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DOCTORAL THESIS

Complements on Log Canonical Fano Varieties and Index Conjecture of Log Calabi-Yau Varieties

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Declaration of Authorship

I, Yanning Xu, declare that this thesis titled, "Complements on Log Canonical Fano Varieties and Index Conjecture of Log Calabi-Yau Varieties" and the work presented in it are my own. I confirm that:

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Date: April 2020

Abstract

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This thesis aims to generalise the theory of complements to log canonical Fano varieties and relate theory of complements to the index conjecture of log Calabi-Yau varieties. We mainly work over an algebraically closed field of characteristic zero, more specifically over $\mathbb{C}$.

We will first introduce some basic background theory for birational geometry, including notion of singularities, pairs, complements. We will then cover some backgrounds of (log) Fano and Calabi-Yau varieties. We will also state the main new results in the introduction.

The majority of work is then split into the following 4 sections: complements on surfaces, complements on log canonical 3-fold, index conjecture for log Calabi-Yau varieties, relative 3-fold complements.

For the section about complements on surfaces, we will firstly cover the known result about the theory for complements and then prove new results about semi-dlt surfaces. We will extend the notion of complements to semi-dlt surfaces and then prove a result for "gluing" complements for semi-dlt surfaces. Then we will move on to prove results about boundedness of complements for global log canonical Fano 3-fold. In other direction, we will prove the index conjecture for log Calabi-Yau varieties in dimension 3 in full generality and then prove some new inductive results towards the conjecture. In the last chapter, I will include proof of the boundedness of complements for log Fano in the relative case in dimension 3. The last chapter is a part of joint work, with Stefano Filipazzi and Joaquin Moraga, where we proved a general theorem of boundedness of complements in dimension 3.

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Contents

Introduction

One key goal in the field of birational geometry is to classify all varieties up to birational equivalence. The so called minimal model program (MMP) is a process and a prediction of such classification. It predicts that all varieties are built up, birationally, from three special types of varieties: Fano varieties (K_X negative varieties), Calabi-Yau varieties (K_X trivial varieties) and varieties of general type (K_X positive varieties). This generalises the classification of smooth projective curves. However, unlike the curve case, in higher dimension, there isn't, in general, a canonical smooth element in each birational equivalence class. However, certain varieties with mild singularities turn out to be nicer in certain regards. Therefore, the minimal model program proposes that we study these varieties with mild singularities, the so called *log canonical singularities*. Therefore, it is very important to understand singularities on the special varieties mentioned above.

A more or less classical result states that smooth Fano varieties in a fixed dimension forms a bounded family. However its counter part in the singular case (the so called BAB conjecture) has been a centre conjecture in the field of birational geometry. Recently, BAB conjecture was proved in full generality by Birkar. Its proof relies on a powerful tool, namely the theory of complements, which is a way to understand and bound singularities on certain special varieties.

The idea of complements originates in Shokurov's paper on the existence of smooth elements in anticanonical systems of Fano threefolds in the 70's in [Sho79]. Given a contraction $X \to Z$ of Fano-type over $z \in Z$, the theory of complements predicts the existence of a positive integer n, so that $|-nK_X/Z|$ contains an element with good singularities around $z \in Z$. Complements were rigorously defined first in [Sho96]. They also introduced some inductive scheme towards the existence of bounded complements for Fano-type varieties in [PS01, PS09]. The boundedness of complements for Fano-type varieties was proved by Birkar in [Bir19], which plays a key role in the proof of the BAB conjecture.

One of the main motivations for the work in this thesis is to generalise the theory of complements from Fano-type varieties to log canonical Fano varieties. In this direction, the author has confirmed the boundedness of complements for log canonical Fano varieties both in the global and the relative cases in dimension 3. In particular, we have the following theorems, which generalise [Bir19] results to the log canonical case in dimension 3.

Theorem 1. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is log canonical of dimension 3,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$.

Theorem 2. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a quasi-projective pair such that*

- *(X, B) is log canonical of dimension 3,*
- *$f : X \to Z$ is a projective contraction,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample over $z \in Z$.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$ over $z \in Z$.

While dealing with log canonical Fano varieties, Calabi-Yau varieties show up naturally. One such example is that a cone over an elliptic curve is a log canonical Fano variety. Therefore, not surprisingly, the theory of complements for log canonical Fano varieties is closely related to the aspects of log Calabi-Yau varieties, in particular, the index conjecture for log Calabi-Yau varieties. In another direction, the author has also worked on the boundedness of log canonical index for log canonical Calabi-Yau varieties. We will prove various inductive results towards this conjecture. In particular, concretely, we have the following theorem.

Theorem 3. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is log canonical of dimension 3,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

Another important aspect when dealing with log canonical singularities is that we have to deal with certain type of varieties that are not normal. Luckily, these varieties are, in some sense, well behaved and form a class of, the so called, *semi log canonical* varieties. Therefore, we need to generalise the theory of complements and index conjecture to these varieties. In this direction, we will show the following theorem.

Theorem 4. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is semi log canonical of dimension 3,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

The main focus of this thesis will be the proofs of the above theorems. The author hopes that the approach in this thesis can be generalised further to higher dimensional cases. In particular, a better understanding of the theory of complements for log canonical Fano varieties and the index conjecture for Calabi-Yau varieties can help to answer problems regarding boundedness of singularities and certain boundedness questions for Calabi-Yau type varieties. For example, as in [FMX19, Corollary 2], authors applied the theory of complements to bound the local index for strict log canonical singularities in dimension 4. Also recently, in [Bir20], the theory of Fano type complements are used to understand boundedness of certain type of Fano type and log Calabi-Yau fibration. We hope that the approach presented in this thesis can maybe be used to extend these results to more general, i.e. non Fano type, cases.

Chapter 1

Background and Preliminary Knowledge

In this chapter, we will introduce some basic terminologies and definitions that we will use throughout the thesis. Also we will state some well-known theorems in birational geometry that are of fundamental importance and are applied repeatedly here. At the end of the chapter, we will state the main results, whose proofs will be the main focus of this thesis.

1.1 Notations and Basic Definitions

Throughout this thesis, we will work over an algebraically closed field k of characteristic zero, which in most cases, can be assumed to be $\mathbb{C}$, the field of complex numbers. All varieties considered here are normal, connected and irreducible, unless otherwise stated. All divisors considered in this thesis will be $\mathbb{Q}$-Weil divisors, unless otherwise stated.

1.1.1 Hyperstandard sets

Let $\mathcal{R}$ be a subset of $[0,1]$. For the rest of this thesis, we will assume that $\mathcal{R}$ will be closed under addition. Then, we define the *set of hyperstandard multiplicities* associated to $\mathcal{R}$ as

$$\Phi(\mathcal{R}) := \left\{ 1 - \frac{r}{m} \middle| \, r \in \mathcal{R}, m \in \mathbb{N} \right\}.$$

When $\mathcal{R} = \{0,1\}$, we call it the set of *standard multiplicities*. Usually, we will assume $0, 1 \in \mathcal{R}$, so that $\Phi(\{0,1\}) \subset \Phi(\mathcal{R})$.

Now, assume that $\mathcal{R} \subset [0,1]$ is a finite set of rational numbers. Notice that the additive closure of $\mathcal{R}$ in $[0,1]$ is also finite and depend only on $\mathcal{R}$. Then, $\Phi(\mathcal{R})$ is a set of rational numbers satisfying the *descending chain condition* (*DCC* in short) whose only accumulation point is 1. We define $I(\mathcal{R})$ to be the smallest positive integer such that $I(\mathcal{R}) \cdot \mathcal{R} \subset \mathbb{N}$. These coefficients will play a natural and important role throughout this thesis.

1.2 Birational Geometry

In this section, we will introduce some basic terminology and convention for birational geometry that we will use throughout this thesis. Most of the details on this section can be found in [KM98].

1.2.1 Contractions

In this thesis a *contraction* is a projective morphism of quasi-projective varieties $f : X \to Z$ with $f_*\mathcal{O}_X = \mathcal{O}_Z$. In particular, f is surjective and has connected fibers. Notice that, if X is normal, then so is Z.

Given any projective morphism $f : X \to Y$ between quasi-projective varieties, we have the *Stein factorization* of $f = h \circ g$, where $X \xrightarrow{g} Y' \xrightarrow{h} Y$ with g being a contraction and h being a finite morphism.

1.2.2 Birational Maps

A *birational map* $f : X \dashleftarrow Y$ is a rational map with a rational inverse between varieties. In particular, X and Y have isomorphic open subsets. A *birational morphism* is a birational map that is also a morphism.

1.2.3 Divisors

Let X be a normal variety, and let M be an $\mathbb{R}$-divisor on X. We denote the coefficient of a prime divisor D in M by $\mu_D M$. If every non-zero coefficient of M belongs to a set $\Phi \subseteq \mathbb{R}$, we write $M \in \Phi$. Writing $M = \sum m_i M_i$ where M_i are the distinct irreducible components, the notation $M^{\geq a}$ means $\sum_{m_i \geq a} m_i M_i$, that is, we ignore the components with coefficients $< a$. One similarly defines $M^{\leq a}$, $M^{>a}$, and $M^{<a}$. Also $\lfloor M \rfloor$ means rounding down all coefficients of M.

Now let $f\colon X \to Z$ be a morphism to a normal variety. We say M is *horizontal* over Z if the induced map $\operatorname{Supp} M \to Z$ is dominant, otherwise we say M is *vertical* over Z.

Again let $f : X \to Z$ be a morphism to a normal variety, and let M and L be $\mathbb{Q}$-Cartier divisors on X. We say $M \sim L$ over Z (resp. $M \sim_{\mathbb{Q}} L$ over Z) if there is a Cartier (resp. $\mathbb{Q}$-Cartier) divisor N on Z such that $M - L \sim f^*N$ (resp. $M - L \sim_{\mathbb{Q}} f^*N$). For a point $z \in Z$, we say $M \sim L$ over z if $M \sim L$ over Z possibly after shrinking Z around z. The properties $M \sim_{\mathbb{Q}} L$ and $M \sim_{\mathbb{Q}} L$ over z are similarly defined.

1.2.4 Divisorial Sheaves

We will also introduce the notion of a divisorial sheaf. Let X be an S2 scheme. A divisorial sheaf is a rank one reflexive sheaf. Note that if L is a divisorial sheaf, then we can define $L^{[m]} := (L^m)^{**}$, since tensor powers of a reflexive sheaf may not be reflexive. We also note that if X is a normal variety, then divisorial sheaves correspond one to one to Weil divisors on X modulo linear equivalence, via a Weil divisor D corresponding to the sheaf $\mathcal{O}_X(D)$.

1.2.5 Pairs and Singularities

Here we will introduce the notion of a pair for normal varieties.

A *sub-pair* (X, B) is the datum of a normal quasi-projective variety and a divisor B such that $K_X + B$ is $\mathbb{Q}$-Cartier. If $B^{\leq 1} = B$, we say that B is a *sub-boundary*, and if in addition $B \geq 0$, we call it *boundary*. A sub-pair (X, B) is called a *pair* if $B \geq 0$. A sub-pair (X, B) is *simple normal crossing* (or *log smooth*) if X is smooth, every irreducible component of $\mathrm{Supp}(B)$ is smooth, and locally analytically $\mathrm{Supp}(B) \subset X$ is isomorphic to the intersection of $r \leq n$ coordinate hyperplanes in $\mathbb{A}^n$. A *log resolution* of a sub-pair (X, B) is a birational contraction $\pi\colon X' \to X$ such that $\mathrm{Ex}(\pi)$ is a divisor and $(X', \pi_*^{-1}\mathrm{Supp}(B) + \mathrm{Ex}(\pi))$ is log smooth. Here $\mathrm{Ex}(\pi) \subset X'$ is the *exceptional locus* of π, i.e., the reduced subscheme of X' consisting of the points where π is not an isomoprhism.

Let (X, B) be a sub-pair, and let $\pi\colon X' \to X$ be a birational contraction from a normal variety X'. Then, we can define a sub-pair (X', B') on X' via the identity

$$K_{X'} + B' = \pi^*(K_X + B),$$

where we assume that $\pi_* K_{X'} = K_X$ as Weil divisors. We call (X', B') the *log pull-back* of (X, B) on X'. The *log discrepancy* of a prime divisor E on X' with respect to (X, B) is defined as $a_E(X, B) := 1 - \mu_E(B')$. We say that a sub-pair (X, B) is *sub-log canonical* (resp. *sub-klt*) if $a_E(X, B) \geq 0$ (resp. $a_E(X, B) > 0$) for every π and every E as above. When (X, B) is a pair, we say that (X, B) is *log canonical* or *klt*, respectively. Notice that, if (X, B) is log canonical (resp. klt), we have $0 \leq B = B^{\leq 1}$ (resp. $0 \leq B = B^{<1}$). Also in the case that $B = 0$, we say X is canonical (resp. terminal) if $a_E(X, 0) \geq 1$ (resp. $a_E(X, B) > 1$) for every π and every E as above. We say (X, B) is *strictly log canonical* if it is log canonical but not klt. The local version of these definitions are defined similarly.

Let (X, B) be a sub-pair. A *non-klt place* is a prime divisor E on a birational model of X such that $a_E(X, B) < 0$. A *non-klt center* is the image of a non-klt place. If $a_E(X, B) = 0$, we say that E is a *log canonical place*, and the corresponding center is said to be a *log canonical center*. The *non-klt locus* $\mathrm{Nklt}(X, B)$ is defined as the union of all the non-klt centers of (X, B). Similarly, the *non-log canonical locus* $\mathrm{Nlc}(X, B)$ is defined as the union of all the non-klt centers of (X, B) that are not log canonical centers. Notice that the non-klt locus of (X, B) is the same as the union of log canonical centers on (X, B) if (X, B) is a log canonical pair.

Given a sub-pair (X, B) and an effective $\mathbb{Q}$-Cartier divisor D, we define the *log canonical threshold* of D with respect to (X, B) as

$$\operatorname{lct}(X, B; D) := \sup\{t \geq 0 | (X, B + tD) \text{ is sub-log canonical}\}.$$

Now we will introduce the notion of a dlt pair. An lc pair (X, B) is called *divisorial log terminal or dlt* if there exists a log resolution $(Y, B_Y) \xrightarrow{f} X$ with $K_Y + B_Y := f^*(K_X + B)$ such that $\mu_E B_Y < 1$ for every exceptional divisor E over X. Such resolutions are sometimes called *Szabo resolutions*. It is well known that the above definition of dlt is equivalent to that there exists a closed subset $Z \subset X$ such that $(X \setminus Z, B|_{X\setminus Z})$ is log smooth and every divisor E on a birational model whose center on X is contained in Z satisfies $a_E(X, B) > 0$. In some sense, a dlt pair is klt pair with some simple normal crossing singularities.

$\mathbb{Q}$-factorial dlt model

Recall that a normal variety is called $\mathbb{Q}$-factorial if every Weil divisor is $\mathbb{Q}$-Cartier. As a corollary of the famous result in [BCHM10], we have the following. If (X, B) is an lc pair, then there exists a $\mathbb{Q}$-factorial dlt model (X', B') and a birational morphism $f : X' \to X$ such that $f^*(K_X + B) := K_{X'} + B'$. Furthermore, the $Ex(f)$ is given by the union of $\lfloor B \rfloor$, i.e. all exceptional divisors appear as coefficient 1 in B. Such (X', B') is called a $\mathbb{Q}$-*factorial dlt model* for (X, B).

1.2.6 B-divisors

This is an introduction to the notion of B-divisors. This is mostly taken from [FMX19, Section 3.9]. Roughly speaking, a B-divisor on X is some divisor defined compatibly on all birational models of X. More precisely, we have the following. We will first define everything rigorously and then use a simplified notation.

Let X be a normal variety, and consider the set of all proper birational morphisms $\pi\colon X_\pi \to X$, where X_π is normal. This is a partially ordered set, where $\pi' \geq \pi$ if π' factors through π. We define the space of *Weil b-divisors* as the inverse limit

$$\mathbf{Div}(X) := \varprojlim_{\pi} \operatorname{Div}(X_\pi), \tag{1.2.1}$$

where $\operatorname{Div}(X_\pi)$ denotes the space of Weil divisors on X_π. Then, we define the space of $\mathbb{Q}$-*Weil b-divisors* $\mathbf{Div}_{\mathbb{Q}}(X) := \mathbf{Div}(X) \otimes \mathbb{Q}$. In the following, by b-divisor we will mean a $\mathbb{Q}$-Weil b-divisor. Equivalently, a b-divisor $\mathbf{D}$ can be described as a (possibly infinite) sum of geometric valuations V_i of $k(X)$ with coefficients in $\mathbb{Q}$,

$$\mathbf{D} = \sum_{i \in I} b_i V_i, \ b_i \in \mathbb{Q},$$

such that for every normal variety X' birational to X, only a finite number of the V_i can be realized by divisors on X'. The *trace* $\mathbf{D}_{X'}$ of $\mathbf{D}$ on X' is defined as

$$\mathbf{D}_{X'} := \sum_{\{i\in I \mid c_{X'}(V_i)=D_i,\ \mathrm{codim}_{X'}(D_i)=1\}} b_i D_i$$

where $c_{X'}(V_i)$ denotes the center of the valuation on X'.

Given a b-divisor $\mathbf{D}$ over X, we say that $\mathbf{D}$ is a *b-$\mathbb{Q}$-Cartier* b-divisor if there exists a birational model X' of X such that $\mathbf{D}_{X'}$ is $\mathbb{Q}$-Cartier on X', and for any model $r\colon X'' \to X'$, we have $\mathbf{D}_{X''} = r^*\mathbf{D}_{X'}$. When this is the case, we will say that $\mathbf{D}$ descends to X' and write $\mathbf{D} = \overline{\mathbf{D}_{X'}}$. We say that $\mathbf{D}$ is *b-effective*, if $\mathbf{D}_{X'}$ is effective for any model X'. We say that $\mathbf{D}$ is *b-nef*, if it is b-$\mathbb{Q}$-Cartier and, moreover, there exists a model X' of X such that $\mathbf{D} = \overline{\mathbf{D}_{X'}}$ and $\mathbf{D}_{X'}$ is nef on X'. The notion of b-nef b-divisor can be extended analogously to the relative case.

Example 1.2.1. Let (X, B) be a sub-pair. The *discrepancy b-divisor* $\mathbf{A}(X, B)$ is defined as follows: on a birational model $\pi\colon X' \to X$, its trace $\mathbf{A}(X, B)_{X'}$ is given by the identity $K_{X'} = \pi^*(K_X + B) + \mathbf{A}(X, B)_{X'}$. Then, the b-divisor $\mathbf{A}^*(X, B)$ is defined by taking its trace $\mathbf{A}^*(X, B)_{X'}$ on X' to be $\mathbf{A}(X, B)_{X'} := \sum_{a_i>-1} a_i D_i$, where $\mathbf{A}(X, B)_{X'} = \sum_i a_i D_i$.

We note that a b-divisor can just be thought of as a uniform system of divisors on all birational models that is compatible with pushforward maps. For simplicity, from now on we will simply use M instead of $\mathbf{M}$ to represent certain b-divisors.

1.2.7 Generalized pairs

Due to technical reasons, it is natural to generalise the notion of a pair to include a B-divisor. These pairs are firstly formally introduced in [BZ16] and nowadays, it is natural to consider these pairs when working with Fano varieties as shown in the famous proof of BAB theorem in [Bir19]. Again most of this section is taken from [FMX19].

A *generalized sub-pair* $(X, B, M)/Z$ over Z is the datum of:

- a normal variety $X \to Z$ projective over Z;
- a divisor B on X;
- a b-$\mathbb{Q}$-Cartier b-divisor M over X which descends to a nef$/Z$ Cartier divisor $M_{X'}$ on some birational model $X' \to X$.

Moreover, we require that $K_X + B + M$ is $\mathbb{Q}$-Cartier. If B is effective, we say that $(X, B, M)/Z$ is a *generalized pair*. The divisor B is called the *boundary part* of $(X, B, M)/Z$, and M is called the *moduli part*. In the definition, we can replace X' with a higher birational model X'' and M with $M_{X''}$ without changing the generalized pair. Whenever $M_{X''}$ descends on X'', then the datum of the rational map $X'' \dashrightarrow X$, B, and $M_{X''}$ encodes all the information of the generalized pair.

Let $(X, B, M)/Z$ be a generalized sub-pair and $\pi\colon Y \to X$ be a projective birational morphism. Then, we may write

$$K_Y + B_Y + M_Y = \pi^*(K_X + B + M_X).$$

Given a prime divisor E on Y, we define the *generalized log discrepancy* of E with respect to $(X, B, M)/Z$ to be $a_E(X, B, M) := 1 - \mathrm{mult}_E(B')$. If $a_E(X, B, M) \geq 0$ for all divisors E over X, we say that $(X, B, M)/Z$ is *generalized sub-log canonical*. Similarly, if $a_E(X, B + M) > 0$ for all divisors E over X and $\lfloor B \rfloor \leq 0$, we say that $(X, B, M)/Z$ is *generalized sub-klt*. When $B \geq 0$, we say that $(X, B, M)/Z$ is *generalized log canonical* or *generalized klt*, respectively.

1.2.8 Canonical Bundle Formula

An important reason that we are interested in the generalised pairs above is that they appear naturally in the so called canonical bundle formula. Canonical bundle formula relates the information of the total space of a fibration to the base of the fibration. We will here cover an overview of canonical bundle formula. For more details, please refer to [FG14a].

An algebraic fiber space $f : X \to Z$ with a given log canonical divisor $K_X + B$ which is $\mathbb{Q}$-linearly trivial over Z is called an *lc-trivial fibration*. The following canonical bundle formula for lc-trivial fibration from [FG14a] will be used.

Theorem 1.2.2. *(*[FG14a, Theorem 1.1], [PS09, Theorem 8.1]*) Let $f : (X, B) \to Z$ be a projective morphism from a log pair to a normal variety Z with connected fibers, B be a $\mathbb{Q}$-boundary divisor. Assume that (X, B) is sub-lc and it is lc on the generic fiber and $K_X + B \sim_{\mathbb{Q}} 0/Z$. Then there exists a boundary $\mathbb{Q}$-divisor B_Z and a $\mathbb{Q}$-divisor M_Z on Z satisfying the following properties.*

(i). $(Z, B_Z + M_Z)$ is a generalized pair,

(ii). M_Z is b-nef.

(iii). $K_X + B \sim_{\mathbb{Q}} f^*(K_Z + B_Z + M_Z)$.

Furthermore, Let I be a positive integer such that $I(K_X + B) \sim 0$ along the generic fiber of f. Then, by [PS09, Construction 7.5]*, we may choose M_Z in its $\mathbb{Q}$-linear equivalence class so that*

$$I(K_X + B) \sim I f^*(K_Z + B_Z + M_Z). \tag{1.2.2}$$

Furthermore, if the general fibers are curves, then I only depends on the coefficients of B and, there exists n, depending only on coefficients of B such that nM_Z is base-point free.

Under the above notations, B_Z is called the *discriminant part* and M_Z is the *moduli part*. We add a few words on the construction of B_Z. For each prime divisor D on Z we let t_D be the lc threshold of f^*D with respect to (X, B) over the generic point of D, that is, t_D is the largest number so that $(X, B + t_D f^*D)$ is sub-lc over the generic point of D. Of course f^*D may not be well-defined everywhere but at least it is defined over the smooth locus of Z, in particular, near the generic point of D, and that is all we need. Next let $b_D = 1 - t_D$, and then define $B_Z = \sum b_D D$ where the sum runs over all the prime divisors on Z.

We also have the following constrains on the coefficients of B_Z and M_Z if the fibers are Fano-type. We will generalise this to some non-Fano type fibrations in later chapters.

Theorem 1.2.3 (Effective Canonical Bundle Formula for Fano-type Fibration)**.** [Bir19, Proposition 6.3] *Let $d \in \mathbb{N}$ be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers, then there exists $q \in \mathbb{N}$ and $\mathfrak{S}$ depending only on $d, \mathfrak{R}$ satisfying the following. Assuming $f : (X, B) \to Z$ is a projective contraction such that*

- *(X, B) is projective lc of dimension d, and* $\dim Z > 0$,
- *$K_X + B \sim_{\mathbb{Q}} 0/Z$ and $B \in \Phi(\mathfrak{R})$,*
- *X is Fano type over some non-empty open subset $U \subset Z$, and*
- *the generic point of each non-klt centre of (X, B) maps into U.*

Then we have
$$q(K_X + B) \sim qf^*(K_Z + B_Z + M_Z),$$
where B_Z and M_Z are the discriminant part and moduli part of canonical bundle formula. Furthermore, we have $B_Z \in \Phi(\mathfrak{S})$ and qM_Z is b-nef b-Cartier. In particular, qM_Z is an integral Weil divisor.

The above theorem is of huge importance because it allows to keep controlling coefficients while using canonical bundle formula. It is expected that the following conjecture (called *semi-ample conjecture*) will hold in the canonical bundle formula.

Conjecture 1.2.4. [PS09, Conjecture 7.13.3] *Let $d \in \mathbb{N}$ be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers, then there exists $q \in \mathbb{N}$ and $\mathfrak{S}$ depending only on $d, \mathfrak{R}$ satisfying the following. Assuming $f : (X, B) \to Z$ is a projective contraction such that*

- *(X, B) is projective lc of dimension d, and* $\dim Z > 0$,
- *$K_X + B \sim_{\mathbb{Q}} 0/Z$ and $B \in \Phi(\mathfrak{R})$.*

Then we have
$$q(K_X + B) \sim qf^*(K_Z + B_Z + M_Z),$$
where B_Z and M_Z are the discriminant part and moduli part of canonical bundle formula. Furthermore, qM_Z is b-base point free.

Remark 1.2.5. By [PS09, Theorem 8.1], the above conjecture holds when the relative dimension is 1 and some other special cases are known when relative dimension is 2.

1.2.9 Minimal Model Program

Throughout this thesis, we will be using the well-known theorems about Minimal Model Program (MMP). Due to the restriction of length, we will not give a detailed overview of all results about minimal model program and all its terminologies. For more details, please refer to [BCHM10].

