Unipotent Elements
in Algebraic Groups
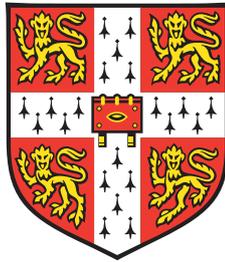
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To My Parents

Declaration

This thesis is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated below.

Chapter 3 is the result of joint research with my collaborator Professor A. Premet. More specifically, all original results contained in Chapter 3 are the result of collaboration, except that Theorem 3.6.2 (also repeated as Theorem 3.1.4) and Corollary 3.7.3 are entirely due to Premet, while Proposition 3.2.7 and Lemma 3.3.5 are my own. In addition, Remarks 3.3.6, 3.6.1(2), 3.6.2 and 3.7.3 are attributed to Premet. All other chapters, together with the appendices, are entirely my own work.

Acknowledgements

Abstract

This thesis is concerned with three distinct, but closely related, research topics focusing on the unipotent elements of a connected reductive algebraic group G, over an algebraically closed field $\Bbbk$, and nilpotent elements in the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$.

The first topic is a determination of canonical forms for unipotent classes and nilpotent orbits of G. Using an original approach, we begin by obtaining a new canonical form for nilpotent matrices, up to similarity, which is symmetric with respect to the non-main diagonal (i.e. it is fixed by the map $f : (x_{i,j}) \mapsto (x_{n+1-j,n+1-i})$), with entries in $\{0, 1\}$. We then show how to modify this form slightly in order to satisfy a non-degenerate symmetric or skew-symmetric bilinear form, assuming that the orbit does not vanish in the presence of such a form. Replacing G by any simple classical algebraic group, we thus obtain a unified approach to computing representatives for nilpotent orbits for all classical groups G. By applying Springer morphisms, this also yields representatives for the corresponding unipotent classes in G. As a corollary, we obtain a complete set of generic canonical representatives for the unipotent classes of the finite general unitary groups $\operatorname{GU}_n(\mathbb{F}_q)$ for all prime powers q.

Our second topic is concerned with unipotent pieces, defined by G. Lusztig in [Unipotent elements in small characteristic, *Transform. Groups* 10 (2005), 449–487]. We give a case-free proof of the conjectures of Lusztig from that paper. This presents a uniform picture of the unipotent elements of G, which can be viewed as an extension of the Dynkin–Kostant theory, but is valid without restriction on p. We also obtain analogous results for the adjoint action of G on its Lie algebra $\mathfrak{g}$ and the coadjoint action of G on $\mathfrak{g}^*$. We also obtain several general results about the Hesselink stratification and $\mathbb{F}_q$-rational structures on G-modules.

Our third topic is concerned with generalised Gelfand-Graev representations of finite groups of Lie type. Let u be a unipotent element

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in such a group G^F and let Γ_u be the associated generalised Gelfand-Graev representation of G^F. Under the assumption that G has a connected centre, we show that the dimension of the endomorphism algebra of Γ_u is a polynomial in q (the order of the associated finite field), with degree given by $\dim C_G(u)$. When the centre of G is disconnected, it is impossible, in general, to parametrise the (isomorphism classes of) generalised Gelfand-Graev representations independently of q, unless one adopts a convention of considering separately various congruence classes of q. Subject to such a convention, we extend our result.

We also present computational data related to the main theoretical results. In particular, tables of our canonical forms are given in the appendices, as well as tables of dimension polynomials for endomorphism algebras of generalised Gelfand-Graev representations, together with the relevant GAP source code.

Contents

Chapter 1

Introduction

We assume that the reader is familiar with the general theory of algebraic groups and the core theory of unipotent elements. Nevertheless, in this introduction we shall review the latter, before outlining the content of the thesis. We shall also review more specialist background material as required.

1.1 Basic facts about unipotent, nilpotent and semisimple elements

1.1.1 Throughout this thesis, G will denote a connected reductive algebraic group over an algebraically closed field $\Bbbk$. We begin by recalling a result of fundamental importance, namely the Jordan-Chevalley decomposition. This says that for all $g \in G$, there exist unique elements g_u, $g_s \in G$ such that $g = g_s g_u = g_u g_s$, and that for any embedding of G as a closed subgroup of some general linear group $\mathrm{GL}_n(\Bbbk)$ (recall that such an embedding is always possible), g_u and g_s correspond to unipotent and semisimple linear transformations respectively. We thus define g to be *unipotent* if $g_s = 1$ and *semisimple* if $g_u = 1$. One could, at this point, justify the study of unipotent elements by the fact that one can answer many questions about general elements of G by answering questions about unipotent and semisimple elements respectively. Whilst this is indeed a powerful and oft-used technique, it is not the full story, since the theory of unipotent elements is a vast and beautiful subject in its own right and has many profound applications

in representation theory.

Also playing a central role in this thesis is the related theory of nilpotent elements in the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$. Identifying G with a closed subgroup of some $\operatorname{GL}_n(\Bbbk)$ again, we define $x \in \mathfrak{g}$ to be *nilpotent* if it is a nilpotent matrix and *semisimple* if it is diagonalisable. Note that nilpotent elements of $\mathfrak{g}$ are also ad-nilpotent elements of $\mathfrak{g}$, but there may be ad-nilpotent elements of $\mathfrak{g}$ which are not nilpotent. This prompts us to bear in mind that our Lie algebras will always be the Lie algebras of a connected reductive algebraic group G and so the notion of nilpotency in intrinsically linked to G. To illustrate, if $\mathfrak{g}$ is the one-dimensional commutative Lie algebra then we may take G to be either the additive group $\Bbbk^+$ *or* the multiplicative group $\Bbbk^\times$. In the former, all elements are nilpotent, while in the latter only 0 is. There is also a version of the Jordan-Chevalley decomposition for $\mathfrak{g}$, which states that for any $x \in \mathfrak{g}$ there exists a unique semisimple element $x_s \in \mathfrak{g}$, and nilpotent element $x_n \in \mathfrak{g}$, such that $x = x_s + n_n$ and $[x_s, x_n] = 0$.

1.1.2 One salient feature of the research described herein is that we have tried to focus on results that are applicable in positive characteristic, often using the (easier) characteristic zero setting as a starting point. One reason for this is to obtain results which may be regarded as 'characteristic-free' in some sense, but also it has allowed us to obtain many results about finite groups of Lie type and their (ordinary) representation theory. We will encounter a number of situations where there is a divergence in behaviour attributable to changing from characteristic zero to positive characteristic during the course of this thesis. This phenomenon is to be expected, of course, as the following basic examples illustrate. Firstly, when $\operatorname{char} \Bbbk = p > 0$ one may characterise unipotent elements as those $g \in G$ such that $g^{p^k} = 1$ for some $k \geq 0$, whereas non-identity unipotent elements have infinite order when $\operatorname{char} \Bbbk = 0$. Also, a simplifying feature of the characteristic zero situation is that an element of a semisimple Lie algebra is, in fact, nilpotent if, and only if, it is ad-nilpotent.

The last remark may lead one to wonder what role G plays in the nilpotent theory. The reason, of course, is that we are interested not only in unipotent and nilpotent elements *per se* but their *orbits* under the $\operatorname{Ad} G$-action. These turn out to have very interesting algebro-geometric properties, and a huge quan-

tity of insightful research has been produced over the past 50 years, which has painted a good picture for us. One of the most important results is the finiteness of unipotent orbits in G, proved in positive characteristic in [Lusztig, 1976]. (The characteristic zero case follows from [Dynkin, 1955] and [Kostant, 1959].) Lusztig's proof is interesting in that it is short and uses the (ordinary) representation theory of finite groups of Lie type in a non-trivial way. The corresponding result for nilpotent orbits was proved in [Holt and Spaltenstein, 1985]. This relies on a computer, however, and no non-computational proof is known to exist at present.

1.1.3 The finite set of unipotent classes can be endowed with a poset structure as follows: For two unipotent classes C_1, C_2 of G, we set

$$C_1 \trianglelefteq C_2 \iff C_1 \subset \overline{C_2},$$

where the bar denotes Zariski closure. For example, it is well-known that the unipotent classes in $\mathrm{GL}_n(\Bbbk)$ are parametrised by the partitions of n via the Jordan canonical form; the poset on unipotent classes described above then agrees with the poset induced by the dominance ordering on the partitions of n under this parametrisation. The set of nilpotent orbits can be endowed with a poset structure in exactly the same manner. These posets have been investigated in detail in [Spaltenstein, 1982] and provide a useful way of visualising the underlying geometry.

1.2 Frobenius endomorphisms

1.2.1 Approximately half of the results in this thesis concern finite groups of Lie type, and so we had better recall a few facts about these. Assume that $\operatorname{char} \Bbbk = p > 0$ and let q be a power of p. Let $n \geq 1$ and consider the map $F_q : \mathrm{GL}_n(\Bbbk) \to \mathrm{GL}_n(\Bbbk)$, defined by

$$F_q : (g_{i,j}) \mapsto (g_{i,j}^q).$$

$$
\begin{array}{c}
\mathrm{F}_4 \\
| \\
\mathrm{F}_4(a_1) \\
| \\
\mathrm{F}_4(a_2) \\
\end{array}
$$

$\mathrm{B}_3 \qquad\qquad \mathrm{C}_3$

$$
\begin{array}{c}
\mathrm{F}_4(a_3) \\
| \\
\mathrm{C}_3(a_1)
\end{array}
$$

$\tilde{\mathrm{A}}_2 + \mathrm{A}_1 \qquad\qquad \mathrm{B}_2$

$\mathrm{A}_2 + \tilde{\mathrm{A}}_1$

$\tilde{\mathrm{A}}_2 \qquad\qquad \mathrm{A}_2$

$$
\begin{array}{c}
\mathrm{A}_1 + \tilde{\mathrm{A}}_1 \\
| \\
\tilde{\mathrm{A}}_1 \\
| \\
\mathrm{A}_1 \\
| \\
1
\end{array}
$$
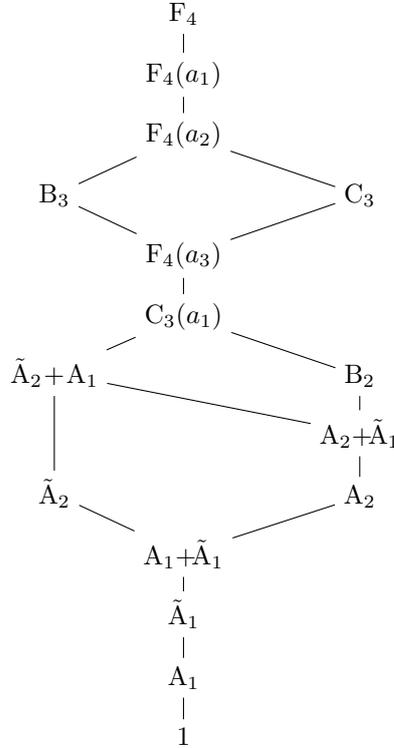
Figure 1.1: The poset structure for F_4 in good characteristic

This is called the *standard Frobenius endomorphism.* For an arbitrary connected reductive group G, we will call a map $F : G \to G$ a *Frobenius endomorphism* if there is an embedding of G as a closed subgroup of some $\mathrm{GL}_n(\Bbbk)$, such that some power of F is a standard Frobenius endomorphism. Whilst succinct, a more useful and general approach is to first define an $\mathbb{F}_q$-*rational structure* on a $\Bbbk$-variety V to be an $\mathbb{F}_q$-variety V_0 such that $V = V_0 \otimes_{\mathbb{F}_q} \Bbbk$. Then the map $F : V \mapsto V$ given by $F_0 \otimes \mathrm{id}$, where F_0 raises the functions on V_0 to the qth power, is called the *Frobenius endomorphism of V*, corresponding to this $\mathbb{F}_q$-rational structure. If such a map is defined on the underlying variety of G and is also a group morphism, then it is a Frobenius endomorphism of G, as defined above. We will often use the language of $\mathbb{F}_q$-rational structures on G in this thesis, in which case we will implicitly have a fixed Frobenius endomorphism in mind. For an accessible account, refer to [Digne and Michel, 1991, Chapter 3].

A Frobenius endomorphism turns out to be a bijective homomorphism of

algebraic groups, but not an isomorphism of algebraic groups. However, it is an isomorphism of abstract groups. The fixed point set

$$G^F = \{g \in G \mid F(g) = g\}$$

is a finite group called a *finite group of Lie type.*

1.2.2 Finite groups of Lie type inherit many nice group-theoretic properties from their ambient connected reductive groups; we again refer the reader to [Digne and Michel, 1991] for details of these, but we recall here several particularly useful and important facts.

The first is the Lang-Steinberg theorem, which states that the self-map on G defined by $g \mapsto g^{-1}F(g)$ is surjective. We will use it several times in our results.

Recall that a connected reductive group G is characterised, up to isomorphism, by its root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$, where $X(T)$ (resp. $Y(T)$) is the character group (resp. cocharacter group) of some maximal torus $T \leq G$, and Σ and $\Sigma^\vee$ are the corresponding root system and coroot system of G respectively, together with a perfect pairing $X(T) \times Y(T) \to \mathbb{Z}$, and bijection $\Sigma \leftrightarrow \Sigma^\vee$. As an extension of this, a finite group of Lie type may be characterised, up to isomorphism, by its extended root datum $(X(T), \Sigma, Y(T), \Sigma^\vee, \tau, q)$, where τ is a permutation of $X(T)$ which fixes Σ and q is a prime power.

We will call a closed subgroup of G *rational* if it is fixed by F. It is well-understood that the structure of G is governed by several important families of subgroups, namely the Borel subgroups, parabolic subgroups, unipotent subgroups, tori and Levi subgroups. By considering rational such subgroups, which are known to exist and behave reasonably well, we may obtain counterparts in the finite groups G^F, which also behave well. One particularly nice consequence of this is the following elegant formula for the order of G^F:

$$\left|G^F\right| = q^{|\Sigma^+|} \left|T^F\right| \sum_{w \in W} q^{\ell(w)}, \tag{1.1}$$

where T is a rational maximal torus of G; W is the corresponding Weyl group with length function ℓ; Σ is the root system with respect to T; and Σ^+ is a set

of positive roots. This follows from a version of the Bruhat decomposition for G^F, using the fact that the order of the unipotent subgroups of G^F are powers of q and that the maximal unipotent subgroups are Sylow p-subgroups and have order $q^{|\Sigma^+|}$. The formula (1.1) is a basic example of a 'polynomial in q'. This is a deceptively subtle concept which, roughly speaking, means a quantity which can be written as a polynomial in the prime power q, such that this polynomial is still valid when we vary q whilst keeping the root system and τ fixed. We will be very interested in polynomials in q in this thesis, especially in Chapters 3 and 4.

1.3 Springer morphisms

1.3.1 Let G_{uni} denote the set of unipotent elements of G and $\mathfrak{g}_{\mathrm{nil}}$ the set of nilpotent elements of $\mathfrak{g}$. It is well known that G_{uni} and $\mathfrak{g}_{\mathrm{nil}}$ are closed irreducible subvarieties of G and $\mathfrak{g}$ respectively, both of dimension equal to the number of roots.

We say that p is *good* for G if p is greater than the coefficient of the highest root in each component of the root system of G, expressed as an integer combination of simple roots. If p is not good then it is said to be *bad*. Since the notion of good and bad primes will be important later, we remind the reader that

$p = 2$ is bad if, and only if, G has a component not of Type A,

$p = 3$ is bad if, and only if, G has a component of exceptional type, and

$p = 5$ is bad if, and only if, G has a component of Type E_8.

For convenience, we will also often consider zero to be a good prime. (Here 'component' refers to an irreducible component of the root system of G.)

Springer has shown (cf. [Springer and Steinberg, 1970, Theorem III.3.12]) that if G is a simple, simply-connected group and char $\Bbbk$ is either zero or a good prime for G then there is a bijective, G-equivariant morphism of varieties

$$\sigma : G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}},$$

which we call a *Springer morphism*.

1.3.2 Despite the restrictiveness of the above hypothesis, this result has useful implications for an arbitrary connected reductive group G, which we now explain. The natural homomorphism $G \to G/Z(G)$ induces a bijection between G_{uni} and $(G/Z(G))_{\mathrm{uni}}$. The latter is semisimple with trivial centre and so there is a bijective homomorphism onto a semisimple group of adjoint type which preserves the unipotent classes. But groups of adjoint type are direct products of simple groups of adjoint type and therefore we may reduce to considering these simple groups of adjoint type separately. Finally, we may move back to the corresponding simple simply-connected group using the restriction of the map we have just mentioned. It follows that we may use Springer morphisms to build a bridge between G_{uni} and $\mathfrak{g}_{\mathrm{nil}}$ for any connected reductive group G, provided, of course, that char $\Bbbk$ is not bad. This bridge is a dimension-preserving bijection from unipotent classes to nilpotent orbits, which respects their geometric structure, and allows one to transfer theorems from one domain to the other. This is especially useful when one considers that $\mathfrak{g}$ is a vector space, thus allowing the full force of linear algebra to be invoked. When p is bad, Springer morphisms do not exist, and in this case even the *number* of unipotent classes and nilpotent orbits generally do not agree.

1.3.3 Assume that char $\Bbbk = p > 0$ is a good prime for G. Given a Frobenius endomorphism $F : G \to G$ and a Springer morphism $\sigma : G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}$, there exists a (possibly non-unique) Frobenius endomorphism on $\mathfrak{g}$ (i.e. a Frobenius endomorphism on $\mathfrak{g}$ as a variety, which is also a morphism of Lie algebras), which we will also denote by F, and which is compatible with the Frobenius endomorphism on G. By compatible, we mean that the following diagram commutes:

$$
\begin{array}{ccc}
G_{\mathrm{uni}} & \xrightarrow{\ F\ } & G_{\mathrm{uni}} \\
{\scriptstyle \sigma}\big\downarrow & & \big\downarrow{\scriptstyle \sigma} \\
\mathfrak{g}_{\mathrm{nil}} & \xrightarrow{\ F\ } & \mathfrak{g}_{\mathrm{nil}}
\end{array}
$$

See [Springer and Steinberg, 1970, Theorem III.3.12] for more details. E.g., if G is a classical matrix group and F is the Frobenius endomorphism which acts by raising matrix entries to the qth-power, then there exists a Springer morphism such that the Frobenius endomorphism defined in the same way on $\mathfrak{g}$ is compatible

with F; see Section 2.5 for explicit examples of such Springer morphisms.

1.4 Classification results

1.4.1 Let G be as before and let G' denote a group with the same root datum as G over the complex numbers and $\mathfrak{g}'$ its Lie algebra. Further assume that G' is a simple adjoint group. Then we have a Springer morphism $\sigma : G'_{\mathrm{uni}} \to \mathfrak{g}'_{\mathrm{nil}}$. (In characteristic zero the unipotent and nilpotent varieties of isogenous simple groups are naturally isomorphic, and so we may drop the requirement that G be simply-connected in this case.) Let $e \in \mathfrak{g}'$ be a nilpotent element. Then the Jacobson-Morozov theorem says that e lies in a subalgebra of $\mathfrak{g}'$ isomorphic to $\mathfrak{sl}_2(\Bbbk)$. (From now on we will refer to a set of standard generators of such a subalgebra as an $\mathfrak{sl}_2$-*triple*.) It was shown in [Kostant, 1959] that this induces a bijection between G'-orbits of nilpotent elements and G'-orbits of subalgebras of $\mathfrak{g}'$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. The latter were determined in [Dynkin, 1955] in terms of weighted Dynkin-diagrams (cf. Theorem 2.2.1), which are Dynkin diagrams with a number from $\{0, 1, 2\}$ attached to each node, although only certain combinations are allowed. Dynkin showed that by considering $\mathfrak{g}'$ as an $\mathfrak{sl}_2(\mathbb{C})$-module, one can naturally define an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathfrak{g}'$. Thus, one obtains a homomorphism of algebraic groups $\mathrm{SL}_2(\mathbb{C}) \to (\mathrm{Aut}\,\mathfrak{g}')^\circ = G'$. (The latter equality holds since G' is adjoint.) Let

$$\tilde{D}_{G'} = \left\{ \omega \in Y(G') \;\middle|\; \begin{array}{c} \exists\, \tilde{\omega} \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G') \\ \text{with } \omega(\xi) = \tilde{\omega} \left[\begin{smallmatrix} \xi & \\ & \xi^{-1} \end{smallmatrix} \right] \end{array} \right\}. \tag{1.2}$$

When G acts on a set X, let X/G denote the set of G-orbits in X. Then we have the following bijection of finite sets:

$$\{\text{unipotent classes of } G'\} \overset{1-1}{\longleftrightarrow} \tilde{D}_{G'} \big/ G'. \tag{1.3}$$

In fact, (1.3) holds even when we relax the assumption that G' is simple and adjoint, by well-known reduction arguments (see, e.g., [Carter, 1993, Chapter 5]).

1.4.2 Whilst (1.3) is a classification of unipotent classes and nilpotent orbits, it is not a very useful one since the right-hand side is not particularly tractable. One can, however, use this to obtain a useful classification using weighted Dynkin diagrams, since to each unipotent class one may attach a unique weighted Dynkin diagram. This process is sometimes referred to as the Dynkin-Kostant classification — we will say more about this in later chapters, where we will use it extensively. One drawback of this as a classification, however, is that there is not a uniform way of describing all possible admissible weighted Dynkin diagrams.

It is worth noting that a more natural approach is given by the Bala-Carter theorem ([Bala and Carter, 1974][1]) which offers a recursive classification of nilpotent orbits. One begins with the notion of a distinguished nilpotent element: a nilpotent element $x \in \mathfrak{g}$ is called *distinguished* if each torus contained in $C_G(x)$ is contained in the centre of G. An orbit itself is called distinguished if all of its elements are distinguished. If an orbit is not distinguished then there exists a Levi subgroup $L \leq G$, with $\dim L < \dim G$, such that its intersection with $\operatorname{Lie} L$ is distinguished for L. By induction one thus reduces the classification of nilpotent orbits to a classification of distinguished nilpotent orbits, which turns out to be a tractable problem. Since we will not use the Bala-Carter theory in any explicit way in this thesis, we will say no more about it, except to recommend the accessible account in [Carter, 1993, Chapter 5].

1.4.3 Now assume that $p = \operatorname{char} \Bbbk \geq 0$. It was shown in [Springer and Steinberg, 1970] that if $p > 3(h-1)$, where h is the Coxeter number of G, then everything described in Subsection 1.4.1 remains true, by essentially the same proofs, since the Jacobson-Morozov theorem is available for such fields. When $p \leq 3(h-1)$ the $\mathfrak{sl}_2$-theory may no longer be available and so an entirely different approach is necessary. However, Pommerening's extension of the Bala-Carter theorem implies that, in fact, this parametrisation extends to any good p. I.e., the set $\tilde{D}_{G'}/G'$ naturally and simultaneously parametrises the unipotent classes of G for all G with

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[1]This was proved under the assumption that $\operatorname{char} \Bbbk > 3(h-1)$. It was subsequently extended to good characteristic in [Pommerening, 1977] and [Pommerening, 1980], although the proof was extremely long and relied on case-by-case analyses. [Jantzen, 2004] was the first to give a uniform proof, which was subsequently significantly shortened first in [Premet, 2003], and then in [Tsujii, 2008].

the same root datum as G', regardless of good characteristic. Since a Springer morphism also exists in good characteristic, we see that $\tilde{D}_{G'}/G'$ also parametrises the nilpotent orbits. In summary, we have the bijections

$$\{\text{nilpotent orbits of } \mathfrak{g}\} \xleftrightarrow{1-1} \{\text{unipotent classes of } G\} \xleftrightarrow{1-1} \tilde{D}_{G'}\big/G', \qquad (1.4)$$

valid whenever char $\Bbbk$ is either zero or a good prime for G. One may therefore parametrise unipotent classes and nilpotent orbits in a characteristic-free manner using $\tilde{D}_{G'}/G'$. Spaltenstein has shown further that this parametrisation preserves the poset structure and dimensions of classes, as well as certain compatibility relations between parabolic subgroups, across all ground fields of good characteristic ([Spaltenstein, 1982, Théorème III.5.2]). We will state this precisely and use it explicitly in Chapter 4.

When p is a bad prime for G, the number of unipotent classes is often greater than $|\tilde{D}_{G'}/G'|$, and, since Springer morphisms do not exist when p is bad, these need not be in bijection with the nilpotent orbits. Both have been determined in all cases, however. (See [Carter, 1993, pp. 180–183] for a bibliographic account.) It turns out that, in all cases, the cardinality of the set G_{uni}/G is less than or equal to that of $\mathfrak{g}_{\mathrm{nil}}/G$.

1.5   Structure of the thesis

Chapter 2 is concerned with obtaining canonical forms for elements of unipotent classes and nilpotent orbits for the classical algebraic groups, as well as $\mathrm{GU}_n(\mathbb{F}_q)$. We begin with some relevant background results in Section 2.1 before obtaining some auxiliary combinatorial results about the Dynkin-Kostant theory for $G = \mathrm{GL}_n(\Bbbk)$ in Section 2.2. This allows us to prove the existence of a non-canonical nilpotent representative with certain special properties. In Section 2.3 we present a combinatorial algorithm which transforms the non-canonical representative into a canonical form for nilpotent $\mathrm{Ad}\,G$-orbits in $\mathfrak{gl}_n(\Bbbk)$, and which is symmetric with respect to the non-main diagonal (i.e. it is fixed by the map $f : (x_{i,j}) \mapsto (x_{n+1-j,n+1-i})$), and has entries in $\{0,1\}$. In Section 2.4 we show how to modify this form slightly in order to satisfy a non-degenerate symmetric

or skew-symmetric bilinear form, assuming that the orbit does not vanish in the presence of such a form. Replacing G by any simple classical algebraic group, we thus obtain a unified approach to computing representatives for nilpotent orbits of all classical Lie algebras. In Section 2.5 we apply Springer morphisms, thus yielding representatives for the corresponding unipotent classes in G. As a corollary, we obtain a generic canonical form for the unipotent classes in the finite general unitary groups $\mathrm{GU}_n(\mathbb{F}_q)$ for all prime powers q. No such form was known until now. Tables of these forms for $2 \leq n \leq 5$ can be found in Appendix A. This chapter is based on [Clarke, 2011b], which is to appear in Mathematical Proceedings of the Cambridge Philosophical Society.

Our goal in Chapter 3 is to prove the conjectures of G. Lusztig about so-called unipotent pieces, which can be viewed as an extension of Dynkin-Kostant theory to bad primes. In Section 3.1 we define unipotent pieces and state the conjectured properties, following [Lusztig, 2005], before outlining various additional related results that we have obtained. We will need an array of tools from geometric invariant theory at our disposal in order to prove Lusztig's conjectures and we will develop these in the subsequent three sections. In Section 3.2 we review the Kempf-Rousseau theory of optimal one parameter subgroups of G, which regards unipotent and nilpotent elements as G-unstable elements in the sense of geometric invariant theory. In Section 3.3 we prove a modified version of the Kirwan-Ness theorem. The original result offers a criterion for optimality of a G-unstable element of a G-module: our modification allows us to formulate this for the action of a parabolic subgroup of G on its unipotent radical, which is not a module in general. Our other main tool for proving Lusztig's conjectures is a famous result of Seshadri concerning G-instability in the group scheme-theoretic setting. We introduce the required machinery in Section 3.4 before stating Seshadri's theorem and then proving a related auxiliary result. In Section 3.5 we prove Lusztig's conjectures for unipotent pieces in a completely case-free manner. In Section 3.6 we obtain two very general results about the Hesselink stratification of G-modules, which are a by-product of the earlier results. We show that, in a precise sense, the stratification of G-modules is independent of base field. We also show that, in the presence of an $\mathbb{F}_q$-rational structure, the cardinality of the fixed-point set of a G-module arising by reduction mod-p from a G'-module may

be regarded as a polynomial in q. Finally, in Section 3.7 we consider the natural analogues of unipotent pieces in $\mathfrak{g}$ and its dual $\mathfrak{g}^*$, which are called nilpotent pieces, before formulating and proving Lusztig's conjectures for these situations too. This chapter is based on the preprint [Clarke and Premet, 2011].

In Chapter 4 we assume that G is a connected reductive algebraic group defined over the finite field $\mathbb{F}_q$, and let F denote the corresponding Frobenius endomorphism, so that G^F is a finite group of Lie type. In Section 4.1 we review key standard concepts such as regular unipotent elements and (ordinary) Gelfand-Graev representations, before defining generalised Gelfand-Graev representations (hereafter GGGRs) and giving an overview of relevant background results. In Section 4.3 we lay down a rigorous framework for the notion of "polynomials in q", which will allow us to formulate our results precisely. In Section 4.4 we use a character formula for GGGRs from [Kawanaka, 1985] for groups of Type A to prove our main result, that the dimension of a generalised Gelfand-Graev module, associated to a unipotent element $u \in G^F$, is a polynomial in q, with degree given by $\dim C_G(u)$ in the case where $G = \mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{GU}_n(\mathbb{F}_q)$. In Section 4.5 we use a character formula from [Lusztig, 1992] to extend this to an arbitrary connected reductive algebraic group G, defined over $\mathbb{F}_q$, provided that q is not too small and the centre of G is connected. When the centre of G is disconnected, it is impossible, in general, to parametrise the (isomorphism classes of) generalised Gelfand-Graev representations independently of q, unless one adopts a convention of considering various congruence classes of q separately. Subject to such a convention we extend our result. This chapter is based on [Clarke, 2011a], which is to appear in Transactions of the American Mathematical Society.

We also include two appendices of related computational content. Appendix A contains explicit tables of the canonical forms which we derive in Chapter 2, followed by source code for a GAP implementation of the algorithm for computing the symmetric canonical form. Appendix B contains tables of the dimensions of endomorphism algebras of GGGRs for $G = \mathrm{GL}_n(\mathbb{F}_q)$ and $\mathrm{GU}_n(\mathbb{F}_q)$. These were also computed in GAP, using an implementation of Kawanaka's character formula, and so we include the relevant source code for this as well.

Chapter 2

Computing unipotent and nilpotent canonical forms: a symmetric approach

2.1 Introduction

The Jordan canonical form for square matrices over an algebraically closed field $\Bbbk$ can be thought of as a canonical form for conjugacy classes of the general linear group $G = \mathrm{GL}_n(\Bbbk)$, or $\mathrm{Ad}\,G$-orbits of the general linear Lie algebra $\mathfrak{gl}_n(\Bbbk)$. More generally, for an element of an algebraic group G, we have the Jordan-Chevalley decomposition; cf. Subsection 1.1.1. This existence result does not, however, yield a method for finding a representative for these unique elements, up to the G-action, in contrast to the Jordan canonical form. A number of algorithms for obtaining explicit representatives have been obtained, though, for unipotent and nilpotent elements, and the purpose of this Chapter is to add a new approach to this list, such that the representatives obtained have useful and interesting properties. Note that if we know an explicit Springer morphism then the problem of finding unipotent class representatives is equivalent to that of finding nilpotent orbit representatives. For all the groups that we consider an explicit Springer morphism is indeed known; cf. Section 2.5.

Let G be of classical type. Gerstenhaber has given a method for computing

representatives of nilpotent orbits of G when the characteristic of the field is not 2; cf. [Gerstenhaber, 1961]. In characteristic zero, Popov has described another one, which is a by-product of his determination of the strata of the nullcone of a linear representation of an algebraic group; cf. [Popov, 2003]. The flavour of these approaches is quite different from the present one though and each relies on an analysis of the intrinsic classical root system, whereas in ours one only ever needs to consider the root system of Type A. In this sense our approach stresses the relationship between nilpotent orbits in $\mathfrak{g}$ and those in the ambient $\mathfrak{gl}_n(\Bbbk)$. We also note that [De Graaf, 2008] contains a probabilistic 'trial and error' algorithm for computing representatives of nilpotent orbits over any algebraically closed field. Whilst the parameters may be set so as to deliver arbitrarily large probability of success, the representatives obtained are not canonical. Using a computer implementation De Graaf has computed representatives of all nilpotent orbits in exceptional Lie algebras.

Our main result (from which the other results are derived using comparatively less effort) is Theorem 2.3.3, which describes a canonical matrix form, which we call the *symmetric form*, for nilpotent $\operatorname{Ad}\operatorname{GL}_n(\Bbbk)$-orbits in $\mathfrak{gl}_n(\Bbbk)$, provided char $\Bbbk \neq 2$. The key features of this form are that it is upper triangular, symmetric with respect to the non-main diagonal (i.e. it is fixed by the map $f : (x_{i,j}) \mapsto (x_{n+1-j,n+1-i})$), and its entries lie in $\{0, 1\}$. The proof of this can be divided into roughly two phases. First we prove the existence of a non-canonical representative which enjoys certain nice properties. For this we use the Dynkin-Kostant classification of nilpotent orbits combined with an algebro-geometric argument. This will allow us to describe a calculus of 'elementary operations' that may be performed on the entries of this representative whilst remaining in the original orbit. These are then applied in the second phase of the proof to give an algorithm with the flavour of Gaussian elimination which puts the representative into the prescribed canonical form. The corresponding canonical forms for the other classical Lie algebras are then derived using short additional phases to this algorithm. We remark that the algorithms used to prove that these canonical forms are indeed representatives of the prescribed orbits are not needed when it comes to the task of merely *writing down* the representative, in the same way that one is able to write down a matrix in Jordan canonical form without having

to know the proof of the corresponding theorem. The algorithm used to write down a matrix in symmetric form is very simple and we include GAP source code for this in Appendix A. Using the explicit Springer morphisms from [Kawanaka, 1985] one is then able to compute corresponding unipotent normal forms in G. Moreover, since these Springer morphisms are compatible with standard Frobenius endomorphisms F_q our unipotent normal forms of G also lie in G^{F_q}.

As we shall see, our symmetric form sometimes does not correspond to the 'correct' orbit in characteristic 2; cf. Remark 2.2.3. However, when this is the case we are still able to find a canonical form fixed by the composite of f and a standard Frobenius endomorphism, F_q. We then show how to use these modified forms to obtain generic canonical forms for the unipotent classes in the finite groups $\mathrm{GU}_n(\mathbb{F}_q)$, for *all* prime powers q. No canonical form for the unipotent classes of $\mathrm{GU}_n(\mathbb{F}_q)$ was known until now.

2.2 The general linear Lie algebra

2.2.1 We shall now show how to obtain a non-canonical orbit representative which enjoys certain nice properties; we will exploit these properties later to obtain a canonical representative. We set $G = \mathrm{GL}_n(k)$ and $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$, but note that there is essentially no difference between this and the special linear case when one is concerned with nilpotent orbits or unipotent classes. (The situation is less straightforward in the finite setting.) We fix, once and for all, a nilpotent element $e \in \mathfrak{g}_{\mathrm{nil}}$ corresponding, by the Jordan canonical form, to some partition $\mu \vdash n$. We denote its $\mathrm{Ad}\, G$-orbit by $\mathcal{O}_e$. Let T and B be the diagonal maximal torus and upper-triangular Borel subgroup of G respectively, and denote the corresponding root system by Σ. Let $\mathfrak{t}$ be the standard diagonal maximal torus of $\mathfrak{g}$, and, for $1 \le i \le n$, let ε_i be the linear map $\mathfrak{t} \to \Bbbk$ which picks out the ith diagonal entry. We may then denote a set of positive roots by $\Sigma^+ = \{\varepsilon_i - \varepsilon_j \mid 1 \leqslant i < j \le n\}$, and let $X_{\varepsilon_i - \varepsilon_j}$ denote the elementary matrix with a 1 in the $(i,j)^{th}$ position and 0 elsewhere. Then the root spaces are of the form $\Bbbk X_{\varepsilon_i - \varepsilon_j}$. We denote the set of simple roots corresponding to Σ^+ by $\Pi = \{\alpha_1, \ldots, \alpha_{n-1}\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

We present the main results from the Dynkin-Kostant-Springer-Steinberg theory as follows (see, e.g., [Kawanaka, 1985, pp. 177–178] and the references there).

Note that, since we will only apply this in the case where Σ is of Type A, an elementary proof is possible, by following, e.g., [Lusztig, 2005, §2].

Theorem. *With the above set-up, there exists a $\mathbb{Z}$-grading*

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

of $\mathfrak{g}$, depending only on $\mathcal{O}_e$, with the following properties.

(i) *Each $\mathfrak{g}_i$ is a sum of root spaces.*

(ii) *We may assume (by replacing by a conjugate if necessary) that $e \in \mathfrak{g}_2$.*

(iii) *$\mathfrak{p}_e = \bigoplus_{i \geq 0} \mathfrak{g}_i$ is the Lie algebra of a standard (block upper-triangular) parabolic subgroup P_e of G.*

(iv) *$\mathfrak{l}_e = \mathfrak{g}_0$ is the Lie algebra of the block-diagonal Levi subgroup L_e of P_e.*

(v) *For $i \geq 1, \mathfrak{u}_{e,i} = \bigoplus_{j \geq i} \mathfrak{g}_i$ is the Lie algebra of a connected normal unipotent subgroup $U_{e,i}$ of P_e. In particular, $U_{e,1}$ is the unipotent radical of P_e.*

(vi) *Each $\mathfrak{g}_i$ is $\operatorname{Ad} L_e$-stable.*

(vii) *$\mathcal{O}_e \cap \mathfrak{g}_2$ is dense in $\mathfrak{g}_2$.*

(viii) *There exists a unique additive function $h_e : \Sigma \to \mathbb{Z}$, fixed by the non-trivial graph automorphism of the Dynkin diagram of G, such that*

    (a) *$h_e(\alpha) \in \{0, 1, 2\}$ for each $\alpha \in \Pi$;*

    (b) *$\mathfrak{g}_i = \bigoplus_{h_e(\alpha)=i} X_\alpha$.*

The Dynkin diagram with nodes labelled by the numbers $h(\alpha_i)$, corresponding to the simple roots, is called the *weighted Dynkin diagram associated to $\mathcal{O}_e$*. We may partition Σ into the following subsets. For $i \in \mathbb{Z}$ set

$$\Sigma_i = \{\alpha \in \Sigma \mid h(\alpha) = i\} = \{\alpha \in \Sigma \mid X_\alpha \subseteq \mathfrak{g}_i\}.$$

2.2.2 We explicitly construct the function h for groups of Type A as follows. Let $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r)$. Then for each μ_i, consider

$$Y_i = \{\mu_i - 1, \mu_i - 3, \ldots, 3 - \mu_i, 1 - \mu_i\}.$$

Viewing $Y = \coprod_i Y_i$ as a multiset of n integers, arrange in decreasing order:

$$Y = \{\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n\}.$$

Then we define h on Π by putting $h(\alpha_i) = \nu_i - \nu_{i+1}$. This uniquely determines h by the additivity property.

In [Shoji, 1998, §2], it is shown how to construct Σ_1. Generalising this we construct Σ_2. Let Π_ε be the set of simple roots with h-weight ε for $\varepsilon = 1, 2$. For $\alpha_i \in \Pi_1$ or Π_2 let a_i be the smallest integer such that $a_i > i$ and $h(\alpha_{a_i}) \neq 0$, and let b_i be the largest integer such that $b_i < i$ and $h(\alpha_{b_i}) \neq 0$. Then we obtain rectangular subsets

$$\Psi_i = \{\varepsilon_s - \varepsilon_t \mid b_i + 1 \leq s \leq i, i + 1 \leq t \leq a_i\}.$$

(If a_i (resp. b_i) does not exist, then set $a_i = n$ (resp. $b_i = 0$).) Then, as observed in [Shoji, 1998, §2], we have a disjoint union

$$\Sigma_1 = \coprod_{\alpha_i \in \Pi_1} \Psi_i.$$

Now we construct Σ_2. A pair $\Psi_i, \Psi_j \subseteq \Sigma_1$ are said to be *adjacent* if $h(\alpha_k) = 0$ whenever $i < k < j$. We will also say that α_i and α_j are adjacent when this is the case. For each adjacent pair Ψ_i, Ψ_j we define another subset $\Psi_{i,j}$ of Σ^+ by

$$\Psi_{i,j} = \{\varepsilon_s - \varepsilon_t \mid b_i + 1 \leq s \leq i, j + 1 \leq t \leq a_j\}.$$

Then we have another disjoint union

$$\Sigma_2 = \left(\coprod_{\alpha_i, \alpha_j \in \Pi_1} \Psi_{i,j} \right) \coprod \left(\coprod_{\alpha_k \in \Pi_2} \Psi_k \right),$$

where the first union is taken over adjacent pairs. We shall call the $\Psi_{i,j}$ and Ψ_k appearing in the above decomposition the *blocks* of Σ_2, and the subspaces $\oplus_{\alpha \in \Psi_{i,j}} \Bbbk X_\alpha$ and $\oplus_{\alpha \in \Psi_k} \Bbbk X_\alpha$ the *blocks* of $\mathfrak{g}_2$.

2.2.3 Clearly $\mathfrak{g}_2 = \oplus_{\alpha \in \Sigma_2} \Bbbk X_\alpha$, and we now have a description of Σ_2. This is a good start, but we will need to collect some more information about $\mathfrak{g}_2$ before proceeding. Define sequences $l_1 \geq l_2 \geq l_3 \geq \ldots$ and $k_1 \geq k_2 \geq k_3 \geq \ldots$ (related to the dual partitions of the purely odd and purely even parts of μ) as follows. For $i \geq 1$ set

$$l_i = \#\{\mu_j \text{ odd} \mid \mu_j \geq 2i - 1\},$$

and

$$k_i = \#\{\mu_j \text{ even} \mid \mu_j \geq 2i\}.$$

Lemma. $\mathfrak{g}_2$ *consists of matrices of the form*

$$x = \begin{pmatrix} \ddots & & & & & & & \\ & A_3 & & & & & & \\ & & A_2 & & & & & \\ & & & A_1 & & & & \\ & & & & C & & & \\ & & & & & B_1 & & \\ & & & & & & B_2 & \\ & & & & & & & B_3 & \\ & & & & & & & & \ddots \end{pmatrix},$$

satisfying the following properties.

