" NF "

Thomas Edward Forster

Pembroke College

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Preface

NF is an unpopular system, and no wonder. However, the criticism usually made of it is ideological rather than mathematical - it is said that NF has no standard model. This goes back to a misleading remark of Rosser and Wang [41] where they also point out that in NF there are classes of Russell-Whitehead ordinals without least members, which proves conclusively that Russell-Whitehead ordinals are not entirely suitable objects with which to conduct ordinal arithmetic in NF. It does not seem to establish much else. A "standard model" is a second-order, alias platonistic, notion, and nobody would recognise one if they saw it any more than anyone will ever know what the power of the continuum is (whatever that means). It seems rather unfair to ignore NF on philosophical grounds when philosophical considerations play so little part in modern, technical, set theory. *

And the technical reasons for studying NF <u>are</u> good: it continually reminds us of the arbitrariness of mathematics and is good practice for formally-minded logicians. With NF ω -incompleteness is an everyday fact of life, not a remote theoretical event on the boundaries of the system as it is in ZF. There are no easy appeals to AC.

NF is said to be counterintuitive, but what does this add up to? 'Intuition' after all, is just the name we give to our way of thinking about our subject - and that is at least as much a result of what we have been taught as of what we were born with (if anything) What mathematicians actually <u>mean</u> (or ought to mean) when they say NF is counterintuitive is that it is not what they are used to. This is not leading up to a grand plea that NF is in any way better, or to use NF as <u>the</u> set theory: notoriously it makes no difference whatever to group theorists, analysts, number theorists topologists etc what set theory they use. No mathematician ever uses set theory because it is useful - it isn't: he/she uses it because it is fun. And NF is just as much fun as ZF

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Chapter 1 is devoted to the cardinal arithmetic of NF. The two main theorems concern consequences of Rosser's Axiom of counting ("Axcount" here) and the curious behaviour of large cardinals with respect to exponentiation.

Chapter 2 is concerned with the model theory of NF and the consistency problem in particular. The two main approaches to it (Scott [42] and Specker [45]) are synthesised and a number of partial results obtained in the direction of Con(NF) - e.g., the consistency of Σ_1^{TST} - ambiguity. Also a new product construction is introduced, a hybrid between possible world semantics and ultraproducts which can be used to give consistency results for certain intuitionist subsystems of NF with the existence of fixed points for a number of inhomogeneous functions.

Chapter 3 discusses speculatively some natural possible ways of enriching NF and examines some consequences of each of the proposals.

The bibliography lists all material on NF known to me whether it is referred to in the body of the thesis or not. A separate bibliography contains other technical literature not directly related to NF but referred to here.

All results are original unless otherwise stated, at any rate, all but one of the proofs are new (Boffa produced a nicer proof of a result of mine which appears here). Prior proofs of results are acknowledged when known to me. Results quoted but without references arise from the activities of the Séminaire Hénéfiste.

I would like to thank here those people whose comments have not been acknowledged in the text, Professor D.S.Scott especially, who, by finding a mistake in an alleged proof of mine of $\sim$Con(NF) made this thesis possible. I would like to thank the other members of the Séminaire Hénéfiste in Belgium for their helpful comments, and also the other members of the Cambridge Logic Seminar.

The Science Research Council kept me for one year of my researches, and my parents for the other two: I am deeply grateful to them.

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* There are some further rude comments in this vein in Lake [21]

I would like to thank Dr. A.R.D. Mathias, my supervisor, for going through the final version of this dissertation with me and suggesting a number of improvements and spotting a number of mistakes.

In affectionate

memory

Marcus Dick † 1971

Catherine Sinton † 1969

who first made me love two of the things that have given me most pleasure: Mathematical Logic and Aotearoa.

Table of Contents

Chapter 0

Definition 0.1 $V =_{df} \{ x : x = x \}$

$\Lambda =_{df} \{ x : x \neq x \}$

Definition 0.2 $\iota' x =_{df} \{x\}$

This gives rise by recursion to $\iota^{n}{}' x$.

Also to $\iota'' x = \{ \iota' y : y \in x \}$

Definition 0.3 $f' x =_{df} (\iota y)(\langle y,x \rangle \in f)$ when f is a variable ranging over functions. This notation will also be used when f is a constant for a class abstract by a mild abuse of notation.

Definition 0.4 $p' x =_{df} \{ y : y \subseteq x \}$

Definition 0.5 $\bar{\bar{x}} =_{df} \{ y : y$ is the same size as $x \}$

$\bar{\bar{x}}$ is a <u>cardinal</u>

Definition 0.6 ${}^{x}y =_{df}$ set of functions from x into y

Definition 0.7 $NC =_{df} \{ \bar{\bar{x}} : x \in V \}$

Definition 0.8 $No'<$ when $<$ is a wellordering is the set of all R orderisomorphic to $<$

Definition 0.9 $NO =_{df} \{ No'< : <$ is a wellordering $\}$

Elements of NO are ordinals

Definition 0.10 $card'\alpha$ for $\alpha \in NO$ is the cardinal of any x such that some wellordering $<$ of x is $\in \alpha$

Definition 0.11 $WC =_{df} \{ card'\alpha : \alpha \in NO \}$

Definition 0.12 $T\overline{\overline{x}} =_{df} \overline{\overline{\iota''x}}$

Definition 0.13 $2^{\overline{\overline{\iota''x}}} =_{df} \overline{\overline{p'x}}$ (Note that 2^{α} is not defined for $\alpha \nleq \overline{\overline{\iota''V}}$

Definition 0.14 $\alpha \leqslant_{*} \beta$ for $\alpha, \beta \in NC$ is to be read.. $x \in \alpha \wedge y \in \beta . \supset . \exists$ map from y onto x.

Definition 0.15 $R|S$ where R and S are relations is $\{ \langle x,y \rangle : (\exists z)(\langle z,y \rangle \in S \wedge \langle x,z \rangle \in R)\}$

Definition 0.16 Beth numbers $\beth_0{}'\alpha = \alpha$.
$\beth_{n+1}{}'\alpha = 2^{\beth_n{}'\alpha}$ $\beth_n$ is $\beth_n{}'\aleph_0$
$\beth_{ult}{}'\alpha$ is the last member of $\varphi'\alpha$ when $\varphi'\alpha$ is finite. (see below)

Definition 0.17 $\varphi'\alpha =_{df} \{ \beta : (\exists n)(\beta = \beth_n{}'\alpha)\}$

Definition 0.18 When α is a cardinal, $\alpha + 1 =_{df} \{ x : (\forall y \in x)(x - \{y\} \in \alpha)\}$

Definition 0.19 $0 =_{df} \{\Lambda\}$ (i.e., $\overline{\overline{\Lambda}}$)

Definition 0.20 $Nn =_{df} \bigcap\{x \subseteq NC : 0 \in x \wedge (\forall m)(m \in x . \supset . m + 1 \in x)\}$

Definition 0.21 $SM =_{df} \{ \alpha : \varphi'\alpha \in Nn\}$ This set was first characterised by Specker [44] and the initials are intended to recall "<u>S</u>pecker<u>M</u>enge".

Definition 0.22 $can(x) \equiv_{df} \overline{\overline{x}} = T\overline{\overline{x}}$

Definition 0.23 $stcan(x) \equiv_{df} \iota\restriction x$ exists. Here ι is the class abstract, the singleton function, as in definition 0.1

The predicates stcan( ) and can( ) will both be applied to cardinals and ordinals by an abuse of notation to mean that the underlying sets are can or stcan.

Definition 0.24 When R is a relation RUSC'R $=_{df}$ $\{ \langle \{x\}, \{y\} \rangle : \langle x,y \rangle \varepsilon R \}$

When R is a wellordering and No'R = α, I shall often write $T\alpha$ to be No'RUSC'R. This is a sufficiently mild abuse of definition 0.12 not to cause any confusion.

Definition 0.25 $^{-}x =_{df} \{ y : y \not\varepsilon x \}$

Definition 0.26 $\aleph$'α (for α ε NC) $=_{df}$ least β ε WC such that $\beta \not\leq \alpha$.

Definition 0.27 μ is a 'least' operator on wellfounded structures.

Definition 0.28 $x \subseteq_f y \equiv_{df} (\exists w \subseteq x)(\bar{\bar{w}} \varepsilon Nn \wedge x - w \subseteq y)$

Definition 0.29 When δ is an n-place relation $\ulcorner \delta(\vec{x}) \urcorner$ is an abbreviation for $\ulcorner \delta(x_1 \ldots x_n) \urcorner$

Definition 0.30 If R is a relation J'R $=_{df}$ $\{ \langle x,y \rangle : x = R''y \}$

Although NF is a <u>set</u> theory I shall sometimes in an informal fashion use the notation $\ulcorner \hat{x}(\ldots) \urcorner$ to do duty for the <u>class</u> of all x such that (....) when $\ulcorner (\ldots) \urcorner$ is unstratified. $\ulcorner \hat{x}\hat{y}(\ldots) \urcorner$ is a class notation for relational abstraction

Definition 0.31

$\text{Axcount} \equiv_{df} (\forall n \in Nn)(n = Tn)$ (Rosser [39])

$\text{Axcount}_{<} \equiv_{df} (\forall n \in Nn)(n < Tn)$

$\text{Axcount}_{>} \equiv_{df} (\forall n \in Nn)(n > Tn)$

NFC is NF + Axcount.

Except where otherwise stated, ordered pairs are Quine ordered pairs as in [27].

Definition 0.32 $\mathbb{N} =_{df} \{ \alpha \in NC : \varphi'\alpha \in n + 1 \}$

(written this way it looks as though $\ulcorner \mathbb{N} \urcorner$ is the result of a definition scheme. It is rather just a shorter way of writing $\ulcorner \gamma'n \urcorner$ if we have defined $\gamma'n$ to be $\{ \alpha \in NC : \varphi'\alpha \in n + 1 \}$. This makes sense of quantification over the $\mathbb{N}$ which will be necessary later on.)

The main content of this chapter is the proof of the arrows in the diagram on p.11. However, there are a number of preparatory lemmas and remarks that need to be got out of the way first.

When α is a cardinal, $\tau'\alpha$ is to be the tree generated by α as follows: $\tau'\alpha = \langle \{\beta : \alpha \in \phi'\beta\} , \{\langle\beta,\gamma\rangle : \alpha \in \phi'\beta \wedge \phi'\gamma \wedge \beta \in \phi'\gamma\}\rangle$

We shall need a rank function defined on these trees, ρ and we shall need to show that ρ is wellfounded. In ZF it is trivial to define a rank function on wellfounded structures. It is possible in NF too, though this is not obvious, as NO is a set and so we must eliminate the possibility that $\tau'\alpha$ is so big that we run out of ordinals. We want to define a function ρ from $\tau'\alpha$ into NO such that $\rho'\beta = \sup\{\rho'\gamma + 1 \ : 2^{\gamma} = \beta\}$ and $\rho'\beta = 1$ iff $\sim(\exists\gamma)(\beta = 2^{\gamma})$. Such a function we can readily construct by recursion. Call it ρ_1. $\ulcorner \alpha = \rho_1{}'\beta \urcorner$ is stratified with α, β having the same type. That is, ρ_1 is a set. We can define $\rho_2, \rho_3 \ldots$ by $\rho_{n+1}{}'\alpha = \rho_n{}'T\alpha$ for each concrete n. It is easy to verify that each ρ_n behaves like a rank function.

If ρ_1 is not defined for some $\tau'\alpha$ because we run out of ordinals this implies that $\overline{\overline{NO}} \leqslant_* {}^{=}(\tau'\alpha)$. This is not obviously impossible. But if ρ_4 is undefined for some $\tau'\alpha$ because we run out of ordinals we have similarly $NO \leqslant_* (\tau' T^4\alpha)$. Now $\tau' T^4\alpha$ is a subset of T^4"NC so we infer $\overline{\overline{NO}} \leqslant_* T^4\overline{\overline{V}}$ This implies $T\overline{\overline{NO}} \leqslant_* T^5\overline{\overline{V}}$ and therefore

$$2^{T\overline{\overline{NO}}} \leqslant T^4\overline{\overline{V}} \qquad \text{but}$$
$$T\overline{\overline{NO}} < 2^{T\overline{\overline{NO}}} \qquad \text{so}$$
$$T\overline{\overline{NO}} < T^4\overline{\overline{V}} \qquad \text{and} \quad NO < T^3V$$

This is impossible, for we can prove by induction on the alephs that if α is an aleph then $T^2\alpha = {}^{=}\{\beta \in NO : \text{card}'\beta = \alpha\}$. Now if $\overline{\overline{NO}} < T^3V$, $\overline{\overline{NO}}$ is $T^2\alpha$ for some aleph α which is not the last aleph (as $\alpha < T\overline{\overline{V}}$). So NO would have to be the same size as some initial segment of itself bounded by an ordinal which is not the last initial ordinal, which is impossible. This establishes that at any rate ρ_4 is defined everywhere as long as $\tau'\alpha$ is always wellfounded. This is what we have to show next. From now on I shall write "$\rho'\alpha$ " instead of "$\rho_4{}'\tau'\alpha$ " (Even though ρ_4 does not exist as a set of ordered pairs, this functional notation does not lead to abuses) We shall need a theorem of Sierpinski's

Proposition 1.0

$$(\forall\alpha \in NC)(2^{2^{2^{\alpha}}} \text{ exists.} \supset . \aleph'\alpha < 2^{2^{2^{\alpha}}})$$

Sierpinski's theorem is provable in ZF without any use of choice. This works in NF as well, but I shall reproduce the proof translated into NF because there are complexities such as the additional premisses required which prevent the provability from being obvious.

Proof:

Fix $x \in \alpha$. The idea is to code wellorderings of subsets of x by the sets of their initial segments. The set of initial segments of a wellordering of a subset of x is a subset of p'x. Thus

$$T\overline{\overline{\{ R : R \text{ wellorders a subset of } x \}}} \leqslant \overline{\overline{p^{2}{}'x}}$$

The object on the left hand side is also the cardinal of

$$\{ \{R\} : R \text{ wellorders a subset of } x \}$$

and we can project this many-one down onto $\{ \beta \in NO : \text{card}'\beta < \aleph'\alpha \}$ by sending $\{R\}$ to the ordinal of which it is a subset. Now $\{ \beta \in NO : \text{card}'\beta < \aleph'\alpha \}$ is of power $T^2\aleph'\alpha$ so this gives

$$T^2\aleph'\alpha \leqslant_* \overline{\overline{p^{2}{}'x}} = 2^{2^{T^2\alpha}} = T^2 2^{2^{\alpha}} \text{ if } 2^{2^{\alpha}} \text{ exists.}$$

Assuming this, we have $\aleph'\alpha \leqslant_* 2^{2^{\alpha}}$

so $\quad T\aleph'\alpha \leqslant_* T2^{2^{\alpha}}$

and $\quad 2^{T\aleph'\alpha} \leqslant 2^{T2^{2^{\alpha}}}$

If $2^{2^{2^{\alpha}}}$ exists, we can rewrite this last line as

$$2^{T\aleph'\alpha} \leqslant T2^{2^{2^{\alpha}}}$$

but anyway $T \aleph'\alpha < 2^{T \aleph'\alpha}$

so $T \aleph'\alpha < T2^{2^{2^{\alpha}}}$

so $\aleph'\alpha < 2^{2^{2^{\alpha}}}$

Lemma 1.1 $(\forall\alpha \in NC)(\mathcal{T}'\alpha$ has no subtree without endpoints)

Proof. Suppose not, and that α is a counterexample, with $[\alpha]$ a subtree without endpoints of $\mathcal{T}'\alpha$. Define $[\alpha]_n$ to be $\{\ \beta \in [\alpha] : \alpha = \beth^{n}{}'\beta\ \}$. Consider the function $f : n \longmapsto \inf(\aleph''[\alpha]_n)$. Let π be the least aleph in the range of f and n_0 the least n such that $f'n = \pi$. Then $\pi = \aleph'\beta$ for some $\beta \in [\alpha]_{n_0}$. But, as $[\alpha]$ is perfect, $\beta = 2^{2^{2^{\gamma}}}$ for some $\gamma \in [\alpha]_{n_0} + 3$ and $\aleph'\gamma < \aleph'\beta$ by Proposition 1.0 so $f'n + 3 < f'n$, contradicting wellfoundedness of the alephs.

Lemma 1.2 (i) $\rho'T\alpha \geqq T\rho'\alpha$

(ii) $\alpha \leqslant \overline{\overline{\iota''V}}. \supset. \ \rho'T\alpha = T\rho'\alpha$

Proof:

When $\alpha \leqslant \overline{\overline{\iota''V}}$ T is an isomorphism between $\mathcal{T}'\alpha$ and $\mathcal{T}'T\alpha$ so we can prove (ii) by an easy induction induction on rank. When $\alpha \nleqslant \overline{\overline{\iota''V}}$, T sends $\mathcal{T}'\alpha$ onto a subset of $\mathcal{T}'T\alpha$ which will be proper if $(\exists\beta \nleqslant \overline{\overline{''V}})(2^{T\beta} = T\alpha)$ as $T\beta$ will then be in $\mathcal{T}'T\alpha$ but $\beta \notin \mathcal{T}'\alpha$. Then the rank of the tree $T''\mathcal{T}'\alpha$ will be $T\rho'\alpha$ but as it may be only a subtree of $\mathcal{T}'T\alpha$, $\rho'T\alpha$ may well be bigger.

Trees of cardinals enable one to give short proofs of results in cardinal arithmetic which can otherwise be quite hard to show. e.g.

Proposition 1.3

(i) $\alpha = 2^{T\alpha} . \supset . \ \alpha$ infinite

(ii) $T\alpha = 2^{\alpha} . \supset . \ \alpha$ infinite

(iii) $(\exists x)(\overline{\overline{x}} < \overline{\overline{\iota''V}} \wedge \overline{\overline{p'x}} = \overline{\overline{V}})$

((i) is a footnote in Specker [44] and is a corollary of a result in Boffa [2] and (iii) was proved independently by Henson [15])

Proof:

(i) compare $\rho'T\alpha$ and $\rho'T^2\alpha$. $T\rho'T\alpha = \rho'T^2\alpha$ but $2^{T^2\alpha} = T\alpha$ so $\rho'T\alpha \geqslant \rho'T^2\alpha + 1 = T\rho'T\alpha + 1$. Now $n \neq Tn + 1$ for any ordinal n so $\rho'T\alpha > \rho'T^2\alpha + 1$ so, as $\rho'T\alpha = \sup \{\rho'\beta + 1 : 2^{\beta} = T\alpha\}$ there must be a β such that $2^{\beta} = T\alpha \wedge \rho'\beta > \rho'T^2\alpha$ so the tree of $T\alpha$ branches. This can never happen if α is finite so the conclusion follows.

(ii) $\rho'2^{\alpha} \geqslant \rho'\alpha + 1$ and $\rho'T\alpha = T\rho'\alpha$ because $\alpha \leqslant \overline{\overline{\iota''V}}$. So $T\rho'\alpha \geqslant \rho'\alpha + 1$ so $T\rho'\alpha > \rho'T\alpha$ because $n \neq Tn + 1$ so the tree branches as before so α is infinite as before.

