On Eigenvectors for Semisimple Elements in Actions of Algebraic Groups

by

Darren John Kenneally
Jesus College
Cambridge

A dissertation submitted for the degree
of Doctor of Philosophy to the
University of Cambridge

Abstract

Let G be a simple simply connected algebraic group defined over an algebraically closed field K and V an irreducible module defined over K on which G acts. Let E denote the set of vectors in V which are eigenvectors for some non-central semisimple element of G and some eigenvalue in K^*. We prove, with a short list of possible exceptions, that the dimension of $\overline{E}$ is strictly less than the dimension of V provided $\dim V > \dim G + 2$ and that there is equality otherwise. In particular, by considering only the eigenvalue 1, it follows that the closure of the union of fixed point spaces of non-central semisimple elements has dimension strictly less than the dimension of V provided $\dim V > \dim G + 2$, with a short list of possible exceptions.

In the majority of cases we consider modules for which $\dim V > \dim G + 2$ where we perform an analysis of weights. In many of these cases we prove that, for any non-central semisimple element and any eigenvalue, the codimension of the eigenspace exceeds $\dim G$. In more difficult cases, when $\dim V$ is only slightly larger than $\dim G + 2$, we subdivide the analysis according to the type of the centraliser of the semisimple element. Here we prove for each type a slightly weaker inequality which still suffices to establish the main result. Finally, for the relatively few modules satisfying $\dim V \leqslant \dim G + 2$, an immediate observation yields the result for $\dim V < \dim B$ where B is a Borel subgroup of G, while in other cases we argue directly.

Preface

Declaration

I hereby declare that this dissertation is not substantially the same as any that I have submitted for a degree, diploma or other qualification at any other university, and that no part of it has been or is currently being submitted for any such degree, diploma or other qualification. I further declare that this dissertation is the result of my own work and contains nothing which is the outcome of work done in collaboration with others, except as specified below.

Chapters 3 and 4 are joint work with Dr. Ross Lawther, and Section 3.6 is solely due to Lawther. All other content is my own work.

Acknowledgements

First I would like to thank my supervisor Dr. Ross Lawther for his guidance, generous support, encouragement and patience, and for elucidating some beautiful mathematics. I am indebted to Prof. Jan Saxl for bringing me to Cambridge in the first place and thank him for his interest in my work.

I have had a wonderful time as a graduate student at Jesus College. I would like to thank all of my supervisees and, indeed, Dr. Stephen Siklos for trusting me to supervise them. I owe much to many people at the University of Warwick, where I was lucky to attend excellent lectures during my undergraduate degree. I would like to thank Prof. Roger Carter in particular for MA5N9: Representation Theory of the General Linear Group. I would like to thank Professors Frank Lübeck and Alex Zalesski for being generous with their time to answer all of my queries.

I would like to thank my family for their help throughout my life, in particular my mother. I thank Jasminder Ghuman for her love, kindness, patience and for giving me confidence when I found it lacking myself.

I would like to thank my office-mate Alex Frolkin for his advice and for listening as I gave my views on the world. Also, thanks to Chris Bowman and Matthew Clarke for their good humour. I have been fortunate to be able to spend time in the company of my friends Dr. Ben Fairbairn and Dr. Olof Sisask; trips to Chicago, Northern Italy and F.Y.R. Macedonia spring to mind. I must thank David Stewart for a most worthwhile hour learning about tensor indecomposable modules. My flat-mates at both Warwick and Cambridge have kept me sane and often done my washing-up. I would like to thank Mark Dunning, Nick Jones and Fabio Viola in particular.

I gratefully acknowledge the funding provided by E.P.S.R.C.

This thesis is dedicated to Dr. John Felton.

Contents

Chapter 1

Introduction

A very powerful way of approaching groups is through their actions. Actions give an insight into the structure of a group by representing it as a group of matrices. We shall be concerned with actions of algebraic groups on vector spaces, both defined over the same algebraically closed field. We study the building blocks of the theory, namely, irreducible actions of simple algebraic groups. We assume that simple algebraic groups are simply connected since this version of the group acts on all of its modules, whereas versions in other isogeny classes do not.

Let G be a simple simply connected algebraic group defined over an algebraically closed field K of characteristic $p \geqslant 0$ and let V be an irreducible G-module also defined over K. In representation theory, important invariants in actions for each group element are eigenvectors and eigenvalues. The eigenspaces with eigenvalue 1 are of interest since they are the fixed point spaces $C_V(g) = \{v \in V \mid gv = v\}$ for each $g \in G$. A recent survey article of Zalesski [29] discusses a number of problems on eigenvalues of elements in representations of algebraic groups and finite Chevalley groups. He points out that the usual method of studying eigenspaces is through the theory of weights of representations of algebraic groups, particularly in relation to semisimple elements.

In this thesis, we consider the question of how likely it is that a vector chosen at random from V will be an eigenvector for some semisimple element of G. We quickly see that we should restrict our attention to non-central semisimple elements since central elements act as scalars. We consider the set E of vectors in V which are eigenvectors for some non-central semisimple element of G and some eigenvalue in K^*. Our aim is to precisely determine when the dimension of its closure is strictly less than $\dim V$. We show, for all but a short list of possible exceptions, that there is a clear dichotomy: the dimension of $\overline{E}$ is strictly less than V whenever $\dim V > \dim G + 2$, otherwise we have equality. In order to establish this result we show in most cases that the codimension of the eigenspace of any non-central semisimple element and any eigenvalue exceeds $\dim G$. In the remaining cases we require an analysis of the centraliser types of semisimple elements.

Fix a non-central semisimple element $s \in G$ and an eigenvalue $\gamma \in K^*$. We shall explain the principal method used to calculate a lower bound for the codimension of the corresponding eigenspace $V_\gamma(s) = \{v \in V \mid sv = \gamma v\}$. Since s is non-central, we deduce from the structure theory of centralisers of semisimple elements that there is a root α in the root system of G such that $\alpha(s) \neq 1$. Each irreducible G-module can be decomposed as a direct sum of weight spaces. Each weight appears in a weight string for the root α and two weights are adjacent in such a string if they differ by α. Thus the weight spaces corresponding to two adjacent weights in a weight string for α cannot both lie in the eigenspace. We shall introduce this later as the adjacency principle in Section 4.2.

As an application of our main result we make a contribution to the study of regular orbits. By restricting our main result above to 1-eigenspaces we deduce that the set of vectors whose stabilisers contain no non-central semisimple element is dense in G whenever $\dim V > \dim G + 2$ except for some possible exceptions. Thus if it can be shown that at least one of these vectors has a stabiliser containing no non-trivial unipotent element, it would follow that such a vector has stabiliser contained in $Z(G)$. Such a vector would lie in a regular orbit when we pass from the action of G to that of G/M, where $M \leqslant Z(G)$ is the kernel of the action of G on V.

One of the motivations for considering regular orbits is the work on base size in the theory of permutation groups by Burness, Liebeck, Shalev, Guralnick, Saxl, et al. Let G be a primitive permutation group on a set Ω. We call a subset $B \subset \Omega$ a *base* for G if the pointwise stabiliser of B in G is trivial, i.e., $\bigcap_{\alpha \in B} G_\alpha = 1$. Denote by $b(G) = b(G, \Omega)$ the minimal size of a base of G. By the O'Nan-Scott theorem [17] G is either almost simple, affine or lies in one of four other classes. If G is affine then $G \leqslant AGL(V)$ where V is a finite-dimensional vector space of order p^n with p prime and $n \in \mathbb{N}$. If we interpret V as the group of translations we may write $G = V \rtimes H$, where $H = G_0$ is an irreducible subgroup of $GL(V)$ and $V \cong (C_p)^n$ is an elementary abelian regular normal subgroup of G. Consider the action of H on V and let $b(H)$ denote the minimal size of a base for this action. We have $b(G) = 1 + b(H)$ and so $b(G) = 2$ if and only if there is a single vector in V whose stabiliser in H is trivial, i.e., if and only if H has a regular orbit in its action on V.

Layout This thesis is organised as follows. In Chapter 2 we assemble the background results required and state the main theorem of this thesis. The next two chapters reflect the dichotomy of the main theorem. In Chapter 3 we state and prove results for small modules of dimension less than or equal to $\dim G + 2$ and in Chapter 4 we provide techniques for large modules, i.e., modules of dimension exceeding $\dim G + 2$. In Chapters 5, 6, 7 and 8 we consider the possible p-restricted irreducible modules for groups of types A_n, B_n, C_n and D_n respectively. In Chapter 9 we consider such modules for groups of exceptional type. In Chapter 10 we examine tensor products with twists of p-restricted irreducible modules for groups of all types thereby completing the investigation. In Chapter 11 we close with a review of the progress we have made and open questions that have arisen. In Appendix A we provide tables containing sets of weights for a specific action as detailed in Lemma 5.13 in Chapter 5.

Chapter 2

Preliminaries and statement of main theorem

In this chapter, we define terms and recall key results from the representation theory of algebraic groups. We shall establish the background to the material addressed in the chapters to follow. We shall finish the chapter by stating the main theorem of this thesis.

2.1 Representation theory

We refer the reader to [1, Chapter 1], [27], [13] and [19] for full details of the theory below.

Let G be a simple simply connected algebraic group over an algebraically closed field of characteristic $p \geqslant 0$. Let T be a maximal torus of rank n and $B \supset T$ be a Borel subgroup. Denote by Φ the root system of G with respect to T, and Φ^+, Φ^- and $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ the positive, negative and simple roots determined by the choice of B. We can analogously denote the coroot system by $\Phi^\vee$ and denote by $\alpha^\vee \in \Phi^\vee$ the coroot corresponding to $\alpha \in \Phi$. Let $(\, , \,)$ denote the associated symmetric bilinear form. By setting $\|\alpha\| = (\alpha, \alpha)^{\frac{1}{2}}$ to be the root length of each $\alpha \in \Phi$ it can be shown that for each Φ either all roots have the same length, or there are two root lengths and Φ is partitioned into two sets; one set denoted Φ_L contains long roots and the other denoted Φ_S contains short roots.

If all roots in Φ are of the same length then we shall regard all roots as short. Each simple algebraic group has an associated connected Dynkin diagram and we shall use the standard Bourbaki labelling adopted by Humphreys in [13, p.58]. Let U denote the unipotent radical of B; there is a semidirect product decomposition $B = UT$. We denote by B^- the opposite Borel subgroup of G with $B \cap B^- = T$ and U^- denotes the unipotent radical of B^-. The one-dimensional connected unipotent proper subgroups of U and U^- correspond to elements of Φ and are called root subgroups; these are denoted X_α where $\alpha \in \Phi$ and we have $U = \prod_{\alpha \in \Phi^+} X_\alpha$.

Let $X = X(T)$ be the character group of G and $Y = Y(T)$ be the cocharacter group of G, both with respect to T and let G_m be the one-dimensional multiplicative group isomorphic to $GL_1(K)$. For $\chi \in X$ and $\gamma \in Y$ since $\chi \circ \gamma \in \mathrm{Hom}(G_m, G_m) \cong \mathbb{Z}$, we have $(\chi \circ \gamma)(\lambda) = \lambda^n$ for some $n \in \mathbb{Z}$ and for all $\lambda \in G_m$; set $\langle \chi, \gamma \rangle = n$. In this way we can define a non-degenerate map $X \times Y \to \mathbb{Z}$ by $(\chi, \gamma) \mapsto \langle \chi, \gamma \rangle$. Let W be the Weyl group with respect to T. The Weyl group acts transitively on roots of the same length in Φ; indeed, W acts faithfully on the lattices X and Y. It is generated by elements w_α for $\alpha \in \Phi$. It is not too hard to show that W is generated by $\{w_{\alpha_i}\}_{\alpha_i \in \Pi}$ and that it is a Coxeter group. Each w_{α_i} acts on X as follows: $w_{\alpha_i}(\chi) = \chi - \langle \chi, \alpha_i^\vee \rangle \alpha_i$ for each $\chi \in X$. Indeed, we have $\langle \chi, \alpha^\vee \rangle = 2\frac{(\chi,\alpha)}{(\alpha,\alpha)}$.

We shall assume from the outset that all representations of G considered are both finite-dimensional and rational. Let V be a finite-dimensional G-module over K with corresponding rational representation $\rho : G \to GL(V)$. Considering V as a T-module, V is diagonalisable in the sense that $V = \bigoplus_{\lambda \in X} V_\lambda$ where $V_\lambda = \{v \in V \mid \rho(t)v = \lambda(t)v \text{ for all } t \in T\}$. We say that λ is a *weight* of the representation ρ whenever $V_\lambda \neq 0$ and we call V_λ the *weight space*. The *multiplicity* of λ in ρ is defined to be $\dim V_\lambda =: m_\lambda$. Since $\dim V_\lambda = \dim V_{w\lambda}$ for all $w \in W$, weights in the same Weyl group orbit have the same multiplicity. Let $\{\omega_1, \ldots, \omega_n\}$ be the basis of $\mathbb{Z}\Phi$ dual to $\Pi^\vee$ with respect to $\langle\, , \rangle$, i.e., $\langle \omega_i, \alpha_j^\vee \rangle = \delta_{ij}$ for $1 \leqslant i, j \leqslant n$. We call $\{\omega_1, \ldots, \omega_n\}$ the *fundamental weights* and since G is simply connected, this is a $\mathbb{Z}$-basis of X. We say that a weight $\lambda \in X$ is *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geqslant 0$ for all i, i.e., λ is a non-negative linear combination of the fundamental weights. We denote the set of dominant weights by X^+. From above we can see that W acts on X via $w_{\alpha_i}(\omega_j) = \omega_j - \delta_{ij}\alpha_i$ and in fact each W-orbit contains a unique dominant weight. The set of weights of V is a union of W-orbits. We can define a partial ordering $\leqslant$ on X whereby $\lambda \leqslant \lambda'$ if and only if $\lambda' - \lambda$ is a non-negative linear combination of simple roots. A dominant weight $\lambda \in X^+$ is said to be *p-restricted* if $0 \leqslant \langle \lambda, \alpha_i^\vee \rangle < p$ for all $i \in [1, n]$. For convenience we shall use the notation $\langle \omega, \alpha \rangle$ for $\langle \omega, \alpha^\vee \rangle$ in subsequent chapters.

We shall write each fundamental weight as a linear combination of the simple roots with coefficients in $\mathbb{Q}^+$: see [13, p.69]. In order to save space we shall denote a weight $\omega = \sum a_i \alpha_i$ by its coefficients a_i, i.e., we shall write $\omega = a_1 a_2 \ldots a_n$.

The following theorem makes it clear that we may study irreducible representations via their highest weight representations.

Theorem 2.1 (Chevalley)**.** *Let G be a semisimple algebraic group over K. Then the following hold.*

(i) *Every irreducible rational G-module V has a unique highest weight $\lambda \in X^+$ with respect to the partial ordering on X^+. In particular the weight space V_λ has multiplicity* 1 *and if $V_\mu \neq 0$ then $\mu \leqslant \lambda$.*

(ii) *Two irreducible modules with the same highest weight λ are isomorphic.*

(iii) *For every $\lambda \in X^+$ there exists an irreducible G-module of highest weight λ.*

We may thus denote an irreducible G-module with highest weight λ by $L(\lambda)$ without confusion.

2.1.1 Duality

If the G-module $L(\lambda)$ of highest weight λ is irreducible then the dual module $L(\lambda)^* = \mathrm{Hom}(L(\lambda), K)$ is again an irreducible G-module. It is not too hard to work out that $L(\lambda)^*$ has highest weight $-w_0.\lambda$ where $w_0 \in W$ is the longest word in the Weyl group and is the unique element mapping the positive roots to the negative roots, i.e., $w_0(\Phi^+) = \Phi^-$. By Chevalley's theorem we have $L(\lambda)^* \cong L(-w_0.\lambda)$. Now $-w_0 = id$ on X when Φ is of type B_n, C_n, D_n (n even), E_7, E_8, F_4 and G_2; in particular, in these cases, an irreducible G-module is self-dual. In the other cases w_0 induces a graph automorphism of the corresponding Dynkin diagrams. Thus, for example, if the root system is of type A_n, the modules $L(a_1\omega_1 + \cdots + a_n\omega_n)$ and $L(a_n\omega_1 + \cdots + a_1\omega_n)$ are dual to each other. It will suffice to consider one of each dual pair since our calculations will be identical in both cases.

2.1.2 Theorems of Lang and Steinberg

Steinberg proved the following fundamental dichotomy.

Theorem 2.2 (Steinberg, [26])**.** *Let G be a simple linear algebraic group and $\sigma : G \to G$ an endomorphism. Then precisely one of the following holds:*

(i) σ is an automorphism of algebraic groups;

(ii) the group $G^\sigma = \{g \in G \mid \sigma(g) = g\}$ of fixed points is finite.

In case (ii) of Theorem 2.2 the endomorphism σ is called a *Frobenius map* of G. The group G^σ is a *finite group of Lie type*.

Given a rational representation $\rho : G \to GL(V)$ and the standard Frobenius map F relative to p (as defined in [1, §1.17]), we can "twist" by the rth power of F to obtain a new representation $\rho \circ F^r : G \to GL(V)$. We shall write $V^{(r)}$ for the associated G-module.

Theorem 2.3 (Steinberg, [26])**.** *Let G be a simply connected semisimple algebraic group over $\overline{\mathbb{F}}_p$ and $\lambda \in X^+$. Write $\lambda = \lambda_0 + p\lambda_1 + p^2\lambda_2 + \ldots + p^r\lambda_r$, the p-adic expansion of λ, where each λ_i is p-restricted and dominant. Then*

$$L(\lambda) \cong L(\lambda_0) \otimes L(\lambda_1)^{(1)} \otimes \ldots \otimes L(\lambda_r)^{(r)}.$$

This theorem allows the study of weights of irreducible rational representations of simple algebraic groups over an algebraically closed field of characteristic p to be reduced to the study of p-restricted weights of such representations. It does not, however, allow us to focus exclusively on p-restricted highest weights λ, since $L(\lambda)$ may have dominant weights lying below λ in the partial ordering which are not p-restricted.

If Φ has two root lengths, let $X(T)_S$ denote the set of weights $\lambda \in X(T)$ such that $\langle \lambda, \alpha \rangle = 0$ for all long roots $\alpha \in \Phi$. Thus the weights in $X(T)_S$ are $\mathbb{Z}$-linear combinations of fundamental weights corresponding to short simple roots, i.e., the set of weights with short support. We analogously define $X(T)_L$.

In Chapters 6 and 7 we shall make use of the following refinement of Theorem 2.3.

Theorem 2.4 (Steinberg, [25])**.** *Let $p = 2$ if G is of type B_n, C_n or F_4 and $p = 3$ if G is of type G_2. If $\lambda \in X(T)$ is a dominant weight then $\lambda = \mu + \nu$ for unique dominant weights $\mu \in X(T)_L$ and $\nu \in X(T)_S$ and $L(\lambda) \cong L(\mu) \otimes L(\nu)$.*

We now state the important Lang-Steinberg theorem [1, §1.17] which shall be required in the next chapter.

Theorem 2.5. *Let G be a connected algebraic group over $K = \overline{\mathbb{F}}_p$. If $F : G \to G$ is a surjective homomorphism such that G^F is finite then the map $L : G \to G$ defined by $L(g) = g^{-1}F(g)$ is surjective.*

2.2 Premet's theorem

We begin this section with a definition. Assume that the characteristic of K is positive. A rational K-representation ρ of G is called *infinitesimally irreducible* if its differential $d\rho$ defines an irreducible representation of the Lie algebra $\mathfrak{g}$.

It has been proven by Curtis in [7] that a rational representation ρ is infinitesimally irreducible if and only if its highest weight λ is p-restricted.

Let $e(\Phi)$ denote the maximum of the squares of the ratios of the lengths of the roots in Φ, so $e(\Phi)$ is 2 if Φ is of type B_n, C_n or F_4, 3 if Φ is of type G_2 and 1 otherwise.

Theorem 2.6 (Premet, [20]). *Suppose that $p > e(\Phi)$. Then the set of weights of an infinitesimally irreducible representation ρ of G with highest weight λ coincides with the set of weights of an irreducible complex representation $\rho_{\mathbb{C}}$ of $\mathfrak{g}$ with the same highest weight.*

Consequently, the set of weights of the irreducible module $L(\lambda)$ is the union of the Weyl group orbits of dominant weights μ with $\mu \leqslant \lambda$ provided that $p > e(\Phi)$.

The theorem proved a conjecture of I.D. Suprunenko who had shown that the result holds for groups of type A_l, $l \geqslant 1$ and for a few other special cases.

The cases in which $p \leqslant e(\Phi)$ certainly need to be excluded. There are numerous counterexamples as we shall see later in Theorem 2.7. The theorem tells us that, provided $p > e(\Phi)$, the weights of $L(\lambda)$ are the same as the weights of the irreducible representation with highest weight λ of a Lie algebra of the same type as G.

Let G be a simple algebraic group of simply connected type over $\overline{\mathbb{F}}_p$. Lübeck has determined (using computational methods) all irreducible G-modules parameterised by p-restricted highest weights in defining characteristic of dimension bounded by a given integer M; details are given in [19]. He details for groups of small rank (A_n, $2 \leqslant n \leqslant 20$; B_n, $2 \leqslant n \leqslant 11$; C_n, $3 \leqslant n \leqslant 11$; D_n, $4 \leqslant n \leqslant 11$) not only the parameterising highest weights and the dimensions of the modules, but also the multiplicities of the dominant weights lower than the highest weight in the partial ordering. For these groups of small rank the bound is given explicitly in Table 1 of [19] and for groups of large rank, $M = \frac{n^3}{8}$ if G is of type A_n and $M = n^3$ otherwise. The information is accessible on Lübeck's website [18]. This explicit information about highest weight modules of bounded dimension in all characteristics is crucial to our work for groups of small rank and the value of M in each case is usually more than sufficient for our requirements.

Lübeck's paper is an extension of work by Gilkey and Seitz in [9] where they considered the exceptional types of Lie algebras and algebraic groups.

For almost all primes p, the weight multiplicities and the dimension of an irreducible module for an algebraic group defined over a field of characteristic $p > 0$ are the same as those of the corresponding simple algebraic group in characteristic 0. This will allow us to state our results for characteristic 0 rather than solely for the modular case when $p > 0$. We note that in characteristic 0 the well-known formula of Freudenthal [13, §22.3] can be used to compute weight multiplicities.

Consider the action of G on $L(\lambda)$ for some $\lambda \in X^+$. Fix $\alpha \in \Phi$, and take some $\mu \in \Pi(\lambda)$ where $\Pi(\lambda)$ denotes the multiset of weights of $L(\lambda)$. The weights in $\Pi(\lambda)$ of the form $\mu + i\alpha$, $i \in \mathbb{Z}$ form a string $\mu - r\alpha, \ldots, \mu, \ldots, \mu + q\alpha$ called the *α-string through μ*. The Weyl group element w_α reverses the string and $r - q = \langle \mu, \alpha \rangle$. We call such a string a *weight string* and we refer the reader to [13, §21.3] for more details. A weight string of *length* k is one consisting of k weights. We say that $\Pi(\lambda)$ is *saturated* if for all $\mu \in \Pi(\lambda)$ and $\alpha \in \Phi$ the weight $\mu - i\alpha \in \Pi(\lambda)$ for each i between 0 and $\langle \mu, \alpha \rangle$. In the cases where the conclusion of Premet's theorem does not hold, the weight strings that are present are not necessarily saturated. The weights that are not present lie in the Weyl group orbit of a dominant weight which is not present for a certain characteristic. As an example, for type B_n the Weyl group orbit of the weight ω_2 for the module $L(\omega_3)$ is not present when the characteristic of K is 2.

At times we shall employ a generalisation of the notion of a weight string. Given two or more orthogonal roots, we take the union of their weight strings; each connected component of the resulting graph is called a *weight net*. For example, if $(G, \lambda) = (A_5, \omega_3)$ then the following is a 2×2 weight net; horizontally there are α_1-strings and vertically α_4-strings. Note that we have omitted a factor of $\frac{1}{5}$ on each coefficient.

2 -1 1 3	-3 -1 1 3
2 -1 1 -2	-3 -1 1 -2

More briefly, rather than state each weight in the weight net above, we use the notation $\cdot$ -1 1 $\cdot$. In general, we shall use dots to stand for any integers that make an expression a weight in $\Pi(\lambda)$.

2.3 Zalesski's theorem

A recent expository paper of Zalesski provides more precise information in the cases where the conclusion of Premet's theorem does not hold.

Theorem 2.7 (Zalesski, [29]). *Let G be a simple algebraic group defined over a field of characteristic p and ϕ a tensor indecomposable p-restricted G-module. Let μ be the highest weight of ϕ. Then the weights of ϕ are the same as the weights of an irreducible representation of a complex Lie algebra of the same type as G unless one of the following holds:*

(i) $p = 2$ *and G is of type B_n;*

(ii) $p = 2$, *G is of type C_n and $\mu = \omega_n$;*

(iii) $p = 2$, *G is of type G_2 and $\mu = \omega_1$;*

(iv) $p = 3$, *G is of type G_2 and $\mu = \omega_2$ or $2\omega_2$;*

(v) $p = 2$, *G is of type F_4 and $\mu = \omega_1$, ω_2 or $\omega_1 + \omega_2$.*

The proof uses the tables of Gilkey and Seitz in [9] for the exceptional types. For type C_n with $p = 2$, Zalesski views weights with short support in C_n as weights inside D_n. We see that if G is of type B_n for $p = 2$ there is no improvement on Premet's theorem. We shall make use of *(ii)* above in Section 7.2.

The next theorem due to Seitz tells us precisely which modules are tensor indecomposable.

Theorem 2.8 (Seitz, [23]). *Let V be a finite-dimensional vector space defined over an algebraically closed field of characteristic p and X a simple closed connected irreducible subgroup of $\mathrm{SL}(V)$. Then V can be expressed as the tensor product $V = V_1 \otimes V_2$ of two non-trivial p-restricted X-modules if and only if V is a p-restricted X-module and the following conditions hold:*

(i) *X has type B_n, C_n, F_4 or G_2 with $p = 2, 2, 2, 3$ respectively;*

(ii) *V_1, V_2 may be arranged such that V_i has highest weight λ_i with $\lambda = \lambda_1 + \lambda_2$ and λ_1 (respectively λ_2) has support on those fundamental dominant weights corresponding to short (respectively long) roots.*

So, for type C_n with $p = 2$ all 2-restricted tensor decomposable modules have the form $L(\nu) \otimes L(\omega_n)$ where $\nu \in X(T)_S$ is a 2-restricted dominant weight with short support.

2.4 Centralisers of semisimple elements

An algebraic group G acts naturally on itself by conjugation via $\operatorname{Int} x : G \to G$, $y \mapsto xyx^{-1}$. Denote the differential of this map by $\operatorname{Ad} x$; this is an automorphism of the Lie algebra $\operatorname{Lie}(G) = \mathfrak{g}$. The map $\operatorname{Ad} : G \to \operatorname{Aut} \mathfrak{g} \subset GL(\mathfrak{g})$ is called the *adjoint representation* of G. It can be shown that $\operatorname{Ad} : GL(n, K) \to GL(n^2, K)$ is a morphism of algebraic groups. In fact $\operatorname{Ad} x$ is conjugation by x when G is a closed subgroup of $GL(n, K)$. For $a \in \mathfrak{g}$ and $x \in G$ recall the following definitions:

$$\begin{aligned}
C_G(a) &= \{g \in G \mid \operatorname{Ad} g.a = a\}, \\
C_{\mathfrak{g}}(a) &= \{b \in \mathfrak{g} \mid [b, a] = 0\}, \\
C_G(x) &= \{g \in G \mid xgx^{-1} = g\}, \text{ and} \\
C_{\mathfrak{g}}(x) &= \{b \in \mathfrak{g} \mid \operatorname{Ad} x.b = b\}.
\end{aligned}$$

The global and infinitesimal centralisers of elements in the Lie algebra are related, specifically the inclusion $\operatorname{Lie}(C_G(a)) \subset C_{\mathfrak{g}}(a)$ for $a \in \mathfrak{g}$ always holds and there is equality in the case that a is a semisimple element.

As we shall see later, it will be important to consider centralisers in the algebraic group of semisimple elements. The most suitable references for this theory are [14, §2] and [1, §3.5].

Since we are assuming that our group G is simply connected, by a result of Steinberg [26, §8] we see that the centraliser in G of any semisimple element is connected.

Proposition 2.9. *Let G be a simply connected simple algebraic group, s a semisimple element and T a maximal torus of G containing s. Then $C_G(s) = \langle T, X_\alpha \mid \alpha(s) = 1\rangle$.*

Thus, up to conjugation, there are only finitely many different centralisers of semisimple elements in G. Moreover $C_G(s)$ is reductive with root system $\Phi_s = \{\alpha \in \Phi \mid \alpha(s) = 1\}$ and Weyl group $W_s = \langle w_\alpha \mid \alpha \in \Phi_s\rangle$.

Recall that a prime p is said to be *bad* for G if p divides a coefficient a_i of some root $\alpha \in \Phi$ where $\alpha = \sum_{\alpha_i \in \Pi} a_i \alpha_i$ is expressed as a linear combination of simple roots. We say that a prime p is *good* for G whenever it is not bad. It is easy to see that for types B_n, C_n and D_n the prime $p = 2$ is bad, for types G_2, F_4, E_6 and E_7 the primes $p = 2, 3$ are bad and for type E_8 the primes $p = 2, 3, 5$ are bad. There are no bad primes for type A_n.

Deriziotis [8] found an elegant characterisation of centralisers of semisimple elements. The possible Dynkin diagrams of the semisimple part of the centraliser of a semisimple element are, except for a subset of the bad primes, given precisely by the possible subdiagrams of the extended Dynkin diagram of G (which is formed from $\tilde{\Pi} = \Pi \cup \{-\alpha_0\}$, where α_0 is the highest root). There are bad primes for which some of these semisimple centralisers do not occur; a precise method for determining these is given in work of Hartley and Kuzucuoğlu [11]. The possible diagrams correspond to all root systems $\Psi \subset \Phi$ which have a basis which is conjugate under the Weyl group to a proper subset of $\tilde{\Pi}$. For example, when G is of type A_n, taking subdiagrams of the extended Dynkin diagram we see that centralisers of semisimple elements are of type $A_{n_1} \cdots A_{n_r} T_l$ where $n_1 \geqslant \cdots \geqslant n_r \geqslant 0$ and $l + \sum n_i = n$. We reserve the letter X to denote the type of the root system of the centraliser of a semisimple element. So if $n = \operatorname{rank} T$ and s is *regular*, i.e., $\dim C_G(s) = n$, which is the least possible dimension, we say that $X = \varnothing$.

We remark that it is necessary to distinguish between the two possible subsystems consisting of two single non-adjacent nodes of the extended Dynkin diagram of types B_n and D_n: the subsystem D_2 formed from nodes corresponding to α_{n-1} and α_n (or, alternatively α_1 and $-\alpha_0$) and the subsystem A_1^2 as before. We make the distinction between the subsystems A_3 and D_3 in types B_n and D_n whenever necessary. Also for type B_n we denote a subsystem of rank one by either A_1 or B_1 in order to distinguish between long roots whereby $\Phi_s = \{\pm\alpha_i\}$ for some $i \neq n$ or short roots $\{\pm\alpha_n\}$. Similarly for type C_n a subsystem of rank one is denoted by either A_1 or C_1 in order to distinguish between short roots whereby $\Phi_s = \{\pm\alpha_i\}$ for some $i \neq n$ or long roots $\{\pm\alpha_n\}$.

2.5 Lie algebras and a result on nilpotent orbits

We shall consider the adjoint representation in the next chapter and therefore we need to recall relevant material from the theory of Lie algebras and state part of a result about nilpotent orbits in good characteristic due to Premet.

The following material is contained in [2] and [12]. Let $\mathrm{Lie}(G) = \mathfrak{g}$ be the Lie algebra of the simple algebraic group G. Recall the Cartan decomposition

$$\mathfrak{g} = C_{\mathfrak{g}}(T) \oplus \sum_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

where $C_{\mathfrak{g}}(T)$ is a Cartan subalgebra and $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid \mathrm{Ad}\, t(x) = \alpha(t)x,\ t \in T\}$. Using the same notation as in [2, Chapter 3] we let $\{h_\alpha,\ \alpha \in \Pi;\ e_\alpha,\ \alpha \in \Phi\}$ denote the *Chevalley basis* for $\mathfrak{g}$ where $h_\alpha \in C_{\mathfrak{g}}(T)$, $[h_\alpha, e_\beta] = A_{\alpha,\beta}e_\beta$ and $A_{\alpha,\beta} = \frac{2(\alpha,\beta)}{(\alpha,\alpha)}$; each element e_β is the root vector corresponding to the root β.

Assume that $\mathfrak{g}$ admits a non-degenerate G-invariant trace form and that p is a good prime for the root system Φ of G. In [21, Theorem A], Premet states that for any nilpotent element $n \in \mathfrak{g}$ there exists a one-parameter subgroup $\lambda \in \mathrm{Hom}(G_m, G)$ such that $\lambda(c).n = c^2 n$, i.e., there is a λ such that n lies in its 2-weight space $\mathfrak{g}(\lambda, 2) = \{x \in \mathfrak{g} \mid \mathrm{Ad}\,\lambda(c).x = c^2 x \text{ for all } c \in K^*\}$.

2.6 Main theorem

We shall henceforth set

$$E = \bigcup_{s\in G_{ss}\setminus Z}\ \bigcup_{\gamma\in K^*} V_\gamma(s),$$

where $V_\gamma(s) = \{v \in V \mid sv = \gamma v\}$, G_{ss} denotes the set of semisimple elements in G and $Z = Z(G)$ denotes the centre of G. We shall list all modules up to duality. This thesis will be concerned with proving the theorem below.

Main Theorem Let G be a simple simply connected algebraic group acting on an irreducible G-module V. If $\dim V \leqslant \dim G + 2$ then

$$\dim \overline{E} = \dim V,$$

with the following possible exceptions:

G	V	n
A_n	$L(3\omega_1)$	2 $(p > 3)$
	$L(\omega_3)$	7
B_n	$L(\omega_1 + \omega_2)$	2 $(p = 5)$
	$L(\omega_6)$	6
C_n	$L(\omega_2)$	$[3, \infty)$
	$L(\omega_n)$	$[5, 6]$ $(p = 2)$
	$L(\omega_3)$	3 $(p \neq 2)$
D_n	$L(\omega_7)$	7

and if $\dim V > \dim G + 2$ then

$$\dim \overline{E} < \dim V,$$

with the following possible exceptions:

G	V	n
A_n	$L(\omega_3)$	8
	$L(\omega_4)$	7
	$L(2\omega_2)$	3 $(p > 2)$
B_n	$L(2\omega_1)$	$[2, \infty)$ $(p > 2)$
	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[2, \infty)$
C_n	$L(\omega_3)$	4 $(p = 3)$
	$L(\omega_4)$	4 $(p \neq 2)$
	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[3, \infty)$
D_n	$L(2\omega_1)$	$[4, \infty)$ $(p > 2)$
	$L(\omega_7)$	8
	$L(\omega_8)$	8
	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[4, \infty)$

The Main Theorem follows from the results in Chapter 3 if $\dim V \leqslant \dim G + 2$, and Theorems 5.1, 6.1, 7.1, 8.1 and 10.1 and Lemmas 9.1, 9.2 and 9.3 if $\dim V > \dim G + 2$.

The proof will be split depending on the dimension of the irreducible G-module, the type of the algebraic group involved and whether the irreducible module on which it acts is parameterised by a p-restricted highest weight or not. In Chapter 3 we deal exclusively with modules of dimension at most $\dim G + 2$; in all later chapters we consider larger modules. In Chapters 5-8 we take in turn G to be one of the classical types A_n, B_n, C_n and D_n and in Chapter 9 we consider together the exceptional types E_6, E_7, E_8, F_4 and G_2. In these chapters we assume that each irreducible G-module is parameterised by a p-restricted highest weight. In Chapter 10 we deal with the remaining modules for all types of simple algebraic groups. These modules are parameterised by highest weights that are not p-restricted; however by Steinberg's tensor product theorem 2.3 they are tensor products of modules with p-restricted highest weights together with a twist.

Since the fixed point spaces $C_V(s)$ are precisely the 1-eigenspaces $V_1(s)$ we have, by inclusion,

$$\dim \overline{\bigcup_{s \in G_{ss} \setminus Z} C_V(s)} \leqslant \dim \overline{\bigcup_{s \in G_{ss} \setminus Z} \bigcup_{\gamma \in K^*} V_\gamma(s)}$$

and hence we obtain the following corollary.

Corollary 2.10. *If G and V are as before and $\dim V \geqslant \dim G + 2$ then there exists a vector $v \in V$ such that $\mathrm{Stab}_{G \setminus Z}(v) \subset G_u$ with the same possible exceptions as above, where G_u denotes the set of unipotent elements in G.*

We remark that if $\dim V < \dim G$ then, by the orbit-stabiliser theorem, any vector must have a non-trivial stabiliser in G, and so must lie in the fixed point space of some element of G. Thus if one wished to show that there exists a regular orbit in the action of G on V one must assume that $\dim V \geqslant \dim G$.

Chapter 3

Small modules

In this chapter we shall consider *small modules*, i.e., irreducible p-restricted G-modules of dimension at most $\dim G+2$ where G is a simple simply connected algebraic group defined over an algebraically closed field. We prove general results for modules V of dimension less than the dimension of a Borel subgroup B of G and for adjoint modules. The only modules with dimension $\dim G + 1$ are $(G, \lambda) = (A_1, 3\omega_1)$ with $p > 3$, $(A_3, \omega_1 + \omega_2)$ with $p = 3$ and $(A_n, \omega_1 + q\omega_1)$ for $n \geqslant 1$. The modules with dimension $\dim G + 2$ are $(G, \lambda) = (A_1, 4\omega_1)$ with $p > 3$, $(A_2, 3\omega_1)$ with $p > 3$ and $(B_2, \omega_1 + \omega_2)$ with $p = 5$. We may therefore focus on modules with dimension strictly between $\dim B$ and $\dim G + 2$.

3.1 General results

We shall begin by considering modules of dimension less than a Borel subgroup of G.

Lemma 3.1. *Let G be a simple simply connected algebraic group, B a Borel subgroup of G and V an irreducible G-module. If $\dim V < \dim B$ then $\bigcup_{s \in G_{ss} \setminus Z} C_V(s) = V$, and moreover $E = V$.*

Proof. By the orbit-stabiliser theorem, for any vector $v \in V$ we have

$$\dim \mathrm{Stab}_G(v) = \dim G - \dim G.v \geqslant \dim G - \dim V > \dim G - \dim B = \dim U$$

where $U = R_u(B)$. Thus it is not possible for $\mathrm{Stab}_G(v)$ to be conjugate to a subgroup of $U.Z$. It follows that the stabiliser in G of any vector contains a non-central semisimple element and hence each vector lies in the fixed point space of a non-central semisimple element. Thus we have $\bigcup_{s\in G_{ss}\setminus Z} C_V(s) = V$, whence $E = V$. □

It clearly follows that the conclusion of our Main Theorem holds for G-modules V satisfying $\dim V < \dim B$ since $E = V$.

The next proposition focuses on the adjoint action of an algebraic group on its associated Lie algebra by conjugation. We provide a complete list of the adjoint modules for each type in the table below.

Type	Module	n	Type	Module
A_n	$L(2\omega_1)$	1 $(p \neq 2)$	E_6	$L(\omega_2)$
	$L(\omega_1 + \omega_n)$	$[2, \infty)$	E_7	$L(\omega_1)$
B_n	$L(2\omega_2)$	2 $(p \neq 2)$	E_8	$L(\omega_8)$
	$L(\omega_2)$	$[3, \infty)$	F_4	$L(\omega_1)$
C_n	$L(2\omega_1)$	$[3, \infty)$ $(p \neq 2)$	G_2	$L(\omega_2)$
D_n	$L(\omega_2)$	$[4, \infty)$		

Proposition 3.2. *Let G be a simple simply connected algebraic group acting on its irreducible adjoint module V. Then E is dense in V, and moreover when p is good for G we have $E = V$.*

Proof. We begin without any conditions on the characteristic of the field p. Let a be a regular semisimple element of $\mathfrak{g}$. The set of such elements is dense in $\mathfrak{g}$. Let T' be a maximal torus with $a \in \mathrm{Lie}(T')$. Recall from Section 2.4 that global and infinitesimal centralisers coincide for semisimple elements, so

$$\mathrm{Lie}(T') = C_{\mathfrak{g}}(a) = \mathrm{Lie}(C_G(a)),$$

whence $C_G(a)^\circ = T'$. Taking $s \in T' \setminus Z$ we see that a is fixed by a non-central semisimple element. Thus all regular semisimple elements of $\mathfrak{g}$ lie in the fixed point space in $\mathfrak{g}$ of some $s \in G_{ss} \setminus Z$. Hence, by passing to the quotient of $\mathfrak{g}$ if necessary, E is dense in V. We note that the adjoint modules for F_4 with $p = 2$ and for G_2 with $p = 3$ are small modules having dimension 26 and 7, respectively.

Assume from now on that p is good for G and let $v \in \mathfrak{g}$. The Lie subalgebra $\langle v \rangle \subset \mathfrak{g}$ is soluble, so lies in a maximal soluble subalgebra which we may take to be $\mathfrak{b} = \mathrm{Lie}(B)$ by applying an element of G. Let the Jordan decomposition of v be $v = t + n$ where t is semisimple, n is nilpotent and $[t, n] = 0$. We may assume without loss of generality that $t \in \mathfrak{t} = \mathrm{Lie}(T)$ since any semisimple element of $\mathfrak{b}$ lies in a Cartan subalgebra of $\mathfrak{b}$, all of which lie in the same G-orbit. Hence we may write $t = \sum_{\alpha \in \Pi} \lambda_\alpha h_\alpha$ with each $\lambda_\alpha \in K$.

Define $\Psi = \{\beta \in \Phi \mid [t, e_\beta] = 0\}$ so that $\beta \in \Psi$ if and only if

$$0 = \sum_{\alpha \in \Pi} \lambda_\alpha [h_\alpha, e_\beta] = \sum_{\alpha \in \Pi} \lambda_\alpha A_{\alpha,\beta}.$$

It is clear that $A_{\alpha,\beta_1+\beta_2} = A_{\alpha,\beta_1} + A_{\alpha,\beta_2}$ so Ψ is a closed subsystem of Φ. Also n may be written as a linear combination of those e_β with $\beta \in \Psi$ since $[t, n] = 0$.

We take the two cases $\Psi \subsetneq \Phi$ and $\Psi = \Phi$ separately. In the former case we take a maximal subsystem $\Psi' \subset \Phi$ containing Ψ. Up to conjugacy we may assume that Ψ' is obtained by removing at least one node from the extended Dynkin diagram of Φ. Recall from Section 2.4 that Ψ' is the root system of the centraliser of a semisimple element $s \in G$. We have that s is non-central by the assumption on Ψ and $s.v = s.t + s.n = v$, i.e., $v \in C_V(s) = V_1(s)$ so $v \in E$.

Thus we are left with the case $\Psi = \Phi$; here $t \in Z(\mathfrak{g})$ since t commutes with $\mathfrak{t}$ and each root vector. Since the irreducible module is $V = \mathfrak{g}/Z(\mathfrak{g})$ by passing to this quotient of $\mathfrak{g}$ we see that v is nilpotent. As remarked in Section 2.5, we know that there is a one-parameter subgroup $\lambda : K^* \to G$ such that for all $c \in K^*$ we have $\lambda(c).v = c^2 v$. Therefore, choose $c \in K^*$ such that $s = \lambda(c)$ is non-central, then $v \in V_{c^2}(s)$ so $v \in E$. $\square$

We remark that if p is a bad prime for G then we need not have $E = V$. We may take for example $G = C_2(K)$ with $p = 2$. The corresponding Lie algebra $\mathfrak{g}$ of type C_2 consists of all 4×4 matrices T satisfying $T^t A + AT = 0$ where

$$A = \left(\begin{smallmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{smallmatrix}\right).$$

Consider the matrix

$$v = \left(\begin{smallmatrix} 0&0&0&1 \\ 0&1&1&0 \\ 0&0&1&0 \\ 0&0&0&0 \end{smallmatrix}\right);$$

it is straightforward to check that $v \in \mathfrak{g}$. If $MvM^{-1} = \lambda v$ for some

$$M = \left(\begin{smallmatrix} a&b&c&d \\ e&f&g&h \\ i&j&k&l \\ m&n&p&q \end{smallmatrix}\right) \in G$$

and $\lambda \in K^*$ then

$$Mv = \left(\begin{smallmatrix} 0&b&b+c&a \\ 0&f&f+g&e \\ 0&j&j+k&i \\ 0&n&n+p&m \end{smallmatrix}\right) = \lambda \left(\begin{smallmatrix} m&n&p&q \\ e+i&f+j&g+k&h+l \\ i&j&k&l \\ 0&0&0&0 \end{smallmatrix}\right) = \lambda v M.$$

Thus we must have $\lambda = 1$ and $a = q \neq 0$, $f = k \neq 0$ and $d, g \in K$; all other entries of M are zero. Since

$$M \in C_2(K) = \{X \in \mathrm{GL}_4(K) \mid X^t J X = J\},$$

where $J = \left(\begin{smallmatrix} 0 & K_2 \\ -K_2 & 0 \end{smallmatrix}\right)$ and $K_2 = \left(\begin{smallmatrix} 0&1 \\ 1&0 \end{smallmatrix}\right)$ we have

$$J = \left(\begin{smallmatrix} a&0&0&0 \\ 0&f&0&0 \\ 0&g&f&0 \\ d&0&0&a \end{smallmatrix}\right) \left(\begin{smallmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&-1&0&0 \\ -1&0&0&0 \end{smallmatrix}\right) \left(\begin{smallmatrix} a&0&0&d \\ 0&f&g&0 \\ 0&0&f&0 \\ 0&0&0&a \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0&0&0&a^2 \\ 0&0&f^2&0 \\ 0&-f^2&0&0 \\ -a^2&0&0 \end{smallmatrix}\right).$$

Therefore $a^2 = f^2 = 1$, i.e., $a = f = 1$ and M is unipotent, hence there is no non-central semisimple element of G for which v is an eigenvector, i.e., $v \notin E$.

If we exclude the adjoint modules, we see from [18] that we need to examine the following modules V with $\dim B \leqslant \dim V \leqslant \dim G$: for type A_n the modules $L(2\omega_1)$ for $n \in [2, \infty)$ with $p > 2$, $L(\omega_3)$ for $n \in [5, 7]$ and $L(\omega_1)$ for $n = 1$; for type B_n the module $L(\omega_n)$ for $n \in [5, 6]$; for type C_n the modules $L(\omega_2)$ for $n \in [3, \infty)$, $L(\omega_3)$ for $n = 3$ with $p \neq 2$ and $L(\omega_n)$ for $n \in [5, 6]$ with $p = 2$; for type D_n the module $L(\omega_n)$ for $n = 7$. For the exceptional groups we observe from Lübeck's tables [18] that there are no irreducible modules V with $\dim B \leqslant \dim V \leqslant \dim G$ other than the adjoint modules.

3.2 Twisted modules

In the next proposition we exploit the fact that $L(\omega_1)$ is the natural module for G of type A_n to deal with $L(\omega_1) \otimes L(\omega_1)^{(q)}$ and $L(\omega_1) \otimes L(\omega_n)^{(q)}$. These modules have dimension $\dim G+1$. Fix a basis of $L(\omega_1)$ and denote it by $v_1, \ldots, v_m$ where $m = n+1 = \dim L(\omega_1) = \dim L(\omega_n)$.

Proposition 3.3. *Let G be a simple algebraic group of type A_n with $n \geqslant 1$ acting on V where we take V to be either $L(\omega_1) \otimes L(\omega_1)^{(q)}$ or $L(\omega_1) \otimes L(\omega_n)^{(q)}$. Then there is a non-empty open set in V each of whose vectors has a non-central semisimple element in its stabiliser; in particular,* $\dim \overline{E} = \dim V$.

Proof. We need to treat the two twisted modules in this proposition separately; the action of $\mathrm{SL}_m(K)$ is different in each case. For any $A \in G$ the action on $L(\omega_1)$ is given by $A : v \mapsto Av$, while the action on $L(\omega_n)$ is given by $A : v \mapsto (A^{-1})^t v$.

The action of $A \in G$ on $L(\omega_1) \otimes L(\omega_1)^{(q)}$ is given as follows: $v \otimes w \mapsto Av \otimes A^{(q)}w$, i.e.,

$$\sum_{i,j} \gamma_{ij} v_i \otimes v_j \mapsto \sum_{i,j} \gamma_{ij} \left(Av_i \otimes A^{(q)} v_j \right) = \sum_{i,j,k,l} \gamma_{ij} a_{ik} a_{jl}^q v_k \otimes v_l,$$

where $Av_i = \sum_k a_{ik} v_k$ and $A^{(q)} v_j = \sum_l a_{jl}^q v_l$. Let $\Gamma = (\gamma_{ij})_{i,j=1}^m$ so the action is given by $\Gamma \mapsto A^t \Gamma A^{(q)}$. In order to show that each Γ has a non-trivial stabiliser it suffices to show that this is true for all Γ in a non-empty open set. Thus assume that each Γ is non-singular.

Consider for the moment the map $GL_m(K) \to GL_m(K)$ given by $\Gamma \mapsto A^{-1} \Gamma A^{(q)}$; this is the action of $GL_m(K)$ on itself by F-conjugation where F is the standard Frobenius map. Given a non-singular matrix Γ, by the Lang-Steinberg theorem 2.5, there exists $B \in GL_m(K)$ such that $\Gamma = B^{-1} F(B) = B^{-1} I F(B)$. Thus Γ lies in the same F-conjugacy class as the identity I. The stabiliser of I in the F-conjugation action consists of the F-stable non-singular matrices $GL_m(q)$. Since the stabiliser of I contains non-central semisimple elements, so does that of Γ. By replacing $F : A \mapsto A^{(q)}$ by the Frobenius map $F\tau$ where $\tau : A \mapsto (A^{-1})^t$ we can similarly deal with the relevant map $\Gamma \mapsto A^t\Gamma A^{(q)}$. If we instead consider the action on $L(\omega_1) \otimes L(\omega_n)^{(q)}$ then the stabiliser of I under $F\tau$ is $GU_m(q)$ which contains non-central semisimple elements, though not necessarily lying in G. By taking elements of the conjugate of $GU_m(K)$ which have determinant 1, we see that the stabiliser of Γ does indeed contain non-central semisimple elements of G. $\square$

3.3 Finite orbit modules

Let G be a connected linear algebraic group defined over an algebraically closed field K of characteristic $p \geqslant 0$ and let V be a finite-dimensional irreducible rational G-module. We shall say that a module is a *finite orbit module* if G has only a finite number of orbits on the set of vectors in V. These are classified in [10, Theorem 1] by Guralnick, Liebeck, Macpherson and Seitz. A G-module V is called a *prehomogeneous space* if G has an open dense orbit on V. It is clear that a finite orbit module is a prehomogeneous space. If G is simple then, by [10, Corollary 1], the converse is also true. A finite orbit module V satisfies $\dim V \leqslant \dim G$ by the orbit-stabiliser theorem, hence is a small module in our sense.

We can use existing literature to obtain the number of orbits, orbit representatives and (useful for our work) the point stabilisers for some of these finite orbit modules.

The irreducible prehomogeneous spaces in characteristic 0 were classified by Sato and Kimura [22], and were classified for positive characteristic by Chen [3]. Chen found that nearly all irreducible prehomogeneous spaces in characteristic $p > 0$ were obtained from the corresponding vector space in characteristic 0 by reducing modulo p. One of the exceptions was the $GL_4(K)$-module $L(\omega_1 + \omega_2)$ for $p = 3$ and this 16-dimensional module was studied in [4], and subsequently [6]. By combining the information in both of these papers we have representatives for the ten orbits in the action, as well as the structure of the point stabilisers of these representatives. By inspection, each vector in $L(\omega_1 + \omega_2)$ for

$p = 3$ is stabilised by a non-central semisimple element in $\mathrm{SL}_4(K)$.

The spin module $L(\omega_5)$ for B_5 is a finite orbit module. If $p \neq 2$ then Igusa details in [15, Proposition 6] the five orbits that occur and the point stabilisers. If $p = 2$ then in fact there are six orbits as detailed in [10, Lemma 2.11]. The information provided in these references allows us to conclude that each vector in the $\mathrm{Spin}_{11}(K)$-module $L(\omega_5)$ is stabilised by a non-central semisimple element.

Thus we have:

Lemma 3.4. *Let G be a simple simply connected algebraic group and $V = L(\lambda)$ an irreducible G-module. If $(G, \lambda) = (A_3, \omega_1 + \omega_2)$ with $p = 3$ or (B_5, ω_5) then we have $E = V$.*

We note that the G-module $V = L(\omega_3)$ for $G = A_n$ with $n \in [5, 7]$ is a finite orbit module. We shall show that $E = V$ for the cases $n = 5$ in 3.4 and $n = 6$ in 3.6; in the latter case point stabilisers have been determined by Chen, Cohen and Wales so we can conclude as above. Some of the possible exceptions for small modules listed in the Main Theorem such as $L(\omega_3)$ for A_7, $L(\omega_3)$ for C_3 with $p \neq 2$ and $L(\omega_7)$ for D_7 are finite orbit modules. We have not been able to find information in the literature to assist with these cases.

3.4 Two specific cases

In this section we shall consider two irreducible G-modules where G is of type A_n. We show that both of these modules are the union of eigenspaces of non-central semisimple elements.

Lemma 3.5. *Let G be a simple simply connected algebraic group of type A_n acting on the irreducible module $V = L(2\omega_1)$ for $n \geqslant 2$ with $p \neq 2$. Then $E = V$.*

Proof. The irreducible module with highest weight $2\omega_1$ is S^2N, the symmetric square of the natural module $N = L(\omega_1)$. For any $0 \neq x \in S^2N$ we may write

$$x = \sum_{i,j=1}^{m} c_{ij} v_i \otimes v_j$$

where $m = n+1$, $c_{ij} = c_{ji}$ for $i \neq j$ and $v_1, \ldots, v_m$ is the standard basis of N. Let $S = \mathrm{Stab}_G(x)$ and assume that S contains no non-central semisimple element. We may assume that $S \subset U.Z$ by conjugating S. Since $\dim S^2N = \frac{1}{2}(n+1)(n+2)$ we have $\dim S \geqslant \frac{1}{2}(n-1)(n+2) = \dim U - 1$ by the orbit-stabiliser theorem. Thus for any subgroup H of U we have $\dim(S \cap H) \geqslant \dim H - 1$. In particular, if we take $H = X_\gamma X_\delta$ where $\gamma = \alpha_2 + \cdots + \alpha_n$ and $\delta = \alpha_1 + \gamma$ then $\dim H = 2$; we have $\dim(S \cap H) \geqslant 1$ so that S contains a non-trivial element $g = x_\gamma(a)x_\delta(b)$ with $(a,b) \neq (0,0)$. It is clear that $gv_i = v_i$ for $i < m$ and $gv_m = v_m + bv_1 + av_2$. Since $g \in S$ and g fixes each $v_i \otimes v_j$ with $i, j < m$ we see that g must fix the vector

$$y = \sum_{i=1}^{m-1} c_{im} \left(v_i \otimes v_m + v_m \otimes v_i\right) + c_{mm} v_m \otimes v_m.$$

By inspecting the coefficients in

$$\begin{aligned} gy - y = \sum_{i=1}^{m-1} c_{im} \left[(av_2 + bv_1) \otimes v_i + v_i \otimes (av_2 + bv_1)\right] \\ + c_{mm}[a(v_2 \otimes v_m + v_m \otimes v_2) + b(v_1 \otimes v_m + v_m \otimes v_1) \\ + a^2(v_2 \otimes v_2) + ab(v_1 \otimes v_2 + v_2 \otimes v_1) + b^2(v_1 \otimes v_1)] \end{aligned}$$

of $v_k \otimes v_i + v_i \otimes v_k$ for $2 < i < m$ and $k \in \{1,2\}$ we see that $c_{im} = 0$. Analogously by inspecting the coefficients of $v_k \otimes v_m + v_m \otimes v_k$ for $k \in \{1,2\}$ we find $c_{mm} = 0$. Now

$$0 \equiv gy - y = 2bc_{1m}(v_1 \otimes v_1) + 2ac_{2m}(v_2 \otimes v_2) + (ac_{1m} + bc_{2m})(v_1 \otimes v_2 + v_2 \otimes v_1)$$

so that $2bc_{1m} = 2ac_{2m} = ac_{1m} + bc_{2m} = 0$ and since $(a,b) \neq (0,0)$ we have $c_{1m} = c_{2m} = 0$. Thus $c_{im} = c_{mi} = 0$ for $1 \leqslant i \leqslant m$, whence $x = \sum_{i,j=1}^{m-1} c_{ij} v_i \otimes v_j$. If $s = \mathrm{diag}(\nu, \ldots, \nu, \nu^{-n})$

for some $\nu \in K^*$ with $\nu^m \neq 1$ then s is a non-central semisimple element with $sx = \nu^2 x$, i.e., x is an eigenvector with eigenvalue ν^2. $\square$

Lemma 3.6. *If $G = \mathrm{SL}_6(K)$ acts on the irreducible module $V = L(\omega_3)$ then $E = V$.*

Proof. The irreducible module $L(\omega_3)$ is $\bigwedge^3 N$, the third exterior power of the natural module. If $n = 5$ then $\dim L(\omega_3) = \dim B = 20 = \operatorname{codim} U$. Take any

$$0 \neq x = \sum_{i<j<k} \lambda_{ijk} v_i \wedge v_j \wedge v_k \in \bigwedge^3 N$$

with $\lambda_{ijk} \in K$ and $1 \leqslant i < j < k \leqslant 6$, and set $S = \mathrm{Stab}_G(x)$. Assume that S contains no non-central semisimple element. We have $\dim U \leqslant \dim S$ by the orbit-stabiliser theorem so we may assume that $U \leqslant S$ by taking conjugates of S, hence $gx = x$ for any $g \in U$.

Let $\beta_1 = \alpha_1 + \cdots + \alpha_5$, $\beta_2 = \alpha_2 + \cdots + \alpha_5$ and $\beta_3 = \alpha_3 + \alpha_4 + \alpha_5$, and consider the action of $x_{\beta_i}(1)$ for each i on the standard basis; it sends $v_j \mapsto v_j$ for $j \neq 6$ and $v_6 \mapsto v_6 + v_i$. Thus we find

$$0 \equiv x_{\beta_1}(1)x - x = \sum_{2 \leqslant i < j \leqslant 5} (\lambda_{ij6} v_i \wedge v_j \wedge v_1)$$

in which case all coefficients λ_{ij6} with $2 \leqslant i < j \leqslant 5$ are zero. Similarly, by applying $x_{\beta_2}(1)$ we see that $\lambda_{1i6} = 0$ for $2 < i \leqslant 5$ and finally applying $x_{\beta_3}(1)$ we see that $\lambda_{126} = 0$. Now x is an eigenvector with eigenvalue ν^3 for the non-central semisimple element $s = \mathrm{diag}(\nu, \nu, \nu, \nu, \nu, \nu^{-5})$ for $\nu^6 \neq 1$. $\square$

3.5 Type A_1

We need to consider the action of $\mathrm{SL}_2(K)$ on the modules $L(\omega_1)$, $L(3\omega_1)$ and $L(4\omega_1)$.

Lemma 3.7. *If $G = \mathrm{SL}_2(K)$ acts on the irreducible module $V = L(\omega_1)$ then $E = V$.*

Proof. The module $L(\omega_1)$ is the natural module N. If $n = 1$ then $\dim G = 3$ and $\dim B = 2$. Consider $0 \neq x = av_1 + bv_2 \in N$ and let $S = \mathrm{Stab}_G(x)$. Suppose that S contains no non-central semisimple element then, as before, we can assume that $U = X_{\alpha_1} \leqslant S$

by conjugating S and so moving x within its orbit. Since $x_{\alpha_1}(1)$ sends $v_1 \mapsto v_1$ and $v_2 \mapsto v_2 + cv_1$ and $x_{\alpha_1}(c)x - x = 0$ we must have $b = 0$. Then x is an eigenvector with eigenvalue ν for $s = \mathrm{diag}(\nu, \nu^{-1}) \in G_{ss} \setminus Z$ for $\nu \neq \pm 1$. $\square$

The module $L(r\omega_1)$ for A_1 with $r \geqslant 1$ can be viewed as the space of homogeneous polynomials of degree r in two variables x and y. The action of $\mathrm{SL}_2(K)$ is given by the following:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .(x^i y^j) = (ax + cy)^i (bx + dy)^j$$

extended by linearity.

Lemma 3.8. *If $G = \mathrm{SL}_2(K)$ acts on the irreducible module $V = L(r\omega_1)$ for $r \in 3, 4$ then $\dim \overline{E} = \dim V$.*

Proof. If $r = 4$, there are five basis vectors x^4, x^3y, x^2y^2, xy^3 and y^4. The semisimple element $s = \left(\begin{smallmatrix} i & 0 \\ 0 & -i \end{smallmatrix}\right)$ where $i^2 = -1$ has eigenspace $V_1(s) = \langle x^4, x^2y^2, y^4 \rangle$ with eigenvalue 1. It is clear that the centraliser $C_G(s)$ of s is the maximal torus T consisting of diagonal matrices. Thus applying diagonal matrices preserves the eigenspace as they just scale the basis elements. We can use the Bruhat decomposition [2, §8.4] to obtain the set of coset representatives for T. Given such a coset representative g, the eigenspace for gsg^{-1} is $gV_1(s)$. Each g is either of the form uh or $uh\dot{w}_{\alpha_1}u'$ where

$$u = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \in U,\ u' = \left(\begin{smallmatrix} 1 & t' \\ 0 & 1 \end{smallmatrix}\right) \in U,\ h = \begin{pmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{pmatrix} \in T \ \text{ and } \ \dot{w}_{\alpha_1} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in N_G(T),$$

and $t, t' \in K$ and $\mu \in K^*$. In the second case, by applying $uh\dot{w}_{\alpha_1}u'$ to $ax^4 + bx^2y^2 + cy^4$ we obtain an expression with five parameters a, b, c, t, t'. Thus the closure of the union of these eigenspaces $\overline{\bigcup_{g \sim s} V_1(g)}$ is five-dimensional, hence $\dim \overline{E} = \dim V$.

Analogously, for $r = 3$ we can reach the same conclusion; here the basis vectors are x^3, x^2y, xy^2 and y^3. Now s has two eigenspaces $V_{-i}(s) = \langle x^3, y^3 \rangle$ and $V_i(s) = \langle x^2y, xy^2 \rangle$. $\square$

3.6 The action of $\mathrm{SL}_7(K)$ on $L(\omega_3)$

In this section we consider the action of $G = \mathrm{SL}_7(K)$ on the irreducible module $V = L(\omega_3)$. As previously mentioned, this is a finite orbit module. The nine orbits and point stabilisers for each orbit have been precisely determined in [5, Table 1]. It is clear from this information that $E = V$ for this module.

We also detail an alternative approach to show that $E = V$ due to Dr. R. Lawther. This method is based on techniques used in the previous two sections.

Recall in general that $\Phi = \Phi(A_n) = \{\epsilon_i - \epsilon_j \mid 1 \leqslant i, j \leqslant n+1\}$. Here we shall write roots in the form $i-j$ rather than $\epsilon_i - \epsilon_j$, where $1 \leqslant i, j \leqslant 7$; assume $\Phi^+ = \{i-j \mid i < j\}$. Let $v_1, \ldots, v_7$ be the standard basis of the natural module of G. Given $i, j, k \leqslant 7$ with $i \neq j \neq k \neq i$, write v_{ijk} for the standard basis vector $v_i \wedge v_j \wedge v_k$ of V, so that $v_{ijk} = v_{jki} = v_{kij} = -v_{ikj} = -v_{kji} = -v_{jik}$; we shall write the unordered triple $\{i, j, k\}$ as simply ijk. This should not cause confusion, as subscripts will never be unordered triples.

Take $v \in V$; write $v = \sum_{i<j<k} \lambda_{ijk} v_{ijk}$ and set $\Delta = \Delta(v) = \{ijk \mid \lambda_{ijk} \neq 0\}$ (note that this is well-defined).

We shall show that $v \in E$; we begin with a series of lemmas showing that this conclusion holds under various hypotheses on Δ. Our first two lemmas are elementary.

Lemma 3.9. *If $|\Delta| \leqslant 6$ then $v \in E$.*

Proof. Take $\lambda \in K^*$ and $p_1, \ldots, p_7 \in \mathbb{Z}$, and let $s = \mathrm{diag}(\lambda^{p_1}, \ldots, \lambda^{p_7})$; then $sv_{ijk} = \lambda^{p_i+p_j+p_k} v_{ijk}$. Thus given $q \in \mathbb{Z}$ we have $sv = \lambda^q v$ if and only if $p_i + p_j + p_k = q$ for all $ijk \in \Delta$; this imposes $|\Delta|$ homogeneous linear conditions on the 8 variables $p_1, \ldots, p_7, q$, in addition to the condition $p_1 + \cdots + p_7 = 0$ which guarantees $s \in G$. Thus provided $|\Delta| \leqslant 6$ there will be a non-zero solution; since $Z(G)$ is finite we may choose λ so that $s \notin Z(G)$. $\square$

For $l \leqslant 7$ write $\Delta_l = \{ijk \mid i, j, k \neq l\}$; given $l, m \leqslant 7$ write $\Delta_{l,m} = \Delta_l \cap \Delta_m$.

Lemma 3.10. *If $\Delta \subseteq \Delta_l$ for some $l \leqslant 7$ then $v \in E$.*

Proof. Take $\lambda \in K^*$ with $\lambda^7 \neq 1$ and let $s = \mathrm{diag}(\lambda^{p_1}, \ldots, \lambda^{p_7})$ where $p_l = -6$ and $p_i = 1$ if $i \neq l$; then $sv = \lambda^3 v$ and $s \notin Z(G)$. $\square$

In the arguments to follow we shall frequently apply root elements or Weyl group elements to modify v and hence Δ; for H a subgroup of G rather than repeating the phrase *applying a suitable element of* H we shall abbreviate this to *applying* H.

Lemma 3.11. *If* $|\Delta \setminus \Delta_6|$, $|\Delta \setminus \Delta_7| \leqslant 1$ *then* $v \in E$.

Proof. By Lemma 3.10 it suffices to treat the case where $|\Delta \setminus \Delta_6| = |\Delta \setminus \Delta_7| = 1$. If $\Delta \setminus \Delta_6 = \Delta \setminus \Delta_7$, there exists $i \leqslant 5$ such that $\Delta \subseteq \Delta_{6,7} \cup \{i67\}$, and then $\mathrm{diag}(1,1,1,1,1,\lambda,\lambda^{-1})v = v$; so we may assume there exist $i,j,k,l \leqslant 5$ such that $\Delta \subseteq \Delta_{6,7} \cup \{ij6, kl7\}$. If $\{i,j\} = \{k,l\}$, applying X_{6-7} we may ensure $ij6 \notin \Delta$, so that $\Delta \subseteq \Delta_6$ and the result follows by Lemma 3.10; so we may assume that $\{i,j\} \neq \{k,l\}$. Given $a \leqslant 5$ with $a \neq i,j$, applying X_{a-6} we may ensure $ija \notin \Delta$; likewise, given $b \leqslant 5$ with $b \neq k,l$, applying X_{b-7} we may ensure that $klb \notin \Delta$. Now if $\{i,j\} \cap \{k,l\} = \varnothing$, take m such that $\{i,j,k,l,m\} = \{1,2,3,4,5\}$, then $\Delta \subseteq \{ikm, ilm, jkm, jlm, ij6, kl7\}$ and the result follows by Lemma 3.9; so we may assume $i = k$, $j \neq l$. Take m,n with $m < n$ such that $\{i,j,l,m,n\} = \{1,2,3,4,5\}$, then $\Delta \subseteq \{jlm, jln, imn, jmn, lmn, ij6, kl7\}$; applying X_{m-n} we may ensure $|\Delta \cap \{jlm, jln\}| \leqslant 1$ and the result follows by Lemma 3.9. $\square$

Lemma 3.12. *If there exist distinct* $l,m,n \leqslant 5$ *such that* $\Delta \subseteq \Delta_{6,7} \cup \{in6, in7 \mid i \leqslant 5, i \neq n\} \cup \{lm6, lm7\}$, *then* $v \in E$.