(i)   *All entries are zero outside the rectangular blocks A_i, B_i, C, and these blocks correspond to the blocks of $\mathfrak{g}_2$.*

(ii) *The entries $x_{i,j}$ inside the blocks may be arbitrary, and for all such entries $i < j$.*

(iii)   *The block structure is symmetric with respect to the non-main diagonal, i.e. it is fixed by the map $f : (x_{i,j}) \mapsto (x_{n+1-j,n+1-i})$.*

(iv)   *Each row (resp. column) intersects at most one block.*

(v)   *The middle block C, which is necessarily square by (iii), exists if, and only if, some μ_i is even.*

(vi)   *The blocks may be partitioned into subsets $I = \{\ldots, A_{i_2}, A_{i_1}, B_{i_1}, B_{i_2}, \ldots\}$ and $J = \{\ldots, A_{j_2}, A_{j_1}, C, B_{j_1}, B_{j_2}, \ldots\}$ with the following properties:*

   (a)   *$J = \emptyset$ if C does not exist.*

   (b)   *Each of I and J is a symmetric block structure with respect to the non-main diagonal.*

   (c)   *A_{i_r} is an $l_{r+1} \times l_r$ matrix (and hence B_{i_r} is an $l_r \times l_{r+1}$ matrix).*

   (d)   *A_{j_r} is a $k_{r+1} \times k_r$ matrix (and hence B_{j_r} is a $k_r \times k_{r+1}$ matrix).*

   (e)   *Set $C = A_{j_0} = B_{j_0}$, $A_{i_0} = B_{i_1}$ and $B_{i_0} = A_{i_1}$. Then row k of x intersects A_{i_r} (resp. $A_{j_r}, B_{i_r}, B_{j_r}$) if, and only if, column k intersects $A_{i_{r+1}}$ (resp. $A_{j_{r+1}}, B_{i_{r-1}}, B_{j_{r-1}}$).*

Proof. The first four parts are clear from the construction. Observe that (v) is equivalent, by symmetry, to there existing a root α of weight 2 which is fixed by the non-trivial graph automorphism $\Pi \to \Pi$. If all μ_i are even then it is easy to see that such a root exists in Π, i.e. the central node of the Dynkin diagram. If at least one, but not all, μ_i are even then the middle part of the sequence of weights of Π is of the form $1, 0, \ldots, 0, 1$, with the two 1s equidistant from the centre. Then α may be taken to be the sum of the roots corresponding to these 1s and the intervening 0s. If all the μ_i are odd then the middle part of the sequence of weights of Π is of the form $2, 0, \ldots, 0, 2, \ldots$, with the two 2s equidistant from the centre. Thus, a root fixed by the graph automorphism cannot have weight 2.

We will now construct the sets I and J. If μ has both odd and even parts choose i to be minimal such that $\mu_i - \mu_{i+1}$ is odd. Then

$$Y_i \cup Y_{i+1} = \{\mu_i - 1, \mu_i - 3, \ldots, \mu_i - (\mu_i - \mu_{i+1}), \mu_{i+1} - 1, \mu_{i+1} - 2, \mu_{i+1} - 3, \ldots,$$

$$3 - \mu_{i+1}, 2 - \mu_{i+1}, 1 - \mu_{i+1}, (\mu_i - \mu_{i+1}) - \mu_i, \dots, 3 - \mu_i, 1 - \mu_i\}.$$

It follows that each integer k, such that $\mu_{i+1} \geq k \geq -\mu_{i+1}$, appears with non-zero multiplicity in Y, and that an integer k such that $\mu_1 - 1 \geq k \geq \mu_{i+1}$, appears with non-zero multiplicity in Y if, and only if, it has the same parity as $\mu_1 - 1$. We thus obtain a symmetric partition of Π into three parts as follows.

$$\Pi = \{\alpha_1, \dots, \alpha_{\hat{\imath}}\} \cup \{\alpha_{\hat{\imath}+1}, \dots, \alpha_{n-\hat{\imath}}\} \cup \{\alpha_{n+1-\hat{\imath}}, \dots, \alpha_{n-1}\},$$

where $\hat{\imath}$ is the largest number such that $\hat{\imath} < (n-1)/2$ and $h(\alpha_i) = 2$. Assume for now that μ does not consist only of even parts. Then the blocks of $\mathfrak{g}_2$ of the form Ψ_i, which are in bijection with the simple roots of weight 2, may be split into two sets of adjacent blocks corresponding to the end parts of the above partition. The blocks of the form $\Psi_{i,j}$ are in bijection with sets of adjacent roots of weight 1 from the middle part. If μ does consist only of even parts then only blocks of the form Ψ_i occur in $\mathfrak{g}_2$.

Let $\alpha_{m_1}, \alpha_{m_2}, \alpha_{m_3}, \dots$ denote the elements of Π_1 with $m_1 \leq m_2 \leq m_3 \leq \cdots$, and set

$$A = \{\Psi_i \mid \alpha_k \in \Pi_2\} \cup \Psi_{m_1,m_2} \cup \Psi_{m_3,m_4} \cup \Psi_{m_5,m_6} \cup \cdots$$

and

$$B = \Psi_{m_2,m_3} \cup \Psi_{m_4,m_5} \cup \Psi_{m_6,m_7} \cup \cdots.$$

It is clear that A and B partition the roots which determine $\mathfrak{g}_2$ and that each is a union of blocks. Set $\{A, B\} = \{I, J\}$ so that J contains the central block. Then (vi) follows from this construction. (See also the example below.) $\qquad \square$

Example. If μ has only odd or only even parts then the elements of Y, disregarding multiplicities, are $\mu_1 - 1, \mu_1 - 3, \dots, 3 - \mu_1, 1 - \mu_1$ and therefore the only weights that can occur are $\{0, 2\}$. In this case all blocks are of the form Ψ_i and we should set $I = \Sigma_2$ and $J = \emptyset$ if all parts are odd and vice versa if all parts are even.

2.2.4 Given an $n \times n$ matrix x and a union of blocks X, we will denote by x_X the matrix obtained by replacing all entries of x which do not correspond to X by zero. If X is a block, we will also refer to x_X as a *block of x*. Our eventual

canonical form will be an element $x \in \mathcal{O}_e \cap \mathfrak{g}_2$, with entries in $\{0, 1\}$, such that $f(x_{A_i}) = x_{B_i}$ for $i \geq 1$, and $f(x_C) = x_C$ if C exists. By the density of $\mathcal{O}_e \cap \mathfrak{g}_2$ in $\mathfrak{g}_2$, we may obtain a representative in any non-empty open set. In what follows, we view $\mathfrak{g}_2$ as an affine space in its own right, and therefore ignore coordinates of $\mathfrak{g}$ outside of $\mathfrak{g}_2$. We will use the following open set.

Let $\mathfrak{M} = 2^{\dim \mathfrak{g}_2} \dim \mathfrak{g}_2^{1/2}$ and then consider all polynomials f_i ($i \in I$, some indexing set) in $\dim \mathfrak{g}_2$ indeterminates of degree at most $\mathfrak{M}$, with coefficients in $\{0, \pm 1\}$. Define $V = V(f_i)$ to be the zero locus of the f_i and let $S = \mathfrak{g}_2 \setminus V$ be the corresponding open set in $\mathfrak{g}_2$. Hence, by density, we may choose a representative $x \in \mathcal{O}_e \cap S$. (The number $\mathfrak{M}$ might seem rather arbitrary here but its nature will become clear later when we consider an algorithm for putting x into symmetric form.)

2.3   An algorithm to obtain canonical forms

2.3.1 In what follows we assume that $x \in \mathcal{O}_e \cap S$ is chosen in the manner explained above, and fixed. We now move on to the second phase of the proof of Theorem 2.3.3. This is based on the fact that the L_e-orbit of x is contained in $\mathcal{O}_e \cap \mathfrak{g}_2$. For this we will need some new terminology. We shall refer to the *rows* and *columns* of the blocks A_i and C as those inherited from the ambient matrix, but it will be convenient for us to invert this definition for the blocks B_i (for $i \geq 1$).

Lemma. *For any block X and any elementary row or column operation on x_X, there exists $l \in L_e$ such that conjugation by l on x agrees with this operation. If $X = A_{i_r}$ (resp. $A_{j_r}, B_{i_r}, B_{j_r}$), then, for row operations, l can be chosen so that it acts trivially on all other blocks except $x_{A_{i_{r+1}}}$ (resp. $x_{A_{j_{r+1}}}, x_{B_{i_{r+1}}}, x_{B_{j_{r+1}}}$), and, for column operations, so that it acts trivially on all other blocks except $x_{A_{i_{r-1}}}$ (resp. $x_{A_{j_{r-1}}}, x_{B_{i_{r-1}}}, x_{B_{j_{r-1}}}$). Furthermore, these pairs of actions are described in Tables 2.1 and 2.2.*

Proof. It is clear from Lemma 2.2.1 that there exists such an elementary matrix l in G. The proof follows from the observation that the non-diagonal, non-zero entries of l correspond to roots of weight zero; thus $l \in L_e$ by Theorem 2.2.1. $\square$

Table 2.1: Duality of operations for x_I (general case)

row operation on $x_{A_{i_r}}$ (resp. $x_{A_{j_r}}, x_{B_{i_r}}, x_{B_{j_r}}$)	column operation on $x_{A_{i_{r+1}}}$ (resp. $x_{A_{j_{r+1}}}, x_{B_{i_{r+1}}}, x_{B_{j_{r+1}}}$)
swap rows a and b	swap columns a and b
multiply row a by λ	multiply column a by λ^{-1}
add λ times row a to row b	add $-\lambda$ times column b to column a

Table 2.2: Duality of operations for x_I (special case)

column operation on $x_{A_{i_1}}$	column operation on $x_{B_{i_1}}$
swap columns a and b	swap columns a and b
multiply column a by λ	multiply column a by λ^{-1}
add λ times column a to column b	add $-\lambda$ times column b to column a

This allows us to consider x_I and x_J separately.

2.3.2 We will now show how to symmetrise x_I. First consider the central pair of blocks of x_I. Rather than using the column numbering from the ambient matrix, we shall translate this for ease of notation. Our set-up is as in Figure 2.1.
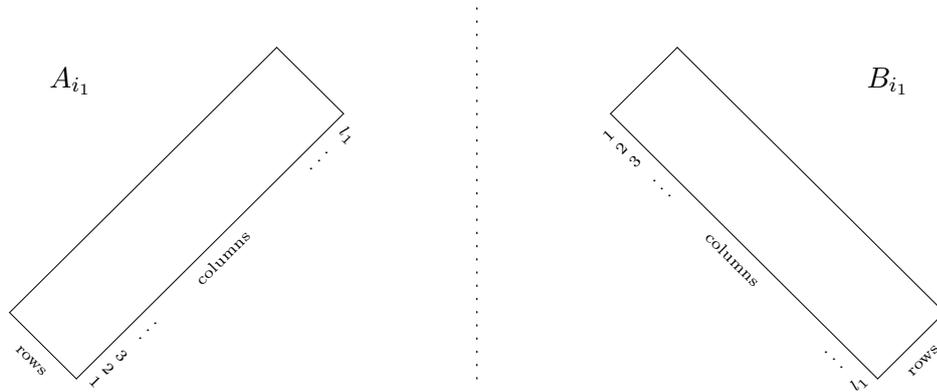
Figure 2.1: Central pair of blocks in x_I

For $m \geq 1$, we define an $m \times m$ matrix J_m as follows: $(J_m)_{i,j} = 1$ if $i+j = m+1$, and 0 otherwise.

Now the column operations are in duality as in Table 2.2. Using these dual operations, together with arbitrary row operations, we may obtain Figure 2.2,

where the dotted lines in the blocks denote diagonal arrays of l_2 ones and the blank space zeros. This is achieved as follows.

1. Perform Gauss-Jordan elimination to put A_{i_1} in the desired form.

2. Perform row operations until the rightmost square of B_{i_1} is J_{l_2}.

3. Using column operations in B_{i_1}, delete all entries not in this rightmost square.
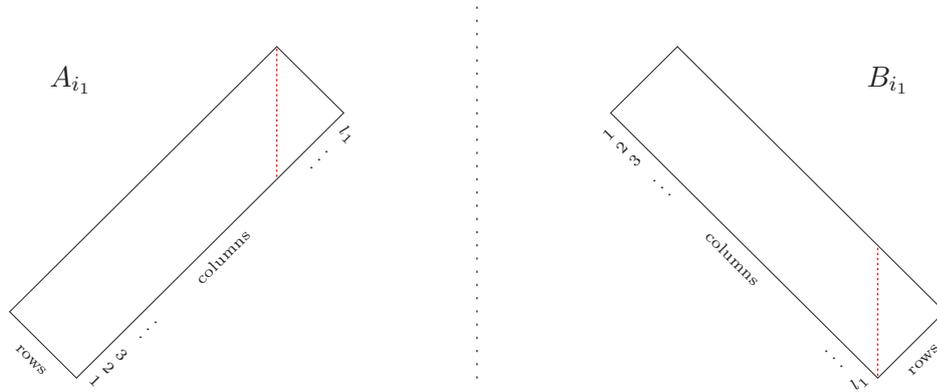
Figure 2.2: Central pair of blocks in x_I

Remark. Implicit in 2 is that the rank of the rightmost $l_2 \times l_2$ sub-matrix of B_{i_1} has remained maximal (i.e. equal to l_2) throughout 1. This is valid because of the way S was constructed. Indeed, first note that the initial representative x has this property, or else a determinant polynomial will be satisfied. Proceeding by induction, assume that we have completed a certain number of steps of the Gauss-Jordan algorithm, and denote the resulting B_{i_1}-component by $x_{B_{i_1}}$ and the resulting rightmost $l_2 \times l_2$ sub-matrix of B_{i_1} by x_Z. Let the inductive hypothesis be that the determinant of every square sub-matrix of $x_{B_{i_1}}$ may be written as a Laurent polynomial in the entries of the initial representative x, with coefficients in $\{0, \pm 1\}$. Letting x_Z' denote our sub-matrix after one more elementary column operation on A_{i_1}, we may write

$$\det x_Z' = \det x_Z + \lambda \det x_Y, \tag{2.1}$$

if the operation results in a column from outside x_Z being added into x_Z, where x_Y is another sub-matrix of $x_{B_{i_1}}$ and λ is as in Table 2.2. Or,

$$\det x'_Z = \lambda \det x_Z, \tag{2.2}$$

if we encounter an internal column operation on x_Z. Then one checks that λ is either ± 1 or a product of entries from A_{i_1} and their inverses, up to sign. It follows that $\det x'_Z$ is a Laurent polynomial in the entries of the initial representative x, by the inductive hypothesis. If this vanishes then we may construct a polynomial in the entries of the initial representative x, with coefficients in $\{0, \pm 1\}$ which also vanishes. This contradicts our choice of set S, since the degree of the polynomial will be lower than the bound $\mathfrak{M}$.

Next, if l_2 is even, then move the columns corresponding to the left half of the copy of J_{l_2} in A_{i_1} to the far left of A_{i_1} using swapping operations. Together with the dual actions on B_{i_1}, we obtain the symmetric Figure 2.3, where the dots denote a diagonal array of $l_2/2$ 1s.
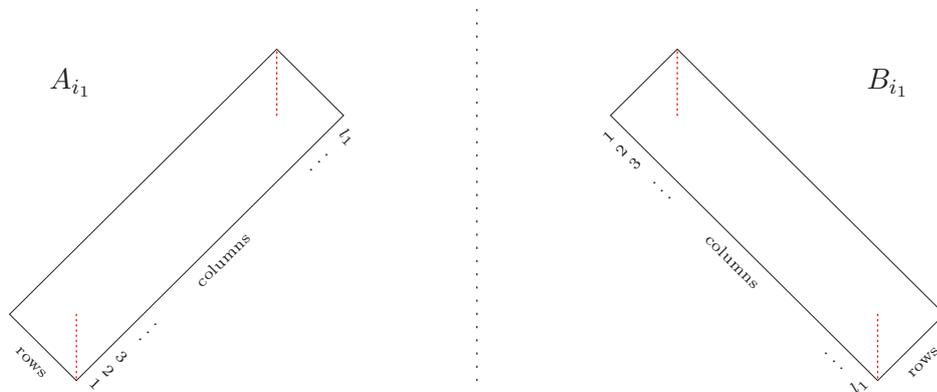
Figure 2.3: Symmetrised central pair of blocks in x_I (l_2 even)

If l_2 is odd, the above step will clearly not work. In this case, we do not seek symmetry immediately. Rather, perform the column swaps using the leftmost $(l_2 - 1)/2$ columns of I in A_{i_1}. The result will be asymmetric, but the only asymmetries will be that column $l_1 + 1 - (l_2 + 1)/2$ of A_{i_1} has a 1 in the middle while column $(l_2 + 1)/2$ of B_{i_1} consists of 0s, and vice versa, as in Figure 2.4, where the filled circle denotes a 1 and the blank circle a 0.
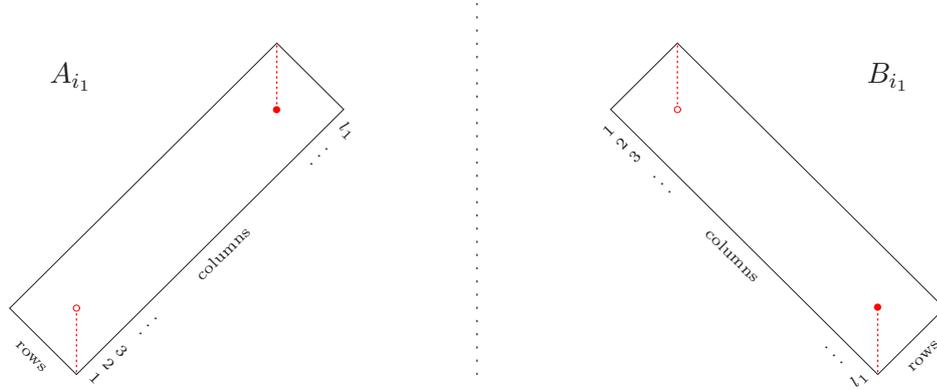
Figure 2.4: Central pair of blocks in x_I (l_2 odd)

For x_I in general we consider the sequence of pairs

$$(A_{i_1}, B_{i_1}), (A_{i_2}, B_{i_2}), (A_{i_3}, B_{i_3}), \ldots \tag{2.3}$$

in order. I.e. we iteratively move out from the centre. We now explain how each of these can be reduced to the case already dealt with. Any elementary column operation on a block in a new pair (A_{i_k}, B_{i_k}) will inevitably induce a row operation on its neighbour (with subscript i_{k-1}, currently described by Figure 2.3 or 2.4), thus knocking it out of canonical form. However, there exists a single elementary column operation on the latter which rectifies this. Then we apply the same process to its neighbour and so on until we reach the other member of (A_{i_k}, B_{i_k}). This creates a duality of operations on (A_{i_k}, B_{i_k}) which agrees with Table 2.2, and so we reduce to the central pair case without damaging the pairs of blocks in between. It might be helpful to view the intervening blocks as a mirror along which one reflects. Eventually, all pairs of blocks will be as in Figure 2.3 or 2.4. To achieve overall symmetry, we must now address those of the form Figure 2.4.

For simplicity of notation consider the central pair case first. We obtain the symmetrised form, Figure 2.5, by the following sequence of operations (together with their duals).

1. Add column $l_1 + 1 - (l_2 + 1)/2$ to column $(l_2 + 1)/2$ in A_{i_1}.

2. Add $1/2$ times column $l_1 + 1 - (l_2 + 1)/2$ to column $(l_2 + 1)/2$ in B_{i_1}.

3. Multiply row $(l_2 + 1)/2$ of B_{i_1} by 2.

4. Multiply column $l_1 + 1 - (l_2 + 1)/2$ of A_{i_1} by 2.

5. One may also need to multiply the central rows of $A_{i_2}, A_{i_3}, A_{i_4}, \ldots$ by 2 depending on the parity of the $l_1, l_2, l_3, \ldots$.

The same method works for pairs as in Figure 2.4 in general, however, in order to exploit the mirror property at each stage we must take the sequence (2.3) in the opposite order. This results in the desired canonical form for x_I.
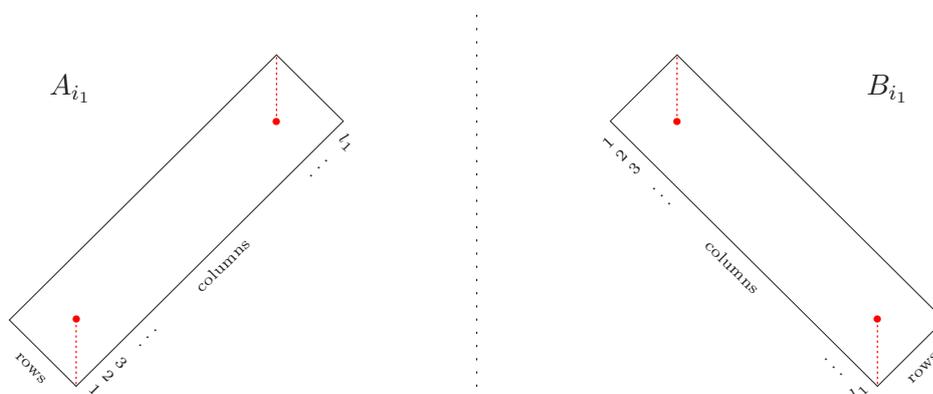
Figure 2.5: Symmetrised central pair of blocks in x_I (l_2 odd)

2.3.3 We will now show how to symmetrise x_J. A slightly ugly part of the symmetrising algorithm in the last section was the fact that asymmetric pairs, as in Figure 2.4, may be part of the central mirror arrangement that is built up, necessitating a second phase to the algorithm. The algorithm for symmetrising x_J also uses a mirror arrangement of intervening blocks, but symmetrising x_J is more straightforward as only one phase is needed. However, the presence of a central square block requires a slightly different calculus of dual operations.

First use arbitrary operations to transform C into J_{k_1}, as in Figure 2.6. Because of this, for any elementary operation on the left of the central block, there exists an elementary operation on the right which cancels it out, and vice versa. Combining this duality with the usual duality on adjacent blocks described in Table 2.1, Table 2.3 gives a duality of column operations on the pair $(x_{A_{j_1}}, x_{A_{j_1}})$

in x_J. Notice that, because of the way we have labelled the columns in Figure 2.6, Tables 2.2 and 2.3 are identical. Now we will show how to obtain Figure 2.6, which acts as a mirror for operations on $(x_{A_{j_2}}, x_{A_{j_2}})$.

Table 2.3: Duality of operations for x_J

column operation on $x_{A_{j_1}}$	**column operation on** $x_{B_{j_1}}$
swap columns a and b	swap columns a and b
multiply column a by λ	multiply column a by λ^{-1}
add λ times column a to column b	add $-\lambda$ times column b to column a

Using Table 2.3, we may symmetrise the pair $(x_{A_{j_1}}, x_{B_{j_1}})$ using the following operations (and their duals). One may check that the result is described by Figure 2.6.

1. Put $x_{A_{j_1}}$ in the desired form using column operations.

2. Obtain the identity matrix on the leftmost part of $x_{B_{j_1}}$ using row operations.

3. Add suitable scalar multiples of the columns of this identity matrix to eliminate the rest of $x_{B_{j_1}}$.
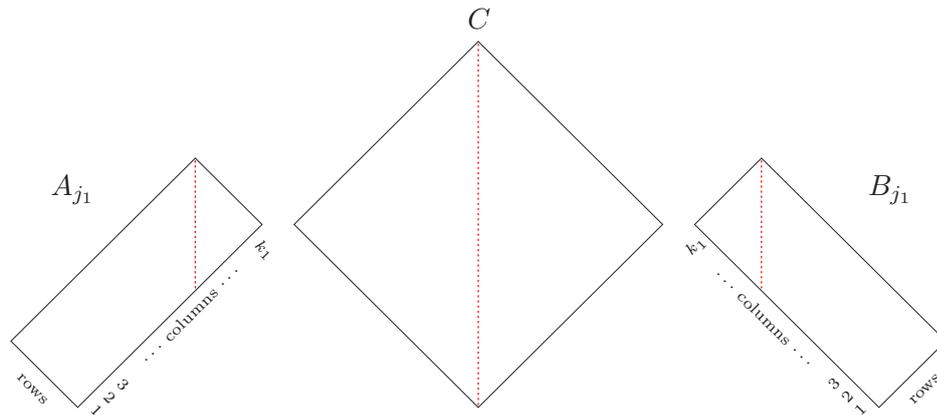
Figure 2.6: Symmetrised central arrangement in x_J

It is clear that this configuration allows Table 2.3 and the above algorithm to be extended to each of the pairs in the sequence

$$(A_{j_1}, B_{j_1}), (A_{j_2}, B_{j_2}), (A_{j_3}, B_{j_3}), \ldots \tag{2.4}$$

in order. This completes the symmetrisation of x_J.

2.3.4 We have now proved our main result, which we state precisely as follows.

Theorem. *Let $\mu \vdash n$, and let $\mathcal{O}_\mu \subset \mathfrak{gl}_n(\Bbbk)$ denote the nilpotent corresponding, via the Jordan canonical form, to μ. Considering the blocks described by Lemma 2.2.1 let x denote the following matrix:*

(i) *The entries of x agree with Figure 2.3 if l_2 is even, and Figure 2.5 if l_2 is odd, on blocks A_{i_1}, B_{i_1} and Figure 2.6 on blocks A_{j_1}, B_{j_1} and C.*

(ii) *For $k \geq 2$, the entries of x corresponding to $A_{i_k}, A_{j_k}, B_{i_k}$ and B_{j_k} are chosen in the same manner as $A_{i_1}, A_{j_1}, B_{i_1}$ and B_{j_1} respectively.*

(iii) *All other entries are zero.*

Then $x \in \mathcal{O}_\mu$.

Example. We illustrate the symmetrising algorithm via the orbit corresponding to the partition $\lambda = (4, 4, 2)$ in $\mathfrak{gl}_{10}(\Bbbk)$. Since all parts of λ are even, $I = \emptyset$, therefore we may skip straight to Subsection 2.3.3. We have three blocks as in the following illustration, $A_{j_1}, C = A_{j_0} = B_{j_0}$, and B_{j_1}. (So $a_{1,1}$ is in the $(1, 3)$-position of x and $b_{3,2}$ is in the $(8, 10)$-position.) Recall that the initial element x was chosen to be in $\mathcal{O}_\lambda$ and the specially constructed open set S. In step (A) we have performed Gauss-Jordan elimination to put C into the desired form. Note that A_{j_1} and B_{j_1} will still have maximal rank because of the way we chose S. In step (B) we have performed column operations to obtain the desired form on A_{j_1}. Recall that every time we perform a column operation on A_{j_1} this induces a row operation on C, which is then put back into the desired from by a suitable column operation on C. B_{j_1} may change in the process, but its rank will remain maximal. Thus, in step (C) we may use row operations on B_{j_1} (remember that the rows of B_{j_1} correspond to columns of the ambient matrix) to obtain the penultimate array shown. Finally, in step (D) we have subtracted $b''_{3,2} \times$ column 3 from column 1 and we have subtracted $b''_{3,1} \times$ column 2 from column 1 in B_{j_1} to obtain the desired form. Of course these will filter through C again, but the effect

on A_{j_1} will be benign — we will have added multiples of column 1 to columns 2 and 3, but column 1 consists of zeros.

$$x = \begin{array}{ccccccccc} a_{1,1} & a_{1,2} & a_{1,3} & & & & & & \\ a_{2,1} & a_{2,2} & a_{2,3} & & & & & & \\ & & & c_{1,1} & c_{1,2} & c_{1,3} & & & \\ & & & c_{2,1} & c_{2,2} & c_{2,3} & & & \\ & & & c_{3,1} & c_{3,2} & c_{3,3} & & & \\ & & & & & & b_{1,1} & b_{1,2} \\ & & & & & & b_{2,1} & b_{2,2} \\ & & & & & & b_{3,1} & b_{3,2} \end{array}$$

$$\xrightarrow{(A)} \begin{array}{ccccccccc} a'_{1,1} & a'_{1,2} & a'_{1,3} & & & & & & \\ a'_{2,1} & a'_{2,2} & a'_{2,3} & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & b'_{1,1} & b'_{1,2} \\ & & & & & & b'_{2,1} & b'_{2,2} \\ & & & & & & b'_{3,1} & b'_{3,2} \end{array}$$

$$\xrightarrow{(B)} \begin{array}{ccccccccc} 0 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & b''_{1,1} & b''_{1,2} \\ & & & & & & b''_{2,1} & b''_{2,2} \\ & & & & & & b''_{3,1} & b''_{3,2} \end{array}$$

$$\xrightarrow{(C)} \begin{array}{ccccccccc} 0 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & 0 & 1 \\ & & & & & & 1 & 0 \\ & & & & & & b'''_{3,1} & b'''_{3,2} \end{array}$$

$$\xrightarrow{(D)} \quad \begin{matrix} 0 & 0 & 1 & & & & & & \\ 0 & 1 & 0 & & & & & & \\ & & & 0 & 0 & 1 & & & \\ & & & 0 & 1 & 0 & & & \\ & & & 1 & 0 & 0 & & & \\ & & & & & & 0 & 1 & \\ & & & & & & 1 & 0 & \\ & & & & & & 0 & 0 & \end{matrix}$$

Remark. In order to symmetrise pairs of blocks described by Figure 2.4, we implicitly assumed that $2 \neq 0$. In fact, this is crucial since, when the characteristic is not 2, the matrix

$$\begin{pmatrix} 1 & 1 & & \\ & & 1 & \\ & & 1 & \end{pmatrix}$$

is the symmetric canonical form corresponding to $(3,1) \vdash 4$. However, in characteristic 2, it has Jordan form $(2,2)$. Using a computer we have also found similar examples for $n = 5, 6$ and 7 in characteristic 2.

2.4 The symplectic and orthogonal Lie algebras

2.4.1 We now consider the other classical algebras. More precisely, let G be a simple classical algebraic group of Type B_l, C_l or D_l, together with the adjoint action on the nilpotent variety $\mathfrak{g}_{\mathrm{nil}}$ of its Lie algebra $\mathfrak{g}$. By considering the natural matrix representation of $\mathfrak{g}$ we may compute the set of elementary divisors of an element of each orbit, thus defining a partition of $2l$ for groups of Type C_l and D_l and of $2l + 1$ for groups of Type B_l. Letting $\mathcal{P}(n)$ denote the set of partitions of n and writing $n = 2l$ or $2l + 1$ accordingly, the corresponding map

$$\{ \text{ orbits of } \mathfrak{g} \} \longrightarrow \mathcal{P}(n)$$

is an injection if G is of Type B_l or C_l, while for a group of Type D_l very even partitions (i.e. those consisting of only even parts, each having even multiplicity)

correspond to two orbits. If G is of Type B_l or D_l then the image consists precisely of those partitions in which even parts occur with even multiplicity, while the image for Type C_l consists of those partitions in which odd parts occur with even multiplicity. We shall refer to the partitions not in the image as *bad*. Using this classification of orbits we may use the notation $\mathcal{O}_\mu$ to denote an orbit corresponding to a partition μ. (It is customary, when μ is very even, to denote the two orbits by $\mathcal{O}'_\mu, \mathcal{O}''_\mu$.)

Let M be an $n \times n$ matrix over $\Bbbk$. Then the set of $n \times n$ matrices X over $\Bbbk$ satisfying the condition

$$X^T M + M X = 0, \tag{2.5}$$

is a Lie algebra under the commutator operation. We construct the classical algebras by selecting a suitable M, following the standard text [Carter, 2005]. (But note that our choices of M differ slightly from those in [Carter, 2005].)

By considering the restriction of $\mathrm{GL}_n(\Bbbk)$-orbits on $\mathfrak{gl}_n(\Bbbk)$ we will show how to obtain a canonical representative for each of these orbits by modifying slightly the symmetric canonical form from Theorem 2.3.3 so that it satisfies (2.5). See the tables in Appendix A for examples.

Clearly it is impossible to modify a canonical element x (as in Theorem 2.3.3) corresponding to a bad partition so that it satisfies (2.5). In view of this we start by observing a characterisation of bad partitions in terms of a feature of the block structure of $\mathfrak{g}_2$. Then, assuming the absence of this feature we present a sequence of elementary operations on x so that x satisfies (2.5). By Lemma 2.2.1 this characterisation is as follows. For Type C_l the bad partitions correspond to the existence of a block in I with an odd number of rows. For Types B_l and D_l they correspond to the existence of a block in J with an odd number of rows.

2.4.2 We shall start by dealing with Type C_l. Here we let

$$M = \begin{pmatrix} & J_l \\ -J_l & \end{pmatrix}. \tag{2.6}$$

Then, writing

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

in terms of $l \times l$ blocks, $X \in \mathfrak{sp}_{2l}(\Bbbk)$ if, and only if, $X_{12} = f(X_{12})$, $X_{21} = f(X_{21})$ and $X_{11} = -f(X_{22})$.

The canonical form x from Theorem 2.3.3 already satisfies the conditions on X_{12} and X_{21}. It suffices, therefore, to change all non-zero entries of X_{22} from 1 to -1. Now the absence of bad partitions means that blocks of the form Figure 2.5 can not occur, and so there is at most one 1 in each row. It follows that we can rescale the non-zero entries of x independently using row operations and thus obtain the desired form.

2.4.3 Now we shall deal with Type D_l. Let $M = J_{2l}$. Then, writing

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{pmatrix},$$

in terms of $l \times l$ blocks, $X \in \mathfrak{so}_{2l}(\Bbbk)$ if, and only if, $X_{12} = -f(X_{12})$, $X_{21} = -f(X_{21})$ and $X_{11} = -f(X_{22})$.

This time more work is required since rescaling alone will not be sufficient to satisfy the condition on X_{12}, if C exists, as it will have non-zero entries fixed by f. We therefore begin by obtaining a new form for C such that $x_C = -f(x_C)$. The new x_C will still be a permutation matrix, although it will now have all entries on the diagonal fixed by f equal to 0. First observe that we may perform operations on the top $k_1 - k_2$ columns of the existing x_C without changing any other entry of x. We may therefore reverse the order of these columns. Similarly, one may perform operations on the bottom $k_2 - k_3$ columns of x_C without changing any other entry of x, via the mirror afforded by B_{j_1}. Hence, we may reverse the order of these columns too. We continue this process until we have transformed x_C into a matrix with copies of (various sized) identity matrices lined up along the diagonal fixed by f, with zeros elsewhere, and the rest of x left unchanged. We may now rescale some of the entries of x_C to -1 so that $x_C = -f(x_C)$, *provided* that each of the numbers $k_1 - k_2, k_2 - k_3, k_3 - k_4, \ldots$ is even, i.e. provided that

μ is not bad.

To finish one just rescales all non-zero entries of the B-blocks from 1 to -1. For this we simply multiply all rows by -1 on $B_{i_1}, B_{i_3}, B_{i_5}, \ldots$ and $B_{j_1}, B_{j_3}, B_{j_5}, \ldots$.

Remark. In the case that μ is very even this only corresponds to one of the two orbits associated to μ.

2.4.4 Lastly, we deal with Type B_l. Let $M = J_{2l+1}$, and write

$$
X = \begin{pmatrix} X_{11} & X_{12} & X_{13} \\ X_{21} & X_{22} & X_{23} \\ X_{31} & X_{32} & X_{33} \end{pmatrix},
$$

where X_{11}, X_{13}, X_{31} and X_{33} are $l \times l$ matrices, X_{12} and X_{32} are $l \times 1$ matrices, X_{21} and X_{23} are $1 \times l$ matrices, and X_{22} is a 1×1 matrix. Hence, $X \in \mathfrak{so}_{2l+1}(\mathbb{k})$ if, and only if, $X_{13} = -f(X_{13})$, $X_{31} = -f(X_{31})$, $X_{22} = 0$, $X_{11} = -f(X_{33})$, $X_{21} = -f(X_{32})$, and $X_{12} = -f(X_{23})$.

The canonical form is obtained in exactly the same manner as for Type D_l.

2.5 Unipotent canonical forms in G and G^F

2.5.1 In this section we explain how to compute canonical forms for unipotent elements corresponding to the nilpotent ones we obtained previously. Assume that $G = \mathrm{GL}_n(\mathbb{k})$ or one of the groups

$$
\{x \in \mathrm{GL}_n(\mathbb{k}) \mid x^T M x = M\},
$$

where M is as in the previous chapter, defined over the field with q elements, where q is a power of a good prime for G. Let F denote the corresponding Frobenius endomorphisms on G and $\mathfrak{g} = \mathrm{Lie}\, G$. (Recall that we require the property $F(g \cdot x) = F(g) \cdot F(x)$ for $g \in G$, $x \in \mathfrak{g}$, where $\cdot$ denotes the adjoint action.) In this setting there exists a Springer morphism $\sigma : G_{\mathrm{uni}} \longrightarrow \mathfrak{g}_{\mathrm{nil}}$ which commutes with the F-actions. In fact, we can write out such a map explicitly for classical groups following [Kawanaka, 1985]. For $G = \mathrm{GL}_n(\mathbb{k})$, together with

F_q, we may take Springer's morphism to be $x \mapsto x - 1$. For Types B, C and D, with untwisted Frobenius endomorphisms, it is easy to check that the Cayley map $x \mapsto (x - 1)(x + 1)^{-1}$ works.

2.5.2 Now we shall focus on the finite unitary groups. Assume, initially, that char $\Bbbk \geq 3$. When G has a disconnected centre, or is of Type B, C or D, the F-stable unipotent classes may split into several G^F-orbits and so there are more unipotent conjugacy classes in G^F than in G. Using a Springer morphism we may therefore map our nilpotent representatives from $\mathfrak{g}$ into G(noting that they are all F-stable) to obtain some unipotent representatives in G^F, but more work would be needed to obtain a *full* set of representatives. However, if $G = \mathrm{GL}_n(\Bbbk)$ then this splitting does not occur for any Frobenius endomorphism. Hence we may compute a full set of representatives for the unipotent classes of G^F in this case. In the case of finite general linear groups then we simply take the elements $x + 1 \in G^F = \mathrm{GL}_n(\mathbb{F}_q)$ where x varies over the symmetric canonical forms from Theorem 2.3.3. Alternatively, the Jordan canonical form also affords a perfectly good set of representatives in this case. The author believes, though, that in the case of the finite unitary groups $G^F = \mathrm{GU}_n(\mathbb{F}_q)$ no canonical form for unipotent elements was known until now.

We will use the following twisted Frobenius endomorphism on $G = \mathrm{GL}_n(\Bbbk)$. For $(g_{i,j}) \in G$, let

$$F((g_{i,j})) = (g^q_{n+1-j,n+1-i})^{-1}. \tag{2.7}$$

We will use the compatible Frobenius endomorphism on $\mathfrak{g} = \mathfrak{gl}_n(\Bbbk)$ given by

$$F((g_{i,j})) = (g^q_{n+1-j,n+1-i}), \tag{2.8}$$

for $(g_{i,j}) \in \mathfrak{g}$. We also note that the map given by

$$F^-((g_{i,j})) = -(g^q_{n+1-j,n+1-i}), \tag{2.9}$$

for $(g_{i,j}) \in \mathfrak{g}$ is also commonly used, and may be more convenient in certain situations. Naturally, we have chosen to use (2.8) as it fixes the representatives obtained in Theorem 2.3.3.

Proposition. *Let $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$. Then the map*

$$\sigma : g \mapsto (g-1)(\alpha - \alpha^q g)^{-1}, \tag{2.10}$$

with inverse

$$\sigma^{-1} : x \mapsto (1 + \alpha^q x)^{-1}(\alpha x + 1),$$

is a G-equivariant bijective morphism $G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}$ which commutes with (2.7) and (2.8).

Proof. The only non-trivial thing to check is that for all $g \in G_{\mathrm{uni}}$, we have $\sigma(F(g)) = F(\sigma(g))$. This can be checked explicitly be writing these Frobenius endomorphisms in terms of familiar matrix operations as follows. We have

$$F(g) = J_n F_q(g^{-T}) J_n,$$

for $g \in G_{\mathrm{uni}}$, where F_q denotes the standard q^{th}-power Frobenius endomorphism, and

$$F(x) = J_n F_q(x^T) J_n,$$

for $x \in \mathfrak{g}_{\mathrm{nil}}$. $\qquad\square$

Corollary. *If $x \in \mathfrak{g}_{\mathrm{nil}}$ is F-stable, then $\sigma^{-1}(x) \in G^F$.*

Remark. If one uses (2.9), the Cayley map is suitable if the characteristic is not 2; a map suitable for any positive characteristic appears in [Kawanaka, 1985].

2.5.3 We may now compute representatives for unipotent classes in the finite general unitary groups, by applying the map σ^{-1} from Proposition 2.5.2 to the canonical forms described by Theorem 2.3.3, provided char $\Bbbk \neq 2$. We may adapt these representatives slightly to obtain a canonical set of representatives valid for arbitrary characteristic. Recall that the algorithm used in the proof Theorem 2.3.3 fails in characteristic 2 when trying to pass from a situation described by Figure 2.4 to one described by Figure 2.5. In fact symmetry, whilst remaining in $\mathfrak{g}_2$, is impossible in some situations, as we saw in Remark 2.3.3.

However, instability under f need not obstruct stability under $F = F_q \circ f$,
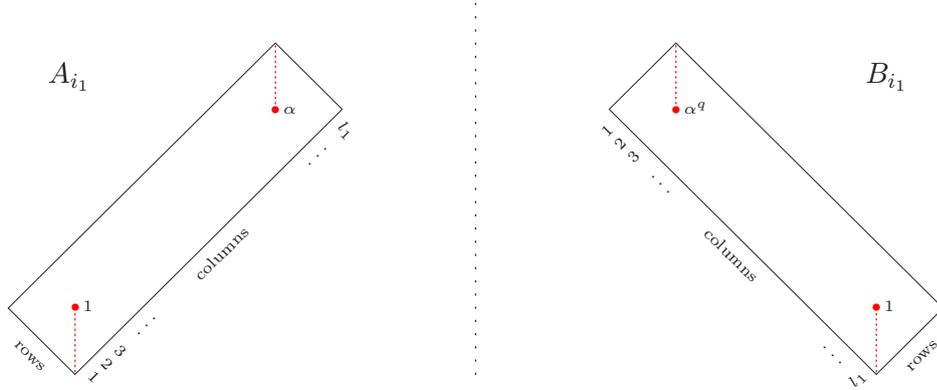
Figure 2.7: F-stable central pair of blocks in x_I (l_2 odd)

which is another way of writing (2.8). We may obtain mere F-stability in all cases as follows, transforming Figure 2.4 to Figure 2.7.

1. Choose $\alpha \in \mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

2. Multiply row $(l_2 + 1)/2$ of B_{i_1} by $1 + \alpha^{q-1}$.

3. Multiply row $(l_2 + 1)/2$ of A_{i_1} by α.

4. Add $\alpha^q(1 + \alpha^{q-1})^{-1}$ times column $l_1 + 1 - (l_2 + 1)/2$ to column $(l_2 + 1)/2$ in B_{i_1}.

5. Add α^{-1} times column $l_1 + 1 - (l_2 + 1)/2$ to column $(l_2 + 1)/2$ in A_{i_1}.

6. One may also need to rescale the central rows of $A_{i_2}, A_{i_3}, A_{i_4}, \ldots$ depending on the parity of the $l_1, l_2, l_3, \ldots$.

