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## ALTERNATIVE ASYMPTOTICS FOR COINTEGRATION TESTS IN LARGE VARS.

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Johansen's (1988, 1991) likelihood ratio test for cointegration rank of a Gaussian VAR depends only on the squared sample canonical correlations between current changes and past levels of a simple transformation of the data. We study the asymptotic behavior of the empirical distribution of those squared canonical correlations when the number of observations and the dimensionality of the VAR diverge to infinity simultaneously and proportionally. We find that the distribution almost surely weakly converges to the so-called Wachter distribution. This finding provides a theoretical explanation for the observed tendency of Johansen's test to find "spurious cointegration". It also sheds light on the workings and limitations of the Bartlett correction approach to the over-rejection problem. We propose a simple graphical device, similar to the scree plot, for a preliminary assessment of cointegration in high-dimensional VARs.

# Alternative asymptotics for cointegration tests in large VARs.

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## Abstract

Johansen’s (1988, 1991) likelihood ratio test for cointegration rank of a Gaussian VAR depends only on the squared sample canonical correlations between current changes and past levels of a simple transformation of the data. We study the asymptotic behavior of the empirical distribution of those squared canonical correlations when the number of observations and the dimensionality of the VAR diverge to infinity simultaneously and proportionally. We find that the distribution almost surely weakly converges to the so-called *Wachter distribution*. This finding provides a theoretical explanation for the observed tendency of Johansen’s test to find “spurious cointegration”. It also sheds light on the workings and limitations of the Bartlett correction approach to the over-rejection problem. We propose a simple graphical device, similar to the scree plot, for a preliminary assessment of cointegration in high-dimensional VARs.

## 1 Introduction

Johansen’s (1988, 1991) likelihood ratio (LR) test for cointegration rank is a very popular econometric technique. However, it is rarely applied to systems of more than three or four variables. On the other hand, there exist many applications involving much larger systems. For example, Davis (2003) discusses a possibility of applying the test to the data on seven aggregated and individual commodity prices to test Lewbel’s (1996) generalization of the Hicks-Leontief composite commodity

theorem. In a recent study of exchange rate predictability, Engel, Mark, and West (2015) contemplate a possibility of determining the cointegration rank of a system of seventeen OECD exchange rates. Banerjee, Marcellino, and Osbat (2004) emphasize the importance of testing for no cross-sectional cointegration in panel cointegration analysis (see Breitung and Pesaran (2008) and Choi (2015)), and the cross-sectional dimension of modern macroeconomic panels can easily be as large as forty.

The main reason why the LR test is rarely used in the analysis of relatively large systems is its poor finite sample performance. Even for small systems, the test based on the asymptotic critical values does not perform well (see Johansen (2002)). For large systems, the size distortions become overwhelming, leading to severe over-rejection of the null in favour of too much cointegration as shown in many simulation studies, including Ho and Sorensen (1996) and Gonzalo and Pitarakis (1995, 1999).

In this paper, we study the asymptotic behavior of the sample canonical correlations that the LR statistic is based on, when the number of observations and the system's dimensionality go to infinity simultaneously and proportionally. We show that the empirical distribution of the squared sample canonical correlations almost surely converges to the so-called *Wachter distribution* which also arises, albeit with different parameters, as the limit of the empirical distribution of the squared sample canonical correlations between two independent high-dimensional white noises (see Wachter (1980)). Our analytical findings explain the observed over-rejection of the null hypothesis by the LR test, shed new light on the workings and limitations of the Bartlett-type correction approach to the problem (see Johansen (2002)), and lead us to propose a very simple graphical device, similar to the scree plot, for a preliminary analysis of the validity of cointegration hypotheses in large vector autoregressions.

The basic framework for our analysis is standard. Consider a  $p$ -dimensional VAR in the error correction form

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Phi D_t + \varepsilon_t, \quad (1)$$

where  $D_t$  and  $\varepsilon_t$  are vectors of deterministic terms and zero-mean Gaussian errors with unconstrained covariance matrix, respectively. The LR statistic for the test of the null hypothesis of no more than  $r$  cointegrating relationships between the  $p$

elements of  $X_t$  against the alternative of more than  $r$  such relationships is given by

$$LR_{r,p,T} = -T \sum_{i=r+1}^p \log(1 - \lambda_i), \quad (2)$$

where  $T$  is the sample size, and  $\lambda_1 \geq \dots \geq \lambda_p$  are the squared sample canonical correlation coefficients between residuals in the regressions of  $\Delta X_t$  and  $X_{t-1}$  on the lagged differences  $\Delta X_{t-i}$ ,  $i = 1, \dots, k-1$ , and the deterministic terms.

In the absence of the lagged differences and deterministic terms, the  $\lambda$ 's are the eigenvalues of  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$ , where  $S_{00}$  and  $S_{11}$  are the sample covariance matrices of  $\Delta X_t$  and  $X_{t-1}$ , respectively, while  $S_{01}$  is the cross sample covariance matrix. More substantively,  $\lambda_1$  is the largest possible squared sample correlation coefficient between arbitrary linear combinations of the entries of  $\Delta X_t$  and the entries of  $X_{t-1}$ ,  $\lambda_2$  is the largest squared correlation among linear combinations restricted to be orthogonal to those yielding  $\lambda_1$ , and so on (see Muirhead (1982), ch. 11).

Johansen (1991) shows that the asymptotic distribution of  $LR_{r,p,T}$  under the asymptotic regime where  $T \rightarrow \infty$  while  $p$  remains fixed, can be expressed in terms of the eigenvalues of a matrix whose entries are explicit functions of a  $p - r$ -dimensional Brownian motion. Unfortunately, for relatively large  $p$ , this asymptotics does not produce good finite sample approximations, as evidenced by the over-rejection phenomenon mentioned above. Therefore, in this paper, we consider a *simultaneous* asymptotic regime  $p, T \rightarrow_c \infty$  where both  $p$  and  $T$  diverge to infinity so that

$$p/T \rightarrow c \in (0, 1], \quad (3)$$

while  $p$  remains no larger than  $T$ . Our Monte Carlo analysis shows that the corresponding asymptotic approximations are relatively accurate even for such small sample sizes as  $p = 10$  and  $T = 20$ .

The basic specification for the data generating process (1) that we consider has  $k = 1$ . In the next section, we discuss extensions to more general VARs with low-rank  $\Gamma_i$  matrices and additional common factor terms. We also explain there that our main results hold independently from whether a deterministic vector  $D_t$  with fixed or slowly-growing dimension is present or absent from the VAR.

Our study focuses on the behavior of the empirical distribution function (d.f.)

of the squared sample canonical correlations,

$$F_{p,T}(\lambda) = \frac{1}{p} \sum_{i=1}^p \mathbf{1}\{\lambda_i \leq \lambda\}, \quad (4)$$

where  $\mathbf{1}\{\cdot\}$  denotes the indicator function. We find that, under the null of  $r$  cointegrating relationships, as  $p, T \rightarrow_c \infty$  while  $r/p \rightarrow 0$ , almost surely (a.s.),

$$F_{p,T}(\lambda) \Rightarrow W(\lambda; c/(1+c), 2c/(1+c)), \quad (5)$$

where  $\Rightarrow$  denotes the weak convergence of d.f.'s (see Billingsley (1995), p.191), and  $W(\lambda; \gamma_1, \gamma_2)$  denotes the *Wachter* d.f. with parameters  $\gamma_1$  and  $\gamma_2$ . The *Wachter distribution* was derived by Wachter (1980) as the limit of the empirical distribution of the eigenvalues of the multivariate beta matrix of growing dimension and degrees of freedom. It has a simple density, which is introduced in the next section, and, for  $\gamma_2 > \gamma_1$  and/or  $\gamma_2 < 1 - \gamma_1$ , point masses at zero and/or one, respectively.

The a.s. weak convergence (5) and the fact that the squared sample canonical correlations are no larger than unity imply the a.s. convergence of averages  $\frac{1}{p} \sum_{i=1}^p f(\lambda_i)$  for any  $f$  which is bounded and continuous on  $[0, 1]$ . By definition, the likelihood ratio statistic scaled by  $1/(pT)$  has this form (with omitted first  $r$  summands), where  $f(\lambda) = -\log(1 - \lambda)$  is continuous but unbounded function. Therefore, (5) can guarantee an a.s. asymptotic lower bound for the scaled LR statistic. For the LR statistic scaled by  $1/p^2$ , we have, almost surely,

$$\lim_{p,T \rightarrow_c \infty} \inf LR_{r,p,T}/p^2 \geq -\frac{1}{c} \int \log(1 - \lambda) dW(\lambda; c/(1+c), 2c/(1+c)). \quad (6)$$

In contrast, we show that, under the (standard) asymptotic regime where  $T \rightarrow \infty$  while  $p$  is held fixed,  $LR_{r,p,T}/p^2$  concentrates around 2 for relatively large  $p$ .<sup>1</sup> A direct calculation reveals that 2 is smaller than the lower bound (6), for all  $c > 0$ , with the gap growing as  $c$  increases. That is, the standard asymptotic distribution of the LR statistic is centered at a too low level, especially for relatively large  $p$ . This explains the tendency of the asymptotic LR test to over-reject the null.

The reason for the poor centering delivered by the standard asymptotic ap-

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<sup>1</sup>Similar to (6), our weak convergence results only guarantee that 2 is a lower bound, but we conjecture that it is also the limit of the scaled LR statistic as first  $T \rightarrow \infty$  and then  $p \rightarrow \infty$ . This conjecture is supported by Monte Carlo evidence.

proximation is that it classifies terms  $(p/T)^j$  in the asymptotic expansion of the likelihood ratio statistic as  $O(T^{-j})$ . When  $p$  is relatively large, such terms can substantially contribute to the finite sample distribution of the statistic, but will be ignored as asymptotically negligible. In contrast, the *simultaneous asymptotics* classifies all terms  $(p/T)^j$  as  $O(1)$ . They are not ignored asymptotically, which improves the centering of the simultaneous asymptotic approximation relative to the standard one.

It is possible to use bound (6), with  $c$  replaced by  $p/T$ , to construct a Bartlett-type correction factor for the standard LR test. As we show below, for  $p/T < 1/3$ , the value of such a theoretical correction factor is very close to the simulation-based factor described in Johansen, Hansen and Fachin (2005). However, for larger  $p/T$ , the values diverge, which may be caused by the fact that Johansen, Hansen and Fachin's (2005) simulations do not consider combinations of  $p$  and  $T$  with  $p/T > 1/3$ , and the functional form that they use to fit the simulated correction factors does not work well uniformly in  $p/T$ .

The weak convergence result (5) can be put to a more direct use by comparing the quantiles of the empirical distribution of the squared sample canonical correlations with the quantiles of the limiting Wachter distribution. Under the null, the former quantiles plotted against the latter ones should form a  $45^\circ$  line, asymptotically. Deviations of such a Wachter quantile-quantile plot from the line indicate violations of the null. Creating Wachter plots requires practically no additional computations beyond those needed to compute the LR statistic, and we propose to use this simple graphical device for a preliminary analysis of cointegration in large VARs.

To the best of our knowledge, our study is the first to derive the limit of the empirical d.f. of the squared sample canonical correlations between random walk  $X_{t-1}$  and its innovations  $\Delta X_t$ . Wachter (1980) shows that  $W(\lambda; \gamma_1, \gamma_2)$  is the weak limit of the empirical d.f. of the squared sample canonical correlations between  $q$ - and  $m$ -dimensional independent Gaussian white noises with the size of the sample  $n$ , when  $q, m, n \rightarrow \infty$  so that  $q/n \rightarrow \gamma_1$  and  $m/n \rightarrow \gamma_2$ . Yang and Pan (2012) show that Wachter's (1980) result holds without the Gaussianity assumption for i.i.d. data with finite second moments.

Our proofs do not rely on those previous results. The values of parameters  $\gamma_1$  and  $\gamma_2$  in (5) imply that the limiting d.f. for the case of  $T$  observations of  $p$ -dimensional random walk and its innovations, that we consider in this paper, is the same as the limiting d.f. for the case of  $T + p$  observations of two independent

white noises - one  $p$ -dimensional and the other  $2p$ -dimensional. It is tempting to think that there exists a deep connection between the two cases, even though we were unable to uncover it so far.

Our paper opens up a new direction for the asymptotic analysis of panel VAR cointegration tests based on the sample canonical correlations. One such test is developed in Larsson and Lyhagen (2007). It generalizes Larsson, Lyhagen, and Lothgren (2001) and Groen and Kleibergen (2003) by allowing for cross-unit cointegration, which is important from the empirical perspective. Larsson and Lyhagen (2007) are reluctant to recommend their test for large VARs and suggest that for the analysis of relatively large panels it may be better to rely on tighter parameterized models, such as that of Bai and Ng (2004). In the recent review of the panel cointegration literature, Choi (2015) expresses a related concern that, with the large number of cross-sectional units, “Larsson and Lyhagen’s test may not work well even with the Bartlett’s correction.”

We speculate that the Larsson-Lyhagen test, as well as Johansen’s LR test, based on the *simultaneous* asymptotics would work well in panels with comparable cross-sectional and temporal dimensions. The results of this paper can be used to describe only the appropriate centering of the corresponding test statistics. The next step would be to derive the *simultaneous* asymptotic distribution of scaled deviations of such statistics from the centering values. We conjecture that the *simultaneous* asymptotic distribution of  $LR_{r,p,T}$  is Gaussian, as is often the case for averages of regular functions of eigenvalues of large random matrices (see Bai and Silverstein (2010) and Paul and Aue (2014)). We are currently undertaking work to validate this conjecture.

The rest of this paper is structured as follows. In Section 2, we prove the convergence of  $F_{p,T}(\lambda)$  to the *Wachter* d.f. and use this result to derive the asymptotic lower bound for  $LR_{r,p,T}$ . Section 3 derives the sequential limit of the empirical d.f. of the squared sample canonical correlations as, first  $T \rightarrow \infty$  and then  $p \rightarrow \infty$ . It then uses differences between the obtained sequential asymptotic limit and the simultaneous limit derived in Section 2 to explain the over-rejection phenomenon, and to design a theoretical Bartlett-type correction factor for the LR statistic in high-dimensional VARs. Section 4 contains a Monte Carlo study that confirms good finite sample properties of the Wachter asymptotic approximation. It also illustrates the proposed Wachter quantile-quantile plot technique using a relatively high-dimensional macroeconomic panel. Section 5 concludes and points out directions for future research. All proofs are given in the Appendix.

## 2 Convergence to the Wachter distribution

Consider the following basic version of (1)

$$\Delta X_t = \Pi X_{t-1} + \Phi D_t + \varepsilon_t \quad (7)$$

with  $d_D$ -dimensional vector of deterministic regressors  $D_t$ . Let  $R_{0t}$  and  $R_{1t}$  be the vectors of residuals from the OLS regressions of  $\Delta X_t$  on  $D_t$ , and  $X_{t-1}$  on  $D_t$ , respectively. Define

$$S_{00} = \frac{1}{T} \sum_{t=1}^T R_{0t} R_{0t}', \quad S_{01} = \frac{1}{T} \sum_{t=1}^T R_{0t} R_{1t}', \quad \text{and} \quad S_{11} = \frac{1}{T} \sum_{t=1}^T R_{1t} R_{1t}', \quad (8)$$

and let  $\lambda_1 \geq \dots \geq \lambda_p$  be the eigenvalues of  $S_{01} S_{11}^{-1} S_{01}' S_{00}^{-1}$ .

The main goal of this section is to establish the a.s. weak convergence of the empirical d.f. of the  $\lambda$ 's to the Wachter d.f., under the null of  $r$  cointegrating relationships, when  $p, T \rightarrow_c \infty$ . The Wachter distribution with d.f.  $W(\lambda; \gamma_1, \gamma_2)$  and parameters  $\gamma_1, \gamma_2 \in (0, 1)$  has density

$$f_W(\lambda; \gamma_1, \gamma_2) = \frac{1}{2\pi\gamma_1} \frac{\sqrt{(b_+ - \lambda)(\lambda - b_-)}}{\lambda(1 - \lambda)} \quad (9)$$

on  $[b_-, b_+] \subseteq [0, 1]$  with

$$b_{\pm} = \left( \sqrt{\gamma_1(1 - \gamma_2)} \pm \sqrt{\gamma_2(1 - \gamma_1)} \right)^2, \quad (10)$$

and atoms of size  $\max\{0, 1 - \gamma_2/\gamma_1\}$  at zero, and  $\max\{0, 1 - (1 - \gamma_2)/\gamma_1\}$  at unity.

We shall assume that model (7) may be misspecified in the sense that the true data generating process is described by the following generalization of (1)

$$\Delta X_t = \Pi X_{t-1} + \sum_{i=1}^{k-1} \Gamma_i \Delta X_{t-i} + \Psi F_t + \varepsilon_t, \quad (11)$$

where  $\varepsilon_t$ ,  $t = 1, \dots, T$ , are still i.i.d.  $N(0, \Sigma)$  with arbitrary  $\Sigma > 0$ ,  $\text{rank } \Pi = r$ , but  $k$  is not necessarily unity, and  $F_t$  is a  $d_F$ -dimensional vector of deterministic or stochastic variables that does not necessarily coincide with  $D_t$ . For example, some of the components of  $F_t$  may be common factors not observed and not modelled

by the econometrician. Further, we do not put any restrictions on the roots of the characteristic polynomial associated with (11). In particular, explosive behavior and seasonal unit roots are allowed. Finally, no constraints on  $F_t$ , and the initial values  $X_{1-k}, \dots, X_0$ , apart from the asymptotic requirements on  $d_F$  and  $k$  as spelled out in the following theorem, are imposed.