1.2.10 Bounded Family

Here we will quickly review some facts about bounded families of couples. A couple (X, B) consists of a normal projective variety X and B a reduced divisor on X. We note that here we are not assuming that $K_X + B$ is $\mathbb{Q}$-Cartier, hence we are not considering pairs.

An example of a couple is $(X, Supp(B))$ for any sub pair (X, B). Let $\mathcal{P}$ be a set of couples. We say $\mathcal{P}$ forms a *bounded family* if there exists finitely many projective morphisms $V^i \to T^i$ and reduced divisor C^i on V^i such that for each $(X, B) \in \mathcal{P}$, there exists i and $t \in T^i$ a closed point, such that there exists an isomorphism $\phi : X \to V^i_t$ with $\phi(B) \leq C^i_t$. Here V^i_t and C^i_t represent fibers over $t \in T^i$. We say that a set of pairs (X, B) is *log bounded* or forms a bounded family if $(X, Supp(B))$ forms a bounded family.

Finally, we note some basic properties of log bounded family. If $\mathcal{P}$ form a bounded family, then the varieties in $\mathcal{P}$ are clearly bounded in complex topology hence bounded topologically. In particular, almost all numerical constant regarding X are bounded (e.g. $b_n(X)$, the n^{th} Betti-number is bounded). Also if (X, B) is log bounded, then there exists a (Y, B_Y) that is also log bounded for some (Y, B_Y), a log resolution of (X, B).

1.3 Fano Varieties

We will introduce the notion of Fano varieties, which will be one of the objects to study in this thesis.

Let (X, B) be a pair and $X \to Z$ be a contraction. We say (X, B) is *log canonical Fano* over Z if it is lc and $-(K_X + B)$ is ample over Z; if $B = 0$ we just say X is Fano over Z. We say X is *of Fano type* over Z if (X, B) is klt and $-(K_X + B)$ is nef and big$/Z$ for some choice of B; it is easy to see this is equivalent to the existence of a big$/Z$ $\mathbb{Q}$-boundary Γ so that (X, Γ) is klt and $K_X + \Gamma \sim_{\mathbb{Q}} 0/Z$.

We will give some examples of log canonical Fano and Fano type varieties.

Example 1.3.1. $X := \mathbb{P}^n$ is Fano type with $B = 0$ for every n.

Example 1.3.2. Here we give an example of log canonical Fano but not Fano type. This is a well-known construction. One such reference is in [Gon09, Construction 5.1]. Let $n \in \mathbb{N}$ and S be a smooth (or terminal) variety with $K_S \sim_{\mathbb{Q}} 0$. Let $S \subset \mathbb{P}^n$ be some projective normal embedding. Let X_0 be the cone over S and $\phi : X \to X_0$ be the blow-up at the vertex. It can be shown that $X = \mathbb{P}_S(\mathcal{O}_S \bigoplus \mathcal{O}_S(-H))$, where H is a hyperplane section on $S \subset \mathbb{P}^n$ and the ϕ-exceptional divisor E is isomorphic to S. It can be shown that $\phi^*(K_{X_0}) = K_X + E$, hence X_0 is log canonical. Also it can be shown by [Gon09, Construction 5.1], that $-K_{X_0} \sim_{\mathbb{Q}} \mathcal{O}_{X_0}(1)$ is ample. Therefore $(X_0, 0)$ is a log canonical Fano variety that is not Fano type since it is not klt.

Classification of Fano varieties is one of the key topics in birational geometry. Recently, we have the following theorem (so called the BAB conjecture).

Theorem 1.3.3. [Bir16, Theorem 1.1] *Let d be a natural number and ϵ be a positive real number. Then the projective varieties X such that*

- *(X, B) is ϵ-lc of dimension d for some boundary B,*
- *$-(K_X + B)$ is nef and big*

forms a bounded family.

If we have some control over the coefficients of B, we have the following stronger result, which will play a key role later in the thesis.

Theorem 1.3.4. [Bir20, Theorem 1.3] *Let d be a natural number and ϵ, δ be a positive real number. Then the set of pairs (X, B) such that*

- *(X, B) is ϵ-lc of dimension d,*
- *$B \geq \delta$,*
- *$K_X + B \sim_{\mathbb{Q}} 0$ and B is big*

form a log bounded family.

For Fano type or log canonical Fano varieties, it is natural to consider the anti-pluricanonical linear system $|-n(K_X + B)|$, and the study of these linear systems is precisely summed up in the idea of complements.

1.4 Complements

Now we are ready to introduce complements, the centre of interest in this thesis. It is of centre of interest in the field of minimal model program. It is used and played a crucial role in the recent famous proof of BAB theorem in [Bir19]. The following definitions are taken from [FMX19]. Let (X, B) be a log canonical pair, $X \to T$ a contraction, and n a positive integer. We say that the divisor B^+ is a $\mathbb{Q}$-*complement* over $t \in T$ if the following conditions hold over some neighborhood of $t \in T$:

(i) (X, B^+) is a log canonical pair;

(ii) $K_X + B^+ \sim_{\mathbb{Q}} 0$ over $t \in T$; and

(iii) $B^+ \geq B$.

Furthermore, we say that B^+ is an n-*complement* for (X, B) over $t \in T$ if the following stronger version of condition (ii) holds:

(ii)′ $n(K_X + B^+) \sim 0$ over $t \in T$.

In particular, if B^+ is an n-complement, nB is an integral Weil divisor.

Remark 1.4.1. Notice that more general complements, where the above condition (iii) is weakened, are used in the literature. See for example [Bir19, 2.18]. Since in this thesis condition (iii) is always satisfied, we will use this stronger definition of complement, in order to avoid redundant terminology and notation. The complements defined here are sometimes called good complements in the literature.

Remark 1.4.2. If $T = Spec\ \mathbb{C}$ is a single point, then we call it the *global case*, else we refer to it as the *relative case* . Also when T is not mentioned explicitly, we assume that we are talking about the global case.

Remark 1.4.3. We also note that if B^+ is an n-complement for (X, B), then indeed, we have
$$0 \leq n(B^+ - B) \sim -n(K_X + B).$$
In particular, we have $n(B^+ - B) \in |-n(K_X + B)|$ and it is a "nice" element in the linear system since $(X, B + (B^+ - B))$ is still lc. Therefore the complements can be viewed as "nice" elements in the plur-anticanonical log linear system.

Now we give some basic examples of complements.

Example 1.4.4. Let $X := \mathbb{P}^1$, and P, Q, R, S be 4 distinct points on X. Let $B := \frac{1}{3}(P + Q + R)$, then a 3-complement for (X, B) is $B^+ := B + S$. Indeed, we have $3(K_X + B^+) \sim 0$ and (X, B^+) is lc.

Example 1.4.5. Let X be a normal variety such that $K_X \sim_{\mathbb{Q}} 0$. Assuming $nK_X \sim 0$, we see that $(X, 0)$ is n-complemented with $B^+ = 0$. This rather trivial example will be of interest later in this thesis.

Following the work of Birkar [Bir19], we can extend the notion of complement to generalized pairs. Let $(X, B, M)/Z$ be a generalized log canonical pair, $X \to T$ a contraction over Z, and n a positive integer. We say that the divisor B^+ is a $\mathbb{Q}$-*complement* over $t \in T$ if the following conditions hold over some neighborhood of $t \in T$:

(i) (X, B^+, M) is a generalized log canonical pair;

(ii) $K_X + B^+ + M_X \sim_{\mathbb{Q}} 0$ over $t \in T$; and

(iii) $B^+ \geq B$.

As above, we say that B^+ is an n-*complement* for $(X, B, M)/Z$ over $t \in T$ if the following stronger version of condition (ii) holds:

(ii)$'$ $n(K_X + B^+ + M_X) \sim 0$ over $t \in T$.

In particular, if B^+ is an n-complement, and nM is an integral b-divisor, then nB is an integral Weil divisor.

We will state the famous theorem on the existence of boundedness of complements for Fano type varieties as in [Bir19].

Theorem 1.4.6. [Bir19, Theorem 1.8] *Let d be a natural number and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a pair and $X \to Z$ is a contraction such that*

- *(X, B) is lc of dimension d and* $\dim Z > 0$,
- *$B \in \Phi(\mathfrak{R})$,*
- *X is of Fano type over Z, and*
- *$-(K_X + B)$ is nef over Z.*

Then for any point $z \in Z$, there is an n-complement $K_X + B^+$ of $K_X + B$ over z.

Theorem 1.4.7. [Bir19, Theorem 1.10] *Let d and p be natural numbers and $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $d, p,$ and $\mathfrak{R}$ satisfying the following. Assume $(X', B' + M')$ is a projective generalized polarised pair with data $\phi\colon X \to X'$ and M such that*

- *$(X', B' + M')$ is generalized lc of dimension d,*
- *$B' \in \Phi(\mathfrak{R})$ and pM is b-Cartier,*
- *X' is of Fano type, and*
- *$-(K_{X'} + B' + M')$ is nef.*

Then there is an n-complement $K_{X'} + B'^+ + M'$ of $K_{X'} + B' + M'$.

The goal and focus of the majorities to this thesis will be to generalise the above theorem to varieties that possibly are not Fano-type.

1.5 Calabi-Yau Varieties

As shown in example 1.3.2, log canonical Fano varieties that are not Fano type are closely related to K_X trivial varieties. Therefore, another centre object of interest for this thesis will be the so called log Calabi-Yau varieties. Hence we will introduce some basics of Calabi-Yau varieties here. In later chapters, we will focus on some properties in more detail.

We will use a very general sense of Calabi-Yau varieties in this thesis. We say (X, B) is *log Calabi-Yau*/Z if (X, B) is lc and $K_X + B \sim_{\mathbb{Q}} 0/Z$. If $B = 0$, then we say X is *Calabi-Yau*/Z. The minimal n such that $n(K_X + B) \sim 0/Z$ is called the index of the pair (X, B).

Example 1.5.1. Clearly if (X, B) is log canonical Fano, then there exists boundary B' such that (X, B') is log Calabi-Yau. Examples of non Fano-type Calabi-Yau varieties include elliptic curves, K3 surfaces, etc.

Also the index above is closely related to the boundedness of complements, in the sense that if (X, B) is log Calabi-Yau with index m, then m is clearly the minimal n such that an n-complement exists for the pair (X, B) (since all $\mathbb{Q}$-complements for a log Calabi-Yau pair are trivial). One important fact of log Calabi-Yau pairs is the following theorem on the coefficients of the boundary. The following theorem is called ACC theorem for numerically trivial pairs.

Theorem 1.5.2. [HMX14, Thoerem D] *Fix a positive integer n and a set $I \subset [0, 1]$, which satisfies the DCC. Then there is a finite set $I_0 \subset I$ with the following property:*

If (X, Δ) is a log canonical pair such that

- *X is projective of dimension n,*
- *the coefficients of Δ belong to I, and*
- *$K_X + \Delta$ is numerically trivial,*

then the coefficients of Δ belong to I_0.

We also have the following for generalised pairs. When $(X', B' + M')$ is representing a generalised pair, we sometimes write M as the divisor in the b-divisor class on a sufficiently high birational model of X'. Then in the spirit of this notation, we have the following theorem.

Theorem 1.5.3. [BZ16, Theorem 1.6] *Let Λ be a DCC set of nonnegative real numbers and d a natural number. Then there is a finite subset $\Lambda_0 \subset \Lambda$ depending only on Λ, d such that if $(X', B' + M')$ and M satisfy*
(i) $(X', B' + M')$ is generalized lc of dimension d,
(ii) $M = \sum \mu_j M_j$ where M_j are nef Cartier divisors and $\mu_j \in \Lambda$,
(iii) $\mu_j = 0$ if $M_j \equiv 0$,
(iv) the coefficients of B' belong to Λ, and
(v) $K_{X'} + B' + M' \equiv 0$,
then the coefficients of B' and the μ_j belong to Λ_0.

Notice that a more or less trivial consequence of the above theorem is the following corollary.

Corollary 1.5.4. *Let Λ be a DCC set of nonnegative real numbers and d a natural number. Then there is a finite subset $\Lambda_0 \subset \Lambda$ depending only on Λ such that if $(X', B' + M')$ and M satisfy*

(i) $(X', B' + M')$ is generalized lc of dimension d,

(ii) $M = \sum \mu_j M_j$ where M_j are nef Cartier divisors and $\mu_j \in \Lambda$,

(iii) the coefficients of B' belong to Λ, and

(iv) $K_{X'} + B' + M' \equiv 0$,

then the coefficients of B' belong to Λ_0.

Proof. Here we simply apply Theorem 1.5.3 and replace M by

$$N := \sum_{M_j \not\equiv 0} \mu_j M_j.$$

Then the conclusion follows from Theorem 1.5.3. $\square$

1.6 B-birational Maps and B-representations

We introduce the notion of *B-birational* as in [Fuj00]. Let (X,B), (X',B') be sub-pairs, we say $f : (X,B) \dashrightarrow (X',B')$ is *B-birational* if there is a common resolution $\alpha : (Y,B_Y) \to (X,B)$, $\beta : (Y,B_Y) \to (X',B')$ such that $K_Y + B_Y = \alpha^*(K_X + B) = \beta^*(K_{X'} + B')$ and a commuting diagram as the following.

$$\begin{tikzcd} & (Y,B_Y) \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ (X,B) \arrow[rr, dashed, "f"] & & (X',B') \end{tikzcd}$$

Let

$$Bir(X,B) := \{f | f : (X,B) \dashrightarrow (X,B) \text{ is B-birational}\}.$$

Let n be a positive integer such that $n(K_X + B)$ is Cartier. Then we define

$$\rho_n : Bir(X,B) \to Aut(H^0(X, n(K_X + B)))$$

to be the representation of the natural action of $Bir(X,B)$ on $H^0(X, n(K_X + B))$ by pulling back sections.

We have the following theorem on B-birational representations.

Theorem 1.6.1. [FG14b, Theorem 1.1] *Let (X,B) be a projective dlt pair with $n(K_X + B) \sim 0$ and n being even, then $\rho_n(Bir(X,B))$ is finite.* □

We also note that the above notions of B-birational map and B-representation easily generalise to potentially disconnected pairs, (i.e. a disjoint union of lc pairs).

We will discuss properties of this B-representation in much more detail in later chapters. This representation turns out to be the key to study complements on log canonical Fano varieties. In particular, it will be a vital tool when we consider gluing sections on certain "nice" non-normal pairs, which we will introduce in the next section.

1.7 Slc Pairs

One key difficulty when working with log canonical singularities is that we are forced to deal with certain non-normal varieties. Nevertheless, they behave nicely in some sense. It turns out we need to work with a broader class of pairs called semi log canonical pairs. We will introduce the basic definition and give some basic yet crucial examples here. More details of slc and sdlt pairs will be introduced in much more detail later.

First we introduce demi-normal schemes as in [Kol13]. A *demi-normal scheme* is a reduced scheme that is S2 and normal crossing in codimensional 1. Let Δ be an effective $\mathbb{Q}$-divisor whose support does not contain any irreducible components of the conductor of X. The pair (X, Δ) is called a *semi-log canonical pair* (an *slc pair*, for short) if

- $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and
- (X', Θ) is log canonical, where $\pi : X' \to X$ is the normalization and $K_{X'} + \Theta = \pi^*(K_X + \Delta)$.

Remark 1.7.1. The definition in [FG14b] is compatible with the definition used in [Kol13]. Furthermore, we remark that we will only be interested in such X that is an algebraic variety and pure n-dimensional.

Remark 1.7.2.
1. Note that $\Theta = \Delta' + D'$, where D' is the conductor on the normalisation and Δ' is the divisorial part of the preimage of Δ on X'. For more detailed treatment, see [Kol13, Chapter 5]. We will also discuss this in more detail in Chapter 2.
2. We also note that if $X := \cup X_i$ where X_i are irreducible components of X, and let $X'_i \to X_i$ be their normalisations, then we have $X' = \sqcup X'_i$.

Similarly one defines a *semi-divisorial log terminal pair* (an *sdlt pair*, for short). An slc pair (X, B) is said to be *sdlt* if the normalisation (X', Θ) is dlt in the usual sense and $\pi : X'_i \to X_i$ is isomorphism. Note that here we are using the definition in [Fuj00] instead of [Kol13]. This is fine since we will only use semi dlt in the following setting. We remark that if (X, B) is a usual dlt pair, then $(\lfloor B \rfloor, \mathrm{Diff}(B - \lfloor B \rfloor))$ is semi-dlt. Also for sdlt pair (X, B) it is clear that $(K_X + B)|_{X_i} = K_{X_i} + \Theta_i$, where $\Theta_i := \Theta|_{X_i}$. We also have the following examples of demi-normal schemes, sdlt and slc pairs.

Example 1.7.3. A nodal curve given by $y^2z = x^2(x - z)$ in $\mathbb{P}^2$ is an example of a demi-normal scheme. However, a cuspidal curve $y^2z = x^3$ in $\mathbb{P}^2$ is not a demi-normal scheme.

Example 1.7.4. Let (X, B) be a dlt pair and let $S := \lfloor B \rfloor$, then $(S, \mathrm{Diff}(B - S))$ is sdlt.

Example 1.7.5. [FG14b, Example 2.6] Let (X, B) be $\mathbb{Q}$-factorial lc with B a $\mathbb{Q}$-divisor. Let $S := \lfloor B \rfloor$ and assume $(X, B - \epsilon S)$ is klt for $0 < \epsilon << 1$. Then $(S, \mathrm{Diff}(B - S))$ is slc. In particular, if S_i is an irreducible component of S, then $(S_i, \mathrm{Diff}(B - S_i))$ is also slc.

1.8 Statement of the Main Results

In this section of the thesis, I will state the main results that will be presented in this thesis. Firstly, I will state a few conjectures.

Conjecture 1.8.1. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension d,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample.*

Then there is an n complement $K_X + B^+$ of $K_X + B$.

Also we have the following conjecture in the relative case.

Conjecture 1.8.2. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension d,*
- *$f : X \to Z$ is a contraction,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample over $z \in Z$.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$ over $z \in Z$.

In a related direction, we have the boundedness of index conjecture.

Conjecture 1.8.3. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension d,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

Also we have the similar version of index conjecture, but for slc pairs.

Conjecture 1.8.4. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is slc of dimension d,*

- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

As we will see the index conjecture plays an important role in the proof of log canonical Fano complements conjecture. Now the remaining of the thesis will be devoted to the proofs of the following theorems. We will show that all of the conjectures above hold in dimension $d \leq 3$.

Theorem 1.8.5. [Xu19a, Theorem 1.2] *Conjecture 1.8.1 holds when $d \leq 3$.*

Theorem 1.8.6. [FMX19, Proposition 10.1] *Conjecture 1.8.2 holds when $d = 3$.*

Theorem 1.8.7. [Xu19b] *Conjecture 1.8.3 holds when $d \leq 3$. Also Conjecture 1.8.3 holds when $d = 4$ with $B \neq 0$ and (X, B) klt . Conjecture 1.8.4 holds when $d \leq 2$.*

Theorem 1.8.8. [Xu19b] *Conjecture 1.8.4 holds when $d = 3$ and Conjecture 1.8.3 holds when $d = 4$ and $B \neq 0$.*

Remark 1.8.9. It is well-known that Conjecture 1.8.3 holds in dimension 2. One possible reference would be [PS09]. We will include the proof later for completeness.

The proof will be following from [Xu19a], [Xu19b] and [FMX19]. Finally, the following theorem will be stated and only given outline of the proof. It is proved in [FMX19] and due to the length and amount of technicalities in the proof, this thesis will only go through a brief summary and outline of the proof.

Theorem 1.8.10. [FMX19, Theorem 1] *Let $\Lambda \subset \mathbb{Q}$ be a set satisfying the descending chain condition with rational accumulation points. There exists a natural number n only depending on Λ which satisfies the following. Let $X \to T$ be a projective contraction between normal quasi-projective varieties so that*

- *(X, B) is a log canonical 3-fold,*
- *(X, B) is $\mathbb{Q}$-complemented over $t \in T$, and*
- *the coefficients of B belong to Λ.*

Then, up to shrinking T around t, we can find

$$\Gamma \sim_T -n(K_X + B)$$

such that $(X, B + \Gamma/n)$ is a log canonical pair, i.e. there exists an n-complement $B^+ := B + \Gamma/n$ for $K_X + B$ over $t \in T$.

Notice that although the result in Theorem 1.8.10 is much stronger than the results in Theorem 1.8.5 and Theorem 1.8.6. However, the proof of Theorem 1.8.10 uses both the results and proofs of Theorem 1.8.5 and Theorem 1.8.6, which is the main results that we will present in this thesis. The remaining of this thesis will be devoted to the proof of the above new theorems.

Chapter 2

Sdlt Complements and Surface Complements

The goal of this chapter is to develop many necessary theories for later chapters. As an immediate application, we will tackle the main conjectures in the surface case, i.e. Conjecture 1.8.1 and Conjecture 1.8.4 in dimension 2. Firstly, we will revisit the notion of divisorial adjunction for pairs. Then we will discuss some properties of sdlt complements.

Most of this chapter contains material in [Xu19a]. Background material is taken mostly from [Kol13], [Bir19] and [Fuj00]. Most of the new results proved by the author are taken from [Xu19a].

2.1 Adjunction

In this section, we review several kinds of adjunction and prove some adjunction formulae, especially for surfaces, which will be needed in the subsequent sections. In general, adjunction is relating the (log) canonical divisors of two varieties that are somehow related. We are particularly interested in how the (hyperstandard) coefficients of the boundaries are related.

We will start with a quick review on the theory of divisorial adjunction.

2.1.1 Divisorial Adjunction on a Prime Divisor

We briefly review [Bir19, Section 3.1]. We consider adjunction for a prime divisor on a variety.

Let (X', B') be a pair such that $K_{X'} + B'$ is $\mathbb{Q}$-Cartier with log resolution $\phi : (X, B) \to (X', B')$ where $K_X + B = \phi^*(K_{X'} + B')$. Assume that S' is the normalisation of a component of B' with coefficient 1, and that S is its birational transform on X. Let $B_S = (B - S)|_S$. We get

$$K_S + B_S = (K_X + B)|_S.$$

Let ψ be the induced morphism $S \to S'$ and let $B_{S'} = \psi_* B_S$. Then we get

$$K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}$$

which we refer to as *divisorial adjunction*. Note that $K_S + B_S = \psi^*(K_{S'} + B_{S'})$.

Remark 2.1.1. We sometimes write $B_{S'}$ as $\mathrm{Diff}(S', B' - S')$. If the reference for S' is clear, we sometime just write $\mathrm{Diff}(B' - S')$. Also note that if C' is $\mathbb{Q}$-Cartier that doesn't contain S', then $\mathrm{Diff}(B' + C' - S') = \mathrm{Diff}(B' - S') + C'|_{S'}$.

Remark 2.1.2. We also note that $K_{S'} + B_{S'}$ is determined up to linear equivalence and $B_{S'}$ is determined as a $\mathbb{Q}$-Weil divisor.

Remark 2.1.3. This definition is the same as the definition given in [Kol13, Chapter 4].

Remark 2.1.4. Assume (X', B') is lc. Then the coefficients of $B_{S'}$ belong to $[0, 1]$ [[BZ16], Remark 4.8] and we have $(S', B_{S'})$ is also lc.

We also have a version of inversion of adjunction.

Lemma 2.1.5 (inversion of adjunction,[Kol13])**.** *Let (X', B') be a $\mathbb{Q}$-factorial pair with $K_{X'} + B'$ being $\mathbb{Q}$-Cartier. Assume S' is a component of B' with coefficient 1 and assume S' is klt. Let*

$$K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}$$

be given by adjunction. If $(S', B_{S'})$ is lc, then (X', B') is lc near S'.

The next lemma is quite a well-known result, for example see [KA92, Proposition 16.6], [MP03, Lemma 4.3]. We will state it as in [Bir19, Lemma 3.3]. It allows us to control coefficients when applying divisorial adjunction.

Lemma 2.1.6. [KA92, Proposition 16.6][MP03, Lemma 4.3][Bir19, Lemma 3.3] *Let $\mathfrak{R} \subset [0, 1]$ be a finite set of rational numbers. Then there is a finite set of rational numbers $\mathfrak{S} \subset [0, 1]$ depending only on $\mathfrak{R}$ satisfying the following. Assume*

- *(X', B') is lc of dimension d,*
- *S' is the normalisation of a component of $\lfloor B' \rfloor$,*
- *$B' \in \Phi(\mathfrak{R})$, and*
- *$(S', B_{S'})$ is the pair determined by adjunction*

$$K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}.$$

Then $B_{S'} \in \Phi(\mathfrak{S})$.

We will later define a similar notion for slc adjunction.

2.2 Properties of Slc and Sdlt Pairs and Adjunction

Here we will review some fundamental properties of slc and sdlt pairs. The goal here is to develop a rigorous approach for adjunction, which we will use later to prove a new form of adjunction to sdlt pairs. Most of this section is based on [Kol13, Chapter 5] and [Fuj00].

2.2.1 Divisors on Deminormal Schemes

Starting with the standard notation, let (X, B) be an slc pair with normalisation X'. The *conductor ideal* $\mathcal{H}om_X(\pi_*\mathcal{O}_{X'}, \mathcal{O}_X) \subset \mathcal{O}_X$ is the largest ideal sheaf on X that is also an ideal sheaf on X'. Therefore, it defines two subschemes $D \subset X$ and $D' \subset X'$, which are called *conductor divisors*.

Then we can write the normalisation $\pi : (X', B' + D') \to (X, B)$, where B' is the divisorial part of the inverse image of B and D' is the conductor on X'. We will firstly introduce the notion of divisors on X. There is a closed subset $Z \subset X$ of codimension at least 2 such that $X_0 := X \backslash Z$ has only regular and normal crossing points. We denote $j : X_0 \to X$ to be the inclusion. Let C be an integral Weil divisor on X such that its support doesn't contain any irreducible components of D, then

$$\mathcal{O}_X(C) := j_*(O_{X_0}(C|_{X_0}))$$

is a divisorial sheaf since $C|_{X_0}$ is Cartier on X_0 [Kol13, Section 5.6.3]. It is also clear that $\mathcal{O}_X(C)^{[n]}$ is the sheaf corresponding to the Weil divisor nC since X is $S2$. Similarly we can define K_X to be the pushforward of the canonical divisor on X_0. Now for C as above, we can confuse the notation of $\mathcal{O}_X(mK_X + C)$ with $\omega_X^{[m]}(C)$.