In Appendix A we have computed tables of all of the symmetric forms for nilpotent matrices from this chapter corresponding to partitions of n, for $n = 2, 3, 4$ and 5, together with the unipotent canonical forms for $\mathrm{GU}_n(\mathbb{F}_q)$. In the interest of a generic approach we have used the F-stable form of Figure 2.7 even when char $\Bbbk \neq 2$ in these tables.

Chapter 3

Unipotent elements in small characteristic

3.1 Introduction and statement of results

3.1.1 In what follows $\Bbbk$ will denote an algebraically closed field of arbitrary characteristic $p \geq 0$, unless stated otherwise. Recall that we denote by G' a connected reductive group over the complex numbers with the same root datum as G, with Lie algebra denoted by $\mathfrak{g}'$. As we discussed in Subsection 1.4.1 there exists a uniform classification of unipotent classes and nilpotent orbits when char $\Bbbk$ is good, and the set $\tilde{D}_{G'}/G'$ may be used as a parameter set in all cases. Moreover, the parametrisation respects many important geometric properties of the set of orbits and classes, such as their poset structures. When char $\Bbbk$ is a bad prime for G, however, this breaks down, as then even the *number* of classes and orbits do not agree with $|\tilde{D}_{G'}/G'|$ in general.

A case-by-case classification of unipotent classes and nilpotent orbits does exist in the case of bad primes, however, and the poset structures have been computed explicitly in [Spaltenstein, 1982]. These posets are not entirely dissimilar to the corresponding posets in good characteristic, and indeed one may observe that the latter may be identified with sub-posets of the former. In [Lusztig, 2005] Lusztig has devised the notion of unipotent pieces in an attempt to unify these disparate good and bad prime pictures at a geometric level. The salient features

of unipotent pieces, which we define shortly, are that they should be unions of unipotent classes, which are parametrised by $\tilde{D}_{G'}/G'$, so that in good characteristic they are precisely the unipotent classes, but in bad characteristic, although they may consist of more than one class, they should exhibit the same geometric features as the corresponding unipotent classes in good characteristic.

3.1.2 Following [Lusztig, 2005] we now define unipotent pieces. First note that $Y(G)/G$ is naturally isomorphic to $Y(G')/G'$. (Indeed, in each case we may restrict to one parameter subgroups of a fixed maximal torus, say T and T', since all maximal tori are conjugate. Then the orbits are precisely the Weyl group orbits on the $\mathbb{Z}$-modules $Y(T), Y(T')$, which can be identified unambiguously.) We let $\tilde{D}_G$ denote the unique G-stable subset of $Y(G)$ whose image in $Y(G')/G'$ corresponds to $\tilde{D}_{G'}/G'$ under this bijection; cf. (1.2). Corresponding to $\tilde{D}_G$ we define D_G to be the set of sequences $\triangle = (G_0^\triangle \supset G_1^\triangle \supset G_2^\triangle \supset \cdots)$ of closed connected subgroups of G such that for some $\omega \in \tilde{D}_G$ we have

$$\operatorname{Lie} G_i^\triangle = \left\{ x \in \mathfrak{g} \;\middle|\; \lim_{\xi \to 0} \xi^{1-i} (\operatorname{Ad} \omega(\xi))x = 0 \right\}.$$

The notion of a limit here is defined as follows. If $f : \Bbbk^\times \to V$ is a morphism of varieties and $v \in V$, then we use the notation $\lim_{\xi \to 0} f(\xi) = v$ to mean that f may be extended to a morphism $\tilde{f} : \Bbbk \to V$ such that $\tilde{f}(0) = v$.

The obvious map $\tilde{D}_G \to D_G$ induces a bijection $\tilde{D}_G/G \xrightarrow{\sim} D_G/G$ on the set of G-orbits. Assume that $\omega \in \tilde{D}_G$ corresponds to some $G_0^\triangle$, and T is a maximal torus of $G_0^\triangle$ containing $\operatorname{Im}\omega$, and let Σ denote the root system of G relative to T. Then one can show that

$$G_0^\triangle = \langle T, U_\alpha \mid \alpha \in \Sigma, \ \langle \alpha, \omega \rangle \geq 0 \rangle, \text{ and}$$

$$G_i^\triangle = \langle U_\alpha \mid \alpha \in \Sigma, \ \langle \alpha, \omega \rangle \geq i \rangle \text{ for } i \geq 1 \ ,$$

where the U_α are the root subgroups of G relative to T. From this characterisation we see that $G_0^\triangle$ is a parabolic subgroup of G, with unipotent radical $G_1^\triangle$, and that $G_i^\triangle$ is normalised by $G_0^\triangle$ for any $i \geq 0$.

For any G-orbit $\blacktriangle \in D_G/G$, let $\tilde{H}^\blacktriangle = \bigcup_{\triangle \in \blacktriangle} G_2^\triangle$. It is straightforward to

see that each set $\tilde{H}^{\blacktriangle}$ is a closed irreducible variety stable under the conjugation action of G; see Lemma 3.5.2. We now define

$$H^{\blacktriangle} = \tilde{H}^{\blacktriangle} \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'},$$

where the union is taken over all $\blacktriangle' \in D_G/G$ such that $\tilde{H}^{\blacktriangle'} \subsetneq \tilde{H}^{\blacktriangle}$. The subsets $H^{\blacktriangle}$ are called the *unipotent pieces* of G. We also define

$$X^{\triangle} = G_2^{\triangle} \bigcap H^{\blacktriangle},$$

for each $\triangle \in D_G$, where $\blacktriangle$ is the G-orbit of $\triangle$. Since $H^{\blacktriangle}$ is the complement of finitely many non-trivial closed subvarieties of $\tilde{H}^{\blacktriangle}$, it is open and dense in $\tilde{H}^{\blacktriangle}$, hence it is locally closed in G_{uni}. The subset $H^{\blacktriangle}$ is G-stable since its complement in $\tilde{H}^{\blacktriangle}$ is. Consequently, $X^{\triangle}$ is open and dense in $G_2^{\triangle}$, and stable under conjugation by $G_0^{\triangle}$.

3.1.3 Lusztig has stated the following five properties in [Lusztig, 2005] and conjectured that they should hold for all connected reductive groups G over algebraically closed fields.

$\mathfrak{P}_1$. The sets $X^{\triangle}$ ($\triangle \in D_G$) form a partition of G_{uni}, i.e. $G_{\text{uni}} = \bigsqcup_{\triangle \in D_G} X^{\triangle}$.

$\mathfrak{P}_2$. For every $\blacktriangle \in D_G/G$ the sets $X^{\triangle}$ ($\triangle \in \blacktriangle$) form a partition of $H^{\blacktriangle}$. More precisely, $H^{\blacktriangle}$ is a fibration over $\blacktriangle$ with smooth fibres isomorphic to $X^{\triangle}$ ($\triangle \in \blacktriangle$).

$\mathfrak{P}_3$. The locally closed subsets $H^{\blacktriangle}$ ($\blacktriangle \in D_G/G$) form a (finite) partition of G_{uni}, i.e. $G_{\text{uni}} = \bigsqcup_{\blacktriangle \in D_G/G} H^{\blacktriangle}$.

$\mathfrak{P}_4$. For any $\triangle \in D_G$ we have that $G_3^{\triangle} X^{\triangle} = X^{\triangle} G_3^{\triangle} = X^{\triangle}$.

$\mathfrak{P}_5$. Suppose $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$ and let $F : G \to G$ be the Frobenius endomorphism corresponding to a split $\mathbb{F}_q$-rational structure with $q - 1$ sufficiently divisible. Let $\triangle \in D_G$ be such that $F(G_i^{\triangle}) = G_i^{\triangle}$ for all $i \geq 0$ and let $\blacktriangle$ be the G-orbit of $\triangle \in D_G$. Then there exist polynomials $\varphi^{\blacktriangle}(t)$ and $\psi^{\triangle}(t)$

in $\mathbb{Z}[t]$ with coefficients independent of p such that $\varphi^{\blacktriangle}(q) = |H^{\blacktriangle}(\mathbb{F}_q)|$ and $\psi^{\vartriangle}(q) = |X^{\vartriangle}(\mathbb{F}_q)|$.

When p is good, properties $\mathfrak{P}_1$– $\mathfrak{P}_4$ follow from Pommerening's classification; see [Jantzen, 2004], [Pommerening, 1977], [Pommerening, 1980]. It is proved in [Lusztig, 2005], [Lusztig, 2008] and [Lusztig, 2011] that $\mathfrak{P}_1$– $\mathfrak{P}_5$ hold for classical groups (any p) by a case-by-case analysis. For groups of type E (any p) properties $\mathfrak{P}_1$– $\mathfrak{P}_5$ can be deduced from [Mizuno, 1980], although this is unsatisfactory since the extensive computations which the results of that paper are based on are largely omitted, and these results are known to contain many misprints. As mentioned in [Lusztig, 2005, p. 451] it is desirable to have an independent verification of properties $\mathfrak{P}_1$– $\mathfrak{P}_5$ for groups of type E.

More recently Lusztig has introduced natural analogues of the unipotent pieces $X^{\vartriangle}$ ($\vartriangle \in D_G$) and $H^{\blacktriangle}$ ($\blacktriangle \in D_G/G$) for the adjoint G-module $\mathfrak{g}$ and its dual $\mathfrak{g}^*$ and called them *nilpotent pieces* of $\mathfrak{g}$ and $\mathfrak{g}^*$. Replacing G_{uni} by the nilpotent varieties $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ (see Subsection 3.2.1) he conjectured that properties $\mathfrak{P}_1$– $\mathfrak{P}_5$ should hold for them as well. We stress that the G-modules $\mathfrak{g}$ and $\mathfrak{g}^*$ are *very* different when $p = 2$ and G is of type B, C or F_4 and when $p = 3$ and G is of type G_2. In all other cases there exists a G-equivariant bijection $\mathcal{N}_{\mathfrak{g}} \overset{\sim}{\to} \mathcal{N}_{\mathfrak{g}^*}$ which restricts to a bijection between the corresponding nilpotent pieces and induces a $1-1$ correspondence between the orbit sets $\mathcal{N}_{\mathfrak{g}}/G$ and $\mathcal{N}_{\mathfrak{g}^*}/G$; see [Premet and Skryabin, 1999, §5.6] for more details. It is worth mentioning that the coadjoint action of G on $\mathfrak{g}^*$ plays a very important role in studying irreducible representations of the Lie algebra $\mathfrak{g}$.

Remark. In $\mathfrak{P}_5$ one must implicitly identify the sets D_G whilst letting q vary, i.e. by either introducing the language of group schemes or, equivalently, fixing a maximally split maximal torus of G and choice of positive roots for each algebraically closed ground field. We will take the former approach in this Chapter. Later, in Chapter 4 we will also wish to consider quantities which are given by polynomials in q, but the situation there is more subtle and so, in Section 4.3, we set up a rigorous framework within which to formulate precise statements about polynomials in q.

3. UNIPOTENT ELEMENTS IN SMALL CHARACTERISTIC

In [Lusztig, 2008], [Lusztig, 2011] and [Lusztig, 2010], it is proved that properties $\mathfrak{P}_1$ – $\mathfrak{P}_5$ hold for $\mathcal{N}_{\mathfrak{g}}$ when G is a classical group and for $\mathcal{N}_{\mathfrak{g}^*}$ in the case where G is a group of type C. Very recently the coadjoint case for groups of type B was settled by Ting Xue, a former PhD student of Lusztig; see [Xue, 2011]. In proving $\mathfrak{P}_1$ – $\mathfrak{P}_5$ for classical groups Lusztig and Xue relied on intricate counting arguments involving linear algebra in characteristic 2 and combinatorics.

The main goal of this chapter is to give a uniform proof of the following using Hesselink's theory of the stratification of nullcones.

Theorem. *Let G be a reductive group over an algebraically closed field of characteristic $p \geq 0$ and $\mathfrak{g} = \operatorname{Lie} G$. Let $\mathcal{G}$ be one of G, $\mathfrak{g}$ or $\mathfrak{g}^*$ and write $X^{\triangle}(\mathcal{G})$ for the piece $X^{\triangle}$ of $\mathcal{G}$ labelled by $\triangle \in D_G$. Then $\mathfrak{P}_1$–$\mathfrak{P}_5$ hold for $\mathcal{G}$. In particular, the centraliser in G of any element in $X^{\triangle}(\mathcal{G})$ is contained in $G_0^{\triangle}$.*

We mention for completeness that the definition of nilpotent pieces used by Lusztig and Xue for G classical differs formally from Lusztig's original definition in [Lusztig, 2005] which we follow. However, Theorem 3.1.3 implies that both definitions give rise to the same partitions of $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$; see Remark 3.7.3 for more details. It is far from clear whether the definition of Lusztig and Xue can be used for exceptional groups in arbitrary characteristic.

Remark. Regarding $\mathfrak{P}_2$, Lusztig has also predicted that each piece $H^{\blacktriangle}(\mathcal{G})$ is a smooth variety and there exists a G-equivariant fibration $f\colon H^{\blacktriangle}(\mathcal{G}) \twoheadrightarrow \blacktriangle \cong G/G_0^{\triangle}$ such that $f^{-1}(\triangle) \cong X^{\triangle}$ for all $\triangle \in \blacktriangle$. We stress that this conjecture remains open, in general, and we know of no counterexamples.

It is well-known that the sets G_{uni}, $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$ coincide with the subvarieties of G-unstable elements of the G-varieties G, $\mathfrak{g}$ and $\mathfrak{g}^*$, respectively (we assume that G acts on itself by conjugation). Therefore each set admits a natural stratification coming from the Kempf–Rousseau theory, which we review in Section 3.2. In fact, such a stratification was defined in [Hesselink, 1979] for any affine G-variety V with a distinguished point $*$ fixed by the action of G. It is often referred to as the *Hesselink stratification* of the variety of Hilbert nullforms of V. In Section 3.5 we show that every piece $H^{\blacktriangle}(\mathcal{G})$ coincides with a Hesselink stratum of $\mathcal{G}$ and conversely every Hesselink stratum of $\mathcal{G}$ has the form $H^{\blacktriangle}(\mathcal{G})$ for a unique

▲ ∈ D_G/G. We also identify the subsets $X^\triangle(\mathcal{G})$ ($\triangle \in D_G$) with the blades of the variety of nullforms of $\mathcal{G}$. (As in the theorem we assume here that $\mathcal{G}$ is one of G, $\mathfrak{g}$ or $\mathfrak{g}^*$.)

In order to relate the pieces $H^\blacktriangle(\mathcal{G})$ (▲ ∈ D_G/G) with Hesselink strata we first upgrade certain reductive subgroups of G involved in the Kempf–Ness criterion for optimality of one parameter subgroups to reductive $\mathbb{Z}$-group schemes split over $\mathbb{Z}$, and then make use of a well-known result from [Seshadri, 1977] on invariants of reductive group schemes. This is done in Section 3.4. After relating unipotent and nilpotent pieces with Hesselink strata we deduce rather quickly that $\mathfrak{P}_1$– $\mathfrak{P}_4$ hold for G, $\mathfrak{g}$ and $\mathfrak{g}^*$.

3.1.4 Proving that $\mathfrak{P}_5$ holds for G, $\mathfrak{g}$ and $\mathfrak{g}^*$ requires more effort. Since our arguments involve induction on the rank of the group we have to look at a much larger class of finite-dimensional rational G-modules.

Let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme split over $\mathbb{Z}$ and suppose that $\Bbbk$ contains an algebraic closure of $\mathbb{F}_p$. Set $G' = \mathfrak{G}(\mathbb{C})$ and $G = \mathfrak{G}(\Bbbk)$. We say that a G-module V is *admissible* if there is a finite-dimensional G'-module V' and an admissible lattice $V'_{\mathbb{Z}}$ in V' such that $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Recall that a $\mathbb{Z}$-lattice in V' is called *admissible* if it is stable under the action of the distribution algebra $\mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G})$; see [Jantzen, 1987] for more details. For any p^{th} power q we may regard the finite vector space $V(\mathbb{F}_q) = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{F}_q$ as an $\mathbb{F}_q$-form of the $\Bbbk$-vector space V.

Since G is a reductive group, the invariant algebra $\Bbbk[V]^G$ is generated by finitely many homogeneous polynomial functions $f_1, \ldots, f_m$ on V. The G-*nullcone* of V, denoted $\mathcal{N}_{G,V}$ or simply $\mathcal{N}_V$, is defined as the zero locus of $f_1, \ldots, f_m$ in V. We set $\mathcal{N}_V(\mathbb{F}_q) = \mathcal{N}_V \cap V(\mathbb{F}_q)$.

Theorem. *For every admissible G-module V there is a polynomial $n_V(t) \in \mathbb{Z}[t]$ such that $|\mathcal{N}_V(\mathbb{F}_q)| = n_V(q)$ for all $q = p^l$. The polynomial $n_V(t)$ depends only on the G'-module V', but not on the choice of an admissible lattice $V'_{\mathbb{Z}}$, and is the same for all primes $p \in \mathbb{N}$.*

In fact, a more general version of Theorem 3.1.4 is established in Subsection 3.6.2 which takes care of non-split Frobenius actions on G. Property $\mathfrak{P}_5$ for $\mathcal{N}_{\mathfrak{g}}$ and $N_{\mathfrak{g}^*}$ now follows almost at once since both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible

G-modules; see Section 3.7. Proving $\mathfrak{P}_5$ for G_{uni} requires some extra work; see Corollary 3.7.3. Theorem 3.1.4 enables us to show that the classical results of Steinberg and Springer on the cardinality of $G_{\mathrm{uni}}(\mathbb{F}_q)$ and $\mathcal{N}_{\mathfrak{g}}(\mathbb{F}_q)$ respectively, are equivalent. It also enables us to compute the cardinality of $\mathcal{N}_{\mathfrak{g}^*}(\mathbb{F}_q)$ thereby generalising a recent result of Lusztig proved for G classical; see [Lusztig, 2010] and [Xue, 2011].

Corollary. *Let* $N = \dim G - \operatorname{rank} G$. *Then* $|\mathcal{N}_{\mathfrak{g}}(\mathbb{F}_q)| = |\mathcal{N}_{\mathfrak{g}^*}(\mathbb{F}_q)| = q^N$ *for any* p^{th} *power* q *and any prime* $p \in \mathbb{N}$.

Once we observe that both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible G-modules coming from the adjoint G'-module $\mathfrak{g}'$, Corollary 3.1.4 becomes a consequence of Steinberg's formula $|G_{\mathrm{uni}}(\mathbb{F}_q)| = q^N$ and the existence for $p \gg 0$ of a G-equivariant isomorphism between $\mathcal{N}_{\mathfrak{g}}$ and G_{uni}; cf. Subsection 4.4.4. Indeed, Theorem 3.1.4 then ensures that the polynomial $n_{\mathfrak{g}}(t) = n_{\mathfrak{g}^*}(t)$ has coefficients independent of p.

3.2 The Kempf–Rousseau theory

Although much of this theory goes back to [Mumford, 1965], [Kempf, 1978] and [Rousseau, 1978], our set-up here is inspired by [Hesselink, 1979], [Slodowy, 1989] and [Tsujii, 2008].

3.2.1 Let V be a pointed G-variety, i.e. a G-variety with a distinguished point $* \in V$ fixed by the action of G. We will assume further that V is non-singular at $*$, although many results still hold even when $*$ is singular. Let H be a closed reductive subgroup of G. Then a point $v \in V$ is called *H-unstable* if there exists some $\lambda \in Y(H)$ such that $\lim_{\xi \to 0} \lambda(\xi) \cdot v = *$. Otherwise we say that v is *H-semistable*.

Theorem. (The Hilbert-Mumford criterion; see [Mumford et al., 1994]) *The following are equivalent.*

(i) *v is H-unstable.*

(ii) *$f(v) = 0$ for each regular function $f \in \Bbbk[V]^H$ which vanishes at $*$.*

(iii) $* \in \overline{H \cdot v}$.

The set of all G-unstable elements is called the *nullcone*, denoted $\mathcal{N}_V$. It is well-known that $\Bbbk[V]^H$ is generated (as a $\Bbbk$-algebra with 1) by finitely many elements, and so $\mathcal{N}_V$ is Zariski-closed in V. (In positive characteristic this requires the Mumford conjecture proved in [Haboush, 1975].) If we take $V = \mathfrak{g}$, with adjoint G-action and $* = 0$, then in all characteristics $\mathcal{N}_{\mathfrak{g}} = \mathfrak{g}_{\mathrm{nil}}$. Similarly, if $V = G$, with the conjugation action and $* = 1_G$, then in all characteristics $\mathcal{N}_G = G_{\mathrm{uni}}$.

3.2.2 Let $\psi : X \to Y$ be a morphism of affine varieties, and let $\psi^* : \Bbbk[Y] \to \Bbbk[X]$ be its comorphism. Let $y \in Y$ and let I_y be the maximal ideal of y in $\Bbbk[Y]$. We define the *coordinate ring* of the schematic fibre $\psi^{-1}(y)$ to be $\Bbbk[X]/\psi^*(I_y)\Bbbk[X]$ (cf. [Eisenbud, 1995, §14.3]). Now let $v \in V$ and $\lambda \in Y(G)$. If $\lim_{\xi \to 0} \lambda(\xi) \cdot v = *$ and $v \neq *$, then the fibre of the extended morphism at $*$ has coordinate ring $\Bbbk[T]/(T^m)$ for some m, where T is an indeterminant.

We now define a function which can be used to measure instability. Given $\lambda \in Y(G)$ we define a function $m(-, \lambda) \colon V \longrightarrow \mathbb{N} \sqcup \{\pm\infty\}$ as follows:

$$
m(v, \lambda) = \begin{cases} -\infty & \text{if } \lim_{\xi \to 0} \lambda(\xi) \cdot v \text{ does not exist;} \\ 0 & \text{if } \lim_{\xi \to 0} \lambda(\xi) \cdot v = v' \neq *; \\ m \text{ (as above)} & \text{if } \lim_{\xi \to 0} \lambda(\xi) \cdot v = * \ (v \neq *) ; \\ +\infty & \text{if } v = *. \end{cases}
$$

Note that $v \in V$ is H-unstable if, and only if, $m(v, \lambda) \geq 1$ for some $\lambda \in Y(H)$. For a set $X \subset V$ we also define $m(X, \lambda) = \inf_{v \in X} m(v, \lambda)$, and say that X is *uniformly unstable* if $m(X, \lambda) \geq 1$ for some $\lambda \in Y(G)$.

3.2.3 Let $\lambda \in Y(G)$. We let G (resp. $\mathbb{Z}$) act on $Y(G)$ by $g \cdot \lambda : \xi \mapsto g\lambda(\xi)g^{-1}$ (resp. $n\lambda : \xi \mapsto \lambda(\xi)^n$) for all $\xi \in \Bbbk$. The identity element of G will be denoted

by 1_G. We define some subgroups of G associated to λ as follows:

$$
\begin{aligned}
P(\lambda) &= \left\{ g \in G \ \middle| \ \lim_{\xi \to 0} \lambda(\xi) g \lambda(\xi)^{-1} \text{ exists} \right\}, \\
L(\lambda) &= C_G(\operatorname{Im} \lambda), \\
U(\lambda) &= \left\{ g \in G \ \middle| \ \lim_{\xi \to 0} \lambda(\xi) g \lambda(\xi)^{-1} = 1_G \right\}.
\end{aligned}
$$

Let T be a maximal torus of $L(\lambda)$ (and therefore a maximal torus of G). If Σ is the root system of G relative to T, then

$$
\begin{aligned}
P(\lambda) &= \langle T, U_\alpha \mid \alpha \in \Sigma, \ \langle \alpha, \lambda \rangle \geq 0 \rangle, \\
L(\lambda) &= \langle T, U_\alpha \mid \alpha \in \Sigma, \ \langle \alpha, \lambda \rangle = 0 \rangle, \\
U(\lambda) &= \langle U_\alpha \mid \alpha \in \Sigma, \ \langle \alpha, \lambda \rangle \geq 1 \rangle.
\end{aligned}
$$

Hence $P(\lambda)$ is a parabolic subgroup of G with unipotent radical $U(\lambda)$. The following is now a straightforward exercise.

Lemma. *Let $v \in V$ and $\lambda \in Y(G)$. Then $m(g \cdot v, \lambda) = m(v, g \cdot \lambda) = m(v, \lambda)$ for all $g \in P(\lambda)$. In particular, for $i \geq 0$, the set of $v \in V$ such that $m(v, \lambda) \geq i$ is $P(\lambda)$-invariant.*

3.2.4 We define the set of *virtual one parameter subgroups* of G as follows. Let

$$
Y_{\mathbb{Q}}(G) = (\mathbb{N} \times Y(G))/\sim,
$$

where $\sim$ is the equivalence relation on $\mathbb{N} \times Y(G)$ such that $(n, \lambda) \sim (m, \mu)$ if, and only if, $n\mu = m\lambda$. Note that $Y(G)$ is naturally a subset of $Y_{\mathbb{Q}}(G)$ and the action of G on $Y(G)$ naturally induces an action on $Y_{\mathbb{Q}}(G)$. If T is a torus, then $Y(T)$ is a free $\mathbb{Z}$-module, and so $Y_{\mathbb{Q}}(T) \cong Y(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ may be regarded as a $\mathbb{Q}$-vector space. We extend our measure of instability to $Y_{\mathbb{Q}}(G)$ as follows. For $\lambda \in Y_{\mathbb{Q}}(G)$, we have that $n\lambda \in Y(G)$ for some $n \in \mathbb{N}$ and so we may define $m(v, \lambda) = n^{-1} m(v, n\lambda)$.

A *squared norm mapping* on $Y_{\mathbb{Q}}(G)$ is a G-invariant function $q : Y_{\mathbb{Q}}(G) \to \mathbb{Q}_{\geq 0}$

whose restriction to $Y_{\mathbb{Q}}(T)$ for any maximal torus T is a positive definite quadratic form. By an averaging trick (cf. [Springer, 1998, §7.1.7]) one can always define a W-invariant positive definite quadratic form q on $Y_{\mathbb{Q}}(T)$. For an arbitrary $\lambda \in Y_{\mathbb{Q}}(G)$, let $g \in G$ be such that $g \cdot \lambda \in Y_{\mathbb{Q}}(T)$. Then define $q(\lambda) = q(g \cdot \lambda)$. One checks that this defines a squared norm mapping on $Y_{\mathbb{Q}}(G)$ by observing that the G-orbits on $Y_{\mathbb{Q}}(G)$ restrict to the W-orbits on $Y_{\mathbb{Q}}(T)$. We define a map $\| \cdot \|_q : Y_{\mathbb{Q}}(G) \to \mathbb{R}_{\geq 0}$ by $\|\lambda\|_q = \sqrt{q(\lambda)}$ for all $\lambda \in Y_{\mathbb{Q}}(G)$, which we call a *norm* on $Y_{\mathbb{Q}}(G)$. From now on we will fix such a norm, and drop the subscript q. Let $X \subset V$ and $\lambda \in Y(G) \setminus \{0\}$. We say that λ is *optimal* for X if

$$\frac{m(X,\lambda)}{\|\lambda\|} \geq \frac{m(X,\mu)}{\|\mu\|} \quad \text{for all } \mu \in Y(G) \setminus \{0\}.$$

If $v \in V$ then, for ease of notation, we will often identify it with the set $\{v\}$ and thus talk about one parameter subgroups which are optimal for v. Usually the notion of optimality depends on the norm, but in the special case that $V = \mathfrak{g}_{\text{nil}}$ or G_{uni}, with adjoint or conjugation action respectively, or when V is a G-module, it is independent of the norm by [Hesselink, 1978, Theorem 7.2]. Note that if λ is optimal for some set, then so is any non-zero scalar multiple of λ. It will be convenient therefore to have a canonical way of choosing an element in $(\mathbb{Q}^{\times}\lambda) \cap Y(G)$ and for this we use the following notion from [Slodowy, 1989]. We say that λ is *primitive* if we cannot write $\lambda = n\mu$ for any integer $n \geq 2$ and $\mu \in Y(G)$. If $X \subset V$ is uniformly unstable, we let Δ_X denote the set of all primitive elements in $Y(G)$ which are optimal for X.

Remark. Hesselink has defined a similar set in [Hesselink, 1979], denoted $\Delta(X)$. This corresponds to a canonical choice for optimal *virtual* one parameter subgroups. Let $\lambda \in \Delta_X$. Then $\Delta(X) = \frac{1}{m(X,\lambda)}\Delta_X$. We will need to use both sets later. To avoid confusion we will use $\tilde{\Delta}_X$ to denote $\Delta(X)$, except in Subsection 3.6.1, where it would be cumbersome to do so.

Theorem. ([Kempf, 1978], [Rousseau, 1978]) *Let $X \subset V$ be uniformly unstable.*

(i) *We have $\Delta_X \neq \emptyset$ and there exists a parabolic subgroup $P(X)$ in G such that $P(X) = P(\lambda)$ for all $\lambda \in \Delta_X$.*

(ii) *We have $\Delta_X = \{g \cdot \lambda \mid g \in P(X)\}$ for any $\lambda \in \Delta_X$.*

(iii) *If T is a maximal torus of $P(X)$, then $Y(T) \cap \Delta_X$ contains exactly one element, which we denote by $\lambda_T(X)$.*

(iv) *For any $g \in G$ we have that $\Delta_{g \cdot X} = g \Delta_X g^{-1}$ and $P(g \cdot X) = g P(X) g^{-1}$. The stabiliser $G_X = \{g \in G \mid g \cdot X = X\}$ is contained in $P(X)$.*

3.2.5 We now restrict to the special case where V is a finite-dimensional rational G-module with, as usual, $* = 0$. Let T be a maximal torus of G with Weyl group W. A very useful set of tools for analysing the T-instability and optimality of subsets of V are certain polytopes in $Y_{\mathbb{Q}}(T)$ defined in terms of weights of the T-action on V. Let $X_{\mathbb{Q}}(T) = X(T) \otimes_{\mathbb{Z}} \mathbb{Q}$, and let $(\ ,\ )$ be a W-invariant inner product on $Y_{\mathbb{Q}}(T)$ induced by the norm $\|\cdot\|$. Then there is a $\mathbb{Q}$-linear isomorphism $\phi_T \colon X_{\mathbb{Q}}(T) \to Y_{\mathbb{Q}}(T)$ defined uniquely by the relation $\langle \chi, \lambda \rangle = (\phi_T(\chi), \lambda)$ for all $\chi \in X_{\mathbb{Q}}(T)$ and $\lambda \in Y_{\mathbb{Q}}(T)$.

Consider the weight space decomposition $V = \bigoplus_{\chi \in X(T)} V_\chi$ of V with respect to T, where

$$V_\chi = \{v \in V \mid t \cdot v = \chi(t) v \text{ for all } t \in T\}.$$

Then for any $v \in V$ we may uniquely write $v = \sum_{\chi \in X(T)} v_\chi$ with $v_\chi \in V_\chi$. If $X \subset V$, we define $S_T(X) = \{\chi \in X(T) \mid v_\chi \neq 0 \text{ for some } v \in X\}$, and let $K_T(X)$ denote the convex hull (or Newton polytope) of $\phi_T(S_T(X))$ in $Y_{\mathbb{Q}}(T)$. Then we have the following.

Lemma. (Cf. [Slodowy, 1989]) *Let $X \subset V$ and T be a maximal torus of G.*

(i) *If $\lambda \in Y(T)$, then $m(X, \lambda) = \min_{\mu \in \phi_T(S_T(X))} (\mu, \lambda) = \min_{\mu \in K_T(X)} (\mu, \lambda)$.*

(ii) *There exists a unique element $\mu_T(X) \in K_T(X)$ of minimal norm.*

(iii) *The set X is uniformly T-unstable if, and only if, $\mu_T(X) \neq 0$, in which case we have that $\|\mu_T(X)\|^2 = m(X, \mu_T(X))$.*

(iv) *If X is T-unstable and $\lambda_T(X)$ is the unique primitive scalar multiple of $\mu_T(X)$, then $\Delta_{X,T} = \{\lambda_T(X)\}$.*

3.2.6 Resume the more general assumption that V is a G-variety. For $i \geq 0$ and $\lambda \in Y_{\mathbb{Q}}(G)$, we denote by $V(\lambda)_i$ be the set of elements $v \in V$ with $m(v, \lambda) \geq i$, a closed subvariety of V. Let $X \subset V$ be uniformly unstable and suppose that $\lambda \in \Delta_X$ and $k = m(X, \lambda)$. Then we define the *saturation* of X to be the set $S(X) = V(\Delta_X)_k$. This is well-defined by Theorem 3.2.4(ii) and Lemma 3.2.3. We call a set *saturated* if it is uniformly unstable and equal to its own saturation.

Assume, temporarily, again that V is a G-module with $* = 0$. We may grade V, with respect to λ, as a direct sum of subspaces

$$V(\lambda, i) = \left\{ v \in V \mid \lambda(\xi) \cdot v = \xi^i v \text{ for all } \xi \in \Bbbk^{\times} \right\},$$

for $i \in \mathbb{Z}$. Then a saturated set $X \subset V$ may be written as

$$X = V(\Delta_X)_k = \bigoplus_{i \geq k} V(\lambda, i),$$

where $\lambda \in \Delta_X$ and $k = m(X, \lambda)$. Letting T be a maximal torus of $C_G(\operatorname{Im} \lambda)$, it is not hard to see that the $V(\lambda, i)$ are sums of weight subspaces of V. More precisely,

$$X = \bigoplus_{\langle \chi, \lambda \rangle \geq k} V_{\chi}.$$

Since all maximal tori of G are conjugate and V has finitely many T-weights, the number of conjugacy classes of saturated subsets of V is finite.

The following result of Hesselink shows that the description of saturated sets in the general situation, in which V is a G-variety, may be reduced to the above consideration. (Note that since $*$ is G-invariant, the tangent space $\mathrm{T}_* V$ naturally becomes a G-module.)

Proposition. ([Hesselink, 1979, Proposition 3.8]) *If X is a saturated subset of V, then $\mathrm{T}_* X$ is a saturated subset of $\mathrm{T}_* V$ which is isomorphic to X and satisfies $\Delta_{\mathrm{T}_* X} = \Delta_X$. The application of T_* is a bijection from the class of saturated subsets of V to the class of saturated subsets of $\mathrm{T}_* V$.*

In particular, the saturated sets in the adjoint action of G on itself are connected unipotent subgroups.

By virtue of Proposition 3.2.6 we may implicitly identify a saturated set with its tangent space, so that Lemma 3.2.5 now makes sense for arbitrary saturated

sets. We now gather some basic facts about saturated sets that will be useful later. First we need the following definitions. Given a uniformly G-unstable subset X of V we define

$$\|X\| = \min\left\{\|\mu_T(g \cdot X)\| : g \in G, \ 0 \notin K_T(g \cdot X)\right\}.$$

Note that $\|X\|$ is the minimal distance from the origin to a point in a finite union of polytopes of the form $K_T(g \cdot X)$ for *some* $g \in G$, and it is independent of the choice of T. It follows from Lemma 3.2.5 that

$$\|X\| = \inf\{\|\lambda\| : \ \lambda \in Y(G), \ m(X, \lambda) \geq 1\}$$

(cf. [Hesselink, 1979], p. 143).

Lemma. *Let X and Y be uniformly unstable subsets of V.*

(i) $S(X)$ *is uniformly unstable,* $\Delta_{S(X)} = \Delta_X$ *and* $\tilde{\Delta}_{S(X)} = \tilde{\Delta}_X$.

(ii) $\tilde{\Delta}_X = \tilde{\Delta}_Y$ *if, and only if,* $Y \subset S(X)$ *and* $\|X\| = \|Y\|$.

(iii) $X \subset S(X) = S(S(X))$.

(iv) *If* $X \subset Y$*, then* $\|X\| \geq \|Y\|$.

(v) *If* $g \in G$*, then* $g \cdot S(X) = S(g \cdot X)$.

Proof. This is a straightforward exercise. Cf. [Hesselink, 1979, Lemma 2.8]. $\square$

3.2.7 Following [Hesselink, 1979, §4] now define some equivalence relations on $\mathcal{N}_V$. For $x, y \in \mathcal{N}_V$ we set

$$
\begin{aligned}
x \approx y &\Leftrightarrow \tilde{\Delta}_x = \tilde{\Delta}_y; \\
x \sim y &\Leftrightarrow \tilde{\Delta}_{g \cdot x} = \tilde{\Delta}_y \text{ for some } g \in G.
\end{aligned}
$$

We call an equivalence class $[v] = \{x \mid x \approx v\}$ a *blade* and an equivalence class $G[v] = \{x \mid x \sim v\}$ a *stratum*. Hesselink gives the following description of blades and strata.

Lemma. *Let $v \in \mathcal{N}_V$. Then*

(i) $[v] = \{x \in S(v) : \|x\| = \|v\|\}$.

(ii) $[v]$ *is open and dense in* $S(v)$.

(iii) $GS(v)$ *is an irreducible closed subset of* $\mathcal{N}_V$.

(iv) $G[v] = \{x \in GS(v) : \|x\| = \|v\|\}$.

(v) $G[v]$ *is open and dense in* $GS(v)$.

We will eventually show that when $V = G_{\mathrm{uni}}$ the strata are precisely Lusztig's unipotent pieces. To that end the following result will be crucial.

Proposition. *Let $v \in V$. Then*

$$G[v] = GS(v) \setminus \bigcup GS(v'),$$

where the union is taken over all saturated sets $S(v')$ such that $GS(v') \subsetneq GS(v)$.

Proof. Let $v, v' \in \mathcal{N}_V$ be such that $GS(v') \subseteq GS(v)$. To prove the proposition, it is sufficient to show that $GS(v') = GS(v)$ if, and only if, $\|v\| = \|v'\|$.

Suppose that $GS(v') = GS(v)$. Then there exists $g \in G$ such that $g \cdot v' \in S(v)$. Hence $\|v'\| = \|g \cdot v'\| \geq \|S(v)\| = \|v\|$ by Lemma 3.2.6. Similarly we can find $h \in G$ such that $h \cdot v \in S(v')$ and deduce that $\|v'\| \leq \|v\|$, and thus $\|v'\| = \|v\|$.

Conversely, suppose that $\|v'\| = \|v\|$. Since $GS(v') \subseteq GS(v)$, there exists $g \in G$ such that $g \cdot v' \in S(v)$. Then Lemma 3.2.6(ii) yields $\tilde{\Delta}_{g \cdot v'} = \tilde{\Delta}_v$, and so $S(g \cdot v') = S(v)$. Hence $g \cdot S(v') = S(v)$ by Lemma 3.2.6(v). It follows that $GS(v') = GS(v)$. $\square$

3.3   A modification of the Kirwan–Ness theorem

3.3.1 Let $\lambda \in Y(G) \setminus \{0\}$ and let T be a maximal torus of G containing $\mathrm{Im}\,\lambda$. (This is equivalent to T being a maximal torus of $L(\lambda)$.) Then we define

$$
\begin{aligned}
T^\lambda &= \langle \mathrm{Im}\,\mu \mid \mu \in Y(T),\ (\mu, \lambda) = 0 \rangle, \\
L^\perp(\lambda) &= \langle T^\lambda,\ \mathcal{D}L(\lambda) \rangle.
\end{aligned}
$$

Note that $L^\perp(\lambda)$ is independent of the choice of T since $(gTg^{-1})^\lambda = gT^\lambda g^{-1}$ for all $g \in G$. Also, T^λ is a subtorus of T and $L^\perp(\lambda) = T^\lambda \cdot \mathcal{D}L(\lambda)$ is a connected reductive group by [Springer, 1998, Corollary 2.2.7], [Borel, 1991, §IV.14.2].

3.3.2 We now restrict to the special case where V is a G-module with, as usual, $* = 0$. In [Slodowy, 1989], [Popov and Vinberg, 1994], [Tsujii, 2008] the following generalisation of the Kirwan–Ness theorem is proved.

Theorem. (Cf. [Kirwan, 1984], [Ness, 1984]) *Let $v \in V \setminus \{0\}$ and $\lambda \in Y(G) \setminus \{0\}$. Assume that $k = m(v, \lambda) \geq 1$ and write $v = \sum_{i \geq k} v_i$ with $v_i \in V(\lambda, i)$ (and $v_k \neq 0$). Then λ is optimal for v if, and only if, v_k is $L^\perp(\lambda)$-semistable.*

Our goal is to obtain an analogous result for the conjugation action of G on the unipotent variety. Our proof is modelled on the proof in [Tsujii, 2008] of the above result. We will need the following lemmas from [Slodowy, 1989] and [Tsujii, 2008] for this task.

3.3.3 We continue to assume that V is a G-module with $* = 0$. It follows from [Borel, 1991, Proposition 8.2(c)] that an element of $X_{\mathbb{Q}}(T^\lambda)$ may be lifted to an element of $X_{\mathbb{Q}}(T)$. In fact, $X_{\mathbb{Q}}(T^\lambda)$ may be naturally identified with the orthogonal projection of $X_{\mathbb{Q}}(T)$ onto the hyperplane $\{\chi \in X_{\mathbb{Q}}(T) \mid (\chi, \lambda) = 0\}$. The following lemma shows that this projection behaves well with respect to optimality.

Lemma. (Cf. [Slodowy, 1989]) *Let $\lambda \in Y(G) \setminus \{0\}$ and $v \in V(\lambda, k)$ for some $k \in \mathbb{N}$. If T is a maximal torus of G containing $\mathrm{Im}\,\lambda$ then $\mu_{T^\lambda}(v) = \mu_T(v) - \frac{k}{(\lambda, \lambda)}\lambda$.*

3.3.4 We continue to assume that V is a G-module with $* = 0$. The following is the key lemma used in the proof of Theorem 3.3.2.