Proof. Applying $\langle w_{1-2}, w_{2-3}, w_{3-4}, w_{4-5}\rangle$ we may assume $(l,m,n) = (1,2,5)$. Applying X_{6-7} and $\langle w_{6-7}\rangle$ we may ensure $127 \notin \Delta$; likewise applying X_{3-4} and $\langle w_{3-4}\rangle$ we may ensure $357 \notin \Delta$, and applying X_{1-2} and $\langle w_{1-2}\rangle$ we may ensure $157 \notin \Delta$. Suppose $457 \in \Delta$; applying $X_{2-4}X_{6-7}$ we may ensure $257, 456 \notin \Delta$, and applying $X_{1-2}X_{2-3}X_{1-3}$ we may ensure $|\Delta \cap \{156, 256, 356\}| \leqslant 1$. Now, by Lemma 3.11, we may assume $126 \in \Delta$ and there exists $i \leqslant 3$ with $\Delta \cap \{156, 256, 356\} = \{i56\}$. If $i = 3$, applying

$$X_{5-6}X_{3-6}X_{2-6}X_{1-6}X_{3-7}X_{2-7}X_{1-7}$$

we may ensure $125, 123, 235, 135, 345, 245, 145 \notin \Delta$, so $\Delta \subseteq \{124, 134, 234, 126, 356, 457\}$ and the result follows by Lemma 3.9; if instead $i \in \{1, 2\}$, applying

$$X_{5-6}X_{4-6}X_{3-6}X_{3-7}X_{2-7}X_{1-7}$$

we may ensure $125, 124, 123, 345, 245, 145 \notin \Delta$, so $\Delta \subseteq \{134, 234, 135, 235, 126, i56, 457\}$ and $\mathrm{diag}(\lambda^3, \lambda^3, \lambda^{-4}, \lambda^3, \lambda^3, \lambda^{-4}, \lambda^{-4})v = \lambda^2 v$. So we may assume $457 \notin \Delta$; by Lemma 3.10 we may assume $257 \in \Delta$, else $\Delta \subseteq \Delta_7$; now applying X_{6-7} we may ensure $256 \notin \Delta$. If $356, 456 \notin \Delta$ then applying X_{2-5} we may ensure $|\Delta \cap \{126, 156\}| \leqslant 1$ and the result follows by Lemma 3.11; so we may assume $\Delta \cap \{356, 456\} \neq \varnothing$, and applying $\langle w_{3-4} \rangle$ we may assume $456 \in \Delta$. Now applying $X_{3-4}X_{1-4}$ we may ensure $356, 156 \notin \Delta$. If $126 \notin \Delta$ the result follows by Lemma 3.11, so we may assume $126 \in \Delta$; applying $X_{5-6}X_{4-6}X_{3-6}X_{2-6}X_{1-6}$ we may ensure $125, 124, 345, 245, 145 \notin \Delta$, so $\Delta \subseteq \{123, 134, 234, 135, 235, 126$ and $\mathrm{diag}(\lambda^3, \lambda^3, \lambda^{-4}, \lambda^3, \lambda^3, \lambda^{-4}, \lambda^{-4})v = \lambda^2 v$ □

Lemma 3.13. *If there exist distinct $m, n \leqslant 5$ such that $\Delta \subseteq \Delta_{6,7} \cup \{im6, in6 \mid i \leqslant 5, i \neq m, n\} \cup \{mn6\} \cup \{in7 \mid i \leqslant 5, i \neq n\}$, then $v \in E$.*

Proof. Applying $\prod_{\substack{i<j\leqslant 5 \\ i,j\neq m,n}} X_{i-j}$ we may ensure that there exists $l \leqslant 5$ with $l \neq m, n$ such that $\Delta \cap \{im6 \mid i \leqslant 5, i \neq m, n\} \subseteq \{lm6\}$, and the result follows by Lemma 3.12. □

Lemma 3.14. *If there exists $l \leqslant 5$ such that $\Delta \cap \{il6, il7 \mid i \leqslant 5, i \neq l\} = \Delta \cap \{i67 \mid i \leqslant 5\} = \varnothing$, then $v \in E$.*

Proof. Applying $\langle w_{1-2}, w_{2-3}, w_{3-4}, w_{4-5} \rangle$ we may assume $l = 5$; by Lemma 3.10 we may assume $\Delta \not\subseteq \Delta_7$, and applying $\langle w_{1-2}, w_{2-3}, w_{3-4} \rangle$ we may assume $347 \in \Delta$. Now applying $X_{2-3}X_{1-3}X_{2-4}X_{1-4}X_{2-7}X_{1-7}$ we may ensure $247, 147, 237, 137, 234, 134 \notin \Delta$; if $126, 127 \notin \Delta$ the result follows by Lemma 3.13 with $(m, n) = (3, 4)$, so we may assume there exists $k \in \{6, 7\}$ with $12k \in \Delta$. Now applying $X_{4-k}X_{3-k}$ we may ensure $123, 124 \notin \Delta$, so $\Delta \subseteq \{ij5, ij6 \mid i, j \leqslant 4\} \cup \{127, 347\}$ and $\mathrm{diag}(\lambda^3, \lambda^3, \lambda^3, \lambda^3, \lambda^{-4}, \lambda^{-4}, \lambda^{-4})v = \lambda^2 v$. □

We may now proceed to our general argument.

Theorem 3.15. *Let $G = \mathrm{SL}_7(K)$ act on the irreducible module $V = L(\omega_3)$. Then $E = V$.*

Proof. Recall that $U = \prod_{\alpha\in\Phi^+} X_\alpha$ and let $U' = \prod_{i<6} X_{i-6} \prod_{j<7} X_{j-7}$, so that $\dim U = 21$ and $\dim U' = 11$. Suppose $\mathrm{Stab}_G(v)^\circ$ is unipotent; applying G we may assume that $\mathrm{Stab}_G(v)^\circ \subseteq U$. Since $\dim G = 48$ while $\dim V = 35$, we must have $\dim \mathrm{Stab}_G(v)^\circ \geqslant 13$; thus $\dim(\mathrm{Stab}_G(v)^\circ \cap U') \geqslant 3$. From now on we shall assume merely that $v \in V$ is such that $\dim(\mathrm{Stab}_G(v)^\circ \cap U') \geqslant 3$, and we shall show that this condition implies that $v \in E$; this will allow us to apply $\langle w_{1-2}, w_{2-3}, w_{3-4}, w_{4-5}\rangle$ wherever it is convenient to do so, since this subgroup of W preserves U'.

Order the roots of U' as

$$5-6 \prec 4-6 \prec 3-6 \prec 2-6 \prec 1-6 \prec 6-7 \prec 5-7 \prec 4-7 \prec 3-7 \prec 2-7 \prec 1-7.$$

Given $u \in U'$ we may write $u = \prod x_\alpha(t_\alpha)$. If α is the first root in the order given for which $t_\alpha \neq 0$, we say that u *begins with* α. Let $\Gamma = \{\gamma \in \Phi^+ \mid$ there exists $u \in \mathrm{Stab}_{U'}(v)$ beginning with $\gamma\}$.

Observe that if $u \in U'$ and $\mu \neq 0$, then u and $I + \mu(u - I)$ fix the same vectors in V. Thus for all $\gamma \in \Gamma$ we may choose $g_\gamma \in \mathrm{Stab}_{U'}(v)$ such that $g_\gamma = x_\gamma(1) \prod_{\beta\succ\gamma} x_\beta(t_\beta)$ for some t_β. Moreover, if $\gamma, \gamma' \in \Gamma$ and $\gamma' \succ \gamma$ we may replace g_γ by $g_\gamma - t_{\gamma'}(g_{\gamma'} - I)$ to ensure that the projection of g_γ on $X_{\gamma'}$ is trivial; we call this *adjusting* g_γ *by* $g_{\gamma'}$. Taking successively the $\gamma' \in \Gamma$ with $\gamma' \succ \gamma$ in increasing order, we may obtain a unique element $g_\gamma \in x_\gamma(1) \prod_{\substack{\beta\succ\gamma \\ \beta\notin\Gamma}} X_\beta$ lying in $\mathrm{Stab}_{U'}(v)$. It is then clear that

$$\mathrm{Stab}_{U'}(v) = \left\{ I + \sum_{\gamma\in\Gamma} t_\gamma(g_\gamma - I) \mid t_\gamma \in K \text{ for all } \gamma \in \Gamma \right\};$$

so $\dim \mathrm{Stab}_{U'}(v) = |\Gamma|$, whence $|\Gamma| \geqslant 3$.

Thus we may take distinct γ_1, γ_2, $\gamma_3 \in \Gamma$; given $i \leqslant 3$ we shall write simply g_i for g_{γ_i}. We shall consider three cases, which between them cover all possibilities:

(I) $(\gamma_1, \gamma_2, \gamma_3) = (a-7, b-7, c-7)$ for distinct a, b, c;

(II) $(\gamma_1, \gamma_2) = (a-7, b-6)$ for distinct a, b, and γ_3 is either $c-7$ for $c \neq a$ or $c-6$ for $c \neq b$;

(III) $(\gamma_1, \gamma_2, \gamma_3) = (a-6, b-6, c-6)$ for distinct a, b, c.

In the following we shall write $[g, ijk]$ to mean the condition that the ijk-component of $gv - v$ is zero.

Suppose that (I) holds. We may assume $a < b < c$; applying

$$\prod_{i<a} X_{i-a} \prod_{\substack{j<b \\ j \neq a}} X_{j-b} \prod_{\substack{k<c \\ k \neq a,b}} X_{k-c}$$

we may assume $g_1 = x_{a-7}(1)$, $g_2 = x_{b-7}(1)$ and $g_3 = x_{c-7}(1)$. Given distinct $i, j \leqslant 6$, if $i, j \neq a$ then $[g_1, ija]$ which implies that $ij7 \notin \Delta$, if $i, j \neq b$ then $[g_2, ijb]$ which implies that $ij7 \notin \Delta$, and if $i, j \neq c$ then $[g_3, ijc]$ which implies that $ij7 \notin \Delta$. Thus $\Delta \subseteq \Delta_7$ and the result follows by Lemma 3.10.

Suppose (II) holds. Applying $\prod_{i<a} X_{i-a} \prod_{j<b} X_{j-b}$ we may assume $g_1 = x_{a-7}(1)$ and $g_2 \in x_{b-6}(1) \prod_{j \neq a} X_{j-7}$. Given $i \leqslant 5$ with $i \neq b$ we have $[g_2, ib7]$ which implies that $i67 \notin \Delta$.

First suppose $c < 6$; then according as $\gamma_3 = c-6$ or $c-7$, either $[g_3, bc7]$ or $[g_3, bc6]$ which implies that $b67 \notin \Delta$, so $\Delta \cap \{i67 \mid i \leqslant 5\} = \varnothing$. Given distinct $i, j \leqslant 5$ with $i, j \neq a$, $[g_1, ija]$ which implies that $ij7 \notin \Delta$. Thus, if $a = 6$ we have $\Delta \subseteq \Delta_7$ so the result follows by Lemma 3.10; we may therefore assume $a \leqslant 5$, and $\Delta \subseteq \Delta_7 \cup \{ia7 \mid i \leqslant 5, i \neq a\}$. Now given distinct $i, j \leqslant 5$ with $i, j \neq a, b$, we have $[g_2, ijb]$ which implies that $ij6 \notin \Delta$; so $\Delta \subseteq \Delta_6 \cup \{ia6, ib6 \mid i \leqslant 5, i \neq a, b\} \cup \{ab6\}$, and the result follows by Lemma 3.13 with $(m, n) = (b, a)$.

So we may suppose instead $c = 6$, so that $\gamma_3 = 6 - 7$; applying $\prod_{k<6} X_{k-6}$ we may assume $g_3 = x_{6-7}(1)$. Given distinct $i, j \leqslant 5$, we have $[g_3, ij6]$ which implies that $ij7 \notin \Delta$; now given distinct $i, j \leqslant 5$ with $i, j \neq b$, we have $[g_2, ijb]$ which implies that $ij6 \notin \Delta$; so $\Delta \subseteq \Delta_{6,7} \cup \{ib6 \mid i \leqslant 5, i \neq b\} \cup \{b67\}$. Thus if $b67 \notin \Delta$ we have $\Delta \subseteq \Delta_7$ and the result follows by Lemma 3.10; if instead $b67 \in \Delta$, applying $\prod_{\substack{i \leqslant 5 \\ i \neq b}} X_{i-7}$ we may ensure

$\Delta \subseteq \Delta_{6,7} \cup \{b67\}$ and the result follows by Lemma 3.11.

Finally suppose (III) holds. Applying $\langle w_{1-2}, w_{2-3}, w_{3-4}, w_{4-5}\rangle$ we may assume $(a, b, c) = (1, 2, 3)$. We have $[g_2, 127]$ which implies that $167 \notin \Delta$, while if $2 \leqslant i \leqslant 5$ we have $[g_1, i17]$ which implies that $i67 \notin \Delta$. Thus $\Delta \cap \{i67 \mid i \leqslant 5\} = \varnothing$. For $p \leqslant 3$ write $g_p = x_{p-6}(1)\prod_{q<7} x_{q-7}(t_{pq})$. If there exists $p \leqslant 3$ with $t_{p6} \neq 0$, then given distinct $i, j \leqslant 5$ we have $[g_p, ij6]$ which implies that $ij7 \notin \Delta$, so $\Delta \subseteq \Delta_7$ and the result follows by Lemma 3.10. Therefore, we may assume that $t_{16} = t_{26} = t_{36} = 0$.

First suppose there exist $p \leqslant 3$ and $q \in \{4, 5\}$ with $t_{pq} \neq 0$. Applying $\langle w_{1-2}, w_{2-3}\rangle$ (and permuting g_1, g_2, g_3 accordingly) and $\langle w_{4-5}\rangle$ we may assume $t_{15} \neq 0$; applying $X_{4-5}X_{3-5}X_{2-5}X_{1-5}$ and T we may assume $t_{15} = 1$ and $t_{11} = t_{12} = t_{13} = t_{14} = 0$, so that $g_1 = x_{1-6}(1)x_{5-7}(1)$; adjusting g_2 and g_3 by g_1 and applying $X_{1-2}X_{1-3}$ we may assume $t_{25} = t_{35} = 0$. Now $[g_1, 123]$, $[g_1, 124]$, $[g_1, 134]$, $[g_1, 235]$, $[g_1, 245]$ and $[g_1, 345]$ imply that $236, 246, 346, 237, 247, 347 \notin \Delta$.

Now if there exists $p' \in \{2, 3\}$ with $t_{p'4} \neq 0$, applying $\langle w_{2-3}\rangle$ (and permuting g_2, g_3 accordingly) we may assume $t_{24} \neq 0$; applying $X_{3-4}X_{2-4}X_{1-4}$ and T we may assume $t_{24} = 1$ and $t_{21} = t_{22} = t_{23} = 0$, so that $g_2 = x_{2-6}(1)x_{4-7}(1)$. Now $[g_2, 123]$, $[g_2, 235]$, $[g_2, 134]$ and $[g_2, 345]$ imply that $136, 356, 137, 357 \notin \Delta$, and the result follows by Lemma 3.14 with $l = 3$. So we may assume $t_{24} = t_{34} = 0$. Now applying $X_{1-5}X_{6-7}$ so as to preserve g_1 we may assume $t_{33} = 0$. We then have $[g_3, 134]$ and $[g_3, 345]$ implying that $146, 456 \notin \Delta$, whence $[g_1, 145]$ which implies that $147 \notin \Delta$. Now if $457 \notin \Delta$ the result follows by Lemma 3.14 with $l = 4$, so we may assume $457 \in \Delta$; then $[g_2, 145]$ which implies that $t_{21} = 0$, $[g_2, 245]$ which implies that $t_{22} = 0$, $[g_2, 345]$ which implies that $t_{23} = 0$, so $g_2 = x_{2-6}(1)$; $[g_3, 145]$ which implies that $t_{31} = 0$ and $[g_3, 245]$ which implies that $t_{32} = 0$, so $g_3 = x_{3-6}(1)$. Now $[g_2, 123]$, $[g_2, 125]$, $[g_2, 235]$, $[g_3, 123]$ and $[g_3, 235]$ imply that $136, 156, 356, 126, 256 \notin \Delta$, so $\Delta \subseteq \Delta_6$ and the result follows by Lemma 3.10.

Thus we may assume that $t_{14} = t_{15} = t_{24} = t_{25} = t_{34} = t_{35} = 0$.

Next suppose there exist distinct $p, q \leqslant 3$ with $t_{pq} \neq 0$. Applying $\langle w_{1-2}, w_{2-3}\rangle$ (and permuting g_1, g_2, g_3 accordingly) we may assume $t_{13} \neq 0$; applying $X_{2-3}X_{1-3}$ and T and adjusting g_3 by g_2 and g_1 we may assume $t_{13} = 1$ and $t_{11} = t_{12} = 0$, so that $g_1 = x_{1-6}(1)x_{3-7}(1)$; adjusting g_2 by g_1 and applying X_{1-2} we may assume $t_{23} = 0$; applying $X_{1-3}X_{6-7}$ so as to preserve g_1, and adjusting g_3 by g_1, we may assume $t_{22} = 0$. Now $[g_1, 124]$, $[g_1, 125]$, $[g_1, 145]$, $[g_1, 234]$, $[g_1, 235]$ and $[g_1, 345]$ imply that $246, 256, 456, 247, 257, 457 \notin \Delta$; now $[g_2, 124]$, $[g_2, 234]$, $[g_2, 125]$ and $[g_2, 235]$ imply that $146, 346, 156, 356 \notin \Delta$; now $[g_1, 134]$ and $[g_1, 135]$ imply that $147, 157 \notin \Delta$. Thus we have $\Delta \subseteq \Delta_{6,7} \cup \{126, 136, 236, 127, 137, 237, 347, 357\}$. If $357 \notin \Delta$ the result follows by Lemma 3.14 with $l = 5$, so we may assume $357 \in \Delta$; but now applying X_{4-5} we may ensure $347 \notin \Delta$ and the result follows by Lemma 3.14 with $l = 4$.

Thus we may assume that $t_{12} = t_{13} = t_{21} = t_{23} = t_{31} = t_{32} = 0$. Applying X_{6-7} we may assume $t_{33} = 0$, so $g_3 = x_{3-6}(1)$. Given distinct $i, j \leqslant 5$ with $i, j \neq 3$, we have $[g_3, ij3]$ which implies that $ij6 \notin \Delta$. If there exists $p \leqslant 2$ with $t_{pp} \neq 0$, applying $\langle w_{1-2} \rangle$ (and permuting g_1, g_2 accordingly) we may assume $t_{11} \neq 0$; now $[g_1, 124]$, $[g_1, 125]$ and $[g_1, 145]$ imply that $247, 257, 457 \notin \Delta$; so

$$\Delta \subseteq \Delta_{6,7} \cup \{i36 \mid i \leqslant 5, i \neq 3\} \cup \{i17, i37 \mid i \leqslant 5, i \neq 1, 3\} \cup \{137\}.$$

Applying $\langle w_{6-7} \rangle$ the result follows by Lemma 3.13 with $(m, n) = (1, 3)$. Thus we may assume $t_{11} = t_{22} = 0$, so $g_1 = x_{1-6}(1)$ and $g_2 = x_{2-6}(1)$. Hence $[g_2, 123]$, $[g_1, 123]$, $[g_1, 134]$ and $[g_1, 135]$ imply that $136, 236, 346, 356 \notin \Delta$, so $\Delta \subseteq \Delta_6$ and the result follows by Lemma 3.10. $\square$

For type A_7 the module $L(\omega_3)$ remains as the only possible exception to the conclusion of the Main Theorem for modules of type A_n with dimension less than $\dim G$.

Chapter 4

Techniques for large modules

The aim of this chapter is to lay out the main theory which we shall apply in the later chapters for *large modules*, i.e., modules V for which $\dim V > \dim G + 2$; recall that our aim is to show that $\dim \overline{E} < \dim V$. Indeed, for the remainder of this thesis we shall be solely concerned with large modules. We shall prove that if a certain inequality dependent on the centraliser type of a non-central semisimple element is satisfied for all such types then the required conclusion holds. We explain the adjacency principle which is used to obtain lower bounds on the codimension of an eigenspace. We also give criteria which we use to obtain a relatively short list of modules which require further consideration.

4.1 Analysis of centraliser types

We begin by noting that [12, §26.2] shows that the elements of the finite centre $Z = Z(G)$ of G are semisimple. We shall require a division of the non-central semisimple elements according to the type of the centraliser. Let I denote the set of possible types of centralisers of non-central semisimple elements (see Section 2.4). By Proposition 2.9 and Deriziotis' theorem, there are a finite number of types X of centralisers of non-central semisimple elements formed by taking all possible subdiagrams of the extended Dynkin diagram of G. Thus I is a finite set. For each $X \in I$ we set

$$S_X = \{s \in G_{ss} \mid C_G(s) \text{ of type } X\}$$

so that $G_{ss} \setminus Z = \bigcup_{X \in I} S_X$. Hence if we can prove for each $X \in I$ that

$$\dim \overline{\bigcup_{s \in S_X} \bigcup_{\gamma \in K^*} V_\gamma(s)} < \dim V$$

then the conclusion of the Main Theorem holds for V.

The next result will be used repeatedly in subsequent chapters. We use the notation $\dim X$ to denote the dimension of the centraliser of a semisimple element with root system of type X and set $\operatorname{codim} X = \dim G - \dim X$; also set $\operatorname{codim} V_\gamma(t) = \dim V - \dim V_\gamma(t)$ for any $t \in T$. Also we write $a \sim b$ if and only if $a, b \in G$ are conjugate in G.

We consider the following two conditions. For a fixed centraliser type $X \in I$ if the inequality

$$\operatorname{codim} V_\gamma(t) > \operatorname{codim} X$$

holds for all $t \in T \cap S_X$ and $\gamma \in K^*$ then we shall say that

$$(\lozenge) \text{ is satisfied for } X.$$

If the inequality

$$\operatorname{codim} V_\gamma(s) > \dim G$$

holds for all $s \in G_{ss} \setminus Z$ and $\gamma \in K^*$ then we shall say that

$$(\dagger) \text{ holds.}$$

Certainly if the stronger condition $(\dagger)$ holds then condition $(\lozenge)$ is satisfied for all $X \in I$. This avoids a case-by-case analysis of centraliser types.

Theorem 4.1. *Let G be a simple simply connected algebraic group and let V be an* *irreducible G-module, both defined over an algebraically closed field K. If ($\Diamond$) holds for some $X \in I$, then*

$$\dim \overline{\bigcup_{s \in S_X} \bigcup_{\gamma \in K^*} V_\gamma(s)} < \dim V.$$

Proof. Consider the set of weights $\{\lambda_i \mid 1 \leqslant i \leqslant l\}$ of the module (ignoring multiplicities) ordered in some way. The set of all $l \times l$ matrices with rows and columns indexed by this ordered set of weights whose entries are all zeros or ones is clearly finite. Given an element $s \in T$, we define such a matrix $M(s)$ as follows: for each pair (i, j) with $i, j \in [1, l]$, we set the (i, j)th entry of $M(s)$ to be one if $\lambda_i(s) = \lambda_j(s)$ and zero otherwise. In other words, an element $s \in T$ has a collection of weight spaces and the possible equality of weights for this element is determined by the matrix $M(s)$ of which there are only finitely many.

Now take $X \in I$ and assume that ($\Diamond$) holds for X. By the above, given $t \in T \cap S_X$ there are finitely many possibilities for the collection of eigenspaces of t, i.e., there exist $t_1, \ldots, t_m \in T \cap S_X$ such that if $t \in T \cap S_X$ then t has the same eigenspaces as some t_i. In particular we have the equality

$$\bigcup_{s \in S_X} \bigcup_{\gamma \in K^*} V_\gamma(s) = \bigcup_{i=1}^{m} \bigcup_{s \sim t_i} \bigcup_{\gamma \in K^*} V_\gamma(s).$$

Thus it suffices to show that for all $i \leqslant m$ we have

$$\dim \overline{\bigcup_{s \sim t_i} \bigcup_{\gamma \in K^*} V_\gamma(s)} < \dim V.$$

Now any $s \in {t_i}^G$ has the same eigenvalues as t_i, and there are only finitely many of these. Thus by reversing the order of the two unions, we see that it suffices to show that for all $i \leqslant m$ and all $\gamma \in K^*$ we have

$$\dim \overline{\bigcup_{s \sim t_i} V_\gamma(s)} < \dim V.$$

We claim that in fact for all $t \in T$ and $\gamma \in K^*$ we have

$$\dim \overline{\bigcup_{s \sim t} V_\gamma(s)} \leqslant \dim t^G + \dim V_\gamma(t). \tag{$\star$}$$

Assuming that we can prove ($\star$), then taking $t = t_i$ we have

$$\dim {t_i}^G + \dim V_\gamma(t_i) = \operatorname{codim} X + \dim V - \operatorname{codim} V_\gamma(t_i) < \dim V$$

as required.

In order to prove ($\star$) we begin by closely following the proof of [16, Proposition 1.14]. Given $\gamma \in K^*$, set

$$Y_\gamma = \{(g, v) \in G \times V \mid gv = \gamma v\}.$$

If $\pi_\gamma, \phi : G \times V \to V$ are the morphisms defined by

$$\pi_\gamma(g, v) = \gamma v, \qquad \phi(g, v) = gv$$

then $Y_\gamma = \{(g, v) \in G \times V \mid \pi_\gamma(g, v) = \phi(g, v)\}$, so Y_γ is a closed subvariety of $G \times V$.

Now for $t \in T$ define

$$Y_{\gamma,t} = \{(t^g, v) \mid g \in G,\ v \in V,\ t^g v = \gamma v\}.$$

Then $Y_{\gamma,t}$ is a variety, and the map $Y_{\gamma,t} \to t^G$ given by $(t^g, v) \mapsto t^g$ has fibres of dimension $V_\gamma(t)$, so

$$\dim Y_{\gamma,t} = \dim t^G + \dim V_\gamma(t).$$

In order to establish ($\star$) it therefore suffices to prove

$$\dim \overline{\bigcup_{s \sim t} V_\gamma(s)} \leqslant \dim Y_{\gamma,t}.$$

Now we have the morphism $\pi : G \times V \to V$ defined by $\pi(g, v) = v$. If we restrict π to $Y_{\gamma,t}$, we have a morphism whose image is $\bigcup_{s \sim t} V_\gamma(s)$. Let the irreducible components of $Y_{\gamma,t}$

be $A_1, \ldots, A_k$. Then for each i, $\overline{\pi(A_i)}$ is a closed set containing $\pi(A_i)$; so $\bigcup_{i=1}^k \overline{\pi(A_i)}$ is a closed set containing $\bigcup_{i=1}^k \pi(A_i) = \pi\left(\bigcup_{i=1}^k A_i\right) = \pi(Y_{\gamma,t})$, and hence $\overline{\pi(Y_{\gamma,t})} \subseteq \bigcup_{i=1}^k \overline{\pi(A_i)}$. On the other hand, for all i we have $\pi(A_i) \subseteq \pi(Y_{\gamma,t})$, so $\overline{\pi(A_i)} \subseteq \overline{\pi(Y_{\gamma,t})}$ and hence $\bigcup_{i=1}^k \overline{\pi(A_i)} \subseteq \overline{\pi(Y_{\gamma,t})}$. Thus $\overline{\pi(Y_{\gamma,t})} = \bigcup_{i=1}^k \overline{\pi(A_i)}$.

Now, by [24, Lemma 1.9.1(iii)], since A_i is irreducible so is $\overline{\pi(A_i)}$, and $\dim A_i \geqslant \dim \overline{\pi(A_i)}$. Thus

$$\dim Y_{\gamma,t} = \max_{1 \leqslant i \leqslant k} \dim A_i \geqslant \max_{1 \leqslant i \leqslant k} \dim \overline{\pi(A_i)} = \dim \bigcup_{i=1}^{k} \overline{\pi(A_i)} = \dim \overline{\pi(Y_{\gamma,t})} = \dim \overline{\bigcup_{s \sim t} V_\gamma(s)}$$

as required to prove $(\star)$. $\square$

We shall prove that for a given G and irreducible module V either $(\dagger)$ holds or $(\Diamond)$ is satisfied for all $X \in I$ with a list of possible exceptions. Given $X \in I$ denote by d_λ^X the minimal codimension of $V_\gamma(s)$ for any $s \in S_X$ and $\gamma \in K^*$. We denote $\operatorname{codim} X = |\Phi(G)| - |\Phi(X)|$ by e_λ^X. Clearly our aim when we cannot show $(\dagger)$ holds will be to show that d_λ^X exceeds e_λ^X for all $X \in I$.

4.2 Adjacency principle

Recall that our aim for large modules is to show that $\dim E < \dim V$. We shall explain a method which we use to enable us to determine a lower bound for the codimension of an eigenspace $V_\gamma(s)$ for any $s \in G_{ss} \setminus Z$ and $\gamma \in K^*$.

Consider any $s \in G_{ss} \setminus Z$ which we may assume lies in T (by conjugation). Recall in Proposition 2.9 that $C_G(s)$ is generated by root subgroups X_β such that $\beta(s) = 1$ along with the torus T. For any $\gamma \in K^*$ the eigenspace $V_\gamma(s)$ is a sum of weight spaces. Since s is not central there must be a root $\alpha \notin \Phi_s$ in which case $\mu(s) \neq (\mu+\alpha)(s)$ for any weight μ and so $V_\gamma(s)$ cannot contain both V_μ and $V_{\mu+\alpha}$; we shall refer to this as the *adjacency principle* in later chapters. It is clear that we can naturally extend the adjacency principle to the consideration of weight nets.

The following situation arises fairly often. Suppose that $\langle \mu, \alpha \rangle$ only takes the values 0 or ± 1, so all weight strings are of length 1 or 2. Assume that there are k mutually orthogonal roots $\alpha_1, \ldots, \alpha_k$ not in Φ_s. We see that all weight nets that occur are of the form $m_1 \times \cdots \times m_k$ where $m_i \in \{1, 2\}$ for each $i \in [1, k]$. Therefore, for all weight nets other than those which are $1 \times \cdots \times 1$, there exists $\alpha \in \{\alpha_1, \ldots, \alpha_k\}$ such that the weight net is a disjoint union of α-strings of length 2. Hence at least half of the weights not orthogonal to each α_i for $i \in [1, k]$ cannot lie in the eigenspace.

We see that the situation is slightly different for even characteristic. Consider the weight string $\mu, \mu - \alpha, \cdots, \mu - k\alpha$. If we assume that $p = 2$ and the weight μ lies in $V_\gamma(s)$ then $\mu - 2^j\alpha$ for $j \geqslant 0$ and $2^j \leqslant k$ cannot lie in $V_\gamma(s)$. This follows since $\alpha(s) = 1$ if and only if $2^j\alpha(s) = 1$ for $j \geqslant 0$. Thus the lower bound we obtain on the contribution from the weight string to the codimension of the eigenspace is at least as great for $p = 2$ as it is for $p \neq 2$.

We shall find it convenient to introduce some notation and terminology for the consideration of weights with a given centraliser type. Define a *cluster* to be a maximal set of weights such that the difference between any two is a linear combination of roots in Φ_s. The set of weight spaces corresponding to the weights in any cluster all lie in the same eigenspace. For example, consider $SL_{10}(K)$ acting on $L(\omega_3)$ with centraliser type $X = A_3$. If we assume that Φ_s has simple roots $\{\alpha_1, \alpha_2, \alpha_3\}$ then there is a cluster of size 6 consisting of the following weights:

7 14 11 8 15 12 9 6 3	7 4 11 8 15 12 9 6 3	7 4 1 8 15 12 9 6 3
-3 4 11 8 15 12 9 6 3	-3 4 1 8 15 12 9 6 3	-3 -6 1 8 15 12 9 6 3

Note that we have omitted a factor of $\frac{1}{10}$ on each coefficient. For brevity we shall denote such a cluster by the string $\cdots$ 8 15 12 9 6 3 where the dots stand for any integers which make the expression a weight in $\Pi(\omega_3)$. We define a *clique* to be a collection of clusters where any two clusters contain weights which differ by a root not in Φ_s. We shall present a clique as a list of clusters in a box. Continuing the example above, the following is a clique:

$\cdots$	8	15	12	9	6	3
$\cdots$	8	5	12	9	6	3
$\cdots$	8	5	2	9	6	3
$\cdots$	8	5	2	-1	6	3
$\cdots$	8	5	2	-1	-4	3
$\cdots$	8	5	2	-1	-4	-7

Given a centraliser type X we wish to show that $\dim \overline{\bigcup_{s \in S_X} \bigcup_{\gamma \in K^*} V_\gamma(s)} < \dim V$ where $S_X = \{s \in G_{ss} \mid C_G(s) \text{ of type } X\}$. If $s \in T$ then $V_\gamma(s)$ is a sum of weight spaces and the set of weights concerned is a union of clusters. By the adjacency principle we see that at most one cluster in each clique may contribute to the dimension of the eigenspace $V_\gamma(s)$ for any $s \in G_{ss} \setminus Z$ and $\gamma \in K^*$.

Henceforth, unless it is otherwise clear from the context, we shall fix $s \in G_{ss} \setminus Z$ and $\gamma \in K^*$ and so consider a specific eigenspace $V_\gamma(s)$. By the statement *a weight μ lies in the eigenspace* $V_\gamma(s)$ we mean that the weight space V_μ lies in $V_\gamma(s)$. Since $\Pi(\lambda)$ denotes the multiset of all weights of a G-module $V = L(\lambda)$ *including* multiplicities, we have $|\Pi(\lambda)| = \dim V$. We set

$$\Lambda = \Lambda(s, \gamma) = \{\mu \in \Pi(\lambda) \mid \mu(s) \neq \gamma\}$$

to be the multiset of weights with multiplicities such that the corresponding weight spaces do not lie in the eigenspace, so $|\Lambda| = \operatorname{codim} V_\gamma(s)$ and, of course, $|\Pi(\lambda) \setminus \Lambda| = \dim V_\gamma(s)$.

We shall use repeatedly the following strategy. Firstly, we use weight strings to obtain a lower bound for $|\Lambda|$. We then use this value to determine which possibilities, if any, for X need to be considered further. For each such centraliser type we arrange the weights into clusters and then cliques to obtain an improved lower bound for $\dim V - \dim V_\gamma(s)$. In most cases this will exceed e_λ^X.

4.3 Criteria for modules

This next proposition will allow us (whenever the conclusion of Premet's theorem holds) to list the irreducible modules V for which we cannot immediately conclude that $\dim E < \dim V$, by providing a criterion by which (†) is satisfied. We begin Chapters 5-8 using this proposition to show in most cases (†) holds and to list the highest weights of modules requiring further consideration. These are provided in tables giving the highest weight λ of the irreducible module and the rank n of the simple algebraic group concerned.

Recall from Chapter 2 we use the convention that if all roots in Φ have the same length then we shall regard them as having short length.

Proposition 4.2. *Let G be a simple simply connected algebraic group with root system Φ acting on $L(\lambda)$. Let $\mu = \sum_{j=1}^{n} a_j\omega_j \leqslant \lambda$ be a dominant weight and assume that $p > e(\Phi)$. Set $\Psi = \langle \alpha_i \mid a_i = 0\rangle \subset \Phi$ and $r_\Psi = |W(\Phi) : W(\Psi)|\frac{|\Phi_S \setminus \Psi_S|}{2|\Phi_S|}$. Then $|\Lambda(s,\gamma)| \geqslant \sum r_\Psi$, where the sum runs over the dominant weights of $L(\lambda)$. Moreover, if $\sum r_\Psi > \dim G$ then* (†) *holds.*

Proof. For any non-central semisimple element s there is at least one root $\alpha \notin \Phi_s$. If Φ consists of roots with different lengths then we may as well assume that α is short. For, if all short roots of Φ lie in Φ_s then all long roots would also lie in Φ_s.

From the definition of the subsystem Ψ we see that $\alpha \in \Psi$ if and only if $\langle \mu, \alpha\rangle = 0$, i.e., μ is orthogonal to α. Thus, the roots that lie in $\Phi_S \setminus \Psi_S$ are the short roots not orthogonal to μ. The stabiliser of μ in $W(\Phi)$ is the parabolic subgroup generated by the reflections in the simple roots α_i for which $a_i = 0$; this is precisely $W(\Psi)$. Hence the Weyl group orbit $W.\mu$ has size $|W(\Phi) : W(\Psi)|$.

We note that for each $\alpha' \in \Phi_S$, the number of weights $\mu' \in W.\mu$ not orthogonal to α' is constant. This follows by observing that a weight μ' is orthogonal to α' if and only if $w\mu'$ is orthogonal to $w\alpha'$ and W acts transitively on the short roots. The number of pairs $(\mu', \alpha') \in W.\mu \times \Phi_S$ with $\langle \mu', \alpha'\rangle \neq 0$ is $|W(\Phi) : W(\Psi)|\,|\Phi_S \setminus \Psi_S|$. Thus, for a fixed $\alpha \in \Phi_S$, the number of weights $\mu' \in W.\mu$ such that $\langle \mu', \alpha\rangle \neq 0$ is

$$|W(\Phi) : W(\Psi)|\,\frac{|\Phi_S \setminus \Psi_S|}{|\Phi_S|} = 2r_\Psi.$$

Henceforth we shall fix a short root $\alpha \notin \Phi_s$. Recall that the set of weights of $L(\lambda)$ is a union of Weyl group orbits and each Weyl group orbit contains a unique dominant weight (which we use as an orbit representative). We take the weights of $L(\lambda)$ and arrange them in α-strings. Each weight occurs in a string of odd or even length.

In an α-string of odd length $2k+1$ there is precisely one weight orthogonal to α; this weight occurs in position $k+1$. The weights in position t and $2k+2-t$ both lie in the same Weyl group orbit of a dominant weight (via $w_\alpha \in W$). On the other hand, in an α-string of even length $2k$ there are no weights orthogonal to α. The weights in position t and $2k+1-t$ both lie in the same Weyl group orbit of a dominant weight. So weights not orthogonal to α occur in pairs lying in the same Weyl group orbit of a dominant weight.

As stated above, for a dominant weight μ, the number of weights $\mu' \in W.\mu$ not orthogonal to α is $2r_\Psi$. Thus, for every dominant weight, the number of weights of $L(\lambda)$ not orthogonal to α is $\sum 2r_\Psi$, where the sum runs over the dominant weights of $L(\lambda)$. In an even string of length $2k$, all $2k$ weights are counted and in an odd string of length $2k+1$, we see that $2k$ of the $2k+1$ weights are counted in this calculation.

We now apply the adjacency principle to each α-string to obtain a lower bound for the codimension of the eigenspace. We shall assume that all weights have multiplicity one. (If an irreducible module has dominant weights with larger multiplicities then the lower bound for the codimension of the eigenspace may in fact be greater.) For an α-string of even length $2k$ the minimal contribution to the codimension of the eigenspace is k since there are k pairs of weights adjacent in the α-string. Similarly, for an α-string of odd length $2k+1$ the minimal contribution to the codimension of the eigenspace is k. We can arrange all but one of the weights into pairs of weights adjacent in the α-string. The remaining weight may or may not contribute to the codimension of the eigenspace; we are unable to conclude either way. Thus, for all α-strings, the minimal contribution to $|\Lambda(s,\gamma)|$ is $\sum r_\Psi$, where the sum runs over the dominant weights of $L(\lambda)$. $\square$

In Chapter 10 we shall consider tensor products of irreducible G-modules in order to complete the proof of the Main Theorem for modules parameterised by highest weights that are not p-restricted. In this case we require the next proposition which provides a criterion which we shall use to show that for all but a short list of modules the conclusion of the Main Theorem holds. For the remaining modules further study is required to show that either (†) holds or ($\Diamond$) is satisfied for all $X \in I$.

Proposition 4.3. *Let G be a simple simply connected algebraic group and let $V = U \otimes W$ where U and W are irreducible G-modules. Suppose that each eigenspace in U of each non-central semisimple element has codimension at least d. If $d \dim W > \dim G$ then* (†) *holds.*

Proof. Given $s \in G_{ss} \setminus Z$ we decompose U and W into eigenspaces for s, i.e., we may write $U = \bigoplus_\sigma U_\sigma(s)$ and $W = \bigoplus_\tau W_\tau(s)$ where we sum over the eigenvalues $\sigma \in K^*$ and $\tau \in K^*$ of s on U and W respectively. Thus we can write the eigenspace of s on V with eigenvalue ρ in the form

$$V_\rho(s) = \bigoplus_{(\sigma,\tau)} U_\sigma(s) \otimes W_\tau(s)$$

where the sum is over pairs (σ, τ) with $\sigma\tau = \rho$.

Fix an eigenvalue $\rho \in K^*$ of s on V. Since by assumption each eigenspace in U has codimension at least d, for each $\tau \in K^*$ we have $\dim U_{\rho\tau^{-1}}(s) \leqslant \dim U - d$ and hence

$$\dim \left(U_{\rho\tau^{-1}}(s) \otimes W_\tau(s)\right) \leqslant (\dim U - d) \dim W_\tau(s).$$

By summing over τ we obtain

$$\dim V_\rho(s) \leqslant (\dim U - d) \dim W = \dim V - d \dim W,$$

i.e., $\operatorname{codim} V_\rho(s) \geqslant d \dim W$ and the result follows. □

In particular, since $d \geqslant 1$, this proposition implies that we need only to consider modules V where the dimension of both factors U and W is at most that of G.

Chapter 5

Groups of type A_n

We are now ready to begin the proof of the Main Theorem. Throughout this chapter we shall examine the action of non-central semisimple elements in a simple simply connected algebraic group G of type $\Phi = A_n$ on an irreducible module $V = L(\lambda)$ parameterised by a p-restricted weight λ. We shall obtain a list of irreducible modules for which $\dim \overline{E} = \dim V$ is possible and show that (†) holds for all others. For the remaining modules we further analyse the weights involved. If we are unable to show (†) holds then we find the possible root system types of centralisers of non-central semisimple elements and use (◊) to draw the necessary conclusion. Recall that modules are listed up to duality. Thus the dual of any module considered should be regarded as also being considered.

We shall prove the following result.

Theorem 5.1. *Let G be a simple simply connected algebraic group of type A_n acting on an irreducible module $V = L(\lambda)$ where λ is p-restricted. If $\dim V \leqslant \dim G + 2$ then $E = V$ with the possible exceptions of $L(3\omega_1)$ for $n \in [1, 2]$ with $p > 3$, $L(4\omega_1)$ for $n = 1$ with $p > 3$, $L(\omega_1 + \omega_2)$ for $n = p = 3$ and $L(\omega_3)$ for $n = 7$; if instead $\dim V > \dim G + 2$ then $\dim \overline{E} < \dim V$ with the possible exceptions of $L(\omega_3)$ for $n = 8$, $L(\omega_4)$ for $n = 7$ and $L(2\omega_2)$ for $n = 3$ with $p > 2$.*

This theorem is a consequence of the lemmas which follow in later sections.

5.1 Initial survey

Let $\mu = \sum_{i=1}^{n} a_i\omega_i \leqslant \lambda$ be a dominant weight. Recall from Section 2.1.1 that we shall be considering weights up to duality. We begin by using Proposition 4.2 to derive six conditions (i)-(vi) on the coefficients a_i of μ under which (†) holds assuming that n is sufficiently large. Indeed, once we have shown for a given weight μ that (†) holds then the same will be true when we add any positive linear combination of fundamental weights. The minimal contribution to the codimension of an eigenspace for $\mu' = \mu + \sum_{i=1}^{n} c_i\omega_i$ where $c_i \geqslant 0$ for $1 \leqslant i \leqslant n$ with subsystem Ψ' is at least that for μ with subsystem Ψ as can be seen by comparing the sizes of the root systems Ψ and Ψ' and also the orders of their Weyl groups; the inequality $r_{\Psi'} \geqslant r_\Psi$ holds.

We shall then use these six conditions and Premet's theorem to determine a short list of irreducible G-modules requiring further consideration, i.e., those modules for which more work is required to show that either (†) holds or ($\Diamond$) is satisfied for all centraliser types of non-central semisimple elements.

Proposition 5.2. *Suppose that $\mu = \sum_{i=1}^{n} a_i\omega_i \leqslant \lambda$ is a dominant weight, and that at least one of the following conditions holds:*

(i) $n \geqslant 6$ *and* $a_j, a_k \neq 0$ *for some* $j \in [3, n-3]$ *and* $k \in [j+2, n]$*;*

(ii) $n \geqslant 5$ *and* $a_j, a_k \neq 0$ *for some* $j \in [2, n-3]$ *and* $k \in [j+2, n-1]$*;*

(iii) $n \geqslant 10$ *and* $a_k \neq 0$ *for some* $k \in [5, n-4]$*;*

(iv) $n \geqslant 13$ *and either* $a_4 \neq 0$ *or* $a_{n-3} \neq 0$*;*

(v) $n \geqslant 7$ *and* $a_4, a_{n-k} \neq 0$ *for some* $k \in [0, n-5]$*;*

(vi) $n \geqslant 6$ *and* $a_k, a_{k+1} \neq 0$ *for some* $k \in [3, n-3]$*.*

Then (†) holds.

Proof. First assume that both $a_3 \neq 0$ and $a_n \neq 0$; then the subsystem $\Psi = \langle \alpha_i \mid a_i = 0 \rangle$ is contained in $\Phi(A_2A_{n-4})$. We find that $|W(\Phi) : W(\Psi)| \geqslant \frac{1}{6}(n+1)n(n-1)(n-2)$ and

$|\Phi \setminus \Psi| \geqslant 2(4n-9)$. Therefore $r_\Psi \geqslant r_{A_2A_{n-4}} = \frac{1}{6}(n-1)(n-2)(4n-9)$ which exceeds $\dim G = n(n+2)$ for $n \geqslant 6$ and (†) is satisfied. If we instead assume that both $a_j \neq 0$ and $a_k \neq 0$ for $3 \leqslant j < k \leqslant n$, $k > j+1$ and $j \neq n-2$ then $\Psi \subset \Phi(A_{j-1}A_{k-j-1}A_{n-k})$. Since $r_{A_{j-1}A_{k-j-1}A_{n-k}} \geqslant r_{A_2A_{n-4}}$ for such values of j and k we see that condition (i) ensures that (†) holds.

We treat condition (v) similarly. If both $a_4 \neq 0$ and $a_n \neq 0$ then $\Psi \subset \Phi(A_3A_{n-5})$ and so $r_\Psi \geqslant r_{A_3A_{n-5}} = \frac{1}{24}(n-1)(n-2)(n-3)(5n-16)$; this exceeds $\dim G$ for $n \geqslant 7$. If we assume that both $a_4 \neq 0$ and $a_{n-k} \neq 0$ for $k \in [0, n-5]$ then $\Psi \subset \Phi(A_3A_{n-5-k}A_k)$ and since $r_{A_3A_{n-5-k}A_k} \geqslant r_{A_3A_{n-5}}$ for $k \in [0, n-5]$ we see that (†) holds.

Next suppose that both $a_2 \neq 0$ and $a_{n-1} \neq 0$. Then $\Psi \subset \Phi(A_1A_{n-4}A_1)$ and so $r_\Psi \geqslant r_{A_1A_{n-4}A_1} = (n-1)(n-2)^2$ which exceeds $\dim G$ for $n \geqslant 5$. If we now assume that both $a_j \neq 0$ and $a_k \neq 0$ where $1 < j < k < n$ and $k > j+1$ then $\Psi \subset \Phi(A_{j-1}A_{k-j-1}A_{n-k})$ and we can use the fact that $r_{A_{j-1}A_{k-j-1}A_{n-k}} \geqslant r_{A_1A_{n-4}A_1}$ to conclude that (†) holds here for $n \geqslant 5$. Thus condition (ii) ensures that (†) holds.

Assume that $a_k \neq 0$ for $k \in [4, n-3]$ so that $\Psi \subset \Phi(A_{k-1}A_{n-k})$ and we find that $r_\Psi \geqslant r_{A_{k-1}A_{n-k}} = \binom{n-1}{k-1}$. If we take $k = 5$ then we see $r_\Psi \geqslant \dim G$ for $n \geqslant 10$. For $k \in [5, n-4]$ since $r_{A_{k-1}A_{n-k}} \geqslant r_{A_4A_{n-5}}$ we see that (†) holds and as such we cannot have $a_k \neq 0$. If $k = 4$ we have $r_\Psi \geqslant \dim G$ for $n \geqslant 13$. Thus conditions (iii) and (iv) both ensure that (†) holds.

Finally we treat condition (vi); assume that both $a_k \neq 0$ and $a_{k+1} \neq 0$ for $k \in [3, n-3]$. Then we have $\Psi \subset \Phi(A_{k-1}A_{n-k-1})$ and $r_\Psi \geqslant r_{A_{k-1}A_{n-k-1}} = \frac{(n-1)!}{k!(n-k)!}(nk - k^2 + n)$ which exceeds $\dim G$ for $n \geqslant 6$. This completes the proof of the proposition. □

We shall split the initial analysis of weights into three sections according as $n \geqslant 10$, $n \in [5, 9]$ or $n \in [1, 4]$.

The modules that have dimension at most $\dim G + 2$ are the following: $L(\omega_1)$ for $n \geqslant 1$, $L(3\omega_1)$ for $n \in [1, 2]$ with $p > 3$, $L(4\omega_1)$ for $n = 1$ with $p > 3$, $L(\omega_1 + \omega_2)$ for $n = p = 3$, $L(\omega_2)$ and $L(2\omega_1)$ ($p > 2$) for all n, the module $L(\omega_3)$ for $n \in [5, 7]$, and the adjoint module $L(2\omega_1)$ for $n = 1$ with $p > 2$ and $L(\omega_1 + \omega_n)$ for $n \geqslant 2$. Recall from

Section 3.1 that we need only consider modules with dimension greater than or equal to $\dim B$, where B is a Borel subgroup of G and those that are not adjoint modules. In type A_n, we have $\dim B = |\Phi^+(G)| + \operatorname{rank} T = \frac{1}{2}n(n+3)$ in which case, for $n \geqslant 2$, the natural module $L(\omega_1)$ with dimension $n+1$ and $L(\omega_2)$ with dimension $\frac{1}{2}n(n+1)$ are too small.

Lemma 5.3. *Suppose that $n \geqslant 10$. If $\dim V > \dim G + 2$ then $(\dagger)$ holds except possibly for the modules with highest weight $3\omega_1$ $(p > 3)$, $\omega_1+\omega_2$, ω_3, $\omega_2+\omega_n$ and $2\omega_1+\omega_n$ $(p > 2)$ for $n \geqslant 10$ and ω_4, $\omega_1+\omega_3$, $2\omega_2$ $(p > 2)$, $2\omega_1+\omega_2$ $(p > 2)$ and $4\omega_1$ $(p > 3)$ for $n \in [10, 12]$.*

Proof. Assume that $n \geqslant 13$. By (iv) of Proposition 5.2 any dominant weight $\mu \leqslant \lambda$ is of the form $\mu = \mu_1 + \mu_2$ where $\mu_1 = \sum_{i=1}^{3} a_i\omega_i$ and $\mu_2 = \sum_{i=n-2}^{n} a_i\omega_i$. Set $m_1 = \sum_{i=1}^{3} ia_i$ and $m_2 = \sum_{i=1}^{3} ia_{n+1-i}$; we shall assume that $m_1 \geqslant m_2$ since we are considering modules up to duality. From [13, p.69] we can see that $\mu_1 = \sum_{i=1}^{3} a_i\omega_i > \sum_{i=1}^{3}(a_i - b_i)\omega_i + \omega_j$ where $0 \leqslant b_i \leqslant a_i$ and $j = \sum_{i=1}^{3} ib_i$. The expression $\sum_{i=1}^{3} b_i\omega_i - \omega_j$ can be written as a non-negative linear combination of simple roots; the coefficient of each α_k is clearly positive for all $k \in [1, 3]$, is $(j-k)(n+1)$ for $k \in [4, j-1]$ and is zero otherwise. If $m_1 \geqslant 4$ then we can take b_i, $i = 1, 2, 3$ such that $4 \leqslant j \leqslant 6$. Thus

$$\mu' = \mu - \left(\sum_{i=1}^{3} b_i\omega_i - \omega_j\right) < \mu \leqslant \lambda$$

is a dominant weight less than the highest weight and if we write $\mu' = \sum_{i=1}^{n} a_i'\omega_i$ then $a_k' \neq 0$ for some $k \in [4, 6]$. Thus $(\dagger)$ is satisfied by condition (iv), so we may assume that $m_1 \leqslant 3$ and $m_2 \leqslant 3$.

Suppose that $m_1 = 3$ so μ_1 is one of ω_3, $\omega_1 + \omega_2$ or $3\omega_1$. If $a_3 \neq 0$ then $\mu_1 = \omega_3$ and, by condition (i) of Proposition 5.2, we must have $m_2 = 0$ for $n \geqslant 6$. In any case since $3\omega_1 > \omega_1 + \omega_2 > \omega_3$ we can appeal to Premet's theorem to conclude that $m_2 = 0$. Thus the three weights ω_3, $\omega_1 + \omega_2$ and $3\omega_1$ require further consideration.

Next suppose that $m_1 = 2$ so μ_1 is either ω_2 or $2\omega_1$. Assume that $\mu_1 = \omega_2$ and $m_2 = 2$, so either $\mu_2 = \omega_{n-1}$ or $2\omega_n$. If $\mu_2 = \omega_{n-1}$ then by condition (ii) of Proposition 5.2 we see that $(\dagger)$ holds for $n \geqslant 5$ and if $\mu_2 = 2\omega_n$ then $(\dagger)$ holds again by condition (ii) since

$\omega_{n-1} < 2\omega_n$. If $\mu_1 = 2\omega_1$ then again (†) holds when $m_2 = 2$ since $\omega_2 < 2\omega_1$. Hence $m_2 \leqslant 1$ and the weights that require further consideration are $\omega_2 + \omega_n$ and $2\omega_1 + \omega_n$ since, as we have already mentioned, the modules $L(\omega_2)$ and $L(2\omega_1)$ have dimension no larger than $\dim G$.

Lastly, if $m_1 = 1$ then $m_2 \leqslant 1$ in which case the possible weights are ω_1 which has dimension less than $\dim B$ and $\omega_1 + \omega_n$ which is the adjoint module.

Now assume that $10 \leqslant n \leqslant 12$. As above we see from condition (iii) of Proposition 5.2 that any dominant weight $\mu \leqslant \lambda$ is of the form $\mu = \mu_1 + \mu_2$ where $\mu_1 = \sum_{i=1}^{4} a_i\omega_i$ and $\mu_2 = \sum_{i=n-3}^{n} a_i\omega_i$. Set $M_1 = \sum_{i=1}^{4} ia_i$ and $M_2 = \sum_{i=1}^{4} ia_{n+1-i}$. In the same way as before if $M_1 > 4$ we can take b_i for $i \in [1,4]$ with $0 \leqslant b_i \leqslant a_i$ and $5 \leqslant j \leqslant 8$ where $j = \sum_{i=1}^{4} ib_i$. Thus

$$\mu' = \mu - \left(\sum_{i=1}^{4} b_i\omega_i - \omega_j\right) < \mu \leqslant \lambda$$

is a dominant weight less than the highest weight with $a'_k \neq 0$ for $k \in [5,8]$ where $\mu' = \sum_{i=1}^{n} a'_i\omega_i$. We need to look further at the cases $n = 11$ with $j = 8$ and $n = 10$ with $j \in [7,8]$ since condition (iii) of Proposition 5.2 cannot be used here. Suppose that $n = 11$. The only weight for which $j = 8$ and $a'_k = 0$ for $k \in [5,7]$ is $2\omega_4$. Here $\omega_8 < 2\omega_4$ and $a_i = b_i$ for $i \in [1,4]$. Similarly for $n = 10$ the only weights for which $j \in [7,8]$ and $a'_k = 0$ for $k \in [5,6]$ are $2\omega_4$ and $\omega_3 + \omega_4$; however we do not need to consider the latter weight by condition (vi) of Proposition 5.2. Indeed, since $2\omega_4 > \omega_3 + \omega_5$ we can use (ii) to conclude that (†) holds for $2\omega_4$. Thus we can assume that $M_1 \leqslant 4$ and, by considering modules up to duality, $M_2 \leqslant 4$.

If $M_1 = 4$ then μ is one of ω_4, $\omega_1 + \omega_3$, $2\omega_2$, $2\omega_1 + \omega_2$ or $4\omega_1$. We may assume that $M_2 = 0$ by condition (v) of Proposition 5.2 and Premet's theorem since if $\nu \in \{\omega_1+\omega_3, 2\omega_2, 2\omega_1+\omega_2, 4\omega_1\}$ then $\omega_4 < \nu$. Thus the weights requiring further consideration are ω_4, $\omega_1 + \omega_3$, $2\omega_2$, $2\omega_1 + \omega_2$ and $4\omega_1$. Similarly if $M_1 = 3$ then we may assume that $M_2 = 0$ by (i) and Premet's theorem. The weights requiring further consideration here are ω_3, $\omega_1 + \omega_2$ and $3\omega_1$. If $M_1 = 2$ then (ii) implies that $M_2 \leqslant 1$. Therefore it remains to consider the weights $\omega_2 + \omega_n$ and $2\omega_1 + \omega_n$. □

Lemma 5.4. *Suppose that* $\dim V > \dim G + 2$. *If* $5 \leqslant n \leqslant 9$ *then* (†) *holds except possibly for the modules with highest weights* $\omega_1 + \omega_3$, $2\omega_1 + \omega_3$ $(p > 2)$, $\omega_2 + \omega_3$, $2\omega_1$ $(p > 2)$, $3\omega_1$ $(p > 3)$, $4\omega_1$ $(p > 3)$, $2\omega_2$ $(p > 2)$, $\omega_1 + \omega_2$, $2\omega_1 + \omega_2$ $(p > 2)$, $3\omega_1 + \omega_2$ $(p > 3)$, $2\omega_1 + \omega_n$ $(p > 2)$ *and* $\omega_2 + \omega_n$ *for* $n \in [5, 9]$, $\omega_1 + 2\omega_2$ $(p > 2)$ *and* $5\omega_1$ $(p > 5)$ *for* $n = 5$, ω_3 *for* $n \in [8, 9]$, ω_4 *for* $n \in [7, 9]$ *and* ω_5 *for* $n = 9$.

Proof. Suppose that $a_3 \neq 0$ and $n \geqslant 6$. By condition (i) of Proposition 5.2 we can assume that $a_k = 0$ for $k \in [5, n]$ and by condition (vi) of Proposition 5.2 we may assume that $a_4 = 0$. We can take $a_3 \leqslant 1$ by condition (ii) of Proposition 5.2 since $2\omega_3 > \omega_2 + \omega_4$. Similarly, we can assume by (ii) that $a_i \neq 0$ for at most two $i \in [1, 3]$ as $\omega_1 + \omega_2 + \omega_3 > \omega_2 + \omega_4$. If $a_1 \neq 0$ and $a_2 = 0$ we have $a_1 \leqslant 2$ by (ii) since $3\omega_1 + \omega_3 > \omega_2 + \omega_4$ and if $a_1 = 0$ and $a_2 \neq 0$ we have $a_2 = 1$ since $2\omega_2 + \omega_3 > \omega_1 + 2\omega_3$. Thus for $n \geqslant 6$ we must consider further the weights $\omega_1 + \omega_3$, $2\omega_1 + \omega_3$ and $\omega_2 + \omega_3$.

Suppose that $a_3 \neq 0$ and $n = 5$. We can only use condition (ii) of Proposition 5.2 in this case. If at least three coefficients of μ have non-zero coefficients then Ψ is contained in either $\Phi(A_2)$ or $\Phi(A_1^2)$. In the former case we see by (ii) that (†) holds for $\omega_1 + \omega_2 + \omega_3$ since $\omega_2 + \omega_4 < \omega_1 + \omega_2 + \omega_3$. Similarly for the weight $\omega_3 + \omega_4 + \omega_5$. In the latter case (†) holds using the calculation in (ii) of Proposition 5.2 as $r_{A_1^2} \geqslant r_{A_1^3}$. Thus we can assume that at most two coefficients of μ are non-zero. We can take $a_3 = 1$ by (ii) since $2\omega_3 > \omega_2 + \omega_4$. If $a_1 \neq 0$ we can take $a_1 \leqslant 2$ by (ii) since $\omega_3 + 3\omega_1 > 2\omega_3$ and if $a_2 \neq 0$ we can take $a_2 = 1$ since $2\omega_2 + \omega_3 > \omega_1 + 2\omega_3$. Therefore for $n = 5$ we must consider further the weights $\omega_1 + \omega_3$, $2\omega_1 + \omega_3$ and $\omega_2 + \omega_3$.

Suppose that $a_4 \neq 0$ and $n \geqslant 7$ (otherwise we are in an earlier case). We may assume that $a_4 = 1$ by (ii) since $2\omega_4 > \omega_3 + \omega_5$. We may also assume that $a_k = 0$ for $k = 3$ by (vi), for $k = 2$ by (ii) and $k \in [5, n]$ by (v) of Proposition 5.2. If $a_1 \neq 0$ then $\Psi \subset \Phi(A_2 A_{n-4})$ and as in (i) we find that $r_\Psi > \dim G$. We note therefore that (†) certainly holds if $a_4 \neq 0$ and any other coefficient of μ is non-zero. Hence we need to consider further the weight ω_4 for $n \in [7, 9]$.

Suppose that $a_5 \neq 0$ and $n = 9$. We may assume that $a_5 = 1$ by (ii) since $2\omega_5 > \omega_4 + \omega_6$. We may assume that $a_k = 0$ for $k \in [2, 3]$ by (ii), for $k = 4$ by (vi) and for $k = 1$ by an argument as in (v) since in this case we have $\Psi \subset \Phi(A_3A_4)$. By duality we may also assume that $a_k = 0$ for $k \in [6, 9]$. Hence we need to consider the weight ω_5 for $n = 9$ further.

Now suppose that $a_2 \neq 0$. Assume $a_k \neq 0$ for some $k \neq 2$. We have dealt with the cases $a_k \neq 0$ for $k \in [3, 5]$ above. Indeed, we may assume by condition (ii) of Proposition 5.2 that if $a_2 \neq 0$ then $a_k = 0$ for $k \in [4, n-1]$.

If $a_1 \geqslant 2$, $a_2 = 0$ and $a_k \neq 0$ for any $k \in [3, n-1]$ then (†) holds by the previous paragraph as $2\omega_1 > \omega_2$. Thus we may assume that $a_1 = 1$ if $a_2 = 0$ and $a_k \neq 0$ for $k \in [3, n-1]$ whence, up to duality, we are in an earlier case.

Thus we are left to consider weights of the form $\mu = a_1\omega_1 + a_2\omega_2 + a_n\omega_n$. We quickly see that $a_n \leqslant 1$ by (ii) and Premet's theorem since $\omega_{n-1} < 2\omega_n$. We shall treat the cases $a_n = 0$ and $a_n = 1$ separately.

Case I: $a_n = 0$. If both $a_1 \neq 0$ and $a_2 \neq 0$ then we may assume that $a_1 \leqslant 3$, $a_2 \leqslant 3$ and $a_1 + a_2 \leqslant 4$ by (ii) since $2\omega_1 + 2\omega_2 > \omega_2 + \omega_4$; thus we may assume that we do not have $a_1 = a_2 = 2$. From the analysis above we see that (†) holds for $\omega_1 + \omega_2 + \omega_4$ so the same is true for $\omega_1 + 3\omega_2 > \omega_1 + \omega_2 + \omega_4$. Note that by the calculation in (i) of Proposition 5.2, we need to consider further the weight $\omega_1 + 2\omega_2$ only for $n = 5$ since $\omega_1 + 2\omega_2 > \omega_1 + \omega_4$. The weights $\omega_1 + \omega_2$, $2\omega_1 + \omega_2$ and $3\omega_1 + \omega_2$ require further consideration.

If $a_1 \neq 0$ and $a_2 = 0$ we may assume that $a_1 \leqslant 5$ by (ii) since $6\omega_1 > \omega_2 + \omega_4$. We need only consider $5\omega_1$ for $n = 5$ by the calculation in (i) since $5\omega_1 > \omega_1 + \omega_4$. The weights $3\omega_1$ and $4\omega_1$ require further consideration for $n \in [5, 9]$.

If $a_1 = 0$ and $a_2 \neq 0$ we may assume that $a_2 \leqslant 2$ by (ii) since $3\omega_2 > \omega_2 + \omega_4$. However the modules $L(\omega_2)$ and $L(2\omega_1)$ both have dimension less than $\dim G$.

Case II: $a_n = 1$. Suppose that $a_n = 1$. Then we can assume that at most one of a_1 and a_2 is non-zero for $n \geqslant 6$ by (i) since $\omega_1 + \omega_2 + \omega_n > \omega_3 + \omega_n$ and for $n = 5$ we find that $r_{A_2} = 48 > 35 = \dim G$. If $\mu = a_1\omega_1 + \omega_n$ then $a_1 \leqslant 2$ since $3\omega_1 + \omega_n > \omega_1 + \omega_2 + \omega_n$. If $\mu = a_2\omega_2 + \omega_n$ we can assume that $a_2 = 1$ by the calculation in (ii) since $2\omega_2 + \omega_n > \omega_1 + \omega_3 + \omega_n$. Hence further work is needed for the modules with highest weights $\omega_2 + \omega_n$ and $2\omega_1 + \omega_n$. □

We shall now consider the low rank cases. The modules outstanding from the following lemma will be considered later in Section 5.5.

Lemma 5.5. *Suppose that* $1 \leqslant n \leqslant 4$. *If* $\dim V > \dim G + 2$ *then* (†) *holds except possibly for the modules with highest weights* $5\omega_1$ *and* $6\omega_1$ *for* $n = 1$ *with* $p > 5$, $3\omega_1$ *for* $n \in [3, 4]$ *with* $p > 3$, $4\omega_1$ *for* $n \in [2, 4]$ *with* $p > 3$, $2\omega_1 + \omega_2$ *for* $n \in [2, 4]$ *with* $p > 2$, $2\omega_2$ $(p > 2)$, $\omega_1 + \omega_2$ *and* $2\omega_1 + \omega_n$ $(p > 2)$ *for* $n \in [3, 4]$, $\omega_2 + \omega_3$ *and* $\omega_1 + \omega_3$ *for* $n = 4$, *and* $3\omega_1 + \omega_2$ $(p > 3)$ *and* $2\omega_1 + 2\omega_2$ $(p > 2)$ *for* $n = 2$.

Proof. We calculate the value r_Ψ for all possible $\Psi \subset \Phi$ and use Premet's theorem to show that (†) holds for all but a small number of weights.

Case I: $n = 4$. We have $\dim G = 24$ and $r_\Psi = 60$, 27, 12, 7, 3 or 1 according as $\Psi = \varnothing$, A_1, A_1^2, A_2, A_2A_1 or A_3. Let $\mu = \sum_{i=1}^{4} a_i\omega_i \leqslant \lambda$ be a dominant weight. Since $r_{A_1} > \dim G$ we see that (†) holds if we have $a_i \neq 0$ for more than two $i \in [1, 4]$ and by Premet's theorem (†) holds for any weight $\mu' < \mu$ with more than two coefficients non-zero.