**Theorem 1** *Suppose that the data are generated by (11), and let  $\lambda_i$ ,  $i = 1, \dots, p$ , be the eigenvalues of  $S_{01}S_{11}^{-1}S'_{01}S_{00}^{-1}$ , where  $S_{ij}$  are as defined in (8). Further, let  $F_{p,T}(\lambda)$  be the empirical d.f. of the  $\lambda$ 's, and let  $\Gamma = [\Gamma_1, \dots, \Gamma_{k-1}]$ . If*

$$\frac{1}{p} (d_D + d_F + r + k + \text{rank } \Gamma) \rightarrow 0 \quad (12)$$

*as  $p, T \rightarrow_c \infty$  while  $p$  remains no larger than  $T$ , then, almost surely,*

$$F_{p,T}(\lambda) \Rightarrow W(\lambda; c/(1+c), 2c/(1+c)). \quad (13)$$

Condition (12) requires the number  $d_D$  of deterministic regressors in the econometrician's model (7), the dimensionality  $d_F$  of  $F_t$ , the number  $r$  of the cointegrating relationships under the null, the order  $k$  of the data generating VAR, and the dimensionality of the union of the column spaces of the matrix coefficients on “further lags” in (11) to be either fixed or growing less than proportionally to the dimensionality  $p$  or, equivalently, to the sample size  $T$ . This condition rules out situations where some or all lags which are omitted from the econometrician's model (7) have full rank coefficients  $\Gamma_i$ . The simplest special situation where (12) is clearly satisfied corresponds to the pure random walk data  $\Delta X_t = \varepsilon_t$ .

The reason why the limit of the empirical d.f.  $F_{p,T}(\lambda)$  does not change when the data generating process (11) changes so that (12) remains true is that the corresponding changes in the matrix  $S_{01}S_{11}^{-1}S'_{01}S_{00}^{-1}$  have rank that is less than proportional to  $p$  (and to  $T$ ). By the so-called rank inequality (Theorem A43 in Bai and Silverstein (2010)), the Lévy distance between the empirical d.f. of eigenvalues corresponding to versions of  $S_{01}S_{11}^{-1}S'_{01}S_{00}^{-1}$  that differ by a matrix of rank  $R$  is no larger than  $R/p$ , which converges to zero as  $p, T \rightarrow_c \infty$ . Since the Lévy distance metrizes the weak convergence (see Billingsley (1995), problem 14.5), the limiting d.f. is not affected. For further details, see the proof of Theorem 1 in the Appendix.

**Remark 2** *In standard cases where  $D_t$  is represented by  $(1, t)$ , it is customary to*

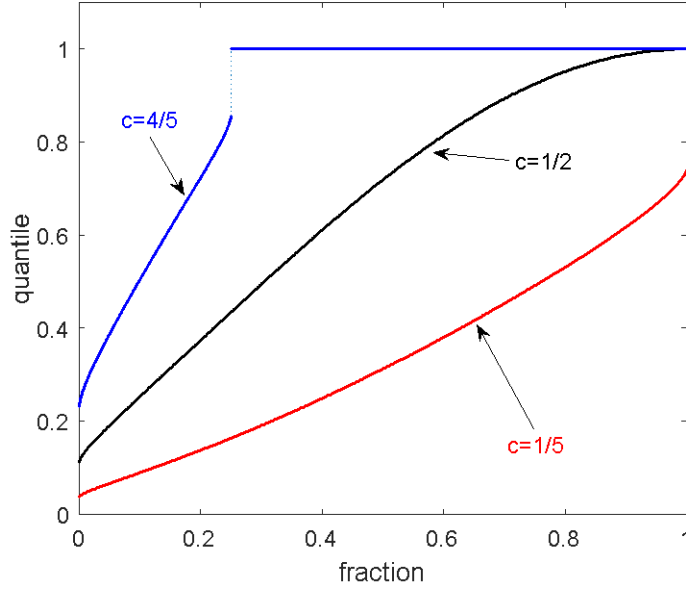


Figure 1: Quantile functions of  $W(\lambda; c/(1+c), 2c/(1+c))$  for  $c = 1/5$ ,  $c = 1/2$ , and  $c = 4/5$ .

impose restrictions on  $\Phi$  so that there is no quadratic trend in  $X_t$  (see Johansen (1995), ch. 6.2). Then, the LR test of the null of  $r$  cointegrating relationships is based on the eigenvalues of  $S_{01}^* S_{11}^{*-1} S_{01}^{*'} S_{00}^{*-1}$ , defined similarly to  $S_{01} S_{11}^{-1} S_{01}' S_{00}^{-1}$  by replacing  $X_{t-1}$  with  $(X_{t-1}', t)'$  and regressing  $\Delta X_t$  and  $(X_{t-1}', t)'$  on constant only to obtain  $R_{0t}$  and  $R_{1t}$ . The empirical distribution function of so modified eigenvalues still converges to  $W(\lambda; c/(1+c), 2c/(1+c))$  because the difference between matrices  $S_{01}^* S_{11}^{*-1} S_{01}^{*'} S_{00}^{*-1}$  and  $S_{01} S_{11}^{-1} S_{01}' S_{00}^{-1}$  has small rank.

Figure 1 shows quantile plots of the Wachter distribution with parameters  $\gamma_1 = c/(1+c)$  and  $\gamma_2 = 2c/(1+c)$  for different values of  $c$ . For  $c = 1/5$ , the dimensionality of the data constitutes 20% of the sample size. The corresponding Wachter limit of  $F_{p,T}(\lambda)$  is supported on  $[0.04, 0.74]$ . In particular, we expect  $\lambda_1$  be larger than 0.7 for large  $p, T$  even in the absence of any cointegrating relationships. For  $c = 1/2$ , the upper boundary of support of the Wachter limit is unity. This accords with Gonzalo and Pitarakis' (1995, Lemma 2.3.1) finding that as  $T/p \rightarrow 2$ ,  $\lambda_1 \rightarrow 1$ . For  $c = 4/5$ , the Wachter limit has mass 3/4 at unity.

Wachter (1980) derives  $W(\lambda; \gamma_1, \gamma_2)$  as the weak limit of the empirical d.f. of eigenvalues of the  $p$ -dimensional beta<sup>2</sup> matrix  $B_p(n_1/2, n_2/2)$  with  $n_1, n_2$  degrees

<sup>2</sup>For the definition of the multivariate beta see Muirhead (1982), p.110.

of freedom as  $p, n_1, n_2 \rightarrow \infty$  so that  $p/n_1 \rightarrow \gamma_1/\gamma_2$  and  $p/n_2 \rightarrow \gamma_1/(1 - \gamma_2)$ . The eigenvalues of multivariate beta matrices are related to many important concepts in multivariate statistics, including canonical correlations, multiple discriminant ratios, and MANOVA. In particular, the squared sample canonical correlations between  $q$ - and  $m$ -dimensional independent Gaussian samples of size  $n$  are jointly distributed as the eigenvalues of  $B_q(m/2, (n - m)/2)$ , where  $q \leq m$  and  $n \geq q + m$ . Therefore, their empirical d.f. weakly converges to  $W(\lambda; \gamma_1, \gamma_2)$  with  $\gamma_1 = \lim q/n$  and  $\gamma_2 = \lim m/n$ , as mentioned above. Since the squared canonical correlations in Theorem 1 are between random walk and its innovations rather than independent white noises, the convergence to the Wachter distribution came to us as a pleasant surprise.

In the context of multiple discriminant analysis, Wachter (1976b) proposes to use a quantile-quantile (qq) plot, where the multiple discriminant ratios are plotted against quantiles of  $W(\lambda; \gamma_1, \gamma_2)$ , as a simple graphical method that helps one “recognize hopeless from promising analyses at an early stage.” A plot that clearly deviates from the  $45^\circ$  line suggests that the data are at odds with the null hypothesis of the homogeneous population, and a further analysis of the heterogeneity is useful. Nowadays, such qq plots are called *Wachter plots* (see Johnstone (2001)).

Theorem 1 implies that the Wachter plot can be used as a simple preliminary assessment of cointegration hypotheses in large VARs. As an illustration, Figure 2 shows a Wachter plot of the simulated sample squared canonical correlations corresponding to a 20-dimensional VAR(1) model (7) with  $\Pi = \text{diag}\{-I_3, 0 \times I_{17}\}$  so that there are three white noise and seventeen random walk components of  $X_t$ . No deterministic terms are included. We set  $T = 200$  so that  $c = 1/10$ . The graph clearly shows three canonical correlations that destroy the  $45^\circ$  line fit, so that the null hypothesis of no cointegration is compromised.

Theorem 1 does not provide any explanation to the fact that exactly three canonical correlations deviate from the  $45^\circ$  line in Figure 2. To interpret deviations of the Wachter plots from the  $45^\circ$  line, it is desirable to investigate behavior of  $F_{p,T}(\lambda)$  under various alternatives. So far, we were able to obtain a clear result only for the “extreme” alternative, where  $X_t$  is a vector of independent white noises. Under such an alternative,

$$F_{p,T}(\lambda) \Rightarrow W(\lambda; c/(2 - c), 1/(2 - c)). \quad (14)$$

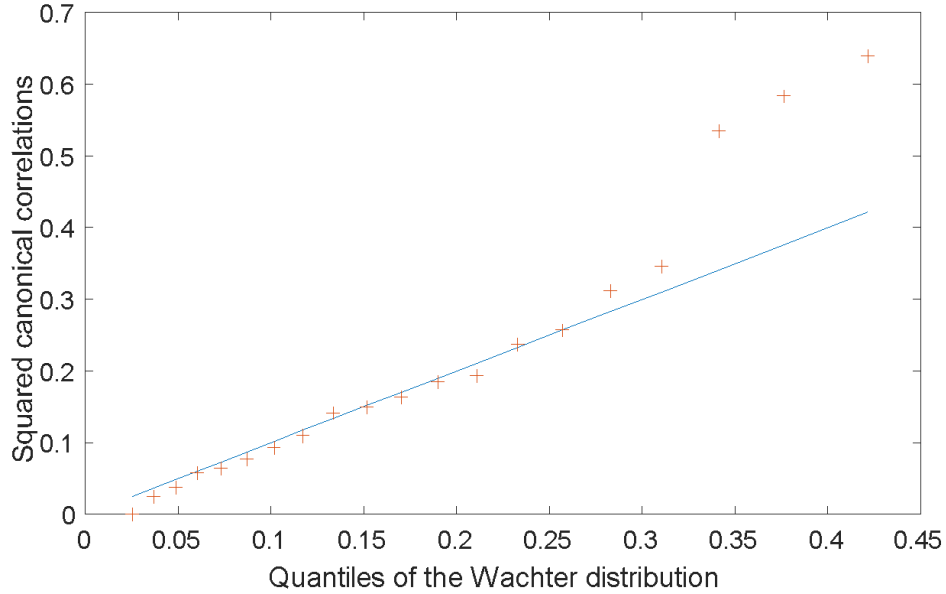


Figure 2: Wachter plot of the squared canonical correlations corresponding to 20-dimensional series with 3 components being white noises and the other components being independent random walks.  $p = 20, T = 200$ .

We plan to publish a full proof of this and some related results elsewhere.

Interestingly, for  $c = 1/2$ , the Wachter limits (13) and (14) corresponding to random walk and white noise nulls, respectively, coincide. Hence, as  $c$  approaches  $1/2$ , not only the largest sample canonical correlation converges to one and the LR test breaks down, but also the Wachter plot loses the ability to differentiate between opposite cointegration hypotheses. For smaller values of  $c$ , however, the Wachter limits (13) and (14) become well separated. We provide Monte Carlo analysis of the behavior of  $F_{p,T}(\lambda)$  under some alternative hypotheses in Section 4 below.

The almost sure weak convergence of  $F_{p,T}(\lambda)$  established in Theorem 1 implies the almost sure convergence of bounded continuous functionals of  $F_{p,T}(\lambda)$ . An example of such a functional is the scaled Pillai-Bartlett statistic for the null of no more than  $r$  cointegrating relationships (see Gonzalo and Pitarakis (1995))

$$\frac{1}{Tp} PB_{r,p,T} = \frac{1}{p} \sum_{j=r+1}^p \lambda_j,$$

which is asymptotically equivalent to the LR statistic under the standard asymptotic regime where  $p$  is fixed and  $T \rightarrow \infty$ . Since, by definition,  $\lambda_j \in [0, 1]$ , we

have

$$\frac{1}{Tp} PB_{r,p,T} = \int f(\lambda) dF_{p,T}(\lambda) - \frac{1}{p} \sum_{j=1}^r \lambda_j, \quad (15)$$

where  $f$  is the bounded continuous function

$$f(\lambda) = \begin{cases} 0 & \text{for } \lambda < 0 \\ \lambda & \text{for } \lambda \in [0, 1] \\ 1 & \text{for } \lambda > 1. \end{cases}$$

As long as  $r/p \rightarrow 0$  as  $p, T \rightarrow_c \infty$ , the second term on the right hand side of (15) converges to zero. Therefore, Theorem 1 implies that  $PB/(Tp)$  almost surely converges to  $\int f(\lambda) dW(\lambda; c/(1+c), 2c/(1+c))$ . A direct calculation based on (9), which we report in the Supplementary Appendix, yields the following corollary.

**Corollary 3** *Under the assumptions of Theorem 1, as  $p, T \rightarrow_c \infty$ , almost surely,*

$$PB_{r,p,T}/(Tp) \rightarrow 2c/(1+c) + \max\{0, 2 - 1/c\}.$$

A similar analysis of the LR statistic is less straightforward because

$$\frac{1}{Tp} LR_{r,p,T} = -\frac{1}{p} \sum_{j=r+1}^p \log(1 - \lambda_j),$$

and  $\log(1 - \lambda)$  is unbounded on  $\lambda \in [0, 1]$ . In fact, for  $c > 1/2$ ,  $LR_{r,p,T}$  is ill-defined because a non-negligible proportion of the squared sample canonical correlations exactly equal unity. However for  $c < 1/2$ , we can obtain the almost sure asymptotic lower bound on  $LR_{r,p,T}/(Tp)$ . Note that for such  $c$ , the upper bound of the support of  $W(\lambda; c/(1+c), 2c/(1+c))$  equals  $b_+ = c(\sqrt{2} - \sqrt{1-c})^{-2} < 1$ . Let

$$\overline{\log}(1 - \lambda) = \begin{cases} 0 & \text{for } \lambda < 0 \\ \log(1 - \lambda) & \text{for } \lambda \in [0, b_+] \\ \log(1 - b_+) & \text{for } \lambda > b_+. \end{cases} \quad (16)$$

Clearly,  $\overline{\log}(1 - \lambda)$  is a bounded continuous function and

$$\frac{1}{Tp} LR_{r,p,T} \geq -\frac{1}{p} \sum_{j=r+1}^p \overline{\log}(1 - \lambda_j).$$

Hence, we have the following a.s. lower bound on  $LR_{r,p,T}/(Tp)$  (the corresponding calculations are reported in the Supplementary Appendix).

**Corollary 4** *Under the assumptions of Theorem 1, for  $c < 1/2$ , as  $p, T \rightarrow_c \infty$ , almost surely,*

$$\lim_{p, T \rightarrow_c \infty} \inf \frac{1}{Tp} LR_{r,p,T} \geq \frac{1+c}{c} \ln(1+c) - \frac{1-c}{c} \ln(1-c) + \frac{1-2c}{c} \ln(1-2c).$$

**Remark 5** *We conjecture that the lower bound reported in the corollary is, in fact, the a.s. limit of  $LR_{r,p,T}/(Tp)$ . To prove this conjecture, one needs to show that  $\lambda_{r+1}$  is almost surely bounded away from unity so that the unboundedness of  $\log(1-\lambda)$  is not consequential. We leave this as an important topic for future research.*

Corollaries 3 and 4 suggest appropriate “centering points” for PB and LR statistics scaled by  $1/(Tp)$  for relatively large and comparable  $p$  and  $T$ . As we show in the next section, the standard asymptotic distribution of the scaled PB and LR statistics are likely<sup>3</sup> to concentrate around very different points when  $p$  becomes large. As will be seen below, this difference sheds new light on the over-rejection phenomenon discussed above and on the workings and limitations of the Bartlett correction for the LR statistic. To study the concentration of the standard asymptotic distributions of the scaled PB and LR statistics as  $p$  grows, we will consider the *sequential* asymptotic regime, where first  $T \rightarrow \infty$ , and then  $p \rightarrow \infty$ .

## 3 Sequential asymptotics and its consequences

### 3.1 Sequential asymptotics

To obtain useful results under the sequential asymptotics, we shall study eigenvalues of the scaled matrix

$$\frac{T}{p} S_{01} S_{11}^{-1} S'_{01} S_{00}^{-1}. \quad (17)$$

Note that under the simultaneous asymptotic regime  $p, T \rightarrow_c \infty$ , the asymptotic behavior of the scaled and unscaled eigenvalues is the same up to the factor  $c^{-1}$ .

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<sup>3</sup>We only establish lower bounds on the concentration points. However, Monte Carlo evidence suggests that these bounds are in fact the points of concentration.

However, as first  $T \rightarrow \infty$  while  $p$  remains fixed, the unscaled eigenvalues converge to zero, while scaled ones do not. We shall denote the empirical d.f. of eigenvalues of the scaled matrix as  $F_{p,T}^{(s)}(\lambda)$ .

Without loss of generality, we focus on the case of simple data generating process

$$\Delta X_t = \varepsilon_t, \quad t = 1, \dots, T, \quad \text{and} \quad X_0 = 0, \quad (18)$$

and on the situation, where the econometrician does not include any deterministic regressors in his or her model, that is  $d_D = 0$ . There is no loss of generality in such simplifications because, as follows from Lemma 11 and the rank inequality used in the proof of Lemma 10 in the Appendix, the Lévy distance between the versions of  $F_{p,T}^{(s)}(\lambda)$  that correspond to the simplified and the general cases is bounded from above by a fixed multiple of  $(d_D + d_F + r + k + \text{rank } \Gamma) / p$ . We shall assume that the latter expression goes to zero as  $p \rightarrow \infty$ . Therefore, whatever the sequential asymptotic limit of  $F_{p,T}^{(s)}(\lambda)$  is under the above simplification, it must also be the sequential asymptotic limit under the general case. For simplicity, in the rest of this section, we shall assume that  $r = 0$ , and will consider statistics  $LR_{0,p,T}$  rather than more general  $LR_{r,p,T}$ .