By [Kol13, Section 5.7.3], we know if mB is a Weil divisor on X, then we have a canonical isomorphism

$$(\pi^*\omega_X^{[m]}(mB))^{**} \cong \omega_{X'}(mD' + mB)$$

and when $m(K_X + B)$ is Cartier, this simplifies to $\pi^*\omega_X^{[m]}(mB) \cong \omega_{X'}(mD' + mB)$, hence we can write $K_{X'} + B' + D' = \pi^*(K_X + B)$. This explains the notation we have in Chapter 1.

2.2.2 Conductor, Involution and Adjunction

Using the notation of an slc pair (X, B) with conductor divisor D and normalisation $(X', B' + D')$. Let D^n be the normalisation of D', then there is a natural Galois involution $\tau : D^n \to D^n$ induced by separating the nodes on X. By divisorial adjunction we can write $K_{D^n} + B_{D^n} := (K_{X'} + B' + D')|_{D^n}$.

Rigorously we mean if we let $\nu : D^n \to X'$ be the natural morphism, and by [Kol13, Chapter 4] , we have a natural isomorphism

$$\mathfrak{R}^n : \nu^*(\omega_{X'}^{[n]}(nB' + nD')) \cong \omega_{D^n}^{[n]}(nB_{D^n}).$$

We also have the following lemma.

Lemma 2.2.1. [Kol13, Proposition 5.38] *Let X be a quasi-projective deminormal scheme and let $\pi : X' \to X$ be normalisation. Let B be a $\mathbb{Q}$-divisor on X and define B', D', D^n, τ as above. Then (X, B) is slc if and only if $(X', B' + D')$ is lc and* $\mathrm{Diff}_{D^n}(B')$ *is τ-invariant.*

the sdlt case

If (X, B) is sdlt, then we can use a much easier notation. Again let $(X', B' + D')$ be its normalisation and $X' := \sqcup X_i$ with X_i being the irreducible components of X. Also denote $K_{X_i} + B_i + D_i := (K_{X'} + B' + D')|_{X_i}$. Then we have

$$D^n := \sqcup_i(\sqcup_{E \text{ a component of } D_i} E).$$

Also note that since $(X', B' + D')$ is dlt, all components of D' are normal therefore the above adjunction is really just the restriction of divisors (for some properly chosen K_{X_i}). In this case the above $\mathfrak{R}^n$ map becomes

$$\mathfrak{R}^n : O_{X'}(nK_{X'} + nB' + nD')|_{D^n} \to \mathcal{O}_{D^n}(nK_{D^n} + nB_{D^n})$$

sending

$$(s_i)_i \mapsto ((s_i|_E)_{E \text{ a component of } D_i})_i \,.$$

Therefore for the dlt case, it is fine to confuse $\mathfrak{R}^n(s)$ with just $s|_{D^n}$.

2.3 Adjunction from Normal pairs to Sdlt Pairs

Now we are ready to define general adjunction to an sdlt boundary. We will prove new results regarding the inversion of adjunction for this part. We will start by talking about the log smooth case, then the strict dlt case and then the general case.

Firstly assume (X, B) is log smooth and B is reduced. Since B is a normal crossing divisor, we have ω_B is a Cartier divisor, hence it corresponds to a Weil divisor K_B, whose support doesn't contain any conductor of B. Following [Kol13, Section 4.2], we see that we have $\omega_X(B) \cong \omega_B$.

2.3.1 Dlt Adjunction

Let (X', B') be a dlt pair, and let $S' := \lfloor B' \rfloor$. Let $f : (X, B) \to (X', B')$ be a log resolution such that any exceptional divisor E of f has $a(E, X', B') > -1$ and $K_X + B = f^*(K_X + B)$. Then we have f is an isomorphism over generic points of S'. Let $S := \lfloor B \rfloor$, and let $g : S \to S'$ be the induced morphism. If we let $B_S := (B - S)|_S$ and $B_{S'} = g_*(B_S)$, we see that we have $K_S + B_S = (K_X + B)|_S$ and we have

$$K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}.$$

Sometimes, we will write $B_{S'}$ as $\mathrm{Diff}(B' - S')$. Notice again we have $K_S + B_S = g^*(K_{S'} + B_{S'})$. It is clear that $(S', B_{S'})$ is a semi-dlt pair, see [FG14a].

If we write $S' := \cup_i S_i$, where S_i are the irreducible components of S', then S_i are normal and the normalisation of S' is simply just $S^\nu := \sqcup S_i$. Let $\pi : S^\nu \to S'$ be the normalisation, let $K_{S^\nu} + B^\nu = \pi^*(K_{S'} + B_{S'})$ and let $B_{S_i} = B^\nu|_{S_i}$. Then it is clear $K_{S_i} + B_{S_i}$ is just the divisorial adjunction of (X, B) on the S_i in the sense that

$$K_{S_i} + B_{S_i} = (K_X + B)|_{S_i}.$$

Hence it is not hard to see that Lemma 2.1.6 and Lemma 2.1.5 also hold for this type of adjunction.

2.3.2 General Case

Consider the case where we have a pair $(X', B' := S' + R')$, with (X', S') dlt, S' reduced and R' not containing any conductor divisor of S'. Then we can define divisorial adjunction similar to the above. We let $R_{S'} := R|_{S'}$ (we note that $R_{S'}$ is indeed a well defined $\mathbb{Q}$-divisor, whose support doesn't contain the components of the conductor divisor on S'). Let $K_{S'} + \mathrm{Diff}(0) := (K_{X'} + S')|_{S'}$ and $B_{S'} := \mathrm{Diff}(0) + R_{S'}$, then we have

$$K_{S'} + B_{S'} = (K_{X'} + B')|_{S'}.$$

Hence we have the following new lemma, which is similar to the inversion of adjunction, but for sdlt adjunction.

Lemma 2.3.1. [Xu19a, Lemma 3.8] *Let $(X', B' := S' + R')$ be a $\mathbb{Q}$-factorial pair with (X', S') dlt, S' reduced and R' not containing any irreducible components of S'. Assume B', R' both $\mathbb{Q}$-divisors and write $K_{S'} + B_{S'} := (K_{X'} + B')|_{S'}$. Assume $(S', B_{S'})$ is slc, then (X', B') is lc near a neighbourhood of S'.*

Proof. Since (X', S') is dlt, then all the irreducible components S_i of S' are normal. Then $\bar{S} := \sqcup S_i \xrightarrow{\pi} S'$ is its normalisation. Let

$$K_{\bar{S}} + \Theta := f^*(K_{S'} + B_{S'}).$$

Then we see that $(\bar{S}, \Theta)$ is lc. Therefore, $(S_i, \Theta_i := \Theta|_{S_i})$ is lc. Notice that we also have

$$K_{S_i} + \Theta_i = (K_{X'} + B')|_{S_i}$$

which is the normal divisorial adjunction. Notice that since (X', S') is dlt, then each S_i is klt. Now we can apply Lemma 2.1.5 and conclude that (X', B') is lc near S_i for each i. Hence (X', B') is lc near S' as claimed. □

The above lemma will be important when lifting complements and trying to deduce properties of singularities of lifts of complements later on. Now we have the notion of adjunction to sdlt pairs. Before we can define complements and show some examples of complements for slc pairs, we first need to have a better understanding of the sections on slc pairs. This will be the focus of the next section.

2.4 Pre-admissible Sections and Admissible Sections

Here we quickly review the definitions of the preadmissible sections and the admissible sections for dlt pairs. Most of this background material is from [Fuj00]. We will continue to use the notation that we have used in Section 2.2.2.

Definition 2.4.1. Let (X, B) be an (possibly disconnected) sdlt pair with dimension n and we assume $m(K_X + B)$ is Cartier and let $(X', B' + D')$ be its normalisation and D^n be the normalisation of D' as in the above section. Then we have the following notation defined inductively on dimension.

1. We say $s \in H^0(X, \mathcal{O}_X(m(K_X + B))$ is preadmissible if
$$s|_{D^n} \in H^0(D^n, \mathcal{O}_{D^n}(m(K_{D^n} + B_{D^n}))$$
is admissible. This set is denoted by $PA(X, m(K_X + B))$.
Here $\mathcal{O}_{D^n}(m(K_{D^n} + B_{D^n})) := O_{X'}(mK_{X'} + mB' + mD')|_{D^n}$

2. We say $s \in H^0(X, \mathcal{O}_X(m(K_X + B))$ is admissible if s is preadmissible and $g^*(s|_{X_j}) = s|_{X_i}$ for every $g : (X_i, B_i + D_i) \dashrightarrow (X_j, B_j + D_j)$ B-birational map, where $X' := \sqcup X_i$. This set of sections is denoted by $A(X, m(K_X + B))$.

Remark 2.4.2.

1. It is clear that if s is admissible, then $s|_{X_i}$ is invariant under B-birational automorphisms for each $(X_i, B_i + D_i)$.

2. See [Gon09, Remark 5.2]. Assume (X, B) is sdlt and $m(K_X + B)$ is Cartier. Let $\pi : (X', B' + D') \to X$ be its normalisation. Then it is clear that $s \in H^0(X, m(K_X + B))$ is (pre-)admissible if and only if $\pi^* s \in H^0(X', m(K'_X + B' + D'))$ is (pre-)admissible.

2.4.1 Descending Sections to Slc Pairs from Normalisation

Firstly we start with stating [Kol13, Proposition 5.8]. For simplicity we will assume that m is even in the original proposition.

Proposition 2.4.3. [Kol13, Proposition 5.8] *Let X be a demi-normal scheme and B a $\mathbb{Q}$-divisor on X such that the support doesn't contain any components of the conductor D. Assume $m(K_X + B)$ is Cartier and assume that m is even. Let $(X', B' + D')$ be its normalisation. Then a section $\phi \in \omega_{X'}^{[m]}(mB' + D')$ descends to a section of $\omega_X^{[m]}(mB)$ if and only if $\mathfrak{R}^m \phi$ is τ-invariant where $\tau : D^n \to D^n$ is the involution and $\mathfrak{R}^m$ is the map defined above in Section 2.2.2.*

It is easy to deduce the following lemma from the above criteria of gluing sections.

Lemma 2.4.4. [Fuj00, Lemma 4.2] *Let (X,B) be an slc pair with $m(K_X + B)$ integral. Let $\pi : (X', B' + D') \to X$ be its normalisation. Let $X' := \sqcup X_i$ and let $K_{X_i} + B_i + D_i := (K_{X'} + B' + D')|_{X_i}$. Let $(Y, B_Y + D_Y)$ be a $\mathbb{Q}$-factorial dlt model of $(X', B' + D')$ in the sense that $Y := \sqcup Y_i$ and let $K_{Y_i} + B_{Y_i} + D_{Y_i} := (K_Y + B_Y + D_Y)|_{Y_i}$ then $(Y_i, B_{Y_i} + D_{Y_i})$ is a dlt model of $(X_i, B_i + D_i)$. Assume that $m(K_Y + B_Y + D_Y)$ is Cartier.*

Now let $s \in PA(Y, m(K_Y + B_Y + D_Y)$, then s descends to a section in $H^0(X, m(K_X + B))$.

Now it is time to show that pre-admissible sections give us a way to pass from linear equivalence to 0 on the normalisation to the linear equivalence to 0 on the slc variety.

Proposition 2.4.5. [Xu19a, Proposition 4.10] *Assume (X, B) is an slc pair with normalisation $\pi : (X', B' + D')$ and let $(Y, B_Y + D_Y)$ be the dlt model for $(X', B' + D')$. Assume that we have $n(K_Y + B_Y + D_Y) \sim 0$ and $0 \neq s \in H^0(Y, n(K_Y + B_Y + D_Y)))$ is pre-admissible. Then $n(K_X + B) \sim 0$.*

Proof. Firstly note that $n(K_Y + B_Y + D_Y) \sim 0$ implies $n(K_X + B)$ is integral. Hence we can apply Lemma 2.4.4 to get a section $0 \neq t \in H^0(X, n(K_X + B))$. Also let s' be the corresponding section on X'. In particular, we see that $n(K_X + B)$ is an integral Weil divisor.

Again let Z be a codimensional 2 subset of X such that $X_0 := X \backslash Z$ only has regular or normal crossing points. Then $n(K_X + B)|_{X_0}$ is Cartier. let $X'_0 := \pi^{-1} X_0$, we have $\mathcal{O}_{X'_0}(n(K_{X'} + B' + D')|_{X'_0}) \cong \mathcal{O}_{X'_0}$ via s', hence we get $\mathcal{O}_{X_0}(n(K_X + B)|_{X_0}) \cong \mathcal{O}_{X_0}$ via t. Therefore we conclude that $\mathcal{O}_X(n(K_X + B)) \cong \mathcal{O}_X$ since X is S2. $\square$

Now with this new language, it is time to introduce the definition of complements for slc and sdlt pairs in the next section.

2.5 Complements for Slc Pairs

Now we are ready to define complements for slc pairs.

Let $f : (X, B) \to Z$ be a projective contraction and $z \in Z$ be a closed point. We say $K_X + B^+$ is an n-complement for $K_X + B$ over z if we have the following properties (potentially after shrinking Z near z):

- (X, B^+) is slc,
- $n(K_X + B^+) \sim 0$ over $z \in Z$, and
- $B^+ \geq B$.

Remark 2.5.1. Notice that this definition guarantees that $B^+ - B'$ is an effective $\mathbb{Q}$-Cartier divisor that doesn't include any irreducible components of the conductor of X.

We will give an easy example of slc complements.

Example 2.5.2. Let X be given by $y^2z = x^2(x - z)$ in $\mathbb{P}^2$ with coordinates $(x : y : z)$ and $B = 0$. This is a nodal curve in $\mathbb{P}^2$. Then $2K_X \sim 0$ and hence K_X is 2-complemented with $B^+ = 0$.

The following is a slightly more involved example.

Example 2.5.3. Let $X = V(xy = 0)$ in $\mathbb{P}^2$ with coordinates $(x : y : z)$ and $B = 0$. Let $P := (0 : 1 : 1)$ and $Q := (1 : 0 : 1)$ be 2 distinct points on X. Then a 1-complement for K_X is $B^+ := P + Q$. Notice that indeed (X, B^+) is slc (in fact it is sdlt). We also have $K_X + B^+ \sim 0$: Indeed X' the normalisation of X is just the disjoint union of $\mathbb{P}^1$. Then it can be easily verified that from Lemma 2.4.4, that the section on X' does descend to a section on X, and hence we have $K_X + B^+ \sim 0$.

Finally, we will state a result that will give us a big picture of how to construct slc complements in general.

Theorem 2.5.4. [Kol13, Theorem 5.38] *Let X be demi-normal and B a $\mathbb{Q}$-divisor on X. Let $(X', B' + D')$ being the normalisation of (X, B). Let D^n be the normalisation of D' and let $\tau : D^n \to D^n$ be the corresponding involution. Then the following are equivalent:*

- *(X, B) is slc.*
- *$(X', B' + D')$is lc and* $\mathrm{Diff}(D^n, B')$ *is τ-invariant.*

Combining all of these, we have the following main result on complements for sdlt pairs, which is a partial step towards [Xu19a, Proposition 6.1].

Proposition 2.5.5. *Let (X, B) be an sdlt pair with $f : (X', B' + D') := \sqcup(X_i, B_i + D_i) \to X$ be its normalisation where D' is the conductor divisor and $\tau : D^n \to D^n$ is the involution, with D^n the normalisation of D'. Let n be an even integer. Assume there is a $\mathbb{Q}$-divisor $R' := \sqcup R_i \geq 0$ with R_i $\mathbb{Q}$-divisor on X_i, such that:*

1. *$n(K_{X'} + B' + R' + D') \sim 0$,*

2. *$(Y', B' + R' + D')$ is lc (hence it implies that R' doesn't contain any components of D'),*

3. *$R'|_{D^n}$ is τ-invariant.*

4. *there exists $0 \neq s \in PA(X', n(K_{X'} + B' + R' + D'))$.*

Then letting R be the pushforward of R' to X, we have $n(K_X + B + R) \sim 0$ and $(X, B + R)$ is still slc. In particular $B^+ := B + R$ is an slc n-complement for (X, B).

Remark 2.5.6. Note that the above proposition essentially gives an essential and necessary condition for constructing complements for sdlt pairs. It says the n-complements on each irreducible component give a global n-complement if the divisors can be glued up in a trivial sense. We also note that the condition (3) is needed since it is satisfied if R' is the pullback of an n-complement from X.

Proof. Firstly, using the assumptions 1,2,3 above, we see that the conditions in Theorem 2.5.4 are satisfied, therefore $(X', B' + R')$ is indeed an slc pair. In particular this implies that R' is $\mathbb{Q}$-Cartier. Furthermore, using condition 4 and by Proposition 2.4.5, we also have $n(K_X + B + R) \sim 0$. Therefore $B^+ := B + R$ is an slc n-complement for (X, B) as required. □

Therefore, we see that in order to construct complements for sdlt pairs, we need to construct complements on each irreducible component in a compatible way (see condition 3 in the above). Now the goal for the next section is to show that subject to some conjectures which we will show for low dimension, condition 4 (i.e. the existence of pre-admissible sections) can be derived from the first 3 conditions (potentially we need to increase n by some bounded number). This will be the focus of the next chapter.

However, before ending the chapter, I will prove some inductive approach to complements and also prove Conjecture 1.8.1 in dimension 2.

2.6 Inductive Approach to Global Complements

This is the main inductive step that we will need to prove complements for log canonical Fano variety in the global case. More technicalities are needed for the relative case, but the global case is quite straight forward. The following proposition and proof are very similar to that of [Bir19, Proposition 6.7]. However, we need to deal with sdlt adjunction and in particular sdlt inversion of adjunction, which is the key results that we have derived from the earlier sections.

Proposition 2.6.1. *Let $\mathfrak{R} \subset [0,1]$ be a finite subset of rationals. Let (X,B) be a $\mathbb{Q}$-factorial dlt pair with $B \in \Phi(\mathfrak{R})$ and $-(K_X+B)$ nef and big. Let $S := \lfloor B \rfloor$ and $\Delta := B - S$, and let $K_S + B_S := (K_X+B)|_S$. We have (S, B_S) is an sdlt pair. Let n be a positive integer such that $n\mathfrak{R} \subset \mathbb{N}$. Let $\Delta_S := B_S - \lfloor B_S \rfloor$, then $B_S = \Delta_S$. Suppose further $(K_S + B_S)$ has an n-complement: More precisely, suppose there is $R_S \geq 0$ and $B_S^+ := B_S + R_S$ such that*

1. *(S, B_S^+) is slc ,*
2. *$n(K_S + B_S^+) \sim 0$.*

Then there is an n-complement $K_X + B^+ := K_X + B + R$ for $K_X + B$ with $R \geq 0$ and $R|_S = R_S$.

Proof. Let $f : (Y, B_Y) \to X$ be a Szabo log resolution of (X,B), i.e. f is an isomorphism over all generic points of all lc centres of (X,B), with $K_Y + B_Y := f^*(K_X+B)$. let $T := B_Y^{=1}$ and we also use $f : T \to S$ as the birational contraction induced by f. Let $\Delta_Y := B_Y - T$ and let $K_T + \Delta_T := (K_Y+T)|_T + \Delta_Y|_T = (K_Y+B_Y)|_T$. We see that $K_T + \Delta_T = f^*(K_S+B_S)$, and since $\Delta_T < 1$, we have $B_S < 1$ and hence $B_S = \Delta_S$, where $\Delta_S := f_*\Delta_T$.

Now let $N := -(K_Y + B_Y)$ and define

$$L := -nK_Y - nT - \lfloor (n+1)\Delta_Y \rfloor = nN + n\Delta_Y - \lfloor (n+1)\Delta_Y \rfloor$$

which is an integral divisor hence Cartier. We see that

$$L - T = K_Y + < (n+1)\Delta_Y > + (n+1)N.$$

Hence we have $H^1(Y, L-T) = 0$ since N is nef and big, $(Y, < (n+1)\Delta_Y >)$ is klt. Therefore we have

$$H^0(Y, L) \twoheadrightarrow H^0(T, L|_T).$$

Now notice that $L|_T = nN|_T + n\Delta_T - \lfloor (n+1)\Delta_T \rfloor$. Since $n(K_S + B_S^+) \sim 0$, pulling back to T, we get $nN|_T = -n(K_T + B_T) \sim f^*(nR_S)$.

Hence $L|_T \sim f^*(nR_S) + n\Delta_T - \lfloor (n+1)\Delta_T \rfloor := G_T$. It is clear that G_T is integral and $G_T > -1$, hence we get $G_T \geq 0$. By the above, there exists $G_Y \geq 0$, an integral divisor with $G_Y|_T = G_T$ and $L \sim G_Y$. Pushing forward to get $-n(K_X+B)+n\Delta-\lfloor(n+1)\Delta\rfloor \sim G \geq 0$, where $G=f_*G_T$, hence we get $n(K_X+B+\frac{1}{n}G-\Delta+\frac{1}{n}\lfloor(n+1)\Delta\rfloor)\sim 0$. Here we remark that since R_S doesn't contain any components of the conductor divisor of S and Δ_T also doesn't contain any components of conductor divisors, we see that G_T also doesn't contain any components of conductor divisor for T. Therefore, G doesn't contain any codim≤ 2 lc centre of (Y,T).

Now let D be a component of Δ with coefficients $1-\frac{r}{m}$ with $r\in\mathfrak{R}$ and $m\in\mathbb{N}$, then $\mu_D(-n\Delta+\lfloor(n+1)\Delta\rfloor)=-n+\frac{rn}{m}+\left\lfloor n+1-\frac{r(n+1)}{m}\right\rfloor$. If $\mu_D(-n\Delta+\lfloor(n+1)\Delta\rfloor)<0$, then we must have $n-\frac{rn}{m}=a+b$, where $a\in\mathbb{N}$ and $0<b<\frac{r}{m}\leq\frac{1}{m}$. This means that $\frac{rn}{m}+b$ is an integer, but this is not possible since $rn\in\mathbb{N}$ by assumption. Hence we have $-n\Delta+\lfloor(n+1)\Delta\rfloor\geq 0$.

Letting $R:=\frac{1}{n}G-\Delta+\frac{1}{n}\lfloor(n+1)\Delta\rfloor\geq 0$, we have $n(K_X+B+R)\sim 0$. Letting $B^+:=B+R$, we see that $n(K_X+B^+)\sim 0$. Also by earlier remarks, we see that R doesn't contain any codim ≤ 2 lc centre of (X,B).

Now $-nf^*(K_X+B+R)=nN+n\Delta_Y-\lfloor(n+1)\Delta_Y\rfloor-G$ since $nN+n\Delta_Y-\lfloor(n+1)\Delta_Y\rfloor-G=L-G\sim 0$. We also have $(nN+n\Delta_Y-\lfloor(n+1)\Delta_Y\rfloor-G)|_T=L|_T-G_T=-n(K_T+B_T)+n\Delta_T-\lfloor(n+1)\Delta_T\rfloor-(f^*(nR_S)+n\Delta_T-\lfloor(n+1)\Delta_T\rfloor)=-nf^*(K_S+B_S+R_S)$. Hence we have

$$(K_X+B+R)|_S=K_S+B_S+R_S.$$

Since (S,B_S+R_S) is slc, we have K_X+B^+ is lc but not klt near S by sdlt inversion of adjunction i.e. Lemma 2.3.1. Now applying connectedness lemma on $-(K_X+B+(1-\epsilon)R)$ (note this is nef and big) for some small $\epsilon>0$, we see that K_X+B+R is lc globally, which proves the proposition. □

Now we will apply this to prove complements for log canonical Fano varieties in dimension ≤ 2 in the next section.

2.7 Complements for Log Canonical Fano Varieties in Dimension ≤ 2

We will finish the chapter by proving some basic and yet important results for boundedness of complements for lc Fano varieties in dimension ≤ 2. Most of this section is taken from [Xu19a].

We start by considering the curve and surface case.

2.7.1 The Case for Curves

Firstly we will consider complements on curves. The following is more or less an obvious fact.

Lemma 2.7.1. *Let $p \in \mathbb{N}$ and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on p and $\mathfrak{R}$ satisfying the following. Assume $(X, B+M)$ is a projective pair such that*

- *$(X, B+M)$ is generalised lc with X a smooth curve,*
- *$B \in \Phi(\mathfrak{R})$, and pM is integral,*
- *$-(K_X + B + M)$ is nef.*
- *$M = 0$ if X is an elliptic curve.*

Then there is an n-complement $K_X + B^+ + M$ of $K_X + B + M$.

Proof. It is clear that X is either an elliptic curve (which implies $B^+ = B = M = 0$ and $n = 1$), or X is a rational curve in which case X is Fano hence we can use Theorem 1.4.7. □

Now we consider the case where (X, B) is an sdlt curve, this means that X itself is a smooth normal crossing curve.

Proposition 2.7.2. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

1. *(X, B) is sdlt curve,*
2. *$B \in \Phi(\mathfrak{R})$, and*
3. *$-(K_X + B)$ is nef.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$.