Lemma. ([Tsujii, 2008, Lemma 2.6]) *Let T be a maximal torus of G and assume that $v \in V \setminus \{0\}$ is T-unstable. Let $k = m(v, \lambda_T(v))$ and $v' \in v + \bigoplus_{i > k} V(\lambda_T(v), i)$. Then $\lambda_T(v) = \lambda_T(v')$.*

3.3.5 We now assume that $V = G_{\mathrm{uni}}$ with $* = 1_G$. Let $\lambda \in Y(G)$ and let T be a maximal torus of $L(\lambda)$ with corresponding G-root system Σ. Recall that for each root $\alpha \in \Sigma$ we denote the corresponding root subgroups by U_α, and we have that

$$R_u(P(\lambda)) = U(\lambda) = \langle U_\alpha \,|\, \alpha \in \Sigma, \ \langle \alpha, \lambda \rangle \geq 1 \rangle,$$

where $R_u(P(\lambda))$ denotes the unipotent radical of $P(\lambda)$. In fact, $U(\lambda)$ is directly spanned by the root subgroups U_α with $\langle \alpha, \lambda \rangle \geq 1$; see [Borel, 1991, §IV.14]. Hence the product morphism

$$\pi : U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n} \longrightarrow \prod\nolimits_{\langle \alpha, \lambda \rangle \geq 1} U_\alpha = U(\lambda)$$

is an isomorphism of varieties, with respect to any choice of ordering

$$\{\alpha_1, \ldots, \alpha_n\} = \{\alpha \in \Sigma \,|\, \langle \alpha, \lambda \rangle \geq 1\},$$

which we now fix once and for all. Moreover, since each of the root subgroups $U_\alpha = \langle x_\alpha(t) \,|\, t \in \Bbbk \rangle$ is isomorphic to the additive group $\Bbbk^+$, this gives an isomorphism $f : U(\lambda) \xrightarrow{\sim} \mathbb{A}^n(\Bbbk)$. Consider $\mathbb{A}^n(\Bbbk)$ as a vector space with basis indexed by the set $\{1, 2, \ldots, n\}$. It becomes a T-module by letting $t \in T$ act on the ith basis vector by scalar multiplication by $\alpha_i(t)$. With respect to this f is T-equivariant. From now on we will implicitly regard $U(\lambda)$ as a T-module.

We define the following $L(\lambda)$-stable closed subvarieties of $U(\lambda)$ for each $i \geq 1$: Let $\{\beta_1, \beta_2, \ldots, \beta_{l(i)}\} = \{\alpha \in \Sigma \,|\, \langle \alpha, \lambda \rangle = i\}$, and set

$$U^i(\lambda) = \pi(U_{\beta_1} \times U_{\beta_2} \times \cdots \times U_{\beta_{l(i)}}).$$

These give a direct product decomposition of $U(\lambda)$ into T-submodules, and we may identify

$$U(\lambda) \cong U^1(\lambda) \times U^2(\lambda) \times \cdots \times U^r(\lambda),$$

for some positive integer r, so that for any $u \in U(\lambda)$ we may write, with uniqueness, $\pi^{-1}(u) = (u_1, u_2, \ldots, u_r)$ with $u_i \in U^i(\lambda)$. For $\lambda \neq 0$ and $u \neq 1_G$ define $m'(u, \lambda) = \min\{i \,|\, u_i \neq 1_G\}$ and $m'(u, \lambda) = +\infty$ for $u = 1_G$. Then we have the following.

Lemma. *Let $\lambda \in Y(G) \setminus \{0\}$ and $u \in U(\lambda)$. Then $m'(u, \lambda) = m(u, \lambda)$.*

Proof. If $u = 1_G$, the statement is obvious, so suppose $u \neq 1_G$. For each root α_i let $m_i = \langle \alpha_i, \lambda \rangle$. Then we have a morphism of varieties $\ell : \mathbb{A}^1(\Bbbk) \to U(\lambda)$ given by $t \longmapsto \lambda(t) u \lambda(t)^{-1}$ for $t \in \Bbbk^\times$ and $\ell(0) = 1_G$. Writing $u = \pi^{-1}(u_{\alpha_1}, u_{\alpha_2}, \ldots, u_{\alpha_n})$ with $u_{\alpha_i} = x_{\alpha_i}(\xi_i) \in U_{\alpha_i}$, we have

$$
\begin{aligned}
\ell(t) &= \pi^{-1}\big(\lambda(t) u_{\alpha_1} \lambda(t)^{-1}, \lambda(t) u_{\alpha_2} \lambda(t)^{-1}, \ldots, \lambda(t) u_{\alpha_n} \lambda(t)^{-1}\big) \\
&= \pi^{-1}\big(x_{\alpha_1}(\xi_1 t^{\langle \alpha_1, \lambda \rangle}), x_{\alpha_2}(\xi_2 t^{\langle \alpha_2, \lambda \rangle}), \ldots, x_{\alpha_n}(\xi_n t^{\langle \alpha_n, \lambda \rangle})\big) \\
&= \pi^{-1}\big(x_{\alpha_1}(\xi_1 t^{m_1}), x_{\alpha_2}(\xi_2 t^{m_2}), \ldots, x_{\alpha_n}(\xi_n t^{m_n})\big).
\end{aligned}
$$

Without loss of generality assume that $m_1 \leq m_2 \leq \cdots \leq m_n$ and $m'(u, \lambda) = m_k$ for some $k \leq n$, so that $\xi_i = 0$ for $i < k$. Then, identifying $\Bbbk[U(\lambda)]$ and $\Bbbk[\mathbb{A}^1(\Bbbk)]$ with the polynomial rings $\Bbbk[T_1, \ldots, T_n]$ and $\Bbbk[T]$ respectively, the comorphism ℓ^* sends $g = g(T_1, \ldots, T_n) \in \Bbbk[U(\lambda)]$ to $g(0, \ldots, 0, \xi_k T^{m_k}, \ldots, \xi_n T^{m_n})$. Hence, if $I = \langle T_1, \ldots, T_n \rangle$ is the maximal ideal of $1_G \in U(\lambda)$, then the ideal $\ell^*(I)$ of the schematic fibre $\ell^{-1}(u)$ is generated by $\xi_k T^{m_k}, \ldots, \xi_n T^{m_n}$. As $\xi_k \neq 0$, it follows that the coordinate ring of the schematic fibre $\ell^{-1}(u)$ equals $\Bbbk[T]/(T^{m_k})$.

Now consider the composition $\mathbb{A}^1(\Bbbk) \xrightarrow{\ell} U(\lambda) \xrightarrow{\iota} G_{\mathrm{uni}}$. If $\iota(1_G) = 1_G$ has maximal ideal I' of $\Bbbk[G_{\mathrm{uni}}]$, then $\iota^*(I') = I$, so $(\iota \circ \ell)^*(I') = \ell^* \circ \iota^*(I') = \ell^*(I)$, which completes the proof. $\qquad\square$

3.3.6 For $i \geq 1$, we set $U_i(\lambda) = \langle U_\alpha \mid \alpha \in \Sigma, \langle \alpha, \lambda \rangle \geq i \rangle$, a connected normal subgroup of $U(\lambda)$. The group $L(\lambda)$ acts rationally on the affine variety $V_i(\lambda) = U_i(\lambda)/U_{i+1}(\lambda) \cong U^i(\lambda)$. The variety $V_i(\lambda)$ is a connected abelian unipotent group. It may be regarded as a vector space over $\Bbbk$ with basis $v_1, \ldots, v_{l(i)}$ consisting of the images of $x_{\beta_1}(1), \ldots, x_{\beta_{l(i)}}(1)$ in $U_i(\lambda)/U_{i+1}(\lambda)$. Our convention here is that $\xi_1 v_1 + \cdots + \xi_{l(i)} v_{l(i)}$ is the image of $\prod_{j=1}^{l(i)} x_{\beta_j}(\xi_j)$ in $U_i(\lambda)/U_{i+1}(\lambda)$ for all $\xi_i \in \Bbbk$. The preceding remarks then imply that the torus $T \subset L(\lambda)$ acts linearly on $V_i(\lambda) \cong U^i(\lambda)$ with the v_j being weight vectors of $V_i(\lambda)$ with respect to T. In view of Chevalley's commutator relations it is straightforward to see that each root subgroup U_α with $\langle \alpha, \lambda \rangle = 0$ acts linearly on $V_i(\lambda)$ as well. It follows that the group $L(\lambda)$ acts linearly and rationally on $V_i(\lambda)$. In other words, each vector space $V_i(\lambda)$ is a rational $L(\lambda)$-module.

3. UNIPOTENT ELEMENTS IN SMALL CHARACTERISTIC

We are now ready to state and prove the following version of the Kirwan–Ness theorem.

Theorem. *Let $u \neq 1_G$ be a unipotent element of G and $\lambda \in Y(G) \setminus \{0\}$. Assume that $u \in U(\lambda)$ and let $k = m(u, \lambda)$. Then λ is optimal for u if, and only if, the image of u in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$ is $L^\perp(\lambda)$-semistable.*

Proof. In proving the theorem we may assume without loss of generality that λ is primitive. We follow Tsujii's arguments from [Tsujii, 2008, Theorem 2.8] very closely.

First suppose λ is optimal for u and let $k = m(u, \lambda)$. Then u lies in the set $U_k(\lambda) \setminus U_{k+1}(\lambda)$ by Lemma 3.3.5. Let $\bar{u}$ denote the image of u in the $L^\perp(\lambda)$-module $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$. We must show that $\bar{u}$ is semistable with respect to all maximal tori of $L^\perp(\lambda)$. Of course, each of these has the form T^λ for some maximal torus T of $L(\lambda)$. In particular, $\lambda \in Y(T)$ and hence $\lambda = \lambda_T(u)$ by our assumption on λ. Note that Lemma 3.2.5 can be used in our present (non-linear) situation in view of Proposition 3.2.6 applied with $G = T$. Then $k = (\mu_T(u), \lambda_T(u))$, so that

$$\mu_T(u) \in \{\mu \in K_T(u) \,|\, (\mu, \lambda_T(u)) = k\} \;=\; K_T(\bar{u}).$$

Therefore $\mu_T(u) = \mu_T(\bar{u})$ and $\lambda_T(u) = \lambda_T(\bar{u})$. Let $\mu \in Y(T) \setminus \{0\}$. Then Lemma 3.2.5 implies that

$$\frac{m(u, \lambda_T(u))}{\|\lambda_T(u)\|} \;=\; \frac{k}{\|\lambda_T(u)\|} \;=\; \frac{m(\bar{u}, \lambda_T(u))}{\|\lambda_T(u)\|} \;=\; \frac{m(\bar{u}, \lambda_T(\bar{u}))}{\|\lambda_T(\bar{u})\|} \;\geq\; \frac{m(\bar{u}, \mu)}{\|\mu\|}.$$

Since $S_T(\bar{u}) \subseteq S_T(u)$ we have that $m(\bar{u}, \mu) \geq m(u, \mu)$. Then $\lambda_T(\bar{u})$ lies in $\Delta_{T,u} = \{\lambda_T(u)\}$, implying that $\mu_{T^\lambda}(\bar{u})$ and λ are proportional; see Lemma 3.3.3. Since λ is orthogonal to $\mu_{T^\lambda}(\bar{u}) \in Y(T^\lambda)$ it must be that $\|\mu_{T^\lambda}(\bar{u})\| = 0$. Hence $\bar{u}$ is T^λ-semistable by Lemma 3.2.5(iii).

Conversely, suppose that $\bar{u}$ is $L^\perp(\lambda)$-semistable. The parabolic subgroups $P(\lambda)$ and $P(u)$ have a maximal torus in common, T' say; see [Humphreys, 1975, Corollary 28.3]. We may choose $w \in U(\lambda)$ with $T = wT'w^{-1} \subset L(\lambda)$ so that $\lambda \in Y(T)$. Then $\bar{u}$ is T^λ-semistable by the assumption and hence $\mu_{T^\lambda}(\bar{u}) = 0$ by Lemma 3.2.5. Applying Lemma 3.3.3 we now get $\mu_T(\bar{u}) = \frac{k}{(\lambda,\lambda)}\lambda$. It follows that $\lambda = \lambda_T(\bar{u})$. We claim that also $\lambda = \lambda_T(wuw^{-1})$.

In order to prove the claim we first recall that $U(\lambda)$ has a T-module structure such that $U_i(\lambda)/U_{i+1}(\lambda) \cong U^i(\lambda)$ as T-modules for all $i \geq 1$; see Subsection 3.3.5. Then $\lambda_T(\bar{u}) = \lambda_T(u_k)$. In view of Lemma 3.3.4, we need to show that the k-component of wuw^{-1} is u_k (which will then be the minimal non-trivial component of wuw^{-1}, by Lemma 3.2.3). Write $u = \prod_{\langle \alpha, \lambda \rangle \geq k} u_\alpha$ and assume that $w = \prod_{i=1}^n x_{\alpha_i}(\zeta_i)$ for some $\zeta_i \in \Bbbk$. Then Chevalley's commutator relations yield

$$
\begin{aligned}
wuw^{-1} \;=\; \prod_{\substack{\alpha \in \Sigma \\ \langle \lambda, \alpha \rangle \geq k}} wu_\alpha w^{-1} &\in \prod_{\substack{\alpha \in \Sigma \\ \langle \lambda, \alpha \rangle \geq k}} \left(u_\alpha \prod_{\substack{i,j>0 \\ i\alpha+j\beta \in \Sigma}} U_{i\alpha+j\beta} \right) \\
&\subseteq \left(\prod_{\substack{\alpha \in \Sigma \\ \langle \lambda, \alpha \rangle \geq k}} u_\alpha \right) \cdot U_{k+1}(\lambda) \;\subseteq\; u_k U_{k+1}(\lambda).
\end{aligned}
$$

Hence $\lambda = \lambda_T(wuw^{-1})$ as claimed. To complete the proof of the theorem note that $T \subset wP(\lambda)w^{-1} = P(wuw^{-1})$, and so $\lambda \in \Delta_{wuw^{-1}} = \Delta_u$ by Theorem 3.2.4. $\qquad \square$

Remark. For each $\beta \in \Sigma$ with $\langle \beta, \lambda \rangle = k$ we let v_β denote the image of $x_\alpha(1)$ in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$ and write X_β for the tangent vector of the root subgroup $U_\beta = \langle x_\beta(t) \mid t \in \Bbbk \rangle$ in $\mathfrak{g} = \operatorname{Lie} G$. Then

$$
(\operatorname{Ad} x_\beta(t))\, y \;\equiv\; y + t[X_\beta, y] \quad \left(\operatorname{mod} \mathfrak{g} \otimes t^2 \Bbbk[t]\right) \qquad (\forall\, y \in \mathfrak{g} \otimes \Bbbk[t]).
$$

The map $v_\beta \mapsto X_\beta$ extends uniquely up to a linear isomorphism between $V_k(\lambda)$ and the subspace $\mathfrak{g}(\lambda, k) = \operatorname{span}\{X_\beta \mid \langle \beta, \lambda \rangle = k\}$ of $\mathfrak{g}$; we call it η_k. Using Chevalley's commutator relations and our definition of the vector space structure on $V_k(\lambda)$ at the beginning of this subsection it is straightforward to see that η_k is an isomorphism of $L(\lambda)$-modules. If G and T are defined over $\mathbb{Z}$, then so is η_k.

3.4 Reductive group schemes and a theorem of Seshadri

We now briefly review reductive group schemes before stating a result of Seshadri which we will need later. For a general reference see [Jantzen, 1987], for example.

3.4.1 For an affine variety X over $\Bbbk$, we say that X is *defined over* $\mathbb{Z}$ if there is an embedding of X into some affine space $\mathbb{A}^n(\Bbbk)$ such that the radical ideal $I(X)$ of X is generated by elements of $\mathbb{Z}[X_1, \ldots, X_n]$. (This is the same as requiring that $\Bbbk[X] \cong \mathbb{Z}[X] \otimes_{\mathbb{Z}} \Bbbk$, where $\mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_n]/(I(X) \cap \mathbb{Z}[X_1, \ldots, X_n])$.) A morphism $\phi : X \to Y$ of $\Bbbk$-varieties defined over $\mathbb{Z}$ is said to be defined over $\mathbb{Z}$ if it can be written in terms of elements of $\mathbb{Z}[X_1, \ldots, X_n]$. (This is the same as requiring that its comorphism restricts to a homomorphism $\phi^* : \mathbb{Z}[Y] \to \mathbb{Z}[X]$ of $\mathbb{Z}$-algebras.)

When X is defined over $\mathbb{Z}$ we may associate to it a reduced affine algebraic $\mathbb{Z}$-scheme, i.e. a functor $\mathfrak{X} : \mathrm{Alg}_{\mathbb{Z}} \to \mathrm{Set}$ such that if A, A' are $\mathbb{Z}$-algebras and $\psi : A \to A'$ is a $\mathbb{Z}$-algebra homomorphism then $\mathfrak{X}(A) = \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[X], A)$ and $\mathfrak{X}(\psi) : \alpha \mapsto \psi \circ \alpha$ for each $\alpha \in \mathrm{Hom}_{\mathbb{Z}\text{-alg}}(\mathbb{Z}[X], A)$. We identify $\mathfrak{X}(A)$ with the set $\{a \in A^n \mid f(a) = 0 \text{ for all } f \in I(X) \cap A[X_1, \ldots, X_n]\}$.

If G is an affine algebraic group over $\Bbbk$, then we say that G is defined over $\mathbb{Z}$ if it is so as a variety and the product and inverse morphisms are defined over $\mathbb{Z}$. (This is the same as requiring that the Hopf algebra structure on $\Bbbk[G]$ restricts to one on $\mathbb{Z}[G]$.) In this case we may associate to it (using Jantzen's terminology) a reduced algebraic $\mathbb{Z}$-group, i.e. a functor $\mathfrak{G} : \mathrm{Alg}_{\Bbbk} \to \mathrm{Grp}$ defined as above, with the group structure on $\mathfrak{G}(A)$ defined via the Hopf algebra structure on $A[G] = \mathbb{Z}[G] \otimes_{\mathbb{Z}} A$ for each $\mathbb{Z}$-algebra A. From now on we call such a functor a $\mathbb{Z}$-*group scheme*. G is said to be $\mathbb{Z}$-*split* if there exists a maximal torus T of G such that there is an isomorphism $T \xrightarrow{\sim} \Bbbk^{\times} \times \cdots \times \Bbbk^{\times}$ which is defined over $\mathbb{Z}$ and the root morphisms of T are defined over $\mathbb{Z}$.

It has been shown by Chevalley that every connected reductive algebraic group over an algebraically closed field $\Bbbk$ may be obtained by extension of scalars from a reduced algebraic $\mathbb{Z}$-group, and that many familiar subgroups and actions are also defined over $\mathbb{Z}$; cf. [Chevalley, 1961]. This allows one to pass information between the characteristic zero and prime characteristic settings; see [Jantzen, 1987]. We will use this to relate optimal one parameter subgroups of groups in arbitrary characteristic to those with the same root system defined over $\mathbb{C}$. This will eventually allow us to use the parameter set $\tilde{D}_{G'}/G'$ from Section 3.1 in arbitrary characteristic.

3.4.2 Let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme and let $\mathfrak{X}$ be a reduced affine algebraic $\mathbb{Z}$-scheme. We will say that $\mathfrak{G}$ *acts* on $\mathfrak{X}$ if, for any $\mathbb{Z}$-algebra A, there is a map $\phi_A : \mathfrak{G}(A) \times \mathfrak{X}(A) \to \mathfrak{X}(A)$, functorial in A, given by polynomials over A. If $\mathfrak{G}$ acts on an affine space $\mathbb{A}^n_{\mathbb{Z}}$ (regarded as a $\mathbb{Z}$-scheme) then we say that this action is linear if, for any $\mathbb{Z}$-algebra A, $g \in \mathfrak{G}(A)$, the map $\phi_A(g) : \mathbb{A}^n_{\mathbb{Z}}(A) \to \mathbb{A}^n_{\mathbb{Z}}(A)$ is A-linear.

We now state a result of Seshadri which allows one to pass information about semistability between characteristics.

Theorem. (Cf. [Seshadri, 1977, Proposition 6]) *Let $\Bbbk$ be an algebraically closed field and let $\mathfrak{G}$ be a reductive $\mathbb{Z}$-group scheme acting linearly on $\mathbb{A}^n_{\mathbb{Z}}$. Suppose that $\mathfrak{X}$ is a $\mathfrak{G}$-stable open subscheme of $\mathbb{A}^n_{\mathbb{Z}}$ and $x \in \mathfrak{X}(\Bbbk)$ is a semistable point. Then there exists a $\mathfrak{G}$-invariant $F \in \mathbb{Z}[\mathbb{A}^n_{\mathbb{Z}}] = \mathbb{Z}[X_1, \dots, X_n]$ such that $F(x) \neq 0$. Furthermore, there is an open subscheme $\mathfrak{X}^{ss}$ of $\mathfrak{X}$ such that for any algebraically closed field $\Bbbk'$, the set $\mathfrak{X}^{ss}(\Bbbk')$ consists of the semistable points of $\mathfrak{X}(\Bbbk')$.*

3.4.3 In the next section we will prove our main result by applying Theorem 3.4.2 to a reductive $\mathbb{Z}$-group scheme associated with $L^{\perp}(\lambda)$. To that end we will now construct such a scheme. From now on assume that we have a fixed reductive $\mathbb{Z}$-group scheme $\mathfrak{G}$, which determines the reductive groups G, G' that we are interested in. In addition, let us fix a maximal torus $\mathfrak{T}$. Then there is a natural identification of the one parameter subgroups of $\mathfrak{T}(\Bbbk)$ as $\Bbbk$ varies. It follows that there is a reductive $\mathbb{Z}$-group scheme $\mathfrak{L}$, the scheme-theoretic centraliser of a one parameter subgroup λ of $\mathfrak{T}$, which gives rise to the groups $L(\lambda)$. The groups $L^{\perp}(\lambda)$ may also be obtained from a reductive $\mathbb{Z}$-group scheme, but since this is not a standard result we will now give an explicit construction.

Recall that a root datum of a connected reductive group, or reductive $\mathbb{Z}$-group scheme, is a quadruple $(X(T), \Sigma, Y(T), \Sigma^{\vee})$, with respect to a fixed maximal torus, together with the perfect pairing $X(T) \times Y(T) \to \mathbb{Z}$ and the associated bijection $\Sigma \to \Sigma^{\vee}$ between the roots and coroots of G with respect to T. If we forget about the fixed torus T and merely regard $X(T)$ and $Y(T)$ as abstract free abelian groups with finite subsets Σ and $\Sigma^{\vee}$ respectively, then the datum is unique and moreover any such abstract root datum gives rise to a connected reductive group, or reductive group $\mathbb{Z}$-scheme. If G' is another such group, or

$\mathbb{Z}$-group scheme, with datum $(X(T'), \Sigma', Y(T'), \Sigma'^\vee)$, then a *homomorphism of root data* is a group homomorphism $f : X(T') \to X(T)$ that maps Σ' bijectively to Σ and such that the dual homomorphism $f^\vee : Y(T) \to Y(T')$ maps $f(\beta)^\vee$ to $\beta^\vee$ for each $\beta \in \Sigma'$. A morphism of algebraic groups $\psi : T \to T'$ is said to be *compatible with the root data* if the induced homomorphism $\psi^* : X(T') \to X(T)$ is a homomorphism of root data.

Proposition. *The connected reductive group $L^\perp(\lambda)$ is a $\mathbb{Z}$-scheme theoretic subgroup of $L(\lambda)$. In other words, if $\mathfrak{L}$ is a $\mathbb{Z}$-group scheme such that $\mathfrak{L}(\mathbb{k}) = L(\lambda)$, then there exists a $\mathbb{Z}$-subgroup scheme $\mathfrak{L}^\perp$ of $\mathfrak{L}$ such that $\mathfrak{L}^\perp(\mathbb{k}) = L^\perp(\lambda)$.*

Proof. Suppose that $(X(T), \Sigma, Y(T), \Sigma^\vee)$ is the root datum of $L(\lambda)$. It follows then that the root datum of $L^\perp(\lambda)$, with respect to the maximal torus T^λ, is $\left(X(T^\lambda), \{\alpha|_{T^\lambda} \mid \alpha \in \Sigma\}, Y(T^\lambda), \Sigma^\vee \right)$. We may also construct reductive $\mathbb{Z}$-group schemes from these data, say $\mathfrak{L}$ (as above) for the former and $\tilde{\mathfrak{L}}^\perp$ for the latter. We now need to construct a subgroup scheme $\mathfrak{L}^\perp$ of $\mathfrak{L}$, isomorphic to $\tilde{\mathfrak{L}}^\perp$ which gives rise to $L^\perp(\lambda)$. We start by showing that T^λ is defined over $\mathbb{Z}$ as a subgroup of T, so that we may construct a $\mathbb{Z}$-group scheme $\mathfrak{T}$ with subgroup scheme $\mathfrak{T}^\lambda$ which give rise to T and T^λ respectively.

We know that T^λ is a subtorus of codimension 1 in T (for it is a connected subgroup of T and $Y(T^\lambda)$ has rank equal to $l - 1$ where $l = \dim T$). Therefore T/T^λ is a 1-dimensional torus. By [Borel, 1991, Corollary 8.3] the natural short exact sequence $1 \to T^\lambda \to T \to T/T^\lambda \to 1$ gives rise to a short exact sequence of character groups $0 \to X(T/T^\lambda) \to X(T) \to X(T^\lambda) \to 0$. Since T/T^λ is a one dimensional torus, its character group $X(T/T^\lambda)$ is generated by one element, say η. By the above η can be regarded as a rational character of T and

$$X(T) \cong \mathbb{Z}\eta \oplus X(T^\lambda). \tag{3.1}$$

(One should keep in mind here that $X(T^\lambda)$ is a free $\mathbb{Z}$-module of rank $l - 1$.) By construction, η vanishes on T^λ.

On the other hand, [Borel, 1991, Proposition 8.2(c)] shows that T^λ coincides with the intersection of the kernels of rational characters of T, say $T^\lambda = \bigcap_{\chi \in A} \ker \chi$ where A is a non-empty subset of $X(T)$. If A contains a character of the form $a\eta + \mu$ for some non-zero $\mu \in X(T^\lambda)$ then $T^\lambda \subseteq \ker \eta \cap \ker \mu$. But then

$\dim T^\lambda \leq l - 2$ because η and μ are linearly independent in $X_{\mathbb{Q}}(T)$. Since this is false, it must be that $A \subseteq \mathbb{Z}\eta$. As a result, $T^\lambda = \ker \eta$.

The above argument is characteristic-free since η can be described as the unique, up to a sign, primitive element of $X(T)$ proportional to λ in $X_{\mathbb{Q}}(T)$, which we identify with $Y_{\mathbb{Q}}(T)$ by means of our W-invariant inner product. In view of (3.1) we may regard η as one of the standard generators of the Laurent polynomial ring $\mathbb{C}[T]$. This implies that $\eta - 1 \in \mathbb{Z}[T]$ generates a prime ideal of $\mathbb{C}[T]$, thus showing that $T^\lambda = \ker \eta$ is defined over $\mathbb{Z}$. This enables us to construct the desired subgroup scheme $\mathfrak{T}^\lambda$ of $\mathfrak{T}$.

The inclusion $\mathfrak{T}^\lambda \subset \mathfrak{T}$ induces a homomorphism of root data, and by [Jantzen, 1987, Proposition II.1.15] (and the proof) there exists an injective homomorphism of $\mathbb{Z}$-group schemes $\iota : \tilde{\mathfrak{L}}^\perp \hookrightarrow \mathfrak{L}$ which agrees on the root subgroups. We may therefore take $\mathfrak{L}^\perp$ to be the functor defined by $A \mapsto \iota(\tilde{\mathfrak{L}}^\perp)(A)$ for any $\mathbb{Z}$-algebra A. We know that this gives rise precisely to $L^\perp(\lambda)$ since the restriction of the functor ι to the root subgroups determines it uniquely by [Jantzen, 1987, II.1.3(10)]. $\quad\square$

3.5 Unipotent pieces in arbitrary characteristic

3.5.1 We will need the following result, due to H. Kraft, during the proof of our next theorem. This was not published by Kraft but the details can be found in [Hesselink, 1978]; see Theorem 11.3 and the remarks in §12. Let (e, h, f) be an $\mathfrak{sl}_2$-triple of $\mathfrak{g}'$ and assume that we have the usual grading on $\mathfrak{g}'$ given by $\mathfrak{g}'(i) = \{x \in \mathfrak{g}' \mid [h, x] = ix\}$ for all $i \in \mathbb{Z}$. Let $\rho : \mathbb{C}^\times \to (\mathrm{Aut}\,\mathfrak{g}')^\circ$ be defined by $\rho(\xi)x = \xi^i x$ if $x \in \mathfrak{g}'(i)$. It follows that there is a one parameter subgroup $\lambda' \in Y(G')$ such that $\rho = \mathrm{Ad} \circ \lambda'$. We then say that λ' is *adapted to* e. (For full details see [Springer and Steinberg, 1970, §E, p. 238].) If $\nu \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$, then we define $\nu_* \in Y(G')$ by composing ν with the map $\xi \mapsto \left[\begin{smallmatrix} \xi & \\ & \xi^{-1} \end{smallmatrix}\right]$.

Theorem. (H. Kraft, unpublished) *The following are true.*

(i) *Let $e \in \mathfrak{g}'_{\mathrm{nil}}$ and assume that $\lambda' \in Y(G')$ is a one parameter subgroup adapted to e. Then $\frac{1}{2}\lambda' \in \tilde{\Delta}_e$.*

(ii) *Let $u \in G'_{\mathrm{uni}}$ and assume that we have $\nu \in \mathrm{Hom}(\mathrm{SL}_2(\mathbb{C}), G')$ such that $\nu\left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix}\right] = u$. Then $\frac{1}{2}\nu_* \in \tilde{\Delta}_u$.*

3.5.2 We now turn our attention to the conjugation action of G on itself, that is we assume that $V = G_{\mathrm{uni}}$ and $* = 1_G$. Recall the subsets $X^\triangle$ ($\triangle \in D_G$) and $H^\blacktriangle$ ($\blacktriangle \in D_G/G$) introduced in Subsection 3.1.2.

Lemma. *Each set $\tilde{H}^\blacktriangle$ is a closed irreducible variety stable under the conjugation action of G.*

Proof. It is clear that the set $\tilde{H}^\blacktriangle$ is G-stable. To see that it is closed, consider the set

$$\mathcal{S} = \left\{ (gG_0^\triangle, x) \mid g^{-1}xg \in G_2^\triangle \right\} \subset G/G_0^\triangle \times \tilde{H}^\blacktriangle.$$

If we show that $\mathcal{S}$ is closed, then $\tilde{H}^\blacktriangle$ is closed since it is the image under the second projection of a closed set, and $G/G_0^\triangle$ is a complete variety. In fact it is sufficient to show that $\mathcal{S}' = \{(g,x) \mid g^{-1}xg \in G_2^\triangle \}$ is closed in $G \times G$. Indeed, $\mathcal{S}$ is isomorphic to the image of $\mathcal{S}'$ under the quotient map $\eta : G \times G \to G/G_0^\triangle \times G$ and it is explained in [Steinberg, 1974, p. 67], for instance, that η maps closed subsets of $G \times G$ consisting of complete cosets of $G_0^\triangle \times \{1_G\}$ to closed subsets of $G/G_0^\triangle \times G$. The set $\mathcal{S}'$ is closed as it is the inverse image of $G_2^\triangle$ under the conjugation morphism $G \times G \to G$. Finally, the set $\tilde{H}^\blacktriangle$ is irreducible since the product map $G \times G_2^\triangle \to \tilde{H}^\blacktriangle$ is a surjective morphism from an irreducible variety. $\qquad\square$

Next we show that the sets from Subsection 3.1.2 defined by Lusztig are precisely the sets from Subsection 3.2.7 defined by Hesselink.

Theorem. *The following are true.*

  (i) *The sets $G_2^\triangle$ ($\triangle \in D_G$) are the saturated sets of G_{uni}.*

  (ii) *The sets $H^\blacktriangle$ ($\blacktriangle \in D_G/G$) are the strata of G_{uni}.*

  (iii) *The sets $X^\triangle$ ($\triangle \in D_G$) are the blades of G_{uni}.*

Furthermore, if $\tilde{\Delta}_G$ denotes the subset of $Y(G)$ consisting of elements which are in some $\tilde{\Delta}_X$, for a uniformly unstable set X, then $\tilde{\Delta}_G = \frac{1}{2}\tilde{D}_G$.

Proof. Let $\triangle \in D_G$, and assume that $\mu \in Y(G)$ is associated to $\triangle$ under the natural map described in Subsection 3.1.2. Assume that $\omega \in Y(G')$ comes from the

same $\mathbb{Z}$-scheme theoretic one parameter subgroup of $\mathfrak{T}$ as μ. (Then $G\mu$ is identified with $G'\omega$ under the canonical bijection $Y(G)/G \leftrightarrow Y(G')/G'$.) So there exists $\tilde{\omega} \in \operatorname{Hom}(\operatorname{SL}_2(\mathbb{C}), G')$ such that $\tilde{\omega}_* = \omega$, as in (1.2). Now let $u' = \tilde{\omega} \left[\begin{smallmatrix} 1 & 1 \\ & 1 \end{smallmatrix} \right] \in G'$. Then $\frac{1}{2}\omega \in \tilde{\Delta}_{u'}$ by Theorem 3.5.1(ii).

Recall that $U(\omega)$ is the unipotent radical of $(G')_0^\triangle = L(\omega)$ and let $U_k(\omega)$ have the same meaning as in Subsection 3.3.6. Let $\bar{u}'$ denote the image of u' in $V_2(\omega) = U_2(\omega)/U_3(\omega)$. Recall that $V_2(\omega) \cong \mathfrak{g}'(\omega, 2)$ as $L^\perp(\omega)$-modules; see Remark 3.3.6. By Theorem 3.3.6 the vector $\bar{u}'$ is $L^\perp(\omega)$-semistable. Since $V_2(\omega) \cong \mathfrak{g}'(\omega, 2)$ and the action on it by $L^\perp(\omega)$ are defined over $\mathbb{Z}$ there exists an affine scheme $\mathcal{V}_2(\omega)_{ss}$, acted on by $\mathfrak{L}^\perp$, such that $\mathcal{V}_2(\omega)_{ss}(\mathbb{C}) = V_2'(\omega)_{ss}$. (One should keep in mind here that $L^\perp(\omega) = \mathfrak{L}^\perp(\mathbb{k})$ thanks to Proposition 3.4.3.) Since $\bar{u}' \in V_2(\omega)$, applying Theorem 3.4.2 shows that $\mathcal{V}_2(\omega)_{ss}$ has content over any algebraically closed field. So over $\mathbb{k}$, there exists $\bar{u} \in V_2(\mu) \cong \mathfrak{g}(\mu, 2)$ which is $L^\perp(\mu)$-semistable. Let u be a preimage of $\bar{u}$ in $U_2(\mu)$. By applying Theorem 3.3.6 again we see that μ is optimal for u. Also, since $\frac{1}{2}\omega \in \tilde{\Delta}_{u'}$, we see that $\frac{1}{2}\mu \in \tilde{\Delta}_u$. Hence $G_2^\triangle = U_2(\mu)$ is a saturated set.

Conversely, suppose that S is a non-trivial saturated set in G_{uni}. We may assume that $S = S(u)$ for some unipotent element $u \neq 1_G$; see Lemma 3.2.6(ii), for example. Let $\lambda \in \Delta_u$ and $k = m(\lambda, u)$. Then $S = U_k(\lambda)$. Replacing u by a G-conjugate we may assume further that $\lambda \in Y(T)$. As before, we identify $Y(T)$ and $Y(T')$. Let $\bar{u}$ denote the image of u in $V_k(\lambda) = U_k(\lambda)/U_{k+1}(\lambda)$. Theorem 3.3.6 then implies that $\bar{u} \in V_k(\lambda)$ is $L^\perp(\lambda)$-semistable. Since $V_k(\lambda) \cong \mathfrak{g}(\lambda, k)$ as $L^\perp(\lambda)$-modules by Remark 3.3.6, we may again obtain an affine scheme $\mathcal{V}_k(\lambda)_{ss}$, defined over $\mathbb{Z}$ and acted on by $\mathfrak{L}^\perp$, such that $\mathcal{V}_k(\lambda)_{ss}(\mathbb{k}) = V_k(\lambda)$. Applying Theorem 3.4.2 we again see that $\mathcal{V}_k(\lambda)_{ss}$ has content over any algebraically closed field, and may therefore find $e' \in \mathfrak{g}'(\lambda, k)_{ss} \cong \mathcal{V}_k(\lambda)_{ss}(\mathbb{C})$; see Remark 3.3.6.

By applying Theorem 3.3.6 we see that λ is optimal and primitive for e'. Since we are now in characteristic zero, the Jacobson–Morozov theorem yields that there exist $f', h' \in \mathfrak{g}'$ such that (e', h', f') is an $\mathfrak{sl}_2$-triple. Now let $\lambda' \in \operatorname{Hom}(\operatorname{SL}_2(\mathbb{C}), G')$ be such that $\lambda'_* \in Y(G')$ is adapted to e', so that $e' \in \mathfrak{g}'(\lambda'_*, 2)$. Applying Theorem 3.5.1 we see that $\frac{1}{2}\lambda'_* \in \tilde{\Delta}_{e'}$. Hence $P(\frac{1}{2}\lambda'_*) = P(\lambda) = P(e')$. Since all maximal tori in $P(e') = L(\lambda) \cdot R_u(P(e'))$ are conjugate we can find $g \in R_u(P(e'))$ such that $\operatorname{Im}(\lambda'_*)$ and $g(\operatorname{Im}\lambda)g^{-1}$ lie in the same maximal torus, T say. Note that

$g \cdot \lambda$ is optimal for $(\mathrm{Ad}\, g)\, e' \in e' + \sum_{i>k} \mathfrak{g}'(\lambda, i)$. Applying Lemma 3.3.4 we see that $g \cdot \lambda$ is optimal for e' as well. Then $g \cdot \lambda \in \mathbb{Q}^{\times} \lambda'_*$ by Theorem 3.2.4(iii). It is well-known that $\lambda'_* \in \tilde{D}_{G'}$ (see, e.g., [Carter, 1993, Proposition 5.5.6]), hence $g^{-1} \cdot \lambda'_* \in \tilde{D}_{G'}$. But $g^{-1} \cdot \lambda'_* = \lambda$ if λ'_* is primitive and $g^{-1} \cdot \lambda'_* = 2\lambda$ otherwise. So we conclude that $\frac{2}{k}\lambda \in \tilde{D}_{G'}$ in all cases. Then, associating a suitable $\triangle \in D_G$ to $\frac{2}{k}\lambda$, we have that $S = U_2(\frac{2}{k}\lambda) = G_2^{\triangle}$. This completes the proof of (i). The claim that $\tilde{\Delta}_G = \frac{1}{2}\tilde{D}_G$ also easily follows from these arguments. Part (ii) now follows from (i) and Proposition 3.2.7. Part (iii) then follows from (i) and (ii). $\qquad\square$

3.5.3 We are now in a position to prove one of our main results.

Theorem. *Properties* $\mathfrak{P}_1$ – $\mathfrak{P}_4$ *hold for any connected reductive group over any algebraically closed field. Moreover,* $C_G(u) \subset G_{\triangle}^0$ *for any* $u \in X^{\triangle}$.

Proof. Properties $\mathfrak{P}_1$ and $\mathfrak{P}_3$ are immediate by Theorem 3.5.2 since the blades and strata are equivalence classes on G_{uni}. That the sets $X^{\triangle}$ ($\triangle \in \blacktriangle$) form a partition of $H^{\blacktriangle}$ for any $\blacktriangle \in D_G/G$ is also clear since $H^{\blacktriangle} = \bigsqcup_{\triangle \in \blacktriangle} X^{\triangle}$. Let $g \in G_3^{\triangle}$ and $u \in X^{\triangle}$. Clearly $gu \in G_2^{\triangle}$. Let $\lambda \in \Delta_u$ and let u_k be the minimal component of u with respect to λ. By the commutator relations u_k is also the minimal component of gu with respect to λ. By Theorem 3.3.6 we see that $\Delta_u = \Delta_{gu}$. Now $\|u\|$, $\|gu\|$ are determined by the minimal component with respect to (any) optimal one parameter subgroup. Hence, $\|u\| = \|gu\|$ by Lemma 3.2.6(ii), and so $gu \in H^{\blacktriangle}$ by Proposition 3.2.7(iv) and Theorem 3.5.2. Hence $G_3^{\triangle} X^{\triangle} = X^{\triangle}$. Similarly $X^{\triangle} G_3^{\triangle} = X^{\triangle}$, and so $\mathfrak{P}_4$ holds for G. Since the parabolic subgroup $G_{\triangle}^0 = P(\lambda)$ is optimal for u, Theorem 3.2.4(iv) implies that $C_G(u) \subset G_{\triangle}^0$. $\qquad\square$

3.6 The Hesselink stratification of G-modules

3.6.1 Previously we did not restrict char $\Bbbk$ but for this section and the next it will be convenient to assume that char $\Bbbk = p > 0$. As in Subsection 3.1.4 we denote by $\mathfrak{G}$ a reductive $\mathbb{Z}$-group scheme split over $\mathbb{Z}$ and write $G' = \mathfrak{G}(\mathbb{C})$ and $G = \mathfrak{G}(\Bbbk)$. Then G' and G are connected reductive groups over $\mathbb{C}$ and $\Bbbk$ respectively. Let V' be a finite-dimensional rational G'-module. Given an admissible lattice $V'_{\mathbb{Z}}$ in V' we set $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. We call V an *admissible G-module*. Since the lattice $V'_{\mathbb{Z}}$ is

stable under the action of the distribution $\mathbb{Z}$-algebra $\mathrm{Dist}(\mathfrak{G})$, the $\Bbbk$-vector space V is a module over $\mathrm{Dist}(G) = \mathrm{Dist}(\mathfrak{G}) \otimes_{\mathbb{Z}} \Bbbk$. This gives V a rational G-module structure; see [Jantzen, 1987, §II.1] for more details.

Let $\mathfrak{T}$ be a toral group subscheme of $\mathfrak{G}$ such that $T' = \mathfrak{T}(\mathbb{C})$ is a maximal torus of G' and $T = \mathfrak{T}(\Bbbk)$ is a maximal torus of G. We may and will identify the groups of rational characters $X(T')$ and $X(T)$ and their duals $Y(T')$ and $Y(T)$. The lattice $V'_{\mathbb{Z}}$ decomposes over $\mathbb{Z}$ into a direct sum $V'_{\mathbb{Z}} = \bigoplus_{\mu \in X(T)} V'_{\mathbb{Z},\mu}$ of common eigenspaces for the action of distribution algebra $\mathrm{Dist}(\mathfrak{T}) \subset \mathrm{Dist}(\mathfrak{G})$ and base-changing this direct sum decomposition we obtain the weight space decompositions $V' = \bigoplus_{\mu \in X(T)} V'_{\mu}$ and $V = \bigoplus_{\mu \in X(T)} V_{\mu}$ of V' and V with respect to T' and T respectively; see [Jantzen, 1987, II1.1(2)]. We mention for completeness that $\dim_{\mathbb{C}} V'_{\mu} = \dim_{\Bbbk} V_{\mu}$ for all $\mu \in X(T)$.