Under the above simplifications, Johansen's (1988, 1991) results imply that, as  $T \rightarrow \infty$  while  $p$  is held fixed, the eigenvalues of the scaled matrix (17) jointly converge in distribution to the eigenvalues of

$$\frac{1}{p} \int_0^1 (dB) B' \left( \int_0^1 BB' du \right)^{-1} \int_0^1 B (dB)', \quad (19)$$

where  $B$  is a  $p$ -dimensional Brownian motion. We denote the eigenvalues of (19) as  $\lambda_j^{(\infty)}$ , and their empirical d.f. as  $F_{p,\infty}(\lambda)$ .

It is not unreasonable to expect that, as  $p \rightarrow \infty$ ,  $F_{p,\infty}(\lambda)$  becomes close to the limit of the empirical distribution of eigenvalues of (17) under a simultaneous, rather than sequential, asymptotic regime  $p, T \rightarrow_\gamma \infty$ , where  $\gamma$  is close to zero. We shall denote such a limit as  $F_\gamma(\lambda)$ . This expectation turns out to be correct in the sense that the following theorem holds.

**Theorem 6** *Let  $F_0(\lambda)$  be the weak limit as  $\gamma \rightarrow 0$  of  $F_\gamma(\lambda)$ . Then, as  $p \rightarrow \infty$ ,  $F_{p,\infty}(\lambda)$  weakly converges to  $F_0(\lambda)$ , in probability.*

Importantly, the weak limit  $F_0(\lambda)$  is not the Wachter d.f. Instead, the following proposition holds.

**Proposition 7**  $F_0(\lambda)$  corresponds to a distribution supported on  $[a_-, a_+]$  with

$$a_{\pm} = \left(1 \pm \sqrt{2}\right)^2, \quad (20)$$

and having density

$$f(\lambda) = \frac{1}{2\pi} \frac{\sqrt{(a_+ - \lambda)(\lambda - a_-)}}{\lambda}. \quad (21)$$

A reader familiar with Large Random Matrix Theory (see Bai and Silverstein (2010)) might recognize that  $F_0(\lambda)$  is the cumulative distribution function of the continuous part of a special case of the Marchenko-Pastur distribution (Marchenko and Pastur (1967)). The general Marchenko-Pastur distribution has density

$$f_{MP}(\lambda; \kappa, \sigma^2) = \frac{1}{2\pi\sigma^2\kappa} \frac{\sqrt{(a_+ - \lambda)(\lambda - a_-)}}{\lambda}$$

over  $[a_-, a_+]$  with  $a_{\pm} = \sigma^2(1 \pm \sqrt{\kappa})^2$  and a point mass  $\max\{0, 1 - 1/\kappa\}$  at zero. Density (21) is two times  $f_{MP}(\lambda; \kappa, \sigma^2)$  with  $\kappa = 2$  and  $\sigma^2 = 1$ . The multiplication by two is needed because the mass  $1/2$  at zero is not a part of the distribution  $F_0$ .

Recall that, as  $T \rightarrow \infty$  while  $p$  remains fixed, the LR statistic converges in distribution to  $p$  times the trace of matrix (19):

$$LR_{0,p,T} \xrightarrow{d} p \sum_{j=1}^p \lambda_j^{(\infty)} \text{ as } T \rightarrow \infty. \quad (22)$$

On the other hand, according to Theorem 6, for any  $\delta_1, \delta_2 > 0$  and all sufficiently large  $p$ ,

$$\Pr\left(\frac{1}{p} \sum_{j=1}^p \lambda_j^{(\infty)} \geq \int \lambda dF_0(\lambda) - \delta_1\right) \geq 1 - \delta_2. \quad (23)$$

A direct calculation, which we report in the Supplementary Appendix, shows that  $\int \lambda dF_0(\lambda) = 2$ . Hence, we have the following corollary.

**Corollary 8** *As first  $T \rightarrow \infty$ , and then  $p \rightarrow \infty$ , the lower probability bound on  $LR_{0,p,T}/(2p^2)$  is unity in the following sense. As  $T \rightarrow \infty$  while  $p$  is held fixed,  $LR_{0,p,T}/(2p^2)$  converges in distribution to  $\sum_{j=1}^p \lambda_j^{(\infty)}/(2p)$ . Further, for any  $\delta_1, \delta_2 > 0$  and all sufficiently large  $p$ , the probability that  $\sum_{j=1}^p \lambda_j^{(\infty)}/(2p)$  is no smaller than  $1 - \delta_1$  is no smaller than  $1 - \delta_2$ .*

The reason why we only claim the lower bound on  $LR_{0,p,T}/(2p^2)$  is that Theorem 6 is silent about the behavior of the individual eigenvalues  $\lambda_j^{(\infty)}$ , the largest

of which may, in principle, quickly diverge to infinity. We suspect that 2 is not just the lower bound, but also the probability limit of  $\sum_{j=1}^p \lambda_j^{(\infty)}/p$ , so that the sequential probability limit of  $LR_{0,p,T}/(2p^2)$  is unity. Verification of this conjecture requires more work, similar to that discussed in Remark 5.

Corollary 8 is consistent with the numerical finding of Johansen, Hansen and Fachin (2005, Table 2) that, as  $T$  becomes large while  $p$  is being fixed, the sample mean of the LR statistic is well approximated by a polynomial  $2p^2 + \alpha p$  (see also Johansen (1988) and Gonzalo and Pitarakis (1995)). The value of  $\alpha$  depends on how many deterministic regressors are included in the VAR. Our theoretical result captures only the ‘highest order’ sequential asymptotic behavior of the LR statistic, which remains (bounded below by)  $2p^2$  independent on the number of the deterministic regressors.

Another piece of numerical support for  $2p^2$  being not only the lower bound but also the first order sequential asymptotic approximation to the LR statistic is provided by the tables of the asymptotic critical values for Johansen’s LR test (see, for example, MacKinnon, Haug and Michelis (1999)). The critical values in such tables become uncomfortably large for  $p > 4$ . Of course, the reason for such an unpleasant growth is that those critical values are of order  $2p^2$ .

The transformation

$$LR_{0,p,T} \mapsto LR_{0,p,T}/p - 2p$$

makes the LR statistic ‘well-behaved’ under the sequential asymptotics. The division by  $p$  reduces the ‘second order behavior’ to  $O_P(1)$ , while subtracting  $2p$  eliminates the remaining explosive ‘highest order term’. We report the corresponding transformed 95% critical values alongside the original ones in Table 1.

The transformed critical values resemble 97-99 percentiles of  $N(0,1)$ . Since the LR test is one-sided, the resemblance is coincidental. However, we do expect the sequential asymptotic distribution of the transformed LR statistic (as well as its simultaneous asymptotic distribution) to be normal (possibly with non-zero mean and non-unit variance). Our expectation is based on the fact that  $LR_{0,p,T}/p$  behaves as the eigenvalue average (see (22)), which is a special case of the so-called linear spectral statistic. The asymptotic normality of linear spectral statistics for relatively simple classes of high-dimensional random matrices is a well established result in the Large Random Matrix Theory (see Bai and Silverstein (2010)). Extending it to the linear spectral statistics of matrices of form (19) is left as an important direction for future research.

$p$	Unadjusted CV	$CV/p - 2p$
1	4.13	2.13
2	12.32	2.16
3	24.28	2.09
4	40.17	2.04
5	60.06	2.01
6	83.94	1.99
7	111.79	1.97
8	143.64	1.96
9	179.48	1.94
10	219.38	1.94
11	263.25	1.93
12	311.09	1.92

Table 1: The 95% asymptotic critical values (CV) for Johansen’s LR test. The unadjusted values are taken from the first column of Table II in MacKinnon, Haug and Michelis (1999).

### 3.2 Over-rejection phenomenon, and the Bartlett correction

In this subsection, let us assume that the following conjecture holds.

**Conjecture 9** *The simultaneous and sequential asymptotic lower bounds for the scaled LR statistics derived in Corollaries 4 and 8 represent the corresponding simultaneous and sequential asymptotic limits. Specifically, for  $c < 1/2$ ,*

$$\lim_{p,T \rightarrow c\infty} \frac{1}{2p^2} LR_{0,p,T} = \frac{1+c}{2c^2} \ln(1+c) - \frac{1-c}{2c^2} \ln(1-c) + \frac{1-2c}{2c^2} \ln(1-2c), \quad (24)$$

$$\text{plim}_{p \rightarrow \infty} \lim_{T \rightarrow \infty} \frac{1}{2p^2} LR_{0,p,T} = 1. \quad (25)$$

Figure 3 plots the right hand side of (24) against the value of  $c \in [0, 1/2)$ . As demonstrated by the Monte Carlo analysis of the next section, in finite samples with comparable values of  $p$  and  $T$ , simultaneous asymptotics provides a better approximation to the finite sample behavior of the LR statistic than the sequential asymptotics. Therefore, ‘typical’ finite sample values of the LR statistic are concentrated around the solid line in Figure 3, and above the dashed line, which represents the points of concentration of the ‘standard’ asymptotic critical values for the LR test. In other words, the standard asymptotic distribution of the LR

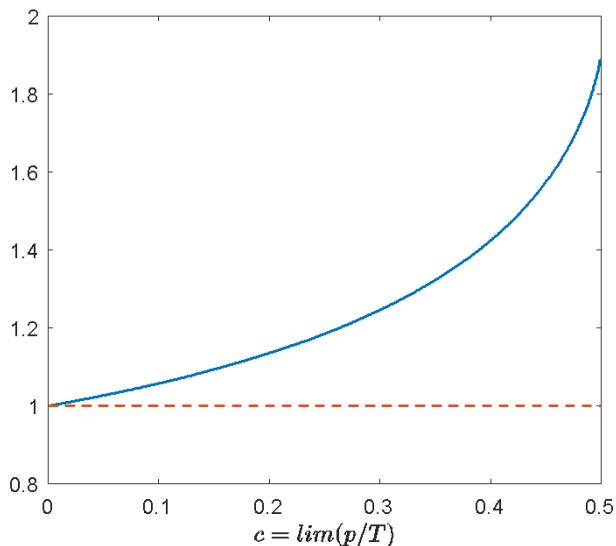


Figure 3: The asymptotic limits (under Conjecture 9) of the scaled LR statistic  $L_{0,p,T}/(2p^2)$ . Dashed line: sequential asymptotic limit. Solid line: simultaneous asymptotic limit.

statistic is centered at a too low level. This leads to the over-rejection of the null of no cointegration.

Gonzalo and Pitarakis (1995) propose an interesting approach to address the problem. Using Monte Carlo, they find that, in contrast to the LR test, the Pillai-Bartlett test based on the PB statistic under-rejects the null. Therefore, they propose to test cointegration hypotheses using the average of the LR and PB statistics. According to Corollary 3, under the simultaneous asymptotics  $PB/(2p^2) \rightarrow 1/(1+c)$ , almost surely. This convergence holds independent on whether Conjecture 9 is true or not.

The fact that  $(1+c)^{-1}$  is smaller than one, explains the under-rejection of the test based on the PB statistic. More interestingly, the average of the simultaneous asymptotic limits of the LR and PB statistics (divided by  $2p^2$ ) turns out to be numerically close to one, and hence to the point of the concentration of the standard critical values (divided by  $2p^2$ ), at least for  $c < 1/3$ . Figure 4 shows such an average. This explains the much better performance of the  $(LR+PB)/2$  test relative to the LR test in Gonzalo and Pitarakis' (1995) Monte Carlo experiments.

A more systematic and popular approach to addressing the over-rejection problem is based on the Bartlett-type correction of the LR statistic. It was explored in much detail in various important studies, including Johansen (2002). The idea

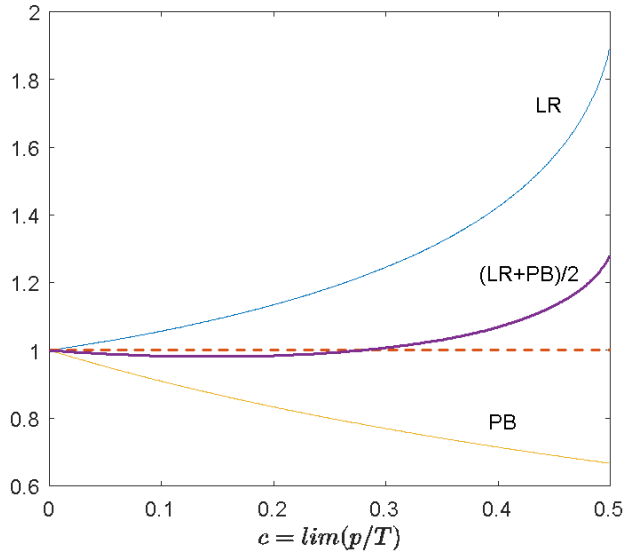


Figure 4: The almost sure limits (under Conjecture 9) of the scaled LR, PB, and  $(LR+PB)/2$  statistics under the simultaneous asymptotic regime.

is to scale the LR statistic so that its finite sample distribution better fits the asymptotic distribution of the unscaled statistic. Specifically, let  $E_{p,\infty}(LR)$  be the mean of the asymptotic distribution under the fixed- $p$ , large- $T$  asymptotic regime. Then, if the finite sample mean,  $E_{p,T}(LR)$ , satisfies

$$E_{p,T}(LR) = E_{p,\infty}(LR) \left( 1 + \frac{a(p)}{T} + o\left(\frac{1}{T}\right) \right), \quad (26)$$

the scaled statistic is defined as  $LR / (1 + a(p)/T)$ . By construction, the match between the scaled mean and the original asymptotic mean is improved by an order of magnitude. Although, as shown by Jensen and Wood (1997) in the context of unit root testing, the match between higher moments does not improve by an order of magnitude, it may become substantially better (see Nielsen (1997)).

A theoretical analysis of the adjustment factor  $1 + a(p)/T$  can be rather involved. In general,  $a(p)$  will depend not only on  $p$ , but also on all the parameters of the VAR. However, for Gaussian VAR(1) without deterministic terms, under the null of no cointegration,  $a(p)$  depends only on  $p$ .

For  $p = 1$ , the exact expression for  $a(p)$  was derived in Larsson (1998). Given the difficulty of the theoretical analysis of  $a(p)$ , Johansen (2002) proposes to numerically evaluate the Bartlett correction factor  $BC_{p,T} \equiv E_{p,T}(LR) / E_{p,\infty}(LR)$  by simulation. Johansen, Hansen and Fachin (2005) simulate  $BC_{p,T}$  for various values

of  $p \leq 10$  and  $T \leq 3000$  and fit a function of the form

$$BC_{p,T}^* = \exp \left\{ a_1 \frac{p}{T} + a_2 \left( \frac{p}{T} \right)^2 + \frac{1}{T} \left[ a_3 \left( \frac{p}{T} \right)^2 + b \right] \right\}$$

to the obtained results. For relatively large values of  $T$ , the term  $\frac{1}{T} \left[ a_3 \left( \frac{p}{T} \right)^2 + b \right]$  in the above expression is small. When it is ignored, the fitted function becomes particularly simple:

$$\widetilde{BC}_{p,T} = \exp \left\{ 0.549 \frac{p}{T} + 0.552 \left( \frac{p}{T} \right)^2 \right\}.$$

Our simultaneous and sequential asymptotic results shed light on the workings of  $\widetilde{BC}_{p,T}$ . Given that Conjecture 9 holds,

$$\frac{\lim_{p,T \rightarrow c\infty} LR_{0,p,T}}{p \lim_{T \rightarrow \infty, p \rightarrow \infty} LR_{0,p,T}} = \frac{1+c}{2c^2} \ln(1+c) - \frac{1-c}{2c^2} \ln(1-c) + \frac{1-2c}{2c^2} \ln(1-2c).$$

Therefore, for non-negligible  $p/T$ , we expect  $BC_{p,T}$  to be well approximated by

$$\widehat{BC}_{p,T} = \frac{1+\hat{c}}{2\hat{c}^2} \ln(1+\hat{c}) - \frac{1-\hat{c}}{2\hat{c}^2} \ln(1-\hat{c}) + \frac{1-2\hat{c}}{2\hat{c}^2} \ln(1-2\hat{c}),$$

where  $\hat{c} = p/T$  is the finite sample analog of  $c$ .

Figure 5 superimposes the graphs of  $\widehat{BC}_{p,T}$  and  $\widetilde{BC}_{p,T}$  as functions of  $\hat{c}$ . For  $p/T \leq 0.3$ , there is a strikingly good match between the two curves, with the maximum distance between them 0.0067. For  $p/T > 0.3$  the quality of the match quickly deteriorates. This can be explained by the fact that all  $p, T$ -pairs used in Johansen, Hansen and Fachin's (2005) simulations are such that  $p/T < 0.3$ .

Further, the good match between  $\widetilde{BC}_{p,T}$  and  $\widehat{BC}_{p,T}$  observed for  $p/T < 0.3$  would be impossible had Johansen, Hansen and Fachin's (2005) specified the Bartlett correction factor as a linear function of  $p/T$ . Note that the standard theoretical choice for the Bartlett correction factor,  $1 + a(p)/T$  from (26), can be viewed as a linear function of  $p/T$  with a slope possibly varying with  $p$ . This is obvious when  $a(p)/T$  is represented as  $\frac{p}{T}\beta(p)$  with  $\beta(p) = a(p)/p$ . Figure 5 shows that such theoretical correction factors cannot work well uniformly with respect to  $p/T$ . Uniformly good correction factors must include terms  $(p/T)^j$  with  $j > 1$ . Under the fixed- $p$ , large- $T$  asymptotics, such terms are of lower order than  $1/T$ , but under the simultaneous asymptotics, they are of order  $O(1)$ .