Proof. Using Lemma 2.7.1, we see that on each irreducible component, there is an n-complement for some n depending only on $\mathfrak{R}$. We can assume n is even. Now it is clear that we can choose complements such that they are disjoint from the double point locus of X (this is clear in the elliptic curve case and for the rational curve case, this follows from the fact that any two points on $\mathbb{P}^1$ are linearly equivalent). Notice that either X is an elliptic curve or a cycle of $\mathbb{P}^1$ or a chain of $\mathbb{P}^1$. In the first two cases, we have $K_X \sim 0$ and hence the result is trivial. For the last case, we can simply choose complements on the end of 2 chain as in Example 2.5.3. An alternative way (more theoretic way) to prove the results can be found in the next chapter. $\square$

2.7.2 Complements for Log Fano Surfaces

Theorem 2.7.3. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X', B') is a pair such that*

- *X' is a projective surface, (X', B') is lc,*
- *$B' \in \Phi(\mathfrak{R})$, and*
- *$-(K_{X'} + B')$ is ample.*

Then there is an n-complement $K_{X'} + B'^+$ of $K_{X'} + B'$ such that $B'^+ \geq B'$.

We will also show that

Theorem 2.7.4. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X', B') is a pair , such that*

- *X' is a projective surface, (X', B') is lc,*
- *$B' \in \Phi(\mathfrak{R})$, and*
- *$-(K_{X'} + B')$ is nef and big.*

Then there is an n-complement $K_{X'} + B'^+$ of $K_{X'} + B'$.

Proof of Theorem 2.7.4. By taking small $\mathbb{Q}$-factorization, we can assume (X', B') is $\mathbb{Q}$-factorial dlt. Let $S := \lfloor B' \rfloor$ and let $K_S + B_S := (K_{X'} + B')|_S$, we see that (S, B_S) is sdlt and $-(K_S + B_S)$ is nef. Also by Lemma 2.1.6, we see that there is, $\mathfrak{S}$, a finite subset of $\mathbb{Q} \cap [0,1]$, depending only on $\mathfrak{R}$, such that $B_S \in \Phi(\mathfrak{S})$. Hence we see that $K_S + B_S$ has an n-complement $K_S + B_S^+$ with $B_S^+ \geq B_S$ for some bounded n depending only on $\mathfrak{R}$. We can also assume $n\mathfrak{R} \in \mathbb{N}$. Now we are done by applying Proposition 2.6.1. $\square$

Proof of Theorem 2.7.3. This follows from Theorem 2.7.4 and considering a $\mathbb{Q}$-factorial dlt model of (X', B'). $\square$

Chapter 3

Index Conjecture for Log Calabi-Yau Pairs

This chapter is dedicated to the following conjectures. Most of this chapter is from [Xu19b].

Conjecture 3.0.1. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension d,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

Conjecture 3.0.2. *Let d be a natural number and $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is slc of dimension d,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$K_X + B \sim_{\mathbb{Q}} 0$.*

Then $n(K_X + B) \sim 0$.

Remark 3.0.3. We note that it is obvious that the above 2 conjectures hold in $d = 1$.

Our goal is to prove Theorem 1.8.7 in this chapter. Also we will show its relation with complements as well in this chapter. This chapter will be divided into 2 general themes. Firstly, we will prove some inductive statement regarding the above 2 conjectures. Then we will spend some time developing some results in lower dimension regarding the results. Finally, we will illustrate the relationship with theory of complements.

Remark 3.0.4. By Theorem 1.5.2, it suffices to prove the above conjecture when coefficients of B belong to $\mathfrak{R}$, a finite set of rationals. Therefore, for the rest of the chapter, we will assume coefficients of B always lie inside some finite set.

3.1 Inductive Approach for Normal Pairs

This section will be devoted to developing inductive approach of the conjectures. Firstly, we will state 3 further conjectures: either $B = 0$ and X is klt, or (X, B) klt and $B \neq 0$, or (X, B) is strictly lc (i.e. not klt).

Conjecture 3.1.1. *Let d be a natural number. Then there exists n, a positive integer, depending only on d such that if X is a klt Calabi-Yau of dimension d, i.e. $K_X \sim_{\mathbb{Q}} 0$, then $nK_X \sim 0$.*

Notice that the above conjecture is really related to the mld near 1 for Calabi-Yau varieites.

Conjecture 3.1.2. *Let d be a natural number and $\mathfrak{R}$ be a finite set of rationals. Then there exists n, a positive integer, depending only on $d, \mathfrak{R}$ such that if (X, B) is a klt log Calabi-Yau of dimension d with $B > 0$, i.e. $K_X + B \sim_{\mathbb{Q}} 0$, and $B \in \mathfrak{R}$ then $n(K_X + B) \sim 0$.*

Conjecture 3.1.1 together with Conjecture 3.1.2 are called the index conjecture for klt log Calabi-Yau pairs.

Conjecture 3.1.3. *let d be a natural number and $\mathfrak{R}$ be a finite set of rationals. Then there exists n, a positive integer, depending only on $d, \mathfrak{R}$ such that if (X, B) is a stricly log canonical log Calabi-Yau of dimension d with $K_X + B \sim_{\mathbb{Q}} 0$ and $B \in \mathfrak{R}$, then $n(K_X + B) \sim 0$.*

Clearly Conjecture 1.8.3 is equivalent to Conjecture 3.1.1 + Conjecture 3.1.2 + Conjecture 3.1.3. We have the following inductive results in this thesis, i.e. index conjecture in lower dimension for slc pairs implies the strict lc index conjecture for normal pairs.

Theorem 3.1.4. *Assuming Conjecture 1.8.4 holds in dimension $\leq d - 1$, then Conjecture 3.1.3 holds in dimension d.*

We have the following results.

Theorem 3.1.5. *Assuming Conjecture 3.1.1 in dimension $\leq d - 1$, then Conjecture 3.1.2 holds in dimension $\leq d$.*

In particular, combining the above, we have the following theorem.

Theorem 3.1.6. *Assuming Conjecture 3.1.1 holds in dimension $\leq d$ and Conjecture 1.8.4 holds in dimension $\leq d - 1$, then Conjecture 1.8.3 holds in dimension $\leq d$.*

Proof. This follows from Theorem 3.1.5 and Theorem 3.1.4. $\square$

Therefore, we show that the Conjecture 1.8.3 is equivalent to the absolute klt Calabi-Yau index in the same dimension case (i.e. $B = 0$ and X is klt) and slc index conjecture in low dimension.

The remaining of this section will be devoted to the proofs of Theorem 3.1.5 and Theorem 3.1.4.

3.1.1 Proof of Theorem 3.1.5

Firstly, we show a more or less well-known result. We will include its proof for reader's convenience.

Lemma 3.1.7. [HX14, Proposition 3.1] *Let d be a natural number and $\mathfrak{R}$ be a finite set of rationals. Then there exists ϵ depending only on $d, \mathfrak{R}$ such that if (X, B) is a pair such that (X, B) is klt, $B \in \mathfrak{R}$ and $K_X + B \sim_{\mathbb{Q}} 0$, then (X, B) is ϵ-klt.*

Proof. We use ACC Theorem as in [HMX14]. Suppose the lemma is false, then there exists a sequence $\epsilon_i \to 0$ such that we have a pair (X_i, B_i) klt of dimension d with $B_i \in \mathfrak{R}$, $K_{X_i} + B_i \sim_{\mathbb{Q}} 0$ and there exists a divisor E_i such that $a(E_i, X_i, B_i) = \epsilon_i$. In particular, by considering the plt blowup extracting divisor E_i, there exists a birational morphism $f_i : X_i' \to X_i$ such that $K_{X_i'} + B_i + (1 - \epsilon_i)E_i = f_i^*(K_{X_i} + B_i)$. Also we have $K_{X_i'} + B_i + (1 - \epsilon_i)E_i \sim_{\mathbb{Q}} 0$. This is a contradiction to Theorem 1.5.2. □

Here we need to use the following Theorem from [DCS16].

Theorem 3.1.8. [DCS16, Theorem 3.2] *Let (X, B) be a klt Calabi-Yau pair with $B > 0$. Then there exists a birational contraction $\pi : X \dashrightarrow X'$ to a klt Calabi-Yau pair $(X', B' := \pi_* B)$, $B' > 0$ and a tower of morphisms*

$$X' = X_0 \xrightarrow{p_0} X_1 \xrightarrow{p_1} X_2 \xrightarrow{p_2} \dots \xrightarrow{p_{k-1}} X_k$$

with $k \geq 1$ such that

- *for any $1 \leq i < k$ there exists a boundary $B_i \neq 0$ on X_i and (X_i, B_i) is a klt Calabi-Yau Pair.*
- *for any $0 \leq i \leq k$, the morphism p_i is a K_{X_i} Mori fiber space, and*
- *either $\dim X_k = 0$, i.e. $X_k = pt$ or $\dim X_k > 0$ and $K_{X_k} \sim_{\mathbb{Q}} 0$*

Now it follows easily from [Bir20, Theorem 1.4] that we have the following.

Proposition 3.1.9. *Let d be a natural number and $\mathfrak{R}$ be a finite set of rationals. Let $\mathcal{F}$ be the set of (X, B), dimension d klt Calabi-Yau pair (i.e. (X, B) klt with $K_X + B \sim_{\mathbb{Q}} 0$) with $B \in \mathfrak{R}$ such that the corresponding $X_k = pt$ as in the above theorem. Let $\mathcal{F}'$ be the corresponding set of pairs of the form (X', B'). Then $\mathcal{F}'$ forms a bounded family hence there is an integer n depending only on $d, \mathfrak{R}$ such that $n(K_X + B) \sim 0$ for any $(X, B) \in \mathcal{F}$.*

Proof. Notice that, by using the above lemma, we are done by applying [Bir20, Theorem 1.4] as all conditions are satisfied. □

Now we are ready to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. By the above proposition, it suffices to consider (X, B) such that after applying Theorem 3.1.8, we end up with $\dim X_k > 0$ and $K_{X_k} \sim_{\mathbb{Q}} 0$. Note that since we only care about the index, we can replace X with X'. We will denote $p : X \to Z := X_k$ to be the composition of all the p_i.

Firstly, we note that p is a contraction. Let $(F, B_F := B|_F)$ be the general fiber of p. Firstly we note that restricting the morphism p_i to F, we can deduce that (F, B_F) belongs to a bounded family depending only on $d, \mathfrak{R}$ using Proposition 3.1.9. Hence, there exists r depending only on $d, \mathfrak{R}$ such that $r(K_F + B_F) \sim 0$. We can apply canonical bundle formula and get

$$r(K_X + B) \sim rf^*(K_Z + B_Z + M_Z).$$

We can deduce that $K_Z + B_Z + M_Z \sim_{\mathbb{Q}} 0$. However $K_Z \sim_{\mathbb{Q}} 0$. This implies that $B_Z = 0$ and $M_Z \equiv 0$. Now we can apply [Flo14, Theorem 1.3], to deduce that there exists m depending only on $d, \mathfrak{R}$ such that $mK_Z \sim 0$. We write $n := mr$. Hence, we deduce that $n(K_X + B) \sim nf^*K_Z$. Now since $\dim Z < \dim X$, by hypothesis, we know that there is a bounded q depending only on d, such that $qK_Z \sim 0$. By replacing n with nq, we can deduce that $n(K_X + B) \sim 0$ for some n depending only on $d, \mathfrak{R}$. □

3.1.2 Proof of Theorem 3.1.4

Proof of Theorem 3.1.4. Since (X, B) is lc but not klt, it suffices to assume that, after replacing (X, B) with a $\mathbb{Q}$-factorial dlt model, (X, B) is dlt and $\lfloor B \rfloor \neq 0$. Choose $\epsilon > 0$, a small rational number and run an MMP on $K_X + B - \epsilon \lfloor B \rfloor \sim_{\mathbb{Q}} -\epsilon \lfloor B \rfloor$. Since it is not pseudo-effective, we will end up with a Mori-Fiber space $f : X' \to V$ with $\dim V < \dim X$. Now replacing X by X', we can assume $X = X'$. Now we consider different cases of the dimension of V.

1. $\dim V = 0$: In this case, X is a Fano type since $K_X + B - \epsilon \lfloor B \rfloor$ is anti-ample globally and $(X, B - \epsilon \lfloor B \rfloor)$ is klt by construction. By boundedness of complements applying on $K_X + B$, we see that there is a bounded n depending only on $d, \mathfrak{R}$ such that $n(K_X + B) \sim 0$.

2. $\dim V = \dim X - 1$: In this case, we see that the general fiber of $f : X \to V$ is a rational curve. By canonical bundle formula and since X is Fano type over V, there exists $\mathfrak{S} \subset [0, 1]$ finite and q depending only on $d, \mathfrak{R}$ such that
$$q(K_X + B) \sim qf^*(K_V + B_V + M_V),$$
where $B_V \in \mathfrak{S}$ and qM_V is b-Cartier. Note that since $\dim V = \dim X - 1$, by [PS09], we can assume that, by replacing q by a bounded multiple, we have qM_V is b-effective-base-point-free. Hence by possibly replacing $\mathfrak{S}$, we can write
$$q(K_X + B) \sim qf^*(K_V + B_V),$$
where $B_V \in \mathfrak{S}$. Now applying Conjecture 1.8.3 in lower dimension, we get there exists a bounded n depending only on $\mathfrak{S}$, which depends only on $d, \mathfrak{R}$, such that $n(K_V + B_V) \sim 0$. Therefore $nq(K_X + B) \sim 0$, which proves the result.

3. $\dim V > 0$ and $\dim V \leq \dim X - 2$: Firstly, we see that $f : X \to V$ is a Fano-type fibration. In this case, the general fiber has dimension ≥ 2. Now since $\epsilon \lfloor B \rfloor$ is ample over V, there exists $S \in \lfloor B \rfloor$ that is horizontal over V. By divisorial adjunction, we can write
$$K_S + B_S = (K_X + B)|_S.$$
Note that here S may not be normal, but (S, B_S) is slc by Example 1.7.5. Note that coefficients of B_S lie in a finite set that depends only on $\mathfrak{R}$ by Lemma 2.1.6. Firstly, we argue that $S \to V$ is a contraction.

   By stein-factorization, we can write $S \xrightarrow{f'} V' \xrightarrow{g} V$, where f' is contraction and g is finite. Since V is normal, it suffices to show the degree of g is one. Suppose that the degree of g is $m > 1$. Let $x \in V$ be a general closed point and let F be the general fiber above x in $X \to V$. Say $x_1, x_2, \ldots, x_m$ are the pre-images of x in V' and let G_i be the fiber of x_i

in S. Note that G_i are all disconnected. Now $S|_F = \cup G_i$, note that $S|_F$ is well-defined since F is a general fiber and S is horizontal over V. However, $S|_F$ is an ample divisor on F by construction. Since $\dim F \geq 2$, all ample divisors on F need to be connected, which is a contradiction.

Now, by abuse of notation, we can also call $f : S \to V$. Then by Theorem 1.2.3, there exists q and $\mathfrak{S}$ depending only on $d = \dim X$ and $\mathfrak{R}$ (coefficients of B), such that

$$q(K_X + B) \sim f^*(q(K_V + B_V + M_V)),$$

where $B_V \in \Phi(\mathfrak{S})$ and qM_V is an integral Weil b-Cartier b-nef divisor.

Let V_{sm} be the smooth locus of V. Then we can assume $q(K_V + B_V + M_V)|_{V_{sm}}$ is Cartier by Theorem 1.2.3: Indeed, we have $q(K_V + B_V + M_V) \sim_{\mathbb{Q}} 0$ Now we can apply Corollary 1.5.4 to deduce that there exists a finite set $\mathfrak{T}$ depending only on $\mathfrak{S}$ such that $B_V \in \mathfrak{T}$. Therefore, by potentially replacing q by a bounded multiple, we can assume qB_V is an integral Weil divisor, and therefore $q(K_V + B_V + M_V)$ is an integral Weil divisor and hence $q(K_V + B_V + M_V)|_{V_{sm}}$ is Cartier on V_{sm}.

Hence we get $q(K_X + B)|_{f^{-1}(V_{sm})}$ is Cartier by canonical bundle formula. Now restricting to S, (since S is horizontal), we have $q(K_S + B_S)|_{f^{-1}(V_{sm})}$ is also Cartier and we have

$$q(K_S + B_S)|_{(f^{-1}(V_{sm}))} \sim f^*(q(K_V + B_V + M_V)|_{V_{sm}}).$$

By hypothesis, we can assume Conjecture 1.8.4 in lower dimension. In particular, (after possibly replacing q by a bounded multiple), we can assume that $q(K_S + B_S) \sim 0$. Hence there is a rational function α such that $q(K_S + B_S) = div(\alpha)$. We can assume that $div(\alpha)$ is vertical over V. Using that V is normal and $f_*\mathcal{O}_S = \mathcal{O}_V$, we have $\alpha|_{f^{-1}(V_{sm})} = \beta_{sm} \circ f$ for some rational function β_{sm} on V_{sm}. Hence we get $q(K_V + B_V + M_V)|_{V_{sm}} \sim 0$ via β_{sm}, and therefore $q(K_V + B_V + M_V) \sim 0$ (since V is normal hence S2 and $V \backslash V_{sm}$ have codimension at least 2). Hence we have $q(K_X + B) \sim 0$ by canonical bundle formula.

□

Remark 3.1.10. We remark that here we only need a much weaker form of Conjecture 1.8.4 in the sense that we only use the Conjecture 1.8.4 when the variety is irreducible. However, the author believes that the difficulties in proving Conjecture 1.8.4 in full generality would be similar to proving it in the above special case when X is irreducible.

3.2 Boundedness of B-representation for Klt Calabi-Yau Pairs

In this section, we will relate the boundedness of B-representation to Conjecture 1.8.4. The conjecture that we will introduce is the so-called boundedness of B-representation for Klt Calabi-Yau Pairs. We will again take an inductive approach for this section. We will prove some concrete results in later sections of this chapter. This section is mainly taken from [Xu19a, Section 5].

Conjecture 3.2.1 (Boundedness of B-representation for Klt Calabi-Yau Pairs**).** *Let n, d be natural numbers and let (X, B) be a d-dimensional projective pair such that (X, B) is dlt and $n(K_X + B) \sim 0$. Then $Bir(X, B)$ denotes the B-birational automorphism group of (X, B). Then there exist $m, M \in \mathbb{N}$ depending only on n, d such that $\rho_m(Bir(X, B))$ has size bounded by M, where $\rho_m : Bir(X, B) \to H^0(X, m(K_X + B))$ denotes the natural action by pulling back sections.*

The goal of the section is to prove the following:

Proposition 3.2.2. *Assuming Conjecture 3.2.1 in dimension $\leq d-1$ and Conjecture 1.8.3 in dimension $\leq d$, Conjecture 1.8.4 holds in dimension d.*

Remark 3.2.3. The proposition 3.2.2 is essentially the same as [Xu19a, Theorem 1.5].

Remark 3.2.4. We note that by [FG14b, Theorem 1.1], $\rho_m(Bir(X, B))$ is a finite group.

Firstly, we will aim to relate the notion of B-representation with admissible sections.

3.2.1 From Boundedness of B-representation to Existence of Admissible Sections

We now make the following observation. If the above conjecture holds, then we in fact have an admissible section in some bounded multiple of $K_X + B$. i.e. We have the following conjecture on admissible sections.

Conjecture 3.2.5. *If (X, B) is connected klt of dimension $\leq d$ and assume $n(K_X + B) \sim 0$, then there exists a constant $N(n, d) > 0$, such that there is an admissible section in $A(X, N(K_X + B))$.*

For technical reasons, we also need the following conjecture for not necessarily connected klt pairs.

Conjecture 3.2.6. *If (X, B) is klt (not necessarily connected) of dimension $\leq d$ and assume $n(K_X + B) \sim 0$, then there exists a constant $N(n, d) > 0$, such that there is a nonzero admissible section in $A(X, N(K_X + B))$.*

We can relate the conjectures by the following result.

Proposition 3.2.7. *Assuming Conjecture 3.2.1 holds in dimension d, then Conjecture 3.2.5 holds in dimension d.*

Proof. Let (X, B), m, N be as in Conjecture 3.2.1. Let $G := \rho_m(Bir(X, B))$, and let $0 \neq s \in H^0(X, m(K_X + B))$. Then $|G| \leq M$. Then clearly

$$t := \prod_{\sigma \in G} \sigma^* s \in H^0(X, m|G|(K_X + B))$$

is $Bir(X, B)$ invariant. Therefore $A(X, m|G|(K_X + B))$ is non-trivial. Also we note that if $A(X, l(K_X + B))$ is not trivial, then $A(X, lp(K_X + B))$ is not trivial for every $p \in \mathbb{N}$. Now since m, $|G|$ are bounded, the result follows by taking $N(d, n) = (mM)!$. □

Also we relate the connected case to the disconnected case.

Proposition 3.2.8. *Assuming Conjecture 3.2.5 holds in dimension d. then Conjecture 3.2.6 also holds in dimension d.*

Proof. Let $N = N(d, n)$ be as in Conjecture 3.2.5. Let $(X, B) = \sqcup(X_i, B_i)$ be a potentially disconnected klt log Calabi-Yau pair of dimension d. Let $s := (\lambda_1 s_1, \lambda_2 s_2, ..) \in H^0(X, N(K_X + B))$, where $s_i \in A(X_i, N(K_{X_i} + B_i))$ and $\lambda_i \in \mathbb{C}$. Now let $G := \rho_N(Bir(X, B))$, which is a finite group by [FG14b]: Indeed, We can see that, by the choice of N, if X has l components, then for all $g \in G$, $\rho_N(g)^{l!} = I \in GL(H^0(X, N(K_X + B)))$. Therefore by the well-known Burnside Theorem, G has finite order since G is a subgroup of general linear group with finite index.

Define

$$t := \sum_{\sigma \in G} \sigma(s).$$

Then $t \in A(X, N(K_X + B))$ by construction. Hence it suffices to show we can choose λ_i such that t is not zero in all components. To this end, by considering orbits of the action, we can assume $Bir(X, B)$ acts on X_i transitively, i.e. for each i, j there is $g \in Bir(X, B)$ mapping X_i into X_j.

Consider $H^0(X, N(K_X + B))$ as a vector space over $\mathbb{C}$ with basis $\{(0.., s_i, .., 0)\}$. We notices that $\rho_N(g)$ can be expressed as a matrix such that the entries on the diagonal are either 0 or 1 due to the fact that $s_i \in A(X_i, N(K_{X_i} + B_i))$. Hence we see that $\sum_{\sigma \in G} \sigma$ is not the zero matrix. Hence there exists some $\lambda_i \in \mathbb{C}$ such that t is not zero on all components. Then since $Bir(X, B)$ acts transitively and t is G-invariant, we see that t is non-zero in all components. □

Therefore, it is sufficient to relate the existence of (potentially disconnected) klt admissible sections to boundedness of slc Calabi-Yau index.

3.2.2 Boundary of Calabi-Yau Pairs

We take a slight detour here to discuss some well-known facts about boundary for Calabi-Yau dlt pairs. Most of this is take from [Fuj00]. Notice that the assumption of MMP in [Fuj00] is well known nowadays after the famous paper of [BCHM10]. For completeness, we will include a brief proof here.

Proposition 3.2.9. *(*[Fuj00, Proposition 2.1],[Gon10, Claim 5.3]*) Let (X, B) be an n-dimensional connected $\mathbb{Q}$-factorial dlt pair. Assume $K_X + B \sim_{\mathbb{Q}} 0$. Then one of the following holds.*

1. *$\lfloor B \rfloor$ is connected.*

2. *$\lfloor B \rfloor$ has 2 connected components B_1, B_2 and there is a rational morphism $(X, B) \dashrightarrow (V, P)$ with general fiber $\mathbb{P}^1$, such that $(V, 0)$ is lt and (V, P) is $\mathbb{Q}$-factorial lc of dimension $n-1$. Furthermore, there is horizontal components S_i in B_i such that $(S_i, \mathrm{Diff}(B - S_i)) \dashrightarrow (V, P)$ is B-birational.*

To show the proposition we first show two easy facts about Mori fiber space.

Lemma 3.2.10. *Let (X, B) be a $\mathbb{Q}$-factorial lc n-fold with $n \geq 2$ and $\lfloor B \rfloor \neq 0$ and $(X, B - \epsilon \lfloor B \rfloor)$ is klt for some small positive rational number ϵ. Let $f : X \to R$ be a projective surjective morphism with connected fibers such that $K_X + B \sim_{\mathbb{Q}} 0/R$. Assume that there is a $(K_X + B - \epsilon \lfloor B \rfloor)$ Mori fiber space $g : X \to V$ over R with* $\dim V = n - 1$*. Let B_h be the horizontal part of $\lfloor B \rfloor$. Then one of the following holds.*

1. *$B_h = D_1$, which is irreducible and degree $[D_1 : V] = 2$.*

2. *$B_h = D_1$, which is irreducible and degree $[D_1 : V] = 1$.*

3. *$B_h = D_1 + D2$, which is irreducible and degree $[D_i : V] = 1$.*

Furthermore, in cases (1) and (3), the number of connected components of $\lfloor B \rfloor \cap f^{-1}(r)$ is at most 2 and in case (2), $\lfloor B \rfloor \cap f^{-1}(r)$ is connected for every $r \in R$. Also, $(V, 0)$ is lt and (V, P) is $\mathbb{Q}$-factorial lc $(n-1)$ fold for some P, such that $K_{D_i} + \mathrm{Diff}(B - D_i) = g^*(K_V + P)$ for $i = 1, 2$. In case (1), there is a B-birational involution $i : (D_1, \mathrm{Diff}(B - D_1)) \dashrightarrow (D_1, \mathrm{Diff}(B - D_1))$ over V such that $i^2 =$ id. In case (3)*, there is a crepant birational involution $j : (D_1, \mathrm{Diff}(B - D_1)) \dashrightarrow (D_2, \mathrm{Diff}(B - D_2))$ over V.*

Proof of Lemma 3.2.10. We firstly note that the general fiber is $\mathbb{P}^1$. Since $\lfloor B \rfloor$ is ample over R, we have $B_h \neq 0$. Also since $K_X + B \sim_{\mathbb{Q}} 0$, restricting to the general fiber, we see that $deg(B_H, V) \leq 2$. Hence we see that (1),(2),(3) are the only possibilities. The divisor P can be constructed using generalised adjunction. We can get a $\mathbb{Q}$-divisor P on V such that $K_{D_i} + \mathrm{Diff}(B - D_i) = g^*(K_V + P)$. In particular (V, P) is lc. We note that V is $\mathbb{Q}$-factorial since X is and g is extremal.
Finally, in cases (1) and (3), the part about involution follows from the fact that $deg(B_H, V) = 2$.

