The Fukaya category, exotic forms and exotic autoequivalences

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A thesis submitted for the degree of

Doctor of Philosophy

at the University of Cambridge

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I also state that my dissertation is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other University. I further state that no part of my dissertation has already been or is being concurrently submitted for any such degree, diploma or other qualification.

Summary of the thesis
"The Fukaya category, exotic forms and exotic autoequivalences"
by Richard Mark Harris:

A symplectic manifold is a smooth manifold M together with a choice of a closed non-degenerate two-form. Recent years have seen the importance of associating an A_∞-category to M, called its Fukaya category, in helping to understand symplectic properties of M and its Lagrangian submanifolds. One of the principles of this construction is that automorphisms of the symplectic manifold should induce autoequivalences of the derived Fukaya category, although precisely what autoequivalences are thus obtained has been established in very few cases.

Given a Lagrangian $V \cong \mathbb{CP}^n$ in a symplectic manifold (M, ω), there is an associated symplectomorphism ϕ_V of M. In Part I, we define the notion of a $\mathbb{CP}^n$-object in an A_∞-category $\mathcal{A}$, and use this to construct algebraically an A_∞-functor Φ_V, which we prove induces an autoequivalence of the derived category $D\mathcal{A}$. We conjecture that Φ_V corresponds to the action of ϕ_V and prove this in the lowest dimension $n = 1$. We also give examples of symplectic manifolds for which this twist can be defined algebraically, but corresponds to no geometric automorphism of the manifold itself: we call such autoequivalences exotic.

Computations in Fukaya categories have also been useful in distinguishing certain symplectic forms on exact symplectic manifolds from the "standard" forms. In Part II, we investigate the uniqueness of so-called exotic structures on certain exact symplectic manifolds by looking at how their symplectic properties change under small nonexact deformations of the symplectic form. This allows us to distinguish between two exotic symplectic forms on $T^*S^3 \cup 2$-handle, even though the standard symplectic invariants such as their Fukaya category and their symplectic cohomology vanish. We also exhibit, for any n, an exact symplectic manifold with n distinct, exotic symplectic structures, which again cannot be distinguished by symplectic cohomology or by the Fukaya category.

Acknowledgements. My thanks firstly go to my PhD supervisor Ivan Smith for providing me with a constant stream of interesting questions and for many helpful discussions, as well as for his patience with an occasionally wayward pupil.

The inspiration behind Part II of this thesis, that exotic structures might be distinguished by deformations of the symplectic form, follows a suggestion of Paul Seidel. I am grateful for being given this idea to explore and, more generally, it should be clear to anyone familiar with Seidel's work that I owe him a considerable intellectual debt.

Jonny Evans and Jack Waldron were there at the beginning to help me find my feet in a new field of research. Along the way, I have benefited particularly from conversations with Mohammed Abouzaid and Samuel Lisi. I would also like to thank the anonymous referee from the Journal of Topology for their careful reading of Part II, for pointing out several errors and for suggestions on improving the exposition. Throughout the time of my research, I was supported by an EPSRC scholarship and later by European Research grant ERC-2007-StG-205349.

Finally, I wish to thank my friends and family for providing me with support, encouragement and distraction, all in the appropriate proportions.

Contents

Chapter 1

Introduction

This thesis comprises two parts: in the first I construct a class of autoequivalences of triangulated A_∞-categories, and in the second I develop a new way of distinguishing between symplectic forms on exact symplectic manifolds, which the standard invariants do not suffice to distinguish.

The underlying theme here is to understand a symplectic manifold in terms of its Fukaya category, so in this introduction, I shall explain what this object is. I shall also give some background to explain the context of my results and include some discussion on the questions that these results themselves pose.

1.1 Fukaya categories

We recall that a symplectic manifold (M, ω) is given by a smooth manifold M^{2n} together with a closed non-degenerate 2-form ω. In particular, this picks out the set of Lagrangian submanifolds, embedded submanifolds L^n such that $\omega|_L = 0$. Following Gromov [21] and Floer [16] the main approach to studying the topology of symplectic manifolds and their Lagrangian submanifolds is via studying the spaces of J-holomorphic curves contained in M, for some compatible almost complex structure J. A particularly sophisticated way of encoding this information is the Fukaya category.

Under favourable circumstances, we can assign to a symplectic manifold (M, ω) an A_∞-category called its Fukaya category $\mathcal{F}(M, \omega)$. The precise definition usually depends on context, in part because for many years many aspects of the theory were poorly understood; however, following the monumental work of Fukaya, Oh, Ohta and Ono [18], there is more of a general framework in place. Here we shall present a heuristic definition of $\mathcal{F}(M,\omega)$ in order that the results of this thesis can be better put in some sort of context. We shall not give too many technical details: the major references are [18] and [39], although the relevant background material from the theory of A_∞-categories can be found in Chapter 2 and an in-depth discussion of some of the holomorphic curve theory can be found in Chapter 8.

Firstly, we choose some almost-complex structure J on M and suppose that $2c_1(M) = 0$. This means that we can pick some quadratic complex volume form η^2, which gives a trivialization of the bicanonical bundle $K_M^{\otimes 2}$. Let $Gr(TM)$ denote the bundle of Grassmanians of Lagrangian subspaces of TM and consider the associated square-phase map

$$\alpha_M \colon Gr(TM) \to S^1, \ \ \alpha_M(\Lambda) = \frac{\eta^2(v_1 \wedge \cdots \wedge v_n)}{|\eta^2(v_1 \wedge \cdots \wedge v_n)|}.$$

The Fukaya category $\mathcal{F}(M,\omega)$ has objects given by *Lagrangian branes* $L^\flat = (L, \widetilde{\alpha}_L, \$_L)$. Here

- L is a closed Lagrangian submanifold of M, which admits a *grading* (see below).

- $\widetilde{\alpha}_L$ is a grading: for any Lagrangian submanifold L, there is an obvious map $s_L \colon L \to Gr(TM)$ defined by $s_L(x) = T_xL$. A grading of L is a lift of $\alpha_M \circ s_L \colon L \to S^1$ to a map $\widetilde{\alpha}_L \colon L \to \mathbb{R}$. When this lifts exists, it allows us to put a $\mathbb{Z}$-grading on the hom-spaces of $\mathcal{F}(M,\omega)$ [37].

- $\$_L$ is some Spin structure on L. This will be used to define orientations on the moduli spaces of holomorphic curves that are involved in defining the A_∞-maps, so that we can define $\mathcal{F}(M,\omega)$ over fields of arbitrary characteristic. If we were happy to forgo this, we could work just in characteristic 2. There are, however, weaker conditions than the existence of Spin structures in which we can still define such orientations, see Remark 5.10 or [18].

Given any two such branes $(L_0^\flat, L_1^\flat)$, we define some perturbation datum $(H, \mathbf{J})$ associated to this pair. Here $H = (H_t)$ is a Hamiltonian such that the time-1 Hamiltonian flow $\phi_H^1(L_0)$ intersects L_1 transversely, and $\mathbf{J} = (J_t)_{t\in[0,1]}$ is some family of almost complex structures on M that are chosen generically to satisfy certain transversality conditions, so that the moduli spaces of holomorphic curves that we consider below are smooth manifolds (see Chapter 8 for more precise details).

We can now define $\hom_{\mathcal{F}(M,\omega)}(L_0^\flat, L_1^\flat)$ as the vector space $\oplus_{x\in\phi^1(L_0)\pitchfork L_1}\Lambda_{\mathbb{R}}\langle x\rangle$, the so-called Floer cochain complex $CF(L_0, L_1)$. Here $\Lambda_{\mathbb{R}}$ is the real Novikov field

$$\Lambda_{\mathbb{R}} = \left\{\sum_r a_r q^r : r, a_r \in \mathbb{R}, r \to \infty, \#\{a_r \neq 0 : r < E\} < \infty \text{ for all } E\right\}$$

of power series in the formal parameter q. We may use the gradings $\widetilde{\alpha}_{L_i}$ to put a $\mathbb{Z}$-grading on $\hom_{\mathcal{F}(M,\omega)}(L_0^\flat, L_1^\flat)$ [37].

For points $y_0, \ldots, y_d$ with $y_i \in \hom(L_{i-1}, L_i)$ and $y_0 \in \hom(L_0, L_d)$, we can, for some appropriately defined $\mathbf{J}$, consider the moduli space $\mathcal{M}_{\mathbf{J}}^d(y_0, \ldots, y_d)$ of $\mathbf{J}$-holomorphic maps $u\colon \mathbb{D}\setminus\{p_0, \ldots, p_d\} \to M$ from the disc with $d+1$ boundary punctures and equipped with *strip-like ends* into M, such that the ends converge to the y_i and the boundary components are sent to the Lagrangians L_i. This allows us to define d-multilinear maps on our hom-spaces

$$\mu^d(y_d, \ldots, y_1) = \sum_{u\in\mathcal{M}^d} q^{E(u)} y_0,$$

where the sum is taken over the curves u in the zero-dimensional part of our moduli space $\mathcal{M}^d$ and $E(u) = \int \|\partial_s u\|^2$ is the *energy* of the curve u. It is important to note that this sum may be infinite. However, Gromov compactness ensures that there will only be finitely many curve classes below any given energy level E, so that these maps are well-defined over $\Lambda_{\mathbb{R}}$. Now, in favourable circumstances (for example, when $\pi_2(M) = \pi_2(M, L_i) = 0$), analysing the boundaries of compactifications of the one-dimensional parts of these moduli spaces, gives us an infinite family of relations called the A_∞-relations:

$$\sum_{m,n}(-1)^{\maltese_n}\mu^{d-m+1}(a_d,\ldots,a_{n+m+1},\mu^m(a_{n+m},\ldots,a_{n+1}),a_n,\ldots,a_1)=0, \quad (1.1)$$

where $\maltese_n = |a_1|+\cdots+|a_n|-n$ and $|a_i|$ denotes the grading of a_i. In particular, μ^1 is a differential so we can consider the so-called Donaldson category $H(\mathcal{F}(M,\omega))$, the cohomological category of $\mathcal{F}(M,\omega)$ such the morphism spaces are now the familiar Floer cohomology groups. However this process forgets any information contained in the higher-order terms, so instead we consider the *derived Fukaya category* $D\mathcal{F}(M,\omega)$, a triangulated category obtained from $\mathcal{F}(M,\omega)$ by a purely algebraic process that is recalled in Chapter 2. We may also consider $D^\pi\mathcal{F}(M,\omega)$ the Karoubi (or idempotent) completion of $D\mathcal{F}(M,\omega)$, which incorporates additional objects for idempotent endomorphisms. The derived Fukaya category is one of the main objects of study in Kontsevich's celebrated Homological Mirror Symmetry conjecture [26] and is the main object we shall use to study symplectic manifolds in this thesis.

Of course, much of the above turns out to be too naïve in general, so we make some more technical remarks.

Recall that, in order for the A_∞-relations to hold, we look at compactifications of our moduli spaces of curves. Standard results say that compactifying adds so-called broken solutions along with curves with bubble components, either sphere bubbles on the interior or disc bubbles on the boundary. The A_∞-relations (1.1) follow from the case when there is breaking but no bubbling, so we want to say that we can remove this potential bubbling obstruction. The issue of sphere bubbles is often not serious: dimensional formulae show that spheres will sometimes appear in sufficiently high codimension that they can be avoided by a judicious choice of $\mathbf{J}$, for instance in the Calabi-Yau case (see Chapter 8).

Disc bubbles, however, are a more serious issue. The moduli space $\mathcal{M}(L;\beta)$ of unparametrized holomorphic discs in the homotopy class $\beta\in\pi_2(M,L)$ has expected dimension

$$n+m(\beta)-3,$$

where $m(\beta)$ denotes the Maslov index of β. If we now consider the moduli space of discs with one boundary puncture $\mathcal{M}_1(L;\beta)$, there is an evaluation map $ev\colon \bar{\mathcal{M}}_1(L;\beta) \to L$ and we can define

$$\mu^0(L) = \sum_{\beta \in \pi_2(M,L)} q^{\omega(\beta)} ev_*([\bar{\mathcal{M}}_1(L;\beta)]).$$

Now, when we examine whether μ^1 is a differential, we find that

$$\mu^1(\mu^1(x)) = \mu^2(\mu^0(L_1), x) - \mu^2(x, \mu^0(L_0)).$$

If $\mu^0(L_0) = \mu^0(L_1)$ and this element is central (for example, some multiple of the fundamental class), then there are no problems, but this will not be true in general. Furthermore, for higher orders we now get a series of relations like (1.1), except they now include extra μ^0 terms - this is often called a *curved* A_∞-category. One potential solution is to deform the μ^d on $\mathcal{F}(M,\omega)$: choose $b_i \in CF^1(L_i, L_i)$ and let

$$\mu_b^d(a_d, \ldots, a_1) = \sum \mu^{d+l}(b_d, \ldots, b_d, a_d, b_{d-1}, \ldots, b_{d-1}, a_{d-1}, \ldots, a_1, b_0, \ldots, b_0).$$

If $\mu^0 + \mu^1(b_i) + \mu^2(b_i, b_i) + \ldots = 0$, then the μ_b^d define a new uncurved A_∞-category structure on $\mathcal{F}(M,\omega)$ and we may proceed as before. Such b_i are called *bounding cochains*. If they exist we can still form the Fukaya category and the Lagrangians are said to be *weakly unobsructed*. This however does not cover all cases, so there are still occasions when we cannot hope to define a nonempty category $\mathcal{F}(M,\omega)$.

Nevertheless, if we have an exact symplectic manifold $(M, \omega = d\theta)$ and restrict attention to exact Lagrangians (meaning that $\theta|_L = df$), then these obstruction issues disappear trivially by Stokes' theorem. Here we should also impose some sort of convexity condition at infinity (see the definition of Liouville manifold in Chapter 6) in order to stop holomorphic curves escaping to infinity. In such a situation, the use of maximum principles yields global energy bounds on holomorphic curves, so that one can set $q = 1$ in our Novikov field and work over $\mathbb{R}$, or even just $\mathbb{Z}$ if we wish. One of the main technical difficulties in Part II lies in trying to make a proof that works in the exact case carry over to a nonexact setting. Then we do need to worry about bubbling issues, although we manage to find more elementary ways around these problems without needing to discuss bounding cochains as above (see Chapter 8.7).

We also remark that this approach to defining $\mathcal{F}(M,\omega)$ involves a lot of choices: to define the moduli spaces $\mathcal{M}^d$ we have to choose Hamiltonians and families of almost complex structures. It is a delicate issue to show that we can choose such data consistently in defining the moduli spaces $\mathcal{M}^d$ for each d so that we can establish the A_∞-relations, and to show that the result is (up to quasi-equivalence) independent of choices (see, for example, [39, Chapter 12] for a discussion of this). We shall not need to worry about such considerations in what follows.

1.2 Symplectomorphism groups

In order for $\mathcal{F}(M,\omega)$ to be a useful symplectic invariant, it must behave well under symplectomorphisms $\phi \in Aut(M,\omega)$. Note first that this will not be true for general symplectomorphisms, due to the extra data involved in setting up $\mathcal{F}(M,\omega)$: for example, we must restrict to symplectomorphisms that preserve the quadratic volume form η^2, and so our trivialization of $K_M^{\otimes 2}$, up to homotopy. We call the subgroup of such symplectomorphisms that preserve all our required structure $Aut^c(M,\omega)$.

Given $\phi \in Aut^c(M,\omega)$, and a Lagrangian brane $L^\flat \in M$, we see that $\phi(L)$ automatically inherits a natural brane structure. Similarly, by considering $\phi^* J$, ϕ will have a well-defined action on the A_∞-structure, so that we do get a map $\Phi\colon \mathcal{F}(M,\omega) \to \mathcal{F}(M,\omega)$ induced from ϕ. We also observe that Hamiltonian symplectomorphisms lie in Aut^c and, by the Hamiltonian invariance of the whole Floer theory package, we see that, once we pass to the derived category $D\mathcal{F}(M,\omega)$, these should act trivally. More generally, there should be a canonical map

$$Aut^c(M,\omega)/Ham(M,\omega) \to Auteq(D\mathcal{F}(M,\omega))/\langle[1]\rangle \tag{1.2}$$

where on the right we quotient out by the shift autoequivalence.

However, as far as the action of this map on specific symplectomorphisms goes, one of the few nontrivial results we have to date comes from Dehn twists: given a Lagrangian sphere $V \subset M$, together with a choice of diffeomorphism

$f: V \to S^n$, there is a symplectomorphism τ_V called the *Dehn twist* about V [38] (the definition of τ_V requires certain choices, but the result is well-defined in $Aut^c(M,\omega)/Ham(M,\omega)$). Algebraically, we can also define the notion of a spherical object V in an A_∞-category $\mathcal{A}$ and define a related functor $T_V : D\mathcal{A} \to D\mathcal{A}$. Seidel [39, 38] has proven that, given another Lagrangian L, $\tau_V L$ and $T_V L$ give rise to isomorphic objects in $D\mathcal{F}(M,\omega)$ (from this point onward we drop the $\flat$ signs and brane terminology as it forms no serious part of our discussion in what follows). It is expected that ongoing work on Lagrangian correspondences should imply that the functors τ_V and T_V are in fact canonically isomorphic in $H^0(fun(\mathcal{F}(M,\omega), \mathcal{F}(M,\omega)))$.

The existence of Dehn twists relies on the fact that the geodesic flow on the round sphere is periodic, and there is a related construction that defines "twist" symplectomorphisms for any Lagrangian submanifold admitting a metric with periodic geodesic flow [37]. In Part I, we focus attention on the projective twist ϕ_V associated to a Lagrangian $V \cong \mathbb{CP}^n$.

In Chapter 3, we shall define the notion of a $\mathbb{CP}^n$-object in an A_∞-category and in the case when $\mathcal{A}$ is a triangulated A_∞-category (see Chapter 2) we use V to define a functor $\Phi_V : \mathcal{A} \to \mathcal{A}$. In Chapter 4 we prove Theorem 4.1 stating that, given a $\mathbb{CP}^n$-object in a cohomologically finite A_∞-category, Φ_V is an autoequivalence of $D\mathcal{A}$.

This result is the first step towards proving the following conjecture:

Conjecture 1.1. *Given a Lagrangian $V \cong \mathbb{CP}^n$ and another Lagrangian L in $\mathcal{F}(M,\omega)$, $\phi_V L$ and $\Phi_V L$ give rise to isomorphic objects in $D\mathcal{F}(M,\omega)$.*

We stress that a proof of this conjecture would likely require a substantial further analysis: for the parallel argument required to bridge the gap in the spherical case, see [39]. We can however verify this conjecture in the case of a $\mathbb{CP}^1$-twist by exploiting the relation

$$\tau_V^2 = \phi_V \tag{1.3}$$

in $Aut^c(M,\omega)/Ham(M,\omega)$. Combining this with Seidel's result on spherical twists means that in this dimension we need only show that Φ_V and T_V^2 give isomorphic functors on $D\mathcal{F}(M,\omega)$. This is proven in Chapter 5.

Related results to these have been obtained by Huybrechts and Thomas [23], who introduce the notion of $\mathbb{P}^n$-objects and $\mathbb{P}^n$-twist functors for the derived category $D(X)$ of some smooth projective variety X. Our construction is modelled on theirs and our results should be thought of as being "mirror" versions.

Finally, in Chapter 5, we also show that there exist symplectic manifolds containing a Lagrangian V with $H^*(V) \cong H^*(\mathbb{CP}^n)$, where we can still define Φ_V, but such that this functor has no preimage under (1.2), so that Φ_V has no geometric representative. We call such autoequivalences *exotic autoequivalences* of our symplectic manifold.

1.3 Lefschetz fibrations

Part II of this thesis concerns the uniqueness of exact symplectic structures on Liouville domains (see Chapter 6 for the definition), an area which has seen considerable recent development. In many situations, such as those coming from cotangent bundles or affine varieties, a Liouville domain M carries what is considered to be a "standard" symplectic form. As we shall recap in the next section, there are now known to be many examples of Liouville domains with exact symplectic forms which are not Liouville equivalent to the standard ones. Any such form will be called "exotic". However it is first worthwhile making some remarks on the role Fukaya categories play in illuminating this problem and the broader question of how to compute the Fukaya category of a given symplectic manifold and what geometric data we can extract from it.

In general, computing $\mathcal{F}(M, \omega)$ is a difficult task. Extracting information from $\mathcal{F}(M, \omega)$ that cannot seemingly be obtained by more elementary means seems usually to rely on either of two sources: Seidel's result relating Dehn and spherical twists, or via importing some algebraic geometry in cases where some version of mirror symmetry has been proven.

Part of the problem is that knowing about some small collection of Lagrangians does not usually provide us with much information about some other arbitrary Lagrangian. What one wants are results about when a collection of Lagrangians generate (or split-generate) the Fukaya category. Such results do exist in the context of exact Lefschetz fibrations (similar results have also been established for cotangent bundles [1] but will not be used here). A general theory is currently under development by Abouzaid, Fukaya, Oh, Ohta and Ono.

In Chapter 6, a more involved definition of Lefschetz fibration will be given but, briefly, a Lefschetz fibration $\pi\colon M \to \mathbb{C}$ is a symplectic fibration with isolated singularities of A_1 type. In such a situation, one can calculate in $\mathcal{F}(M,\omega)$ by means of dimensional reduction: questions about $\mathcal{F}(M,\omega)$ are reduced to questions about the Fukaya category of a smooth fibre $\mathcal{F}(F,\omega|_F)$, and the monodromy data of our fibration. This is the general form of the argument in Chapter 9. Also, inside a Lefschetz fibration, there is a collection of *noncompact* Lagrangians called the *Lefschetz thimbles* Δ_i and these thimbles generate the derived Fukaya category (here we allow a slightly more general class of objects than before).

These results mean that we can relate the theory of Fukaya categories to various other holomorphic curve theories. One particular such theory is the symplectic cohomology of a Liouville manifold, which may briefly be defined as

$$SH^*(M) = \varinjlim_k HF^*(H_k).$$

Here M has a *cylindrical end* looking like $Y \times [0,\infty)$ and H_k is a Hamiltonian of the form $H_k(y,r) = kr + \textit{constant}$ near infinity, and the maps involved are monotone continuation maps (for more details see for example [42]).

For any Liouville domain, and a suitable definition of $\mathcal{F}(M,\omega)$, there is an open-closed string map [2]

$$\mathcal{OC}\colon HH_*(\mathcal{F}(M,\omega)) \to SH^*(M),$$

where the Hochschild homology of an A_∞-category may be defined as $HH_*(\mathcal{A}) = H(\hom_{fun(\mathcal{A},\mathcal{A})}(\mathrm{id},\mathrm{id}))$. In general, it remains conjectural that $\mathcal{OC}$ is an isomorphism, but it has been proven for exact Lefschetz fibrations [12, Appendix]. This is one point of contact between Fukaya categories and the work of Bourgeois-Ekholm-Eliashberg on symplectic and contact homology [12].

1.4 Exotic symplectic structures

With this in place, we shall now briefly recap the major results in this area.

Historically, Gromov [21] was the first to exhibit a nonstandard exact symplectic structure on Euclidean space, although, whereas the standard symplectic structure is Liouville, Gromov's is not known to be (see Chapter 6 for the relevant definitions). The first exotic structures on $\mathbb{R}^{4n}$ (for $4n \geq 8$) known to be Liouville were discovered by Seidel-Smith [41], later extended by McLean [31] to cover all even dimensions greater than 8. McLean actually found infinitely many such pairwise-distinct nonstandard symplectic structures on T^*M for any manifold M with $\dim_{\mathbb{R}} \geq 4$, which were all distinguished by considering their symplectic cohomology $SH^*(M)$.

More recently, Fukaya categorical techniques have been used by Maydanskiy-Seidel [28] (refining earlier work of Maydanskiy [27]) to find exotic symplectic structures on T^*S^n (for $n \geq 3$). These are shown to be nonstandard by proving that they contain no homologically essential exact Lagrangian S^n, in contrast to the zero-section for the standard symplectic form, and in contrast to McLean's examples. Similar results have also been obtained using the work of Bourgeois-Ekholm-Eliashberg [12] again using symplectic/contact cohomology-type invariants. Such results have been further extended by Abouzaid-Seidel [4] to show the existence of infinitely many distinct exotic structures on any affine variety of real dimension ≥ 6, again distinguished using symplectic cohomology.