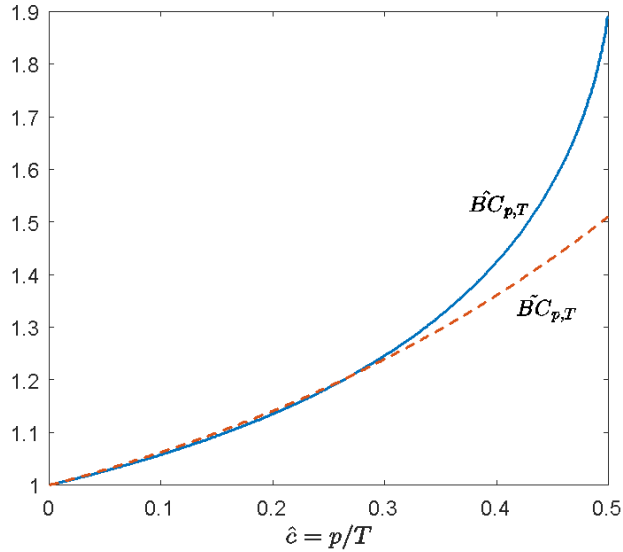


Figure 5: Bartlett correction factors as functions of  $p/T$ . Solid line: the factor based on simultaneous asymptotics. Dashed line: numerical approximation from Johansen, Hansen and Fachin (2005).

Although the Bartlett-type correction approach may deliver good results for high-dimensional systems with carefully chosen correction factor, we believe that tests based on the simultaneous asymptotics of the appropriately scaled and centered LR statistic would be preferable for relatively large  $p$ .

## 4 Monte Carlo and some examples

In this section, we describe results of small-scale Monte Carlo experiments that assess the finite sample quality of the Wachter asymptotic approximation. In addition, we illustrate the Wachter qq plot technique using a macroeconomic dataset of relatively high dimensions.

### 4.1 Monte Carlo experiments

First, we generate pure random walk data with zero starting values for  $p = 10, T = 100$  and  $p = 10, T = 20$ . Throughout this section, the analysis is based on 1000 Monte Carlo replications. The generated random walk data are ten-dimensional so that there are ten corresponding squared sample canonical correlations,  $\lambda$ . Figure 6 shows the Tukey boxplots summarizing the MC distribution of each of the  $\lambda_i$ ,

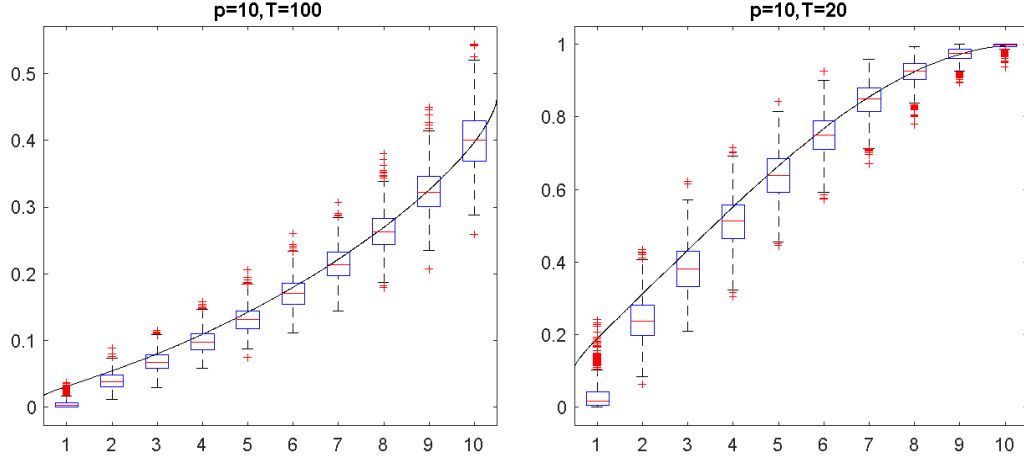


Figure 6: The Tukey boxplots for 1000 MC simulations of ten sample squared canonical correlations corresponding to pure random walk data. The boxplots are superimposed with the quantile function of the Wachter limit.

$i = 1, \dots, 10$  (sorted in the ascending order throughout this section). The boxplots are superimposed with the quantile function of the Wachter limit with  $c = 1/10$  for the left panel and  $c = 1/2$  for the right panel. Precisely, for  $x = i$ , we show the value the  $100(i - 1/2)/p$  quantile of the Wachter limit. For  $i = 1, 2, \dots, 10$ , these are the 5-th, 15-th, ..., 95-th quantiles of  $W(\lambda; c/(1+c), 2c/(1+c))$ .

Even for such small values of  $p$  and  $T$ , the theoretical quantiles track the location of the MC distribution of the empirical quantiles very well. The smallest sample canonical correlation is an exception. Its distribution lies mostly below the corresponding theoretical quantile.

The dispersion of the MC distributions around the theoretical quantile is quite large for the chosen small values of  $p$  and  $T$ . To see how such a dispersion changes when  $p$  and  $T$  increase while  $p/T$  remains fixed, we generated pure random walk data with  $p = 20, T = 200$  and  $p = 100, T = 1000$  for  $p/T = 1/10$ , and with  $p = 20, T = 40$  and  $p = 100, T = 200$  for  $p/T = 1/2$ . Instead of reporting the Tukey boxplots, we plot only the 5-th and 95-th percentiles of the MC distributions of the  $\lambda_i$ ,  $i = 1, \dots, p$  against  $100(i - 1/2)/p$  quantiles of the corresponding Wachter limit. The plots are shown on Figure 7.

We see that the  $[5\%, 95\%]$  ranges of the MC distributions of  $\lambda_i$  are still considerably large for  $p = 20$ . These ranges become much smaller for  $p = 100$ . Interestingly, the distribution of  $\lambda_1$  remains below the Wachter limit even for  $p = 100$ . This does not contradict our theoretical results because a weak limit of the empirical distri-

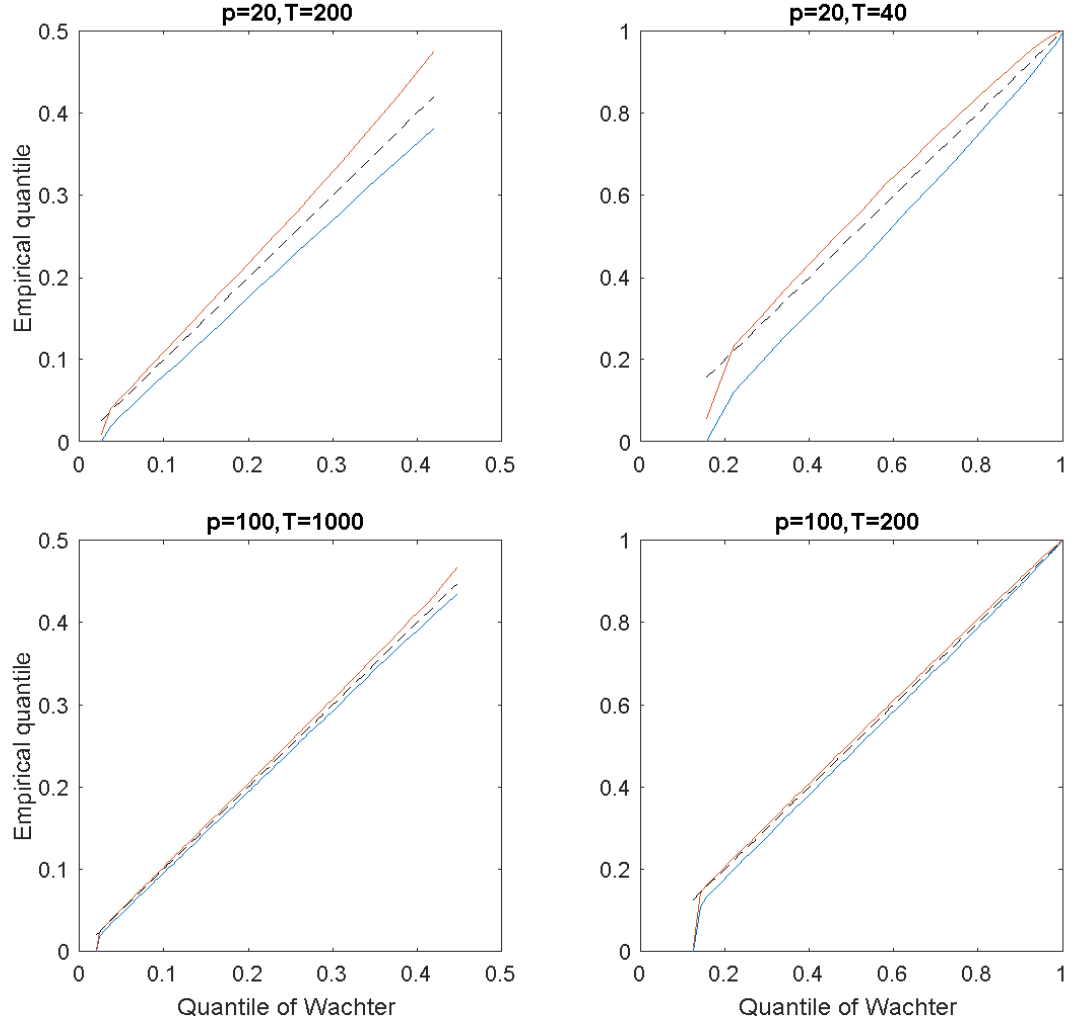


Figure 7: The qq Wachter plots for pure random walk data. The dashed line is the 45° line. The solid lines are the 5-th and the 95-th percentiles of the MC distributions of  $\lambda_i$ , which are plotted against  $100(i-1/2)/p$  quantiles of the Wachter limit

bution of  $\lambda$ 's is not affected by an arbitrary change in a finite (or slowly growing) number of them. In fact, we find it somewhat surprising that only the distribution of  $\lambda_1$  is not well-aligned with the derived theoretical limit. Our proofs are based on several low rank alterations of the matrix  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$ , and there is nothing in them that guarantee that only one eigenvalue of  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$  behaves in a “special” way. In future work, it would be interesting to investigate the behavior of  $\lambda_1$  and other extreme eigenvalues of  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$  theoretically.

Next, we explore the effect of the deterministic regressor on the quality of the Wachter approximation. We generate data with and without constant in the data generating process (11). That is, we consider two cases:  $F_t = 1$  and  $F_t = 0$ . The coefficient  $\Psi$  on  $F_t$  is a  $N(0, I_p)$  vector independent across different MC replications. We also consider two models (7) contemplated by the econometrician: one with  $D_t = 1$ , and the other with  $D_t = 0$ . If  $F_t \neq D_t$ , the econometrician's model is misspecified. Figure 8 shows the Wachter plots similar to those reported in Figure 7. The dimensions of the data are  $p = 20$  and  $T = 100$ .

If the data generating process (DGP) contains constant ( $F_t = 1$ ), but the econometrician does not include it in his or her model, then the largest  $\lambda$ ,  $\lambda_p$ , start to significantly deviate from the  $45^\circ$  line on the Wachter plot (lower right panel). If the econometrician's model is over-specified (lower left panel), there are no dramatic deviations from the line.

Our next Monte Carlo experiment simulates data that are not random walk. Instead, the data are stationary VAR(1) with zero mean, zero initial value, and  $\Pi = \rho I_p$ . We consider three cases of  $\rho$ : 0, 0.5, and 0.95. Figure 9 shows the Wachter plots with solid lines representing 5th and 95-th percentiles of the MC distributions of  $\lambda_i$  plotted against the  $100(i - 1/2)/p$  quantiles of the corresponding Wachter limit. The dashed line correspond to the null case where the data are pure random walk (shown for comparison).

The lower panel of the figure corresponds to the most persistent alternative with  $\rho = 0.95$ . Samples with  $p = 20$  seem to be too small to generate substantial differences in the behavior of Wachter plots under the null and under such persistent alternatives. The less persistent alternative with  $\rho = 0.5$  is easily discriminated against by the Wachter plot for  $p/T = 1/10$  (left panel). The discrimination power of the plot for  $p/T = 1/5$  (central panel) is weaker. For  $p/T = 1/2$  there is still some discrimination power left, but the location of the Wachter plot under alternative “switches” the side relative to the  $45^\circ$  line.

The plots easily discriminate against white noise ( $\rho = 0$ ) alternative for  $c =$

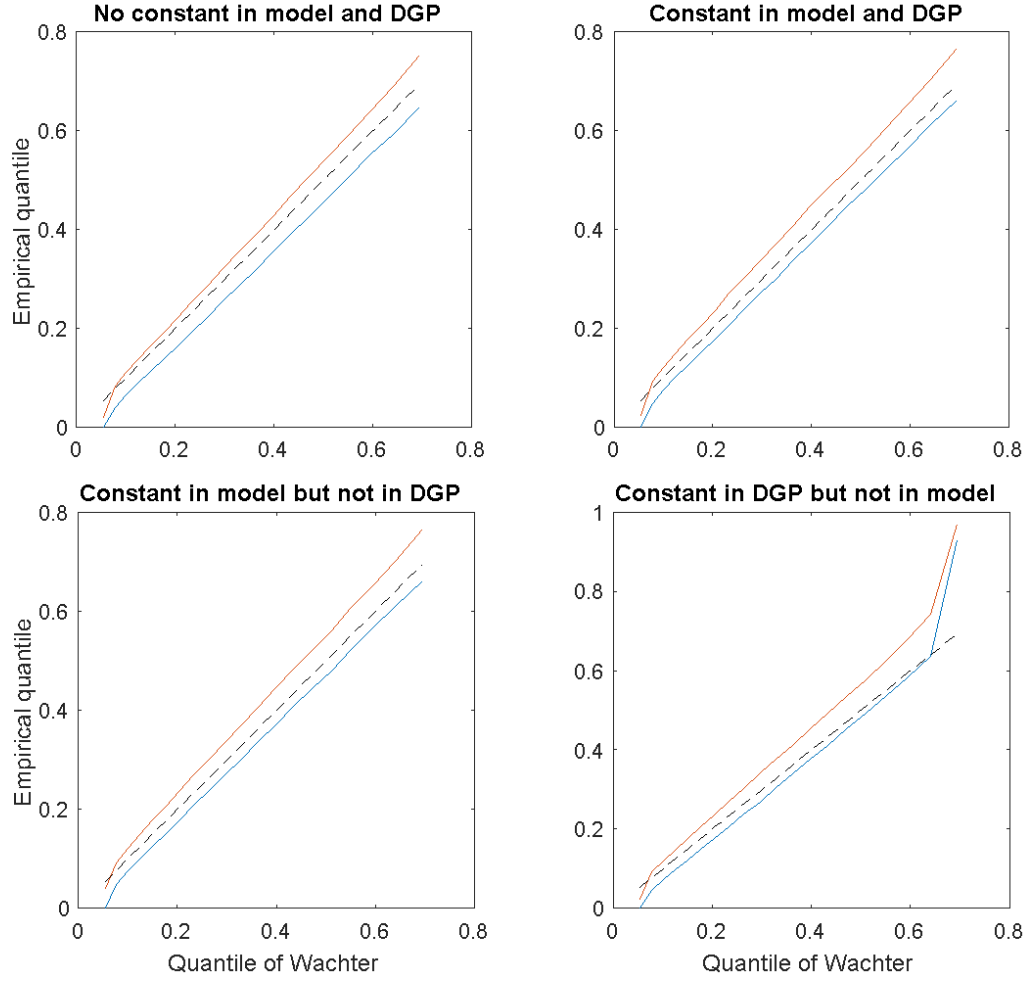


Figure 8: The qq Wachter plots for  $p = 20$  and  $T = 100$ . The data generating process (DGP) is (11) with  $k = 1$ ,  $\Pi = 0$ , and either  $F_t = 1$  (constant in DGP) or  $F_t = 0$  (no constant in GDP). The econometrician's model is (7) with  $\Pi = 0$  and either  $D_t = 1$  (constant in model) or  $D_t = 0$  (no constant in model).

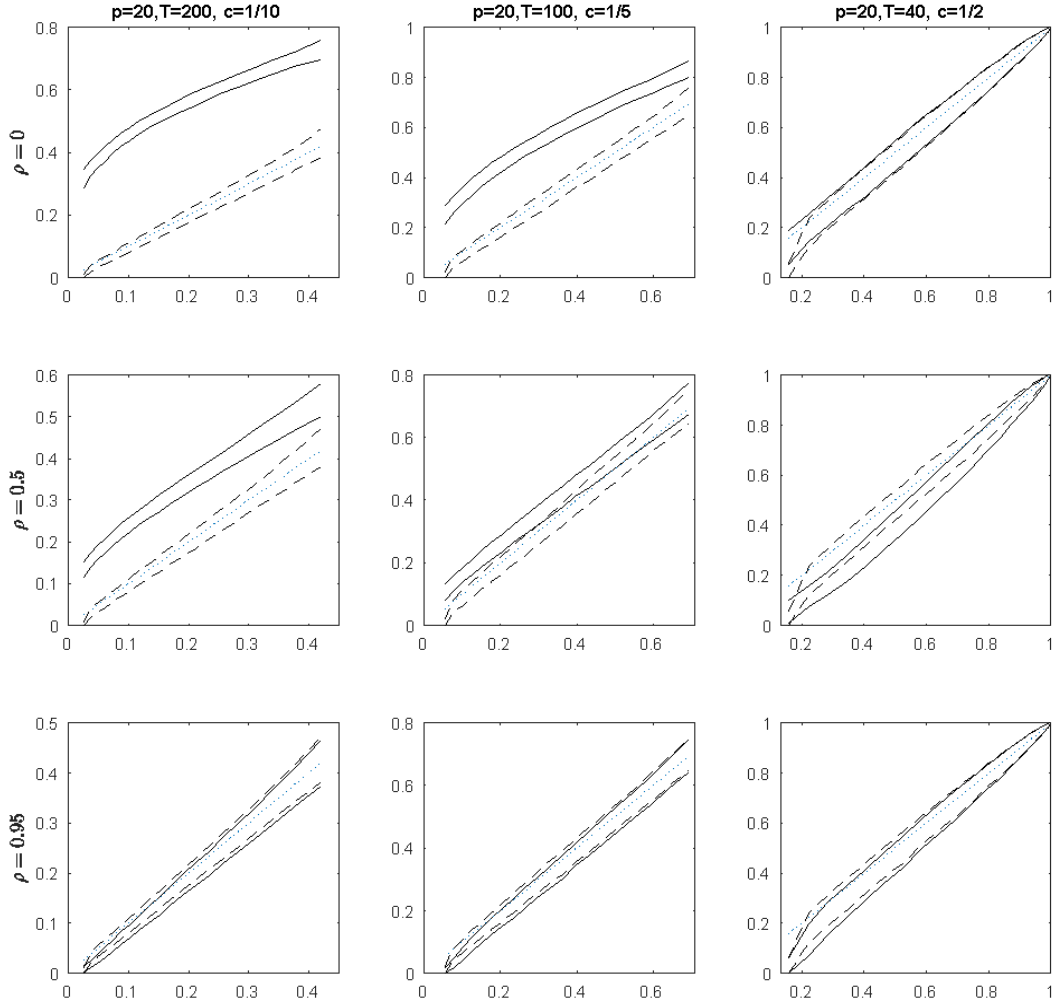


Figure 9: The qq Wachter plots for stationary data  $X_t = \rho X_{t-1} + \varepsilon_t$ . Solid lines: 5 and 95 percentiles of the MC distribution of  $\lambda_i$  plotted against  $100(i - 1/2)/p$  quantile of the Wachter limit. Dashed lines correspond to 5 and 95 percentiles of the MC distribution of  $\lambda_i$  for pure random walk data (the null).