First let us assume that $a_3 = a_4 = 0$. If $a_1 = 0$ also then we may assume that $a_2 \leqslant 2$ due to the fact that $\omega_1 + \omega_2 + \omega_3 < 3\omega_2$. Thus we must consider further the weight $2\omega_2$. Similarly if $a_2 = 0$ then $a_1 \leqslant 4$; we have

$$5\omega_1 > 3\omega_1 + \omega_2 > \omega_1 + 2\omega_2 > 2\omega_1 + \omega_3 > \omega_2 + \omega_3$$

so we find by Premet's theorem that $|\Lambda| \geqslant 1 + 7 + 7 + 12 + 12 = 39$. Thus the weights $3\omega_1$ and $4\omega_1$ require further consideration. If both a_1 and a_2 are non-zero then we may assume that $a_1 \leqslant 2$ and $a_2 \leqslant 1$ since (†) is satisfied for both $\omega_1 + 2\omega_2$ and $3\omega_1 + \omega_2$ as can be seen from the calculation for $5\omega_1$ above. Thus the weights $\omega_1 + \omega_2$ and $2\omega_1 + \omega_2$ require further consideration.

Now let us assume that $\mu = a_p\omega_p + a_q\omega_q$ where $p \in \{1, 2\}$, $q \in \{3, 4\}$ and $a_p \geqslant a_r > 0$. Let us suppose first that $q = 3$. If $p = 2$ we may assume that $a_2 \leqslant 1$ from the remark above since $2\omega_2 + \omega_3 > \omega_1 + \omega_2 + \omega_4$. If $p = 1$ we may assume that $a_1 \leqslant 1$ since $2\omega_1 + \omega_3 > \omega_2 + \omega_3 > \omega_1 + \omega_4$ so that $|\Lambda| \geqslant 12 + 12 + 7 = 31$, hence (†) holds. Now suppose that $q = 4$. If $p = 2$ then we may assume that $a_2 \leqslant 1$ since $2\omega_2+\omega_4 > \omega_1+\omega_3+\omega_4$. If $p = 1$ then we may assume that $a_1 + a_4 \leqslant 3$; we have $3\omega_1 + \omega_4 > \omega_1 + \omega_2 + \omega_4$ and since $2\omega_1 + 2\omega_4 > 2\omega_1 + \omega_3$ we can use Premet's theorem and the calculation above for $2\omega_1+\omega_3$. Recall that we do not consider the modules $L(\omega_1)$, $L(\omega_2)$ and $L(2\omega_1)$ since they have dimension smaller than $\dim G$. Note that we shall concern ourselves with $L(\omega_1+\omega_3)$ rather than $L(\omega_2 + \omega_4)$ as we are considering modules up to duality. Thus the weights requiring further consideration are $\omega_1 + \omega_3$, $\omega_2 + \omega_3$ and $2\omega_1 + \omega_4$.

Case II: $n = 3$. We have $\dim G = 15$ and $r_\Psi = 12$, 5, 2, or 1 according as $\Psi = \varnothing$, A_1, A_1^2 or A_2. Suppose first that $a_2 = a_3 = 0$. Since $\nu < 5\omega_1$ for $\nu \in \{3\omega_1+\omega_2, \omega_1+2\omega_2, 2\omega_1+\omega_3\}$ we may assume that $a_1 \leqslant 4$. Thus further consideration is required for the weights $3\omega_1$ and $4\omega_1$. Suppose that $\mu = a_1\omega_1 + a_3\omega_3$ with $a_1 \geqslant a_3 > 0$. Since $\omega_1 + \omega_2 + \omega_3 < 3\omega_1 + \omega_3$ and $\nu < 2\omega_1 + 2\omega_3$ for $\nu \in \{\omega_2 + 2\omega_3, 2\omega_1 + \omega_2, 2\omega_2\}$ we can assume that $a_1 \leqslant 2$ and $a_3 = 1$. Thus we need to consider further only the weight $2\omega_1 + \omega_3$ since the module with highest weight $\omega_1 + \omega_3$ is the adjoint module.

Now suppose that $\mu = \sum_{i=1}^{3} a_i\omega_i$ with $a_2 \neq 0$. We may assume that $a_3 = 0$ as (†) is satisfied if $a_i \neq 0$ for each $i \in [1, 3]$. This is because $\nu < \omega_1 + \omega_2 + \omega_3$ for each $\nu \in \{2\omega_3, 2\omega_1, \omega_2\}$. Consequently we may assume that $a_2 \leqslant 2$ since $\omega_1+\omega_2+\omega_3 < 3\omega_2$. If $a_2 = 2$ then $a_1 = 0$ since $\nu < \omega_1 + 2\omega_2$ for each $\nu \in \{2\omega_1 + \omega_3, \omega_2 + \omega_3, \omega_1\}$. If $a_2 = 1$ then $a_1 \leqslant 2$ since $\omega_1 + 2\omega_2 < 3\omega_1 + \omega_2$ and we have just observed that (†) holds for $\omega_1 + 2\omega_2$. Thus we are left to consider further the weights $\omega_1 + \omega_2$, $2\omega_1 + \omega_2$ and $2\omega_2$.

Case III: $n = 2$. Here $\dim G = 8$ and we find that $r_\varnothing = 3$ and $r_{A_1} = 1$. If $\mu = a_1\omega_1$ then we may assume that $a_1 \leqslant 4$ since $\nu < 5\omega_1$ for each $\nu \in \{3\omega_1 + \omega_2, \omega_1 + 2\omega_2, 2\omega_1, \omega_2\}$. Thus the weights $3\omega_1$ and $4\omega_1$ require further consideration. Suppose that $\mu = a_1\omega_1+a_2\omega_2$ with $a_1 \geqslant a_2 > 0$. Then we have $a_1 \leqslant 3$ and $a_2 \leqslant 2$ with $a_1 + a_2 < 5$ since $\nu < 3\omega_1 + 2\omega_2$ for each $\nu \in \{\omega_1 + 3\omega_2, 2\omega_1 + \omega_2\}$ and $\eta < 4\omega_1 + \omega_2$ for each $\eta \in \{2\omega_1 + 2\omega_2, \omega_1 + \omega_2\}$. We shall need to consider further the weights $2\omega_1 + \omega_2$, $3\omega_1 + \omega_2$ and $2\omega_1 + 2\omega_2$.

Case IV: $n = 1$. We have $\dim G = 3$; here of course $r_\varnothing = 1$. By listing the eight weights for $L(7\omega_1)$ we see, using the adjacency principle, for any $\gamma \in K^*$ and $s \in G_{ss} \setminus Z$ that $|\Lambda| \geqslant 4$. Thus we need to further investigate the weights $k\omega_1$ for $k \in [5, 6]$ since

$L(k\omega_1)$ has dimension $k+1$. □

5.2 Weight string analysis

From the analysis in the previous sections, the remaining p-restricted highest weights λ for $n \geqslant 1$ appear in Table 5.1. We reference the lemma in which each module is treated and the conclusion drawn. Although it is not stated explicitly in the table we shall assume that the characteristic is such that each weight is p-restricted. Thus, for example, we list $3\omega_1$ for $n \in [3, \infty)$ under the assumption that $p > 3$. If $\mu = \sum_{i=1}^{n} a_i\omega_i$ then we shall say that μ has level j where $j = \sum_{i=1}^{n-1} ia_i + a_n$. We shall order the weights in the table and organise the lemmas in the next section according to levels of weights.

It is worthwhile to remark that the dimension of each module above depends on characteristic; this is not clear from Table 5.1.

We shall now begin a sequence of lemmas working through the possible modules. In each of the succeeding lemmas, we take $s \in G_{ss} \setminus Z$ and $\gamma \in K^*$. As detailed in Section 4.1, we aim to show that (†) is satisfied for each $s \in G_{ss} \setminus Z$ or else that (◊) holds for each $t \in S_X \cap T$ and $X \in I$. First we consider irreducible modules parameterised by weights with level 5.

Lemma 5.6. *Let G act on an irreducible module V where we take V to be one of $L(\omega_2 + \omega_3)$, $L(2\omega_1 + \omega_3)$ $(p > 2)$ and $L(3\omega_1 + \omega_2)$ $(p > 3)$ for $n \in [5, 9]$ and $L(\omega_1 + 2\omega_2)$ $(p > 2)$ and $L(5\omega_1)$ $(p > 5)$ for $n = 5$. Then, in each case, (†) holds.*

Proof. In this lemma, the weights concerned are related by the partial ordering

$$5\omega_1 > 3\omega_1 + \omega_2 > \omega_1 + 2\omega_2 > 2\omega_1 + \omega_3 > \omega_2 + \omega_3 > \omega_1 + \omega_4 > \omega_5.$$

Consider the module with highest weight $\omega_2+\omega_3$. We find for this module that there are at least $r_{A_1A_{n-3}} = \frac{1}{2}(n-1)(3n-4)$ weights not in the eigenspace $V_\gamma(s)$. Since $\omega_1+\omega_4 < \omega_2+\omega_3$, by Premet's theorem 2.6 the weights in the Weyl group orbit $W.(\omega_1+\omega_4)$ occur as weights of $L(\omega_2 + \omega_3)$ so we may include their contribution $r_{A_2A_{n-4}} = \frac{1}{6}(n-1)(n-2)(4n-9)$

λ	n	Lemma
$\omega_2+\omega_n$	$[8,\infty)$	5.8 (†)
	$[5,7]$	5.8 (◊)
$2\omega_1+\omega_n$	$[5,\infty)$	5.8 (†)
	4	5.17 (†)
	3	5.18 (◊)
ω_3	$[14,\infty)$	5.12 (◊)
	$[9,13]$	5.13 (◊)
	8	5.13 (◊)[a]
$\omega_1+\omega_2$	$[5,\infty)$	5.14 (◊)
	4	5.17 (◊)
	3 ($p\neq 3$)	5.21 (◊)
$3\omega_1$	$[5,\infty)$	5.14 (◊)
	4	5.17 (◊)
	3	5.19 (◊)
ω_4	$[11,12]$	5.9 (†)
	$[9,10]$	5.9 (◊)
	8	5.10 (◊)
	7	5.11 (◊)[b]
$\omega_1+\omega_3$	$[5,12]$	5.7 (†)
	4	5.16 (◊)
$2\omega_2$	$[6,12]$	5.7 (†)
	5 ($p\neq 2,3$)	5.7 (†)
	5 ($p=3$)	5.7 (◊)
	4	5.16 (◊)
	3	5.20 (◊)[c]

λ	n	Lemma
$2\omega_1+\omega_2$	$[5,12]$	5.7 (†)
	4	5.16 (†)
	3	5.18 (†)
	2	5.23 (◊)
$4\omega_1$	$[5,12]$	5.7 (†)
	4	5.16 (◊)
	3	5.18 (◊)
	2	5.23 (◊)
ω_5	9	5.9 (†)
$\omega_2+\omega_3$	$[5,9]$	5.6 (†)
	4 ($p\neq 3$)	5.15 (†)
	4 ($p=3$)	5.15 (◊)
$2\omega_1+\omega_3$	$[5,9]$	5.6 (†)
$\omega_1+2\omega_2$	5	5.6 (†)
$3\omega_1+\omega_2$	$[5,9]$	5.6 (†)
	2 ($p\neq 3,5$)	5.22 (†)
	2 ($p=5$)	5.22 (◊)
$5\omega_1$	5	5.6 (†)
	1	5.24 (◊)
$2\omega_1+2\omega_2$	2 ($p\neq 2,5$)	5.22 (†)
	2 ($p=5$)	5.22 (◊)
$6\omega_1$	1	5.24 (◊)

Except for: [a] $X=A_2^3$ and A_4A_3, [b] $X=A_3^2$, [c] $X=A_1^2$.

Table 5.1: Possible weights in type A_n, $n\geqslant 1$

to $|\Lambda|$. Thus there are at least $r_{A_2A_{n-4}}+r_{A_1A_{n-3}}$ weights not in the eigenspace, which is greater than $\dim G$ for $n\in[5,9]$, as required. The other modules are each above $\omega_2+\omega_3$ in the partial ordering. Hence, by using Premet's theorem, (†) holds for these also for $n\in[5,9]$. □

In the next lemma we deal with weights with level 4. We find here that simply counting weight strings is insufficient to conclude that $|\Lambda|$ is large enough to satisfy (†). We employ the information [18] on multiplicities of weights to assist our analysis.

Lemma 5.7. *Let G act on the irreducible module V where we take V for $n\in[5,12]$ to* *be one of $L(\omega_1+\omega_3)$, $L(2\omega_2)$ $(p>2)$, $L(2\omega_1+\omega_2)$ $(p>2)$ and $L(4\omega_1)$ $(p>3)$. Then in each case (†) holds unless $n=5$ and $p=3$ for $L(2\omega_2)$ in which case (◊) is satisfied for all $X \in I$.*

Proof. We are considering the four irreducible modules with highest weights satisfying the partial ordering

$$4\omega_1 > 2\omega_1+\omega_2 > 2\omega_2 > \omega_1+\omega_3 > \omega_4.$$

We wish to show that for each non-central $s \in G_{ss}$ and $\gamma \in K^*$ the condition (†) holds in each case. By similar calculations to those above, the module with highest weight $\omega_1+\omega_3$ for $n \in [5,12]$ has at least $r_{A_1A_{n-3}} = \frac{1}{2}(n-1)(3n-4)$ weights $\mu \in W.(\omega_1+\omega_3)$ in $|\Lambda|$. Indeed, $\omega_4 < \omega_1+\omega_3$, so we can employ Premet's theorem to add an extra $r_{A_3A_{n-4}} = \binom{n-1}{3}$ weights from $W.\omega_4$ to $|\Lambda|$. We see that (†) is satisfied unless $n \in [5,6]$; we shall return to these two cases shortly.

Take $\alpha \notin \Phi_s$; the module with highest weight $2\omega_2$ for $n \in [5,12]$ has at least $n-1$ weights $\mu \in W.(2\omega_2)$ in weight strings of the form $\mu, \mu-\alpha, \mu-2\alpha$, hence at least $r_{A_1A_{n-2}} = n-1$ weights in Λ. Using Premet's theorem to include weight strings from the Weyl group orbits $W.\omega_4$ and $W.(\omega_1+\omega_3)$ we are done unless $n=5$ since $|\Lambda| \geqslant \binom{n-1}{3} + \frac{1}{2}(n-1)(3n-4)+(n-1)$ satisfies (†). If $n=5$ then $|\Lambda| \geqslant 4+22+4=30$, hence the module $L(2\omega_2)$ requires further consideration in this case.

For the highest weight $2\omega_1+\omega_2$, we need only consider the case $n=5$ from the previous paragraph and Premet's theorem. By adding together the various values of r_Ψ as we run through the subsystems $\Psi = A_3$, A_3A_1, A_2A_1 and A_4 corresponding to the four dominant weights $2\omega_1+\omega_2$, $2\omega_2$, $\omega_1+\omega_3$ and ω_4 we find that $|\Lambda| \geqslant 9+4+22+4 = 39 > 35 = \dim G$.

For $4\omega_1$ we are done by the previous calculations using Premet's theorem 2.6.

It remains for us to consider the modules $L(\omega_1+\omega_3)$ for $n \in [5,6]$ and $L(2\omega_2)$ for $n=5$.

First we turn to the module with highest weight $\omega_1+\omega_3$ for $n=5$; it has dimension 90 for characteristic 2 and 105 otherwise.

In Figure 5.1 on the left we tabulate data about the weights from Lübeck's tables.

In the first column we index the Weyl group orbits of dominant weights listed in the second column; the third column gives the size of each of these orbits. The last columns give the multiplicities m_ω of each weight in the orbit depending on the characteristic of K. On the right we tabulate the weight strings that occur and a lower bound for their contribution to the codimension of the eigenspace. We denote by l the minimal possible contribution to the codimension from each weight string for the possible characteristics. The various contributions are summed and a lower bound for $|\Lambda|$ is provided in the last row and column. We shall provide further explanation shortly. A weight labelled μ_i is one which lies in the Weyl group orbit of the weight indexed by i.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	$\omega_1 + \omega_3$	60	1	1
1	ω_4	15	3	2

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	16		
$\mu_2\ \mu_2$	16	16	16
$\mu_2\ \mu_1\ \mu_2$	6	12	12
μ_1	1		
$\mu_1\ \mu_1$	4	12	8
Lower bound on $\lvert\Lambda\rvert$		40	36

Figure 5.1: $(\lambda, n) = (\omega_1 + \omega_3, 5)$

We give full details of the weight string calculations. We calculate $|W(A_1A_2)| = 12$, $|W(A_3A_1)| = 48$ and $|W(A_5)| = 720$. For each $\mu \in W.(\omega_1 + \omega_3)$, there are 8, 8 and 3 roots $\alpha \in \Phi(A_5)$ such that $\langle \mu, \alpha \rangle = 0$, 1 and 2. Similarly there are 14 or 8 roots $\alpha \in \Phi(A_5)$ according as $\langle \mu_1, \alpha \rangle = 0$ or 1. Fix some $\alpha \in \Phi(A_5)$. We see that there are $\frac{\frac{720}{12}.8}{30} = 16$ weights $\mu \in W.(\omega_1 + \omega_3)$ with α-string μ, 16 weights with α-string $\mu, \mu - \alpha$ and $\frac{\frac{720}{12}.3}{30} = 6$ weights with α-string $\mu, \mu - \alpha, \mu - 2\alpha$; these strings are of the form μ_2 in the first case, μ_2, μ_2 in the second and μ_2, μ_1, μ_2 in the third. There are $\frac{\frac{720}{48}.14}{30} = 7$ weights $\mu_1 \in W.\omega_4$ with $\langle \mu_1, \alpha \rangle = 0$; six of these occur in the α-strings of length 3 above, leaving just one with α-string simply μ_1. There are $\frac{\frac{720}{48}.8}{30} = 4$ weights $\mu_1 \in W.\omega_4$ with α-string μ_1, μ_1.

We may assume that $\alpha \notin \Phi_s$. There are 16 strings of the form μ_2, μ_2, i.e., there are 16 weight strings consisting of two weights both in the Weyl group orbit of $\omega_1 + \omega_3$ which differ by α. Given such a string, the adjacency principle states that the weight spaces corresponding to both weights cannot both lie in the eigenspace by the choice of α. Thus for each weight string μ_2, μ_2 at least one of the two weight spaces cannot lie in the eigenspace and there is a contribution to $|\Lambda|$ of at least $\dim V_{\mu_2} = m_{\mu_2} = 1$ for all characteristics. There are 16 such strings, so the contribution l to $|\Lambda|$ is at least 16. Similarly for each of the 4 weight strings of the form μ_1, μ_1 at least one of the weight spaces corresponding to these weights cannot lie in the eigenspace. Since $\dim V_{\mu_1} = 3$ or 2 when $p \neq 2$ or $p = 2$ the minimal contribution to $|\Lambda|$ from such weight strings is either 12 if $p \neq 2$ or 8 if $p = 2$.

There are 6 weight strings of the form μ_2, μ_1, μ_2, i.e., there are 6 weight strings consisting of three weights $\mu - 2\alpha\ \mu - \alpha\ \mu$ where the first and third weights lie in $W.(\omega_1 + \omega_3)$ and the second lies in $W.\omega_4$. We wish to find the minimal contribution of each such weight string to $|\Lambda|$. For a given weight string we cannot have two weight spaces corresponding to two adjacent weights in the root string (i.e., those differing by a root outside Φ_s) both lying in the eigenspace. In order to obtain the maximum possible dimension of the eigenspace (and so the minimal possible codimension of the eigenspace), for each weight string we must either have the weight spaces corresponding to both $\mu - 2\alpha$ and μ contained in the eigenspace or else the weight space corresponding to the weight $\mu - \alpha$ in $W.\omega_4$ contained in the eigenspace. When $p \neq 2$ the minimal contribution from the 6 weight strings to $|\Lambda|$ is 18 in the first case and 12 in the second case; clearly the minimal contribution occurs in the second case. Suppose that $p = 2$. If the weight $\mu - 2\alpha$ lies in the eigenspace then the weights $\mu - \alpha$ and μ both cannot lie in the eigenspace by the adjacency principle and the fact that $2\alpha(s) = 1$ if and only if $\alpha(s) = 1$. Thus, for each weight string of length three, precisely one of the weights $\mu - 2\alpha$, $\mu - \alpha$ or μ lies in the eigenspace. By taking these possibilities in turn, we see that the contribution from the 6 weight strings to $|\Lambda|$ is 18, 12 or 18, repectively. Thus the minimal contribution to $|\Lambda|$ occurs when the weight $\mu - \alpha$ in each of the 6 weight strings lies in the eigenspace.

The weight spaces with weight string of the form μ_2 may or may not lie in the eigenspace. So we cannot draw a conclusion regarding the contribution to $|\Lambda|$. The same holds for the weight strings of the form μ_1.

We present this information in Figure 5.1 and adding together the minimal contributions l from the various weight strings we find that we have $|\Lambda| \geqslant 36$ or $|\Lambda| \geqslant 40$ according as $p = 2$ or $p \neq 2$. In either case we have $|\Lambda| > \dim G$.

We carry out analogous calculations for the module with highest weight $\omega_1 + \omega_3$ for $n = 6$. We see in Figure 5.2 that $|\Lambda| > \dim G$ for all p, so in this case we are done.

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	$\omega_1 + \omega_3$	105	1	1
1	ω_4	35	3	2

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	35		
$\mu_2\ \mu_2$	25	25	25
$\mu_2\ \mu_1\ \mu_2$	10	20	20
μ_1	5		
$\mu_1\ \mu_1$	10	30	20
Lower bound on $\lvert\Lambda\rvert$		75	65

Figure 5.2: $(\lambda, n) = (\omega_1 + \omega_3, 6)$

Now we consider $2\omega_2$ for $n = 5$. Note that we do not consider $p = 2$ here since we are assuming that highest weights are p-restricted. We perform similar calculations to those above and display the results in the right-hand table in Figure 5.3. We find that $|\Lambda| > \dim G$ except in characteristic 3.

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 2, 3$	$p = 3$
3	$2\omega_2$	15	1	1
2	$\omega_1 + \omega_3$	60	1	1
1	ω_4	15	2	1

Weight strings	No. of strings	l	
		$p \neq 2, 3$	$p = 3$
μ_3	7		
$\mu_3\ \mu_2\ \mu_3$	4	4	4
μ_2	12		
$\mu_2\ \mu_2$	16	16	16
$\mu_2\ \mu_1\ \mu_2$	6	12	6
μ_1	1		
$\mu_1\ \mu_1$	4	8	4
Lower bound on $\lvert\Lambda\rvert$		40	30

Figure 5.3: $(\lambda, n) = (2\omega_2, 5)$

From the fourth and fifth columns in the left-hand table in Figure 5.3 we note that when $p \neq 2, 3$ the added information that $m_{\omega_4} = 2$ allows us to conclude that $|\Lambda| \geqslant 40$; yet we must argue further for $p = 3$ when $m_{\omega_4} = 1$.

If $p = 3$, we show in Figure 5.3 that $(\lozenge)$ is satisfied unless $X = \varnothing$ since $e^{\varnothing}_{2\omega_2} = 30$. If $X = \varnothing$ then, by listing the 90 weights for this module, we can arrange some of these into cliques (as given below) in order to show that $(\dagger)$ is satisfied. Note that we have omitted a factor of $\frac{1}{3}$ on each coefficient.

4	2	6	4	2
4	2	3	4	2
4	2	3	1	2
4	2	3	1	-1

1	5	6	4	2
1	5	3	4	2
1	5	3	1	2
1	5	3	1	-1

4	8	6	4	2
4	5	6	4	2
4	5	3	4	2
4	5	3	1	2
4	5	3	1	-1

1	2	3	4	2
1	-1	3	4	2
1	-1	0	4	2
1	-1	0	1	2

-2	2	6	4	2
-2	2	3	4	2
-2	2	3	1	2
-2	2	3	1	-1

By taking these cliques and the corresponding cliques obtained by reversing the order of the coefficients and changing signs, we see that $d^{\varnothing}_{2\omega_2} \geqslant 2(4+4.3) = 32 > 30 = e^{\varnothing}_{2\omega_2}$. $\square$

The next lemma differs from previous ones as the highest weights of the irreducible modules concerned depend on the rank n. As usual we calculate the minimum contribution to the codimension of the eigenspace for each weight string. If some of the weights have multiplicity greater than one (as indicated in Lübeck's tables for $n \leqslant 20$), then the corresponding contribution may in fact be greater.

Lemma 5.8. *Let G act on one of the irreducible modules $L(\omega_2 + \omega_n)$ and $L(2\omega_1 + \omega_n)$ ($p > 2$) both for $n \in [5, \infty)$. Then (†) holds for both modules, unless $n \in [5, 7]$ for $L(\omega_2 + \omega_n)$, in which case (◊) is satisfied for all $X \in I$.*

Proof. Consider the module with highest weight $\omega_2 + \omega_n$ for general n. As displayed in the bottom table in Figure 5.4, we have $|\Lambda| \geqslant \frac{1}{2}(3n^2 - 7n + 6)$, (note that by Premet's theorem we know that each weight in $W.\omega_1$ has multiplicity at least one). This is larger than $\dim G$ for $n \geqslant 11$. Thus we need only consider the possibilities $n \in [5, 10]$ and we may use the information about multiplicities in the tables to calculate better lower bounds for the codimension.

i	ω	$\lvert W.\omega\rvert$	m_ω		
			all p, $n \geqslant 21$	$p \nmid n \leqslant 20$	$p \mid n \leqslant 20$
2	$\omega_2 + \omega_n$	$\frac{1}{2}(n+1)n(n-1)$	1	1	1
1	ω_1	$n+1$	$\geqslant 1$	$n-1$	$n-2$

Weight strings	No. of strings	l		
		all p, $n \geqslant 21$	$p \nmid n \leqslant 20$	$p \mid n \leqslant 20$
μ_2	$\frac{1}{2}(n-1)(n^2-5n+8)$			
$\mu_2\ \mu_2$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$
$\mu_2\ \mu_1\ \mu_2$	$n-1$	$n-1$	$2(n-1)$	$2(n-1)$
$\mu_1\ \mu_1$	1	1	$n-1$	$n-2$
Lower bound on $\lvert\Lambda\rvert$		$\frac{1}{2}(3n^2-7n+6)$	$\frac{3}{2}n(n-1)$	$\frac{1}{2}(3n^2-3n-2)$

Figure 5.4: $\lambda = \omega_2 + \omega_n$ for $n \geqslant 5$

Certainly for all p we have $|\Lambda| \geqslant \frac{1}{2}(3n^2 - 3n - 2)$ and (†) is satisfied for $n \geqslant 8$. Furthermore, for $n \in [6, 7]$ since $|\Phi(G)| < |\Lambda|$ in these cases (◊) is satisfied for all $X \in I$. If $n = 5$ and $p \neq 5$ then $|\Lambda| \geqslant 30$ and $X = \varnothing$ is the only centraliser type requiring further consideration. The same is true for $n = p = 5$ since $|\Lambda| \geqslant 29$ here.

We use the fact that there are certainly two orthogonal roots not in Φ_s; take roots α_1, α_5 and compute the 66 weights for this module explicitly. We provide weight nets below. Note that we have omitted a factor of $\frac{1}{6}$ on each coefficient.

1×2 and 2×1 nets: $\cdot$ 10 9 8 $\cdot$, $\cdot$ 4 9 2 $\cdot$, $\cdot$ -2 3 8 $\cdot$, $\cdot$ 4 -3 2 $\cdot$, $\cdot$ -2 3 -4 $\cdot$, $\cdot$ -8 -3 2 $\cdot$, $\cdot$ -2 -9 -4 $\cdot$, $\cdot$ -8 -9 -10 $\cdot$

2×2 nets: $\cdot$ 4 9 8 $\cdot$, $\cdot$ 4 3 8 $\cdot$, $\cdot$ 4 3 -4 $\cdot$, $\cdot$ 4 -3 -4 $\cdot$, $\cdot$ -8 -3 -4 $\cdot$, $\cdot$ -8 -9 -4 $\cdot$

These weight nets do not involve any weights from $W.\omega_1$. We give below the weight nets that do involve such weights (which we embolden):

$$\begin{array}{ccc} 5\ 4\ 3\ 2\ {-5} & \mathbf{5\ 4\ 3\ 2\ 1} & 5\ 4\ 3\ 2\ 7 \\ {-1}\ 4\ 3\ 2\ {-5} & \mathbf{{-1}\ 4\ 3\ 2\ 1} & {-1}\ 4\ 3\ 2\ 7 \end{array} \qquad \begin{array}{cc} 5\ {-2}\ {-3}\ {-4}\ {-5} & 5\ {-2}\ {-3}\ {-4}\ 1 \\ \mathbf{{-1}\ {-2}\ {-3}\ {-4}\ {-5}} & \mathbf{{-1}\ {-2}\ {-3}\ {-4}\ 1} \\ {-7}\ {-2}\ {-3}\ {-4}\ {-5} & {-7}\ {-2}\ {-3}\ {-4}\ 1 \end{array}$$

$$\begin{array}{ccc} & 5\ {-2}\ {-3}\ 2\ 1 & \\ {-1}\ {-2}\ {-3}\ 2\ {-5} & \mathbf{{-1}\ {-2}\ {-3}\ 2\ 1} & {-1}\ {-2}\ {-3}\ 2\ 7 \\ & {-7}\ {-2}\ {-3}\ 2\ 1 & \end{array} \qquad \begin{array}{ccc} & 5\ {-2}\ 3\ 2\ 1 & \\ {-1}\ {-2}\ 3\ 2\ {-5} & \mathbf{{-1}\ {-2}\ 3\ 2\ 1} & {-1}\ {-2}\ 3\ 2\ 7 \\ & {-7}\ {-2}\ 3\ 2\ 1 & \end{array}$$

So we see that when $p \neq 2, 5$ we have $|\Lambda| \geqslant 6 + 6 + 4 + 4 + 8 + 2.6 = 40$, when $p = 2$ we have $|\Lambda| \geqslant 42$ (the increase derives from the 3×2 and 2×3 weight nets) and when $p = 5$ we have $|\Lambda| \geqslant 5 + 5 + 3 + 3 + 8 + 2.6 = 36$; in any case $|\Lambda|$ exceeds $\dim G$, so we are done.

Since $\omega_2 + \omega_n < 2\omega_1 + \omega_n$ and the value of m_{ω_1} for $L(2\omega_1 + \omega_n)$ is at least the corresponding value of m_{ω_1} for $L(\omega_2 + \omega_n)$ when $n \leqslant 20$, we can use Premet's theorem 2.6 and the calculation of the number of weight strings in Figure 5.4 to conclude for $n \geqslant 8$ that (†) holds for $L(2\omega_1 + \omega_n)$. In fact we see in Figure 5.5 that (†) also holds for $n \in [5, 7]$.

$\square$

5.3 Centraliser analysis for $L(\omega_4)$ and $L(\omega_5)$

In this section we examine the irreducible modules with highest weight ω_4 for $n \in [7, 12]$ and ω_5 for $n = 9$. For convenience we shall introduce some notation for weights in the Weyl group orbit of a fundamental weight.

Notation Consider a fundamental weight ω_k with $1 \leqslant k \leqslant n$ written as a sum of simple roots. If we write a fundamental weight as a string of the coefficients in such a sum

i	ω	$\lvert W.\omega \rvert$	m_ω		
			all p, $n+2 \geqslant 23$	$2 \neq p \nmid n+2 \leqslant 22$	$2 \neq p \mid n+2 \leqslant 22$
3	$2\omega_1 + \omega_n$	$(n+1)n$	1	1	1
2	$\omega_2 + \omega_n$	$\frac{1}{2}(n+1)n(n-1)$	1	1	1
1	ω_1	$n+1$	$\geqslant 1$	n	$n-1$

Weight strings	No. of strings	l		
		all p, $n+2 \geqslant 23$	$2 \neq p \nmid n+2 \leqslant 22$	$2 \neq p \mid n+2 \leqslant 22$
μ_3	$(n-1)(n-2)$			
$\mu_3\ \mu_3$	$n-1$	$n-1$	$n-1$	$n-1$
$\mu_3\ \mu_2\ \mu_3$	$n-1$	$n-1$	$n-1$	$n-1$
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	2	$n+1$	n
μ_2	$\frac{1}{2}(n-1)(n-2)(n-3)$			
$\mu_2\ \mu_2$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$	$\frac{3}{2}(n-1)(n-2)$
$\mu_2\ \mu_1\ \mu_2$	$n-1$	$n-1$	$2(n-1)$	$2(n-1)$
Lower bound on $\lvert\Lambda\rvert$		$\frac{1}{2}(3n^2-3n+4)$	$\frac{n}{2}(3n+1)$	$\frac{1}{2}(3n^2+n-2)$

Figure 5.5: $\lambda = 2\omega_1 + \omega_n$ for $n \geqslant 5$

and imagine a zero coefficient on either end of this expression we see that we can encode a fundamental weight by an ordered string of plus signs and minus signs depending on whether the difference between successive coefficients is positive or negative. Thus ω_k can be written as k plus signs and $n-k+1$ minus signs. The effect of the Weyl group is to permute the signs in the string, so all weights in the Weyl group orbit of ω_k can be written as all possible strings of plus and minus signs as described.

We can see when two different weights in $W.\omega_k$ differ by a root precisely if they have $k-1$ plus signs in the same position. Suppose we have two weights which (when written in plus-minus notation) are identical except the first has a plus in the i^{th} position and the second in the j^{th} position, then, assuming $i < j$, the difference between the first and second weights is $\alpha_i + \cdots + \alpha_{j-1}$. So, if two weights in different clusters have $k-1$ plus signs in the same position, then at most one of the clusters of weight spaces can lie in $\Pi(V) \setminus \Lambda$, provided the difference between the weightsis a linear combination of roots outside Φ_s.

Later we shall compare sizes of clusters of weights in cliques; in this case it is helpful to write a cluster as a string of plus and minus signs partitioned by bars and concluded with a colon to indicate the centraliser type. Thus the bar is used to separate non-trivial simple systems in X and the colon signifies that the signs which follow lie in positions unaffected by reflection in any root in X. For example, if the centraliser type is A_2A_1

then $++-|+-:---$ denotes a cluster consisting of six weights lying in $W.\omega_3$ with $n = 7$, i.e., the cluster consists of the weights $++-+----$, $+-++----$, $-+++----$, $++--+---$, $+-+-+---$ and $-++-+---$. As a shorthand we shall underline a string of plus and minus signs in a cluster to denote all clusters which occur containing all possible arrangements of the underlined symbols. For instance, if $X = A_2$ we shall use $++-:\underline{+----}$ to denote the five clusters with exactly one plus sign in the last five positions.

In the next lemma, we shall employ the following strategy. Suppose that G is of type A_n acting on the G-module $L(\lambda)$. The initial lower bound for $|\Lambda|$ may be such that ($\Diamond$) is satisfied for all but a few subsystems $X \in I$ of low rank but does not exceed $\dim G$ (so we cannot immediately conclude that (†) holds). We can certainly assume that there are two orthogonal roots outside of Φ_s for the remaining subsystems. We make the general observation that either $X = A_{n-1}$ or there are two orthogonal roots outside Φ_s. If we can show that $d_\lambda^{A_{n-1}} > \dim G$ and subsequently show that $|\Lambda| > \dim G$ by taking two orthogonal roots outside Φ_s, then we can conclude that (†) holds.

Lemma 5.9. *Let G act on the irreducible module $V = L(\omega_k)$ for $k \in [4, 5]$. Then for $k = 4$ when $n \in [11, 12]$ and for $k = 5$ when $n = 9$ the condition (†) holds, otherwise ($\Diamond$) is satisfied for all $X \in I$.*

Proof. Consider the module with highest weight ω_k in A_n. We may assume that there is a root $\alpha \notin \Phi_s$ so there are $r_{A_{k-1}A_{n-k}} = \binom{n-1}{k-1}$ weights $\mu \in W.\omega_k$ with weight string of the form $\mu, \mu - \alpha$. By the adjacency principle we see that each weight string of length 2 contains at least one weight space that cannot lie in the eigenspace $V_\gamma(s)$ for each $k \in [4, 5]$.

Consider the Weyl group orbit of ω_4; it has size $\binom{n+1}{4}$. For $n = 12$ we have $|\Lambda| \geqslant \binom{11}{3} = 165$. Since $|\Phi(G)| = 156$ we see that ($\Diamond$) is satisfied for all centraliser types and the result holds. We wish to show that (†) holds, so we need to show that it holds both when $X = A_{11}$ and when there are two orthogonal roots outside Φ_s. (It is clear that the only centraliser type where there are not two orthogonal roots outside Φ_s is $X = A_{11}$.) Suppose that $X = A_{11}$ with simple roots $\alpha_1, \ldots, \alpha_{11}$; there are two clusters, the first containing weights whose last sign is plus and the second containing those whose last sign is minus; so the clusters sizes are $\binom{12}{3} = 220$ and $\binom{12}{4} = 495$, whence $d_{\omega_4}^{A_{11}} \geqslant 220 > \dim G$.

We can therefore assume that there are at least two orthogonal roots lying outside Φ_s; take them to be α_1 and α_{12}. In particular, for $n = 12$, we have

$$\omega_4 = 9\ 18\ 27\ 36\ 32\ 28\ 24\ 20\ 16\ 12\ 8\ 4,$$

omitting a factor of $\frac{1}{13}$ on each coefficient. As described above we can write this as a string of four plus signs followed by nine minus signs and we obtain all possible strings as the Weyl group interchanges plus signs and minus signs. Thus we find that we have a convenient combinatorial approach to counting the number of weights orthogonal to α_1 and α_{12}, i.e., those weights which begin and end with either two plus signs or two minus signs. Firstly we see that there are $\binom{11}{2} + \binom{11}{4}$ weights orthogonal to α_1: those having two plus signs and those having two minus signs at the beginning of the string. Hence all other weights are in pairs differing by α_1, giving codimension $|\Lambda| > \frac{1}{2}(715 - 385) = 165$. (Of course this agrees with our calculation above.) Then we see that there are $1 + 2\binom{9}{2} + \binom{9}{4}$ weights orthogonal to α_1 and α_{12}. All other weights occur in pairs differing by α_1 or α_{12}, so, by the adjacency principle, $|\Lambda| \geqslant \frac{1}{2}(715 - 199) = 258 > \dim G$.

The case $n = 11$ follows similarly. We have $|\Lambda| \geqslant \binom{10}{3} = 120$, hence we are done if $|\Phi(X)| > 12$ by ($\Diamond$). Suppose that $X = A_{10}$ with simple roots $\alpha_1, \ldots, \alpha_{10}$; there are two clusters, the first containing weights whose last sign is plus and the second containing those whose last sign is minus; so the clusters sizes are $\binom{11}{3} = 165$ and $\binom{11}{4} = 330$, whence $d_{\omega_4}^{A_{10}} \geqslant 165 > \dim G$.

Thus assume that there are two orthogonal roots lying outside Φ_s; take them to be α_1 and α_{11}. We can now write ω_4 as a string of four plus signs followed by eight minus signs. We calculate that there are $\binom{10}{2} + \binom{10}{4}$ weights in $W.\omega_4$ orthogonal to α_1 giving codimension $|\Lambda| > \frac{1}{2}(495 - 255) = 120$ and there are $1 + 2\binom{8}{2} + \binom{8}{4}$ weights orthogonal to α_1 and α_{11}. All other weights occur in pairs differing by α_1 or α_{11}, so, by the adjacency principle, $|\Lambda| \geqslant \frac{1}{2}(495 - 127) = 184 > \dim G$.

For $n = 10$ we have $|\Lambda| \geqslant \binom{9}{3} = 84$ and we have the result when $|\Phi(X)| > 26$ using ($\Diamond$). Thus there will certainly be two orthogonal roots not in Φ_s, namely α_1 and α_{10}. There are $1+2\binom{7}{2}+\binom{7}{4}$ weights in $W.\omega_4$ orthogonal to both α_1 and α_{10}, all other weights occurring in pairs differing by α_1 or α_{10}, hence $|\Lambda| \geqslant \frac{1}{2}(330-78) = 126 > \dim G$. We conclude that ($\Diamond$) is satisfied for all $X \in I$.

For $n = 9$ we have $|\Phi(G)| = 90$. From our initial calculation above we have $|\Lambda| \geqslant 56$ and ($\Diamond$) shows that we may take two orthogonal roots α_1, α_9 not in Φ_s. This improves our lower bound of the codimension of the eigenspace since there are $1+2\binom{6}{2}+\binom{6}{4} = 46$ weights in $W.\omega_4$ orthogonal to both α_1 and α_9; we find that $|\Lambda| \geqslant \frac{1}{2}\left(\binom{10}{4}-46\right) = 82$. As a consequence, we may assume that there are three orthogonal roots outside Φ_s; taking the third to be α_5 we obtain $|\Lambda| \geqslant \frac{1}{2}\left(\binom{10}{4}-(1+3\binom{4}{2}+3)\right) = 94$, whence ($\Diamond$) is satisfied for all $X \in I$.

Now consider the module with highest weight ω_5 when $n = 9$. We show that (†) holds. We have $|\Lambda| \geqslant 70$ and so we are done if $|\Phi(X)| > 20$ using ($\Diamond$). Suppose that $X = A_8$ with simple roots $\alpha_1, \dots, \alpha_8$; there are two clusters, the first containing weights whose last sign is plus and the second containing those whose last sign is minus; so both clusters have size $\binom{9}{4} = 126$, whence $d^{A_8}_{\omega_5} \geqslant 126 > \dim G$.

Thus assume there are two orthogonal roots lying outside of the centraliser, take α_1 and α_9. We view weights in $W.\omega_5$ as ordered strings of five plus signs and five minus signs. We count $\binom{6}{1}+2\binom{6}{3}+\binom{6}{5}$ weights orthogonal to both α_1 and α_9, so $|\Lambda| \geqslant 100$. The conclusion follows since $\dim G = 99$. □

It remains for us to consider $L(\omega_4)$ when $n \in [7, 8]$. Note that $\dim L(\omega_4) = |W.\omega_4| = 70$ or 126 according to whether $n = 7$ or 8.

Lemma 5.10. *Let G act on the irreducible module $V = L(\omega_4)$ for $n = 8$. Then ($\Diamond$) holds.*

Proof. Consider $L(\omega_4)$ with $n = 8$. We know that $|\Lambda| \geqslant r_{A_3A_4} = \binom{7}{3} = 35$ and we are done for all types of centralisers satisfying $|\Phi(X)| > 37$. For the remaining types, there will always be two orthogonal roots not lying in Φ_s. Assume that α_1 and α_3 are two orthogonal roots outside Φ_s. Since there are 26 weights orthogonal to them both, we have

$|\Lambda| \geqslant \frac{1}{2}(126-26) = 50$. Repeating this procedure we are done for $|\Phi(X)| > 22$ by ($\Diamond$), and for the remaining possibilities for X there are three orthogonal roots not in Φ_s, which we may take to be α_1, α_3 and α_5. There are 12 weights orthogonal to these roots, so $|\Lambda| \geqslant 57$. Repeating this, we may take four orthogonal roots not in Φ_s and can improve the lower bound to $|\Lambda|$. Indeed, we find that $|\Lambda| \geqslant 60$ and consequently we are done if $|\Phi(X)| > 12$. This leaves eleven centraliser types each requiring individual attention. They are $X = \varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, A_2^2 and A_3.

For each type we arrange the weights into clusters and then form cliques. As described in Section 4.2, in each clique the weight spaces corresponding to the weights constituting at most one cluster can lie in the eigenspace. For each clique, the lower bound l for the contribution to $|\Lambda|$ is therefore the sum of all the cluster sizes except the largest. The tables below show clusters arranged in cliques and calculate the minimal contribution l of each clique to $|\Lambda|$. Note that we do not provide all clusters in the tables that follow, only a selection. For each centraliser type X we find from the table that the codimension exceeds $e_{\omega_4}^X$.

$X = A_3$

Clique	Cluster size	l	Clique	Cluster size	l
$+++-:\underline{+----}$	4	16	$++--:--\underline{++-}$	6	12
$++--:+\underline{+---}$	6	18	$+---:++\underline{+--}$	4	8
$++--:-+\underline{+--}$	6	12			

For $X = A_3$, we see that $d_{\omega_4}^{A_3} \geqslant 66 > 60 = e_{\omega_4}^{A_3}$.

$X = A_2^2$

Clique	Cluster size	l	Clique	Cluster size	l
$++-\mid+--:\underline{+--}$	9	18	$++-\mid++-:---$	9	27
$+--\mid+--:\underline{++-}$	9	18	$+--\mid++-:\underline{+--}$	9	

For $X = A_2^2$, we see that $d_{\omega_4}^{A_2^2} \geqslant 63 > 60 = e_{\omega_4}^{A_2^2}$.

$X = A_2A_1^3$

Clique	Cluster size	l	Clique	Cluster size	l
$++-\mid+-\mid+-\mid--:$	12	24	$+--\mid+-\mid++\mid--:$	6	6
$++-\mid+-\mid--\mid+-:$	12		$+--\mid--\mid++\mid+-:$	6	
$++-\mid--\mid+-\mid+-:$	12		$+--\mid+-\mid--\mid++:$	6	6
$+--\mid++\mid+-\mid--:$	6	6	$+--\mid--\mid+-\mid++:$	6	
$+--\mid++\mid--\mid+-:$	6				

If $X = A_2A_1^3$ there is a cluster of weights $+--\vert+-\vert+-\vert+-:$ of size 24. If the weights in this cluster lie in $\Pi(V) \setminus \Lambda$, then none of the 72 weights in the table above can lie in $\Pi(V) \setminus \Lambda$. On the other hand, if this cluster lies in Λ then $d_{\omega_4}^{A_2A_1^3} \geqslant 24 + 42 = 66 > 60 = e_{\omega_4}^{A_2A_1^3}$.

$X = A_2A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++-\vert+-\vert+-:--$	12	24	$++-\vert++\vert--:--$	3	6
$+--\vert+-\vert+-:\underline{+-}$	12		$+--\vert++\vert--:\underline{+-}$	3	
$++-\vert+-\vert--:\underline{+-}$	6	6	$++-\vert--\vert++:--$	3	6
$++-\vert--\vert+-:\underline{+-}$	6	6	$+--\vert--\vert++:\underline{+-}$	3	
$+--\vert\underline{++\vert+-}:--$	6	6	$---\vert+-\vert++:\underline{+-}$	2	2
$+--\vert\underline{+-\vert--}:++$	6	6	$---\vert++\vert+-:\underline{+-}$	2	2

For $X = A_2A_1^2$, we see that $d_{\omega_4}^{A_2A_1^2} \geqslant 64 > 62 = e_{\omega_4}^{A_2A_1^2}$.

$X = A_2A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$++-\vert+-:\underline{+---}$	6	18	$++-\vert++:----$	3	12
$+--\vert--:\underline{+++-}$	3	9	$+--\vert++:\underline{+---}$	3	
$+--\vert+-:-\underline{++-}$	6	12	$+--\vert+-:+\underline{+--}$	6	12
$++-\vert--:+\underline{+--}$	3	6			

For $X = A_2A_1$, we see that $d_{\omega_4}^{A_2A_1} \geqslant 69 > 64 = e_{\omega_4}^{A_2A_1}$.

$X = A_2$		
Clique	Cluster size	l
$++-:\underline{++----}$	3	36
$+--:\underline{+++---}$	3	48

For $X = A_2$, we see that $d_{\omega_4}^{A_2} \geqslant 84 > 66 = e_{\omega_4}^{A_2}$.

$X = A_1^4$					
Clique	Cluster size	l	Clique	Cluster size	l
$+-\vert+-\vert+-\vert+-:-$	16	32	$\underline{+-\vert+-}\vert++\vert\underline{--}:-$	4	8
$\underline{+-\vert+-\vert+-\vert--}:+$	8		$\underline{+-\vert+-\vert--}\vert++:-$	4	8
$++\vert\underline{+-\vert+-\vert--}:-$	4	8	$++\vert\underline{+-\vert--\vert--}:+$	2	4
$\underline{+-}\vert++\vert\underline{+-\vert--}:-$	4	8			

For $X = A_1^4$, we see that $d_{\omega_4}^{A_1^4} \geqslant 68 > 64 = e_{\omega_4}^{A_1^4}$.

$X = A_1^3$					
Clique	Cluster size	l	Clique	Cluster size	l
$+-\mid+-\mid+-:\underline{+--}$	8	16	$\underline{+-\mid--\mid--}:+++$	2	4
$\underline{++\mid+-\mid+-}:---$	4	8	$++\mid+-\mid--:\underline{+--}$	2	4
$+-\mid+-\mid--:\underline{++-}$	4	8	$++\mid--\mid+-:\underline{+--}$	2	4
$+-\mid--\mid+-:\underline{++-}$	4	8	$+-\mid++\mid--:\underline{+--}$	2	4
$--\mid+-\mid+-:\underline{++-}$	4	8	$--\mid++\mid+-:\underline{+--}$	2	4

For $X = A_1^3$, we see that $d_{\omega_4}^{A_1^3} \geqslant 68 > 66 = e_{\omega_4}^{A_1^3}$.

$X = A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$+-\mid+-:\underline{++---}$	4	32	$--\mid+-:\underline{+++--}$	2	16
$+-\mid--:\underline{+++--}$	2	16	$++\mid+-:\underline{+----}$	2	8

For $X = A_1^2$, we see that $d_{\omega_4}^{A_1^2} \geqslant 72 > 68 = e_{\omega_4}^{A_1^2}$.

$X = A_1$ (and $X = \varnothing$)					
Clique	Cluster size	l	Clique	Cluster size	l
$+-:+\underline{++----}$	2	24	$++:\underline{++-----}$	1	8
$+-:-+\underline{++---}$	2	16	$--:++\underline{++---}$	1	8
$+-:--+\underline{++--}$	2	8			

It is clear that the lower bounds for $d_{\omega_4}^X$ calculated in the table above are the same whether $X = \varnothing$ or A_1; in both cases we find $d_{\omega_4}^X \geqslant 74$ which exceeds both $e_{\omega_4}^{\varnothing} = 72$ and $e_{\omega_4}^{A_1} = 70$. $\square$

Finally in this section we consider the module $L(\omega_4)$ with $n = 7$.

Lemma 5.11. *Let G act on the irreducible module $V = L(\omega_4)$ for $n = 7$. Then ($\diamondsuit$) is satisfied for each $X \in I \setminus \{A_3^2\}$.*

Proof. We have $\dim L(\omega_4) = 70$ for $n = 7$ and we can use the exact same iterative process as we did for $n = 8$ to show that we are done provided that $|\Phi(X)| > 24$. There are seventeen centraliser types remaining to be investigated. They are (with $e_{\omega_4}^X$ given in brackets) as follows: $\varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_2, A_2A_1, $A_2A_1^2$, A_2^2, $A_2^2A_1$, A_3, A_3A_1, $A_3A_1^2$, A_3A_2, A_3^2, A_4 and A_4A_1.

$X = A_4A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$++++-\mid--:-$	5	15	$++---\mid++:-$	10	10
$+++--\mid+-:-$	20		$++---\mid+-:+$	20	
$+++--\mid--:+$	10		$+----\mid++:+$	5	

In the table above we arrange the clusters into cliques and we see that $d_{\omega_4}^{A_4A_1} \geqslant 25$. If the 20 weights in the cluster $++---|+-:+$ are in $\Pi(V) \setminus \Lambda$ then $d_{\omega_4}^{A_4A_1} \geqslant 45 > 34 = e_{\omega_4}^{A_4A_1}$. For, we exclude the weights in the clusters $+++--|+-:-$, $+++--|--:+$, $++---|++:-$ and $+----|++:+$ from the eigenspace since these clusters all contain at least one weight differing in precisely two positions from at least one of the weights in $++---|+-:+$. If we do not have the weights in $++---|+-:+$ then the table above shows that $d_{\omega_4}^{A_4A_1} \geqslant 35$.

If $X = A_4$ the clusters that occur are easy to see from those given above. There are two cliques (corresponding to the left and right hand sides of the table) which both guarantee a contribution of at least 25 to $|\Lambda|$, hence $d_{\omega_4}^{A_4} \geqslant 50 > 36 = e_{\omega_4}^{A_4}$.

$X = A_3A_2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++--\mid++-:-$	18	18	$++++\mid---:-$	1	5
$++--\mid+--:+$	18		$+++-\mid+--:-$	12	
$+---\mid+++:-$	4	5	$+++-\mid---:+$	4	
$+---\mid++-:+$	12				
$----\mid+++:+$	1				

From the arrangement of clusters into cliques in the table above, we have $d_{\omega_4}^{A_3A_2} \geqslant 28$. If the weights in the cluster $++--|++-:-$ are in $\Pi(V) \setminus \Lambda$ then the clusters $+++-|+--:-$, $+---|+++:-$ and $+---|++-:+$ do not contribute to the eigenspace, hence $d_{\omega_4}^{A_3A_2} \geqslant 47 > 38 = e_{\omega_4}^{A_3A_2}$. We can conclude similarly for the weights in $++--|+--:+$. If neither cluster of size 18 contributes to the eigenspace, then $d_{\omega_4}^{A_3A_2} \geqslant 46$.

$X = A_3A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++++\mid--\mid--:$	1	9	$++--\mid+-\mid+-:$	24	14
$+++-\mid+-\mid--:$	8		$++--\mid--\mid++:$	6	
$+++-\mid--\mid+-:$	8		$+---\mid+-\mid++:$	8	
$++--\mid++\mid--:$	6	6	$----\mid++\mid++:$	1	
$+---\mid++\mid+-:$	8				

If the weights in the cluster of size 24 lie in $\Pi(V) \setminus \Lambda$ then, by excluding other clusters, we find $d_{\omega_4}^{A_3 A_1^2} \geqslant 44 > 40 = e_{\omega_4}^{A_3 A_1^2}$. Otherwise, as can be seen from the table above, $d_{\omega_4}^{A_3 A_1^2} \geqslant 45$.

$X = A_3 A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$++++\mid--:--$	1	9	$++--\mid+-:+-$	12	26
$+++-\mid+-:--$	8		$++--\mid+-:-+$	12	
$+++-\mid--:+-$	4		$++--\mid--:++$	6	
$+++-\mid--:-+$	4		$+---\mid+-:++$	8	
$++--\mid++:--$	6	8	$----\mid++:++$	1	
$+---\mid++:+-$	4				
$+---\mid++:-+$	4				

From the table we see that $d_{\omega_4}^{A_3 A_1} \geqslant 43 > 42 = e_{\omega_4}^{A_3 A_1}$.

$X = A_3$					
Clique	Cluster size	l	Clique	Cluster size	l
$++++:----$	1	13	$+---:\underline{+++-}$	4	13
$+++-:\underline{+---}$	4		$----:++++$	1	
$++--:+\underline{+--}$	6	12			
$++--:-\underline{++-}$	6	12			

From the table we see that $d_{\omega_4}^{A_3} \geqslant 50 > 44 = e_{\omega_4}^{A_3}$.

$X = A_2^2 A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$+++\mid+--\mid--:$	3	3	$+--\mid+++\mid--:$	3	2
$+++\mid---\mid+-:$	2		$---\mid+++\mid+-:$	2	
$++-\mid++-\mid--:$	9	27	$++-\mid---\mid++:$	3	3
$++-\mid+--\mid+-:$	18		$+--\mid+--\mid++:$	9	
$+--\mid++-\mid+-:$	18		$---\mid++-\mid++:$	3	

If we assume in turn that the weights in the clusters of size 18 are in $\Pi(V) \setminus \Lambda$, we obtain $d_{\omega_4}^{A_2^2 A_1} \geqslant 46 > 42 = e_{\omega_4}^{A_2^2 A_1}$ for each. If these two clusters do not contribute to the eigenspace then we have $d_{\omega_4}^{A_2^2 A_1} \geqslant 43$ as required.

$X = A_2^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$+++\mid+--:--$	3	21	$+--\mid++-:+-$	9	21
$++-\mid++-:--$	9		$+--\mid++-:-+$	9	
$++-\mid+--:+-$	9		$+--\mid+--:++$	9	
$++-\mid+--:-+$	9		$---\mid++-:++$	3	
$+--\mid+++:--$	3	2	$+++\mid---:\underline{+-}$	1	1
$---\mid+++:\underline{+-}$	1				

From the table we see that $d_{\omega_4}^{A_2^2} \geqslant 45 > 44 = e_{\omega_4}^{A_2^2}$.

$X = A_2A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$+++\vert+-\vert--:-$	2	3	$++-\vert+-\vert--:+$	6	6
$+++\vert--\vert+-:-$	2		$++-\vert--\vert+-:+$	6	
$+++\vert--\vert--:+$	1		$++-\vert--\vert++:-$	3	3
$++-\vert++\vert--:-$	3	3	$+--\vert--\vert++:+$	3	
$+--\vert++\vert--:+$	3		$++-\vert+-\vert+-:-$	12	24
$---\vert++\vert++:-$	1	3	$+--\vert++\vert+-:-$	6	
$---\vert++\vert+-:+$	2		$+--\vert+-\vert++:-$	6	
$---\vert+-\vert++:+$	2		$+--\vert+-\vert+-:+$	12	

If either of the clusters of size 12 are in $\Pi(V)\setminus\Lambda$, we see that $d_{\omega_4}^{A_2A_1^2} \geqslant 49 > 46 = e_{\omega_4}^{A_2A_1^2}$. Therefore, assuming that these weights are in Λ we have that the minimum contribution of the largest clique above becomes 30. Thus $d_{\omega_4}^{A_2A_1^2} \geqslant 48$.

$X = A_2A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$+++\vert+-:---$	2	3	$+--\vert++:\underline{+--}$	3	6
$+++\vert--:\underline{+--}$	1		$+--\vert+-:\underline{++-}$	6	15
$++-\vert++:---$	3	15	$+--\vert--:+++$	3	
$++-\vert+-:\underline{+--}$	6		$---\vert++:\underline{++-}$	1	3
$++-\vert--:\underline{++-}$	3	6	$---\vert+-:+++$	2	

From the table above we have $d_{\omega_4}^{A_2A_1} \geqslant 48 = e_{\omega_4}^{A_2A_1}$. If the weights in the cluster $+++\vert+-:---$ are in $\Pi(V)\setminus\Lambda$, then none of the clusters $++-\vert+-:\underline{+--}$ can lie in $\Pi(V)\setminus\Lambda$, so $d_{\omega_4}^{A_2A_1} \geqslant 51$. Otherwise $d_{\omega_4}^{A_2A_1} \geqslant 49$ and we are done.

$X = A_2$					
Clique	Cluster size	l	Clique	Cluster size	l
$+++:\underline{+----}$	1	4	$+--:++\underline{+--}$	3	6
$++-:+\underline{+---}$	3	9	$+--:+-\underline{++-}$	3	6
$++-:-+\underline{+--}$	3	6	$+--:-\underline{+++-}$	3	9
$++-:--\underline{++-}$	3	6	$---:\underline{++++-}$	1	4

From the table, by arranging the clusters into cliques, we have $d_{\omega_4}^{A_2} \geqslant 50 = e_{\omega_4}^{A_2}$. We can improve the codimension by 3 by observing that at most two of the 10 clusters from $+--:\underline{+++--}$ correspond to weights lying in $\Pi(V) \setminus \Lambda$. (In the table we were assuming that at most three such clusters were contributing to the eigenspace dimension.) If we assume without loss that the weights in the cluster $+--:+++--$ lie in $\Pi(V) \setminus \Lambda$, this forces all clusters but those of the form $+--:\underline{+--}++$ to lie outside the eigenspace. But only one of the clusters in $+--:\underline{+--}++$ can contribute to the eigenspace.

$X = A_1^4$					
Clique	Cluster size	l	Clique	Cluster size	l
$++\mid+-\mid+-\mid--:$	4	12	$--\mid++\mid+-\mid+-:$	4	8
$+-\mid++\mid+-\mid--:$	4		$--\mid+-\mid++\mid+-:$	4	
$+-\mid+-\mid++\mid--:$	4		$--\mid+-\mid+-\mid++:$	4	
$+-\mid+-\mid+-\mid+-:$	16		$++\mid++\mid--\mid--:$	1	
$++\mid+-\mid--\mid+-:$	4	8	$++\mid--\mid++\mid--:$	1	
$+-\mid++\mid--\mid+-:$	4		$++\mid--\mid--\mid++:$	1	
$+-\mid+-\mid--\mid++:$	4		$--\mid++\mid++\mid--:$	1	
$++\mid--\mid+-\mid+-:$	4	8	$--\mid++\mid--\mid++:$	1	
$+-\mid--\mid++\mid+-:$	4		$--\mid--\mid++\mid++:$	1	
$+-\mid--\mid+-\mid++:$	4				

If the cluster of size 16 lies in $\Pi(V) \setminus \Lambda$, then all clusters with size greater than one cannot be in $\Pi(V) \setminus \Lambda$, so $d_{\omega_4}^{A_1^4} \geqslant 48 = e_{\omega_4}^{A_1^4}$. We are done, unless the six weight spaces corresponding to the remaining six clusters are in $\Pi(V) \setminus \Lambda$. However, this implies that $2\alpha_4(s) = 2\alpha_6(s) = 1$, i.e., $\alpha_4(s) = \alpha_6(s) = -1$ for $s \in S_{A_1^4}$, so $\alpha_4 + \alpha_5 + \alpha_6 \in \Phi_s$ and $X \neq A_1^4$. Now using the table above and assuming that the cluster of size 16 does not contribute to the dimension of the eigenspace, we have $d_{\omega_4}^{A_1^4} \geqslant 48$ and we improve the value of the codimension by at least one using the previous observation about the six weights constituting the clusters of size one.

$X = A_1^3$					
Clique	Cluster size	l	Clique	Cluster size	l
$++\mid+-\mid+-:--$	4	20	$+-\mid+-\mid--:++$	4	8
$+-\mid++\mid+-:--$	4		$+-\mid--\mid+-:++$	4	
$+-\mid+-\mid++:--$	4		$--\mid+-\mid+-:++$	4	
$+-\mid+-\mid+-:+-$	8		$--\mid++\mid+-:+-$	2	2
$+-\mid+-\mid+-:-+$	8		$--\mid+-\mid++:+-$	2	
$++\mid+-\mid--:+-$	2	2	$--\mid++\mid+-:-+$	2	2
$+-\mid++\mid--:+-$	2		$--\mid+-\mid++:-+$	2	
$++\mid+-\mid--:-+$	2	2	$++\mid++\mid--:--$	1	
$+-\mid++\mid--:-+$	2		$++\mid--\mid++:--$	1	
$++\mid--\mid+-:+-$	2	2	$++\mid--\mid--:++$	1	
$+-\mid--\mid++:+-$	2		$--\mid++\mid++:--$	1	
$++\mid--\mid+-:-+$	2	2	$--\mid++\mid--:++$	1	
$+-\mid--\mid++:-+$	2		$--\mid--\mid++:++$	1	

From the table above, we see that $d_{\omega_4}^{A_1^3} \geqslant 40$. By similar arguments as for the case $X = A_1^4$ we are done if any one of the clusters of sizes 8 or 4 contribute to the dimension of the eigenspace. Hence, assuming that none of these contribute, it is easy to show $d_{\omega_4}^{A_1^3} \geqslant 51 > 50 = e_{\omega_4}^{A_1^3}$.

$X = A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++\mid++:----$	1	7	$--\mid++:\underline{++--}$	1	4
$++\mid+-:\underline{+---}$	2		$+-\mid--:\underline{+++-}$	2	6
$+-\mid++:\underline{+---}$	2	6	$--\mid+-:\underline{+++-}$	2	7
$+-\mid+-:\underline{++--}$	4	16	$--\mid--:++++$	1	
$++\mid--:\underline{++--}$	1	4			

From the table above we see that $d_{\omega_4}^{A_1^2} \geqslant 50$. Suppose that the two clusters in the clique $+-\mid+-:\underline{++--}$ lie in $\Pi(V) \setminus \Lambda$; otherwise we are done since $e_{\omega_4}^{A_1^2} = 52$. Then the cliques $++\mid+-:\underline{+---}$ and $+-\mid++:\underline{+---}$ do not contribute to the dimension of the eigenspace, so $d_{\omega_4}^{A_1^2} \geqslant 53$.

If $X = A_1$, we may separate the weights occurring into three types: there are $\binom{6}{2} = 15$ weights of the form $++:\underline{++----}$, $\binom{6}{3} = 20$ pairs of weights in clusters of the form $+-:\underline{+++---}$ and $\binom{6}{2} = 15$ weights of the form $--:\underline{++++--}$. We wish to find an upper bound for the dimension of the eigenspace. The first and third types can have at most three weights lying in $\Pi(V) \setminus \Lambda$. The second type can contribute at most eight weights (from four clusters) to the dimension of the eigenspace. In particular, assume by using the Weyl group that the cluster $+-:+++---$ contributes to the eigenspace, in which case it rules out any cluster with two plus signs in the third, fourth and fifth positions. (If we have $+-:---+++$ also, then this rules out all of the remaining weights of the second type, so only two clusters contribute in this case.) From the remaining weights suppose without loss of generality that the cluster $+-:+--++-$ also contributes to the eigenspace. This rules out all but four clusters and we can have at most the weights of two of these lying in $\Pi(V) \setminus \Lambda$, i.e., we either have the two clusters $+-:-+-+-+$ and $+-:--+-++$ or $+-:--++-+$ and $+-:-+--++$ contributing to the eigenspace. Hence there are at most eight weights in $\Pi(V) \setminus \Lambda$ and $d_{\omega_4}^{A_1} \geqslant 62 > 54 = e_{\omega_4}^{A_1}$.

Finally, suppose that $X = \varnothing$. In order to conclude that ($\Diamond$) holds we shall show that $|\Pi(\omega_4) \setminus \Lambda| < 14$ since $e^{\varnothing}_{\omega_4} = 56$. Indeed, we see from the 14 cliques below of size 5 that $|\Pi(V) \setminus \Lambda| \leqslant 14$.

$++++----$ $+++-+---$ $++-++---$ $+-+++---$ $-++++---$	$++--++--$ $++--+-+-$ $++---++-$ $+---+++-$ $-+--+++-$	$+-+-+-+-$ $+-+-+--+$ $+-+---++$ $+---+-++$ $--+-+-++$	$+++--+--$ $+++----+$ $++---+-+$ $+-+--+-+$ $-++--+-+$	$+-++-+--$ $+-++--+-$ $+-+--++-$ $+--+-++-$ $--++-++-$	$+--+++--$ $+--++--+$ $+--+-+-+$ $+---++-+$ $---+++-+$	$++-+--+-$ $++-+---+$ $++----++$ $+--+--++$ $-+-+--++$
$----++++$ $---+-+++$ $--+--+++$ $-+---+++$ $+----+++$	$--++--++$ $--++-+-+$ $--+++--+$ $-+++---+$ $+-++---+$	$-+-+-+-+$ $-+-+-++-$ $-+-+++--$ $-+++-+--$ $++-+-+--$	$---++-++$ $---++++-$ $--+++-+-$ $-+-++-+-$ $+--++-+-$	$-+--+-++$ $-+--++-+$ $-+-++--+$ $-++-+--+$ $++--+--+$	$-++---++$ $-++--++-$ $-++-+-+-$ $-+++--+-$ $+++---+-$	$--+-++-+$ $--+-+++-$ $--++++--$ $-++-++--$ $+-+-++--$

Suppose without loss of generality that $++++----$ is in $\Pi(V) \setminus \Lambda$. This rules out 16 weights which have three plus signs in any of the first four positions, i.e., the weights $\underline{+++-}\,\underline{+---}$. We shall consider separately the two cases that the weight $----++++$ either lies in or not in the eigenspace.