□

Lemma 3.2.11. *Let (X, B) be a $\mathbb{Q}$-factorial lc n-fold with $n \geq 2$, $\lfloor B \rfloor \neq 0$ and $(X, B - \epsilon \lfloor B \rfloor)$ is klt for some small positive rational number ϵ. Let $f : X \to R$ be a projective surjective morphism with connected fibers such that $K_X + B \sim_{\mathbb{Q}} 0/R$. Assume that there is a $(K_X + B - \epsilon \lfloor B \rfloor)$ Mori fiber space $g : X \to V$ over R. Then either $\lfloor B \rfloor \cap f^{-1}(r)$ is connected for every $r \in R$ or $\dim V = n - 1$ (i.e. we are in the case of Lemma 3.2.10).*

Proof. Firstly, let's assume $\lfloor B \rfloor \cap f^{-1}(r)$ is not connected for some $r \in R$, this means that $\lfloor B \rfloor \cap g^{-1}(v)$ is not connected for some $v \in V$. Noting that $K_X + B \sim_{\mathbb{Q}} 0/R$ we see that $\lfloor B \rfloor$ is g-ample, and hence B_h, the horizontal part of $\lfloor B \rfloor$, is g-ample. Since $\rho(X/V) = 1$, we derive that $B_h \cap g^{-1}(v)$ is connected unless the general fiber is $\mathbb{P}^1$. Now it is also clear that since g is extremal, the vertical part of $\lfloor B \rfloor$ is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on V, hence we get $\lfloor B \rfloor \cap g^{-1}(v)$ is connected for each $v \in V$ as claimed. □

Proof of Prop 3.2.9. Now run an MMP on $K_X + B - \epsilon \lfloor B \rfloor$ for some $\epsilon > 0$ small rational number. Since $K_X + B \sim_{\mathbb{Q}} 0$, we know that we will terminate with a Mori Fiber space $g : X' \to V/R$ with $X \dashrightarrow X'$ a sequence of flips and divisorial contractions. Let B' be the pushforward of B to X'. Notice that since $\lfloor B \rfloor$ is relatively ample for each divisorial contraction and flip, we know that the number of connected components of $\lfloor B \rfloor$ doesn't change during MMP. We can replace (X, B) with (X', B'), since all conditions are preserved (because we have $K_X + B \sim_{\mathbb{Q}} 0$). Notice now (X', B') may not be dlt but it is still lc and $\mathbb{Q}$-factorial. We can finish the proof using Lemma 3.2.11 and Lemma 3.2.10. □

Now with this setup we are ready to show the main result of the section.

3.2.3 From Admissible Sections to Slc Index Conjecture

The goal of this subsection is to show Proposition 3.2.2. We will follow similar ideas as in [Fuj00].

Now we will show 2 statements using induction. Although, we will only apply them for surfaces and curves. The proofs mostly follows the same route as in [Fuj00]. However, since in [Fuj00] and [Gon10], there is no boundedness consideration, at certain steps in the proof, we need to take extra care and use results that we proved earlier, for example, in the proof that A_d implies B_d below.

Proposition 3.2.12. *($\mathbf{A_d}$) Assuming Conjecture 3.2.5 in dimension $\leq d - 1$. Let (X, B) be a (not necessarily connected) projective dlt pair of dimension d, with $m(K_X + B) \sim 0$ and m being even. Also assume that $mN(d - 1, m) | n$ where $N(d - 1, m)$ is as in Conjecture 3.2.5, then $PA(X, n(K_X + B))$ is non-trivial.*

Proposition 3.2.13. *($\mathbf{B_d}$) Assuming Conjecture 3.2.5 in dimension $\leq d$. Let (X, B) be a (not necessarily connected) projective dlt pair of dimension d, with $m(K_X + B) \sim 0$ and m being even. Also assume that $mN(d, m) | n$ where $N(d, m)$ is as in Conjecture 3.2.5, then $A(X, n(K_X + B))$ is non-trivial.*

Before we show the above two propositions, we will show the lemma below.

Lemma 3.2.14 ([Fuj00] Proposition 4.5, [Gon10] Claim 5.4). *Assume (X, B) is a projective dlt pair (not necessarily connected) with $n(K_X + B) \sim 0$ and n is even. Assume $s \in A(\lfloor B \rfloor, n(K_X + B)|_{\lfloor B \rfloor})$ is non-zero.*
Then there exists a nonzero $t \in PA(X, n(K_X + B))$ such that $t|_{\lfloor B \rfloor} = s$.

Proof. This proof follows the same route as [Fuj00, Proposition 4.5]. Note that the lemma is trivial if $\lfloor B \rfloor = 0$, hence we assume $\lfloor B \rfloor \neq 0$. It is clear by definition it suffices to show there is $t \in H^0(X, n(K_X + B))$ such that $t|_{\lfloor B \rfloor} = s$. Therefore, we can assume that X is connected. By Prop 3.2.9, we have either $\lfloor B \rfloor$ is connected or has 2 connected components. If $\lfloor B \rfloor$ is connected, then $H^0(X, n(K_X + B)) \to H^0(\lfloor B \rfloor, \mathcal{O}_{\lfloor B \rfloor}(n(K_X + B)|_{\lfloor B \rfloor}))$ is injective, hence isomorphism since both are 1 dimensional. In this case, we see that the lemma is clear.

Now we assume the $\lfloor B \rfloor$ has 2 connected components, B_1, B_2. In this case we see that X is generically a $\mathbb{P}^1$ bundle over (V, P). More precisely, there is a sequence of flips and divisorial contractions $\phi : X \to X'$ and a Mori fiber space $g : (X', B') \to (V, P)$ such that the general fiber of g is $\mathbb{P}^1$ and $K_X + B = g^*(K_V + P)$. We also remark that $(\lfloor B' \rfloor, \mathrm{Diff}(B' - \lfloor B' \rfloor))$ is slc by Example 1.7.5. Also there are 2 connected components of B', B'_1, B'_2, and each component has an irreducible component D_i such that $g_i := g|_{D_i} : (D_i, \mathrm{Diff}(B' - D_i)) \to (V, P)$ is B-birational. Now it is easy to see that

$$H^0(X, n(K_X + B)) \cong H^0(X', n(K_{X'} + B')).$$

Also we have as in [FG14b, Remark 2.15],

$$H^0(\lfloor B \rfloor, n(K_X + B)|_{\lfloor B \rfloor}) \cong H^0(\lfloor B' \rfloor, n(K_{X'} + B')|_{\lfloor B' \rfloor}).$$

Hence it suffices to treat (X', B'). Now let $s \in A(\lfloor B' \rfloor, n(K_{X'} + B')|_{\lfloor B' \rfloor})$, and we write B'_h and B'_v to be the horizontal and vertical parts of $\lfloor B' \rfloor$ with respect to V. From Proposition 3.2.9, we see that $s|_{D_i}$ is birational invariant in particular, it descends to a section $t \in H^0(V, n(K_V + P))$. We note that $n(K_{D_i} + \mathrm{Diff}(B' - D_i)) \sim 0$ is Cartier, hence we get $n(K_V + P) \sim 0$ and in particular is Cartier. Now since g is contraction and hence we get $H^0(X', n(K_{X'} + B')) \cong H^0(V, n(K_V + P))$, therefore t lifts to a section $w \in H^0(X', n(K_{X'} + B'))$. It suffices to show $w|_{\lfloor B' \rfloor} = s$ as remarked before.

Firstly, We note that $w|_{D_i}$ and $s|_{D_i}$ are different by at most $(-1)^m$ by [Fuj00] and [Kol13] using the theory of $\mathbb{P}^1$ linked lc centres. Hence, since we assume n is even, we have the desired claim on B'_h. Next we check on B'_v. It is clear that $B'_v = \sum_i g^*(F_i)$ for some F_i irreducible divisor in $\lfloor P \rfloor$. Let $E_i := g^*(F_i)$. We will show $s|_{E_i} = w|_{E_i}$. We let Θ_i be an irreducible component of $E_i \cap D_1$ that dominants F_i (can always do this since E_i intersects D_i non-trivially). In particular we see that $g|_{\Theta_i} : \Theta_i \to F_i$ is dominant. Hence we have the following diagram.

$$\begin{array}{ccc} H^0(E_i, n(K_{X'}+B')|_{E_i}) & \xrightarrow{|_{\Theta_i}} & H^0(\Theta_i, n(K_{X'}+B')|_{\Theta_i}) \\ \cong\uparrow & & i\uparrow \\ H^0(D_i, n(K_V+P)|_{F_i}) & \xrightarrow{id} & H^0(D_i, n(K_V+P)|_{F_i}) \end{array}$$

The right vertical map is injective since $\Theta_i \to D_i$ is dominant, and left vertical map is isomorphism since D_i is seminormal and $g|_{E_i}$ has connected fibers. Since we have $s|_{\Theta_i} = w|_{\Theta_i}$ by the horizontal part argument. Hence we have $s|_{E_i} = w|_{E_i}$, which proves the claim. □

Now we show the following lemma.

Lemma 3.2.15. [Fuj00, Proposition 4.7] *Let (X, B) be a connected projective dlt pair with $n(K_X + B) \sim 0$ where n is even. Assuming $\lfloor B \rfloor \neq 0$, then $PA(X, n(K_X + B)) = A(X, n(K_X + B))$.*

Firstly, we will state a well known lemma about crepant birational maps.

Lemma 3.2.16. [Fuj00, Proposition 4.7 Claim A_n, B_n] *(Also see Lemma 2.16 in* [FG14b]*)*
Let $f : (X, B) \dashrightarrow (X', B')$ be a B-birational map between projective dlt pairs. Let S be an lc center of (X, B) such that $K_S + B_S := (K_X + B)|_S$. Let $\alpha : (Y, B_Y) \to (X, B)$, $\beta : (Y, B_Y) \to (X', B')$ be a common log resolution such that $K_Y + B_Y = \alpha^*(K_X + B) = \beta^*(K_{X'} + B')$. Then we can find an lc centre V of (X, Δ) contained in S with $K_V + B_V := (K_X + B)|_V$, an lc center T of (Y, B_Y) with $K_T + B_T := (K_Y + B_Y)|_T$ and an lc centre V' of (X', B') with $K_{V'} + B_{V'} := (K_V + B_V)|_{V'}$ such that the following holds.

1. *$\alpha|_T : (T, B_T) \to (V, B_V), \beta|_T : (T, B_T) \to (V', B_{V'})$ are B-birational morphisms. Hence $\beta|_T \circ \alpha|T^{-1} : (B, B_V) \dashrightarrow (V', B_{V'})$ is B-birational.*

2. *$H^0(S, m(K_S + B_S)) \cong H^0(V, m(K_V + B_V))$ by the natural restriction where $m \in \mathbb{N}^+$ such that $m(K_X + B)$ is Cartier.* □

Now we will use the above lemma to prove Lemma 3.2.15.

Proof of Lemma 3.2.15. It is clear from the definition that $A(X, n(K_X + B)) \subset PA(X, n(K_X + B))$. Hence it suffices to show $PA(X, n(K_X + B)) \subset A(X, n(K_X + B))$. Let $s \in PA(X, n(K_X + B))$, we need to show for any $g \in Bir(X, B)$, $g^*(s) = s$. Since $H^0(n(K_X + B))$ is 1 dimensional, it suffices to show $(g^*s)|_{\lfloor B \rfloor} = s|_{\lfloor B \rfloor}$ (since $H^0(X, m(K_X + B)) \to H^0(\lfloor B \rfloor, \mathcal{O}_{\lfloor B \rfloor}(m(K_X + B)|_{\lfloor B \rfloor}))$ is injective).

Let $g \in Bir(X, B)$ and let $\alpha, \beta : (Y, B_Y) \to (X, B)$ be a Szabo log resolution such that $\alpha := g \circ \beta$, i.e. α, β are isomorphisms above the generic points of all lc centres of (X, B). Let $\Theta := B_Y^{=1}$, then by standard theory $\Theta \to \lfloor B \rfloor$

has connected fibers and hence we have $\alpha_*\mathcal{O}_\Theta = \beta_*\mathcal{O}_\Theta = \mathcal{O}_{\lfloor B \rfloor}$. Then α^*, β^* induce isomorphisms from

$$H^0(\lfloor B \rfloor, \mathcal{O}_{\lfloor B \rfloor}(n(K_X+B)|_{\lfloor B \rfloor})) \cong H^0(\Theta, \mathcal{O}_\Theta(n(K_Y+B_Y)|_\Theta)).$$

Now let E be an irreducible component of Θ and let S be its birational transform on X, which is an irreducible component of $\lfloor B \rfloor$, such that E dominates S, Then it suffices to show $(\alpha^* s)|_E = (\beta^* s)|_E$.

Now we apply Lemma 3.2.16, we see that we can find lc centre V contained in S and T an lc centre for (Y, B_Y), such that all the conditions are satisfied as in Lemma 3.2.16. Note we can take $V' = V \subset S$. Then we have $\alpha|_T^*(s|_V) = \beta|_T^*(s)(s|_V) \in H^0(T, n(K_T+B_T))$ since $s \in PA(X, n(K_X+B))$. However we have

$$H^0(E, n(K_E+B_E)) \cong H^0(S, n(K_S+B_S)) \cong H^0(V, n(K_V+B_V)) \cong H^0(T, n(K_T+B_T)).$$

Hence we have $\alpha^*(s)|_E = \beta^*(s)|_E$. Since E is arbitrary, we have $\alpha^*(s|_{\lfloor B \rfloor}) = \beta^s(s|_{\lfloor B \rfloor})$ on Θ. Hence we get $g^*s|_{\lfloor B \rfloor} = s|_{\lfloor B \rfloor}$, which proves the lemma. □

We are now ready to show the above 2 propositions. We will first show B_{d-1} implies A_d.

Proof of B_{d-1} implies A_d. This is precisely Proposition 3.2.14. □

Finally we show A_d implies B_d.

Proof of A_d implies B_d. We will construct a non-trivial element in $A(X, n(K_X+B))$. Let $G = \rho_n(Bir(X, B))$, which is finite. We can wlog (X_i, B_i) in fact can be put into 2 different classes: We say (X_i, B_i) is of type 1, if $\lfloor B_i \rfloor \neq 0$, we denote these pairs as $(X_{i,1}, B_{i,1})$. If (X_i, B_i) is klt, i.e. $\lfloor B_i \rfloor = 0$, then we say this has type 2 and write $(X_{i,2}, B_{i,2})$. Using this notation, we can assume

$$(X, B) = (\sqcup_i (X_{i,1}, B_{i,1})) \sqcup (\sqcup_i (X_{i,2}, B_{i,2})).$$

It is clear that $Bir(X, B)$ maps type 1 into type 1 and type 2 into type 2. Also $G' := Bir(\sqcup(X_{i,2}, B_{i,2})) \subset G$ is also finite. To this end, we write $s = (s_1, s_2, s_3, .., t_1, t_2, ...) \in PA(X, n(K_X+B))$, where $s_i \in A(X_{i,1}, n(K_{X_{i,1}} + B_{i,1}))$ by Lemma 3.2.15 and $(t_i)_i \in A(X_{i,2}, n(K_{X_{i,2}} + B_{i,2})$ by our assumption on Conjecture 3.2.6. We can now think of G as acting on (s_i) and (t_i) separately. Now let $s = (s_i, t_i)$ be the above denoting an element in $PA(X, n(K_X+B))$.

Step 1: We firstly claim that (s_i) is G-invariant. let $\sigma \in G$ be represented by $g \in Bir(X, B)$, the claim is true if g maps $X_{i,1}$ into $X_{i,1}$ for all i since $s_i \in A(X_{i,1}, n(K_{X_{i,1}} + B_{i,1}))$. Therefore, we can assume g maps $X_{i,1}$ to $X_{j,1}$, with $i \neq j$. It suffices to show $g^*(s_j) = s_i$, where we view $g|_{X_{i,1}} : (X_{i,1}, B_{i,1}) \to (X_{j,1}, B_{j,1})$ as a B-birational map. Now since we are in type 1, we can assume that $\lfloor B_{i,1} \rfloor \neq 0$. Let S be an lc centre of $(X_{i,1}, B_{i,1})$, we can apply Lemma 3.2.16, we see that we can find lc centre V of $(X_{i,1}, B_{i,1})$ contained in S and V'

of $(X_{j,1}, B_{j,1})$, such that g induces a B-Birational map from $g' : (V, B_V) \dashrightarrow (V', B_{V'})$, where $K_V + B_V := (K_{X_{i,1}+B_{i,1}})|_V$ and $K_{V'} + B_{V'} := (K_{X_{j,1}+B_{j,1}})|_{V'}$. Also it is clear that $H^0(X_{i,1}, n(K_{X_{i,1}+B_{i,1}})) \to H^0(V, n(K_V + B_V))$ is injective hence an isomorphism. Hence we have the following commutative diagram.

$$\begin{array}{ccc}
H^0(X_{j,1}, n(K_{X_{j,1}+B_{j,1}})) & \xrightarrow[g^*]{\sim} & H^0(X_{i,1}, n(K_{X_{i,1}+B_{i,1}})) \\
|_{V'}\downarrow\wr & & |_V\downarrow\wr \\
H^0(V', n(K_{V'} + B_{V'})) & \xrightarrow[g'^*]{\sim} & H^0(V, n(K_V + B_V))
\end{array}$$

Also since V, V' have the same codimension in $X_{i,1}$ and $X_{j,1}$, using the definition that s_i, s_j are pre-admissible, we see that $g'^*(s_j|_{V'}) = s_i|_V$, which using the above isomorphism, we get $g^*(s_j) = s_i$ as claimed.

Step 2: Now we deal with the type 2 case. This is done by our assumption that Conjecture 3.2.6 holds in dimension d. $\square$

Remark 3.2.17. We remark that if we only assume Conjecture 3.2.5 in dimension $d-1$, then B_d also holds if no connected component of (X, B) is klt, i.e. if all (X_i, B_i) has $\lfloor B_i \rfloor \neq 0$.

Now we are ready to prove Proposition 3.2.2.

Proof of Proposition 3.2.2. Let (X, B) be an slc pair such that $K_X + B \sim_{\mathbb{Q}} 0$, $B \in \Phi(\mathfrak{R})$. Using Theorem 1.5.2 and possibly replacing $\mathfrak{R}$, we can assume $B \in \mathfrak{R}$. Let $(X', B') := \sqcup(X'_i, B'_i) \to (X, B)$ be its normalisation and let $(Y, \Theta) := (Y_i, \Theta_i) \to (X', B')$ be a dlt model. Then we have $B' \in \mathfrak{R}$ and hence $\Theta \in \mathfrak{R}$. By assumption, we can assume that there is a bounded n such that $n(K_Y + \Theta) \sim 0$ and $n(K_X + B)$ is integral. Notice that by Proposition 3.2.7, we can assume the statement in Conjecture 3.2.5. Hence by Proposition 3.2.12, possibly replacing n by a bounded multiple, we can find a pre-admissible section $s \in PA(Y, n(K_Y + \Theta))$. By Proposition 2.4.5, we get $n(K_X + B) \sim 0$, as required. $\square$

3.3 Low Dimensional Results and Proof of Theorem 1.8.7

Here we collect some low dimensional results. First, we start with boundedness of B-representation for curves.

Proposition 3.3.1. *Conjecture 3.2.1 holds for curves.*

Proof of Conjecture 3.2.5 for curves. Let (X, B) be an lc curve where $n(K_X + B) \sim 0$. We see that X is either a rational curve or elliptic curve. We can assume that n is even. In either case, we claim that $|G|$, where $G := \rho_n(Bir(X, B))$, is bounded depending only on n: If X is a rational curve, then we see that $2n \geq |SuppB| \geq 3$, and hence $Bir(X, B) = Aut(X, B) \leq 6\binom{2n}{3}$. If X is an elliptic curve then $B = 0$ and it is well known that $\rho_{12}(Aut(X))$ is trivial. (For example, see [KDLJ92, Section 12.2.9.1]). □

Now we also show Conjecture 1.8.3 in dimension 1 and 2.

Proposition 3.3.2. *Conjecture 1.8.3 holds in dimension 1 and 2*

Proof. Dimension 1 case is obvious. Let (X, B) be dimension 2 and $K_X + B \sim_{\mathbb{Q}} 0$ and also assume $B \in \mathfrak{R}$. If $B = 0$, then by classification of Enrique surfaces, we see that $nK_X \sim 0$ for a bounded n (in particular $n \leq 21$). Furthermore, if $B \neq 0$, then the result follows from the result from curves and Theorem 3.1.5 and Theorem 3.1.4. □

Finally, we can show one of the main theorems.

Proof of Theorem 1.8.7. Notice that Conjecture 3.1.1 holds in dimension 3 due to [CJ19, Theorem 1.7]. Since Conjecture 3.2.1 holds for curves, we have Conjecture 1.8.4 holds for dimension 2 by Proposition 3.2.2. Now combining Theorem 3.1.5 and Theorem 3.1.4, we have Conjecture 1.8.3 holds in dimension 3 in full generality. Furthermore, by Theorem 3.1.5, Conjecture 1.8.3 holds in dimension 4 with (X, B) klt and $B \neq 0$. □

We make the following corollary relating to theory of sdlt complements.

Corollary 3.3.3. *Let (X, B) be an sdlt surface with $f : (X', B' + D') := \sqcup(X_i, B_i + D_i) \to X$ being its normalisation where D' is the conductor divisor. Also let $\tau : D^n \to D^n$ be the involution, where D^n is the normalisation of D'. Let n be an even integer. Assume there is a $\mathbb{Q}$-divisor $R' := \sqcup R_i \geq 0$ with R_i $\mathbb{Q}$-divisor on X_i, such that:*

1. *$n(K_{X'} + B' + R' + D') \sim 0$,*

2. *$(Y', B' + R' + D')$ is lc (hence it implies that R' doesn't contain any components of D'),*

3. *$R'|_{D^n}$ is τ-invariant.*

Then there exists m depending only on n and not on (X, B) such that if we let R be the pushforward of R' to X, then we have $m(K_X + B + R) \sim 0$ and $(X, B + R)$ is still slc. In particular $B^+ := B + R$ is an slc n-complement for (X, B).

Proof. This follows from Proposition 2.5.5 and Theorem 1.8.7. $\square$

3.4 Slc 3-fold Index and B-representation for Bounded Family

After discussing some low dimensional application of results proved earlier, we will focus on B-representation for surfaces and bounded family. In particular, we will show the following:

Theorem 3.4.1. *Let $d \in \mathbb{N}$ and $\mathfrak{R} \subset [0,1]$ be a finite set of rationals. Let $\mathcal{P}$ be a bounded family. Then there exists m and n depending only on $d, \mathfrak{R}, \mathcal{P}$ such that if (X,B) is a pair that satisfies the following:*

- *(X,B) projective klt of dimension d,*
- *$K_X + B \sim_{\mathbb{Q}} 0$ and $B \in \mathfrak{R}$,*
- *$(X, Supp(B)) \in \mathcal{P}$.*

Then $|\rho_m(Bir(X,B))| \leq n$ where $\rho_m : Bir(X,B) \to GL(H^0(X, m(K_X+B)))$ is the standard B-representation.

We make the following remark.

Remark 3.4.2. Since $K_X + B \sim_{\mathbb{Q}} 0$, $B \in \mathfrak{R}$ and $(X, Supp(B)) \in \mathcal{P}$, then it is clear that there exists m depending only on $d, \mathfrak{R}, \mathcal{P}$ such that $m(K_X+B) \sim 0$.

Now we will discuss the proof of the above theorem in the next few subsections

3.4.1 Proof of Theorem 3.4.1

We follow the ideas and constructions as in the proof in [FG14b, Proposition 3.5, Remark 3.6,Proposition 3.8], [Gon10, Proposition 4.9],[Uen, Proposition 14.4]. For readers' convenience we will include the construction here.

Proof of Theorem 3.4.1. We will follow the same ideas and notations as in [FG14b, Proposition 3.5]

Step 0: By taking log resolution and potentially replacing $\mathcal{P}$, we may assume that $(X, \mathrm{Supp}(B)) \in \mathcal{P}$ where $m(K_X+B) \sim 0$, (X,B) sub klt of dimension d, X is smooth and $\mathrm{Supp}(B)$ is a simple normal crossing divisor. Note that m here depends only on $\mathfrak{R}$ and $\mathcal{P}$ by remark.

Step 1: We let $0 \neq \omega \in H^0(X, m(K_X+B))$ and $g \in Bir(X,B)$, we will show that if $g^*\omega = \lambda\omega$, then $\lambda^N = 1$ with $N < b_n(Y')$ where b_n is the n^{th} Betti-number and Y' constructed as in the following. Notice that in particular, we have the order of $\rho_m(g)$ is bounded by $b_n(Y')$. Now we will construct Y'. Let $B = B^+ - B^-$, where B^+, B^- are both effective and contain no common components. We consider the projective space bundle

$$\pi : M := \mathbb{P}_X(\mathcal{O}_X(-K_X) \bigoplus \mathcal{O}_X) \to X.$$

Let $\{U_\alpha\}$ be coordinate neighbourhood of X with holomorphic coordinates $(z_\alpha^1, ..., z_\alpha^d)$. We may write w locally as

$$w|_{U_\alpha} = \frac{\phi_\alpha}{\delta_\alpha}(z_\alpha^1 \wedge \cdots \wedge z_\alpha^d)^m,$$

where $\phi_\alpha, \delta_\alpha$ are holomorphic with no common factors and $\frac{\phi_\alpha}{\delta_\alpha}$ has poles at most mB^+.
We may assume that $\{U_\alpha\}$ gives a trivialisation for M, i.e. $M|_{U_\alpha} \cong U_\alpha \times \mathbb{P}^1$. Let $(\xi_\alpha^1 : \xi_\alpha^2)$ be homogeneous coordinate for $\mathbb{P}^1$. Then Set

$$Y_{U_\alpha} := \{(\xi_\alpha^1)^m \delta_\alpha - (\xi_\alpha^2)^m \phi_\alpha = 0\} \subset U_\alpha \times \mathbb{P}^1.$$

It can be verified that $\{Y_{U_\alpha}\}$ patches to give Y. Notice that Y can have singularities and maybe reducible. Let $f : M' \to M$ be a log resolution of $(M, Y \cup \pi^{-1}(\operatorname{Supp} B))$ such that Y' (:= strict transform of Y) is smooth. Then it is shown in [FG14b, Proof of Remark 3.6] and [Uen, Proposition 14.4] that $N < b_n(Y')$.