1/10 and  $c = 1/5$ , but not for  $c = 1/2$ . In accordance to the result that we announced above, and plan to publish elsewhere, the Wachter limit for  $c = 1/2$  approximates equally well the empirical distribution of the squared sample canonical correlations based on random walk and on white noise data.

Results reported in Figure 9 indicate that for relatively small  $p$  and  $p/T$ , Wachter plots can be effective in discriminating against alternatives to the null of no cointegration, where the cointegrating linear combinations of the data are not very persistent. Further, tests of no cointegration hypothesis that may be developed using simultaneous asymptotics would probably need to be two-sided. It is because the location of the Wachter plot under the alternative may “switch sides” relative to the  $45^\circ$  depending on the persistence of the data under the alternative. Finally, cases with  $c$  close to  $1/2$  must be analyzed with much care. For such cases, the behavior of the sample canonical correlations become similar under extremely different random walk and white noise data generating processes. Furthermore, the largest sample canonical correlations are close to unity, which can result in an unstable behavior of the LR statistic.

Our final MC experiment studies the finite sample behavior of the scaled LR statistic  $LR_{0,p,T}/(2p^2)$ . We simulate pure random walk data with  $p = 10$  and  $p = 100$  and  $T$  varying so that  $p/T$  equals  $1/10, 2/10, \dots, 5/10$ . Corollary 4 shows that the simultaneous asymptotic lower bound on  $LR_{0,p,T}/(2p^2)$  has form

$$\frac{1+c}{2c^2} \ln(1+c) - \frac{1-c}{2c^2} \ln(1-c) + \frac{1-2c}{2c^2} \ln(1-2c). \quad (27)$$

Figure 10 shows the Tukey boxplots of the MC distributions of  $LR_{0,p,T}/(2p^2)$  corresponding to  $p/T = 1/10, \dots, 5/10$  with  $p = 10$  (left panel), and  $p = 100$  (right panel). The boxplots are superimposed with the plot of the line representing the above displayed formula for the lower bound (with  $c$  replaced by  $p/T$ ). For the case  $p = 10$ , we also show (horizontal dashed line) the 95% asymptotic critical value (scaled by  $1/(2p^2)$ ) of the standard Johansen trace test taken from MacKinnon et al (1999, Table II). For  $p = 100$ , critical values for the standard test are not available, and we show the dashed horizontal line at unit height instead. This is the sequential asymptotic lower bound on  $LR_{0,p,T}/(2p^2)$  as established in Corollary 8.

The reported results support our conjecture that the simultaneous asymptotic lower bound (27) is, in fact, the simultaneous asymptotic limit of  $LR_{0,p,T}/(2p^2)$  for  $c < 1/2$ . Interestingly, the bound is located near the “center” of the MC distribution of the scaled LR statistic even for the case  $c = 1/2$ .

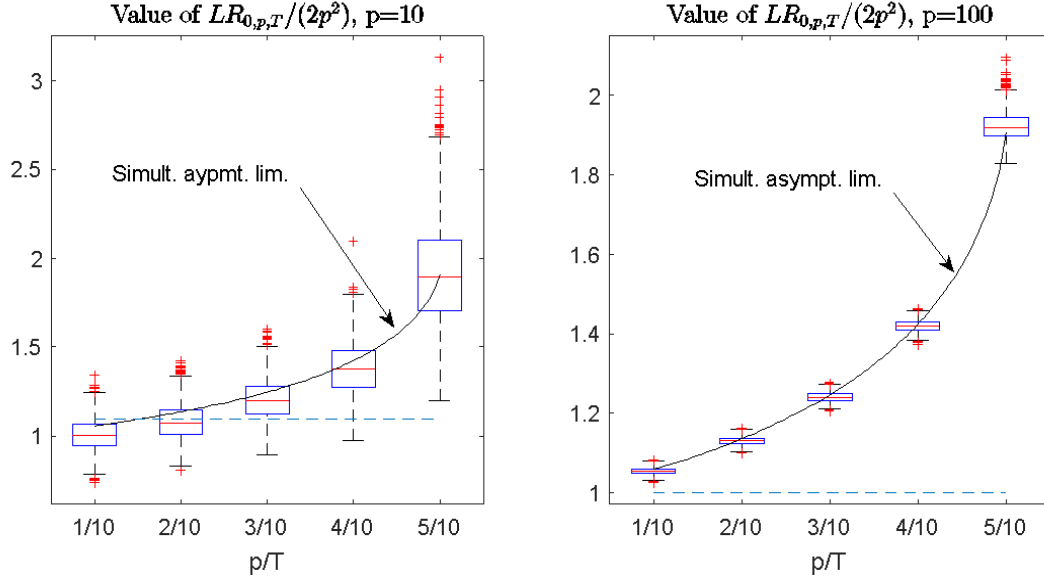


Figure 10: The Tukey boxplots for the MC distributions of  $LR_{0,p,T}/(2p^2)$  for various  $p/T$  ratios. The boxplots are superimposed with the simultaneous asymptotic lower bound on  $LR_{0,p,T}/(2p^2)$ . Dashed line in the left panel correspond to 95% critical value for the standard asymptotic Johansen trace test (taken from MacKinnon et al (1999, Table II)). Dashed line in the right panel has ordinate equal one.

The left panel of Figure 10 illustrates the “over-rejection phenomenon”. The horizontal dashed line that corresponds to the 95% critical value of the standard test is just above the interquartile range of the MC distribution of  $LR_{0,p,T}/(2p^2)$  for  $c = 1/10$ , is below this range for  $c \geq 3/10$ , and is below all 1000 MC replications of the scaled LR statistic for  $c = 5/10$ .

Although the lower bound (27) seems to provide a very good centering point for the scaled LR statistic, the MC distribution of this statistic is quite dispersed around such a center for  $p = 10$ . As discussed above, we suspect that the scaled statistic centered by (27) and appropriately rescaled has Gaussian simultaneous asymptotic distribution. Optimistically, the Tukey plots on Figure 10, that correspond to  $c < 1/2$ , look reasonably symmetric although some skewness is present for the left panel where  $p = 10$ .

## 4.2 Examples

Our first example uses  $T = 103$  quarterly observations (1973q2-1998q4, with the initial observation 1973q1) on bilateral US dollar log nominal exchange rates for

$p = 17$  OECD countries: Australia, Austria, Belgium, Canada, Denmark, Finland, France, Germany, Japan, Italy, Korea, Netherlands, Norway, Spain, Sweden, Switzerland, and the United Kingdom. The data are as in Engel, Mark, and West (2015), and were downloaded from Charles Engel’s website at <http://www.ssc.wisc.edu/~cengel/>. That data are available for a longer time period up to 2008q1, but we have chosen to use only the “early sample” that does not include the Euro period.

Engel, Mark, and West (2015) point out that log nominal exchange rates are well modelled by random walk, but may be cointegrated, which can be utilized to improve individual exchange rate forecasts relative to the random walk forecast benchmark. They propose to estimate the common stochastic trends in the exchange rates by extracting a few factors from the panel. In principle, the number of factors to extract can be determined using Johansen’s test for cointegrating rank, but Engel, Mark, and West (2015) do not exploit this possibility, referring to Ho and Sorensen (1996) that indicates poor performance of the test for large  $p$ .

Figure 11 shows the Wachter plot for the log nominal exchange rate data. The squared sample canonical correlations are computed as the eigenvalues of  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$ , where  $S_{ij}$  are defined as in (8) with  $R_{0t}$  and  $R_{1t}$  being the demeaned changes and the lagged levels of the log exchange rates, respectively. The dashed lines correspond to the 5-th and 95-th percentiles of the MC distribution of the squared canonical correlation coefficients under the null of no cointegration. Precisely, we generated data from model (7) with  $p = 17$ ,  $T = 103$ ,  $\Pi = 0$ ,  $D_t = 1$ , and  $\Phi$  being i.i.d.  $N(0, I_p)$  vectors across the MC repetitions. Log exchange rates for 1973q1 was used as the initial value of the generated series.

The figure shows a mild evidence for cointegration in the data with the largest five  $\lambda$ ’s being close to the corresponding 95-th percentiles of the MC distributions. If we interpret this as the existence of five cointegrating relationships in the data, we would be lead to conclude that there are twelve stochastic trends. Recall, however, that the ability of the Wachter plot to differentiate against highly persistent cointegration alternatives with  $p/T \approx 1/5$  is very low, so there well may be many more cointegrating relationships in the data. Whatever such relationships are, the deviations from the corresponding long-run equilibrium are probably highly persistent as no dramatic deviations from the 45° line are present in the Wachter plot.

Very different Wachter plots (shown in Figure 12) correspond to the log industrial production (IP) index data and the log consumer price index (CPI) data for

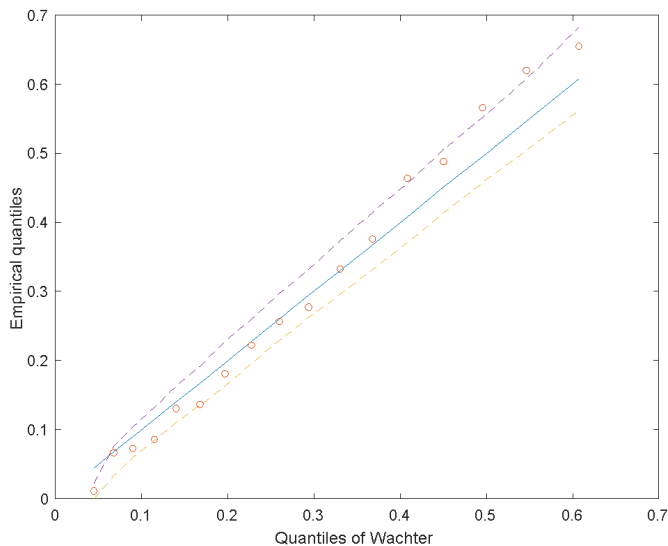


Figure 11: The Wachter plot for the bilateral US log nominal exchange rates of 17 OECD countries. Dashed lines: 5% and 95% quantiles of the MC distribution of the squared sample canonical correlations under the null of no cointegration.

the same countries plus the US. These data are still the same as in Engel, Mark, and West (2015). We used the long sample 1973q2:2008q1 ( $T = 140$ ) because the IP and CPI data are not affected by the introduction of the Euro to the same degree as the exchange rate data. For the CPI data, we included both intercept and trend in model (7) for the first differences because the level data seem to be quadratically trending. The plots clearly indicate that the IP and CPI data are either stationary or cointegrated with potentially many cointegrating relationships, short run deviations from which are not very persistent.

## 5 Conclusion

In this paper, we consider the simultaneous, large- $p$ , large- $T$ , asymptotic behavior of the squared sample canonical correlations between  $p$ -dimensional random walk and its innovations. We find that the empirical distribution of these squared sample canonical correlations almost surely weakly converges to the so-called *Wachter distribution* with parameters that depend only on the limit of  $p/T$  as  $p, T \rightarrow_c \infty$ . In contrast, under the sequential asymptotics, when first  $T \rightarrow \infty$  and then  $p \rightarrow \infty$ , we establish the convergence in probability to the so-called Marchenko-Pastur distribution. The differences between the limiting distributions allow us to explain

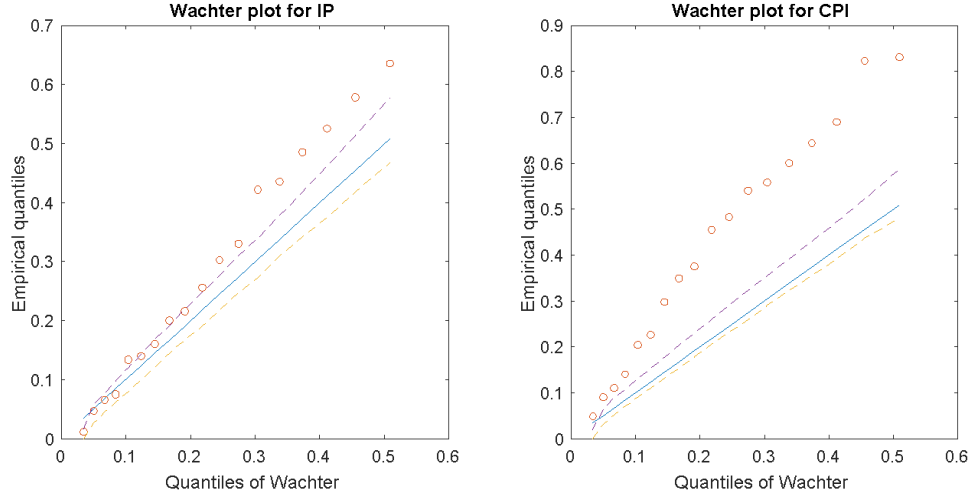


Figure 12: The Wachter plots for the industrial production indices and consumer price indices of 18 OECD countries. Dashed lines: 5% and 95% quantiles of the MC distribution of the squared sample canonical correlations under the null of no cointegration.

from a theoretical point of view the tendency of the LR test for cointegration to severely over-reject the null when the dimensionality of the data is relatively large. Furthermore, we derive a simple analytic formula for the Bartlett-type correction factor in systems with relatively large  $p/T$  ratio.

We propose a quick graphical method, the Wachter plot, for a preliminary analysis of cointegration in large-dimensional systems. The Monte Carlo analysis shows that the quantiles of the Wachter distribution constitute very good centering points for the finite sample distributions of the corresponding squared sample canonical correlations. The quality of the centering is excellent even for such small  $p$  and  $T$  as  $p = 10$  and  $T = 20$ . However, for such small values of  $p$  and  $T$ , the empirical distribution of the squared sample canonical correlation can considerably fluctuate around the Wachter limit. As  $p$  increases to 100, the fluctuations become numerically very small.

Our analysis leaves many open questions. First, it is very important to study the fluctuations of the empirical distribution around the Wachter limit. We conjecture that linear combinations of reasonably smooth functions of the squared sample canonical correlations, including the  $\log(1 - \lambda)$  used by the LR statistic, will be asymptotically Gaussian after appropriate centering and scaling. The centering can be derived from the results obtained in this paper. A proof of the asymptotic Gaussianity would require different methods from those used here. We

are currently investigating this research direction.

Further, it would be important to remove the Gaussianity assumption on the data. We believe that the existence of the finite fourth moments is a sufficient condition for the validity of the Wachter limit. Next, it would be interesting to study the simultaneous asymptotic behavior of a few of the largest sample canonical correlations. This may lead to a modification of Johansen's maximum eigenvalue test.

Another interesting direction of research is to study situations where the number of cointegrating relationships under the null is growing proportionally with  $p$  and  $T$ . The simultaneous asymptotics of the empirical distribution of the squared sample canonical correlations under various alternatives, as well under the null in VAR( $k$ ) with  $k > 1$ , also deserves further study.

Still another, totally different, research direction is to investigate the quality of bootstrap when  $p$  is large. Our own very preliminary analysis indicates that the currently available non-parametric bootstrap procedures (see, for example, Cavaliere, Rahbek, and Taylor (2012)) do not work well for  $p/T$  as large as, say,  $1/3$ . However, further analysis is needed before we can claim any specific results. We hope that this paper opens up an interesting and broad area for future research.

## 6 Appendix

### 6.1 Proof of Theorem 1

#### 6.1.1 Reduction to pure random walk data.

Let  $G(\lambda)$  and  $\tilde{G}(\lambda)$  be distribution functions that may depend on  $p$  and  $T$  and are possibly random. We shall call them asymptotically equivalent if the a.s. weak convergence  $G(\lambda) \Rightarrow F(\lambda)$  to some non-random d.f.  $F(\lambda)$  implies similar a.s. weak convergence for  $\tilde{G}(\lambda)$ , and vice versa. Let  $S_i$  and  $\tilde{S}_i$  with  $i = 0, 1, 2$  be, possibly random, matrices that may depend on  $p$  and  $T$  such that  $S_i$  and  $\tilde{S}_i$  are a.s. positive definite for  $i = 0, 1$ . Below, we shall often refer to the following auxiliary lemma.