Suppose first that $----++++$ lies in $\Pi(V) \setminus \Lambda$. As before, this rules out 16 more weights, namely $\underline{+---}\,\underline{+++-}$. There are 36 weights remaining, all being of the form $\underline{++--}\,\underline{++--}$. Suppose without loss that $++--++--$ lies in $\Pi(V) \setminus \Lambda$. This rules out the 8 weights $++--\,\underline{+-}\,\underline{+-}$ and $\underline{+-}\,\underline{+-}\,++--$. Looking at the cliques above, we must have the weight $--++++--$ in $\Pi(V) \setminus \Lambda$ which rules out $--++--++$ whence $++----++$ is ruled out also. There are now 6 cliques (the 1st, 2nd, 7th, 8th, 9th and 14th reading left-to-right above) with one weight present in $\Pi(V) \setminus \Lambda$ and there are two weights remaining in each of the other 8 cliques. The 6 weights that we are assuming are present in $\Pi(V) \setminus \Lambda$ are invariant under $H = \langle w_{\alpha_1}, w_{\alpha_3}, w_{\alpha_5}, w_{\alpha_7} \rangle$ which has a single orbit on the remaining weights. Therefore by applying a suitable element of H we may assume that $+-+-+-+-$ lies in $\Pi(V) \setminus \Lambda$. This determines which weights in the other seven pairs may lie in $\Pi(V) \setminus \Lambda$. Thus we may have the following configuration of 14 weights lying in $\Pi(V) \setminus \Lambda$.

$++++----$	$++--++--$	$++----++$	$+-+-+-+-$	$+-+--+-+$	$+--+-++-$	$+--++--+$
$----++++$	$--++--++$	$--++++--$	$-+-+-+-+$	$-+-++-+-$	$-++-+--+$	$-++--++-$

Suppose that $s = \mathrm{diag}(\lambda_1, \ldots, \lambda_8) \in S_{\varnothing} \cap T$. Since the pairs $++--++--$ and $++----++$, $+-+-+-+-$ and $+-+--+-+$, and $-++-+--+$ and $-++--++-$ are in $\Pi(V) \setminus \Lambda$ we have

$$(\alpha_5 + 2\alpha_6 + \alpha_7)(s) = (\alpha_5 + \alpha_7)(s) = (\alpha_5 - \alpha_7)(s) = 1.$$

Thus $2\alpha_6(s) = 2\alpha_7(s) = 1$ implying that $\alpha_6(s) = \alpha_7(s) = -1$ since by assumption $\alpha_6(s) \neq 1$. It follows that the root $\alpha_6 + \alpha_7 \in \Phi_s$, hence the configuration of weights is not possible. It follows that if a given weight is present in $\Pi(V) \setminus \Lambda$, then its antipodal weight where plus and minus signs are interchanged cannot lie in $\Pi(V) \setminus \Lambda$.

Now we are assuming that $----++++$ lies in Λ. If no weight in $\Pi(V) \setminus \Lambda$ has two plus signs in two of the first four positions, i.e., all weights $\underline{++--}\ \underline{++--}$ are in Λ then the 4th, 5th, 7th, 9th, 10th and 13th cliques above are ruled out, so there are at most 8 weights in $\Pi(V) \setminus \Lambda$. Thus we may assume that we must have at least one weight of this form, take $++--++--$. This rules out 13 weights $--++--++$, $++--\ \underline{+-}\ \underline{+-}$, $\underline{+-}\,--++\,\underline{+-}$ and $\underline{+-}\ \underline{+-}\,++--$.

Suppose we have no weight of the form $\underline{+-}\ --\ \underline{+-}\ ++$. Then there are only two weights not ruled out in each of the 4th, 8th and 10th cliques above. Consider the 8th clique. We are done if we have no weights in this clique in $\Pi(V) \setminus \Lambda$ so assume that we have one of them. If $---+-+++$ is in $\Pi(V) \setminus \Lambda$ then the two weights in the 10th clique cannot be in $\Pi(V) \setminus \Lambda$. Similarly, if $--+--+++$ is in $\Pi(V) \setminus \Lambda$ then the two weights in the 4th clique cannot be in $\Pi(V) \setminus \Lambda$. We may assume therefore, since H preserves the weights already in $\Pi(V) \setminus \Lambda$ and has only one orbit on the remaining weights, that $+----+++$ lies in $\Pi(V) \setminus \Lambda$. The following 26 weights are not ruled out from lying in $\Pi(V) \setminus \Lambda$.

$\underline{+-}\ \underline{+-}\ +-\ \underline{+-}$	$-+--+-++$	$-+\ \underline{+-}\ -+\ \underline{+-}$	$-+\ \underline{+-}\ --++$	$--+++-\ \underline{+-}$
	$--++++--$	$--\ \underline{+-}\ ++\ \underline{+-}$	$--++-+\ \underline{+-}$	$--\ \underline{+-}\ +-++$

If no weights of the form $\underline{+-}\ \underline{+-}\ +-\ \underline{+-}$ lie in $\Pi(V) \setminus \Lambda$ then there are at most 12 weights in $\Pi(V) \setminus \Lambda$: 3 from the three weights we are assuming are present, at most 2 from the 2nd column, at most 2 from both sets of weights in the 3rd column above, at most 1 from both sets of weights in the 4th column and at most 1 from the 5th column. Therefore we must have at least one of these 8 weights present in $\Pi(V) \setminus \Lambda$. The subgroup $J = \langle w_{\alpha_3}, w_{\alpha_7} \rangle$ preserves the set of weights already in $\Pi(V) \setminus \Lambda$ and has an orbit of size 2 on the remaining weights. Thus we may assume by applying appropriate elements of J that either $+-+-+-+-$ or $-++-+-+-$ lies in $\Pi(V) \setminus \Lambda$. In the former case, there are 20 weights remaining that have not been ruled out from being in $\Pi(V) \setminus \Lambda$. In the 4th clique above only the weight $-++--+-+$ remains which we may assume lies in $\Pi(V) \setminus \Lambda$, but this means that no weight in the 13th clique can lie in $\Pi(V) \setminus \Lambda$. The latter case is slightly more difficult. There are 17 weights remaining which can be rearranged in eight cliques as follows.

$+-+-+--+$	$+--++-+-$	$-+-+-++-$	$-++--+-+$	$-+-++--+$	$--++--++$	$--++++--$	$--++-+-+$
$+--++--+$	$---++++-$	$--++-++-$	$-+-+-+-+$	$-+-+--++$	$---++-++$	$--+++--+$	$--+-++-+$
							$---+++-+$

Hence, assuming that $++++----$, $++--++--$, $+----+++$ and $-++-+-+-$ lie in $\Pi(V) \setminus \Lambda$ together with one weight from each of the above eight cliques, we are done. $\square$

We are unable to show that $d_{\omega_4}^{A_3^2} \geqslant 33$ here. The clusters are shown in the table below. It may be the case that the clusters of size 36 and both of size 1 lie in $\Pi(V) \setminus \Lambda$, in which case we can only conclude that $d_{\omega_4}^{A_3^2} = 32$ and $2\alpha_4(s) = 1$ whence $\alpha_4(s) = -1$. The set $S_{A_3^2}$ consists of matrices $s = \operatorname{diag}(a, a, a, a, b, b, b, b)$ for $a, b \in K^*$ with $a^4 b^4 = 1$ and, as just mentioned, it may be the case that $a = -b$.

$X = A_3^2$	
Cluster	Cluster size
$++++\,|\,----:$	1
$+++-\,|\,+---:$	16
$++--\,|\,++--:$	36
$+---\,|\,+++-:$	16
$----\,|\,++++:$	1

5.4 Centraliser analysis for $L(\omega_3)$

In this section we shall consider the modules $L(\omega_3)$ for $n \geqslant 8$, $L(\omega_1 + \omega_2)$ for $n \geqslant 5$ and $L(3\omega_1)$ for $n \geqslant 5$.

First, suppose that the largest rank of a component in the root subsystem $X \in I$ is at least $n-k+1$ where $n > 2k-2$. We claim that there are at most $k-1$ orthogonal roots not in Φ_s. The possible centraliser types are determined by taking subsystems of the extended Dynkin diagram. Suppose that the root subsystem of largest rank in X is A_{n-k+1} where X contains the simple roots $\alpha_{i+1}, \ldots, \alpha_{n-j}$ for $0 \leqslant i \leqslant k-1$ and $j = k - i - 1$. Then either $\alpha_i \notin \Phi_s$ for $i \geqslant 1$ or $\alpha_{n-j+1} \notin \Phi_s$ for $j \geqslant 1$. A maximal set of orthogonal roots not in Φ_s is

$$\begin{array}{cc}
\alpha_i & \alpha_{n-j+1} \\
\alpha_{i-1}+\alpha_i+\alpha_{i+1} & \alpha_{n-j}+\alpha_{n-j+1}+\alpha_{n-j+2} \\
\vdots & \vdots \\
\alpha_1+\cdots+\alpha_i+\cdots+\alpha_{2i-1} & \alpha_{n-2j+2}+\cdots+\alpha_{n-j+1}+\cdots+\alpha_n
\end{array}$$

There are $i+j=k-1$ such roots.

Lemma 5.12. *Let G act on the irreducible module $L(\omega_3)$ for $n \geqslant 14$. Then ($\Diamond$) holds for each $X \in I$.*

Proof. For a given n, each weight in the Weyl group orbit of ω_3 can be written as an ordered string of 3 plus signs and $n-2$ minus signs; there are $\binom{n+1}{3}$ such weights. The number of weights in $W.\omega_3$ orthogonal to α_1 is $n-1+\binom{n-1}{3}$ so that $|\Lambda| \geqslant \frac{1}{2}(n-1)(n-2)$.

There are two orthogonal roots outside Φ_s provided $X \neq A_{n-1}$. If $X = A_{n-1}$ with simple roots $\alpha_1, \ldots, \alpha_{n-1}$ then there are two clusters, the first containing weights whose last sign is plus and the second containing those whose last sign is minus; so the cluster sizes are $\binom{n}{2}$ and $\binom{n}{3}$, whence $d_{\omega_3}^{A_{n-1}} \geqslant \binom{n}{2} > 2n = e_{\omega_3}^{A_{n-1}}$. Therefore, in this case, ($\Diamond$) holds for $n \geqslant 6$. There are $2(n-3)+\binom{n-3}{3}$ weights orthogonal to both α_1 and α_3 so we obtain an improved lower bound for $|\Lambda|$; we have $|\Lambda| \geqslant n^2-5n+8$.

There are three orthogonal roots not in Φ_s provided $X \neq A_{n-2}$ or a root subsystem properly containing it. However, $e_{\omega_3}^{A_{n-2}} = 2(2n-1)$ so ($\Diamond$) holds in this case for $n \geqslant 8$. Taking pairs of weights differing by α_1, α_3 or α_5 we find that $|\Lambda| \geqslant \frac{1}{2}(3n^2-21n+50)$.

There are four orthogonal roots not in Φ_s provided $X \neq A_{n-3}$ or a root subsystem properly containing it. However, $e_{\omega_3}^{A_{n-3}} = 6(n-1)$ so ($\Diamond$) holds in this case for $n \geqslant 9$. We now find that $|\Lambda| \geqslant 2(n^2-9n+28)$ by considering orthogonal roots α_1, α_3, α_5 and α_7.

We may take five orthogonal roots outside Φ_s provided $X \neq A_{n-4}$ or a root subsystem properly containing it. Since $e_{\omega_3}^{A_{n-4}} = 4(2n-3)$ we find that ($\Diamond$) holds for $n \geqslant 10$. There are $5(n-9) + \binom{n-9}{3}$ weights orthogonal to α_1, α_3, α_5, α_7 and α_9 so that $|\Lambda| \geqslant \frac{5}{2}(n^2 - 11n + 42)$ which exceeds $\dim G$ for $n \geqslant 16$.

We may take a sixth orthogonal root outside the centraliser since $e_{\omega_3}^{A_{n-5}} = 10(n-2)$ so that ($\Diamond$) is satisfied for $X = A_{n-5}$ for $n \geqslant 11$ or a root subsystem properly containing it. Thus assuming that the six orthogonal roots α_1, α_3, α_5, α_7, α_9 and α_{11} are not in Φ_s we find that $|\Lambda| \geqslant \frac{1}{2}\left[\binom{15}{3} - 6.3 - 1\right] = 218$ or $\frac{1}{2}\left[\binom{16}{3} - 6.4 - 4\right] = 266$ according as $n = 14$ or 15. Thus ($\Diamond$) holds for all $X \in I$ when $n \in [14, 15]$. $\square$

We next consider $L(\omega_3)$ for $n \in [8, 13]$. Here we shall be forced to consider in detail the weights in $W.\omega_3$ for many different centraliser types.

Lemma 5.13. *Let G act on the irreducible module $L(\omega_3)$ for $n \in [8, 13]$. Then ($\Diamond$) holds for each $X \in I$ when $n \in [9, 13]$ and for each $X \in I \setminus \{A_2^3, A_4A_3\}$ when $n = 8$.*

Proof. If $n \in [12, 13]$ then, from the previous lemma, we can take six orthogonal roots not in Φ_s. We see that $|\Lambda| \geqslant \frac{1}{2}\left[\binom{13}{3} - 6\right] = 140$ for $n = 12$ and $|\Lambda| \geqslant \frac{1}{2}\left[\binom{14}{3} - 6.2\right] = 176$ for $n = 13$. Thus ($\Diamond$) holds provided $|\Phi(X)| > 16$ or 6 according as $n = 12$ or 13.

If $n = 10$ or 11 we may take five orthogonal roots outside Φ_s in which case $|\Lambda| \geqslant \frac{1}{2}\left[\binom{11}{3} - 5\right] = 80$ if $n = 10$ or $\frac{1}{2}\left[\binom{12}{3} - 10\right] = 105$ if $n = 11$. Thus ($\Diamond$) holds if $|\Phi(X)| > 110 - 80 = 30$ for $n = 10$ or if $|\Phi(X)| > 132 - 105 = 27$ for $n = 11$.

Similarly for $n = 9$ we may take four orthogonal roots α_1, α_3, α_5 and α_7 not in Φ_s. The module has dimension $\binom{10}{3} = 120$ and there are 8 weights orthogonal to the four roots. Thus $|\Lambda| \geqslant 56$ and ($\Diamond$) holds provided $|\Phi(X)| > 90 - 56 = 34$.

If $n = 8$ we wish to take four orthogonal roots not in Φ_s. We require ($\Diamond$) to hold for $X = A_5$ and root subsystems properly containing it. We can take three orthogonal roots, say α_1, α_3 and α_5 to show that $|\Lambda| \geqslant \frac{1}{2}\left[\binom{9}{3} - 10\right] = 37$ so ($\Diamond$) holds provided $|\Phi(X)| > 72 - 37 = 35$. Therefore we need only consider $X = A_5$ and A_5A_1, else $|\Phi(X)|$ is large enough. If $X = A_5$ then the two cliques $+++---:---$, $++----:\underline{+--}$ and $+-----:\underline{++-}$, $------:+++$ show that $d_{\omega_3}^{A_5} \geqslant 58 > 42 = e_{\omega_3}^{A_5}$. If $X = A_5A_1$ then we may again form two cliques indicated in the table below to show that $d_{\omega_3}^{A_5A_1} \geqslant 42 > 40 = e_{\omega_3}^{A_5A_1}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 8,\, A_5A_1)$					
Cliques	Cluster size	l	Cliques	Cluster size	l
$+++---\,\vert\,--:-$	20	35	$+-----\,\vert\,++:-$	6	7
$++----\,\vert\,+-:-$	30		$+-----\,\vert\,+-:+$	12	
$++----\,\vert\,--:+$	15		$------\,\vert\,++:+$	1	

Finally we may also take $\alpha_7 \notin \Phi_s$ and we see that $|\Lambda| \geqslant \frac{1}{2}(84-4) = 40$ whence ($\Diamond$) holds if $|\Phi(X)| > 72 - 40 = 32$. There are 25 centraliser types to consider.

The following table lists the centraliser types requiring consideration for $L(\omega_3)$ with $n \in [8, 13]$.

n	Centraliser types
13	$\varnothing$, A_1, A_1^2, A_1^3, A_2
12	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_1^5, A_1^6; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, $A_2A_1^4$, $A_2A_1^5$, A_2^2, $A_2^2A_1$, $A_2^2A_1^2$; A_3, A_3A_1, $A_3A_1^2$
11	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_1^5, A_1^6; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, $A_2A_1^4$, A_2^2, $A_2^2A_1$, $A_2^2A_1^2$, $A_2^2A_1^3$, A_2^3, $A_2^3A_1$, A_2^4; A_3, A_3A_1, $A_3A_1^2$, $A_3A_1^3$, $A_3A_1^4$, A_3A_2, $A_3A_2A_1$, $A_3A_2A_1^2$, $A_3A_2^2$, $A_3A_2^2A_1$, A_3^2, $A_3^2A_1$; A_4, A_4A_1, $A_4A_1^2$, $A_4A_1^3$, A_4A_2
10	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_1^5; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, $A_2A_1^4$, A_2^2, $A_2^2A_1$, $A_2^2A_1^2$, A_2^3, $A_2^3A_1$; A_3, A_3A_1, $A_3A_1^2$, $A_3A_1^3$, A_3A_2, $A_3A_2A_1$, $A_3A_2A_1^2$, $A_3A_2^2$, A_3^2, $A_3^2A_1$, $A_3^2A_2$; A_4, A_4A_1, $A_4A_1^2$, $A_4A_1^3$, A_4A_2, $A_4A_2A_1$; A_5
9	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4, A_1^5; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, A_2^2, $A_2^2A_1$, $A_2^2A_1^2$, A_2^3; A_3, A_3A_1, $A_3A_1^2$, $A_3A_1^3$, A_3A_2, $A_3A_2A_1$, $A_3A_2^2$, A_3^2, $A_3^2A_1$; A_4, A_4A_1, $A_4A_1^2$, A_4A_2, $A_4A_2A_1$, A_4A_3; A_5, A_5A_1, $A_5A_1^2$
8	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, A_2^2, $A_2^2A_1$, A_2^3; A_3, A_3A_1, $A_3A_1^2$, A_3A_2, $A_3A_2A_1$, A_3^2; A_4, A_4A_1,$A_4A_1^2$, A_4A_2, A_4A_3; A_5, A_5A_1

Table 5.2: Centraliser types requiring consideration for $L(\omega_3)$ with $n \in [8, 13]$

We shall now consider the five centraliser types remaining for $n = 13$.

Suppose $X = A_2$ with simple roots α_1 and α_2. There is a contribution of at least 30 to $|\Lambda|$ from the clique $++-:\underline{+-----------}$. There can be at most five clusters with one plus sign in the first three positions present in $\Pi(V)\backslash\Lambda$, for example $+--:++----------$, $+--:--++--------$, $+--:----++------$, $+--:-------++---$ and $+--:---------++-$. The remaining clusters of this form therefore contribute at least 150 to $|\Lambda|$ so in total we have $d_{\omega_3}^{A_2} \geqslant 180 > 176 = e_{\omega_3}^{A_2}$.

Suppose $X = A_1^3$ with simple roots α_1, α_3 and α_5. Consider cliques with one plus sign in the first six positions. Assuming that it is in one of the first two positions there can be at most four cliques of this form lying in $\Pi(V) \setminus \Lambda$, for example $+-|--|--:++------$, $+-|--|--:--++----$, $+-|--|--:----++--$ and $+-|--|--:------++$. The same is true taking the plus sign in the 3rd and 5th position so there are at least $\left[\binom{8}{2} - 4\right].2.3 = 144$ weights in $|\Lambda|$. The cliques $+-|+-|--:\underline{+-------}$, $+-|--|+-:\underline{+-------}$ and $--|+-|+-:\underline{+-------}$ contribute $7.4.3 = 84$ to $|\Lambda|$, hence $(\lozenge)$ holds since $d_{\omega_3}^{A_1^3} \geqslant 228 > 176 = e_{\omega_3}^{A_1^3}$.

Suppose $X = A_1^2$ with simple roots α_1 and α_3; we have $e_{\omega_3}^{A_1^2} = 178$. Consider the clusters with one plus sign in one of the first two positions and none in the 3rd and 4th position. There can be at most five clusters lying in $|\Lambda|$ since there are at most five distinct weights of this form which do not have two plus signs in the same position. For example we could take $+-|--:++--------$, $+-|--:--++------$, $+-|--:----++----$, $+-|--:------++--$ and $+-|--:--------++$. Thus there are at least 80 weights in $|\Lambda|$ which we can double by considering clusters with one plus sign in one of the 3rd and 4th positions and none in the 1st and 2nd positions. The clique $+-|+-:\underline{+---------}$ contributes a further 36 weights to $|\Lambda|$. Thus we have $d_{\omega_3}^{A_1^2} \geqslant 196 > 178 = e_{\omega_3}^{A_1^2}$.

Suppose $X = A_1$ with simple root α_1. There can be at most six cliques with one plus sign in the first two positions and two plus signs in the remaining positions lying in $\Pi(V) \setminus \Lambda$. There are 66 cliques in total of this form so $|\Lambda| \geqslant 60.2 = 120$. The collections of cliques $--:\underline{+--} + \underline{+--------}$, $--:\underline{+---} + \underline{+-------}$ and $--:\underline{+----} + \underline{+------}$ show that there are at least a further $3.7 + 4.6 + 5.5 = 70$ weights in $|\Lambda|$. Thus we have $d_{\omega_3}^{A_1} \geqslant 190 > 180 = e_{\omega_3}^{A_1}$.

Suppose $X = \varnothing$. The following table shows that $d_{\omega_3}^{\varnothing} \geqslant 197 > 182 = e_{\omega_3}^{\varnothing}$. We shall use the notation $\underline{+--} + \underline{+---------}$ to denote the three cliques $+-- + \underline{+---------}$, $-+- + \underline{+---------}$ and $--+ + \underline{+---------}$. The same notation is employed for the other collections of cliques.

$(\lambda,\, n,\, X) = (\omega_3,\, 13,\, \varnothing)$					
Collection of cliques	Cluster size	l	Collection of cliques	Cluster size	l
$\underline{+--} + \underline{+---------}$	1	27	$\underline{+-----} + \underline{+------}$	1	36
$\underline{+---} + \underline{+--------}$	1	32	$\underline{+------} + \underline{+-----}$	1	35
$\underline{+----} + \underline{+-------}$	1	35	$\underline{+-------} + \underline{+----}$	1	32

The remaining cases for $n \in [9, 12]$ are similar, and Appendix A contains the details for $n = 12$. If $n = 8$ we can show that $(\lozenge)$ holds for all $X \in I \setminus \{A_2^3, A_4A_3\}$; the argument for $X = \varnothing$ is particularly involved. If X is either A_2^3 or A_4A_3 then it is possible that $d_{\omega_3}^X = e_{\omega_3}^X$. In the former case the clusters are given in the table below. It is possible for the cluster of size 27 and the three clusters of size 1 to lie inside the eigenspace so that $d_{\omega_3}^{A_2^3} = 54 = e_{\omega_3}^{A_2^3}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 8,\, A_2^3)$			
Clusters	Cluster size	Clusters	Cluster size
$++ + \mid - - - \mid - - - :$	1	$+ - - \mid - - - \mid + + - :$	9
$+ + - \mid + - - \mid - - - :$	9	$- - - \mid + + + \mid - - - :$	1
$+ + - \mid - - - \mid + - - :$	9	$- - - \mid + + - \mid + - - :$	9
$+ - - \mid + + - \mid - - - :$	9	$- - - \mid + - - \mid + + - :$	9
$+ - - \mid + - - \mid + - - :$	27	$- - - \mid - - - \mid + + + :$	1

In the latter case there are just four clusters $+++--\mid----:$, $++---\mid+---:$, $+----\mid++--:$ and $-----\mid+++-:$ of size 10, 40, 30 and 4, respectively. The second and fourth cluster may lie inside the eigenspace, ruling out the other clusters by the adjacency principle, so $d_{\omega_3}^{A_4A_3} = 40 = e_{\omega_3}^{A_4A_3}$. □

We complete this section by considering both $L(\omega_1 + \omega_2)$ and $L(3\omega_1)$ for $n \in [5, \infty)$.

Lemma 5.14. *Let G act on the irreducible modules $L(\omega_1 + \omega_2)$ and $L(3\omega_1)$ $(p > 3)$ for $n \in [5, \infty)$. Then $(\lozenge)$ holds for each $X \in I$.*

Proof. The result follows for both modules for $n \in [9, \infty)$ by Premet's theorem and the previous lemmas since $\omega_3 < \omega_1 + \omega_2 < 3\omega_1$.

Throughout this lemma for convenience we shall omit a factor of $\frac{1}{n+1}$ on the coefficients of weights. From Lübeck's tables [18] we find that $m_{\omega_3} = 1$ or 2 according as $p = 3$ or $p \neq 3$.

Consider $L(\omega_1 + \omega_2)$ for $n = 8$. Recall the calculation above for $L(\omega_3)$ which shows that we can take four roots outside Φ_s. We may use this if $p \neq 3$ for $L(\omega_1 + \omega_2)$ to conclude that $|\Lambda| \geqslant 80$ so $(\lozenge)$ is satisfied. If $p = 3$ we see from Figure 5.6 that $|\Lambda| \geqslant 36$. Therefore we may assume that there are two orthogonal roots not in Φ_s, say α_1 and α_8.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 3$	$p = 3$
2	$\omega_1 + \omega_2$	$n(n+1)$	1	1
1	ω_3	$\binom{n+1}{3}$	2	1

Weight strings	No. of strings	l	
		$p \neq 3$	$p = 3$
μ_2	$(n-1)(n-2)$		
$\mu_2\ \mu_2$	n	n	n
$\mu_2\ \mu_1\ \mu_2$	$n-1$	$2(n-1)$	$n-1$
μ_1	$(n-1)(n-2)(n-3)$		
$\mu_1\ \mu_1$	$\frac{1}{2}(n-1)(n-2)$	$(n-1)(n-2)$	$\frac{1}{2}(n-1)(n-2)$
Lower bound on $\lvert\Lambda\rvert$		n^2	$\frac{1}{2}n(n+1)$

Figure 5.6: $\lambda = \omega_1 + \omega_2$ for $n \in [3, 20]$

The arrangements of weights below (with a factor of $\frac{1}{9}$ omitted on each coefficient) and the same arrangements with signs negated and coefficients reversed show that $|\Lambda| \geqslant 44$.

15 21 18 15 12 9 6 3 6 21 18 15 12 9 6 3	6 3 18 15 12 9 6 3 -3 3 18 15 12 9 6 3	6 3 9 15 12 9 6 3 -3 3 9 15 12 9 6 3	6 3 0 15 12 9 6 3 -3 3 0 15 12 9 6 3	6 3 9 6 12 9 6 3 -3 3 9 6 12 9 6 3	6 3 0 6 12 9 6 3 -3 3 0 6 12 9 6 3
-3 12 9 6 3 0 -3 3 -3 12 9 6 3 0 -3 -6	6 12 9 6 3 0 -3 3 6 12 9 6 3 0 -3 -6	15 12 9 6 3 0 -3 3 15 12 9 6 3 0 -3 -6	-3 3 9 6 3 0 -3 3 -3 3 9 6 3 0 -3 -6	6 3 9 6 3 0 -3 3 6 3 9 6 3 0 -3 -6	6 3 9 6 3 9 6 3 -3 3 9 6 3 9 6 3
6 3 0 -3 3 0 6 3 -3 3 0 -3 3 0 6 3	6 3 0 6 3 0 6 3 -3 3 0 6 3 0 6 3	6 3 0 -3 -6 9 6 3 -3 3 0 -3 -6 9 6 3	6 3 0 -3 3 9 6 3 -3 3 0 -3 3 9 6 3	6 3 0 -3 12 9 6 3 -3 3 0 -3 12 9 6 3	6 3 0 6 3 9 6 3 -3 3 0 6 3 9 6 3
6 3 9 6 3 0 6 3 -3 3 9 6 3 0 6 3	-3 -6 9 6 3 0 -3 3 -3 -6 9 6 3 0 -3 -6	-3 3 0 6 3 0 -3 3 -3 3 0 6 3 0 -3 -6	6 3 0 6 3 0 -3 3 6 3 0 6 3 0 -3 -6		

Thus ($\Diamond$) is satisfied for $X = A_5$ and larger subsystems of A_8 and we may assume that there are four orthogonal roots, which we take to be α_1, α_3, α_6 and α_8, not in Φ_s. We obtain the following weight nets with their contribution to $|\Lambda|$ given in brackets:

· 3 · 6 3 0 -3 3 (2), · 21 18 15 12 9 6 3 (1), · 12 · 15 12 9 6 3 (3), · 3 · 15 12 9 6 3 (3), · 12 9 6 3 · 6 3 (3), · 12 9 6 12 9 6 3 (1), -3 -6 · 15 12 9 6 3 (1), · 12 9 6 3 0 -3 · (3), · 3 · 6 3 · 6 3 (4), · 3 0 -3 12 9 6 3 (1), -3 -6 · 6 12 9 6 3 (1), · 3 0 -3 3 · 6 3 (2), -3 -6 -9 6 3 · 6 3 (1), · 3 0 -3 -6 · 6 3 (3), · 3 · 6 3 0 -3 · (4), -3 -6 · 6 3 · 6 3 (2), -3 -6 · -3 12 9 6 3 (1) and -3 -6 · -3 3 · 6 3 (2).

Taking these weight nets together with the nets obtained by negating signs and reversing coefficients we have $|\Lambda| \geqslant 76$, so ($\Diamond$) is satisfied for each $X \in I$.

Consider $L(\omega_1 + \omega_2)$ for $n = 7$. From Figure 5.6 and ($\Diamond$) we can assume that there are two orthogonal roots not in Φ_s; we may take these to be α_1 and α_7. We provide weight nets below with weights in $W.\omega_3$ italicised since these have multiplicity 2 when $p \neq 3$. Note that we omit a factor of $\frac{1}{8}$ on each coefficient.

1×2 nets: · 18 15 12 9 6 3, · 2 15 12 9 6 3, · 2 -1 12 9 6 3, · 2 -1 -4 9 6 3, · 2 -1 -4 -7 6 3,
· 2 7 12 9 6 3, · 2 7 4 9 6 3, · 2 7 4 1 6 3, · 2 -1 4 9 6 3, · 2 -1 -4 1 6 3, · 2 -1 4 1 6 3

2×2 nets: · 2 7 4 1 -2 ·, · 2 -1 4 1 -2 ·

1×3 nets: · 10 15 12 9 6 3, · 10 7 12 9 6 3, · 10 7 4 9 6 3, · 10 7 4 1 6 3

2×3 nets: · 10 7 4 1 -2 ·

We remark that the 1×3 and 2×3 weight nets consist of strings of the form μ_2 μ_1 μ_2. Taking these weight nets together with the corresponding ones with weights negated and coefficients reversed we have $|\Lambda| \geqslant 74, 76$ or 44 according as $p \neq 2, 3$, $p = 2$ or $p = 3$. Hence we may assume that $p = 3$. We see that ($\Diamond$) holds for $X = A_4$ and larger subsystems of A_7 so we may assume that α_1, α_3, α_5 and α_7 do not lie in Φ_s. The weight nets (up to negation and reversing coefficients) together with their contribution to the codimension of the eigenspace in brackets are as follows:

· 18 15 12 9 6 3 (1), -3 -6 · 12 9 6 3 (1), · 10 · 12 9 6 3 (3), · 10 7 4 · 6 3 (3), · 2 · 12 9 6 3 (3),

· 10 7 4 1 -2 · (3), · 2 -1 -4 · 6 3 (3), -3 -6 · 4 · 6 3 (3), · 2 · 4 · 6 3 (4) and · 2 · 4 1 -2 · (4).

Thus $|\Lambda| \geqslant 56$ and ($\Diamond$) holds unless $X = \varnothing$. If $X = \varnothing$ then we can arrange the weights into cliques to show that $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 58 > 56 = e^{\varnothing}_{\omega_1+\omega_2}$; we use the eight cliques given below together with the counterparts of the first four which are obtained by reversing the coefficients and negating.

$\boxed{\text{13 18 15 12 9 6 3, 5 18 15 12 9 6 3, . . . , 5 10 7 4 1 -2 -5}}$ $\boxed{\text{13 10 7 12 9 6 3, . . . , 13 10 7 4 1 -2 -5}}$

$\boxed{\text{5 2 15 12 9 6 3, . . . , 5 2 7 4 1 -2 -5}}$ $\boxed{\text{-3 10 15 12 9 6 3, . . . , -3 10 7 4 1 -2 -5}}$

$\boxed{\text{-3 2 7 4 9 6 3, -3 -6 7 4 9 6 3, -3 -6 -1 4 9 6 3, -3 -6 -1 -4 9 6 3, -3 -6 -1 -4 1 -2 3}}$

$\boxed{\text{-3 -6 -9 4 9 6 3, . . . , -3 -6 -9 4 1 -2 -5}}$ $\boxed{\text{5 2 -1 -4 1 6 3, . . . , -3 -6 -9 -12 -7 -2 3}}$

$\boxed{\text{-3 2 -1 -4 1 -2 -5, . . . , -3 -6 -9 -12 1 -2 -5}}$

Consider $L(\omega_1 + \omega_2)$ for $n = 6$. From Figure 5.6 and using condition ($\Diamond$), as before, we can assume that there are two orthogonal roots not in Φ_s; take these to be α_1 and α_6. We provide weight nets below with weights in $W.\omega_3$ italicised. Note that we omit a factor of $\frac{1}{7}$ on each coefficient.

1×2 nets: · 15 12 9 6 3, · 1 12 9 6 3 , · 1 -2 9 6 3, · 1 -2 -5 6 3, *· 1 5 9 6 3,* *· 1 5 2 6 3,* *· 1 -2 2 6 3*

2×2 nets: *· 1 5 2 -1 ·,* *· 1 -2 2 -1 ·*

1×3 nets: · 8 12 9 6 3 , · 8 5 9 6 3, · 8 5 2 6 3

2×3 net: · 8 5 2 -1 ·

Using these weight nets together with the corresponding ones with weights negated and coefficients reversed (although note that the net · *1 -2 2 -1* · is its own counterpart) we have $|\Lambda| \geqslant 52, 54$ or 32 according as $p \neq 2,3$, $p = 2$ or $p = 3$. Hence we may assume that $p = 3$. We see that ($\lozenge$) holds for $X = A_4$ and larger subsystems of A_6 so we may assume that α_1, α_3 and α_6 do not lie in Φ_s. The weight nets are as follows.

$2\times 1\times 1$, $1\times 2\times 1$, $1\times 1\times 2$ nets: · 15 12 9 6 3, -3 -6 · -5 6 3, -3 -6 -9 -12 -1 ·,
· 1 -2 -5 6 3, -3 -6 · 9 6 3, -3 -6 -9 -12 -15 ·

$1\times 2\times 2$, $2\times 1\times 2$, $2\times 2\times 1$ nets: -3 -6 · -5 -1 ·, · 1 -2 -5 -1 ·, · 1 · 2 6 3

$3\times 1\times 1$, $1\times 3\times 1$, $1\times 1\times 3$ nets: · 8 5 2 6 3, -3 -6 · 2 6 3, -3 -6 -9 -12 -8 ·

$1\times 2\times 3,\dots,3\times 2\times 1$ nets: -3 -6 · -5 -8 ·, -3 -6 · 2 -1 ·, · 1 -2 -5 -8 ·, · 1 · 9 6 3, · 8 5 2 -1 ·, · 8 · 9 6 3

$2\times 2\times 2$ net: · 1 · 2 -1 ·

Thus $|\Lambda| \geqslant 37$ and ($\lozenge$) holds unless $X = \varnothing$, A_1 or A_1^2.

Suppose that $X = A_1^2$ with simple roots α_1 and α_6. From the table below, by arranging clusters into cliques, we have $d_{\omega_1+\omega_2}^{A_1^2} \geqslant 39 > 38 = e_{\omega_1+\omega_2}^{A_1^2}$.

$(\lambda,\, n,\, X) = (\omega_1+\omega_2,\, 6,\, A_1^2)$					
Cliques	Cluster size	l	Cliques	Cluster size	l
· 15 12 9 6 3	2	11	· 1 -2 -5 -8 ·	6	11
· 8 12 9 6 3	3		-3 -6 -2 -5 -8 ·	3	
· 8 5 9 6 3	3		-3 -6 -9 -5 -8 ·	3	
· 8 5 2 6 3	3		-3 -6 -9 -12 -8 ·	3	
· 8 5 2 -1 ·	6		-3 -6 -9 -12 -15 ·	2	
· 1 12 9 6 3	2	6	· 1 -2 -5 -1 ·	4	6
· 1 5 9 6 3	2		-3 -6 -2 -5 -1 ·	2	
· 1 5 2 6 3	2		-3 -6 -9 -5 -1 ·	2	
· 1 5 2 -1 ·	4		-3 -6 -9 -12 -1 ·	2	
· 1 -2 9 6 3	2	4	-3 -6 -2 2 6 3	1	1
· 1 -2 2 6 3	2		-3 -6 -9 2 6 3	1	
· 1 -2 2 -1 ·	4				

If $X = A_1$ with simple root α_1 then $d_{\omega_1+\omega_2}^{A_1} \geqslant 41 > 40 = e_{\omega_1+\omega_2}^{A_1}$ from the table below.

$(\lambda,\, n,\, X) = (\omega_1+\omega_2,\, 6,\, A_1)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
· 15 12 9 6 3	2	14	· 1 12 9 6 3	2	2	· 1 5 2 -1 -4	2	2
· 8 12 9 6 3	3		· 1 5 9 6 3	2		· 1 -2 2 -1 -4	2	
· 8 5 9 6 3	3		· 1 5 2 -1 3	2	2	· 1 -2 -5 -8 3	2	3
· 8 5 2 6 3	3		· 1 -2 2 -1 3	2		-3 -6 -2 -5 -8 3	1	
· 8 5 2 -1 3	3		· 1 -2 -5 -8 -4	2	2	-3 -6 -9 -5 -8 3	1	
· 8 5 2 -1 -4	3		· 1 -2 -5 -8 -11	2		-3 -6 -9 -12 -8 3	1	
· 1 5 2 6 3	2	8	-3 -6 -2 2 6 3	1	4	-3 -6 -2 -5 -8 -4	1	4
· 1 -2 2 6 3	2		-3 -6 -9 2 6 3	1		-3 -6 -9 -5 -8 -4	1	
· 1 -2 -5 6 3	2		-3 -6 -9 -5 6 3	1		-3 -6 -9 -12 -8 -4	1	
· 1 -2 -5 -1 3	2		-3 -6 -9 -5 -1 3	1		-3 -6 -9 -12 -15 -4	1	
· 1 -2 -5 -1 -4	2		-3 -6 -9 -5 -1 -4	1		-3 -6 -9 -12 -15 -11	1	

If $X = \varnothing$ then we see that $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 43 > 42 = e^{\varnothing}_{\omega_1+\omega_2}$ from the cliques below, together with the counterparts of the first three cliques given (obtained by reversing the coefficients and negating).

11 15 12 9 6 3, 4 15 12 9 6 3, …, 4 8 5 2 -1 -4

11 8 12 9 6 3, …, 11 8 5 2 -1 -4

-3 8 12 9 6 3, …, -3 8 5 2 -1 -4

4 1 5 9 6 3, -3 1 5 9 6 3, …, -3 -6 -2 2 -1 -4

-3 -6 -9 2 6 3, …, -3 -6 -9 -5 -1 -4

4 1 -2 9 6 3, -3 1 -2 9 6 3

4 1 -2 -5 6 3, …, -3 -6 -2 -5 6 3

4 1 12 9 6 3, -3 1 12 9 6 3

4 1 5 2 6 3, …, -3 -6 5 2 6 3

Finally, consider $L(\omega_1+\omega_2)$ for $n = 5$. We repeat the same procedure as above. From Figure 5.6 and ($\Diamond$) we can take $\alpha_1, \alpha_5 \notin \Phi_s$. We provide weight nets below with weights in $W.\omega_3$ italicised. We have omitted a factor of $\frac{1}{6}$ on each coefficient.

1×2 nets: · 12 9 6 3, · 0 9 6 3 , · 0 -3 6 3, · *0 3 6 3* 2×2 net: · *0 3 0* ·

1×3 nets: · 6 9 6 3 , · 6 3 6 3 2×3 net: · 6 3 0 ·

Using these weight nets together with the corresponding ones with weights negated and coefficients reversed we have $|\Lambda| \geqslant 34, 36$ or 22 according as $p \neq 2, 3$, $p = 2$ or $p = 3$. Thus ($\Diamond$) is satisfied if $p \neq 3$ and we may assume that $p = 3$. We see that ($\Diamond$) holds for $X = A_3$ and larger subsystems of A_5 so we may assume that α_1, α_3 and α_5 do not lie in Φ_s. The weight nets are as follows.

$1\times1\times2$, $1\times2\times1$, $2\times1\times1$ nets: -3 -6 -9 -12 ·, -3 -6 · 6 3, · 12 9 6 3 $2\times2\times2$ net: · 0 · 0 ·

$1\times2\times3, \ldots, 3\times2\times1$ nets: -3 -6 · -6 ·, -3 -6 · 0 ·, · 0 -3 -6 ·, · 0 · 6 3, · 6 3 0 ·, · 6 · 6 3

Thus $|\Lambda| \geqslant 25$ and ($\Diamond$) holds unless $X = \varnothing$, A_1 or A_1^2.

Suppose that $X = A_1^2$ with simple roots α_1 and α_5. From the table below, by arranging clusters into cliques, we have $d_{\omega_1+\omega_2}^{A_1^2} \geqslant 30 > 28 = e_{\omega_1+\omega_2}^{A_1^2}$.

$(\lambda,\, n,\, X) = (\omega_1+\omega_2,\, 5,\, A_1^2)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
· 12 9 6 3	2	5	-3 -6 -3 -6 ·	3	5	· 6 3 0 ·	6	14
· 6 9 6 3	3		-3 -6 -9 -6 ·	3		· 0 3 0 ·	4	
· 6 3 6 3	3		-3 -6 -9 -12 ·	2		· 0 -3 0 ·	4	
· 0 9 6 3	2	2	-3 -6 -3 0 ·	2	2	· 0 -3 -6 ·	6	
· 0 3 6 3	2		-3 -6 -9 0 ·	2				
· 0 -3 6 3	2	1	-3 -6 3 6 3	1	1			
-3 -6 -3 6 3	1		-3 -6 3 0 ·	2				

If $X = A_1$ with simple root α_1 then $d_{\omega_1+\omega_2}^{A_1} \geqslant 29 > 28 = e_{\omega_1+\omega_2}^{A_1}$ from the table below.

$(\lambda,\, n,\, X) = (\omega_1+\omega_2,\, 5,\, A_1)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
· 12 9 6 3	2	11	· 0 -3 -6 -3	2	4	· 0 3 6 3	2	4
· 6 9 6 3	3		-3 -6 -3 -6 -3	1		-3 -6 3 6 3	1	
· 6 3 6 3	3		-3 -6 -9 -6 -3	1		-3 -6 -3 6 3	1	
· 6 3 0 3	3		-3 -6 -9 -12 -3	1		-3 -6 -3 0 3	1	
· 6 3 0 -3	3		-3 -6 -9 -12 -9	1		-3 -6 -3 0 -3	1	
· 0 -3 -6 -9	2	2	· 0 -3 6 3	2	4	· 0 9 6 3	2	4
-3 -6 -3 -6 -9	1		· 0 -3 0 3	2		· 0 3 0 3	2	
-3 -6 -9 -6 -9	1		· 0 -3 0 -3	2		· 0 3 0 -3	2	

If $X = \varnothing$ then $d_{\omega_1+\omega_2}^{\varnothing} \geqslant 32 > 30 = e_{\omega_1+\omega_2}^{\varnothing}$ from the cliques below.

9 12 9 6 3, 3 12 9 6 3, ..., 3 6 3 0 -3

3 0 -3 -6 -3, -3 0 -3 -6 -3, ..., -3 -6 -9 -12 -9

9 6 9 6 3, ..., 9 6 3 0 -3

3 0 -3 -6 -9, ..., -3 -6 -9 -6 -9

3 0 9 6 3, ..., 3 0 3 0 -3

3 0 -3 0 -3, ..., -3 -6 -9 0 -3

-3 6 9 6 3, ..., -3 0 3 0 -3

3 0 -3 0 3, ..., -3 -6 -9 -6 3

3 0 -3 6 3, ..., -3 -6 -3 6 3

The lemma follows for $L(3\omega_1)$ with $n \in [5, 8]$ by Premet's theorem. □

5.5 Groups of low rank

In this section we shall consider in turn the groups of type A_n for $n \in [1, 4]$. The calculations for each irreducible G-module are similar to those when we considered groups of higher rank, so we can withhold the full details of calculations for most of them. It will suffice to exhibit the results of the calculations in tables. As in the previous section, on the left hand side are tables with information on each module pertaining to weight orbits and multiplicities. This information is taken from [18]. On the right hand side are tables detailing the weight strings that occur and their number. From this we can calculate a lower bound for the codimension of the eigenspace.

5.5.1 Case I: $n = 4$

We begin with the module with highest weight $\omega_2 + \omega_3$; we shall give full details of the calculations performed.

Lemma 5.15. *Let G be as above acting on the irreducible module $L(\omega_2 + \omega_3)$. Then (†) holds when $p \neq 3$ and ($\Diamond$) is satisfied for all $X \in I$ when $p = 3$.*

Proof. From Lübeck's tables we see that there are three orbits on weights, with dominant weights $\omega_2 + \omega_3$, $\omega_1 + \omega_4$ and 0; we label these orbits with $i = 2, 1, 0$ respectively.

Given $\mu \in W.(\omega_2+\omega_3)$, we have $\langle \mu, \alpha \rangle = 0, 1$ and 2 for four roots α in each case. Since $|W.(\omega_2 + \omega_3)| = \frac{|W(A_4)|}{|W(A_1^2)|} = 30$, we see that given $\alpha \in \Phi(A_4)$ there are $\frac{30.4}{20} = 6$ weights $\mu \in W.(\omega_2+\omega_3)$ orthogonal to α, there are 6 weights with weight string $\mu, \mu-\alpha$ and there are 6 weights with weight string $\mu, \mu - \alpha, \mu - 2\alpha$. Given $\mu' \in W.(\omega_1 + \omega_4)$, we calculate that $\langle \mu', \alpha \rangle = 0$ and $\langle \mu', \alpha \rangle = 1$ for 6 roots α in each case, and $\langle \mu', \alpha \rangle = 2$ for 1 root α. We have $|W.(\omega_1 + \omega_4)| = \frac{|W(A_4)|}{|W(A_2)|} = 20$. Given $\alpha \in \Phi(A_4)$ there are $\frac{20.6}{20} = 6$ weights $\mu' \in W.(\omega_1+\omega_4)$ orthogonal to α, there are 6 weights $\mu' \in W.(\omega_1+\omega_4)$ with weight string $\mu_1, \mu_1-\alpha$ and there is 1 weight $\mu' \in W.(\omega_1+\omega_4)$ with weight string $\mu', \mu'-\alpha, \mu'-2\alpha$. We note that the weights in $W.(\omega_1+\omega_4)$ orthogonal to α are contained in the weight string of length three when $\langle \mu, \alpha \rangle = 2$. Taking multiplicities into account, from each weight string we find the minimal value of $|\Lambda|$ (or equivalently, the maximum value of $\Pi(\omega_2+\omega_3) \setminus |\Lambda|$). The calculations detailed are displayed in Figure 5.7.

When $p \neq 3$ we find that (†) holds. If $p = 3$ then $|\Lambda| \geqslant 19$ and we see that ($\Diamond$) is satisfied unless $X = \varnothing$. The five cliques given, together with their negative counterparts,

i	ω	$\lvert W.\omega\rvert$	m_ω		
			$p \neq 2,3$	$p=2$	$p=3$
2	$\omega_2+\omega_3$	30	1	1	1
1	$\omega_1+\omega_4$	20	2	2	1
0	0	1	5	4	1

Weight strings	No. of strings	l		
		$p \neq 2,3$	$p=2$	$p=3$
μ_2	6			
$\mu_2\ \mu_2$	6	6	6	6
$\mu_2\ \mu_1\ \mu_2$	6	12	12	6
$\mu_1\ \mu_1$	6	12	12	6
$\mu_1\ \mu_0\ \mu_1$	1	4	4	1
Lower bound on $\lvert\Lambda\rvert$		34	34	19

Figure 5.7: $\lambda = \omega_2 + \omega_3$

show that $d^{\varnothing}_{\omega_2+\omega_3} \geqslant 2.12 = 24 > 20 = e^{\varnothing}_{\omega_2+\omega_3}$.

$$\boxed{\begin{matrix}1&2&2&1\\1&1&2&1\\1&1&1&1\\1&1&1&0\end{matrix}} \quad \boxed{\begin{matrix}0&1&1&1\\0&0&1&1\\0&0&0&1\end{matrix}} \quad \boxed{\begin{matrix}1&0&1&1\\1&0&1&0\\1&0&0&0\end{matrix}} \quad \boxed{\begin{matrix}1&1&0&1\\1&0&0&1\\1&0&-1&0\end{matrix}} \quad \boxed{\begin{matrix}1&2&1&1\\1&2&1&0\\1&1&0&0\\0&1&0&0\end{matrix}}$$

□

In the next lemma we deal with irreducible modules parameterised by highest weights with level 4. We shall see that in low rank cases it is often necessary to list all of the weights that occur for a given module.

Lemma 5.16. *If G acts on $L(2\omega_1+\omega_2)$ $(p>2)$ then $(\dagger)$ holds, and if it acts on $L(4\omega_1)$ $(p>3)$, $L(\omega_1+\omega_3)$ or $L(2\omega_2)$ $(p>2)$ then $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. The calculations involving weight strings for $L(2\omega_1+\omega_2)$ and $L(4\omega_1)$ are contained in Figures 5.8 and 5.9.

i	ω	$\lvert W(\omega)\rvert$	m_ω
			$p \neq 2$
4	$2\omega_1+\omega_2$	20	1
3	$2\omega_2$	10	1
2	$\omega_1+\omega_3$	30	2
1	ω_4	5	3

Weight strings	No. of strings	l
		$p \neq 2$
μ_4	6	
$\mu_4\ \mu_4$	3	3
$\mu_4\ \mu_3\ \mu_4$	1	1
$\mu_4\ \mu_2\ \mu_2\ \mu_4$	3	9
μ_3	3	
$\mu_3\ \mu_2\ \mu_3$	3	6
μ_2	3	
$\mu_2\ \mu_2$	6	12
$\mu_2\ \mu_1\ \mu_2$	3	9
$\mu_1\ \mu_1$	1	3
Lower bound on $\lvert\Lambda\rvert$		43

Figure 5.8: $\lambda = 2\omega_1 + \omega_2$

From the Figures we see that $(\dagger)$ is satisfied for $L(2\omega_1+\omega_2)$ and $(\Diamond)$ holds for all $X \in I$ for $L(4\omega_1)$.

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2,3$
5	$4\omega_1$	5	1
4	$2\omega_1 + \omega_2$	20	1
3	$2\omega_2$	10	1
2	$\omega_1 + \omega_3$	30	1
1	ω_4	5	1

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_5	3	
$\mu_5\ \mu_4\ \mu_3\ \mu_4\ \mu_5$	1	2
μ_4	6	
$\mu_4\ \mu_4$	3	3
$\mu_4\ \mu_2\ \mu_2\ \mu_4$	3	6
μ_3	3	
$\mu_3\ \mu_2\ \mu_3$	3	3
μ_2	3	
$\mu_2\ \mu_2$	6	6
$\mu_2\ \mu_1\ \mu_2$	3	3
$\mu_1\ \mu_1$	1	1
Lower bound on $\lvert\Lambda\rvert$		24

Figure 5.9: $\lambda = 4\omega_1$

Now consider $L(\omega_1 + \omega_3)$ and $L(2\omega_2)$; the usual calculations involving weight strings are contained in Figures 5.10 and 5.11. We note that although $\omega_1 + \omega_3 < 2\omega_2$ it will not suffice to consider only $\omega_1 + \omega_3$ and use Premet's theorem. The multiplicity of each weight in $W.(\omega_4)$ for $L(\omega_1 + \omega_3)$ is higher than that for $L(2\omega_2)$.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	$\omega_1 + \omega_3$	30	1	1
1	ω_4	5	3	2

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	6		
$\mu_2\ \mu_2$	9	9	9
$\mu_2\ \mu_1\ \mu_2$	3	6	6
$\mu_1\ \mu_1$	1	3	2
Lower bound on $\lvert\Lambda\rvert$		18	17

Figure 5.10: $\lambda = \omega_1 + \omega_3$

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,3$	$p = 3$
3	$2\omega_2$	10	1	1
2	$\omega_1 + \omega_3$	30	1	1
1	ω_4	5	2	1

Weight strings	No. of strings	l	
		$p \neq 2,3$	$p = 3$
μ_3	4		
$\mu_3\ \mu_2\ \mu_3$	3	3	3
μ_2	3		
$\mu_2\ \mu_2$	9	9	9
$\mu_2\ \mu_1\ \mu_2$	3	6	3
$\mu_1\ \mu_1$	1	2	1
Lower bound on $\lvert\Lambda\rvert$		20	16

Figure 5.11: $\lambda = 2\omega_2$

Let us consider the module $L(2\omega_2)$; we first assume $p = 3$. We see that $|\Lambda| \geqslant 16$ and $(\Diamond)$ is satisfied if $|\Phi(X)| > 20 - 16 = 4$, so the centraliser types requiring consideration are $X = \varnothing$, A_1 or A_1^2. The weight nets for $X = A_1^2$ are given in Figure 5.12.

Suppose $X = A_1^2$ with simple roots α_1, α_4; we have $e_{2\omega_2}^{A_1^2} = 16$. The weight net diagram (Figure 5.12) displays the clusters of weights. The weights in italics and bold lie

	$6\ 12\ 8\ 4$			
$\bar{4}\ \bar{8}\ \overline{12}\ \bar{6}$		$\bar{4}\,\bar{8}\,\bar{7}\,\bar{6}$ $\bar{4}\,\bar{8}\,\bar{7}\,\bar{1}$		$\bar{4}\ \bar{8}\ \bar{2}\ \bar{6}$ $\bar{4}\,\bar{8}\,\bar{2}\,\bar{1}$ $\bar{4}\ \bar{8}\ \bar{2}\ 4$
	$6\,7\,8\,4$ $1\,7\,8\,4$		$6\,7\,3\,\bar{1}$ $6\,7\,3\,4$ $1\,7\,3\,\bar{1}$ $1\,7\,3\,4$	
		$1\,\bar{3}\,\bar{7}\,\bar{6}$ $1\,\bar{3}\,\bar{7}\,\bar{1}$ $\bar{4}\,\bar{3}\,\bar{7}\,\bar{6}$ $\bar{4}\,\bar{3}\,\bar{7}\,\bar{1}$	$1\,\bar{3}\,3\,\bar{1}$ $1\,\bar{3}\,3\,4$ $\bar{4}\,\bar{3}\,3\,\bar{1}$ $\bar{4}\,\bar{3}\,3\,4$	$1\,\bar{3}\,\bar{2}\,\bar{6}$ $\mathbf{1\,\bar{3}\,\bar{2}\,\bar{1}}$ $1\,\bar{3}\,\bar{2}\,4$ $\bar{4}\,\bar{3}\,\bar{2}\,\bar{6}$ $\mathbf{\bar{4}\,\bar{3}\,\bar{2}\,\bar{1}}$ $\bar{4}\,\bar{3}\,\bar{2}\,4$
	$6\ 2\ 8\ 4$ $1\,2\,8\,4$ $\bar{4}\ 2\ 8\ 4$		$6\,2\,3\,\bar{1}$ $6\,2\,3\,4$ $\mathbf{1\,2\,3\,\bar{1}}$ $\mathbf{1\,2\,3\,4}$ $\bar{4}\,2\,3\,\bar{1}$ $\bar{4}\,2\,3\,4$	$6\ 2\ \bar{2}\ \bar{6}$ $6\,2\,\bar{2}\,\bar{1}$ $6\ 2\ \bar{2}\ 4$ $1\,2\,\bar{2}\,\bar{6}$ $\mathbf{1\,2\,\bar{2}\,\bar{1}}$ $1\,2\,\bar{2}\,4$ $\bar{4}\ 2\ \bar{2}\ \bar{6}$ $\bar{4}\,2\,\bar{2}\,\bar{1}$ $\bar{4}\ 2\ \bar{2}\ 4$

Figure 5.12: Weight nets for $\lambda = 2\omega_2$ and $X = A_1^2$

in $W.(2\omega_2)$ and $W.\omega_4$ respectively, and we set $\overline{x} = -x$ for a coefficient x of a weight. Suppose we have any weight in the cluster $\cdot$ 2 -2 $\cdot$ in $\Pi(V) \setminus \Lambda$, then the clusters

$$\cdot \text{ -3 -2 } \cdot, \ \cdot \text{ 2 3 } \cdot, \ \cdot \text{ 7 3 } \cdot, \ \cdot \text{ -3 -7 } \cdot$$

lie in Λ by the adjacency principle, hence $d_{2\omega_2}^{A_1^2} \geqslant 20$. So assume that this is not the case. Suppose we have a weight from the cluster $\cdot$ 2 3 $\cdot$ in $\Pi(V) \setminus \Lambda$. Then we cannot have any weight in the clusters $\cdot$ -3 3 $\cdot$, $\cdot$ 2 8 $\cdot$, $\cdot$ 7 3 $\cdot$ in $\Pi(V) \setminus \Lambda$, so $d_{2\omega_2}^{A_1^2} \geqslant 9 + 11 = 20$. Hence, we must have the cluster $\cdot$ 2 3 $\cdot$ in Λ and by symmetry, the cluster $\cdot$ -3 -2 $\cdot$ must lie in Λ. We have now eliminated three clusters of size 9, 6 and 6, so we are done.

Suppose $X = A_1$ with simple root α_4; we have $e_{2\omega_2}^{A_1} = 18$. We can have at most one of the clusters in $\Pi(V) \setminus \Lambda$ from each of the following cliques:

1 -3 -7 ·	1 2 -2 ·	6 7 3 ·	6 2 3 ·	-4 2 3 ·	1 2 3 ·	6 7 8 4
-4 -3 -7 ·	1 -3 -2 ·	1 7 3 ·	6 2 -2 ·	-4 2 -2 ·	1 -3 3 ·	1 7 8 4
-4 -8 -7 ·	-4 -3 -2 ·				-4 -3 3 ·	1 2 8 4

If we ignore any weights in $W.(2\omega_2)$ we obtain $d_{2\omega_2}^{A_1} \geqslant 4 + 6 + 2 + 1 + 1 + 4 + 2 = 20$ and we are done.

Suppose that $X = \varnothing$; we have $e_{2\omega_2}^{\varnothing} = 20$. We can form the following seven cliques:

6 7 8 4	6 2 3 4	-4 2 3 4	**1 2 3 4**	1 2 -2 4	-4 -8 -2 -1	1 2 -2 -6
1 7 8 4	6 2 3 -1	-4 2 3 -1	**1 2 3 -1**	1 -3 -2 4	-4 -8 -7 -1	1 -3 -2 -6
1 2 8 4	6 2 -2 -1	-4 2 -2 -1	**1 2 -2 -1**	-4 -3 -2 4	-4 -8 -7 -6	-4 -3 -2 -6
			1 -3 -2 -1			
			-4 -3 -2 -1			

We can see from the weight net above that the 12 remaining weights in the orbit of $\omega_1 + \omega_3$ (those of the form $\cdot$ -3 -7 $\cdot$, $\cdot$ 7 3 $\cdot$ and $\cdot$ -3 3 $\cdot$) can be taken in pairs, with each weight in a pair differing by a root. Here at least one of the two weights in each pair of weights is in Λ implying that $d_{2\omega_2}^{\varnothing} \geqslant 22$.

When $p \neq 2, 3$, the 5 weights in the orbit $W.(\omega_4)$ have multiplicity 2 and we know that $|\Lambda| \geqslant 20$ whence the only centraliser type requiring further consideration is $X = \varnothing$.

Clearly the same calculation as above for characteristic 3 deals with this possibility.

Finally, consider the module $L(\omega_1+\omega_3)$. We find that $|\Lambda| \geqslant 18$ or 17 according as $p \neq 2$ or $p = 2$. The condition $(\Diamond)$ is satisfied unless $X = \varnothing$ or A_1 for all characteristics. We can use the exact same analysis as above for both of these possibilities since we avoided using weights from the orbit $W.(2\omega_2)$. $\square$

The next lemma deals with weights with level 3 and will complete the investigation for groups of type A_4.

Lemma 5.17. *If G acts on $L(2\omega_1+\omega_4)$ $(p > 2)$ then $(\dagger)$ holds, and if it acts on $L(\omega_1+\omega_2)$ or $L(3\omega_1)$ $(p > 3)$ then $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. We perform the usual calculations for these modules. Considering weight strings for $L(\omega_1 + \omega_2)$ we obtain Figure 5.13.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 3$	$p = 3$
2	$\omega_1 + \omega_2$	20	1	1
1	ω_3	10	2	1

Weight strings	No. of strings	l	
		$p \neq 3$	$p = 3$
μ_2	6		
$\mu_2\ \mu_2$	4	4	4
$\mu_2\ \mu_1\ \mu_2$	3	6	3
μ_1	1		
$\mu_1\ \mu_1$	3	6	3
Lower bound on $\lvert\Lambda\rvert$		16	10

Figure 5.13: $\lambda = \omega_1 + \omega_2$

When $p = 3$, $(\Diamond)$ is satisfied provided that $|\Phi(X)| > 20 - 10 = 10$ and so we need to consider the centraliser types $X = \varnothing$, A_1, A_1^2, A_2 and A_2A_1; when $p \neq 3$ only the first three of these possibilities need to be examined. Indeed, it will suffice to consider the case $p = 3$. We shall omit a factor of $\frac{1}{5}$ on each coefficient of a weight. First suppose that $X = A_2A_1$ with simple roots α_1, α_3 and α_4; there are four clusters in this case. We list these with cluster size in brackets after each of them:

$$\cdot\ 9\ 6\ 3\ (2),\ \ \cdot\ 4\ \cdot\ \cdot\ (9),\ \ \cdot\ \text{-}1\ \cdot\ \cdot\ (12)\ \text{ and }\ \text{-}3\ \text{-}6\ \cdot\ \cdot\ (7).$$

Assuming that the largest cluster $\cdot$ -1 $\cdot$ $\cdot$ is in $\Pi(V) \setminus \Lambda$ means that $d^{A_2A_1}_{\omega_1+\omega_2} \geqslant 16$ by the adjacency principle; this exceeds $e^{A_2A_1}_{\omega_1+\omega_2} = 12$. Otherwise, applying the adjacency principle to the clusters $\cdot$ 9 6 3 and $\cdot$ 4 $\cdot$ $\cdot$ we see that $d^{A_2A_1}_{\omega_1+\omega_2} \geqslant 12 + 2 = 14$, so $(\Diamond)$ is satisfied.

If $X = A_2$ with simple roots α_3 and α_4 then the clusters are

$$7\ 9\ 6\ 3\ (1),\ 2\ 9\ 6\ 3\ (1),\ 7\ 4 \cdot \cdot\ (3),\ 2\ 4 \cdot \cdot\ (3),\ \text{-}3\ 4 \cdot \cdot\ (3),$$
$$2\ \text{-}1 \cdot \cdot\ (6),\ \text{-}3\ \text{-}1 \cdot \cdot\ (6)\ \text{ and }\ \text{-}3\ \text{-}6 \cdot \cdot\ (7).$$

We can form three cliques with the clusters of size one, the clusters of sizes six and seven and two of the clusters of size three. Hence $d^{A_2}_{\omega_1+\omega_2} \geqslant 1 + 12 + 3 = 16 > 14 = e^{A_2}_{\omega_1+\omega_2}$.

If $X = A_1^2$ with simple roots α_1 and α_4 then $e^{A_1^2}_{\omega_1+\omega_2} = 16$ and there are nine clusters

$$\cdot\ 9\ 6\ 3\ (2),\ \cdot\ 4\ 6\ 3\ (3),\ \cdot\ 4\ 1 \cdot\ (6),\ \cdot\ \text{-}1\ 6\ 3\ (2),\ \cdot\ \text{-}1\ 1 \cdot\ (4),$$
$$\cdot\ \text{-}1\ \text{-}4 \cdot\ (6),\ \text{-}3\ \text{-}6\ 1 \cdot\ (2), \text{-}3\ \text{-}6\ \text{-}4 \cdot\ (3)\ \text{ and }\ \text{-}3\ \text{-}6\ \text{-}9 \cdot\ (2).$$

If the cluster $\cdot$ 4 1 $\cdot$ lies in $\Pi(V) \setminus \Lambda$ then the clusters

$$\cdot\ 9\ 6\ 3,\ \ \cdot\ 4\ 6\ 3,\ \ \cdot\ \text{-}1\ 1 \cdot\ \text{ and }\ \cdot\ \text{-}1\ \text{-}4 \cdot$$

contribute to the codimension of the eigenspace. The remaining four clusters form two cliques so we have $d^{A_1^2}_{\omega_1+\omega_2} \geqslant 19$. Thus we may assume that $\cdot$ 4 1 $\cdot$ lies in Λ and we may assume the same for $\cdot$ -1 -4 $\cdot$ and we have $d^{A_1^2}_{\omega_1+\omega_2} \geqslant 18$ by forming cliques with six of the remaining clusters, namely $\cdot$ 9 6 3 and $\cdot$ 4 6 3, $\cdot$ -1 6 3 and $\cdot$ -1 1 $\cdot$ and finally -3 -6 -4 $\cdot$ and -3 -6 -9 $\cdot$.

If $X = A_1$ with $\Phi_s = \{\pm\alpha_4\}$ then the following cliques show that $d^{A_1}_{\omega_1+\omega_2} \geqslant 19 > 18 = e^{A_1}_{\omega_1+\omega_2}$.

Clique	Cluster size	l	Clique	Cluster size	l
7 9 6 3	1	2	2 4 6 3	1	1
7 4 6 3	1		-3 4 6 3	1	
7 4 1 ·	2		2 -1 6 3	1	1
2 9 6 3	1	3	-3 -1 6 3	1	
2 4 1 ·	2		2 -1 -4 ·	3	8
-3 4 1 ·	2		-3 -1 -4 ·	3	
2 -1 1 ·	2	4	-3 -6 -4 ·	3	
-3 -1 1 ·	2		-3 -6 -9 ·	2	
-3 -6 1 ·	2				

Similarly, if $X = \varnothing$ then the following cliques show that $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 22 > 20 = e^{\varnothing}_{\omega_1+\omega_2}$.

7 4 6 3	7 9 6 3	-3 -1 1 3	2 -1 -4 3	2 -1 -4 -7	7 4 1 3	-3 4 6 3	-3 -1 6 3
2 4 6 3	2 9 6 3	-3 -1 1 -2	-3 -1 -4 3	-3 -1 -4 -7	7 4 1 -2	-3 4 1 3	-3 -6 1 3
2 -1 6 3	2 4 1 3	-3 -1 -4 -2	-3 -6 -4 3	-3 -6 -4 -7	2 -1 -4 -2	-3 4 1 -2	-3 -6 1 -2
2 -1 1 3	2 4 1 -2	-3 -6 -4 -2	-3 -6 -9 -2	-3 -6 -9 -7			
2 -1 1 -2							

Now we shall consider the module $L(3\omega_1)$; the calculations are shown in Figure 5.14.

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2,3$
3	$3\omega_1$	5	1
2	$\omega_1+\omega_2$	20	1
1	ω_3	10	1

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_3	3	
$\mu_3\ \mu_2\ \mu_2\ \mu_3$	1	2
μ_2	6	
$\mu_2\ \mu_2$	3	3
$\mu_2\ \mu_1\ \mu_2$	3	3
μ_1	1	
$\mu_1\ \mu_1$	3	3
Lower bound on $\lvert\Lambda\rvert$		11

Figure 5.14: $\lambda = 3\omega_1$

If the centraliser type satisfies $9 < |\Phi(X)|$ then we are done; otherwise the centraliser types requiring further consideration are the same as for $L(\omega_1+\omega_2)$ when $p = 3$. Since $\omega_1+\omega_2 < 3\omega_1$, the same analysis for $L(\omega_1+\omega_2)$ when $p = 3$ above deals with all the centraliser types requiring consideration.