(iii) Apply (i) to get $\beta \neq \overline{\overline{\iota^2{}''V}} \quad 2^{\beta} = \overline{\overline{\iota''V}}$. β is $\overline{\overline{\iota''x}}$ for some x, and any such x satisfies (iii)

The ranking function also enables us to relate the complexity of $\rho'\alpha$ to the size of α. If we define rank as is customary in ZF, as a function sending endpoints to 0 and sending x to $\sup\{\rho'y + 1 : y$ precedes $x\}$ we can prove the following

Remark 1.4

$$(\forall\alpha)(\rho'\alpha \text{ limit}. \supset . \ T^4\aleph'\alpha \geqslant \aleph_{\iota\rho'\alpha})$$

The T^4 is there to make the remark stratified. Proof is then by a trivial induction on rank. We shall need the following observations:

Lemma 1.5

(i) $\alpha \leqslant \beta . \supset . \ \overline{\overline{\varphi'\alpha}} \geqslant \overline{\overline{\varphi'\beta}}$

(ii) $\overline{\overline{\varphi' T\alpha}} \geqslant T \ \overline{\overline{\varphi'\alpha}} + 1$

(iii) $\overline{\overline{\varphi'\aleph'\alpha}} > \overline{\overline{\varphi'\alpha}} - 3$

(iv) $\alpha \ \varepsilon \ \tau'\overline{\overline{V}} . \supset . \ \overline{\overline{\varphi' T\alpha}} = T \ \overline{\overline{\varphi'\alpha}} + 1$

(v) $\alpha = T\alpha . \supset . \ \alpha \not\varepsilon \ \tau'\overline{\overline{V}}$

Proof

(i) We first show by induction on n that $\beth_n'\alpha \leqslant \beth_n'\beta$. For $n = 1$ we are given it by the antecedent. Assume it holds for $n = k$, to prove it holds for $n = k + 1$ we need only $\gamma \leqslant \delta . \supset . \ 2^\gamma \leqslant 2^\delta$ which is trivial. Now let n be such that $\beth_n'\beta = \beth_{ult}'\beta$. Then $\beth_n'\alpha \leqslant \beth_{ult}'\beta$. So the function

$f : \beth_m{}'\beta \mapsto \beth_m{}'\alpha$ for $m < n$ is a 1 - 1 map from $\varphi'\beta$ into $\varphi'\alpha$. (Note that we can still prove (i) if we weaken the premiss to $\alpha \leqslant_* \beta \wedge 2^\alpha$ exists)

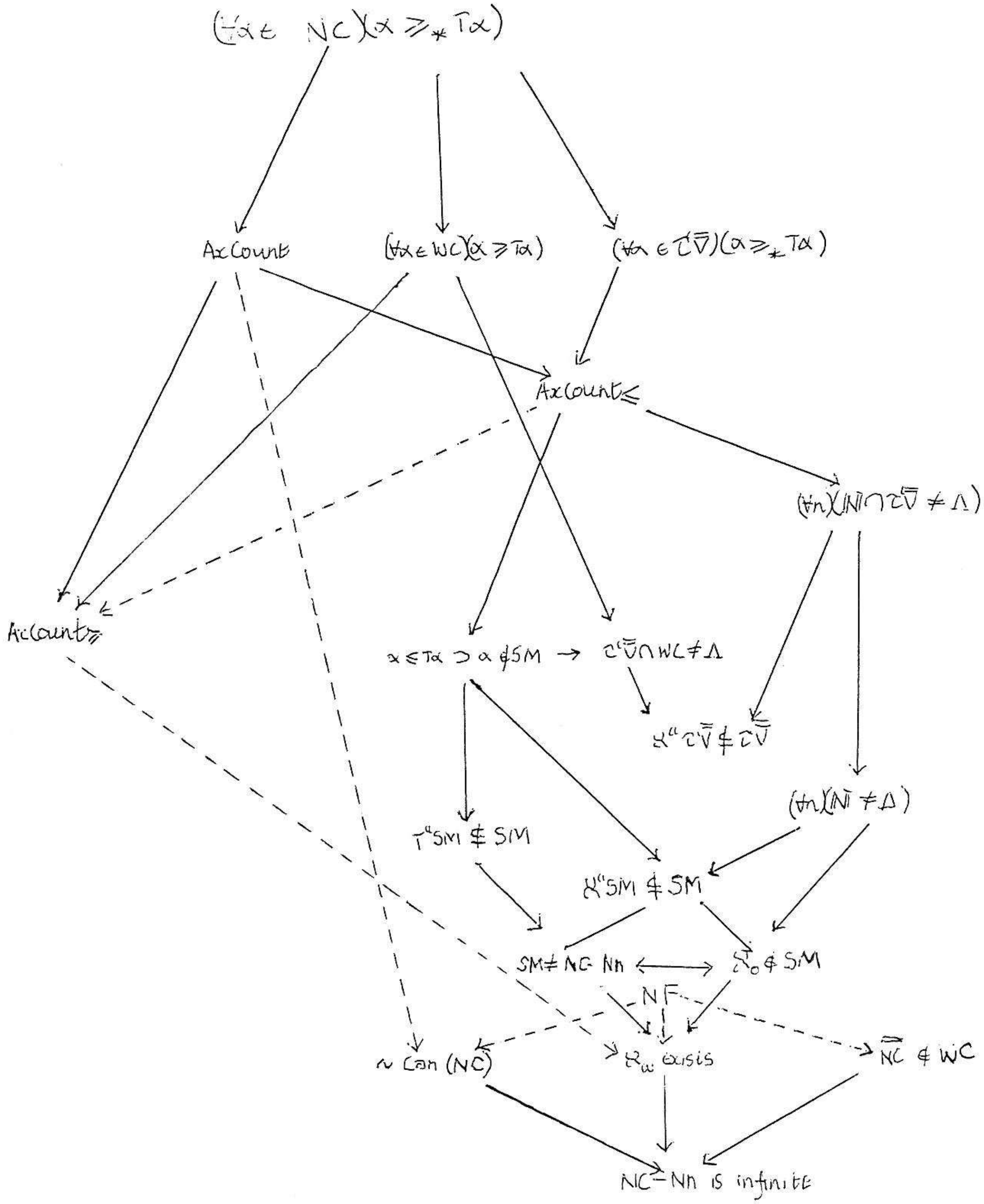
(ii) We prove by induction on n that $\beth_{Tn}{}'T\alpha = T\beth_n{}'\alpha$ as long as the powers exist. So if $\beth_{ult}{}'\alpha$ is $\beth_n{}'\alpha$ then $T\beth_n{}'\alpha = \beth_{Tn}{}'T\alpha$. But this cardinal is $\leqslant \overline{\overline{\iota''V}}$ and so is not $\beth_{ult}{}'T\alpha$. Thus there is at least one more cardinal in $\varphi'T\alpha$ so $\overline{\overline{\varphi'T\alpha}} > T\overline{\overline{\varphi'\alpha}} + 1$.

(iii) Corollary of (i) and Proposition 1.0

(iv) Proof as in (ii), only $\beth_{ult}{}'\alpha = \overline{\overline{V}}$ so $T\beth_{ult}{}'\alpha = \overline{\overline{\iota''V}}$ so T must send $\varphi'\alpha$ onto precisely $\varphi'T\alpha - \{\beth_{ult}{}'T\alpha\}$

(v) Corollary of (iv) Suppose $\alpha \in \mathcal{T}'\overline{\overline{V}} \wedge \alpha = T\alpha$ then $\overline{\overline{\varphi'T\alpha}} = T\overline{\overline{\varphi'\alpha}} + 1 = T\overline{\overline{\varphi'T\alpha}} + 1$ which contradicts the fact that for no ordinal n is n = Tn + 1.

The diagram on the following page constitutes Theorem 1.6



(Dotted arrows are conjectural)

The following arrows are nontrivial

$(\forall n)(\ \mathbb{N} \neq \Lambda) \quad \rightarrow \quad \aleph``SM \not\subseteq SM$

$(\forall n)(\ \mathbb{N} \cap \tau'\overline{\overline{V}} \neq \Lambda) \quad \rightarrow \quad \aleph``\tau'\overline{\overline{V}} \not\subseteq \tau'\overline{\overline{V}}$

Proof:

look at $\inf(\aleph``SM)$ (resp. $\inf(\aleph``\tau'\overline{\overline{V}}$) There can be no last n such that $(\exists\alpha \in \mathbb{N})(\aleph'\alpha = \inf \aleph``SM)$ (resp. $(\exists\alpha \in \mathbb{N} \cap \tau'\overline{\overline{V}})(\aleph'\alpha = \inf \aleph``\tau'\overline{\overline{V}})$ otherwise by Proposition 1.0 the number of nonempty $\mathbb{N}$ (resp. $\mathbb{N} \cap \tau'\overline{\overline{V}}$) is finite. So arbitrarily late $\mathbb{N}$ (resp $\mathbb{N} \cap \tau'\overline{\overline{V}}$) contain cardinals whose alephs are minimal. So, by lemma 1.5 (iii), $\varphi'\inf\aleph``SM$ (resp. $\varphi'\inf\aleph``\tau'\overline{\overline{V}}$) must be arbitrarily large.

$(\forall\alpha)(\alpha \leqslant T\alpha . \supset . \alpha \notin SM) \quad \rightarrow \quad T``SM \not\subseteq SM$

Proof:

Look at inf $(SM \cap WC)$. Call it β. Assume the antecedent. Then $T\beta < \beta$ so $T\beta \notin SM$ so $T``SM \not\subseteq SM$. The conclusion tells us that being a member of SM is not a satisfactory criterion for being a large cardinal.

$(\forall\alpha)(\alpha \leqslant T\alpha . \supset . \alpha \notin SM) \quad \rightarrow \quad \aleph``SM \not\subseteq SM$

Proof:

Take $\inf\aleph``\tau'\overline{\overline{V}}$. Call it β. $\beta \leqslant T\beta$ because $\tau'\overline{\overline{V}}$ is closed under T, so $\beta \notin SM$.

$(\forall\alpha)(\alpha \leqslant T\alpha . \supset . \alpha \notin SM) \quad \rightarrow \quad \tau'\overline{\overline{V}} \cap WC = \Lambda$

Proof:

Reductio ad absurdum. Assume the antecedent. Let β be inf $\tau'\overline{\overline{V}}\cap WC$. Then $T\beta \in \tau'\overline{\overline{V}}\cap WC$ as both $\tau'\overline{\overline{V}}$ and WC are closed under T. So, by minimality of β, $\beta \leqslant T\beta$ so $\beta \notin SM$, and a fortiori $\beta \notin \tau'\overline{\overline{V}}$ contradicting assumption.

$\aleph_0 \notin SM. \rightarrow \aleph_\omega$ exists.

Proof:

If $\aleph_0 \notin SM$, then $\beth_n$ exists for all finite n. So, by Sierpinski's theorem, $\aleph_n$ exists for all n. Now $WC \neq Nn \cup \{\aleph_n : n \in Nn\}$ otherwise WC would be closed under $\breve{T}$, which it isn't. So there must be more alephs, the first of which will be $\aleph_\omega$.

$(\forall \alpha \in WC)(\alpha \geqslant T\alpha) \rightarrow \tau'\overline{\overline{V}}\cap WC = \Lambda$

Proof:

Suppose the consequent is false. Let γ be the least aleph in $\tau'\overline{\overline{V}}$. Then $T\gamma \in \tau'\overline{\overline{V}}$ so $\gamma \leqslant T\gamma$ by minimality, so, by the antecedent, $\gamma = T\gamma$, contradicting Lemma 1.5 (v).

Assume Axcount$_\geqslant$. Then either

(i) $\tau{}'\overline{\overline{V}} \cap WC = \Lambda$

or (ii) $\tau{}'\overline{\overline{V}} \cap WC \neq \Lambda$

In case (ii) argue: let α be $(\mu\alpha\ \varepsilon\tau{}'\overline{\overline{V}} \cap WC)$ Then $\alpha < T\alpha$ by minimality and lemma 1.5 (v). $\alpha = \aleph_\beta$ for some β. Either this β is infinite, in which case $\aleph_\omega$ exists, or it is finite, in which case the inference $\beta < T\beta$ will contradict Axcount$_\geqslant$, so

Axcount$_\geqslant$ $.\supset.$ $\aleph_\omega$ exists v $\tau{}'\overline{\overline{V}} \cap WC = \Lambda$. (**)

The technique behind (**) is to find some sentence φ which implies the existence of some ordinal $\beta < T\beta$. Then we throw in Axcount$_\geqslant$ to show that β must be infinite and infer Axcount$_\geqslant$$.\supset.$ $\aleph_\omega$ exists v $\sim\varphi$. Unfortunately there are not all that many φ. The hope is, however, that eventually we will be able to find enough such φ for the conjunction of their negations to imply '$\aleph_\omega$ exists '. Then we will have Axcount$_\geqslant$ $\rightarrow$ $\aleph_\omega$ exists.

$(\forall\alpha \in NC)(\alpha \geqslant_* T\alpha)$ trivially implies Axcount$_\geqslant$, as $\geqslant_*$ is $\geqslant$ when restricted to the integers.

$(\forall\alpha \in \tau'\bar{\bar{V}})(\alpha \geqslant_* T\alpha) \to$ Axcount$_\leqslant$

Proof:

This is in two steps. (i) Let $n \in Nn$ be such that $(\exists\alpha \in \tau'\bar{\bar{V}} \cap \mathbb{N})(T\alpha \leqslant_* \alpha)$. We have $\overline{\overline{\varphi'T\alpha}} = T\overline{\overline{\varphi'\alpha}} + 1 = n + 1$. And, by $T\alpha \leqslant_* \alpha$ we have $\overline{\overline{\varphi'T\alpha}} \geqslant \overline{\overline{\varphi'\alpha}} = n$ (note to lemma 1.5 (i)) whence $n \leqslant Tn + 1$ and $n \leqslant Tn$.

(ii) Now we use $(\forall\alpha \in \tau'\bar{\bar{V}})(\alpha \geqslant_* T\alpha)$ to show that all $n \in Nn$ satisfy the hypothesis of (i). Suppose there is a last n such that $\tau'\bar{\bar{V}} \cap \mathbb{N} \neq \Lambda$. Fix it. Pick $\alpha \in \tau'\bar{\bar{V}} \cap \mathbb{N}$. We have $\overline{\overline{\varphi'T\alpha}} = Tn + 1$ and $Tn + 1 \leqslant Tn$ by maximality of n. But this is not possible if $\alpha \geqslant_* T\alpha$ so there is no last n, so $(\forall n \in Nn)(\tau'\bar{\bar{V}} \cap \mathbb{N} \neq \Lambda)$.

Now, putting (i) and (ii) together we infer Axcount$_\leqslant$, and combining both with the statement at the top of this page proves $(\forall\alpha \in NC)(\alpha \geqslant_* T\alpha) \to$ Axcount.

A close look at the proof will persuade the reader that $(\forall\alpha \in NC)(\alpha \geqslant_* T\alpha)$ is stronger than is necessary to infer Axcount. $2^\alpha \geqslant_* 2^{T\alpha}$ or $\beth_n{}'\alpha \geqslant_* \beth_n{}'T\alpha$ would do just as well for any n. Even $\beth_n{}'\alpha \geqslant_* \beth_m{}'T\alpha$ for any m, n. - at the cost of a more complicated proof, would give us Axcount$_\leqslant$.

$$T''SM \not\subseteq SM, \quad \aleph''SM \not\subseteq SM \;\rightarrow\; SM \neq NC - Nn$$

Proof:

NC - Nn is closed under T, and $\aleph'\alpha$ is infinite when α is, so if SM is not closed under T and $\aleph$ it must be distinct from NC - Nn.

$$SM \neq NC - Nn \;\leftrightarrow\; \aleph_0 \notin SM$$

Proof:

R → L is trivial. For L → R observe that if there is an infinite $\alpha \notin SM$ we must have, by Sierpinski's theorem, $\aleph_0 < 2^{2^{2^{\alpha}}}$ whence $\aleph_0 \notin SM$.

$$\sim Can(NC), \quad \overline{\overline{NC}} \notin WC \;\rightarrow\; NC - Nn \text{ is infinite}$$

Proof:

By contraposition. If NC - Nn is finite, then NC is the union of a finite set and a countable set and will therefore be countable. So it is wellordered and cantorian, contradicting the hypotheses.

(Note that the weaker hypothesis $\sim$stcan(NC) is already a theorem of NF.)

$\text{Axcount}_{\leqslant} \rightarrow (\forall\alpha)(\ \alpha \leqslant T\alpha . \supset . \ \alpha \not\in SM\ )$

Proof:

By reductio ad absurdum. $\alpha \leqslant T\alpha$ gives us $\overline{\overline{\varphi'\alpha}} \geqslant \overline{\overline{\varphi'T\alpha}}$ and anyway we have $\overline{\overline{\varphi'T\alpha}} \geqslant \overline{\overline{T\varphi'\alpha}} + 1$ which gives $\overline{\overline{\varphi'\alpha}} > \overline{\overline{T\varphi'\alpha}}$ which contradicts $\text{Axcount}_{\leqslant}$

As a corollary of this we have

stcan(α). $\supset$. $\alpha \not\in SM$, because either (i) α is finite, in which case $\alpha \not\in SM$, or (ii) α is infinite, in which case we infer Axcount and a fortiori $\text{Axcount}_{\leqslant}$ which we can then use to prove $\alpha \not\in SM$ as above.

I remarked earlier that we can prove $\rho'T\alpha = T\rho'\alpha$ only for $\alpha \not\in 0$, because it might be that some $\alpha \in \mathbb{O}$ is of rank 0 but that $T\alpha$ is of huge rank because, although $T\alpha = 2^{T\beta}$, $\alpha \neq 2^{\beta}$ because $\beta \in \mathbb{O}$. For such a β we have $\beta \in \mathbb{O}$ $T\beta \in 2$. We may ask: can we actually show that this <u>happens</u>? If it does, for which n can we find β such that $\beta \in \mathbf{O}$ $T\beta \in \mathbb{N}$? It turns out that assuming either that cardinals behave "very well" or that they behave "very badly", give us the same answer, essentially that for all standard integers n there is

$\alpha \in \mathbf{O}$ with $T\alpha \in \mathbb{N}$. We can also prove that if $T''SM \notin SM$ then $(\exists\alpha \in \mathbf{O})(T\alpha \notin SM)$ but that is less significant.

Theorem 1.7 (i)

$$(\exists n)(\mathbb{N} = \Lambda). \supset . (\forall n \in Nn)(\mathrm{stcan}(n) \supset (\exists\alpha \in \mathbb{O})(T\alpha \in \mathbb{N}))$$

Proof:

We assume the hypothesis and prove the contrapositive of the conclusion. Write $\rho_{\mathbf{N}}$ for $\sup(\rho''\mathbf{N})$. Assume the antecedent. Then $\rho_{\mathbf{N}} = \underline{\underline{k - n}}$ where k is the last m such that $M \neq \Lambda$. $k > Tk$. Suppose every $\alpha \in \mathbf{N}$ is $\leq \overline{\overline{\iota^2{}''V}}$. Look at $\sup(\rho'' \breve{T}''\mathbb{N})$. As $\alpha \leq \overline{\overline{\iota''V}}. \supset . T\rho'\alpha = \rho'T\alpha$ we infer $\sup(\rho'' \breve{T}''\mathbf{N}) = \breve{T}(k - n)$ so $\rho_{\mathbb{1}} = k - 1 \geqslant \breve{T}(k - n)$

$$Tk - 1 \geqslant k - n$$
$$n - 1 \geqslant k - Tk$$

So n is not strongly cantorian. So if we assume that n was strongly cantorian we can infer that not every $\alpha \in \mathbf{N}$ is $\overline{\overline{\iota^2{}''V}}$, so $\breve{T}\alpha \in \mathbb{O}$ for some $\alpha \in \mathbf{N}$.