Theorem. *The following are true.*

(i) *Let $\mathcal{S}'$ and $\mathcal{S}$ denote the collections of saturated sets of V' and V associated with the one parameter subgroups in $Y(T')$ and $Y(T)$ respectively. There exists a collection $\mathfrak{S}$ of $\mathrm{Dist}(\mathfrak{T})$-stable direct summands of $V'_{\mathbb{Z}}$ such that*

$$\mathcal{S}' = \{S \otimes_{\mathbb{Z}} \mathbb{C} \mid S \in \mathfrak{S}\} \ \ and \ \ \mathcal{S} = \{S \otimes_{\mathbb{Z}} \Bbbk \mid S \in \mathfrak{S}\}.$$

(ii) *For every $S \in \mathfrak{S}$ we have that $\Delta(S \otimes_{\mathbb{Z}} \mathbb{C}) \cap Y_{\mathbb{Q}}(T') = \Delta(S \otimes_{\mathbb{Z}} \Bbbk) \cap Y_{\mathbb{Q}}(T)$.*

(iii) *The strata of V are parametrised by those of V'.*

(iv) *The parametrisation from (iii) respects the dimensions of the strata. In particular, the dimensions of the nullcones of V' and V agree.*

Proof. (i) Let $v' \in V'$ and $v \in V$ be unstable relative to T' and T respectively. Let λ' and λ be the sole elements of $\tilde{\Delta}_{v',T'}$ and $\tilde{\Delta}_{v,T}$ respectively. Then we have $S(v') = \bigoplus_{\langle \mu, \lambda' \rangle \geq 1} V'_{\mu}$ and $S(v) = \bigoplus_{\langle \mu, \lambda \rangle \geq 1} V_{\mu}$. As we mentioned earlier, for every $\mu \in X(T)$ we have that $V'_{\mu} = V_{\mu,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$ and $V_{\mu} = V_{\mu,\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Since the sets of weights of V' and V in $X(T') = X(T)$ coincide, part (i) follows.

(ii) Let $S \in \mathfrak{S}$. Our proof of part (i) and Remark 3.2.4 then show that $S = V'(\lambda)_k \cap V'_{\mathbb{Z}}$ for some $\lambda \in Y(T') = Y(T)$ and some positive integer k.

Put $\mathfrak{L}^\perp = \mathfrak{L}^\perp(\lambda)$ and consider the actions of $\mathfrak{L}^\perp(\mathbb{C})$ and $\mathfrak{L}^\perp(\Bbbk)$ on $V'(k, \lambda)$ and $V(k, \lambda)$ respectively. By Theorem 3.4.2, there is an open subscheme $\mathcal{V}(\lambda, k)_{ss}$ of $V_\mathbb{Z}(\lambda, k) = V'(k, \lambda) \cap V'_\mathbb{Z}$ with the property that $\mathcal{V}(\lambda, k)_{ss}(\mathbb{C})$ is the set of $\mathfrak{L}^\perp(\mathbb{C})$-semistable vectors of $V'(k, \lambda)$ and $\mathcal{V}(\lambda, k)_{ss}(\Bbbk)$ is the set of $\mathfrak{L}^\perp(\Bbbk)$-semistable vectors of $V(k, \lambda)$. On the other hand, Theorem 3.3.2 tells us that λ is optimal for an element in $V'(\lambda)_k$ (resp. in $V(\lambda)_k$) if, and only if, $\mathcal{V}(\lambda, k)_{ss}(\mathbb{C}) \neq \emptyset$ (resp. $\mathcal{V}(\lambda, k)_{ss}(\Bbbk) \neq \emptyset$). This shows that either both sets $\Delta(S \otimes_\mathbb{Z} \mathbb{C}) \cap Y_\mathbb{Q}(T)$ and $\Delta(S \otimes_\mathbb{Z} \Bbbk) \cap Y_\mathbb{Q}(T)$ are empty or there exists a natural number $m = m(S)$ such that

$$\Delta(S \otimes_\mathbb{Z} \mathbb{C}) \cap Y_\mathbb{Q}(T) = \Delta(S \otimes_\mathbb{Z} \Bbbk) \cap Y_\mathbb{Q}(T) = \frac{1}{m}\lambda.$$

This proves part (ii).

(iii) Consider a stratum $G'[v] \subset V'$. Without loss of generality we may assume that the blade $[v]$ is T'-unstable, since all maximal tori are conjugate in G'. Then part (ii) gives us a blade $[w] \subset V$ corresponding to $[v]$. Since all maximal tori in G are conjugate as well, part (ii), in conjunction with our discussion in Subsection 3.2.7, shows that any stratum $G[w] \subset V$ is obtained by the above construction in a unique way. Then the map $G'[v] \mapsto G[w]$ defines the required parametrisation.

(iv) With $[v] \subset V'$ and $[w] \subset V$ as above we have that

$$\dim_\mathbb{C} G'[v] = \dim_\mathbb{C} G' - \dim_\mathbb{C} P(v) + \dim_\mathbb{C} S(v)$$

and

$$\dim_\Bbbk G[w] = \dim_\Bbbk G - \dim_\Bbbk P(w) + \dim_\Bbbk S(w)$$

by [Hesselink, 1979, Proposition 4.5(c)]. It follows from part (i) that we have $\dim_\mathbb{C} S(v) = \dim_\Bbbk S(w)$, whilst the equality $\dim_\mathbb{C} P(v) = \dim_\Bbbk P(w)$ follows from the definition of $P(\lambda)$ in Section 3.2.3. Hence $\dim_\mathbb{C} G'[v] = \dim_\Bbbk G[w]$, as required.

Since the set of T'-weights of V' is finite, so is the set $\{K_T(v') \mid v' \in V'\}$. Then Lemma 3.2.5 implies that the number of $S \in \mathfrak{S}$ with $\Delta(S \otimes_\mathbb{Z} \mathbb{C}) \cap Y_\mathbb{Q}(T) \neq \emptyset$ is finite, too. In view of our earlier remarks we now get $\dim_\mathbb{C} \mathcal{N}_{V'} = \dim_\Bbbk \mathcal{N}_V$. $\quad\square$

Remark. 1. In general, different lattices $V'_\mathbb{Z}$ may give rise to non-isomorphic G-

modules. On the other hand, the theorem implies that the stratification does *not* depend on the choice of lattice and is independent of the (algebraically closed) field.

2. Let $E(\lambda)$ denote the finite dimensional irreducible G-module of highest weight $\lambda \in X(T)$. Then it is well-known that λ is a dominant weight and there exists an admissible lattice, $V''_{\mathbb{Z}}(\lambda)$, in the irreducible finite dimensional $\mathfrak{g}'$-module $V'(\lambda)$ of highest weight λ such that $E(\lambda)$ is isomorphic to a submodule of the G-module $V''_{\mathbb{k}}(\lambda) = V''_{\mathbb{Z}}(\lambda) \otimes_{\mathbb{Z}} \mathbb{k}$; see [Steinberg, 1968, §12, Exercise after Theorem 39]. If $\nu \in Y(G)$ is optimal for a G-unstable vector $v \in E(\lambda)$, then the definition in Subsection 3.2.4 shows that it remains so for v regarded as a vector of $V''_{\mathbb{k}}(\lambda)$. Therefore the Hesselink strata of $E(\lambda)$ are precisely the intersections of those of $V'_{\mathbb{k}}(\lambda)$ with $E(\lambda)$. Now Theorem 3.6.1(iii) implies the Hesselink strata of $E(\lambda)$ are parametrised by a subset of the Hesselink strata of the $\mathfrak{g}'$-module $V'(\lambda)$.

3.6.2 In this subsection we assume that $\mathbb{k}$ is an algebraic closure of $\mathbb{F}_p$. Keeping the notation of Subsection 3.4.3 we assume that $(X(T), \Sigma, Y(T), \Sigma^{\vee})$ is the root datum of the reductive group scheme $\mathfrak{G}$. Let $G = \mathfrak{G}(\mathbb{k})$ and write $x_\alpha(t)$ for Steinberg's generators of the unipotent root subgroups U_α of G; see [Steinberg, 1968]. Choose a basis of simple roots Π in Σ and denote by $Y^+(T)$ the Weyl chamber in $Y(T)$ associated with Π. (It consists of all $\mu \in Y(T)$ such that $\langle \alpha, \mu \rangle \geq 0$ for all $\alpha \in \Pi$.) Let τ be an automorphism of the lattice $X(T)$ and denote by τ^* the natural action of τ on $Y(T) = \mathrm{Hom}_{\mathbb{Z}}(X(T), \mathbb{Z})$. Assume further that τ preserves both Σ and Π and τ^* preserves $\Sigma^{\vee}$. Finally, assume that the quadratic form q from Subsection 3.2.4 is invariant under τ^*.

Now fix a pth power $q = p^l$. Then it is well-known that τ gives rise to a Frobenius endomorphism $F = F(\tau, l) \colon x \mapsto x^F$, of the algebraic $\mathbb{k}$-group $G = \mathfrak{G}(\mathbb{k})$. The endomorphism F is uniquely determined by the following properties:

1. $(\tau\eta)(x^F) = \eta(x)^q$ for all $\eta \in X(T)$ and $x \in T$;

2. $\lambda(t)^F = (\tau^*\lambda)(t^q)$ for all $\lambda \in Y(T)$ and $t \in \mathbb{k}^\times$;

3. $x_\alpha(t)^F = x_{\tau\alpha}(t^q)$ for all $\alpha \in R$ and $t \in \mathbb{k}$;

see [Digne and Michel, 1991, Theorem 3.17] for instance. Let V be an admissible G-module endowed with an action of F such that

$$g(v)^F = g^F(v^F) \quad \text{for all } g \in G \text{ and } v \in V. \tag{3.2}$$

As usual we require that the action of F is q-*linear*, that is $(\lambda v)^F = \lambda^q v^F$ for all $\lambda \in \Bbbk$ and $v \in V$, and that each vector in V is fixed by a sufficiently large power of F. In this situation one knows that the fixed point space V^F is an $\mathbb{F}_q$-form of V. In particular, $\dim_{\mathbb{F}_q} V^F = \dim_{\Bbbk} V$; see [Digne and Michel, 1991, Corollary 3.5]. We mention, for use later, that there is a natural q-linear action of F on the dual space V^*, compatible with that of G (recall that G acts on V^* via $(g \cdot \xi)(v) = \xi(g^{-1} \cdot v)$ for all $g \in G$, $\xi \in V^*$, $v \in V$). Since V^F is an $\mathbb{F}_q$-form of V, the dual space $(V^F)^*$ contains a $\Bbbk$-basis of V^*, say $\xi_1, \ldots, \xi_m$. Then every $\xi \in V^*$ can be uniquely expressed as a linear combination $\xi = \sum_{i=1}^m \lambda_i \xi_i$ with $\lambda_i \in \Bbbk$ and we can define $F \colon V^* \to V^*$ by setting $\xi^F = \sum_i \lambda_i^q \xi_i$. Verifying (3.2) for this action of F reduces to showing that $g^{-1} g^F(\xi) = \xi$ for all $\xi \in (V^F)^*$ and $g \in G$, which is clear because $(g^{-1})^F g(v) = v$ for all $v \in V^F$.

There are many reasons to be interested in the cardinality of the finite set $\mathcal{N}_V{}^F = \mathcal{N}_V \cap V^F$, and here we can offer the following general result.

Theorem. *Under the above assumptions on F and V there exists a polynomial $n_V(t) \in \mathbb{Z}[t]$ such that $|\mathcal{N}_V{}^F| = n_V(q)$ for all $q = p^l$. The polynomial $n_V(t)$ depends only on V' and τ, but not on the choice of an admissible lattice $V'_{\mathbb{Z}}$, and is the same for all primes $p \in \mathbb{N}$.*

Proof. Let $\Lambda(V)$ denote the set of pairs (λ, k) where $\lambda \in Y^+(T)$ is primitive and k is a positive integer such $\mathcal{V}(\lambda, k)_{ss}(\Bbbk) \neq \emptyset$ (the notation of Subsection 3.6.1). Set $\Lambda(V, \tau) = \{(\lambda, k) \in \Lambda(V) \mid \tau^*\lambda = \lambda\}$ and define

$$\mathcal{H}(\lambda, k) = G \cdot \left(\mathcal{V}(\lambda, k)_{ss}(\Bbbk) \oplus \bigoplus_{i > k} V(\lambda, i) \right),$$

the Hesselink stratum associated with $(\lambda, k) \in \Lambda(V)$. Now recall that we have $\mathcal{V}(\lambda, k)_{ss}(\Bbbk) = V(\lambda, k) \setminus \mathcal{N}_{V(\lambda, k)}$ where $\mathcal{N}_{V(\lambda, k)}$ is the set of all $L^{\perp}(\lambda)$-unstable

vectors of $V(\lambda, k)$. To ease notation we set

$$V(\lambda, \geq k)_{ss} = \mathcal{V}(\lambda, k)_{ss}(\Bbbk) \oplus \bigoplus_{i>k} V(\lambda, i).$$

If $\mu \in Y(G)$ is optimal for a non-zero vector $v \in \mathcal{N}_V{}^F$, then so is μ^F, forcing $P(v) = P(\mu) = P(\mu^F) = P(v)^F$. So the optimal parabolic subgroup of v is F-stable. But then $P(v)$ contains an F-stable Borel subgroup which, in turn, contains an F-stable maximal torus of G; we shall call it T_1. Since both T and T_1 are F-stable maximal tori contained in F-stable Borel subgroups of G, there is an element $g_1 \in G^F$ such that $T_1 = g_1^{-1}Tg_1$; see [Digne and Michel, 1991, 3.15]. Then $Y(T)$ contains an optimal one parameter subgroup for $g_1(v) \in V^F$, say μ_1. Lemma 3.2.5(iv) yields $\tau^*\mu_1 = \mu_1$. Since the unipotent radical $U(\mu_1)$ of $P(\mu_1)$ is contained in the Borel subgroup of G associated with our basis of simple roots Π, we see that $\mu_1 \in Y^+(T)$.

Now suppose $v \in \mathcal{H}(\lambda, k)^F$, so that $v = gw$ for some $w \in V(\lambda, \geq k)_{ss}$ and $g \in G$. Let $g_1 \in G^F$ and $\mu_1 \in Y^+(T)$ be as above (so that μ_1 is optimal for $v_1 = g_1(g'w) \in V^F$). Note that $T \subset L(\mu_1) \subset P(v_1)$. We may assume without loss of generality that μ_1 is primitive in $Y(G)$. Since w and v_1 are in the same Hesselink stratum of V it must be that $G \cdot \Delta_{v_1} = G \cdot \Delta_w$. This yields the equality $(G \cdot \mu_1) \cap Y(T) = (G \cdot \lambda) \cap Y(T)$ which, in turn, implies that that μ_1 and λ are conjugate under the action of the Weyl group W on $Y(T)$. Since both λ and μ_1 are in $Y^+(T)$, we get $\mu_1 = \lambda$.

As a result, we deduce that $\tau^*\lambda = \lambda$. Hence both $P(\lambda)$ and $V(\lambda, \geq k)_{ss}$ are F-stable. Applying [Hesselink, 1979, Proposition 4.5(b)] now yields that $g^F \in gP(w)$. We choose in G^F a set of representatives $\mathfrak{X}(\lambda, \tau, q)$ for $G^F/P(\lambda)^F$, so that

$$|\mathfrak{X}(\lambda, \tau, q)| = |G^F/P(\lambda)^F|.$$

As $P(\lambda)$ is an F-stable connected group, the Lang–Steinberg theorem shows that $g^{-1}g^F = x^{-1}x^F$ for some $x \in P(v)$; see [Digne and Michel, 1991, Theorem 3.10] for instance. Then $gx^{-1} \in P(\lambda)^F$ and hence no generality will be lost by assuming that $g \in \mathfrak{X}(\lambda, \tau, q)$.

According to [Hesselink, 1979, Proposition 4.5(b)] there is an F-equivariant bijection between the fibre product $G \times^{P(\lambda)} V(\lambda, \geq k)_{ss} \cong (G/P(\lambda)) \times V(\lambda, \geq k)_{ss}$

and the stratum $\mathcal{H}(\lambda, k)$. Since $v \in V^F$ and $g \in G^F$ we have that $g(w^F) = gw$, which shows that $w \in V(\lambda, \geq k)_{ss}{}^F$. As a consequence,

$$|\mathcal{H}(\lambda, k)| = |\mathfrak{X}(\lambda, \tau, q)| \cdot |V(\lambda, \geq k)_{ss}{}^F| = f_{\tau,\lambda}(q) \cdot q^{N(\lambda, k)} \left(q^{n(\lambda, k)} - |\mathcal{N}_{V(\lambda, k)}{}^F| \right) \tag{3.3}$$

where $f_{\tau,\lambda}(q) = |\mathfrak{X}(\lambda, \tau, q)| = |G^F/P(\lambda)^F|$, $N(\lambda, k) = \sum_{i>k} \dim V(\lambda, i)$, and $n(\lambda, k) = \dim V(\lambda, k)$.

After these preliminary remarks we are going to prove our theorem by induction on the rank of G. If $\operatorname{rank} G = 0$, then $G = \{1_G\}$ and hence $\Bbbk[V]^G = \Bbbk[V]$. Therefore $\mathcal{N}_V{}^F = \{0\}$ and we can take 1, a constant polynomial, as $n_V(t)$. Now suppose that $\operatorname{rank} G > 0$ and our theorem holds for all connected reductive groups of rank $< \operatorname{rank} G$. Since for every $\lambda \in \Lambda(V, \tau)$ we have that $\operatorname{rank} L^\perp(\lambda) < \operatorname{rank} G$ and each $L^\perp(\lambda)$-module $V(\lambda, i)$ is admissible by our discussion in Subsection 3.6.1, there exist polynomials $n_{V(\lambda,i)}(t) \in \mathbb{Z}[t]$ with coefficients independent of p and our choice of an admissible lattice $V'_\mathbb{Z}(\lambda, i)$ in $V'(\lambda, i)$ such that $|\mathcal{N}_{V(\lambda,i)}{}^F| = n_{V(\lambda,i)}(q)$.

Next we note that for every $\lambda \in Y(T)$ with $\tau^*\lambda = \lambda$ there is a polynomial $f_{\tau,\lambda} \in \mathbb{Z}[t]$ with coefficients independent of p such that $f_{\tau,\lambda}(q) = |G^F/P(\lambda)^F|$ for all p^{th} powers q and all p. Indeed, it is immediate from [Digne and Michel, 1991, Proposition 3.19(ii)] that $f_{\tau,\lambda}$ can be chosen as a quotient $a_{\tau,\lambda}/b_{\tau,\lambda}$ of two coprime polynomials $a_{\tau,\lambda}, b_{\tau,\lambda} \in \mathbb{Z}[t]$ with coefficients independent of p. Since $f_{\tau,\lambda}(q) \in \mathbb{Z}$ for infinitely many $q \in \mathbb{Z}$, it must be that $\deg b_{\tau,\lambda} = 0$. Therefore $f_{\tau,\lambda} \in \mathbb{Q}[t]$. On the other hand, G^F/P^F is the set of $\mathbb{F}_q$-rational points a smooth projective variety defined over $\mathbb{F}_p$. Applying [Goodwin and Röhrle, 2009a, Lemma 2.12] one obtains that $f_{\tau,\lambda} \in \mathbb{Z}[t]$, as stated.

Putting everything together we now get

$$
\begin{aligned}
|\mathcal{N}_V{}^F| &= 1 + \sum_{(\lambda, k) \in \Lambda(V, \tau)} |\mathcal{H}(\lambda, k)^F| \\
&= 1 + \sum_{(\lambda, k) \in \Lambda(V, \tau)} f_{\lambda, \tau}(q) \cdot q^{N(\lambda, k)} \left(q^{n(\lambda, k)} - n_{V(\lambda, i)}(q) \right).
\end{aligned}
$$

Since the data $\left\{ \big(n(\lambda, k), N(\lambda, k) \big) \mid (\lambda, k) \in \Lambda(V, \tau) \right\}$ arrives unchanged from the G'-module V' and is independent of p by Theorem 3.6.1, this is a polynomial in q with integer coefficients independent of p and the choice of admissible lattice. $\quad\square$

Remark. In the notation of Subsection 3.6.1, the distribution algebra $\mathrm{Dist}_{\mathbb{Z}}(\mathfrak{G})$ acts naturally on the $\mathbb{Z}$-algebra $\mathbb{Z}[V'_{\mathbb{Z}}]$ and we may consider the invariant algebra of this action, which coincides with $\mathbb{Z}[V'_{\mathbb{Z}}]^{\mathfrak{G}}$. According to [Seshadri, 1977, §II], the algebra $\mathbb{Z}[V'_{\mathbb{Z}}]^{\mathfrak{G}}$ is generated over $\mathbb{Z}$ by finitely many homogeneous elements. The ideal of $\mathbb{Z}[V'_{\mathbb{Z}}]$ generated by these elements defines a closed subscheme of the affine scheme $\mathrm{Spec}\,\mathbb{Z}[V'_{\mathbb{Z}}]$ which we denote by $\mathcal{N}(V'_{\mathbb{Z}})$. It follows from [Seshadri, 1977, Proposition 6(2)] that for any prime $p \in \mathbb{N}$ the nullcone $\mathcal{N}_V$ coincides with the variety of closed points of the affine $\Bbbk$-scheme $\mathcal{N}(V'_{\mathbb{Z}}) \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\,\Bbbk$. At this point Theorem 3.6.2 shows that the affine $\mathbb{Z}$-scheme $\mathcal{N}(V'_{\mathbb{Z}})$ is *strongly polynomial-count* in the terminology of N. Katz. Applying [Katz., 2008, Theorem 1(3)] we now deduce that the polynomial $n_V(t)$ from Theorem 3.6.2 is closely related with the *E-polynomial* $E(\mathcal{N}_{V'}; x, y) = \sum_{i,j} e_{i,j} x^i y^j \in \mathbb{Z}[x,y]$ of the complex algebraic variety $\mathcal{N}_{V'}$. More precisely, we have that $E(\mathcal{N}_{V'}; x, y) = n_V(xy)$ as polynomials in x, y; see [Katz., 2008, p. 618] for more details. This shows that the coefficients of $n_V(t)$ are determined by Deligne's mixed Hodge structure on the compact cohomology groups $H_c^k(\mathcal{N}_{V'}, \mathbb{Q})$.

Define $n'_V(t) = (n_V(t) - 1)/(t - 1)$. As $n'_V(q) = \mathrm{Card}\left\{ \mathbb{F}_q^{\times} v \mid v \in \mathcal{N}_V{}^F, \ v \neq 0 \right\}$ for all p^{th} powers q, it is straightforward to see that $n'_V(t)$ is a polynomial in t. The long division algorithm then shows that $n'_V(t) \in \mathbb{Z}[t]$. We conjecture that the polynomial $n'_V(t)$ has *non-negative* coefficients. This conjecture holds true for $\mathfrak{G} = \mathbf{SL}_2$ where one can compute $n'_V(t)$ explicitly for any admissible G-module V. The details are left as an exercise for the interested reader.

3.7 Nilpotent pieces in $\mathfrak{g}$ and $\mathfrak{g}^*$

3.7.1 We now define nilpotent pieces in the Lie algebra $\mathfrak{g}$ completely analogously to the definition of unipotent pieces, that is, we partition $\mathfrak{g}_{\mathrm{nil}} = \mathcal{N}_{\mathfrak{g}}$ into locally closed G-stable pieces, indexed by the unipotent classes in $G' = \mathfrak{G}(\mathbb{C})$. For convenience, we now allow $\mathrm{char}\,\Bbbk = p \geq 0$. For $\triangle \in D_G$ and $i \geq 0$ we define $\mathfrak{g}_i^{\triangle} = \mathrm{Lie}\,G_i^{\triangle}$. For any G-orbit $\blacktriangle \in D_G$, let $\tilde{H}^{\blacktriangle}(\mathfrak{g}) = \bigcup_{\triangle \in \blacktriangle} \mathfrak{g}_2^{\triangle}$. This is a closed irreducible G-stable variety by the proof of Lemma 3.1.2. We define the *nilpotent*

pieces of $\mathfrak{g}$ to be the sets

$$H^{\blacktriangle}(\mathfrak{g}) = \tilde{H}^{\blacktriangle}(\mathfrak{g}) \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'}(\mathfrak{g}),$$

where the union is taken over all $\blacktriangle' \in D_G/G$ such that $\tilde{H}^{\blacktriangle'}(\mathfrak{g}) \subsetneq \not\supseteq \tilde{H}^{\blacktriangle}(\mathfrak{g})$. We also define

$$X^{\triangle}(\mathfrak{g}) = \mathfrak{g}_2^{\triangle} \bigcap H^{\blacktriangle}(\mathfrak{g}),$$

for each $\triangle \in D_G$, where $\blacktriangle$ is the G-orbit of $\triangle$. Since $H^{\blacktriangle}(\mathfrak{g})$ is the complement of finitely many non-trivial closed subvarieties of $\tilde{H}^{\blacktriangle}(\mathfrak{g})$, it is open and dense in $\tilde{H}^{\blacktriangle}(\mathfrak{g})$, hence it is locally closed in $\mathfrak{g}_{\mathrm{nil}}$. The subset $H^{\blacktriangle}(\mathfrak{g})$ is G-stable since its complement in $\tilde{H}^{\blacktriangle}(\mathfrak{g})$ is. Consequently, $X^{\triangle}(\mathfrak{g})$ is open and dense in $\mathfrak{g}_2^{\triangle}$, and stable under the adjoint action of $G_0^{\triangle}$.

Recall from Subsections 3.5.1 and 3.5.2 that for any $\triangle \in D_G$ there is an element $g \in G$ and a one parameter subgroup $\omega \in Y(T) = Y(T')$, coming from a rational homomorphism $\mathrm{SL}_2(\mathbb{C}) \to G'$, such that $\frac{1}{2}\omega \in \tilde{\Delta}_x$ for some $x \in \mathfrak{g}'(2, \omega)$ and $\mathfrak{g}_k^{\triangle} = \bigoplus_{i \geq k} \mathfrak{g}(i, g \cdot \omega)$ for all $k \in \mathbb{Z}$. Note that different $g \in G$ with this property have the same image in $G_0^{\triangle}/G$. Given $\mu \in Y(G)$ and $i \in \mathbb{Z}$ we denote by $\mathfrak{g}^*(i, \mu)$ the subspace in $\mathfrak{g}^*$ consisting of all linear functions that vanish on each $\mathfrak{g}(j, \mu)$ with $j \neq -i$. Now define $(\mathfrak{g}^*)_k^{\triangle} = \bigoplus_{i \geq k} \mathfrak{g}^*(i, g \cdot \omega)$, for $k \in \mathbb{Z}$. The preceding remark shows that this is independent of the choice of $g \in G$ and therefore the subspaces $(\mathfrak{g}^*)_k^{\triangle}$ are well-defined.

In a completely analogous way we now define the *nilpotent pieces* of the dual space $\mathfrak{g}^*$. For any G-orbit $\blacktriangle \in D_G$, we let $\tilde{H}^{\blacktriangle}(\mathfrak{g}^*) = \bigcup_{\triangle \in \blacktriangle} (\mathfrak{g}^*)_2^{\triangle}$, a closed irreducible G-stable subset of $\mathfrak{g}^*$, and put

$$H^{\blacktriangle}(\mathfrak{g}^*) = \tilde{H}^{\blacktriangle}(\mathfrak{g}^*) \setminus \bigcup_{\blacktriangle'} \tilde{H}^{\blacktriangle'}(\mathfrak{g}^*),$$

where the union is taken over all $\blacktriangle' \in D_G/G$ with $\tilde{H}^{\blacktriangle'}(\mathfrak{g}^*) \subsetneq \not\supseteq \tilde{H}^{\blacktriangle}(\mathfrak{g}^*)$. We define

$$X^{\triangle}(\mathfrak{g}^*) = (\mathfrak{g}^*)_2^{\triangle} \bigcap H^{\blacktriangle}(\mathfrak{g}^*),$$

for each $\triangle \in D_G$. Arguing as before we observe that each $H^{\blacktriangle}(\mathfrak{g}^*)$ is a G-stable, locally closed subset of $\mathcal{N}_{\mathfrak{g}^*}$. Hence $X^{\triangle}(\mathfrak{g}^*)$ is open and dense in $\mathfrak{g}_2^{\triangle}$, and stable

under the coadjoint action of $G_0^\triangle$.

3.7.2 In the next two subsections we study the nullcone $\mathcal{N}_{\mathfrak{g}^*}$ associated with the coadjoint action of G on the dual space $\mathfrak{g}^* = \mathrm{Hom}_{\Bbbk}(\mathfrak{g}, \Bbbk)$. Recall that $(g \cdot \xi)(x) = \xi((\mathrm{Ad}\, g^{-1})x)$ for all $g \in G$, $x \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. It is immediate from the Hilbert–Mumford criterion (our Theorem 3.2.1) that $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ if and only if ξ vanishes on the Lie algebra of a Borel subgroup of G. The nilpotent linear functions $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ play an important role in the study of the centre of the enveloping algebra $U(\mathfrak{g})$ and were first investigated in our setting in [Kac and Weisfeiler, 1976]. In characteristic zero the Killing form induces a G'-equivariant isomorphism $\mathfrak{g}' \cong (\mathfrak{g}')^*$. However, in positive characteristic it may happen that $\mathfrak{g} \not\cong \mathfrak{g}^*$ as G-modules.

We first assume that the group G is simple and simply connected. Rather than study $\mathfrak{g}^*$ directly, we will present a slightly different construction which will allow us to combine Theorems 3.6.1 and 3.6.2 with classical results from [Dynkin, 1955] and [Kostant, 1959] on $\mathcal{N}_{\mathfrak{g}'}$. As before, we fix a set of simple roots Π in Σ and denote the corresponding set of positive roots by Σ^+. Let $\mathcal{C}' = \{X_\alpha, H_\beta \mid \alpha \in \Sigma, \beta \in \Pi\}$ be a Chevalley basis of $\mathfrak{g}'$ and denote by $\mathfrak{g}'_{\mathbb{Z}}$ the $\mathbb{Z}$-span of $\mathcal{C}'$ in $\mathfrak{g}$. Then the following equations hold in $\mathfrak{g}'_{\mathbb{Z}}$:

(i) $[H_\alpha, X_\beta] = \langle \beta, \alpha \rangle X_\beta$ for all $\alpha \in \Pi$, $\beta \in \Sigma$;

(ii) $[X_\beta, X_{-\beta}] = H_\beta$ for all $\beta \in \Pi$, where $H_\beta = \mathrm{d}_e \beta^\vee$ is an integral linear combination of $H_\alpha = \mathrm{d}_e \alpha^\vee$ with $\alpha \in \Pi$;

(iii) $[X_\alpha, X_\beta] = N_{\alpha,\beta} X_{\alpha+\beta}$ if $\alpha + \beta \in \Sigma$, where $N_{\alpha,\beta} = \pm(q+1)$ and q is the maximal integer for which $\beta - q\alpha \in \Sigma$;

(iv) $[X_\alpha, X_\beta] = 0$ if $\alpha + \beta \notin \Sigma$;

see [Steinberg, 1968, §1], for example. As usual, $\langle \alpha, \beta \rangle = 2(\alpha, \beta)/(\alpha, \alpha)$, where $(\ ,\ )$ is a scalar product on the $\mathbb{R}$-span of Π, invariant under the action of the Weyl group W of Σ. We may assume, by rescaling if necessary, that $(\alpha, \alpha) = 2$ for every short root α of Σ. Let $\tilde{\alpha}$ denote the maximal root, and α_0 the maximal short root in Σ^+ respectively, and set $d = (\tilde{\alpha}, \tilde{\alpha})/(\alpha_0, \alpha_0)$. Recall that a prime

$p \in \mathbb{N}$ is called *special* for Σ if $d \equiv 0 \pmod{p}$. The special primes are 2 and 3. To be precise, 2 is special for Σ of type B_ℓ, C_ℓ, $\ell \geq 2$, and F_4, whilst 3 is special for Σ of type G_2.

Since G is assumed to be simply connected, we have that $\mathfrak{g} = \operatorname{Lie} G = \mathfrak{g}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ (cf. [Borel, 1970, §2.5] or [Jantzen, 1991, §1.3]). Also, the distribution algebra $\operatorname{Dist}_{\mathbb{Z}}(\mathfrak{G})$ identifies canonically with the unital $\mathbb{Z}$-subalgebra of the universal enveloping algebra $U(\mathfrak{g}')$ generated by all $X_\beta^n / n!$ with $\beta \in \Sigma$ and $n \in \mathbb{N}$. The algebra $U_{\mathbb{Z}}$ is known as *Kostant's $\mathbb{Z}$-form* of $U(\mathfrak{g})$ and was first introduced in [Kostant, 1966]. Thus, a $\mathbb{Z}$-lattice $V'_{\mathbb{Z}}$ in a finite-dimensional $\mathfrak{g}'$-module V' is admissible if, and only if, it is invariant under all operators $X_\alpha^n / n!$ ($n \in \mathbb{N}$) under the obvious action of $U(\mathfrak{g}')$ on V'. For instance, $\mathfrak{g}'_{\mathbb{Z}}$ itself is admissible, since $\mathfrak{g}'_{\mathbb{Z}} = U_{\mathbb{Z}} \cdot X_{\tilde{\alpha}}$.

We now recall very briefly how admissible lattices give rise to rational G-modules. Let $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. Since $\operatorname{Dist}_{\Bbbk}(G) = \operatorname{Dist}_{\mathbb{Z}}(\mathfrak{G}) \otimes_{\mathbb{Z}} \Bbbk = U_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$, the action of $U_{\mathbb{Z}}$ on $V'_{\mathbb{Z}}$ gives rise to a representation of $\operatorname{Dist}_{\Bbbk}(G)$ on $\operatorname{End}_{\Bbbk} V$, and hence to a rational linear action of G on V; see [Jantzen, 1987, §§II.1.12, II.1.20] for more details. Given $X \in U_{\mathbb{Z}}$ we denote the induced linear transformations on $V'_{\mathbb{Z}}$ and V by $\rho_{\mathbb{Z}}(X)$. We then define invertible linear transformations $x_\beta(t) = \sum_{n \geq 0} t^n \rho_{\mathbb{Z}}(X_\beta^n / n!)$ on V, for each $\beta \in \Sigma$, where $t \in \Bbbk$. (Note that the sum is finite since the X_β act nilpotently on V'.) The set $\{ x_\beta(t) \,|\, \beta \in \Sigma,\, t \in \Bbbk \}$ generates a Zariski-closed, connected subgroup $G(V)$ of $\operatorname{GL}(V)$. Since G is simply connected and hence a universal Chevalley group in the sense of [Steinberg, 1968], the linear group $G(V)$ is a homomorphic image of G. For any admissible lattice $V'_{\mathbb{Z}}$ in a finite-dimensional $\mathfrak{g}'$-module V', we thus obtain a G-module structure on $V = V'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$.

Now we define a symmetric bilinear form $\langle\ ,\ \rangle \colon \mathfrak{g}'_{\mathbb{Z}} \times \mathfrak{g}'_{\mathbb{Z}} \to \mathbb{Z}$ by setting

$$
\begin{aligned}
\langle X_\alpha, X_\beta \rangle &= 0 & &\text{if } \alpha + \beta \neq 0, \\
\langle H_\alpha, H_\beta \rangle &= \frac{4d(\alpha, \beta)}{(\alpha, \alpha)(\beta, \beta)} & &\text{for all } \alpha, \beta \in \Sigma, \\
\langle X_\alpha, X_{-\alpha} \rangle &= \frac{2d}{(\alpha, \alpha)} & &\text{for all } \alpha \in \Sigma,
\end{aligned}
$$

and extending to $\mathfrak{g}'_{\mathbb{Z}}$ by $\mathbb{Z}$-bilinearity. Note that this is well-defined, since the condition $(\alpha_0, \alpha_0) = 2$ ensures that the image is indeed in $\mathbb{Z}$; see the tables in

[Bourbaki, 1975]. Obviously we may extend $\langle\ ,\ \rangle$ to symmetric bilinear forms $\langle\ ,\ \rangle_{\mathbb{C}}$ on $\mathfrak{g}' = \mathfrak{g}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$, and $\langle\ ,\ \rangle_{\Bbbk}$ on $\mathfrak{g} = \mathfrak{g}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$.

In is proved in [Premet, 1997, p. 240] that the bilinear form $\langle\ ,\ \rangle_{\mathbb{C}}$ is a scalar multiple of the Killing form κ of $\mathfrak{g}' = \operatorname{Lie} G'$. In particular, $\langle\ ,\ \rangle_{\mathbb{C}}$ is G'-invariant. This, in turn, implies that

$$\langle X(u), v \rangle = \langle u, X^{\top}(v) \rangle \quad \text{for all} \ \ u, v \in V'_{\mathbb{Z}} \ \text{ and } \ X \in U_{\mathbb{Z}}, \tag{3.4}$$

where $\top$ stands for the canonical anti-automorphism of $U(\mathfrak{g})$. Since $x^{\top} = -x$ for all $x \in \mathfrak{g}'$, it is straightforward to see that $\top$ preserves the $\mathbb{Z}$-form $U_{\mathbb{Z}}$ of $U(\mathfrak{g}')$. (In fact, the map $\top \colon U_{\mathbb{Z}} \to U_{\mathbb{Z}}$ is nothing but the antipode of the Hopf algebra $U_{\mathbb{Z}} = \operatorname{Dist}_{\mathbb{Z}}(\mathfrak{G})$.) As a consequence, the bilinear form $\langle\ ,\ \rangle_{\Bbbk}$ on $\mathfrak{g} = \operatorname{Lie} G$ is G-invariant.

Lemma. *If p is non-special for Σ, then the radical of $\langle\ ,\ \rangle_{\Bbbk}$ coincides with the centre $\mathfrak{z}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. If p is special for $\mathfrak{g}$, then $\operatorname{Rad} \langle\ ,\ \rangle_{\Bbbk} \nsubseteq \mathfrak{z}(\mathfrak{g})$.*

Proof. The first statement of the lemma is [Premet, 1997, Lemma 2.2(ii)]. For the second statement, we note that the image of X_{α_0} in $\mathfrak{g} = \left(\mathfrak{g}'_{\mathbb{Z}}/p\mathfrak{g}'_{\mathbb{Z}}\right) \otimes_{\mathbb{F}_p} \Bbbk$ lies in the radical of $\langle\ ,\ \rangle_{\Bbbk}$, but not in the centre of $\mathfrak{g}$. (Recall that G is assumed to be simply connected.) $\qquad\square$

The lemma hints at the fact that $\mathfrak{g}$ and $\mathfrak{g}^*$ are similar as G-modules if p is non-special, but very different if p is special. Nevertheless, as we will see, we may construct an alternative admissible lattice $\mathfrak{g}''_{\mathbb{Z}} \subset \mathfrak{g}'$ which gives rise to another G-module $\mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ such that $\langle\ ,\ \rangle$ induces a non-degenerate pairing between $\mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{g}$ in all cases. This will enable us to identify the G-modules $\mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{g}^*$.

3.7.3 We define $\mathfrak{g}''_{\mathbb{Z}} = \{x \in \mathfrak{g}' \mid \langle x, y \rangle \in \mathbb{Z}, \ \forall\ y \in \mathfrak{g}_{\mathbb{Z}}\}$, a $\mathbb{Z}$-lattice in $\mathfrak{g}'$. It is immediate from (3.4) that $\mathfrak{g}''_{\mathbb{Z}}$ is an admissible lattice. Consequently, we obtain a G-module structure on the vector space $\mathfrak{g}'_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$. We also obtain a G-invariant pairing

$$\langle\ ,\ \rangle_{\Bbbk}^* \colon \ \mathfrak{g} \times \left(\mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk\right) \ \longrightarrow \ \Bbbk. \tag{3.5}$$

We will now exhibit a basis of $\mathfrak{g}''_{\mathbb{Z}}$ dual to our Chevalley basis $\mathcal{C}'$, with respect to $\langle\ ,\ \rangle$. Thus, we will show that the pairing $\langle\ ,\ \rangle_{\Bbbk}^*$ is non-degenerate. Let $\mathfrak{t}'$ be the

Cartan subalgebra of $\mathfrak{g}'$ spanned by $\{H_\alpha \mid \alpha \in \Pi\}$. Let $\{H'_\alpha \mid \alpha \in \Pi\}$ be the dual basis of $\mathfrak{t}'$ with respect to the restriction of $\langle \ , \ \rangle_\mathbb{C}$ to $\mathfrak{t}'$. (These may be thought of as the fundamental weights of the dual root system $\Sigma^\vee$.) This extends to a basis

$$\mathcal{C} = \{H'_\alpha \mid \alpha \in \Pi\} \bigsqcup \{X_\beta \mid \beta \in \Sigma \text{ long}\} \bigsqcup \{(1/d)X_\beta \mid \beta \in \Sigma \text{ short}\}$$

of $\mathfrak{g}$ which is dual to our Chevalley basis $\mathcal{C}'$ with respect to $\langle \ , \ \rangle_\mathbb{C}$. Specifically, the corresponding pairing of basis elements is as follows:

$$
\begin{array}{lll}
H_\alpha & \leftrightarrow \quad H'_\alpha & \text{if } \alpha \in \Pi, \\
X_\beta & \leftrightarrow \quad X_{-\beta} & \text{if } \beta \in \Sigma \text{ is long}, \\
X_\beta & \leftrightarrow \quad (1/d)X_{-\beta} & \text{if } \beta \in \Sigma \text{ is short}.
\end{array}
$$

Moreover, it is easy to check that $\mathcal{C}$ is a $\mathbb{Z}$-basis of $\mathfrak{g}''_\mathbb{Z}$, as required. Since the lattice $\mathfrak{g}''_\mathbb{Z}$ is admissible, we see that the bases $\mathcal{C}' \otimes 1$ of $\mathfrak{g} = \mathfrak{g}'_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk$ and $\mathcal{C} \otimes 1$ of $\mathfrak{g}''_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk$ are dual to each other with respect to $\langle \ , \ \rangle_\Bbbk^*$. This shows that $\mathfrak{g}$ and $\mathfrak{g}^* \cong \mathfrak{g}''_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk$ are admissible G-modules associated with different admissible lattices in $\mathfrak{g}'$.

Now suppose that G is semisimple and simply connected. Then G is a direct product of simple, simply connected groups and the above arguments carry over to G in a straightforward fashion. In particular, (3.5) is still available for a suitable choice of an admissible lattice $\mathfrak{g}''_\mathbb{Z} \subset \mathfrak{g}'$ and $\mathfrak{g}^* \cong \mathfrak{g}''_\mathbb{Z} \otimes_\mathbb{Z} \Bbbk$ as G-modules.