Finally, considering $L(2\omega_1+\omega_4)$ we see in Figure 5.15 that (†) holds for all characteristics. □

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,3$	$p=3$
3	$2\omega_1+\omega_4$	20	1	1
2	$\omega_2+\omega_4$	30	1	1
1	ω_1	5	4	3

Weight strings	No. of strings	l	
		$p \neq 2,3$	$p=3$
μ_3	6		
$\mu_3\ \mu_3$	3	3	3
$\mu_3\ \mu_2\ \mu_3$	3	3	3
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	5	4
μ_2	3		
$\mu_2\ \mu_2$	9	9	9
$\mu_2\ \mu_1\ \mu_2$	3	6	6
Lower bound on $\lvert\Lambda\rvert$		26	25

Figure 5.15: $\lambda = 2\omega_1+\omega_4$

5.5.2 Case II: $n = 3$

There are six modules to consider here. We shall start with the four modules of highest dimension.

Lemma 5.18. *If G acts on $L(2\omega_1+\omega_2)$ $(p > 2)$ then (†) holds, and if it acts on $L(2\omega_1+\omega_3)$ $(p > 2)$ or $L(4\omega_1)$ $(p > 3)$ then $(\Diamond)$ is satisfied for each $X \in I$.*

Proof. It suffices to perform calculations involving weight strings. These are detailed for each module in Figures 5.16-5.18. Considering $L(2\omega_1+\omega_2)$ we see that $|\Lambda| > \dim G$ so (†)

is satisfied and considering $L(2\omega_1+\omega_3)$ and $L(4\omega_1)$ we see in each case that $|\Lambda| > |\Phi(G)|$ so that $(\diamondsuit)$ is satisfied for all centraliser types.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,5$	$p = 5$
3	$2\omega_1+\omega_3$	12	1	1
2	$\omega_2+\omega_3$	12	1	1
1	ω_1	4	3	2

Weight strings	No. of strings	l	
		$p \neq 2,5$	$p = 5$
μ_3	2		
$\mu_3\ \mu_3$	2	2	2
$\mu_3\ \mu_2\ \mu_3$	2	2	2
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	4	3
$\mu_2\ \mu_2$	3	3	3
$\mu_2\ \mu_1\ \mu_2$	2	4	4
Lower bound on $\lvert\Lambda\rvert$		15	14

Figure 5.16: $\lambda = 2\omega_1+\omega_3$

i	ω	$W.\omega$	m_ω
			$p \neq 2,3$
4	$4\omega_1$	4	1
3	$2\omega_1+\omega_2$	12	1
2	$2\omega_2$	6	1
1	$\omega_1+\omega_3$	12	1
0	0	1	1

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_4	2	
$\mu_4\ \mu_3\ \mu_2\ \mu_3\ \mu_4$	1	2
μ_3	2	
$\mu_3\ \mu_3$	2	2
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	4
μ_2	1	
$\mu_2\ \mu_1\ \mu_2$	2	2
$\mu_1\ \mu_1$	2	2
$\mu_1\ \mu_0\ \mu_1$	1	1
Lower bound on $\lvert\Lambda\rvert$		13

Figure 5.17: $\lambda = 4\omega_1$

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2$
3	$2\omega_1+\omega_2$	12	1
2	$2\omega_2$	6	1
1	$\omega_1+\omega_3$	12	2
0	0	1	3

Weight strings	No. of strings	l
		$p \neq 2$
μ_3	2	
$\mu_3\ \mu_3$	2	2
$\mu_3\ \mu_2\ \mu_3$	1	1
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	6
μ_2	1	
$\mu_2\ \mu_1\ \mu_2$	2	4
$\mu_1\ \mu_1$	2	4
$\mu_1\ \mu_0\ \mu_1$	1	3
Lower bound on $\lvert\Lambda\rvert$		20

Figure 5.18: $\lambda = 2\omega_1+\omega_2$

$\square$

Lemma 5.19. *If G acts on the irreducible module $L(3\omega_1)$ $(p > 3)$ then $(\diamondsuit)$ is satisfied for all $X \in I$.*

Proof. From Figure 5.19 we see that $|\Lambda| \geqslant 7$ for $p \neq 2,3$.

If $|\Phi(X)| > 5$ then $(\diamondsuit)$ is satisfied so $X = \varnothing$, A_1, A_1^2 are the centraliser types requiring further consideration. First consider the case $X = A_1^2$ and note that $e_{3\omega_1}^{A_1^2} = 8$. There are

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2,3$
3	$3\omega_1$	4	1
2	$\omega_1+\omega_2$	12	1
1	ω_3	4	1

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_3	2	
$\mu_3\ \mu_2\ \mu_2\ \mu_3$	1	2
μ_2	2	
$\mu_2\ \mu_2$	2	2
$\mu_2\ \mu_1\ \mu_2$	2	2
$\mu_1\ \mu_1$	1	1
Lower bound on $\lvert\Lambda\rvert$		7

Figure 5.19: $\lambda = 3\omega_1$

four clusters of the form $\cdot$ 6 3 (4), $\cdot$ 2 $\cdot$ (6), $\cdot$ -2 $\cdot$ (6) and -3 -6 $\cdot$ (4), where the number in brackets indicates the size of the cluster and we omit a factor of $\frac{1}{4}$ on each coefficient. If a cluster of size 6 is in $\Pi(V) \setminus \Lambda$ then the adjacency principle shows that $d_{3\omega_1}^{A_1^2} \geqslant 10$. So no cluster of size 6 can be present and we are done.

If $X = A_1$ with simple root α_3 then there are 10 clusters. Note that $e_{3\omega_1}^{A_1} = 10$. If the cluster of size 4 is in $\Pi(V) \setminus \Lambda$ then both clusters of size 3 must be in Λ and the other clusters can be arranged in cliques as shown below so $d_{3\omega_1}^{A_1} \geqslant 11$. Assuming that the cluster of size 4 is in Λ, the clusters of size 3 form a clique and together with the other cliques we find $d_{3\omega_1}^{A_1} \geqslant 12$.

-3 -6 3	1 -2 3	-3 -2 3	1 6 3	9 6 3	5 2 3
-3 -6 -1	1 -2 -1	-3 -2 -1			5 2 -1
-3 -6 -5	1 -2 -5	-3 -2 -5	-3 6 3	5 6 3	
-3 -6 -9					1 2 3
			3 2 3		1 2 -1
			-3 2 -1		

If $X = \varnothing$ the following cliques show that $d_{3\omega_1}^{\varnothing} \geqslant 15 > 12 = e_{3\omega_1}^{\varnothing}$.

9 6 3	1 6 3	1 2 3	1 -2 3	1 -2 -5
5 6 3	-3 6 3	1 2 -1	-3 -2 3	-3 -2 -5
5 2 3	-3 2 3	1 -2 -1	-3 -6 3	-3 -6 -5
5 2 -1	-3 2 -1	-3 -2 -1	-3 -6 -1	-3 -6 -9

□

The final two irreducible modules have dimensions very close to $\dim G$ and detailed consideration of the weights is required.

Lemma 5.20. *If G acts on the irreducible module $V = L(2\omega_2)$ $(p > 2)$ then $(\Diamond)$ is satisfied for all $X \in I \setminus \{A_1^2\}$.*

Proof. As detailed in Figure 5.20, when $p \neq 2,3$ we have $\lvert\Lambda\rvert \geqslant 8$ and when $p = 3$ we have $\lvert\Lambda\rvert \geqslant 7$. In both cases, $(\Diamond)$ holds unless the centraliser types are $X = \varnothing$, A_1 or A_1^2.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,3$	$p = 3$
2	$2\omega_2$	6	1	1
1	$\omega_1+\omega_3$	12	1	1
0	0	1	2	1

Weight strings	No. of strings	l	
		$p \neq 2,3$	$p = 3$
μ_2	2		
$\mu_2\ \mu_1\ \mu_2$	2	2	2
$\mu_1\ \mu_1$	4	4	4
$\mu_1\ \mu_0\ \mu_1$	1	2	1
Lower bound on $\lvert\Lambda\rvert$		8	7

Figure 5.20: $\lambda = 2\omega_2$

First suppose that $X = \varnothing$ in which case $e^{\varnothing}_{2\omega_2} = 12$. Suppose for any characteristic that the weight 0 is in the eigenspace. This rules out any root lying in the eigenspace, i.e., the twelve weights in $W.(\omega_1+\omega_3)$. If any of the remaining six weights are in Λ then we are done. So let us assume that the weights

1 2 1, -1 -2 -1, 1 0 1, -1 0 -1, 1 0 -1 and -1 0 1

are also in the eigenspace. However, this is not possible since $X = \varnothing$ so $\alpha_1+\alpha_2 \notin \Phi_s$. Thus the weight 0 cannot lie in the eigenspace. We arrange all of the other weights in cliques as shown below.

$$\boxed{\begin{array}{c}1\ 2\ 1\\1\ 1\ 1\\1\ 1\ 0\end{array}}\quad \boxed{\begin{array}{c}\text{-1}\ \ 0\ 1\\\text{-1}\ \text{-1}\ 0\end{array}}\quad \boxed{\begin{array}{c}1\ \ 0\ 1\\0\ \ 0\ 1\\0\ \text{-1}\ 0\end{array}}\quad \boxed{\begin{array}{c}1\ 0\ \ 0\\1\ 0\ \text{-1}\end{array}}\quad \boxed{\begin{array}{c}0\ 1\ 1\\0\ 1\ 0\\\text{-1}\ 0\ 0\end{array}}\quad \boxed{\begin{array}{c}0\ 0\ \text{-1}\\\text{-1}\ 0\ \text{-1}\end{array}}\quad \boxed{\begin{array}{c}0\ \text{-1}\ \text{-1}\\\text{-1}\ \text{-1}\ \text{-1}\\\text{-1}\ \text{-2}\ \text{-1}\end{array}}$$

When $p \neq 2,3$ this shows that $d^{\varnothing}_{2\omega_2} \geqslant 13$. When $p = 3$ we need to argue further as $d^{\varnothing}_{2\omega_2} \geqslant 12$. We shall show that all weights in the fifth clique lie in Λ, raising the codimension by one as required. If the weight 0 1 0 is in $\Pi(V) \setminus \Lambda$, then we cannot have any weight in the sixth clique in $\Pi(V) \setminus \Lambda$, so $d^{\varnothing}_{2\omega_2} \geqslant 13$. Similarly if the weight -1 0 0 is in $\Pi(V) \setminus \Lambda$, then we cannot have any weight in the second clique. If the weight 0 1 1 is in $\Pi(V) \setminus \Lambda$ then considering the first clique the weights 1 2 1 and 1 1 1 cannot lie in $\Pi(V) \setminus \Lambda$. If the weight 1 1 0 is in Λ then we are done, and if it is not then it rules out both weights in the fourth clique.

Now suppose that $X = A_1$ with simple root α_1; note that $e^{A_1}_{2\omega_2} = 10$. Arrange the weights into clusters as shown with weights of $W.(2\omega_2)$ given in italics and the zero weight given in bold.

$$\boxed{\mathit{1\ 2\ 1}}\quad \boxed{\begin{array}{c}1\ 1\ 1\\0\ 1\ 1\end{array}}\quad \boxed{\begin{array}{c}1\ 1\ 0\\0\ 1\ 0\end{array}}\quad \boxed{\begin{array}{c}\mathit{1\ 0\ 1}\\0\ 0\ 1\\\mathit{\text{-1}\ 0\ 1}\end{array}}\quad \boxed{\begin{array}{c}1\ 0\ 0\\\mathbf{0\ 0\ 0}\\\text{-1}\ 0\ 0\end{array}}\quad \boxed{\begin{array}{c}\mathit{1\ 0\ \text{-1}}\\0\ 0\ \text{-1}\\\mathit{\text{-1}\ 0\ \text{-1}}\end{array}}\quad \boxed{\begin{array}{c}0\ \text{-1}\ 0\\\text{-1}\ \text{-1}\ 0\end{array}}\quad \boxed{\begin{array}{c}0\ \text{-1}\ \text{-1}\\\text{-1}\ \text{-1}\ \text{-1}\end{array}}\quad \boxed{\mathit{\text{-1}\ \text{-2}\ \text{-1}}}$$

First suppose that the weights in the fifth cluster are in $\Pi(V) \setminus \Lambda$ for all possible characteristics. Then none of the other clusters containing weights in $W.(\omega_1 + \omega_3)$ can be in $\Pi(V) \setminus \Lambda$. Thus $d_{2\omega_2}^{A_1} \geqslant 14$ and we can assume that the fifth cluster does not lie in $\Pi(V) \setminus \Lambda$. When $p \neq 2, 3$ suppose that the weights in the fourth cluster are in $\Pi(V) \setminus \Lambda$. This means that the second and seventh clusters cannot be in $\Pi(V) \setminus \Lambda$. We can have one of the first and third clusters and one of the sixth and eighth clusters lying in $\Pi(V) \setminus \Lambda$. Thus $d_{2\omega_2}^{A_1} \geqslant 4+2+2+1+2 = 11$ as required. Using the same analysis when $p = 3$ we only obtain $d_{2\omega_2}^{A_1} \geqslant 10$. However, we cannot have both the third and sixth clusters in $\Pi(V) \setminus \Lambda$. Therefore for all p we may assume that the fourth cluster is in Λ (and by symmetry we may assume the same for the sixth cluster). The first, second and third clusters form a clique as do the seventh, eighth and ninth. Thus $d_{2\omega_2}^{A_1} \geqslant 15$ or 16 depending on whether $p = 3$ or $p \neq 2, 3$. □

Suppose in the previous Lemma that $X = A_1^2$ with simple roots α_1, α_3. There are five cliques: $\cdot$ 2 $\cdot$ and $\cdot$ -2 $\cdot$ each containing one weight, $\cdot$ 1 $\cdot$ and $\cdot$ -1 $\cdot$ with four weights each and $\cdot$ 0 $\cdot$ containing nine weights. If we can show that the weights in the largest clique are in Λ then we are done as $e_{2\omega_2}^{A_1^2} = 8$ here. If the clique $\cdot$ 0 $\cdot$ lies in $\Pi(V) \setminus \Lambda$ then both cliques of size four are in Λ. We are done unless both 1 2 1 and -1 -2 -1 lie in $\Pi(V) \setminus \Lambda$ which would imply that $2\alpha_2(s) = 1$, i.e., $\alpha_2(s) = -1$ for any $s \in S_{A_1^2}$. Thus for scalar multiples λs of the diagonal matrix $s = \mathrm{diag}(1, 1, -1, -1)$ we have $\dim V_\gamma(s) = 11$ or 12 depending on whether $p = 3$ or $p \neq 2, 3$. In this case our methods are insufficient.

Lemma 5.21. *If G acts on the irreducible module $L(\omega_1 + \omega_2)$ for $p \neq 3$ then $(\Diamond)$ holds for all $X \in I$.*

Proof. From Figure 5.21 we see that $(\Diamond)$ is satisfied provided $|\Phi(X)| > 3$.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 3$	$p = 3$
2	$\omega_1 + \omega_2$	12	1	1
1	ω_3	4	2	1

Weight strings	No. of strings	l	
		$p \neq 3$	$p = 3$
μ_2	2		
$\mu_2\ \mu_2$	3	3	3
$\mu_2\ \mu_1\ \mu_2$	2	4	2
$\mu_1\ \mu_1$	1	2	1
Lower bound on $\lvert\Lambda\rvert$		9	6

Figure 5.21: $\lambda = \omega_1 + \omega_2$

Considering the case $X = \varnothing$ we obtain the following cliques (with weights in $W.\omega_3$ written in italics and a factor of $\frac{1}{4}$ omitted).

$$\boxed{\begin{array}{c} \mathit{1\ \ 2\ \ 3} \\ \mathit{1\ \ 2\ \text{-}1} \\ \mathit{1\ \text{-}2\ \text{-}1} \\ \mathit{\text{-}3\ \text{-}2\ \text{-}1} \end{array}} \quad \boxed{\begin{array}{c} 5\ 6\ 3 \\ 1\ 6\ 3 \end{array}} \quad \boxed{\begin{array}{c} 5\ 2\ \ 3 \\ 5\ 2\ \text{-}1 \end{array}} \quad \boxed{\begin{array}{c} \text{-}3\ \ 2\ 3 \\ \text{-}3\ \text{-}2\ 3 \end{array}} \quad \boxed{\begin{array}{c} \text{-}3\ \ 2\ \text{-}1 \\ \text{-}3\ \text{-}2\ \text{-}5 \end{array}} \quad \boxed{\begin{array}{c} 1\ \text{-}2\ \text{-}5 \\ \text{-}3\ \text{-}6\ \text{-}5 \end{array}} \quad \boxed{\begin{array}{c} 1\ \text{-}2\ \ 3 \\ \text{-}3\ \text{-}6\ \text{-}1 \end{array}}$$

Thus $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 12 = e^{\varnothing}_{\omega_1+\omega_2}$. Suppose that *123* is in $\Pi(V) \setminus \Lambda$. This excludes any weight in the second and fourth clique so $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 14$. Repeating for the other weights in $W.\omega_3$ we reach the same conclusion and if we assume that all weights in the first clique are in Λ we are done.

When $X = A_1$ with simple root α_4 we arrange the weights into clusters. There are three clusters containing weights in $W.\omega_3$. Assuming that each of these in turn lies in $\Pi(V) \setminus \Lambda$ we find that $d^{A_1}_{\omega_1+\omega_2} \geqslant 12 > 10 = e^{A_1}_{\omega_1+\omega_2}$ as required. If none of these clusters lie in $\Pi(V) \setminus \Lambda$ we reach the same conclusion. $\square$

We note that $L(\omega_1 + \omega_2)$ for $n = p = 3$ is a small module since it has dimension 16.

5.5.3 Case III: $n = 2$

There are five irreducible modules in this subsection. The main difficulty occurs when the dimension of the module being considered is close to that of G.

Lemma 5.22. *If G acts on the irreducible modules $L(2\omega_1+2\omega_2)$ $(p > 2)$ and $L(3\omega_1+\omega_2)$ $(p > 3)$ then, for both modules, $(\dagger)$ holds for $p \neq 5$ and $(\Diamond)$ is satisfied for each $X \in I$ otherwise.*

Proof. For these modules it suffices just to perform the initial calculations (as shown in the Figures 5.22 and 5.23) to show that $(\dagger)$ holds, or that $(\Diamond)$ holds for all $X \in I$. $\square$

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,5$	$p = 5$
4	$2\omega_1 + 2\omega_2$	6	1	1
3	$3\omega_1$	3	1	1
2	$3\omega_2$	3	1	1
1	$\omega_1 + \omega_2$	6	2	1
0	0	1	3	1

Weight strings	No. of strings	l	
		$p \neq 2,5$	$p = 5$
$\mu_4\ \mu_3\ \mu_4$	1	1	1
$\mu_4\ \mu_2\ \mu_4$	1	1	1
$\mu_4\ \mu_1\ \mu_0\ \mu_1\ \mu_4$	1	4	2
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	3	2
$\mu_2\ \mu_1\ \mu_1\ \mu_2$	1	3	2
Lower bound on $\lvert\Lambda\rvert$		12	8

Figure 5.22: $\lambda = 2\omega_1 + 2\omega_2$

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 3,5$	$p = 5$
4	$3\omega_1 + \omega_2$	6	1	1
3	$\omega_1 + 2\omega_2$	6	1	1
2	$2\omega_1$	3	2	1
1	ω_2	3	2	1

Weight strings	No. of strings	l	
		$p \neq 3,5$	$p = 5$
$\mu_4\ \mu_4$	1	1	1
$\mu_4\ \mu_3\ \mu_3\ \mu_4$	1	2	2
$\mu_4\ \mu_2\ \mu_1\ \mu_2\ \mu_4$	1	4	2
$\mu_3\ \mu_2\ \mu_3$	1	2	1
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	3	2
Lower bound on $\lvert\Lambda\rvert$		12	8

Figure 5.23: $\lambda = 3\omega_1 + \omega_2$

Lemma 5.23. *If G acts on the irreducible modules $L(2\omega_1 + \omega_2)$ $(p > 2)$ and $L(4\omega_1)$ $(p > 3)$ then in both cases $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. From Figures 5.24 and 5.25 we see for both modules that we only need to deal with the possibility that the centraliser has type $X = \varnothing$. This can be done easily. $\square$

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p \neq 2$
3	$2\omega_1 + \omega_2$	6	1
2	$2\omega_2$	3	1
1	ω_1	3	2

Weight strings	No. of strings	l
		$p \neq 2$
$\mu_3\ \mu_3$	1	1
$\mu_3\ \mu_2\ \mu_3$	1	1
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	2
$\mu_2\ \mu_1\ \mu_2$	1	1
Lower bound on $\lvert\Lambda\rvert$		5

Figure 5.24: $\lambda = 2\omega_1 + \omega_2$

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p \neq 2,3$
4	$4\omega_1$	3	1
3	$2\omega_1 + \omega_2$	6	1
2	$2\omega_2$	3	1
1	ω_1	3	1

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_4	1	
$\mu_4\ \mu_3\ \mu_2\ \mu_3\ \mu_4$	1	2
$\mu_3\ \mu_3$	1	1
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	1	2
$\mu_2\ \mu_1\ \mu_2$	1	1
Lower bound on $\lvert\Lambda\rvert$		6

Figure 5.25: $\lambda = 4\omega_1$

We are unable to draw the same conclusion about the action of $\mathrm{SL}_3(K)$ on $L(3\omega_1)$. Here the dimension of the module may be the same as the dimension of the closure of the set of vectors which are eigenvectors for some non-central semisimple element of G and some eigenvalue.

5.5.4 Case IV: $n = 1$

It is relatively straightforward in this case to describe the weights which occur; for details see [13, §7.2]. The $\mathrm{SL}_2(K)$-module $L(m\omega_1)$ with $m \in [0, p-1]$ has dimension $m + 1$

and weights $m\omega_1, (m-2)\omega_1, \ldots, -m\omega_1$. We see from Lemma 5.5 that we need only be concerned with modules having dimensions 6 and 7.

Lemma 5.24. *If $G = \mathrm{SL}_2(K)$ acts on the irreducible modules $L(r\omega_1)$ for $r \in [5,6]$ with $p > 5$ then $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. Consider the module $L(6\omega_1)$. The weights are $6\omega_1$, $4\omega_1$, $2\omega_1$, 0, $-2\omega_1$, $-4\omega_1$ and $-6\omega_1$. By the adjacency principle we have, for any $\gamma \in K^*$ and $s \in G_{ss} \setminus Z$, that $|\Lambda| \geqslant 3$ and so $(\Diamond)$ is satisfied for all $X \in I$. The same is true for the module $L(5\omega_1)$. $\square$

Chapter 6

Groups of type B_n

In this chapter we shall assume that G is a simple simply connected algebraic group of type B_n with $n \geqslant 2$ defined over an algebraically closed field K and $V = L(\lambda)$ is an irreducible G-module with p-restricted highest weight λ. It will be sensible to treat the cases $p \neq 2$ and $p = 2$ separately since the conclusion of Premet's theorem does not hold in the latter case.

We shall prove the following result.

Theorem 6.1. *Let $G = \mathrm{Spin}_{2n+1}(K)$ act on $V = L(\lambda)$. If $\dim V \leqslant \dim G + 2$ then $\dim \overline{E} = \dim V$ with the possible exceptions of $L(\omega_1 + \omega_2)$ for $n = 2$ with $p = 5$ and $L(\omega_n)$ for $n \in [5, 6]$; if instead $\dim V > \dim G + 2$ then $\dim \overline{E} < \dim V$ with the possible exception of $L(2\omega_1)$ for $n \geqslant 2$ with $p \neq 2$.*

This theorem is a consequence of the results to follow in later sections.

6.1 Initial survey

Assume that $p \neq 2$. We shall deal with the case $p = 2$ later in Section 6.2. Consider $\mu \leqslant \lambda$ where $\mu = \sum_{i=1}^{n} a_i \omega_i$ is a dominant weight. We shall begin by obtaining conditions on the coefficients a_i in order to show that (†) is satisfied for n large enough. This will allow us later to list modules which will require further consideration. Recall that there is a short root outside the root system of the centraliser of a non-central semisimple element and in calculations to follow we shall be considering short roots only: see Section 4.3.

It is useful to note that for $1 < k \neq n$ we have $\omega_{k-1} < \omega_k$; this is clear since each fundamental weight is of the form $\omega_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i\,(\alpha_i + \cdots + \alpha_n)$ for $i \in [1, n-1]$. Also we have $\omega_{n-1} < 2\omega_n$ since $\omega_n = \frac{1}{2}(\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n)$. In this section and the next, to avoid any confusion, we shall occasionally use the notation Λ_λ rather than just Λ to emphasise that we are considering the module $L(\lambda)$.

Proposition 6.2. *Suppose that $\mu = \sum_{i=1}^{n} a_i\omega_i \leqslant \lambda$ is a dominant weight, and at least one of the following conditions holds:*

(i) *$n \geqslant 6$ and $a_k \neq 0$ for some $k \in [4, n-1]$;*

(ii) *$n \geqslant 9$ and $a_n \neq 0$;*

(iii) *$n \geqslant 5$ and $a_i, a_n \neq 0$ for some $i \in [1, 3]$;*

(iv) *$n = 5$ and $a_i, a_4 \neq 0$ for some $i \in [1, 3]$;*

(v) *$n \geqslant 5$ and $a_i, a_3 \neq 0$ for some $i \in [1, 2]$.*

Then (†) holds.

Proof. We shall use Proposition 4.2 repeatedly. Suppose $a_k \neq 0$ for some $k \in [4, n-1]$. Then $\Psi = \langle \alpha_i \mid a_i = 0 \rangle$ is contained in $\Phi(A_{k-1}B_{n-k})$ and

$$r_\Psi \geqslant r_{A_{k-1}B_{n-k}} = \frac{1}{2}\,\frac{2^n n!}{k!\;2^{n-k}(n-k)!}\,\frac{2n - 2(n-k)}{2n} = 2^{k-1}\binom{n}{k}\frac{k}{n}.$$

We have $r_{A_3B_{n-4}} = \frac{4}{3}(n-1)(n-2)(n-3) > n(2n+1) = \dim G$ for $n \geqslant 6$, hence (†) holds. Since $r_{A_{k-1}B_{n-k}} \geqslant r_{A_3B_{n-4}}$ for $k \in [4, n-1]$, we see that condition (i) ensures that (†) holds.

Similarly for condition (ii). If $a_n \neq 0$ then $r_\Psi \geqslant r_{A_{n-1}} = 2^{n-1} > \dim G$ for $n \geqslant 9$.

Suppose that both $a_1 \neq 0$ and $a_n \neq 0$. Then $r_\Psi \geqslant r_{A_{n-2}} = 2^{n-1}n$ which exceeds $\dim G$ for $n \geqslant 5$. It is clear that $r_{A_2A_{n-4}} > r_{A_1A_{n-3}} > r_{A_{n-2}}$ for $n \geqslant 5$ so we may not have both $a_i \neq 0$ for $i \in [1, 3]$ and $a_n \neq 0$. Thus condition (iii) ensures that (†) holds.

For condition (iv) we need to calculate $r_{A_2B_1} = 128$ and $r_{A_1^2B_1} = 192$ for $n = 5$; these both exceed $\dim B_5(K) = 55$.

Suppose that both $a_2 \neq 0$ and $a_3 \neq 0$. Then (†) holds for $n \geqslant 5$ since then we have $r_{A_1B_{n-3}} = 6(n-1)(n-2) > \dim G$. The same holds if both $a_1 \neq 0$ and $a_3 \neq 0$. $\square$

The modules of dimension at most $\dim G + 2$ are $L(\omega_1)$ and $L(\omega_n)$ for $n \in [2, 6]$, the adjoint modules $L(\omega_2)$ for $n \geqslant 3$ and $L(2\omega_2)$ for $n = 2 \neq p$, and $L(\omega_1 + \omega_2)$ for $n = 2$ with $p = 5$. The only modules with dimensions strictly between $\dim B = n(n+1)$ and $\dim G + 2$ are $L(\omega_n)$ for $n \in [5, 6]$ and $L(\omega_1 + \omega_2)$ for $n = 2$ with $p = 5$.

Lemma 6.3. *Suppose that* $\dim V > \dim G + 2$. *If* $n \geqslant 2$ *then* (†) *holds except possibly for the modules* $L(2\omega_1)$ $(p > 2)$, $L(\omega_1 + \omega_2)$ *and* $L(3\omega_1)$ $(p > 3)$ *for* $n \in [2, \infty)$, $L(\omega_3)$ *for* $n \in [4, \infty)$, $L(\omega_n)$ *for* $n \in [7, 8]$, $L(2\omega_n)$ $(p > 2)$ *for* $n \in [3, 4]$, $L(\omega_1 + \omega_3)$ *for* $n = 3$, *and* $L(\omega_1 + 2\omega_2)$ $(p > 2)$, $L(3\omega_2)$ $(p > 3)$ *and* $L(4\omega_2)$ $(p > 3)$ *for* $n = 2$.

Proof. We shall split the following analysis of weights into five parts according as $n \geqslant 6$ or $n = 5, 4, 3$ or 2.

Case I: $n \geqslant 6$. By conditions (i), (ii) and (iii) of Proposition 6.2 we may assume that $\mu = \sum_{i=1}^{3} a_i\omega_i$ if $n \geqslant 6$ or $\mu = a_n\omega_n$ if $n \in [6, 8]$ since otherwise (†) holds. Moreover, in the latter case, we may assume that $a_n = 1$ because $2\omega_n > \omega_{n-1}$ and (†) holds for $L(\omega_{n-1})$. The spin module $L(\omega_6)$ for $n = 6$ has dimension 64; this is less than $\dim B_6(K) = 78$ so we need only to consider further the spin modules $L(\omega_n)$ for $n \in [7, 8]$. So we may assume that $\mu = \sum_{i=1}^{3} a_i\omega_i$ for $n \geqslant 6$. Set $m = \sum_{i=1}^{3} ia_i$. We follow a similar argument to Lemma 5.3. If $m \geqslant 4$ then we claim that we can find b_i with $0 \leqslant b_i \leqslant a_i$ for each $i \in [1, 3]$ such that

$$\mu' = \mu - \left(\sum_{i=1}^{3} b_i\omega_i - \omega_j \right) < \mu \leqslant \lambda,$$

where $j = \sum_{i=1}^{3} ib_i$ and $4 \leqslant j \leqslant 6$. Now if $m \geqslant 4$ we need only consider the following seven weights: $4\omega_1$, $2\omega_1 + \omega_2$, $\omega_1 + \omega_3$, $2\omega_2$ $(m = 4)$; $\omega_1 + 2\omega_2$, $\omega_2 + \omega_3$ $(m = 5)$; and $2\omega_3$ $(m = 6)$. This is because all other weights have coefficients that are at least that of one of these seven weights, hence if we can find b_i for each of the seven, we can do the same for any other. In fact, if $n > 6$, since the seven weights μ with $m \in [4, 6]$ each satisfy $\mu > \omega_m$ we can take $a_i = b_i$ and $j = m$, whence the claim follows. If $n = 6$, the claim follows as before but if $m = 6$ we must use the partial ordering $2\omega_3 > 2\omega_6 > \omega_5$.

Let us take for example the weight $2\omega_1 + 3\omega_3$. If $n > 6$ then, since $2\omega_3 > \omega_6$, we have $2\omega_1 + 3\omega_3 > 2\omega_1 + \omega_3 + \omega_6$; here $j = 6$, $b_1 = a_1 = 2$, $b_2 = a_2 = 0$ and $b_3 = 1 < 3 = a_3$. If $n = 6$ then $2\omega_1 + 3\omega_3 > 2\omega_1 + \omega_3 + \omega_5$; the only difference here is that we take $j = 5$.

We may assume that $m \leqslant 3$ since otherwise the claim above shows that we can find a weight $\mu' < \mu$ such that $a'_j \neq 0$ and $j \in [4, 6]$ for $n > 6$ or $j \in [4, 5]$ for $n = 6$; hence, by Premet's theorem and condition (i) of Proposition 6.2, (†) holds. Thus for $n \geqslant 6$ we need to consider further the irreducible modules with highest weights $2\omega_1$, ω_3, $\omega_1 + \omega_2$ and $3\omega_1$.

Case II: $n = 5$. If $a_4 \neq 0$ then (†) holds since $\omega_4 > \omega_3$ and, applying Premet's theorem, we have $|\Lambda_{\omega_4}| \geqslant r_{A_3B_1} + r_{A_2B_2} = 32 + 24 = 56$. Similarly, if $a_5 \neq 0$ then (†) holds since $2\omega_5 > \omega_4$ and the dimension of the spin module $L(\omega_5)$ is smaller than $\dim G$. Thus we may assume that $\mu = \sum_{i=1}^{3} a_i\omega_i$. Suppose that at least two a_i for $i \in [1, 3]$ are non-zero. By condition (v) of Proposition 6.2 we may assume that $\mu = a_1\omega_1 + a_2\omega_2$ since (†) holds otherwise. Since $2\omega_1 + \omega_2 > \omega_1 + \omega_3$ and $\omega_1 + 2\omega_2 > \omega_2 + \omega_3$ and (†) holds for both $\omega_1 + \omega_3$ and $\omega_2 + \omega_3$ (as we previously noted), we need only investigate further the case $a_1 = a_2 = 1$.

Assume that only one a_i is non-zero for $i \in [1, 3]$. If $a_1 \neq 0$ we may assume that $a_1 \leqslant 3$ since $4\omega_1 > 2\omega_1 + \omega_2$. The condition (†) holds if $a_2 \neq 0$ since $2\omega_2 > \omega_1 + \omega_3$ and $L(\omega_2)$ is the adjoint module. Similarly if $a_3 \neq 0$ we may assume that $a_3 = 1$ since $2\omega_3 > \omega_2 + \omega_3$. Thus we must consider further the irreducible modules with highest weights $\omega_1 + \omega_2$, $2\omega_1$, $3\omega_1$ and ω_3.

Case III: $n = 4$. Assume that at least three coefficients of μ are non-zero. Since the smallest r_Ψ in this case occurs for $\Psi = B_1$ and $r_{B_1} = 72$, we see that (†) holds. Suppose that precisely two a_i for $i \in [1, 4]$ are non-zero. If both $a_1 \neq 0$ and $a_3 \neq 0$ then (†) holds since $\omega_1 + \omega_3 > \omega_1 + \omega_2$ and $r_{A_1B_1} + r_{B_2} = 36 + 12 = 48$. The same holds true if both $a_1 \neq 0$ and $a_4 \neq 0$ since $\omega_1 + \omega_4 > \omega_4$ and $r_{A_2} + r_{A_3} = 32 + 8 = 40$, and hence if both $a_2 \neq 0$ and $a_3 \neq 0$ since $\omega_2 + \omega_3 > \omega_1 + 2\omega_4$. Since $r_{A_1^2} = 48$ we cannot have both $a_2 \neq 0$

and $a_4 \neq 0$ and hence we cannot have both $a_3 \neq 0$ and $a_4 \neq 0$ since $\omega_3 + \omega_4 > \omega_2 + \omega_4$. Finally, if both $a_1 \neq 0$ and $a_2 \neq 0$ then (†) holds if $a_1 + a_2 > 2$ since $2\omega_1 + \omega_2 > \omega_1 + \omega_3$ and $\omega_1 + 2\omega_2 > \omega_2 + \omega_3$.

Assume now that only one coefficient of μ is non-zero. If $a_1 \neq 0$ then we may assume that $a_1 \leqslant 3$ since $4\omega_1 > 2\omega_1 + \omega_2$. If $a_2 \neq 0$ then (†) holds since $2\omega_2 > \omega_1 + \omega_3$ and $L(\omega_2)$ is the adjoint module which we do not consider by assumption. If $a_3 \neq 0$ we may assume that $a_3 = 1$ since $2\omega_3 > \omega_1 + \omega_3$. Similarly, if $a_4 \neq 0$ we may assume that $a_4 = 2$ since $3\omega_4 > \omega_3 + \omega_4$ and the spin module $L(\omega_4)$ only has dimension 16. Thus further investigation is required for the irreducible modules with highest weights $2\omega_1$, ω_3, $\omega_1+\omega_2$, $3\omega_1$ and $2\omega_4$.

Case IV: $n = 3$. Since $r_{\varnothing} = 24$ we can assume that at most two a_i for $i \in [1, 3]$ are non-zero. By performing quick calculations we have $r_{A_1} = 12$, $r_{B_1} = 8$ and $r_{A_1B_1} = r_{A_2} = 4$. Consider the case that precisely two coefficients of μ are non-zero. We see that (†) holds if both a_2 and a_3 are non-zero since $\omega_2 + \omega_3 > \omega_1 + \omega_3$. If a_1 and a_2 are non-zero then the partial ordering $\omega_1 + 2\omega_2 > 2\omega_1 + \omega_2 > 2\omega_2 > \omega_1 + 2\omega_3$ shows that (†) holds unless $a_1 = a_2 = 1$. Similarly, if a_1 and a_3 are non-zero then the partial orderings $\omega_1 + 2\omega_3 > \omega_1 + \omega_2 > 2\omega_3$ and $2\omega_1 + \omega_3 > \omega_2 + \omega_3$ together show that (†) holds unless $a_1 = a_3 = 1$. Now assume that only one coefficient of μ is non-zero. If $a_1 \neq 0$ then we may assume that $a_1 \leqslant 3$ since $4\omega_1 > 2\omega_1 + \omega_2$ and we saw previously that (†) holds for $2\omega_1+\omega_2$. Similarly, if $a_2 \neq 0$ then (†) holds since $2\omega_2 > \omega_1+2\omega_3$ and $L(\omega_2)$ is the adjoint module, and if $a_3 \neq 0$ then we may assume that $a_3 = 2$ since $3\omega_3 > \omega_2 + \omega_3$ and the spin module $L(\omega_3)$ has dimension less than G. Thus we need to consider further the weights $\omega_1 + \omega_2$, $\omega_1 + \omega_3$, $2\omega_1$, $3\omega_1$ and $2\omega_3$.

Case V: $n = 2$. We calculate $r_{\varnothing} = 4$, $r_{A_1} = 2$ and $r_{B_1} = 1$. Assume that both a_1 and a_2 are non-zero. Since $2\omega_1 + \omega_2 > 3\omega_2 > \omega_1 + \omega_2 > \omega_2$ and $\omega_1 + 3\omega_2 > 2\omega_1 + \omega_2 > 3\omega_2 > \omega_1 + \omega_2 > \omega_2$, we need only consider further the weights with $a_1 = 1$ and $a_2 \leqslant 2$. Assume that $a_1 \neq 0$ and $a_2 = 0$. We may take $a_1 \leqslant 3$ since $4\omega_1 > 2\omega_1 + 2\omega_2$; this follows from the partial ordering for $2\omega_1 + \omega_2$ just given together with Premet's theorem. Assume now that $a_1 = 0$ and $a_2 \neq 0$. We may take $a_2 \leqslant 4$ since $5\omega_2 > \omega_1 + 3\omega_2$. Thus we need to consider further the weights $\omega_1+\omega_2$, $\omega_1+2\omega_2$, $2\omega_1$, $3\omega_1$, $3\omega_2$ and $4\omega_2$ for $n=2$. $\square$

6.2 Even characteristic

In this section we deal with the case $p=2$.

Lemma 6.4. *Let $G = \mathrm{Spin}_{2n+1}(K)$ act on $V = L(\lambda)$ for $p=2$ and $\dim V > \dim G + 2$. Then (†) holds with the possible exceptions of $L(\omega_3)$ for $n \in [4,\infty)$, $L(\omega_1+\omega_2)$ for $n \geqslant 2$, $L(\omega_n)$ for $n \in [7,8]$, $L(\omega_4)$ for $n=5$ and $L(\omega_1+\omega_3)$ for $n=3$.*

Proof. We write the 2-restricted highest weight $\lambda = \sum_{i=1}^{n} a_i\omega_i$ with $a_i \in \{0,1\}$ for all $i \in [1,n]$. We shall treat ω_n differently from the other fundamental weights since $\omega_i = \alpha_1+2\alpha_2+\cdots+(i-1)\alpha_{i-1}+i\,(\alpha_i+\cdots+\alpha_n)$ for $i<n$ and $\omega_n = \frac{1}{2}\,(\alpha_1+2\alpha_2+\cdots+n\alpha_n)$. We shall assume in turn that the number of non-zero a_i is one, then two, and then three or more.

Suppose that only one of the a_i is non-zero; we may assume that $i \geqslant 3$ as otherwise the module has dimension at most $\dim G$. We have $|W.\omega_i| = 2^i\binom{n}{i}$.

The weights of the module $L(\omega_n)$ are precisely the 2^n weights in $W.\omega_n$. Since this is true for all p we may refer back to the consideration of this module in the previous section where we see that we need to further examine $L(\omega_n)$ for $n \in [7,8]$.

Henceforth we may assume that $i<n$. We see that $\langle\omega_i,\alpha\rangle = 0$ or ± 2 for short roots $\alpha \in \Phi$. Moreover, $\langle\omega_i,\alpha\rangle = 2$ for i such α. Therefore, for a fixed short root α there are σ_i weights $\mu \in W.\omega_i$ with weight string containing the weights μ $\mu+2\alpha$ (but possibly not $\mu+\alpha$), where $\sigma_i = 2^{i-1}\binom{n-1}{i-1}$. Now if $4 \leqslant i \leqslant n-1$ then, since $\sigma_i \geqslant \sigma_4 = 2^3\binom{n-1}{3} > n(2n+1) = \dim G$ for $n \geqslant 6$, it remains for us to consider $L(\omega_4)$ for $n=5$ as well as $L(\omega_3)$ for $n \geqslant 4$.

Next suppose that precisely two coefficients a_i of λ are non-zero. Consider the module $L(\omega_i+\omega_j)$ where
$$|W.(\omega_i+\omega_j)| = \frac{2^j n!}{i!(j-i)!(n-j)!}.$$
Assume that $1 \leqslant i < j \leqslant n-1$ so that $\langle\omega_i+\omega_j,\alpha\rangle = 0, \pm 2$ or ± 4 for short roots $\alpha \in \Phi$.

Moreover, $\langle \omega_i + \omega_j, \alpha \rangle = 2$ for $j - i$ short roots and $\langle \omega_i + \omega_j, \alpha \rangle = 4$ for i short roots. Thus, for a given α there are τ_{ij} weights $\mu \in W.(\omega_i + \omega_j)$ with weight string containing the weights μ $\mu + 2\alpha$ or μ $\mu + 4\alpha$ where

$$\tau_{ij} = \frac{|W.(\omega_i + \omega_j)|(j - i + i)}{2n} = \frac{2^{j-1} j(n-1)\cdots(n-j+1)}{i!(j-i)!}.$$

As usual we take $\alpha \notin \Phi_s$ and subsequently apply the adjacency principle. We may do this since $p = 2$ whence $2^k \alpha(s) = 1$ if and only if $\alpha(s) = 1$.

Assume that $(i, j) \neq (1, 2)$. It is clear that $\tau_{ij} \geqslant \tau_{13} = \tau_{23} = 6(n-1)(n-2)$. Since this exceeds $\dim G$ for $n \geqslant 5$, the modules $L(\omega_1 + \omega_3)$ and $L(\omega_2 + \omega_3)$ for $n = 4$ as well as $L(\omega_1 + \omega_2)$ for $n \geqslant 3$ require further consideration.

Now take $j = n$ in which case $\langle \omega_i + \omega_n, \alpha \rangle = \pm 1$ or ± 3. We concentrate on the former case since then we may apply the adjacency principle. There are $n - i$ short roots α such that $\langle \omega_i + \omega_n, \alpha \rangle = 1$. Thus, for a given short root α there are θ_i weights $\mu \in W.(\omega_i + \omega_n)$ with weight string containing μ $\mu + \alpha$ where $\theta_i = 2^{n-1}\binom{n-1}{i}$. Now $\theta_i \geqslant \theta_1 = 2^{n-1}(n-1)$ for $i \in [1, n-2]$ which exceeds $\dim G$ for $n \geqslant 5$. If $i = n - 1$ we conclude for $n \geqslant 5$ by applying Proposition 4.3. Therefore it remains to consider $\omega_1 + \omega_n$ for $n \in [2, 4]$. Both $\theta_2 = 2^{n-2}(n-1)(n-2)$ and $\theta_3 = \frac{2^{n-2}}{3}(n-1)(n-2)(n-3)$ exceed $\dim G$ for $n \geqslant 5$ so we must later deal with $L(\omega_2 + \omega_n)$ for $n \in [3, 4]$ and $L(\omega_3 + \omega_4)$ for $n = 4$.

We can deal with the modules $L(\omega_i + \omega_j)$ for $n = 4$ with $(i, j) \neq (1, 2)$ and $L(\omega_2 + \omega_3)$ for $n = 3$ by referring to Lübeck's tables [18]. First assume that $n = 4$. Consider $L(\omega_1 + \omega_4)$ where we see that there are no dominant weights below $\omega_1 + \omega_4$ in the partial ordering appearing with multiplicity zero, i.e., the conclusion of Premet's theorem holds. Since $\omega_1 + \omega_4 > \omega_4$ and we calculate $r_{A_2} = 32$ and $r_{A_3} = 8$ we conclude that (†) holds for this module. Indeed, by the same reasoning the same conclusion holds for the modules $L(\omega_3 + \omega_4)$ and $L(\omega_2 + \omega_4)$; we only need to calculate that $r_{A_1^2} = 48$. We see from the tables that the conclusion of Premet's theorem holds for neither $L(\omega_1 + \omega_3)$ nor $L(\omega_2 + \omega_3)$. We see that $\omega_1 + \omega_3 > 2\omega_4$ and $m_{2\omega_4} \neq 0$ for $L(\omega_1 + \omega_3)$ from the tables and we calculate $|W.(\omega_1 + \omega_3)| = 96$ and $|W.(2\omega_4)| = 16$. There are three short roots α for which $\langle \omega_1+\omega_3, \alpha\rangle = 2$ or 4 and we have $\langle 2\omega_4, \alpha\rangle = 2$ for all the positive short roots. Thus, for a given α, there are $\frac{96.3}{8} = 36$ weights $\mu \in W.(\omega_1+\omega_3)$ with weight string containing $\mu\ \mu+2\alpha$ or $\mu\ \mu+4\alpha$. Similarly there are $\frac{16.4}{8} = 8$ weights $\nu \in W.(2\omega_4)$ with weight string containing $\nu\ \nu+2\alpha$. Thus $|\Lambda| \geqslant 44$ whence (†) holds. In an exactly analogous way we can handle $L(\omega_2+\omega_3)$ by using the Weyl group orbit of $\omega_1+2\omega_4$ which occurs with non-zero multiplicity in this module.

Assume now that $n = 3$ and consider the module $L(\omega_2+\omega_3)$. From Lübeck's tables we find that the conclusion of Premet's theorem holds for this module. Since $\omega_2+\omega_3 > \omega_1+\omega_3$ and $r_{A_1} = 12$ we have $|\Lambda| \geqslant 24 > \dim G$, so (†) holds.

Suppose that three coefficients of λ are non-zero and that $n \geqslant 4$. Assume first that $a_n = 0$. Then $\langle \lambda, \alpha\rangle = \pm 2, \pm 4$ or ± 6 for short roots α and, as before, we are only interested in the first two of these possibilities. For a fixed α the minimum number of weights $\mu \in W.\lambda$ such that $\langle \mu, \alpha\rangle = 2$ or 4 occurs for $\lambda = \omega_1+\omega_2+\omega_3$; this is $2^3(n-1)(n-2)$ which exceeds $\dim G$ for $n \geqslant 4$. We notice that the Weyl group orbit of a dominant weight increases in size as more of its coefficients (when written as a $\mathbb{Z}$-linear combination of fundamental weights) are non-zero. It follows, therefore, that (†) holds for all dominant weights $\lambda \in X(T)$ with three or more coefficients a_i non-zero for $i \neq n$.

Assume instead that $a_n = 1$, so $\lambda = \omega_i+\omega_j+\omega_n$ where $1 \leqslant i < j \leqslant n-1$. Then $\langle \lambda, \alpha\rangle = \pm 1, \pm 3$ or ± 5 for short roots α and we can only apply the adjacency principle in the first of these possibilities. The minimum number of $\mu \in W.\lambda$ with weight string $\mu\ \mu+\alpha$ occurs for either $\lambda = \omega_1+\omega_{n-1}+\omega_n$ or $\omega_{n-2}+\omega_{n-1}+\omega_n$; this is $2^{n-1}(n-1)$ which exceeds $\dim G$ for $n \geqslant 5$. Therefore (†) holds for all weights λ with three or more coefficients a_i non-zero including $a_n \neq 0$ provided $n \geqslant 5$.

It remains to consider the irreducible modules $L(\omega_1+\omega_2+\omega_3+\omega_4)$, $L(\omega_1+\omega_2+\omega_4)$, $L(\omega_1+\omega_3+\omega_4)$ and $L(\omega_2+\omega_3+\omega_4)$ all for $n = 4$ and $L(\omega_1+\omega_2+\omega_3)$ for $n = 3$.

Consider $L(\omega_1+\omega_2+\omega_3+\omega_4)$ for $n = 4$. We have $\langle \omega_1+\omega_2+\omega_3+\omega_4, \alpha\rangle = 1$ for one short root only, namely α_4. For $\alpha = \alpha_4$ there are $\frac{2^4 4!}{2.4} = 48$ weights $\mu \in W.\left(\sum_{i=1}^4 \omega_i\right)$ with weight string $\mu\ \mu+\alpha$. Thus (†) holds for this module since $|\Lambda| \geqslant 48 > 36 = \dim G$. Recall that, by Theorem 2.4, the remaining four modules are tensor decomposable and are of the form $L(\nu)\otimes L(\omega_n)$ where $\nu \in X(T)_L$. For $L(\omega_n)$ we find that $\langle \omega_n, \alpha\rangle = 1$ for all positive short roots α. Thus, if we fix a short root α and take this to lie outside Φ_s then all weights occur in pairs differing by α, i.e., we have $|\Lambda_{\omega_n}| \geqslant 2^{n-1}$. (We shall study this module further in Lemma 6.7.) We appeal to Proposition 4.3 for the remaining modules. From Lübeck's tables [18] we find that $\dim L(\omega_1+\omega_3) = 246$ and $\dim L(\omega_2+\omega_3) = 784$ for $n = 4$, and $\dim L(\omega_1+\omega_2) = 64$ or 160 according as $n = 3$ or 4. Thus, for each module we have $|\Lambda_{\omega_n}| \dim L(\nu) > \dim G$, whence (†) holds. □

6.3 Weight string analysis

We list in Table 6.1 the weights which require further consideration following Lemmas 6.3 and 6.4.

λ	n	Lemma
$2\omega_1$	$[2,\infty)$ $(p \neq 2)$	-
ω_3	$[4,\infty)$	6.5 (◊)
$\omega_1+\omega_2$	$[6,\infty)$	6.5 (◊)
	$[3,5]$	6.5 (†) $p \neq 3$, (◊) $p = 3$
	2 $(p \neq 5)$	6.5 (◊)
$3\omega_1$	$[6,\infty)$ $(p > 3)$	6.5 (◊)
	$[2,5]$ $(p > 3)$	6.5 (◊)
ω_n	$[7,8]$	6.7 (◊)
ω_4	5 $(p = 2)$	6.6 (◊)
$2\omega_n$	4 $(p \neq 2)$	6.9 (†)
	3 $(p \neq 2)$	6.9 (◊)
$\omega_1+\omega_3$	3	6.8 (†) $p \neq 7$, (◊) $p = 7$
$\omega_1+2\omega_2$	2 $(p \neq 2)$	6.8 (†) $p \neq 2, 3$, (◊) $p = 3$
$3\omega_2$	2 $(p > 3)$	6.8 (◊)
$4\omega_2$	2 $(p > 3)$	6.8 (†)

Table 6.1: Possible weights in type B_n for $n \geqslant 2$

We begin by considering the modules $L(\omega_3)$ for $n \geqslant 4$ and $L(\omega_1+\omega_2)$ and $L(3\omega_1)$ for $n \geqslant 2$.

Recall from [2, p.47] that for the root system of type B_n we have $\Phi = \Phi_S \cup \Phi_L$ where $\Phi_L = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant n\}$ consists of $2n(n-1)$ long roots and $\Phi_S = \{\pm\epsilon_i \mid 1 \leqslant i \leqslant n\}$

consists of $2n$ short roots. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [1, n-1]$ and $\alpha_n = \epsilon_n$ be the simple roots. Therefore we have $\omega_i = \sum_{k=1}^{i} \epsilon_k$ for $i < n$ and $\omega_n = \frac{1}{2}\sum_{k=1}^{n} \epsilon_k$. We shall need this for the next lemma.

Lemma 6.5. *Let G act on the irreducible module V where we take V to be one of $L(\omega_3)$ for $n \geqslant 4$, $L(\omega_1 + \omega_2)$ for $n \geqslant 2$, or $L(3\omega_1)$ for $n \geqslant 2$ with $p > 3$. Then $(\Diamond)$ is satisfied for all $X \in I$ unless $V = L(\omega_1 + \omega_2)$ for $n \in [3, 5]$ with $p \neq 3$ in which case $(\dagger)$ holds or for $n = 2$ with $p = 5$ in which case the module is small.*

Proof. If $p = 2$ then the conclusion of Premet's theorem does not hold. For $L(\omega_3)$ we see from Lübeck's tables that when $n \leqslant 11$ we have $m_{\omega_2} = 0$. The tables also show that $m_{\omega_1} \neq 0$; in order to conclude that this is true for $n > 11$ we can adapt the arguments in Section 2.4 of [29]. Zalesski's technique involves rewriting weights in C_n as weights in D_n and shows in Lemma 12 of his paper that for a dominant weight $\mu = \sum_{i=1}^{n} a_i\omega_i$ in C_n, provided $a_n = 0$, two dominant weights $\lambda < \mu$ in C_n remain dominant when considered in D_n and the partial ordering is preserved. Then, since the conclusion of Premet's theorem holds for groups of type D_n in characteristic 2, he concludes that λ appears with non-zero multiplicity in $\Pi(\mu)$ in C_n. We rewrite the weights ω_1 and ω_3 in B_n as weights in D_n and apply the theory in Section 2.4 in order to conclude that each weight in the Weyl group orbit of ω_1 appears with non-zero multiplicity in $\Pi(\omega_3)$ in B_n. Analogously, for $L(\omega_1+\omega_2)$ with $p = 2$ we can conclude that both m_{ω_3} and m_{ω_1} are non-zero.

Consider the module $L(\omega_3)$ for $n \geqslant 4$. The usual calculations are detailed in Figure 6.1.

Assume that $p \neq 2$. We see from Figure 6.1 that for $n \geqslant 4$ we have $|\Lambda| \geqslant 2(n-1)^2$. We may assume that there are two orthogonal roots (α short and β long) not in Φ_s since $(\Diamond)$ holds for both $X = B_{n-1}$ ($e_{\omega_3}^{B_{n-1}} = 4n - 2$) and D_n ($e_{\omega_3}^{D_n} = 2n$). It is straightforward to check that $\langle \omega_3, \alpha \rangle = 0$ or ± 2 and $\langle \omega_3, \beta \rangle = 0$, ± 1 or ± 2. We note that for a root system of type B_n there are $2n.2(n-1)(n-2)$ orthogonal pairs of roots α short, β long.

We have $\langle \omega_3, \alpha \rangle = 2$ and $\langle \omega_3, \beta \rangle = 2$ for 3 pairs of roots α, β with $(\alpha, \beta) = 0$. In more detail, the 3 pairs of orthogonal long and short roots are ϵ_i and $\epsilon_j + \epsilon_k$ with

i	ω	$\lvert W.\omega\rvert$	m_ω		
			$n \leqslant 11$, $p \neq 2$	$n \geqslant 12$, $p \neq 2$	$p = 2$
3	ω_3	$\frac{4}{3}n(n-1)(n-2)$	1	1	1
2	ω_2	$2n(n-1)$	1	1	0
1	ω_1	$2n$	$n-1$	1	$\geqslant 1$
0	0	1	n	1	0

Weight strings	No. of strings	l		
		$n \leqslant 11$, $p \neq 2$	$n \geqslant 12$, $p \neq 2$	$p = 2$
μ_3	$\frac{4}{3}(n-1)(n-2)(n-3)$			
$\mu_3\ \mu_2\ \mu_3$	$2(n-1)(n-2)$	$2(n-1)(n-2)$	$2(n-1)(n-2)$	$2(n-1)(n-2)$
$\mu_2\ \mu_1\ \mu_2$	$2(n-1)$	$4(n-1)$	$2(n-1)$	
$\mu_1\ \mu_0\ \mu_1$	1	n	1	1
Lower bound on $\lvert\Lambda\rvert$		$2n^2-n$	$2n^2-4n+3$	$2n^2-6n+5$

Figure 6.1: $\lambda = \omega_3$ for $n \geqslant 4$

$i, j, k \in [1, 3]$, $j < k$ and $i \neq j \neq k \neq i$. There are $4n(n-1)(n-2)$ triples $(\mu; \alpha, \beta)$ with $\langle \mu, \alpha\rangle = \langle \mu, \beta\rangle = 2$ for orthogonal roots $\alpha \in \Phi_S$, $\beta \in \Phi_L$ and $\mu \in W.\omega_3$. For a given pair α, β there is 1 weight $\mu \in W.\omega_3$ such that $\langle \mu, \alpha\rangle = \langle \mu, \beta\rangle = 2$. Thus there is a 3×3 weight net.

We have $\langle \omega_3, \alpha\rangle = 0$ and $\langle \omega_3, \beta\rangle = 2$ for $6(n-3)$ pairs of roots α, β with $(\alpha, \beta) = 0$. These pairs of orthogonal long and short roots are $\epsilon_i + \epsilon_j$ with $1 \leqslant i < j \leqslant 3$ and $\pm\epsilon_k$ with $k \in [4, n]$. There are $8n(n-1)(n-2)(n-3)$ triples $(\mu; \alpha, \beta)$ with $\langle \mu, \alpha\rangle = 0$ and $\langle \mu, \beta\rangle = 2$ for orthogonal roots $\alpha \in \Phi_S$, $\beta \in \Phi_L$ and $\mu \in W.\omega_3$. For a given pair α, β there are $2(n-3)$ weights $\mu \in W.\omega_3$ such that $\langle \mu, \alpha\rangle = 0$ and $\langle \mu, \beta\rangle = 2$. Thus there are $2(n-3)$ weight nets of size 1×3; it is easy to check that these take the form $\mu_3\ \mu_1\ \mu_3$.

Similarly we calculate that there are $4(n-3)$ weight nets of size 3×2, $4(n-3)(n-4)$ weight nets of size 1×2 and $2 + 2(n-3)(n-4)$ weight nets of size 3×1 (which take the form $\mu_3\ \mu_2\ \mu_3$ as indicated by Figure 6.1). We need not consider weights $\mu \in W.\omega_3$ with $\langle \mu, \alpha\rangle = \langle \mu, \beta\rangle = 0$ since there is no contribution to $|\Lambda|$ from 1×1 weight nets. So from weight nets containing $\mu \in W.\omega_3$ we find that

$$|\Lambda| \geqslant 4 + 2(n-3) + 4(n-3) + 4(n-3)(n-4) + 2 + 2(n-3)(n-4) = 6n^2 - 28n + 36.$$

Now we consider the remaining weight nets for $L(\omega_3)$, i.e., those containing no weights from $W.\omega_3$. As before, we can check that $\langle \omega_2, \alpha\rangle = 0$ or ± 2 and $\langle \omega_2, \beta\rangle = 0$, ± 1 or ± 2. It is not possible to find pairs of orthogonal roots α and β with $\langle \omega_2, \alpha\rangle = \langle \omega_2, \beta\rangle = 2$. We find that there are two 3×2 and $2(n-3)$ weight nets of size 3×1. All other weight nets and strings are included in those found previously.

In summary, the weight nets are provided below.

$$\begin{array}{cccccc}
\begin{matrix} \mu_3 & \mu_2 & \mu_3 \\ \mu_1 & \mu_0 & \mu_1 \\ \mu_3 & \mu_2 & \mu_3 \end{matrix} &
\begin{matrix} & \mu_3 & \\ \mu_2 & \mu_1 & \mu_2 \\ & \mu_3 & \end{matrix} &
\begin{matrix} \mu_3 & \mu_2 & \mu_3 \\ \mu_3 & \mu_2 & \mu_3 \\ & & \end{matrix} &
\begin{matrix} \mu_3 \\ \mu_3 \\ \end{matrix} &
\begin{matrix} \mu_3 & \mu_2 & \mu_3 \\ & & \\ & & \end{matrix} &
\begin{matrix} \mu_2 & \mu_1 & \mu_2 \\ \mu_2 & \mu_1 & \mu_2 \\ & & \end{matrix} \\
1 & 2(n-3) & 4(n-3) & 4(n-3)(n-4) & 2(n^2-7n+13) & 2
\end{array}$$

Since all weights have multiplicity at least one we find $|\Lambda| \geqslant 6n^2 - 28n + 42 > 2n^2 = |\Phi(G)|$ for $n \geqslant 5$. We are therefore left to consider $L(\omega_3)$ for $n = 4$; here we know that $m_{\omega_1} = 3$ and $m_0 = 4$, so using the weight nets above we have $|\Lambda| \geqslant 38 > 32 = |\Phi(B_4)|$. Thus $(\Diamond)$ is satisfied for all $X \in I$ provided $p \neq 2$.

Assume now that $p = 2$. From Figure 6.1 we see that $|\Lambda| \geqslant 2n^2 - 6n + 5$. If $X = D_n$ then $(\Diamond)$ is satisfied for $n \geqslant 4$. If $X = B_{n-1}$ then $(\Diamond)$ is satisfied provided $n \geqslant 5$. For $n = 4$ when $X = B_3$ there are three clusters. They are $1 \cdots$, $0 \cdots$ and $-1 \cdots$ of sizes 14, 20 and 14 respectively. Since $e_{\omega_3}^{B_3} = 14$ we see the $(\Diamond)$ is satisfied for B_3. Therefore we can assume that there are two orthogonal roots, one long and one short, not in Φ_s. Using the weight nets described above for $p \neq 2$ and the fact that $\alpha(s) = 1$ if and only if $2\alpha(s) = 1$ we see that $|\Lambda| \geqslant 6n^2 - 32n + 49$. This exceeds $|\Phi(G)|$ for $n \geqslant 6$.

We are left to consider $L(\omega_3)$ for $n \in [4, 5]$ with $p = 2$. From Lübeck's tables we see that $m_{\omega_1} = 2$ in both cases. Using the weight nets above we see that if $n = 5$ then $|\Lambda| \geqslant 46$ so that $(\Diamond)$ is satisfied provided $|\Phi(G)| > 4$, and if $n = 4$ then $|\Lambda| \geqslant 22$ so that $(\Diamond)$ is satisfied provided $|\Phi(G)| > 10$. Thus, if $n = 5$ we are left to consider $X = \varnothing$, A_1, B_1, A_1^2, A_1B_1 and D_2 and if $n = 4$ we are left with $X = \varnothing$, A_1, A_1^2, D_2, D_2A_1, A_2, B_1, B_1A_1, B_1D_2, B_1A_2, B_2 and B_2A_1. We shall provide the clusters and cliques for the six remaining cases when $n = 5$; we omit the calculations for $n = 4$.

Let $X = A_1^2$ with simple roots α_1 and α_3. By using the cliques tabulated below together with the negative counterparts of the cliques in the second column we have $d_{\omega_3}^{A_1^2} \geqslant 48 > 46 = e_{\omega_3}^{A_1^2}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, A_1^2)$					
Cliques	Cluster size	l	Cliques	Cluster size	l
$\cdot$ 1 1 1 1	4	16	$\cdot$ 1 $\cdot$ 2 1	4	4
0 0 $\cdot$ 1 1	4		$\cdot$ 1 $\cdot$ 1 1	4	
0 0 0 0 1	2		$\cdot$ 1 $\cdot$ 0 1	4	4
0 0 0 0 -1	2		$\cdot$ 1 $\cdot$ 0 -1	4	
0 0 $\cdot$ -1 -1	4		1 2 $\cdot$ 3 3	2	2
$\cdot$ -1 -1 -1 -1	4		$\cdot$ 1 2 3 3	2	
$\cdot$ 0 $\cdot$ 1 1	4	4	1 2 2 2 3	1	3
$\cdot$ 0 $\cdot$ -1 -1	4		1 2 2 2 1	1	
$\cdot$ 0 0 0 1	2	2	1 2 2 1 1	1	
$\cdot$ 0 0 0 -1	2		1 2 1 1 1	1	

Let $X = A_1B_1$ with simple roots α_1 and α_5. By using the cliques tabulated below together with the negative counterparts of the cliques in the second column we have $d_{\omega_3}^{A_1B_1} \geqslant 48 > 46 = e_{\omega_3}^{A_1B_1}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, A_1B_1)$					
Cliques	Cluster size	l	Cliques	Cluster size	l
$\cdot$ 1 1 1 1	2	16	$\cdot$ 1 1 0 $\cdot$	4	4
0 0 1 1 1	1		$\cdot$ 1 0 0 $\cdot$	4	
0 0 0 1 1	1		1 2 2 2 $\cdot$	2	6
0 0 0 0 $\cdot$	2		$\cdot$ 1 2 2 $\cdot$	4	
0 0 0 -1 -1	1		$\cdot$ 1 1 2 $\cdot$	4	
0 0 -1 -1 -1	1		1 2 3 3 3	1	2
$\cdot$ -1 -1 -1 -1	2		1 2 2 3 3	1	
$\cdot$ 0 1 1 1	2	8	$\cdot$ 1 2 3 3	2	
$\cdot$ 0 0 1 1	2				
$\cdot$ 0 0 0 $\cdot$	4				
$\cdot$ 0 0 -1 -1	2				
$\cdot$ 0 -1 -1 -1	2				

Let $X = D_2$ with simple root α_1 and the longest root α_0. By using the cliques tabulated below together with the negative counterparts of the cliques in the second column we have $d_{\omega_3}^{D_2} \geqslant 48 > 46 = e_{\omega_3}^{D_2}$. Note that we provide a representative weight from each cluster.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, D_2)$					
Cliques	Cluster size	l	Cliques	Cluster size	l
1 1 1 1 1	8	24	1 0 1 1 1	2	4
0 0 1 1 1	4		1 0 0 1 1	2	
0 0 0 1 1	4		1 0 0 0 1	2	
0 0 0 0 1	4		1 1 1 0 1	4	4
0 0 0 0 -1	4		1 1 0 0 1	4	
0 0 0 -1 -1	4		1 1 1 0 -1	4	4
0 0 -1 -1 -1	4		1 1 0 0 -1	4	

Let $X = B_1$ with simple root α_5. By using the cliques tabulated below together with the negative counterparts of the cliques in the second and third columns we have $d^{B_1}_{\omega_3} \geqslant 50 > 48 = e^{B_1}_{\omega_3}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, B_1)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
1 1 1 1 1	2	16	1 1 1 0 ·	2	4	1 2 3 3 3	1	3
0 1 1 1 1	2		1 1 0 0 ·	2		1 2 2 3 3	1	
0 0 1 1 1	2		1 0 0 0 ·	2		1 1 2 3 3	1	
0 0 0 1 1	2		1 1 1 2 ·	2	4	0 1 2 3 3	1	
0 0 0 0 ·	4		0 1 1 2 ·	2		1 2 2 1 1	1	2
0 0 0 -1 -1	2		0 0 1 2 ·	2		1 1 2 1 1	1	
0 0 -1 -1 -1	2		1 2 2 2 ·	2	4	0 1 2 1 1	1	
0 -1 -1 -1 -1	2		1 1 2 2 ·	2				
-1 -1 -1 -1 -1	2		0 1 2 2 ·	2				

Let $X = A_1$ with simple root α_1. By using the cliques tabulated below together with the negative counterparts of the cliques in the first and second columns we have $d^{A_1}_{\omega_3} \geqslant 50 > 48 = e^{A_1}_{\omega_3}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, A_1)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
1 2 3 3 3	1	2	· 0 1 1 1	2	4	· 1 1 1 1	4	16
1 2 2 3 3	1		· 0 0 1 1	2		· 0 1 1 1	2	
· 1 2 3 3	2		· 0 0 0 1	2		· 0 0 1 1	2	
1 2 2 2 3	1	3	· 1 1 0 1	2	2	· 0 0 0 1	2	
· 1 2 3 3	2		· 1 0 0 1	2		· 0 0 0 -1	2	
· 1 1 2 3	2		· 1 1 0 -1	2	2	· 0 0 -1 -1	2	
· 1 1 2 1	2	2	· 1 0 0 -1	2		· 0 -1 -1 -1	2	
· 1 0 1 1	2					· -1 -1 -1 -1	4	
· 1 2 2 1	2	2						
· 1 2 1 1	2							

Let $X = \varnothing$; by using the cliques tabulated below together with their negative counterparts we have $d^{\varnothing}_{\omega_3} \geqslant 52 > 50 = e^{\varnothing}_{\omega_3}$.