In Part II, we shall consider six-dimensional symplectic manifolds of the types considered by Maydanskiy [27] and Maydanskiy-Seidel [28]. In [28], infinitely many ways are presented of constructing a nonstandard T^*S^3, but the question of whether all these constructions actually yield symplectically distinct manifolds is left open. We shall not answer that question, but instead we shall consider what happens when we add a 2-handle to such an exotic T^*S^3. The result will be diffeomorphic to a manifold constructed in [27], which again contains no exact Lagrangian S^3.

Specifically, we shall consider the manifolds X_1, X_2 given by the diagrams in Figure 1.1. The meaning of such diagrams will be explained in Chapter 6. Briefly, our main method of constructing symplectic manifolds E^6 will be as Lefschetz fibrations over $\mathbb{C}$. To run this construction, the input data consists of a symplectic manifold M^4 and an ordered collection of Lagrangian spheres in M^4 (see Lemma 6.2). In Figure 1.1, we can associate to each path some Lagrangian sphere in a 4-dimensional Milnor fibre $\{x^2 + y^2 + p(z) = 0\} \subset \mathbb{C}^3$ for a suitable polynomial $p(z)$. This Milnor fibre and the collection of spheres is our required data.

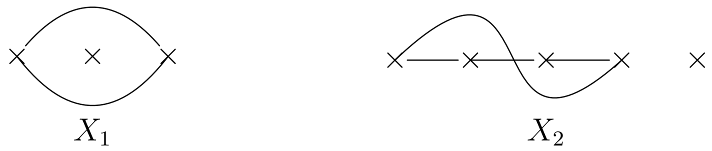


Figure 1.1:

These spaces are diffeomorphic (in fact they are both diffeomorphic to $T^*S^3 \cup$ 2-handle). There is a standard way of attaching a 2-handle to T^*S^3 [43] such that we still get an exact symplectic manifold containing an exact Lagrangian sphere inherited from the zero-section. However, neither X_1 nor X_2 contains such a sphere, so are considered exotic. In addition, X_1, X_2 both have vanishing symplectic cohomology. This is not proved in [27, 28] and so we include this calculation in Chapter 11; it has the consequence (already proven for X_1 in [27]) that X_1 and X_2 actually contain no exact Lagrangian submanifolds (such symplectic manifolds are sometimes called "empty"). Despite the usual collection of invariants being insufficient to distinguish these two manifolds, we shall nevertheless prove

Theorem 1.2. *X_1 and X_2 are not symplectomorphic.*

We shall then extend our methods to prove

Theorem 1.3. *Pick any $n \geq 1$. Then there exists a manifold M (diffeomorphic to T^*S^3 with n 2-handles attached), and exact symplectic forms $\omega_1, \ldots, \omega_{n+1}$ on it such that, with respect to each ω_i, (M, ω_i) is Liouville and contains no exact Lagrangian submanifolds, but such that there exists no diffeomorphism ϕ of M such that $\phi^*\omega_i = \omega_j$ for $i \neq j$.*

The main technique used is to consider what happens after a nonexact deformation of the symplectic structure. For any 2-form $\beta \in H^2(X_i; \mathbb{R})$, we can consider an arbitrarily small nonexact deformation of ω to $\omega + \epsilon\beta$. If this new form is still symplectic, we can look at the symplectic properties of these new symplectic manifolds. (In the case of X_1 and X_2 above $H^2(X_i; \mathbb{R}) = \mathbb{R}$, so Moser's argument tells us that the way we can perform such a deformation is essentially unique, in a sense which will be made precise in Chapter 10.) We discover that, after an arbitrarily small deformation, X_1 (which with our original exact form contains no Lagrangian S^3) does in fact contain such a sphere, in a nonzero homology class, an interesting phenomenon in its own right which is explained in Chapter 7.

In contrast, after such a deformation, X_2 still contains no homologically essential Lagrangian sphere. The proof of this fact requires rerunning the argument of [28], except that somewhat more care needs to be exercised in the use of Floer cohomology groups, owing to the nonexactness of our deformed situation. This is the content of Chapter 9.

In general, given a symplectic manifold M (satisfying certain topological assumptions), we can consider the set $\Gamma_1 \subset \mathbb{P}(H^2(M; \mathbb{R}))$, of directions in which we get no homologically essential Lagrangian sphere inside M after an arbitrarily small deformation of the symplectic form in that direction. We show that this is a symplectic invariant, which completes the proof of Theorem 1.2. Finally, in Chapter 12 these ideas are extended to prove Theorem 1.3.

1.5 Discussion of results

1.5.1 Part I

The first obvious goal is to prove Conjecture 1.1 that $\phi_V L$ and $\Phi_V L$ give isomorphic objects in $D\mathcal{F}(M, \omega)$. I have not yet identified the precise details necessary to verify this, but a proof should follow similar lines to the parallel argument in the spherical case: we want to find a map

$$\hom_{\mathcal{F}(M,\omega)}(Z, \Phi_V L) \to \hom_{\mathcal{F}(M,\omega)}(Z, \phi_V L)$$

for any Z, and prove that it a quasi-isomorphism, or equivalently that it's mapping cone is acyclic. There should be some algebraic criteria, similar to [38, Lemma 5.3], saying when a collection of elements from and maps between the hom-spaces between the Lagrangians V, L and $\phi_V L$ will define such a map. Geometrically, this data would hopefully come from counting holomorphic sections of some Morse-Bott fibrations as in [39, Chapter 17]. In the spherical case, the acyclicity of the mapping cone is equivalent to Seidel's long exact sequence.

Similar twist maps to ϕ_V for $V \cong \mathbb{CP}^n$ exist for Lagrangian $\mathbb{RP}^n$s and $\mathbb{HP}^n$s since they are compact symmetric spaces of rank one and so admit metrics whose geodesic flow is periodic [10]. The results of Part I can easily be reinterpreted in these contexts: we leave it to the interested reader the make the necessary minor adjustments (although we do remark that in the case of $\mathbb{RP}^n$ one has to work in characteristic 2 to avoid sign issues). The key feature is that $\mathbb{RP}^n$ and $\mathbb{HP}^n$ both have cohomology rings which are truncated polynomial algebras (again only in characteristic 2 for $\mathbb{RP}^n$); indeed, this is necessary for the geodesic flow to be periodic by a theorem of Bott [11].

There is an interesting algebraic counterpart to this observation: the construction of [23] has been extended by Grant to give a great many autoequivalences of derived categories [20]. He works in the setting of the bounded derived category $D^b(A)$ of modules over a finite dimensional symmetric k-algebra A and proves that, given P a projective A-module whose endomorphism algebra $\mathrm{End}_A(P)^{op}$ is *periodic*, then there is a related autoequivalence Ψ_P of $D^b(A)$. Here we say that a k-algebra E is periodic if there exists an integer $n \geq 1$ and an exact sequence

$$P_{n-1} \xrightarrow{d_{n-1}} P_{n-2} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0$$

of projective E^{en}-modules ($E^{en} = E \otimes_k E^{op}$) such that $\mathrm{coker}\, d_1 \cong E$ and $\ker d_{n-1} \cong E$. This includes the case when the endomorphism algebra is a truncated polynomial ring. It would be interesting to try to understand if there is any geometric motivation for the other autoequivalences that Grant constructs.

1.5.2 Part II

It seems that Theorem 1.2 suggests the following picture: there is a 1-parameter family X_1^t of symplectic manifolds with the property that $\mathcal{F}(X_1^t, \omega_t)$ is trivial for $t = 0$ but nontrivial for $t > 0$ (to the extent that $\mathcal{F}(X_1^t, \omega_t)$ can be defined for $t > 0$, an issue we only partly address). This differs from the picture one usually finds in deformation theory, where we expect to observe some sort of upper-semicontinuity, as in the Semicontinuity Theorem of Grothendieck [9]. It should be said that, as far as I am aware, the general theory of deformations of A_∞-categories has not received that much attention; in particular, I am not aware if there is a widely-accepted notion of a flat deformation in this context.

Perhaps the key here is that we are working with a Fukaya category made up only of *embedded* Lagrangian submanifolds, and we should widen our scope and include certain non-embedded Lagrangians. Although the exact X_1 contains no exact embedded Lagrangians, it does contain some singular Lagrangian cycle, topologically an S^3 with an S^1 collapsed to a point, so having an isolated T^2-cone singularity. Joyce considers a similar situation in [24, Section 3]. Here he looks at how special Lagrangians in Calabi-Yaus can degenerate and identifies three families of embedded special Lagrangian 3-submanifold that asymptotically approach this cone. However these families bound holomorphic discs so we would expect their Floer theory to be obstructed in the sense of [18]. Perhaps we should consider some *bulk deformation* [18] of the Fukaya category $\mathcal{F}(X_1, \omega)$, where we count only those holomorphic curves passing through some 4-cycle Poincaré dual to the cohomology class in which we perturb our symplectic form. This will not change anything in our non-exact scenario, but possibly would mean that some sense can be made of the Floer theory of this singular Lagrangian cone.

1.5.3 Fragility of symplectomorphisms

It seems appropriate to mention here one of the motivations behind Theorem 1.2, even if the proof proceeds along different lines. We recall Maydanskiy's example from Figure 1.1. In Figure 1.2 the solid lines correspond to Lagrangian spheres A and B inside a four-dimensional A_2 Milnor fibre M_2. There are also Lagrangian spheres L and R over the dotted lines, and [39, Lemma 16.13] says that the

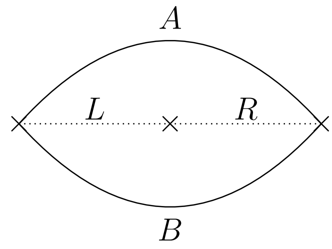


Figure 1.2:

spheres A and B are related to each other by $\tau_R^2(A) = B$. In particular this means they are differentiably isotopic, reflecting the fact that this Dehn twist lies in the kernel

$$\pi_0 Aut(M, \omega) \to \pi_0 Diff(M),$$

but a Floer-theoretic argument in [27] shows that τ_R^2 and the identity are not isotopic through symplectomorphisms. There is, however, another interesting phenomenon that one sometimes observes in symplectomorphism groups as we deform the symplectic form.

Definition 1.4. ([40]) *Let f be a symplectomorphism with respect to a given symplectic form ω. We say that f is potentially fragile if there is a smooth family ω_s of symplectic forms, $s \in [0, s_0)$ for some $s_0 > 0$, and a smooth family f_s of diffeomorphisms such that $f_s^* \omega_s = \omega_s$ with the following properties*

- *$(f_0, \omega_0) = (f, \omega)$;*
- *for all $s > 0$, f_s is isotopic to the identity inside $Aut(M, \omega_s)$.*

If, in addition, f is not isotopic to the identity in $Aut(M, \omega)$, we say that f is fragile.

Proposition 1.5. ([40, Corollary 1.3]) *For a Lagrangian $V \cong S^2$, the Dehn twist τ_V is potentially fragile.*

This now suggests the following picture: X_1 is built from the data (M_2, A, B). We should try to deform M_2 and consider the deformed data (M_2^t, A^t, B^t), which we use to build X_1^t. Now note that, for $t > 0$, $A^t \cong B^t$ in $\mathcal{F}(M_2^t, \omega_t)$, which will have the consequence that X_1^t will contain a Lagrangian sphere. I was unable to proceed by this approach, but to explain why it is useful to derive an explicit formula for the perturbed Dehn twist (Seidel's proof of Proposition 1.5 is more indirect).

The standard Dehn twist is defined using the following model for T^*S^2:

$$T^*S^2 = \left\{(u,v) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \|v\| = 1, \langle u,v\rangle = 0\right\} \tag{1.4}$$

equipped with the exact symplectic form $\omega = du \wedge dv$.

We take a Hamiltonian $h = \|u\|$ which is defined away from $u = 0$. The Hamiltonian vector field defined by $\omega(X_h, \cdot) = dh(\cdot)$ is

$$X_h = \frac{u_i}{\|u\|}\partial v_i - v_i \|u\| \partial u_i,$$

and we need to solve $\frac{d}{dt}\sigma_t(u,v) = X_h \circ \sigma_t(u,v)$ to find the flow $\sigma_t(u,v)$ of this vector field. A direct calculation verifies that

$$\sigma_t(u,v) = \begin{pmatrix} u\cos t - v\|u\|\sin t \\ v\cos t + \frac{u}{\|u\|}\sin t \end{pmatrix}$$

is the correct flow. We define the Dehn twist by taking the time-π flow, joining up with the antipodal map on the zero-section and applying a cut-off function to undo the flow away from the zero-section [40].

We now perturb our original symplectic form ω by $\omega^s = \omega + s\pi^*\eta$ where η is the standard area form on S^2 and $\pi\colon T^*S^2 \to S^2$ is the standard projection. Viewing $S^2 \subset \mathbb{R}^3$ we have

$$\eta = v_1 dv_2 \wedge v_3 + v_2 dv_3 \wedge v_1 + v_3 dv_1 \wedge v_2.$$

Seidel [40] now proves that there is a Hamiltonian S^1-action coming from the Hamiltonian

$$h = \|-sv - u \times v\| = \sqrt{s^2 + \|u\|^2},$$

which leads to

$$dh = \frac{u_i}{h}du_i - \frac{v_i\|u\|^2}{h}dv_i.$$

To find the flow $\sigma_t(u,v) = \begin{pmatrix} \alpha_t(u,v) \\ \beta_t(u,v) \end{pmatrix}$ of the associated Hamiltonian vector field we must solve the differential equation

$$\begin{pmatrix} \dot{\alpha}_t(u,v) \\ \dot{\beta}_t(u,v) \end{pmatrix} = \begin{pmatrix} \frac{\|u\|^2}{h}\beta_t(u,v) + \frac{s}{h}\beta_t(u,v) \times \alpha_t(u,v) \\ \frac{-1}{h}\alpha_t(u,v) \end{pmatrix}. \tag{1.5}$$

where $\times$ denotes the cross-product in $\mathbb{R}^3$. Differentiating the first equation and inserting the second quickly leads to

$$\ddot{\alpha}(t) = -\alpha(t),$$

which means we have to consider

$$\begin{aligned} \alpha(t)(u,v) &= A(u,v)\cos t + B(u,v)\sin t, \\ \beta(t)(u,v) &= \frac{B(u,v)}{h}\cos t - \frac{A(u,v)}{h}\sin t + C(u,v). \end{aligned} \tag{1.6}$$

The initial conditions at time $t = 0$ give

$$\begin{aligned} A &= u, \\ \frac{B}{h} + C &= v. \end{aligned}$$

By plugging the equation for β (1.6) into (1.5), we get the extra conditions below: clearly $\{u, v, u \times v\}$ form an orthogonal basis for $\mathbb{R}^3$ and it's easy to show that B and C have no u-component. This means that

$$\begin{aligned} B &= \kappa v + \lambda u \times v, \\ C &= \mu v + \nu u \times v, \end{aligned}$$

and we obtain the equations

$$\begin{aligned}\kappa + \mu h &= h,\\ \lambda + \nu h &= 0,\\ \kappa\mu + \lambda\nu &= 0,\\ \kappa^2 + \lambda^2 &= \|u\|^2 .\end{aligned}$$

This leads to the flow

$$\begin{aligned}\alpha_t(u, v) &= u\cos t + v\frac{\|u\|^2}{h}\sin t + u \times v\frac{s\,\|u\|}{h}\sin t,\\ \beta_t(u, v) &= -u\frac{1}{h}\sin t + v\left(\frac{\|u\|^2}{h^2}\cos t + \frac{s^2}{h^2}\right) + u \times v\left(\frac{s\,\|u\|}{h^2}\cos t - \frac{s\,\|u\|}{h^2}\right).\end{aligned}$$

Now that we have this, there are two things to look at:

- when $u = 0$, we get $\alpha(t) = 0$, $\beta(t) = v$.
- when $t = \pi$, we get $\alpha(t) = -u$, $\beta(t) = v\left(\frac{s^2 - \|u\|^2}{s^2 + \|u\|^2}\right)$.

Also, fix some compact set $K \subset T^*S^2 \setminus S^2$ and observe that as $s \to 0$, the perturbed Dehn twist tends to the unperturbed one on K but on the zero-section itself we get the identity and not the antipodal map from the exact case. It is this lack of global convergence everywhere that makes it difficult to implement an approach to Theorem 1.2 along the lines above.

Part I

Projective twists in A_∞-catgeories

Chapter 2

A_∞-categories

Here we recall the basic background material on A_∞-categories that we shall need. Sign conventions differ throughout the literature, but all our signs and notation come from [39], to which we direct the reader who finds the treatment in this chapter too brief.

2.1 Categories

Fix some coefficient field $\mathbb{K}$. An A_∞-category $\mathcal{A}$ consists of a set of objects $Ob\mathcal{A}$ as well as a finite-dimensional $\mathbb{Z}$-graded $\mathbb{K}$-vector space $\hom_{\mathcal{A}}(X,Y)$ for any pair of objects X, Y, and composition maps $(\mu^d_{\mathcal{A}})_{d\geq 1}$,

$$\mu^d_{\mathcal{A}}\colon \hom_{\mathcal{A}}(X_{d-1},X_d)\otimes\cdots\otimes\hom_{\mathcal{A}}(X_0,X_1)\to\hom_{\mathcal{A}}(X_0,X_d)[2-d],$$

which satisfy the A_∞-relations

$$\sum_{m,n}(-1)^{\maltese_n}\mu^{d-m+1}_{\mathcal{A}}(a_d,\ldots,a_{n+m+1},\mu^m_{\mathcal{A}}(a_{n+m},\ldots,a_{n+1}),a_n,\ldots,a_1)=0. \tag{2.1}$$

Here $\maltese_n=|a_1|+\cdots+|a_n|-n$ and by $[k]$ we mean a shift in grading *down* by k.

The opposite category of $\mathcal{A}$, denoted $\mathcal{A}^{opp}$, has the same objects as $\mathcal{A}$ and $\hom_{\mathcal{A}^{opp}}(X,Y)=\hom_{\mathcal{A}}(Y,X)$, but composition is reversed:

$$\mu^d_{\mathcal{A}^{opp}}(a_d,\ldots,a_1)=(-1)^{\maltese_d}\mu^d_{\mathcal{A}}(a_1,\ldots,a_d).$$

The A_∞-relations in particular mean that $\mu^1_{\mathcal{A}}(\mu^1_{\mathcal{A}}(\cdot)) = 0$ so we can consider the cohomological category $H(\mathcal{A})$, which has the same objects as $\mathcal{A}$ and has morphism spaces $\hom_{H(\mathcal{A})}(X, Y) = H(\hom_{\mathcal{A}}(X, Y), \mu^1_{\mathcal{A}})$ with (associative) composition

$$[a_2] \cdot [a_1] = (-1)^{|a_1|}[\mu^2_{\mathcal{A}}(a_2, a_1)].$$

We call $\mathcal{A}$ *cohomologically unital* (c-unital for short) if $H(\mathcal{A})$ has identity morphisms (so is a category in the standard sense). Although this is perhaps not the most natural notion in the context of A_∞-categories, all categories considered in this thesis will be assumed to be c-unital, since Fukaya categories always carry cohomological units for geometric reasons.

There is another notion of unitality that is helpful to consider although Fukaya categories in general do not satisfy it: we say $\mathcal{A}$ is strictly unital if, for each X, there is an element $e_X \in \hom^0(X, X)$ such that

- $\mu^1(e_X) = 0$;
- $(-1)^{|a|}\mu^2(e_X, a) = a = \mu^2(a, e_X)$ for $a \in \hom(X_0, X_1)$;
- $\mu^d(a_{d-1}, \dots, e_X, \dots, a_1) = 0$ for all $d \geq 3$.

This is useful because every c-unital A_∞-category is quasi-equivalent to a strictly unital one [39, Lemma 2.1].

2.2 Functors

An A_∞-functor $\mathcal{F} \colon \mathcal{A} \to \mathcal{B}$ consists of a map $\mathcal{F} \colon Ob\mathcal{A} \to Ob\mathcal{B}$ and maps

$$\mathcal{F}^d \ : \ \hom_{\mathcal{A}}(X_{d-1}, X_d) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \hom_{\mathcal{B}}(\mathcal{F}X_0, \mathcal{F}X_d)[1-d]$$

for all $d \geq 1$, which are required to satisfy

$$\sum_r \sum_{s_1+\cdots+s_r=d} \mu^r_{\mathcal{B}}(\mathcal{F}^{s_r}(a_d, \dots, a_{d-s_r+1}), \dots, \mathcal{F}^{s_1}(a_{s_1}, \dots, a_1)) = \sum_{m,n} (-1)^{\maltese_n} \mathcal{F}^{d-m+1}(a_d, \dots, \mu^m_{\mathcal{A}}(a_{n+m}, \dots, a_{n+1}), a_n, \dots, a_1). \quad (2.2)$$

$\mathcal{F}$ induces a functor $H\mathcal{F}\colon H(\mathcal{A}) \to H(\mathcal{B})$ by $[a] \mapsto [\mathcal{F}^1(a)]$. We call a functor $\mathcal{F}$ between c-unital categories c-unital if $H\mathcal{F}$ is unital. All functors in this thesis will be assumed to be c-unital. We say $\mathcal{F}$ is *cohomologically full and faithful* if $H\mathcal{F}$ is full and faithful, and we say $\mathcal{F}$ is a *quasi-equivalence* if $H\mathcal{F}$ is an equivalence.

The set of A_∞-functors $\mathcal{F}\colon \mathcal{A} \to \mathcal{B}$ can itself be considered as the objects of an A_∞-category $fun(\mathcal{A}, \mathcal{B})$ (or more specifically $nu - fun(\mathcal{A}, \mathcal{B})$ for "non-unital functors" if we make no assumptions about units). We shall only need this in the following specific context.

2.3 A_∞-modules

We first note that any dg category can be considered as an A_∞-category with $\mu^d = 0$ for $d \geq 3$. In particular, for a given A_∞-category $\mathcal{A}$, we can consider A_∞-functors from $\mathcal{A}^{opp}$ to the category of chain complexes Ch over $\mathbb{K}$. We call such functors A_∞-modules over $\mathcal{A}$. Such functors can be thought of as the objects of a new A_∞-category $\mathcal{Q} = mod(\mathcal{A}) = fun(\mathcal{A}^{opp}, Ch)$.

An A_∞-module $\mathcal{M}\colon \mathcal{A} \to Ch$ assigns a graded vector space $\mathcal{M}(X)$ to all $X \in Ob\mathcal{A}$ and, in this specific setting, we follow [39] in changing notation of (2.2) slightly so that we have maps

$$\mu^d_{\mathcal{M}}\colon \mathcal{M}(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}(X_0)[2-d]$$

satisfying

$$\sum_{m,n} (-1)^{\maltese_n} \mu^{n+1}_{\mathcal{M}}(\mu^{d-n}_{\mathcal{M}}(b, a_{d-1}, \ldots, a_{n+1}), \ldots, a_1) + \sum_{m,n} (-1)^{\maltese_n} \mu^{d-m+1}_{\mathcal{M}}(b, a_{d-1}, \ldots, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), a_n, \ldots, a_1) = 0. \quad (2.3)$$

The morphism space $\hom^r_{\mathcal{Q}}(\mathcal{M}_0, \mathcal{M}_1)$ in degree r is made up of so-called *pre-module homomorphisms* $t = (t^d)_{d\geq 1}$ where

$$t^d\colon \mathcal{M}_0(X_{d-1}) \otimes \hom_{\mathcal{A}}(X_{d-2}, X_{d-1}) \otimes \cdots \otimes \hom_{\mathcal{A}}(X_0, X_1) \to \mathcal{M}_1(X_0)[r-d+1].$$

The composition maps in $\mathcal{Q}$ are

$$\begin{aligned}
&\left(\mu^1_{\mathcal{Q}} t\right)^d (b, a_{d-1}, \dots, a_1) = \qquad (2.4)\\
&\qquad \sum (-1)^{\ddagger} \mu^{n+1}_{\mathcal{M}_1}(t^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)\\
&\qquad + \sum (-1)^{\ddagger} t^{n+1}(\mu^{d-n}_{\mathcal{M}_0}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1)\\
&\qquad + \sum (-1)^{\ddagger} t^{d-m+1}(b, a_{d-1}, \dots, \mu^m_{\mathcal{A}}(a_{n+m}, \dots, a_{n+1}), \dots, a_1);\\
&\left(\mu^2_{\mathcal{Q}}(t_2, t_1)\right)^d (b, a_{d-1}, \dots, a_1) = \qquad (2.5)\\
&\qquad \sum (-1)^{\ddagger} t_2^{n+1}(t_1^{d-n}(b, a_{d-1}, \dots, a_{n+1}), a_n, \dots, a_1);
\end{aligned}$$

and $\mu^d_{\mathcal{Q}} = 0$ for $d \geq 3$. Here $\ddagger = |a_{n+1}| + \dots + |a_{d-1}| + |b| - d + n + 1$. We stress that the fact that higher composition maps vanish is not true for more general A_∞-functor categories, but rather reflects the dg nature of Ch.