Theorem. *Let G be a connected reductive group over an algebraically closed field $\Bbbk$ of characteristic $p \geq 0$ and let $\mathcal{G}$ be $\mathfrak{g}$ or $\mathfrak{g}^*$. If $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$, assume further that we have a Frobenius endomorphism $F \colon G \to G$ corresponding to an $\mathbb{F}_q$-rational structure of G. Then $\mathfrak{P}_1$– $\mathfrak{P}_5$ hold for $\mathcal{G}$ and the stabiliser G_x of any element $x \in X^\triangle(\mathcal{G})$ is contained in the parabolic subgroup $G_0^\triangle$ of G.*

Proof. Let U be an F-stable maximal connected unipotent subgroup of G. It follows from the Hilbert–Mumford criterion (our Theorem 3.2.1) that $\mathcal{N}_\mathfrak{g} = (\mathrm{Ad}\, G) \cdot \mathfrak{u}$ where $\mathfrak{u} = \mathrm{Lie}\, U$. Since $U \subset \mathcal{D}G$, we have that $\mathcal{N}_\mathfrak{g} \subseteq \mathcal{N}_{\bar{\mathfrak{g}}}$ where $\bar{\mathfrak{g}} = \mathrm{Lie}\, \mathcal{D}G$. As any $\xi \in \mathcal{N}_{\mathfrak{g}^*}$ vanishes on a Borel subalgebra of $\mathfrak{g}$, the restriction map $\mathfrak{g}^* \to \bar{\mathfrak{g}}^*$, $\xi \mapsto \xi|_{\bar{\mathfrak{g}}}$, induces a G-equivariant injection $\eta \colon \mathcal{N}_{\mathfrak{g}^*} \to \mathcal{N}_{\bar{\mathfrak{g}}^*}$. But η is, in fact, a

bijection since every linear function on $\mathfrak{u}$ can be extended to a nilpotent linear function on $\mathfrak{g}$.

Let $\tilde{G}$ be a semisimple, simply connected group isogeneous to $\mathcal{D}G$. Let $\iota\colon \tilde{G} \to \mathcal{D}G$ be an isogeny and let $\tilde{U}$ be the connected unipotent subgroup of $\tilde{G}$ with $\iota(\tilde{U}) = U$. Let $\tilde{\mathfrak{g}} = \operatorname{Lie}\tilde{G}$ and $\tilde{\mathfrak{u}} = \operatorname{Lie}\tilde{U}$. Then $\mathrm{d}_e\iota\colon \tilde{\mathfrak{g}} \to \bar{\mathfrak{g}}$ maps $\tilde{\mathfrak{u}}$ isomorphically onto $\mathfrak{u}$ and induces a $\tilde{G}$-equivariant bijection between $\mathcal{N}_{\tilde{\mathfrak{g}}}$ and $\mathcal{N}_{\bar{\mathfrak{g}}} = \bar{\mathfrak{g}}_{\mathrm{nil}}$. Let $\tilde{T}$ be a maximal torus of $\tilde{G}$ normalising $\tilde{\mathfrak{u}}$ and $T = \iota(\tilde{T})$, a maximal torus of G normalising $\mathfrak{u}$. We regard $\mathfrak{u}^*$ and $\tilde{\mathfrak{u}}^*$ as subspaces of $\bar{\mathfrak{g}}^*$ and $\tilde{\mathfrak{g}}^*$ respectively, by imposing that every $\xi \in \mathfrak{u}^*$ vanishes on the T-invariant complement of $\mathfrak{u}$ in $\mathfrak{g}$ and every $\tilde{\xi} \in \tilde{\mathfrak{u}}^*$ vanishes on the $\tilde{T}$-invariant complement of $\tilde{\mathfrak{u}}$ in $\tilde{\mathfrak{g}}$. Then the linear map $(\mathrm{d}_e\iota)^*\colon \bar{\mathfrak{g}}^* \to \tilde{\mathfrak{g}}^*$ induced by $\mathrm{d}_e\iota$ restricts to a linear isomorphism between $\mathfrak{u}^*$ and $\tilde{\mathfrak{u}}^*$. Since the map $(\mathrm{d}_e\iota)^*$ is $\tilde{G}$-equivariant, it induces a natural bijection between $\mathcal{N}_{\tilde{\mathfrak{g}}^*} = (\operatorname{Ad}^*\tilde{G}) \cdot \tilde{\mathfrak{u}}^*$ and $\mathcal{N}_{\mathfrak{g}^*} = (\operatorname{Ad}^*G) \cdot \mathfrak{u}^*$. It is clear from our description of F in Subsection 3.6.2 that there is a Frobenius endomorphism $\tilde{F}\colon \tilde{G} \to \tilde{G}$ such that $\iota \circ \tilde{F} = F|_{\mathcal{D}G}$. Furthermore, $\tilde{T}$ and $\tilde{U}$ can be chosen to be $\tilde{F}$-stable.

The above discussion shows that in proving the theorem we may assume that the group G is semisimple and simply connected. Then both $\mathfrak{g}$ and $\mathfrak{g}^*$ are admissible G-modules. More precisely, $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ and $\mathfrak{g}^* = \mathfrak{g}''_{\mathbb{Z}} \otimes_{\mathbb{Z}} \Bbbk$ for some admissible lattices $\mathfrak{g}'_{\mathbb{Z}}$ and $\mathfrak{g}''_{\mathbb{Z}}$ in $\mathfrak{g}'$. Then Theorem 3.6.1 shows that $\mathfrak{P}_1 - \mathfrak{P}_3$ for $\mathfrak{g}^*$ can be reduced to showing the corresponding statements for $\mathfrak{g}$, together with [Hesselink, 1978, Proposition 4.5], which shows that for every $\blacktriangle \in D_G/G$ there is a surjective morphism $H^{\blacktriangle} \to G/G_0^{\triangle}$ whose fibres are exactly the blades $X^{\triangle}$ with $\triangle \in \blacktriangle$.

The proof of $\mathfrak{P}_1 - \mathfrak{P}_3$ for $\mathfrak{g}$ is completely analogous to the proof of Theorem 3.5.2. Of course it is much easier since we may use Tsujii's result (Theorem 3.3.2) in its original form, and there is no need for Section 3.3.

In order to show that $\mathfrak{P}_4$ holds for $\mathcal{G}$ it suffices to establish that for every $x \in X^{\triangle}(\mathcal{G})$ the optimal parabolic subgroup $P(x)$ coincides with $G_{\triangle}^0$. Also, the inclusion $G_x \subset G_{\triangle}^0$ follows from Theorem 3.2.4(iv).

It remains to show that $\mathfrak{P}_5$ holds for $\mathcal{G}$, so suppose from now on that $\Bbbk$ is an algebraic closure of $\mathbb{F}_p$ and $F = F(\tau, l)$ where $q = p^l$; see Subsection 3.6.2. As explained there, we have a natural q-linear action of F on $\mathfrak{g}^*$ compatible with the

coadjoint action of G. We adopt the notation introduced in the course of proving Theorem 3.6.2. It follows from Theorem 3.5.1 that the set $\Lambda(\mathfrak{g}, \tau) = \Lambda(\mathfrak{g}^*, \tau)$ consists of all pairs $(\lambda'_{\blacktriangle}, k)$ such that $\lambda'_{\blacktriangle} \in Y^+(T)$ is primitive, $k \in \{1, 2\}$ and $\frac{2}{k}\lambda'_{\blacktriangle}$ is adapted by a suitable nilpotent element in the adjoint G'-orbit labelled by $\blacktriangle$. Then (3.3) yields

$$
\begin{aligned}
\varphi^{\blacktriangle}_{\mathcal{G}}(q) &= |H^{\blacktriangle}(\mathcal{G})^F| = f_{\tau, \lambda'_{\blacktriangle}}(q) \cdot q^{N(\lambda'_{\blacktriangle}, k)}\left(q^{n(\lambda'_{\blacktriangle}, k)} - |\mathcal{N}_{\mathcal{G}(\lambda'_{\blacktriangle}, k)}{}^F|\right) \\
&= f_{\tau, \lambda'_{\blacktriangle}}(q) \cdot q^{N(\lambda'_{\blacktriangle}, k)}\left(q^{n(\lambda'_{\blacktriangle}, k)} - n_{\mathcal{G}(\lambda'_{\blacktriangle}, k)}(q)\right).
\end{aligned}
$$

If $\triangle \in \blacktriangle$ is such that $F(G^{\triangle}_i) = G^{\triangle}_i$ for all $i \geq 0$, then the proof of Theorem 3.6.2 also yields that $\tau^*(\lambda'_{\blacktriangle}) = \lambda'_{\blacktriangle}$ and

$$
\psi^{\triangle}_{\mathcal{G}}(q) = |X^{\triangle}(\mathcal{G})^F| = q^{N(\lambda'_{\blacktriangle}, k)}\left(q^{n(\lambda'_{\blacktriangle}, k)} - n_{\mathcal{G}(\lambda'_{\blacktriangle}, k)}(q)\right).
$$

As the $L^{\perp}(\lambda'_{\blacktriangle})$-modules $\mathfrak{g}(\lambda'_{\blacktriangle}, k)$ and $\mathfrak{g}^*(\lambda'_{\blacktriangle}, k)$ come from different admissible lattices of the $(\mathfrak{L}^{\perp}(\lambda'_{\blacktriangle}))(\mathbb{C})$-module $\mathfrak{g}'(\lambda'_{\blacktriangle}, k)$, applying Theorem 3.6.2 shows that $\psi^{\triangle}_{\mathfrak{g}}(q) = \psi^{\triangle}_{\mathfrak{g}^*}(q)$ are polynomials in q with integer coefficients independent of p. This, in turn, implies that so are $\varphi^{\blacktriangle}_{\mathfrak{g}}(q) = \varphi^{\blacktriangle}_{\mathfrak{g}^*}(q)$, completing the proof. $\qquad\square$

Corollary. *Let G be a connected reductive group defined over an algebraic closure of $\mathbb{F}_p$ and assume that we have a Frobenius endomorphism $F\colon G \to G$ corresponding to an $\mathbb{F}_q$-rational structure on G. Then $\mathfrak{P}_5$ holds for G.*

Proof. Let $\triangle \in D_G$ be such $F(G^{\triangle}_i) = G^{\triangle}_i$ for all $i \geq 0$ and let $\blacktriangle$ be the orbit of $\triangle$ in D_G/G. Then $gG^{\triangle}_0 g^{-1} = P(\lambda'_{\blacktriangle})$ and $gG^{\triangle}_i g^{-1} = U_i(\lambda'_{\blacktriangle})$ for some $g \in G$, where $i \geq 1$. If s is the order of τ^*, then there exists $r \in \mathbb{N}$ with $r \equiv 1 \pmod{s}$ such that $X^{\triangle}(G)^{F^r} \neq \emptyset$. Then $H^{\blacktriangle}(G)^{F^r} \neq \emptyset$ and the argument used in the proof of Theorem 3.6.2 shows that $\tau^*(\lambda'_{\blacktriangle}) = \tau^{*r}(\lambda'_{\blacktriangle}) = \lambda'_{\blacktriangle}$. Since $\tau^*(\lambda'_{\blacktriangle}) = \tau^{*r}(\lambda^*_{\blacktriangle})$ by our choice of r, we see that $P(\lambda'_{\blacktriangle})$ is F-stable. Hence $g^F G^{\triangle}_0 (g^F)^{-1} = gG_0 g^{-1}$ forcing $g^{-1}g^F \in N_G(G^{\triangle}_2) = G^{\triangle}_0$. As $G^{\triangle}_0$ is connected and F-stable, the Lang–Steinberg theorem shows that $g^{-1}g^F = x^{-1}x^F$ for some $x \in G^{\triangle}_0$; see [Digne and Michel, 1991, Theorem 3.10]. Replacing g by gx^{-1} we thus may assume that $g \in G^F$. In conjunction with Theorems 3.3.6 and 3.5.2 this shows that

$$
\left|X^{\triangle}(G)^F\right| = \left|\pi^{-1}\left(V_2(\lambda'_{\blacktriangle})_{ss}{}^F\right)\right| \tag{3.6}
$$

where $V_2(\lambda'_\blacktriangle)_{ss}$ stands for the set of all $L^\perp(\lambda'_\blacktriangle)$-semistable vectors of the $L(\lambda'_\blacktriangle)$-module $V_2(\lambda'_\blacktriangle) = U_2(\lambda'_\blacktriangle)/U_3(\lambda'_\blacktriangle)$ and $\pi\colon U_2(\lambda'_\blacktriangle)^F \to V_2(\lambda'_\blacktriangle)^F$ is the map induced by the canonical homomorphism $U_2(\lambda'_\blacktriangle) \twoheadrightarrow V_2(\lambda'_\blacktriangle)$. Now the argument used in the proof of Theorem 3.6.2 yields

$$\left|H^\blacktriangle(G)^F\right| = \left|G^F/P(\lambda'_\blacktriangle)^F\right| \cdot \left|\pi^{-1}\big(V_2(\lambda'_\blacktriangle)_{ss}^F\big)\right|. \tag{3.7}$$

In view of Remark 3.3.6 we have that

$$\left|V_2(\lambda'_\blacktriangle)_{ss}^F\right| = \left|\mathfrak{g}(\lambda'_\blacktriangle, 2)_{ss}^F\right|. \tag{3.8}$$

Since the group $U_3(\lambda'_\blacktriangle)$ is connected and F-stable, the Lang–Steinberg theorem shows that for every $v \in V_2(\lambda'_\blacktriangle)_{ss}^F$ there is an element $\tilde{v} \in V_2(\lambda'_\blacktriangle)_{ss}^F$ such that $\pi(\tilde{v}) = v$. From this it is immediate that

$$\pi^{-1}(v) = \tilde{v} \cdot U_3(\lambda'_\blacktriangle)^F \qquad \big(\forall\, v \in V_2(\lambda'_\blacktriangle)_{ss}^F\big). \tag{3.9}$$

Combining (3.6), (3.8) and (3.9) we obtain that

$$\left|X^\vartriangle(G)^F\right| = \left|\pi^{-1}\big(V_2(\lambda'_\blacktriangle)_{ss}^F\big)\right| = \left|\mathfrak{g}(\lambda'_\blacktriangle, 2)_{ss}^F\right| \cdot \left|U_3(\lambda'_\blacktriangle)^F\right|. \tag{3.10}$$

As we know by Remark 3.3.6, for each $i \geq 3$ the connected abelian group $V_i(\lambda'_\blacktriangle) = U_i(\lambda'_\blacktriangle)/U_{i+1}(\lambda'_\blacktriangle)$ is a vector space over $\Bbbk$ isomorphic to $\mathfrak{g}(\lambda_\blacktriangle, i)$. Since $\tau^*\lambda'_\blacktriangle = \lambda'_\blacktriangle$, it is equipped with a q-linear action of F. Therefore

$$\left|V_i(\lambda'_\blacktriangle)^F\right| = q^{\dim \mathfrak{g}(\lambda'_\blacktriangle, i)}, \quad i \geq 3; \tag{3.11}$$

see [Digne and Michel, 1991, Corollary 3.5], for example. Since every group $U_i(\lambda'_\blacktriangle)$ with $i \geq 3$ is connected and F-stable, the Lang–Steinberg theorem yields that for every $u \in V_i(\lambda'_\blacktriangle)^F$ there exists $\tilde{u} \in U_i(\lambda'_\blacktriangle)^F$ whose image in $V_i(\lambda'_\blacktriangle)^F$ equals u. This, in turn, implies that every quotient $V_i(\lambda'_\blacktriangle)^F$ with $i \geq 3$ has a section in $U_i(\lambda'_\blacktriangle)^F$; we call it $\widetilde{V}_i(\lambda'_\blacktriangle)$. Then

$$\left|U_3(\lambda'_\blacktriangle)^F\right| = \prod_{i \geq 3} \left|\widetilde{V}_i(\lambda'_\blacktriangle)^F\right|. \tag{3.12}$$

Together (3.10), (3.11) and (3.12) show that

$$\left| X^{\triangle}(G)^F \right| = \left| \pi^{-1}\left(V_2(\lambda'_{\blacktriangle})_{ss}{}^F \right) \right| = \left(q^{\dim \mathfrak{g}(\lambda'_{\blacktriangle}, 2)} - \left| \mathcal{N}_{\mathfrak{g}(\lambda'_{\blacktriangle}, 2)}{}^F \right| \right) \cdot q^{\dim \mathfrak{g}(\lambda'_{\blacktriangle}, \geq 3)}.$$

As a result, $\left| X^{\triangle}(G)^F \right| = \left| X^{\triangle}(\mathfrak{g})^F \right| = \psi^{\triangle}_{\mathfrak{g}}(q)$ for every $\triangle$ as above. Now (3.7) yields $\left| H^{\blacktriangle}(G)^F \right| = \left| H^{\blacktriangle}(\mathfrak{g})^F \right| = \varphi^{\blacktriangle}_{\mathfrak{g}}(q)$. In view of Theorem 3.7.3 this implies that $\mathfrak{P}_5$ holds for G. $\hfill\square$

Remark. 1. In the appendix to [Lusztig, 2011] and more recently in [Lusztig, 2010], Lusztig and Xue proposed a definition of nilpotent pieces for classical groups which avoids the partial ordering of nilpotent orbits. Given $\triangle \in D_G$ choose $g \in G$ as in Subsection 3.7.1 and define $\mathfrak{g}_2^{\triangle!}$ to be the set of all $x = \sum_{i \geq 2} x_i \in \mathfrak{g}_2^{\triangle}$ with $x_i \in \mathfrak{g}(i, g \cdot \omega)$ and $C_G(x_2) \subset G_2^{\triangle}$. Similarly, let $(\mathfrak{g}^*)_2^{\triangle!}$ be the set of all $\xi = \sum_{i \geq 2} \xi_i \in (\mathfrak{g}^*)_2^{\triangle}$ with $\xi_i \in \mathfrak{g}^*(i, g \cdot \omega)$ such that the stabiliser of ξ in G is contained in $G_0^{\triangle}$. According to the definition of Lusztig and Xue, the blades and nilpotent pieces of $\mathfrak{g}$ are

$$\left\{ \mathfrak{g}_2^{\triangle!} \mid \triangle \in D_G \right\} \quad \text{and} \quad \left\{ (\mathrm{Ad}\, G) \cdot \mathfrak{g}_2^{\triangle!} \mid \blacktriangle \in D_G/G \right\}$$

respectively, whilst the blades and nilpotent pieces of $\mathfrak{g}^*$ are

$$\left\{ (\mathfrak{g}^*)_2^{\triangle!} \mid \triangle \in D_G \right\} \quad \text{and} \quad \left\{ (\mathrm{Ad}^*\, G) \cdot (\mathfrak{g}^*)_2^{\triangle!} \mid \blacktriangle \in D_G/G \right\}$$

respectively, where $\triangle$ is implicitly taken to be a representative of $\blacktriangle$ in each case. Lusztig and Xue proved that for G classical these subsets stratify $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$. On the other hand, Theorem 3.7.3 implies that $X^{\triangle}(\mathfrak{g}) \subseteq \mathfrak{g}_2^{\triangle!}$ and $X^{\triangle}(\mathfrak{g}^*) \subseteq (\mathfrak{g}_2^*)^{\triangle!}$ for every $\triangle \in D_G$. But equality must hold in each case because the blades, too, stratify the nullcones. This shows that for G classical both definitions lead to the same stratifications of $\mathcal{N}_{\mathfrak{g}}$ and $\mathcal{N}_{\mathfrak{g}^*}$.

2. The proof of Corollary 3.7.3 shows that for any $p > 0$ there exists a bijection between $G_{\mathrm{uni}}{}^F$ and $\mathfrak{g}_{\mathrm{nil}}{}^F$ which maps every non-empty subset $X^{\triangle}(G)^F$ onto $X^{\triangle}(\mathfrak{g})^F$ and every non-empty subset $H^{\blacktriangle}(G)^F$ onto $H^{\blacktriangle}(\mathfrak{g})^F$.

3. It follows from [Seshadri, 1977, Proposition 6(2)] that for every $\blacktriangle \in D_G/G$ there is a homogeneous regular function $f_{\blacktriangle} \in \mathbb{Z}[\mathfrak{g}'_{\mathbb{Z}}(\lambda'_{\blacktriangle}, 2)]$ invariant under the

natural action of the group scheme $\mathfrak{L}^{\perp}(\lambda'_{\blacktriangle})$ and such that for any algebraically closed field $\Bbbk$ the variety $\mathcal{N}_{\mathfrak{g}(\lambda'_{\blacktriangle}, 2)}$ coincides with the zero locus of the image of $f_{\blacktriangle}$ in $\Bbbk[\mathfrak{g}(\lambda'_{\blacktriangle}, 2)] = \mathbb{Z}[\mathfrak{g}'_{\mathbb{Z}}(\lambda'_{\blacktriangle}, 2)] \otimes_{\mathbb{Z}} \Bbbk$; see [Premet, 2003, §2.4] for a related discussion.

Chapter 4

The endomorphism algebra of generalised Gelfand-Graev representations

4.1 Introduction and background material

4.1.1 In order to define Gelfand-Graev representations and their generalisations, we shall need to recall some facts about regular unipotent elements of G, following [Digne and Michel, 1991]. Heuristically, a regular element is one which is least likely to commute with other elements, or, more precisely, if the dimension of its centraliser is minimal. In fact, it is easy to show that this dimension always equals the rank of G. The theory of regular elements of G is very important and has numerous applications in representation theory. It can largely be reduced to the study of regular unipotent elements of G since, if $x = su$ is the Jordan decomposition of an element $x \in G$. Then x is regular if, and only if, u is regular in $C_G^\circ(s)$. However, it can be difficult to work with regular unipotent elements, and the only known proof that they even exist is highly non-trivial.

Of fundamental importance is the fact that all regular unipotent elements are conjugate. Moreover, this regular unipotent class is an open dense subset of G_{uni}. It is thus the unique maximal element of the poset structure on unipotent classes. If G has a connected centre and p is a good prime for G, then the centraliser $C_G(u)$

is connected for any regular unipotent element u. We will use this fact several times in this chapter.

Of course, we can define regular nilpotent elements in a completely analogous manner, and all of the above facts hold for these too.

4.1.2 Now assume that G is defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F. Fix a rational Borel subgroup B of G and a rational maximal torus $T \leq B$, so that we get a rational unipotent subgroup $U = R_u(B)$ and Levi decomposition $B = U \rtimes L$. Closely related to regular unipotent elements are regular characters of U^F, which are used to construct Gelfand-Graev characters. We now explain how to construct and classify the regular characters of U^F before defining Gelfand-Graev characters and describing some of their basic properties.

Let Π denote a basis of the root system Σ relative to T, and Σ^+ the corresponding set of positive roots. Recall that the Frobenius endomorphism F acts on Π via an automorphism τ of $X(T)$. (This action is non-trivial only for non-split groups G^F, such as the unitary groups. In those cases the action can be identified with a non-trivial graph automorphism of the Dynkin diagram of G.) As we have seen in the previous chapter U may be written as a product $U = \prod_{\alpha \in \Sigma^+} U_\alpha$ for any order on the positive roots. Then the derived group of U may be written as $\mathcal{D}U = \prod_{\alpha \in \Sigma - \Pi} U_\alpha$, using Chevalley's commutator relations. We would first like to describe the structure of the abelian group $U^F/(\mathcal{D}U)^F \cong (U/\mathcal{D}U)^F$. As we have seen earlier $U/\mathcal{D}U$ is isomorphic to a direct product of the images of the U_α for $\alpha \in \Pi$. The action of F then permutes these summands. Explicitly, let $\mathcal{O}$ denote an orbit of τ in Π, and $U_\mathcal{O}$ the image of $\prod_{\alpha \in \mathcal{O}} U_\alpha$ in $U/\mathcal{D}U$. Then $U_\mathcal{O}$ is F-stable and $U_\mathcal{O}^F$ is isomorphic to $U_\alpha^{F^{|\mathcal{O}|}}$ for any $\alpha \in \mathcal{O}$. Furthermore, the F- and T-actions are compatible with the natural isomorphism of varieties $U_\mathcal{O} \cong \prod_{\alpha \in \mathcal{O}} U_\alpha$. Since root subgroups are isomorphic to the additive group of $\Bbbk$, it follows that

$$U^F/(\mathcal{D}U)^F \cong \mathbb{F}_{q^{|\mathcal{O}_1|}}^+ \times \cdots \times \mathbb{F}_{q^{|\mathcal{O}_k|}}^+, \tag{4.1}$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_k$ are the τ-orbits in Π. We may now define a linear character of U^F to be *regular* if the following hold:

(i) Its restriction to the subgroup $(\mathcal{D}U)^F$ is trivial (i.e. it is lifted from a

character of the group in (4.1)).

(ii) Its restriction to $U_{\mathcal{O}}^F$ for any τ-orbit $\mathcal{O}$ in Π is non-trivial.

With very few exceptions, $(\mathcal{D}U)^F = \mathcal{D}(U^F)$, in which case (i) above is not necessary as this is automatic for linear characters. (E.g. [Howlett, 1974, Lemma 7] lists the three exceptions for quasi-simple groups.) Also note that the action of T^F on $\text{Irr}(U^F)$ preserves regular characters.

We now come to our main definition. Let ψ be a regular character of U^F. Then the induction $\text{Ind}_{U^F}^{G^F} \psi$ is called a *Gelfand-Graev character of G^F*. One of the most celebrated results on Gelfand-Graev characters is the following, due to Steinberg. (Special cases were proved previously by Gelfand and Graev, and Yokonuma.)

Theorem. (Cf. [Steinberg, 1968, Theorem 49]) *Gelfand-Graev characters are multiplicity-free.*

The idea in Steinberg's proof is to construct the endomorphism algebra of a Gelfand-Graev representation and then show that it is abelian.

We shall now determine when two Gelfand-Graev characters coincide, and classify the distinct ones. First note that by Clifford theory,

$$\text{Ind}_{U^F}^{G^F} \psi = \text{Ind}_{U^F}^{G^F} \psi'$$

if, and only if, ψ and ψ' are T^F-conjugate. So the task is reduced to determining the T^F-orbits of regular characters. This leads us to the following well-known result, which we prove here since the proof in [Digne and Michel, 1991] is somewhat sketchy.

Proposition. *The Gelfand-Graev characters are in bijection with the F-twisted conjugacy classes of $Z(G)/Z(G)^\circ$. In particular, there is only one if the centre of G is connected.*

Proof. Assume that G is semisimple; the general case is similar. By our observation above our task is reduced to showing that the T^F-orbits of regular characters of U^F are in bijection with the F-twisted conjugacy classes of $Z(G)/Z(G)^\circ$. First

consider $U/\mathcal{D}U$ as a direct sum of the images of U_α for simple α. We may refer to those elements which have no non-trivial direct summand as the *regular elements* of $U/\mathcal{D}U$. The regular elements of $U^F/(\mathcal{D}U)^F = (U/\mathcal{D}U)^F$ are in bijection with the regular characters of U^F by (4.1). Moreover, they are also isomorphic as T^F-sets. Therefore, it is sufficient to prove the result for the former. We will show that any two regular elements of $U/\mathcal{D}U$ are T-conjugate. Then the result follows from the Lang-Steinberg theorem, and the fact that the stabiliser of a regular element is $Z(G)$.

Let $l = |\Pi|$. The action of T on

$$U/\mathcal{D}U \cong \Bbbk^+ \times \cdots \times \Bbbk^+ \qquad (l \text{ times})$$

may be thought of as a homomorphism of algebraic groups

$$\phi : T \to H = \Bbbk^\times \times \cdots \times \Bbbk^\times \qquad (l \text{ times}),$$

given by $\phi(t) = (\alpha_1(t), \dots, \alpha_l(t))$. It is sufficient, therefore, to show that ϕ is surjective. Now, by their simplicity, the elements of Π are linearly independent in the character group $X(T)$, thought of as a free $\mathbb{Z}$-module. So $\prod \alpha_i{}^{m_i}$ $(m_i \in \mathbb{Z})$ are distinct (abstract group) homomorphisms from T to $\Bbbk^\times$, and as such (see, e.g., [Humphreys, 1975, Lemma 16.1]), they are linearly independent as elements of $\Bbbk[T]$. On the other hand, if we denote the coordinate functions of H by γ_i, then we have that $\phi^*(\gamma_i) = \alpha_i$ for each i. But $k[H]$ has basis $\prod \gamma_i{}^{m_i}$ $(m_i \in \mathbb{Z})$, which shows that ϕ^* is injective, and hence ϕ is dominant. Being a homomorphism of algebraic groups, ϕ must also be surjective. $\qquad\qquad\square$

4.1.3 We keep the same notation as in Subsection 4.1.2 and we additionally let F denote a Frobenius endomorphism on $\mathfrak{g}$ which is compatible with the one on G and that p is a good prime for G. We follow the construction of generalised Gelfand-Graev representations (hereafter GGGRs) in [Kawanaka, 1985]. The basic idea is to associate various GGGRs to each nilpotent $\mathrm{Ad}\,G$-orbit of $\mathfrak{g}$, such that the the GGGRs associated with the regular orbit are precisely the ordinary Gelfand-Graev representations. We note that Kawanaka states a list of properties [Kawanaka, 1985, (1.1.1)] which he says must hold in order to define GGGRs, but

later shows these to be unnecessary in [Kawanaka, 1986].

For a nilpotent element $e \in \mathfrak{g}^F$ we associate a $\mathbb{Z}$-grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ as in Theorem 2.2.1. We denote by U_i the unipotent subgroups $U_{e,i}$ from Chapter 2 and let $\mathfrak{u}_i = \operatorname{Lie} U_i$. Using the explicit choice of Springer morphisms σ from [Kawanaka, 1985] and [Kawanaka, 1986], we have that $\sigma(U_i) = \mathfrak{u}_i$ for $i \geq 1$, and the 'log-like' property for $u \in U_i, v \in U_j, i, j \geq 1$,

$$\sigma(uv) - \sigma(u) - \sigma(v) \in \mathfrak{u}_{i+j}. \tag{4.2}$$

For $x \in \mathfrak{g}$, we let $x \mapsto x^*$ denote the F-stable opposition automorphism, i.e. an involutive automorphism such that $\mathfrak{t}^* = \mathfrak{t}$ and $\mathfrak{u}_i^* = \mathfrak{u}_{-i}$ for $i \geq 1$, where $\mathfrak{t} = \operatorname{Lie} T$. We let $\langle \, , \, \rangle$ denote an F- and G-stable symmetric bilinear form on $\mathfrak{g}$ such that

$$\langle x, [y, z] \rangle = \langle [x, y], z \rangle,$$

for $x, y, z \in \mathfrak{g}$, and

$$X_\alpha^\perp = \mathfrak{t} \oplus \sum_{\beta \in \Sigma \setminus \alpha} \Bbbk X_{-\beta}.$$

(For a simple Lie algebra, clearly the only option is the Killing form, up to a scalar.) Then define

$$\lambda : \mathfrak{g} \to k \quad \text{by} \quad \lambda(x) = \langle e^*, x \rangle.$$

Then the composition

$$U_2 \xrightarrow{\sigma} \mathfrak{u}_2 \xrightarrow{\lambda} k \tag{4.3}$$

is a homomorphism of algebraic groups. The final ingredient required is a 'once-and-for-all fixed' non-trivial additive character ψ of $\mathbb{F}_q$. Since both λ and σ are F-stable, so we may define a linear character

$$\Delta : U_2^F \xrightarrow{\sigma} \mathfrak{u}_2^F \xrightarrow{\lambda} \mathbb{F}_q \xrightarrow{\psi} \bar{\mathbb{Q}}_l. \tag{4.4}$$

One may check that the skew-symmetric bilinear form $(,)$ on $\mathfrak{g}_1$, defined by

$$(x, y) = \langle e^*, [x, y] \rangle \tag{4.5}$$

is non-degenerate. It follows that $\dim \mathfrak{g}_1$ is even and that there exists an F-stable subspace $\mathcal{S} \leq \mathfrak{g}_1$ of dimension $(\dim \mathfrak{g}_1)/2$, such that for all $x, y \in \mathcal{S}$,

$$\langle e^*, [x, y] \rangle = 0.$$

(Hence, $\mathcal{S}$ is a Lagrangian subspace.) We then define a subalgebra

$$\mathfrak{u}_{1.5} = \mathcal{S} \oplus \mathfrak{u}_2.$$

It follows that $U_{1.5} = \sigma^{-1}(\mathfrak{u}_{1.5})$ is an F-stable subgroup of G, and, furthermore,

$$U_2 \leq U_{1.5} \leq U_1,$$

and

$$\left[U_1^F : U_{1.5}^F\right] = \left[U_{1.5}^F : U_2^F\right]. \tag{4.6}$$

Using properties of p-groups it is not hard to show that Δ is extendible to a linear character $\Delta^\sim$ of $U_{1.5}^F$. Then the representation Γ_e of G^F induced from $\Delta^\sim$ is called the *generalised Gelfand-Graev representation* of G^F associated to e.

We denote the character of Γ_e by γ_e. If e' is in the same G^F-orbit as e then $\gamma_e = \gamma_{e'}$. Also, γ_e does not depend on the choice of $\mathcal{S}$ or extension $\Delta^\sim$. The name is justified by the fact that if e is a regular nilpotent element then Γ_e is indeed the usual Gelfand-Graev representation from Subsection 4.1.2. Our convention will be to consider all representations of G^F to be over $\bar{\mathbb{Q}}_l$, to remind us of the connections with Deligne–Lusztig theory, which will soon become apparent. By means of a suitable Springer morphism we may also define GGGRs associated to rational unipotent elements, whence we will use the notation Γ_u, for $u \in G_{\text{uni}}^F$.

4.2   The endomorphism algebra of a GGGR

As we have seen, we may associate a generalised Gelfand-Graev representation to each unipotent class of a finite group of Lie type G^F. These representations have deep connections with the geometry of the unipotent classes of G, and have been key tools in the ongoing programme to determine the ordinary character tables of

all finite groups of Lie type, due to their relationship with Green functions, and thus Deligne–Lusztig characters. Character formulas for GGGRs have performed a key role in this regard, since they allow one to deduce information about irreducible constituents. In the seminal article [Kawanaka, 1985] where GGGRs are defined for the first time Kawanaka has given character formulas for GGGRs in terms of Green polynomials in the case of general linear and unitary groups. Inspired by this, Lusztig has obtained a similar formula, valid for an arbitrary finite group of Lie type (with p sufficiently large), expressed in terms of intersection cohomology complexes of closures of unipotent classes with coefficients in various local systems; [Lusztig, 1992]. Using the fact that the inner product of a complex character with itself is equal to the dimension of its endomorphism algebra, in this Chapter we study the dimensions of endomorphism algebras of GGGRs using these character formulas.

Naturally, the prime power q features heavily in these character formulas and in such situations it is useful to think of q as a variable. However, as we saw in the previous Chapter, care is needed in order to formulate precise and meaningful statements. This notion is central in the theory of finite groups of Lie type as it allows generic behaviour to be observed, i.e. behaviour which is independent of the associated finite field. The set-up for this chapter is inspired by the set-up in the recent papers [Goodwin and Röhrle, 2009b] and [Goodwin and Röhrle, 2009a], in which the notion of polynomials in q is used extensively. It was pointed out by the referee of the paper on which this chapter is based that this set-up goes back to [Broué et al., 1993]; the authors use the term *generic finite reductive group* there to refer to the family of groups obtained by varying q.

Under a certain assumption on the root datum of G (ensuring that the centre of G, together with all groups with the same root datum but with a possibly different associated prime power q, are connected), we shall explain what we mean by the statement "the dimension of a generalised Gelfand-Graev module is a polynomial in q", before proving it. We will also show that the degree of this polynomial is given by the dimension of the centraliser (in G) of a unipotent element in G^F from the class associated with that module. (For groups with a component of Type E_8 we may need to dichotomise our set-up, depending on the value of q modulo 3.) When G has a disconnected centre we cannot

even *parametrise* the generalised Gelfand-Graev characters independently of q in general, and so we cannot hope for such a clean statement here. However, similar behaviour *is* exhibited in the disconnected centre case, and we have found that a suitable way to capture this is to consider q as a variable, but whilst only allowing certain congruence classes of q. Subject to this restriction we extend the aforementioned results to this situation too.

As we have seen, Steinberg has shown that (ordinary) Gelfand-Graev representations are multiplicity-free (although this was proved previously for split groups in [Yokonuma, 1967] and [Yokonuma, 1968]). Moreover, when the centre is connected they contain $|Z(G)^F|q^l$ irreducible constituents. O. Brunat has generalised this formula to the disconnected centre case; see [Brunat, 2010, Theorem 3.5]. His formula is of the form $|Z(G)^{\circ F}|f$ where f is a polynomial in q. So when the centre of G is connected this number may be viewed as a polynomial in q of degree $\operatorname{rank} G$, the latter agreeing with the dimension of the centraliser of a regular unipotent element. This fact can also be extended to the disconnected centre case using Brunat's formula, provided one adopts a suitable convention to control the number $|Z(G)^{\circ F}|$ as q varies. Now clearly, for ordinary Gelfand-Graev representations, this number is also the dimension of the endomorphism algebra, hence the results of this chapter are a generalisation of this property of the Steinberg-Yokonuma-Brunat formula to generalised Gelfand-Graev representations.

The author is grateful to Cédric Bonnafé for pointing out some interesting results on the dimension of the endomorphism algebra of a GGGR which appear in the recent PhD thesis of O. Dudas, obtained using only elementary methods; cf. [Dudas, 2010, Chapter 3]. Let d_u be the dimension of the endomorphism algebra of the GGGR associated to an F-stable unipotent element u, where F is a Frobenius endomorphism associated to a split $\mathbb{F}_q$-rational structure. Dudas has shown, for instance, that if G is a quasi-simple adjoint group, and if u lies in the minimal unipotent class (i.e. the unique non-trivial unipotent class $C_{\min}$ such that $\overline{C_{\min}} = C_{\min} \cup \{1_G\}$), then

$$d_u = \frac{|C_G(u)^F||\overline{C_{\min}}^F|}{|U_1^F||U_2^F|},$$

where U_1, U_2 are certain F-stable closed unipotent subgroups associated to u by the Dynkin-Kostant theory; cf. Theorem 2.2.1. However, our tables in Appendix B imply that this closed formula is *not* true in general, for other unipotent classes. More generally, he shows that for an arbitrary F-stable unipotent element u,

$$d_u \frac{|U_1^F||U_2^F|}{|C_G(u)^F|} \in \mathbb{N}.$$

He also shows that

$$\lim_{q \to \infty} q^{-\dim C_G(u)} d_u = 1,$$

which is verified by our main result.

This chapter is organised as follows. In Section 4.3 we lay down the rigorous foundation necessary to formulate precise statements involving polynomials in q. In Section 4.4 we prove our main result in the special case of general linear and unitary groups, using Kawanaka's character formula. In Section 4.5 we extend this to a much more general setting, using Lusztig's character formula. Although Section 4.4 is essentially a special case of Section 4.5, it is considerably easier to understand, conceptually, the ingredients of the former, and may serve to illuminate the latter.

4.3   Polynomials in q

4.3.1 Recall that a connected reductive algebraic group is uniquely determined by its root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$, and that a related finite group of Lie type is uniquely determined by the following; cf. [Digne and Michel, 1991, Theorem 3.17].

1. A root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$.

2. A prime power q.

3. An automorphism τ of $X(T)$ which preserves Σ.

Now suppose we have a quantity attached to a fixed choice of data 1 and 3 above, which is a function of various prime powers q. If there exists a polynomial $f \in \mathbb{Q}[x]$ such that the quantity is given by $f(q)$ for those q under consideration,

then we say that this quantity is a *polynomial in q*. In the same spirit, we may also talk about quantities which are *independent of q*. When we write G we will sometimes mean a fixed group with an associated fixed prime power q (sometimes the notation $G(q)$ is used), but sometimes, by abuse, we will talk about properties of G which are independent of q, in which case we are, strictly speaking, referring to properties of the root datum, together with τ.

For the statement "$\dim \operatorname{End}_{\overline{\mathbb{Q}}_l G^F} \Gamma_u$ is a polynomial in q" to be meaningful, we first need to establish that the number of distinct generalised Gelfand-Graev characters is fixed as we vary q, and that they can be parametrised independently of q. But since the generalised Gelfand-Graev characters are parametrised by the unipotent classes of G^F we shall focus on the latter. Clearly we will need to fix, once and for all, a root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$, and automorphism τ.

Before we begin the discussion of unipotent classes we fix some data which will play a part both in the current situation and later on when we will wish to compare G^F-classes of certain rational subgroups of G, across different values of q. We fix a maximally split maximal torus $T \leq G$ for each prime power, and a simple system $\Pi \subset \Sigma$ such that $\tau(\Pi) = \Pi$. Then this uniquely determines a rational Borel subgroup $B \supset T$.

In order to parametrise geometric unipotent classes and F-stable geometric unipotent classes independently of q we apply an idea of N. Spaltenstein of using a group G' with the same root datum as G but over a field of characteristic zero, as a reference point. (We could, alternatively, employ the language of group schemes again, as in Chapter 3, but since we can get away without using such heavy machinery here we will do so.) We let T' be a maximal torus of G' and B' a Borel subgroup of G' containing T'. For this discussion we restrict our attention to q which are powers of a good prime for our fixed root datum. (Recall that this restriction is also necessary to define generalised Gelfand-Graev representations.) Spaltenstein has shown that there exists a map $\pi_G : G'_{\mathrm{uni}}/G' \to G_{\mathrm{uni}}/G$, which is characterised by the following three properties:

(i) It is an isomorphism of posets.

(ii) It preserves the dimensions of classes.

(iii) It satisfies certain compatibility relations between parabolic subgroups in G and G' containing B and B' respectively.

Cf. [Spaltenstein, 1982, Théorème III.5.2].

Since this map is uniquely defined we may use it as a means of parametrising the geometric unipotent classes of G independently of q. Now the Frobenius endomorphism F on G may be written as $F = F_q \circ F_0 = F_0 \circ F_q$ where F_q is determined by the multiplication by q map on the character group of T, whilst F_0 is an automorphism of G of finite order, determined uniquely by τ. Then F_0 determines also a map F_0' on G' in the same manner. Clearly the maps F and F_0' induce permutations on G_{uni}/G and G_{uni}'/G' respectively and, moreover, the following diagram commutes:

$$
\begin{array}{ccc}
G_{\mathrm{uni}}'/G' & \xrightarrow{\pi_G} & G_{\mathrm{uni}}/G \\
\downarrow{\scriptstyle F_0'} & & \downarrow{\scriptstyle F} \\
G_{\mathrm{uni}}'/G' & \xrightarrow{\pi_G} & G_{\mathrm{uni}}/G
\end{array}
$$

(The only non-trivial thing to show here is that F_q acts trivially on G_{uni}/G, a fact which follows from the Springer correspondence; cf. [Geck and Malle, 2000, p. 24].) It follows that our parametrisation of geometric unipotent classes respects the F-action, and thus we have a q-independent parametrisation of F-stable unipotent classes of G.