Step 3: We finish the proof by noting that if (X, B) is log bounded. Say (X, B) is the fiber over $t \in T$ of the morphism $f : V \to T$. Then we can choose $\{U_\alpha\}$ and holomorphic coordinates uniformly in some analytic neighbourhood of $t \in T_i$, and such construction would work for a family of varieties (by considering the relative projective space bundle for the relative canonical divisor). Since the construction is analytic, which preserves topological boundedness, we derive that Y' is topologically bounded, hence $b_n(Y')$ is bounded. Therefore we get that $\rho_m(Bir(X, B))$ is uniformly bounded depending only on $\mathcal{P}$. □

3.5 Application and Proof of Theorem 1.8.8

The goal here is to prove slc index conjecture for dimension 3. Firstly, we will show the following:

Theorem 3.5.1. *Conjecture 3.2.1 holds in dimension 2. i.e. the following holds:*

Let $\mathfrak{R} \subset [0, 1]$ be a finite set of rationals. Then there exists n depending only on $\mathfrak{R}$ such that if (X, B) is a pair that satisfies the following:

- *(X, B) projective klt of dimension 2,*
- *$K_X + B \sim_{\mathbb{Q}} 0$ and $B \in \mathfrak{R}$.*

Let m be such that $m(K_X + B) \sim 0$, then $|\rho_m(Bir(X, B))| \leq n$ where $\rho_m : Bir(X, B) \to GL(H^0(X, m(K_X + B)))$ is the standard B-representation.

Proof. Firstly, we note that m can be chosen depending only on $\mathfrak{R}$. By considering the terminal model of (X, B) we can assume that X is terminal. In particular X is smooth.

If $B = 0$, then we are done by applying the same arguments as in [Fuj01, Proposition 3.6] and noting that all smooth surfaces with $K_X \sim_{\mathbb{Q}} 0$ have bounded second Betti number. Notice that this is not surprising since it is well-known that K_X trivial surfaces are topologically bounded. (Although they don't belong to a bounded family in the algebraic sense.)

If $B \neq 0$, then running an MMP on $K_X + B - \epsilon B$ ends in a Mori fiber space where the base is a rational or elliptic curve by canonical bundle formula. Then by [Bir20, Theorem 1.4], the set of these (X, B) is log bounded. Then we are done by applying Theorem 3.4.1. □

This allows us to prove one of the main theorems.

Proof of Theorem 1.8.8. By Theorem 3.5.1 and Proposition 3.3.1, Conjecture 3.2.1 holds in dimension ≤ 2. Also by Theorem 1.8.7, Conjecture 1.8.3 holds in dimension 3. Finally, we get the claimed result by applying Proposition 3.2.2 and Theorem 3.1.4. □

Chapter 4

Complements for Global Log Canonical Fano Threefolds

This chapter will be devoted to the proof of one of the main theorems, Theorem 1.8.5. For completeness, we will state the theorem again here.

Theorem 4.0.1. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a projective pair such that*

- *(X, B) is lc of dimension 3,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$.

The strategy is the following: We assume (X, B) is dlt by passing to a dlt model. If $\lfloor B \rfloor = 0$, then (X, B) is klt and hence Fano type and we are done by Theorem 1.4.7. If $\lfloor B \rfloor \neq 0$, then we apply adjunction and try to construct an sdlt complement. Then we lift the complement to a global one.

Before giving the proof, we need to do some preparation work. This chapter is mainly based on [Xu19a].

4.1 Complements for Finite Morphisms Between Curves

Here, we will show some basic properties of complements for finite morphisms. Given a finite morphism $f : X \to Y$ between normal projective varieties, we say f is *Galois* if $K(X)/f^*(K(Y))$ is a Galois extension, where $K(X), K(Y)$ are the function field of X, Y respectively.

We have the following result.

Lemma 4.1.1. *Let $\mathfrak{R} \subset [0,1]$ be a finite subset of rationals. Let $f : C \to T$ be a finite morphism between smooth curves. Assume f is Galois. Let $B_C \geq 0$ be a $\mathbb{Q}$-divisor on C such that $-(K_C + B_C)$ is ample and assume $B_C \in \Phi(\mathfrak{R})$. Also assume $K_C + B_C$ is $Gal(C/T)$ invariant. Then there exists $B_T \geq 0$ such that $B_T \in \Phi(\mathfrak{R})$ and $K_C + B_C = f^*(K_T + B_T)$. In particular, there is an integer n, depending only on $\mathfrak{R}$ such that there is an n-complement $K_T + B_T + R_T$ of $K_T + B_T$ with $R_T \geq 0$. In particular, if $R_C := R_T|_C$, then $K_C + B_C + R_C$ is an n-complement of $K_C + B_C$ and R_C is $Gal(C/T)$ invariant.*

Proof. We apply the Riemann-Hurwitz formula. We have $K_C = f^*(K_T) + \sum_{Q \in C}(e_Q - 1)Q$ where e_Q is the ramification index at Q. Now since f is Galois, we see that $e_Q = e_{Q'}$ if $f(Q) = f(Q')$. Hence we can define $e_P := e_{Q \in f^{-1}P}$ for $P \in T$, which is well defined. It is clear that we have $f^*P = e_P \sum_{Q:f(Q)=P} Q$. Furthermore, the above formula becomes

$$K_C = f^*(K_T + \sum_{P \in T}(1 - \frac{1}{e_P})P).$$

Now since B_C is $Gal(C/T)$ invariant, we can write $B_C = \sum_{P \in T} a_P(\sum_{Q:f(Q)=P} Q)$, where $a_P \in \Phi(\mathfrak{R})$. Hence we have $B_C = f^*(\sum_{P \in T} \frac{a_P}{e_P} P)$. Hence we have

$$K_C + B_C = f^*(K_T + \sum_{P \in T}(1 - \frac{1 - a_P}{e_P})P) =: f^*(K_T + B_T).$$

Now if $a_P = 1 - \frac{r}{m}$ for some $r \in \mathfrak{R}$ and $m \in \mathbb{N}$, then we have $\mu_P(B_T) = 1 - \frac{r}{me_P} \in \Phi(\mathfrak{R})$. The last part of the claim is clear by taking a general n-complement on C to be the pullback of an n-complement on T. $\square$

4.2 Complements on Surfaces Fibred over Curves

Here we will discuss various properties of surface complements over curves. This will be crucial when constructing complements for log canonical Fano threefolds. The goal of this section is to understand various ways to get complements on surfaces. In particular, we need a version of effective canonical bundle formula for elliptic fibration, which is our first goal of the section

4.2.1 Effective Canonical Bundle Formula for Surfaces over Curves

We first start by stating a result by Shokurov.

Proposition 4.2.1. [PS09, Theorem 8.1] *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists $q \in \mathbb{N}$ depending only on $\mathfrak{R}$ satisfying the following. Assume (S,B) is a pair and $f\colon S \to C$ a contraction such that*

- *S is a projective surface, (S,B) is lc, C is a curve, and*
- *$K_S + B \sim_{\mathbb{Q}} 0/C$ and $B_h \in \Phi(\mathfrak{R})$ where B_h is the horizontal part of B over C.*

Then we can write

$$q(K_S + B) \sim qf^*(K_C + B_C + M_C)$$

where B_C and M_C are the discriminant and moduli parts of adjunction, and the moduli divisor $qM_C \geq 0$ is base point free and Cartier. In particular, since C is a curve, we can assume that qM_C is effective and Cartier with support in general position.

We will use it to show the following result on relative complements.

Lemma 4.2.2. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural $n \in \mathbb{N}$ depending only on $\mathfrak{R}$ satisfying the following. Assume (S,B) is a pair and $f\colon S \to C$ a contraction such that*

- *S is a surface, (S,B) is lc, C is a rational curve, and*
- *$K_S + B \sim_{\mathbb{Q}} 0/C$ and $B \in \Phi(\mathfrak{R})$.*

Then for any point $z \in C$, there is an n-complement $K_S + B^+$ of $K_S + B$ over z such that $B^+ \geq B$.

Proof. Firstly, C is normal hence it is smooth. let $z \in C$ and let $D := f^*z$ be a Cartier divisor. Write

$$q(K_S + B) \sim qf^*(K_C + B_C + M_C)$$

as in Proposition 4.2.1. Note that qK_C, qM_C are both Cartier hence integral. Now let $t := lct(D, S, B)$ be the log canonical threshold of D with respect to $K_S + B$, we see that $\mu_z(B_C) = 1 - t$ from the definition of canonical bundle formula. This is because t is also the log canonical threshold of D with respect to $K_S + B$ over z (as z is Cartier and D is supported in the fiber over z). Hence if we let $B^+ := B + tD$, we see that (S, B^+) is lc by definition and B^+ has the same horizontal components as B. Hence we get

$$q(K_S + B^+) \sim qf^*(K_C + B_C^+ + M_C),$$

where $B_C^+ := B_C + tz$. Hence we get B_C^+ is Cartier near z as $\mu_z(B_C^+) = 1$. Therefore, we get $q(K_C + B_C + M_C) \sim 0$ in an open neighbourhood of z, hence we get $q(K_X + B^+) \sim 0$ over an open neighbourhood of z. Now q is the bounded n that we are looking for. The last claim is clear. □

We are now ready to show the following proposition that is similar to Theorem 1.2.3.

Proposition 4.2.3. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exist $q \in \mathbb{N}$ and a finite set of rational numbers $\mathfrak{S} \subset [0,1]$ depending only on $\mathfrak{R}$ satisfying the following. Assume (S, B) is a pair and $f\colon S \to C$ a contraction such that*

- *S is a projective surface, (S, B) is lc, C is a rational curve, and*
- *$K_S + B \sim_{\mathbb{Q}} 0/C$ and $B \in \Phi(\mathfrak{R})$.*

Then we can write

$$q(K_S + B) \sim qf^*(K_C + B_C + M_C).$$

where B_C and M_C are the discriminant and moduli parts of adjunction, $B_C \in \Phi(\mathfrak{S})$, and the moduli divisor $qM_C \geq 0$ is effective, semiample, and Cartier.

Proof. Most parts of the claim are the same as Proposition 4.2.1 except the coefficients of B_C. We will show this using n-complements. The question is local on C. Pick any $z \in C$. Let $B^+ \geq B$ be an n-complement $K_X + B$ over z as in Lemma 4.2.2. We see that

$$q(K_S + B) \sim qf^*(K_C + B_C + M_C).$$

Now we have $\mu_z(B_C) = 1 - t$, where $B^+ = B + tf^*z$. Let F be a component of f^*z and let $l := \mu_F(f^*z) \in \mathbb{N}$ and let $b := \mu_F(B)$. We have $b = 1 - \frac{r}{m}$ for some $r \in \mathfrak{R}$ and $m \in \mathbb{N}$. Hence we have $\mu_F(B^+) = 1 - \frac{r}{m} + tl$. Also we have $n\mu_F(B^+) \in \mathbb{N}$. Therefore we get

$$n \geq n(1 - \frac{r}{m} + tl) \in \mathbb{N}.$$

If $t = 0$, then we have nothing to prove. Hence we can assume $t > 0$, so by letting $a := n(1 - \frac{r}{m} + tl)$, we have

$$t = \frac{\frac{a}{n} - 1 + \frac{r}{m}}{l}.$$

Now if $a = n$, then it is clear that $1 - t = 1 - \frac{r}{ml} \in \Phi(\mathfrak{R})$. If $\frac{a}{n} < 1$, we have $\frac{r}{m} > 1 - \frac{a}{n} \geq \frac{1}{n}$ as $t > 0$. Now since $r \leq 1$, we have $\frac{1}{m} > \frac{1}{n}$. Therefore $m < n$, and hence there are only finitely many choices for $\frac{r}{m}$, and hence only finitely many choices for $\frac{a}{n} - 1 + \frac{r}{m}$. Denote this set union $\mathfrak{R}$ to be $\mathfrak{S}$, we see that $\mu_z(B_C) = 1 - t \in \Phi(\mathfrak{S})$, which proves the claim. □

4.2.2 Complements on Surface Fibred Curves

Here we consider global complements on surfaces when given a fibration to a curve. In particular, we will show the following.

Proposition 4.2.4. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number q depending only on $\mathfrak{R}$ satisfying the following. Assume (X, B) is a pair, such that*

- *X is a projective surface, (X, B) is dlt,*
- *$B \in \Phi(\mathfrak{R})$, and*
- *There is a contraction $f : X \to C$ such that $K_X + B \sim_{\mathbb{Q}} 0/C$ and C is rational curve.*
- *$-(K_X + B)$ is nef.*

We write $q(K_X + B) \sim qf^*(K_C + B_C + M_C)$, where $B_C \in \Phi(\mathfrak{S})$, qM_C is effective semiample Cartier divisor and $q, \mathfrak{S}$ are as in Proposition 4.2.3 depending only on $\mathfrak{R}$. Then any pq-complement $K_C + B_C^+ + M_C$ of $K_C + B_C + M_C$ with $B_C^+ \geq B_C$ lifts to a pq-complement $K_X + B^+$ of $K_X + B$ with $B^+ := B + f^*(B_C^+ - B_C) \geq B$. In particular, $K_X + B$ has an n-complement for some n depending only on $\mathfrak{R}$.

Proof. By Lemma 2.7.1, $K_C + B_C + M_C$ has p-complement for some p depending only on $\mathfrak{R}$. Let $B_C^+ := B_C + D_C$, where $D_C \geq 0$, be such a p-complement. Then letting $n = pq$, we have

$$n(K_X + B + f^*(D_C)) \sim nf^*(K_C + B_C^+ + M_C) \sim 0.$$

Hence it suffices to show $(X, B + D)$ is lc where $D := f^*D_C$. This follows from the fact that $(C, B_C^+ + M_C)$ is generalised lc: Indeed, consider a log resolution of $(X, B + D)$, $g : (Y, B_Y + D_Y) \to X$, where $K_Y + B_Y = g^*(K_X + B)$ and $D_Y := g^*D$. Now if $(X, B + D)$ is not lc, then there exist an irreducible component E such that $\mu_E(B_Y + D_Y) > 1$, hence $\mu_E(D_Y) > 0$, which means E is vertical over C. But by the property and the definition of the canonical bundle formula, we see that $K_X + B + D \sim_{\mathbb{Q}} f^*(K_C + B_C^+ + M_C)$ is the canonical bundle formula for the pair $(X, B + D)$, hence if E is mapped to $z \in C$, then $\mu_z(B_C^+) > 1$, which is a contradiction. □

Remark 4.2.5. The key in the above result is that any complement from base will lift to a complement on the top.

Remark 4.2.6. We note that if $E \in \lfloor B \rfloor$ is a vertical lc centre for (X, B) over C, say, mapping to $z \in C$, then it is clear that $\mu_z B_C = 1$ and hence $(f^*(B_C^+ - B_C))|_E = 0$. This is to say the surface complement is trivial along vertical lc centres.

However, we need a more delicate result when considering gluing of complements. The above result will suffice when there is no horizontal lc centres. We need to consider the case with horizontal lc centres more carefully in the next lemma. Firstly, we will state and prove an easy fact for the criteria for log canonical.

Lemma 4.2.7. *Let $X \to S$ be a projective contraction from a normal surface to a smooth curve S. Let (X, B) be a dlt pair such that $K_X + B \sim_{\mathbb{Q},S} 0$. Let E be an irreducible component of $B^{=1}$ such that it is horizontal over S. Let (E, B_E) be defined by $K_E + B_E = (K_X + B)|_E$. Let $R_S \geq 0$ be a $\mathbb{Q}$-divisor on S, $R := R_S|_X$ be its pull-back on X, and $R_E := R_S|_E$. Assume, furthermore, that $(E, B_E + R_E)$ is log canonical. Then $(X, B + R)$ is log canonical.*

Proof. This is a local question, hence we can work over $s \in S$. Also, we may assume that $\operatorname{mult}_s R_S > 0$, as the conclusion is trivial over s otherwise. To derive a contradiction, we can assume that $(X, B + R)$ is not log canonical near the fiber over s, i.e., there exists a vertical non-klt center $Z \subset X_s$ mapping to s that is not a log canoncial center. However, $(E, B_E + R_E)$ is log canonical. Hence, by inversion of adjunction, $(X, B + R)$ is log canonical near a neighbourhood of E. Now, since $\operatorname{mult}_s(R_S) > 0$ and $(E, B_E + R_E)$ is log canonical, (E, B_E) is klt near $X_s \cap E$. Therefore, (X, B) is plt near $X_s \cap E$. Thus, by considering $(X, \Omega := B + aR)$ for some $a < 1$ very close to 1, we see that (X, Ω) is plt near $X_s \cap E$. Therefore, $\operatorname{Nklt}(X, B + aR)$ is disconnected over s. Indeed, Z is disjoint from E over s. Hence, by [HH19, Theorem 1.2], we see that $(X, B + aR)$ is plt near the fiber over s, which is a contradiction. □

Remark 4.2.8. We note that the above lemma also works in the local case near $s \in S$ since the proof is local.

The above theorem is crucial when trying to show the following. Notice that the following is similar to but not the same as canonical bundle formula since there is no arbitrary moduli part and the boundary divisor on the base is determined as a $\mathbb{Q}$-divisor. Here notice that we are using the notation of $|_X$ to potentially mean the pull-back to X. The following result is quite technical and specific. Its use will be clear in later sections of this chapter.

Proposition 4.2.9. *Let $\mathfrak{R}$ be a finite set of rationals. Let $f : X \to C$ be a projective surjective morphism (maybe not a contraction) from a normal surface to a smooth curve C. Assume the following holds*

- *(X, B) is a dlt pair such that $K_X + B \sim_{\mathbb{Q}} f^*A$, where $-A$ is ample on C*
- *$B \in \Phi(\mathfrak{R})$.*
- *Let D be an irreducible component of $B^{=1}$ such that it is horizontal over S.*

- *Let (D, B_D) be defined by $K_D + B_D = (K_X + B)|_D$ and assume that we can define, by Riemann Hurwitz, (C, B_C) to be such that $K_E + B_E = (K_C + B_C)|_E$.*

Furthermore there exists q depending only on $\mathfrak{R}$ such that $q(K_X + B) \sim q(K_C + B_C)|_X$.

In particular, if $R_C \geq 0$ is such that $K_C + B_C + R_C$ an nq-complement for $K_C + B_C$, then letting $R := R_C|_X$, we have $K_X + B + R$ is an nq-complement for $K_X + B$.

Proof. If we let $X \to E \to C$ be the stein-factorization, then by restricting to the general fiber of $X \to E$, we have $D \to E$ has degree 1 or 2.

By Proposition 4.2.3, there exists bounded q such that

$$q(K_X + B) \sim qL = q(K_E + B_E + M_E)|_X,$$

where B_E and M_E are the discriminant and the moduli parts of the canonical bundle formula, respectively, and L is a vertical divisor over E.

Then we have that

$$q(K_D + B_D) \sim qL|_S = q(K_E + B_E + M_E)|_D.$$

Hence, we derive that $q(K_E + B_E + M_E)|_D \sim q(K_C + B_C)|_D$ since $K_E + B_E = (K_C + B_C)|_E$. Now, all curves here are rational curves since $(K_D + B_D) \sim_{\mathbb{Q}} f^*A$, which is anti-ample, therefore D is a rational curve, and hence so are E and C. Therefore, we see that by replacing q by $2q$, we have $q(K_E + B_E + M_E) \sim q(K_C + B_C)|_E$. This will prove the first part of the proposition. The claim about complements follows from Lemma 4.2.7 and Lemma 4.1.1. □

Remark 4.2.10. Notice that at the end of the proof, we have used the following more or less obvious fact:
Let $f : \mathbb{P}^1 \to \mathbb{P}^1$ be a degree 2 map, and if D, E are $\mathbb{Q}$-divisors on $\mathbb{P}^1$ such that $f^*D \sim f^*E$, then $2D \sim 2E$: Indeed, if $f^*(D - E) \sim 0$, this implies that $deg(D - E) = 0$ and $D - E$ has coefficients in $\frac{1}{2}\mathbb{Z}$. Therefore $2D - 2E$ is integral hence Cartier, and $deg(2D - 2E) = 0$. Hence we have $2D - 2E \sim 0$ since Cartier divisor of degree 0 are linear equivalent to 0 on $\mathbb{P}^1$.

With this setup we are almost ready to prove the boundedness of complements for log canonical Fano threefolds.

4.3 Spring and Source of Log Canonical Centre

For the last bit of preparation for the proof of the main theorem. We need to quickly cover some important facts about springs and sources of Log canonical centres. Most of this small section is taken from [Kol13, Theorem-Definition 4.45]. For completeness, we will state the theorem here.

Proposition 4.3.1. [Kol13, Theorem 4.45] *Let $f : (X, B) \to (X', B')$ be a $\mathbb{Q}$-factorial dlt model, with (X', B') lc. Let Z be an lc centre of (X', B'), and let W be a minimal lc centre of (X, B) that dominates Z. Let $K_W + B_W := (K_X + B)|_W$, and let $W \to Z_s \to Z$ be the stein factorization. Then the isomorphism class of Z_s is independent of the choices of W and X, and Z_s is called the spring of Z. Also the B-birational class of (W, B_W) is also independent of W and X and is called the source of Z. Furthermore, $Z_s \to Z$ is Galois and $Bir_Z(W, B_W) \to Gal(Z_s/Z)$ is surjective. In particular, (W, B_W) is $Gal(Z_s/Z)$ invariant. Here Bir_Z denotes the the set of B-birational automorphisms that preserve Z.*

In particular we have the following very important classification of exceptional divisors for threefold log canonical Fano varieties.

Remark 4.3.2. Let (X', B') be an lc threefold with $-(K_{X'} + B')$ ample. Let $f : (X, B) \to X$ be a dlt model. Then let S be an irreducible component of $\lfloor B \rfloor$. Write $K_S + B_S := (K_X + B)|_S$. Then one of the following holds:

- $f(S)$ is a surface and hence $f|_S$ is birational.
- $f(S)$ is a point and therefore $K_S + B_S \sim_{\mathbb{Q}} 0$.
- $f(S)$ is a curve C' with normalisation C, then one of the following holds:
  - $\lfloor B_S \rfloor$ has no horizontal component over C'. In particular, (S, B_S) is a minimal lc centre over C' and therefore the B-birational class of (S, B_S) is independent from the choice of S. Furthermore, for any other $S' \in \lfloor B \rfloor$, with $f(S') = C'$, the same property holds.
  - There exists $E \in \lfloor B_S \rfloor$ horizontal over C'. Let $K_E + B_E := (K_S + B_S)|_E$. In particular, E is the minimal lc centre over C'. In this case, E is the spring of C' and hence is determined (up to isomorphism) independent from the choice of S. Also (E, B_E) is the source of C' and therefore the B-birational class of (E, B_E) depends only on C' and not on the choice of S. Furthermore, $E \to C$ is Galois between smooth curves (as E is normal) and (E, B_E), (which only depends on C') is $Gal(E/C)$ invariant. In particular, by Riemann Hurwitz, there exists a well defined pair (C, B_C) such that $K_E + B_E = (K_C + B_C)|_E$.

Finally we note that the last case in the above classification is precisely the assumption as in Proposition 4.2.9, which is why the Proposition is very important.

4.4 Proof of Theorem 1.8.5

Now we are ready to show the proof of the main theorem of the paper. For convenience of the reader, we will restate our result here.

Theorem 4.4.1 (Theorem 1.8.5)**.** *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on $\mathfrak{R}$ satisfying the following. Assume (X', B') is a projective pair such that*

- *(X', B') is lc of dimension 3,*
- *$B' \in \Phi(\mathfrak{R})$, that is, the coefficients of B' are in $\Phi(\mathfrak{R})$, and*
- *$-(K_{X'} + B')$ is ample.*

Then there is an n complement $K_{X'} + B'^+$ of $K_{X'} + B'$.

Proof of Theorem 1.8.5. Let (X', B') be as in the Theorem. If (X', B') is klt, then X' is Fano type and we are done by Theorem 1.4.7. Therefore from now on, we will assume that (X', B') is not klt. Let $f : (X, B) \to (X', B')$ be a $\mathbb{Q}$-factorial dlt model of (X', B') with $K_X + B = f^*(K_{X'} + B')$. Let $S := \lfloor B \rfloor$ and write $K_S + B_S := (K_X + B)|_S$. Then by Prop. 2.6.1, it suffices to show $K_S + B_S$ has an n-complement with n depending only on $\mathfrak{R}$. First we will show there are complements on each irreducible component of S and then we will use Corollary 3.3.3 to show they glue to give a complement for $K_S + B_S$. We note that $B_S \in \Phi(\mathfrak{S})$ with S a finite set of rationals depending only on $\mathfrak{R}$. We note that S is connected by connectedness theorem since $-(K_X + B)$ is nef and big.

Step 0: We first settle the case when S is irreducible. In this case, let T' be the image of S on X' and let $S \xrightarrow{g} T \to T'$ be the stein factorization. Then we see that either T is dimension 0, in which case we have $K_S + B_S \sim_{\mathbb{Q}} 0$ or T is a curve or a surface. We can apply Theorem 1.8.7 or Proposition 4.2.4 or Proposition 2.7.3, to show there is an n-complement with n depending only on $\mathfrak{R}$ for $K_S + B_S$. Now we are done by applying Prop. 2.6.1. Hence from now on, we will assume S has multiple irreducible components.