$(\lambda,\, n,\, X) = (\omega_3,\, 5,\, \varnothing)$								
Cliques	Cluster size	l	Cliques	Cluster size	l	Cliques	Cluster size	l
1 1 1 1 1	2	8	1 2 2 2 1	1	2	1 2 3 3 3	1	3
0 1 1 1 1	2		1 2 2 1 1	1		1 2 2 3 3	1	
0 0 1 1 1	2		1 2 1 1 1	1		1 1 2 3 3	1	
0 0 0 1 1	2		1 0 1 1 1	1	2	0 1 2 3 3	1	
0 0 0 0 1	2		1 0 0 1 1	1		1 1 1 2 1	1	2
1 2 2 2 3	1	2	1 0 0 0 1	1		0 1 1 2 1	1	
1 1 2 2 3	1		1 1 1 2 3	1	2	0 0 1 2 1	1	
0 1 2 2 3	1		0 1 1 2 3	1		1 1 1 0 -1	1	2
1 1 1 0 1	1	2	0 0 1 2 3	1		0 1 1 0 -1	1	
0 1 1 0 1	1		1 1 2 2 1	1	1	0 0 1 0 -1	1	
0 0 1 0 1	1		0 1 2 2 1	1				

Suppose that $n \geqslant 6$. Using the work for $L(\omega_3)$ above together with Premet's theorem, we can conclude for $L(\omega_1+\omega_2)$ for all p and $L(3\omega_1)$ for $p \neq 2$ that $(\diamondsuit)$ is satisfied for all $X \in I$.

Consider $L(\omega_1+\omega_2)$ for $n \leqslant 5$. If $n \in [4,5]$ then Figure 6.2 shows that $(\dagger)$ holds unless $p = 3$ when $(\diamondsuit)$ is satisfied for each $X \in I$.

i	ω	$\lvert W.\omega\rvert$	m_ω			
			$p \neq 2, 3,\ p \neq 5 = n$	$p = 3$	$p = 5 = n$	$p = 2$
5	$\omega_1+\omega_2$	$4n(n-1)$	1	1	1	1
4	ω_3	$\frac{4}{3}n(n-1)(n-2)$	2	1	2	2
3	$2\omega_1$	$2n$	1	1	1	0
2	ω_2	$2n(n-1)$	2	1	2	0
1	ω_1	$2n$	$2n-1$	n	8	$2n-2$
0	0	1	$2n-1$	$n-1$	8	0

Weight strings	No. of strings	l			
		$p \neq 2, 3,\ p \neq 5 = n$	$p = 3$	$p = 5 = n$	$p = 2$
μ_5	$4(n-1)(n-2)$				
$\mu_5\ \mu_3\ \mu_5$	$2(n-1)$	$2(n-1)$	$2(n-1)$	$2(n-1)$	$2(n-1)$
$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	$2(n-1)$	$8(n-1)$	$4(n-1)$	$8(n-1)$	$4(n-1)$
μ_4	$\frac{4}{3}(n-1)(n-2)(n-3)$				
$\mu_4\ \mu_2\ \mu_4$	$2(n-1)(n-2)$	$4(n-1)(n-2)$	$2(n-1)(n-2)$	$4(n-1)(n-2)$	$4(n-1)(n-2)$
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	$2n+1$	$n+1$	$2n$	$2n-2$
Lower bound on $\lvert\Lambda\rvert$		$4n^2-1$	$2n^2+n-1$	$4n^2-2$	$4n(n-1)$

Figure 6.2: $\lambda = \omega_1+\omega_2$ for $n \in [4,5]$

If $n = 3$ then Figure 6.3 shows that $(\dagger)$ holds unless $p = 3$ in which case $(\diamondsuit)$ is satisfied for each $X \in I$.

i	ω	$\lvert W.\omega\rvert$	m_ω		
			$p \neq 2,3$	$p=3$	$p=2$
5	$\omega_1+\omega_2$	24	1	1	1
4	$2\omega_3$	8	2	1	2
3	$2\omega_1$	6	1	1	0
2	ω_2	12	2	1	0
1	ω_1	6	5	2	4
0	0	1	5	1	0

Weight strings	No. of strings	l		
		$p \neq 2,3$	$p=3$	$p=2$
μ_5	8			
$\mu_5\ \mu_3\ \mu_5$	4	4	4	4
$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	4	16	8	8
$\mu_4\ \mu_2\ \mu_4$	4	8	4	8
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	7	3	4
Lower bound on $\lvert\Lambda\rvert$		35	19	24

Figure 6.3: $(\lambda, n) = (\omega_1+\omega_2, 3)$

If $n=2$ then Figure 6.4 shows that $(\lozenge)$ is satisfied for each $X \in I$ other than $X = \varnothing$ for $p \neq 2, 5$, and A_1 and B_1 for $p=5$.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 5$	$p=5$
2	$\omega_1+\omega_2$	8	1	1
1	ω_2	4	2	1

Weight strings	No. of strings	l		
		$p \neq 2,5$	$p=5$	$p=2$
$\mu_2\ \mu_2$	2	2	2	2
$\mu_2\ \mu_1\ \mu_1\ \mu_2$	2	6	4	8
Lower bound on $\lvert\Lambda\rvert$		8	6	10

Figure 6.4: $(\lambda, n) = (\omega_1+\omega_2, 2)$

If $X = \varnothing$ and $p \neq 5$ then the following cliques show that $d^{\varnothing}_{\omega_1+\omega_2} \geqslant 9$ in which case $(\lozenge)$ holds. Note that the weights in $W.\omega_2$ have multiplicity 2 and are $\frac{1}{2}$ 1, $\frac{1}{2}$ 0, -$\frac{1}{2}$ 0 and -$\frac{1}{2}$ -1.

$$\boxed{\begin{array}{c} \frac{1}{2}\ 1 \\ \frac{1}{2}\ 0 \\ -\frac{1}{2}\ 0 \end{array}} \quad \boxed{\begin{array}{c} \frac{1}{2}\ -1 \\ -\frac{1}{2}\ -1 \\ -\frac{1}{2}\ -2 \end{array}} \quad \boxed{\begin{array}{c} \frac{3}{2}\ 2 \\ \frac{3}{2}\ 1 \end{array}} \quad \boxed{\begin{array}{c} \frac{1}{2}\ 2 \\ -\frac{1}{2}\ 1 \end{array}} \quad \boxed{\begin{array}{c} -\frac{3}{2}\ -1 \\ -\frac{3}{2}\ -2 \end{array}}$$

If $p=5$ then the module is small since it has dimension $\dim G + 2$. Indeed, we can see for $X \in \{\varnothing, A_1, B_1\}$ there are possible configurations of weights such that $d^X_{\omega_1+\omega_2} = e^X_{\omega_1+\omega_2}$.

Now consider $L(3\omega_1)$ for $n \in [3,5]$. From Figure 6.5 below we see that we may take two orthogonal roots (α short and β long) not in Φ_s. If $n=3$ the Weyl group orbit $W.(2\omega_3)$ is present in place of $W.\omega_3$ and $2n+3$ is not a prime.

We see that $\langle 3\omega_1, \alpha\rangle$ can take the value 0 or ± 6 and $\langle 3\omega_1, \beta\rangle$ can take the value 0 or ± 3. Similarly we can have $\langle \omega_1+\omega_2, \alpha\rangle = 0$, ± 2 or ± 4 and $\langle \omega_1+\omega_2, \beta\rangle = 0$, ± 1, ± 2 or ± 3 though for a fixed pair α, β there are some pairs of these values that are not possible. It is sufficient to only consider the weight nets formed from weights in $W.(3\omega_1)$ and $W.(\omega_1+\omega_2)$; these are detailed below and the number of each such is given.

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 2, 3, 2n+3$	$p = 2n+3$
6	$3\omega_1$	$2n$	1	1
5	$\omega_1 + \omega_2$	$4n(n-1)$	1	1
4	ω_3 $(2\omega_3)$	$\frac{4}{3}n(n-1)(n-2)$	1	1
3	$2\omega_1$	$2n$	1	1
2	ω_2	$2n(n-1)$	1	1
1	ω_1	$2n$	n	$n-1$
0	0	1	n	$n-1$

Weight strings	No. of strings	l	
		$p \neq 2, 3, 2n+3$	$p = 2n+3$
μ_6	$2(n-1)$		
$\mu_6\ \mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3\ \mu_6$	1	$n+2$	$n+1$
μ_5	$4(n-1)(n-2)$		
$\mu_5\ \mu_3\ \mu_5$	$2(n-1)$	$2(n-1)$	$2(n-1)$
$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	$2(n-1)$	$4(n-1)$	$4(n-1)$
μ_4	$\frac{4}{3}(n-1)(n-2)(n-3)$		
$\mu_4\ \mu_2\ \mu_4$	12	24	24
Lower bound on $\lvert\Lambda\rvert$		$2n^2+n$	$2n^2+n-1$

Figure 6.5: $\lambda = 3\omega_1$ for $n \in [3, 5]$

μ_6	$\mu_6\ \mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3\ \mu_6$	μ_5	$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	$\mu_5\ \mu_3\ \mu_5$	$\mu_5\ \mu_3\ \mu_5$	μ_5	μ_5
μ_5		μ_1	$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$		$\mu_4\ \mu_2\ \mu_4$		μ_4	μ_5
μ_6		μ_1			$\mu_5\ \mu_3\ \mu_5$		μ_5	
		μ_5						
2	1	2	2	$2(n-3)$	2	$2(n-3)$	$4(n-3)$	$4(n-3)$

Therefore $|\Lambda| \geqslant 19n - 20$ for $p \neq 2, 3$ or $2n+3$ and $|\Lambda| \geqslant 19n - 25$ for $p = 2n+3$. In either case $(\Diamond)$ is satisfied for all $X \in I$ for $n \in [3, 5]$.

If $n = 2$ then we show in Figure 6.6 that $(\Diamond)$ holds for all $X \in I$. $\square$

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 2, 3, 7$	$p = 7$
5	$3\omega_1$	4	1	1
4	$\omega_1 + 2\omega_2$	8	1	1
3	$2\omega_1$	4	1	1
2	$2\omega_2$	4	1	1
1	ω_1	4	2	1
0	0	1	2	1

Weight strings	No. of strings	l	
		$p \neq 2, 3, 7$	$p = 7$
μ_5	2		
$\mu_5\ \mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3\ \mu_5$	1	4	3
$\mu_4\ \mu_3\ \mu_4$	2	2	2
$\mu_4\ \mu_2\ \mu_1\ \mu_2\ \mu_4$	2	4	4
Lower bound on $\lvert\Lambda\rvert$		10	9

Figure 6.6: $(\lambda, n) = (3\omega_1, 2)$

We note that we are unable to draw conclusions for $L(\omega_1 + \omega_2)$ and $L(3\omega_1)$ with $n \in [4, 5]$ by appealing to Premet's theorem since the multiplicities of weights in $W.\omega_1$ and 0 in $L(\omega_1 + \omega_2)$ and $L(3\omega_1)$ are not always greater than those for the corresponding weights in $L(\omega_3)$. Indeed, using the multiplicities given for $W.\omega_1$ and 0 in $L(\omega_1 + \omega_2)$ and $L(3\omega_1)$ and calculating the codimension of the eigenspace as above for $L(\omega_3)$ does not suffice either.

In the next lemma we consider a module that requires treatment for $p = 2$ only.

Lemma 6.6. *Let G act on the irreducible module $V = L(\omega_4)$ for $n = 5$ with $p = 2$. Then $(\Diamond)$ holds for all $X \in I$.*

Proof. We can use Lübeck's tables [18] to see that the conclusion of Premet's theorem does not hold for this module, in particular, we have $m_{\omega_3} = m_{\omega_1} = 0$. In Figure 6.7 we carry out the usual calculations to identify the weight strings for V and find that $|\Lambda| \geqslant 48$.

i	ω	$\lvert W.\omega \rvert$	m_ω $p=2$
4	ω_4	80	1
3	ω_3	80	0
2	ω_2	40	2
1	ω_1	10	0
0	0	1	4

Weight strings	No. of strings	l $p=2$
μ_4	16	
$\mu_4\ \mu_3\ \mu_4$	32	32
$\mu_3\ \mu_2\ \mu_3$	24	
$\mu_2\ \mu_1\ \mu_2$	8	16
$\mu_1\ \mu_0\ \mu_1$	1	
Lower bound on $\lvert\Lambda\rvert$		48

Figure 6.7: $(\lambda, n) = (\omega_4, 5)$

Thus we may certainly take two orthogonal roots, one short and one long, outside Φ_s. Set $x = \langle \omega_4, \alpha \rangle$ and $y = \langle \omega_4, \beta \rangle$. Then x can take the values 0 or ± 2 and y can take the values 0, ± 1 or ± 2. The weight strings of length three for $y = 2$ are of the form μ_4 μ_2 μ_4. There are 12 pairs of roots with $(x, y) = (2, 2)$, 24 with $(x, y) = (2, 1)$, 24 with $(x, y) = (2, 0)$, 12 with $(x, y) = (0, 2)$ and 24 with $(x, y) = (0, 0)$. There are 240 pairs of mutually orthogonal short and long roots in Φ. Thus, for a fixed pair α, β there are $\frac{80.12}{240} = 4$ weights $\mu \in W.\omega_4$ with $(x, y) = (2, 2)$. Taking the horizontal differences between weights to be α and vertical β we see, therefore, that there are four 3×3 weight nets. However, since the weights in $W.\omega_3$ do not appear for $p = 2$ we shall instead consider these four nets as 2×3 weight nets with a difference 2α between two adjacent horizontal weights. Analogously, we find eight 2×2, eight 2×1 and four 1×3 weight nets. There are eight 1×1 weight nets but these do not contribute to $|\Lambda|$. By considering weights in $W.\omega_2$ we find two 2×2 and eight 1×2 weight nets. There is one 1×3 weight net of the form μ_2 μ_0 μ_2. Therefore we have $|\Lambda| \geqslant 4.5 + 8.2 + 8.1 + 4.2 + 2.4 + 1.4 + 8.2 = 80$, so $(\Diamond)$ holds. $\square$

We have not been able to draw any useful conclusions about the irreducible module $L(2\omega_1)$ with $p \neq 2$. The main difficulty is that $\dim L(2\omega_1)$ is only slightly larger than $\dim G$ and most $\mu \in \Pi(2\omega_1)$ satisfy $\langle \mu, \alpha \rangle = 0$ for a given short root $\alpha \notin \Phi_s$. Thus most weights lie in strings of length 1, so we can only show that $|\Lambda| \geqslant 2n$.

We next consider the spin module $L(\omega_n)$ for $n \in [7, 8]$. We may label weights in $W.\omega_n$ by strings consisting of n plus and minus signs. This is done by considering the coefficients of a weight $\mu \in W.\omega_n$, imagining a zero coefficient preceding the original coefficients of μ and then placing a plus sign in position i of the string if the $(i+1)$st coefficient is larger than the ith coefficient for $i \in [0, n-1]$; otherwise there is a minus sign. This simplifies the task of determining whether the difference between two weights is a root and to determine clusters. Two weights differ by $\alpha_i + \cdots + \alpha_j$ for $j < n$ if they are identical except a plus and minus sign have been interchanged in the ith and $(j+1)$st positions. Two weights differ by α_n if they are identical except one has a plus sign in the nth position of the string and the other has a minus sign. It is clear, therefore, that if we compare two weights with one having two plus signs and the other having two minus signs in the ith and jth positions (otherwise identical) then they differ by the root $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_n$. (Note that an analogous method for labelling weights in $W.\omega_n$ is available for types C_n when $p = 2$ and D_n.)

Lemma 6.7. *Let G act on the irreducible module $L(\omega_n)$ for $n \in [7, 8]$. If $n \in [7, 8]$ then $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. It is clear that $\langle \omega_n, \alpha \rangle = 1$ for each of the n positive short roots α. Since $|W(B_n) : W(A_n)| = 2^n$ there are $2^n n$ pairs (μ, α) for $\mu \in W.\omega_n$ and $\alpha \in \Phi_S$ with weight string of the form $\mu\ \mu - \alpha$. Thus, for a fixed short root α there are 2^{n-1} weight strings of the form $\mu\ \mu - \alpha$, i.e., all weights of $L(\omega_n)$ appear in pairs differing by α. By the adjacency principle we have $|\Lambda| \geqslant 2^{n-1}$.

First assume $n = 8$; then we have $|\Lambda| \geqslant 128$, so $(\Diamond)$ holds unless $X = \varnothing$. Indeed, if $X = \varnothing$ then we see that $d_{\omega_8}^{\varnothing} \geqslant 129 > 128 = e_{\omega_8}^{\varnothing}$ by considering the clique below formed from two pairs of weights with weights in a pair differing by α_8. Note that we have omitted a factor of $\frac{1}{2}$.

1 2 3 4 5 4 3 4, 1 2 3 4 5 4 3 2 and -1 0 1 2 3 2 1 2, -1 0 1 2 3 2 1 0

Now assume $n = 7$; then we have $|\Lambda| \geqslant 64$. Since $|\Phi(B_7)| = 98$, the condition $(\Diamond)$ holds for all $X \in I$ satisfying $|\Phi(X)| > 34$. It remains to go through a list of 72 centraliser types listed below. The calculations are routine. □

Centraliser types
$\varnothing$, A_1, A_1^2, A_1^3, D_2, D_2A_1, $D_2A_1^2$, B_1, B_1A_1, $B_1A_1^2$, $B_1A_1^3$, $B_1A_1^2D_2$, $B_1A_1D_2$, D_2B_1, A_2, A_2A_1, $A_2A_1^2$, A_2D_2, $A_2D_2A_1$, A_2B_1, $A_2B_1A_1$, $A_2B_1D_2$, A_3, A_3A_1, A_3D_2, A_3A_2, A_3B_1, $A_3B_1A_1$, $A_3B_1D_2$, A_2^2, $A_2^2B_1$, A_3D_3, D_3, D_3A_1, $D_3A_1^2$, D_3B_1, $D_3A_1B_1$, D_3A_2, $D_3A_2B_1$, A_4, A_4A_1, A_4D_2, A_5, A_5B_1, D_4, D_4A_1, D_4B_1, $D_4A_1B_1$, D_4A_2, B_2, B_2A_1, $B_2A_1^2$, B_2D_2, $B_2D_2A_1$, B_2A_2, $B_2A_2A_1$, $B_2A_2D_2$, B_2A_3, B_2D_3, $B_2D_3A_1$, B_2A_4, B_2D_4, B_3, B_3A_1, $B_3A_1^2$, B_3D_2, $B_3D_2A_1$, B_3A_2, B_3A_3, B_3D_3, B_4, B_4A_1

Table 6.2: Centraliser types requiring consideration for $L(\omega_7)$ with $n = 7$

We remark that since all weights in $W.\omega_n$ appear in pairs differing by a fixed short root there is no benefit to taking further roots not in Φ_s and forming weight nets as we have done in previous lemmas.

We shall now deal with the remaining modules which all have low rank.

Lemma 6.8. *Let G act on the irreducible module V where we take V to be one of $L(3\omega_2)$ $(p > 3)$, $L(4\omega_2)$ $(p > 3)$ or $L(\omega_1 + 2\omega_2)$ $(p \neq 2)$ with $n = 2$, or $L(\omega_1 + \omega_3)$ with $n = 3$. Then $(\dagger)$ holds for $L(4\omega_2)$ with $n = 2$, $L(\omega_1 + 2\omega_2)$ with $n = 2$ and $p \neq 2, 3$ and for $L(\omega_1 + \omega_3)$ with $n = 3$ and $p \neq 7$; otherwise $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. Consider the modules $L(3\omega_2)$ and $L(4\omega_2)$ for $n = 2$. In Figure 6.8 we show that $(\dagger)$ holds and in Figure 6.9 we show that $(\Diamond)$ is satisfied for each $X \in I$ since $|\Phi(B_2)| = 8$.

We consider the module $L(\omega_1 + 2\omega_2)$ for $n = 2$ in Figure 6.10 below. We see that $(\dagger)$ holds if $p \neq 2, 3$ and that $(\Diamond)$ is always satisfied if $p = 3$.

Next we consider the module $L(\omega_1 + \omega_3)$ for $n = 3$. We see in Figure 6.11 that $(\dagger)$ holds if $p \neq 2, 7$ and that $(\Diamond)$ is always satisfied if $p = 7$ since $|\Phi(B_3)| = 18$. □

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p \neq 2,3$
5	$4\omega_2$	4	1
4	$\omega_1 + 2\omega_2$	8	1
3	$2\omega_1$	4	1
2	$2\omega_2$	4	2
1	ω_1	4	2
0	0	1	3

Weight strings	No. of strings	l
		$p \neq 2,3$
$\mu_5\ \mu_4\ \mu_3\ \mu_4\ \mu_5$	2	4
$\mu_4\ \mu_2\ \mu_1\ \mu_2\ \mu_4$	2	8
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	4
Lower bound on $\lvert\Lambda\rvert$		16

Figure 6.8: $(\lambda, n) = (4\omega_2, 2)$

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p \neq 2,3$
3	$3\omega_2$	4	1
2	$\omega_1 + \omega_2$	8	1
1	ω_2	4	2

Weight strings	No. of strings	l
		$p \neq 2,3$
$\mu_3\ \mu_2\ \mu_2\ \mu_3$	2	4
$\mu_2\ \mu_1\ \mu_1\ \mu_2$	2	6
Lower bound on $\lvert\Lambda\rvert$		10

Figure 6.9: $(\lambda, n) = (3\omega_2, 2)$

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 2,3$	$p = 3$
4	$\omega_1 + 2\omega_2$	8	1	1
3	$2\omega_1$	4	1	1
2	$2\omega_2$	4	2	1
1	ω_1	4	3	2
0	0	1	3	1

Weight strings	No. of strings	l	
		$p \neq 2,3$	$p = 3$
$\mu_4\ \mu_3\ \mu_4$	2	2	2
$\mu_4\ \mu_2\ \mu_1\ \mu_2\ \mu_4$	2	8	4
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	5	3
Lower bound on $\lvert\Lambda\rvert$		15	9

Figure 6.10: $(\lambda, n) = (\omega_1 + 2\omega_2, 2)$

i	ω	$\lvert W.\omega \rvert$	m_ω	
			$p \neq 7$	$p = 7$
2	$\omega_1 + \omega_3$	24	1	1
1	ω_3	8	3	2

Weight strings	No. of strings	l		
		$p \neq 2,7$	$p = 7$	$p = 2$
$\mu_2\ \mu_2$	8	8	8	8
$\mu_2\ \mu_1\ \mu_1\ \mu_2$	4	16	12	20
Lower bound on $\lvert\Lambda\rvert$		24	20	28

Figure 6.11: $(\lambda, n) = (\omega_1 + \omega_3, 3)$

Lemma 6.9. *Let G act on the irreducible module $L(2\omega_n)$ $(p \neq 2)$ for $n \in [3,4]$. If $n = 4$ then $(\dagger)$ holds and if $n = 3$ then $(\Diamond)$ holds for all $X \in I$.*

Proof. If $n = 4$ then Figure 6.12 shows that $(\dagger)$ certainly holds since $\dim G = 36$.

If $n = 3$ then $(\Diamond)$ holds provided $|\Phi(X)| > 3$. Therefore we need to consider separately the possibilities $X = \varnothing$, A_1 or B_1.

We first note that $W.\omega_1$ consists of the short roots and $W.\omega_2$ the long roots hence it is easy to identify in which Weyl group orbit a particular weight lies.

Suppose that $X = \varnothing$. If $000 \in \Pi(V) \setminus \Lambda$ then no root lies in the eigenspace, i.e., $\Phi(B_3) \subset \Lambda$. Since there are 18 roots, 6 of which have multiplicity two (the short roots), we have $d^{\varnothing}_{2\omega_3} \geqslant 24 > 18 = e^{\varnothing}_{2\omega_3}$. Now assume that $000 \in \Lambda$. Suppose that a long root lies in $\Pi(V) \setminus \Lambda$, without loss of generality take 100. Then the weights

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2$
4	$2\omega_4$	16	1
3	ω_3	32	1
2	ω_2	24	2
1	ω_1	8	3
0	0	1	6

Weight strings	No. of strings	l
		$p \neq 2$
μ_4 μ_3 μ_4	8	8
μ_3 μ_2 μ_3	12	24
μ_2 μ_1 μ_2	6	18
μ_1 μ_0 μ_1	1	6
Lower bound on $\lvert\Lambda\rvert$		56

Figure 6.12: $(\lambda, n) = (2\omega_4, 4)$

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2$
3	$2\omega_3$	8	1
2	ω_2	12	1
1	ω_1	6	2
0	0	1	3

Weight strings	No. of strings	l
		$p \neq 2$
μ_3 μ_2 μ_3	4	4
μ_2 μ_1 μ_2	4	8
μ_1 μ_0 μ_1	1	3
Lower bound on $\lvert\Lambda\rvert$		15

Figure 6.13: $(\lambda, n) = (2\omega_3, 3)$

1 1 2, 1 1 1, 1 0 1, 1 1 0, 1 0 -1, 0 -1 0, 0 -1 -1 and 0 -1 -2

lie in Λ. We can arrange some of the remaining weights into the following three cliques:

1 2 2, 0 1 2, 0 1 1; 0 0 -1, -1 -1 -1, -1 -1 -2; and 1 2 3, 0 0 1.

Thus we have $d_{2\omega_3}^{\varnothing} \geqslant 3 + 10 + 6 = 19$. We now suppose that a short root lies in $\Pi(V) \setminus \Lambda$, without loss take this to be 0 0 1. Then the following weights lie in Λ: 1 2 3, 1 1 2, 0 1 2, 1 1 1, 1 0 1, 0 1 1, -1 0 1, 0 -1 0, 0 -1 -1, -1 -1 0, -1 -1 -1, -1 -2 -1; so $d_{2\omega_3}^{\varnothing} \geqslant 3 + 16 = 19$.

Suppose that $X = B_1$ with short root α_3. The weights in $\Pi(2\omega_3)$ can be arranged into nine clusters as follows.

1 2 3	1 1 2	0 1 2	1 0 1	0 0 1	-1 0 1	0 -1 0	-1 -1 0	-1 -2 -1
1 2 2	1 1 1	0 1 1	1 0 0	0 0 0	-1 0 0	0 -1 -1	-1 -1 -1	-1 -2 -2
1 2 1	1 1 0	0 1 0	1 0 -1	0 0 -1	-1 0 -1	0 -1 -2	-1 -1 -2	-1 -2 -3

By reading from left to right, the 2nd, 3rd, 7th and 8th clusters have size 4 and the 5th cluster has size 7. If the 5th cluster lies in Λ then $d_{2\omega_2}^{B_1} \geqslant 28$ since all other clusters contain roots. Assume therefore that the 5th cluster lies in $\Pi(V) \setminus \Lambda$. The 1st, 2nd and 3rd and the 7th, 8th and 9th clusters form two cliques so $d_{2\omega_2}^{B_1} \geqslant 7 + 14 = 21$. Thus ($\Diamond$) holds.

Suppose that $X = A_1$ with long root α_1; here $e_{2\omega_3}^{A_1} = 16$. Consider the following seven clusters.

1 1 2	1 1 1	1 0 1	1 0 0	1 0 -1	0 -1 -1	0 -1 -2
0 1 2	0 1 1	0 0 1	0 0 0	0 0 -1	-1 -1 -1	-1 -1 -2
		-1 0 1	-1 0 0	-1 0 -1		

Assume that $000 \in \Pi(V) \setminus \Lambda$. Then all clusters above not containing 000 lie in Λ, whence $d_{2\omega_3}^{A_1} \geqslant 20$ and we are done. Instead let us assume that $000 \in \Lambda$; we can form three cliques from the 1st, 2nd and 3rd clusters and the 5th, 6th and 7th clusters. Thus $d_{2\omega_3}^{A_1} \geqslant 5 + 6 + 6 = 17$. $\square$

Chapter 7

Groups of type C_n

In this chapter we shall assume that G is a simple simply connected algebraic group of type C_n defined over an algebraically closed field K and $V = L(\lambda)$ is an irreducible G-module with p-restricted highest weight λ. As in the previous chapter, it will be sensible to treat the cases $p \neq 2$ and $p = 2$ separately since the conclusion of Premet's theorem does not hold in the latter case. We may assume that $n \geqslant 3$ since we studied $B_2(K)$ in the previous chapter and $B_2(K) \cong C_2(K)$.

We shall prove the following result.

Theorem 7.1. *Let $G = \mathrm{Sp}_{2n}(K)$ act on $V = L(\lambda)$. If $\dim V \leqslant \dim G + 2$ then $\dim \overline{E} = \dim V$ with the possible exceptions of $L(\omega_2)$ for $n \in [3, \infty)$, $L(\omega_n)$ for $n \in [5, 6]$ and $p = 2$, and $L(\omega_3)$ for $n = 3$ and $p \neq 2$; if instead $\dim V > \dim G + 2$ then $\dim \overline{E} < \dim V$ with the possible exceptions of $L(\omega_4)$ with $n = 4$ and $p \neq 2$, and $L(\omega_3)$ with $n = 4$ and $p = 3$.*

This theorem is a consequence of the results to follow in later sections.

7.1 Initial survey

Assume throughout this section that $p \neq 2$. Consider $\mu \leqslant \lambda$ where $\mu = \sum_{i=1}^{n} a_i \omega_i$ is a dominant weight. We shall begin by obtaining conditions on the coefficients a_i in order to show that (†) is satisfied for n large enough. This will allow us later to list modules which will require further consideration. Recall that we may assume that there is a short root outside the root system of the centraliser of a non-central semisimple element and in calculations to follow we shall be considering short roots only: see Section 4.3.

It is useful to note that $\omega_{k-2} < \omega_k$ for $k \geqslant 3$; this is clear since each fundamental weight is of the form $\omega_i = \alpha_1 + 2\alpha_2 + \cdots + (i-1)\alpha_{i-1} + i\left(\alpha_i + \cdots + \alpha_{n-1} + \frac{1}{2}\alpha_n\right)$ for each $i \in [1, n]$.

Proposition 7.2. *Suppose that $\mu = \sum_{i=1}^{n} a_i\omega_i \leqslant \lambda$ is a dominant weight, and at least one of the following conditions holds:*

(i) $n \geqslant 9$ and $a_k \neq 0$ for some $k \in [3, n]$;

(ii) $n \in [7, 8]$ and $a_k \neq 0$ for some $k \in [4, n]$;

(iii) $n = 6$ and either $a_4 \neq 0$ or $a_6 \neq 0$;

(iv) $n \geqslant 5$ and $a_i, a_j \neq 0$ for some $i \in [1, 3]$ and $j \in [n-1, n]$;

(v) $n = 4$ and $a_i, a_3 \neq 0$ for some $i \in [1, 2]$.

Then (†) holds.

Proof. Assume that $a_k \neq 0$ for some $k \in [3, n]$. Then $\Psi = \langle \alpha_i \mid a_i = 0 \rangle$ is contained in $\Phi(A_{k-1}C_{n-k})$ and

$$r_\Psi \geqslant r_{A_{k-1}C_{n-k}} = 2^{k-2}k\binom{n}{k}\frac{4n-3k-1}{n(n-1)}.$$

If $k = 4$ then $r_{A_3C_{n-4}} \geqslant \dim G = n(2n+1)$ for $n \geqslant 6$ and $r_{A_3C_{n-4}} \leqslant r_{A_{k-1}C_{n-k}}$ for $k \in [4, n-2]$. Thus (†) holds for $n \geqslant 6$ provided that $a_k \neq 0$ for $k \in [4, n-2]$. Also, by taking $k = 3$ and $k = n-1$, we find that $r_{A_2C_{n-3}} > \dim G$ if $n \geqslant 9$ and $r_{A_{n-2}C_1} > \dim G$ if $n \geqslant 7$. If $k = n$ then $r_{A_{n-1}} > \dim G$ whenever $n \geqslant 10$; moreover, if $n \in [6, 9]$ and $a_n \neq 0$ then (†) holds by Premet's theorem since $\omega_{n-2} < \omega_n$.

Suppose that both $a_1 \neq 0$ and $a_n \neq 0$. Then $\Psi \subset \Phi(A_{n-2})$ and $r_\Psi \geqslant r_{A_{n-2}} = 2^{n-2}(n+2)$ exceeds $\dim G$ for $n \geqslant 5$. Furthermore, since $r_{A_{n-2}} \leqslant r_{A_1A_{n-3}} \leqslant r_{A_2A_{n-4}}$, we cannot have $a_n \neq 0$ and either a_2 or a_3 non-zero.

We calculate $r_{A_2A_{n-5}C_1} = \frac{2^{n-4}}{3}(n-2)(n-3)(n^2+7n-26)$, $r_{A_1A_{n-4}C_1} = 2^{n-4}(n-2)(n^2+5n-14)$ and $r_{A_{n-3}C_1} = 2^{n-3}(n^2+3n-6)$ so that (†) holds if both a_3 and a_{n-1} are non-zero for $n \geqslant 5$ and if both a_i and a_{n-1} are non-zero with $i \in [1,2]$ for $n \geqslant 4$. $\square$

The proposition above very much restricts the possible highest weights for our irreducible modules.

The modules that have dimension at most $\dim G + 2$ are as follows: $L(\omega_1)$, $L(\omega_2)$ and the adjoint module $L(2\omega_1)$ for $n \geqslant 3$, and the module $L(\omega_3)$ for $n = 3$. Only $L(\omega_1)$ for $n \geqslant 3$ has dimension less than $\dim B = n(n+1)$ so we are left to consider $L(\omega_2)$ for $n \geqslant 3$ and $L(\omega_3)$ for $n = 3$.

Lemma 7.3. *Suppose that* $\dim V > \dim G + 2$. *If* $n \geqslant 3$ *then* (†) *holds except possibly for the modules with highest weight* ω_4 *for* $n \in [4,5]$, $\omega_1 + \omega_3$ *for* $n = 3$, ω_5 *for* $n = 5$, ω_3 *for* $n \in [4,8]$, *and both* $\omega_1 + \omega_2$ *and* $3\omega_1$ $(p > 3)$ *for* $n \in [3,4]$.

Proof. We shall split the analysis of weights to follow into six parts according as $n \geqslant 9$, $n \in [7,8]$ or $n = 6, 5, 4$ or 3.

Case I: $n \geqslant 9$. By condition (i) of Proposition 7.2 we may assume that $\mu = a_1\omega_1 + a_2\omega_2$. Let us set $m_1 = \sum_{i=1}^{2} ia_i$ so that, as in Lemma 5.3, if $m_1 \geqslant 3$ then for some b_i with $0 \leqslant b_i \leqslant a_i$ for each $i \in [1,2]$ we have

$$\mu' = \mu - \left(\sum_{i=1}^{2} b_i\omega_i - \omega_j\right) < \mu \leqslant \lambda,$$

where $j = \sum_{i=1}^{2} ib_i$ and $3 \leqslant j \leqslant 4$. Since the coefficient a'_j of μ' is non-zero, by applying Premet's theorem, we may assume that $m_1 \leqslant 2$. Hence the possible weights μ correspond to modules with dimension at most $\dim G$.

Case II: $n \in [7,8]$. By condition (ii) of Proposition 7.2, we may assume that $a_k = 0$ for $k \in [4,n]$, whence $\mu = \sum_{i=1}^{3} a_i\omega_i$. Set $M_1 = \sum_{i=1}^{3} ia_i$; if $M_1 \geqslant 4$ then, as in Lemma 5.3 there exist b_i for each $i \in [1,3]$ satisfying $0 \leqslant b_i \leqslant a_i$ such that

$$\mu' = \mu - \left(\sum_{i=1}^{3} b_i\omega_i - \omega_j\right) < \mu \leqslant \lambda,$$

where $j = \sum_{i=1}^{3} ib_i$ and $4 \leqslant j \leqslant 6$. Since $a'_j \neq 0$ in μ' we may assume, by Premet's theorem, that $M_1 \leqslant 3$.

Considering the weight $\omega_1 + \omega_2$ we calculate $r_{A_2C_{n-3}} = 2(n-2)(2n-5)$ and $r_{C_{n-2}} = 4(2n-3)$ since $\omega_1 + \omega_2 > \omega_3$. Thus we have $|\Lambda| \geqslant 44 + 90 = 134$ for $n = 7$ and $|\Lambda| \geqslant 52 + 132 = 184$ for $n = 8$, and (†) holds in both cases. Therefore we must consider further the weight ω_3 for $n \in [7, 8]$.

Case III: $n = 6$. By conditions (iii) and (iv) of Proposition 7.2 we may assume that $\mu = \sum_{i=1}^{3} a_i\omega_i + a_5\omega_5$ and that we do not have both $a_i \neq 0$ and $a_5 \neq 0$ for $i \in [1, 3]$. Thus, if $a_5 \neq 0$ then $\mu = a_5\omega_5$. In fact (†) always holds in this case since $\omega_5 > \omega_3$ and $r_{A_4C_1} + r_{A_2C_3} = 64 + 56 = 120 > \dim G$. We may now assume that $\mu = \sum_{i=1}^{3} a_i\omega_i$. Suppose first that precisely two coefficients of μ are non-zero. The condition (†) holds if a_1 and a_3 are non-zero since $\omega_1 + \omega_3 > \omega_4$, and if a_1 and a_2 are non-zero since $\omega_1 + \omega_2 > \omega_3$ and $r_{C_4} + r_{A_2C_3} = 36 + 56 = 92$. Similarly if a_2 and a_3 are non-zero since $\omega_2 + \omega_3 > \omega_1 + \omega_2$. In particular this shows that at most one coefficient of μ can be non-zero. We can use Premet's theorem and the partial orderings $3\omega_1 > \omega_1 + \omega_2$, $2\omega_2 > \omega_4$ and $2\omega_3 > \omega_1 + \omega_3$ to deduce that we need only to consider further the weight ω_3 for $n = 6$.

Case IV: $n = 5$. By condition (iv) of Proposition 7.2 we may assume that we do not have both $a_i \neq 0$ and $a_j \neq 0$ for $i \in [1, 3]$ and $j \in [4, 5]$. If both a_4 and a_5 are non-zero then (†) holds since $r_{A_3} = 56$. If $a_4 \neq 0$ then we may assume $a_4 = 1$ since $2\omega_4 > \omega_2 + \omega_4$. Similarly, if $a_5 \neq 0$ then we may assume $a_5 = 1$ since $2\omega_5 > \omega_3 + \omega_5$.

We may now assume that $\mu = \sum_{i=1}^{3} a_i\omega_i$. If both a_2 and a_3 are non-zero then (†) holds since $r_{A_1C_2} = 102$; the same is true if both a_1 and a_3 are non-zero. If both a_1 and a_2 are non-zero then (†) holds by Premet's theorem since $\omega_1 + \omega_2 > \omega_3$ and we have $r_{C_3} + r_{A_2C_2} = 28 + 30 = 58$. Thus we can assume that precisely one coefficient of μ is non-zero. We must have $a_1 \leqslant 2$ since $3\omega_1 > \omega_1 + \omega_2$. Also if $a_2 \neq 0$ then $a_2 = 1$ since $2\omega_2 > \omega_1 + \omega_3$ and if $a_3 \neq 0$ then $a_3 = 1$ since $2\omega_3 > \omega_2 + \omega_4$. Therefore we must consider further ω_3, ω_4 and ω_5 for $n = 5$.

Case V: $n = 4$. By condition (v) of Proposition 7.2 we see that (†) holds if both $a_i \neq 0$ and $a_3 \neq 0$ for $i \in [1, 2]$. Suppose that both a_1 and a_4 are non-zero. Then $\Psi \subset \Phi(A_2)$

and $r_{A_2} = 24$. We see that (†) holds since $r_{C_2} = 20$ and $\omega_1 + \omega_4 > \omega_1 + \omega_2$. Similarly we are done if both a_2 and a_4 are non-zero since then $\Psi \subset \Phi(A_1^2)$ and $r_{A_1^2} = 40$. Therefore we can have at most two coefficients of μ non-zero and if two coefficients are non-zero then either $a_1 \neq 0$ and $a_2 \neq 0$ or $a_3 \neq 0$ and $a_4 \neq 0$. The condition (†) holds in the latter case since $\omega_3 + \omega_4 > \omega_2 + \omega_3$. In the former case we must have $a_1 = a_2 = 1$ since $2\omega_1 + \omega_2 > \omega_1 + \omega_3$ and $\omega_1 + 2\omega_2 > \omega_2 + \omega_3$.

Assume that precisely one coefficient of μ is non-zero. If $a_1 \geqslant 4$ then (†) holds since $4\omega_1 > \omega_1 + \omega_3$. Similarly if $a_i \geqslant 2$ for $i \in [2, 4]$ since $2\omega_4 > 2\omega_3 > 2\omega_2 > \omega_1 + \omega_3$. Therefore we must consider further $\omega_1 + \omega_2$, $3\omega_1$, ω_3 and ω_4.

Case VI: $n = 3$. If both a_2 and a_3 are non-zero then (†) holds by Premet since $\omega_2 + \omega_3 > \omega_1 + \omega_2$ and we have $r_{A_1} = 10$ and $r_{C_1} = 12$. In particular we see that at most two coefficients of μ can be non-zero. If both a_1 and a_3 are non-zero then (†) holds if $a_1 + a_3 > 2$ since $\omega_1 + 2\omega_3 > \omega_2 + \omega_3$ and $2\omega_1 + \omega_3 > \omega_2 + \omega_3$. The weight $\omega_1 + \omega_3$ satisfies $\omega_1 + \omega_3 > 2\omega_1 > \omega_2$ but $r_{A_1} + r_{C_2} + r_{A_1C_1} = 10 + 2 + 5 = 17$ which does not exceed $\dim G$. If both a_1 and a_2 are non-zero then (†) holds if $a_1 + a_2 > 2$ since $\omega_1 + 2\omega_2 > \omega_2 + \omega_3$ and since $2\omega_1 + \omega_2 > \omega_1 + \omega_3$. Now assuming that only one coefficient of μ is non-zero we may take $a_1 \leqslant 3$ since $4\omega_1 > 2\omega_1 + \omega_2$ in which case (†) holds, $a_2 = 1$ by above since $2\omega_2 > \omega_1 + \omega_3$ and $a_3 = 0$ since $2\omega_3 > 2\omega_2$ and $L(\omega_3)$ has dimension less than $\dim G$. We must consider further $\omega_1 + \omega_2$, $\omega_1 + \omega_3$ and $3\omega_1$. □

7.2 Even characteristic

Assume now that $p = 2$. By Zalesski's Theorem 2.7 and Theorem 2.8 any 2-restricted weight with short support can be treated in the same way as the previous section since the conclusion of Premet's theorem holds true and the associated module is tensor indecomposable. Thus (†) holds for all 2-restricted modules with short support except those listed in Lemma 7.3.

The tensor decomposable modules take the form $L\left(\sum_{i=1}^{n-1} a_i\omega_i\right) \otimes L(\omega_n)$ where each $a_i \in \{0, 1\}$. Note that this module is isomorphic to $L\left(\sum_{i=1}^{n-1} a_i\omega_i + \omega_n\right)$ by Theorem 2.4.

We shall use Proposition 4.3 to show that (†) holds for these modules. In Lemma 7.3 we have shown that (†) holds for all but a short list of modules (most of which are 2-restricted with short support). Therefore, after applying Proposition 4.3 it remains to investigate the modules $L(\mu) \otimes L(\omega_n)$ where μ is one of ω_4 for $n = 5$, ω_3 for $n \in [4, 8]$, and $\omega_1 + \omega_2$ for $n \in [3, 4]$ (the relevant exceptions in Lemma 7.3), as well as ω_1 and ω_2 both for $n \in [3, \infty)$ (since $L(\omega_1)$ and $L(\omega_2)$ have dimensions at most $\dim G$).

Lemma 7.4. *If $G = \mathrm{Sp}_{2n}(K)$ acts on a tensor decomposable p-restricted module V then (†) holds except possibly if $V = L(\omega_1) \otimes L(\omega_n)$ for $n \in [3, 4]$.*

Proof. We shall apply Proposition 4.3 to each of the tensor decomposable modules. First we notice that, in any case, we may assume $n \leqslant 6$ since $\dim L(\omega_n) > \dim G$ otherwise.

Consider $L(\omega_1 + \omega_2) \otimes L(\omega_n)$ for $n \in [3, 4]$. Recall that the conclusion of Premet's theorem holds for $\omega_1 + \omega_2 \in X(T)_S$. We find that

$$|\Lambda_{\omega_1+\omega_2}| \geqslant r_{C_{n-2}} = \frac{1}{2}\frac{2^n n!}{2^{n-2}(n-2)!}\frac{2n(n-1) - 2(n-2)(n-3)}{2n(n-1)} = 4(2n-3),$$

whence $|\Lambda_{\omega_1+\omega_2}| \dim L(\omega_n) \geqslant 2^{n+2}(2n-3)$; this exceeds $\dim G$ for $n \in [3, 4]$.

Consider $L(\omega_4) \otimes L(\omega_5)$ for $n = 5$. Since $|\Lambda_{\omega_4}| \geqslant r_{A_3C_1} = 28$ and $\dim L(\omega_5) = 32$ we have that (†) holds.

Consider $L(\omega_2) \otimes L(\omega_n)$ for $n \in [3, 6]$. We find that $|\Lambda_{\omega_2}| \geqslant r_{A_1C_{n-2}} = 4n - 7$ so $|\Lambda_{\omega_2}| \dim L(\omega_n) > \dim G$ for $n \in [3, 6]$. We deal with $L(\omega_3) \otimes L(\omega_n)$ for $n \in [4, 6]$ analogously since $r_{A_2C_{n-3}} = 2(n-2)(2n-5) > r_{A_1C_{n-2}}$.

Finally, consider the module $L(\omega_1) \otimes L(\omega_n)$ for $n \in [3, 6]$. We have $|\Lambda_{\omega_n}| \geqslant r_{A_{n-1}} = 2^{n-2}$, whence $|\Lambda_{\omega_n}| \dim L(\omega_1) \geqslant 2^{n-1}n$ exceeds $\dim G$ for $n \geqslant 5$. □

We must examine $L(\omega_n)$ for $p = 2$ and $n \in [7, \infty)$; this module has dimension 2^n. We have $\langle \omega_n, \alpha \rangle = 0$ or ± 2 for short roots $\alpha \in \Phi(C_n)$ and in particular, $\langle \omega_n, \alpha \rangle = 2$ for $\frac{1}{2}n(n-1)$ short roots. Thus, for a fixed short root α there are $\frac{2^{n-1}n(n-1)}{2n(n-1)} = 2^{n-2}$ weights $\mu \in W.\omega_n$ with weight string containing μ $\mu + 2\alpha$. Since $p = 2$ we obtain $|\Lambda| \geqslant 2^{n-2}$ by the adjacency principle; this exceeds $\dim G$ for $n \geqslant 10$. Hence, in the next section, it remains for us to consider $L(\omega_n)$ for $p = 2$ and $n \in [7, 9]$.

7.3 Weight string analysis

We list in Table 7.1 the weights which require further consideration.

λ	n	Lemma
ω_4	5	7.6 (†) ($p \neq 3$), (◊) ($p = 3$)
	4 ($p \neq 2$)	7.6 (◊)[a]
$\omega_1 + \omega_n$	4 ($p = 2$)	7.5 (†)
	3	7.5 (†)
ω_5	5 ($p \neq 2$)	7.6 (◊)
ω_3	$[7, 8]$	7.7 (†)
	$[5, 6]$	7.7 (◊)
	4	7.7 (◊)[b]
$\omega_1 + \omega_2$	$[3, 4]$	7.5 (†)
$3\omega_1$	$[3, 4]$ ($p \neq 2, 3$)	7.5 (†)
ω_n	$[7, 9]$ ($p = 2$)	7.8 (◊)

Except for: [a] $X = A_1^2$, A_3 and C_2^2,
[b] $X = A_3$ if $p = 3$.

Table 7.1: Possible weights in type C_n for $n \geqslant 3$

Lemma 7.5. *Let G act on the irreducible module V where we take V to be one of $L(\omega_1 + \omega_3)$ for $n = 3$, $L(\omega_1 + \omega_4)$ for $n = 4$ and $p = 2$, $L(3\omega_1)$ for $n \in [3, 4]$ and $p > 3$, or $L(\omega_1 + \omega_2)$ for $n \in [3, 4]$. Then (†) holds in each case.*

Proof. Consider $L(\omega_1 + \omega_3)$ with $n = 3$. In the last column of the second table in Figure 7.1 we are using fact that if $p = 2$ then $2\alpha(s) = 1$ if and only if $\alpha(s) = 1$. We see that (†) holds for all p.

Now consider $L(\omega_1 + \omega_4)$ for $n = 4$ and $p = 2$. By ignoring the weights in $W.\omega_1$ and $W.(\omega_1 + \omega_2)$ (since they occur with multiplicity zero) we see from Figure 7.2 below that (†) holds.

Consider the module $L(3\omega_1)$ for $n \in [3, 4]$; we have $3\omega_1 > \omega_1 + \omega_2 > \omega_3 > \omega_1$. It is more convenient to take these two low rank cases separately since certain weight strings simply do not occur for $n = 3$.

i	ω	$\lvert W.\omega \rvert$	m_ω		
			$p \neq 2,3$	$p = 3$	$p = 2$
3	$\omega_1 + \omega_3$	24	1	1	1
2	$2\omega_1$	6	1	1	0
1	ω_2	12	3	2	2
0	0	1	4	3	0

Weight strings	No. of strings	l		
		$p \neq 2,3$	$p = 3$	$p = 2$
μ_3	4			
$\mu_3\ \mu_3$	4	4	4	4
$\mu_3\ \mu_2\ \mu_3$	2	2	2	2
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	4	16	12	16
$\mu_2\ \mu_2$	2	2	2	
μ_1	2			
$\mu_1\ \mu_0\ \mu_1$	1	4	3	2
Lower bound on $\lvert\Lambda\rvert$		28	23	24

Figure 7.1: $(\lambda, n) = (\omega_1 + \omega_3, 3)$

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p = 2$
4	$\omega_1 + \omega_4$	64	1
3	$\omega_1 + \omega_2$	48	0
2	ω_3	32	2
1	ω_1	8	0

Weight strings	No. of strings	l
		$p = 2$
μ_4	16	
$\mu_4\ \mu_4$	8	8
$\mu_4\ \mu_3\ \mu_4$	8	8
$\mu_4\ \mu_2\ \mu_2\ \mu_4$	8	32
$\mu_3\ \mu_3$	10	
$\mu_3\ \mu_2\ \mu_3$	8	
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	
$\mu_2\ \mu_1\ \mu_2$	4	8
Lower bound on $\lvert\Lambda\rvert$		56

Figure 7.2: $(\lambda, n) = (\omega_1 + \omega_4, 4)$

If $n = 4$ the usual calculations with weight strings shows that $|\Lambda| \geqslant 2r_{C_3} + r_{C_2} + r_{A_2C_1} = 4 + 20 + 12 = 36 = \dim G$. We show in Figure 7.3 that (†) holds by using weight multiplicities.

i	ω	$\lvert W.\omega \rvert$	m_ω
			$p \neq 2,3$
4	$3\omega_1$	8	1
3	$\omega_1 + \omega_2$	48	1
2	ω_3	32	1
1	ω_1	8	4

Weight strings	No. of strings	l
		$p \neq 2,3$
μ_4	4	
$\mu_4\ \mu_3\ \mu_3\ \mu_4$	2	4
μ_3	8	
$\mu_3\ \mu_3$	8	8
$\mu_3\ \mu_2\ \mu_3$	8	8
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	10
$\mu_2\ \mu_2$	8	8
$\mu_2\ \mu_1\ \mu_2$	4	8
Lower bound on $\lvert\Lambda\rvert$		46

Figure 7.3: $(\lambda, n) = (3\omega_1, 4)$

Similarly, if $n = 3$ we have $|\Lambda| \geqslant 2r_{C_2} + r_{C_1} + r_{A_2} = 4 + 12 + 2 = 18 = |\Phi(G)|$, and we show in Figure 7.4 that (†) holds by using weight multiplicities.

Consider $L(\omega_1 + \omega_2)$ for $n \in [3, 4]$. In Figures 7.5 and 7.6 for $n = 4$ and $n = 3$ we show that (†) holds for all characteristics. $\square$

Recall from [2, p.47] that for the root system of type C_n we have $\Phi = \Phi_S \cup \Phi_L$ where $\Phi_S = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant n\}$ consists of $2n(n-1)$ short roots and $\Phi_L = \{\pm 2\epsilon_i \mid 1 \leqslant$

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2, 3$
4	$3\omega_1$	6	1
3	$\omega_1 + \omega_2$	24	1
2	ω_3	8	1
1	ω_1	6	3

Weight strings	No. of strings	l
		$p \neq 2, 3$
μ_4	2	
$\mu_4\ \mu_3\ \mu_3\ \mu_4$	2	4
$\mu_3\ \mu_3$	4	4
$\mu_3\ \mu_2\ \mu_3$	4	4
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	8
$\mu_2\ \mu_1\ \mu_2$	2	4
Lower bound on $\lvert\Lambda\rvert$		24

Figure 7.4: $(\lambda, n) = (3\omega_1, 3)$

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 3$	$p = 3$
3	$\omega_1 + \omega_2$	48	1	1
2	ω_3	32	2	1
1	ω_1	8	6	4

Weight strings	No. of strings	l		
		$p \neq 2, 3$	$p = 3$	$p = 2$
μ_3	8			
$\mu_3\ \mu_3$	10	10	10	10
$\mu_3\ \mu_2\ \mu_3$	8	16	8	16
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	14	10	16
$\mu_2\ \mu_2$	8	16	8	16
$\mu_2\ \mu_1\ \mu_2$	4	16	8	16
Lower bound on $\lvert\Lambda\rvert$		72	44	74

Figure 7.5: $(\lambda, n) = (\omega_1 + \omega_2, 4)$

$i \leqslant n\}$ consists of $2n$ long roots. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [1, n-1]$ and $\alpha_n = 2\epsilon_n$ be the simple roots. Therefore we have $\omega_i = \sum_{k=1}^{i} \epsilon_k$. We shall need this for the next lemma.

Lemma 7.6. *Let G act on the irreducible module V where we take V to be one of $L(\omega_5)$ for $n = 5$ with $p \neq 2$, $L(\omega_4)$ for $n = 4$ with $p \neq 2$ or $L(\omega_4)$ for $n = 5$. If $V = L(\omega_4)$ for $n = 5$ with $p \neq 3$ then (†) holds. If V is either $L(\omega_5)$ for $n = 5$ with $p \neq 2$ or $L(\omega_4)$ for $n = 5$ with $p = 3$ then ($\Diamond$) is satisfied for all $X \in I$. If $V = L(\omega_4)$ for $n = 4$ with $p \neq 2$ then ($\Diamond$) holds for all $X \in I \setminus \{A_1^2, A_3, C_2^2\}$.*

Proof. Consider the module $L(\omega_4)$ for $n = 5$. In Figure 7.7 we see that (†) holds unless $p = 3$.

If $p = 3$ then ($\Diamond$) holds for each $X \in I$ such that $|\Phi(X)| > 9$. We may therefore assume that there are two orthogonal roots, α (short) and β (long), outside Φ_s. It is straightforward to check that $\langle \omega_4, \alpha \rangle = 0$, ± 1 or ± 2 and $\langle \omega_4, \beta \rangle = 0$ or ± 1. We note that for a root system of type C_n there are $2n.2(n-1)(n-2)$ orthogonal pairs of roots α short, β long, so 240 such pairs if $n = 5$.

We have $\langle \omega_4, \alpha \rangle = 2$ and $\langle \omega_4, \beta \rangle = 1$ for 12 pairs of roots α, β with $(\alpha, \beta) = 0$. In more detail, the 12 pairs of orthogonal long and short roots are $2\epsilon_i$ and $\epsilon_j + \epsilon_k$ with $i, j, k \in [1, 4]$, $j < k$ and $k \neq i \neq j$. There are $80.12 = 960$ triples $(\mu; \alpha, \beta)$ with $\langle \mu, \alpha \rangle = 2$, $\langle \mu, \beta \rangle = 1$

i	ω	$\|W.\omega\|$	m_ω		
			$p \neq 3, 7$	$p = 7$	$p = 3$
3	$\omega_1 + \omega_2$	24	1	1	1
2	ω_3	8	2	2	1
1	ω_1	6	4	3	3

Weight strings	No. of strings	l			
		$p \neq 2, 3, 7$	$p = 7$	$p = 3$	$p = 2$
$\mu_3\ \mu_3$	6	6	6	6	6
$\mu_3\ \mu_2\ \mu_3$	4	8	8	4	8
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	2	10	8	8	12
$\mu_2\ \mu_1\ \mu_2$	2	8	8	4	8
Lower bound on $\|\Lambda\|$		32	28	22	34

Figure 7.6: $(\lambda, n) = (\omega_1 + \omega_2, 3)$

i	ω	$\|W.\omega\|$	m_ω		
			$p \neq 2, 3$	$p = 2$	$p = 3$
2	ω_4	80	1	1	1
1	ω_2	40	2	2	1
0	0	1	5	4	1

Weight strings	No. of strings	l		
		$p \neq 2, 3$	$p = 2$	$p = 3$
μ_2	24			
$\mu_2\ \mu_2$	16	16	16	16
$\mu_2\ \mu_1\ \mu_2$	12	24	24	12
μ_1	2			
$\mu_1\ \mu_1$	12	24	24	12
$\mu_1\ \mu_0\ \mu_1$	1	4	4	1
Lower bound on $\|\Lambda\|$		68	68	41

Figure 7.7: $(\lambda, n) = (\omega_4, 5)$

for orthogonal roots $\alpha \in \Phi_S$, $\beta \in \Phi_L$ and $\mu \in W.\omega_4$. For a given pair α, β there are $\frac{960}{240} = 4$ weights $\mu \in W.\omega_4$ such that $\langle \mu, \alpha \rangle = 2$ and $\langle \mu, \beta \rangle = 1$. Thus there are four 3×2 weight nets and similarly we calculate that there are four 3×1, eight 2×2 and eight 1×2 weight nets. We cannot obtain 2×1 weight nets since $\langle \omega_4, \alpha \rangle = 1$ and $\langle \omega_4, \beta \rangle = 0$ is not possible and we need not consider weights $\mu \in W.\omega_4$ with $\langle \mu, \alpha \rangle = \langle \mu, \beta \rangle = 0$ since there is no contribution to $|\Lambda|$ from 1×1 weight nets. So from weight nets containing $\mu \in W.\omega_4$ we find that $|\Lambda| \geqslant 4.3 + 4.1 + 8.2 + 8.1 = 40$.

Now we consider the remaining weight nets for $L(\omega_4)$, i.e., those containing $\nu \in W.\omega_2$ and no weights from $W.\omega_4$. As before, we can check that $\langle \omega_2, \alpha \rangle = 0$, ± 1 or ± 2 and $\langle \omega_2, \beta \rangle = 0$ or ± 1. It is not possible to find pairs of orthogonal roots α and β with $\langle \omega_2, \alpha \rangle = 2$ and $\langle \omega_2, \beta \rangle = 1$. We find that there are two 2×2 and eight 2×1 weight nets and there is one 3×1 weight net (which has the form $\mu_1\ \mu_0\ \mu_1$). The contribution to $|\Lambda|$ from the weight nets formed from weights $\nu \in W.\omega_2$ with $\langle \nu, \alpha \rangle = 0$ and $\langle \nu, \beta \rangle = 1$ was counted earlier since these nets appear in the 3×2 weight nets above. Therefore $|\Lambda| \geqslant 40 + 2.2 + 8.1 + 1.1 = 53$ and $(\diamondsuit)$ is satisfied for all $X \in I$.

Consider the module $L(\omega_5)$ for $n = 5$. When $p = 2$ the module has dimension less than $\dim G$, so we assume that $p \neq 2$. There are three dominant weight orbits for this module; they are related as follows: $\omega_5 > \omega_3 > \omega_1$. If $p \neq 2, 3$ then $m_{\omega_5} = m_{\omega_3} = 1$ and $m_{\omega_1} = 2$ whereas if $p = 3$ then $m_\nu = 1$ for each $\nu \in \Pi(\omega_5)$. Thus in the latter case we do not gain anything from considering weight multiplicities. We use Proposition 4.3 to see that $|\Lambda| \geqslant r_{A_4} + r_{A_2C_2} + r_{C_4} = 8 + 30 + 2 = 40$. The condition ($\Diamond$) is satisfied for all $X \in I$ with $|\Phi(X)| > 10$ and it is clear that we may assume that there are two orthogonal roots (one long and one short) not in Φ_s.

As before we may calculate the number of weight nets, however we arrange the weights in $\Pi(\omega_5)$ into weight nets provided explicitly below (ordered by nets of the same size: 1×2, 2×2, 3×2, 2×1, 3×1 and one 3×2 weight net that does not have a negative counterpart since it is unchanged under negation) where the difference between horizontal weights is α_1 and vertical weights is α_5. We omit the eight 1×1 weight nets.

$$\begin{array}{ccccc}
1\,2\,3\,4\,\tfrac{5}{2} & 1\,2\,3\,2\,\tfrac{3}{2} & 1\,2\,2\,2\,\tfrac{3}{2} & 1\,2\,1\,2\,\tfrac{3}{2} & 1\,2\,1\,0\,\tfrac{1}{2} \\
1\,2\,3\,4\,\tfrac{3}{2} & 1\,2\,3\,2\,\tfrac{1}{2} & 1\,2\,2\,2\,\tfrac{1}{2} & 1\,2\,1\,2\,\tfrac{1}{2} & 1\,2\,1\,0\,\text{-}\tfrac{1}{2}
\end{array}$$

$$\begin{array}{cc@{\qquad}cc@{\qquad}cc@{\qquad}cc}
1\,1\,2\,2\,\tfrac{3}{2} & 0\,1\,2\,2\,\tfrac{3}{2} & 1\,1\,1\,2\,\tfrac{3}{2} & 0\,1\,1\,2\,\tfrac{3}{2} & 1\,1\,1\,0\,\tfrac{1}{2} & 0\,1\,1\,0\,\tfrac{1}{2} & 1\,1\,0\,0\,\tfrac{1}{2} & 0\,1\,0\,0\,\tfrac{1}{2} \\
1\,1\,2\,2\,\tfrac{1}{2} & 0\,1\,2\,2\,\tfrac{1}{2} & 1\,1\,1\,2\,\tfrac{1}{2} & 0\,1\,1\,2\,\tfrac{1}{2} & 1\,1\,1\,0\,\text{-}\tfrac{1}{2} & 0\,1\,1\,0\,\text{-}\tfrac{1}{2} & 1\,1\,0\,0\,\text{-}\tfrac{1}{2} & 0\,1\,0\,0\,\text{-}\tfrac{1}{2}
\end{array}$$

$$\begin{array}{ccc@{\qquad}ccc}
1\,0\,1\,2\,\tfrac{3}{2} & 0\,0\,1\,2\,\tfrac{3}{2} & \text{-}1\,0\,1\,2\,\tfrac{3}{2} & 1\,0\,1\,0\,\tfrac{1}{2} & 0\,0\,1\,0\,\tfrac{1}{2} & \text{-}1\,0\,1\,0\,\tfrac{1}{2} \\
1\,0\,1\,2\,\tfrac{1}{2} & 0\,0\,1\,2\,\tfrac{1}{2} & \text{-}1\,0\,1\,2\,\tfrac{1}{2} & 1\,0\,1\,0\,\text{-}\tfrac{1}{2} & 0\,0\,1\,0\,\text{-}\tfrac{1}{2} & \text{-}1\,0\,1\,0\,\text{-}\tfrac{1}{2}
\end{array}$$

$$\begin{array}{cc@{\qquad}cc@{\qquad}cc@{\qquad}cc@{\qquad}cc}
1\,1\,2\,3\,\tfrac{3}{2} & 0\,1\,2\,3\,\tfrac{3}{2} & 1\,1\,2\,1\,\tfrac{1}{2} & 0\,1\,2\,1\,\tfrac{1}{2} & 1\,1\,1\,1\,\tfrac{1}{2} & 0\,1\,1\,1\,\tfrac{1}{2} & 1\,1\,0\,1\,\tfrac{1}{2} & 0\,1\,0\,1\,\tfrac{1}{2} & 1\,1\,0\,\text{-}1\,\text{-}\tfrac{1}{2} & 0\,1\,0\,\text{-}1\,\text{-}\tfrac{1}{2}
\end{array}$$

$$\begin{array}{ccc@{\qquad}ccc}
1\,0\,1\,1\,\tfrac{1}{2} & 0\,0\,1\,1\,\tfrac{1}{2} & \text{-}1\,0\,1\,1\,\tfrac{1}{2} & 1\,0\,0\,1\,\tfrac{1}{2} & 0\,0\,0\,1\,\tfrac{1}{2} & \text{-}1\,0\,0\,1\,\tfrac{1}{2}
\end{array}$$

$$\begin{array}{ccc}
1\,0\,0\,0\,\tfrac{1}{2} & 0\,0\,0\,0\,\tfrac{1}{2} & \text{-}1\,0\,0\,0\,\tfrac{1}{2} \\
1\,0\,0\,0\,\text{-}\tfrac{1}{2} & 0\,0\,0\,0\,\text{-}\tfrac{1}{2} & \text{-}1\,0\,0\,0\,\text{-}\tfrac{1}{2}
\end{array}$$

If we take the weight nets above together with the corresponding negative versions for all but the last weight net then, by the adjacency principle, we have $|\Lambda| \geqslant 55$ if $p = 3$ and $|\Lambda| \geqslant 62$ if $p \neq 2, 3$. Thus ($\Diamond$) is satisfied for all $X \in I$.