4.3.2 We now turn our attention to the unipotent classes of G^F. For this we will need to assume that $X/\mathbb{Z}\Sigma$ is torsion-free. Let C be an F-stable unipotent class in G and fix an F-stable point $u \in C$. Then, by the Lang-Steinberg theorem, the G^F-orbits in C^F are parametrised by the F-conjugacy classes of $A(u) = C_G(u)/C_G^\circ(u)$. Explicitly, this is done as follows: For each F-conjugacy class in $A(u)$, choose a representative a; then choose $g_a \in G$ such that $g_a^{-1}F(g_a) = \dot{a}$ for some representative $\dot{a}$ of a in $C_G(u)$; then $\{g_a u g_a^{-1}\}$ is a set of representatives for the G^F-conjugacy classes in C^F.

The next step is to make a special choice for u for which the F-action on $A(u)$ is as simple as possible and so that we have a more canonical reference point in C for when we vary q. Thankfully, this is possible by an idea of T. Shoji. If G

is simple modulo its centre and Σ is not of type E_8 then u may be chosen to be a so-called *split element* (proved in [Shoji, 1983], [Shoji, 1982], and [Beynon and Spaltenstein, 1984]; see also the survey [Shoji, 1987, §5]). We omit the definition here, but suffice to say that split elements in C comprise a unique G^F-conjugacy class and, if u is a split element, F acts trivially on $A(u)$. Thus, the G^F classes in C^F correspond canonically to the conjugacy classes of $A(u)$. In fact, suppose $u' \in G'$ is a unipotent element whose G'-class agrees with the G-class of u under Spaltenstein's map, then we may write down an explicit bijection between the conjugacy classes of $A(u)$ and $A(u')$. (See [McNinch and Sommers, 2003, p. 336] for the details of this bijection, although this is based on earlier work in [Mizuno, 1980] and [Alekseevskii, 1979].) In this manner we can label the unipotent classes, and hence the generalised Gelfand-Graev characters, of G^F independently of q.

4.3.3 We now consider groups of type E_8. Following [Kawanaka, 1986, Proposition 1.2.1] we dichotomise the situation according to whether q is congruent to 1 or -1 modulo 3. (Note that we do not consider the case where q is a power of 3 since this is a bad prime for E_8.) From now on, when we refer to "polynomials in q" or "treating q as a variable" we tacitly assume that we have fixed one or the other of these situations. In fact we only really need this distinction when u is in the geometric unipotent class with Bala-Carter label $E_8(b_6)$ (corresponding to Mizuno label $D_8(a_3)$). In this case there is a G^F-class of split elements if q is congruent to 1 modulo 3 (and hence the G^F-classes may be parametrised independently of q as before), but split elements do not exist in this class if q is congruent to -1 modulo 3. However, it is possible to deal with this case explicitly. (E.g. it is known that there are precisely three G^F-classes and that their class sizes are given by different polynomials so we could, for instance, label these classes by these known polynomials.) This distinction is also implicitly used in Section 4.5 since Green functions, which appear there, have an analogous q-dependence issue for groups of type E_8.

4.3.4 By a well-known process of reduction in the theory of unipotent classes of reductive groups we may also lift the assumption that G be simple modulo its centre. (See the standard text [Carter, 1993, Ch. 5] for a general treatment,

and [Goodwin and Röhrle, 2009a, p. 7] for a discussion relevant to the current context.)

In summary, we have the following:

Proposition. *Fix a root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$ and automorphism τ of $X(T)$ which preserves Σ, and assume that $X/\mathbb{Z}\Sigma$ is torsion-free. Then we may parametrise the unipotent classes (and therefore the generalized Gelfand-Graev characters) of all finite groups of Lie type which have this data independently of the associated prime power q, provided q is a power of a good prime.*

4.3.5 We will also make use of the following important result, sometimes implicitly.

Proposition. *With the set-up of Proposition 4.3.4, let R be a set of q-independent labels for the unipotent classes of these groups. For each power q of a good prime and each $r \in R$, let $u_{r,q}$ be a representative of the corresponding unipotent class. Then, allowing q to vary, the order of the centraliser of $u_{r,q}$ is a polynomial in q.*

Proof. This is [Goodwin and Röhrle, 2009b, Proposition 3.3]. The proof appeals to the Lusztig-Shoji algorithm for computing Green functions. □

4.4 Type A: Kawanaka's formula

4.4.1 In this section we set $G = \mathrm{GL}_n(\Bbbk)$ and endow it with a split or non-split $\mathbb{F}_q$-rational structure, with corresponding Frobenius endomorphism F, so that G^F is $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{GU}_n(\mathbb{F}_q)$. Using [Kawanaka, 1985, (3.2.14)] we prove the following:

Theorem. *Let G and F be as above, $u \in G^F$ a unipotent element, and Γ_u the corresponding generalised Gelfand-Graev representation. Then the dimension of the endomorphism algebra $\mathrm{End}_{\bar{\mathbb{Q}}_l G^F} \Gamma_u$ is a monic polynomial in q with rational coefficients. Moreover, its degree is given by the dimension of the centraliser $C_G(u)$.*

Note that we need no condition on p here since all primes are good. Before we can state Kawanaka's formula we must first explain the ingredients from

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[Kawanaka, 1985]. The unipotent classes of G are parametrised by the partitions of n, via the Jordan normal form, and the rational points of such a class comprise a single G^F-class. We may therefore denote a typical generalised Gelfand-Graev character by γ_λ, where $\lambda \vdash n$. We may also use partitions to label the non-zero values of generalised Gelfand-Graev characters since they are known to vanish on non-unipotent elements. We shall therefore adopt the convention of writing $\gamma_\lambda(\mu)$ for the character value of γ_λ on the class corresponding to μ.

4.4.2 We now introduce the necessary notation to state Kawanaka's character formula. We set $\varepsilon = 1$ or -1 depending on whether F is split or not, respectively. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r) \vdash n$ then we define

$$n(\lambda) = \sum_i (i-1)\lambda_i.$$

With reference to a fixed rational maximal torus T, we may, by the Lang-Steinberg theorem, label the G^F-classes of rational maximal tori by the F-classes of the Weyl group $W = N_G(T)/T \cong \mathrm{S}_n$. If T is the diagonal maximal torus then F acts trivially on W and therefore we may label these by the classes of W and, thus, by the partitions of n. (Here we have chosen the Frobenius endomorphism defining $\mathrm{GU}_n(\mathbb{F}_q)$ to be $F(g) = F_q(g^{-T})$, where $g \in G$ and F_q is a standard Frobenius endomorphism. Note that this is different from our earlier choice (2.7), but it is well-known that the resulting fixed-point groups are G-conjugate.) With this set-up we denote representatives of the G^F-classes of rational maximal tori by T_λ for $\lambda \vdash n$, and define

$$W_\lambda = (N_G(T_\lambda)/T_\lambda)^F.$$

For $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{r(\lambda)}) \vdash n$, we set

$$\mathrm{sgn}_\varepsilon(\lambda) = \varepsilon^{\lfloor n/2 \rfloor}(-1)^{n+r(\lambda)},$$

and

$$e_\lambda(t) = \prod_i (1 - t^{\lambda_i}).$$

$Q_\lambda^\mu(t) \in \mathbb{Z}[t]$ will denote the Green polynomial with parameters $\lambda, \mu \vdash n$ (cf. [Macdonald, 1979]). Finally, we define a rational function

$$X_\lambda^\mu(t) = t^{n(\mu)} Q_\lambda^\mu(t^{-1}).$$

In fact this is a polynomial of degree $n(\mu)$ by [Macdonald, 1979, Chapter III, §7]. Now we may state Kawanaka's formula.

Theorem. ([Kawanaka, 1985, (3.2.14)]) *With the above set-up,*

$$\gamma_\mu(\lambda) = \varepsilon^{n(\mu)} \sum_{\rho \vdash n} |W_\rho|^{-1} \operatorname{sgn}_\varepsilon(\rho) q^n e_\rho((\varepsilon q)^{-1}) X_\rho^\mu(\varepsilon q) Q_\rho^\lambda(\varepsilon q).$$

4.4.3 Recall that a Green function of G^F is the restriction of a Deligne-Lusztig virtual character $R_{T,\theta}$ to G_{uni}^F. In the case of general linear groups these are simply Green polynomials, and for unitary groups they are of the form $Q_\lambda^\mu(-q)$. We shall need the following orthogonality formula for Green functions. (See, e.g., [Shoji, 1987].) Let λ^F denote the unipotent class in G^F corresponding to $\lambda \vdash n$. Then

$$|G^F|^{-1} \sum_{\lambda \vdash n} |\lambda^F| Q_\rho^\lambda(\varepsilon q) Q_\pi^\lambda(\varepsilon q) = \frac{|N_G(T_\rho, T_\pi)^F|}{|T_\rho^F||T_\pi^F|}, \tag{4.7}$$

where $N_G(T_\rho, T_\pi) = \{n \in G \,|\, n^{-1} T_\rho n = T_\pi\}$.

4.4.4 Now $\dim \operatorname{End}_{\bar{\mathbb{Q}}_l G^F} \Gamma_\mu$ may be written as

$$\begin{aligned}
\langle \gamma_\mu, \gamma_\mu \rangle &= |G^F|^{-1} \sum_{\lambda \vdash n} |\lambda^F| \left(\sum_{\rho \vdash n} |W_\rho|^{-1} \operatorname{sgn}_\varepsilon(\rho) q^n e_\rho((\varepsilon q)^{-1}) X_\rho^\mu(\varepsilon q) Q_\rho^\lambda(\varepsilon q) \right) \\
&\quad \times \left(\sum_{\pi \vdash n} |W_\pi|^{-1} \operatorname{sgn}_\varepsilon(\pi) q^n e_\pi((\varepsilon q)^{-1}) X_\pi^\mu(\varepsilon q) Q_\pi^\lambda(\varepsilon q) \right),
\end{aligned}$$

which is a polynomial in q. Indeed, it is easy to check that it is a rational function of the form $f/|G^F|$ for some $f \in \mathbb{Q}[q]$. (We consider $|G^F|$ as a polynomial in q here.) So $f(q)/|G^F| \in \mathbb{Z}$ for infinitely many q. By applying the division algorithm

we may write $f = g|G^F| + r$, where $\deg r < \deg |G^F|$. It follows that for some integer c, $cr(q)/|G^F| \in \mathbb{Z}$ for all q. But the limit as $q \to \infty$ is 0, so $r = 0$ and the claim follows. We label this argument by ♣ as we shall reuse it later. The fact that conjugacy class sizes are given by polynomials in q can be deduced by applying ♣ and the orbit-stabiliser theorem to Proposition 4.3.5.

Now we will show that the degree of this polynomial is $\dim C_G(u)$, where u is in the class corresponding to μ by the Jordan normal form. For $\mathrm{GL}_n(\Bbbk)$ it is known that $\dim C_G(u) = n + 2n(\mu)$. (Adapt, e.g., the corresponding result, [Geck, 2003, Proposition 2.6.1], for $\mathrm{SL}_n(\Bbbk)$.) To make the derivation tidier we define an equivalence relation on $\mathbb{Q}[q]$, denoted $\approx$, by setting $f \approx g$ if $\deg f = \deg g$, for $f, g \in \mathbb{Q}[q]$. Under this relation the above expression is equivalent to

$$
q^{-n^2+n+2n(\mu)} \sum_{\lambda \vdash n} |\lambda^F| \left(\sum_{\rho \vdash n} \frac{q^n e_\rho((\varepsilon q)^{-1})(-1)^{r(\rho)}}{|W_\rho|} Q_\rho^\lambda(\varepsilon q) \right)
$$

$$
\times \left(\sum_{\pi \vdash n} \frac{(-1)^{r(\pi)}}{|W_\pi|} Q_\pi^\lambda(\varepsilon q) \right)
$$

$$
\approx q^{-n^2+n+2n(\mu)} \sum_{\rho, \pi \vdash n} \frac{q^n e_\rho((\varepsilon q)^{-1})(-1)^{r(\rho)+r(\pi)}}{|W_\rho||W_\pi|} \sum_{\lambda \vdash n} |\lambda^F| Q_\pi^\lambda(\varepsilon q) Q_\rho^\lambda(\varepsilon q)
$$

$$
\approx q^{-n^2+n+2n(\mu)} \sum_{\rho \vdash n} \frac{q^n e_\rho((\varepsilon q)^{-1})|G^F|}{|W_\rho||T_\rho^F|} \quad \text{(by (4.7))}
$$

$$
\approx q^{n+2n(\mu)} \sum_{\rho \vdash n} \frac{1}{|W_\rho|} \approx q^{n+2n(\mu)},
$$

where we have used the fact that $|T_\rho^F| = q^n e_\rho((\varepsilon q)^{-1})$. So, to complete the proof of Theorem 4.4.1, it remains to show that $\dim \mathrm{End}_{\bar{\mathbb{Q}}_l G^F} \Gamma_\mu$ is monic. Indeed, observe that the coefficient of the leading term has been preserved in the above until the last line. Using the fact (see, e.g., [Kawanaka, 1985]) that $W_\rho \cong C_{\mathrm{S}_n}(\rho)$, the centraliser in the symmetric group of an element of cycle type ρ, we see that

$$
\sum_{\rho \vdash n} \frac{1}{|W_\rho|} = \sum_{\rho \vdash n} \frac{|\mathrm{cl}(\rho)|}{n!} = 1, \tag{4.8}
$$

which completes the proof of Theorem 4.4.1.

4.5 The general case: Lusztig's formula

4.5.1 Inspired by Kawanaka's work, Lusztig has derived another character formula ([Lusztig, 1992, Theorem 7.3]), valid for any connected reductive group, but assuming that p is large enough in the sense of [Lusztig, 1992] throughout this section. I.e., large enough so that the Jacobson-Morozov theorem holds, and that the log map may be used as a Springer morphism. Lusztig's formula, however, is rather more geometric and is given in terms of intersection cohomology complexes of closures of unipotent classes with coefficients in various local systems. We will use it to prove the following.

Theorem. *Let G be a connected reductive group, defined over $\mathbb{F}_q$, with root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$ and Frobenius endomorphism F. Let $u \in G^F$ be a unipotent element and let Γ_u be the corresponding generalised Gelfand-Graev representation. Then, assuming that $X/\mathbb{Z}\Sigma$ is torsion-free, the dimension of the endomorphism algebra $\mathrm{End}_{\bar{\mathbb{Q}}_l G^F} \Gamma_u$ is a monic polynomial in q with rational coefficients. Moreover, its degree is given by the dimension of the centraliser $C_G(u)$.*

4.5.2 In [Lusztig, 1992] a generalised Gelfand-Graev representation Γ_ϕ is associated to each homomorphism $\phi : \mathfrak{sl}_2 \to \mathfrak{g}$, where $\mathfrak{g} = \mathrm{Lie}\, G$. However, for convenience we will use the equivalent notation $\Gamma_e = \Gamma_\phi$, where $e \in \mathfrak{g}$ is the nilpotent element which is the image under ϕ of the matrix with 1 in the $(1,2)$-position and 0 elsewhere. This is defined on certain rational points of the nilpotent variety $\mathfrak{g}_{\mathrm{nil}}$ of $\mathfrak{g}$, but can be made into a usual generalised Gelfand-Graev representation as follows. For any Frobenius endomorphism on G there exists an $\mathrm{Ad}\, G$-compatible Frobenius endomorphism on $\mathfrak{g}$. (We denote the Frobenius endomorphism on $\mathfrak{g}$ also by F. Hence, the domain of Γ_e is $\mathfrak{g}_{\mathrm{nil}}^F$.) Furthermore, by [Springer and Steinberg, 1970, Theorem 3.2], there exists a Springer morphism (i.e. a G-equivariant bijective morphism of varieties) $\sigma : G_{\mathrm{uni}} \to \mathfrak{g}_{\mathrm{nil}}$ which is compatible with these Frobenius endomorphisms. For the purposes of this chapter, then, we can and will identify unipotent and nilpotent elements via σ. Then for $u \in G_{\mathrm{uni}}^F$, $x \in G^F$,

$$\Gamma_u(x) = \begin{cases} \Gamma_{\sigma(u)} \circ \sigma(x) & \text{if } x \in G_{\mathrm{uni}}^F, \\ 0 & \text{otherwise.} \end{cases}$$

4.5.3 The basic parameter set used in [Lusztig, 1992] is the set $\mathscr{I}$ of all pairs $(\mathcal{O}, \mathscr{F})$, where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and $\mathscr{F}$ is an irreducible, G-equivariant, $\bar{\mathbb{Q}}_l$-local system on $\mathcal{O}$, up to isomorphism. By starting with a closed subgroup L of G, which is the Levi subgroup of some parabolic subgroup of G, we may obtain another parameter set in the same manner. In this way, we obtain triples $(L, \mathfrak{o}, \mathscr{L})$, where $\mathfrak{o}$ is a nilpotent $\operatorname{Ad} L$-orbit in $\mathfrak{l} = \operatorname{Lie} L$, and $\mathscr{L}$ is an irreducible, L-equivariant, $\bar{\mathbb{Q}}_l$-local system on $\mathfrak{o}$, given up to isomorphism. Using the generalised Springer correspondence we can partition the set $\mathscr{I}$ into blocks, and to each block we may associate (the G-orbit of) such a triple (a *cuspidal* triple).

By [Lusztig, 1992, §4] the elements of these blocks are naturally parametrised by the irreducible characters of the Weyl group $W_L = N_G(L)/L$. There is a single block associated with the maximal tori, since all maximal tori are G-conjugate and contain only the trivial unipotent class. We shall call this block the *principal block*, by analogy with Harish-Chandra theory.

4.5.4 In each class C fix a representative u once and for all, and consider the component group $A(u) = C_G(u)/C_G^\circ(u)$. This acts naturally on the stalk $\mathscr{F}_u$ of a G-equivariant local system $\mathscr{F}$ on C, and thus gives rise to a finite-dimensional $\bar{\mathbb{Q}}_l$-representation of $A(u)$. On the other hand, if $\rho \in \operatorname{Irr} A(u)$ then we may obtain the irreducible G-equivariant local system $\operatorname{Hom}_{A(u)}(\rho, \pi_* \bar{\mathbb{Q}}_l)$, where

$$\pi : G/C_G(u)^\circ \to G/C_G(u) \cong C$$

is a finite étale covering with group $A(u)$ (cf. [Shoji, 1988, p. 74]). We shall denote by $\mathcal{N}^G$ the set of all pairs (C, ψ), where C is a unipotent class in G and $\psi \in \operatorname{Irr} A(u)$. By the above, $\mathscr{I}$ may be naturally identified with $\mathcal{N}^G$, using a Springer morphism.

4.5.5 Assume now that G is defined over $\mathbb{F}_q$, with Frobenius endomorphism F. Then F acts on $\mathscr{I}$ by

$$F : (\mathcal{O}, \mathscr{F}) \to (F^{-1}(\mathcal{O}), F^*(\mathscr{F})),$$

where $F^*(\mathscr{F})$ is an inverse image of local systems. If C is F-stable and $u \in C^F$, we also let F act on $\mathcal{N}^G$ in the obvious manner, and then the F-actions are compatible with our identification of $\mathscr{I}$ and $\mathcal{N}^G$. If C is not F-stable, more care is needed to describe the action. (But it is still respected by F.) The correspondence between blocks and triples $(L, \mathfrak{o}, \mathscr{L})$ also respects the F-action (cf. [Lusztig, 1986a, 24.2]). Thus, F permutes the blocks. In fact only the F-stable blocks feature in Lusztig's character formula, so we will not be interested in blocks which are not F-stable. (Note that the principal block is always F-stable.)

4.5.6 Following the natural parametrisation of the elements of a block $\mathscr{I}_0$ (associated with a triple $(L, \mathfrak{o}, \mathscr{L})$) by the irreducible characters of W_L, we set

$$\operatorname{Irr} W_L = \{V_\iota \mid \iota \in \mathscr{I}_0\}$$

(where the V_ι are regarded as modules). For $\iota = (\mathcal{O}, \mathscr{F}) \in \mathscr{I}_0$ we define $\operatorname{supp}(\iota) = \mathcal{O}$. With respect to a fixed rational Levi subgroup L, we may parametrise the G^F-orbits of the rational Levi subgroups which are G-conjugate to L by the F-classes of W_L, using the Lang-Steinberg theorem, in the same manner as for maximal tori. We let $\mathcal{Z}_{L_w}$ denote the centre of the Levi subgroup L_w corresponding to the F-class of $w \in W_L$. (In fact $\mathcal{Z}_{L_w}$ is connected if $Z(G)$ is by [Digne and Michel, 1991, Lemma 13.14].) We also let $\mathfrak{z}(\mathfrak{l})$ denote the centre of $\mathfrak{l} = \operatorname{Lie} L$.

Theorem. ([Lusztig, 1992, Theorem 7.3]) *Let G be a connected reductive group with a split $\mathbb{F}_q$-structure, and let $\mathscr{I}_0$ be an F-stable block and let $(\gamma_e)_{\mathscr{I}_0} : \mathfrak{g}^F \to \bar{\mathbb{Q}}_l$ be the function defined by*

$$\sum_{\iota, \iota', \iota_1 \in \mathscr{I}_0} q^{f'(\iota, \iota_1)} \zeta^{-1} |W_L|^{-1} \sum_{w \in W_L} \operatorname{Tr}(w, V_\iota) \operatorname{Tr}(w, V_{\hat{\iota}_1}) |\mathcal{Z}_{L_w}^{\circ F}| P'_{\iota', \iota} \overline{\mathcal{Y}_{\iota'}(-f)} \mathfrak{X}_{\iota_1},$$

where

$$
\begin{aligned}
f'(\iota, \iota_1) &= -\dim \operatorname{supp}(\iota_1)/2 + \dim \operatorname{supp}(\iota)/2 \\
&\quad -(\dim(\operatorname{Ad} G)e)/2 + (\dim \mathfrak{g}/\mathfrak{z}(\mathfrak{l}))/2,
\end{aligned}
$$

ζ *is a certain fourth root of 1 and* $\iota \mapsto \hat{\iota}$ *is a certain bijection* $\mathscr{I}_0 \to \mathscr{I}_0$ *(both defined in [Lusztig, 1992]). Then*

$$\gamma_e = \sum_{\mathscr{I}_0} (\gamma_e)_{\mathscr{I}_0},$$

where $\mathscr{I}_0$ *runs over the set of all F-stable blocks.*

Remark. The functions $\mathfrak{X}_\iota, \mathfrak{Y}_\iota$ are analogous to the Green polynomials in Theorem 4.4.2 (in fact, related to generalised Green functions). These are certain nilpotently supported functions $\mathfrak{g}^F \to \bar{\mathbb{Q}}_l$, and the $P'_{\iota',\iota}$ are related combinatorial objects (cf. [Lusztig, 1992, §§6.4 – 6.6]). Much is known about these in the special case that $\mathscr{I}_0$ is the principal block and we shall exploit this information in the course of the proof of Theorem 4.5.1.

We will now prove Theorem 4.5.1 under the assumption that G has a split $\mathbb{F}_q$ structure. Since this is equivalent to τ acting trivially on Π, it follows that all geometric unipotent classes are F-stable. We show how to remove this assumption in the next subsection.

4.5.7 In addition to the set-up of Section 4.3 we must show that the F-stable blocks may be parametrised independently of q, in order to establish a rigorous foundation for the proof of Theorem 4.5.1. Before we consider blocks, however, we first describe a treatment of Levi subgroups which is independent of q. Recall that, with respect to fixed data $(X(T), \Sigma, Y(T), \Sigma^\vee, \tau)$, we have fixed a choice of maximally split maximal torus for each prime power q, and that we have fixed a simple system $\Pi \subset \Sigma$ (such that $\tau(\Pi) = \Pi$) so that a rational Borel subgroup $B \supset T$ is determined. For each subset $J \subset \Pi$ such that $\tau(J) = J$ we let P_J denote the corresponding standard parabolic subgroup containing B, and L_J the unique Levi subgroup of P_J containing T. Since both T and B are F-stable, so are P_J and L_J. As mentioned above the G^F-orbits of F-stable Levi subgroups conjugate to L_J are parametrised by the F-classes of W_{L_J}. We have thus parametrised all F-stable Levi subgroups of G (up to G^F-action) independently of q.

4.5.8 Now we move on to consider blocks. As mentioned a block is F-stable precisely when the corresponding triple $(L, \mathfrak{o}, \mathscr{L})$ is F-stable. I.e. its image under F is in the same G-orbit. Any such L is G-conjugate to some L_J so we may assume that our triple is $(L_J, \mathfrak{o}, \mathscr{L})$. Since the F-action on unipotent classes is independent of q, the same is true of nilpotent orbits, since Springer morphisms respect the F-action. So, in order to parametrise the F-stable blocks independently of q it just remains to parametrise the irreducible, L-equivariant, $\bar{\mathbb{Q}}_l$-local systems independently of q. But, as mentioned earlier, these are naturally parametrised by the elements of $\operatorname{Irr} A(u)$, for any u such that $\sigma(u) \in \mathfrak{o}$. If we choose u to be a split element then we can thus obtain such a q-independent parametrisation, analogous to the parametrisation of unipotent classes of G^F considered earlier. In the special case that split elements do not exist we may, by our previous discussion in Section 4.3, choose u to be from a G^F-class corresponding to a fixed q-independent label, which is sufficient for the current task. Note that in all cases F acts trivially on $A(u)$. (For u split this is clear; for the other case see [Kawanaka, 1986, 1.2.1]). Thus F preserves the isomorphism classes of the corresponding local systems. In summary we have the following.

Lemma. *The F-stable blocks are parametrised independently of q.*

4.5.9 For the remainder of this subsection we assume that G has a split $\mathbb{F}_q$-structure. Define, for ι, ι' in the same F-stable block,

$$\omega_{\iota,\iota'} = |W_L|^{-1} q^{-\operatorname{codim} \mathcal{O}/2 - \operatorname{codim} \mathcal{O}'/2 + \dim \mathfrak{z}(\mathfrak{l})} \sum_{w \in W_L} \operatorname{Tr}(w, V_\iota) \operatorname{Tr}(w, V_{\iota'}) \frac{|G^F|}{|\mathcal{Z}_{L_w}^{\circ F}|}$$

where $\mathcal{O} = \operatorname{supp}(\iota)$ and $\mathcal{O}' = \operatorname{supp}(\iota')$. Set $\omega_{\iota,\iota'} = 0$ if ι, ι' are in different blocks (cf. [Lusztig, 1992, §6.5]). Also define

$$\alpha_{\iota,\iota'} = \sum_{w \in W_L} \operatorname{Tr}(w, V_\iota) \operatorname{Tr}(w, V_{\iota'}) |\mathcal{Z}_{L_w}^{\circ F}|.$$

One may check, using the relations from [Lusztig, 1992, §§6.5, 6.6] that

$$\sum_{x \in \mathfrak{g}_{\mathrm{nil}}^F} \overline{\mathcal{X}_\iota(x)} \mathcal{X}_{\iota'}(x) = \omega_{\iota,\iota'}.$$

It follows that the class functions $(\gamma_e)_{\mathscr{I}_0}$ are mutually orthogonal. We may therefore write

$$\langle \gamma_e, \gamma_e \rangle = \sum_{\mathscr{I}_0} \langle (\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0} \rangle,$$

summing over the F-stable blocks. Each summand may be written as follows.

$$|G^F|^{-1} \sum_{\iota, \iota_1, j, j_1 \in \mathscr{I}_0} q^{f'(\iota, \iota_1) + f'(j, j_1)} |W_L|^{-2} \overline{\alpha_{\iota, \hat{\iota}_1}} \alpha_{j, \hat{j}_1} \omega_{\iota_1, j_1}$$

$$\times \sum_{\iota', j' \in \mathscr{I}_0} P'_{\iota', \iota} \overline{\mathcal{Y}_{\iota'}(-f)} P'_{j', j} \mathcal{Y}_{j'}(-f). \tag{4.9}$$

4.5.10 From now on we will assume that $X/\mathbb{Z}\Sigma$ is torsion-free so that we may begin talking about polynomials in q as in Section 4.3. $|\mathcal{Z}_{L_w}^{\circ F}|$ can be seen to be a polynomial in q by, e.g., [Carter, 1993, p.74].

Lemma. *Under the assumptions of Theorem 4.5.1,* $\dim \operatorname{End}_{\overline{\mathbb{Q}}_l G^F} \Gamma_e$ *is a polynomial in q with rational coefficients, and, for each block $\mathscr{I}_0$,*

$$|G^F| \langle (\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0} \rangle$$

is a Laurent polynomial in q with rational coefficients.

Proof. By Lemma 4.5.8 we may consider the contribution from each F-stable block independently. The second statement is clear with respect to the top line of (4.9) if the $\omega_{\iota, j}$ are polynomials in q for any $\iota, j \in \mathscr{I}_0$. But this follows from Argument ♣, used with the fact (cf. [Lusztig, 1992, §6.5]) that the $\omega_{\iota, j}$ are integers. The corresponding statement for the $P'_{\iota, j}$ then follows from the fact that they are defined ([Lusztig, 1992, p. 151]) in terms of the $\omega_{\iota, j}$. Now let $\iota = (\mathcal{O}, \mathscr{F}) \in \mathscr{I}_0$ and assume (as we may, since G has a split $\mathbb{F}_q$-structure) that $\mathcal{O}$ is F-stable, and let $x \in \mathcal{O}^F$ and $u = \sigma^{-1}(x)$. Then $\mathcal{Y}_\iota(x)$ is defined (cf. [Lusztig, 1992, §6]) in terms of the action of $A(u)$ on the stalk $\mathscr{F}_u$ at u, and a certain choice of scalar multiple. Subject to the conventions of Section 4.3 $\mathscr{F}_u$ and the action of $A(u)$ on it are independent of q. So it remains to check that this scalar multiple can be chosen independently of q. One way of seeing this is via [Geck,

1999, (2.2)], where the scalar is uniquely determined by a choice of extension of the character of $A(u)$ corresponding to $\mathscr{F}$ to the semidirect product $A(u)\langle F \rangle$. Hence, the scalar can be chosen independently of q. The second statement then follows by applying Argument ♣.   □

4.5.11 Define the degree of a Laurent polynomial $\sum \alpha_i t^i$ to be the largest integer i such that $\alpha_i \neq 0$. Then by Lemma 4.5.10, in order to prove Theorem 4.5.1 it is sufficient to consider

$$\deg \left(|G^F| \left\langle (\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0} \right\rangle \right)$$

for the various blocks. The following lemma describes some properties of the degrees of polynomials involved in (4.9) in the case that $\mathscr{I}_0$ is the principal block.

Lemma. *Let ι, ι', ι_0 belong to the principal block, with $\mathrm{supp}(\iota_0)$ equal to the regular nilpotent orbit, and let $n = \mathrm{rank}\, G$. Then the following hold.*

(i) ι_0 *is unique with this property.*

(ii) $\deg \omega_{\iota,\iota'} \leq (\dim \mathrm{supp}(\iota) + \dim \mathrm{supp}(\iota'))/2$, *with equality if $\iota = \iota'$.*

(iii) $\deg \alpha_{\iota,\iota'} \leq n$, *with equality if, and only if, $\iota = \iota'$.*

(iv) $\sum_{\iota' \in \mathscr{I}_0} P'_{\iota',\iota} \mathcal{Y}_{\iota'}(-f) = 1$ *if $\iota = \iota_0$.*

(v) $f'(\iota', \iota) \leq f'(\iota_0, \iota)$, *with equality if, and only if, $\iota' = \iota_0$.*

Proof. If u is regular then $A(u) = 1$ since $C_G(u)$ is connected ([Carter, 1993, Proposition 5.1.6]). This shows that the regular nilpotent class can only appear in one element of $\mathscr{I}$. Moreover, it must be in the principal block, by the Springer correspondence (cf. [Carter, 1993, §12.6]). This proves (i). (ii) is equivalent to showing that

$$\deg \sum_{w \in W_L} \mathrm{Tr}(w, V_\iota) \, \mathrm{Tr}(w, V_{\iota'}) \frac{|G^F|}{|\mathcal{Z}_{L_w}^{\circ F}|} \leq \dim G - n, \qquad (4.10)$$

with equality if $\iota = \iota'$. Since we are in the principal block, $\mathcal{Z}_{L_w}^{\circ F} = T_w^F$, and its order is always a polynomial in q of degree n. (See, e.g., [Carter, 1993, Chapter

2].) Then the required statement is an elementary exercise in character theory and properties of sums of rational functions. (iii) follows by a similar argument.

Since $\mathscr{I}_0$ is the principal block we may use the theory of Green functions (as opposed to generalised Green functions), and for this we refer to [Shoji, 1987, §5]. We may define related class functions associated with irreducible characters of the Weyl group as follows. For $\chi \in \mathrm{Irr}(W)$, let

$$Q_\chi = |W|^{-1} \sum_{w \in W} \chi(w) Q_{T_w}.$$

Then, for $\iota \in \mathscr{I}_0$, the principal block, we have

$$Q_\iota = \sum_{j \in \mathscr{I}_0} P_{j,\iota} \mathcal{Y}_j,$$

where $\chi \leftrightarrow \iota$ is the Springer correspondence. Since the element of $\mathrm{Irr}(W)$ corresponding to ι_0 by the Springer correspondence is the trivial character, the expression in (iv) may be written as $Q_1(-f)(q^{-1})$. But $Q_1 = 1$ by, e.g., [Digne and Michel, 1991, Proposition 12.13].

(v) follows from the fact that the dimension of the regular nilpotent orbit is strictly greater than that of the others. $\qquad\square$

4.5.12 We may now deduce the following.

Proposition. *Let $\mathscr{I}_0$ be the principal block. Then*

$$\deg \left(|G^F| \left\langle (\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0} \right\rangle \right) = \dim G + \dim C_G(u).$$

Proof. We will show that the required degree is obtained by a careful choice of parameters ι, ι_1, j, j_1. We choose $\iota = j = \iota_0$, the unique parameter corresponding to the regular orbit, and then choose $\iota_1 = j_1$ such that $\hat{\iota}_1 = \iota$. (This is a unique choice since $\hat{\ }$ is a bijection.) With this fixed choice we see, by Lemma 4.5.11, that the degree is given by

$$2f'(\iota_0, \iota_1) + \deg \alpha_{\iota_0, \hat{\iota}_1} + \deg \alpha_{\iota_0, \hat{\iota}_1} + \deg \omega_{\iota_1, \iota_1}$$
$$= \dim G + \dim \mathrm{supp}(\iota_0) + n - \dim(\mathrm{Ad}\,G)e.$$

It is known that the regular orbit has dimension $\dim G - n$. It follows that we obtain the required degree. To complete the proof, note that by Lemma 4.5.11, any deviation from this choice of ι, ι_1, j, j_1 gives rise to a polynomial of strictly lower degree. $\qquad\square$

4.5.13 One can check (using (4.9)) that $|G^F|\langle(\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0}\rangle$ is, in fact, monic. But this is not sufficient to prove Theorem 4.5.1 since it may be the case that the leading terms from non-principal blocks exceed or annihilate this contribution. We shall, however, show that this is impossible, by describing some of the features of non-principal blocks. The associated Levi subgroups will no longer be maximal tori and so we have

$$\deg |\mathcal{Z}_{L_w}^{\circ F}| = \operatorname{rank} L_w = \dim \mathfrak{z}(\mathfrak{l}) =: m < n. \tag{4.11}$$

Lemma. *Let ι, ι' belong to the non-principal block $\mathscr{I}_0$. Then the following hold.*

(i) *No element of $\mathscr{I}_0$ is supported by the regular orbit.*

(ii) $\deg \omega_{\iota,\iota'} \leq (\dim \operatorname{supp}(\iota) + \dim \operatorname{supp}(\iota'))/2$, *with equality if $\iota = \iota'$.*

(iii) $\deg \alpha_{\iota,\iota'} \leq m$, *with equality if, and only if $\iota = \iota'$.*

Proof. This follows from (4.11) and the proof of Lemma 4.5.11. $\qquad\square$

4.5.14 We may now deduce, for a non-principal block $\mathscr{I}_0$, that the degree of a typical term of

$$|G^F|\,\langle(\gamma_e)_{\mathscr{I}_0}, (\gamma_e)_{\mathscr{I}_0}\rangle$$

with respect to (4.9) is not greater than

$$f'(\iota, \iota_1) + f'(j, j_1) + \deg \alpha_{\iota,\hat{\iota}_1} + \deg \alpha_{j,\hat{\iota}_1} + \deg \omega_{\iota_1,j_1},$$

but this is strictly less than $\dim G + \dim C_G(u)$. This completes the proof of Theorem 4.5.1.

4.6 Groups with a non-split $\mathbb{F}_q$-structure

4.6.1 In the previous subsection the analysis of (4.9) was made easier by the fact that the geometric unipotent classes were fixed by the Frobenius action. (In particular it was possible to deduce useful information about the $\mathcal{Y}_\iota$.) For non-split groups we may reduce to a situation where the only functions $\mathcal{Y}_\iota$ that appear are such that $\mathrm{supp}(\iota)$ is F-stable. We do this by considering a transposed version of generalised Gelfand-Graev characters as follows (see [Lusztig, 1992, §7.5]) .

Let $\mathcal{O}$ be an F-stable nilpotent orbit such that $\iota = (\mathcal{O}, \mathscr{F})$ belongs to an F-stable block $\mathscr{I}_0$, and let $x_1, \ldots, x_r$ be representatives for the G^F-classes in $\mathcal{O}^F$. Let a denote the order of $C_{G^F}(x_i)$, for some i. (Clearly a is independent of i.) Also, let a_i denote the order of $C_{G^F}(x_i)^F$. Then we define

$$\gamma_\iota = \sum_{i=1}^r a a_i^{-1} \mathcal{Y}_\iota(x_i) \gamma_{x_i}.$$

We also have

$$\gamma_{x_i} = a^{-1} \sum_k \overline{\mathcal{Y}_k(x_i)} \gamma_k,$$

where the sum is taken over the F-fixed k such that $\mathrm{supp}(k) = \mathcal{O}$. Hence we may write

$$\langle \gamma_{x_i}, \gamma_{x_i} \rangle = a^{-2} \sum_{k,l} \overline{\mathcal{Y}_k(x_i)} \mathcal{Y}_l(x_i) \langle \gamma_k, \gamma_l \rangle. \tag{4.12}$$

4.6.2 In [Lusztig, 1992] there is a formula for the γ_ι, valid for split groups, which is analogous to the one in Theorem 4.5.6. Lusztig also hints at how to alter various formulas contained in [Lusztig, 1992] to make them valid for non-split groups. This is carried out explicitly in [Digne et al., 2003], from which we borrow the following formula ([Digne et al., 2003, (6.1)]):

$$\gamma_\iota = \sum_{\iota, \iota_1 \in \mathscr{I}_0} q^{f'(\iota, \iota_1)} \zeta^{-1} a |W_L|^{-1} \sum_{w \in W_L} \mathrm{Tr}(wF, \tilde{V}_\iota) \, \mathrm{Tr}(wF, \tilde{V}_{\hat{\iota}_1}) |\mathcal{Z}_L^{\circ wF}| P'_{\iota_0, \iota} \varepsilon_{\iota_1} \mathcal{X}_{\iota_1}.$$

Here, $\tilde{V}_\iota$ are certain extensions of V_ι to modules for the group $W_L \langle F \rangle$. In [Lusztig, 1986b, §2.2] they are chosen in an explicit and unique way, which is

independent of q. Lusztig calls them the *preferred extensions*. By [Digne et al., 2003, Remark 3.6], the scalar $\varepsilon_{\iota_1} = \pm 1$ is determined by a preferred extension and thus is also independent of q.

Thus, we have

$$\langle \gamma_k, \gamma_l \rangle = |G^F|^{-1} \sum_{\iota, \iota_1, j, j_1 \in \mathscr{I}_0} q^{f'(\iota, \iota_1) + f'(j, j_1)} a^2 |W_L|^{-2} \overline{\alpha_{\iota, \hat{\iota}_1}} \alpha_{j, \hat{j}_1} \omega_{\iota_1, j_1}$$

$$\times \varepsilon_{\iota_1} {P'}_{k, \iota} \varepsilon_{j_1} {P'}_{l, j}, \tag{4.13}$$

where

$$\alpha_{\iota, j} = \sum_{w \in W_L} \operatorname{Tr}(wF, \tilde{V}_\iota) \operatorname{Tr}(wF, \tilde{V}_{\hat{\iota}_1}) |\mathcal{Z}_L^{\circ wF}|$$

this time.

By combining (4.12) and (4.13) we obtain an expression for the dimension of a generalised Gelfand-Graev representation in which no function $\mathcal{Y}_\iota$ appears unless $\operatorname{supp}(\iota)$ is F-stable. Theorem 4.5.1 now follows for non-split groups by applying the same argument that we used for the split case to this expression.

4.7 Groups with a disconnected centre

4.7.1 In this subsection we consider what happens when we remove the assumption that $X/\mathbb{Z}\Sigma$ is torsion-free. In this situation $Z(G)$ may have a disconnected centre. (In fact this occurs for $G = G(p^m)$ precisely when $X/\mathbb{Z}\Sigma$ has no p'-torsion.) It should be clear from the discussion in Section 4.3 that the parametrisation of F-stable geometric unipotent classes is still independent of q. The difficulty arises when we try to parametrise, independently of q, the G^F-classes of rational points in some C^F, where C is an F-stable geometric unipotent class. In fact this is impossible in general since even the number of G^F classes in C^F may depend on q. We now explain a way of getting around this difficulty.

First note that split elements still exist and are unique up to G^F-conjugacy. (We may ignore the E_8 difficulty this time as there is only one isogeny class associated with this group and it has the property that $X/\mathbb{Z}\Sigma$ is torsion-free.) For a given finite group of Lie type G^F, with data $(X(T), \Sigma, Y(T), \Sigma^\vee, \tau)$ and

prime power q_0, set $D = D(G^F)$ to be the set of prime powers q_1 such that the component groups of F-stable geometric classes of $G(q_1)$ are isomorphic to those of $G(q_0)$ (via Spaltenstein's map), and furthermore that the F-action respects these isomorphisms. Then if $f \in \mathbb{Q}[x]$ is a polynomial such that some quantity associated with $(X(T), \Sigma, Y(T), \Sigma^\vee, \tau)$ is given by $f(q)$ for all $q \in D$, then we say that this quantity is a *polynomial in q on D*.