Step 1: We first consider complements on curves. Let T be an irreducible one dimensional lc centre on $K_X + B$. Write $K_T + B_T := (K_X + B)|_T$ (note this is well defined up to sign by [Kol13, Section 4.18] and we will always assume n to be even). We note that the coefficients of B_T lie in $\Phi(\mathfrak{F})$ with $\mathfrak{F}$ depending only on $\mathfrak{R}$. Then either T is contracted by f or the image of T is a curve.

1. If T is contracted, then $K_T + B_T \sim_{\mathbb{Q}} 0$ and hence $n(K_T + B_T) \sim 0$ for some n depending only on $\mathfrak{R}$. In this case, we let $R_T = 0$ be an n-complement for (T, B_T).
2. If T is not contracted by f, let T' be its image on X, then we have $K_T + B_T \sim_{\mathbb{Q}} f|_T^*(-A)$ for some A ample on T'. Hence we see that T, T' are rational curves. since T is the minimal lc centre dominating T', we see that $T \to T'$ is Galois. Furthermore, by [Kol13, Theorem–Definition

4.45], If $(\hat{T}, B_{\hat{T}})$ is another one dimensional lc centre on (X, B) dominating T', then (T, B_T) is naturally B-birational to $(\hat{T}, B_{\hat{T}})$ in the sense that we have a commutative diagram

$$\begin{tikzcd} (T, B_T) \arrow[rr, "\sigma"] \arrow[dr, "f|_T"'] & & (\hat{T}, B_{\hat{T}}) \arrow[dl, "f|_{\hat{T}}"] \\ & T' & \end{tikzcd}$$

We note that σ is not unique.
If we let $\bar{T}$ be the normalisation of T', then $T \to \bar{T}$ is also Galois. Also note that $Gal(T/\bar{T}) = Gal(T/T')$. Now since $Bir(T, B_T) \to Gal(T/T')$ is surjective, we see that $K_T + B_T$ is $Gal(T/\bar{T})$ invariant. Therefore, by Riemann Hurwitz, we can define $(\bar{T}, B_{\bar{T}})$ such that $K_T + B_T = (K_{\bar{T}} + B_{\bar{T}})$. Notice that $(\bar{T}, B_{\bar{T}})$ is well-defined and independent from the choice of T. Furthermore, by Lemma 4.1.1 we can assume that $B_{\bar{T}} \in \Phi(\mathfrak{T})$. Therefore, also by Lemma 4.1.1, there exists n, depending only on $\mathfrak{T}$ (which in turn, only depends on $\mathfrak{R}$), such that there exists n-complements $K_{\bar{T}} + B_{\bar{T}} + R_{\bar{T}}$, with $R_{\bar{T}} \geq 0$ such that $K_T + B_T + R_T$ is an n-complement for $K_T + B_T$ with $R_T := R_{\bar{T}}|_T$. Notice that this defines R_T for all T mapping to T'. Also we note that R_T is $Gal(T/T')$ invariant.

Hence now for each T, dimension 1 irreducible lc centre of (X, B), we have constructed an n-complement $K_T + B_T + R_T$ with $R_T \geq 0$ and R_T is disjoint from any other lc centre on (X, B).

Step 2: Now let $S := \cup S_i$, where S_i are the irreducible components of S. Then S_i are mapped to either points, curves or surfaces on X'. We distinguish the 3 cases. Let W be a general S_i.

1. If W is mapped to a point on X', then $K_W + B_W := (K_X + B)|_W \sim_{\mathbb{Q}} 0$, and the coefficients of B_W are in $\Phi(\mathfrak{S})$ for some finite set of rationals $\mathfrak{S}$ depending only on $\mathfrak{R}$. Hence by Theorem 1.8.7, there is n, depending only on $\mathfrak{R}$, such that $n(K_W + B_W) \sim 0$, in this case, the n-complement $R_W = 0$.

2. If W is mapped to a surface on X', then $-(K_W + B_W)$ is nef and big. Let $V := \lfloor B_W \rfloor$ and $K_V + B_V := (K_W + B_W)|_V$, we see that V is an sdlt curve and $-(K_V + B_V)$ is nef. In particular, for each irreducible component of V, we have already created an n-complement with n depending only $\mathfrak{R}$ such that they are disjoint from the non-normal locus of V. Hence by Proposition 2.7.2, we have already found an n-complement for $K_V + B_V$ in the form of $K_V + B_V + R_V$, with $R_V \geq 0$. Therefore, by Proposition 2.6.1, we can lift these complements to an n-complement $K_W + B_W + R_W$ for $K_W + B_W$ such that $R_W|_W := R_V$, i.e. for each irreducible component T in V, we have $R_W|_T = R_T$ defined as above.

3. The last case is that W is mapped to a curve T' with normalisation T. Let $f : W \xrightarrow{g} C \to T$ be the stein factorization. By Prop. 4.2.3, we can

find q depending only on $\mathfrak{R}$, such that

$$q(K_W + B_W) \sim q(K_C + B_C + M_C).$$

We note that $-(K_C + B_C + M_C)$ is $\mathbb{Q}$-linearly equivalent to the pullback of an ample divisor on T', hence we see that C is a smooth rational curve. Now we split into further cases depending on B_h, the horizontal over C part of $\lfloor B_W \rfloor$.

(a) Case 1: $B_h = 0$, then by Lemma 2.7.1,we can simply choose any n-complement $K_C + B_C + R_C + M_C$ for $K_C + B_C + M_C$. Using Proposition 4.2.4, it lifts to an n-complement $K_W + B_W + R_W$ for $K_W + R_W$ with $R_C \geq 0$ and $R_W := g^*(R_C)$. Note that in this case, for any D, an irreducible component of $\lfloor B_W \rfloor$, we have $R_W|_D = 0$ since if D is mapped to $z \in C$, then $\mu_z(B_C) = 1$ and hence $\mu_z(R_C) = 0$.

(b) Case 2: $B_h \neq 0$. By Remark 4.3.2, we can apply Proposition 4.2.9.Notice that by step 2, we have already constructed an n-complement $K_T + B_T + R_T$ for $K_T + B_T$. Now apply Proposition 4.2.9, after replacing n by a possibly bounded multiple, we see that $K_W + B_W + R_W$ is an n-complement for $K_W + B_W$ with $R_W := R_T|_W$. Also we can easily see that $R_W|_D = 0$ for any D, an irreducible vertical component of $\lfloor B_W \rfloor$, by similar reasons as in (a). In particular, by construction, we have $R_W|_{B_h} = R_{B_h}$ as constructed in step1.

Now summing up, we have found n, depending only on $\mathfrak{R}$ such that for each W, an irreducible component of S, there is an n-complement $K_W + B_W + R_W$ for $K_W + B_W$ with $R_W \geq 0$ and for each irreducible component T in $\lfloor B_W \rfloor$, we have $R_W|_T = R_T$ defined above in step 1.

Step 3 : We are now done by applying Proposition 3.3.3 and Proposition 2.6.1 again. More precisely, by Proposition 3.3.3, by potentially replacing n by a bounded multiple, we can get an n-complement for $K_S + B_S$, which will lift to an n-complement for $K_X + B$ by Proposition 2.6.1. Pushing forward to X', we get an n-complement for $K_{X'} + B'$, which finishes the proof. $\square$

Chapter 5

Complements for Relative Log Canonical Fano Threefolds

This chapter will be devoted to the proof of one of the main theorems, Theorem 1.8.6. We will also give a quick sketch of the slightly more general result as in Theorem 1.8.10. For completeness, we will state the theorem again here. Most of this chapter is taken from [FMX19], which is a joint work with Stefano Filipazzi and Joaquin Moraga.

Theorem 5.0.1. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rational numbers. Then there exists a natural number n depending only on d and $\mathfrak{R}$ satisfying the following. Assume (X, B) is a quasi-projective pair such that*

- *(X, B) is lc of dimension 3,*
- *$f : X \to Z$ is a projective contraction,*
- *$B \in \Phi(\mathfrak{R})$, that is, the coefficients of B are in $\Phi(\mathfrak{R})$, and*
- *$-(K_X + B)$ is ample over $z \in Z$.*

Then there is an n-complement $K_X + B^+$ of $K_X + B$ over $z \in Z$.

We remark that since the result is relative around $z \in Z$, we may shrink Z around z freely if necessary.

There are a lot of technicalities in the proof. Hence we will first have a section to prepare for all the results that are needed for the proof. One key difference to the global case is that we will need to apply a more general form of vanishing theorem, called Injectivity Theorem. This theorem allows much more freedom as we will see in the proof. Another key difference is that we will need to use Kollár gluing theorem instead of pre-admissible sections to glue sections on sdlt surfaces. We will give a quick review of Kollár gluing theory as well in the next section.

5.1 Preparation for the Proof

5.1.1 Kollár Gluing Theorem

Kollár developed a theory of quotients by finite equivalence relations [Kol13, Chapter 9]. In particular, it is a powerful tool to study a semi-log canonical pair (X, B) via its normalization $(X', B' + D')$. In particular, the technique is used in the proof of Theorem 2.5.4. Also in [HX13, HX16], this technique is used to show the following result for semi-log canonical pairs.

Theorem 5.1.1. [HX16, Theorem 1.4] *Let (X, B) be a semi-log canonical pair, $f\colon X \to S$ be a projective morphism, $\nu\colon X^\nu \to X$ be the normalization. Write $\nu^*(K_X + B) = K_{X^\nu} + B^\nu + D^\nu$, where D^ν is the double locus. If $K_{X^\nu} + B^\nu + D^\nu$ is semi-ample over S, then $K_X + B$ is semi-ample over S.*

The idea of the proof of Theorem 5.1.1 is the following. Assume for simplicity that $S = \operatorname{Spec}(\mathbb{C})$, $B = 0$, and $(X^\nu, B^\nu + D^\nu) = (X_1, D_1) \sqcup (X_2, D_2)$, where each D_i is normal and irreducible. A section of $\mathcal{O}_{X^\nu}(m(K_{X^\nu} + D^\nu))$ descends to a section of $\mathcal{O}_X(mK_X)$ if its restriction to D^ν is invariant under the involution τ that exchanges D_1 and D_2, as in Proposition 2.4.3. Therefore, in order to show $|\mathcal{O}_X(mK_X)|$ is base point free, i.e. separates x_1 and x_2 for any $x_1 \neq x_2$, it suffices to find two sections $s_1, s_2 \in H^0(\mathcal{O}_{X^\nu}(m(K_{X^\nu} + D^\nu)))$ that separate the preimages of x_1 and x_2 and such that each $s_i|_{D^\nu}$ is τ-invariant.

The theory of finiteness of B-representations (i.e. [FG14b, Theorem 3.15]) guarantees that we can find the needed τ-invariant sections in $|\mathcal{O}_{X^\nu}(m(K_{X^\nu} + D^\nu))|$ for some m. As we are interested in n-complements for a bounded n, we need an effective version of this approach. Therefore, we need to prove the following.

Proposition 5.1.2. *Let $X \to T$ be a contraction such that the pair (X, B) is semi-dlt with $\dim X \leq 2$. Let $(X^\nu, B^\nu + D^\nu)$ be the normalisation of (X, B). Assume that we have $n(K_{X^\nu} + B^\nu + D^\nu) \sim_T 0$, and $n(K_{X^\nu} + B^\nu + D^\nu)$ is Cartier. Then, there exists m, only depending on n such that $m(K_X + B) \sim_T 0$.*

Remark 5.1.3. We have already proved this proposition in the case that T is a point, see Theorem 1.8.7. Here the key difficulty is coming from the relative setting.

Remark 5.1.4. We note that the above proposition is more or less trivial when $\dim X = 1$ since in this case, T is either a point (i.e., we are in the projective case) or T is X, which the claim follows trivially. We also note that it is shown in [HX16, Theorem 1.4] that such an m exists, and here we need to bound m depending only on n and $\mathfrak{R}$.

Proof of Proposition 5.1.2. By Remark 5.1.4, we can assume $\dim(X) = 2$. Therefore, if we denote the double locus of $X^\nu \to X$ by D^ν, the components of D^ν are curves. In particular, by Proposition 3.3.1, we can choose m depending only on n such that $\rho_m(\operatorname{Bir}(Z, B_Z))$ is trivial for all Z irreducible components of D^ν, which is the key for our proof. In particular, we have $(M|_{D^\nu})$ is Cartier, where we set $M := m(K_{X^\nu} + B^\nu + D^\nu)$.

Now, we follow the proof in [HX16, Theorem 1.4]. We consider the morphism $f\colon X^\nu \to T$, and we have that $M = m(K_{X^\nu} + B^\nu + D^\nu) \sim_T 0$ is Cartier. Hence, f is the morphism induced by $n(K_{X^\nu} + B^\nu + D^\nu)$ over T. Let H be the (very ample over T) line bundle on T such that $f^*H = n(K_{X^\nu} + B^\nu + D^\nu)$. Let $p_X\colon X_M \to X$ and $p_T\colon T_H \to T$ be the total spaces of the line bundles of M and H, respectively. Define $D_M := p_X^{-1}(D^\nu)$, $Y := f(D^\nu)$, and $Y_H := p_T^{-1}(Y)$. We see that the involution $\tau\colon D^\nu \to D^\nu$ induces a set relation on $Y_H \to T_H$. Now following [HX13, Section 3.2], we see that the quotient A with respect to $Y_H \to T_H$ exists. This implies that there is a line bundle A on T whose pullback to X is $m(K_X + B)$. In particular, this implies that $m(K_X + B) \sim_T 0$. $\square$

5.1.2 Some Notes on Curves

Firstly we will introduce the notion of semi-normality.

Semi-normal curves

Let X be a scheme, and let $f\colon X' \to X$ be a finite morphism. The morphism f is a *partial semi-normalization* if X' is reduced, each point $x \in X$ has exactly one preimage $x' := f^{-1}(x)$, and $f^*\colon k(x) \to k(x')$ is an isomorphism. A scheme X is called *semi-normal* if every partial semi-normalization $f\colon X' \to X$ is an isomorphism. In particular, a semi-normal scheme is reduced. Over an algebraically closed field, a curve singularity $(0 \in C)$ is semi-normal if and only if it is analytically isomorphic to the union of n coordinate axes in $\mathbb{A}^n$ [Kol13, Example 10.12]. We will only be interested in semi-normal curves.

The reason that we are interested in semi-normal curves is because of the following result: [Kol13, Section 4.20] If (X, B) is an lc pair, then any union of lc centres of (X, B) is semi-normal.

5.1.3 Some Remarks on Curves

Here we need to state some more or less trivial result for finite morphism between curves. The following remark is important when considering relative complements on curves. It will be used a few times in the proof later, so for readers' convenience, we will state it here.

Remark 5.1.5. Let C be a smooth curve. Let $P_1, \ldots, P_n$ be n closed points on C. Then, for any Cartier divisor D on C, we have $D \sim 0$ in a neighbourhood of P_i, for all i. Indeed, let Q be an arbitrary point on C, away from P_i for all i. Then, for a sufficiently large m, $D + mQ$ is very-ample. Hence, we can find $0 \leq R \sim D + mQ$ such that P_i is not in $\mathrm{Supp}(R)$ for all i. Hence, we get $D \sim R - mQ \sim 0$, in a neighbourhood of P_i. Note that the exact same result holds in the relative case via a finite map $f\colon C \to E$ over another curve E, where E is irreducible but not necessarily smooth.

5.1.4 Injectivity Theorem

We have the following injectivity theorem, which we will use for the proof. It is quite technical.

Theorem 5.1.6. [Fuj17, Theorem 2.12] *Let (X, Γ) be a log smooth pair with* $\operatorname{coeff}(\Gamma) \subset [0, 1]$. *Let $\phi\colon X \to T$ be a proper morphism between schemes. Let ϵ be a positive rational number. Let L be a Cartier divisor on X. Let S be an effective Cartier divisor on X, which does not contain any log canonical centre of (X, Γ). Assume that*

(1) $L \sim_{\mathbb{Q},T} K_X + \Gamma + N$;

(2) N is a $\mathbb{Q}$-divisor that is semi-ample over T; and

(3) $\epsilon N \sim_{\mathbb{Q},T} S + \overline{S}$, where $\overline{S}$ is an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor which doesn't contain any log canonical centre of (X, Γ) in its support.

Then the natural map

$$R^q\phi_*(\mathcal{O}_X(L)) \to R^q\phi_*(\mathcal{O}_X(L + S))$$

is injective for every q.

Remark 5.1.7. We notice that the above theorem is essentially a generalisation of relative Kodaria type vanishing for the simple normal crossing case. It is in general quite difficult to apply this vanishing due the technical condition (3). Notice that (3) is always true if (X, Γ) is klt and when N is ample, we recover the standard Kodiara type vanishing theorem.

Now we will give a quick reason why the above vanishing would be useful in the proof here. Since we are working with right derived function of the push forward functor, we will give a quite detailed explanation here. It is more or less straight forward for people that are familiar with such statements.

Remark 5.1.8. Assuming the same notation as in the above vanishing theorem and let $t \in T$ be a closed point such that $t \in \phi(S)$. Consider the exact sequence

$$0 \to \mathcal{O}_X(L) \to \mathcal{O}_X(L + S) \to \mathcal{O}_S(L|_S) \to 0.$$

By pushing forward via ϕ and taking the right derived function, we arrive at the following exact sequence

$$\phi_*(\mathcal{O}_X(L + S)) \to \phi_*(\mathcal{O}_S(L|_S)) \to R^1\phi_*(\mathcal{O}_X(L)) \to R^1\phi_*(\mathcal{O}_X(L + S)).$$

Therefore, if $R^1\phi_*(\mathcal{O}_X(L)) \to R^1\phi_*(\mathcal{O}_X(L + S))$ is injective, then

$$\phi_*(\mathcal{O}_X(L + S)) \to \phi_*(\mathcal{O}_S(L|_S))$$

is a surjective morphism of sheaves. Now by the definition of push forward of sheaves and the fact that a surjective morphism of sheaves is surjective on sections on any sufficiently small neighbourhood of t, we have that, potentially by shrinking T near t,

$$H^0(X, \mathcal{O}_X(L+S)) \to H^0(S, \mathcal{O}_S(L|_S))$$

is surjective, which is exactly what we need to lift sections.

As we have mentioned, criteria (3) is quite technical. Luckily, we can easily satisfy it with the following lemma. Due to the technical nature of this subject, the following proof is also quite technical. However the idea is simple, if (X, B) is lc with $-(K_X + B)$ ample$/T$, then if $f : (X', B') \to X$ is a dlt model of (X, B) and let $N' := -K_{X'} + B'$, then in some sense N' is ample outside $Ex(f)$. A more precise statement is the following.

Lemma 5.1.9. *Let $\phi\colon X \to T$ be a projective morphism of normal quasi-projective varieties. Let (X, B) be a log canonical pair, with $-(K_X + B)$ ample over T. Let $\pi\colon X' \to X$ be a $\mathbb{Q}$-factorial dlt modification of (X, B), and define $N' := -\pi^*(K_X + B)$. Then, we can write*

$$N' \sim_{\mathbb{Q},T} A + D,$$

where A is ample over T, and D is an effective divisor which is semi-ample over T outside $\mathrm{Ex}(\pi)$.

Proof. First, we prove that the relative augmented base locus of N' is contained in $\mathrm{Ex}(\pi)$. Let A be an ample divisor on X', H be a very ample divisor on T. Fix a rational number $0 < \epsilon \ll 1$ such that $\mathbb{B}_+(N'/T) = \mathbb{B}(N' - \epsilon A/T)$. Then, we have

$$\mathbb{B}_+(N'/T) = \bigcap_{m \in \mathbb{N}} \bigcap_{n \in \mathbb{N}} \mathrm{Bs}\left|m(N' - \epsilon A) + n\phi^* H\right|.$$

On the other hand, we may choose m and n so that the Cartier divisor $mN' + n\phi^*H$ is big and nef on X'. Moreover, we may further assume that $|mN' + n\phi^*H|$ defines an isomorphism on the complement of $\mathrm{Ex}(\pi)$. By [BCL14, Theorem A], we conclude that

$$\mathbb{B}_+(mN' + n\phi^*H) \subset \mathrm{Ex}(\pi).$$

This latter inclusion implies that for ϵ small enough, we have

$$\mathrm{Bs}\left|mN' + n\phi^*H - m\epsilon A\right| \subset \mathrm{Ex}(\pi).$$

Thus, we conclude that $\mathbb{B}_+(N'/T) \subset \mathrm{Ex}(\pi)$. By the above inclusion, we conclude that we may write

$$N' \sim_{\mathbb{Q},T} A + D,$$

where A is ample over T, and the base locus of D is contained in $\mathrm{Ex}(\pi)$. We conclude the claim by replacing D by some general element in its relative $\mathbb{Q}$-linear system. $\square$

5.2 Relative Sdlt Surface Complements

Notice that here we need to prove a specific type of existence of complements for sdlt surfaces in the relative case. Notice that since we are still interested in the threefold case, Remark 4.3.2 still applies. In the light of the remark, we have the following condition. It may seem a bit strange and technical, but this is exactly what we need before lifting complements.

By Remark 4.3.2, we need to prove the existence of semi-dlt relative complements in the following setting, which we will call Condition A.

Definition 5.2.1 (Condition A). Let $(X, B) \to S \to T$ be surjective morphisms between (not necessarily normal) quasi-projective varieties, and let $\mathcal{R} \subset [0,1]$ be a finite set of rational numbers. Assume that $X \to T$ is a contraction, and let $t \in T$ be a closed point. We say that the contraction satisfies *Condition A* if the following holds:

- (X, B) is a semi-dlt surface that is $\mathbb{Q}$-complemented over the closed point $t \in T$;
- the coefficients of B belong to $\Phi(\mathcal{R})$;
- S is a possibly reducible semi-normal curve; and
- T is either a possibly reducible semi-normal curve, or $T = \{t\}$.

Moreover, given any irreducible component X_1 of X, we assume that one of the following occurs:

1. X_1 is mapped to the closed point $s \in S$ (where s maps to t), and $K_{X_1} + B_1 \sim_{\mathbb{Q}} 0$;
2. X_1 is mapped to S_1, a curve in S, S_1 is mapped to t, and $K_{X_1} + B_1 \sim_{\mathbb{Q}} f^*A$, where $f\colon X_1 \to S_1$ and $-A$ is globally ample on S_1; or
3. X_1 is mapped onto S_1, a curve on S, and S_1 is mapped onto T, and $K_{X_1} + B_1 \sim_{\mathbb{Q},S_1} 0$.

Furthermore, if (2) or (3) occurs (that is, X_1 is mapped onto a curve $S_1 \subset S$), we assume the following condition:

- let E be a component of $\lfloor B_1 \rfloor$ that dominates S_1. Then, $E \to C$ is a Galois finite morphism, where C is the normalisation of S_1, $K_E + B_E := (K_{X_1} + B_1)|_E$ is $\mathrm{Gal}(E/C)$ invariant, and the pair (E, B_E) is (up to B-birational automorphism) only dependent on the choice of such S_1 and independent of the choice of X_1.

Then we have the following theorem, which is the main result of this section.

Proposition 5.2.2. [FMX19, Proposition 9.4] *Let $\mathcal{R} \subset [0,1]$ be a finite set of rational numbers. Then, there exists a natural number n only depending on $\mathcal{R}$ which satisfies the following. Let $X \to S \to T$ be a projective contraction between* *quasi-projective varieties that satisfies Condition A. Then, up to shrinking T around t, we can find*

$$\Gamma \sim_T -n(K_X + B),$$

such that $(X, B + \Gamma/n)$ is a log canonical pair.

Proof. We will first treat the case when T is a semi-normal curve. We split the proof in two main steps. We first show how to create complements on each component of X_i and then show that they can be glued together to form a global complement.

Step 1: We consider each case above separately using the same numbering as in the definition of Condition A. In this step, we will prove the existence of an n-complement on the component.

1. Assuming X_1 is mapped to $s \in S$, then we have $K_{X_1} + B_1 \sim_{\mathbb{Q}} 0$. Hence, by Theorem 1.8.7, there exists a bounded n, such that $n(K_{X_1} + B_1) \sim 0$. In particular, the complement is trivial. Also, we note that any complement of X_1 will be trivial on any irreducible component of $\lfloor B_1 \rfloor$.

2. In this case, we apply the canonical bundle formula. Notice that, in this case, the curve S_1 is projective. Therefore, we can consider global complements. We split into 2 further cases for gluing: this is because we will construct complements differently depending on the different cases and we need these specific constructions for gluing the sdlt complements later.

   (a) The first case is where $\lfloor B_1 \rfloor$ doesn't contain any horizontal component mapping onto S_1. Applying the canonical bundle formula, we get there exists a positive integer q depending only on $\mathcal{R}$ such that
   $$q(K_{X_1} + B_1) \sim qf^*(K_{S_1} + B_{S_1} + M_{S_1}),$$
   where here we possibly replace S_1 by its normalisation and its finite cover in the Stein factorization of $X_1 \to S_1$. Furthermore, by Proposition 4.2.3, we can assume that qM_{S_1} is Cartier, base point free, and the coefficients of B_{S_1} belong to $\Phi(\mathcal{S})$, where $\mathcal{S} \subset [0,1]$ is a finite set of rational numbers only depending on $\mathcal{R}$ by Theorem 4.2.3. Now, since $-(K_{S_1} + B_{S_1} + M_{S_1})$ is ample and M_{S_1} is nef, we conclude that S_1 is a rational curve. Hence, there exists an $R_{S_1} \geq 0$ such that
   $$q(K_{S_1} + B_{S_1} + R_{S_1} + M_{S_1}) \sim 0,$$
   possibly after replacing q by a bounded multiple. Pulling R_{S_1} back and letting $R_1 := f^*R_{S_1}$, we get
   $$q(K_{X_1} + B_1 + R_1) \sim 0.$$
   Furthermore, it is clear that $(X_1, B_1 + R_1)$ is log canonical from the canonical bundle formula.