If $\mu^1_{\mathcal{Q}} t = 0$, we say that t is a *A_∞-module homomorphism.* In this situation, we have a map $H(t)\colon H(\mathcal{M}_0(X)) \to H(\mathcal{M}_1(X))$ for all X, given by $[b] \mapsto [(-1)^{|b|} t^1(b)]$, where here $H(\mathcal{M}(X))$ is the cohomology of $\mathcal{M}(X)$ computed with respect to the differential $\partial(b) = (-1)^{|b|}\mu^1_{\mathcal{M}}(b)$.

Lemma 2.1. ([39, Lemma 1.16]) *Suppose the A_∞-module homomorphism $t \in \hom_{\mathcal{Q}}(\mathcal{M}_0, \mathcal{M}_1)$ is such that the induced maps $H(t)\colon H(\mathcal{M}_0(X)) \to H(\mathcal{M}_1(X))$ are isomorphisms for all X. Then, left composition with t induces a quasi-isomorphism $\hom_{\mathcal{Q}}(\mathcal{M}_1, \mathcal{N}) \to \hom_{\mathcal{Q}}(\mathcal{M}_0, \mathcal{N})$ and a similar result holds for right composition.*

Corollary 2.2. *Under the above hypotheses, $[t]$ is an isomorphism in $H(\mathcal{Q})$.*

Given $Y \in \mathcal{A}$, there is an associated A_∞-module $\mathcal{Y} \in \mathcal{Q}$ where

$$\mathcal{Y}(X) = \hom_{\mathcal{A}}(X, Y), \quad \mu^d_{\mathcal{Y}} = \mu^d_{\mathcal{A}}.$$

This forms part of an A_∞-functor $\ell\colon \mathcal{A} \to \mathcal{Q}$ called the *Yoneda embedding.* Given $t \in \hom_{\mathcal{A}}(Y, Z)$, $\ell^1(t) \in \hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{Z})$ is the morphism

$$\left(\ell^1(t)\right)^d (b, a_{d-1}, \dots, a_1) = \mu^{d+1}_{\mathcal{A}}(t, b, a_{d-1}, \dots, a_1),$$

and the higher order parts of the functor ℓ are defined similarly. ℓ is cohomologically full and faithful [39, Corollary 2.13].

2.4 Twisted complexes

Given $\mathcal{A}$ we can form a new category $\Sigma\mathcal{A}$ called the *additive enlargement* of $\mathcal{A}$ whose objects are formal sums

$$X = \bigoplus_{i \in I} V_i \otimes X_i,$$

where I is some finite set, the V_i are finite-dimensional graded vector spaces and X_i are objects of $\mathcal{A}$.

$$\hom_{\Sigma\mathcal{A}}\left(\bigoplus_{i\in I} V_i \otimes X_i, \bigoplus_{j\in J} W_j \otimes Y_j\right) = \bigoplus_{i,j} \hom_{\mathbb{K}}(V_i, W_j) \otimes \hom_{\mathcal{A}}(X_i, Y_j),$$

and we write morphisms $a \in \hom_{\Sigma\mathcal{A}}(X, Y)$ as $\alpha^{ji} \otimes x^{ji}$ where α^{ji} and x^{ji} are matrices of morphisms in $\hom_{\mathbb{K}}(V_i, W_j)$, $\hom_{\mathcal{A}}(X_i, Y_j)$ respectively. The composition maps are given by

$$\mu^d_{\Sigma\mathcal{A}}(a_d, \ldots, a_1) = \sum (-1)^{\triangleleft} \alpha_d \cdots \alpha_1 \otimes \mu^d_{\mathcal{A}}(x_d, \ldots, x_1)$$

where $\triangleleft = \sum_{p<q} |\alpha_p^{i_p, i_{p-1}}| \cdot (|x_q^{i_q, i_{q-1}}| - 1)$. $\mathcal{A}$ clearly sits inside $\Sigma\mathcal{A}$ as a full A_∞-subcategory once an object X is mapped to $\mathbb{K} \otimes X$, with $\mathbb{K}$ given grading zero.

A twisted complex in $\mathcal{A}$ is an object X of $\Sigma\mathcal{A}$, together with a differential $\delta_X \in \hom^1_{\Sigma\mathcal{A}}(X, X)$ which satisfies the following conditions:

- δ_X is strictly lower-triangular with respect to some filtration on X. By "filtration" here we mean a finite decreasing collection of subcomplexes $F^i X$ such that the induced differential on $F^k X / F^{k+1} X$ is zero [39, Section 3l];
- $\sum_d \mu^d_{\Sigma\mathcal{A}}(\delta_X, \ldots, \delta_X) = 0$.

Given this we can define new composition maps

$$\mu^d_{Tw\mathcal{A}}(a_d,\dots,a_1) = \sum_{i_0,\dots,i_d} \mu^{d+i_0+\dots+i_d}_{\Sigma\mathcal{A}} \left(\underbrace{\delta_{X_d},\dots\delta_{X_d}}_{i_d}, a_d, \underbrace{\delta_{X_{d-1}},\dots,\delta_{X_{d-1}}}_{i_{d-1}}, a_{d-1},\dots,a_1, \underbrace{\delta_{X_0},\dots,\delta_{X_0}}_{i_0} \right).$$

The sum is taken over all $i_j \geq 0$, but the conditions on δ_X imply that this is a finite sum and that moreover the A_∞-relations (2.1) still hold. $\Sigma\mathcal{A}$ sits inside $Tw\mathcal{A}$ as a full A_∞-subcategory given by those twisted complexes with zero differential.

We may relate $Tw\mathcal{A}$ and $\mathcal{Q}$ using the diagram below. $\mathcal{I}$ is the obvious inclusion functor and $\mathcal{I}^*$ is the induced pullback. The reader may find the appropriate formulae in [39].

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ell} & \mathcal{Q} \\ \downarrow{\scriptstyle \mathcal{I}} & & \uparrow{\scriptstyle \mathcal{I}^*} \\ Tw\mathcal{A} & \xrightarrow{\ell} & mod(Tw\mathcal{A}). \end{array} \tag{2.6}$$

We shall denote the resulting map from $Tw\mathcal{A}$ by $\mathcal{Q}$ by $\tilde{\ell}$.

2.5 Tensor products and shifts

Working in the larger categories $Tw\mathcal{A}$ and $\mathcal{Q}$ allows us perform many familiar algebraic constructions not necessarily possible in $\mathcal{A}$. As an example, take a chain complex (Z, ∂) and an A_∞-module $\mathcal{M} \in \mathcal{Q}$ and define a new A_∞-module $Z \otimes \mathcal{M} \in \mathcal{Q}$ by

$$\begin{aligned} &(Z \otimes \mathcal{M})(X) = Z \otimes \mathcal{M}(X), \\ &\mu^1_{Z\otimes\mathcal{M}}(z \otimes b) = (-1)^{|b|-1}\partial(z) \otimes b + z \otimes \mu^1_{\mathcal{M}}(b), \\ &\mu^d_{Z\otimes\mathcal{M}}(z \otimes b, a_{d-1},\dots,a_1) = z \otimes \mu^d_{\mathcal{M}}(b, a_{d-1},\dots,a_1) \text{ for } d \geq 2. \end{aligned} \tag{2.7}$$

As a special case of this, consider $Z = \mathbb{K}$, a one-dimensional chain complex concentrated in degree -1 and with trivial differential. We shall denote $Z \otimes \mathcal{M}$ by $S\mathcal{M}$ and call it the *shift* of $\mathcal{M}$. Similarly we have $S^\sigma\mathcal{M}$ for any $\sigma \in \mathbb{Z}$ and we have a canonical isomorphism

$$\hom_{H(\mathcal{Q})}(\mathcal{Y}, S^\sigma \mathcal{Z}) = \hom_{H(\mathcal{Q})}(\mathcal{Y}, \mathcal{Z})[\sigma].$$

When $\mathcal{A}$ is strictly unital, we can do a similar thing with twisted complexes. Given $(X, \delta_X) \in Tw\mathcal{A}$ and a chain complex (Z, ∂), we can form the twisted complex

$$\left(Z \otimes X, \mathrm{id} \otimes \delta_X + \widetilde{\partial} \otimes e_X\right),$$

where $\widetilde{\partial}(z) = (-1)^{|z|-1}\partial(z)$. We can also do shifts here: $S^\sigma Y = \mathbb{K}[\sigma] \otimes Y$

Remark 2.3. ([39, Remark 3.2]) *Given a chain complex (Z, ∂), we can form a new chain complex given by $H(Z)$ with trivial differential. By choosing a linear map that picks a chain representative for each cohomology class, we can define a map $H(Z) \otimes \mathcal{M} \to Z \otimes \mathcal{M}$ and Corollary 2.2 says that this will in fact induce an isomorphism in $H(\mathcal{Q})$.*

2.6 Evaluation maps

Given $V \in \mathcal{A}$ and $\mathcal{Y} \in \mathcal{Q}$ we have an evaluation morphism

$$\begin{aligned} &ev\colon \mathcal{Y}(V) \otimes \mathcal{V} \to \mathcal{Y}, \\ &ev^d(y \otimes v, a_{d-1}, \ldots, a_1) = \mu_{\mathcal{Y}}^{d+1}(y, v, a_{d-1}, \ldots, a_1). \end{aligned} \tag{2.8}$$

In the strictly unital case, we can also define this for twisted complexes. In order to define $ev\colon \hom_{Tw\mathcal{A}}(V, Y) \otimes V \to Y$, we require that ev be an element of $\hom_{Tw\mathcal{A}}(V, Y)^\vee \otimes \hom_{Tw\mathcal{A}}(V, Y)$. To do this, choose a homogeneous basis $\{b_i\}$ of $\hom_{Tw\mathcal{A}}(Y, V)$ and let $\{\beta_i\}$ be the dual basis. Now let $ev = \sum \beta_i \otimes b_i$. It is easy to verify that the two maps correspond under $\tilde{\ell}$, so we shall feel justified in abusing notation and referring to both as ev since it will always be clear in which setting we are working.

We can also define a dual evaluation map $ev^\vee\colon Y \to \hom_{Tw\mathcal{A}}(Y, V)^\vee \otimes V$ given by $ev^\vee = \sum \gamma_j \otimes c_j$ where again $\{c_j\}$ is a basis for $\hom_{Tw\mathcal{A}}(Y, V)$ and $\{\gamma_i\}$ is the dual basis.

2.7 Cones and triangles

Given $t\colon \mathcal{M}_0 \to \mathcal{M}_1$ a degree zero module homomorphism, we can form the mapping cone $\mathcal{C} = Cone(t)$ given by

$$\begin{aligned} &\mathcal{C}(X) = \mathcal{M}_0(X)[1] \oplus \mathcal{M}_1(X), \\ &\mu^d_{\mathcal{C}}\left(\begin{pmatrix} b_0 \\ b_1 \end{pmatrix}, a_{d-1}, \dots, a_1\right) = \begin{pmatrix} \mu^d_{\mathcal{M}_0}(b_0, a_{d-1}, \dots, a_1) \\ \mu^d_{\mathcal{M}_1}(b_1, a_{d-1}, \dots, a_1) + t^d(b_0, a_{d-1}, \dots, a_1) \end{pmatrix}. \end{aligned} \tag{2.9}$$

The cone $\mathcal{C}$ comes with module homomorphisms ι and π which fit into the following diagram in $H(\mathcal{Q})$

$$\begin{array}{ccc} \mathcal{M}_0 & \xrightarrow{[t]} & \mathcal{M}_1 \\ & {\scriptstyle [\pi]}\nwarrow {\scriptstyle [1]} & \downarrow {\scriptstyle [\iota]} \\ & & \mathcal{C}. \end{array}$$

Any triangle in $H(\mathcal{A})$ quasi-isomorphic to one of the above form under the Yoneda embedding is called *exact*.

Likewise the cone of $t\colon X \to Y$ in $Tw\mathcal{A}$ for a degree zero cocycle t is given by

$$Cone(t) = \left(SX \oplus Y, \begin{pmatrix} S(\delta_X) & 0 \\ -S(t) & \delta_Y \end{pmatrix}\right).$$

We call an A_∞-category $\mathcal{A}$ triangulated if every morphism $[t]$ fits into some exact triangle and $\mathcal{A}$ is closed under all shifts, positive and negative.

Proposition 2.4. ([39, Proposition 3.14]) *If $\mathcal{A}$ is a triangulated A_∞-category, then $H^0(\mathcal{A})$ is triangulated in the classical sense. Moreover, for $\mathcal{F}$ an A_∞-functor between triangulated A_∞-categories, $H\mathcal{F}$ is an exact functor of triangulated categories.*

For a given $\mathcal{A}$, we can consider the triangulated A_∞-subcategory $\widetilde{\mathcal{Q}} \subset \mathcal{Q}$ generated by the image of the Yoneda embedding. We call $H^0(\widetilde{\mathcal{Q}})$ the *derived category* of $\mathcal{A}$, which we denote $D\mathcal{A}$. Equivalently, we may define $D\mathcal{A}$ as $H^0(Tw\mathcal{A})$.

Chapter 3

$\mathbb{CP}^n$-twists

In the interests of legibility, we introduce the shorthand $\mathbf{a}_{d-1}$ for $a_{d-1}, \ldots, a_1$.

Huybrechts and Thomas [23], motivated by mirror symmetry, introduced the notion of a $\mathbb{P}^n$-object P in the derived category $D(X)$ of a smooth projective variety X. They showed that there are associated twists Φ_P of $D(X)$ which are in fact autoequivalences. We reinterpret their construction in our setting.

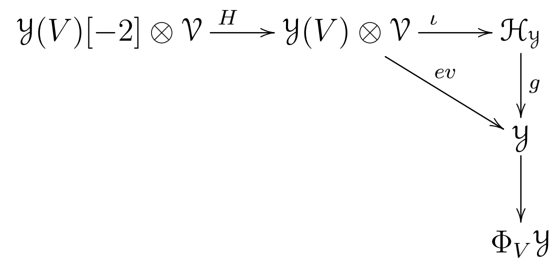
Definition 3.1. *A $\mathbb{CP}^n$-object is a pair (V, h) where $V \in Ob\mathcal{A}$ and $h \in \hom^2(V, V)$ such that*

- $\mu^1_{\mathcal{A}} h = 0$;
- $\hom_{H(\mathcal{A})}(V, V) \cong \mathbb{K}[h]/h^{n+1}$ *as a graded ring;*
- *There exists a map* $\int \colon \hom^{2n}_{H(\mathcal{A})}(V, V) \to \mathbb{K}$ *such that, for any X, the resulting bilinear map* $\hom^{2n-k}_{H(\mathcal{A})}(X, V) \times \hom^k_{H(\mathcal{A})}(V, X) \to \hom^{2n}_{H(\mathcal{A})}(V, V) \to \mathbb{K}$ *is nondegenerate.*

We shall often just refer to a $\mathbb{CP}^n$-object by V since, following Remark 2.3, the choice of h will be irrelevant up to quasi-equivalence.

To define our twist, we imitate the construction in [23]. Take some $\mathbb{CP}^n$-object

V and consider the following diagram



(3.1)

where here $\mathcal{H}_{\mathcal{Y}}$ is $Cone(H)$ and $\Phi_V \mathcal{Y}$ is $Cone(g)$.

Here ev is the evaluation map (2.8) and we define the other maps by

$$H^1(y \otimes v) = (-1)^{|y|+|v|}\mu^2_{\mathcal{Y}}(y,h) \otimes v + (-1)^{|y|-1} y \otimes \mu^2_{\mathcal{V}}(h,v),$$
$$H^d(y \otimes v, \mathbf{a}_{d-1}) = (-1)^{|y|-1} y \otimes \mu^{d+1}_{\mathcal{V}}(h, v, \mathbf{a}_{d-1}) \text{ for } d \geq 2.$$

and

$$g^d\left(\begin{pmatrix} y_1 \otimes v_1 \\ y_2 \otimes v_2 \end{pmatrix}, \mathbf{a}_{d-1}\right) = \mu^{d+1}_{\mathcal{Y}}(y_2, v_2, \mathbf{a}_{d-1}) + (-1)^{|y_1|-1}\mu^{d+2}_{\mathcal{Y}}(y_1, h, v_1, \mathbf{a}_{d-1}).$$

Lemma 3.2. *H and g are $\mu^1_{\mathcal{Q}}$-closed.*

Proof. This is a direct calculation. Using (2.4) and (2.7), we see that

$$\begin{aligned}
&\left(\mu^1_{\mathcal{Q}} H\right)^d (y \otimes v, \mathbf{a}_{d-1}) = \\
&y \otimes \begin{pmatrix} \sum_n (-1)^{\ddagger_n + |y|-1} \mu^{n+1}_{\mathcal{V}}(\mu^{d-n+1}(h, v, a_{d-1}, \ldots, a_{n+1}), \ldots, a_1) \\ + \sum_n (-1)^{\ddagger_n + |y|-1} \mu^{n+2}_{\mathcal{V}}(h, \mu^{d-n}(v, a_{d-1}, \ldots, a_{n+1}), \ldots, a_1) \\ + \sum_{m,n} (-1)^{\ddagger_n + |y|-1} \mu^{d-m+2}_{\mathcal{V}}(h, v, a_{d-1}, \ldots, \mu^m_{\mathcal{A}}(a_{n+m}, \ldots, a_{n+1}), \ldots, a_1) \end{pmatrix} \\
&+ \left((-1)^{\ddagger_0 + |y| - 1 + |\mu^{d+1}(h,v,\mathbf{a}_{d-1})|-1} + (-1)^{\ddagger_{d-1} + |v| - 1 + |y| - 2}\right) \mu^1_{\mathcal{Y}}(y) \otimes \mu^{d+1}_{\mathcal{V}}(h, v, \mathbf{a}_{d-1}) \\
&+ \left((-1)^{\ddagger_{d-1} + |y| + |v|} + (-1)^{\ddagger_0 + |y| + |\mu^d(v, \mathbf{a}_{d-1})|}\right) \mu^2_{\mathcal{Y}}(y, h) \otimes \mu^d_{\mathcal{V}}(v, \mathbf{a}_{d-1}).
\end{aligned}$$

The terms involving $\mu^1_{\mathcal{Y}}(y)$ and $\mu^2_{\mathcal{Y}}(y,h)$ cancel, and inside the big bracket, we find precisely the terms from the A_∞-relation (2.1) except for the term involving $\mu^d_{\mathcal{V}}(\mu^1_{\mathcal{A}}(h), v, \mathbf{a}_{d-1})$. But, by assumption, $\mu^1_{\mathcal{A}}(h) = 0$ so this term vanishes.

The proof for g is similar:

$$\left(\mu^1_{\mathcal{Q}} g\right)^d \left(\left(\begin{array}{c} y_1 \otimes v_1 \\ y_2 \otimes v_2 \end{array} \right), \mathbf{a}_{d-1} \right) =$$

$$\sum_n (-1)^{\ddagger_n + |y_1| - 1} \mu^{n+1}_{\mathcal{Y}} (\mu^{d-n+2}_{\mathcal{Y}} (y_1, h, v_1, a_{d-1}, \dots, a_{n+1}), \dots, a_1)$$
$$+ \sum_n (-1)^{\ddagger_n + |y_1| - 1} \mu^{n+3}_{\mathcal{Y}} (y_1, h, \mu^{d-n}_{\mathcal{V}} (v_1, a_{d-1}, \dots, a_{n+1}), \dots, a_1)$$
$$+ (-1)^{\ddagger_{d-1} + |\mu^1(y_1)| - 1 + |v_1| - 1} \mu^{d+2}_{\mathcal{Y}} (\mu^1_{\mathcal{Y}}(y_1), h, v_1, a_{d-1}, \dots, a_1)$$
$$+ \sum_{m,n} (-1)^{\ddagger_n + |y_1| - 1} \mu^{d-m+3}_{\mathcal{Y}} (y_1, h, v_1, a_{d-1}, \dots, \mu^m_{\mathcal{A}} (a_{n+m}, \dots, a_{n+1}), \dots, a_1)$$

$$+ \sum_n (-1)^{\ddagger_n} \mu^{n+1}_{\mathcal{Y}} (\mu^{d-n+1}_{\mathcal{Y}} (y_2, v_2, a_{d-1}, \dots, a_{n+1}), \dots, a_1)$$
$$+ \sum_n (-1)^{\ddagger_n} \mu^{n+2}_{\mathcal{Y}} (y_2, \mu^{d-n}_{\mathcal{V}} (v_2, a_{d-1}, \dots, a_{n+1}), \dots, a_1)$$
$$+ (-1)^{\ddagger_{d-1} + |v_2| - 1} \mu^{d+1}_{\mathcal{Y}} (\mu^1_{\mathcal{Y}}(y_2), v_2, a_{d-1}, \dots, a_1)$$
$$+ \sum_{m,n} (-1)^{\ddagger_n} \mu^{d-m+2}_{\mathcal{Y}} (y_2, v_2, a_{d-1}, \dots, \mu^m_{\mathcal{A}} (a_{n+m}, \dots, a_{n+1}), \dots, a_1)$$

$$+ \sum (-1)^{\ddagger_n + |y_1| - 1} \mu^{n+2}_{\mathcal{Y}} (y_1, \mu^{d-n+1}_{\mathcal{V}} (h, v_1, a_{d-1}, \dots, a_{n+1}), \dots, a_1)$$
$$+ (-1)^{\ddagger_{d-1} + |y_1| + |v_1|} \mu^{d+1}_{\mathcal{Y}} (\mu^2_{\mathcal{Y}} (y_1, h), v_1, a_{d-1}, \dots, a_1).$$

Here the final two lines come from the presence of H in the μ^d maps in $Cone(H)$ as in (2.9). Again we find all the terms from (2.3) except for those involving $\mu^1_{\mathcal{A}}(h)$, so the above sum vanishes.

□

Concretely, $\Phi_V \mathcal{Y} = (\mathcal{Y}(V) \otimes \mathcal{V}) \oplus (\mathcal{Y}(V)[1] \otimes \mathcal{V}) \oplus \mathcal{Y}$ and

$$\mu^1_{\Phi_V \mathcal{Y}} \left(\begin{array}{c} y_1 \otimes v_1 \\ y_2 \otimes v_2 \\ \\ y_3 \end{array} \right) = \left(\begin{array}{c} (-1)^{|v_1|-1} \mu^1_{\mathcal{Y}}(y_1) \otimes v_1 + y_1 \otimes \mu^1_{\mathcal{V}}(v_1) \\ (-1)^{|v_2|-1} \mu^1_{\mathcal{Y}}(y_2) \otimes v_2 + (-1)^{|y_1|+|v_1|} \mu^2_{\mathcal{Y}}(y_1, h) \otimes v_1 \\ + y_2 \otimes \mu^1_{\mathcal{V}}(v_2) + (-1)^{|y_1|-1} y_1 \otimes \mu^2_{\mathcal{V}}(h, v_1) \\ \mu^1_{\mathcal{Y}}(y_3) + \mu^2_{\mathcal{Y}}(y_2, v_2) + (-1)^{|y_1|-1} \mu^3_{\mathcal{Y}}(y_1, h, v_1) \end{array} \right)$$

and, for $d \geq 2$,

$$\mu^d_{\Phi_V \mathcal{Y}}\left(\begin{pmatrix} y_1 \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \end{pmatrix}, \mathbf{a}_{d-1}\right)$$
$$= \begin{pmatrix} y_1 \otimes \mu^d_{\mathcal{V}}(v_1, \mathbf{a}_{d-1}) \\ y_2 \otimes \mu^d_{\mathcal{V}}(v_2, \mathbf{a}_{d-1}) + (-1)^{|y_1|-1} y_1 \otimes \mu^{d+1}_{\mathcal{V}}(h, v_1, \mathbf{a}_{d-1}) \\ \mu^d_{\mathcal{Y}}(y_3, \mathbf{a}_{d-1}) + \mu^{d+1}_{\mathcal{Y}}(y_2, v_2, \mathbf{a}_{d-1}) + (-1)^{|y_1|-1} \mu^{d+2}_{\mathcal{Y}}(y_1, h, v_1, \mathbf{a}_{d-1}) \end{pmatrix}.$$

3.1 $\mathbb{CP}^n$-twist functor

We want to upgrade Φ_V to a functor $\Phi_V \colon \mathcal{Q} \to \mathcal{Q}$ and so, having described the effect of Φ_V on objects, we must describe how it acts on morphisms.

