq \operatorname{cov}_{\text{soc}}(A)$.

Proof This follows immediately from Remark 2 concerning $\operatorname{rank}_{\text{soc}}$ and the definition of $\operatorname{cov}_{\text{soc}}$. □

[1] In fact here we only work with the extreme rays $\{v_n v_n^T : n \in \mathbb{N}\}$, see Sect. 4 for the implication of this.

A simple but crucial fact concerning *soc*-coverings that we will use is the following: in any *soc*-covering of A of the form (10), each matrix M_l must satisfy $M_l[i,j]=0$ whenever $A[i,j]=0$. This is because the matrices $M_1,\ldots,M_k$ are all entrywise nonnegative.

We are now ready to state our main result.

Theorem 3 *Consider a sequence* (A_n) *of matrices of sparsity pattern given in* (9). *Then for any* $n_0\geq 2$ *we have* $\operatorname{cov}_{\text{soc}}(A_{3n_0^2})\geq \operatorname{cov}_{\text{soc}}(A_{n_0})+1$. *As a consequence* $\operatorname{cov}_{\text{soc}}(A_n)\to+\infty$ *when* $n\to+\infty$.

The proof of our theorem rests on a key lemma concerning the sparsity pattern of any term in a *soc*-covering of A_n.

Lemma 1 (Main) *Let* n *be such that* $n\geq 3n_0^2$ *for some* $n_0\geq 2$. *Assume* $M\in\mathbb{R}^{\binom{n}{2}\times n}$ *satisfies* $\operatorname{rank}_{\text{soc}}(M)=1$ *and* $M[e,j]=0$ *for all* $e\in\binom{n}{2}$ *and* $j\in[n]$ *such that* $j\in e$. *Then there is a subset* C *of* $[n]$ *of size at least* n_0 *such that the submatrix* $M[\binom{C}{2},C]$ *is identically zero.*

Before proving this lemma, we show how this lemma can be used to easily prove Theorem 3.

Proof of Theorem 3 Let $n=3n_0^2$ and consider a *soc*-covering of $A_n\overset{supp}{=}M_1+\cdots+M_r$ of size $r=\operatorname{cov}_{\text{soc}}(A_n)$ (note that we have of course $r\geq 1$ since A_n is not identically zero). By Lemma 1 there is a subset C of $[n]$ of size n_0 such that $M_1[\binom{C}{2},C]=0$. It thus follows that we have $A_n[\binom{C}{2},C]\overset{supp}{=}M_2[\binom{C}{2},C]+\cdots+M_r[\binom{C}{2},C]$. Also note that $A_n[\binom{C}{2},C]\overset{supp}{=}A_{n_0}$. It thus follows that A_{n_0} has a *soc*-covering of size $r-1$ and thus $\operatorname{cov}_{\text{soc}}(A_{n_0})\leq\operatorname{cov}_{\text{soc}}(A_{3n_0^2})-1$. This completes the proof. □

For completeness we show how Theorem 1 follows directly from Theorem 3.

Proof of Theorem 1 Since for any $n\geq 1$, A_n is a submatrix of the slack matrix of $\mathbf{S}^3_+$, Theorem 3 shows that the slack matrix of $\mathbf{S}^3_+$ does not admit any $\mathcal{Q}^k$-factorization for finite k. This shows, via Theorem 2, that $\mathbf{S}^3_+$ does not have a $\mathcal{Q}^k$-lift for any finite k. □

The rest of the section is devoted to the proof of Lemma 1.

Proof of Lemma 1 Let $M\in\mathbb{R}^{\binom{n}{2}\times n}$ and assume that M has a factorization $M_{e,j}=\langle a_e,b_j\rangle$ where $a_e,b_j\in\mathcal{Q}$ for all $e\in\binom{[n]}{2}$ and $j\in[n]$, and that $M_{e,j}=0$ whenever $j\in e$.

Let $E_0:=\{e\in\binom{[n]}{2}:a_e=0\}$ be the set of rows of M that are identically zero and let $E_1=\binom{[n]}{2}\setminus E_0$. Similarly for the columns we let $S_0:=\{j\in[n]:b_j=0\}$ and $S_1=[n]\setminus S_0$.

In the next lemma we use the sparsity pattern of A_n together with Proposition 1 to infer additional properties on the sparsity pattern of M.

Lemma 2 *Let* C *be a connected component of the graph with vertex set* S_1 *and edge set* $E_1(S_1)$ *(where* $E_1(S_1)$ *consists of elements in* E_1 *that connect only elements of* S_1*). Then necessarily* $M[\binom{C}{2},C]=0$.