Consider the module $L(\omega_4)$ for $n = 4$. If $p = 2$ then $\dim L(\omega_4) = 16 < 20 = \dim B$ so we may assume that $p \neq 2$. We may use the available information about weight multiplicities to show that $|\Lambda| \geqslant 13$ or 14 according as $p = 3$ or $p \neq 2, 3$. As usual ($\Diamond$) is satisfied for $X \in I$ satisfying $|\Phi(X)| > 18$ or 19; the remaining centraliser types are $\varnothing$, A_1, A_1^2, A_2, A_3, C_1, C_1A_1, C_1A_2, C_1^2, $C_1^2A_1$, C_2, C_2A_1, C_2C_1, C_2^2 and C_3. It is straightforward to show that ($\Diamond$) holds for all but three of these. If X is one of A_3, C_2^2 or A_1^2 then we may have equality in ($\Diamond$) so this module for $p \neq 2$ is a possible exception in Theorem 7.1. $\square$

Lemma 7.7. *Let G act on the irreducible module $L(\omega_3)$ for $n \in [4, 8]$. If $n \in [7, 8]$ then ($\dagger$) holds and if $n \in [5, 6]$ or $n = 4$ with $p \neq 3$ then ($\Diamond$) holds for each $X \in I$. If $n = 4$ with $p = 3$ then ($\Diamond$) holds for all $X \in I \setminus \{A_3\}$.*

Proof. Consider the module $L(\omega_3)$ with $n \in [4, 8]$. It is necessary to separate the case $n = 4 = p+1$ from the others in the right-hand table of Figure 7.8 because $m_{\omega_1} = 1$ here. We see that for $n \in [7, 8]$ the condition ($\dagger$) holds.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \nmid n-1$	$p \mid n-1$
2	ω_3	$\frac{4}{3}n(n-1)(n-2)$	1	1
1	ω_1	$2n$	$n-2$	$n-3$

Weight strings	No. of strings	l		
		$p \nmid (n-1)$	$p \mid (n-1) \neq 3$	$p \mid (n-1) = 3$
μ_2	$\frac{4}{3}(n-2)(n^2-7n+15)$			
$\mu_2\ \mu_2$	$4(n-2)(n-3)$	$4(n-2)(n-3)$	$4(n-2)(n-3)$	8
$\mu_2\ \mu_1\ \mu_2$	$2(n-2)$	$4(n-2)$	$4(n-2)$	4
$\mu_1\ \mu_1$	2	$2(n-2)$	$2(n-3)$	2
Lower bound on $\lvert\Lambda\rvert$		$2(n-2)(2n-3)$	$2(n-1)(2n-5)$	14

Figure 7.8: $\lambda = \omega_3$ for $n \in [4, 8]$

Suppose that we can find two orthogonal roots, α (short) and β (long), outside Φ_s. We see that $\langle \omega_3, \alpha \rangle = 0$, ± 1 or ± 2 and $\langle \omega_3, \beta \rangle = 0$ or ± 1. We note that for a root system of type C_n there are $4n(n-1)(n-2)$ orthogonal pairs of roots α short and β long.

We have $\langle \omega_3, \alpha \rangle = 2$ and $\langle \omega_3, \beta \rangle = 1$ for 3 pairs of roots α, β with $(\alpha, \beta) = 0$. These 3 pairs of orthogonal long and short roots are $\epsilon_2 + \epsilon_3$ and $2\epsilon_1$, $\epsilon_1 + \epsilon_3$ and $2\epsilon_2$, and $2\epsilon_3$ and $\epsilon_1 + \epsilon_2$. There are $\frac{4}{3}n(n-1)(n-2).3$ triples $(\mu; \alpha, \beta)$ with $\langle \mu, \alpha \rangle = 2$, $\langle \mu, \beta \rangle = 1$ for orthogonal roots $\alpha \in \Phi_S$, $\beta \in \Phi_L$ and $\mu \in W.\omega_3$. For a given pair α, β there is one weight $\mu \in W.\omega_4$ such that $\langle \mu, \alpha \rangle = 2$ and $\langle \mu, \beta \rangle = 1$. Thus there is one 3×2 weight net and similarly we calculate that there are $2(n-3)$ 3×1, $4(n-3)$ 2×2, $4(n-3)(n-4)$ 2×1 and $2 + 2(n-3)(n-4)$ 1×2 weight nets.

Now we consider the remaining weight nets for $L(\omega_3)$, i.e., those containing $\nu \in W.\omega_1$ and no weights from $W.\omega_3$. As before, we can check that $\langle \omega_1, \alpha \rangle = 0$ or ± 1 and $\langle \omega_1, \beta \rangle = 0$

or ± 1. It is not possible to find pairs of orthogonal roots α and β with $\langle \omega_1, \alpha \rangle = \langle \omega_1, \beta \rangle = 1$. We find that there are two 2×1 weight nets. The contribution to $|\Lambda|$ from the weight net formed from weights $\nu \in W.\omega_1$ with $\langle \nu, \alpha \rangle = 0$ and $\langle \nu, \beta \rangle = 1$ was counted earlier since this net appears in the 3×2 weight net above.

If $n = 6$ then from Figure 7.8 we see that $|\Lambda| \geqslant 72$ or 70 according as $p \neq 5$ or $p = 5$. There are certainly two orthogonal roots, one short and one long, outside Φ_s. Therefore, if $p = 5$ we have $|\Lambda| \geqslant 14 + 24 + 24 + 12 + 5 + 6 = 85$. This exceeds $\dim G$ and the same is true for $p \neq 5$ since the weights in $W.\omega_1$ have larger multiplicity.

If $n = 5$ we see from Figure 7.8 that $|\Lambda| \geqslant 40$ or 42 according as $p = 2$ or $p \neq 2$ and if $n = 4$ then $|\Lambda| \geqslant 14$ or 20 according as $p = 3$ or $p \neq 3$. Thus ($\diamondsuit$) is satisfied for $n = 5$ if $|\Phi(X)| > 8$ or 10 according as $p \neq 2$ or $p = 2$ and for $n = 4$ if $|\Phi(X)| > 12$ or 18 according as $p \neq 3$ or $p = 3$. In fact if $n = 4$ and $p = 3$ then all $X \in I$ are possible except C_3C_1 and C_4.

We provide the subsystems X in Table 7.2 below that remain to be dealt with to show that ($\diamondsuit$) is satisfied for all $X \in I$. For $n = 5$ there are three subsystems which need consideration for $p = 2$ only; these are presented at the end of the $n = 5$ row. Similarly with $n = 4$ there are two subsystems to be treated for $p = 3$ only.

n	Centraliser type
5	$\varnothing$, A_1, C_1, A_1^2, A_1C_1, C_1^2, $A_1C_1^2$, $A_1^2C_1$, A_2, A_2A_1, A_2C_1, C_2; $A_2C_1^2$, C_2A_1, C_2C_1
4	$\varnothing$, A_1, C_1, A_1^2, A_1C_1, C_1^2, $A_1C_1^2$, A_2, A_2C_1, C_2, C_2A_1, C_2C_1, A_3; C_2^2, C_3

Table 7.2: Centraliser types requiring consideration for $L(\omega_3)$ with $n \in [4, 5]$

If $n = 5$ we may assume that there are two orthogonal roots, α (short) and β (long), outside Φ_s. Using the weight nets calculated above we find that $|\Lambda| \geqslant 49$ or 47 according as $p \neq 2$ or $p = 2$. We see that ($\diamondsuit$) is satisfied unless $X = \varnothing$ (for all p), or A_1 or C_1 for $p = 2$.

We may take three orthogonal roots outside Φ_s, two short roots and one long root as in Lemma 7.6. Set $x = \langle \omega_3, \alpha \rangle$, $y = \langle \omega_3, \beta \rangle$ and $z = \langle \omega_3, \gamma \rangle$ where α and γ are short roots and β is a long root. Note that there are the same number of triples of roots α, β, γ taking values (a, b, c) and (c, b, a). We find that there are 12 triples of such roots with $(x, y, z) = (2, 1, 0)$, 48 with $(x, y, z) = (1, 1, 1)$, 48 with $(x, y, z) = (0, 1, 0)$, 24 with $(x, y, z) = (2, 0, 1)$ and 48 with $(x, y, z) = (1, 0, 0)$. Since $|W.\omega_3| = 80$ and there are 960 triples of two short roots and one long root (mutually orthogonal) we calculate that, for a given triple α, β, γ there is one weight $\mu \in W.\omega_3$ with $3 \times 2 \times 1$ weight net, one weight with $1 \times 2 \times 3$ weight net, four weights with $2 \times 2 \times 2$ weight nets, four weights with $1 \times 2 \times 1$ weight net, two weights with $3 \times 1 \times 2$ weight net, two weights with $2 \times 1 \times 3$ weight net, four weights with $2 \times 1 \times 1$ weight net and four weights with $1 \times 1 \times 2$ weight net. Note that weight nets with side of length 3 here consist of weight strings of the form $\mu_2\ \mu_1\ \mu_2$.

Since $\langle \omega_1, \alpha \rangle = \langle \omega_1, \gamma \rangle = 0$ and $\langle \omega_1, \beta \rangle = 1$ for 96 triples of roots α, β, γ and $W.\omega_1 = 10$ we see, by taking such a triple, that there is only one weight with $1 \times 2 \times 1$ weight net. This means that the string $\mu_1\ \mu_1$ (where the weights in this string differ by β) in the $3 \times 2 \times 1$ and the $1 \times 2 \times 3$ weight net are the same. These should be combined to form a three-dimensional plus shape consisting of eight weights in $W.\omega_3$ occurring with multiplicity 1 and two weights in $W.\omega_1$ occurring with multiplicity 3 or 2 according as $p \neq 2$ or $p = 2$.

If we take α, β, $\gamma \notin \Phi_s$ then we have $|\Lambda| \geqslant 7$ or 8 from the plus-shaped net according as $p \neq 2$ or $p = 2$ and $|\Lambda| \geqslant 10$ for each of the $3 \times 1 \times 2$ and $2 \times 1 \times 3$ weight nets for any p. Thus $|\Lambda| \geqslant 55$ or 56 according as $p \neq 2$ or $p = 2$. In both cases ($\diamondsuit$) is satisfied for all $X \in I$.

If $n = 4$ with $p \neq 3$ then we can take two orthogonal roots outside Φ_s as before to show that $|\Lambda| \geqslant 22$, which is a small improvement. If $p = 3$ we need to deal with the cases $X = C_2^2$ and C_3 before we can assume that there are two orthogonal roots, one short and one long, not in Φ_s. It is straightforward to run through each of the possibilities for $n = 4$ in Table 7.2 with $p \neq 3$ to show that ($\diamondsuit$) is satisfied. However, if $p = 3$ it is only possible to deal with all $X \in I \setminus \{A_3\}$. Suppose that $X = A_3$ with simple roots α_1, α_2 and α_3. There are four cliques and weights in a clique are characterised by the coefficient of α_4; this can be one of $\frac{3}{2}$, $\frac{1}{2}$, $-\frac{1}{2}$ or $-\frac{3}{2}$. The cliques have size 4, 16, 16 and 4 in the order just mentioned. It is certainly possible that $d_{\omega_3}^{A_3} = 20 = e_{\omega_3}^{A_3}$, hence $2\alpha_4(s) = 1$ and we must have $\alpha_4(s) = -1$. $\square$

We end this chapter by considering the spin module $L(\omega_n)$ for $n \in [7,9]$ with $p = 2$. We may label weights in $W.\omega_n$ by strings consisting of n plus and minus signs. We do this by considering the coefficients of a weight $\mu \in W.\omega_n$, imagining a zero coefficient preceding the original coefficients of μ and then placing a plus sign in the ith position of the string if the $(i+1)$st coefficient is larger than the ith coefficient for $i \in [0, n-2]$ and we place a plus sign in the nth position if twice the nth coefficient is larger than the $(n-1)$st coefficient; otherwise there is a minus sign. This simplifies the task of determining whether the difference between two weights is a root and to determine clusters. Two weights differ by $2(\alpha_i + \cdots + \alpha_j)$ for $j < n$ if they are identical except that a plus and minus sign have been interchanged in the ith and $(j+1)$st positions. Recall that $p = 2$ so $2\alpha(s) = 1$ if and only if $\alpha(s) = 1$. Two weights differ by α_n if they are identical except one has a plus sign in the nth position of the string and the other has a minus sign. It is clear, therefore, that if we compare two weights with one having two plus signs and the other having two minus signs in the ith and jth positions (otherwise identical) then they differ by $2(\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-1} + \alpha_n)$.

Lemma 7.8. *Let G act on the irreducible modules $L(\omega_n)$ for $n \in [7,9]$ with $p = 2$. If $n \in [7,9]$ then $(\Diamond)$ is satisfied for all $X \in I$.*

Proof. Recall that we have $|\Lambda| \geqslant 2^{n-2}$ by the last paragraph in Section 7.2. If $n = 9$ then $|\Lambda| \geqslant 128$ and $\dim G = 171$, whence we quickly see that we may assume that there are two orthogonal roots, one long and one short, not in Φ_s. If $n = 8$ then $|\Lambda| \geqslant 64$ and $\dim G = 136$; we see that there are two orthogonal roots not in Φ_s unless $X = C_4C_4$. However, if $X = C_4C_4$ then, since $p = 2$, the semisimple element s is a scalar multiple of the identity, i.e., s is central.

Assume that we can take two orthogonal roots α short and β long not in the Φ_s. We have $\langle \omega_n, \alpha \rangle = 0$ or ± 2 and $\langle \omega_n, \beta \rangle = \pm 1$. We may calculate the number of 1×2 and 2×2 weight nets, but it is quicker to notice that all weights in $W.\omega_n$ occur in pairs in the weight nets; therefore $|\Lambda| \geqslant 2^{n-1}$. If $n = 9$ we now find that $(\Diamond)$ is satisfied for all $X \in I$

since $|\Lambda| \geqslant 256 > 162 = |\Phi(G)|$. If $n = 8$ then ($\Diamond$) holds for all $X \in I$ except $X = \varnothing$ since $|\Lambda| \geqslant 128 = |\Phi(G)|$. Suppose that $X = \varnothing$ and arrange the weights into weight nets where the horizontal difference between weights in the same net is α_1 and the vertical difference α_8, say. Clearly $d^{\varnothing}_{\omega_8} \geqslant 128$ since the horizontal (or vertical) difference between pairs of weights are roots and all weights appear in such pairs. However, at most one of the four weights $++++++--+$, $+++++++---$ and $++-++--+$, $++-++---$ can lie in the eigenspace and these occur in two pairs as indicated, whence $d^{\varnothing}_{\omega_8} \geqslant 129$.

If $n = 7$ then in order to take two orthogonal roots as before, we need to deal with the cases $X = C_5C_2$ and $X = C_4C_3$. However, in these cases, we see that s is a scalar multiple of the identity, hence central. Therefore, $|\Lambda| \geqslant 64$ and $\dim G = 105$. It remains to show that ($\Diamond$) holds for the 60 centraliser types listed in the table below; this is routine.

Centraliser type
$\varnothing$, A_1, A_1^2, A_1^3, A_2, A_2A_1, $A_2A_1^2$, A_2^2, A_3, A_3A_1, A_3A_2, A_4, A_4A_1, A_5, C_1, C_1A_1, $C_1A_1^2$, $C_1A_1^3$, C_1A_2, $C_1A_2A_1$, $C_1A_2^2$, C_1A_3, $C_1A_3A_1$, C_1A_4, C_1A_5, C_1^2, $C_1^2A_1$, $C_1^2A_1^2$, $C_1^2A_2$, $C_1^2A_2A_1$, $C_1^2A_3$, $C_1^2A_4$, C_2, C_2A_1, $C_2A_1^2$, C_2C_1, $C_2C_1A_1$, $C_2C_1A_1^2$, C_2A_2, $C_2A_2A_1$, $C_2A_2C_1$, C_2A_3, $C_2A_3C_1$, C_2A_4, C_2^2, $C_2^2A_1$, $C_2^2A_2$, C_3, C_3A_1, $C_3A_1^2$, C_3C_1, $C_3C_1A_1$, C_3A_2, $C_3A_2C_1$, C_3A_3, C_3C_2, $C_3C_2A_1$, C_4, C_4A_1, C_4C_1

Table 7.3: Centraliser types requiring consideration for $L(\omega_7)$ with $n = 7$

□

Chapter 8

Groups of type D_n

Throughout this chapter we shall assume that G is a simple simply connected algebraic group of type D_n defined over an algebraically closed field K and $V = L(\lambda)$ is an irreducible G-module with p-restricted highest weight λ. Recall that we list modules up to duality.

We shall prove the following result.

Theorem 8.1. *Let $G = \mathrm{Spin}_{2n}(K)$ act on $V = L(\lambda)$. If $\dim V \leqslant \dim G + 2$ then $\dim \overline{E} = \dim V$ with the possible exception of $L(\omega_7)$ for $n = 7$; if instead $\dim V > \dim G+2$ then $\dim \overline{E} < \dim V$ with the possible exceptions of $L(2\omega_1)$ for $n \geqslant 4$ with $p \neq 2$, and both $L(\omega_7)$ and $L(\omega_8)$ for $n = 8$.*

This theorem is a consequence of the lemmas which follow in later sections.

8.1 Initial survey

Consider $\mu \leqslant \lambda$ where $\mu = \sum_{i=1}^{n} a_i\omega_i$ is a dominant weight. We use Proposition 4.2 repeatedly in this section. We shall begin by obtaining conditions on the coefficients a_i in order to show that (†) is satisfied for n large enough. This will allow us later to list modules which will require further consideration and in some cases we can use the information on weight multiplicities in Lübeck's tables to show that (†) holds.

Proposition 8.2. *Suppose that* $\mu = \sum_{i=1}^{n} a_i\omega_i \leqslant \lambda$ *is a dominant weight, and at least one of the following conditions holds:*

(i) $n \geqslant 6$ *and* $a_k \neq 0$ *for some* $k \in [4, n-2]$, $n \geqslant 8$ *and* $a_3 \neq 0$, *and* $n \geqslant 11$ *and either* $a_{n-1} \neq 0$ *or* $a_n \neq 0$;

(ii) $n \geqslant 7$ *and* $a_{n-1}, a_n \neq 0$;

(iii) $n \geqslant 5$ *and either* $a_k, a_{n-1} \neq 0$ *or* $a_k, a_n \neq 0$ *for some* $k \in [2, n-2]$;

(iv) $n \geqslant 7$ *and either* $a_1, a_{n-1} \neq 0$ *or* $a_1, a_n \neq 0$;

(v) *If* $n \geqslant 6$ *then* $m_1 = \sum_{i=1}^{3} ia_i \leqslant 3$.

Then (†) *holds.*

Proof. Let us first assume that at least one coefficient $a_k \neq 0$. Suppose that $a_k \neq 0$ for $k \in [1, n-2]$. Then $\Psi = \langle \alpha_i \mid a_i = 0 \rangle \subset \Phi(A_{k-1}D_{n-k})$ and so

$$r_\Psi \geqslant r_{A_{k-1}D_{n-k}} = 2^{k-2}k\binom{n}{k}\frac{4n-3k-1}{n(n-1)}.$$

If $k = 3$ then $r_{A_2D_{n-3}} = 2(n-2)(2n-5) > \dim G$ for $n \geqslant 8$, if $k = 4$ then $r_{A_3D_{n-4}} = \frac{2}{3}(n-2)(n-3)(4n-13) > \dim G$ for $n \geqslant 6$ and if $k = n-2$ then $r_{A_{n-3}D_2} = 2^{n-5}(n+5)(n-2) > \dim G$ for $n \geqslant 6$. We see that for $n \geqslant 6$ the value of $r_{A_{k-1}D_{n-k}}$ is at least $r_{A_3D_{n-4}}$ for each $k \in [4, n-2]$.

Suppose that $a_{n-1} \neq 0$ (or equivalently $a_n \neq 0$). Then $\Psi \subset \Phi(A_{n-1})$ and $r_\Psi \geqslant r_{A_{n-1}} = 2^{n-3}$ which exceeds $\dim G$ for $n \geqslant 11$. If both $a_{n-1} \neq 0$ and $a_n \neq 0$ then $\Psi \subset \Phi(A_{n-2})$ and $r_\Psi \geqslant r_{A_{n-2}} = 2^{n-3}(n+2)$ exceeds $\dim G$ for $n \geqslant 7$. Thus, (†) may not be satisfied if $n \geqslant 11$ and $a_k \neq 0$ for $k \in [1, 2]$, if $8 \leqslant n \leqslant 10$ and $a_k \neq 0$ for $k \in [1, 2] \cup [n-1, n]$ and if $6 \leqslant n \leqslant 7$ and $a_k \neq 0$ for $k \in [1, 3] \cup [n-1, n]$.

For $n \geqslant 6$, if $m_1 = \sum_{i=1}^{3} ia_i \geqslant 4$ then we claim that we can find $\{b_i\}_{i=1}^{n}$ such that $\mu' = \sum_{j=1}^{n} b_j\omega_j < \mu$ with $b_4 \neq 0$. It suffices to check this for the weights $4\omega_1$, $2\omega_2$, $2\omega_1 + \omega_2$, $\omega_1 + \omega_3$ with $m_1 = 4$, the weights $\omega_1 + 2\omega_2$ and $\omega_2 + \omega_3$ with $m_1 = 5$ and $2\omega_3$ with $m_1 = 6$

since the coefficients a_1, a_2 and a_3 of each of these weights are the smallest values that need to be checked for any μ. Indeed all weights ν listed with $m_1 = 4$ satisfy $\nu > \omega_4$, both weights τ with $m_1 = 5$ listed satisfy $\tau > \omega_1 + \omega_4$, and $2\omega_3 > \omega_2 + \omega_4$. Hence we can assume that $m_1 \leqslant 3$, otherwise (†) holds.

Next suppose that $a_2 \neq 0$ and $a_{n-1} \neq 0$. We have $\Psi \subset \Phi(A_1 A_{n-3})$ so that $r_\Psi \geqslant 2^{n-4}(n^2 + 3n - 8)$ which exceeds $\dim G$ for $n \geqslant 5$. Similarly for $a_2 \neq 0$ and $a_n \neq 0$. Since $r_{A_{k-1}A_{n-k-1}} \geqslant r_{A_1 A_{n-3}}$ for $2 \leqslant k \leqslant n-2$ and $n \geqslant 5$ we see that (†) is satisfied when $a_k \neq 0$ and $a_{n-1} \neq 0$ (or $a_n \neq 0$) for such values of k and n.

Finally, suppose that $a_1 \neq 0$ and $a_{n-1} \neq 0$. Here $\Psi \subset \Phi(A_{n-2})$ and, as we have already seen, $r_\Psi \geqslant \dim G$ for $n \geqslant 7$ in which case (†) is satisfied. Similarly for $a_1 \neq 0$ and $a_n \neq 0$. □

We shall repeatedly use these conditions in the next three lemmas in order to obtain a list of weights for values of $n \geqslant 4$ when (†) may not hold.

We shall not consider the natural module $L(\omega_1)$ for $n \geqslant 4$ since $\dim L(\omega_1) = 2n < n^2 = \dim B$, i.e., the module is too small. Recall that the adjoint module is $L(\omega_2)$ for $n \in [4, \infty)$. The modules $L(\omega_{n-1})$ and $L(\omega_n)$ need not be considered for $n \in [5, 6]$ since $\dim L(\omega_{n-1}) = \dim L(\omega_n) = 2^{n-1} < n^2$ for these values of n. The module $L(\omega_7)$ for $n = 7$ has dimension 64 which is less than $\dim G = 91$.

Lemma 8.3. *Let $\mu = \sum_{i=1}^n a_i \omega_i$ be a dominant weight with $\mu \leqslant \lambda$. If at least three a_i are non-zero for $n \geqslant 6$ then (†) is satisfied.*

Proof. If $n \geqslant 8$ then we can have both a_1 and a_2 non-zero and at most one of a_{n-1} and a_n non-zero; all other coefficients can be assumed to be zero. However we cannot have both $a_1 \neq 0$ and one of $a_t \neq 0$ for $t \in \{n-1, n\}$.

If $n = 7$ we can again have at most one of a_6 and a_7 non-zero and since $m_1 \leqslant 3$ we have at most two a_i for $i \in [1, 3]$ non-zero. Thus at least one of a_2 and a_3 are non-zero and one of a_t for $t \in [6, 7]$ is non-zero, hence (†) is satisfied.

If $n = 6$ we may have both $a_5 \neq 0$ and $a_6 \neq 0$. Since $m_1 \leqslant 3$ we can have at most two $a_i \neq 0$ for $i \in [1, 3]$. If two $a_i \neq 0$ for $i \in [1, 3]$ then we must have $a_1 = a_2 = 1$ and if we have either $a_5 \neq 0$ or $a_6 \neq 0$ then (†) holds. Similarly, if only one $a_i \neq 0$ for $i \in [1,3]$ then we must have $a_1 \neq 0$ since (†) is satisfied if $a_s \neq 0$ and $a_t \neq 0$ for $s \in [2,3]$ and $t \in [5,6]$. Therefore we have $a_i \neq 0$ for $i \in \{1,5,6\}$; however $\omega_1+\omega_5+\omega_6 > \omega_1+\omega_3 > \omega_4$ and since (†) is satisfied for weights with $a_4 \neq 0$ we can use Premet's theorem for $\omega_1+\omega_5+\omega_6$. □

We can now assume for $n \geqslant 6$ that at most two a_i are non-zero.

Lemma 8.4. *Suppose that* $\dim V > \dim G + 2$. *If* $n \geqslant 6$ *then* (†) *holds except possibly for the modules with highest weights* $2\omega_1$ *for* $n \in [6,\infty)$ *with* $p \neq 2$, ω_{n-1} *for* $n \in \{8,10\}$, ω_n *for* $n \in [8,10]$ *and* ω_3 *for* $n = 6$.

Proof. Assume that $n \geqslant 11$. We have $m_1 \leqslant 3$ and we may assume that $a_k = 0$ for $k \in [3,n]$. The remaining weights to be considered are $2\omega_1$, $\omega_1+\omega_2$ and $3\omega_1$. However, by Premet, we need not consider either $\omega_1+\omega_2$ or $3\omega_1$ since ω_3 lies beneath both of these in the partial ordering.

If $8 \leqslant n \leqslant 10$ we again have $m_1 \leqslant 3$ and we can assume $a_3 = 0$ but we may also have either $a_{n-1} = 1$ or $a_n = 1$ (but not both). We cannot have either a_{n-1} or $a_n \geqslant 2$ as then there is a weight $\mu' < \mu$ with $a'_{n-2} \neq 0$ in which case (†) is satisfied. We may not have both $a_s \neq 0$ and $a_t \neq 0$ for $s \in \{1,2\}$ and $t \in \{n-1,n\}$. The remaining weights to be considered are the same three for $n \geqslant 11$ as well as ω_{n-1} and ω_n. If $n = 7$ we may have $a_3 \neq 0$, but since $m_1 \leqslant 3$ the only weights other than those for $8 \leqslant n \leqslant 10$ to be considered are $3\omega_1$, $\omega_1+\omega_2$ and ω_3.

Consider the weight ω_3 for $n = 7$. Since $\omega_3 > \omega_1$ we can use Premet's theorem to include the contribution to $|\Lambda|$ from the weights in $W.\omega_1$. We see in the case $n = 7$ that $r_{A_2D_{n-3}} + r_{D_{n-1}} = 2(n-2)(2n-5)+2$ just exceeds $\dim G$. Hence (†) also holds for the weights $3\omega_1$ and $\omega_1+\omega_2$ which lie above ω_3 in the partial ordering.

If $n = 6$ we need to consider weights which have both $a_5 \neq 0$ and $a_6 \neq 0$; in such a case $a_5 = a_6 = 1$ since $\omega_4 < 2\omega_s$ for $s \in \{5,6\}$ and we may assume that $a_4 = 0$. We can have both $a_1 \neq 0$ and $a_s \neq 0$ but not both $a_2 \neq 0$ and $a_s \neq 0$ for $s \in \{5,6\}$. In this case we are forced to have $a_s = 1$ as just argued and $a_1 < 2$ since $\omega_2+\omega_s < 2\omega_1+\omega_s$. Thus, we must consider further the weights $\omega_5+\omega_6$, $\omega_1+\omega_5$ and $\omega_1+\omega_6$ together with ω_5 and ω_6 and the relevant weights satisfying $m_1 \leqslant 3$ and $a_j=0$ for $j \in [4,6]$.

Consider the weight $\omega_1+\omega_6$. We find for type D_n that $r_{A_{n-2}} = 2^{n-3}(n+2)$ and $r_{A_{n-1}} = 2^{n-3}$ so using the fact that $\omega_5 < \omega_1+\omega_6$ together with Premet's theorem we see for $n=6$ that $r_{A_4}+r_{A_5} = 72 > 66 = \dim G$, i.e., (†) holds. Similarly we can use $\omega_6 < \omega_1+\omega_5$ for the module with highest weight $\omega_1+\omega_5$. Also for $n=6$ we can use the ordering $\omega_5+\omega_6 > \omega_3$ to see that $r_{A_4}+r_{A_2D_3} = 64+56 = 120$ which exceeds $\dim G$.

Consider $\omega_1+\omega_2$ for $n=6$. We have $r_{D_{n-2}} = 4(2n-3)$ and $\omega_1+\omega_2 > \omega_3 > \omega_1$ so that $r_{D_4}+r_{A_2D_3}+r_{D_5} = 36+56+2 = 94$ which exceeds $\dim G$. By this calculation we need not consider $3\omega_1$ for $n=6$ any further.

We note that $\dim L(\omega_{n-1}) = \dim L(\omega_n) = 2^{n-1} < \dim G$ for $n \in [4,7]$ so, by duality, we need only consider $L(\omega_{n-1})$ for $n \in \{8,10\}$ and $L(\omega_n)$ for $n \in [8,10]$. □

We are left to investigate the low rank cases $n=4$ and 5.

Lemma 8.5. *Suppose that* $\dim V > \dim G + 2$. *If* $n \in [4,5]$ *then* (†) *holds except possibly for the modules with highest weights* $2\omega_1$ *for* $n \in [4,5]$ *with* $p \neq 2$, ω_3 *for* $n=5$, *both* $\omega_1+\omega_{n-1}$ *and* $\omega_1+\omega_n$ *for* $n \in [4,5]$, *and both* $2\omega_4$ *and* $2\omega_5$ *for* $n=5$ *with* $p \neq 2$.

Proof. If $n=5$ we cannot have three or more coefficients $a_i \neq 0$. The smallest possible value of r_Ψ occurs for A_1^2, however, $r_{A_1^2} = 216$ which certainly exceeds $\dim G = 45$. Therefore, assume that at most two a_i are non-zero. We cannot have both $a_s \neq 0$ and $a_t \neq 0$ for $s \in \{2,3\}$ and $t \in \{4,5\}$. Suppose that both $a_1 \neq 0$ and $a_t \neq 0$ for $t \in \{4,5\}$. Since $\omega_2+\omega_t < 2\omega_1+\omega_t$ we must have $a_1 = 1$. Using Premet's theorem, since $\omega_1+\omega_3 < \omega_1+2\omega_t$ and $r_{A_1^3} = 102$ we see that (†) is satisfied and the only weights left to be considered are $\omega_1+\omega_4$ and $\omega_1+\omega_5$.

Suppose that only one of either $a_4 \neq 0$ or $a_5 \neq 0$. Since $\omega_3+\omega_4 < 3\omega_4$ and $\omega_3+\omega_5 < 3\omega_5$ and $r_{A_2A_1} = 64$ we are left to consider the weights $2\omega_4$ and $2\omega_5$.

Similarly, we cannot have both $a_4 \neq 0$ and $a_5 \neq 0$ since $\omega_2 < \omega_4+\omega_5$ and $r_{A_1A_3}+r_{A_3} = 18+28 = 46 > 45 = \dim G$. We can assume from now on that $a_4 = a_5 = 0$. We can have neither $\omega_2+\omega_3$ nor $\omega_1+\omega_3$ since $r_{A_1^3} = 102$. Similarly we cannot have $\omega_1+\omega_2$ since

$\omega_3 < \omega_1 + \omega_2$ and $r_{A_3} + r_{A_2 A_1^2} = 28 + 30 = 58$. Therefore we cannot have any two of a_i for $i \in [1, 3]$ non-zero. We can assume that $a_1 \leqslant 2$ since $3\omega_1 > \omega_1 + \omega_2$. The following ordering of weights $2\omega_3 > 2\omega_2 > \omega_1 + \omega_3$ means we may assume that $a_2 \leqslant 1$ and $a_3 \leqslant 1$ since $r_{A_1^3} = 102$. Thus the weights remaining to be considered are $2\omega_1$ and ω_3.

Consider the case $n = 4$. If at least three $a_i \neq 0$ we see that the smallest value of r_Ψ occurs, of course, for $\Psi = A_1$ in which case $r_{A_1} = 44$; this exceeds $\dim G = 28$ and (†) is satisfied. We can assume that at most two a_i are non-zero. First suppose that $a_2 \neq 0$. Since $2\omega_2 > \omega_1 + \omega_3 + \omega_4$ we can use Premet's theorem to conclude that (†) holds and therefore we may assume that $a_2 < 2$. We do not need to consider $\omega_1 + \omega_2$ further since $\omega_1 + \omega_2 > \omega_3 + \omega_4$ and we calculate $r_{A_1^2} = 20$ and $r_{A_2} = 12$. By symmetry we see that if $a_2 = 1$ then $a_1 = a_3 = a_4 = 0$. Now suppose that $a_2 = 0$. As we have just seen, we cannot have $3\omega_1$ since $3\omega_1 > \omega_1 + \omega_2$. Similarly we cannot have the weight $2\omega_1 + \omega_3$ since $2\omega_1 + \omega_3 > \omega_2 + \omega_3$. Thus the weights which require further consideration are $2\omega_1$, $\omega_1 + \omega_3$ and $\omega_1 + \omega_4$. □

The following table displays (up to duality) the highest weights of irreducible p-restricted G-modules which require further consideration.

λ	n	Lemma
$2\omega_1$	$[4, \infty)$ $(p \neq 2)$	-
ω_{n-1}	10	8.7 (†)
	8	8.7 (◊)[a]
ω_n	10	8.7 (†)
	$[8, 9]$	8.7 (◊)[a]
ω_3	6	8.6 (†)
	5	8.6 (◊)
$\omega_1 + \omega_{n-1}$	5	8.6 (†)
	4	8.6 (◊)
$\omega_1 + \omega_n$	5	8.6 (†)
	4	8.6 (◊)
$2\omega_4$	5 $(p \neq 2)$	8.6 (†)
$2\omega_5$	5 $(p \neq 2)$	8.6 (†)

[a] Except for $X = A_7$ when $n = 8$

Table 8.1: Possible weights in type D_n for $n \geqslant 4$

8.2 Weight string analysis

We shall begin with the modules which require the consideration of few, if any, centraliser types.

Lemma 8.6. *Let G act on the irreducible module V where we take V to be one of $L(\omega_1 + \omega_{n-1})$ or $L(\omega_1 + \omega_n)$ for $n \in [4, 5]$, $L(2\omega_4)$ or $L(2\omega_5)$ for $n = 5$ with $p \neq 2$, or $L(\omega_3)$ for $n \in [5, 6]$. Then (†) holds for $L(2\omega_4)$, $L(2\omega_5)$, $L(\omega_1 + \omega_4)$ and $L(\omega_1 + \omega_5)$ for $n = 5$, and $L(\omega_3)$ for $n = 6$, otherwise ($\Diamond$) holds for each $X \in I$.*

Proof. We perform the usual calculations for $L(2\omega_5)$ with $n = 5$ and display them in Figure 8.1. Since $\dim D_5(K) = 45$ we see that (†) holds. Similarly for $L(2\omega_4)$ with $n = 5$.

i	ω	$\lvert W.\omega\rvert$	m_ω
			$p \neq 2$
3	$2\omega_5$	16	1
2	ω_3	80	1
1	ω_1	10	3

Weight strings	No. of strings	l
		$p \neq 2$
μ_3	8	
$\mu_3\ \mu_3$	4	4
μ_2	20	
$\mu_2\ \mu_2$	24	24
$\mu_2\ \mu_1\ \mu_2$	6	12
$\mu_1\ \mu_1$	2	6
Lower bound on $\lvert\Lambda\rvert$		46

Figure 8.1: $(\lambda, n) = (2\omega_5, 5)$

The use of weight multiplicities if $n = 6$ for the module $L(\omega_3)$ in Figure 8.2 shows that (†) holds. If $n = 5$ then Figure 8.3 shows that ($\Diamond$) holds for all $X \in I$ except for $X = \varnothing$ with $p = 2$.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	ω_3	160	1	1
1	ω_1	12	5	4

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	48		
$\mu_2\ \mu_2$	48	48	48
$\mu_2\ \mu_1\ \mu_2$	8	16	16
$\mu_1\ \mu_1$	2	10	8
Lower bound on $\lvert\Lambda\rvert$		74	72

Figure 8.2: $(\lambda, n) = (\omega_3, 6)$

Thus we can take two orthogonal roots not in Φ_s; we consider pairs of such roots not of the form $\epsilon_i + \epsilon_j$ and $\epsilon_i - \epsilon_j$. We shall calculate the number of weight nets. Set $x = \langle \omega_3, \alpha \rangle$ and $y = \langle \omega_3, \beta \rangle$. We note that there are the same number of pairs of orthogonal roots with $(x, y) = (a, b)$ and (b, a) provided $a \neq b$. There are 12 pairs of orthogonal roots α, β such

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	ω_3	80	1	1
1	ω_1	10	4	2

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	20		
$\mu_2\ \mu_2$	24	24	24
$\mu_2\ \mu_1\ \mu_2$	6	12	12
$\mu_1\ \mu_1$	2	8	4
Lower bound on $\lvert\Lambda\rvert$		44	40

Figure 8.3: $(\lambda, n) = (\omega_3, 5)$

that $(x, y) = (2, 1)$, 12 pairs such that $(x, y) = (2, 0)$, 48 pairs such that $(x, y) = (1, 1)$ and 24 pairs such that $(x, y) = (1, 0)$. Thus, for a particular pair α, β there are $\frac{80.12}{480} = 2$ weights $\mu \in W.\omega_3$ such that $(x, y) = (2, 1)$, i.e., there are two 3×2 weight nets. Similarly, we find that there are two 2×3 weight nets, two 3×1 and two 1×3 weight nets, eight 2×2 weight nets, and four 2×1 and four 1×2 weight nets. Note that weight strings of length 3 occurring in the weight nets are of the form $\mu_2\ \mu_1\ \mu_2$ (taking the notation of Figure 8.3).

There are 96 pairs of orthogonal roots with $\langle\omega_1, \alpha\rangle = \langle\omega_1, \beta\rangle = 0$. So, for a given pair of roots α and β, there are $\frac{10.96}{480} = 2$ weights $\mu \in W.\omega_1$ orthogonal to both α and β. Thus the two 3×1 and two 1×3 weight nets above actually form two weight nets which have a plus-shape. The same weight in $W.\omega_1$ occurs in one of the 1×3 and one of the 3×1 weight nets.

Thus $\lvert\Lambda\rvert \geqslant 56$ or 52 according as $p \neq 2$ or $p = 2$ whence $(\lozenge)$ always holds.

Next we consider $L(\omega_1 + \omega_n)$ for $n \in [4, 5]$. If $n = 5$ then Figure 8.4 shows that $(\dagger)$ holds, whereas if $n = 4$ further work is required to show that $(\lozenge)$ is satisfied for $X = \varnothing$ and A_1 for all p and D_2 for $p = 2$ as is evident from Figure 8.5. We note that the subsystem A_1^2 does not occur here.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 5$	$p = 5$
2	$\omega_1 + \omega_5$	80	1	1
1	ω_4	16	4	3

Weight strings	No. of strings	l	
		$p \neq 5$	$p = 5$
μ_2	24		
$\mu_2\ \mu_2$	20	20	20
$\mu_2\ \mu_1\ \mu_2$	8	16	16
$\mu_1\ \mu_1$	4	16	12
Lower bound on $\lvert\Lambda\rvert$		52	48

Figure 8.4: $(\lambda, n) = (\omega_1 + \omega_5, 5)$

If $X = \varnothing$ then the following cliques and their negatives show that $d^{\varnothing}_{\omega_1+\omega_4} \geqslant 32$ or 26 according as $p \neq 2$ or $p = 2$. Therefore $(\lozenge)$ is satisfied since $e^{\varnothing}_{\omega_1+\omega_4} = 24$. Note that the

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2$	$p = 2$
2	$\omega_1 + \omega_4$	32	1	1
1	ω_3	8	3	2

Weight strings	No. of strings	l	
		$p \neq 2$	$p = 2$
μ_2	8		
$\mu_2\ \mu_2$	8	8	8
$\mu_2\ \mu_1\ \mu_2$	4	8	8
$\mu_1\ \mu_1$	2	6	4
Lower bound on $\lvert\Lambda\rvert$		22	20

Figure 8.5: $(\lambda, n) = (\omega_1 + \omega_4, 4)$

first cluster below consists of weights in $W.\omega_3$.

$$\boxed{\begin{array}{cccc} \frac{1}{2} & 1 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 0 & \frac{1}{2} \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{2} \end{array}} \quad \boxed{\begin{array}{cccc} \frac{3}{2} & 2 & 1 & \frac{3}{2} \\ \frac{1}{2} & 2 & 1 & \frac{3}{2} \\ \frac{1}{2} & 1 & 1 & \frac{3}{2} \end{array}} \quad \boxed{\begin{array}{cccc} \frac{3}{2} & 2 & 1 & \frac{1}{2} \\ \frac{1}{2} & 2 & 1 & \frac{1}{2} \\ \frac{1}{2} & 1 & 1 & -\frac{1}{2} \end{array}} \quad \boxed{\begin{array}{cccc} \frac{1}{2} & 0 & 1 & \frac{1}{2} \\ -\frac{1}{2} & 0 & 1 & \frac{1}{2} \\ -\frac{1}{2} & -1 & 0 & \frac{1}{2} \end{array}} \quad \boxed{\begin{array}{cccc} \frac{3}{2} & 1 & 1 & \frac{1}{2} \\ \frac{3}{2} & 1 & 0 & \frac{1}{2} \end{array}}$$

In the table below we handle the case $X = A_1$ with simple root α_1. There are six cliques involving weights in $W.\omega_3$ as can be seen in the right-hand table in Figure 8.5. The table below shows the arrangement of some clusters into cliques; taking them and the corresponding negative cliques yields $d^{A_1}_{\omega_1+\omega_4} \geqslant 28$ or 24 according as $p \neq 2$ or $p = 2$ which in either case exceeds $e^{A_1}_{\omega_1+\omega_4} = 22$.

$X = A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$\cdot\ 2\ 1\ \frac{3}{2}$	2	2	$\cdot\ 1\ 1\ \frac{1}{2}$	5/4	10/8
$\cdot\ 2\ 1\ \frac{1}{2}$	2		$\cdot\ 1\ 0\ \frac{1}{2}$	5/4	
$\cdot\ 0\ 1\ \frac{1}{2}$	2	2	$\cdot\ 0\ 0\ \frac{1}{2}$	6/4	
$\cdot\ 0\ 1\ -\frac{1}{2}$	2				

We are left with $X = D_2$ with simple roots α_3 and α_4. There are two cliques consisting of six weights and containing two weights in $W.\omega_3$. Using the clusters given in the table below together with their negatives, we see that $d^{D_2}_{\omega_1+\omega_4} \geqslant 28$ or 24 according as $p \neq 2$ or $p = 2$. Since $e^{D_2}_{\omega_1+\omega_4} = 20$ we have that $(\diamondsuit)$ is satisfied.

$X = D_2$								
Clique	Cluster size	l	Clique	Cluster size	l	Clique	Cluster size	l
$\frac{1}{2}\ 1\ \cdot\ \cdot$	10/8	10/8	$\frac{3}{2}\ 2\ 1\ \cdot$	2	2	$\frac{1}{2}\ 2\ 1\ \cdot$	2	2
$\frac{1}{2}\ 0\ \cdot\ \cdot$	10/8		$\frac{3}{2}\ 1\ \cdot\ \frac{1}{2}$	2		$-\frac{1}{2}\ 1\ \cdot\ \frac{1}{2}$	2	

Similarly for $L(\omega_1 + \omega_{n-1})$ with $n \in [4, 5]$. $\square$

Our methods do not suffice for $L(2\omega_1)$. Using weight multiplicities the best we can show is that $\lvert\Lambda\rvert \geqslant 4(n-1)$. This is not conducive to further progress.

It remains to consider the spin modules $L(\omega_n)$ for $n \in [8,10]$ and $L(\omega_{n-1})$ for $n \in \{8,10\}$. We recall that $\Phi(D_n) = \{\pm\epsilon_i \pm \epsilon_j \mid 1 \leqslant i < j \leqslant n\}$. Let $\alpha_i = \epsilon_i - \epsilon_{i+1}$ for $i \in [1, n-2]$, $\alpha_{n-1} = \epsilon_{n-1} - \epsilon_n$ and $\alpha_n = \epsilon_{n-1} + \epsilon_n$ be the simple roots.

Lemma 8.7. *Let G act on the irreducible modules $L(\omega_{n-1})$ for $n \in \{8,10\}$ and $L(\omega_n)$ for $n \in [8,10]$. Then (†) holds for both $L(\omega_9)$ and $L(\omega_{10})$ with $n = 10$, (◊) is satisfied for all $X \in I$ for $L(\omega_9)$ with $n = 9$, and (◊) is satisfied for all $X \in I \setminus \{A_7\}$ for both $L(\omega_7)$ and $L(\omega_8)$ with $n = 8$.*

Proof. Recall that $\dim L(\omega_n) = 2^{n-1}$ and $r_{A_{n-1}} = 2^{n-3}$. We shall explain how we may write each weight in this module as a string of n plus and minus signs. Consider the coefficients b_i of a weight μ written as a linear combination of simple roots $\mu = \sum_{i=1}^{n} b_i\alpha_i$ and setting $b_0 = 0$ we imagine a zero coefficient preceding b_1. If $b_i > b_{i-1}$ for $1 \leqslant i \leqslant n-2$ and $i = n$ there is a plus sign in position i; otherwise there is a minus sign. There is a plus sign in position $n-1$ if $b_{n-2} < b_{n-1} + b_n$; otherwise there is a minus sign. Suppose that two weights when written as a string of plus and minus signs are identical apart from the ith and jth positions where the first weight has a plus sign in position i and minus sign in position j and vice-versa for the second weight. Then the difference between them is the root $\alpha_i + \cdots \alpha_{j-1}$. Consider two weights identical except one has two plus signs and the other has two minus signs in the $(n-1)$st and nth positions. The difference between these two weights is α_n. Therefore, the difference between two weights that are identical other than one has two plus signs and the other has two minus signs in the ith and jth positions is the root $\alpha_i + \cdots + \alpha_{j-1} + 2\alpha_j + \cdots + 2\alpha_{n-2} + \alpha_{n-1} + \alpha_n$. We note that there are always an even number of minus signs in the string of plus and minus signs for a given weight. There is one weight consisting of n plus signs (and no minus signs), $\binom{n}{2}$ weights with $n-2$ plus signs, $\binom{n}{4}$ weights with $n-4$ plus signs, etc.

First assume that $n = 10$ so that $|\Lambda| \geqslant 128$. There is a dichotomy here: either $X = A_9$ or we can take the two orthogonal roots $\epsilon_1 + \epsilon_{10} = \alpha_1 + \cdots + \alpha_8 + \alpha_{10}$ and $\epsilon_1 - \epsilon_{10} = \alpha_1 + \cdots + \alpha_9$ not in Φ_s.

In the former case we provide the clusters that occur in the table below. It is clear that $d^{A_9}_{\omega_{10}} \geqslant 210 > 190 = \dim G$.

$X = A_9$	
Cluster	Cluster size
$++++++++++:$	1
$++++++++--:$	45
$++++++----:$	210
$++++------:$	210
$++--------:$	45
$----------:$	1

In the latter case, taking the two orthogonal roots outside Φ_s, each weight occurs in a pair differing by one of these two roots; given below are the four types of pairs. Note that weights in a pair have identical signs in the second to eighth positions.

$$\begin{array}{cccc} +\bullet\bullet\bullet\bullet\bullet\bullet\bullet++ & +\bullet\bullet\bullet\bullet\bullet\bullet\bullet-- & +\bullet\bullet\bullet\bullet\bullet\bullet\bullet+- & +\bullet\bullet\bullet\bullet\bullet\bullet\bullet-+ \\ -\bullet\bullet\bullet\bullet\bullet\bullet\bullet+- & -\bullet\bullet\bullet\bullet\bullet\bullet\bullet-+ & -\bullet\bullet\bullet\bullet\bullet\bullet\bullet++ & -\bullet\bullet\bullet\bullet\bullet\bullet\bullet-- \end{array}$$

Thus by the adjacency principle we find that $|\Lambda| \geqslant 256 > 190 = \dim G$ so (†) holds.

Assume that $n = 9$ so that $|\Lambda| \geqslant 64$. We can always take the two orthogonal roots $\epsilon_1 + \epsilon_9$ and $\epsilon_1 - \epsilon_9$ outside Φ_s unless $X = A_8$. However, as can be seen from the table below, for $X = A_8$ if the cluster $+++++----$ lies in or not in the eigenspace, (◊) is satisfied.

$X = A_8$	
Cluster	Cluster size
$+++++++++:$	1
$+++++++--:$	36
$+++++----:$	126
$+++------:$	84
$+--------:$	9

Now by taking the two orthogonal roots not in Φ_s we have $|\Lambda| \geqslant \frac{1}{2}\dim L(\omega_9) = 128$ as before. Thus (◊) holds for all $X \in I$ satisfying $|\Phi(X)| > 144 - 128 = 16$. The remaining centraliser types are detailed in Table 8.2 and can be dealt with easily.

Assume that $n = 8$ so $|\Lambda| \geqslant 32$. As above there are two cases: either $X = A_7$ or the orthogonal roots $\epsilon_1 + \epsilon_8$ and $\epsilon_1 - \epsilon_8$ are not in Φ_s. In the latter case we have $|\Lambda| \geqslant \frac{1}{2}\dim L(\omega_8) = 64$ so (◊) is satisfied unless $|\Phi(X)| > 112 - 64 = 48$ and the remaining centraliser types are given in Table 8.2. These are reasonably straightforward to go through. In the former case we arrange the 128 weights in $W.\omega_8$ into five clusters in the table below.

$X = A_7$	
Cluster	Cluster size
$++++++++:$	1
$++++++--:$	28
$++++----:$	70
$++------:$	28
$--------:$	1

It is possible that $d_{\omega_8}^{A_7} = e_{\omega_8}^{A_7} = 56$ by assuming that the second and fourth clusters in the table below lie in Λ. Thus we cannot conclude that ($\Diamond$) holds for $X = A_7$.

n	Centraliser types
9	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4; A_2, A_2A_1, $A_2A_1^2$, $A_2A_1^3$, A_2^2, $A_2^2A_1$; A_3, A_3A_1, $A_3A_1^2$; D_2, D_2A_1, $D_2A_1^2$, $D_2A_1^3$, D_2A_2, $D_2A_2A_1$, $D_2A_2A_1^2$, $D_2A_2^2$, D_2A_3, D_2^2, $D_2^2A_1$, $D_2^2A_1^2$, $D_2^2A_2$, $D_2^2A_2A_1$; D_3, D_3A_1, $D_3A_1^2$, D_3D_2
8	$\varnothing$, A_1, A_1^2, A_1^3, A_1^4; A_2, A_2A_1, $A_2A_1^2$, A_2^2, $A_2^2A_1$; A_3, A_3A_1, $A_3A_1^2$, A_3A_2, A_3^2; A_4, A_4A_1, A_4A_2; A_5, A_5A_1; A_6; D_2, D_2A_1, $D_2A_1^2$, $D_2A_1^3$, D_2A_2, $D_2A_2A_1$, $D_2A_2^2$, D_2A_3, D_2A_4, D_2A_5, D_2^2, $D_2^2A_1$, $D_2^2A_1^2$, $D_2^2A_2$, $D_2^2A_3$; D_3, D_3A_1, $D_3A_1^2$, D_3A_2, $D_3A_2A_1$, D_3D_2, $D_3D_2A_1$, $D_3D_2A_2$, D_3A_3, D_3A_4, D_3^2, $D_3^2A_1$; D_4, D_4A_1, $D_4A_1^2$, D_4A_2, D_4A_3, D_4D_2, $D_4D_2A_1$, D_4D_3, D_4^2; D_5, D_5A_1, D_5A_2, D_5D_2

Table 8.2: Centraliser types requiring consideration for $L(\omega_n)$ with $n \in [8, 9]$

$\square$

Chapter 9

Groups of exceptional type

In this chapter we consider the groups of exceptional type acting on p-restricted irreducible modules. We shall use Proposition 4.2 repeatedly in order to show that (†) holds for nearly all modules V with $\dim V > \dim G + 2$. If there are roots of different lengths we shall use the standard notation $\tilde{A}_j$ to denote the root subsystem A_j for $j \in [1, n]$ consisting of short roots. We remark that if G is a simple algebraic group of type E_8, F_4 or G_2 then there is only one group in the isogeny class, i.e., the adjoint and simply connected group are the same. However, for all other simple algebraic groups there is a need to distinguish between the simply connected group and other isogenous groups such as the adjoint group.

9.1 Types E_6, E_7 and E_8

Lemma 9.1. *Consider the simple simply connected algebraic group G of type E_n for $n \in \{6, 7, 8\}$ defined over an algebraically closed field K acting on an irreducible G-module with p-restricted highest weight λ. If $\dim V > \dim G + 2$ then (†) holds.*

Proof. Consider a dominant weight $\mu = \sum_{i=1}^{n} a_i \omega_i$ with $\mu \leqslant \lambda$. Suppose that $a_k \neq 0$ for some $k \in [1, n]$. Then $\Psi = \langle \alpha_i \mid a_i = 0 \rangle \subset \Psi_k = \langle \alpha_i \mid i \neq k \rangle$ and $r_\Psi \geqslant r_{\Psi_k}$ for each k. The possible Ψ_k and values of r_{Ψ_k} for each k and n are given in the table below.

	Type E_8		Type E_7		Type E_6	
k	Ψ_k	r_{Ψ_k}	Ψ_k	r_{Ψ_k}	Ψ_k	r_{Ψ_k}
1	D_7	702	D_6	33	D_5	6
2	A_7	6624	A_6	192	A_5	21
3	A_1A_6	28224	A_1A_5	752	A_1A_4	75
4	$A_2A_1A_4$	213696	$A_2A_1A_3$	4240	$A_2A_1A_2$	290
5	A_4A_3	104832	A_4A_2	1600	A_4A_1	75
6	D_5A_2	122220	D_5A_1	252	D_5	6
7	E_6A_1	2324	E_6	12		
8	E_7	57				

Assume that $n = 8$. Clearly r_Ψ exceeds $\dim G = 248$ for $k \neq 8$ so (†) holds. Thus we need only consider weights of the form $\mu = a_8\omega_8$. We quickly see that only $a_8 = 1$ is possible by using Premet's theorem since $\omega_7 < 2\omega_8$. The module of highest weight ω_8 is the adjoint module.

Now assume that $n = 7$ in which case $\dim G = 133$. We see that if $a_k \neq 0$ for $k \in [2, 6]$ then (†) holds and we need only consider modules of the form $\mu = a_1\omega_1 + a_7\omega_7$. However we have $2\omega_1 > \omega_3$, $2\omega_7 > \omega_6$ and $\omega_1 + \omega_7 > \omega_2$. Therefore, the only weights remaining are ω_1 and ω_7 which correspond to the adjoint module and a 56-dimensional module $L(\omega_7)$; recalling that for type E_7 the dimension of a Borel subgroup is 70 finishes the analysis for type E_7.

Finally, if $n = 6$ then $\dim G = 78$ and we can only conclude from the table above that we may assume that $a_4 = 0$ as otherwise (†) holds. Note that there is a non-trivial graph automorphism for E_6 so we need only consider modules up to duality. Suppose only one of the $a_k \neq 0$ for $k \neq 4$. If $a_1 \geqslant 3$ then (†) holds since $3\omega_1 > \omega_1 + \omega_2 > \omega_3$ and the same is true if $a_3 \geqslant 2$ since $2\omega_3 > \omega_4$. By duality we may assume that $a_6 \leqslant 2$ and $a_5 \leqslant 1$. We are done if $a_2 \geqslant 2$ since $2\omega_2 > \omega_4$. It can be easily checked that $2\omega_1 > \omega_3 > \omega_6$ and from the table we see that $r_{A_1A_4} = 75$ and $r_{D_5} = 6$. This shows that (†) holds for the irreducible modules with highest weights ω_3 and $2\omega_1$. We are left with the irreducible modules $L(\omega_2)$ and (up to duality) $L(\omega_1)$; however the former is the adjoint module and we do not consider the latter since $\dim L(\omega_1) = 27 < \dim B = 42$.

Suppose now we have at least two $a_k \neq 0$. The subsystem of E_6 of rank 4 with largest

Weyl group and the most roots is D_4. Thus in order to obtain the smallest value of r_Ψ when Ψ has rank at most 4 we must take $\Psi = D_4$. Since $r_{D_4} = 90$ exceeds $\dim G$ we may assume that only one a_k is non-zero; we have just dealt with this situation. $\square$

9.2 Type F_4

Recall that we need only consider short roots when there are two different root lengths.

Lemma 9.2. *Consider the simple algebraic group $G = F_4(K)$ acting on an irreducible G-module with p-restricted highest weight λ. If $\dim V > \dim G + 2$ then (†) holds.*

Proof. We need not consider the module $L(\omega_4)$ which has dimension 25 or 26 according as $p = 3$ or $p \neq 3$ nor the adjoint module $L(\omega_1)$. Let $\mu = \sum_{i=1}^{4} a_i\omega_i \leqslant \lambda$ be a dominant weight. We assume first that $p \neq 2$ in order to use Premet's theorem to obtain conditions on the coefficients of μ. Suppose at least two $a_k \neq 0$. Then $\operatorname{rank}\Psi \leqslant 2$ and the possible subsystems of rank 2 in F_4 are $\tilde{A}_2$, A_2, $A_1\tilde{A}_1$ and B_2; the smallest r_Ψ occurs for $\Psi = B_2$ where $r_{B_2} = \frac{1}{2}\frac{1152}{2^3}\frac{(24-4)}{24} = 60$ which exceeds $\dim G = 52$. So we are left to consider modules with one non-zero coefficient. We do not need to consider further any weight of the form $a_1\omega_1$ since $2\omega_1 > \omega_1 + \omega_4$ and as mentioned above we cannot have $a_1 = 1$. Similarly if $a_3 \neq 0$ then (†) holds by Premet's theorem since $\omega_3 > \omega_1 > \omega_4$ and we find $r_{A_2\tilde{A}_1} = 44$, $r_{C_3} = 6$ and $r_{B_3} = 9$. If $a_4 \neq 0$ and $a_4 > 1$ then (†) holds since $2\omega_4 > \omega_3$. Again, we cannot have $a_2 \neq 0$ since $\omega_2 > \omega_1 + \omega_4$. Thus the result holds for $p \neq 2$.

Now assume $p = 2$. We need to take care in this case as weight strings may not be saturated, hence we must rely on the information in [18] and [9]. However neither of these references provide information about $\omega_2 + \omega_3$ for $p = 2$ or weights with at least three coefficients non-zero. We shall use the fact that, for a given weight μ and short root $\alpha \in \Phi$, if $\langle \mu, \alpha \rangle = \pm 2^m$ for $m \geqslant 0$ then the adjacency principle applies because $\alpha(s) = 1$ if and only if $2^m\alpha(s) = 1$ for any $s \in G_{ss} \setminus Z$.

First suppose that only one $a_k \neq 0$. We need only consider the module $L(\omega_2)$ for $p = 2$ by the first sentence of the lemma and since there is a graph automorphism of the group for $p = 2$. We see from Figure 9.1 that (†) holds for $p = 2$. We have included all characteristics to highlight the situation for weight strings when the conclusion of Premet's theorem does not hold; we can ignore all μ_i with $i \in \{1, 3, 5\}$ for $p = 2$ as these weights occur with multiplicity zero.

i	ω	$\lvert W.\omega\rvert$	m_ω		
			$p \neq 2,3$	$p = 3$	$p = 2$
6	ω_2	96	1	1	1
5	$\omega_1 + \omega_4$	144	1	1	0
4	$2\omega_4$	24	3	3	2
3	ω_3	96	4	4	0
2	ω_1	24	10	9	4
1	ω_4	24	13	12	0
0	0	1	26	22	6

Weight strings	No. of strings	l		
		$p \neq 2,3$	$p = 3$	$p = 2$
μ_6	24			
$\mu_6\ \mu_4\ \mu_6$	24	48	48	24
$\mu_6\ \mu_3\ \mu_2\ \mu_3\ \mu_6$	12	96	96	24
$\mu_5\ \mu_5$	24	24	24	0
$\mu_5\ \mu_4\ \mu_5$	6	12	12	0
$\mu_5\ \mu_3\ \mu_3\ \mu_5$	24	120	120	0
$\mu_5\ \mu_2\ \mu_1\ \mu_2\ \mu_5$	6	90	84	24
$\mu_4\ \mu_3\ \mu_4$	8	32	32	16
$\mu_4\ \mu_1\ \mu_0\ \mu_1\ \mu_4$	1	26	24	4
$\mu_3\ \mu_1\ \mu_1\ \mu_3$	8	136	128	0
Lower bound on $\lvert\Lambda\rvert$		584	568	92

Figure 9.1: $\lambda = \omega_2$

Suppose that two $a_k \neq 0$. Suppose that $\mu = \omega_3 + \omega_4$ in which case we have $\langle \mu, \alpha\rangle = 1$ for four short roots (0010, 0001, 0110, and 1110), $\langle \mu, \alpha\rangle = 2$ for three short roots (0011, 0111 and 1111) and $\langle \mu, \alpha\rangle = 4$ for one short root (1231). Thus, for a given $\alpha \in \Phi_S$, there are $\frac{192.8}{24} = 64$ weights $\mu \in W(\Phi)$ that appear in weight strings along with $\mu - 2^m\alpha$ for some $m \in [0, 2]$, so there is a contribution of at least 64 to $|\Lambda|$ and (†) is satisfied. Using Lübeck's tables [18] we can use this contribution also for the weights $\omega_1 + \omega_3$ and $\omega_2 + \omega_4$ since $\omega_3 + \omega_4$ appears with non-zero multiplicity lower in the partial ordering of both of these weights. We calculate in an exactly similar way that $\omega_2 + \omega_3$, $\omega_1 + \omega_2$ and $2\omega_3$ contribute at least 84, 48 and 36 to $|\Lambda|$. Since by [18] we know that $2\omega_3$ occurs in $L(\omega_1 + \omega_2)$ with a non-zero multiplicity, (†) holds. From [18] we see that ω_3, ω_1 and ω_4 occur in $L(\omega_1 + \omega_4)$ with non-zero multiplicity and that the same is true for ω_1 and ω_4 in $L(\omega_3)$. Therefore we can use the same calculation above for $L(\omega_3)$ with $p \neq 2$ to conclude that (†) holds for $L(\omega_1 + \omega_4)$.

Suppose that three a_k are non-zero. If $\mu = \omega_1 + \omega_3 + \omega_4$ then the number of short roots $\alpha \in \Phi(F_4)$ such that $\langle \mu, \alpha\rangle = 1$, 2 or 4 is three (0010, 0001 and 0110), two (0011 and 0111) or one (1111), respectively. Thus, for a given $\alpha \in \Phi_S$, there are $\frac{576.6}{24} = 144$ weights $\mu \in W(\Phi)$ that appear in weight strings along with $\mu - 2^m\alpha$ for some $m \in [0, 2]$

implying that there is a contribution of at least 144 to $|\Lambda|$ by the adjacency principle. The calculations for $\omega_1+\omega_2+\omega_3$, $\omega_1+\omega_2+\omega_4$ and $\omega_2+\omega_3+\omega_4$ and $\omega_1+\omega_2+\omega_3+\omega_4$ are analogous, and also show that (†) holds. □

We remark that we could have used Zalesski's theorem for the module $L(\omega_3+\omega_4)$ with $p=2$ since it has zero support on the long roots whence it is tensor indecomposable. The same is not true though for $L(\omega_1+\omega_2)$ with $p=2$ even though it has zero support on the short roots; this is one of the exceptions for type F_4 in Theorem 2.7.

9.3 Type G_2

In this section we consider the action of $G_2(K)$ on irreducible p-restricted modules.

Lemma 9.3. *Consider the simple algebraic group $G=G_2(K)$ acting on an irreducible G-module with p-restricted highest weight λ. If $\dim V > \dim G+2$ then (†) holds except for the modules $L(2\omega_1)$ for $p\neq 2$ and $L(2\omega_2)$ for $p=3$ where ($\Diamond$) is satisfied.*

Proof. The conditions in the statement of the lemma mean that we are not considering the modules $L(\omega_1)$ and $L(\omega_2)$ since $\dim G=14$. We calculate $r_\varnothing=6$, $r_{\tilde{A}_1}=2$ and $r_{A_1}=3$. We shall assume for the moment that $p\neq 2,3$. Let $\mu=a_1\omega_1+a_2\omega_2\leqslant\lambda$ be a dominant weight. If $a_2=0$ then we may assume that $a_1=2$ since $3\omega_1>\omega_1+\omega_2>2\omega_1>\omega_2>\omega_1$ in which case $|\Lambda|\geqslant 17$. Similarly, if $a_1=0$ then (†) holds by Premet's theorem if $a_2\geqslant 2$ since $2\omega_2>3\omega_1$.

Now suppose that both $a_1\neq 0$ and $a_2\neq 0$. Since

$$\omega_1+2\omega_2>2\omega_1+\omega_2>2\omega_2>\omega_1+\omega_2>\omega_2$$

we see that (†) holds if $a_1+a_2>2$ using the values of r_Ψ given above and Premet's theorem. We shall consider the module $L(\omega_1+\omega_2)$ later (where we use weight multiplicities) since the partial ordering $\omega_1+\omega_2>2\omega_1>\omega_2>\omega_1$ only shows that $|\Lambda|\geqslant 6+3+2+3=14$.

Assume that $p=2$; the only weight we need to consider is $\omega_1+\omega_2$ since the other two 2-restricted modules are too small as we have mentioned.

Assume that $p = 3$. Then the weights $\omega_1 + \omega_2$, $2\omega_1$ and $2\omega_2$ require further analysis. We see from [18] that the same ordering of weights used for $p > 3$ shows that (†) holds for $\omega_1 + 2\omega_2$ and $2\omega_1 + \omega_2$. The remaining 3-restricted weight $2\omega_1 + 2\omega_2$ is only listed by Lübeck on his website for $p = 5$ and 7 as otherwise the dimension of the module exceeds 500. However the required information is contained in the paper of Gilkey and Seitz in [9, p.413] from which we see that for $p > 2$ all dominant weights μ with $\mu \leqslant 2\omega_1 + 2\omega_2$ occur with non-zero multiplicity. Thus (†) holds as $2\omega_1 + 2\omega_2 > \omega_1 + 2\omega_2$. We remark that for $p = 3$ the multiplicities of dominant weights below $2\omega_1 + 2\omega_2$ in the partial ordering are the same as those obtained using Freudenthal's formula [13, p.124] for characteristic 0.

It remains to consider the modules $L(2\omega_1)$ for $p \neq 2$, $L(\omega_1 + \omega_2)$ for all p and $L(2\omega_2)$ for $p = 3$.

We begin with $L(2\omega_1)$ and detail the calculations involving weight strings in Figure 9.2.

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2,7$	$p = 7$
3	$2\omega_1$	6	1	1
2	ω_2	6	1	1
1	ω_1	6	2	2
0	0	1	3	2

Weight strings	No. of strings	l	
		$p \neq 2,7$	$p = 7$
$\mu_3\ \mu_2\ \mu_3$	2	2	2
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	4	4
$\mu_2\ \mu_1\ \mu_1\ \mu_2$	2	6	6
Lower bound on $\lvert\Lambda\rvert$		12	12

Figure 9.2: $\lambda = 2\omega_1$

We see that (◊) is satisfied for all centraliser types except when $X = \varnothing$. It suffices to consider only the case $p = 7$. We assume in turn that the weight 0 is either in or not in the eigenspace. If 0 is in the eigenspace then all weights in $W.\omega_1$ or $W.\omega_2$ lie in Λ, so $d^{\varnothing}_{2\omega_1} \geqslant 6 + 2.6 = 18 > 12 = e^{\varnothing}_{2\omega_1}$. If 0 is not in the eigenspace then the two cliques shown below together with their negative counterparts show that $d^{\varnothing}_{2\omega_1} \geqslant 2 + 2.4 + 2.2 = 14$. The weights in $W.\omega_1$ are italicised since they have multiplicity 2.