4.7.2 For given data $(X(T), \Sigma, Y(T), \Sigma^\vee, \tau)$ it should be possible to write down an explicit list of the possible sets D, for simple simply connected groups at least. The discussion on [Goodwin and Röhrle, 2009a, p. 8] would be a good starting point. We briefly illustrate what might happen by means of an example; cf. [Goodwin and Röhrle, 2009a, Example 2.6]. When $G = \mathrm{SL}_3$, with standard Frobenius map, and $u \in C$, where C is the regular unipotent class, there are three possibilities for D, depending on the congruence of q modulo 3. When q is a power of 3, $A(u) = 1$. When q is congruent to 1 modulo 3, $A(u) \cong \mathbb{Z}/3\mathbb{Z}$, with F acting trivially. When q is congruent to 2 modulo 3, $A(u) \cong \mathbb{Z}/3\mathbb{Z}$, but this time F acts non-trivially.

4.7.3 We now state and prove the main result of this subsection.

Theorem. *Let G be a connected reductive group, defined over $\mathbb{F}_q$, with root datum $(X(T), \Sigma, Y(T), \Sigma^\vee)$ and Frobenius endomorphism F, and let $D = D(G^F)$ be as above. Let $u \in G^F_{\mathrm{uni}}$ and let Γ_u be the corresponding generalised Gelfand-Graev representation. Then the dimension of the endomorphism algebra $\mathrm{End}_{\overline{\mathbb{Q}}_l G^F} \Gamma_u$ is a monic polynomial in q on D with rational coefficients. Moreover, its degree is given by the dimension of the centraliser $C_G(u)$.*

Proof. The proof essentially works in the same way as for groups with a connected centre. We will assume here, for simplicity, that G has a split $\mathbb{F}_q$-structure, although the non-split generalisation carries over to this case, as before. First note that the statement of the theorem is meaningful, in the sense that the generalised Gelfand-Graev characters are naturally parametrised independently of $q \in D$, by the above discussion. Next, note that Lemmas 4.5.8 and 4.5.10 still hold in this situation. Indeed, the set $\mathscr{I}$ and its partition into blocks, as well as the F-action

on it, only depends on $(X(T), \Sigma, Y(T), \Sigma^\vee)$, and the component groups, together with their F-action. The analysis of (4.9) required in the proof of Lemma 4.5.10 poses no new difficulty, although the reason why $|\mathcal{Z}_{L_w}^{\circ F}|$ is a polynomial in q is somewhat deeper in this case (see, e.g., [Carter, 1993, pp.73,74]).

The statement of Lemma 4.5.11 remains true here also, although establishing the truth of (i) requires a different argument since it may not be the case that $A(u) = 1$ for a regular unipotent element any more. However, (i) is equivalent to there only being one Springer representation associated with the regular nilpotent orbit. But this is clear from Springer's construction (as described in, e.g., [Carter, 1993, §12.6]). This fact also means that the special parameters chosen in the proof of Lemma 4.5.12 may be chosen again in the current situation and thus this result, and the monic property, remain true too. Finally, Lemma 4.5.13, (i) may not be true in the current situation, but (ii) and (iii) are true and these are, in fact, sufficient to deduce that polynomials associated with non-principal blocks have degree strictly less than $\dim G + \dim C_G(u)$. □

Appendix A: Computing canonical forms using GAP

We now present tables of all of the nilpotent canonical forms from Chapter 2 which correspond to partitions of n for $n \in \{2, 3, 4, 5\}$, together with the unipotent canonical forms for $\mathrm{GU}_n(\mathbb{F}_q)$, as in Subsection 2.5.3. These were computed using the computer algebra package GAP [GAP Group, 2008]. We omit the trivial orbits/classes, corresponding to the partition $[1, \ldots, 1]$. We begin with $n = 2$. Recall that α may be taken to be any element in $\mathbb{F}_{q^2} \setminus \mathbb{F}_q$.

Partition	Jordan form	Symmetric form	$\mathfrak{sp}_2(\mathbb{k})$	$\mathrm{GU}_2(\mathbb{F}_q)$
[2]	$\begin{pmatrix} & 1 \\ & \end{pmatrix}$	$\begin{pmatrix} & 1 \\ & \end{pmatrix}$	$\begin{pmatrix} & 1 \\ & \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha - \alpha^q \\ & 1 \end{pmatrix}$

Next we have $n = 3$. Notice the gap in the $[2, 1]$-row, owing to the fact that the corresponding class in $\mathfrak{gl}_3(\mathbb{k})$ vanishes upon restriction to $\mathfrak{so}_3(\mathbb{k})$.

Partition	Jordan form	Symmetric form	$\mathfrak{so}_3(\mathbb{k})$	$\mathrm{GU}_3(\mathbb{F}_q)$
[3]	$\begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$	$\begin{pmatrix} & 1 & \\ & & 1 \\ & & \end{pmatrix}$	$\begin{pmatrix} & 1 & \\ & & -1 \\ & & \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha - \alpha^q & \alpha^{2q} - \alpha^{q+1} \\ & 1 & \alpha - \alpha^q \\ & & 1 \end{pmatrix}$
[2, 1]	$\begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}$	$\begin{pmatrix} & 1 & \\ & & \\ & & \end{pmatrix}$		$\begin{pmatrix} 1 & & \alpha - \alpha^q \\ & 1 & \\ & & 1 \end{pmatrix}$

On Pages 110 and 111 we present the tables for $n = 4$ and 5 respectively. Recall that for the very even partition $[2, 2]$ only one of the two orbits in $\mathfrak{so}_4(\mathbb{k})$ is represented.

Partition	Jordan form	Symmetric form	$\mathfrak{sp}_4(\Bbbk)$	$\mathfrak{so}_4(\Bbbk)$	$GU_4(\mathbb{F}_q)$
[4]	$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ 1 & & & \end{pmatrix}$	$\begin{pmatrix} & & & 1 \\ & & 1 & \\ & 1 & & \\ -1 & & & \end{pmatrix}$		$\begin{pmatrix} 1 & & & \\ \alpha - \alpha^q & 1 & & \\ \alpha^{2q} - \alpha^{q+1} & \alpha - \alpha^q & 1 & \\ \alpha^{2q+1} - \alpha^{3q} & \alpha^{2q} - \alpha^{q+1} & \alpha - \alpha^q & 1 \end{pmatrix}$
[3,1]	$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} & & 1 & \\ & 1 & & \\ 1 & & & \\ & & & 1 \end{pmatrix}$		$\begin{pmatrix} & & 1 & \\ & & & 1 \\ 1 & & & \\ & -1 & & \\ & & -1 & \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ \alpha - \alpha^q & 1 & & \\ \alpha^{2} - \alpha^{q+1} & \alpha - \alpha^q & 1 & \\ \alpha^{3q} - \alpha^{q+2} & \alpha^{q+1} - \alpha^{2q} & \alpha - \alpha^q & 1 \end{pmatrix}$
[2,2]	$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & 1 \\ & & 1 & \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & & -1 \\ & & 1 & \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ \alpha - \alpha^q & & 1 & \\ & \alpha - \alpha^q & & 1 \end{pmatrix}$
[2,1,1]	$\begin{pmatrix} 1 & 1 & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} & 1 & & \\ 1 & & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$			$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ \alpha - \alpha^q & & & 1 \end{pmatrix}$

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Partition	Jordan form	Symmetric form	$\mathfrak{so}_3(\Bbbk)$	$GU_5(\mathbb{F}_q)$
[5]	$\begin{pmatrix} 1 & 1 & & & \\ & 1 & 1 & & \\ & & 1 & 1 & \\ & & & 1 & 1 \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha-\alpha^q & \alpha^{2q}-\alpha^{q+1} & \alpha^{2q}-\alpha^{q+1} & \alpha^{4q}-\alpha^{3q+1} \\ & 1 & \alpha-\alpha^q & \alpha^{2q}-\alpha^{q+1} & \alpha^{2q+1}-\alpha^{3q} \\ & & 1 & \alpha-\alpha^q & \alpha^{2q}-\alpha^{q+1} \\ & & & 1 & \alpha-\alpha^q \\ & & & & 1 \end{pmatrix}$
[4,1]	$\begin{pmatrix} 1 & 1 & & \\ & 1 & 1 & \\ & & 1 & 1 \\ & & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$		$\begin{pmatrix} 1 & \alpha-\alpha^q & \alpha^{2q}-\alpha^{q+1} & \alpha^{2q}-\alpha^{q+1} \\ & 1 & \alpha-\alpha^q & \alpha^{2q+1}-\alpha^{3q} \\ & & 1 & \alpha-\alpha^{q+1} \\ & & & 1 \end{pmatrix}$
[3,2]	$\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$		$\begin{pmatrix} 1 & \alpha-\alpha^q & \alpha^{2q}-\alpha^{q+1} \\ & 1 & \alpha-\alpha^q \\ & & 1 \end{pmatrix}$
[3,1,1]	$\begin{pmatrix} 1 & 1 & \\ & 1 & 1 \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 & -1 \\ & 1 & -1 \\ & & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha-\alpha^q & \alpha^2-\alpha^{q+1} & \alpha^{3q}-\alpha^{q+2} \\ & 1 & \alpha-\alpha^q & \alpha^{q+1}-\alpha^{2q} \\ & & 1 & \alpha-\alpha^q \\ & & & 1 \end{pmatrix}$
[2,2,1]	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & -1 \\ & -1 \end{pmatrix}$	$\begin{pmatrix} 1 & \alpha-\alpha^q \\ & 1 \end{pmatrix}$
[2,1,1,1]	$\begin{pmatrix} 1 \end{pmatrix}$			$\begin{pmatrix} 1 & \alpha-\alpha^q \\ & 1 \end{pmatrix}$

APPENDIX A: COMPUTING CANONICAL FORMS USING GAP

We now present a GAP function `SymmetricForm(` *list* `)` for computing the symmetric form matrix from Theorem 2.3.4 corresponding to any given partition of a natural number, written as a list of its parts. Functions for the other normal forms can also be created rather easily by using this function.

```
# we begin by constructing the functions l_i and k_i from Page 18

l:=function(p,i)
  local r, j;
  r:=0;
  for j in [1..Length(p)] do
  if IsOddInt(p[j]) and p[j] >= 2*i - 1 then
    r:=r+1;
    fi;
  od;
  return r;
end;

k:=function(p,i)
  local r, j;
  r:=0;
  for j in [1..Length(p)] do
  if IsEvenInt(p[j]) and p[j] >= 2*i then
    r:=r+1;
    fi;
  od;
  return r;
end;

# now we define the function which computes the symmetric form
# corresponding to a partition
```

```
SymmetricForm:=function(p)
  local I, J, n, M, i, j;
  I:=[]; J:=[];
  n:=Sum(p);
  M:=NullMat(n,n);


# first we'll compute J: the top right and bottom left corner of
# each of the blocks C, B_{j_1}, B_{j_2}, B_{j_3}, ... is stored

if k(p,1) <> 0 then
  J[1]:=[[(n-l(p,1))/2,(n+l(p,1))/2+1],[(n-l(p,1))/2
  -(k(p,1)-1),(n+l(p,1))/2+1+(k(p,1)-1)]];
  i:=1;
  while k(p,i+1) <> 0 do
    J[i+1]:=[[J[i][2][2],J[i][2][2]+(l(p,i+1)+1)],
    [J[i][2][2]-(k(p,i)-1),J[i][2][2]+(l(p,i+1)+1)+(k(p,i+1)-1)]];
    i:=i+1;
    od;
  fi;


# now we'll compute I: the top right and bottom left corner of each
# of the blocks B_{i_1}, B_{i_2}, B_{i_3}, ... is stored

if l(p,2) <> 0 then
  I[1]:=[[(n+l(p,1))/2,(n+l(p,1))/2+k(p,1)+1],
  [(n+l(p,1))/2-(l(p,1)-1),(n+l(p,1))/2+k(p,1)+1+(l(p,2)-1)]];
  i:=1;
  while l(p,i+2) <> 0 do
    I[i+1]:=[[I[i][2][2],I[i][2][2]+(k(p,i+1)+1)],
    [I[i][2][2]-(l(p,i+1)-1),I[i][2][2]+(k(p,i+1)+1)+(l(p,i+2)-1)]];
    i:=i+1;
    od;
  fi;
```

APPENDIX A: COMPUTING CANONICAL FORMS USING GAP

```
# starting with our n by n matrix of zeros M we now insert 1s in
# the appropriate places for all blocks in J

if k(p,1) <> 0 then
  for j in [0..k(p,1)-1] do
    M[J[1][2][1]+j][J[1][2][2]-j]:=1;
    od;
  i:=1;
  while k(p,i+1) <> 0 do
    for j in [0..k(p,i+1)-1] do
      M[J[i+1][2][1]+j][J[i+1][2][2]-j]:=1; # 1s for the B-blocks
      M[n+1-(J[i+1][2][2]-j)][n+1-(J[i+1][2][1]+j)]:=1; # A-blocks
      od;
    i:=i+1;
    od;
  fi;

# we now insert 1s in the appropriate places for all blocks in I

i:=0;
while l(p,i+2) <> 0 do
  for j in [0..Int((l(p,i+2)+1)/2)-1] do
    M[I[i+1][2][1]+j][I[i+1][2][2]-j]:=1;# B-blocks
    M[I[i+1][1][1]-j][I[i+1][1][2]+j]:=1; # B-blocks
    M[n+1-(I[i+1][2][2]-j)][n+1-(I[i+1][2][1]+j)]:=1; # A-blocks
    M[n+1-(I[i+1][1][2]+j)][n+1-(I[i+1][1][1]-j)]:=1; # A-blocks
    od;
  i:=i+1;
  od;

  return(M);
end;
```

Appendix B: Computing $\dim \operatorname{End}_{\bar{\mathbb{Q}}_l G^F} \Gamma_u$ using GAP

We now present tables listing the dimensions of the endomorphism algebras of all GGGRs for the general linear and general unitary groups, up to rank 7. These were computed in GAP using the character formula from [Kawanaka, 1985], together with data from the CHEVIE library; cf. [Geck et al., 1996].

Group	Class	Dimension of the endomorphism algebra
$\mathrm{GL}_2(q)$	$[1,1]$	$q(q+1)(q-1)^2$
	$[2]$	$q(q-1)$
$\mathrm{GL}_3(q)$	$[1,1,1]$	$q^3(q^2+q+1)(q+1)(q-1)^3$
	$[2,1]$	$(q^3+q^2-1)(q-1)^2$
	$[3]$	$q^2(q-1)$
$\mathrm{GL}_4(q)$	$[1,1,1,1]$	$q^6(q^2+q+1)(q^2+1)(q+1)^2(q-1)^4$
	$[2,1,1]$	$q(q^5+q^4+q^3-q-1)(q+1)(q-1)^3$
	$[2,2]$	$(q^5-q^2+1)(q+1)(q-1)^2$
	$[3,1]$	$q(q^3+q^2-1)(q-1)^2$
	$[4]$	$q^3(q-1)$
$\mathrm{GL}_5(q)$	$[1,1,1,1,1]$	$q^{10}(q^4+q^3+q^2+q+1)(q^2+q+1)(q^2+1)(q+1)^2(q-1)^5$
	$[2,1,1,1]$	$q^3(q^7+q^6+q^5+q^4-q^2-q-1)(q^2+q+1)(q+1)(q-1)^4$
	$[2,2,1]$	$q(q^8+q^7+q^6-q^4-2q^3-q^2+q+1)(q+1)(q-1)^3$
	$[3,1,1]$	$(q^4+q^2-1)(q^3+q^2-1)(q+1)(q-1)^3$
	$[3,2]$	$q(q^6+q^5-q^3-q^2+1)(q-1)^2$
	$[4,1]$	$q^2(q^3+q^2-1)(q-1)^2$
	$[5]$	$q^4(q-1)$

Group	Class	Dimension of the endomorphism algebra	
$\mathrm{GL}_6(q)$	$[1,1,1,1,1,1]$	$q^{15}(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)^2(q^2 + 1)(q^2 - q + 1)$ $(q + 1)^3(q - 1)^6$	[continued...]
	$[2,1,1,1,1]$	$q^6(q^9 + q^8 + q^7 + q^6 + q^5 - q^3 - q^2 - q - 1)(q^2 + q + 1)$ $(q^2 + 1)(q + 1)^2(q - 1)^5$	[continued...]
	$[2,2,1,1]$	$q^3(q^{11} + q^{10} + 2q^9 + q^8 + q^7 - q^6 - 2q^5 - 3q^4 - 2q^3 + q + 2)$ $(q + 1)^2(q - 1)^4$	[continued...]
	$[2,2,2]$	$q^4(q^8 - q^4 - q^3 + q + 1)(q^2 + q + 1)(q + 1)(q - 1)^3$	
	$[3,1,1,1]$	$q(q^{10} + q^9 + q^8 + q^7 - q^5 - 2q^4 - 2q^3 + q + 1)(q^2 + q + 1)$ $(q + 1)(q - 1)^4$	[continued...]
	$[3,2,1]$	$(q^{11} + 2q^{10} + 2q^9 + q^8 - q^7 - 3q^6 - 4q^5 - 2q^4 + 3q^3$ $+ 3q^2 - 1)(q - 1)^3$	[continued...]
	$[3,3]$	$q^2(q^7 - q^4 + 1)(q + 1)(q - 1)^2$	
	$[4,1,1]$	$q(q^4 + q^2 - 1)(q^3 + q^2 - 1)(q + 1)(q - 1)^3$	
	$[4,2]$	$q^2(q^6 + q^5 - q^3 - q^2 + 1)(q - 1)^2$	
	$[5,1]$	$q^3(q^3 + q^2 - 1)(q - 1)^2$	
	$[6]$	$q^5(q - 1)$	
$\mathrm{GL}_7(q)$	$[1,1,1,1,1,1]$	$q^{21}(q^6 + q^5 + q^4 + q^3 + q^2 + q + 1)(q^4 + q^3 + q^2 + q + 1)$ $(q^2 + q + 1)^2(q^2 - q + 1)(q^2 + 1)(q + 1)^3(q - 1)^7$	[continued...]
	$[2,1,1,1,1,1]$	$q^{10}(q^{11} + q^{10} + q^9 + q^8 + q^7 + q^6 - q^4 - q^3 - q^2 - q - 1)$ $(q^4 + q^3 + q^2 + q + 1)(q^2 + q + 1)(q^2 + 1)(q + 1)^2(q - 1)^6$	[continued...]
	$[2,2,1,1,1]$	$q^5(q^{15} + q^{14} + 2q^{13} + 2q^{12} + 2q^{11} + q^{10} - 2q^8 - 3q^7 - 4q^6$ $- 3q^5 - q^4 + 2q^2 + 2q + 1)(q^2 + q + 1)(q + 1)^2(q - 1)^5$	[continued...]
	$[2,2,2,1]$	$q^3(q^{15} + q^{14} + q^{13} + q^{12} - q^{10} - 2q^9 - 3q^8 - 2q^7$ $+ 2q^5 + 3q^4 + 2q^3 - q - 1)(q^2 + q + 1)(q + 1)(q - 1)^4$	[continued...]
	$[3,1,1,1,1]$	$q^3(q^{13} + q^{12} + q^{11} + q^{10} + q^9 - q^7 - 2q^6 - 2q^5 - 2q^4$ $+ q^2 + q + 1)(q^2 + q + 1)(q^2 + 1)(q + 1)^2(q - 1)^5$	[continued...]
	$[3,2,1,1]$	$q(q^{15} + 2q^{14} + 3q^{13} + 3q^{12} + 2q^{11} - 3q^9 - 6q^8 - 7q^7$ $- 5q^6 + q^5 + 6q^4 + 6q^3 + q^2 - 2q - 1)(q + 1)(q - 1)^4$	[continued...]
	$[3,2,2]$	$q(q^{14} + q^{13} + q^{12} - q^{10} - 2q^9 - 2q^8 - q^7 + q^6$ $+ 3q^5 + 2q^4 - 2q^2 - q + 1)(q + 1)(q - 1)^3$	[continued...]
	$[3,3,1]$	$q(q^{12} + q^{11} + q^{10} - q^8 - 2q^7 - 2q^6 - q^5 + 2q^4 + 3q^3$ $+ q^2 - q - 1)(q + 1)(q - 1)^3$	[continued...]
	$[4,1,1,1]$	$(q^{12} + q^{11} + q^{10} + q^9 - q^7 - 2q^6 - 3q^5 + 2q^3 + 2q^2 - 1)$ $(q^2 + q + 1)(q + 1)(q - 1)^4$	[continued...]
	$[4,2,1]$	$q(q^{11} + 2q^{10} + 2q^9 + q^8 - q^7 - 3q^6 - 4q^5 - 3q^4 + 2q^3$ $+ 4q^2 + q - 1)(q - 1)^3$	[continued...]
	$[4,3]$	$q^3(q^8 + q^7 - q^5 - q^4 + 1)(q - 1)^2$	
	$[5,1,1]$	$q^2(q^4 + q^2 - 1)(q^3 + q^2 - 1)(q + 1)(q - 1)^3$	
	$[5,2]$	$q^3(q^6 + q^5 - q^3 - q^2 + 1)(q - 1)^2$	
	$[6,1]$	$q^4(q^3 + q^2 - 1)(q - 1)^2$	
	$[7]$	$q^6(q - 1)$	

APPENDIX B: COMPUTING GGGR ENDOMORPHISM
ALGEBRA DIMENSIONS USING GAP

Group	Class	Dimension of the endomorphism algebra
$\mathrm{GU}_2(q)$	$[1,1]$	$q(q+1)^2(q-1)$
	$[2]$	$q(q+1)$
$\mathrm{GU}_3(q)$	$[1,1,1]$	$q^3(q^2-q+1)(q+1)^3(q-1)$
	$[2,1]$	$(q^3-q^2+1)(q+1)^2$
	$[3]$	$q^2(q+1)$
$\mathrm{GU}_4(q)$	$[1,1,1,1]$	$q^6(q^2+1)(q^2-q+1)(q+1)^4(q-1)^2$
	$[2,1,1]$	$q(q^5-q^4+q^3-q+1)(q+1)^3(q-1)$
	$[2,2]$	$(q^5+q^2-1)(q+1)^2(q-1)$
	$[3,1]$	$q(q^3-q^2+1)(q+1)^2$
	$[4]$	$q^3(q+1)$
$\mathrm{GU}_5(q)$	$[1,1,1,1,1]$	$q^{10}(q^4-q^3+q^2-q+1)(q^2+1)(q^2-q+1)(q+1)^5(q-1)^2$
	$[2,1,1,1]$	$q^3(q^7-q^6+q^5-q^4+q^2-q+1)(q^2-q+1)(q+1)^4(q-1)$
	$[2,2,1]$	$q(q^8-q^7+q^6-q^4+2q^3-q^2-q+1)(q+1)^3(q-1)$
	$[3,1,1]$	$(q^4+q^2-1)(q^3-q^2+1)(q+1)^3(q-1)$
	$[3,2]$	$q(q^6-q^5+q^3-q^2+1)(q+1)^2$
	$[4,1]$	$q^2(q^3-q^2+1)(q+1)^2$
	$[5]$	$q^4(q+1)$

We now present an example of the GAP code that we used to compute these tables. The first part contains data extracted from the CHEVIE library specific to the group, in this case $\mathrm{GU}_4(q)$. The second part contains functions which do not need to be changed when we vary the group.

```
q:=Indeterminate(Rationals, "q");

######### DATA INPUT ###################

GRank:= 4;
# the rank of the group

epsilon:=-1;
# set to 1 for GL and -1 for GU

GSize:= (q+1)^4*(q-1)^2*(q^2+1)*q^6*(q^2-q+1);
# the cardinality of the group
```

Group	Class	Dimension of the endomorphism algebra	
	$[1,1,1,1,1,1]$	$q^{15}(q^4 - q^3 + q^2 - q + 1)(q^2 + q + 1)(q^2 + 1)(q^2 - q + 1)^2$	[continued...]
		$(q+1)^6(q-1)^3$	
	$[2,1,1,1,1]$	$q^6(q^9 - q^8 + q^7 - q^6 + q^5 - q^3 + q^2 - q + 1)(q^2 + 1)$	[continued...]
		$(q^2 - q + 1)(q+1)^5(q-1)^2$	
	$[2,2,1,1]$	$q^3(q^{11} - q^{10} + 2q^9 - q^8 + q^7 + q^6 - 2q^5 + 3q^4 - 2q^3 + q - 2)$	[continued...]
		$(q+1)^4(q-1)^2$	
	$[2,2,2]$	$q^4(q^8 - q^4 + q^3 - q + 1)(q^2 - q + 1)(q+1)^3(q-1)$	
GU$_6(q)$	$[3,1,1,1]$	$q(q^{10} - q^9 + q^8 - q^7 + q^5 - 2q^4 + 2q^3 - q + 1)(q^2 - q + 1)$	[continued...]
		$(q+1)^4(q-1)$	
	$[3,2,1]$	$(q^{11} - 2q^{10} + 2q^9 - q^8 - q^7 + 3q^6 - 4q^5 + 2q^4 + 3q^3$	[continued...]
		$-3q^2 + 1)(q+1)^3$	
	$[3,3]$	$q^2(q^7 + q^4 - 1)(q+1)^2(q-1)$	
	$[4,1,1]$	$q(q^4 + q^2 - 1)(q^3 - q^2 + 1)(q+1)^3(q-1)$	
	$[4,2]$	$q^2(q^6 - q^5 + q^3 - q^2 + 1)(q+1)^2$	
	$[5,1]$	$q^3(q^3 - q^2 + 1)(q+1)^2$	
	$[6]$	$q^5(q+1)$	
	$[1,1,1,1,1,1]$	$q^{21}(q^6 - q^5 + q^4 - q^3 + q^2 - q + 1)(q^4 - q^3 + q^2 - q + 1)$	[continued...]
		$(q^2 + q + 1)(q^2 + 1)(q^2 - q + 1)^2(q+1)^7(q-1)^3$	
	$[2,1,1,1,1,1]$	$q^{10}(q^{11} - q^{10} + q^9 - q^8 + q^7 - q^6 + q^4 - q^3 + q^2 - q + 1)$	[continued...]
		$(q^4 - q^3 + q^2 - q + 1)(q^2 + 1)(q^2 - q + 1)(q+1)^6(q-1)^2$	
	$[2,2,1,1,1]$	$q^5(q^{15} - q^{14} + 2q^{13} - 2q^{12} + 2q^{11} - q^{10} + 2q^8 - 3q^7 + 4q^6$	[continued...]
		$-3q^5 + q^4 - 2q^2 + 2q - 1)(q^2 - q + 1)(q+1)^5(q-1)^2$	
	$[2,2,2,1]$	$q^3(q^{15} - q^{14} + q^{13} - q^{12} + q^{10} - 2q^9 + 3q^8 - 2q^7$	[continued...]
		$+2q^5 - 3q^4 + 2q^3 - q + 1)(q^2 - q + 1)(q+1)^4(q-1)$	
	$[3,1,1,1,1]$	$q^3(q^{13} - q^{12} + q^{11} - q^{10} + q^9 - q^7 + 2q^6 - 2q^5 + 2q^4$	[continued...]
		$-q^2 + q - 1)(q^2 + 1)(q^2 - q + 1)(q+1)^5(q-1)^2$	
	$[3,2,1,1]$	$q(q^{15} - 2q^{14} + 3q^{13} - 3q^{12} + 2q^{11} - 3q^9 + 6q^8 - 7q^7$	[continued...]
		$+5q^6 + q^5 - 6q^4 + 6q^3 - q^2 - 2q + 1)(q+1)^4(q-1)$	
GU$_7(q)$	$[3,2,2]$	$q(q^{14} - q^{13} + q^{12} - q^{10} + 2q^9 - 2q^8 + q^7 + q^6$	[continued...]
		$-3q^5 + 2q^4 - 2q^2 + q + 1)(q+1)^3(q-1)$	
	$[3,3,1]$	$q(q^{12} - q^{11} + q^{10} - q^8 + 2q^7 - 2q^6 + q^5 + 2q^4 - 3q^3$	[continued...]
		$+q^2 + q - 1)(q+1)^3(q-1)$	
	$[4,1,1,1]$	$(q^{12} - q^{11} + q^{10} - q^9 + q^7 - 2q^6 + 3q^5 - 2q^3 + 2q^2 - 1)$	[continued...]
		$(q^2 - q + 1)(q+1)^4(q-1)$	
	$[4,2,1]$	$q(q^{11} - 2q^{10} + 2q^9 - q^8 - q^7 + 3q^6 - 4q^5 + 3q^4 + 2q^3$	[continued...]
		$-4q^2 + q + 1)(q+1)^3$	
	$[4,3]$	$q^3(q^8 - q^7 + q^5 - q^4 + 1)(q+1)^2$	
	$[5,1,1]$	$q^2(q^4 + q^2 - 1)(q^3 - q^2 + 1)(q+1)^3(q-1)$	
	$[5,2]$	$q^3(q^6 - q^5 + q^3 - q^2 + 1)(q+1)^2$	
	$[6,1]$	$q^4(q^3 - q^2 + 1)(q+1)^2$	
	$[7]$	$q^6(q+1)$	

```
GClasses:=[[1,1,1,1], [2,1,1], [2,2], [3,1], [4]];
# the partitions corresponding to the possible unipotent classes


GreenPolyTable:= [[(q^2+1)*(q^2+q+1)*(q+1)^2,
    -(q-1)*(q+1)*(q^2+1)*(q^2+q+1),
    (q^2+1)*(q^2+q+1)*(q-1)^2,  (q^2+1)*(q-1)^2*(q+1)^2,
    -(q+1)*(q^2+q+1)*(q-1)^3],
  [(q+1)*(3*q^2+2*q+1),  -q^3+q^2+q+1,  -(q-1)*(q^2+1),
    -(q-1)*(q+1),  (q+1)*(q-1)^2],
  [(q+1)*(2*q+1),  q+1,  2*q^2-q+1,  -(q-1)*(q+1),  -q+1],
  [3*q+1,  q+1,  -q+1,  1,  -q+1] ,
  [1,  1,  1,  1,  1]];
# Green polynomials, where the order is so that the (i,j)th entry
# of the array is Q^{GClasses[i]}_{GClasses[j]};
# note that the same table is used for both GL and GU


ClassSizes:=  [1, (q+1)*(q-1)*(q^2+1)*(q^2-q+1),
  (q+1)^2*(q-1)*(q^2+1)*q*(q^2-q+1),
  (q^2+1)*(q-1)^2*(q+1)^2*q^2*(q^2-q+1),
  (q^2-q+1)*q^3*(q^2+1)*(q-1)^2*(q+1)^3];
# listed in the same order as GClasses


##########################################


GreenPolysq:=function(l,r)
  return GreenPolyTable[Position(GClasses,l)][Position(GClasses,r)];
end;
# function which calls the (l,r)th entry from GreenPolyTable


GreenPolysepsilonq:=function(l,r)
  if GreenPolysq(l,r) = 1 then
  return 1;
  else
```

```
    return Value(GreenPolysq(l,r), epsilon*q);
  fi;
end;
# function which returns GreenPolysq(l,r) evaluated at epsilon*q

nLambda:=function(l)
  local i, n;
  n:=0;
  for i in [1..Length(l)] do
    n := n + (i-1)*l[i];
    od;
  return n;
end;
# cf. Page 92

GreenX:=function(m,l)
  local f;
  if GreenPolysq(m,l) = 1 then
    f:= q^(nLambda(m));
  else
    f:= Value(GreenPolysq(m,l),q^(-1))*q^(nLambda(m));
  fi;
  if f = 1 then
    return 1;
  else
  return Value(f, epsilon*q);
  fi;
end;
# cf. Page 93
# this function is evaluated at epsilon*q, rather than q

eLambda:=function(l)
  local i, e;
```

```
  e:=1;
  for i in [1..Length(l)] do
    e := e*( 1 - q^(l[i]) );
  od;
  return e;
end;
# cf. Page 93

W:=function(l)
  local i, j;
  i:=1; j:=1;
  while j <= Maximum(l) do;
    i:=i*(j^Number(l, n-> n = j))*Factorial(Number(l, n-> n = j));
    j:=j+1;
  od;
  return i;
end;
# cf. Page 92

sgn:=function(l)
  return epsilon^(Int(GRank/2))*(-1)^(GRank+Length(l));
end;
# cf. Page 92

GGGR:=function(m,l)
  local i, r;
  i:=0;
  for r in GClasses do
    i:=i+(sgn(r)*(q^GRank)*Value(eLambda(r),epsilon*q^(-1))
      *GreenX(m,r)*GreenPolysepsilonq(l,r))/(W(r));
    od;
  return i*epsilon^(nLambda(m));
end;
```

```
# cf. Theorem 4.4.2

InnerProd:=function(m,r)
  local i, l;
  i:=0;
  for l in GClasses do
    i:=i + ClassSizes[Position(GClasses,l)]*GGGR(m, l)*GGGR(r, l);
  od;
  return i/GSize;
end;
# inner product formula for GGGRs

for i in GClasses do
  Print(i);
  Print("\n");
  Print(Reversed(Factors(InnerProd(i,i))));
  Print("\n");
  od;
# this lists each class together with the list of irreducible
# factors of the dimension polynomial of the corresponding
# endomorphism algebra
```

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List of symbols

G connected reductive group, 1

G' group over $\mathbb{C}$ with the same root datum as G, 8

G^F finite group of Lie type, 5

G_{uni} set of unipotent elements of G, 6

$\mathrm{GL}_n(\Bbbk)$ general linear group, 3

$\mathrm{GU}_n(\mathbb{F}_q)$ general unitary group, 10

1_G identity element of G, 45

$\Bbbk[V]^G$ G-invariant algebra of V, 42

$\mathcal{Z}_{L_w}$ centre of the Levi subgroup L_w, 98

$A(u)$ component group of $C_G(u)$, 90

$C_G(x)$ centraliser of x in G, 9

C_{min} minimal unipotent class, 87

$L(\lambda)$ Levi subgroup associated to λ, 45

L_e Levi subgroup of P_e, 16

$P(\lambda)$ parabolic subgroup associated to λ, 45

$P(X)$ parabolic subgroup associated to X, 46

P_e parabolic subgroup associated to e, 16

$U(\lambda)$ unipotent subgroup associated to λ, 45

U_α root subgroup of G, 38

$U_{e,i}$	unipotent subgroup of P_e, 16
W	Weyl group, 5
W_L	quotient of $N_G(L)$ by L, 97
$X(G)$	character group of G, 5
X/G	set of G-orbits of X, 8
$Y(G)$	cocharacter group of G, 5
$Y^+(T)$	Weyl chamber in $Y(T)$, 65
$Y_{\mathbb{Q}}(G)$	virtual one parameter subgroups of G, 45
$\mathfrak{g}$	Lie algebra of G, 2
$\mathfrak{g}'$	Lie algebra of G', 8
$\mathfrak{gl}_n(\Bbbk)$	general linear Lie algebra, 13
$\mathfrak{g}^*$	dual of $\mathfrak{g}$
$\mathfrak{g}^*(i,\mu)$	the $\mathfrak{g}(j,\mu)$-vanishing elements of $\mathfrak{g}^*$ for $j \neq -i$, 70
$\mathfrak{g}_i$	sum of root spaces of weight i, 16
$\mathfrak{g}_{\mathrm{nil}}$	set of nilpotent elements of $\mathfrak{g}$, 6
$\mathfrak{l}_e$	Lie algebra of L_e, 16
$\mathfrak{t}$	Cartan subalgebra of $\mathfrak{g}$, 73
$\mathfrak{z}(\mathfrak{g})$	centre of $\mathfrak{g}$, 73
$\mathcal{O}$	nilpotent $\mathrm{Ad}\,G$-orbit in $\mathfrak{g}$, 97
$\mathcal{O}_e$	$\mathrm{Ad}\,G$-orbit of e, 15
$\mathfrak{p}_e$	Lie algebra of P_e, 16
$\mathfrak{sl}_n(\Bbbk)$	special linear Lie algebra, 8
$\mathfrak{so}_n(\Bbbk)$	special orthogonal Lie algebra, 32
$\mathfrak{sp}_n(\Bbbk)$	symplectic Lie algebra, 32
$\mathfrak{u}_{e,i}$	Lie algebra of $U_{e,i}$, 16

H_β	element of a Chevalley basis of $\mathfrak{g}$, 71
X_α	root element in $\mathfrak{g}$, 15
η_k	$L(\lambda)$-module isomorphism, 55
Σ	root system of G, 5
Σ^+	positive roots of Σ, 5
$\Sigma^\vee$	coroot system of G, 5
Σ_i	roots of weight i, 16
Π	simple roots of Σ, 15
Ψ_i, $\Psi_{i,j}$	blocks of Σ_2, 17
τ	permutation of $X(T)$ which fixes Σ, 5
$\mathfrak{G}$	reductive $\mathbb{Z}$-group scheme, 42
$\mathfrak{L}$, $\mathfrak{L}^\perp$	scheme-theoretic analogues of $L(\lambda)$ and $L^\perp(\lambda)$ in $\mathfrak{G}$, 57
$\mathfrak{T}$, $\mathfrak{T}^\lambda$	scheme-theoretic analogues of T and T^λ in $\mathfrak{G}$, 57
$\mathfrak{X}$	reduced affine algebraic scheme, 56
$\mathfrak{X}^{ss}$	semistable subscheme of $\mathfrak{X}$, 57
$\mathrm{Dist}_\mathbb{Z}(\mathfrak{G})$	distribution algebra of $\mathfrak{G}$, 42
$H^\blacktriangle$	unipotent piece, 39
$H^\blacktriangle(\mathfrak{g})$, $H^\blacktriangle(\mathfrak{g}^*)$	nilpotent pieces, 70
$\tilde{H}^\blacktriangle$	union of certain $G_2^\triangle$, 38
$\tilde{H}^\blacktriangle(\mathfrak{g})$, $\tilde{H}^\blacktriangle(\mathfrak{g}^*)$	nilpotent versions of $\tilde{H}^\blacktriangle$, 69
$X^\triangle$	unipotent blades, 39
$X^\triangle(\mathfrak{g})$, $X^\triangle(\mathfrak{g}^*)$	nilpotent blades, 70
$G_i^\triangle$	certain unipotent subgroups of G, 38
$(\mathfrak{g}^*)_k^\triangle$	sum of $\mathfrak{g}^*(i, g \cdot \omega)$ for $i \geq k$, 70
$\mathfrak{g}_i^\triangle$	Lie algebra of $G_i^\triangle$, 69

$\mathcal{G}$ one of G, $\mathfrak{g}$ or $\mathfrak{g}^*$, 41

$\varphi^{\blacktriangle}$, $\psi^{\triangle}$ cardinality polynomials for rational points of $H^{\blacktriangle}$ and $X^{\triangle}$, 40

D_G set of sequences $\triangle = (G_0^{\triangle} \supset G_1^{\triangle} \supset G_2^{\triangle} \supset \cdots)$, 38

Γ_e GGGR associated to e, 85

γ_e character of Γ_e, 85

γ_{λ} GGGR character for $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{GU}_n(\mathbb{F}_q)$, 93

Γ_{ϕ} GGGR associated to ϕ, 96

$[v]$ blade, 49

$G[v]$ stratum, 49

Δ_X, $\Delta(X)$, $\tilde{\Delta}_X$ sets of optimal one parameter subgroups for X, 46

$\mu_T(X)$, $\lambda_T(X)$ canonical choices of optimal one parameter subgroups, 47

$m(v, \lambda)$ measure of instability, 44

λ' adapted one parameter subgroup, 59

$\mathcal{N}_V$ nullcone of V, 44

$L^{\perp}(\lambda)$ group generated by T^{λ} and $\mathcal{D}L(\lambda)$, 51

$S(X)$ saturation of X, 48

T^{λ} subtorus of T perpendicular to λ, 51

$U^i(\lambda)$ product of root subgroups of $U(\lambda)$ of λ-weight i, 52

$U_i(\lambda)$ group generated by $U^k(\lambda)$, for $k \geq i$, 53

$V(\lambda)_i$ set of $v \in V$ with $m(v, \lambda) \geq i$, 48

$V(\lambda, i)$ set of $v \in V$ such that $\lambda(\xi) \cdot v = \xi^i v$ for all $\xi \in \mathbb{k}^{\times}$, 48

$V_i(\lambda)$ quotient of $U_i(\lambda)$ by $U_{i+1}(\lambda)$, 53

$Q_{\lambda}^{\mu}(t)$ Green polynomial with parameters λ, μ, 94

$X_{\lambda}^{\mu}(t)$ polynomial associated with $Q_{\lambda}^{\mu}(t)$, 94

λ^F unipotent class in $\mathrm{GL}_n(\mathbb{F}_q)$ or $\mathrm{GU}_n(\mathbb{F}_q)$, 94

$\mathrm{sgn}_\varepsilon(\lambda)$	combinatorial invariant of the partition λ, 93
$e_\lambda(t)$	combinatorial invariant of the partition λ, 94
$n(\lambda)$	combinatorial invariant of the partition λ, 93
T_λ	rational maximal torus of $\mathrm{GL}_n(\Bbbk)$, 93
W_λ	F-fixed points of the Weyl group determined by T_λ, 93
$\mathcal{N}^G$	set of pairs (C, ψ), 97
$\mathscr{F}$	irreducible G-equivariant $\bar{\mathbb{Q}}_l$-local system on $\mathcal{O}$, 97
$\mathscr{F}_u$	stalk of $\mathscr{F}$ at u, 97
$\hat{\iota}$	image of ι under a certain permutation of $\mathscr{I}_0$, 99
$\mathcal{X}_\iota, \mathcal{Y}_\iota$	generalisations of Green polynomials, 99
$\mathscr{I}$	set of pairs $(\mathcal{O}, \mathscr{F})$, 97
$\mathrm{supp}(\iota)$	the support of ι, 98
$f'(\iota, \iota_1)$	function of dimensions of supports, 99
$P'_{\iota', \iota}$	combinatorial objects related to $\mathcal{X}_\iota$ and $\mathcal{Y}_\iota$, 99
$*$	G-fixed point, 43
$\mathbb{F}_q$	field with q elements, 4
$\Bbbk$	algebraically closed field, 1
$\Bbbk^+$	additive group of $\Bbbk$, 2
$\Bbbk^\times$	multiplicative group of $\Bbbk$, 2
$\overline{V}$	Zariski closure of a variety V, 3
π_G	Spaltenstein's map, 89
σ	Springer morphism, 6
$\tilde{D}_{G'}$	certain subset of $Y(G')$, 8
F	Frobenius endomorphism, 4
F_q	standard Frobenius endomorphism, 3