Firstly we set $\Phi^d_V = 0$ for $d \geq 2$, so that Φ_V is in fact a dg functor and, given $t \in \hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{Z})$, $\hat{t} = \Phi_V(t)$ has first order part

$$\hat{t}^1 \begin{pmatrix} y_1 \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} (-1)^{|v_1|+|t|} t^1(y_1) \otimes v_1 \\ (-1)^{|v_2|-1} t^1(y_2) \otimes v_2 + (-1)^{|y_1|+|v_1|} t^2(y_1, h) \otimes v_1 \\ t^1(y_3) + t^2(y_2, v_2) + (-1)^{|y_1|-1} t^3(y_1, h, v_1) \end{pmatrix}$$

and, for $d \geq 2$,

$$\hat{t}^d\left(\begin{pmatrix} y_1 \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \end{pmatrix}, \mathbf{a}_{d-1}\right)$$
$$= \begin{pmatrix} 0 \\ 0 \\ t^d(y_3, \mathbf{a}_{d-1}) + t^{d+1}(y_2, v_2, \mathbf{a}_{d-1}) + (-1)^{|y_1|-1} t^{d+2}(y_1, h, v_1, \mathbf{a}_{d-1}) \end{pmatrix}.$$

Lemma 3.3. *Φ_V is an A_∞-functor.*

Proof. The condition we need to verify is (2.2), which here reduces to the two conditions

$$\mu^1_{\mathcal{Q}}(\Phi^1_V(t_1)) = \Phi^1_V(\mu^1_{\mathcal{Q}}(t_1)),$$
$$\mu^2_{\mathcal{Q}}(\Phi^1_V(t_2), \Phi^1_V(t_1)) = \Phi^1_V(\mu^2_{\mathcal{Q}}(t_2, t_1)),$$

since $\mu^d_{\mathcal{Q}} = 0$ for $d \geq 3$. Both are straightforward calculations. □

Proposition 3.4. $\Phi_V \mathcal{V} \cong S^{-2n}\mathcal{V}$. *Also, if* $\hom_{H(\mathcal{A})}(V, Y) = 0$, *then* $\Phi_V Y \cong Y$.

We first recall a basic algebraic lemma that we shall need.

Lemma 3.5. *If* $f : V \to W$ *is a map of chain complexes such that* f *is surjective and* $\ker f$ *is acyclic, then* f *is a quasi-isomorphism.*

Proof of Proposition 3.4. Following Remark 2.3, we may replace the $\hom_{\mathcal{A}}(V, Y)$ terms in $\Phi_V \mathcal{Y}$ with $\hom_{H(\mathcal{A})}(V, Y)$. This vector space has a basis given by $e_V, h, \ldots, h^n$ so that we replace $\Phi_V \mathcal{V}$ with the quasi-isomorphic

$$\bigoplus_{i=0}^{n} h^i[-2i]\mathcal{V} \oplus \bigoplus_{i=0}^{n} h^i[-2i+1]\mathcal{V} \oplus \mathcal{V}.$$

There is a module homomorphism π_1 that to first-order is a projection annihilating the summands $e_V[1]\mathcal{V} \oplus \mathcal{V}$ and has higher-order terms zero. We want to apply Corollary 2.2 to π_1, so let $\partial(b) = (-1)^{|b|}\mu^1_{\Phi_V \mathcal{Y}}(b)$. Now, if an element $(0, e_V \otimes v_2, v_3)$ of the kernel of π_1 is ∂-closed, then $\partial(0, -e_V \otimes v_3, 0) = (0, e_V \otimes v_2, v_3)$, so by Lemma 3.5, π_1 is a quasi-isomorphism.

This means $\Phi_V \mathcal{V}$ is quasi-isomorphic to the image of π_1,

$$\bigoplus_{i=0}^{n} h^i[-2i]\mathcal{V} \oplus \bigoplus_{i=1}^{n} h^i[-2i+1]\mathcal{V}.$$

We can project once more so as to kill the summands $e_V \mathcal{V} \oplus h[-1]\mathcal{V}$. A similar argument shows that this is a quasi-isomorphism. By repeating this process, removing pairs of summands by a series of projection quasi-isomorphisms, we can remove everything except $h^n[-2n]\mathcal{V}$.

The second fact is trivial from the definition of Φ_V. □

Remark 3.6. *These results coincide with what one finds geometrically: namely that* ϕ_V *acts on itself by a shift in grading by* $2n$ *[37], and if* W *and* V *are disjoint Lagrangians, then we can arrange that* ϕ_V *is supported in a region disjoint from* W *so that* ϕ_V *has no effect on* W*.*

Chapter 4

Φ_V is a quasi-equivalence

In the case where $\mathcal{A}$ itself is a triangulated A_∞-category, the discussion in [39, Section 3d] shows that we can define an A_∞-functor, which we shall also denote Φ_V, on $\mathcal{A}$ itself in such a way that

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\ell} & \mathcal{Q} \\ \downarrow{\scriptstyle \Phi_V} & & \downarrow{\scriptstyle \Phi_V} \\ \mathcal{A} & \xrightarrow{\ell} & \mathcal{Q} \end{array} \tag{4.1}$$

commutes (up to isomorphism in $H^0(fun(\mathcal{A}, \mathcal{Q}))$). In this chapter we shall prove

Theorem 4.1. *If V is a $\mathbb{CP}^n$-object in a cohomologically finite A_∞-triangulated category $\mathcal{A}$, then $\Phi_V\colon \mathcal{A} \to \mathcal{A}$ is a quasi-equivalence.*

To prove this, it will be useful to have an explicit formula for Φ_V on the level of twisted complexes. In order to do this, we have to assume that $\mathcal{A}$ is strictly unital. The more general c-unital case later will be discussed later. The diagram (4.1) can now be augmented to the following:

$$\begin{array}{ccccc} \mathcal{A} & \longrightarrow & Tw\mathcal{A} & \xrightarrow{\tilde{\ell}} & \mathcal{Q} \\ \vdots{\scriptstyle \Phi_V} & & \downarrow{\scriptstyle \Phi_V} & & \downarrow{\scriptstyle \Phi_V} \\ \mathcal{A} & \longrightarrow & Tw\mathcal{A} & \xrightarrow{\tilde{\ell}} & \mathcal{Q}. \end{array} \tag{4.2}$$

We shall define an A_∞-functor Φ_V on $Tw\mathcal{A}$ such that the righthand square precisely commutes. Then, in the case where $\mathcal{A}$ is triangulated, the inclusion into

$Tw\mathcal{A}$ is a quasi-equivalence so can be inverted [39, Theorem 2.9], which allows us to pullback Φ_V to $\mathcal{A}$.

In order to imitate the construction of Chapter 3 we need to define a map $H\colon \hom_{Tw\mathcal{A}}(V,Y)\otimes V \to \hom_{Tw\mathcal{A}}(V,Y)\otimes V$. This means H must be an element of $\mathrm{End}_{\mathbb{K}}(\hom_{Tw\mathcal{A}}(V,Y)) \otimes \hom_{Tw\mathcal{A}}(V,V)$. Let $\bar{h} \in \mathrm{End}_{\mathbb{K}}(\hom_{Tw\mathcal{A}}(V,Y))$ be the linear map $a \mapsto \mu^2(a,h)$, and define

$$H = \bar{h} \otimes e_V - \mathrm{id} \otimes h.$$

With this we can consider the diagram

$$\begin{array}{ccccc} \hom_{Tw\mathcal{A}}(V,Y)[-2]\otimes V & \xrightarrow{H} & \hom_{Tw\mathcal{A}}(V,Y)\otimes V & \xrightarrow{\iota} & \mathcal{H}_Y \\ & & & \searrow^{ev} & \downarrow g \\ & & & & Y \\ & & & & \downarrow \\ & & & & \Phi_V Y. \end{array} \tag{4.3}$$

As in (3.1), $\mathcal{H}_Y = Cone(H)$ and $\Phi_V Y = Cone(g)$, where now g is now given by ev on the second summand of $\mathcal{H}_Y$ and zero on the first summand. It is straightforward to verify that the above diagram becomes (3.1) under $\tilde{\ell}$. We have now defined a twisted complex

$$\Phi_V Y = \left(\begin{matrix} \hom_{Tw\mathcal{A}}(V,Y)\otimes V \\ \oplus \hom_{Tw\mathcal{A}}(V,Y)[1]\otimes V \\ \oplus Y \end{matrix}, \begin{pmatrix} \delta_{\hom_{Tw\mathcal{A}}(V,Y)\otimes V} & 0 & 0 \\ -S^2(H) & -\delta_{\hom_{Tw\mathcal{A}}(V,Y)\otimes V} & 0 \\ 0 & -S(ev) & \delta_Y \end{pmatrix} \right). \tag{4.4}$$

Also, given $t \in \hom_{Tw\mathcal{A}}(Y,Z)$, we get $\Phi_V t \in \hom_{Tw\mathcal{A}}(\Phi_V Y, \Phi_V Z)$ given with respect to the above splittings by

$$\begin{pmatrix} (-1)^{|t|}\bar{t}\otimes e_V & 0 & 0 \\ \overset{\triangle}{t}\otimes e_V & \bar{t}\otimes e_V & 0 \\ 0 & 0 & t \end{pmatrix},$$

where $\bar{t}\colon \hom_{Tw\mathcal{A}}(V,Y) \to \hom_{Tw\mathcal{A}}(V,Z)$ is given by $a \mapsto (-1)^{|a|}\mu^2(t,a)$ and $\overset{\triangle}{t}$

denotes the map $a \mapsto \mu^3(t, a, h)$. This now defines an A_∞-functor Φ_V on $Tw\mathcal{A}$ which has only first-order terms (it is a dg functor). We leave it to the reader to check that the righthand square in (4.2) commutes.

4.1 Adjoints

One of the benefits of making the assumption of strict unitality and working with twisted complexes is that is easy now to identify an adjoint twist functor to Φ_V. We recall that, given a pair of functors $F : \mathcal{D} \to \mathcal{C}$ and $G : \mathcal{C} \to \mathcal{D}$, we say that F is left adjoint to G (and G is right adjoint to F) if there are isomorphisms $\hom_{\mathcal{C}}(FY, X) \cong \hom_{\mathcal{D}}(Y, GX)$ which are natural in X and Y.

Consider the following diagram

$$\begin{array}{ccccc} & & & & S^{-1}Y \\ & & & \swarrow^{ev^\vee} & \downarrow g^\vee \\ \hom_{Tw\mathcal{A}}(Y,V)[1]^\vee \otimes V & \xrightarrow{H^\vee} & \hom_{Tw\mathcal{A}}(Y,V)[-1]^\vee \otimes V & \xrightarrow{\iota} & Cone(H^\vee) \\ & & & & \downarrow \\ & & & & Cone(g^\vee) \end{array} \tag{4.5}$$

Define $h^\vee\colon \hom_{Tw\mathcal{A}}(Y,V)[-2]^\vee \to \hom_{Tw\mathcal{A}}(Y,V)^\vee$ by $h^\vee(\eta)(a) = \eta(\mu^2(h,y))$. Now let $H^\vee = h^\vee \otimes e_V - \mathrm{id} \otimes h$ and $g^\vee = (0, ev^\vee)$. We define $\mathcal{H}_Y^\vee = Cone(H^\vee)$ and $\Phi_V^\vee Y = Cone(g^\vee)$. $\Phi_V^\vee Y$ is given by the twisted complex

$$\left(\begin{array}{l} Y \\ \oplus \hom_{Tw\mathcal{A}}(Y,V)[2]^\vee \otimes V \\ \oplus \hom_{Tw\mathcal{A}}(Y,V)[-1]^\vee \otimes V \end{array}, \begin{pmatrix} \delta_Y & 0 & 0 \\ 0 & \delta_{\hom_{Tw\mathcal{A}}(V,Y)^\vee \otimes V} & 0 \\ ev^\vee & H^\vee & \delta_{\hom_{Tw\mathcal{A}}(V,Y)^\vee \otimes V} \end{pmatrix} \right).$$

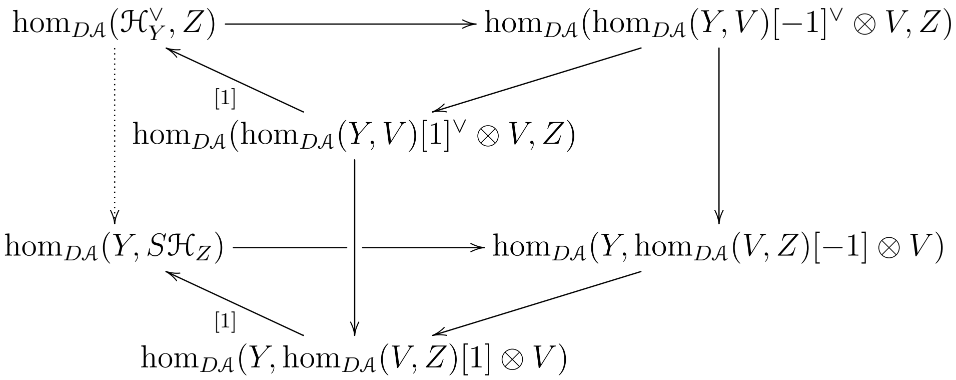
Given $t \in \hom_{Tw\mathcal{A}}(Y, Z)$, we similarly get $\Phi_V^\vee t \in \hom_{Tw\mathcal{A}}(\Phi_V^\vee Y, \Phi_V^\vee Z)$, so that $\Phi_V^\vee$ is a (dg) functor on $Tw\mathcal{A}$.

Proposition 4.2. *$H\Phi_V^\vee$ is both left and right adjoint to $H\Phi_V$.*

Proof. We first prove that $H\Phi_V^\vee$ is left adjoint to $H\Phi_V$. We want to show there are isomorphisms

$$\hom_{D\mathcal{A}}(\Phi_V^\vee Y, Z) \cong \hom_{D\mathcal{A}}(Y, \Phi_V Z)$$

that are natural in $D\mathcal{A}$. By applying the exact functors $\hom_{D\mathcal{A}}(-, Z)$ to (4.5) and $\hom_{D\mathcal{A}}(Y, -)$ to (4.3), we get long exact sequences, natural in $D\mathcal{A}$,

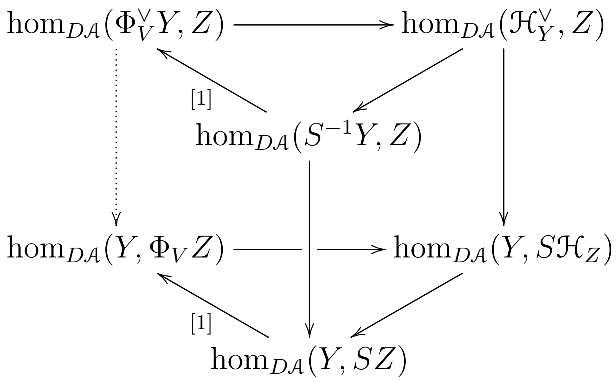


Here the vertical isomorphisms come from the natural identities

$$\begin{aligned}
\hom_{D\mathcal{A}}(\hom_{D\mathcal{A}}(Y,V)^\vee \otimes V, Z) &= \hom_{D\mathcal{A}}(Y,V)^{\vee\vee} \otimes \hom_{D\mathcal{A}}(V,Z) \\
&= \hom_{D\mathcal{A}}(Y,V) \otimes \hom_{D\mathcal{A}}(V,Z) \\
&= \hom_{D\mathcal{A}}(Y, \hom_{D\mathcal{A}}(V,Z) \otimes V)
\end{aligned}$$

so that $\hom_{D\mathcal{A}}(\mathcal{H}_Y^\vee, Z) \cong \hom_{D\mathcal{A}}(Y, \mathcal{H}_Z)$ naturally (note that this requires that $\mathcal{A}$ be cohomologically finite). This proves in particular that the functor assigning Y to $\mathcal{H}_Y^\vee$ is left adjoint to the functor sending Y to $\mathcal{H}_Y$ (these functors are defined by the obvious restriction of the above construction).

Similarly we have



and therefore

$$\hom_{D\mathcal{A}}(\Phi_V^\vee Y, Z) \cong \hom_{D\mathcal{A}}(Y, \Phi_V Z)$$

naturally. Proving right adjointness is similar. □

With the existence of adjoints proven, the rest of the proof of Theorem 4.1 is an exercise in the abstract machinery of triangulated categories.

4.2 Spanning classes

A nontrivial collection Ω of objects in a triangulated category $\mathcal{D}$ is called a spanning class if, for all $B \in \mathcal{D}$, we have

- If $\hom_{\mathcal{D}}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.
- If $\hom_{\mathcal{D}}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \simeq 0$.

Given an object $A \in \mathcal{D}$, we denote by $A^\perp = \{B : \hom^*_{\mathcal{D}}(A, B) = 0\}$ and can define ${}^\perp A$ similarly.

Lemma 4.3. *For a $\mathbb{CP}^n$-object $V \in \mathcal{A}$, $\{V\} \cup V^\perp$ is a spanning class in $D\mathcal{A}$.*

Proof. Suppose we have B such that $\hom_{D\mathcal{A}}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$. Then putting $A = V$ shows that $B \in V^\perp$. Therefore, in particular, $\hom_{D\mathcal{A}}(B, B[i]) = 0$ for all i so that $B \simeq 0$. For the other condition, note that, by the definition of $\mathbb{CP}^n$-object, $\hom_{D\mathcal{A}}(V, A) = 0$, if and only if $\hom_{D\mathcal{A}}(A, V) = 0$. □

4.3 Equivalence

We now appeal to the following theorem of Bridgeland [13, Theorem 2.3]

Theorem 4.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between $\mathbb{K}$-linear triangulated categories such that F has a left and a right adjoint. Then F is fully faithful if and only if there exists some spanning class $\Omega \subset \mathcal{C}$ such that, for all objects $K, L \in \Omega$ and all $i \in \mathbb{Z}$ the natural homomorphism*

$$F\colon \hom_{\mathcal{C}}(K, L[i]) \to \hom_{\mathcal{D}}(F(K), F(L[i]))$$

is an isomorphism

For the spanning class from Lemma 4.3, this condition follows immediately from Proposition 3.4, so Φ_V is cohomologically full and faithful.

To show that it is an quasi-equivalence, let $\mathcal{B} \subset Tw\mathcal{A}$ be the full A_∞-subcategory of objects isomorphic to $\Phi_V Y$ for some Y. Since Φ_V maps exact triangles in $H(Tw\mathcal{A})$ to exact triangles in $H(Tw\mathcal{A})$, $\mathcal{B}$ is actually a triangulated A_∞-category. On the other hand, from Proposition 3.4, $V \in \mathcal{B}$ and so (4.3) shows that $\mathcal{B}$ generates $Tw\mathcal{A}$. This means that the inclusion $\mathcal{B} \to Tw\mathcal{A}$ must be a quasi-equivalence, which implies that Φ_V is also a quasi-equivalence.

So far we have only dealt with the case when $\mathcal{A}$ is strictly unital. In the c-unital case, the standard trick [39, Section 2] is to pass to a quasi-equivalent A_∞-category $\widetilde{\mathcal{A}}$ which is strictly unital and such that

$$\begin{array}{ccc} D\mathcal{A} & \xrightarrow{H\Phi_V} & D\mathcal{A} \\ \downarrow \cong & & \downarrow \cong \\ D\widetilde{\mathcal{A}} & \xrightarrow{H\widetilde{\Phi_V}} & D\widetilde{\mathcal{A}} \end{array}$$

commutes (up to isomorphism). Then we can apply our result from the strictly unital case to complete the proof of Theorem 4.1.

Chapter 5

Some geometric consequences

5.1 The connection with spherical objects

As we mentioned in the Introduction, it would require a more substantial analysis to verify that Φ_V does in fact represent the categorical version of ϕ_V. However, in the lowest dimension when $V \cong \mathbb{CP}^1$, this can be done by using Seidel's result resulting geometric Dehn twists and algebraic spherical twists [39], and the relationship (1.3).

We shall first recall the basic facts about spherical objects and spherical twists [39, Section 5].

Definition 5.1. *An object $V \in \mathcal{A}$ is called spherical of dimension n if*

- $\hom_{H(\mathcal{A})}(V, V) \cong \mathbb{K}[t]/t^2$.
- *There exists a map* $\int \colon \hom^n_{H(\mathcal{A})}(V, V) \to \mathbb{K}$ *such that, for all X, the resulting bilinear map* $\hom^{n-k}_{H(\mathcal{A})}(X, V) \times \hom^k_{H(\mathcal{A})}(V, X) \to \hom^n_{H(\mathcal{A})}(V, V) \to \mathbb{K}$ *is nondegenerate.*

Definition 5.2. *Given an object V, the twist map T_V is defined by $\mathcal{T}_V\mathcal{Y} = Cone(ev)$.*

This forms part of a functor $T_V \colon \mathcal{Q} \to \mathcal{Q}$ where, given $t \in \hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{Z})$, $\tilde{t} =$

$T_V(t)$ has first order part

$$\tilde{t}^1 \begin{pmatrix} y_1 \otimes v \\ y_2 \end{pmatrix} = \begin{pmatrix} (-1)^{|v|-1} t^1(y_1) \otimes v \\ t^1(y_2) + t^2(y_1, v) \end{pmatrix}$$

and

$$\tilde{t}^d \left(\begin{pmatrix} y_1 \otimes v \\ y_2 \end{pmatrix}, \mathbf{a}_{d-1} \right) = \begin{pmatrix} 0 \\ t^d(y_2, \mathbf{a}_{d-1}) + t^{d+1}(y_1, v, \mathbf{a}_{d-1}) \end{pmatrix}.$$

If $\mathcal{A}$ is triangulated, we may define the functor T_V on $\mathcal{A}$ and Seidel proves the following lemma:

Lemma 5.3. ([39, Lemma 5.11]) *Given a spherical object V in a c-finite triangulated A_∞-category $\mathcal{A}$, the spherical twist T_V is a quasi-equivalence of $\mathcal{A}$.*

Theorem 5.4. *When V is a $\mathbb{CP}^1$-object (so is also a spherical object of dimension 2), T_V^2 and Φ_V give rise to isomorphic functors on $D\mathcal{A}$.*

Proof. $T_V(T_V\mathcal{Y}) = (\mathcal{Y}(V) \otimes \mathcal{V}(V)[2] \otimes \mathcal{V}) \oplus (\mathcal{Y}(V)[1] \otimes \mathcal{V}) \oplus (\mathcal{Y}(V)[1] \otimes \mathcal{V}) \oplus \mathcal{Y}$ with

$$\mu^1_{T_V^2\mathcal{Y}} \begin{pmatrix} y_1 \otimes q \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \otimes v_3 \\ y_4 \end{pmatrix}$$

$$= \begin{pmatrix} (-1)^{|v_1|+|q|} \mu^1(y_1) \otimes q \otimes v_1 + (-1)^{|v_1|-1} y_1 \otimes \mu^1(q) \otimes v_1 + y_1 \otimes q \otimes \mu^1(v_1) \\ (-1)^{|v_2|-1} \mu^1(y_2) \otimes v_2 + y_2 \otimes \mu^1(v_2) + (-1)^{|v_1|-1} \mu^2(y_1, q) \otimes v_1 \\ (-1)^{|v_3|-1} \mu^1(y_3) \otimes v_3 + y_3 \otimes \mu^1(v_3) + y_1 \otimes \mu^2(q, v_1) \\ \mu^1(y_4) + \mu^2(y_2, v_2) + \mu^2(y_3, v_3) + \mu^3(y_1, q, v_1) \end{pmatrix}$$

and

$$\mu^d_{T^2_V \mathcal{Y}}\left(\begin{pmatrix} y_1 \otimes q \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \otimes v_3 \\ y_4 \end{pmatrix}, \mathbf{a}_{d-1}\right)$$
$$= \begin{pmatrix} y_1 \otimes q \otimes \mu^d(v_1, \mathbf{a}_{d-1}) \\ y_2 \otimes \mu^d(v_2, \mathbf{a}_{d-1}) \\ y_3 \otimes \mu^d(v_3, \mathbf{a}_{d-1}) + y_1 \otimes \mu^{d+1}(q, v_1, \mathbf{a}_{d-1}) \\ \mu^d(y_4, \mathbf{a}_{d-1}) + \mu^{d+1}(y_2, v_2, \mathbf{a}_{d-1}) + \mu^{d+1}(y_3, v_3, \mathbf{a}_{d-1}) + \mu^{d+2}(y_1, q, v_1, \mathbf{a}_{d-1}) \end{pmatrix}$$

for $d \geq 2$.