**Lemma 10** *If, almost surely,  $\frac{1}{p} \text{rank}(S_i - \tilde{S}_i) \rightarrow 0$  as  $p, T \rightarrow_c \infty$  for  $i = 0, 1, 2$ , then  $G(\lambda)$  and  $\tilde{G}(\lambda)$  are asymptotically equivalent, where  $G(\lambda)$  and  $\tilde{G}(\lambda)$  are the empirical d.f. of eigenvalues of  $S_2 S_1^{-1} S_2' S_0^{-1}$  and  $\tilde{S}_2 \tilde{S}_1^{-1} \tilde{S}_2' \tilde{S}_0^{-1}$ , respectively.*

**Proof of Lemma 10.** Let  $R = \text{rank} \left( S_2 S_1^{-1} S_2' S_0^{-1} - \tilde{S}_2 \tilde{S}_1^{-1} \tilde{S}_2' \tilde{S}_0^{-1} \right)$ . The a.s. convergence  $\frac{1}{p} \text{rank} \left( S_i - \tilde{S}_i \right) \rightarrow 0$  implies the a.s. convergence  $R/p \rightarrow 0$ . On the other hand, by the rank inequality (Theorem A43 in Bai and Silverstein (2010)),  $\mathcal{L} \left( G, \tilde{G} \right) \leq R/p$ , where  $\mathcal{L} \left( G, \tilde{G} \right)$  is the Lévy distance between  $G(\lambda)$  and  $\tilde{G}(\lambda)$ . Recall that the Lévy distance metrizes the weak convergence. Therefore, the almost sure convergence  $\mathcal{L} \left( G, \tilde{G} \right) \rightarrow 0$  yields the asymptotic equivalence of  $G(\lambda)$  and  $\tilde{G}(\lambda)$ .  $\square$

Now, let  $S_0 = S_{00}$ ,  $S_1 = S_{11}$ , and  $S_2 = S_{01}$ , and let

$$\tilde{S}_0 = \frac{1}{T} \sum_{t=1}^T \Delta X_t \Delta X_t', \quad \tilde{S}_1 = \frac{1}{T} \sum_{t=1}^T X_{t-1} X_{t-1}', \quad \text{and} \quad \tilde{S}_2 = \frac{1}{T} \sum_{t=1}^T \Delta X_t X_{t-1}'.$$

Since  $R_{0t}$  and  $R_{1t}$ , which enter the definition (8) of  $S_{ij}$ , are the residuals in the regressions of  $\Delta X_t$  on  $D_t$  and  $X_{t-1}$  on  $D_t$ , respectively, we have

$$\max_{i=0,1,2} \text{rank} \left( S_i - \tilde{S}_i \right) \leq d_D.$$

By assumption,  $d_D/p \rightarrow 0$  as  $p, T \rightarrow_c \infty$ , so that by Lemma 10,  $F_{p,T}(\lambda)$  is asymptotically equivalent to the empirical d.f. of eigenvalues of  $\tilde{S}_2 \tilde{S}_1^{-1} \tilde{S}_2' \tilde{S}_0^{-1}$ . Therefore, we may and will replace  $R_{0t}$  and  $R_{1t}$  in the definitions (8) of  $S_{ij}$  by  $\Delta X_t$  and  $X_{t-1}$ , respectively, without loss of generality. Furthermore, scaling  $S_{ij}$  by  $T$  does not change the product  $S_{01} S_{11}^{-1} S_{01}' S_{00}^{-1}$ , and thus, in the rest of the proof, we shall work with

$$S_{00} = \sum_{t=1}^T \Delta X_t \Delta X_t', \quad S_{01} = \sum_{t=1}^T \Delta X_t X_{t-1}', \quad \text{and} \quad S_{11} = \sum_{t=1}^T X_{t-1} X_{t-1}'. \quad (28)$$

Next, we show that, still without loss of generality, we may replace the data generated process (11) by pure random walk with zero initial value. Indeed, let  $X = [X_{-k+1}, \dots, X_T]$ , where  $X_{-k+1}, \dots, X_0$  are arbitrary and  $X_t$  with  $t \geq 1$  are generated by (11). Further, let  $\tilde{X}_{-k+1}, \dots, \tilde{X}_0$  be zero vectors,  $\tilde{X}_t = \sum_{s=1}^t \varepsilon_s$  for  $t \geq 1$ , and  $\tilde{X} = [\tilde{X}_{-k+1}, \dots, \tilde{X}_T]$ .

**Lemma 11**  $\text{rank} \left( X - \tilde{X} \right) \leq 2(r + \text{rank} \Gamma + k + d_F).$

A proof of this lemma is given in the Supplementary Appendix. It is based on the representation of  $X_t$  as a function of the initial values,  $\varepsilon$  and  $F$  (see Theorem 2.1 in Johansen (1995)), and requires only elementary algebraic manipulations.

Lemmas 11 and 10 together with the assumption (12) imply that replacing  $\Delta X_t$  and  $X_{t-1}$  in (28) by  $\Delta \tilde{X}_t$  and  $\tilde{X}_{t-1}$ , respectively, does not change the weak limit of  $F_{p,T}(\lambda)$ . Hence, in the rest of the proof of Theorem 1, without loss of generality, we shall assume that the data are generated by

$$\Delta X_t = \varepsilon_t, \quad t = 1, \dots, T, \quad \text{with } X_0 = 0. \quad (29)$$

### 6.1.2 Block-diagonalization

Assuming that  $\lambda$ 's are the eigenvalues of  $S_{01}S_{11}^{-1}S_{01}'S_{00}^{-1}$  with  $S_{ij}$  satisfying (28) and (29), we can interpret them as the squared sample canonical correlations between lagged values of a random walk  $X_{t-1}$  and its current innovations  $\varepsilon_t$ . Since the sample canonical correlations are invariant with respect to the multiplication of the data by any invertible matrix, we assume without loss of generality that the variance of  $\varepsilon_t$  equals  $\Sigma = I_p/T$ . Further, we assume that  $T$  is even. The case of odd  $T$  can be analyzed similarly, and we omit it to save space.

Let  $\varepsilon = [\varepsilon_1, \dots, \varepsilon_T]$  and let  $U$  be the upper-triangular matrix with ones above the main diagonal and zeros on the diagonal. Then  $\varepsilon U = [X_0, \dots, X_{T-1}]$ , so that

$$S_{00} = \varepsilon \varepsilon', \quad S_{01} = \varepsilon U' \varepsilon', \quad \text{and} \quad S_{11} = \varepsilon U U' \varepsilon'. \quad (30)$$

We shall show that the empirical d.f. of the  $\lambda$ 's,  $F_{p,T}(\lambda)$ , is asymptotically equivalent to the empirical d.f.  $\hat{F}_{p,T}(\lambda)$  of eigenvalues of  $CD^{-1}C'A^{-1}$ , where

$$C = \varepsilon \Delta_2' \varepsilon', \quad D = \varepsilon \Delta_1 \varepsilon', \quad \text{and} \quad A = \varepsilon \varepsilon',$$

$\Delta_1$  is a diagonal matrix,

$$\Delta_1 = \text{diag} \left\{ r_1^{-1} I_2, \dots, r_{T/2}^{-1} I_2 \right\}, \quad (31)$$

and  $\Delta_2$  is a block-diagonal matrix,

$$\Delta_2 = \text{diag} \left\{ r_1^{-1} (R_1 - I_2), \dots, r_{T/2}^{-1} (R_{T/2} - I_2) \right\}. \quad (32)$$

Here  $I_2$  is the 2-dimensional identity matrix, and  $r_j, R_j$  are defined as follows. Let

$\theta = -2\pi/T$ . Then for  $j = 1, 2, \dots$

$$r_{j+1} = 2 - 2 \cos j\theta, \quad R_{j+1} = \begin{pmatrix} \cos j\theta & -\sin j\theta \\ \sin j\theta & \cos j\theta \end{pmatrix},$$

whereas  $r_1 = 4$ ,  $R_1 = -I_2$ .

**Lemma 12** *The distribution functions  $F_{p,T}(\lambda)$  and  $\hat{F}_{p,T}(\lambda)$  are asymptotically equivalent.*

**Proof of Lemma 12.** Let  $V$  be the circulant matrix (see Golub and Van Loan (1996, p.201)) with the first column  $v = (-1, 1, 0, \dots, 0)'$ . Direct calculations show that  $UV = I_T - le'_T$  and  $VU = I_T - e_1l'$ , where  $e_j$  is the  $j$ -th column of  $I_T$ , and  $l$  is the vector of ones. Using these identities, it is straightforward to verify that

$$U = (V + e_1e'_1)^{-1} - le'_1, \quad \text{and} \quad (33)$$

$$UU' = (V'V - (e_1 - e_T)(e_1 - e_T)' + e_Te'_T)^{-1} - ll'. \quad (34)$$

Now, let us define

$$C_1 = \varepsilon (U + le'_1)' \varepsilon' \quad \text{and} \quad D_1 = \varepsilon (UU' + ll') \varepsilon'.$$

Using identities (30) for  $S_{ij}$  and Lemma 10, we conclude that  $F_{p,T}(\lambda)$  is asymptotically equivalent to  $F_{p,T}^{(1)}(\lambda)$ , where  $F_{p,T}^{(1)}(\lambda)$  is the empirical d.f. of the eigenvalues of  $C_1 D_1^{-1} C_1' A^{-1}$ .

Further, (33) and (34) yield

$$\begin{aligned} C_1 &= \varepsilon (V + e_1e'_1)^{-1} \varepsilon' \quad \text{and} \\ D_1 &= \varepsilon (V'V - (e_1 - e_T)(e_1 - e_T)' + e_Te'_T)^{-1} \varepsilon'. \end{aligned}$$

Applying Lemma 10 one more time, we obtain the asymptotic equivalence of  $F_{p,T}^{(1)}(\lambda)$  and  $F_{p,T}^{(2)}(\lambda)$ , where  $F_{p,T}^{(2)}(\lambda)$  is the empirical d.f. of the eigenvalues of  $C_2 D_2^{-1} C_2' A^{-1}$  with

$$C_2 = \varepsilon V^{-1} \varepsilon' \quad \text{and} \quad D_2 = \varepsilon (V'V)^{-1} \varepsilon'. \quad (35)$$

As is well known (see, for example, Golub and Van Loan (1996), chapter 4.7.7),  $T \times T$  circulant matrices can be expressed in terms of the discrete Fourier transform

matrices

$$\mathcal{F} = \{\exp(i\theta(s-1)(t-1))\}_{s,t=1}^T$$

with  $\theta = -2\pi/T$ . Precisely,

$$V = \frac{1}{T} \mathcal{F}^* \text{diag}(\mathcal{F}v) \mathcal{F}, \text{ and } V'V = \frac{1}{T} \mathcal{F}^* \text{diag}(\mathcal{F}w) \mathcal{F},$$

where  $w = (2, -1, 0, \dots, 0, -1)'$  and the star superscript denotes transposition and complex conjugation. For the  $s$ -th diagonal elements of  $\text{diag}(\mathcal{F}v)$  and  $\text{diag}(\mathcal{F}w)$ , we have

$$\text{diag}(\mathcal{F}v)_s = -1 + \exp\{i\theta(s-1)\}, \text{ and } \text{diag}(\mathcal{F}w)_s = 2 - 2\cos(s-1)\theta.$$

Note that  $\text{diag}(\mathcal{F}w)_s = \text{diag}(\mathcal{F}w)_{T+2-s}$  for  $s = 2, 3, \dots$ . If  $T$  is even, as we assumed above, then there are  $T/2 - 1$  pairs  $(s, T+2-s)$ , and there is one pair  $(1, T/2+1)$  that correspond to

$$\text{diag}(\mathcal{F}w)_1 = 0, \text{diag}(\mathcal{F}w)_{T/2+1} = 4.$$

Define a permutation matrix  $P$  so that the equal diagonal elements of  $P' \text{diag}(\mathcal{F}w) P$  are grouped in adjacent pairs. Precisely, let  $P = \{p_{st}\}$ , where

$$p_{st} = \begin{cases} 1 & \text{if } t = 2s - 1 \text{ for } s = 1, \dots, T/2 \\ 1 & \text{if } t = 2(T - s + 2) \bmod T \text{ for } s = T/2 + 1, \dots, T \\ 0 & \text{otherwise} \end{cases}$$

and let  $W$  be the unitary matrix

$$W = \begin{pmatrix} I_2 & 0 \\ 0 & I_{T/2} \otimes Z \end{pmatrix} \text{ with } Z = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix},$$

where  $\otimes$  denotes the Kronecker product. Further, let  $Q = \frac{1}{\sqrt{T}} W P' \mathcal{F}$ . As is easy to check,  $Q$  is an orthogonal matrix. Furthermore,

$$V = Q' (\Delta_2^{-1} + 2e_1 e_1') Q, \text{ and } VV' = Q' (\Delta_1^{-1} - 4e_1 e_1') Q,$$

where  $\Delta_1$  and  $\Delta_2$  are as defined in (31) and (32). Combining this with (35) and using Lemma 10 once again, we obtain the asymptotic equivalence of  $F_{p,T}^{(2)}(\lambda)$  and

$F_{p,T}^{(3)}(\lambda)$ , where  $F_{p,T}^{(3)}(\lambda)$  is the empirical d.f. of the eigenvalues of  $C_3 D_3^{-1} C_3' A^{-1}$  with

$$C_3 = \varepsilon Q' \Delta_2 Q \varepsilon' \text{ and } D_3 = \varepsilon Q' \Delta_1 Q \varepsilon'.$$

Because of the rotational invariance of the Gaussian distribution, the distributions of  $\varepsilon Q'$  and  $\varepsilon$  are the same. Hence,  $F_{p,T}^{(3)}(\lambda)$  is asymptotically equivalent to  $\hat{F}_{p,T}(\lambda)$ , and thus,  $\hat{F}_{p,T}(\lambda)$  is asymptotically equivalent to  $F_{p,T}(\lambda)$ .  $\square$

### 6.1.3 A system of equations for the Stieltjes transform

Our proof of the almost sure weak convergence of  $\hat{F}_{p,T}(\lambda)$  to the Wachter distribution consists of showing that the Stieltjes transform of  $\hat{F}_{p,T}(\lambda)$ ,

$$\hat{m}_{p,T}(z) = \int \frac{1}{\lambda - z} \hat{F}_{p,T}(d\lambda), \quad (36)$$

almost surely converges pointwise in  $z \in \mathbb{C}^+ = \{\zeta : \Im \zeta > 0\}$  to the Stieltjes transform  $m(z)$  of the Wachter distribution. To establish such a convergence, we show that, if  $m$  is a limit of  $\hat{m}_{p,T}(z)$  along any *subsequence* of  $p, T \rightarrow_c \infty$ , then it must satisfy a system of equations with unique solution given by  $m(z)$ . The almost sure convergence of  $\hat{F}_{p,T}(\lambda)$  (and thus, also of  $F_{p,T}(\lambda)$ ) to the Wachter distribution follows then from the Continuity Theorem for the Stieltjes transforms (see, for example, Corollary 1 in Geronimo and Hill (2003)).

We shall write  $\hat{m}$  for the Stieltjes transform  $\hat{m}_{p,T}(z)$  to simplify notation. Let

$$M = CD^{-1}C' - zA \text{ and } \tilde{M} = C'A^{-1}C - zD. \quad (37)$$

Then by definition (36),  $\hat{m}$  must satisfy the following equations

$$\hat{m} = \frac{1}{p} \text{tr} [AM^{-1}], \quad (38)$$

$$\hat{m} = \frac{1}{p} \text{tr} [D\tilde{M}^{-1}]. \quad (39)$$

Let us study the above traces in detail. Define

$$\varepsilon_{(j)} = [\varepsilon_{2j-1}, \varepsilon_{2j}], \quad j = 1, \dots, T/2.$$

We now show that the traces in (38) and (39) can be expressed as functions of the terms having form  $\varepsilon'_{(j)} \Omega_j \varepsilon_{(j)}$ , where  $\Omega_j$  is independent from  $\varepsilon_{(j)}$ . Then, we argue

that

$$\varepsilon'_{(j)} \Omega_j \varepsilon_{(j)} - \frac{1}{T} \text{tr} [\Omega_j] I_2$$

a.s. converge to zero, and use this fact to derive equations that the limit of  $\hat{m}$ , if it exists, must satisfy.

First, consider (38). Note that

$$\frac{1}{p} \text{tr} [AM^{-1}] = \frac{1}{p} \sum_{j=1}^{T/2} \text{tr} [\varepsilon'_{(j)} M^{-1} \varepsilon_{(j)}]. \quad (40)$$

Let us introduce new notation:

$$\begin{aligned} \Delta_{1j} &= r_j^{-1} I_2, \quad \Delta_{2j} = r_j^{-1} (R_j - I_2), \\ C_j &= C - \varepsilon_{(j)} \Delta'_{2j} \varepsilon'_{(j)}, \quad D_j = D - \varepsilon_{(j)} \Delta_{1j} \varepsilon'_{(j)}, \\ A_j &= A - \varepsilon_{(j)} \varepsilon'_{(j)}, \quad \text{and } M_j = C_j D_j^{-1} C'_j - z A_j. \end{aligned}$$

In addition, let

$$\begin{aligned} s_j &= \varepsilon'_{(j)} D_j^{-1} \varepsilon_{(j)}, \quad u_j = \varepsilon'_{(j)} D_j^{-1} C'_j M_j^{-1} \varepsilon_{(j)}, \\ v_j &= \varepsilon'_{(j)} M_j^{-1} \varepsilon_{(j)}, \quad \text{and} \\ w_j &= \varepsilon'_{(j)} D_j^{-1} C'_j M_j^{-1} C_j D_j^{-1} \varepsilon_{(j)}. \end{aligned}$$

A straightforward algebra that involves multiple use of the Sherman-Morrison-Woodbury formula (see Golub and Van Loan (1996), p.50)

$$(V + XWY)^{-1} = V^{-1} - V^{-1}X (W^{-1} + YV^{-1}X)^{-1} YV^{-1}, \quad (41)$$

and the identity

$$\Delta_{2j} \Delta'_{2j} = \Delta'_{2j} \Delta_{2j} = \Delta_{1j}, \quad (42)$$

establishes the following equality

$$\varepsilon'_{(j)} M^{-1} \varepsilon_{(j)} = v_j - [v_j, u'_j] \Omega_j [v_j, u'_j]', \quad (43)$$

where

$$\Omega_j = \begin{pmatrix} \frac{1}{1-z} I_2 + v_j & \frac{1}{1-z} r_j \Delta'_{2j} + u'_j \\ \frac{1}{1-z} r_j \Delta_{2j} + u_j & \frac{z}{1-z} r_j I_2 - s_j + w_j \end{pmatrix}^{-1}.$$

A derivation of (43) can be found in the Supplementary Appendix.