We get a similar result on assumption of "good" behaviour on the part of NC:

Theorem 1.7 (ii)

$$\mathrm{Axcount}_{\leq} \supset (\forall n > 0)(\exists\alpha \in \mathbf{O})(T\alpha \in \mathbb{N})$$

Proof:

Notice first that $\mathrm{Axcount}_{\leq} \supset (\forall n)(\mathbf{N} \neq \Lambda)$. For suppose $\mathrm{Axcount}_{\leq}$ and that n_0 is the last n such that $\mathbb{N} \neq \Lambda$. Then $\exists\alpha \in \mathbf{N}_0$ such that $\phi'T\alpha$ is of power $> Tn_0$ which is greater than n_0 by $\mathrm{Axcount}_{\leq}$. So n_0 was not least. Now we assume $\mathrm{Axcount}_{\leq}$ and make a nonce definition on alephs:

$$F(\beta) \quad \mathrm{iff}_{df} \ (\forall n)(\exists\alpha \in \mathbb{N})(\beta \leq \alpha)$$

Note that $\sim F(\breve{T}\overline{\overline{NO}})$. Otherwise trivially $\breve{T}\overline{\overline{NO}} \leq T\overline{\overline{V}}$ and $\overline{\overline{NO}}$ is $T^2\alpha$ for some α, which we have seen earlier is impossible (p 5). We shall need

$$\mathrm{Axcount}_{\leq} \supset (\forall\beta)(\beta \leq T\beta. \supset . F(\beta))$$

Proof:

We prove by induction on n that for all n there is $m \geqslant n$ such that $\exists\alpha \in \tau\overline{\overline{V}} \cap \mathbb{M} \wedge \beta \leq \alpha$, as long as $\beta \leq T\beta$. For n = 0 it is trivial, because $\beta \leq \overline{\overline{V}}$. Now if $\beta \leq \alpha$ and $\alpha \in \mathbb{N} \cap \tau'\overline{\overline{V}}$ then $\beta \leq T\beta \leq T\alpha$ which is in $\tau'\overline{\overline{V}} \cap \mathbb{TN} + \mathbb{1}$ and by $\mathrm{Axcount}_{\leq}$ $Tn + 1 > n$.

Now set

$$\kappa = (\mu\aleph)(\sim F(\aleph))$$

By the above we have $\kappa > T\kappa$ so let $A = \{\beta \in SM : \kappa \leq \beta\}$. Because $\sim F(\kappa)$ there is a maximal n such that $\mathbb{N}$ meets A but because $F(T\kappa)$ there is no maximal N which meets T"A which is $\{\beta \in SM : T\kappa \leq \beta\} - 0$. Now define $\beth_{ult}{}'\alpha$ to be $\bigcup(\phi'\alpha \cap \mathbb{O})$. Then $\beth_{ult}{}''A \subseteq \mathbb{O}$ and $T''\beth_{ult}{}''A$ meets arbitrarily late $\mathbb{N}$, which proves the theorem.

This question is interesting not merely for its own sake, but also for the possibilities it raises of coding subsets of Nn with large cardinals. For example, we might well speculate what the following class might turn out to be: (Of course this phase of the discussion would have to be conducted in ML) $\hat{n}(\exists m)(T^{m}{}'\alpha \in \mathbb{N})$. It may be that by judiciously picking α we can code in this highly unstratified way subsets of Nn whose existence is not otherwise provable by more normal ZF-type methods. Of course we would probably not be able to prove sethood theorems for them.

The foregoing results enable us to show that many further weaker forms of the axiom of choice fail in NFC than can be shown to fail in NF. We shall see later that the first aleph α for which $\sim AC_{\alpha}$ (if there is one) will satisfy $\alpha \leq T\alpha$ and is therefore not otherwise obviously pathological. We can prove worse than this, however:

Theorem 1.8 (NFC)

$$(\forall n \in Nn)(\forall \alpha \in WC)(\overline{\overline{TV}} \nleq \beth_n{}'\alpha)$$

Lemma 1.9 (NF)

$$T\,''\bigcup_{\alpha \geqslant \overline{\overline{\iota''V}}} (\tau'\alpha) \subseteq \bigcup_{\alpha \geqslant \overline{\overline{\iota''V}}} (\tau'\alpha)$$

Proof of lemma:

suppose $\varphi'\beta$ has n members, the n^{th} of which is $\geqslant \overline{\overline{\iota''V}}$. Then $\varphi'T\beta$ has $\geqslant Tn + 1$ members. Let $\overline{\overline{x}}$ be the n^{th} member of $\varphi'\beta$ and then $T\overline{\overline{x}}$ is the Tn^{th} member of $\varphi'T\beta$ and the $Tn + 1$st is $2^{T\overline{\overline{x}}}$ which is $\overline{\overline{p'x}}$. Now $\overline{\overline{\iota''V}} \leqslant \overline{\overline{x}} . \supset . \overline{\overline{\iota''V}} \leqslant \overline{\overline{p'x}}$ so we are done.

Proof of theorem;

suppose per contra that $\bigcup_{\alpha \geqslant \overline{\overline{\iota''V}}} (\tau'\alpha)$ contained an aleph. Let α be the least such. By lemma 1.9 and minimality of α we have $\alpha \leqslant T\alpha$, which contradicts $\alpha \in SM$ in the presence of Axcount.

Comparability of cardinals fails quite early on, even in NF

Theorem 1. 10 (NFC)

$$(\exists\alpha)(\alpha \not\leqslant \aleph'\alpha = \aleph'T\alpha)$$

Proof:

Pick $\alpha \in \tau'\overline{\overline{V}}$ such that $\aleph'\alpha$ is $\inf(\aleph''\tau'\overline{\overline{V}})$ We have also $\aleph'\alpha \leqslant \aleph'T\alpha$ (intuitively this means that $\aleph'\alpha$ is small) but $\alpha \not\leqslant \aleph'\alpha$ otherwise α is an aleph which is impossible as $\alpha \in \tau'\overline{\overline{V}}$.

The prime ideal theorem is probably the strongest consequence of AC that has not yet been refuted. If we could find an equivalent of it in the language of cardinal arithmetic we could probably attack it by methods resembling those above.

One open problem in the theory of cardinals in NF is the possible range of relationships between α and $T\alpha$. We know that quite often $\alpha = T\alpha$ (e.g. when $\alpha = \aleph_0$) and that sometimes $\alpha > T\alpha$ (e.g. when $\alpha = \bar{\bar{V}}$) The two other possibilities of $\alpha < T\alpha$ and $\alpha \nleqslant T\alpha \nleqslant \alpha$ are not known to be realised. As we have seen, the assumption $(\forall\alpha)(\alpha \geqslant_* T\alpha)$ entails Axcount and is therefore unprovable in NF. The same is not as yet known to hold for the weaker version $(\forall\alpha \notin Nn)(\alpha \nless T\alpha)$

Theorem 1.11

$(\exists\alpha \notin Nn)(\alpha < T\alpha) \wedge (NC - Nn \text{ is finite}). \supset . \sim Axcount_{\geqslant}$

Proof:

Pick α infinite, $\alpha < T\alpha$. Then $\{ \beta \in NC - Nn : \beta \leqslant \alpha \}$ is a proper subset of $\{ \beta \in NC - Nn : \beta \leqslant T\alpha\}$ and by NC - Nn is finite is smaller and finite so $(\exists n \in Nn)(n < Tn)$ contradicting $Axcount_{\geqslant}$.

If we are looking for an α incomparable with $T\alpha$ the best hunting ground is probably $\tau`\bar{\bar{V}}$.

Theorem 1.12

$Axcount_{\geqslant} \wedge NC - Nn \text{ finite}. \supset. (\exists\alpha \in \tau`\bar{\bar{V}})(T\alpha \nleqslant \alpha \nleqslant T\alpha)$

Lemma 1.13

$(\forall\alpha \notin Nn)(\alpha \nless T\alpha) \wedge NC - Nn$ is finite. $\supset$.

$(\exists\alpha$ minimal in $\tau`\bar{\bar{V}})(T\alpha \nleqslant \alpha \nleqslant T\alpha)$

Proof of lemma:

As $NC - Nn$ is finite there is α minimal in $\tau`\bar{\bar{V}}$. By minimality of $\alpha$ and closure of $\tau`\bar{\bar{V}}$ under T we infer $T\alpha \nleqslant \alpha$. Also since $\beta = T\beta . \supset . \beta \notin \tau`\bar{\bar{V}}$ we have $\alpha \neq T\alpha$. And $\alpha < T\alpha$ is ruled out by the hypothesis, so α and $T\alpha$ are incomparable.

Putting the proof of lemma 1.13 and theorem 1.11 together we obtain a proof of theorem 1.12.

Consideration of $\tau`\bar{\bar{V}}$ can help us with questions like $(\exists\alpha)(\alpha \neq T\alpha \wedge \aleph`\alpha = \aleph`T\alpha)$?

Theorem 1.14

$(\exists\alpha)(\alpha \neq T\alpha \wedge \aleph`\alpha = \aleph`T\alpha) \vee (\exists\alpha)(\alpha < T\alpha)$

Proof:

Look at $\inf(\aleph``\tau`\bar{\bar{V}})$ or $\beta$ for short. As before we have $\beta \leqslant T\beta$ so either $\beta < T\beta$ in which case we have the right hand disjunct, or $\beta = T\beta$ in which case pick some $\alpha \in \tau`\bar{\bar{V}}$ such that $\beta = \aleph`\alpha$. Then $\alpha \neq T\alpha$ but $\aleph`\alpha = \beta$ and $\aleph`T\alpha = T\aleph`\alpha = T\beta = \beta$ so we have the left hand disjunct.

Theorem 1.15

Axcount$_{\geqslant} . \supset . \aleph_\omega$ exists $\vee (\exists\alpha)(\sim can(\aleph`2^{\alpha}) \wedge can(\aleph`\alpha))$

Proof:

Let γ be $\inf(\aleph''\tau'\overline{\overline{TV}})$. Then $\gamma \leqslant T\gamma$. Now either $\gamma < T\gamma$ in which case Axcount $_{\geqslant}$ $.\supset.$ $\aleph_\omega$ exists, or $\gamma = T\gamma$. In this second case we argue: let n_o be the least n such that $(\exists\alpha \in \mathbb{N}\cap\tau'T\overline{\overline{V}})(\aleph'\alpha = \gamma)$ Fix such an α. Note that $\overline{\overline{\varphi' T\alpha}} = Tn_o + 1$ so, by minimality of n_o, $n_o \leqslant Tn_o + 1$, whence $n_o \leqslant Tn_o$. Axcount $_{\geqslant}$ implies that n_o is strongly cantorian, as it implies that $n_o = Tn_o$, and if $m < n_o$ such that $m \neq Tm$ then either $Tm > m$ or $T(n_o - m) > (n_o - m)$ both of which contradict Axcount $_{\geqslant}$. Now look at $\varphi'\alpha$. It is strongly cantorian, and so also will $\{\langle\beta,\aleph'\beta\rangle : \beta \in \varphi'\alpha\}$ be. We want $(\mu\beta \in \varphi'\alpha)(\sim can(\aleph'\beta))$. Although this is defined by an unstratified expression we can use the fact that stcan$(\varphi'\alpha)$ to locate it. In these circumstances it is evident that we have functions f and g such that

$$\langle\beta, \aleph'\beta\rangle \overset{f}{\longmapsto} \{\langle\beta, \aleph'\beta\rangle\} \overset{g}{\longmapsto} \langle T\beta, T\aleph'\beta\rangle \quad \text{so}$$

$g|f : \langle\beta, \aleph'\beta\rangle \mapsto \langle T\beta, T\aleph'\beta\rangle$. The β we want is the last β such that the second component of $g|f\ '\langle\beta,\aleph'\beta\rangle$ is equal to the second component of $\langle\beta,\aleph'\beta\rangle$, and this condition is stratified. There must be a last such as $\sim can(\aleph'\overline{\overline{TV}})$. This gives us a β such that $can(\aleph'\beta) \wedge \sim can(\aleph'2^\beta)$.

It is well known that any nonstandard model of arithmetic must contain a copy of the rationals. I shall show below that if Axcount fails we can find a <u>definable</u> subclass of Nn of the order type of the rationals.

Let n be a <u>difference number</u> if it is m - Tm for some m ε Nn, m > Tm or Tm - m for m ε Nn, m < Tm. We shall need the following

Lemma 1.16 $(\forall m \in Nn)(\forall p \in Nn)(m \neq Tm \wedge stcan(p) \rightarrow p \mid |m - Tm|)$

Proof:

$n = m - Tm$. Let p be strongly cantorian. The p < m. we have

$m \equiv m_1 \pmod{p}$

$Tm \equiv Tm_1 \pmod{Tp}$ but $p = Tp$ and $m_1 = Tm_1$ so subtract:

$m - Tm \equiv 0 \pmod{p}$ but p was an arbitrary strongly cantorian integer, so the lemma follows. Then to prove

Theorem 1.17 (NF + ~Axcount)

∃ subclass of Nn isomorphic to the "true" rationals

we argue as follows: Pick any difference number n. Then if m_1, m_2 are strongly cantorian integers $\frac{n \cdot m_1}{m_2}$ is an integer, and the class of all integers obtained in this way from n is obviously a copy of the 'actual' rationals.

The proof of lemma 1.16 uses the fact that any integer earlier than a strongly cantorian one is also cantorian (indeed strongly cantorian) We may ask if any integer earlier than a cantorian integer is cantorian (with 'ordinal' instead of 'integer' this is Henson's axiom CS [15]) This question turns out to be related to one also asked by Specker, namely, are Axcount $\geqslant$ and Axcount $\leqslant$ equivalent ?

Remark 1.18

$(\forall n \in Nn)(\exists m > n)(m = Tm) . \supset . \text{Axcount}_{\geqslant} \equiv \text{Axcount}_{\leqslant}$

Proof:

Pick $n \neq Tn$ (if there is one - if there is not then Axcount holds and there is nothing to prove) and $m = Tm$, $m > n$. Then if $n > Tn$, $m - n$ must be smaller than $m - Tn = T(m - n)$. Mutatis mutandis for $n < Tn$. Either way, if one of the weaker forms of Axcount fails, so must the other, on this hypothesis.

As well as cantorian and strongly cantorian there is a family of properties closely related to the latter and of approximately equal strength. stcan(x), we remember, is defined to hold iff $\iota \restriction x$ exists. If φ is a function which has values n types higher than its arguments for some $n > 1$, we must consider the collection of its fixed points (if any) and the question of whether or not it is a set. For example, suppose $\{ x : x = \{ y : x \in y \} \}$ is a nonempty set, call it A, we show easily enough that $\iota^2 \restriction A$ exists, but does this prove stcan(A)? In general, if φ is a function whose values are n types higher than its arguments the set A (if it is a set) of fixed points (if any) will have the property that $\iota^n \restriction A$ exists, because the function sending $\iota^n{}'x$ to $\varphi' x$ will be definable by a stratified condition and so will exist, but for $x \in A$, $\varphi' x = x$, so this function is precisely $\iota^n \restriction A$. Does this imply stcan(A)? Probably for no n is this provable (though we can show by elementary arithmetic that if $\iota^m \restriction A$ and $\iota^n \restriction A$ exist and m and n are coprime then stcan(A) - there are parallels here with the axiom scheme Amb^n to be introduced in chapter 2.§4). However, we have an analogue of

$(\forall x)(\bar{\bar{x}} \notin Nn \wedge \mathrm{stcan}(x) \;.\supset.\; \mathrm{stcan}(\aleph'\bar{\bar{x}}))$

namely

$(\forall x)(\bar{\bar{x}} \notin Nn \wedge \iota^n \restriction x \text{ exists} \;.\supset.\; \mathrm{stcan}(\aleph'\bar{\bar{x}}))$

Proof: As before, $\iota^n \restriction y$ exists for any finite y, so $T^n \restriction \{ m : m < p \}$ exists for all $p \in Nn$, so $T^n \restriction Nn$ exists; then we prove by induction on Nn that this relation is the identity. So $(\forall p \in Nn)(p = T^n p)$. Now if $p > Tp$, $Tp > T^2p > T^3p > \ldots T^np$ (mutatis mutandis for $<$) so $(\forall p \in Nn)(p = Tp)$. Basically this is the proof that the existence of an infinite strongly cantorian set is equivalent to Axcount: to prove the sentence above read ' of cardinality $<$ $\aleph'\overline{\overline{x}}$ ' for 'finite' and the proof goes through. i.e., when we can wellorder everything, the existence of $\iota^n \upharpoonright x$ is not merely implied by stcan(x) but is actually equivalent to it, for each n. Note that this is a theorem <u>scheme</u>, as we cannot quantify over the n.

It is an open problem in ZF whether or not there is an uncountable regular cardinal. This question is open in NF too, but because of the existence of pathological cardinals in NF there is a remote chance that these foreign methods might make the problem easier to solve. A related, but more difficult problem is "Is there an uncountable weak inaccessible in NF?" As we shall see later, this would enable us to prove con(ZF). The hope arises from the various definable pathological alephs. At the moment is is open whether or not $\overline{\overline{WC - Nn}}$ is regular (or even limit - that would imply 'NC - Nn is infinite' which is open). $\inf(WC \cap SM)$, $\inf \aleph``\tau'\overline{\overline{V}}$, and $\inf \aleph``SM$ could all turn out to have suitable closure properties, and it is relatively easy to show that the first of these is limit. ($\alpha^+ < 2^{2^{2^\alpha}}$ when $\alpha \in WC$ so $\alpha \notin SM . \supset . \alpha^+ \notin SM$) In the absence of Axcount we do not know how to prove even that $\inf(WC \cap SM)$ is uncountable. We could try proving that it is of uncountable cofinality if we could show A: $\alpha + \beta \in SM . \supset . \alpha \in SM \vee \beta \in SM$, as this would mean that NC - SM - Nn was an ideal in $\langle NC - Nn, \leqslant \rangle$. If it turns out to be an ideal and, say, an α-complete ideal, then the cofinality of $\inf(WC \cap SM)$ would be at least α. However we still lack even A. Other alephs which could have useful closure properties are $(\mu\alpha \in WC)(2^{\alpha} \notin WC)$ if there is one such, (which is open) and $(\mu\alpha)(\sim AC_{\alpha})$. ($AC_{\alpha}$ is "Every set of size α has a selection set" In this case α is assumed to be an aleph) If there is one such it satisfies $\alpha < T\alpha$ because $AC_{T\alpha} . \supset . AC_{\alpha}$ (If we seek a selection set for a set x of power α, we use $AC_{T\alpha}$ to get a selection set for $\{ \iota "y : y \in x \}$ which is of power $T\alpha$ and then take its sumset.) Also $(\mu\alpha)(\sim AC_{\alpha})$ is obviously regular.

There is no shortage of other open problems: there is no positive consistency result yet for any consequence of AC. Nobody has yet refuted the Prime Ideal theorem. It is not known whether or not can(NC) or $\overline{\overline{NC}} \in WC$ or whether or not $\overline{\overline{NO}}$ is a beth number. Specker has asked what partition properties $\overline{\overline{V}}$ has, but still lists as open problem number 1 to prove that NC - Nn is infinite.

The curious behaviour of the axiom of choice mentioned in the last paragraph but one is one of the hangovers of the Theory of Types in NF, that objects of higher type code more information than corresponding objects of lower type. We can "code" in $T\alpha$ enough information for $AC_{T\alpha}$ to imply AC_{α} but not vice versa, despite the fact that $T\alpha$ is usually no bigger than α. Similarly, it is easy to define an ordered pair of x and y two types higher than x or y but very difficult to define one the same type as x and y. This is possible in NF only because of the axiom of infinity and does not work in NFU. To define an ordered pair one type lower is impossible, as otherwise $f : x \to \{\langle x, \Lambda\rangle\}$ sends V into ι"V contradicting Cantor's theorem.

This trade-off between stratification and information is probably of great importance but its significance is not well understood. If we wish to construct a model for Z in NF + $\aleph_0 \notin$ SM the obvious way to do it is to take $V_{\omega+\omega}$, or, at least, Nn, p'Nn, p^2'Nn..... That way we pick canonical sets of the right size, but the process cannot be described in a stratified way. The other way would be to take the (by hypothesis) infinite $\varphi'\aleph_0$, pick one set from each cardinal, label them appropriately and then define an ε-relation in the obvious way. This is stratified all right, but it needs AC_ω. (As we shall see later, NF + $\aleph_0 \notin$ SM does easily prove con(Z) but by a completely different method). One would like to say that AC_ω expedites the proof by giving more 'information' about $\varphi'\aleph_0$ and even to say, perhaps, that AC fails because it gives "too much" information. In one sense, of course, this is trivially true: what we are in need of is a notion of set-theoretical information that makes formal sense of these intuitions.