Without loss of generality we may assume that $\mathcal{V}(V)$ is two-dimensional with basis $\{e_V, h\}$ so that we may write $\mathcal{Y}(V) \otimes \mathcal{V}(V)[2]$ as a direct sum $e[2]\mathcal{Y}(V) \oplus h\mathcal{Y}(V)$ and denote by π_h the projection onto the second summand (without any correcting sign factor).

For all $\mathcal{Y}$, we now define maps $\alpha_{\mathcal{Y}} \colon T^2_V \mathcal{Y} \to \Phi_V \mathcal{Y}$ by

$$\alpha^1_{\mathcal{Y}} \begin{pmatrix} y_1 \otimes q \otimes v_1 \\ y_2 \otimes v_2 \\ y_3 \otimes v_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} (-1)^{|v_1|}\pi_h(y_1 \otimes q) \otimes v_1 \\ (-1)^{|y_2|+|v_2|} y_2 \otimes v_2 + (-1)^{|y_3|+|v_3|} y_3 \otimes v_3 \\ (-1)^{|y_4|-1} y_4 \end{pmatrix},$$

and, given $t \in \hom_{\mathcal{Q}}(\mathcal{Y}, \mathcal{Z})$, we now have the diagram

$$\begin{array}{ccc} T^2_V \mathcal{Y} & \xrightarrow{\tilde{\tilde{t}}} & T^2_V \mathcal{Z} \\ \downarrow{\scriptstyle \alpha_{\mathcal{Y}}} & & \downarrow{\scriptstyle \alpha_{\mathcal{Z}}} \\ \Phi_V \mathcal{Y} & \xrightarrow{\hat{t}} & \Phi_V \mathcal{Z}, \end{array}$$

and the following are easily checked:

- $\mu^1_{\mathcal{Q}}(\alpha_{\mathcal{Y}}) = 0$ for all $\mathcal{Y}$;
- By a similar argument to the proof of Proposition 3.4, $\alpha_{\mathcal{Y}}$ is a quasi-isomorphism for all $\mathcal{Y}$;

- $(-1)^{|\tilde{\tilde{t}}|}\mu^2_{\mathcal{Q}}\left(\alpha_{\mathcal{Z}}, \tilde{\tilde{t}}\right) = (-1)^{|\alpha_{\mathcal{Y}}|}\mu^2_{\mathcal{Q}}\left(\hat{t}, \alpha_{\mathcal{Y}}\right)$.

This suffices to prove that there is a natural isomorphism between the two functors in $D\mathcal{A}$. □

Corollary 5.5. *In light of* (1.3)*, Conjecture 1.1 holds in the case of a* $\mathbb{CP}^1$*-object.*

5.2 Exotic autoequivalences

Suppose we have a symplectic manifold (M, ω) and a Lagrangian $V \subset M$ which satisfies the classical ring isomorphism $HF^*(V, V) \cong H^*(V)$. Then if V has the same cohomology ring as $\mathbb{CP}^n$ we can form the projective twist Φ_V of $D\mathcal{F}(M)$ even if V is not itself diffeomorphic to $\mathbb{CP}^n$. However, in this case we would not expect to find a geometric representative of Φ_V as we do not expect to find a metric on V with periodic geodesic flow. We shall prove that there are indeed situations as above where no such geometric twist exists (we call such an autoequivalence exotic). The argument in this section is very similar to that in [6, Proposition 2.17] and we refer the reader there for a more precise discussion of the technical issues underpinning the definition of the Fukaya category in this situation.

Take some manifold V such that $H^*(V) \cong k[h]/h^{n+1}$ as a ring but such that $\pi_1(V)$ is nontrivial (for example we could take the connect sum of $\mathbb{CP}^n$ and some homology sphere Σ^{2n}). Consider the disc cotangent bundle D^*V and add a Weinstein handle [43] to cap off the Legendrian S^{2n-1} bounding some cotangent fibre. The result is an exact symplectic manifold $M = D^*V \# D^*S^{2n}$, which contains Lagrangians $Y \cong S^{2n}$ and V, and results of [6] say that (for some suitable definition of the Fukaya category) $\mathcal{F}(M)$ is generated (not merely split-generated) by these two Lagrangians. Moreover, here we have the identity $HF^*(V, V) \cong H^*(V)$.

Proposition 5.6. *In this situation there is no geometric representative* ϕ_V *of* Φ_V.

We first fix the coefficient field $\mathbb{K}$ we shall use to define our Fukaya category: let $\iota\colon \widetilde{V} \to V$ denote the universal cover and fix some $\mathbb{K}$ such that $char(\mathbb{K})$ divides the index of ι (so that $char(\mathbb{K})$ is arbitrary when the index is infinite).

Now suppose that such a geometric morphism ϕ_V exists. Then there will be a Lagrangian submanifold $L = \phi_V(Y)$ which is represented by the twisted complex

$$V \xrightarrow{h} V[1] \xrightarrow{x} Y, \tag{5.1}$$

where the arrows denote the terms in the differential as in (4.4) (if necessary we pass to a quasi-equivalent, strictly unital $\widetilde{\mathcal{F}}(M)$ so that we may work with twisted complexes as in Chapter 4). Here we observe that $HF^*(V,Y) = \mathbb{K}$ generated by their one point of intersection x. The objects of $\mathcal{F}(M)$ are all closed Lagrangians, but $\mathcal{F}(M)$ embeds as a full category of some *wrapped Fukaya category* $\mathcal{W}(M)$, which includes nonclosed Lagrangians such as cotangent fibres. Let $\pi\colon \widetilde{M} \to M$ be the cover induced by $\iota\colon \widetilde{V} \to V$. Results of [3, Section 6] now say that there exists a pullback Fukaya category $\mathcal{W}(\widetilde{M};\pi)$ with the following properties:

Theorem 5.7. *There is a wrapped Fukaya category $\mathcal{W}(\widetilde{M};\pi)$ which comes with a pullback functor*

$$\pi^*\colon \mathcal{W}(M) \to \mathcal{W}(\widetilde{M};\pi)$$

which acts on objects L of $\mathcal{W}(M)$ by taking the total inverse image $\pi^{-1}(L) \subset \widetilde{M}$ and such that the map on morphisms

$$HF^*(L,L) \to HF^*(\pi^{-1}(L), \pi^{-1}(L))$$

agrees with the classical pullback on cohomology whenever $L \subset M$ is closed. Moreover, deck transformations of π act by autoequivalences of $\mathcal{W}(\widetilde{M};\pi)$.

So when we pullback the twisted complex (5.1) under π, we get a new twisted complex in $\mathcal{W}(\widetilde{M};\pi)$:

$$\widetilde{V} \xrightarrow{0} \widetilde{V}[1] \longrightarrow \pi^{-1}(Y),$$

where the first differential is zero by our choice of $\mathbb{K}$. This means that, up to shifts, we get the splitting

$$\pi^{-1}(L) \cong \widetilde{V} \oplus \left(\widetilde{V}[1] \to \pi^{-1}(Y)\right). \tag{5.2}$$

Also, $\pi^{-1}(L) = \coprod_\alpha \widetilde{L}_\alpha$ where all the components are related in $\mathcal{W}(\widetilde{M};\pi)$ by deck transformations of π. By looking at the rank of $HW^0(\widetilde{V}, \widetilde{V}) = HF^0(\widetilde{V}, \widetilde{V}) = \mathbb{K}$ we see that $\widetilde{V}$ is an indecomposable object of the category, as is each $\widetilde{L}_\alpha$.

We now work in $D^\pi \mathcal{W}(\widetilde{M}; \pi)$, the idempotent completion of $D\mathcal{W}(\widetilde{M}; \pi)$ [39, Chapter 4], where we can appeal to the following lemma.

Lemma 5.8. *If $X = \oplus X_i$ is a direct sum of indecomposable objects in $D^\pi \mathcal{W}(\widetilde{M}; \pi)$ and Y is a indecomposable summand of X, then Y must be isomorphic to one of the X_i.*

Proof. By considering inclusion and projection morphisms, we see that the composition $X \to Y \to X$ is idempotent. This splits as a direct sum of idempotents $X_i \to Y \to X_i$. When one of these is nonzero it means that, either the composition is the identity or that, having taken idempotent completion, X_i admits a nontrivial summand. In the first instance, $Y \to X_i \to Y$ is then idempotent, so again the composition is either the identity or Y admits a nontrivial decomposition. As X_i and Y are assumed indecomposable, we conclude that X_i and Y must be isomorphic. □

Therefore, in order to show that the twisted complex in the right-hand side of (5.2) cannot arise as the pullback of a geometric Lagrangian and that therefore ϕ_V cannot exist, it suffices to prove

Lemma 5.9. *$\widetilde{V}[1] \to \pi^{-1}(Y)$ is not quasi-isomorphic in $\mathcal{W}(\widetilde{M}; \pi)$ to a direct sum of objects obtained from $\widetilde{V}$ by deck transformations.*

Proof. Pick a cotangent fibre to one of the components of $\pi^{-1}(Y)$ and consider its Floer cohomology with these two twisted complexes. In the case of $\widetilde{V}$ the rank will be zero; in the case of $\widetilde{V}[1] \to \pi^{-1}(Y)$ the rank will be 1. □

Remark 5.10. *This argument requires that we may freely choose our coeffiecient field for $\mathcal{F}(M)$. To do this one usually restricts attention to* spin *Lagrangians so that we can orient the moduli spaces of holomorphic curves used to define our A_∞-maps. However, following [18], it is enough that our Lagrangians be* relatively spin*, meaning that there is some class $st \in H^*(M, \mathbb{Z}/2)$ such that $st|_L = w_2(L)$, which clearly holds here. Therefore the above argument will still work in the case where n is even.*

Part II

Distinguishing between exotic symplectic structures

Chapter 6

Lefschetz fibrations

In this chapter, we recall the standard notions of Picard-Lefschetz theory. The treatment here largely follows that of [39, Part III], but we shall adapt the presentation there to include certain nonexact symplectic manifolds, as we want to consider arguments involving nonexact deformations of our symplectic form.

Let (M, ω) be a noncompact symplectic manifold. We say (M, ω) is *convex at infinity* if there exists a contact manifold (Y, α) which splits M into two parts: a relatively compact set M^{in}; and M^{out}, which is diffeomorphic to the positive symplectization of (Y, α) where, in a neighbourhood of Y, we have a 1-form θ satisfying $d\theta = \omega$ and $\theta|_Y = \alpha$. Such a contact manifold is canonically identified up to contactomorphism. If θ can be defined on the whole of M, we call (M, θ) a *Liouville manifold.*

Given a compact symplectic manifold with boundary M such that, in a neighbourhood of the boundary, we have a primitive θ of the symplectic form which makes the boundary contact, we say M has *convex boundary.* If θ is defined everywhere, (M, θ) is usually called a *Liouville domain.* Given a symplectic manifold with convex boundary, we can complete M canonically to get a symplectic manifold convex at infinity,

$$\widehat{M} = M \cup_{\partial M} [0, \infty) \times \partial M,$$

with forms $\widehat{\theta} = e^r \theta$ and $\widehat{\omega} = d\widehat{\theta}$ on the collar, where r denotes the coordinate on $[0, \infty)$.

6.1 Definition

Let (E, ω) be a compact symplectic manifold with corners such that, near the boundary, $\omega = d\theta$ for some form θ which makes the codimension 1 strata contact, and let $\pi \colon E \to S$ be a proper map to a compact Riemann surface with boundary such that the following conditions hold:

- There exists a finite set $E^{crit} \subset E$ such that $D\pi_x$ is a submersion for all $x \notin E^{crit}$, and such that $D^2\pi_x$ is nondegenerate for all $x \in E^{crit}$, which means that locally we can find charts such that $\pi(z) = \sum z_i^2$. We denote by S^{crit} the image of E^{crit} and require that $S^{crit} \subset S \setminus \partial S$. We also assume, for sake of notational convenience, that there is at most 1 element of E^{crit} in each fibre.

- For all $z \notin S^{crit}$ the fibre $E_z = \pi^{-1}(z)$ becomes a symplectic manifold with convex boundary with respect to $\omega|_{E_z}$. This means that we get a splitting of tangent spaces
$$TE_x = TE_x^h \oplus TE_x^v,$$
where the vertical part TE_x^v is the kernel $\ker(D\pi_x)$ and the horizontal part TE_x^h is the orthogonal complement of TE_x^v with respect to ω.

- At every point $x \in E$ such that $z = \pi(x) \in \partial S$, we have $TS = T(\partial S) + D\pi(TE_x)$. This implies that $\pi^{-1}(\partial S)$ is a boundary stratum of E of co-dimension 1, which we shall call the *vertical boundary*, denoted $\partial^v E$. The union of boundary faces of E not contained in $\partial^v E$ we shall call the *horizontal boundary* of E, denoted $\partial^h E$.

- If F is a boundary face of E not contained in $\partial^v E$, then $\pi|_F \colon F \to S$ is a smooth fibration, which implies that any fibre is smooth near its boundary. We also want the horizontal boundary $\partial^h E$ to be horizontal with respect to the above splitting, so that parallel transport (see below) will be well-defined along the boundary.

Definition 6.1. *If all the above holds we call (E, π, ω) a compact convex Lefschetz fibration. For ease of notation, in what follows we shall often call (E, π, ω) simply a Lefschetz fibration, suppressing the extra adjectives.*

The splitting of tangent spaces into horizontal and vertical subspaces means that we have a connection over $S \setminus S^{crit}$, and hence *symplectic parallel transport* maps. In other words, for a path $\gamma\colon [0,1] \to S$ which misses S^{crit}, our connection defines a symplectomorphism $\phi_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$.

There is a method [31] of completing E to a symplectic manifold $\widehat{E}$ which is convex at infinity, such that we get a map $\widehat{\pi}\colon \widehat{E} \to \widehat{S}$ to the completion of the base. When S is a disc $\mathbb{D}$, this is done as follows: firstly, the horizontal boundary $\partial^h E$ is just $\partial M \times \mathbb{D}$, where M is a smooth fibre, and we can attach $\partial M \times [0,\infty) \times \mathbb{D}$ to $\partial^h E$ in the same as we complete a symplectic manifold with convex boundary. This gives us a new manifold we shall call E_1 and we can extend π to π_1 on E_1 in the obvious way. Now consider $\pi_1^{-1}(\partial\mathbb{D}) = N$. Attach to this $N \times [0,\infty)$ and call the resulting manifold $\widehat{E}$, over which we can extend π_1 to $\widehat{\pi}$. More details can be found in [31, Section 2]. This map $\widehat{\pi}$ restricts to π on the subsets corresponding to E and S and outside we have a local model looking like the completion of the mapping cone for some symplectic map μ which we shall call the *outer monodromy* of the Lefschetz fibration E. Given this, we shall also talk in this thesis about Lefschetz fibrations over $\mathbb{C}$, which are understood to be the completions of Lefschetz fibrations over some disc $\mathbb{D}_R \subset \mathbb{C}$, in the sense of Definition 6.1.

6.2 Vanishing cycles

We can use the parallel transport maps to introduce the notion of a *vanishing cycle*. Choose an embedded path $\gamma\colon [0,1] \to S$ such that $\gamma^{-1}(S^{crit}) = \{1\}$. We can consider the set of points which tend to the critical point $y = \gamma(1)$ under our parallel transport maps

$$V_\gamma = \left\{ x \in E_{\gamma(0)} : \lim_{t \to 1} \phi_{\gamma|_{[0,t]}}(x) = y \right\}.$$

This set V_γ is called the *vanishing cycle* associated to the *vanishing path* γ. The vanishing cycle is actually a Lagrangian sphere in the fibre [38] and if we take the *Lefschetz thimble*, the union of the images of the vanishing cycle as we move along γ together with the critical point, we get a Lagrangian ball Δ_γ in the total space E. In fact, Δ_γ is the unique embedded Lagrangian ball that lies over γ. These vanishing cycles come together with the extra datum of a "framing" [38, Lemma 1.14], meaning a parametrization $f\colon S^n \to V$. Here, two framings f_1, f_2 are equivalent if $f_2^{-1} f_1$ can be deformed inside $Diff(S^n)$ to an element of $O(n+1)$, but this framing information is irrelevant in the dimensions in which we work, so shall neglect to mention framings in what follows.

6.3 Constructing Lefschetz fibrations

Given a Lefschetz fibration (E, π), we can pick a smooth reference fibre E_z and a collection of vanishing paths γ_i, one for each critical point, which all finish at z, but which are otherwise disjoint. This then gives us a symplectic manifold $M = E_z$ and a collection of vanishing cycles $V_i \subset M$ associated to the γ_i. For our purposes, in constructing symplectic manifolds, it is important to note that we can go the other way as in the following lemma, taken from [39, Lemma 16.9] but with unnecessary assumptions of exactness removed.

Lemma 6.2. *Suppose we have a collection $(V_1, \ldots, V_m)$ of (framed) Lagrangian spheres in a symplectic manifold M with convex boundary. On the disc $\mathbb{D}$, choose a base point z, and a distinguished basis of vanishing paths $\gamma_1, \ldots, \gamma_m$ all of which have one endpoint at z. Then there is a compact convex Lefschetz fibration $\pi\colon E \to \mathbb{D}$, whose critical values are precisely the endpoints $\gamma_1(1), \ldots, \gamma_m(1)$; this comes with an identification $E_z = M$, under which the (framed) vanishing cycles V_{γ_k} correspond to V_k.*

This will be the technique used to construct the symplectic manifolds considered in this thesis. However, in order to do this, we need to identify a collection of Lagrangian spheres in a given symplectic manifold M. In the case where M itself admits a Lefschetz fibration, we shall do this by considering matching cycles.

6.4 Matching cycles

Consider a Lefschetz fibration $\pi\colon M \to S$ and an embedded path $\gamma\colon [0,1] \to S$ such that $\gamma^{-1}(S^{crit}) = \{0, 1\}$. In the fibre $\pi^{-1}(\gamma(\frac{1}{2}))$, we get two vanishing cycles, one coming from either endpoint. If they agree, then parallel transport allows us to glue the two thimbles together to obtain a smooth Lagrangian sphere $V \subset M$. We shall call γ a *matching path*, and V the associated *matching cycle*.

In this thesis we shall usually work in situations where the vanishing cycles do agree exactly so that we do get matching cycles, but occasionally we will have the situation where the two vanishing cycles are not equal, but are merely Hamiltonian isotopic. In this situation we may appeal to the following result of [8, Lemma 8.4]:

Lemma 6.3. *Let (M, ω) be a symplectic manifold with a Lefschetz fibration $\pi\colon M \to \mathbb{C}$ and let $\gamma\colon [0,1] \to \mathbb{C}$ be a path such that $\gamma^{-1}(S^{crit}) = \{0,1\}$. Suppose that the two vanishing cycles $V_0, V_1 \subset M_{\gamma(\frac{1}{2})}$ coming from either end of this path are Hamiltonian isotopic for some compactly supported Hamiltonian H_s defined on the fibre $M_{\gamma(\frac{1}{2})}$. Then M contains a Lagrangian sphere homotopic to $\Delta_0 \cup \Delta_1 \cup \mathrm{im}(H_s(V_0))$.*

Matching cycles will be used for our main method of construction. We take a symplectic manifold (M, ω) equipped with a Lefschetz fibration and consider an ordered collection of matching paths. In favourable circumstances these will give rise to a family of framed Lagrangian spheres $(V_1, \dots, V_n) \subset M$ and we now apply Lemma 6.2 to construct a new Lefschetz fibration (E, π).

6.5 Maydanskiy's examples

Figure 6.1 shows the examples considered in [27]. Although higher-dimensional examples are also considered in [27], the meaning of all such diagrams in this thesis is that we take the symplectic manifold M^4 built according to Lemma 6.2 by taking fibre T^*S^1 and vanishing cycles given by the zero-section, one for each cross. The lines in Figure 6.1 are then matching paths which yield the spheres required to apply Lemma 6.2 again to obtain E^6. The fact that the paths in Figure 6.1 actually do give matching cycles will for us be a consequence of the method of construction considered in the next chapter.

Maydanskiy [27] proves that the two symplectic manifolds in Figure 6.1 are diffeomorphic (they are both $T^*S^3 \cup$ 2-handle) but are not symplectomorphic.

X_1' -contains a Lagrangian S^3 $\quad$ X_1 -contains no Lagrangian S^3

Figure 6.1:

X_1' is just T^*S^3 with a Weinstein 2-handle attached as in [43] and contains an exact Lagrangian sphere inherited from the zero-section of T^*S^3. In contrast, X_1 contains no exact Lagrangian submanifolds, and so is considered exotic.

One way of thinking about this intuitively is that the manifolds are diffeomorphic because one can construct a smooth isotopy taking the top matching cycle in X_1 and moving it over the critical point in the middle to get X_1'. The reason this fails to work symplectically is that we are free to move our cycles only by Hamiltonian isotopies, and we will not then be able to avoid the central critical point (since we cannot displace the zero-section of T^*S^1), although the actual proof in [27] has to make use of more sophisticated Floer-theoretic arguments.

Chapter 7

Deformations of symplectic structures

Definition 7.1. *Let* (E, ω) *be a symplectic manifold. By a deformation of the symplectic structure* (E, ω) *we shall mean a smooth 2-form* Ω *on* $\tilde{E} = E \times [0, 1]$ *such that*

- $\Omega|_t$ *is symplectic on each* $E \times \{t\}$
- $\Omega|_0 = \omega$
- $\iota_v \Omega = 0$ *for any* $v \in \ker(D\rho)$ *where* ρ *is the projection* $\tilde{E} \to E$.

This is equivalent to a smooth 1-parameter family of symplectic forms $\{\omega_t : t \in [0, 1]\}$ *on* E *such that* $\omega_0 = \omega$. *We shall denote by* $(\tilde{E}^t, \omega_t)$ *the symplectic manifold* $(E \times \{t\}, \Omega|_t)$.

We shall consider X_1, the exotic example of Maydanskiy from the previous chapter. In this chapter, we shall prove

Theorem 7.2. *There is a deformation* $\tilde{X}_1$ *of* X_1 *such that, for all* $t > 0$, $\tilde{X}_1^t$ *contains a Lagrangian sphere.*

7.1 Constructing a deformation of X_1

The fibres of Maydanskiy's examples are A_2 Milnor fibres. For our purposes, which crucially rely on matching paths defining genuine matching cycles without having to rely on Lemma 6.3, we shall work with the specific model as below.

Let M be the affine variety defined by

$$M = \left\{ z_1^2 + z_2^2 = (z_3 - 1)(z_3 - 2)(z_3 - 3) \right\} \subset \mathbb{C}^3$$

equipped with symplectic form ω, which is the restriction of the standard symplectic form on $\mathbb{C}^3$. We may restrict to some compact set $M^{in} \subset M$ ($M^{in} \subset B_R \subset \mathbb{C}$ for some sufficiently large R), such that M^{in} is a Liouville domain which becomes a Lefschetz fibration in the sense of Definition 6.1 once we project onto the z_3-coordinate [39, Section 19b]. It has three critical values, at 1, 2 and 3.

There is a homologically essential Lagrangian sphere A living over the straight-line path joining the two critical points at 1 and 2, which is given by the part of the real locus $M_{\mathbb{R}}$ living over this path. This sphere is precisely the matching cycle associated to that line. We can do the same with the part of $M \cap \mathbb{R}\langle x_3, y_1, y_2 \rangle$ living over the interval $[2, 3]$ to find another Lagrangian sphere B and we shall take A and B to define our standard basis of $H_2(M; \mathbb{R}) = \mathbb{R}^2$.

The manifold M carries an S^1-action given by

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \mapsto \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and the symplectic form ω is invariant under this action.