Proof We first show using Proposition 1 that all the vectors $\{b_j\}_{j\in C}$ are necessarily collinear. Let $j_1, j_2 \in S_1$ such that $e = \{j_1, j_2\} \in E_1$. Note that since $M_{e,j_1} = M_{e,j_2} = 0$ then we have, by Proposition 1 that b_{j_1} and b_{j_2} are collinear. It is easy to see thus now that if j_1 and j_2 are connected by a path in the graph $(S_1, E_1(S_1))$ then b_{j_1} and b_{j_2} must be collinear.

We thus get that all the columns of M indexed by C must be proportional to each other, and so they must have the same sparsity pattern. Now let $e \in \binom{C}{2}$. If $a_e = 0$ then $M[e, C] = 0$ since the entire row indexed by e is zero. Otherwise if $a_e \neq 0$ let $e = \{j_1, j_2\}$ with $j_1, j_2 \in C$. Since, by assumption, $M_{e,j_1} = 0$ it follows that for any $j \in C$ we must have $M_{e,j} = 0$, i.e., $M[e, C] = 0$. This is true for any $e \in \binom{C}{2}$ thus we get that $M[\binom{C}{2}, C] = 0$. □

To complete the proof of Lemma 1 assume that $n \geq 3n_0^2$ for some $n_0 \geq 2$. We need to show that there is a subset C of $[n]$ of size at least n_0 such that $M[\binom{C}{2}, C] = 0$.

First note that if the graph $(S_1, E_1(S_1))$ has a connected component of size at least n_0 then we are done by Lemma 2. Also note that if S_0 has size at least n_0 we are also done because all the columns indexed by S_0 are identically zero by definition.

In the rest of the proof we will thus assume that $|S_0| < n_0$ and that the connected components of $(S_1, E_1(S_1))$ all have size $< n_0$. We will show in this case that E_0 necessarily contains a clique of size at least n_0 (i.e., a subset of the form $\binom{C}{2}$ where $|C| \geq n_0$) and this will prove our claim since all the rows in E_0 are identically zero by definition. The intuition is as follows: the assumption that $|S_0| < n_0$ and that the connected components of $(S_1, E_1(S_1))$ have size $< n_0$ mean that the graph $(S_1, E_1(S_1))$ is very sparse. In particular this means that E_1 has to be small which means that $E_0 = E_1^c$ must be large and thus it must contain a large clique.

More precisely, to show that E_1 is small note that it consists of those edges that are either in $E_1(S_1)$ or, otherwise, they must have at least one node in $S_1^c = S_0$. Thus we get that

$$|E_1| \leq |E_1(S_1)| + |S_0|(n-1) \leq |E_1(S_1)| + (n_0 - 1)(n-1).$$

where in the second inequality we used the fact that $|S_0| < n_0$. Also since the connected components of $(S_1, E_1(S_1))$ all have size $< n_0$ it is not difficult to show that $|E_1(S_1)| < n_0 n/2$ (indeed if we let $x_1, \ldots, x_k$ be the size of each connected component we have $|E_1(S_1)| \leq \frac{1}{2}\sum_{i=1}^k x_i^2 < \frac{1}{2}\sum_{i=1}^k n_0 x_i \leq \frac{1}{2} n_0 n$). Thus we get that

$$|E_1| \leq \frac{n_0 n}{2} + (n_0 - 1)(n-1) \leq \left(\frac{3}{2}n_0 - 1\right) n$$

Thus this means, since $E_0 = \binom{[n]}{2} \setminus E_1$:

$$|E_0| \geq \binom{n}{2} - \left(\frac{3}{2}n_0 - 1\right) n > \frac{n^2}{2} - \frac{3}{2} n_0 n$$

We now invoke a result of Turán to show that E_0 must contain a clique of size at least n_0:

Theorem 4 (Turán, see e.g., [2]) *Any graph on n vertices with more than* $\left(1 - \frac{1}{k}\right)\frac{n^2}{2}$ *edges contains a clique of size* $k + 1$.

By taking $k = n_0 - 1$ we see that E_0 contains a clique of size n_0 if

$$\frac{n^2}{2} - \frac{3}{2}n_0 n > \left(1 - \frac{1}{n_0 - 1}\right)\frac{n^2}{2}$$

This simplifies into

$$n > 3n_0(n_0 - 1)$$

which is true for $n \geq 3n_0^2$. □

4 Slices of the 3 × 3 positive semidefinite cone

Hankel slice The proof given in the previous section actually shows the following more general statement.