Chapter 2

The Consistency problem for NF

§1 Introduction and definitions

The reader is assumed familiar with the Simple Theory of types (hereinafter 'TST') and with the fact that TST can be expressed indifferently as a many-sorted theory or as a one-sorted theory with type predicates. It is usually discussed here in the latter form. 'TNT' refers to the Theory of Negative types, which is like TST except in having its types indexed by $\mathbb{Z}$ rather than ω. Like TST, TNT when discussed here will be in its one-sorted guise. Unless otherwise stated, both theories are assumed to be axiomatised without choice or infinity.

Definition 2.0

If M is a model of TST, M^* is the result of relabelling all types in M with indices one lower. Thus M^* has a type -1 as well. M^*_- is M^* with this extra type cut off. If M is a model of TNT then M^* is defined analagously. M^n and M^n_- are the results of applying the respective above operations n times to M.

Definition 2.1

If φ is an expression in the language of type theory then φ^* is the result of raising all type indices in φ by one. Similarly φ^n is the result of applying the * operation n times to φ. When the variable of lowest type in φ is of type 0 I shall often write φ as φ_0 and φ_0^n as φ_n.

As a result of these two definitions above we have $M^* \models \varphi$ iff $M \models \varphi^*$ Throughout this chapter I shall be mainly concerned with finding $M \models$ TST or $M \models$ TNT with the property that M has an automorphism (if $M \models$ TNT) or an endomorphism (if $M \models$ TST) that sends the objects of type n onto the objects of type n + 1. Such an auto- (endo-) morphism will be referred to as a tsau (<u>t</u>ype-<u>s</u>hifting <u>au</u>tomorphism). For $M \models$ TST it is evident that M posesses a tsau iff M is isomorphic to $\underline{M^*}$.

Definition 2.2

When φ is a stratified expression we say that φ is (n,m)-stratifiable when the following happens; φ is stratified if we can assign every variable in φ an integer as follows: pick a variable at random and give all its occurrences the same integer. Then, whenever we see '$x \in y$' and x has been assigned n, y must be assigned n+1 at all its occurrences. We may assume that such trivial relettering as is necessary has been carried out. This process may end without all variables having been assigned an integer. If it does, start again on the remainder of the formula. The number of times we have to go through this process in order to assign every variable an integer is the m in ' (n,m)-stratifiable', and the number of integers we eventually use is the n. Clearly when φ is (n,m)-stratifiable with $m > 1$ there are many nontrivially distinct stratification assignments for φ, some of which may may use more types than others. We shall need a canonical one. If φ is (n,m)-stratifiable we see that the variables in φ are divided into m disjoint batches each of which can be given a stratification assignment independently of the others. The canonical assignment we want is that which assigns to each variable at each stage the lowest integer which is compatible with the rules. This has the advantage of ensuring that we use the smallest possible number of integers in forming the stratification assignment. We can now make

Definition 2.3

If φ is a stratified expression of the language of set theory, φ' is the result of giving every variable in φ an index as in a stratification assignment and then incorporating the indices as type predicates. i.e., if x has been given the index n, conjoin '$T_n(x)$' to every atomic proposition mentioning x.

The starting point for all subsequent work on the model theory of NF was Specker's theorem [45], [46]. It is stated here in a rather more generalised form than in [45], [46]: Specker's theorem is in fact Corollary 2.6.

Metatheorem 2.4

Let S be a theory with primitives =, ε. Then define S^n_π as follows: It has primitive predicates =, ε, $T_r(\ )$ for each $r \leqslant n$ and the axioms

Axiom scheme 1 $\quad T_r(x) \wedge x \in y. \supset . T_{r+1}(y)$

" " 2 $\quad T_r(x) \wedge y = \pi' x. \supset . T_{r+1}(y)$

Axiom 3 $\quad x \in y. \equiv . \pi' x \in \pi' y$

Axiom scheme 4a $\quad (\forall x)(\exists ! y)(\ y = \pi' x)$ (where x is of type $< n$)

b $\quad (\forall y)(\exists ! x)(\ y = \pi' x)$ (where y is of type > 0)

" " 5 $\quad \varphi^\pi$ for φ an axiom of S, where φ^π is defined as follows: write out φ in primitive notation, replace every occurrence of $\ulcorner x \in y \urcorner$ by $\ulcorner T_0(x) \wedge T_0(y) \wedge x \in \pi' y \urcorner$

Then S and S^n_π are equiconsistent.

Proof;

If S^n_π is consistent obviously S is consistent because $S^n_\pi \vdash \varphi^\pi$ whenever $S \vdash \varphi$ by induction on the rules of inference so any contradiction in S can be reproduced in S^n_π.

For the converse, let $\langle V, \varepsilon, =\rangle$ be a model for S. Then construct a model for S_{π}^{n} as follows: let the universe of the model be $V \times \{ r : r \leqslant n \}$, the " ε " relation will be $\langle x, r\rangle \ \varepsilon \ \langle y, r + 1\rangle$ iff $x \ \varepsilon \ y$. $T_r(x)$ will be defined to hold iff the second component of x is r. Equality will be defined as in the model for S. It is trivial to verify that this is a model for S_{π}^{n}.

Q. E. D.

Now let S be a theory with primitive predicates =, ε all of whose axioms are stratified. Define S_{*}^{n} as follows: It has primitive predicates ε, =, $T_r(\ )$ for $r < n$ and the axioms

Axiom scheme 1 $T_r(x) \wedge x \ \varepsilon \ y . \supset . \ T_{r+1}(y)$

2 φ' for φ an axiom of S

3 $\varphi \equiv \varphi^*$ for all φ

Metatheorem 2.5

S_{π}^{n} and S_{*}^{n+1} are equiconsistent for $n \geqslant n_o$ where n_o is the least m s.t. every axiom of S is (m, k) - stratifiable (note that S_{*}^{n} is not defined otherwise)

Proof:

(i) $\mathrm{Con}(S_{\pi}^{n}) \rightarrow \mathrm{Con}(S_{*}^{n+1})$

For this we need to show that we can eliminate all occurrences of π from φ^{π}, in fact, that $S_{\pi}^{n} \vdash \varphi^{\pi} \equiv \varphi'$ for $n > n_o$. This we do as follows: if the variable y in φ^{π} is assigned k in the stratification assignment then whenever we have ' $y \varepsilon \pi' z$ ' we replace this by ' $\pi^{k}{}' y \varepsilon \pi^{k+1}{}' z$ to which it is equivalent by virtue of Axiom 3. And we replace all occurrences of ' $T_0(y)$ ' by ' $T_k(\pi^{k}{}' y)$'. That way y occurs in the amended φ^{π} only as $\pi^{k}{}' y$. So we can now replace all occurrences of '$\pi^{k}{}' y$' by a new variable w. This is a trivial relettering and cannot alter the truth-value of the sentence. We can do this simultaneously for all variables in φ^{π} and the result is easily seen to be φ'. So $S^n_* \subseteq S^n_{\pi}$ so $S^{n+1}_* \subseteq S^{n+1}_{\pi}$ but S^{n+1}_{π} and S^n_{π} are equiconsistent by Metatheorem 2.4 so (i) follows.

For (ii) $\mathrm{Con}\,(S^{n+1}_*) \rightarrow \mathrm{Con}\,(S^n_{\pi})$ we argue: If S^{n+1}_* is consistent it has a saturated model M. Then M^*_- is a model for S^n_*. Also M with its top type chopped off is a model for S^n_*. Both these structures are saturated. They are elementarily equivalent in view of the axiom scheme $\varphi \equiv \varphi^*$ so they are isomorphic. Pick an isomorphism Υ. Υ must send the type 0 of M^*_- onto the type 0 of the truncated version of M but that means it is an isomorphism from type n of M onto type n + 1 of M. Then it is trivial to verify that Υ satisfies the π - axioms in the definition of S^n_{π} from which we infer that $M^*_- \models S^n_{\pi}$ whence follows (ii).

Corollary 2.6 (Specker [45], [46])

NF is equiconsistent with TST plus the axiom scheme $\sum_{\varphi} (\varphi \equiv \varphi^*)$

Proof:

All axioms of NF are stratified so NF^{ω}_* is precisely TST plus the axiom scheme $\sum_{\varphi} (\varphi \equiv \varphi^*)$ Q. E. D.

The proof given here, although short, is highly nonconstructive. One would expect that there would be a proof-theoretic demonstration of the same result. M. Crabbé has exhibited one such. (unpublished, but see [6])

Corollary 2.7 †

Every stratified theorem of any consistent theory with only stratified axioms has a proof which contains no unstratified expressions.

Proof:

Suppose not and that φ is a counterexample. Then φ' is independent of S^n_* and so $S^n_* \cup \{\sim \varphi'\}$ is consistent so by Metatheorems 2.4 and 2.5 $S \cup \{\sim \varphi\}$ is consistent. (Again, Crabbé has a proof-theoretic demonstration of this for NF).

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† For the special case of NF this is implicit in Orey [22].

Now if we apply Metatheorem 2.4 to unstratified extensions of NF we see that any such system is equivalent to TST plus a number of axioms governing the behaviour of the tsau π, one for each unstratified axiom in the extension. It is reasonably clear that if $M \models$ TST posesses even one tsau it must posess a lot, as any permutation γ of the set of objects of type 0 in M can be extended to an automorphism of the whole model by setting $\gamma'x = \gamma''x$ for x of type > 0. Then, since the relative product of an automorphism and endomorphism is another endomorphism, γ must generate a new tsau. Let $M = \langle V, \varepsilon, = \rangle$ be a model of NF. Let R(M) be the model obtained from M as in the proof of Metatheorem 2.4. Let γ be a permutation of M which is in M. Then γ extends to an automorphism of R(M) since it trivially gives a permutation of the objects of type 0 and any such can be extended to an automorphism of R(M) as indicated above. If π is the canonical tsau $\pi : \langle x, n\rangle \longmapsto \langle x, n + 1\rangle$ then $\pi|\gamma$ is obviously another tsau so we can construct another model of NF by reversing the process, but using the new tsau in place of the old, setting $M' = \langle T_0, =, \varepsilon^{R(M)}|\gamma^{R(M)}\rangle$. T_0 is a copy of V coded in ordered pairs. If we decode M' from this we see that it is isomorphic to $\langle V, =, \varepsilon|\gamma\rangle$. This boils down to the fact that if we have a model $\langle V, =, \varepsilon\rangle$ of NF we can form a new one with the same universe by reading $'x \in \Upsilon'y'$ instead of $'x \in y'$ throughout. The rewritten version of the axiom of extensionality will be verified as Υ is 1 - 1, and the rewritten axiom scheme of abstraction will be verifed because all the rewritten versions of axioms are equivalent to other old axioms. It we write a set abstract instead of the unexpanded term Υ we can control the results we get. In this case the expression inside the set abstract must be stratified of course. The proof that $\langle V, =, \varepsilon|\Upsilon\rangle \models$ NF when Υ is a definable permutation is due to Scott [42] (as is the idea of using permutations in the first place) who used them to show the consistency and independence of $(\exists x)(x = \{x\})$ modulo NF. This sentence is highly unstratified, as of course it must be: it is only unstratifed sentences whose truth we can affect by manipulating the tsaus. $(\exists x)(x = \{x\})$ corresponds to the following condition on π : $(\exists x)(T_0(x) \wedge \pi'x = \{x\})$. Scott only considered permutations which were definable by stratified expressions in NF, and in these cases the relative consistency proof can be carried out in NF. Henson [16] pointed out that the permutation method could be used on any stratified extension of NF. If the need arose,

we could prove that for any stratified theory S whatever, when $M = \langle V, \varepsilon \rangle \models S$ and $R(M) =$
$\langle\langle V \times \{n\} : n \leqslant m\rangle, \{\langle\langle x, n\rangle, \langle y, n+1\rangle\rangle : x \varepsilon y\}, =,$
$\{\langle\langle x, n\rangle, \langle x, n+1\rangle\rangle : n \leqslant m \wedge x \varepsilon V\}\rangle \models S^m_\pi$ that the model got from $R(M)$ by having $\{\langle\langle x, n\rangle, \langle x, n+1\rangle\rangle : n \leqslant m \wedge x \varepsilon V\} \mid \gamma^{R(M)}$ instead of the canonical tsau is also a model of S^m_π when γ is a permutation of V but there seems little call for anything this general at the moment.

We could also extend the theorems of this section in various ways to set theories with classes by adding an axiom scheme of class existence at each type to the type theory where $\hat{x}(\varphi)$exists if φ contains occurrences of π but is still stratified (in the extended sense where one adds to the algorithm for finding out whether or not a formula is stratified the rule that π'y is always one type higher than y) This may later be used to attack ML but at the moment it is the consistency problem for NF which is more pressing.

§2. Permutations.

For the purposes of this section the reader is assumed familiar with the presentation in Scott [42]. When we write out φ^{π} in primitive notation it does not look very much like φ in primitive notation except when φ is very simple because if φ is, say, 3-stratifiable and x is a variable of type 2 x might sometimes appear as x and sometimes as π'x. It would be nice if we had new permutation π_0 π_1 π_2.... such that π_0 is the identity π_1 is π and that φ^{π} is equivalent to φ with x (if it is of type n) replaced by π_n'x throughout, and similarly for other variables in φ. It is simple to see that the condition we want on this sequence of derived permutations is that $(\forall y)(\forall x)(x \in \pi'y \equiv \pi_n'x \in \pi_{n+1}'y)$. This enables us to define the sequence of π_n by recursion on n (notice that this recursion is taking place in the metalanguage, as it has to, since it is not stratified) A little computation will verify that the definition we want is

$$\pi_{n+1} =_{df} (j'\pi_n)|\pi \qquad \text{or, equivalently}$$
$$\pi_{n+1} =_{df} (j^{n}{}'\pi)|\pi_n$$

It is now simple to verify that if $\varphi(x, y)$ is stratified with x and y of types n and m respectively, then φ^{π} is equivalent to $\varphi(\pi_n'x, \pi_m'y)$ The π_n were devised by Henson [16]

If x is a transitive set $x \subseteq p'x$, so $\bar{\bar{x}} \leqslant 2^{T\bar{\bar{x}}}$. We would like the converse to be true, that is, if $\alpha \in NC$ satisfies $\alpha \leqslant 2^{T\alpha}$ then α has a transitive member. For any given cardinal α we can prove this consistent.

Let $\varphi(\ )$ be a stratified expression such that $NF \vdash (\exists! x)(\varphi(x)) \wedge (\forall x)(\ \varphi(x) \supset . x \in NC \wedge x \leqslant 2^{Tx})$. Then there is a permutation model in which the unique cardinal such that φ contains a transitive member.

Proof:

Let α be the cardinal such that φ, and let $x \in \alpha$ be such that $x \cap p'x = \Lambda$. We must prove that we can find such a set x for any cardinal α. This involves close attention to the structure of Quine ordered pairs. Let us define

$$X =_{df} \{\ w : 0 \in w\ \}$$

$$\gamma_1 {}' x =_{df} (x - Nn) \cup \{\ n+1 : n \in x\ \}$$

$$\gamma_2 {}' x =_{df} (x - Nn) \cup \{\ n+1 : n \in x\ \} \quad \{0\}$$

$$\langle x, y \rangle =_{df} \gamma_1 {}'' x \cup \gamma_2 {}'' y$$

This is the standard definition of Quine ordered pairs. Finally define

$$X_1 =_{df} X \times \{\Lambda\} = \{\ \gamma_1 {}'' x : x \in X\ \}$$

We want to show that $p'X_1$ is disjoint from X_1. Let y be a subset of X_1. y is $\{\ \gamma_1 {}'' x : x \in Y\ \}$ for some subset Y of X. If y is also a member of X_1 y must be $\langle w, \Lambda \rangle$ for some w. $\langle w, \Lambda \rangle = \gamma_1 {}'' w$ so $0 \notin^2 y$ by definition of γ_1. If $x \in X$, $0 \in x$. $\gamma_1 {}' 0 = 0$ so $0 \in \gamma_1 {}'' x$. So if y is a subset of X_1 we infer that $0 \in^2 y$. So X_1 is disjoint from its power set. Clearly this property will be inherited by any subset of X_1. As X_1 is of the same power as the universe we can find sets of any desired size that are disjoint from their power sets.

To return to the main proof. We have $x \in \alpha$ such that $x \cap p'x = \Lambda$. By hypothesis there is a bijection f from x onto a subset of $p'x$. We can suppose that x itself is not in the range of f. Define a permutation π by $\pi'z = f'z$ for $z \in x$. $\pi'z = \breve{f}'z$ for $x \in f''x$ and $\pi'z = z$ otherwise. In the permutation model x is transitive.

$$(\forall y)(y \in^2 x \supset y \in x)^{\pi}$$

$$(\forall y)((\exists z)(y \in z \wedge z \in x) \supset y \in x)^{\pi}$$

$$(\forall y)((\exists z)(y \in \pi'z \wedge z \in \pi'x) \supset y \in \pi'x)$$

By construction of π we have $\pi'x = x$ and $\pi'z$ is a subset of x whenever z is a member of x.

Unfortunately there does not seem to be enough structure on the cardinals for us to perform this construction simultaneously on all cardinals and thereby prove the consistence modulo NF of

$$(\forall \alpha \in NC)(\alpha \leqslant 2^{T\alpha} \supset (\exists x \in \alpha)(\cup x \subseteq x))$$

However we can do something like it in the finite case. Order the finite sets of integers by setting $x < y$ iff_{df} the sup of $x - y$ is less than the sup of $y - x$. Let f be the function that enumerates this sequence.

Define π by $\pi'n = f'n$, $\pi'f'n = n$ for $n \in Nn$ and $\pi'n = n$ otherwise. We want $\pi'' \{ m : m < n \} \subseteq p'\{ m : m < n \}$ for $\{ m : m < n \}$ to be transitive in the permutation model and for this it is sufficient that $n < 2^{Tn - 1} + 1$, so we have

$$\mathrm{Con}(NF) \rightarrow$$
$$\mathrm{Con}(NF + (\forall n \in Nn)(n \leqslant 2^{Tn - 1} + 1 \supset (\exists x \in n)(\cup x \subseteq x)))$$

The following observations put great restrictions on the applicability of the permutation method to consistency questions in NF

Metatheorem 2.2.1

Every sentence in the n^{th} order theory of relational types is invariant, for each n.

Sketch of proof:

I shall only give a sketch as the idea is very simple and the execution very laborious. The crucial fact is that if x is a permutation and α is a relational type then $(j^{n}{}'x)'\alpha = \alpha$ for n sufficiently large. (This theorem also applies to cardinal arithmetic and if α is a cardinal $n > 2$ is enough to ensure that $(j^{n}{}'x)'\alpha = \alpha$.) From now on let us fix n to be some integer which is sufficiently large. It then follows that if Y is a set of (sets of..)^k relational types then $(j^{n + k}{}'x)'Y = Y$. Relational types are fixed because they are closed under isomorphism and $(j^{n}{}'x)'\alpha$ is $(j^{n-1}{}'x)''\alpha$ which is the set of relations isomorphic under $(j^{n-1}{}'x)$ to things in α, in other words, α (n big enough, of course). Let us now take a simple unstratified example in cardinal arithmetic to see how we can show all such expressions invariant.

$$(\exists\alpha)(\alpha \in NC \wedge \alpha > T^2\alpha \wedge \alpha \not< T\alpha \not< \alpha)^\pi$$

For n sufficiently large this is equivalent to

$$(\exists\alpha)(\pi_n{}'\alpha \in NC \wedge \pi_{n+2}{}'\alpha > T^2\pi_n{}'\alpha \wedge \pi_{n+1}{}'\alpha \not< T\pi_n{}'\alpha \not< \pi_{n+1}{}'\alpha)$$

Now reletter $\pi_n{}'\alpha$ as α. The expression becomes

$$(\exists\alpha)(\alpha \in NC \wedge (j^{n+1}{}'\pi | j^{n}{}'\pi)'\alpha > T^2\alpha \wedge (j^{n}{}'\pi)'\alpha \not< T\alpha \not< (j^{n}{}'\pi)'\alpha)$$

Since this is true for n large enough, we can pick n to be the n which satisfies the condition on the preceding page, so

$$(j^{n+1}{}'\pi)|(j^{n}{}'\pi)'\alpha = (j^{n+1}{}'\pi)'\alpha = \alpha$$

and our sentence becomes

$$(\exists\alpha)(\alpha \in NC \wedge \alpha > T^2\alpha \wedge \beta\alpha \not< T\alpha \not< \alpha$$

which is what we started with.