(b) Now assume that D is a component in $\lfloor B_1 \rfloor$ mapping onto S_1. Let C be the normalisation of S_1. We see that by assumption we have $D \to C$ is Galois. Notice that here both D and C are smooth curves. Let $K_D + B_D := (K_{X_1} + B_1)|_D$, where $B_D \in \Phi(\mathcal{S})$and $\mathcal{S} \subset [0,1]$ is a finite subset of rational numbers depending only on $\mathcal{R}$. By Condition A and Remark 4.3.2, we see that there exists $B_C \in \Phi(\mathcal{S})$, such that $K_D + B_D = (K_C + B_C)|_D$.

Furthermore, we see that by Proposition 4.2.9 there exists a bounded q such that $q(K_{X_1} + B_1) \sim q(K_C + B_C)|_{X_1}$.

We note that such $K_C + B_C$ is in fact determined independent of the choice of S_1 by [Kol13, Theorem 4.45 (5)]. Hence, since $K_C + B_C$ is anti-ample, there exists an $R_C \geq 0$ such that $q(K_C + B_C + R_C) \sim 0$. Letting $R_1 := R_C|_{X_1}$, we see that $q(K_{X_1} + B_1 + R_1) \sim 0$, possibly after replacing q by a bounded multiple. However, we still need to show that $(X_1, B_1 + R_1)$ is log canonical. By Lemma 4.1.1, we can show that, possibly by replacing q, we can assume that $(D, B_D + R_D)$ is log canonical, where $R_D := R_C|_D$. Then, we are done by Lemma 4.2.7.

3. This case is almost the same as the previous one. Again, we split it into two further cases to discuss.

(a) The first case is where $\lfloor B_1 \rfloor$ does not contain any horizontal component mapping onto S_1. Applying the canonical bundle formula, we get there exists a positive integer q, depending only on $\mathcal{R}$, such that $q(K_{X_1} + B_1) \sim q(K_{S_1} + B_{S_1} + M_{S_1})|_{X_1}$. Here, we possibly replace S_1 by its normalisation and the Stein factorization of $X_1 \to S_1$. Furthermore, by Proposition 4.2.3, we can assume that qM_{S_1} is Cartier and the coefficients of B_{S_1} belong to $\Phi(\mathcal{S})$, where $\mathcal{S} \subset [0,1]$ is a finite set depending only on $\mathcal{R}$. Now, let $\{s_1, \dots, s_n\}$ be the preimage of $t \in T$ in S_1. We can define

$$R_{S_1} := (1 - \mathrm{mult}_{s_1}(B_{S_1}))s_1 + \cdots + (1 - \mathrm{mult}_{s_n}(B_{S_1}))s_n.$$

By Remark 5.1.5, we see that $q(K_{S_1} + B_{S_1} + R_{S_1} + M_{S_1}) \sim 0$ over a neighbourhood of t. Hence, if we let $R_1 := R_{S_1}|_{X_1}$, we get $q(K_{X_1} + B_1 + R_1) \sim 0$ over a neighbourhood of t. We note here by the above linear equivalence,we mean that $\mathcal{O}_{X_1}(q(K_{X_1} + B_1 + R_1)) \cong f^*\mathcal{O}_T$ around a neighbourhood of t. Furthermore, note that, by inversion of adjunction in canonical bundle formula, $(X_1, B_1 + R_1)$ is log canonical.

(b) Now assume that D is a component in $\lfloor B_1 \rfloor$ mapping onto S_1. Let C be the normalisation of S_1. By assumption, we have that $D \to C$ is Galois. Notice that here both D and C are smooth curves. Set $K_D + B_D := (K_{X_1} + B_1)|_D$. By Condition A, we see that there exists $B_C \in \Phi(\mathcal{R})$, such that $K_D + B_D = (K_C + B_C)|_D$. Furthermore, by similar argument as in (2b), there exists a bounded q such that

$$q(K_{X_1} + B_1) \sim q(K_C + B_C)|_{X_1}.$$

We note that, by the assumptions, such $K_C + B_C$ is in fact determined, independent of the choice of S_1. Hence, we can define $R_C := (1 - \mathrm{mult}_{c_1}(B_C))c_1 + \cdots + (1 - \mathrm{mult}_{c_n}(B_C))c_k \geq 0$, where $\{c_1, \ldots c_k\}$ is the preimage of t on C. Then, possibly by shrinking around t, by Remark 5.1.5, it follows that $q(K_C + B_C + R_C) \sim 0$, where $R_1 := R_C|_{X_1}$. Then, we see that $q(K_{X_1} + B_1 + R_1) \sim 0$, possibly after replacing q by a bounded multiple and shrinking around t. Furthermore, by Lemma 4.2.7 and Lemma 4.1.1, we see that $(X_1, B_1 + R_1)$ is log canonical.

Step 2: Now we consider gluing these complements together. Firstly, we note that, up to shrinking around t, each complement R_1 constructed in Step 1 is such that $\mathcal{O}_{X_1}(n(K_{X_1} + B_1 + R_1)) \sim f^*\mathcal{O}_T$, i.e., each $n(K_{X_1} + B_1 + R_1)$ is linearly equivalent to the pull-back of the structure sheaf on T. Furthermore, it can be verified that, given X_1, X_2, two different irreducible components of X, and E being a component of $X_1 \cap X_2$, the complements R_1 and R_2 agree along E.
Indeed, we have the following cases. Let R_1 and R_2 be two complements that we have constructed in Step 1 on X_1 and X_2, respectively. Now, if E is mapped to a point on S, then it is clear that $R_1|_E = 0 = R_2|_E$, since $R_1|_E \geq 0$ and $K_E + B_E \sim_{\mathbb{Q}} 0$. On the other hand, if E is mapped onto S_1, an irreducible component of S, it follows from the construction that $R_1|_E = R_2|_E$, since they are both pull-backs of a fixed well-defined divisor on C by considering the finite Galois map $E \to C$, where C is the normalisation of S_1.

Now, we are done applying Proposition 5.1.2. Indeed, we define R^ν on X^ν (the normalisation of X) to be such that $R^\nu|_{X_i} = R_i$ as above. Notice by the definition, we have $n(K_{X^\nu} + B^\nu + D^\nu + R^\nu) \sim 0$, where D^ν is the conductor. In particular, by [Kol13, Theorem 5.39], we see that R is also $\mathbb{Q}$-Cartier where R is the pushforward of R^ν to X. Therefore $(X, B + R)$ is indeed an slc pair. Now we are done by applying Proposition 5.1.2 to the slc pair $(X, B + R)$ and deducing that there is a bounded n depending only on $\mathfrak{R}$ such that $n(K_X + B + R) \sim 0$ over t.

Step 3: Now we deal with the case, where T is a single point. The proof is exactly the same as in the case where T is a semi-normal curve, except that we only have Case 1 and Case 2. $\square$

5.3 Proof of Theorem 1.8.6

Finally, we are ready to prove the last main theorem of this thesis.

Proof of Theorem 1.8.6. The strategy follows the proof of [Bir19, Proposition 8.1]. The general idea is to create complements on the sdlt surfaces and then apply injectivity theorem to lift complements to the total space. We proceed in several steps as in [Bir19, Proposition 8.1].

Step 1: In this step, we define some birational models of X and set some notations.
Let $f\colon X'' \to X$ be a log resolution of the pair (X, B) and let $X' \to X$ be the corresponding $\mathbb{Q}$-factorial dlt model. We can assume that (X, B) is strictly lc over t, hence we can assume that a log canonical place, whose center on T is t, is extracted. Furthermore, we can assume that $X'' \dashrightarrow X'$ is a morphism. Let (X', B') and (X'', B'') denote the pullbacks of (X, B) on X' and X'', respectively. We can further assume that $W'' := \lfloor B''^{\geq 0} \rfloor$ and $W' := \lfloor B'^{\geq 0} \rfloor$ are birational, i.e. all components of $\lfloor B''^{\geq 0} \rfloor$ map birationally to its image on $\lfloor B'^{\geq 0} \rfloor$. We will use the notation that if Ω'' is a divisor on X'', then Ω' and Ω will denote the push-forwards on X' and X, respectively. By assumption, $N := -(K_X + B)$ is ample over T, hence, $N'' := -(K_{X''} + B'')$ is nef and big over T. Also define $\Delta'' := B'' - W''$, and $S'' := W'' - \pi_*^{-1}\lfloor B \rfloor$. Observe that S'', W'' are integral Weil divisors on X'', and therefore they are Cartier since X'' is smooth.

Step 2: In this step, we show that $S' \to T'$ is a contraction, where $T' \subset T$ denotes the image of S' in T.

Note that we have the following exact sequence

$$0 \to \mathcal{O}_{X'}(-S') \to \mathcal{O}_{X'} \to \mathcal{O}_{S'} \to 0.$$

We get the following sequence

$$\phi_*\mathcal{O}_{X'} \to \phi_*\mathcal{O}_{S'} \to R^1\phi_*\mathcal{O}_{X'}(-S')$$

that is exact in the middle, where we set $\phi\colon X' \to T$. By Lemma 5.1.9, we can write

$$\begin{aligned} -S' &= K_{X'} + B' - S' + N' \\ &\sim_{\mathbb{Q},T} K_{X'} + B' - S' + (1-\epsilon)N' + \epsilon A' + \epsilon D', \end{aligned}$$

where A' is ample over T, and D' is an effective divisor that is semi-ample over T outside of $\mathrm{Ex}(X' \to X)$. By the property of $\mathbb{Q}$-factorial dlt models, we have that all the log canonical centers of (X', B') that are contained in $\mathrm{Ex}(X' \to X)$ are contained in S'. Therefore, if we pick $0 < \epsilon \ll 1$, by Lemma 5.1.9, the pair $(X', B' - S' + \epsilon D')$ is dlt. Moreover, since $\epsilon A'$ is ample over T, we may pick δ small enough such that $(X', B' - S' - \delta\lfloor \pi_*^{-1} B \rfloor + \epsilon D')$ is klt, and $\epsilon A' + \delta\lfloor \pi_*^{-1} B \rfloor$ is ample over T. Hence, we may write

$$-S' \sim_{\mathbb{Q},T} (K_{X'} + B' - S' - \delta\lfloor \pi_*^{-1} B \rfloor + \epsilon D') + ((1-\epsilon)N' + \epsilon A' + \delta\lfloor \pi_*^{-1} B \rfloor),$$

where the first summand is log divisor of a klt pair, and the second one is a divisor that is ample over T. Applying the relative version of Kawamata–Viehweg vanishing, we conclude that

$$R^1\phi_*\mathcal{O}_{X'}(S') = 0.$$

Thus, $\phi_*\mathcal{O}_{X'} \to \phi_*\mathcal{O}_{S'}$ is surjective. Let $S' \to S'_0 \to T$ be the Stein factorization of $S' \to T$, and write $\phi_0 \colon S'_0 \to T$ for the induced morphism. Then we have that $\phi_*\mathcal{O}_{X'} \to \phi_*\mathcal{O}_{S'} = \phi_{0*}\mathcal{O}_{S'_0}$ is surjective. The morphism $\mathcal{O}_T \to \phi_{0*}\mathcal{O}_{S'_0}$ factors as $\mathcal{O}_T \to \mathcal{O}_{T'} \to \phi_{0*}\mathcal{O}_{S'_0}$. Hence, we conclude that $\mathcal{O}_{T'} \to \phi_{0*}\mathcal{O}_{S'_0}$ is surjective. Since $S'_0 \to T'$ is finite, then $\mathcal{O}_{T'} \to \phi_{0*}\mathcal{O}_{S'_0}$ is indeed an isomorphism. Hence, $S'_0 \to T'$ is an isomorphism. Therefore we can conclude that $S' \to T'$ is a contraction.

Step 3: In this step, we consider adjunction and complements on S'.
By adjunction [Xu19a, 3.7.1], we can define a semi-dlt surface via

$$(K_{X'} + B')|_{S'} = K_{S'} + B_{S'}.$$

By Remark 4.3.2, this pair satisfies the conditions of Proposition 5.2.2. Indeed, by [Xu19a, 3.7.1], there exists a finite set of rational numbers $\mathcal{S} \subset [0,1]$, only depending on $\mathcal{R}$, such that the coefficients of $B_{S'}$ belong to $\Phi(\mathcal{S})$. Therefore, by potentially replacing $\mathfrak{R}$, we can apply Proposition 5.2.2. Hence, by Proposition 5.2.2, $(S', B_{S'})$ has a bounded n-complement $B^+_{S'} = B_{S'} + R_{S'}$ over $t \in T$, i.e., $n(K_{S'} + B_{S'} + R_{S'}) \sim_T 0$, after possibly shrinking around $t \in T$. Fix n for the rest of the proof. Up to taking a bounded multiple only depending on n, we may assume that $I(\mathcal{R})$ divides n. From now on the goal is to lift this complement to X'.

Step 4: In this step, we introduce some line bundles on X'' that are suitable for the use of vanishing theorems.
On X'', consider the integral, hence Cartier divisor

$$L'' := -nK_{X''} - nW'' - \lfloor (n+1)\Delta'' \rfloor.$$

The choice is motivated as follows: our goal is to lift the complement $B^+_{S'}$ from S' to X'. Since X' may be singular, we need to work on the smooth model X''

to use the appropriate vanishing theorems. Observe that we may write

$$\begin{aligned}
L'' =& - nK_{X''} - nW'' - \lfloor (n+1)\Delta'' \rfloor \\
=& K_{X''} + W'' + (n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor - (n+1)K_{X''} - (n+1)W'' - (n+1)\Delta'' \\
=& K_{X''} + W'' + (n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor + (n+1)N'' \\
=& K_{X''} + B'' + n\Delta'' - \lfloor (n+1)\Delta'' \rfloor + (n+1)N'' \\
=& n\Delta'' - \lfloor (n+1)\Delta'' \rfloor + nN''.
\end{aligned}$$

Hence, we can write

$$L'' - S'' = K_{X''} + (W'' - S'') + (n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor + (n+1)N''.$$

Step 5: In this step, we introduce divisors Φ'' and Λ'' on X'' and study their properties.
Let Φ'' be the unique integral divisor on X'' so that

$$\Lambda'' := (W'' - S'') + (n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor + \Phi''$$

is a boundary, (X'', Λ'') is dlt, and $\lfloor \Lambda'' \rfloor = W'' - S''$. We note that Φ'' exists and is unique since all negative coefficients of $(W'' - S'') + (n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor$ are in the range $(-1, 0)$. By the choices of X'' and X', it follows that Φ'' is supported on $\mathrm{Ex}(X'' \to X')$ and shares no components with W''.

Step 6: In this step, we apply Theorem 5.1.6 to $L'' - S'' + \Phi''$.
Recall that N'' is semi-ample over T. Let F'' be an effective divisor on X'' that is exceptional and anti-ample for $X'' \to X$. Hence, $N'' - \epsilon F''$ is ample over T for $0 < \epsilon \ll 1$. Observe that (X'', Λ'') is a log smooth pair. By the choice of X'', $(X'', \mathrm{Supp}(\Lambda'' + S'' + F''))$ is also log smooth. Since $\lfloor \Lambda'' \rfloor = W'' - S''$, $\mathrm{Supp}(S'' + F'')$ contains no log canonical center of (X'', Λ''). Notice that we have

$$L'' - S'' + \Phi'' = K_{X''} + \Lambda'' + (n+1)N''.$$

To apply Theorem 5.1.6, we are left with checking that the third condition of the statement holds. Fix $0 < \delta \ll 1$, so that $N'' - \epsilon F'' - \delta S''$ is ample over T. Then, we may write $N'' - \epsilon F'' - \delta S'' \sim_{\mathbb{Q},T} G'' \geq 0$, where G'' contains no log canonical center of (X'', Λ''). Hence, we have a $\mathbb{Q}$-linear relation

$$N'' \sim_{\mathbb{Q},T} G'' + \epsilon F'' + \delta S'',$$

where $G'' + \epsilon F''$ is an effective divisor that does not contain any log canonical center of (X'', Λ''). Also $\mathrm{Supp}(S'')$ also doesn't contain any log canonical center of (X'', Λ''). Thus, by Theorem 5.1.6, we deduce that there is an injection

$$R^1\psi_*\mathcal{O}_{X''}(L'' - S'' + \Phi'') \to R^1\psi_*\mathcal{O}_{X''}(L'' + \Phi'').$$

Here, ψ denotes the morphism $X'' \to T$. Therefore we have a surjection

$$\psi_*\mathcal{O}_{X''}(L'' + \Phi'') \to \psi_*\mathcal{O}_{X''}((L'' + \Phi'')|_{S''}).$$

Since the surjectivity of a morphism between sheaves implies the surjectivity on a sufficiently small local neighbourhood. After potentially shrinking T around $t \in T$, we have

$$H^0(\mathcal{O}_{X''}(L'' + \Phi'')) \to H^0(\mathcal{O}_{S''}((L'' + \Phi'')|_{S''}), \tag{5.3.1}$$

as desired.

Step 7: In this step, we introduce some divisors on S''.
In Step 3, we constructed an n-complement $B^+_{S'} = B_{S'} + R_{S'}$ for $(S', B_{S'})$ over $t \in T$. Notice that $R_{S'}$ is a $\mathbb{Q}$-Cartier divisor not containing any irreducible component of the conductor of $(S', B_{S'})$. We have a birational morphism of possibly reducible algebraic varieties $S'' \to S'$. Furthermore, by construction, every irreducible component of S'' maps birationally onto its image in S'. Therefore, $R_{S'}$ does not contain the image of any component of S'' on S', and its pull-back $R_{S''}$ on S'' is well-defined. Now, we have

$$n(K_{S''} + B_{S''} + R_{S''}) \sim_T 0.$$

By construction, we have $B_{S''} = (B'' - S'')|_{S''}$, and the restriction preserves the coefficients, as we are in a log smooth setting. Removing the contribution of $(W'' - S'')|_{S''}$, which is integral, we realize that $n(\Delta_{S''} + R_{S''})$ is integral, where we have $\Delta_{S''} := \Delta''|_{S''}$. We define

$$G_{S''} := nR_{S''} + n\Delta_{S''} - \lfloor (n+1)\Delta_{S''} \rfloor + \Phi_{S''},$$

where we have $\Phi_{S''} := \Phi''|_{S''}$. By definition, $G_{S''}$ is an integral divisor, and $nR_{S''} + \Phi_{S''}$ is effective. We claim that $G_{S''}$ is effective. Indeed, it suffices to show that the coefficients of $n\Delta_{S''} - \lfloor (n+1)\Delta_{S''} \rfloor$ are strictly greater than -1 as $G_{S''}$ is integral. Since we are in the log smooth case, we may write

$$n\Delta_{S''} - \lfloor (n+1)\Delta_{S''} \rfloor = ((n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor - \Delta'')|_{S''},$$

where the summand $(n+1)\Delta'' - \lfloor (n+1)\Delta'' \rfloor$ is effective. As the coefficients of Δ'' are strictly less than 1 by construction, it follows that the coefficients of $-\Delta''$ are strictly greater than -1. In particular, $G_{S''}$ is effective.

Step 8: In this step, we lift $G_{S''}$ to X''.
We have $N_{S''} := N''|_{S''} = -(K_{X''} + B'')|_{S''} = -(K_{S''} + B_{S''})$. Then, it follows that $nR_{S''} \sim_T nN_{S''}$. By shrinking T around t, in the following we may drop T in the linear equivalence. In particular, we have $nR_{S''} \sim nN_{S''}$ and therefore, by definition, we have

$$0 \leq G_{S''} \sim nN_{S''} + n\Delta_{S''} - \lfloor (n+1)\Delta_{S''} \rfloor + \Phi_{S''}.$$

Then, observe that in step 4, we showed

$$L'' := n\Delta'' - \lfloor (n+1)\Delta'' \rfloor + nN''.$$

Therefore, we have

$$L_{S''} := L''|_{S''} = n\Delta_{S''} - \lfloor (n+1)\Delta_{S''} \rfloor + nN_{S''}.$$

Hence, we conclude that

$$0 \leq G_{S''} \sim L_{S''} + \Phi_{S''}.$$

Thus, by the surjectivity of Equation (5.3.1), there exists $0 \leq G'' \sim L'' + \Phi''$ on X'' such that $G''|_{S''} = G_{S''}$ and G'' is integral.

Step 9: In this step, we study G', the push-forward of G'' to X', and we introduce $(B')^+$,which we will soon show that it is a complement for (X', B'). By the definition of L'', we get

$$0 \leq G'' \sim -nK_{X''} - nW'' - \lfloor (n+1)\Delta'' \rfloor + \Phi''.$$

Let G' be the push-forward of G'' to X'. Then, as Φ'' is exceptional for $X'' \to X'$, we have

$$0 \leq G' \sim -nK_{X'} - nW' - \lfloor (n+1)\Delta' \rfloor. \tag{5.3.2}$$

Then, we can define

$$nR' := G' + \lfloor (n+1)\Delta' \rfloor - n\Delta' \sim -n(K_{X'} + B'),$$

where the linear equivalence follows from Equation (5.3.2). By assumption, the coefficients of Δ' are in $\Phi(\mathcal{R})$, it follows from easy arithmetic that nR' is indeed effective. Then, we can define $(B')^+ := B' + R'$. Notice that by construction, we have that $n(K_{X'} + B'^+) \sim 0$. Therefore, it suffices to show (X', B'^+) is lc over $t \in T$.

Step 10: In this step, we will show that (X', B'^+) is log canonical by applying connectedness principle.
First, we show that $R'|_{S'} = R_{S'}$.
Let

$$nR'' := G'' - \Phi'' + \lfloor (n+1)\Delta'' \rfloor - n\Delta'' \sim L'' + \lfloor (n+1)\Delta'' \rfloor - n\Delta'' = nN'' \sim_{\mathbb{Q},X} 0.$$

As R'' pushes forward to R', it follows that R'' is the pull-back of R'. Observe that $R''|_{S''} = R_{S''}$. Hence, we have $R'|_{S'} = R_{S'}$. This implies the equality

$$K_{S'} + B_{S'} + R_{S'} = (K_{X'} + B' + R')|_{S'} = (K_{X'} + (B')^+)|_{S'}.$$

Therefore, by inversion of adjunction [Xu19a, Lemma 3.8], the pair $(X', (B')^+)$ is log canonical in a neighbourhood of S'. If $(X', B' + R')$ is not log canonical in a neighbourhood of $\phi^{-1}(t)$, then we can write $\mathrm{Nklt}(X', B' + R') = \mathrm{Supp}(\lfloor B' \rfloor) \cup Z_1' \cup Z_2'$, where Z_1' is a union of log canonical centers, and $\mathrm{Nlc}(X', B' + R') = Z_2'$. As $(X', (B')^+)$ is log canonical in a neighborhood of S', we have $\mathrm{Supp}(S') \cap Z_2' = \emptyset$. Then, fix $0 < \alpha \ll 1$, so that $\mathrm{Nklt}(X', B' + (1-\alpha)R') = \mathrm{Supp}(\lfloor B' \rfloor) \cup Z_2'$, and $\mathrm{Nlc}(X', B' + (1-\alpha)R') = Z_2'$. We notice that

$$-(K_{X'} + B' + (1-\alpha)R') \sim_{\mathbb{Q},T} \alpha R' \sim_{\mathbb{Q},T} -\alpha(K_{X'} + B')$$

is nef and big over T. Also by assumption, we have that $\mathrm{Nklt}(X', B' + (1-\alpha)R')$ is disconnected along $\phi^{-1}(t)$. Therefore, since $\dim X = 3 < 4$, we can apply [HH19, Theorem 1.2], and derive that $(X', B' + (1-\alpha)R')$ is plt in a neighbourhood of $\phi^{-1}(t)$ which is a contradiction since $(X', B' + (1-\alpha)R')$ is not lc at Z_2. Therefore this shows that $(X', B' + R')$ is log canonical along $\phi^{-1}(t)$, which concludes the proof. □

5.4 Some Corollaries and Consequences

Here we state some applications of the main result in this chapter. In particular, we note that the relative case included the identity case and the question about complements becomes the question about local index for strictly log canonical singularities. Therefore, we have the following.

Corollary 5.4.1. *Let $\mathfrak{R} \subset [0,1]$ be a finite set of rationals numbers. There exists a natural number n only depending on $\mathfrak{R}$ which satisfies the following. Let X be a projective normal quasi-projective varieties, $(x \in X)$ a closed point.*

- *$(x \in X, B)$ is a log canonical 3-fold,*
- *$(x \in X, B)$ is strictly log canonical near x,*
- *the coefficients of B belong to $\Phi(\mathfrak{R})$.*

Then, perhaps after shrinking X near x, $n(K_X + B) \sim 0$.

Proof. We apply Theorem 1.8.6 in the case where $T = X$ and $f : X \to T$ is the identity map. Notice that since (X, B) is strictly log canonical near x, any local n-complement is disjoint from $x \in X$. Therefore, by shrinking X near x, we see that there is a bounded n, depending only on $\mathfrak{R}$, such that $n(K_X + B) \sim 0$. $\square$

5.5 Further Research Questions

Here we briefly list some related questions in the field:

- This thesis mainly deals with the threefold case. Therefore it is natural to try to generalise the approach to higher dimension. However as shown in Example 1.3.2, the boundedness of complements for global log canonical Fano variety requires the boundedness of canonical index for klt Calabi-Yau with $K_X \sim_{\mathbb{Q}} 0$. Therefore, one may attempt to look for a result in the theory of complements in this direction in all dimensions assuming the boundedness of certain canonical index for klt Calabi-Yau varieties.

- Following the above point, another natural question to consider is the boundedness of canonical index for klt Calabi-Yau with $K_X \sim_{\mathbb{Q}} 0$. This conjecture seems to have relationship with ACC for mld for Calabi-Yau when X is not canonical. In the case where X is canonical or terminal, maybe the use of analytic techniques is better suited to tackle the problem.

- Another interesting topic is the boundedness of B-representations for Calabi-Yau manifolds in higher dimension. This seems hard and is somehow inter-related to the index conjecture for Calabi-Yau varieties.

- The effective-canonical bundle formula in higher dimension is also of great interest, in particular, the control the coefficients of the discriminant part and control the base point free index of the moduli part. These conjectures in higher dimension are mostly open in the case of Calabi-Yau but non Fano type fibration. Understanding a good version of effective canonical bundle formula is essential for further study of complements in higher dimension.

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