Every smooth fibre is of the form $z_1^2 + z_2^2 = \lambda$ for some nonzero $\lambda = se^{i\alpha}$ and we observe that such a fibre is preserved by the S^1-action, which in particular means that the parallel transport map associated to a path γ is S^1-equivariant. This fibre is symplectomorphic to T^*S^1, where the model we use for T^*S^1 is

$$T^*S^1 = \left\{ (q, p) \in \mathbb{R}^2 \times \mathbb{R}^2 : \|q\| = 1 \ , \ \langle q, p \rangle = 0 \right\}.$$

The symplectomorphism is defined as follows: let $\hat{z} = ze^{-i\alpha/2}$ and map

$$z \mapsto \left(\frac{\Re(\hat{z})}{\|\Re(\hat{z})\|}, -\Im(\hat{z})\|\Re(\hat{z})\| \right).$$

Note that, for each fibre, the S^1-orbits are mapped to level sets $\|p\| = constant$ so, given that the parallel transport maps are S^1-equivariant, the vanishing cycle associated to any vanishing path will itself correspond to such a level set.

We shall deform the symplectic structure by introducing 2-forms which are intended to resemble area forms supported near the equators of A and B. We therefore consider the 2-form on $\mathbb{C}^3 \setminus i\mathbb{R}^3$,

$$\eta = g_\epsilon\left(\frac{x}{\|x\|}\right)\left(\frac{x_1}{\|x\|^3}dx_2 \wedge dx_3 + \frac{x_2}{\|x\|^3}dx_3 \wedge dx_1 + \frac{x_3}{\|x\|^3}dx_1 \wedge dx_2\right)$$

where $g_\epsilon(x) = g_\epsilon(x_3)$ denotes a cutoff function for the x_3-coordinate which has $\mathrm{supp}(g_\epsilon) \subset \{|x_3| < \epsilon\}$.

As η is defined using only coordinates on the real slice $\mathbb{R}^3 \setminus \{0\}$ and annihilates the radial direction, this is a closed form on $\mathbb{C}^3 \setminus i\mathbb{R}^3$. We shall choose ϵ such that $\epsilon < \frac{1}{8R}$, and apply a translation $x \mapsto x + (0, 0, 3/2)$. It is easy to show that η is now well-defined on M, so that in the Lefschetz fibration $M^{in} \to \mathbb{D}_R$, η is a closed, S^1-equivariant 2-form supported in the region lying over $\{|x_3 - 3/2| < 1/4\}$ and the sphere A has some nonzero area with respect to η.

Moreover, we can rescale η so that $\omega + \eta$ is still symplectic on M^{in}, since the property of being symplectic is an open condition and M^{in} is compact. Also, since M is an A_2 Milnor fibre, its boundary ∂M is topologically the quotient of S^3 by a $\mathbb{Z}/3$ action and therefore $H^2(\partial M; \mathbb{R}) = 0$. This means that, perhaps after rescaling η again, M^{in} will still have contact boundary.

We repeat the above procedure to obtain another closed 2-form η' on M^{in}, defined now using the coordinates y_1, y_2, x_3 which is again S^1-equivariant and is supported over $\{|x_3 - 5/2| < 1/4\}$ and has the property that

$$\eta(A) = -\eta'(B).$$

We denote by ω_t the 2-forms $\omega + t(\eta + \eta')$ for $t \in [0, 1]$, all of which make M^{in} symplectic with convex boundary.

Remark 7.3. *Such a construction can be generalized: choose a finite collection* *of distinct points* $p_1, \ldots, p_{n+1} \in \mathbb{R}$ *and consider the affine variety*

$$M_{\mathbf{p}} = \left\{ z_1^2 + z_2^2 = \prod_i (z_3 - p_i) \right\} \subset \mathbb{C}^3,$$

which will be diffeomorphic to the A_n *Milnor fibre, with a basis of* $H_2(M_{\mathbf{p}})$ *given by the spheres* A_i *living over the straightline path joining* p_i *and* p_{i+1}*. We may construct a deformation of the symplectic structure on* $M_{\mathbf{p}}$ *by adding on 2-forms which are supported on strips lying between the critical points as above.*

7.2 Obstructions to forming matching cycles are purely homological

We now consider the path γ_0 in Figure 7.1, going from 1 to 3 in $\mathbb{C}$. We would like this to define a genuine matching cycle, with respect to the parallel transport maps coming from $\omega_t = \omega + t(\eta + \eta')$ for $t \in [0, 1]$. However, we may no longer get a genuine Lefschetz fibration in the sense of Chapter 6, since the horizontal boundary may no longer be horizontal with respect to our splitting. This means that parallel transport cannot be done near $\partial^h M$, but we shall not need this: our vanishing cycles stay within a region away from the boundary, since deforming the symplectic form will only change the parallel transport maps by a small amount.

Therefore, for any given t, the path γ_0 gives us two circles in the central fibre which we know correspond to level sets $\|p\| = \textit{constant}$. (In Figure 7.1, the fibres shown at the top are those living over the path γ_0.) These two circles enclose some chain S_t in the fibre over $\gamma_0(\frac{1}{2})$, and the sum of this chain and the two thimbles is homologous to $[A] + [B]$, so therefore has symplectic area 0 with respect to ω_t. Since the vanishing thimbles are Lagrangian, this means that the chain $S_t \subset T^*S^1$ must also have zero symplectic area, and therefore S_t must in fact be empty. In other words, we get a genuine matching cycle for all t, which we denote V_0^t. We can do likewise for the path γ_1 to obtain the matching cycle V_1^t.

By the same argument, for any $t > 0$ we can take a straightline path given by the interval $[1, 3]$, which goes over the central critical point at 2, and say that this

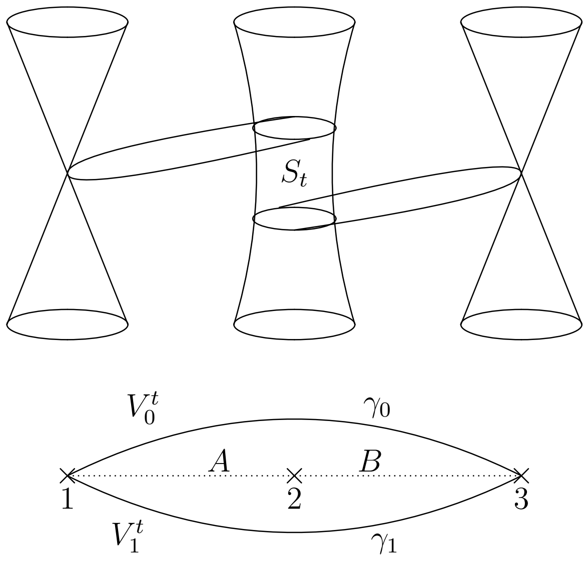


Figure 7.1:

too will define a matching cycle: in the central nonsmooth fibre we shall either get, by S^1-symmetry, the critical point or some circle. However, if we obtained the critical point, then we would have found a Lagrangian in a homology class of positive symplectic area. Which smooth component this circle lives in depends on whether we choose to give the class A positive or negative area.

Therefore, for $t > 0$, we can take a smooth family of paths interpolating between the two matching paths and get a smooth family $(V_s^t)_{s\in[0,1]}$ of Lagrangian S^2s joining the two matching cycles. This has the following standard consequence.

Lemma 7.4. *For $t > 0$, V_0^t and V_1^t are Hamiltonian isotopic.*

Proof. We can identify some neighbourhood of V_0^t with T^*S^2 and, for $0 \leq s \leq s_0$ for some small s_0, V_s^t will correspond to the graph of some 1-form α_s. Since V_s^t is Lagrangian, $d\alpha_s = 0$, and therefore $\alpha_s = df_s$ since $H^2(V_s^t;\mathbb{R}) = 0$. We can moreover choose these f_s smoothly. A direct calculation shows that $H(x,s) = \frac{d}{ds}(f_s(\rho(x)))$ is a Hamiltonian yielding an isotopy between V_0^t and $V_{s_0}^t$, where here $\rho\colon T^*S^2 \to S^2$ is the standard projection map. We can patch together such isotopies to get from V_0^t to V_1^t, and then apply some cutoff function to make our Hamiltonian to be compactly supported. $\square$

7.3 X_1 contains a Lagrangian sphere after deformation

We are now in a position to prove Theorem 7.2. To do this, we shall establish a deformation version of Lemma 6.2. This is stated below in the case where there is just one vanishing cycle, since the general case follows from gluing together such examples.

Suppose we have $\tilde{M}$, a deformation of the symplectic structure (M, ω), such that $\tilde{M}^t$ has convex boundary for all t, and suppose that we also have $\tilde{V} \subset \tilde{M}$, which is the image of an embedding of $S^n \times [0,1]$ such that, for all t, we get a Lagrangian sphere $\tilde{V}^t \subset (\tilde{M}^t, \omega_t)$.

Then, by Lemma 6.2, we can construct a Lefschetz fibration $E^t \to \mathbb{D}$ from $\tilde{M}^t$ and $\tilde{V}^t$ for each t. We want the family E^t to comprise a deformation of (E^0, ω_E).

Proposition 7.5. *In the above situation, we can construct a bundle of symplectic manifolds $\tilde{E} \to [0,1]$, such that each fibre $\tilde{E}^t$ has convex boundary and comes with an identification $\tilde{E}^t_z \cong \tilde{M}^t$, under which the vanishing cycle V_γ corresponds to $\tilde{V}^t$. After applying a trivialization of this bundle which is the identity over 0, this is a deformation of (E^0, ω_E).*

Proof. We closely follow [38, Proposition 1.11]. First we need a neighbourhood theorem, whose proof follows the same argument as that of the standard Lagrangian neighbourhod theorem [29].

Lemma 7.6. *Let $(\tilde{M}, \Omega)$ be a deformation of (M, ω). Suppose we have $\tilde{V} \subset \tilde{M}$ an embedding of $V \times [0,1]$ such that, for all t, we get a Lagrangian $\tilde{V}^t \subset (\tilde{M}^t, \omega_t)$. Then there exists a neighbourhood $\mathcal{N} \subset T^*V \times [0,1]$ of the zero-section $V \times [0,1]$ and a neighbourhood $\mathcal{U} \subset \tilde{M}$ of $\tilde{V}$ and a diffeomorphism $\phi\colon \mathcal{N} \to \mathcal{U}$ such that $\phi^*\Omega = \beta$ where β is the 2-form on $T^*V \times [0,1]$ given by the pullback of the standard symplectic form on T^*V.*

In our case, we may assume our neighbourhood $\mathcal{N}$ in Lemma 7.6 is of the form $\mathcal{N} = T^*_{\leq\lambda} S^n \times [0,1]$ for some $\lambda > 0$, where $T^*_{\leq\lambda} S^n$ denotes the disc cotangent bundle with respect to the standard metric on T^*S^n. Given this, we follow [38, Proposition 1.11] which starts by considering the local Lefschetz model $q\colon \mathbb{C}^{n+1} \to \mathbb{C}$, $q(z) = \sum z_i^2$. We also consider the function $h(z) = \|z\|^4 - |q(z)|^2$.

When we restrict to $W \subset \mathbb{C}^{n+1}$ cut out by the inequalities $h(x) \leq 4\lambda^2$ and $|q(z)| \leq 1$, we get a compact Lefschetz fibration $\pi_W\colon W \to \mathbb{D}$. As explained in [38], W comes together with an identification $\psi\colon \pi_W^{-1}(1) \to T^*_{\leq\lambda}S^n$, a neighbourhood $Y \subset W$ of $\partial_h W$, a neighbourhood Z of $\partial(T^*_{\leq\lambda}S^n)$ in $T^*_{\leq\lambda}S^n$ and a diffeomorphism $\Psi\colon Y \to \mathbb{D} \times Z$ which fibres over $\mathbb{D}$ and agrees with ψ on $Y \cap \pi_W^{-1}(1)$. Let $\tilde{W} = W \times [0,1]$ and, by taking the product with $[0,1]$, consider the corresponding $\tilde{Y}, \tilde{Z}, \tilde{\psi}, \tilde{\Psi}$.

Now define $\tilde{M}_-$ to be $\tilde{M} \setminus (\phi(\mathcal{N} \setminus \tilde{Z}))$ and consider

$$\tilde{E} = \mathbb{D} \times \tilde{M}_- \cup_\sim \tilde{W},$$

where the identification made identifies $\tilde{Y}$ with $\mathbb{D} \times \phi(\tilde{Z})$ through $(id \times \phi) \circ \tilde{\Psi}$. This now has all the required properties. □

Proof of Theorem 7.2. Using Proposition 7.5, we can construct a deformation $\tilde{X}_1$ of Maydanskiy's exotic example X_1 and we want to say that we have a Lagrangian sphere $L^t \subset \tilde{X}_1^t$ for all $t > 0$. $\tilde{X}_1^t$ admits a Lefschetz fibration with two critical points. We take a path joining the two critical points in the Lefschetz fibration on X_1. If we choose the vanishing paths γ in Proposition 7.5 such that they join together smoothly, then the concatenation of these paths is smooth and yields two vanishing cycles in the central fibre, which are precisely just V_0^t and V_1^t from Lemma 7.4, which we know are Hamiltonian isotopic for all $t > 0$. We then just apply Lemma 6.3 to find a Lagrangian sphere. □

Remark 7.7. *As $t \to 0$, the Lagrangian spheres L^t degenerate to some singular Lagrangian cycle, which is worse than immersed. In fact, topologically it looks like S^3 with some S^1 in it collapsed to a point. Presumably, pseudoholomorphic curve theory with respect to this cycle is very badly behaved, so that a Floer theory along the lines of [7] cannot be made to work here, although see [24] for some analysis of holomorphic discs on certain similar special Lagrangian cones.*

Chapter 8

Floer cohomology

To consider X_2 and adapt the arguments presented in [28], we shall need to consider the Lagrangian Floer cohomology $HF(L_0, L_1)$ of two transversely intersecting Lagrangian submanifolds in some symplectic manifold (M, ω). To define this, one has to pick a generic family of almost complex structures $\mathbf{J} = (J_t)$, which are usually required to be compatible with ω, in the sense that $g_t(u, v) = \omega(u, J_t v)$ defines a Riemannian metric. However, we shall want to consider J_t which are ω-tame except on a small neighbourhood of $L_0 \cap L_1$, where here J_t is still ω-compatible. (ω-tame means that $\omega(u, J_t u) > 0$ for all nonzero u.) We shall show that, given any such family of almost complex structures $\mathbf{J} = (J_t)$, there exists $\tilde{\mathbf{J}} = (\tilde{J}_t)$ arbitrarily close to it, with the same properties, such that $HF(L_0, L_1)$ can be defined with respect to $(\tilde{J}_t)$. The key point is that we are using Cauchy-Riemann type operators with totally real boundary conditions, so all the relevant elliptic regularity theory can still be applied.

Remark 8.1. *The content of this chapter, that we can relax the condition on the almost complex structures to define Floer cohomology is probably already known to experts, but we are unaware of any written account of this in the literature.*

8.1 Setup

Let (M, ω) be a symplectic manifold of dimension $2n$ and let L_0, L_1 be two Lagrangian submanifolds which intersect transversely. For each intersection x, fix some small open set U_x around x such that $L_0 \cap L_1 \cap U_x = \{x\}$. Assume moreover that the U_x are disjoint. Pick some family $\mathbf{J} = (J_t)$ of smooth almost complex structures which tame ω (this in particular implies that the L_k are totally real), and which are ω-compatible on each U_x.

We note here for future reference the following lemma due to Frauenfelder [17].

Lemma 8.2. *Let (M^{2n}, J) be an almost complex manifold and $L^n \subset M$ a totally real submanifold. Then there exists a Riemannian metric g on M such that*

- *$g(J(p)v, J(p)w) = g(v, w)$ for $p \in M$ and $v, w \in T_pM$,*
- *$J(p)T_pL$ is the orthogonal complement of T_pL for every $p \in L$,*
- *L is totally geodesic with respect to g.*

Let Σ denote the holomorphic strip $\mathbb{R} \times [0, 1] \subset \mathbb{C}$. Given a map $u\colon \Sigma \to M$, we can consider the $\bar{\partial}_{\mathbf{J}}$ operator defined by

$$\bar{\partial}_{\mathbf{J}} u(s, t) = \partial_s u(s, t) + J_t(s, t)\partial_t u(s, t).$$

We care about holomorphic maps, which are just those such that $\bar{\partial}_{\mathbf{J}} u = 0$ and we define the energy of any map u to be $E(u) = \int \|\partial_s u\|^2$.

Let $\mathcal{M}_{\mathbf{J}}$ denote the set of holomorphic u as above which also satisfy the boundary conditions $u(s, 0) \in L_0$, $u(s, 1) \in L_1$ as well as $E(u) < \infty$. It is proved in [36] that any such map must have the property that

$$\lim_{s \to \pm\infty} u(s, t) = x^{\pm},$$

where $x^{\pm}$ are intersection points in $L_0 \cap L_1$. Moreover, the convergence near the ends is exponential in a suitable sense about which we shall say more later. We define $\mathcal{M}_{\mathbf{J}}(x, y)$ to be the space of finite-energy trajectories as above which converge to x and y at the ends.

We want to examine the properties of $\mathcal{M}_{\mathbf{J}}(x, y)$ and, in particular, determine when it is a smooth manifold, so we follow the standard procedure of Floer [16], in exhibiting $\mathcal{M}_{\mathbf{J}}(x, y)$ as the zero set of some Fredholm section of a Banach bundle.

Much of what follows is already contained in Floer's original work [16], but we shall recall the main details for the reader's convenience.

8.2 Banach manifolds

Let $kp > 2$. We can consider the Sobolev space $L^p_{k;loc}(\Sigma, M)$ and define

$$\mathcal{P}^p_k = \left\{u \in L^p_{k;loc}(\Sigma, M) : u(s,0) \in L_0,\ u(s,1) \in L_1\right\}.$$

Let $\Sigma_\rho = \{z \in \Sigma : |\Re z| < \rho\}$. The topology on $\mathcal{P}^p_k$ is defined using the basis of open sets given by

$$\mathcal{O}_{u,\rho,\epsilon} = \{v \in \mathcal{P}^p_k : v = \exp_u \xi \text{ on } \Sigma_\rho \text{ and } \|\xi\|_{k,p} < \epsilon \text{ for } p < \rho\}.$$

Here $u \in \mathcal{P}^p_k$ and $\rho, \epsilon > 0$.

For our present purposes, and in order to ensure that we do in fact get a Banach manifold, we shall need to restrict to a subset of $\mathcal{P}^p_k$ with nice behaviour near intersection points $x \in L_0 \pitchfork L_1$. Consider

$$\mathcal{P}^p_k(\cdot, x) = \left\{u \in \mathcal{P}^p_k : \exists\rho > 0, \exists\xi \in L^p_{k;loc}(\Sigma, T_xM), u(s,t) = \exp_x \xi(s,t) \forall s > \rho\right\}.$$

In other words, we restrict attention to maps which, at one end, look like the exponentiation of some vector field. We impose a similar condition at the other end to define $\mathcal{P}^p_k(x, \cdot)$, and then consider $\mathcal{P}^p_k(x, y)$.

For $u \in \mathcal{P}^p_k$, u^*TM is an $L^p_{k;loc}$-bundle, so we can talk about sections which are locally of $L^p_{k;loc}$-type. We shall introduce the shorthand $L^p_k(u) = L^p_{k;loc}(u^*TM)$ and we may also consider

$$W^p_k(u) = \left\{\xi \in L^p_k(u) : \xi(s,0) \in T_{u(s,0)}L_0, \xi(s,1) \in T_{u(s,1)}L_1\right\},$$

so here we have tangent pointing along the Lagrangian boundary.

We can also consider spaces of sections $W^q_l(u)$ and $L^q_l(u)$ of different regularity provided that $l \leq k$ and

$$l - \frac{2}{q} \leq k - \frac{2}{p}. \tag{8.1}$$

Theorem 8.3. ([16, Theorem 3]) *Let $p \geq 1$ and $kp > 2$. Then $\mathcal{P}^p_k(x, y)$ is a smooth Banach manifold and its tangent space at u is given by $T_u\mathcal{P}^p_k(x, y) = W^p_k(u)$.*

To show this is a Banach manifold, Floer uses a system of charts based on the exponential map. Accordingly, pick a family of metrics (g_t) such that L_k is totally geodesic with respect to g_k, as in Lemma 8.2.

Define

$$\exp\colon \Sigma \times TM \to M,$$
$$\exp(s, t, x, v) = \exp_{g_t}(x, v).$$

Let ι denote the minimal injectivity radius of the metrics g_t and define

$$U_u = \{\xi \in W^p_k(u) : \|\xi\|_\infty < \iota\}.$$

On a noncompact manifold M, we will not necessarily have $\iota > 0$. However, in our cases, this will hold since all our symplectic manifolds are geometrically bounded at infinity.

The charts are now given by

$$\exp_u\colon U_u \to V_u = \exp_u(U_u),$$
$$\exp_u(\xi)(s, t) = \exp(s, t, u(s, t), \xi(s, t)).$$

It is because of this system of charts that we restricted the convergence conditions at the ends in defining $\mathcal{P}^p_k(x, y)$. The proof of above theorem is technical but makes no use of the symplectic structure.

Moreover we may also consider Banach bundles $\mathcal{W}^q_l \to \mathcal{P}^p_k(x, y)$ and $\mathcal{L}^q_l \to \mathcal{P}^p_k(x, y)$, with fibres modelled on $W^q_l(u)$ and $L^q_l(u)$ respectively, provided that the regularity condition (8.1) holds.

The same proof as in [16] shows that $\bar{\partial}_{\mathbf{J}}$ is a smooth section of $\mathcal{L}^p_{k-1}$. In fact, since $\bar{\partial}_{\mathbf{J}}$ is a real Cauchy-Riemann operator with totally real boundary conditions [30, Appendix C] $\bar{\partial}_{\mathbf{J}}$ is a Fredholm operator. We denote its linearization at u by $E_u = D\bar{\partial}_{\mathbf{J}}(u)\colon W^p_k \to L^p_{k-1}$.

We now consider the zero-set of the section $\bar{\partial}_{\mathbf{J}}$. It is shown in [36] that if $u \in \mathcal{M}(x, y)$, then u has the right convergence conditions at the ends to be an element of $\mathcal{P}^p_k(x, y)$ and moreover these sets are locally homeomorphic. Moreover, any solution to $\bar{\partial}_{\mathbf{J}} u = 0$ will in fact be smooth, using elliptic bootstrapping techniques. This is proved in [16] for ω-compatible $\mathbf{J}$, and this proof carries over in region U_x, and elsewhere it follows from [30, Proposition 3.1.9]. Therefore the zero set of $\bar{\partial}_{\mathbf{J}}$ is precisely $\cup_{x,y} \mathcal{M}_{\mathbf{J}}(x, y)$.

8.3 Fredholm theory

This zero set will not always be a manifold, but we shall show that we can always perturb $\mathbf{J} = (J_t)$ to some arbitrarily close $\tilde{\mathbf{J}} = (\tilde{J}_t)$ such that the corresponding moduli space $\mathcal{M}_{\tilde{\mathbf{J}}}$ is in fact a manifold. To do this, we need to have some space which represents the possible perturbations of $\mathbf{J}$.

The space of ω-tame J is a Fréchet manifold whose tangent space at J is given by smooth sections of $\mathrm{End}(TM, J, \omega)$, which is defined to be the bundle over M whose fibre at x is the space of linear maps $Y \colon T_x M \to T_x M$ such that $YJ + JY = 0$. In order that we may have a Banach manifold, not a Fréchet one, we use the following argument of Floer [16].

Pick any sequence of positive real numbers (ϵ_k) and define

$$\|Y\|_\epsilon = \sum \epsilon_k \max_x |D^k Y(x)|.$$

Denote by $C^\infty_\epsilon(M, \mathrm{End}(TM, J, \omega))$ those Y with finite $\|\cdot\|_\epsilon$ norm. This is a Banach manifold. Floer [16] proves that there is a sequence (ϵ_k) that tends to zero sufficiently quickly that $C^\infty_\epsilon(M, \mathrm{End}(TM, J, \omega))$ is dense in $L^2(M, \mathrm{End}(TM, J, \omega))$.