If we encounter difficulties in proving some desirable expression φ we can always at least try (if it is unstratified) to use some permutation to prove it consistent, that is, try to prove $(\exists\pi)(\varphi^\pi)$. The same may apply again, inviting us to consider the sequence φ, $(\exists\pi)(\varphi^\pi)$, $(\exists\delta)((\exists\pi)(\varphi^\pi)^\delta)$ ... and so on.

Each member of the sequence implies all later ones and if we can prove one, however late, all the earlier ones are consistent. The burden of the result below is that this process gets us nowhere.

Metatheorem 2.2.2

For all φ, $(\exists\pi)(\pi$ is a permutation and $\varphi^{\pi})$ is invariant.

Proof:

To obtain φ^{π} replace every occurrence of '$x \in y$' in φ by '$x \in \pi 'y$'. To obtain $(\varphi^{\pi})^{\delta}$ replace every occurrence of '$x \in y$' in φ by '$(x \in \pi 'y)^{\delta}$' which is equivalent to $\delta_n 'x \in (\delta_{n+1} '\pi)|\delta_{n+1} 'y$. In '$(\exists\pi)(\varphi^{\pi} \wedge \pi$ is a permutation$)^{\delta}$' "$(\pi$ is a permutation$)^{\delta}$" is "$\delta_{n+1} '\pi$ is a permutation" for n big enough so, as we are binding π with a quantifier we can reletter this as " $\delta_n 'x \in \pi|\delta_{n+1} 'y$. Now, by the result so far, every variable has at least δ_n in front of it so we can reletter them all, getting $x \in \pi|(j^n '\delta)'y$. So far we have established that in order to obtain $(\exists\pi)(\pi$ is a permutation $\wedge\ \varphi^{\pi})^{\delta}$ we replace every occurrence of '$x \in y$ ' in φ by ' $x \in \pi|(j^n '\delta)'y$ and bind the π with $\exists$. Now $\pi|(j^n '\delta)$ is a permutation of V iff π is so we can simplify again, getting '$x \in \pi 'y$' and the δ has disappeared. This saves us the bother of developing a theory of iterated permutations.

§.3 Combinatorics

One might be forgiven for expecting that if we start with M a model of TST and take the set of M^n for $n \in \mathbb{Z}$ and then form an ultraproduct modulo a nonprincipal ultrafilter on $\mathbb{Z}$ (or, mutatis mutandis, the set of $M^{\underline{n}}$ for $n \in \omega$) that we would get a model in which all types are elementarily equivalent as the fact that the ultrafilter is nonprincipal would ensure that the ultraproduct is moderately impartial over its factors, and $M^n \models \varphi$ iff $M \models \varphi^n$. Of course this can never tell us anything new, as Łoś's theorem determines the behaviour of ultraproducts too thoroughly, and, if it worked at all, it would give us a consistency proof not only of NF but also of NF + AC as well.

However, this approach is not totally sterile, as the axiom scheme we are trying to prove the consistency of is not finitely axiomatisable. We would like $\mathbb{M} = \prod_{n \in \omega} M^{\underline{n}} / \mathcal{U}$ to be a model in which all types are elem-entarily equivalent, i.e.,

$$\text{A:} \quad \mathbb{M} \models \varphi_0 \quad \text{iff} \quad (\forall n)(\mathbb{M} \models \varphi_n)$$

$$\begin{aligned} \text{Now} \quad \mathbb{M} \models \varphi_0 \quad &\text{iff} \quad \{ m : M^{\underline{m}} \models \varphi_0 \} \in \mathcal{U} \\ &\text{iff} \quad \{ m : M \models \varphi_m \} \in \mathcal{U} \\ \text{and} \quad \mathbb{M} \models \varphi_n \quad &\text{iff} \quad \{ m : M^{\underline{m}} \models \varphi_n \} \in \mathcal{U} \\ &\text{iff} \quad \{ m : M \models \varphi_{m+n} \} \in \mathcal{U} \end{aligned}$$

so A is equivalent to

$$\{ m : M \models \varphi_m \} \in \mathcal{U} \equiv (\forall n)(\{ m : M \models \varphi_{n+m} \} \in \mathcal{U})$$

The nearest we can get to this at the moment is when $\mathcal{U}$ is either Ramsey, or what Hindman [53] calls an "almost downward translation invariant" ultrafilter. This second is an ultrafilter satisfying

$$x \in \mathcal{U} . \supset . \{ n : \{ m - n : m \in x \} \in \mathcal{U} \} \in \mathcal{U}$$

and for $\mathcal{U}$ such an ultrafilter we have

$$\{ m : M \models \varphi_m \} \in \mathcal{U} . \supset . \{ n : \{ m : M \models \varphi_{m+n} \} \in \mathcal{U} \} \in \mathcal{U}$$

In fact we have replaced '$\forall n$' by 'on a set in $\mathcal{U}$' and satisfied the weaker version of the implication from left to right. In the case where $\mathcal{U}$ is Ramsey then for each φ $(\exists X \in \mathcal{U})(\forall n,m \in X)(M \models \varphi_n \equiv \varphi_m)$ (by using the partition properties of Ramsey ultrafilters). Now we can use the fact that if $\mathcal{U}$ is Ramsey, $F \subseteq \mathcal{U}$, $\bar{\bar{F}} = \aleph_0$ then $\exists X' \in \mathcal{U}$ $X' \subseteq_f X$ for every $X \in F$, to get

$$(\exists X \in \mathcal{U})(\forall \varphi)(\exists y \subseteq_f X)(\forall n,m \in y)(M \models \varphi_n \equiv \varphi_m)$$

§ 4. Subschemes of typical ambiguity.

There are at least three natural ways of weakening the axiom scheme $\sum_{\varphi}(\varphi \equiv \varphi^*)$ of typical ambiguity, viz.:

(i) Cutting down on the set of φ for which we assert $(\varphi \equiv \varphi^*)$

(ii) Amending '$\varphi \equiv \varphi^*$' to '$\varphi \equiv \varphi^n$' for some $n > 1$

(iii) Restricting the axiom scheme to an initial segment of the theory.

The motivation is to find possibly weaker schemes whose consistency may be provable. In (i) it is obvious how the weakness comes about. (ii) is in fact a special case of (i) (though I shall not discuss it as such) because $\varphi \equiv \varphi^n$ is equivalent by propositional logic to $\gamma \equiv \gamma^*$ where γ is $(\varphi \equiv (\varphi^* \equiv (\varphi^{**} \equiv (\ldots \equiv \varphi^{n-1}))))$. The motivation for (iii) is that we can then non-trivially ask the question 'Is this subscheme finitely axiomatisable?' However, if we do not, at the same time, restrict the scheme on the lines of (i) or (ii) the answer will still be no, because, by a result of Grishin [9] we only need the axiom scheme of typical ambiguity restricted to the first five types in order to get something equiconsistent with NF, and any initial segment of TST plus finitely many instances of the axiom scheme of typical ambiguity can be proved consistent in NF. (Orey [22]) However, if ambiguity for Σ_3^{LPC} sentences should turn out to be finitely axiomatisable when restricted in this way to some initial segment then we can prove it consistent in NF. It is of course a trivial corollary of Orey's result that the full scheme is not finitely axiomatisable when restricted along these lines so any subscheme which is must be strictly weaker. (ii) is not on the face of it a very serious weakening: true, I see no way of proving $\sum_{\varphi}(\varphi \equiv \varphi^*)$ from $\sum_{\varphi}(\varphi \equiv \varphi^n)$ with $n > 1$ but $\sum_{\varphi}(\varphi \equiv \varphi^n)$ does suffice to refute AC for any n, and therefore generates all the pathological consequences of the axiom scheme at present known.

I shall use the following notation:

Definition 2.8 $\mathrm{Amb}^n(\Gamma) \upharpoonright T_{<m}$ is the axiom scheme of typical ambiguity restricted to sentences φ of the form $\gamma \equiv \gamma^n$ with $\gamma \in \Gamma$, where no φ contains variables of type $\geq m$. When $n = 1$, Γ = set of all expressions, or $m = \omega$, they will be omitted. Thus $\mathrm{Amb}^n(\Delta_2)$ below is just $\sum_{\varphi \in \Delta_2}(\varphi \equiv \varphi^n)$.

In what follows $\sum_n^S$, $\prod_n^S$, Δ_n^S refer to the classification of formulae in the Levy [55] hierarchy for theory S. $\sum_n^{LPC}$, $\prod_n^{LPC}$, Δ_n^{LPC} refer to the ordinary hierarchy on the lower predicate calculus.

The following six observations complement each other.

Metatheorem 2.9

(i) For all n, TST $\mathrm{Amb}^n(\Delta_2^{TST}) \vdash \sim AC$

(ii) In TST, all expressions are Δ_2^{TST}

(iii) TST $\vdash \mathrm{Amb}(\Delta_1^{TST})$

(iv) $\mathrm{Con}(TST) \rightarrow \mathrm{Con}(TST + \mathrm{Amb}(\Sigma_1^{TST}))$

(v) $\mathrm{Amb}^n(\Delta_2^{TST}) \vdash \mathrm{Amb}^n$

(vi) $\mathrm{Amb}^n(\Sigma_1^{TST}) \vdash \mathrm{Amb}(\Sigma_1^{TST})$

Proof:

(v) follows from (ii). To prove (ii), take any expression φ. Assume without loss of generality that all its quantifiers have been brought to the front so it is $\ulcorner (Q_1 x^n)(Q_2 y^m)(\ldots)(\delta(x^n, y^m, \ldots) \urcorner$ where n, m, ... are the types of x, y, Restrict each Q_i to some arbitrary variable x^{m+1} thus $(Q_i x^m \in x^{m+1})$. φ is then equivalent to both of the following

$(\forall x^{n+1} y^{m+1} \ldots)(x^{n+1} = V_{n+1} \quad y^{m+1} = V_{m+1} \ldots\ldots$
$. \supset . (Q x^n \in x^{n+1})(Q_2 y^m \in y^{m+1})(\ldots)\delta(x^n, y^m, \ldots))$
and $(\exists x^{n+1} y^{m+1} \ldots\ldots)(x^{n+1} = V_{n+1} \wedge y^{m+1}$
$V_{m+1} \wedge \ldots\ldots\ldots \wedge (Q_1 x^n \in x^{n+1})(Q_2 y \quad \in y^{m+1})(\ldots.$

$\ldots)((x^n, y^m, \ldots))$. Now $\ulcorner x^{i+1} = V_{i+1}\urcorner$ is Π_1^{TST} and we can bring all quantifiers to the front by the usual devices of the lower predicate calculus, so φ is $\Sigma_2^{TST} \cap \Pi_2^{TST} = \Delta_2^{TST}$.

This proof, due to Boffa, is an improvement on my original version: this applies also to the proof of Metatheorem 2.11

(i)

$\mathrm{Amb}^n(\Delta_2)$ proves $\sim AC$

Proof

Define $SM_n =_{df}$ SM at type n

$\alpha_n =_{df} \inf(SM_n)$

α_n is defined if we have AC. We shall want:

(1) $T\alpha_n = \alpha_{n+1}$

(2) $T''SM_n \subseteq SM_{n+1}$

Proof of (2) Let $\alpha \in SM_n$ and let β be the last member of $\phi`\alpha$. Then $\beta > \overline{\overline{\iota``V_{k-1}}}$ so $T\beta > \overline{\overline{\iota^2``V_n}}$. So, as in Chapter 1, $\phi`T\beta$ has 1 or 2 members. Thus if $\phi`\alpha$ has k members, $\phi`T\alpha$ has $Tk+1$ or $Tk+2$ members, and conversely. k is finite iff $Tk+1$ and $Tk+2$ are so $T\alpha \in SM_{n+1}$.

Proof of (1) $\alpha_{n+1} \leqslant T\alpha_n$ follows from (2). If $\alpha_{n+1} < T\alpha_n$ then $\breve{T}\alpha_{n+1} < \alpha_n$ contradicting minimality of α_n. This is because $\breve{T}\alpha_{n+1} \in SM_{n+1}$ by the converse of the argument in the proof of (2) above, as $\alpha_{n+1} \leqslant T\alpha_n$. So $\alpha_{n+1} = T\alpha_n$.

Notice that $\overline{\overline{}}(\phi`\beta)$ and $\overline{\overline{}}(\phi`T\beta)$ must have different residues mod 3. We can express this in TST since $\phi`T\beta$ is $T''\phi`\beta$ plus one or two more members. This will apply to show that $\overline{\overline{\phi`\alpha_n}}$ and $\overline{\overline{\phi`\alpha_{n+1}}}$ have different residues mod 3. Now " $\overline{\overline{}}(\phi`\alpha_n) \equiv i \pmod 3$ " is a sentence in TST for each concrete i and n, so, by applying Amb to the cases $i = 0, 1, 2$ we can compel $\overline{\overline{}}(\phi`\alpha_n)$ and $\overline{\overline{}}(\phi`\alpha_{n+1})$ to have the same residues mod 3 contradicting the above. For this we use the following instances of Amb:

$$\overline{\overline{}}(\phi`\alpha_n) \equiv i \pmod 3) . \equiv . \overline{\overline{}}(\phi`\alpha_{n+1}) \equiv i \pmod 3 \quad \text{for each } i < 3$$

This is a reproduction of Specker's proof of $\sim AC$ in TST with Amb. If we wish to reproduce the proof in TST with Amb^n instead we define α_n and SM_n as before and infer

$\overline{\overline{\phi`\beta}}$ and $\overline{\overline{\phi`T\beta}}$ have different residues mod 3

$\overline{\overline{\phi`\beta}}$ and $\overline{\overline{\phi`T^2\beta}}$ 6

$\overline{\overline{\phi`\beta}}$ ... $\overline{\overline{\phi`T^3\beta}}$ 9

. . . . .

$\overline{\overline{\phi`\beta}}$ ... $\overline{\overline{\phi T^m\beta}}$ 3m

and then, for $i < 3m$ " $\overline{\overline{}}(\phi`\alpha_n) \equiv i \pmod{3m}$ " is a sentence of TST. We then apply $\mathrm{Amb}^m$ to compel $\overline{\overline{}}(\phi`\alpha_n)$ and $\overline{\overline{}}(\phi`\alpha_{n+m})$ to have the same residues mod 3m and so derive a contradiction, on an analogy with Amb above using

$$\overline{\overline{}}(\phi`\alpha_n) \equiv i \pmod{3m} . \equiv . \overline{\overline{}}(\phi`\alpha_{n+m}) \equiv i \pmod{3m} \quad \text{for each } i < 3m.$$

(iii) Let M be a model of TST. Define $h: M \to M^*$ by $h'x = \{x\}$ for $T_o(x)$ and $h'x = h''x$ otherwise. The range of h is transitive and clearly $x \in y$. $h'x \in h''y = h'y$ so h is Δ_o^{TST}-elementary. So if φ is a Δ_o^{TST} expression with one free variable we have

$$(\exists x)(\varphi(x)). \equiv . (\exists h'x)(\varphi^*(h'x))$$

whence

$$(\exists x)(\varphi(x)). \supset . (\exists x)(\varphi^*(x))$$

so Σ_1^{TST} sentences generalise upwards, so Π_1^{TST} sentences generalise downwards, so Δ_1^{TST} sentences are absolute.

To prove (iv) we continue this line of argument: $M \models \varphi \equiv \varphi^*$ as long as φ is Σ_1^{TST} so φ_n can change truth-value at most once - i.e., $\{ n : M \models \varphi_n \equiv \varphi_{n+1} \}$ is cofinite. Now take a shifting ultraproduct, $\prod_{n<\omega} M^n_- / \mathcal{U}$ for $\mathcal{U}$ nonprincipal. That will satisfy $\mathrm{Amb}(\Sigma_1^{TST})$

(vi) Assume $\mathrm{Amb}^n(\Sigma_1^{TST})$ Let φ_o be Σ_1^{TST} then by $\mathrm{Amb}^n(\Sigma_1^{TST})$ we have $\varphi_n \equiv \varphi_o$. But as φ is Σ_1^{TST} we have anyway $\varphi_o \supset \varphi_1 \supset \varphi_2 \supset \varphi_3 \ldots.$ so we have $\varphi_1 \supset \varphi_n$, $\varphi_n \supset \varphi_o$ so $\varphi_1 \supset \varphi_o$

So. $\varphi_o \equiv \varphi_1$. But φ was an arbitrary Σ_1^{TST} expression.

Metatheorem 2.10

In TNT, $\mathrm{Amb}^n(\Sigma_1^{TNT})$ is not finitely axiomatisable.

Proof:

Suppose it were, and that it were equivalent to some sentence φ. We have easily $M \models \mathrm{Amb}^n(\Sigma_1^{TNT})$ iff $M^* \models \mathrm{Amb}^n(\Sigma_1^{TNT})$ so $M \models \varphi \equiv \varphi^*$ Thus if $M \models \sim\varphi$, for all n, $M^n \models \sim\varphi$ so $\prod_{n<\omega} M^n/\mathcal{U} \models \sim\varphi$ contradicting (iv) above.

Unfortunately this does not tell us that $\mathrm{Amb}^n(\Sigma_1^{TST})$ is not finitely axiomatisable in TST as in that case we have $\varphi \supset \varphi^*$ but not the converse, as $\mathrm{Amb}^n(\Sigma_1^{TST})$ might hold on a terminal segment of M. Still less does it tell us that $\mathrm{Amb}^n(\Sigma_1^{TST}) \upharpoonright T_{<k}$ is not finitely axiomatisable for any k, which is the case that we really want. Crabbé's recent result that any model of TST in which all types are actually infinite satisfies Amb((2,n)-strat) (i.e., ambiguity for sentences which are (2,n)-stratifiable) has as a corollary that Amb((2,n)-strat) is not finitely axiomatisable either, as the property of being actually infinite is not finitely axiomatisable. (though it does show that Amb((2,n)-strat) can be given an axiomatisation in which all variables are of type 0.

Note that the proof of Metatheorem 2.9 (ii) above can be extended to give

Metatheorem 2.11

In NF, every expression is Δ_2^{NF}.

I am endebted to Maurice Boffa for pointing this out to me: I had originally proved it only for stratified expressions.

Ad 2.9 (iii), (iv) above: one uses in these two the fact that if φ is Σ_1^{TST} then TST $\vdash \varphi \supset \varphi^*$. Note that the converse does not hold, for if 'φ_n' is 'there is a nonprincipal ultrafilter on V_n' then TST $\vdash \varphi_n \supset \varphi_{n+1}$ but φ_n is definitely Δ_2^{TST}.

The natural way to construct models of TNT is to start with some $M \models$ TST and then form $\prod_{n \in \omega} M^n/\mathcal{U}$ for some nonprincipal $\mathcal{U}$. Any model thus constructed will satisfy Amb(Σ_1^{TST}). Question: can we find $M \models$ TNT that does not satisfy Amb(Σ_1^{TST})?