Theorem 5 *Let* $\mathcal{H}$ *denote the cone of* 3×3 *positive semidefinite* Hankel *matrices:*

$$\mathcal{H} = \{X \in \mathbf{S}^3_+ : X_{13} = X_{22}\}.$$

Assume K *is a convex cone that is "sandwiched" between* $\mathcal{H}$ *and* $\mathbf{S}^3_+$, *i.e.,* $\mathcal{H} \subseteq K \subseteq \mathbf{S}^3_+$. *Then* K *does not have a second-order cone representation.*

Proof The proof follows from the observation that the matrices A_n considered in Sect. 3 (see Eq. (8)) are actually submatrices of the *generalized slack matrix* of the pair of nested cones $(\mathcal{H}, \mathbf{S}^3_+)$, the definition of which we now recall (see e.g., [9, Definition 6]): The *generalized slack matrix* of a pair of convex cones (K_1, K_2) with $K_1 \subseteq K_2$ is a matrix whose rows are indexed by $\text{ext}(K_2^*)$ (the valid linear inequalities of K_2) and its columns indexed by $\text{ext}(K_1)$ and is defined by

$$S_{K_1,K_2}[x, y] = \langle x, y\rangle \quad x \in \text{ext}(K_2^*), y \in \text{ext}(K_1).$$

When $K_1 = K_2$ this is precisely the slack matrix of $K_1 = K_2$. The following theorem is a generalization of Theorem 2 and can be proved using very similar arguments (see e.g., [9, Proposition 7]).

Theorem 6 (Generalization of Theorem 2 to nested cones) *Let* K_1, K_2 *be two convex cones with* $K_1 \subseteq K_2$, *and assume there exists a convex cone* K *with a* $\mathcal{Q}^k$*-lift such that* $K_1 \subseteq K \subseteq K_2$. *Then* S_{K_1,K_2} *has a* $\mathcal{Q}^k$*-factorization.*

The main observation is to see that the vector v_i defined in (6) satisfies $v_i v_i^T \in \mathcal{H}$, and so this shows that A_n defined in Equation (8) is a submatrix of the generalized slack matrix of the pair $(\mathcal{H}, \mathbf{S}^3_+)$. Since $\text{rank}_{\text{soc}}(A_n)$ grows to infinity with n, this gives the desired result. □

The dual cone of $\mathcal{H}$ is the cone of nonnegative quartic polynomials on the real line (see e.g., [3, Section 3.5]). It thus follows that the latter is also not second-order cone representable using the second-order cone. More generally we can prove:

Corollary 1 *Let $\Sigma_{n,2d}$ be the cone of polynomials in n variables of degree at most $2d$ that are sums of squares. Then $\Sigma_{n,2d}$ is not second-order cone representable except in the case $(n, 2d) = (1, 2)$.*

For the proof we recall that in the cases $n = 1$ (univariate polynomials) and $2d = 2$ (quadratic polynomials), nonnegative polynomials are sums of squares.

Proof For $2d = 2$, $\Sigma_{n,2d}$ is the cone of nonnegative quadratic polynomials in n variables. By homogenization, this cone is linearly isomorphic to $\mathbf{S}^{n+1}_+$, the cone of nonnegative quadratic forms in $n+1$ variables. By Theorem 1, this shows that $\Sigma_{n,2}$ is not second-order cone representable for $n \geq 2$. The case $(n, 2d) = (1, 2)$ is clearly second-order cone representable because $\mathbf{S}^2_+$ is linearly isomorphic to the second-order cone.

If $2d \geq 4$ then the cone of nonnegative quartic polynomials on the real line can be obtained as a section of $\Sigma_{n,2d}$ by setting to zero the coefficients of some appropriate monomials. This shows that $\Sigma_{n,2d}$ is not second-order cone representable when $2d \geq 4$. $\square$

Other slices of $\mathbf{S}^3_+$ that are second-order cone representable There are certain slices of $\mathbf{S}^3_+$ of codimension 1 that are, on the other hand, known to admit a second-order cone representation. For example the following second-order cone representation of the slice $\{X \in \mathbf{S}^3_+ : X_{11} = X_{22}\}$ appears in [7]:

$$\begin{bmatrix} t & a & b \\ a & t & c \\ b & c & s \end{bmatrix} \succeq 0 \iff \exists u, v \in \mathbb{R} \text{ s.t. } \begin{bmatrix} t+a & b+c \\ b+c & u \end{bmatrix} \succeq 0, \quad \begin{bmatrix} t-a & b-c \\ b-c & v \end{bmatrix} \succeq 0, \quad u + v = 2s. \tag{11}$$

(The 2×2 positive semidefinite constraints can be converted to second-order cone constraints using (1)). To see why (11) holds note that by applying a congruence transformation by $\frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{smallmatrix}\right]$ on the 3×3 matrix on the left-hand side of (11) we get that

$$\begin{bmatrix} t & a & b \\ a & t & c \\ b & c & s \end{bmatrix} \succeq 0 \iff \begin{bmatrix} t+a & 0 & b+c \\ 0 & t-a & b-c \\ b+c & b-c & 2s \end{bmatrix} \succeq 0.$$

The latter matrix has an arrow structure and thus using results on the decomposition of matrices with chordal sparsity pattern [1,6,10] we get the decomposition (11).

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