4 2	*2 1*
3 2	*1 1*
3 1	*1 0*

Figure 9.3 shows that (†) is satisfied for $L(\omega_1 + \omega_2)$ for all p.

Consider the module $L(2\omega_2)$. We see from Figure 9.4 that if $p = 3$ then (◊) holds except when $X = \varnothing$. (The case $p \neq 2, 3$ is included in the figure to highlight that the

i	ω	$\lvert W.\omega\rvert$	m_ω		
			$p \neq 3, 7$	$p = 3$	$p = 7$
4	$\omega_1 + \omega_2$	12	1	1	1
3	$2\omega_1$	6	2	2	1
2	ω_2	6	2	1	1
1	ω_1	6	4	3	2
0	0	1	4	1	2

Weight strings	No. of strings	l		
		$p \neq 3, 7$	$p = 3$	$p = 7$
$\mu_4\ \mu_4$	2	2	2	2
$\mu_4\ \mu_3\ \mu_2\ \mu_3\ \mu_4$	2	8	6	4
$\mu_4\ \mu_2\ \mu_1\ \mu_1\ \mu_2\ \mu_4$	2	14	10	8
$\mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3$	1	8	5	4
Lower bound on $\lvert\Lambda\rvert$		32	23	18

Figure 9.3: $\lambda = \omega_1 + \omega_2$

conclusion of Premet's theorem does not hold when $p = 3$.)

i	ω	$\lvert W.\omega\rvert$	m_ω	
			$p \neq 2, 3$	$p = 3$
6	$2\omega_2$	6	1	1
5	$3\omega_1$	6	1	1
4	$\omega_1 + \omega_2$	12	1	0
3	$2\omega_1$	6	2	0
2	ω_2	6	3	2
1	ω_1	6	3	0
0	0	1	5	3

Weight strings	No. of strings	l	
		$p \neq 2, 3$	$p = 3$
μ_6	2		
$\mu_6\ \mu_4\ \mu_3\ \mu_2\ \mu_3\ \mu_4\ \mu_6$	2	10	4
$\mu_5\ \mu_4\ \mu_4\ \mu_5$	2	4	2
$\mu_4\ \mu_2\ \mu_1\ \mu_1\ \mu_2\ \mu_4$	2	14	4
$\mu_5\ \mu_3\ \mu_1\ \mu_0\ \mu_1\ \mu_3\ \mu_5$	1	8	2
Lower bound on $\lvert\Lambda\rvert$		36	12

Figure 9.4: $\lambda = 2\omega_2$

If $X = \varnothing$ we can arrange the weights into clusters using the fact that $\alpha(s) = 1$ if and only if $3\alpha(s) = 1$ for any $s \in G_{ss} \setminus Z$. We consider separately the cases when the weight 0 lies and does not lie in the eigenspace. We shall italicise weights in $W.\omega_2$ since these occur with multiplicity 2. If 0 is in the eigenspace then the weights *3 2*, *3 1*, *0 1*, 3 0 and their negatives lie in Λ, so $d^{\varnothing}_{2\omega_2} \geqslant 14 > 12 = e^{\varnothing}_{2\omega_2}$. If 0 is not in the eigenspace then the three cliques below show that $d^{\varnothing}_{2\omega_2} \geqslant 3 + 2 + 4 + 4 = 13$, as required.

6	4
6	3
3	3

3	*2*
3	*1*
0	*1*

0	*-1*
-3	*-1*
-3	*-2*

$\square$

Chapter 10

Twisted modules

In this chapter we consider simple simply connected algebraic groups defined over an algebraically closed field K of characteristic $p > 0$ acting on tensor products with twists of irreducible p-restricted modules; we shall call these *twisted modules*. Recall that we need only consider irreducible modules $V = U \otimes W^{(q)}$ where both $\dim U$ and $\dim W$ are less than $\dim G$; this quickly follows from Proposition 4.3. It is also clear that we need not consider modules that are tensor products with twists of three or more p-restricted irreducible modules. For classical groups the p-restricted modules of lowest dimension are the natural modules and the dimension of the tensor product of two of these for any simple algebraic group exceeds $\dim G$.

We shall prove the following result.

Theorem 10.1. *Let G be a simple simply connected algebraic group acting on an irreducible module $V = L(\mu) \otimes L(\nu)^{(q)}$ where μ and ν are p-restricted and q is a power of p, and $\dim V > \dim G + 2$. Then $\dim \overline{E} < \dim V$ with the possible exceptions of the action of a group of type B_n, C_n or D_n on $L(\omega_1) \otimes L(\omega_1)^{(q)}$.*

This theorem is a consequence of the lemmas which follow in later sections.

10.1 Initial survey

We shall use Proposition 4.3 to show that (†) holds for all but a small list of irreducible modules. If $V = U \otimes W^{(q)}$ is such a module then we investigate weight strings for both U and W and then combine the weight strings together to form weight nets; in this way we can find a lower bound for the codimension of any eigenspace of V. Consider the G-module $L(\lambda)^{(q)}$ where $q = p^r$ with $r > 0$ (note that we used slightly different notation in Theorem 2.3). The weights of this module are just $q\mu$ where $\mu \in \Pi(\lambda)$. In some sense, the twist by q does not affect the adjacency principle since the equation $q\mu(s) = q(\mu + \alpha)(s)$ for some $s \in G_{ss} \setminus Z(G)$ is equivalent to $q\alpha(s) = \alpha(s) = 1$. In particular in the tensor product of two p-restricted modules the twist by q may be applied to either module without affecting our analysis. We choose, therefore, without loss of generality to apply the twist to the p-restricted irreducible module of larger dimension. We note that the action of a group on $U \otimes V^{(q)}$ is not necessarily the same as that of the same group on $U^* \otimes V^{(q)}$; there is an example of this in Section 3.2.

Proposition 10.2. *Let G be a simple simply connected algebraic group and let $V = U \otimes W^{(q)}$ where U and W are irreducible p-restricted G-modules. Then* (†) *holds except possibly for the irreducible modules given in Table 10.1.*

Proof. Case I: A_n. Suppose that G is of type A_n for $n \geqslant 1$, so $\dim G = n(n+2)$. If $n = 1$ we need to consider the modules $L(\omega_1) \otimes L(2\omega_1)^{(q)}$ and $L(2\omega_1) \otimes L(2\omega_1)^{(q)}$. If $n \geqslant 2$ then we need to consider the p-restricted modules $L(\omega_1)$ of dimension $n+1$, $L(\omega_2)$ of dimension $\frac{1}{2}n(n+1)$, $L(2\omega_1)$ of dimension $\frac{1}{2}(n+1)(n+2)$, the adjoint module $L(\omega_1+\omega_n)$ of dimension $n(n+2)$ if $p \nmid n+1$ and $n(n+2) - 1$ otherwise and $L(\omega_3)$ for $n \in [5, 7]$ of dimension $\frac{1}{6}n(n^2 - 1)$. We calculate lower bounds (in the same order just used) for the codimension of a given eigenspace for these five modules: these are $r_{A_{n-1}} = 1$, $r_{A_1 A_{n-2}} = n - 1$, $r_{A_{n-1}} + r_{A_1 A_{n-2}} = n$ (since $2\omega_1 > \omega_2$), $r_{A_{n-2}} = 2n - 1$ and $r_{A_2 A_{n-3}} = \frac{1}{2}(n-1)(n-2)$.

We see that (†) holds for $L(\omega_1) \otimes L(\omega_1 + \omega_n)^{(q)}$ since $r_{A_{n-2}} \dim L(\omega_1) > \dim G$ for $n \geqslant 2$. Since $L(\omega_1+\omega_n)$ is the adjoint module and $|\Lambda| > 1$ for all modules to be considered except $L(\omega_1)$ (which we have just dealt with), we see that (†) holds for any twisted module involving $L(\omega_1+\omega_n)$. Also (†) holds for $L(\omega_1)\otimes L(\omega_3)^{(q)}$ since $r_{A_2A_{n-3}}\dim L(\omega_1) > \dim G$ for $n\in[5,7]$ and $L(\omega_2)\otimes L(\omega_3)^{(q)}$ since $r_{A_1A_{n-2}}\dim L(\omega_3) > \dim G$ for $n\in[5,7]$, whence it also holds for $L(\omega_3)\otimes L(\omega_3)^{(q)}$. The modules $L(2\omega_1)\otimes L(2\omega_1)^{(q)}$ and $L(2\omega_1)\otimes L(\omega_3)^{(q)}$ both satisfy (†) since $(r_{A_{n-1}}+r_{A_1A_{n-2}})\dim L(2\omega_1) > \dim G$ for $n\geqslant 2$ and $r_{A_2A_{n-3}}\dim L(2\omega_1) > \dim G$ for $n\in[5,7]$. Similarly (†) holds for the modules $L(\omega_2)\otimes L(2\omega_1)^{(q)}$ for $n\geqslant 3$ since $r_{A_1A_{n-2}}\dim L(2\omega_1) > \dim G$ and $L(\omega_2)\otimes L(\omega_2)^{(q)}$ for $n\geqslant 4$ since $r_{A_1A_{n-2}}\dim L(\omega_2) > \dim G$.

We shall need to investigate further $U\otimes V^{(q)}$ where $U\in\{L(\omega_1),L(\omega_n)\}$ and $V\in\{L(\omega_2),L(\omega_{n-1})\}$ for $n\geqslant 3$, $U\otimes V^{(q)}$ where $U\in\{L(\omega_1),L(\omega_n)\}$ and $V\in\{L(2\omega_1),L(2\omega_n)\}$ for $n\geqslant 1$, $L(\omega_2)\otimes L(\omega_2)^{(q)}$ for $n=3$ and $L(2\omega_1)\otimes L(2\omega_1)^{(q)}$ for $n=1$.

Case II: B_n. Suppose now that G is of type B_n for $n\geqslant 2$. We need to consider the p-restricted modules $L(\omega_1)$ for $n\geqslant 2$ which has dimension $2n+1-\delta_{2p}$, $L(\omega_n)$ for $n\in[2,6]$ which has dimension 2^n, the module $L(\omega_2)$ for $n\geqslant 3$ which has dimension $n(2n+1)=\dim G$ for $p\neq 2$ and at least $2n(n-1)$ for $p=2$ (this is the adjoint module for $n\geqslant 3$) and $L(2\omega_2)$ for $n=2\neq p$ which has dimension 10.

We calculate lower bounds for the codimension of a given eigenspace for each of the modules given. First, for the module $L(\omega_1)$ we have $|\Lambda|\geqslant r_{B_{n-1}}=1$ and for the spin module $L(\omega_n)$ we have $|\Lambda|\geqslant r_{A_{n-1}}=2^{n-1}$. We find that $|\Lambda|\geqslant r_{A_1B_{n-2}}+r_{B_{n-1}}=2n-1$ for the module $L(\omega_2)$ if $p\neq 2$ and $|\Lambda|\geqslant r_{B_{n-1}}=2n-2$ if $p=2$. For $L(2\omega_2)$ with $n=2\neq p$ the weights $2\omega_2>\omega_1>0$ occur, where $|W.(2\omega_2)|=|W.\omega_1|=4$ and $m_0=2$ so $|\Lambda|\geqslant r_{B_1}+r_{A_1}=2+1=3$.

We see that (†) holds for $L(\omega_1)\otimes L(\omega_2)^{(q)}$ for $n\geqslant 3$ since $(2n-1-\delta_{2p})\dim L(\omega_1) > \dim G$ and for $L(\omega_1)\otimes L(2\omega_2)^{(q)}$ for $n=2\neq p$ since $3\dim L(\omega_1)=12>10=\dim G$. Indeed, (†) holds for all tensor products of modules involving the adjoint modules $L(\omega_2)$ for $n\geqslant 3$ and $L(2\omega_1)$ for $n=2\neq p$. Also, (†) holds for both $L(\omega_1)\otimes L(\omega_n)^{(q)}$ and $L(\omega_n)\otimes L(\omega_n)^{(q)}$ for $n\geqslant 3$ since both $r_{A_{n-1}}\dim L(\omega_1)$ and $r_{A_{n-1}}\dim L(\omega_n)=2^{2n-1}$ exceed $\dim G$ unless $n=2$.

Hence we shall need to investigate further the modules $L(\omega_1)\otimes L(\omega_2)^{(q)}$ and $L(\omega_2)\otimes L(\omega_2)^{(q)}$ for $n=2$, and $L(\omega_1)\otimes L(\omega_1)^{(q)}$ for $n\geqslant 2$.

Case III: C_n. Suppose that G is of type C_n for $n \geqslant 3$, so $\dim G = n(2n+1)$. We need to consider the p-restricted modules $L(\omega_1)$ which has dimension $2n$, $L(\omega_2)$ which has dimension $2n(n-1)$, the adjoint module $L(2\omega_1)$ for $p \neq 2$, $L(\omega_n)$ for $n \in [4,6]$ and $p=2$ which has dimension 2^n and $L(\omega_3)$ for $n=3$ which has dimension 8 for $p=2$ and 14 for $p \neq 2$ since here $m_{\omega_1} \neq 0$.

As before we calculate lower bounds for the codimension of a given eigenspace for each of the modules given. First, for the module $L(\omega_1)$ we have $|\Lambda| \geqslant r_{C_{n-1}} = 2$ and for the spin module $L(\omega_n)$ we have $|\Lambda| \geqslant r_{A_{n-1}} = 2^{n-2}$. For $L(\omega_3)$ with $n=3$ and $p \neq 2$ we have $|\Lambda| \geqslant 2+2 = 4$ by including the contribution from $W.\omega_1$. The module $L(\omega_2)$ satisfies $|\Lambda| \geqslant r_{A_1C_{n-2}} = 4n-7$ and for the adjoint module $L(2\omega_1)$ we have $|\Lambda| \geqslant r_{C_{n-1}} + r_{A_1C_{n-2}} = 4n-5$.

It is clear that (†) holds for any twisted module involving $L(2\omega_1)$. We see that (†) holds for $L(\omega_1) \otimes L(\omega_2)^{(q)}$ with $n \geqslant 3$ since $r_{A_1C_{n-2}} \dim L(\omega_1) > \dim G$ whence also for $L(\omega_2) \otimes L(\omega_2)^{(q)}$ with $n \geqslant 3$, for $L(\omega_2) \otimes L(\omega_3)^{(q)}$ with $n=3$ since $2\dim L(\omega_2) = 24 > \dim G$, for $L(\omega_2) \otimes L(\omega_n)^{(q)}$ with $n \in [4,6]$ since $r_{A_{n-1}} \dim L(\omega_2) > \dim G$ and for $L(\omega_n) \otimes L(\omega_n)^{(q)}$ with $n \in [4,6]$ since $r_{A_{n-1}} \dim L(\omega_n) = 2^{2n-2} > \dim G$. The condition (†) holds if $n=3$ and $p \neq 2$ for $L(\omega_1) \otimes L(\omega_3)^{(q)}$ since $4 \dim L(\omega_1) > 21$ whence for $L(\omega_3) \otimes L(\omega_3)^{(q)}$ since $\dim L(\omega_1) < \dim L(\omega_3)$ here and finally for $L(\omega_1) \otimes L(\omega_n)^{(q)}$ with $n \in [5,6]$ and $p=2$ since $r_{A_{n-1}} \dim L(\omega_1) > \dim G$.

We shall need to investigate further the modules $L(\omega_3) \otimes L(\omega_3)^{(q)}$ for $n=3$ and $p=2$, the module $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [3,4]$ and $p=2$, and $L(\omega_1) \otimes L(\omega_1)^{(q)}$ for $n \geqslant 3$.

Case IV: D_n. Suppose that G is of type D_n for $n \geqslant 4$, so $\dim G = n(2n-1)$. We need to consider the p-restricted modules $L(\omega_1)$ for $n \geqslant 4$ which has dimension $2n$, the adjoint module $L(\omega_2)$ which has dimension $n(2n-1)$ if $p \neq 2$ and $n(2n-1)-(2,n)$ if $p=2$ and the spin module $L(\omega_n)$ for $n \in [4,7]$ with dimension 2^{n-1}.

We calculate that for the module $L(\omega_1)$ we have $|\Lambda| \geqslant r_{D_{n-1}} = 2$, for $L(\omega_2)$ we have $|\Lambda| \geqslant r_{A_1D_{n-1}} = 4n-7$ and for $L(\omega_n)$ with $n \in [4,7]$ we have $|\Lambda| \geqslant r_{A_{n-1}} = 2^{n-3}$. The condition (†) holds for any twisted module including the adjoint module. It also holds for $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [6,7]$ since $r_{A_{n-1}} \dim L(\omega_1) > \dim G$. However, we need to further investigate the modules $L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$ and $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [4, 5]$, and $L(\omega_1) \otimes L(\omega_1)^{(q)}$ for $n \geqslant 4$.

Case V: Exceptional types. If $G = (E_6)_{sc}(K)$ then we need to consider the p-restricted modules $L(\omega_1)$ (and $L(\omega_6)$) which has dimension 27 and the adjoint module $L(\omega_2)$ which has dimension $78 - \delta_{3p}$. We see that for $L(\omega_1)$ we have $|\Lambda| \geqslant r_{D_5} = 6$ and for $L(\omega_2)$ we have $|\Lambda| \geqslant r_{A_5} = 21$. It is clear from Proposition 4.3 since $\dim G = 78$ that (†) holds for all twisted modules.

If $G = (E_7)_{sc}(K)$ then we need to consider the p-restricted modules $L(\omega_7)$ which has dimension 56 and the adjoint module $L(\omega_1)$ which has dimension $133 - \delta_{2p}$. For $L(\omega_7)$ we have $|\Lambda| \geqslant r_{E_6} = 12$ and for $L(\omega_2)$ we have $|\Lambda| \geqslant r_{D_6} = 33$. Again, it is clear from Proposition 4.3 since $\dim G = 133$ that (†) holds in all cases here.

If $G = E_8(K)$ then we need only to consider the adjoint module $L(\omega_8)$ which has dimension 248. We have $|\Lambda| \geqslant r_{E_7} = 57$ so we are done immediately by Proposition 4.3.

If $G = F_4(K)$ then we need to consider $L(\omega_4)$ which has dimension $26 - \delta_{3p}$ and the adjoint module $L(\omega_1)$ which has dimension 52 if $p \neq 2$ and 26 if $p = 2$. We calculate $r_{B_3} = 9$ and $r_{C_3} = 6$ so we are done.

Finally, if $G = G_2(K)$ then the modules to be considered are $V_1 = L(\omega_1)$ which has dimension $7 - \delta_{2p}$ and $V_2 = L(\omega_2)$ which has dimension 7 if $p = 3$ and 14 if $p \neq 3$. We find that $|\Lambda_1| \geqslant r_{A_1} = 3$ and $|\Lambda_2| \geqslant r_{\tilde{A}_1} = 2$ so (†) holds for $V_1 \otimes V_1^{(q)}$ since $|\Lambda_1| \dim V_1 > \dim G$ and for $V_1 \otimes V_2^{(q)}$ since $|\Lambda_1| \dim V_2 > \dim G$. The module $V_2 \otimes V_2^{(q)}$ requires further consideration if $p = 3$ since $|\Lambda_2| \dim V_2 = 2.7 = 14$ here; if $p \neq 3$ then (†) holds. □

We list in Table 10.1 the irreducible twisted modules which require further consideration.

10.2 Weight net analysis

We shall begin with the modules $L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$ and $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [4, 5]$.

Type	Module	n	Lemma
A_n	$L(\omega_1) \otimes L(\omega_2)^{(q)}$	$[7, \infty)$	10.5 (†)
		$[3, 6]$	10.5 (◊)
	$L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$	$[7, \infty)$	10.5 (†)
		$[3, 6]$	10.5 (◊)
	$L(\omega_n) \otimes L(\omega_2)^{(q)}$	$[7, \infty)$	10.5 (†)
		$[3, 6]$	10.5 (◊)
	$L(\omega_n) \otimes L(\omega_{n-1})^{(q)}$	$[7, \infty)$	10.5 (†)
		$[3, 6]$	10.5 (◊)
	$L(\omega_1) \otimes L(2\omega_1)^{(q)}$	$[3, \infty)$	10.5 (†)
		$[1, 2]$	10.5 (◊)
	$L(\omega_1) \otimes L(2\omega_n)^{(q)}$	$[3, \infty)$	10.5 (†)
		$[1, 2]$	10.5 (◊)
	$L(\omega_n) \otimes L(2\omega_1)^{(q)}$	$[3, \infty)$	10.5 (†)
		$[1, 2]$	10.5 (◊)
	$L(\omega_n) \otimes L(2\omega_n)^{(q)}$	$[3, \infty)$	10.5 (†)
		$[1, 2]$	10.5 (◊)
	$L(\omega_2) \otimes L(\omega_2)^{(q)}$	3	10.6 (†)
	$L(2\omega_1) \otimes L(2\omega_1)^{(q)}$	1	10.6 (†)
B_n	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[2, \infty)$	-
	$L(\omega_1) \otimes L(\omega_2)^{(q)}$	2	10.6 (◊)
	$L(\omega_2) \otimes L(\omega_2)^{(q)}$	2	10.6 (◊)
C_n	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[3, \infty)$	-
	$L(\omega_1) \otimes L(\omega_n)^{(q)}$	$[3, 4]$ $(p = 2)$	10.6 (†)
	$L(\omega_3) \otimes L(\omega_3)^{(q)}$	3 $(p = 2)$	10.6 (†)
D_n	$L(\omega_1) \otimes L(\omega_1)^{(q)}$	$[4, \infty)$	-
	$L(\omega_1) \otimes L(\omega_5)^{(q)}$	5	10.3 (†)
	$L(\omega_1) \otimes L(\omega_4)^{(q)}$	5	10.3 (†)
	$L(\omega_1) \otimes L(\omega_4)^{(q)}$	4	10.3 (◊)
	$L(\omega_1) \otimes L(\omega_3)^{(q)}$	4	10.3 (◊)
G_2	$L(\omega_2) \otimes L(\omega_2)^{(q)}$	$(p = 3)$	10.4 (†)

Table 10.1: Possible twisted modules for all Lie types

Lemma 10.3. *Let $G = \mathrm{Spin}_{2n}(K)$ act on $L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$ and $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [4, 5]$. In both cases, if $n = 5$ then (†) holds and if $n = 4$ then (◊) holds for all $X \in I$.*

Proof. First consider the module $L(\omega_1)$. There are $2(n-1)(n-2)$ or $2(n-1)$ roots $\alpha \in \Phi(D_n)$ according as $\langle \omega_1, \alpha \rangle = 0$ or 1. Thus, for a given α there are $\frac{2n.2(n-1)(n-2)}{2n(n-1)} = 2(n-2)$ weights $\mu \in W.\omega_1$ with weight string μ and similarly 2 weights $\mu \in W.\omega_1$ with weight string μ $\mu+\alpha$.

For the module $L(\omega_n)$ there are $n(n-1)$ or $\frac{1}{2}n(n-1)$ roots $\alpha \in \Phi(D_n)$ according as $\langle \omega_n, \alpha \rangle = 0$ or 1. Thus, for a given α there are $\frac{2^{n-1}.n(n-1)}{2n(n-1)} = 2^{n-2}$ weights $\mu \in W.\omega_n$ with weight string μ and similarly 2^{n-3} weights $\mu \in W.\omega_n$ with weight string μ $\mu+\alpha$.

By combining these two calculations, given $\alpha \notin \Phi_s$ the weight nets for $L(\omega_1) \otimes L(\omega_n)^{(q)}$ where $\mu \in W.\omega_1$ and $\mu' \in W.\omega_n$ are given below along with the number of each type of weight net.

$\mu+q\mu'$	$\mu+q\mu'$	$\mu+q(\mu'+\alpha)$	$\mu+q\mu'$	$\mu+q\mu'$	$\mu+q(\mu'+\alpha)$
			$(\mu+\alpha)+q\mu'$	$(\mu+\alpha)+q\mu'$	$(\mu+\alpha)+q(\mu'+\alpha)$
$2^{n-1}(n-2)$	$2^{n-2}(n-2)$		2^{n-1}	2^{n-2}	

Thus we have $|\Lambda| \geqslant 2^{n-2}(n-2)+2^{n-1}+2^{n-1} = 2^{n-2}(n+2)$ which exceeds $\dim G$ for $n=5$ so (†) holds here and equals $|\Phi(D_n)| = 24$ for $n=4$ in which case we are left to show that ($\Diamond$) holds for $X = \varnothing$.

The 8 weights in $W.\omega_1$ are as follows.

$1\ 1\ \frac{1}{2}\ \frac{1}{2}$ $\quad$ $0\ 1\ \frac{1}{2}\ \frac{1}{2}$ $\quad$ $0\ 0\ \frac{1}{2}\ \frac{1}{2}$ $\quad$ $0\ 0\ \frac{1}{2}\ \text{-}\frac{1}{2}$ $\quad$ $0\ 0\ \text{-}\frac{1}{2}\ \frac{1}{2}$ $\quad$ $0\ 0\ \text{-}\frac{1}{2}\ \text{-}\frac{1}{2}$ $\quad$ $0\ \text{-}1\ \text{-}\frac{1}{2}\ \text{-}\frac{1}{2}$ $\quad$ $\text{-}1\ \text{-}1\ \text{-}\frac{1}{2}\ \text{-}\frac{1}{2}$

Similarly, the 8 weights in $W.\omega_4$ are as follows.

$\frac{1}{2}\ 1\ \frac{1}{2}\ 1$ $\quad$ $\frac{1}{2}\ 1\ \frac{1}{2}\ 0$ $\quad$ $\frac{1}{2}\ 0\ \frac{1}{2}\ 0$ $\quad$ $\text{-}\frac{1}{2}\ 0\ \frac{1}{2}\ 0$ $\quad$ $\frac{1}{2}\ 0\ \text{-}\frac{1}{2}\ 0$ $\quad$ $\text{-}\frac{1}{2}\ 0\ \text{-}\frac{1}{2}\ 0$ $\quad$ $\text{-}\frac{1}{2}\ \text{-}1\ \text{-}\frac{1}{2}\ 0$ $\quad$ $\text{-}\frac{1}{2}\ \text{-}1\ \text{-}\frac{1}{2}\ \text{-}1$

If we write the 64 weights $\mu+q\mu'$ where $\mu \in W.\omega_1$, $\mu' \in W.\omega_4$ of $L(\omega_1) \otimes L(\omega_4)^{(q)}$ in an 8×8 grid then it is clear that in each row and each column at least 6 weights lie in Λ; thus $d^{\varnothing}_{\omega_1+q\omega_4} = 48 > 24 = e^{\varnothing}_{\omega_1+q\omega_4}$. The lemma follows for $L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$ with $n \in [4,5]$ similarly. $\square$

In the next lemma we shall deal with the only twisted module requiring consideration for a group of exceptional type.

Lemma 10.4. *Consider the simple algebraic group $G = G_2(K)$ acting on $L(\omega_2) \otimes L(\omega_2)^{(q)}$ with $p=3$. Then (†) holds.*

Proof. When $p=3$ the weights in $W.\omega_1$ appear with multiplicity zero and the weight 0 appears with multiplicity one in $L(\omega_2)$. The short roots in $\Phi(G_2)$ are $\pm\alpha_1$, $\pm(\alpha_1+\alpha_2)$ and $\pm(2\alpha_1+\alpha_2)$ and we may assume that one of these does not lie in Φ_s. We see that $\langle \omega_2, \alpha \rangle = 0$ for $\alpha \in \{\pm\alpha_1\}$ and $\langle \omega_2, \alpha \rangle = 3$ for $\alpha \in \{\alpha_1+\alpha_2, 2\alpha_1+\alpha_2\}$. Thus, given a short root α, there are $\frac{6.2}{6} = 2$ weights $\mu \in W.\omega_2$ with $\langle \mu, \alpha \rangle = 0$ and similarly 2 weights $\mu \in W.\omega_2$ with $\langle \mu, \alpha \rangle = 3$. If $p \neq 3$ there are two weight strings of the form $\mu_2\ \mu_1\ \mu_1\ \mu_2$ where $\mu_i \in W.\omega_i$ for $i = 1, 2$. Although $m_{\omega_1} = 0$ we may still use the adjacency principle for $p = 3$ by taking $\alpha \notin \Phi_s$ and using the fact that $\mu(s) = (\mu - 3\alpha)(s)$ if and only if $\alpha(s) = 1$. If μ, $\mu' \in W.\omega_2$ then the weight nets that occur for $L(\omega_2) \otimes L(\omega_2)^{(q)}$ are as follows.

$$\begin{array}{cccccc}
\mu + q\mu' & \mu + q\mu' & \mu + q(\mu' + 3\alpha) & \mu + q\mu' & \mu + q\mu' & \mu + q(\mu' + 3\alpha) \\
 & & & (\mu + 3\alpha) + q\mu' & (\mu + 3\alpha) + q\mu' & (\mu + 3\alpha) + q(\mu' + 3\alpha) \\
4 & 4 & & 4 & 4 &
\end{array}$$

Thus we have $|\Lambda| \geqslant 4.1 + 4.1 + 4.2 = 16 > \dim G = 14$ as required. $\square$

Recall from Section 5.3 that for type A_n the weights in $W.\omega_k$ with $k \in [1, n]$ can be represented by strings of length $n+1$ consisting of k plus signs and $n-k+1$ minus signs. Also elements of the Weyl group act by permuting plus and minus signs.

Lemma 10.5. *Let $G = \mathrm{SL}_{n+1}(K)$ act on $U \otimes V^{(q)}$ where $U \in \{L(\omega_1), L(\omega_n)\}$ and $V \in \{L(\omega_2), L(\omega_{n-1})\}$ for $n \in [3, \infty)$, and $U \otimes V^{(q)}$ where $U \in \{L(\omega_1), L(\omega_n)\}$ and $V \in \{L(2\omega_1), L(2\omega_n)\}$ for $n \in [1, \infty)$. Then in the former case (†) holds for $n \in [7, \infty)$ and (◊) is always satisfied for $n \in [3, 6]$, and in the latter case (†) holds for $n \in [3, \infty)$ and (◊) is always satisfied for $n \in [1, 2]$.*

Proof. We begin by recalling that $\dim L(\omega_1) = n + 1$, $\dim L(\omega_2) = \frac{1}{2}n(n + 1)$ and $\dim L(2\omega_1) = \frac{1}{2}(n + 1)(n + 2)$. Following the usual calculations we find that, for a given $\alpha \in \Phi(A_n)$, there are $n - 1$ weights $\mu \in W.\omega_1$ with weight string μ and there is 1 weight $\mu \in W.\omega_1$ with weight string $\mu\ \mu + \alpha$. Similarly, there are $1 + \frac{1}{2}(n - 2)(n - 1)$ weights $\nu \in W.\omega_2$ with weight string ν and there are $n - 1$ weights $\nu \in W.\omega_2$ with weight string $\nu\ \nu + \alpha$. Finally, considering $L(2\omega_1)$, there are $n-1$ weights $\eta \in W.(2\omega_1)$ with weight string η and there is 1 weight $\eta \in W.(2\omega_1)$ with weight string $\eta\ \eta + \alpha\ \eta + 2\alpha$ where $\eta + \alpha \in W.\omega_2$.

Consider the module $L(\omega_1) \otimes L(\omega_2)^{(q)}$; taking $\alpha \notin \Phi_s$ we count the number of weights that occur as singletons, i.e., those weights of the form $\mu + q\nu$ with $\langle \mu, \alpha \rangle = \langle \nu, \alpha \rangle = 0$. There are $(n-1) + \frac{1}{2}(n-2)(n-1)^2$ such weights. The remaining weights of $L(\omega_1) \otimes L(\omega_2)^{(q)}$

all occur in pairs so we find that $|\Lambda| \geqslant \frac{1}{2}\left(\frac{1}{2}n(n+1)^2 - (n-1) - \frac{1}{2}(n-2)(n-1)^2\right) = \frac{1}{2}(3n^2 - n + 2)$. This exceeds $\dim G$ for $n \geqslant 7$ so (†) holds.

We may now take two orthogonal roots outside the root system Φ_s of $C_G(s)$ since ($\Diamond$) is satisfied if $X = A_{n-1}$ for $n \geqslant 3$. Assume without loss that these two roots are α_1 and α_3. We see that there are $\binom{n-3}{1} = n-3$ weights in $W.\omega_1$ orthogonal to both of these roots and similarly there are $2 + \binom{n-3}{2}$ weights in $W.\omega_2$ orthogonal to them both (those weights of the form $++--- \cdots -$, $--++- \cdots -$ and $----\underline{++-\cdots-}$). Thus we now find that $|\Lambda| \geqslant \frac{1}{2}\left(\frac{1}{2}n(n+1)^2 - 2(n-3) - \frac{1}{2}(n-4)(n-3)^2\right) = 3n^2 - 9n + 12$ which exceeds $\dim G$ for $n \geqslant 5$. If $n = 4$ then $|\Lambda| \geqslant 24 = \dim G$ so ($\Diamond$) is satisfied for all $X \in I$ and if $n = 3$ then $|\Lambda| \geqslant 12 = \Phi(A_3)$ so we are left to consider the possibility $X = \varnothing$ here.

Now if $n = 3$ then the weights in $W.\omega_1$ are 3 2 1, -1 2 1, -1 -2 1 and -1 -2 -3 where we have omitted a factor of $\frac{1}{4}$ on each coefficient. Clearly the difference between any two of these is a root in A_3. Suppose that $X = \varnothing$; if we arrange the 24 weights $\mu + q\nu$ with $\mu \in W.\omega_1$, $\nu \in W.\omega_2$ of $L(\omega_1) \otimes L(\omega_2)^{(q)}$ into a 6×4 grid then, in each column, at least three of the four weights lies in Λ. Thus $d^{\varnothing}_{\omega_1 + q\omega_2} \geqslant 18 > 12 = e^{\varnothing}_{\omega_1 + q\omega_2}$. (We are essentially using Proposition 4.3 here.)

It is clear that the same conclusions for $L(\omega_1) \otimes L(\omega_2)^{(q)}$ with $n \in [3, \infty)$ hold for $L(\omega_1) \otimes L(\omega_{n-1})^{(q)}$, $L(\omega_n) \otimes L(\omega_2)^{(q)}$ and $L(\omega_n) \otimes L(\omega_{n-1})^{(q)}$ with $n \in [3, \infty)$.

Consider the module $L(\omega_1) \otimes L(2\omega_1)^{(q)}$. Since $\omega_2 < 2\omega_1$ we can use Premet's theorem to conclude that (†) holds for $n \geqslant 7$. Using the data at the start of this lemma we find that the weight nets for this module are as follows.

$$\begin{array}{c} \mu+q\nu \\ \\ (n-1)^2 \end{array} \qquad \begin{array}{ccc} \mu+q\nu & \mu+q(\nu+\alpha) & \mu+q(\nu+2\alpha) \\ & & \\ & n-1 & \end{array} \qquad \begin{array}{c} \mu+q\nu \\ (\mu+\alpha)+q\nu \\ n-1 \end{array} \qquad \begin{array}{ccc} \mu+q\nu & \mu+q(\nu+\alpha) & \mu+q(\nu+2\alpha) \\ (\mu+\alpha)+q\nu & (\mu+\alpha)+q(\nu+\alpha) & (\mu+\alpha)+q(\nu+2\alpha) \\ & 1 & \end{array}$$

$$\begin{array}{c} \mu+q\eta \\ \\ \frac{1}{2}(n-2)(n-1)^2 \end{array} \qquad \begin{array}{cc} \mu+q\eta & \mu+q(\eta+\alpha) \\ & \\ (n-1)^2 & \end{array} \qquad \begin{array}{c} \mu+q\eta \\ (\mu+\alpha)+q\eta \\ \frac{1}{2}(n-2)(n-1) \end{array} \qquad \begin{array}{cc} \mu+q\eta & \mu+q(\eta+\alpha) \\ (\mu+\alpha)+q\eta & (\mu+\alpha)+q(\eta+\alpha) \\ & n-1 \end{array}$$

Thus we have $|\Lambda| \geqslant \frac{1}{2}(3n^2 + n + 2)$ which exceeds $\dim G$ for $n \geqslant 3$ and equals $\dim G$ for $n \in [1, 2]$.

We can draw the same conclusions when $n \in [2, \infty)$ for $L(\omega_n) \otimes L(2\omega_1)^{(q)}$, $L(\omega_1) \otimes$

$L(2\omega_n)^{(q)}$ and $L(\omega_n) \otimes L(2\omega_n)^{(q)}$. $\square$

The last lemma of this chapter deals with the remaining six families of modules in Table 10.1.

Lemma 10.6. *Let G be a simple simply connected algebraic group defined over K.*

(i) If G is of type A_n acting on either $L(\omega_2) \otimes L(\omega_2)^{(q)}$ for $n = 3$ or $L(2\omega_1) \otimes L(2\omega_1)^{(q)}$ for $n = 1$ then $(\dagger)$ holds.

(ii) If G is of type B_2 acting on either $L(\omega_1) \otimes L(\omega_2)^{(q)}$ or $L(\omega_2) \otimes L(\omega_2)^{(q)}$ then $(\Diamond)$ is satisfied for all $X \in I$.

(iii) If G is of type C_n with $p = 2$ acting on either $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [3, 4]$ or $L(\omega_3) \otimes L(\omega_3)^{(q)}$ for $n = 3$ then $(\dagger)$ holds.

Proof. Consider the action of $\mathrm{SL}_2(K)$ on $L(2\omega_1) \otimes L(2\omega_1)^{(q)}$. We can arrange the 9 weights of this module in a 3×3 grid as follows.

$$\begin{array}{rrr} 2\omega_1 + q(2\omega_1) & 2\omega_1 + q0 & 2\omega_1 - q(2\omega_1) \\ 0 + q(2\omega_1) & 0 + q0 & 0 - q(2\omega_1) \\ -2\omega_1 + q(2\omega_1) & -2\omega_1 + q0 & -2\omega_1 - q(2\omega_1) \end{array}$$

Clearly, by observing that $2\omega_1 = \alpha_1$, we have $|\Lambda| \geqslant 4 > 3 = \dim G$ so $(\dagger)$ holds.

Consider the action of $\mathrm{SL}_4(K)$ on $L(\omega_2) \otimes L(\omega_2)^{(q)}$. If $\mu, \mu' \in W.\omega_2$ and $\alpha \notin \Phi_s$ then, using the calculations in the previous lemma, the weight nets for this module are as follows.

$$\begin{array}{cccc} \mu + q\mu' & \mu + q\mu' \quad \mu + q(\mu' + \alpha) & \begin{array}{c} \mu + q\mu' \\ (\mu + \alpha) + q\mu' \end{array} & \begin{array}{cc} \mu + q\mu' & \mu + q(\mu' + \alpha) \\ (\mu + \alpha) + q\mu' & (\mu + \alpha) + q(\mu' + \alpha) \end{array} \\ 4 & 4 & 4 & 4 \end{array}$$

Thus $|\Lambda| \geqslant 4 + 4 + 8 = 16 > 15 = \dim G$.

Consider the action of $\mathrm{Spin}_5(K)$ on $L(\omega_1) \otimes L(\omega_2)^{(q)}$. Recall that $\dim L(\omega_2) = 4$ and $\dim L(\omega_1) = 5$ or 4 according as $p \neq 2$ or $p = 2$ (the difference in the latter case is that $m_0 = 0$). For a fixed short root $\alpha \in \Phi(B_2)$ there are 2 weights $\mu \in W.\omega_1$ with weight string μ and 1 weight $\mu \in W.\omega_1$ with weight string μ $\mu + \alpha$ $\mu + 2\alpha$ where the middle weight in the string is 0. Similarly we find that there are 2 weights $\mu' \in W.\omega_2$ with weight string μ' $\mu' + \alpha$.

Assuming that $\alpha \notin \Phi_s$ we can arrange the weights of $L(\omega_1) \otimes L(\omega_2)^{(q)}$ into four 2×1 and two 2×3 or 2×2 weight nets according as $p \neq 2$ or $p = 2$. If the characteristic is odd then $|\Lambda| \geqslant 4.2 + 2.3 = 10 = \dim G$ and if it is even then, using the fact that $2\alpha(s) = 1$ if and only if $\alpha(s) = 1$, we have $|\Lambda| \geqslant 4 + 2.2 = 8 = |\Phi(B_2)|$. It remains to show that $(\Diamond)$ holds for $X = \varnothing$ if $p = 2$. The weights of $L(\omega_1)$ are 1 1, 0 1, 0 -1, -1 -1 and the weights of $L(\omega_2)$ are $\frac{1}{2}$ 1, $\frac{1}{2}$ 0, -$\frac{1}{2}$ 0 and -$\frac{1}{2}$ -1 so if we arrange the 16 weights of $L(\omega_1) \otimes L(\omega_2)^{(q)}$ for $p = 2$ into a 4×4 grid then at least three weights in each row and column lie in Λ. Thus $d^{\varnothing}_{\omega_1+q\omega_2} \geqslant 4.3 = 12 > 8 = e^{\varnothing}_{\omega_1+q\omega_2}$.

Now consider the action of $\mathrm{Spin}_5(K)$ on $L(\omega_2) \otimes L(\omega_2)^{(q)}$. Taking $\mu, \mu' \in W.\omega_2$ and $\alpha \notin \Phi_s$ we find $|\Lambda| \geqslant 8$ since there are four 2×2 weight nets for this module. We just need to consider the case $X = \varnothing$ to show that $(\Diamond)$ holds for all $X \in I$, but this follows immediately as before.

Consider the action of $\mathrm{Sp}_{2n}(K)$ with $p = 2$ on $L(\omega_1) \otimes L(\omega_n)^{(q)}$ for $n \in [3, 4]$. Then $\dim L(\omega_1) = 2n$ and $\dim L(\omega_n) = 2^n$. We have $\langle \omega_n, \alpha \rangle = 0$ or 2 for either $n(n-1)$ or $\frac{1}{2}n(n-1)$ short roots $\alpha \in \Phi(C_n)$. For a given α there are 2^{n-1} weights $\mu \in W.\omega_n$ with weight string μ and 2^{n-2} weights $\mu \in W.\omega_n$ with weight string containing μ $\mu + 2\alpha$. For $L(\omega_1)$ there are $2(n-2)$ weights $\mu' \in W.\omega_1$ with weight string μ' and 2 weights $\mu' \in W.\omega_1$ with weight string μ' $\mu' + \alpha$.

The weight nets for $L(\omega_1) \otimes L(\omega_n)^{(q)}$ with $p = 2$ are as follows where $\alpha \notin \Phi_s$.

$$\begin{array}{ccccccc}
\mu + q\mu' & \quad & \mu + q\mu' & \mu + q(\mu' + 2\alpha) & \quad \mu + q\mu' & \quad & \mu + q\mu' \quad \mu + q(\mu' + 2\alpha) \\
 & & & & (\mu+\alpha) + q\mu' & & (\mu+\alpha) + q\mu' \quad (\mu+\alpha) + q(\mu' + 2\alpha) \\
2^n(n-2) & & 2^{n-1}(n-2) & & 2^n & & 2^{n-1}
\end{array}$$

Therefore we have $|\Lambda| \geqslant 2^{n-1}(n-2) + 2^n + 2^n = 2^{n-1}(n+2)$ which exceeds $\dim G$ for $n = 4$. If $n = 3$ then we have $|\Lambda| \geqslant 20$ whereas $\dim G = 21$ so more work is required. We need to show that $d^X_{\omega_1+q\omega_3} > \dim G$ for $X = A_2$ and $X = C_2$ in order to take two orthogonal roots (one short and one long) outside Φ_s and conclude that $(\dagger)$ holds. Suppose $X = A_2$ with roots α_1 and α_2. The weights in $W.\omega_1$ split into a clique consisting of two clusters $\cdot\ \cdot$ $\frac{1}{2}$ and $\cdot\ \cdot$ -$\frac{1}{2}$ of size 3. By the adjacency principle for $L(\omega_1)$ at least one of these clusters does not lie in the eigenspace. The weights in $W.\omega_3$ split into four clusters 1 2 $\frac{3}{2}$, $\cdot\ \cdot$ $\frac{1}{2}$, $\cdot\ \cdot$ -$\frac{1}{2}$ and -1 -2 -$\frac{3}{2}$ of sizes 1, 3, 3 and 1 respectively. Thus, by the adjacency principle for $L(\omega_3)$ with $p = 2$, the maximum number of weights that can lie in the eigenspace is 3 (from either cluster of size 3). We form a 6×8 grid consisting of weights $\mu + q\mu'$ where $\mu \in W.\omega_1$ and $\mu' \in W.\omega_3$. Arranging these weights into eight clusters (four of the form 3×1 and four of the form 3×3) we find that $d_{\omega_1+q\omega_3} \geqslant 2(1.3 + 3.3 + 1.3) = 30 > 21 = \dim G$.

Suppose that $X = C_2$ with roots α_2 and α_3. The weights in $W.\omega_1$ split into a clique consisting of three clusters 1 1 $\frac{1}{2}$, 0 $\cdot$ $\cdot$ and -1 -1 -$\frac{1}{2}$ of sizes 1, 4 and 1 respectively. The weights in $W.\omega_3$ split into a clique consisting of two clusters 1 $\cdot$ $\cdot$ and -1 $\cdot$ $\cdot$, both of size 4. As before, we form a 6×8 grid consisting of weights $\mu + q\mu'$ where $\mu \in W.\omega_1$ and $\mu' \in W.\omega_3$. Arranging these weights into six clusters (four of the form 1×4 and two of the form 4×4) we find that $d_{\omega_1+q\omega_3} \geqslant 4 + 4 + 16 + 4 = 28 > 21 = \dim G$.

Now we take two orthogonal roots α short and β long outside Φ_s. We see that $\langle \omega_1, \alpha \rangle = 0$ or ±1 and $\langle \omega_1, \beta \rangle = 0$ or ±1; however it is not possible to find two orthogonal roots α, β such that $\langle \omega_1, \alpha \rangle = \langle \omega_1, \beta \rangle = 0$ since $n = 3$. Similarly we have $\langle \omega_3, \alpha \rangle = 0$ or ±2 (in the latter case the weight strings obtained are not saturated) and $\langle \omega_3, \beta \rangle = \pm1$. In particular there are no weights in both $L(\omega_1)$ and $L(\omega_3)$ orthogonal to both a short and long root orthogonal to each other. Thus the weights in $L(\omega_1) \otimes L(\omega_3)^{(q)}$ for $n = 3$ all appear in pairs when arranged into weight nets for some fixed pair of mutually orthogonal short and long roots, whence

$$|\Lambda| \geqslant \frac{1}{2} \dim L(\omega_1) \otimes L(\omega_3)^{(q)} = 24 > 21 = \dim G$$

and (†) holds.

Finally, consider the action of $\mathrm{Sp}_6(K)$ with $p = 2$ on $L(\omega_3) \otimes L(\omega_3)^{(q)}$. The weight nets that occur are similar to those in the previous case except in the vertical direction the difference between two weights is 2α rather than α. We find $|\Lambda| \geqslant 8 + 8 + 2.4 = 24$, so (†) holds. $\square$

It remains to consider $L(\omega_1) \otimes L(\omega_1)^{(q)}$ for types B_n, C_n and D_n. We should be able to adapt our techniques to show for each n that (♢) is satisfied for all $X \in I$. The natural approach is to inductively take mutually orthogonal roots outside Φ_s.

Chapter 11

Concluding remarks

In this final chapter we shall briefly discuss a number of points that have arisen in the course of this thesis.

The Main Theorem presents a clear dichotomy between modules of dimension at most and larger than $\dim G + 2$. It is somewhat surprising and not entirely clear that this should be the dividing line. Indeed, one would perhaps intuitively expect that $\dim G$ is a more likely boundary. There are very few modules with dimension $\dim G+1$ or $\dim G+2$, the most notable is the infinite family $L(\omega_1) \otimes L(\omega_1)^{(q)}$ in type A_n for $n \geqslant 1$; the other modules occur only for a single rank n for small characteristic, or else for $n = 1$.

Our methodology has been to begin each chapter on large modules with an initial survey to decide which modules we cannot yet conclude that (†) holds. We have a reasonably efficient technique to do this and we show that (†) holds for the vast majority of modules. It is only for modules with dimension closer to $\dim G + 2$ that more work is required and we usually show that ($\Diamond$) is satisfied for each $X \in I$ rather than that (†) holds. Indeed, if a module has dimension approximately $2 \dim G$ then we expect to be able to show at least that ($\Diamond$) is satisfied for all $X \in I$.

In some cases we have been able to assume that there are two or more orthogonal roots lying outside the root system Φ_s for some $s \in G_{ss} \setminus Z$ since ($\Diamond$) is satisfied for sufficiently large centraliser types. In these cases, it is subsequently possible to show that the codimension of the eigenspace exceeds $\dim G$. However, we cannot conclude that (†)

holds since there are simply too many large centraliser types X to check to ensure that $d_\lambda^X > \dim G$ always holds.

It would be reasonable to expect that a computer program could be produced to find, for any possible centraliser type, better lower bounds for the codimension of an eigenspace. This would be easier in cases where the plus minus notation for weights can be used, i.e., for modules $L(\omega_k)$ for $k \in [1, n]$ in type A_n, and $L(\omega_n)$ for types B_n, C_n ($p = 2$ only) and D_n. It is far from efficient when there are many centraliser types to check by hand to show that ($\Diamond$) is satisfied for all $X \in I$. For example, there were 60 centraliser types to go through for the module $L(\omega_7)$ for type D_7 with $p = 2$.

We have not made any claims about the size of the codimension of the eigenspace for modules where (†) holds; for modules V where $\dim V > 3 \dim G$ the codimension of the eigenspace would certainly far exceed $\dim G$ with few exceptions. We do not claim that (†) is an optimal condition, rather that it is convenient. It allows us to reach our desired conclusion that $\dim \overline{E} < \dim V$.

Naturally, we would like to have fewer possible exceptions to the Main Theorem. In particular there are three infinite families for which our methods do not appear to be sufficient: $L(\omega_2)$ for $n \in [3, \infty)$ in type C_n, $L(2\omega_1)$ for $n \in [2, \infty)$ with $p \neq 2$ in type B_n and $L(2\omega_1)$ for $n \in [4, \infty)$ with $p \neq 2$ in type D_n. It should be possible to deal with the modules $L(\omega_1) \otimes L(\omega_1)^{(q)}$ for types B_n, C_n and D_n; an approach by inductively taking orthogonal roots not in Φ_s should show that ($\Diamond$) is satisfied for all $X \in I$. Other modules with dimension at most $\dim G$ could possibly be dealt with using the strategy employed by Lawther in Section 3.6, though this would be most difficult in the case $L(\omega_3)$ for $\mathrm{SL}_8(K)$. Many of the remaining modules with dimension larger than $\dim G + 2$ can be dealt with for almost all centraliser types $X \in I$. In cases where $d_\lambda^X = e_\lambda^X$ occurs, it should be possible to say more since we have quite a bit of information. We know the centraliser type of a given non-central semisimple element and precisely which weights lie in the eigenspace.

We have shown that the set of vectors whose stabilisers contain no non-central semisimple element is dense in V. It remains to show that at least one of these vectors has a stabiliser containing no non-trivial unipotent element. Clearly this will require a different strategy to that employed in this thesis for non-central semisimple elements. There has been much work done by Suprunenko on the behaviour of unipotent elements in modular representations of algebraic groups which may be helpful in this regard; she intends to obtain estimates for the dimensions of the 1-eigenspaces $V^u = \{v \in V \mid uv = v\}$ for each unipotent $u \in G$ following on from the work in [28].

It would be helpful to use the existence of a regular orbit for the action of a simple algebraic group to deduce that a primitive permutation group of affine type has minimal base size 2. Indeed, much work has already been done on the almost simple permutation groups, particularly by Burness.

It should be possible to find further applications of the Main Theorem. In particular, we have shown for the majority of modules with dimension exceeding $\dim G + 2$ that (†) holds. This may be useful for problems on eigenvectors in actions of simple algebraic groups, some of which are described in [29].

Dyfal donc a dyr y garreg

Ní heolas go haontios

Appendix A

Cliques for $(G, V) = (A_{12}, L(\omega_3))$

We shall provide cliques for the nineteen centraliser types remaining to be considered for $\mathrm{SL}_{13}(K)$ acting on $L(\omega_3)$ listed in Table 5.2 of Lemma 5.13. We show in each case that $(\lozenge)$ is satisfied.

$X = \varnothing$					
Clique	Cluster size	l	Clique	Cluster size	l
$+\underline{++----------}$	1	60	$--+\underline{++--------}$	1	40
$-+\underline{++---------}$	1	50	$---+\underline{++-------}$	1	32

From the table above we have $d_{\omega_3}^{\varnothing} \geqslant 182 > 156 = e_{\omega_3}^{\varnothing}$.

$X = A_1$		
Clique	Cluster size	l
$+-:\underline{++---------}$	2	100
$--:+\underline{++--------}$	1	40
$--:-+\underline{++-------}$	1	32

From the table above we have $d_{\omega_3}^{A_1} \geqslant 172 > 154 = e_{\omega_3}^{A_1}$.

$X = A_1^2$		
Clique	Cluster size	l
$+-\mid+-:\underline{+--------}$	4	32
$+-\mid--:\underline{++-------}$	2	64
$--\mid+-:\underline{++-------}$	2	64

From the table above we have $d_{\omega_3}^{A_1^2} \geqslant 160 > 152 = e_{\omega_3}^{A_1^2}$.

$X = A_1^3$					
Clique	Cluster size	l	Clique	Cluster size	l
$+-\mid+-\mid--:\underline{+-------}$	4	24	$+-\mid--\mid--:\underline{++------}$	2	36
$+-\mid--\mid+-:\underline{+-------}$	4	24	$--\mid+-\mid--:\underline{++------}$	2	36
$--\mid+-\mid+-:\underline{+-------}$	4	24	$--\mid--\mid+-:\underline{++------}$	2	36

From the table above we have $d_{\omega_3}^{A_1^3} \geqslant 180 > 150 = e_{\omega_3}^{A_1^3}$.

$X = A_1^4$					
Clique	Cluster size	l	Clique	Cluster size	l
$+-\mid--\mid--\mid--:\underline{++---}$	2	16	$+-\mid--\mid+-\mid--:\underline{+----}$	4	16
$--\mid+-\mid--\mid--:\underline{++---}$	2	16	$+-\mid--\mid--\mid+-:\underline{+----}$	4	16
$--\mid--\mid+-\mid--:\underline{++---}$	2	16	$--\mid+-\mid+-\mid--:\underline{+----}$	4	16
$--\mid--\mid--\mid+-:\underline{++---}$	2	16	$--\mid+-\mid--\mid+-:\underline{+----}$	4	16
$+-\mid+-\mid--\mid--:\underline{+----}$	4	16	$--\mid--\mid+-\mid+-:\underline{+----}$	4	16

From the table above we have $d_{\omega_3}^{A_1^4} \geqslant 160 > 148 = e_{\omega_3}^{A_1^4}$.

$X = A_1^5$					
Clique	Cluster size	l	Clique	Cluster size	l
$\underline{+-\mid+-\mid+-\mid--\mid--}:---$	8	64	$\underline{+-\mid+-\mid--\mid--\mid--}:-+-$	4	32
$\underline{+-\mid+-\mid--\mid--\mid--}:+--$	4	32	$\underline{+-\mid+-\mid--\mid--\mid--}:--+$	4	32

From the table above we have $d_{\omega_3}^{A_1^5} \geqslant 160 > 146 = e_{\omega_3}^{A_1^5}$.

$X = A_1^6$					
Clique	Cluster size	l	Clique	Cluster size	l
$\underline{+-\mid+-\mid+-\mid--\mid--\mid--}:-$	8	128	$\underline{+-}\mid++\mid\underline{--\mid--\mid--\mid--}:-$	2	9
$++\mid\underline{+-\mid--\mid--\mid--\mid--}:-$	2	9	$--\mid++\mid--\mid--\mid--\mid--:+$	1	
$++\mid--\mid--\mid--\mid--\mid--:+$	1				

From the table above we have $d_{\omega_3}^{A_1^6} \geqslant 146 > 144 = e_{\omega_3}^{A_1^6}$.

$X = A_2$		
Clique	Cluster size	l
$++-:\underline{+---------}$	3	27
$+--:\underline{++--------}$	3	120
$---:++\underline{+-------}$	1	7

From the table above we have $d_{\omega_3}^{A_2} \geqslant 154 > 150 = e_{\omega_3}^{A_2}$.

$X = A_2A_1$		
Clique	Cluster size	l
$+--\mid+-:\underline{+-------}$	6	42
$+--\mid--:\underline{++------}$	3	72
$---\mid+-:\underline{++------}$	2	48

From the table above we have $d_{\omega_3}^{A_2A_1} \geqslant 162 > 148 = e_{\omega_3}^{A_2A_1}$.

$X = A_2A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++-\mid--\mid+-:-------$	6	42	$+--\mid--\mid--:\underline{++-----}$	3	36
$+--\mid+-\mid+-:-------$	12		$---\mid+-\mid--:\underline{++-----}$	2	24
$+--\mid+-\mid--:\underline{+------}$	6		$---\mid--\mid+-:\underline{++-----}$	2	24
$+--\mid--\mid+-:\underline{+------}$	6	30			

From the table above we have $d_{\omega_3}^{A_2A_1^2} \geqslant 156 > 146 = e_{\omega_3}^{A_2A_1^2}$.

$X = A_2A_1^3$					
Clique	Cluster size	l	Clique	Cluster size	l
$+--\mid\underline{+-\mid+-\mid--}:-----$	12	32	$++-\mid\underline{+-\mid--\mid--}:-----$	6	24
$---\mid+-\mid+-\mid+-:-----$	8		$++-\mid--\mid--\mid--:\underline{+---}$	3	
$+--\mid+-\mid--\mid--:\underline{+---}$	6	18	$---\mid+-\mid+-\mid--:\underline{+---}$	4	12
$+--\mid--\mid+-\mid--:\underline{+---}$	6	18	$---\mid+-\mid--\mid+-:\underline{+---}$	4	12
$+--\mid--\mid--\mid+-:\underline{+---}$	6	18	$---\mid--\mid+-\mid+-:\underline{+---}$	4	12

From the table above we have $d_{\omega_3}^{A_2A_1^3} \geqslant 146 > 144 = e_{\omega_3}^{A_2A_1^3}$.

$X = A_2A_1^4$					
Clique	Cluster size	l	Clique	Cluster size	l
$++-\mid\underline{+-\mid--\mid--\mid--}:--$	6	24	$+--\mid\underline{+-\mid+-\mid--\mid--}:--$	12	48
$++-\mid--\mid--\mid--\mid--:\underline{+-}$	3		$---\mid\underline{+-\mid+-\mid+-\mid--}:--$	8	24
$+--\mid\underline{+-\mid--\mid--\mid--}:+-$	6	18	$---\mid\underline{+-\mid+-\mid--\mid--}:+-$	4	16
$+--\mid\underline{+-\mid--\mid--\mid--}:-+$	6	18			

From the table above we have $d_{\omega_3}^{A_2A_1^4} \geqslant 148 > 142 = e_{\omega_3}^{A_2A_1^4}$.

$X = A_2A_1^5$		
Clique	Cluster size	l
$+--\mid\underline{+-\mid+-\mid--\mid--\mid--:}$	12	96
$---\mid\underline{+-\mid+-\mid+-\mid--\mid--:}$	8	64

From the table above we have $d_{\omega_3}^{A_2A_1^5} \geqslant 160 > 140 = e_{\omega_3}^{A_2A_1^5}$.

$X = A_2^2$		
Clique	Cluster size	l
$+--\mid+--:\underline{+-------}$	9	54
$+--\mid---:\underline{++------}$	3	54
$---\mid+--:\underline{++------}$	3	54

From the table above we have $d_{\omega_3}^{A_2^2} \geqslant 162 > 144 = e_{\omega_3}^{A_2^2}$.

$X = A_2^2A_1$					
Clique	Cluster size	l	Clique	Cluster size	l
$++-\vert+--\vert--:------$	9	63	$+--\vert---\vert+-:\underline{+-----}$	6	24
$+--\vert++-\vert--:------$	9		$---\vert+--\vert+-:\underline{+-----}$	6	24
$+--\vert+--\vert+-:------$	18		$+--\vert---\vert--:\underline{++----}$	3	24
$+--\vert+--\vert--:\underline{+-----}$	9		$---\vert+--\vert--:\underline{++----}$	3	24

From the table above we have $d_{\omega_3}^{A_2^2A_1} \geqslant 159 > 142 = e_{\omega_3}^{A_2^2A_1}$.

$X = A_2^2A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$+--\vert+--\vert\underline{+-\vert--}:---$	18	45	$\underline{+--\vert---}\vert--\vert--:+-+$	3	7
$+--\vert+--\vert--\vert--:\underline{+--}$	9		$---\vert---\vert\underline{+-\vert--}:+-+$	2	
$+--\vert---\vert+-\vert--:\underline{+--}$	6	12	$\underline{+--\vert---}\vert--\vert--:-++$	3	7
$+--\vert---\vert--\vert+-:\underline{+--}$	6	12	$---\vert---\vert\underline{+-\vert--}:-++$	2	
$---\vert+--\vert+-\vert--:\underline{+--}$	6	12	$++-\vert---\vert\underline{+-\vert--}:---$	6	15
$---\vert+--\vert--\vert+-:\underline{+--}$	6	12	$++-\vert---\vert--\vert--:\underline{+--}$	3	
$\underline{+--\vert---}\vert--\vert--:++-$	3	7	$---\vert++-\vert\underline{+-\vert--}:---$	6	15
$---\vert---\vert\underline{+-\vert--}:++-$	2		$---\vert++-\vert--\vert--:\underline{+--}$	3	

From the table above we have $d_{\omega_3}^{A_2^2A_1^2} \geqslant 144 > 140 = e_{\omega_3}^{A_2^2A_1^2}$.

$X = A_3$		
Clique	Cluster size	l
$++--:\underline{+---------}$	6	48
$+---:\underline{++--------}$	4	128

From the table above we have $d_{\omega_3}^{A_3} \geqslant 176 > 144 = e_{\omega_3}^{A_3}$.

$X = A_3A_1$		
Clique	Cluster size	l
$++--\vert--:\underline{+-------}$	6	36
$+---\vert+-:\underline{+-------}$	8	48
$+---\vert--:\underline{++------}$	4	72

From the table above we have $d_{\omega_3}^{A_3A_1} \geqslant 156 > 142 = e_{\omega_3}^{A_3A_1}$.

$X = A_3A_1^2$					
Clique	Cluster size	l	Clique	Cluster size	l
$++--\vert+-\vert--:------$	12	56	$++--\vert--\vert+-:------$	12	44
$+---\vert++\vert--:------$	4		$+---\vert--\vert++:------$	4	
$+---\vert+-\vert+-:------$	16		$+---\vert--\vert+-:\underline{+-----}$	8	
$+---\vert+-\vert--:\underline{+-----}$	8		$+++-\vert--\vert--:------$	4	28
$----\vert+-\vert+-:\underline{+-----}$	4	16	$++--\vert--\vert--:\underline{+-----}$	6	

From the table above we have $d_{\omega_3}^{A_3A_1^2} \geqslant 144 > 140 = e_{\omega_3}^{A_3A_1^2}$.

Bibliography

[1] R.W. Carter, *Finite groups of Lie type*, John Wiley & Sons Inc., New York, 1985.

[2] ______, *Simple groups of Lie type*, Wiley Classics Library, John Wiley & Sons Inc., New York, 1989, Reprint of the 1972 original, A Wiley-Interscience Publication.

[3] Z. Chen, *A classification of irreducible prehomogeneous vector spaces over an algebraically closed field of characteristic p. I*, Chinese Ann. Math. Ser. A **6** (1985), no. 1, 39–48.

[4] ______, *A new prehomogeneous vector space of characteristic* p, Chinese Ann. Math. Ser. B **8** (1987), no. 1, 22–35.

[5] A.M. Cohen and A.G. Helminck, *Trilinear alternating forms on a vector space of dimension* 7, Comm. Algebra **16** (1988), no. 1, 1–25.

[6] A.M. Cohen and D.B. Wales, GL(4)*-orbits in a* 16*-dimensional module for characteristic* 3, J. Algebra **185** (1996), no. 1, 85–107.

[7] C.W. Curtis, *Representations of Lie algebras of classical type with applications to linear groups*, J. Math. Mech. **9** (1960), 307–326.

[8] D.I. Deriziotis, *Centralizers of semisimple elements in a Chevalley group*, Comm. Algebra **9** (1981), no. 19, 1997–2014.

[9] P.B. Gilkey and G.M. Seitz, *Some representations of exceptional Lie algebras*, Geom. Dedicata **25** (1988), no. 1-3, 407–416, Geometries and groups (Noordwijkerhout, 1986).

[10] R.M. Guralnick, M.W. Liebeck, D. Macpherson, and G.M. Seitz, *Modules for algebraic groups with finitely many orbits on subspaces*, J. Algebra **196** (1997), no. 1, 211–250.

[11] B. Hartley and M. Kuzucuoğlu, *Centralizers of elements in locally finite simple groups*, Proc. London Math. Soc. (3) **62** (1991), no. 2, 301–324.

[12] J.E. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag, New York, 1975.

[13] ______, *Introduction to Lie algebras and representation theory*, Graduate Texts in Mathematics, vol. 9, Springer-Verlag, New York, 1978, Second printing, revised.

[14] ———, *Conjugacy classes in semisimple algebraic groups*, Mathematical Surveys and Monographs, vol. 43, American Mathematical Society, Providence, RI, 1995.

[15] J. Igusa, *A classification of spinors up to dimension twelve*, Amer. J. Math. **92** (1970), 997–1028.

[16] R.I. Lawther, M.W. Liebeck, and G.M. Seitz, *Fixed point spaces in actions of exceptional algebraic groups*, Pacific J. Math. **205** (2002), no. 2, 339–391.

[17] M.W. Liebeck, C.E. Praeger, and J. Saxl, *On the O'Nan-Scott theorem for finite primitive permutation groups*, J. Austral. Math. Soc. Ser. A **44** (1988), no. 3, 389–396.

[18] F. Lübeck, *Tables of weight multiplicities*, `http://www.math.rwth-aachen.de/~Frank.Luebeck/chev/WMSmall`.

[19] ———, *Small degree representations of finite Chevalley groups in defining characteristic*, LMS J. Comput. Math. **4** (2001), 135–169 (electronic).

[20] A.A. Premet, *Weights of infinitesimally irreducible representations of Chevalley groups over a field of prime characteristic*, Math. USSR-Sb. **61** (1988), no. 1, 167–183.

[21] ———, *Nilpotent orbits in good characteristic and the Kempf-Rousseau theory*, J. Algebra **260** (2003), no. 1, 338–366.

[22] M. Sato and T. Kimura, *A classification of irreducible prehomogeneous vector spaces and their relative invariants*, Nagoya Math. J. **65** (1977), 1–155.

[23] G.M. Seitz, *The maximal subgroups of classical algebraic groups*, Mem. Amer. Math. Soc. **67** (1987), no. 365, iv+286.

[24] T.A. Springer, *Linear algebraic groups*, Progress in Mathematics, vol. 9, Birkhäuser Boston, Mass., 1981.

[25] R. Steinberg, *Representations of algebraic groups*, Nagoya Math. J. **22** (1963), 33–56.

[26] ———, *Endomorphisms of linear algebraic groups*, Memoirs of the American Mathematical Society, No. 80, American Mathematical Society, Providence, R.I., 1968.

[27] ———, *Lectures on Chevalley groups*, Yale University, New Haven, Conn., 1968, Notes prepared by John Faulkner and Robert Wilson.

[28] I.D. Suprunenko, *The minimal polynomials of unipotent elements in irreducible representations of the classical groups in odd characteristic*, Mem. Amer. Math. Soc. **200** (2009), no. 939, vi+154.

[29] A.E. Zalesski, *On eigenvalues of group elements in representations of simple algebraic groups and finite Chevalley groups*, Acta Appl. Math. (2009), 21pp., (to appear).