That is the end of the easy proofs. A glance through the proofs of 2.9 (iii) and (iv) will show that it all followed from the existence of a Δ_0^{TST} - elementary embedding from $M \to M^*$ whose restriction to initial segments of the model could be discussed inside the model. Thus one can show that $M \models \Sigma_1^{TST}$ sentences generalise upwards and then

$\prod_{i<\omega} M^i/\mathcal{U} \models \mathrm{Amb}(\Sigma_1^{TST})$. In general, a Δ_n^{LPC} - (resp. $\Sigma_n^{LPC} \cup \Pi_n^{LPC}$) -elementary embedding from $M \to M^*$ will show that $M \models \Sigma_n^{LPC}$ (resp. Σ_{n+1}^{LPC}) sentences generalise upwards whence $M \models \mathrm{Amb}(\Delta_n^{LPC})$ (resp. $\mathrm{Amb}(\Delta_{n+1}^{LPC})$ and $\prod_{i<\omega} M^i/\mathcal{U} \models \mathrm{Amb}(\Sigma_n^{LPC} \cup \Pi_n^{LPC})$(resp $\mathrm{Amb}(\Sigma_{n+1}^{LPC} \cup \Pi_{n+1}^{LPC})$. This seems the obvious way to produce stronger results on ambiguity. However there are great difficulties. To prove $\mathrm{Amb}(\Sigma_{n+1}^{LPC})$ consistent on the above plan we would want $M \models$ TST such that for some h all of whose initial segments were in M $M \models (\forall \vec{x})(\varphi(\vec{x}) \equiv \varphi^*(h"\vec{x}))$ for $\varphi \in \Sigma_n^{LPC}$ and at each type. For n sufficiently large this would have extremely powerful consequences. If h preserves enough structure to send sets of ordinals to sets of ordinals then the restriction of h to the ordinals in type k would not be <u>onto</u> the ordinals of type k + 1, otherwise this would give Burali-Forti's paradox, but it must send $No'\leqslant_0$ at type k to $No'\leqslant_0$ at type k + 1, so there must be a first ordinal moved (for this we need that all initial segments of h are coded in M) and any such ordinal must be measurable. So, for some finite n, proving the relative consistency of $\mathrm{Amb}(\Sigma_n^{LPC})$ would need feeding measurables into the construction.

If we could get them out again, having once put them in, by some construction which gave us, for every n

$Amb(\Sigma_n^{LPC}) \rightarrow (\exists h)(\ h : M \rightarrow M^* \quad M \models (\forall \vec{x})(\varphi(\vec{x})$ $\varphi^*(h^{\text{"}}\vec{x}))$ for each $\varphi \in \Sigma_k^{LPC}$ and at each type, with $k \leqslant n$ but increasing with n)

we would have the staggering corollary that

$Con(NF) \rightarrow Con(TST + \exists\, \text{measurable})$

The first ordinal α is measurable because $\{x \subseteq seg_{<}{}'\alpha : \alpha \in h'x\}$ is a normal ultrafilter on $seg_{\leqslant_0}{}'\alpha$. Now this is going to depend on h'x being one type higher than x. If we start with $Amb^n(\Sigma_k^{LPC})$ with $n > 1$ the construction of such an h would not give us a measurable as we cannot construct the normal ultrafilter on α, so the subscheme Amb^n could well turn out to be weaker than the full scheme even though it too is enough to prove $\sim AC$.

There is faint light at the end of the tunnel: it has been a puzzle for some time that AC should fail in NF. We understand the proof of course, but nobody can explain how it comes to be there. However, if, given $Amb(\Sigma_n^{LPC})$ we could weave a Σ_k^{LPC}-elementary embedding from $M \rightarrow M^*$ ($k \leqslant n$ and increasing with n) then Amb would give us a collection of arbitrarily good embeddings from M into M^*....and we know, that in ZF, a non-trivial elementary embedding from the universe into itself contradicts AC (Kunen [54]) . Perhaps a similar argument is waiting to be discovered here.

These results are probably best possible when we do not put strong conditions on our models. However, if we are prepared to look at nonclassical systems we can still get results by this shifting ultraproduct method. Intuitively one might expect it to be easier to find models for intuitionistic versions of NF than for NF tout court, for in proving Con(NF) we seek an $M \models$ TST such that $M \not\models \varphi_n \equiv \varphi_{n+1}$. This is equivalent to $M \not\models ((\varphi_{n+1} \wedge \sim\varphi_n) \vee (\sim\varphi_{n+1} \wedge \varphi_n))$. In the classical case this is highly nontrivial. However, in the intuitionistic case they are not equivalent. If M is a prime intuitionistic model $M \not\models ((\varphi_{n+1} \wedge \sim\varphi_n) \vee ((\sim\varphi_{n+1} \wedge \varphi_n))$ is quite plausible, for otherwise, by primeness, $M \models (\varphi_n \wedge \sim\varphi_{n+1})$ or $M \models (\varphi_{n+1} \wedge \sim\varphi_n)$ either of which is unlikely, as TST $\vdash \varphi_n$ iff TST $\vdash \varphi_{n+1}$ for n sufficiently large and TNT $\vdash \varphi_n$ iff TNT $\vdash \varphi_{n+1}$ anyway. I would conjecture that NF weakened by deleting all axioms containing 'v' and substituting intuitionistic for classical predicate logic is equiconsistent with TST + Ax Inf. The consistency of a related but weaker system is proved below by a modification of the shifting ultraproduct method. I shall develop the technique in some generality before applying it to type theory.

It is well known that Łoś's Theorem applies only to reduced products which are reduced modulo an ultrafilter, as some of the induction steps in the proof depend on the filter in question being maximal. Sometimes this is rather a pity as a reduced product modulo a proper non-maximal ultrafilter can have desirable modeltheoretic properties not available to any ultraproduct. I shall here outline a method of getting the best of both worlds at the cost of having an intuitionistic satisfaction relation on the model instead of a classical one.

Suppose we have a family of models of set theory with primitives ε and $=$ $\{ A_i : i \varepsilon I \}$ then we can define a Kripke model whose 'possible worlds' are nonprincipal filters on I. (This is less natural but more typographically convenient than taking reduced products modulo those filters to be the possible worlds).

Let F, F' be filters over I and let f, g be elements of the direct product $\prod \{ A_i : i \varepsilon I \}$. We can define

$$F \models f \varepsilon g \quad \text{iff}_{df} \quad \{ i : f'i \ \varepsilon \ g'i \} \ \varepsilon \ F$$

$$F \models f = g \quad \text{iff}_{df} \quad \{ i : f'i = g'i \} \ \varepsilon \ F$$

and the recursions for molecular formulae are as usual (e.g. Fitting [51] p 45)

$$F \models \sim \varphi \qquad \text{iff}_{df} \quad (\forall F' \supseteq F)(F' \not\models \varphi)$$

$$F \models \varphi \vee \delta \qquad \text{iff}_{df} \quad F \models \varphi \ \ v \ \ F \models \delta$$

$$F \models \varphi \wedge \delta \qquad \text{iff}_{df} \quad F \models \varphi \ \wedge \ F \models \delta$$

$$F \models \varphi \rightarrow \delta \qquad \text{iff}_{df} \quad (\forall F' \supseteq F)(F' \models \varphi . \supset . \ F' \models \delta)$$

$$F \models (\exists x)(\varphi(x)) \text{iff}_{df} \quad (\exists f)(F \models \varphi(f))$$

$$F \models (\forall x)(\varphi(x)) \text{iff}_{df} \quad (\forall f)(\forall F' \supseteq F)(F' \models \varphi(f))$$

Lastly, we set

$$\models \varphi \ \text{iff}_{df} \quad (\forall F)(F \models \varphi).$$

Let $\{ A_i : i \varepsilon I \}$ be a family of models of set theory and let $\mathbb{F}$ be a collection of nonprincipal filters on I satisfying $(\forall F)(\forall x)(F \varepsilon \mathbb{F} \wedge x \not\varepsilon F \wedge I - x \not\varepsilon F . : \supset$ $: (\exists F', F'' \supseteq F)(x \varepsilon F' \wedge I - x \varepsilon F'' \wedge F' \varepsilon \mathbb{F} \wedge F'' \varepsilon \mathbb{F}))$

then we can make the definition

Definition 2.12 (i) F_o for $\bigcap \mathbb{F}$

(ii) $\prod \{ A_i : i \varepsilon I \} // \mathbb{F}$ is the Kripke model described above with its satisfaction relation.

We can now prove a version of Łoś's theorem for these 'intuitionistic ultraproducts'.

Metatheorem 2.13

Let φ be an expression built up from atomic sentences using only $\rightarrow$, $\wedge$, $\sim$, and the two quantifiers. Then

$(\forall F \;\varepsilon\; \mathbb{F})(\, F \models \varphi \;.\!\equiv\!.\; \{\, i : A_i \models \varphi \,\} \;\varepsilon\; F)$

Proof:

By induction on the composition by quantifiers and connectives. When φ is atomic the proof is provided by the definitions. We have five cases to consider in the proof by induction.

($\wedge$) φ is of the form $\ulcorner \delta \wedge \gamma \urcorner$ and the theorem holds for δ and γ. $F \models \varphi$

iff $F \models \delta \wedge \gamma$

iff $F \models \delta$ and $F \models \gamma$

iff $\{\, i : A_i \models \delta \,\} \;\varepsilon\; F$ and $\{\, i : A_i \models \gamma \,\} \;\varepsilon\; F$.

iff $\{\, i : A_i \models \delta \wedge \gamma \,\} \;\varepsilon\; F$

iff $\{\, i : A_i \models \varphi \,\} \;\varepsilon\; F$

($\sim$) φ is of the form $\ulcorner \sim \delta \urcorner$ and the theorem holds for δ. $F \models \varphi$

iff $F \models \sim \delta$

iff $(\forall F' \supseteq F)(\, F' \not\models \delta)$

iff $\{\, i : A_i \models \quad \delta \,\} \;\varepsilon\; F$ (by use of the condition on $\mathbb{F}$ on the preceding page).

($\exists$) φ is of the form $\ulcorner (\exists x)(\gamma(x)) \urcorner$ and the theorem holds for $\gamma(x)$.

$$F \models \varphi$$

$$\text{iff} \quad F \models (\exists x)(\gamma(x))$$

$$\text{iff} \quad (\exists f)(F \models \gamma(f))$$

$$\text{iff} \quad (\exists f)(\{ i : A_i \models \gamma(f'i) \} \varepsilon F$$

$$\Rightarrow \quad \{ i : A_i \models (\exists x)(\gamma(x)) \} \varepsilon F$$

For the converse, given $\{ i : A_i \models (\exists x)(\gamma(x)) \} \varepsilon F$ we construct $f \varepsilon \prod_{i \varepsilon I} A_i$ by setting $f'i$ to be some x such that $A_i \models \gamma(x)$ when there is one and an arbitrary element otherwise. Then $\{ i : A_i \models \gamma(f'i) \} \varepsilon F$, so $F \models \gamma(f)$ so $F \models (\exists x)(\gamma(x))$

($\rightarrow$) φ is of the form $\ulcorner \delta \rightarrow \gamma \urcorner$ and the theorem holds for δ and γ.

$$F \models \varphi$$

$$\text{iff} \quad F \models \delta \rightarrow \gamma$$

$$\text{iff} \quad (\forall F' \supseteq F)(F' \models \delta . \supset . F' \models \gamma)$$

$$\text{iff} \quad (\forall F' \supseteq F)(\{ i: A_i \models \delta \} \varepsilon F' . \supset . \{ i : A_i \models \gamma \} \varepsilon F')$$

$$\text{iff} \quad (\forall F' \supseteq F)(\{ i: A_i \models \delta \rightarrow \gamma \} \varepsilon F')$$

$$\text{iff} \quad \{ i : A_i \models \delta \rightarrow \gamma \} \varepsilon F$$

($\forall$) φ is of the form $\ulcorner (\forall x)(\gamma(x)) \urcorner$ and the theorem holds for $\gamma(x)$.

$$F \models \varphi$$

$$\text{iff} \quad F \models (\forall x)(\gamma(x))$$

$$\text{iff} \quad (\forall f)(\forall F' \supseteq F)(F' \models \gamma(f))$$

$$\text{iff} \quad (\forall f)(\forall F' \supseteq F)(\{ i : A_i \models \gamma(f'i) \} \varepsilon F')$$

$$\text{iff} \quad (\forall f)(\{ i : A_i \models \gamma(f'i)\} \varepsilon F)$$

iff $\{ i : A_i \models (\forall x)(\gamma(x)) \} \varepsilon F$

(I am endebted to Professor S. Feferman for pointing out to me that this theorem holds for '$\rightarrow$' and '$\forall$' as well. I originally asserted it only for $\exists$, $\sim$ and $\wedge$.)

Corollary 2. 14

For expressions φ not containing 'v',

$\prod_{i \varepsilon I} A_i /\!\!/ \mathbb{F} \models \varphi$ iff $\{ i : A_i \models \varphi \} \varepsilon F_o$

Proof immediate from above.

Theorem 2. 15

When all A_i are identical and F_o is the filter of cofinite subsets of I any permutation of I gives rise to an automorphism of $A^I /\!\!/ \mathbb{F}$

Proof:

Let δ be any permutation of I and π the automorphism of P'I induced by δ. As a trivial consequence of the definitions we have, for all f, g εA^I and for all $F \varepsilon \mathbb{F}$

$$F \models f \varepsilon g \quad \text{iff } \pi"F \models f|\delta \ \varepsilon \ g|\delta$$

$$F \models f = g \quad \text{iff } \pi"F \models f|\delta = g|\delta$$

So we prove by induction on the composition by quantifiers and connectives that

$$F \models \varphi(\vec{x}) \quad \text{iff } \pi"F \models \varphi(\vec{x}\restriction\delta)$$

We now want

$$A^I /\!\!/ \mathbb{F} \models \varphi(\vec{f}) \quad \text{iff } A^I /\!\!/ \mathbb{F} \models \varphi(\vec{f}\restriction\delta)$$

we argue $\quad A^I /\!\!/ \mathbb{F} \models \varphi(\vec{f})$

$$\text{iff} \quad (\forall F \supseteq F_o)(\ F \models \varphi(\vec{f})\ )$$

$$\text{iff} \quad (\forall \pi''F \supseteq \pi''F_o)(\ \ \pi''F \models \varphi(f \upharpoonright \delta))$$

but if F_o is the filter of cofinite subsets of I then $\pi''F_o = F_o$ and any $F \supseteq F_o$ is $\pi''G_1$ and $\breve{}\pi''G_2$ for some G_1 and G_2 so this is equivalent to

$$(\forall F \supseteq F_o)(\ F \models \varphi(f \vec{\upharpoonright} \delta))$$

which is of course $A^I /\!\!/ \mathbb{F} \models \varphi(f \upharpoonright \delta)$

We can now apply the method of Theorem 2.15 to get a model $\mathbb{M}$ of intuitionistic type theory such that $\mathbb{M} \models \varphi$ iff $\mathbb{M} \models \varphi^*$

Theorem 2. 16

If $M \models$ TNT and F_o is the filter of cofinite subsets of $\omega^* + \omega$ then $\mathbb{M} = \prod_{n \in \omega^* + \omega} M^n /\!\!/ \mathbb{F}$ satisfies $\mathbb{M} \models \varphi$ iff $\mathbb{M} \models \varphi^*$

Proof:

The permutation δ of the integers that we want is defined by $\delta\colon\ n \longmapsto n + 1$. Write f^* for $f|\delta$. After the manner of theorem 2.14 we prove $\models f \,\varepsilon\, g$ iff $\models f^* \,\varepsilon\, g^*$, similarly for $=$. So as far as 'ε' and '=' are concerned * is an automorphism. For the type predicates we have $F_o \models T_n(f)$ iff $F_o \models T_{n+1}(f^*)$. So we then get by induction on composition by quantifiers and connectives $\models \varphi(\vec{f})$ iff $\models \varphi^*(\vec{f \restriction \delta})$ whence the theorem.

When we have a model for TNT with a tsau one's reaction is to divide out immediately to obtain a model for NF. However, as the model here is an intuitionistic one the proof of Specker's corollary does not go through. So, rather than divide out $\prod_{i \in \mathbb{Z}} M_i // \mathbb{F}$ by $*$ we divide out instead the submodel $\mathbb{M}$ defined as follows

Definition 2.17 $\mathbb{M} =_{df} \{ f : (\exists n)(F_o \models T_n(f)) \}$

The objects in $\mathbb{M}$ are much better behaved - for example we see easily that for any f in $\mathbb{M}$ we have $F_o \models T_o(f)$ iff $(\forall n)(F_o \models T_n(f^n))$ iff $(\forall n)(\forall F)(F \models T_n(f^n))$ where "f^n" is an abbreviation of " $*^n$'f ". It is easy to check that Metatheorem 2.13 holds for $\mathbb{M}$ as for $\prod_{i \in \mathbb{Z}} M_i // \mathbb{F}$

We are now set to divide out by $*$.

Definition 2.18

If φ is an expression in the language of set theory let $\varphi^{(\mathbb{M}/*)}$ be the result of restricting all bound variables in φ to T_o and writing ' $x \in y*$ ' instead of ' $x \in y$ ' Further, write ' $\mathbb{M}/* \models \varphi$ ' instead of ' $\mathbb{M} \models \varphi^{(\mathbb{M}/*)}$ '

Metatheorem 2.19

If φ is (2,n)-stratifiable then for all $F \in \mathbb{F}$

$F \models \varphi^{(\mathbb{M}/*)}(x, y, \ldots)$ iff $F \models \varphi'(x^n, y^m, \ldots)$

where the n and m ... are the number of asterisks attached to x and y ... and correspond to the type of x, y, ... in φ.

Proof:

When φ is (2,n)-stratifiable and does not contain quantifiers over objects of type 2 there is nothing to prove. Now if it does, we have two cases to consider:

(i) φ is $(\exists x)(\delta(x))$ x of type 2 in φ

$F \models \varphi^{(\mathbb{M}/*)}$

iff $F \models (\exists x)(\delta^{(\mathbb{M}/*)}(x) \wedge T_0(x))$ (Now by induction hyp. $F \models \delta^{(\mathbb{M}/*)}(x)$ iff $F \models \delta'(x^*)$ and $F \models T_0(x)$ iff $F \models T_1(x^*)$ anyway.)

iff $(\exists x)\ F \models \delta^{(\mathbb{M}/*)}(x) \wedge T_0(x)$

iff $(\exists x)\ F \models \delta'(x^*) \wedge T_1(x^*)$

iff $(\exists x^*)\ F \models \delta'(x^*) \wedge T_1(x^*)$

iff $(\exists z)\ F \models \delta'(z) \wedge T_1(z)$

iff $F \models (\exists z)(\delta'(z))$

iff $F \models \varphi'$

(ii) φ is of the form $(\forall x)(\delta(x))$

$F \models \varphi^{(\mathbb{M}/*)}$

iff $F \models (\forall x)(T_0(x) \rightarrow \delta^{(\mathbb{M}/*)}(x))$

iff $(\forall x)(\forall F' \supseteq F)(F' \models T_0(x) .\supset.\ F' \models \delta^{(\mathbb{M}/*)}(x))$

which by induction hyp. is equivalent to

$(\forall x)(\forall F' \supseteq F)(F' \models T_1(x^*) . \supset . F' \models \delta'(x^*))$

$(\forall x^*)(\forall F' \supseteq F)(F' \models T_1(x^*) . \supset . F' \models \delta'(x^*))$

now reletter x^* as z

$(\forall z)(\forall F' \supseteq F)(F' \models T_1(z) . \supset . F' \models \delta'(z)$

$$F \models (\forall z)(T_1(z) \rightarrow \delta'(z))$$

$$F \models \varphi'$$

In the induction step for all other connectives there is nothing to prove.

Metatheorem 2.20

If φ is built up from (2,n)-stratifiable sentences by means of the two quantifiers, conjunction, disjunction and has no ' v ' inside the scope of a universal quantifier, and is stratified, then

$\models \varphi^{(\mathbb{M}/*)}(x, y, \ldots)$ iff $\models \varphi'(x^n, y^m, \ldots)$ where n and m are as above.