Now fix some 1-parameter family $\mathbf{J^0} = (J^0_t)$ of almost complex structures. For a 1-parameter family $\mathbf{Y} = (Y_t)$ of elements of $C^\infty_\epsilon(M, \mathrm{End}(TM, J, \omega))$, we consider the map $f \colon Y_t \mapsto J^0_t \exp(-J^0_t Y_t))$. On some neighbourhood of the zero-section f restricts to a diffeomorphism. Define

$$\mathcal{Y} = \{\mathbf{Y} = (Y_t) \colon \|Y_t\|_\infty < r \text{ and } Y_t(p) = 0 \text{ for } p \in U\},$$

where $U = \cup_x U_x$ is our neighbourhood of the intersection points x and r is chosen small enough such that the restriction of f is a diffeomorphism. Denote by $\mathcal{J}_r(\mathbf{J^0})$ the image of $\mathcal{Y}$ under f. This space represents our space of perturbations of $\mathbf{J^0}$. In what follows, we shall usually consider $\mathbf{J^0}$ to be fixed and write $\mathcal{J}$ instead of $\mathcal{J}_r(\mathbf{J^0})$.

We have a section of Banach manifolds

$$\tilde{\partial} \colon \mathcal{P} \times \mathcal{Y} \to \mathcal{L},$$
$$\tilde{\partial}(u, \mathbf{Y}) = \bar{\partial}_{f(\mathbf{Y})} u.$$

As before, this section is smooth. We want to prove that its linearization is surjective on its zero set. Since $E_u = D\bar{\partial}_{\mathbf{J}}(u)$ is closed, it suffices to prove that the image is dense whenever $\bar{\partial}_{\mathbf{J}} u = 0$. This is proved in the Appendix of [33], which is itself a correction of the argument appearing in [16]. This result makes no assumption of any ω-compatibility condition.

Now the implicit function theorem [30, Theorem A.3.3] says that the *universal Floer moduli space*

$$\mathcal{M}(x, y, \mathcal{J}) = \{(u, \mathbf{J}) : u \in \mathcal{M}_{\mathbf{J}}(x, y)\}$$

is a smooth Banach manifold. Once we have this, we may consider the projection onto the $\mathcal{J}$ factor, which is a Fredholm map and apply the Sard-Smale theorem.

Theorem 8.4 (Sard-Smale)**.** *The set of regular values of a Fredholm map $g \colon A \to B$ between paracompact Banach manifolds is a Baire set in B.*

This shows that there is a second category set $\mathcal{J}_{reg} \subset \mathcal{J}$ of so-called *regular* almost complex structures, such that $\mathcal{M}_{\mathbf{J}}$ is a smooth manifold for $\mathbf{J} \in \mathcal{J}_{reg}$. In particular, this means that there exist regular $\mathbf{J}$ arbitrarily close to $\mathbf{J^0}$. The dimension of this manifold is given by the Fredholm index, which in this case is $|x| - |y|$, the difference of the Maslov indices of the intersections [15]. Note also that $\mathcal{M}_{\mathbf{J}}(x, y)$ carries a free $\mathbb{R}$-action by translation in the s variable and we shall denote the quotient space by $\widehat{\mathcal{M}}_{\mathbf{J}}(x, y)$.

8.4 Compactifications

From this point onward we shall assume that $c_1(M) = 0$. This is independent of the almost complex structure chosen. From the previous section, we now know that, given two intersection points x and y, $\mathcal{M}_{\mathbf{J}}(x, y)$ is a smooth manifold of the correct dimension, provided we pick $\mathbf{J} \in \mathcal{J}_{reg}$. Given some real number E, we can restrict attention to the set $\mathcal{M}^E_{\mathbf{J}}(x, y)$ of Floer trajectories with the energy bound $E(u) < E$. Gromov compactness says that this manifold admits a natural compactification by adding broken trajectories, possibly with bubbles. In order to be able to define Floer cohomology, we shall need to look at the compactifications of these moduli spaces in cases when they have dimension ≤ 2.

We want to prove that we can pick our almost complex structures (J_t) in such a way that we get no bubbling along solutions to the Floer equation. There are two possible types of bubbles: discs appearing on the Lagrangian boundary, and spheres appearing on the interior of some Floer solution. We shall prove that in the case where $c_1(M) = 0$, we can exclude the possibility of sphere bubbles. Disc bubbles are more difficult and no general approach exists to deal with these (in fact such an approach *cannot* exist in all situations as evidenced by the existence of *obstructed* Lagrangians [18]). However, we shall show later that we can avoid such bubbles in some specific cases. To prove that we get no sphere bubbles, we adapt the argument found in [22].

Fix some nonzero homology class $A \in H_2(M; \mathbb{Z})$. For a given J, we can consider the moduli space of simple J-holomorphic maps $v \colon S^2 \to M$ representing the homology class A, which we shall denote $\mathcal{M}_s(A, J)$. We can also take a 1-parameter family $\mathbf{J} = (J_t)$ and consider the space

$$\mathcal{M}_s(A, \mathbf{J}) = \{(t, v) : v \in \mathcal{M}_s(A, J_t)\}.$$

We can also consider the universal moduli space

$$\mathcal{M}_s(A, \mathcal{J}) = \{(t, v, \mathbf{J}) : (t, v) \in \mathcal{M}_s(A, \mathbf{J})\}.$$

This is a smooth Banach bundle and the projection to $\mathcal{J}$ is Fredholm of index $2n + 2c_1(A) + 1$, so that for $\mathbf{J} \in \mathcal{J}'_{reg}$ some second category set of almost complex strcutures, $\mathcal{M}_s(A, \mathbf{J})$ is a smooth manifold of that dimension. The analysis underlying all this is similar to that in the previous section and can be found, for example, in [30]. We also note that $\mathcal{M}_s(A, \mathbf{J})$ admits a free action by the real 6-dimensional reparametrization group of the sphere $G = PSL(2, \mathbb{C})$ and we consider the space $\mathcal{M}_s(A, \mathbf{J}) \times_G S^2$, which, for generic $\mathbf{J}$, is a smooth manifold of dimension $2n + 2c_1(M) - 3$.

By taking the fibre product over $\mathcal{J}$, we can consider

$$\mathcal{N} = \left(\mathcal{M}_s(A, \mathcal{J}) \times_G S^2\right) \times_{\mathcal{J}} (\mathcal{M}(x, y, \mathcal{J}) \times [0, 1])$$

and the map

$$\mathcal{N} \to M \times [0, 1] \times M \times [0, 1]$$

given by

$$([v, z], t, u, t') \mapsto (v(z), t, u(0, t'), t').$$

We want to know the intersection of the image of this map with the diagonal $\Delta_{M\times[0,1]}$. Since $\mathcal{M}_{\mathbf{J}}(x, y)$ carries an $\mathbb{R}$-action, if there is any such intersection, there must be a bubble intersecting a Floer solution u at some $u(0, t)$, since we only care about J_{t_0}-bubbles meeting some Floer solution at time t_0.

For any t, we have an evaluation map $ev_t \colon \mathcal{M}_{\mathbf{J}}(x, y) \to M$ given by $ev_t(u) = u(0, t)$ and a version of Proposition 3.4.2 in [30] says that this map is a submersion for all t. This means that the intersection with the diagonal is transverse, and therefore the space

$$\mathcal{Z} = \{([v, z], t, u, t') : (v(z), t) = (u(0, t'), t')\}$$

is a submanifold of $(\mathcal{M}_s(A, \mathcal{J}) \times_G S^2) \times_{\mathcal{J}} (\mathcal{M}(x, y, \mathcal{J}) \times [0, 1])$ of codimension $2n+1$. This means that the projection $\mathcal{Z} \to \mathcal{J}$ has Fredholm index

$$\begin{aligned} &(2n + 2c_1(A) - 3) + (|x| - |y| + 1) - (2n + 1) \\ = {} & 2c_1(A) + |x| - |y| - 3. \end{aligned}$$

Since we have $c_1 = 0$, this means that for generic $\mathbf{J} = (J_t)$, the 1- and 2-dimensional moduli spaces of Floer solutions (which are needed to define the

Floer differential d and show that $d^2 = 0$) will not intersect any sphere bubbles. Bearing in mind that the compactification of these spaces involves adding broken solutions, possibly with bubbles, the same argument as in [22] shows that we still get no intersection even after compactifying our spaces.

The case of disc bubbles is more difficult and there is no general approach that will work, but if we have chosen appropriate J_0, J_1 such that we get no disc bubbles for our Lagrangians, then picking a generic path of almost complex structures (J_t) interpolating between these two gives a family of (J_t) such that we can in fact define $HF(L_0, L_1)$. This will be discussed more in Chapter 8.7.

8.5 Floer cohomology

We first fix the coefficient field we shall use. Although (subject to certain topological assumptions) the relevant moduli spaces can be oriented so that Floer cohomology can be defined over fields of arbitrary characteristic, we don't need this for our purposes. We therefore introduce the Novikov ring

$$\Lambda_{\mathbb{Z}/2} = \left\{ \sum_r a_r q^r : a_r \in \mathbb{Z}/2, r \in \mathbb{R}, r \to \infty, \#\{a_r \neq 0 : r < E\} < \infty \text{ for all } E \right\}$$

of power series in the formal parameter q as in the Introduction. This is in fact a field.

In order to define Floer cohomology, we define the Floer cochain complex to be

$$CF(L_0, L_1) = \bigoplus_{x \in L_0 \cap L_1} \Lambda_{\mathbb{Z}/2}\langle x \rangle.$$

In the case where $|y| = |x| - 1$, the Floer differential is defined by

$$dy = \sum_{u \in \widehat{\mathcal{M}}_{\mathbf{J}}(x,y)} q^{E(u)} x.$$

For this map to be well-defined over the Novikov ring, for any E, there must be only finitely many terms involving powers of q less than E. This follows from Gromov compactness. When $|y| = |x| - 1$, the compactification of $\widehat{\mathcal{M}}^E_{\mathbf{J}}(x, y)$ can only involve adding bubbles, since breaking cannot occur as the solutions are already of minimal index. But we have shown that we can pick $\mathbf{J}$ such that no bubbling occurs. Therefore the zero-dimensional manifold $\widehat{\mathcal{M}}^E_{\mathbf{J}}(x,y)$ is compact, hence consists of finitely many points.

In order to show that this is in fact a differential (i.e. that $d^2 = 0$), the standard approach is to identify the boundary of the compactification of any 1-dimensional $\widehat{\mathcal{M}}_{\mathbf{J}}(x,z)$ with $\widehat{\mathcal{M}}_{\mathbf{J}}(x,y) \times \widehat{\mathcal{M}}_{\mathbf{J}}(y,z)$, and use the fact that boundary points of a 1-manifold come in pairs. This identification again relies on the fact that no bubbing occurs, which is ensured by the previous section. Once again we stress that we have not yet dealt with disc bubbling, so that the content of this section is incomplete and Floer cohomology will not be properly defined until we do so in Chapter 8.7.

In our setting, where $c_1(M) = 0$, we may also pick a grading so that $HF^*(L_0, L_1)$ becomes a $\mathbb{Z}$-graded group [37].

We also want to define a multiplication map on Floer cohomology. We start by doing this on the chain level.

Consider three Lagrangian submanifolds L_i, $i = 0, 1, 2$ and transverse intersection points $x \in L_0 \cap L_2$, $y \in L_0 \cap L_1$, $z \in L_1 \cap L_2$. Similar to before we may consider the moduli space $\mathcal{M}^2_{\mathbf{J}}(x,y,z)$ of holomorphic curves u from a disc with 3 marked boundary points mapping to M such that the marked boundary points tend to x, y, z and the remainder of the boundary maps to the various Lagrangians (see [39, Section 2] for more specific details). Here $\mathbf{J}$ is a 2-parameter family of almost complex structures $(J_w)_{w\in\mathbb{D}}$ and a similar analysis to the previous section shows that, for a generic choice of $\mathbf{J}$, $\mathcal{M}^2_{\mathbf{J}}(x,y,z)$ is a smooth manifold of dimension $|x| - |y| - |z|$.

We can therefore define

$$\begin{aligned} m &\colon CF(L_1, L_2) \otimes CF(L_0, L_1) \to CF(L_0, L_2), \\ m(z,y) &= \sum_{u \in \mathcal{M}^2_{\mathbf{J}}(x,y,z)} q^{E(u)} x. \end{aligned}$$

in the case where $|x| = |y| + |z|$. We want this to be a chain map so that we get a multiplication on the cohomological level.

Here the standard approach is again to consider the boundary of the compactification of the 1-dimensional part of $\mathcal{M}^2_{\mathbf{J}}(x, y, z)$ (see for example [35]). However, in our case we must once more rule out the possibility of bubbling off of spheres (disc bubbles will be dealt with in Chapter 8.7).

We continue in a similar vein to before and consider the universal moduli space

$$\mathcal{M}^2(x, y, z, \mathcal{J}) = \left\{(u, \mathbf{J}) : u \in \mathcal{M}^2_{\mathbf{J}}(x, y, z)\right\}$$

for an appropriate Banach space $\mathcal{J}$ of 2-parameter families of almost complex structures defined similarly to the previous section. We then consider

$$\mathcal{N}' = \left(\mathcal{M}_s(A, \mathcal{J}) \times_G S^2\right) \times_{\mathcal{J}} \left(\mathcal{M}^2(x, y, z, \mathcal{J}) \times \mathbb{D}\right).$$

By mapping to $M \times \mathbb{D} \times M \times \mathbb{D}$ via $([v, z], w, u, w') \mapsto (v(z), w, u(w'), w')$, we see that $\mathcal{N}'$ contains a submanifold

$$\mathcal{Z}' = \{([v, z], w, u, w') : (v(z), w) = (u(w'), w')\}$$

of codimension $2n + 2$, which represents the intersections between J_w-bubbles and multiplication curves u at point $u(w)$. The projection $\mathcal{Z}' \to \mathcal{J}$ is Fredholm of index

$$\begin{aligned} &(2n + 2c_1(A) - 2) + (|x| - |y| - |z| + 2) - (2n + 2) \\ &= 2c_1(A) + |x| - |y| - |z| - 2. \end{aligned} \tag{8.2}$$

Therefore, for generic $\mathbf{J} = (J_w)$, the 0- and 1-dimensional moduli spaces of such holomorphic discs do not intersect any sphere bubbles (recall that we are assuming $c_1(M) = 0$), so these will not obstruct our multiplication surviving to cohomology.

We shall also want, when defining *wrapped* Floer cohomology, to have a map

$$\Psi_H \colon CF(L_0, L_1) \to CF(L_0, \psi_H(L_1)),$$

where ψ_H is the Hamiltonian isotopy coming from some Hamiltonian $H \colon M \times [0, 1] \to \mathbb{R}$ (when M is noncompact but convex at infinity, we additionally require H to be *monotone*: $\partial_s H_s \leq 0$ [35]). First note that intersection points $y \in L_0 \cap$

$\psi(L_1)$ are in one-to-one correspondence with *Hamiltonian chords* $y\colon [0,1] \to M$ such that $y(0) \in L_0, y(1) \in L_1$, and $\dot{y}(s) = X_H(y(s))$.

For $x \in L_0 \cap L_1$ and $y \in L_0 \cap \psi(L_1)$, we consider the moduli space of *continuation Floer trajectories* $\mathcal{M}^H_{\mathbf{J}}(x,y)$, solutions u to the equation

$$\partial_s v + J_{s,t}(\partial_t v - X_H) = 0$$

on the strip $\mathbb{R} \times [0,1]$ such that $u(\cdot,0) \in L_0$, $u(\cdot,1) \in L_1$, and which converge to the point x at $+\infty$ and to the chord $y(t)$ at $-\infty$. The standard approach [5] shows that, for generic $\mathbf{J} = (J_{s,t})$, this moduli space is a smooth manifold of dimension $|y| - |x|$ and we can define

$$\Psi_H x = \sum_{u \in \mathcal{M}^H_{\mathbf{J}}(x,y)} q^{E(u)} y$$

in the case when $|y| = |x|$. Again the standard argument involving the 1-dimensional part of $\mathcal{M}^H_{\mathbf{J}}(x,y)$ shows that this is a chain map modulo bubbling. But no bubbling of spheres occurs because of the same dimension count as in (8.2) replacing $\operatorname{vdim} \mathcal{M}^2_{\mathbf{J}}(x,y,z)$ with $\operatorname{vdim} \mathcal{M}^H_{\mathbf{J}}(x,y)$: the space $\mathcal{Z}''$ representing intersections between $J_{s,t}$-bubbles and continuation trajectories at $u(s,t)$ has virtual dimension

$$\begin{aligned} &(2n + 2c_1(A) - 2) + (|y| - |x| + 2) - (2n+2) \\ &= 2c_1(A) + |y| - |x| - 2. \end{aligned}$$

Note that we are here using 2-parameter families of almost complex structures on $\mathbb{R}\times[0,1]$ as opposed to the 1-parameter families used in defining d. See Chapter 8.7 for the argument for disc bubbles.

A similar argument shows that Ψ_H intertwines the multiplicative structures on $HF(L_0,L_1)$ and $HF(L_0,\psi_H(L_1))$.

Remark 8.5. *In the case of exact Lagrangians in an exact symplectic manifold, much of the above analysis is unnecessary: exactness means that no bubbles occur in the compactifications of our moduli spaces, and we also get a priori energy bounds independent of u, so we can actually work over $\mathbb{Z}/2$ should we wish.*

8.6 Floer cohomology in Lefschetz fibrations

In the context of a Lefschetz fibration $\pi\colon E \to \mathbb{C}$, we can make a choice of almost complex structures which lends itself well to Floer cohomology calculations.

In some neighbourhood of E^{crit} we pick J to agree with the standard integrable complex structure in the local model $z \mapsto \sum z_i^2$ as in Definition 6.1, which makes ω locally a Kähler form. Away from E^{crit}, we have the splitting

$$T_x E = T_x^h E \oplus T_x^v E$$

where $T_x^v E = \ker(D\pi_x)$ and $T_x^h E \cong T_{\pi(x)}\mathbb{C}$. With respect to this splitting, we choose J that, away from E^{crit}, look like

$$\begin{pmatrix} j & 0 \\ 0 & J^v \end{pmatrix},$$

such that J^v, the vertical part of J, is compatible with ω restricted to the fibre and j is compatible with the standard form on the base. Such a J makes the projection π J-holomorphic, so that Floer solutions in E project to j-holomorphic strips $\pi \circ u\colon \Sigma \to \mathbb{C}$, and we can now use the maximum principle for holomorphic functions to restrict the region in which Floer solutions may appear.

The problem is that such a J will not necessarily be regular, so not be suitable for defining $HF(L_0, L_1)$. In [28], they proceed as follows. They take some small generic perturbation of (J_t) to regular $(\tilde{J}_t)$ such that $(\tilde{J}_t)$ is still ω-compatible, losing in the process the property that π is holomorphic. However, Gromov compactness says that Floer solutions for (J_t) will be close to Floer solutions for $(\tilde{J}_t)$. In order to apply Gromov's compactness theorem for this argument to work, we need some energy bounds, which a priori exist in the setting of [28] as all their manifolds are exact.

We do not have any such energy bounds. Therefore, we perturb J by adding some *horizontal* component to get

$$\tilde{J} = \begin{pmatrix} j & 0 \\ H & J^v \end{pmatrix},$$

where H is some small perturbation that is zero on some neighbourhood of the intersctions of our Lagrangians and such that $\tilde{J}^2 = -1$. Now π is still holomorphic, so we can use maximum principles in the base, but $\tilde{J}$ is no longer compatible with ω. However, for small H, it will still tame ω and we can use the discussion above to say that we can still do Floer cohomology in this setting. The proof that the space of such H is large enough for us to achieve transversality as in Chapter 8.3 can be found in [38, Lemma 2.4].

8.7 Disc bubbles

We have not yet said anything about how to avoid disc bubbles, J-holomorphic maps $w\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$. However, for the purposes of this thesis, we need only consider specific sorts of Lagrangian submanifolds, namely spheres or Lefschetz thimbles in some Lefschetz fibration, with a six-dimensional total space and whose first Chern class vanishes.

In the first instance, it is shown in [44], using techniques inspired by symplectic field theory, that for a Lagrangian sphere in a symplectic manifold of dimension at least 4 with vanishing first Chern class, there exists a J such that no disc bubbles exist. This is proven in [44] only for compatible J, not the larger class of almost complex structures we have considered in this chapter. However, in the next chapter there is only one point at which we need to consider the Floer cohomology of a 3-sphere in the total space of a Lefschetz fibration (Chapter 9.1) and here we don't need to perform the horizontal perturbation trick, so at this point in the argument we can just pick a compatible J for the sphere as usual.

As for thimbles, we start by picking J adapted to our Lefschetz fibration as above. If a disc bubble exists, then by considering the projection to the base, we see that any such bubble must entirely be contained in some fibre of $\pi\colon E \to \mathbb{C}$. The part of the thimble living in this fibre is now just a sphere, so we can arrange for the vertical part J^v of J to be such that we get no bubbles as in the previous paragraph. However, this fails to take into account of the fact that we have a 1-parameter family of such fibres (the vanishing path). In fact, in [44] the relevant Fredholm problem involves a Fredholm operator whose index is bounded from above by -2, so we may in fact generically pick a 1-parameter family of such J

so that we get no bubbles.

Now to complete the definition of the Floer cohomology of two such Lagrangians, we pick appropriate J_0 and J_1 as above and then pick some path $\mathbf{J} = (J_t)$ interpolating between them. A generic perturbation of $\mathbf{J}$, which may be chosen such that the endpoints are fixed will then be suitable. We may do likewise to exclude the possibility of disc bubbles appearing in the compactifications of $\mathcal{M}^2_{\mathbf{J}}(x, y, z)$ and $\mathcal{M}^H_{\mathbf{J}}(x, y)$ (although we now consider 2-parameter families of almost complex structures, we are free to choose that $\mathbf{J}$ be constant along the boundary components of the disc/strip since we can achieve transversality by perturbing $\mathbf{J}$ just on the interior), thus completing the constructions of Chapter 8.5.

Chapter 9

The examples of Maydanskiy-Seidel

Using the same method as explained in Chapter 6, we can construct the six-dimensional symplectic manifold X_2 in Figure 9.1. Its generic fibre is diffeomorphic to the A_{m+1} Milnor fibre M_{m+1} and the Lefschetz fibration $\pi\colon X_2 \to \mathbb{C}$ has $m+1$ critical points corresponding to $m+1$ vanishing cycles in M_{m+1}. The first m, $V_1, \dots, V_m$ come from the straightline matching paths, but V_{m+1} is the sphere associated to the curved path γ_{m+1}. For each critical value x_i, corresponding to V_i, fix some vanishing path $\beta_i\colon [0, \infty) \to \mathbb{C}$ such that $\beta_i(t) = t$ for $t \gg 0$. Let $\Delta_i \subset X_2$ denote the corresponding Lefschetz thimble.

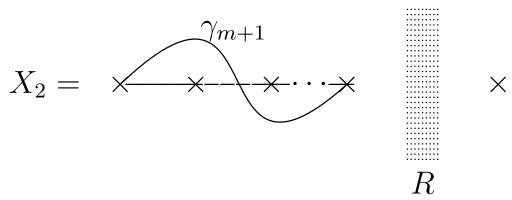


Figure 9.1:

A trivial extension of the argument in [28], which will be recapped in this chapter, shows that X_2 is diffeomorphic to $T^*S^3 \cup$ 2-handle and also contains no Lagrangian sphere L such that $[L] \neq 0$ in $H_2(X_2; \mathbb{Z}/2)$. (We have shown below only one such possible choice of γ_{m+1}; there are infinitely many others for which this is also true [28].) We construct a deformation $\tilde{X}_2$ of this manifold by adding on a closed 2-form supported in the shaded region R, as in Chapter 7, to obtain a family of symplectic manifolds $(\tilde{X}_2^t, \omega_t)$. $c_1(X_2) = 0$ so therefore $c_1(\tilde{X}_2^t) = 0$ for all t. We also note that, after deformation, the V_i will still be Lagrangian in M_{m+1} since they live away from the region R. Also the thimbles Δ_i will stay Lagrangian in $\tilde{X}_2^t$.

In this chapter, we shall prove the following:

Theorem 9.1. *For all $t \in [0,1]$, $\tilde{X}_2^t$ contains no Lagrangian sphere L such that $[L] \neq 0 \in H_2(\tilde{X}_2^t; \mathbb{Z}/2)$.*

The proof of this will essentially just be a repeat of the argument in [28], so we shall not explain all the details fully, instead directing the interested reader to the relevant sections of [28]. However, this proof relies heavily on the technology of Floer cohomology and Fukaya categories. In the original paper, everything is carried out working within the category of exact symplectic manifolds so the analytical issues involved in setting up Floer cohomology are easily overcome. This was why we had to go through the analysis of the previous chapter as we now often have to work in the more problematic nonexact setting. With the results of the previous chapter however, the argument of [28] more or less just carries over, and we only make a few remarks where particular care needs to be exercised.