Let us define

$$\begin{aligned}\hat{s} &= \frac{1}{T} \operatorname{tr} [D^{-1}], \quad \hat{u} = \frac{1}{T} \operatorname{tr} [D^{-1}C'M^{-1}], \\ \hat{v} &= \frac{1}{T} \operatorname{tr} [M^{-1}], \quad \text{and} \\ \hat{w} &= \frac{1}{T} \operatorname{tr} [D^{-1}C'M^{-1}CD^{-1}].\end{aligned}$$

We have the following lemma, where  $\|\cdot\|$  denotes the spectral norm. Its proof is given in the Supplementary Appendix.

**Lemma 13** *For all  $z \in \mathbb{C}^+$ , as  $p, T \rightarrow_c \infty$ , we have*

$$\begin{aligned}\max_{j=1,\dots,T/2} \|s_j - \hat{s}I_2\| &\xrightarrow{a.s} 0, \quad \max_{j=1,\dots,T/2} \|u_j - \hat{u}I_2\| \xrightarrow{a.s} 0 \\ \max_{j=1,\dots,T/2} \|v_j - \hat{v}I_2\| &\xrightarrow{a.s} 0, \quad \max_{j=1,\dots,T/2} \|w_j - \hat{w}I_2\| \xrightarrow{a.s} 0.\end{aligned}$$

The lemma yields an approximation to the right hand side of (43), which we use in (40) and (38) to obtain the following result.

**Proposition 14** *There exists  $\zeta > 0$  such that, for any  $z$  with zero real part,  $\Re z = 0$ , and the imaginary part satisfying  $\Im z > \zeta$ , we have*

$$\hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \frac{f_1(\varphi)}{(1-z)f_1(\varphi) + f_2(\varphi)} d\varphi + o(1), \quad \text{where} \quad (44)$$

$$\begin{aligned}f_1(\varphi) &= (\hat{w} - \hat{s} - 4\sin^2 \varphi) \hat{v} - \hat{u}^2, \\ f_2(\varphi) &= \hat{w} - \hat{s} - 4\sin^2 \varphi (1 - \hat{u} - \hat{v}),\end{aligned}$$

and  $o(1) \xrightarrow{a.s} 0$ , as  $p, T \rightarrow_c \infty$ .

**Proof of Proposition 14.** Consider a  $2 \times 2$  matrix  $\hat{S}_j$  that is obtained from  $\varepsilon'_{(j)} M^{-1} \varepsilon_{(j)}$  by replacing  $s_j, v_j, u_j$  and  $w_j$  in (43) with  $\hat{s}I_2, \hat{v}I_2, \hat{u}I_2$ , and  $\hat{w}I_2$ , respectively. We have

$$\hat{S}_j = \hat{v}I_2 - [\hat{v}I_2, \hat{u}I_2] \hat{\Omega}_j [\hat{v}I_2, \hat{u}I_2]',$$

where

$$\hat{\Omega}_j = \begin{pmatrix} \frac{1}{1-z}I_2 + \hat{v}I_2 & \frac{1}{1-z}r_j\Delta'_{2j} + \hat{u}I_2 \\ \frac{1}{1-z}r_j\Delta_{2j} + \hat{u}I_2 & \frac{z}{1-z}r_jI_2 + (\hat{w} - \hat{s})I_2 \end{pmatrix}^{-1}.$$

A simple algebra and the identity  $\Delta_{2j} + \Delta'_{2j} = -I_2$  yield

$$\hat{\Omega}_j = \frac{1-z}{\delta_j} \tilde{\Omega}_j, \text{ where} \quad (45)$$

$$\tilde{\Omega}_j = \begin{pmatrix} \frac{z}{1-z} r_j I_2 + (\hat{w} - \hat{s}) I_2 & -\frac{1}{1-z} r_j \Delta'_{2j} - \hat{u} I_2 \\ -\frac{1}{1-z} r_j \Delta_{2j} - \hat{u} I_2 & \frac{1}{1-z} I_2 + \hat{v} I_2 \end{pmatrix}, \quad (46)$$

and

$$\delta_j = (\hat{w} - \hat{s})(1 + \hat{v} - z\hat{v}) + r_j(\hat{u} + z\hat{v} - 1) - (1 - z)\hat{u}^2.$$

By definition,

$$\begin{aligned} |\hat{s}| &\leq \frac{p}{T} \|D^{-1}\|, |\hat{u}| \leq \frac{p}{T} \text{tr} \|D^{-1} C' M^{-1}\|, \\ |\hat{v}| &\leq \frac{p}{T} \|M^{-1}\|, \text{ and } |\hat{w}| \leq \frac{p}{T} \text{tr} \|D^{-1} C' M^{-1} C D^{-1}\|. \end{aligned}$$

In the proof of Lemma 13, we show that the norms  $\|D^{-1}\|$ ,  $\|D^{-1} C'\|$ , and  $\|M^{-1}\|$  almost surely remain bounded as  $p, T \rightarrow_c \infty$ . Hence,  $\hat{s}$ ,  $\hat{u}$ ,  $\hat{v}$ , and  $\hat{w}$  are also almost surely bounded. Further, by definition,

$$r_j \Delta_{2j} = R_j - I_2 \text{ and } r_j \Delta'_{2j} = R'_j - I_2,$$

where  $R_j$  is an orthogonal matrix, so that  $\|r_j \Delta_{2j}\|$  and  $\|r_j \Delta'_{2j}\|$  are clearly bounded uniformly in  $j$ . Therefore, the norm of matrix  $\tilde{\Omega}_j$  almost surely remains bounded as  $p, T \rightarrow_c \infty$ , uniformly in  $j$ . Regarding  $\delta_j$ , which appear in the denominator on the right hand side of (45), the Supplementary Appendix establishes the following result.

**Lemma 15** *There exists  $\zeta > 0$  such that, for any  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ , almost surely,*

$$\liminf_{p, T \rightarrow_c \infty} \max_{j=1, \dots, T/2} |\delta_j| > c^2 / (1 - c^2).$$

The above results imply that, for  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ ,  $\|\hat{\Omega}_j\|$  almost surely remains bounded as  $p, T \rightarrow_c \infty$ , uniformly in  $j$ . Therefore, by Lemma 13,

$$\varepsilon'_{(j)} M^{-1} \varepsilon_{(j)} = \hat{S}_j + o(1), \quad (47)$$

where  $o(1) \xrightarrow{a.s.} 0$  as  $p, T \rightarrow_c \infty$ , uniformly in  $j$ .

A straightforward algebra reveals that

$$\hat{S}_j = \frac{(\hat{w} - \hat{s} - r_j) \hat{v} - \hat{u}^2}{\delta_j}.$$

Using this in equations (47) and (40), we obtain

$$\begin{aligned} \hat{m} &= \frac{2}{p} \sum_{j=0}^{T/2-1} \frac{(\hat{w} - \hat{s} - r_{j+1}) \hat{v} - \hat{u}^2}{\delta_{j+1}} + o(1) \\ &= \frac{2}{p} \sum_{j=1}^{T/2-1} \frac{f_1(j\pi/T)}{(1-z)f_1(j\pi/T) + f_2(j\pi/T)} + o(1), \end{aligned}$$

where, in the latter expression, the term corresponding to  $j = 0$  is included in the  $o(1)$  term to take into account the special definition of  $r_1$ .

As follows from Lemma 15 and the boundedness of  $\hat{s}$ ,  $\hat{u}$ ,  $\hat{v}$ , and  $\hat{w}$ , the derivative

$$\frac{d}{d\varphi} \frac{f_1(\varphi)}{(1-z)f_1(\varphi) + f_2(\varphi)}$$

almost surely remains bounded by absolute value as  $p, T \rightarrow_c \infty$ , uniformly in  $\varphi \in [0, 2\pi]$ . Therefore

$$\frac{2}{p} \sum_{j=1}^{T/2-1} \frac{f_1(j\pi/T)}{(1-z)f_1(j\pi/T) + f_2(j\pi/T)} = \frac{2}{\pi c} \int_0^{\pi/2} \frac{f_1(\varphi) d\varphi}{(1-z)f_1(\varphi) + f_2(\varphi)} + o(1).$$

The statement of Proposition 14 now follows by noting that the latter integral is one quarter of the integral over  $[0, 2\pi]$ .  $\square$

A similar analysis of equation (39) gives us another proposition, describing  $\hat{m}$  as function of  $\tilde{s}$ ,  $\tilde{u}$ ,  $\tilde{v}$ , and  $\tilde{w}$ , where

$$\begin{aligned} \tilde{s} &= \frac{1}{T} \operatorname{tr} [A^{-1}], \quad \tilde{u} = \frac{1}{T} \operatorname{tr} [A^{-1} C \tilde{M}^{-1}], \\ \tilde{v} &= \frac{1}{T} \operatorname{tr} [\tilde{M}^{-1}], \quad \text{and} \\ \tilde{w} &= \frac{1}{T} \operatorname{tr} [A^{-1} C \tilde{M}^{-1} C' A^{-1}]. \end{aligned}$$

We omit the proof because it is very similar to that of Proposition 14.

**Proposition 16** *There exists  $\zeta > 0$  such that, for any  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ ,*

we have

$$\hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \frac{g_1}{(1-z)g_1 + g_2(\varphi)} d\varphi + o(1), \text{ where} \quad (48)$$

$$\begin{aligned} g_1 &= (\tilde{w} - \tilde{s} - 1) \tilde{v} - \tilde{u}^2, \\ g_2(\varphi) &= \tilde{v} - 4 \sin^2 \varphi (\tilde{s} + 1 - \tilde{u} - \tilde{w}), \end{aligned}$$

and  $o(1) \xrightarrow{a.s} 0$ , as  $p, T \rightarrow_c \infty$ .

Although we now have two asymptotic equations for  $\hat{m}$ , (44) and (48), they contain many unknowns:  $\hat{s}, \hat{u}, \hat{v}, \hat{w}$ , and the corresponding variables with tildes. The following result establishes simple relationships between the unknowns with hats and tildes.

**Lemma 17** *We have the following three identities*

$$\hat{u} = \tilde{u}, \quad z\tilde{v} + \hat{s} = \hat{w}, \quad \text{and} \quad z\hat{v} + \tilde{s} = \tilde{w}. \quad (49)$$

**Proof of Lemma 17.** The identity  $\hat{u} = \tilde{u}$  is established by the following sequence of equalities

$$\begin{aligned} T\hat{u} &= \text{tr } D^{-1}C'M^{-1} = \text{tr } D^{-1}C' (CD^{-1}C' - zA)^{-1} \\ &= \text{tr } \left( C - zA(C')^{-1}D \right)^{-1} = \text{tr } (C' - zD(C)^{-1}A)^{-1} \\ &= \text{tr } A^{-1}C(C'A^{-1}C - zD)^{-1} = \text{tr } A^{-1}C\tilde{M}^{-1} = T\tilde{u}. \end{aligned}$$

The relationship  $z\tilde{v} + \hat{s} = \hat{w}$  is obtained as follows

$$\begin{aligned} T(z\tilde{v} + \hat{s}) &= \text{tr } D^{-1} \left( zI_p (C'A^{-1}CD^{-1} - zI_p)^{-1} + I_p \right) \\ &= \text{tr } D^{-1} \left( -I_p + C'A^{-1}CD^{-1} (C'A^{-1}CD^{-1} - zI_p)^{-1} + I_p \right) \\ &= \text{tr } D^{-1} \left( I_p - DC^{-1}A(C')^{-1}z \right)^{-1} \\ &= \text{tr } D^{-1}C' (CD^{-1}C' - Az)^{-1} CD^{-1} = T\hat{w}. \end{aligned}$$

The identity  $z\hat{v} + \tilde{s} = \tilde{w}$  is obtained similarly to  $z\tilde{v} + \hat{s} = \hat{w}$  by interchanging the roles of  $D, C$  and  $A, C'$ .  $\square$

The identities (49) imply the following equality

$$(1-z)f_1(\varphi) + f_2(\varphi) = (1-z)g_1 + g_2(\varphi).$$

We denote the reciprocal of the common value of the right and left hand sides of this equality as  $\hat{h}(z, \varphi)$ . A direct calculation shows that

$$\hat{h}(z, \varphi) = \left( (1-z)(z\tilde{v}\hat{v} - \hat{u}^2) + z\tilde{v} + 4\sin^2\varphi(z\hat{v} + \hat{u} - 1) \right)^{-1}, \quad (50)$$

and the asymptotic relationships (44) and (48) can be written in the following form

$$\begin{cases} \hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) \left( (z\tilde{v} - 4\sin^2\varphi)\hat{v} - \hat{u}^2 \right) d\varphi + o(1) \\ \hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) \left( (z\hat{v} - 1)\tilde{v} - \hat{u}^2 \right) d\varphi + o(1) \end{cases}. \quad (51)$$

This can be viewed as an asymptotic system of two equations with four unknowns:  $\hat{m}$ ,  $\tilde{v}$ ,  $\hat{v}$ , and  $\hat{u}$ . We shall now complete the system by establishing the other two asymptotic relationships connecting these unknowns.

Multiplying both sides of the identity

$$MA^{-1} = CD^{-1}C'A^{-1} - zI_p \quad (52)$$

by  $AM^{-1}$ , taking trace, dividing by  $p$ , and rearranging terms, we obtain

$$1 + z\hat{m} = \frac{1}{p} \text{tr} [CD^{-1}C'M^{-1}]. \quad (53)$$

Next, we analyze (53) similarly to the above analysis of (38). That is, first, we note that

$$\frac{1}{p} \text{tr} [CD^{-1}C'M^{-1}] = \frac{1}{p} \sum_{j=1}^{T/2} \text{tr} [\Delta'_{2j}\varepsilon'_{(j)}D^{-1}C'M^{-1}\varepsilon_{(j)}]. \quad (54)$$

Then elementary algebra, based on the Sherman-Morrison-Woodbury formula (41), yields

$$\begin{aligned} \varepsilon'_{(j)}D^{-1}C'M^{-1}\varepsilon_{(j)} &= r_j(r_jI_2 + s_j)^{-1} s_j\Delta_{2j} \left( v_j - [v_j, u'_j] \Omega_j [v_j, u'_j]' \right) \\ &\quad + r_j(r_jI_2 + s_j)^{-1} \left( u_j - [u_j, w_j] \Omega_j [v_j, u'_j]' \right). \end{aligned} \quad (55)$$

Multiplying both sides of (55) by  $\Delta'_{2j}$  and replacing  $s_j$ ,  $u_j$ ,  $v_j$ , and  $w_j$  by  $\hat{s}I_2$ ,  $\hat{u}I_2$ ,  $\hat{v}I_2$ , and  $\hat{w}I_2$ , respectively, yields an asymptotic approximation to  $\Delta'_{2j}\varepsilon'_{(j)}D^{-1}C'M^{-1}\varepsilon_{(j)}$ , which can be used in (54) and (53) to produce the following result. Its proof, as well as the proof of (55), are given in the Supplementary Appendix.

**Proposition 18** *There exists  $\zeta > 0$  such that, for any  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ , we have*

$$1 + z\hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) (2\hat{u} \sin^2 \varphi + z\tilde{v}\hat{v} - \hat{u}^2) d\varphi + o(1), \text{ where} \quad (56)$$

$o(1) \xrightarrow{a.s} 0$ , as  $p, T \rightarrow_c \infty$ .

One might think that the remaining asymptotic relationship can be obtained by using the identity

$$\tilde{M}D^{-1} = C'A^{-1}CD^{-1} - zI_p, \quad (57)$$

which parallels (52). Unfortunately, following this idea delivers a relationship equivalent to (56). Therefore, instead of using (57), we consider the identity

$$\frac{1}{p} \text{tr} [C'M^{-1}] = \frac{1}{p} \text{tr} [DD^{-1}C'M^{-1}], \quad (58)$$

which yields

$$\frac{1}{p} \sum_{j=1}^{T/2} \text{tr} [\Delta_{2j}\varepsilon'_{(j)}M^{-1}\varepsilon_{(j)}] = \frac{1}{p} \sum_{j=1}^{T/2} \text{tr} [\Delta_{1j}\varepsilon'_{(j)}D^{-1}C'M^{-1}\varepsilon_{(j)}]. \quad (59)$$

Then, we proceed as in the above analysis of (54) and (40) to obtain the remaining asymptotic relationship. The proof of the following proposition is given in the Supplementary Appendix.

**Proposition 19** *There exists  $\zeta > 0$  such that, for any  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ , we have*

$$0 = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) (4\hat{v} \sin^2 \varphi + 2\hat{u}) d\varphi + o(1), \text{ where} \quad (60)$$

$o(1) \xrightarrow{a.s} 0$ , as  $p, T \rightarrow_c \infty$ .

Summing up the results in Propositions 14, 16, 18, and 19, the unknowns  $\hat{m}, \hat{v}, \tilde{v}$ , and  $\hat{u}$  must satisfy the following system of asymptotic equations

$$\begin{cases} \hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) ((z\tilde{v} - 4\sin^2 \varphi) \hat{v} - \hat{u}^2) d\varphi + o(1) \\ \hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) ((z\hat{v} - 1) \tilde{v} - \hat{u}^2) d\varphi + o(1) \\ 1 + z\hat{m} = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) (2\hat{u} \sin^2 \varphi + z\tilde{v}\hat{v} - \hat{u}^2) d\varphi + o(1) \\ 0 = \frac{1}{2\pi c} \int_0^{2\pi} \hat{h}(z, \varphi) (4\hat{v} \sin^2 \varphi + 2\hat{u}) d\varphi + o(1) \end{cases} \quad (61)$$

#### 6.1.4 Solving the system

Recall that the unknowns  $\hat{m}$ ,  $\hat{v}$ ,  $\tilde{v}$ , and  $\hat{u}$  in the asymptotic relationships (61) depend on  $p, T$ . The definition (36) of  $\hat{m}$  implies that  $|\hat{m}|$  is bounded by  $(\Im z)^{-1}$ . Further, as shown in the proof of Proposition 14,  $\hat{u}$  and  $\hat{v}$  are a.s. bounded by absolute value, and it can be similarly shown that  $\tilde{v}$  is a.s. bounded by absolute value. Therefore, there exist a subsequence of  $p, T$  along which  $\hat{m}$ ,  $\hat{v}$ ,  $\tilde{v}$ , and  $\hat{u}$  a.s. converge to some limits  $m, v, y$ , and  $u$ .