Proof:

By induction on the composition by quantifiers and connectives. Consider them in turn:

$(\forall)$ Let φ be of the form $\ulcorner(\forall x)(\delta(x))\urcorner$ where $\ulcorner\delta\urcorner$ does not contain ' v '. We have

$$\models \varphi^{(\mathbb{M}/*)}$$

$$\text{iff} \models (\forall x)(T_0(x) \rightarrow \delta^{(\mathbb{M}/*)}(x))$$

by Łos's theorem this is equivalent to

$$\{ n : \underline{M}^n \models (\forall x)(T_0(x) \rightarrow \delta^{(\mathbb{M}/*)}(x) \} \in F_0$$

iff $\forall x \{ n : \underline{M}^n \models T_o(x'n) \to \delta^{(\mathbb{M}/*)}(x'n)\} \in F_o$

Abbreviate this sentence to A. We shall want it to be be equivalent to the sentence B:

$$\forall x \quad \{ n : \underline{M}^n \models T_o(x'n)\} \in F_o \;.\supset.$$
$$\{ n : \underline{M}^n \models \delta^{(\mathbb{M}/*)}(x'n)\} \in F_o$$

A $\to$ B is trivial. For the converse we use the fact that x is standard and that therefore $\{ n : \underline{M}^n \models T_o(x'n)\}$ is finite or cofinite. Suppose B were true and A false. If A is false, $\underline{M}^n \models T_o(x'n)$ must hold infinitely often, so $\{ n : \underline{M}^n \models T_o(x'n)\}$ is infinite and therefore cofinite. As B is to be true, we infer that $\{ n : \underline{M}^n \models \delta^{(\mathbb{M}/*)}(x'n) \}$ is cofinite too, and therefore in F_o. But if a, b are cofinite subsets of ω so is $a \Rightarrow b$ $(-a \cup b)$ so A must have been true, so A iff B. We must now look closely at B. We see it is equivalent to

$$F_o \models T_o(x) \quad \supset \quad F_o \models \delta^{(\mathbb{M}/*)}(x) \qquad \text{iff}$$
$$F \models T_n(x^n) \quad \supset \quad F_o \models \delta'(x^n) \qquad \text{iff}$$
$$F \models T_n(x^n) \quad \supset (\forall F) F' \models \delta'(x^n) \qquad \text{(take the}$$

quantifier to the front) iff

$(\forall F')\ F \models T_n(x^n) \quad \supset \quad F' \models \delta'(x^n)$ (close) iff

$(\forall x)(\forall F)(\forall F')(F \models T_n(x^n) \supset F' \models \delta'(x^n))$ Now substitute F/F'. This step can be reversed because, as x is standard, $F \models T_n(x^n)$ iff $F' \models T_n(x^n)$ for all F, F'.

$\forall x \forall F \quad F \models T_n(x^n) . \supset . \; F \models \delta'(x^n)$

$F_o \models (\forall x)(T_n(x^n) \rightarrow \delta'(x^n))$

$\models \varphi'$

When φ is of the form $\ulcorner (\exists x)(\delta(x)) \urcorner$

$\models \varphi^{(\mathbb{M}/*)}$

iff $\models (\exists x)(T_o(x) \wedge \delta^{(\mathbb{M}/*)}(x))$

If x is of type n in δ we shall want to substitute ' $T_n(x^n)$ ' for ' $T_o(x)$ '. As before we can do this.

iff $\exists x \models T_o(x) \wedge \delta^{(\mathbb{M}/*)}(x)$

iff $\exists x \models T_n(x^n) \wedge \delta^{(\mathbb{M}/*)}(x)$ which by induction hypothesis is equivalent to

$\exists x \models T_n(x^n) \wedge \delta'(x^n)$ reletter

$\exists z \models T_n(z) \wedge \delta'(z)$

$\models (\exists z)(T_n(z) \wedge \delta'(z))$

$\models \varphi'$

When φ is of the form ' $\delta \vee \pi$ '

$\models \varphi^{(\mathbf{M}/*)}$

iff $\models \delta^{(\mathbb{M}/*)} \vee \pi^{(\mathbb{M}/*)}$

Iff $\models \delta^{(\mathbf{M}/*)}$ or $\models \pi^{(\mathbb{M}/*)}$

iff $\models \delta'$ or $\models \pi'$

iff $\models \varphi'$

When the main connective in φ is ' $\wedge$ ' the proof is exactly the same as for ' v ' reading ' $\wedge$ ' for ' v ' throughout.

Definition 2.21

If φ is stratified, contains no occurrence of ' v ' and is built up from (2,n)-stratifiable sentences by means of the two quantifiers and conjunction then φ is permissible.

Metatheorem 2.22

If φ is permissible, then $\mathbb{M}/* \models \varphi(x_1, x_2, x_3 \ldots)$ iff $\{ n : M^n_- \models \varphi(x_1{}'n+k_1, x_2{}'n+k_2, x_3{}'n+k_3 \ldots\ldots) \} \varepsilon F_0$ where k_i is the type of x_i in φ'

Proof: This is an immediate corollary of the preceding metatheorem.

We can now set about identifying the unstratified φ such that $\mathbb{M}/* \models \varphi$. The easiest case is when φ **arises from a stratified** expression δ with two free variables x and y, with y one type higher than x. If $\delta(x, y)$ is permissible we have

$$\mathbb{M}/* \models \delta(x, y) \quad \text{iff} \quad \mathbb{M} \models \delta(x, y^*)$$

(for ease of notation I am considering only the case whe where x is of type 0 in δ) Now substituting x for y we get $\mathbb{M}/* \models \delta(x, x)$ iff $\mathbb{M} \models \delta(x, x^*)$ By results above the RHS is equivalent to $\{ n : M^n_- \models \delta(x'n, x'n+1) \} \varepsilon F_0$. If δ is a relation such that there is an infinite δ-chain in M^0 this shows easily that $\mathbb{M}/* \models \delta$ has a fixed point. In particular, $\mathbb{M}/*$ has Quine atoms, since these are fixed points for ϵ.

Metatheorem 2.22 explains why we can only prove Metatheorem 2.20 for this apparently rather restricted collection of formulæ, as Metatheorem 2.22 has the following corollary

Corollary 2.23

If φ is a permissible expression with two free variables one one type higher than the other, such that there is an infinite φ-chain in M_0 then $\mathbb{M}/*$ $\nvDash \varphi$ has a fixed point.

Proof above.

(note that if we are seriously using this to get consistency results we can assume M_0 to be saturated in which case we need only require that M_0 has arbitrarily long finite φ-chains)

That this cannot work for arbitrary φ was shown by Boffa and Crabbé, who, in response to my conjecture that if NF is consistent it remains so on adjoining, for each φ with two free variables one one type higher than the other where φ has an infinite φ-chain in ML, a fixed point for φ, produced the following counterexample. Let $\varphi(x, y)$ read

$(y = \{x\} \wedge (\forall w)(x \neq \{w\})) \vee ((\exists w)(x = \{w\} \wedge y = \{x, 0, 1\})$

Plainly there is no fixed point for this φ. This explains why we cannot prove Metatheorem 2.18 for sentences containing ' v ' (or '$\sim$' applied to formulae that are not (2-n)-stratifiable, as even if we write the ' ... v --- ' in φ as '$\sim$(... $\wedge$ ----)' contradiction still ensues.

Now the restricted nature of metatheorems 2.20 and 2.22 makes life rather difficult when we wish to deal with ordered pairs in $\mathbf{M}/*$. This easiest way out of this is to start with a model M_0 of TST + AC + Ax Inf in which ordered pairs are taken as primitive, with the following axioms.

$$\ulcorner(\forall x)(\forall y)(\, T_n(x) \wedge T_n(y) . \supset .(\exists z)(\, T_n(z) \quad z = \langle x, y\rangle))\urcorner$$

$$\ulcorner(\forall z)(\, T_n(z). \supset .(\exists x)(\exists y)(\, T_n(x) \wedge T_n(y) \wedge z = \langle x, y\rangle\,))\urcorner$$

Then we can define ordered pairs in $\prod_{i<\omega} M^n_i /\!/ \mathbf{F}$ and $\mathbf{M}$ in the obvious way. When we have done this, $z = \langle x, y\rangle$ is (1,n)-stratifiable rather than (4,n) stratifiable and is quite manageable.

With ordered pairs taken as primitives in this way, expressions such as ' x is a permutation of V' 'x is a wellorder ',' x is an equivalence relation ' become (2,n)-stratifiable. Indeed, some uses of the J function (definition 0.30) are definable by a permissible expression thus: ' y = J'x $\wedge$ y is a permutation of V ' becomes

$(\forall a)(\exists b)(\,\langle a,b\rangle \in y \wedge (\forall w)(w \in a \equiv (\forall u)((\langle u,w\rangle \in x . \supset$ u $\in$ b))) $\wedge$ x is a permutation of V $\wedge$ y is a permutation of V.

Clearly any M $\models$ TST will have an infinite J-chain so any $\mathbb{M}/*$ will have a fixed point for J. It is easy to check that any permutation of V which is a fixed point for J will be an automorphism. (Though we will probably not be able to prove in $\mathbb{M}/*$ that these fixed points will be automorphisms) Indeed there will be a lot of these objects which are 'really' automorphisms and we can characterise the group of them quite adequately by looking at what they arise from: it is isomorphic to the group of type-preserving automorphisms of terminal segments of M, or which is the same thing, the direct limit of G_n for $n < \omega$, where G_n is the group of permutations of the objects of type n.

Axcount fails in $\mathbb{M}/*$. We can establish this easily by noting that "x is finite and not the same size as y" is permissible, and then constructing a function whose value is pointwise finite and increasing in size. Clearly any such f is finite$^{(\mathbb{M}/*)}$ and noncantorian$^{(\mathbb{M}/*)}$. Note, however, that this construction, although it sinks Axcount$_{\leqslant}$, does not apparently sink Axcount$_{\geqslant}$. To be precise, if we construct $f \in \mathbb{M}/*$ which is of power n at type n (considered as an element of the direct product) we show easily that f is larger$^{(\mathbb{M}/*)}$ than ι"$f^{(\mathbb{M}/*)}$. We can do this because there is an increasing ω-

sequence of finite cardinals, but, at least in the presence of AC_ω, no infinite descending sequence of finite cardinals (we need to read "finite" as "dedekind finite" to make finitude a permissible predicate) which we would need to refute $\text{Axcount}_\geqslant$ by this method. This concurs with the experience of ch.1., that $\text{Axcount}_\geqslant$ is much weaker and more difficult to refute than $\text{Axcount}_\leqslant$. The present considerations suggest that $\text{Axcount}_\geqslant$ is in some obscure way tied up with AC_ω in a fashion that $\text{Axcount}_\leqslant$ is not. It would be nice to see this emerge in a more formal manner.

The system for which $\mathbf{M}/_*$ provides us with a consistency proof is thus slightly more than a nonclassical subsystem of NF. Let us call it M. Then the axioms of M are

(i) All sentences φ such that φ is the result of deleting all type indices from an expression γ of TNT where (a) γ is a theorem and (b) γ is built up from 2-stratifiable expressions by means of $\rightarrow$, $\wedge$, $\sim$, $\exists$, and $\forall$.

(ii) All sentences of the form $\ulcorner(\exists x)(\varphi(x, x))\urcorner$ where φ is the result of deleting all type indices from an expression $\ulcorner\gamma(x, y)\urcorner$ of TNT which satisfies (b) above, such that y is one type higher than x in $\gamma(x, y)$ and we can show in the metalanguage for TNT that there is an infinite chain $\langle x_n : n \in \omega\rangle$ such that $\forall n \ \ \gamma(x_n, x_{n+1})$. All this is embedded in intuitionistic predicate calculus. M thus extends intuitionistic NF_2.

The usual proof that the consistency of an intuitionistic system implies the consistency of the corresponding classical system (e.g. [57]) does not work here as it depends on constructing an inner model by recursion on $\in$ which is of course not possible in the Quine systems.

Lastly it is evident that a similar construction can be made for any stratified theory and with similar results. e.g., if str(ZF) is that subtheory of ZF whose axioms are the stratified theorems of ZF and strp(ZF) that subtheory of str(ZF) whose axioms are permissible then we have Con(str(ZF)) $\rightarrow$ Con(strp(ZF) + Every permissible type-raising operation with an ω-chain in str(ZF) has a fixed point. Amusing though this is, it is still a long way from a proof of Con(str(ZF) + $(\exists x)(x = p'x)$) which would give us a proof of Con(NF) and this does not look too implausible.

Chapter 3

Ways of strengthening NF

From the point of view of the classical set theorist NF is an unsatisfactory system in a number of ways. The failure of AC is one, the lack of Axcount another. We are in need of ways of strengthening NF in ways which are both mathematically natural and will admit, sooner or later, of some sort of consistency proof, whether of a formal nature (as of ZFC relative to ZF) or of the nature of appealing to the fact that the new axiom does not essentially change the way in which the system behaves but merely makes it larger (as of ZFI relative to ZF). Henson has proposed an axiom which he calls 'CS': "All wellordered cantorian sets are strongly cantorian". Most of the Séminaire Hénéfiste are uneasy about this axiom. Henson gives it a nice formulation which brings out its analogy with Axcount, but the two are hardly comparable, as Axcount merely requires good behaviour on the part of finite sets, which leaves us with the hope that any 'normal' system strong enough to prove the consistency of NF might also prove Con(NFC). CS on the other hand compels quite large cardinals to be strongly cantorian, and at the same time compels sets which are not obviously large to have partitions of quite large size.

For let α be a cardinal of strongly cantorian rank. Then $\rho''\{ \beta : 2^{\beta} = \alpha \}$ must be cofinal in $\rho'\alpha$. As a consequence if α is a regular strongly cantorian initial ordinal, we have $\text{Card}'\alpha \leqslant_{*} {}^{=}(\tau '\bar{\bar{\bar{V}}})$ (as $\bar{\bar{\bar{V}}}$ is of $>$ stcan rank). Rosser's proposal, AC_{can} (i.e., the axiom of choice for wellordered sets of cantorian sets) is open to the objection that it implies $\sim AC_{wo} \rightarrow$ $(\exists \alpha \in WC)(\alpha < T\alpha)$ (Proof: let α be $(\mu\beta)(\sim AC_{\beta})$. (we have $\alpha \neq T\alpha$ otherwise AC_{α} and $\alpha \leqslant T\alpha$ because $AC_{T\alpha}$ $\rightarrow AC_{\alpha}$)

I feel that too much attention has been given to trying to find axioms that entail good behaviour on the part of small sets, which is to say, to getting small sets to behave as though they were in ZF and not in NF, and that we would be more likely to bring out the peculiar virtues of NF by adding new axioms on <u>all</u> sets. e.g., I would argue for the prime ideal theorem as a new axiom over AC_{can} and for $(\forall\alpha \in NC)(\alpha \geqslant_{*} T\alpha)$ in preference to CS. It is especially important to extend the Quine systems in a uniform way such as this if they are going to continue to be of the tentative interest to category theorists that they are at the moment : the whole point about the appeal they make to category theorists is that they are, in their unvarnished forms, impartial between large and small sets. For more on the Quine systems and category theory see Feferman [7]. Since we are stuck with, for example, the failure of the axiom of foundation, we may as well make a virtue of necessity and start adding illfounded objects with nice properties.

In ZF a lot of the proposed new axioms have the form of fixed point postulates. The axiom of infinity itself is equivalent to "Every continuous function on the ordinals has a fixed point". The existence of Ramsey cardinals (or weakly compact cardinals) is equivalent to the existence of a fixed point for the function $f : \alpha \mapsto (\mu\beta)(\beta \to (\alpha)^{<\omega})$ (or for the function $g : \alpha \mapsto (\mu\beta)(\beta \to (\alpha)^2)$. Even the consistency of NF can be expressed as a fixed point postulate, for let X be the space of isomorphism types of models of TNT with the usual topology. For $M \models$ TNT, let $[M]$ be the equivalence class of M. Then the function * defined by $[M]^* = [M^*]$ is a homeomorphism of M and a fixed point for * is an equivalence class of models of TST + Amb.

I would like to propose an axiom scheme of this nature for NF. We saw at the end of the last chapter that, because of the Boffa-Crabbé counterexample, we cannot suppose that arbitrary type-raising operations have fixed points. However, there is nothing obviously wrong with the following

Axiom Scheme PF : Let φ be an expression as in the hypothesis of Metatheorem 2.23. Then φ has a fixed point.

A number of the consistency results already obtained for NF by Scott's permutation method are of a form stating that a certain type-raising operation has a a fixed point:

$(\exists x)(\ x = \{x\}\ )$ (Scott [42])

$(\exists x \neq V)(\ x = p'x)$ (Henson [16])

and when I suggested the axiom scheme, Hinnion and Pétry proved consistent modulo NF

$(\exists x)(\ x = \{\ y : x \varepsilon y\ \})$

(Boffa has pointed out that the type-raising operation here, $f : x \to \{\ y : x \varepsilon y\ \}$ has the amusing (and potentially useful?) property that $(\forall x,y)(\ x \varepsilon y. \equiv . f'x \varepsilon f'y)$, as does the related operation $g : x \to \{\ y : x \not\varepsilon y\ \}$). I mentioned in the preceding chapter that a fixed point theorem for permissible operations would give us non-trivial automorphisms of the universe. We can go into more detail here. Arguing in NF we note that if π is a permutation of order n, $J'\pi$ is a permutation of order Tn. So $\pi = J'\pi$ is an automorphism of order n = Tn. This implies that any automorphism of V is of cantorian order. Also note that "π is a permutation of order n" is permissible for each n so the proposed scheme would give us a rich structure of automorphisms of V. The sentence "π is a permutation of infinite order" does not appear to be permissible, and PF does not seem at the moment to imply Axcount. The question of the existence of these objects has been raised by others, including Boffa and Hinnion. At the moment all we can prove about nontrivial automorphisms of V is the following

Remark 3.1 (i) If X is a set of automorphisms of V then stcan(X)

(ii) If '$\varphi(x,y)$' is an expression which defines an automorphism of V, then $\{\langle x,y\rangle : \varphi(x,y)\}$ (if it exists) is in the centre of the group of automorphisms.

Proof

(i) is a trivial corollary of ch 1 p 26

(ii) By elementarity, any definable object must be sent to itself by any automorphism, and a fortiori, the same must hold for any definable automorphism.

Another motivation for PF arises from the possibility of using forcing to prove Con(NF). In ZF, whenever we collapse some cardinal α onto $\aleph_0$ say, we take as conditions the finite sequences of ordinals less than α. If our model is M we find that the generic collapsing map meets every function in ${}^{\omega}\alpha^{(M)}$. If something similar happens when we do the rather more complicated collapsing which is constructing a tsau on top of a model of type theory we would expect something like

If f is a type-raising operation, π a tsau, then $f \cap \pi \neq \Lambda$ which is $(\exists x)(\ f'x = \pi'x)$. So if we write '$\varphi(x,y)$' for '$y = f'x$' we have $(\exists x)(\ \varphi(x,\pi'x))$. Now if we decode this into the language of NF as in the proof of Metatheorem 2.5 we find that the model obtained by factoring out by π satisfies $(\exists x)(\varphi(x,x))$. The Boffa-Crabbé counterexample comes to mind at once of course to show that it will not really be this simple, but the idea is there.

Forcing will give us a very nice proof of $\mathrm{Con}(NF_2)$ (the subsystem of NF whose axioms are precisely the (2, n) stratifiable axioms of NF) The consistency of NF_2 is folklore, but the application of forcing to it is not.

Start in a model of ZF satisfying CH, for simplicity's sake. Take the set of forcing conditions to be $[\mathbb{R}]^{<\omega}$ - the set of finite sequences of reals and let $\mathbb{B}$ be the corresponding boolean algebra. Then we let e be the composition of the canonical generic map and set inclusion. Then we can establish easily enough that $\langle p'\omega, e\rangle$ is a model for NF_2. This much is fairly routine - roughly any boolean algebra with as many atoms as elements can be turned into a model of NF_2: what is not routine is that the model obtained in this way from a generic collapse will also contain Quine atoms (objects x such that $x = \{x\}$). This (and similar results on fixed points of (2,n)-stratifiable type-raising operations) will follow from the combinatorial properties of the generic map discussed above.

There is at least one type-raising operations for which a fixed point would give us an inner model for NF. It is permissible and therefore covered by PA. If $\equiv$ is an equivalence relation on V, let $\equiv'$ be defined by $x \equiv' y$ iff_{df} $(\exists f \in 1-1)(f"x = y \wedge (\forall w \in x)(w \equiv f'w))$. Suppose $\equiv$ is a nontrivial fixed point.

Look at $V/\equiv$. Let $[x]$ be the equivalence set of x under $\equiv$. Then $\{ x : [x] = \{x\} \}$ is an inner model.

Before we leave the subject of fixed points I would like to point out that in Str(ZF) (the subtheory of ZF whose axioms are precisely the stratified axioms of ZF) which is a theory to which a lot of the results of Chapter 2 apply, it is enough to find a fixed point for the power set operation to get a model for NF, and while this is obviously impossible in ZF because of Cantor's theorem (even without the axiom of foundation) there seems no reason why it should not be consistent with respect to Str(ZF).† I have hopes that some forcing argument as above, an application of Scott's permutation method or perhaps even the shifting ultraproduct method of Chapter 2, §5 will yield a consistency proof of PF relative to NF. It does not seem to imply Axcount so it is not obviously out of the question, while the natural way in which it arises whets the appetite.

One of the curious features of the Quine systems - as indeed of all set theories with universal sets - is that the failure of aussonderung gives rise to

†For more on str(ZF) see [4] and [5]

subclasses of sets which are not themselves sets. There are obvious parallels here with what happens in ZF when we construct generic objects in the process of extending a model which are small enough to be sets yet are not in the model we started with. Martin's axiom is an attempt to ensure that the universe of classical set theory is closed under a substantial number of forcing extensions. Of course it is in the nature of classical set theory that it can never capture all of them, but in the Quine set theories there is nothing to prevent us (apparently!) postulating this as long as our added generic objects are not required to be sets. We could add

GC (Axiom of "generic closure"): If $\mathbb{B}$ is an atomless c.b.a. then there exists a generic ultrafilter on $\mathbb{B}$.

(GC is of course an axiom for a <u>class</u> theory, and so we must make precise the sense in which $\mathbb{B}$ is a complete b.a. - it has to be complete in the sense that any <u>set</u> of points in it has a sup and an inf.) This ultrafilter will not of course be a set in general, and as with $\mathbb{B}$ itself, when we say that it is V - complete we mean that the inf of any sub<u>set</u> of the ultrafilter is in the ultrafilter.

One immediate consequence of GC is that there is a 1 - 1 enumeration of V against the integers, which achieves an otherwise rather ad hoc suggestion of Quine [32] that we add to ML "Every nonempty class of disjoint sets has a selection class". Boffa has shown (unpublished) that ML + GC is a conservative extension of ML, thus: Start with a model $\langle A, \epsilon \rangle$ of NF. Extend it to a model of ML by letting P'A be the collection of classes. If A is countable, then, by the Rasiowa-Sikorski theorem, all the classes whose existence is required by GC will in fact be present in P'A. Clearly the addition of GC results in an extension of ML which is conservative for sentences not containing quantifiers over proper classes.

GC is unfortunately inconsistent with another potentially useful axiom scheme (at least in the presence of Axcount) namely the axiom scheme of replacement for strongly cantorian sets.

Rep. Stcan : Every image of a strongly cantorian set in a function is a set.

GC and the axiom scheme of replacement for strongly cantorian sets are incompatible in the presence of the axiom of counting because otherwise, by Axcount, both $\aleph_0$ and $\aleph_1$ are strongly cantorian, and so the generic collapse which GC allows us to create as a <u>class</u> will be a <u>set</u> by strong cantorian replacement. Strong cantorian replacement is unattractive and unnatural on the grounds adumbrated at the beginning of this chapter, namely that it tidies up small sets while saying nothing about big sets. But there is no denying that it has some nice consequences.

Metatheorem 3.2

Con(NF + CS + Rep. Stcan) $\rightarrow$ Con(ZF)

Proof:

What follows below is an outline of a general method rather than a specific proof of the above. It is very close to a method used by Orey [24]. However the fuel for his proof is different. The basic tool I shall use is the Gödel F function on the ordinals [52]. It is an unsatisfactory construction in some ways, and, in the form in which Gödel produced it, can be applied only to strongly cantorian ordinals (Hinnion has shown that if R is a wellfounded relation on a strongly cantorian set then there is a permutation model of NF where R is isomorphic to the ε-relation on a transitive set) but there is a way round this. Essentially instead of defining L by recursion on the ordinals we define a relation E ($\subseteq \leqslant_o$) on the ordinals by recursion so that $\langle NO, E\rangle \cong_F \langle L, \varepsilon\rangle$ (more exactly, $\langle \alpha, E\restriction \mathrm{seg}_{\leqslant_o}{}'\alpha\rangle \cong_F \langle F''\mathrm{seg}_{\leqslant_o}{}'\alpha, \varepsilon\restriction F''\mathrm{seg}_{\leqslant_o}{}'\alpha\rangle$ for each ordinal α) and this is easily done by mimicing Gödel's construction. There is a superstition abroad that this construction cannot be executed in NF as the ordinals are not 'really' wellfounded. But all that matters is that the ordinals <u>think</u>

they are wellfounded. It is true that if we iterate the construction up to bad ordinals we get structures that are not isomorphic to 'really' wellfounded models of ZF but that does not affect consistency results.

Order the triples of ordinals $\langle \alpha, \beta, i \rangle$ with $\alpha, \beta \in NO$, $i \leqslant 8$ lexicographically. Let g be the function $NO \to NO \times NO \times \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ that enumerates them. Now define E by recursion as follows:

$$(\forall \alpha)(\alpha \not{E} \; 0)$$

The recursion step on α splits into 9 cases, depending on the third member of the triple coded by α

Case 0. $\alpha = g' \langle \beta, \gamma, 0 \rangle$ for some β, γ

we define $(\forall \delta < \alpha)(\delta \; E \; \alpha)$

Case 1 $\alpha = g' \langle \beta, \gamma, 1 \rangle$ for some β, γ

we define $\delta \; E \; \alpha$. iff $\delta \; E \; \beta \vee \delta \; E \; \gamma$

case 2 $\alpha = g' \langle \beta, \gamma, 2 \rangle$ for some β, γ

we define $\delta \; E \; \alpha$ iff $(\exists \eta \xi)(\delta \overset{(\langle NO, E \rangle)}{=} \langle \eta, \xi \rangle \wedge \delta \; E \; \beta \wedge \eta \; E \; \xi)$

case 3 $\alpha = g' \langle \beta, \gamma, 3 \rangle$ for some β, γ

we define $\delta \; E \; \alpha$ iff $\delta \; E \; \beta \wedge \delta \not{E} \; \gamma$

case 4 $\alpha = g' \langle \beta, \gamma, 4 \rangle$ for some β, γ

we define $\delta \; E \; \alpha$ iff $(\exists \eta \xi)(\delta \overset{(\langle NO, E \rangle)}{=} \langle \eta \xi \rangle \wedge \delta \; E \; \beta \wedge \xi \; E \gamma)$

case 5 $\alpha = g' \langle \beta, \gamma, 5 \rangle$ for some β, γ

we define $\delta \mathrm{E} \alpha$ iff $\delta \mathrm{E} \beta \wedge (\exists \rho)(\langle \eta, \delta \rangle^{\langle NO\ E \rangle} \mathrm{E} \gamma)$

case 6 $\alpha = g' \langle \beta, \gamma, 6 \rangle$ for some β, γ. We define

$\delta \mathrm{E} \alpha$ iff $(\exists \eta \zeta)(\delta = \langle \eta \zeta \rangle^{\langle NO, E \rangle} \delta \mathrm{E} \beta \wedge$
$\langle \zeta \eta \rangle^{\langle NO, E \rangle} \mathrm{E} \gamma)$

case 7 $\alpha = g' \langle \beta, \gamma, 7 \rangle$ for some β, γ. We define

$\delta \mathrm{E} \alpha$ iff $(\exists \eta \zeta \aleph)(\delta = \langle \eta, \zeta \aleph \rangle^{\langle NO, E \rangle} \wedge \delta \mathrm{E} \beta \wedge$
$\langle \eta \aleph \zeta \rangle^{\langle NO, E \rangle} \mathrm{E} \gamma)$

case 8 $\alpha = g' \langle \beta, \gamma, 8 \rangle$ for some β, γ We define

$\delta \mathrm{E} \alpha$ iff $(\exists \eta \zeta \aleph)(\delta = \langle \eta, \zeta, \aleph \rangle^{\langle NO, E \rangle} \wedge \delta \mathrm{E} \beta$
$\langle \aleph, \eta, \zeta \rangle^{\langle NO, E \rangle} \mathrm{E} \gamma)$

It is easily checked that the definition of 'E' in each of the eight cases requires us to look only at earlier ordinals and is therefore a proper recursive definition. Also that E is a homogeneous relation (i.e., in $\ulcorner \alpha \mathrm{E} \delta \urcorner$ we have to give α and δ the same type) We have to ensure that E will be extensional. This can be done either by removing ordinals from the domain of E which are duplicates of earlier ordinals and then collapsing, or by defining '=' in the model in terms of 'E' so that 'E', although not necessarily really extensional, will behave as though it were.

Linden [56] has shown that when $\alpha = \aleph_\gamma$ is an initial ordinal $F'\aleph_\gamma = M_{\aleph_\gamma}$. (For definition of

M see Linden [56]. Also $M_{\aleph_\alpha} = L \cap V_{\omega+\alpha}$ is standard as are (i) $L \cap V_\alpha \models Z$ when α is a limit ordinal $> \omega$. (ii) $L \cap V_\alpha \models ZF$ when α is weakly inaccessible and $> \omega$. Hence the interest in chapter 1 in finding limit and weakly inaccessible alephs. Of course inaccessibility of α is sufficient for $L \cap V_\alpha \models ZF$ but if the universe is fat enough $L \cap V_{\aleph_1} \models ZF$ can happen. It is a very remote possibility that $NF \vdash (\exists \alpha \varepsilon NO)(\ seg_{<_o} {}'\alpha, E \restriction seg_{<_o} {}'\alpha \models ZF)$ However $NF \vdash \aleph_\omega$ exists is possible and this would give $NF \vdash Con(Z)$. If we are interested only in relative consistency we might try another tack. To get models for classical set theories we want basically to find initial segments of NO with sufficiently strong closure properties. If the initial segment is a set we have an absolute consistency result. However we can always use initial segments which are classes should this look promising. The most suitable is usually the proper class of stcan ordinals. That is what I shall use in this case. Since the strongly cantorian alephs are closed under successor we now want merely that they are also closed under singular limits where the cofinal sequences are definable in the language of $\langle NO, E\rangle$ where all expressions have their 'quantifiers' restricted to stcan ordinals.

What we want is that if φ is an expression in the language of $\langle NO, E \rangle$ with all its bound variables restricted to stcan ordinals and δ is a stcan ordinal then $A = \hat{\alpha}\hat{\beta}\varphi(\alpha\beta)"\{ \gamma : \gamma \mathrel{E} \delta \}$ is a <u>set</u> and is coded above by a stcan ordinal. We prove easily that A contains only stcan ordinals, as every ordinal in $\{ \gamma : \gamma \mathrel{E} \delta \}$ is cantorian and φ, being an expression (pace use-mention distinction) in the language of $\langle NO, E \rangle$ must commute with T (this we prove by induction on composition with quantifiers and connectives). Next we use CS to show that everything in A is therefore stcan, and rep. stcan to show that A is a set. Now no sub<u>set</u> of the proper class of stcan ordinals can be cofinal in it, otherwise we fill out the set into an initial segment and we have the class of stcan ordinals is a set, which would give us the Burali-Forti paradox. Therefore A is bounded above by some stcan ordinal α, and, since it is definable, must be coded by some ordinal $< \alpha^+$ (the first initial ordinal $> \alpha$) which will also be strongly cantorian.

That completes the rough outline of the proof. There are other variants of the method - e.g. using precan ordinals (ordinals β which are less than some $\alpha = T\alpha$). None seem to give results in NF tout court.

The difference between Orey's method in [24] and the method here is that Orey's method is conducted in ML with the use of additional postulates about ordinals instead of NF with additional such postulates. The ordinals Orey uses are those whose corresponding orderings have no subclass without a least member. This condition implies strong cantorianness but probably not conversely. Also the elements of Orey's model are not ordinals but sets of ordinals. The common feature to both Orey's construction and the construction here is the use of Gödels recursive generation of L.

Some instances of Rep. stcan are provable in NF already. If φ is an expression with x free, unstratified, but the result of substituting x for y in $\psi(x, y)$ which is stratified but inhomogeneous we have

$$\mathrm{stcan}(z) . \supset . \{x \in z : \phi(x)\} \text{ exists}$$

(The set of such φ is a subset of the expressions "Faiblement stratifiée" of Crabbe [6])

Proof:

Suppose $\varphi(x)$ is the result of substituting x for y in $\psi(x, y)$ where y is n types higher than x. By strong cantorianness of z there is a one-one map h from z onto ι^n"z which is precisely $\iota^n \restriction z$. Then we can write $\ulcorner\varphi(x)\urcorner$ as $\ulcorner\psi(x, \breve{h}'\iota^n{}'x)\urcorner$ which is now stratified and we can use the abstraction scheme to get the subset of z we want.

Replacement for strongly cantorian sets has the agreeable consequence noted above, namely greatly facilitating a relative consistency proof for ZF but is undesirable because of the partiality argument, that it talks preferentialy of small sets not large ones.

Question: Is there a natural impartial extension of NF that proves Con(ZF)?

[1] Beneš. V.E. A Partial Model for NF. J.S.L. vol 19 (1954) pp 197 - 200

[2] Boffa. M Sets Equipollent to their power sets in NF J.S.L. Vol 40(1975)no 1

[3] Boffa. M. Entre NF et NFU. C.R. Acad. Sci. Paris sér. A. t 277 Oct 1973

[4] Boffa. M. Stratified formulas in ZF. Bull. Pol. Acad. Sci. sér. Maths, Astron, Phys. vol 19 no 4 1971

[5] Coret. J. Sur les cas stratifiés du schema de Remplacement. C.R. Acad. Sci. Paris sér. A vol 271 (1970) p 57

[6] Crabbé. M Types Ambigus. C. R. Acad Sci Paris Sér A vol 280 Jan 1975 pp 1-3

[7] Feferman . S Some Formal systems for the unlimited Theory of Structures and Categories. Unpublished m/s circulated 1974.

[8] Grishin. V.N. The consistency of a Fragment of NF. Dokl. Akad Nauk Vol 189 1969 no 2 pp 1387-9

[9] Гришин. В. Н. Метод стратификаций в теорий множеств Москва 1973

[10] Гришин. В. Н. Непротиворечивость некоторо фрагмента системы Куайна Д.А.Н. СССР 189 1969 2 стр 241 - 243

[11] Гришин. В. Н. О некоторых фрагментах системы Куайна "Иссмедования по математической лингвистике, математиской логике и информационным языкам" Москва 1972 стр 200 - 212

[12] Гришин. В. Н. Эквивалентность системы Куайна одному ее фрагменту "Научно-техническая Информация" сер. 2 Информационные процессы и системы 1 1972 стр 22 - 34

[13] Гришин. В. Н. Исследование некоторых вариантов системы Куайна "Н. Т. И." сер 2, 5 1973 стр 34 - 37

[14] Hailperin. T. A set of Axioms for Logic. J.S.L. vol 9 (1944) pp 1 - 19.

[15] Henson. C.W. Type-raising Operations in NF. J.S.L. vol 38 (1973) pp 59 - 68.

[16] Henson. C.W. Permutation methods applied to NF. ibid pp 68 - 76.

[17] Henson. C.W. Finite Sets in Quine's New Foundations. J.S.L. vol 34 (1969) pp 589 - 596.

[18] Henson. C.W. Ph.D. Thesis. M.I.T. 1968

[19] Hinnion. R. Sur Les Modeles de NF. C.R. Acad Sci Paris Ser A - B vol 275 (1972) A 567

[20] Jensen. R.B. On the Consistency of a slight (?) Modification of NF. Synthese vol 19 (1968-9) pp 250 - 263.

[21] Lake. J. Some Topics in Set Theory. Ph.D. Thesis Bedford College 1973.

[22] Orey. S. New Foundations and the Axiom of Counting. Duke Math. J. vol 31 (1964) pp 655-660.

[23] Orey. S. Formal Development of Ordinal Number Theory. J.S.L. vol 20 (1955) pp 95 - 104.

[24] Orey. S. On the relative Consistency of Set Theory. J.S.L. vol 21 (1956) pp 280 - 290.

[25] Pétry. A. A propos des Individus dans les "New Foundations" de Quine. C.R. Acad. Sci. Paris vol 279 (Oct 1974) pp

[26] Quine. W.V.O. New Foundations in From a Logical Point of View. Harper and Row.

[27] Quine. W.V.O. On ordered pairs and relations in Selected Logic Papers. Random House.

[28] Quine. W.V.O. On ω-inconsistency. ibid.

[29] Quine. W.V.O. On an Application of Tarski's Theory of Truth. ibid.

[30] Quine. W.V.O. On Cantor's Theorem. J.S.L. vol 2 (1937) pp 120 - 124.

[31] Quine. W.V.O. On the Consistency of NF. Proc. Nat. Acad. Sci. U.S.A. vol 37 pp 538 - 540.

[32] Quine. W.V.O. Mathematical Logic. 3rd Edition. Harper and Row.

[33] Quine. W.V.O. Set Theory and its Logic. Harper and Row 1963.

[34] Rosser. J.B. On the Consistency of NF. J.S.L. vol 4 (1939) pp 15 - 24.

[35] Rosser. J.B. The Burali-Forti paradox. J.S.L. vol 7 (1942) pp

[36] Rosser. J.B. The relative Strength of Zermelo's Set Theory and NF. Proc. Intern. Cong. Math. Amsterdam. 1954. Vol 111. pp 289 - 294.

[37] Rosser. J.B. The Axiom of Infinity in NF. J.S.L. vol 17 (1952) pp 238 - 242.

[38] Rosser. J.B. Deux Esquisses de Logique. Paris 1953.

[39] Rosser. J.B. Logic for Mathematicians. McGraw-Hill 1953

[40] Rosser. J.B. Review of Specker [44] below. J.S.L vol (1954) p 127.

[41] Rosser. J.B. and Wang. H. Non-standard Models for Formal Logic. J.S.L. vol 15 (1950) pp 113 - 129.

[42] Scott. D.S. Quine's individuals. Logic, Methodology and Philosophy of Science ed Nagel. Stanford 1962

[43] Scott. D.S. Review of Specker [45] . Math. Rev. vol 21 (1960) p 1026.

[44] Specker. E.P. The Axiom of Choice in Quine's 'New Foundations for Mathematical Logic' Proc. Nat. Acad. Sci. U.S.A. vol 39 (1953) pp 972 - 975.

[45] Specker. E.P. Dualität. Dialectica vol 12 1958 pp 451 - 465.

[46] Specker. E.P. Typical Ambiguity. Logic, Methodology and Philosophy of Science. ed Nagel. Stanford 1962.

[47] Stanley. R.L. Simplified Foundations for Mathematical Logic. J.S.L. vol 20 (1955) pp 123 - 139.

[48] Wang. H. The Categoricity question of certain grand Logics. Math. Zeitschr. vol 59 (1953) pp 47 - 56.

[49] Wang. H. A Formal System of Logic. J.S.L. vol 15 (1950) pp 25 - 32.

[50] Wang. H. Negative Types. Mind. vol 61 (1952) pp 366 - 368.

11

[51] Fitting. M.C. Intuitionistic Logic, Model Theory and Forcing. North Holland 1969.

[52] Gödel. K. The consistency of the Continuum hypothesis. Princeton University Press, Princeton 1940

[53] Hindman. N. The existence of certain Ultrafilters on N and a conjecture of Graham and Rothschild. Proc. Amer. Maths. Soc. Vol 36 no2 Dec. 1972 pp 341-346.

[54] Kunen. K Elementary Embeddings and Infinitary Combinatorics. J.S.L. vol 36 (1971) pp 407 - 413.

[55] Levy. A. A Hierarchy of Formulas in Set Theory. American Mathematical Society Memoirs vol 57 1965.

[56] Linden. T. A. Equivalences between Gödel's definitions of constructibility. in "Sets, models and recursion theory" Ed Crossley North Holland 1967.

[57] "Extending Gödel's negative interpretation to ZF" William C Powell. Mimeographed typescript 1974