These limits must satisfy a non-asymptotic system of equations

$$\begin{cases} m = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) ((zy - 4 \sin^2 \varphi) v - u^2) d\varphi \\ m = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) ((zv - 1) y - u^2) d\varphi \\ 1 + zm = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) (2u \sin^2 \varphi + zvy - u^2) d\varphi \\ 0 = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) (2v \sin^2 \varphi + u) d\varphi \end{cases}, \quad (62)$$

where

$$h(z, \varphi) = [(1 - z)(zvy - u^2) + zy + 4 \sin^2 \varphi (zv + u - 1)]^{-1}.$$

Let us consider, until further notice, only such  $z$  that have zero real part,  $\Re z = 0$ , and the imaginary part satisfying  $\Im z > \zeta$ , for some  $\zeta > 0$ . Let us solve system (62) for  $m$ . Adding two times the last equation to the first one, and subtracting the second equation we obtain

$$0 = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) (y + 2u) d\varphi. \quad (63)$$

Note that  $\int_0^{2\pi} h(z, \varphi) d\varphi \neq 0$ . Otherwise, from the second equation of (62), we have  $m = 0$ , which cannot be true because  $\hat{m}$  is the Stieltjes transform of the empirical distribution of the squared canonical correlations, all of which lie between zero and one. Indeed, clearly, for any  $0 \leq \lambda \leq 1$  and  $z$  with  $\Re z = 0$ ,

$$\Im \left( \frac{1}{\lambda - z} \right) = \frac{\Im z}{\lambda^2 + (\Im z)^2} \geq \frac{\Im z}{1 + (\Im z)^2}.$$

Therefore,  $\Im \hat{m} \geq \Im z / (1 + (\Im z)^2)$ , and  $\hat{m}$  cannot converge to  $m = 0$ .

Since  $\int_0^{2\pi} h(z, \varphi) d\varphi \neq 0$ , (63) yields

$$y + 2u = 0 \quad (64)$$

with  $y \neq 0$  and  $u \neq 0$  (if one of them equals zero, the other equals zero too, and  $m = 0$  by the second equation of (62), which is impossible). Since  $u \neq 0$ , the last equation implies that  $v \neq 0$  as well.

Further, subtracting from the third equation the sum of  $z$  times the second and  $u/v$  times the last equation, and using (64), we obtain

$$1 = \frac{1}{2\pi c} \int_0^{2\pi} h(z, \varphi) \frac{u}{v} (2zv + u) (zv - v - 1) d\varphi. \quad (65)$$

This equation, together with the second equation of (62) yield

$$m = \frac{v(2zv + u - 2)}{(1 + v - zv)(2zv + u)}. \quad (66)$$

Next, for the integrand in the last equation of (62), we have

$$\begin{aligned} h(z, \varphi) (2v \sin^2 \varphi + u) &= \frac{1}{2} \frac{v}{zv + u - 1} \\ &+ h(z, \varphi) \frac{u}{2} \left( \frac{(1 - z)v(2zv + u) + 2(2zv + u - 1)}{zv + u - 1} \right). \end{aligned} \quad (67)$$

This assumes that

$$zv + u - 1 \neq 0. \quad (68)$$

If not, then

$$h(z, \varphi) = [(1 - z)(zvy - u^2) + zy]^{-1}$$

would not depend on  $\varphi$  and the last equation of (62) would imply that  $u + v = 0$ . The latter equation and the equality  $zv + u - 1 = 0$  would yield  $v = -(1 - z)^{-1}$ , which when combined with the second equation of (62) would give us  $m = -c^{-1}(1 - z)^{-1}$ , which cannot be true because  $m$ , being a limit of  $\hat{m}$ , must satisfy  $\Im m \geq 0$  for  $\Im z > 0$ .

Equations (65), (67), and the last equation of (62) imply that

$$u = \frac{2c}{2c - 1 - (1 - z)v(1 - c)} - 2zv. \quad (69)$$

Combining this with (66) yields

$$m = v \frac{1 - c}{c}. \quad (70)$$

Finally, elementary calculations given in the Supplementary Appendix show that

$$\left( \frac{1}{2\pi} \int_0^{2\pi} \frac{1}{x + 2 \sin^2 \varphi} d\varphi \right)^2 = \frac{1}{x(x+2)}, \quad (71)$$

where  $x \in \mathbb{C} \setminus [-2, 0]$ . Using (71), (65), and the definition of  $h(z, \varphi)$ , we obtain the following relationship

$$\begin{aligned} & \left( \frac{2cv(zv + u - 1)}{u(2zv + u)(zv - v - 1)} \right)^2 \\ &= \frac{4(zv + u - 1)^2}{u((1 - z)(-2zv - u) - 2z)(-u + uz + 2)(u + 2vz - 2)}, \end{aligned} \quad (72)$$

that holds as long as

$$\frac{u((1 - z)(-2zv - u) - 2z)}{2(zv + u - 1)} \in \mathbb{C} \setminus [-2, 0].$$

The latter inclusion holds because otherwise  $h(z, \varphi)$  is not a bounded function of  $\varphi$ , which would contradict Lemma 15.

Using (69) in (72), and simplifying, we find that there exist only three possibilities. Either

$$v = -\frac{1}{1 - z}, \quad (73)$$

or

$$1 - (c + cz - 1)v + z(1 - z)(1 - c)v^2 = 0, \quad (74)$$

or

$$\frac{c}{1 - c} - (c + cz - z)v + z(1 - z)(1 - c)v^2 = 0. \quad (75)$$

Equation (73) cannot hold because otherwise, (70) would imply that  $\Im m < 0$ , which is impossible as argued above. Equation (74) taken together with (69) implies that

$$u + zv - 1 = 0,$$

which was ruled out above. This leaves us with (75), so that, using (70), we get

$$m = \frac{-(z - c - cz) \pm \sqrt{(z - c - cz)^2 - 4c(1 - z)z}}{2z(1 - z)c}. \quad (76)$$

For  $z \in \mathbb{C}^+$  with  $\Re z = 0$ , the imaginary part of the right hand side of (76) is

negative when ‘ $-$ ’ is used in front of the square root. Here we choose the branch of the square root, with the cut along the positive real semi-axis, which has positive imaginary part. Since  $\Im m$  cannot be negative, we conclude that

$$m = \frac{-(z - c - cz) + \sqrt{(z - c - cz)^2 - 4c(1 - z)z}}{2z(1 - z)c}. \quad (77)$$

But the right hand side of the above equality is the value of the limit of the Stieltjes transforms of the eigenvalues of the multivariate beta matrix  $B_p(p, (T - p)/2)$  as  $p, T \rightarrow_c \infty$ . This can be verified directly by using the formula for such a limit, given for example in Theorem 1.6 of Bai, Hu, Pan and Zhou (2015). As follows from Wachter (1980), the weak limit of the empirical distribution of the eigenvalues of the multivariate beta matrix  $B_p(p, (T - p)/2)$  as  $p, T \rightarrow_c \infty$  equals  $W(\lambda; c/(1 + c), 2c/(1 + c))$ .

Equation (77) shows that, for  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ , any converging subsequence of  $\hat{m}$  converges to the same limit. Hence,  $\hat{m}$  a.s. converges for all  $z$  with  $\Re z = 0$  and  $\Im z > \zeta$ . Note that  $\hat{m}$  is a sequence of bounded analytic functions in the domain  $\{z : \Im z > \delta\}$ , where  $\delta$  is an arbitrary positive number. Therefore, by Vitaly’s convergence theorem (see Titchmarsh (1939), p.168)  $\hat{m}$  a.s. converges to  $m$ , described by (77), for any  $z \in \mathbb{C}^+$ . The almost sure convergence of  $\hat{F}_{p,T}(\lambda)$  (and thus, also of  $F_{p,T}(\lambda)$ ) to the Wachter distribution follows from the Continuity Theorem for the Stieltjes transforms (see, for example, Corollary 1 in Geronimo and Hill (2003)).

## 6.2 Proof of Theorem 6

First, let us show that the weak limit  $F_0(\lambda)$  of  $F_\gamma(\lambda)$  as  $\gamma \rightarrow 0$  exists and equals the continuous part of the Marchenko-Pastur distribution with density (21). By definition and Theorem 1,  $F_\gamma(\lambda)$  is the (scaled) Wachter d.f.  $W(\gamma\lambda; \gamma/(1 + \gamma), 2\gamma/(1 + \gamma))$ . Therefore, by (9) and (10), the density,  $f_\gamma(\lambda)$ , and the boundaries of the support,  $[\hat{b}_-, \hat{b}_+]$ , of the distribution  $F_\gamma$  equal

$$f_\gamma(\lambda) = \frac{1 + \gamma}{2\pi} \frac{\sqrt{(\hat{b}_+ - \lambda)(\lambda - \hat{b}_-)}}{\lambda(1 - \gamma\lambda)}, \text{ and}$$

$$\hat{b}_\pm = \left(\sqrt{2} \mp \sqrt{1 - \gamma}\right)^{-2}.$$

As  $\gamma \rightarrow 0$ ,  $\hat{b}_{\pm} \rightarrow a_{\pm}$ , where  $a_{\pm} = (1 \pm \sqrt{2})^2$  as in (20), and  $f_{\gamma}(\lambda)$  converges to the density given by (21). This implies the weak convergence of  $F_{\gamma}(\lambda)$  to  $F_0(\lambda)$  with  $F_0$  supported on  $[a_-, a_+]$  and having density (21).

To establish the theorem, it remains to show that, as  $p \rightarrow \infty$ ,  $F_{p,\infty}(\lambda)$  weakly converges to  $F_0(\lambda)$ , in probability. Recall that the weak convergence is metrized by the Lévy distance  $\mathcal{L}(\cdot, \cdot)$ . We need to show that for any  $\delta > 0$ , there exists  $p_0$  such that (s.t.) for all  $p > p_0$ ,

$$\Pr(\mathcal{L}(F_0, F_{p,\infty}) < \delta) > 1 - \delta. \quad (78)$$

Let  $\gamma > 0$  be so small that

$$\mathcal{L}(F_0, F_{\gamma}) < \delta/4. \quad (79)$$

For any  $p$ , let  $T_{\gamma}$  be the smallest even integer satisfying  $p/T_{\gamma} \leq \gamma$ . That is,

$$T_{\gamma} = \min_{T \in 2\mathbb{Z}} \{T : p/T \leq \gamma\}.$$

For any  $T_{\infty} > T_{\gamma}$ , by the triangle inequality, we have

$$\mathcal{L}(F_0, F_{p,\infty}) \leq \mathcal{L}(F_0, F_{\gamma}) + \mathcal{L}(F_{\gamma}, F_{p,T_{\gamma}}) + \mathcal{L}(F_{p,T_{\gamma}}, F_{p,T_{\infty}}) + \mathcal{L}(F_{p,T_{\infty}}, F_{p,\infty}), \quad (80)$$

where  $F_{p,T_{\gamma}}$  and  $F_{p,T_{\infty}}$  denote the empirical distributions of eigenvalues of

$$\frac{T}{p} C D^{-1} C' A^{-1}, \quad (81)$$

with  $T = T_{\gamma}$  and  $T = T_{\infty}$ , respectively.

By Theorem 1,  $\mathcal{L}(F_{\gamma}, F_{p,T_{\gamma}})$  a.s. converges to zero as  $p \rightarrow \infty$ . Therefore, for all sufficiently large  $p$ , we have

$$\Pr(\mathcal{L}(F_{\gamma}, F_{p,T_{\gamma}}) < \delta/4) > 1 - \delta/4. \quad (82)$$

Further, as shown by Johansen (1988, 1991), for any  $p$ , as  $T_{\infty} \rightarrow \infty$ , the eigenvalues of (81) with  $T = T_{\infty}$  jointly converge in distribution to those of

$$\frac{1}{p} \int_0^1 (dB) B' \left( \int_0^1 B B' du \right)^{-1} \int_0^1 B (dB)'. \quad (83)$$

Therefore, for any  $p$  and all sufficiently large  $T_\infty$ , we have

$$\Pr(\mathcal{L}(F_{p,T_\infty}, F_{p,\infty}) < \delta/4) > 1 - \delta/4. \quad (84)$$

Let us denote the sum of  $\mathcal{L}(F_0, F_\gamma)$ ,  $\mathcal{L}(F_\gamma, F_{p,T_\gamma})$ , and  $\mathcal{L}(F_{p,T_\infty}, F_{p,\infty})$  as  $\mathcal{L}_{\gamma,p,T_\infty}$ . By (80), we have

$$\mathcal{L}(F_0, F_{p,\infty}) \leq \mathcal{L}_{\gamma,p,T_\infty} + \mathcal{L}(F_{p,T_\gamma}, F_{p,T_\infty}). \quad (85)$$

Inequalities (79), (82), and (84) show that for any  $\delta > 0$ , there exists  $\gamma_\delta > 0$  such that (s.t.) for any positive  $\gamma < \gamma_\delta$ , there is a  $p_\gamma$  s.t. for any  $p > p_\gamma$ , there is a  $T_p$  s.t. for any  $T_\infty > T_p$

$$\Pr(\mathcal{L}_{\gamma,p,T_\infty} < 3\delta/4) > 1 - \delta/2. \quad (86)$$

The subscripts in  $\gamma_\delta$ ,  $p_\gamma$  and  $T_p$  signify dependence on the value of the corresponding parameter. Inequalities (86) and (85) would establish (78) as long as we are able to show that for any  $\delta > 0$ , there exists  $\tilde{\gamma}_\delta > 0$  s.t. for any positive  $\gamma < \tilde{\gamma}_\delta$ , there is a  $\tilde{p}_\gamma$  s.t. for any  $p > \tilde{p}_\gamma$  and any  $\tilde{T}_p$ , there exists  $T_\infty > \tilde{T}_p$  s.t.

$$\Pr(\mathcal{L}(F_{p,T_\gamma}, F_{p,T_\infty}) < \delta/4) > 1 - \delta/2. \quad (87)$$

Let us denote  $\xi = \sqrt{T}\varepsilon$ , where  $\varepsilon$  is a  $p \times T$  matrix with i.i.d.  $N(0, 1/T)$  entries, as defined in Section 2. We shall assume that, as  $p, T$  change,  $\xi$  represents  $p \times T$  sections of a fixed infinite array of i.i.d. standard normal random variables. Consider

$$M_{p,T} = \frac{T}{p} \left( \frac{\xi\xi'}{T} \right)^{-1/2} \frac{\xi\Delta_2'\xi'}{T} \left( \frac{\xi\Delta_1\xi'}{T} \right)^{-1} \frac{\xi\Delta_2\xi'}{T} \left( \frac{\xi\xi'}{T} \right)^{-1/2}.$$

So defined matrix  $M_{p,T}$  is identical to the real symmetric matrix  $\frac{T}{p}A^{-1/2}CD^{-1}C'A^{-1/2}$ . The above definition is formulated in terms of  $\xi$  to clarify that  $M_{p,T}$  depends on  $T$  not only via the term  $T/p$ , but also through  $A, C$ , and  $D$ . Note that  $F_{p,T_\gamma}$  and  $F_{p,T_\infty}$  are the empirical distributions of eigenvalues of  $M_{p,T_\gamma}$  and  $M_{p,T_\infty}$ , respectively. The following lemma is established in the Supplementary Appendix.

**Lemma 20** *For any  $\tau > 0$  there exists  $\gamma_\tau > 0$  s.t. for any positive  $\gamma < \gamma_\tau$ , there is a  $\tilde{p}_\gamma$  s.t. for any  $p > \tilde{p}_\gamma$  and any  $\tilde{T}_p$ , there exists  $T_\infty > \tilde{T}_p$  s.t. with probability larger than  $1 - \tau$ ,  $M_{p,T_\gamma} - M_{p,T_\infty}$  can be represented as the sum of two real symmetric*

matrices  $S$  and  $R$ ,

$$M_{p,T_\gamma} - M_{p,T_\infty} = S + R,$$

where  $\|S\| \leq K\sqrt{\gamma}$ ,  $\text{rank } R \leq \tau p$ , and  $K$  is an absolute constant.

Finally, let  $F_{SR}$  be the empirical distribution of eigenvalues of  $M_{p,T_\gamma} - S = M_{p,T_\infty} + R$ . Then, by Theorem A45 (norm inequality) of Bai and Silverstein (2010),

$$\mathcal{L}(F_{p,T_\gamma}, F_{SR}) \leq \|S\| \leq K\sqrt{\gamma},$$

whereas by their Theorem A43 (rank inequality),

$$\mathcal{L}(F_{SR}, F_{p,T_\infty}) \leq \frac{1}{p} \text{rank } R \leq \tau.$$

Therefore, by Lemma 20 and the triangle inequality, for any  $\tau > 0$  there exists  $\gamma_\tau > 0$  s.t. for any positive  $\gamma < \gamma_\tau$ , there is a  $\tilde{p}_\gamma$  s.t. for any  $p > \tilde{p}_\gamma$  and any  $\tilde{T}_p$ , there exists  $T_\infty > \tilde{T}_p$  s.t.

$$\Pr(\mathcal{L}(F_{p,T_\gamma}, F_{p,T_\infty}) < \tau + K\sqrt{\gamma}) > 1 - \tau.$$

For  $\tau = \delta/8$ , this inequality implies (87) with  $\tilde{\gamma}_\delta = \min\{\gamma_\tau, (\delta/8K)^2\}$ . Combining (87) with (86) yields (78), which completes the proof.

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