In what follows, we shall denote by $HF_t^*(L_0, L_1)$ the Floer cohomology computed with respect to ω_t in any situations where there is likely to be confusion about the symplectic form being used.

9.1 Wrapped Floer cohomology

We start by defining a variant of Floer cohomology, wrapped Floer cohomology. Following [28], we shall not need to define this in the level of generality found in [5, 35], but instead restrict to a simpler (and, in our setting, equivalent) definition which is well-suited to Lefschetz fibrations.

Given a Lefschetz fibration $\pi\colon E \to \mathbb{C}$, we consider a Hamiltonian $H\colon E \to \mathbb{R}$ of the form $H(y) = \psi(\frac{1}{2}|\pi(y)|^2)$ where $\psi\colon \mathbb{R} \to \mathbb{R}$ is such that $\psi(r) = 0$ for $r < 1/2$

and $\psi'(r) = 1$ for $r \gg 0$. Let Φ^α denote the time-α flow of this Hamiltonian and, given some Lagrangian L, we define $L^\alpha = \Phi^\alpha(L)$.

We can now define the wrapped Floer cohomology of a Lagrangian L and a thimble Δ (where, in order to exclude bubbling of discs as mentioned previously, L is either a sphere or another thimble) to be the direct limit of Floer cohomology groups

$$HW_t^*(L, \Delta) = \varinjlim_k HF_t^*(L, \Delta^{2\pi k+\epsilon})$$

for some very small $\epsilon > 0$. The maps involved in this direct limit are the continuation maps from Chapter 8.5.

We will actually need to perform an extra small Hamiltonian isotopy in addition to Φ^α in order to ensure transversality of intersections but will suppress further mention of this. For our purposes, it is not necessary to identify our Floer groups canonically so the details of how we do this are irrelevant for what follows.

To prove Theorem 9.1, suppose for sake of contradiction that there does exist a Lagrangian sphere $L \subset \tilde{X}_2^t$ such that $[L] \neq 0$ in $H_*(\tilde{X}_2^t; \mathbb{Z}/2)$. $\tilde{X}_2^t$ is topologically T^*S^3 with a 2-handle attached, and it is shown in [28, Section 9] that $L \cdot \Delta_{m+1} \neq 0$ for such a sphere . This intersection number is the Euler characteristic of the Floer cohomology group $HF_t^*(L, \Delta_{m+1})$. Given the compactness of L, this group is equal to the wrapped Floer cohomology group $HW_t^*(L, \Delta_{m+1})$ (we may choose to start "wrapping" outside some compact set containing L) and $HW_t^*(L, \Delta_{m+1})$ is itself a module over the unital ring $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$, where the multiplication maps here are the images under the direct limit of the multiplication defined in Chapter 8.5. Thus we conclude

Lemma 9.2. *If such a Lagrangian sphere exists, then* $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$.

The rest of this chapter is devoted to proving that $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) = 0$ to provide the required contradiction.

9.2 From total space to fibre

If we consider the directed system of groups used to define $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$, we see that each step introduces new intersection points as the path over which our wrapped Lefschetz thimble lives wraps round the base once more. Choose our family of almost complex structures (J_t) as in Chapter 8.6. In [28], they establish the existence of a spectral sequence computing the wrapped Floer cohomology of any two thimbles, which carries over in our setting in light of the discussion of Chapter 8. When we consider $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$, this spectral sequence yields the following long exact sequence

$$\begin{array}{ccc} HF_t^*(\Delta_{m+1}, \Delta_{m+1}^{\epsilon}) & \longrightarrow & HF_t^*(\Delta_{m+1}, \Delta_{m+1}^{2\pi+\epsilon}) \\ & \nwarrow^{\sigma} & \downarrow \\ & & HF_t^*(\mu(V_{m+1}), V_{m+1}), \end{array}$$

where the bottom group is calculated in the fibre E_z and μ denotes the outer monodromy of the Lefschetz fibration. Lemma 6.2 allows us to identify some particular fibre $E_{z'}$ with the manifold M included in the data of this lemma. We may arrange that $z = z'$.

In particular, since the unit in $HW_t^*(\Delta_{m+1}, \Delta_{m+1})$ arises as the image of $1 \in HF_t^*(\Delta_{m+1}, \Delta_{m+1}^{\epsilon}) = \Lambda_{\mathbb{Z}/2}$, the map σ must be zero. By analysing the curves involved in defining the map σ [28, Section 5] and comparing to the maps involved in Seidel's the long exact sequence [38], we can, by Poincaré duality, identify the map σ with an element of $HF_t^0(V_{m+1}, \mu(V_{m+1}))$, which we shall also denote by σ.

Lemma 9.3. ([28, Proposition 5.1]) *If $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$, then σ vanishes.*

9.3 Fukaya categories

We now shift attention to the Fukaya category of the fibre $\mathcal{F}(E_z)$, and introduce two related categories.

The first is a directed A_∞-category $\mathcal{A}$, which has as objects the finite set

$\{V_1, \ldots, V_m\}$ and morphisms

$$hom_{\mathcal{A}}(V_i, V_j) = \begin{cases} (\mathbb{Z}/2)e_i & \text{for } i = j \\ (\mathbb{Z}/2)f_i & \text{for } i = j-1 \\ 0 & \text{otherwise,} \end{cases}$$

where the degrees are chosen to be $|e_i| = 0$ and $|f_i| = 1$. This category is chosen to reflect the fact that we have an A_m configuration of Lagrangian spheres $V_i \subset M_m$ coming from the straightline paths in Figure 9.1, where the only points of intersection are between adjacent spheres and the gradings can be chosen in a nice way. This determines the higher-order A_∞-structure, namely that the only nontrivial higher products are given by $\mu^2(e_i, e_i) = e_i$ and $\mu^2(f_i, e_i) = f_i = \mu^2(e_{i+1}, f_i)$.

The second variant of the Fukaya category we shall consider is the A_∞-category $\mathcal{B}$, which is the subcategory of the Fukaya category $\mathcal{F}(E_z)$ generated by the following collection of Lagrangian submanifolds

$$V_1, \ldots, V_m, V_{m+1}, \tau_{V_m}(V_{m+1}), \tau_{V_{m-1}}\tau_{V_m}(V_{m+1}), \ldots, \tau_{V_1} \ldots \tau_{V_m}(V_{m+1}).$$

In [28], there is no need to restrict attention specifically to $\mathcal{B}$ and we can happily work with the whole Fukaya category $\mathcal{F}(E_z)$, even though as above we do not strictly need to. However, all the objects in $\mathcal{B}$ are disjoint from the region R where ω_t is nonexact and we can use maximum principles to ensure that all pseudoholomorphic curves between these objects also do not enter the region R. This means there is no extra analysis to do in defining the A_∞-category $\mathcal{B}$ as we are essentially just in an exact setting.

In what follows, we shall also want to use Seidel's long exact sequence in Floer cohomology [38]. Part of the proof of this long exact sequence in [38] relies on a spectral sequence argument coming from a filtration on Floer cochain groups given by the symplectic action functional. Seidel needs to upgrade this $\mathbb{R}$-filtration to some $\mathbb{Z}$-subfiltration in order to show that a certain mapping cone is acyclic, which can be done since the action spectrum will be discrete for finitely many exact Lagrangians in an exact symplectic manifold. In $\mathcal{B}$, this argument remains valid since maximum principles mean that we are considering the same holomorphic curves with the same actions as in the exact case, although this approach would not work in general.

We can consider the "derived" versions of $\mathcal{A}$ and $\mathcal{B}$ defined via twisted complexes as $D\mathcal{A} = H^0(Tw\mathcal{A})$ and $D\mathcal{B} = H^0(Tw\mathcal{B})$ [39]. There is a canonical (up to quasi-isomophism) functor $\iota : \mathcal{A} \to \mathcal{B}$ which on the derived level extends to an exact functor $D\iota : D\mathcal{A} \to D\mathcal{B}$.

On the level of derived Fukaya categories $D\mathcal{B}$, thanks to the result of Seidel [39] relating algebraic and geometric twisting operations, σ corresponds to an element $S \in hom_{D\mathcal{B}}(V_{m+1}, T_{V_1} \cdots T_{V_m} V_{m+1})$. If σ vanishes S must too, so, looking at exact triangles in $D\mathcal{B}$, this means that

$$V_{m+1}[1] \oplus T_{V_1} \cdots T_{V_m} V_{m+1} \cong Cone(S),$$

so we wish to understand $C = Cone(S)$.

Given all this, the next lemma is pure algebra.

Lemma 9.4. ([28, Proposition 6.2]) *If $S = 0$, then V_{m+1} is isomorphic to a direct summand of an object lying in the image of the functor $D\iota\colon D\mathcal{A} \to D\mathcal{B}$.*

9.4 Contradiction

The fibre E_z itself admits a Lefschetz fibration as pictured at the start of this chapter, such that the matching cycles of interest arise from matching paths $\gamma_1, \ldots, \gamma_{m+1}$. By assumption, γ_{m+1} is not isotopic to γ_i for $1 \le i \le m$ within the class of paths which avoid the critical values except possibly at their endpoints.

Lemma 9.5. ([28, Lemma 7.2]) *For $1 \le i \le m$, and for all $t \in [0, 1]$, the image of the product map*

$$HF_t^*(V_{m+1}, V_i) \otimes HF_t^*(V_i, V_{m+1}) \to HF_t^*(V_{m+1}, V_{m+1}) \cong H^*(V_{m+1}; \Lambda_{\mathbb{Z}/2})$$

does not contain the identity in its image.

As in [28], this is proved by considering the auxiliary Lagrangian $L_\xi \cong S^1 \times \mathbb{R}$ associated to the path ξ in Figure 9.2. The key point is that, since γ_i is not isotopic to γ_{m+1}, we can draw ξ so that it intersects γ_{m+1} but is disjoint from γ_i (here we have drawn only two of the matching paths, γ_{m+1} and γ_i, to avoid clutter).

It is proven in [25] that $\dim HF_t^*(L_\xi, V_{m+1}) > 0$, whereas clearly we have $\dim HF_t^*(L_\xi, V_i) = 0$. As before, we may choose ξ to lie away from the region R where our deforming 2-form is supported since, by assumption, this also true for the paths γ_j, so once more we may use maximum principles to restrict all Floer solutions to a region of M_{m+1} where ω_t is exact.

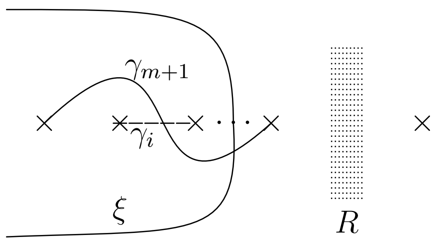


Figure 9.2:

Suppose we have elements $a_1 \in HF_t^*(V_{m+1}, V_i)$ and $a_2 \in HF_t^*(V_i, V_{m+1})$ such that $a_2 \cdot a_1 \in H^0(V_{m+1})$, the invertible part of this ring.

This then means that the composition

$$HF_t^*(L_\xi, V_{m+1}) \xrightarrow{a_1\cdot} HF_t^*(L_\xi, V_i) \xrightarrow{a_2\cdot} HF_t^*(L_\xi, V_{m+1})$$

is an isomorphism, which is a contradiction.

Once we have this, we can complete the proof of Theorem 9.1, the remainder of which carries over directly from [28] as it is essentially just algebra.

Suppose that $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) \neq 0$. Then V_{m+1} is contained in the image of $D\iota : D\mathcal{A} \to D\mathfrak{B}$. Say that V_{m+1} occurs as a direct summand of C in the image. Then, in particular

$$hom_{D\mathfrak{B}}(C, V_{m+1}) \otimes hom_{D\mathfrak{B}}(V_{m+1}, C) \to hom_{D\mathfrak{B}}(V_{m+1}, V_{m+1}) \cong H^*(S^n; \Lambda_{\mathbb{Z}/2})$$

contains the identity in its image as we can consider the maps corresponding to projection and inclusion with respect to this summand. However, thanks to the particularly simple form of $\mathcal{A}$, there exists a classification of twisted complexes in $\mathcal{A}$, following from Gabriel's theorem [19]. It says that any twisted complex is isomorphic to a direct sum of (possibly shifted copies of) the basic complexes C_{kl}

$$C_{kl} = \begin{cases} W_i = \mathbb{Z}/2 & \text{for } k \leq i < l \text{ concentrated in degree 0} \\ W_i = 0 & \text{otherwise} \\ \delta_{i+1,i} = f_i & \text{for } k \leq i < l \\ \delta_{ij} = 0 & \text{otherwise.} \end{cases}$$

However, by repeated application of our Lemma 9.5 above, we derive a contradiction, since the terms in the C_{kl} correspond geometrically to V_i involved there. This completes the proof that $HW_t^*(\Delta_{m+1}, \Delta_{m+1}) = 0$, and therefore, by Lemma 9.2, there cannot exist a homologically essential Lagrangian sphere in $\tilde{X}_2^t$.

Chapter 10

Distinguishing X_1 and X_2

10.1 Moser for symplectic manifolds convex at infinity

Take a symplectic manifold (M, ω) which is convex at infinity. Recall that this means that there is a relatively compact set M^{in} such that on a neighbourhood of the boundary ∂M^{in} we have a 1-form θ such that $d\theta = \omega$ and $\theta|_{\partial M^{in}}$ is a contact 1-form, and that $M \setminus M^{in}$ looks like the positive symplectization of ∂M^{in} according to $\theta|_{\partial M^{in}}$.

Suppose that we have a family of cohomologous 2-forms $(\omega_t)_{t\in[0,1]}$ which make M^{in} a symplectic manifold with convex boundary. We can complete (M^{in}, ω_t) to a family $(M, \widehat{\omega}_t)$ of noncompact symplectic manifolds with cohomologous symplectic forms all convex at infinity. We want to prove a version of Moser's theorem [32] in this setting.

Lemma 10.1. *The family (M, ω_t) above are all symplectomorphic, by symplectomorphisms modelled on contactomorphisms at infinity.*

Proof. We follow the standard argument, but need to pay attention to possible problems arising from the noncompactness of M. Since the ω_t are all cohomologous, we pick σ_t such that

$$\frac{d}{dt}\omega_t = d\sigma_t.$$

Then, Moser's theorem follows from considering the flow ψ_t defined by integrating the vector fields Y_t determined by

$$\sigma_t + \iota(Y_t)\omega_t = 0,$$

although we need to be careful that we can actually integrate Y_t all the way to time 1. This can be done because our forms are all cylindrical at infinity, so the vector fields obtained above will all scale according to e^r as we move in the r-direction along the collar. This bound is enough to ensure we can integrate to a flow. □

10.2 Proof of Theorem 1.2

To prove Theorem 1.2 we just apply Lemma 10.1 in our case. Let ω_1, ω_2 be the exact forms induced on X_1, X_2 respectively and suppose, for a contradiction, that there exists a diffeomorphism $\phi\colon X_2 \to X_1$ such that $\phi^*(\omega_1) = \omega_2$.

Then we also consider the deforming 2-forms η_2 and $\phi^*(\eta_1)$ defined on X_2 and by rescaling we may assume without loss of generality that these two 2-forms are cohomologous (since $H^2(X_i;\mathbb{R}) = \mathbb{R}$). We now consider the family of 2-forms on X_2

$$\Omega_t = (1-t)(\omega_2 + \eta_2) + t\phi^*(\omega_1 + \eta_1) = \omega_2 + t\phi^*(\eta_1) + (1-t)\eta_2.$$

There exists some compact subset X_2^{in} which is an interior for X_2 with respect to $\Omega_0 = \omega_2 + \eta_2$, and by the compactness of both X_2^{in} and its boundary, we can say that, after perhaps once more rescaling η_1 and η_2 if necessary, Ω_t makes X_2^{in} a symplectic manifold with convex boundary for all t. However, Ω_t is not necessarily cylindrical for all t so we now change our family Ω_t, by replacing $\Omega_t|_{X_2^{out}}$ with the completion of $\Omega_t|_{X_2^{in}}$ to get a new family of cohomologous symplectic forms $\tilde{\Omega}_t$ on $X_2^{in} \cup_{\partial X_2^{in}} [0,\infty) \times \partial X_2$, which are all cylindrical on the collar. Therefore, by Lemma 10.1, $(X_2^{in}, \tilde{\Omega}_t)$ are all symplectomorphic.

However, we can choose X_2^{in} sufficiently large that it contains the image $\phi^{-1}(L)$ of the Lagrangian sphere exhibited in Section 7. This is a contradiction of Theorem 9.1.

Chapter 11

Symplectic cohomology vanishes

In this chapter, we digress from the main theme and discuss symplectic cohomology. All symplectic manifolds considered in this chapter will be exact and we shall work with $\mathbb{Z}/2$-coefficients. As mentioned in the Introduction, symplectic cohomology is one of the standard invariants used to examine and distinguish Liouville domains. We prove that the symplectic cohomology $SH^*(X_i;\mathbb{Z}/2)$ of X_1 and X_2 both vanish, thereby showing that this invariant does not suffice to distinguish between the examples of this thesis, and so a different approach such as that of this thesis truly is needed.

We shall not define symplectic cohomology here; an appropriate definition may be found in [42], for example. We shall instead refer to two results from [4]. In the formulation of these two lemmas, we consider the Liouville domain E to be built from fibre M and the collection of vanishing cycles $(V_1,\dots,V_r)$ according to Lemma 6.2. We denote by Δ_i the Lefschetz thimble associated to V_i in the corresponding Lefschetz fibration $\pi\colon E \to \mathbb{C}$.

Lemma 11.1. ([4, Property 2.3]) *For a Liouville domain E, constructed from $(M;V_1,\dots,V_m)$, $SH^*(E)=0$ if and only if $HW^*(\Delta_i,\Delta_i)=0$ for all i.*

Lemma 11.2. ([4, Property 2.5]) *Consider a Liouville domain E, constructed from $(M;V_1,\dots,V_m)$ and let E' be the Liouville domain built from $(M;V_2,\dots,V_m)$. Let Δ_i,Δ_i' be the Lefschetz thimbles in E,E' respectively. If $HW^*(\Delta_1,\Delta_1)=0$ and $HW^*(\Delta_i',\Delta_i')=0$ for all i, then $HW^*(\Delta_i,\Delta_i)=0$ for all i.*

We also note that if $SH^*(E;\mathbb{Z}/2) = 0$, then E cannot contain any exact Lagrangian submanifolds [42].

Lemma 11.1 suffices to prove that Maydanskiy's exotic examples [27] have vanishing symplectic cohomology, as do the exact symplectic manifolds X_n^j considered in Chapter 12. We now prove that the exotic examples of Maydanskiy-Seidel, as well as their versions obtained from adding a 2-handle in the way described in Chapter 9 have vanishing symplectic cohomology. Take some exotic example X_0 from [28], as in Figure 9.1, but without the extra rightmost critical point.

The proof in [28], as outlined in Chapter 9, shows that $HW^*(\Delta_{m+1}, \Delta_{m+1}) = 0$. We apply Lemma 11.2 in this setting, and remark that this lemma still holds if we remove the final vanishing cycle instead of the first. If we restrict to the A_m configuration of vanishing cycles $(V_1, \ldots, V_m)$ in Figure 9.1, then X_0' is just isomorphic to the standard ball. This means that if we compute $HW^*(\Delta_i', \Delta_i')$, we get zero as all the Floer groups involved in the definition of $HW^*(\Delta_i', \Delta_i')$ will vanish. This suffices to prove that $HW^*(\Delta_i, \Delta_i) = 0$ for all i, and so $SH^*(X_0) = 0$.

We construct the manifold X_2 of Chapter 9 by adding a 2-handle to M_m. However, because this handle is added away from all the vanishing cycles, we can just view this as a subcritical handle added to X_0, as opposed to a critical one added to M_m since X_0 is a product fibration in the region where the handle is attached. Cieliebak's result [14] says that $SH^*(X_2) = SH^*(X_0)$ is still zero. In particular we have

Theorem 11.3. *X_2 and X_0 are both empty as exact symplectic manifolds, in the sense of containing no exact Lagrangian submanifolds.*

Remark 11.4. *It is sometimes possible to define symplectic cohomology with respect to some nonexact symplectic form. Ritter [34] shows that, if one performs a nonexact deformation of the exact symplectic form, then this is the same as computing the symplectic cohomology of the original structure, but with coefficients in some twisted Novikov bundle: $SH^*(M, d\theta + \eta) = SH^*(M, d\theta; \underline{\Lambda}_{\tau\eta})$. This has implications for the existence of exact Lagrangians and it would be interesting to compare the results of this thesis with this viewpoint.*

Chapter 12

Many inequivalent exotic symplectic forms

12.1 An invariant

We shall now extend the ideas of Chapter 10 in order to prove Theorem 1.3. Suppose we have a symplectic manifold (E, ω) which is convex at infinity and such that the map $H^2(E;\mathbb{R}) \to H^2(\partial E;\mathbb{R})$ is zero. Then, given any cohomology class $\eta \in H^2(E;\mathbb{R})$, we can construct a deformation of E in the sense of Chapter 9 in the direction of η, in other words $[\omega_t] = [\omega + t\epsilon\eta] \in H^2(E;\mathbb{R})$ for some small $\epsilon > 0$.

Suppose in addition that (E, ω) contains no homologically essential Lagrangian sphere. We denote by $\Gamma_1(E, \omega)$ the set of directions $l \in \mathbb{P}(H^2(E;\mathbb{R}))$ such that, after constructing a "small" deformation of (E, ω) in direction l, we *still have no* homologically essential Lagrangian sphere. The Moser-type argument from Chapter 10 says that this set is well-defined (up to projective linear equivalence).

We can likewise consider the invariants $\Gamma_k(E, \omega)$, which are the set of k-planes P_k in the Grassmanian $Gr(H^2(E;\mathbb{R}))$, such that we get no homologically essential Lagrangian sphere *for every* direction l contained in P_k. These are again invariants up to the correct notion of linear equivalence, and so in particular, if we get a finite set of such planes, the cardinality of $\Gamma_k(E, \omega)$ is invariant.

12.2 The construction

We now extend the construction of Maydanskiy [27] to exhibit, for any $n \geq 1$, a Liouville manifold which admits $n+1$ symplectic forms ω_k all of which have no homologically essential exact Lagrangian sphere (in fact which have vanishing symplectic cohomology $SH^*(E, \omega_k; \mathbb{Z}/2)$ and therefore no exact Lagrangian submanifolds), but such that there exists no diffeomorphism ϕ of E such that $\phi^*\omega_i = \omega_j$ for $i \neq j$.

Take the points $0, 1, \ldots, n+1 \in \mathbb{C}$ and consider two paths in $\mathbb{C}$ as in Figure 12.1. The first γ_0 joins the extreme crosses and goes over all the others. We have some choice in the second path and denote by γ_j the path which goes below the points $1, \ldots, j$ and then over $j+1, \ldots, n$. (We include here the possibility that the second path actually goes over all central crosses and in this case just consider it to be another copy of γ_0.)

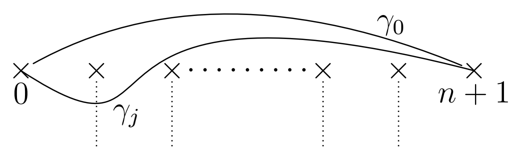


Figure 12.1:

With the same conventions as before, having made our choice of γ_j, we can associate to Figure 12.1 the 6-dimensional manifold (X_n^j, ω_j), which is diffeomorphically T^*S^3 with n 2-handles attached. It is the total space of a Lefschetz fibration whose generic fibre is the A_{n+1} Milnor fibre, which we shall denote M_{n+1}. Associated to each dotted line we get a Lagrangian 2-ball $B_i \subset M_{n+1}$ for $1 \leq i \leq n$, and we denote by V_0 and V_j the two matching paths associated to the paths γ_0 and γ_j. If $\gamma_j = \gamma_0$, the 6-manifold we obtain clearly contains a Lagrangian S^3, coming from the zero-section of T^*S^3. We shall denote by Δ_0, Δ_j the Lefschetz thimbles associated to the two critical points of the Lefschetz fibration $\pi\colon X_n^j \to \mathbb{C}$.

$H_2(M_{n+1}; \mathbb{R}) \cong \mathbb{R}^{n+1}$ and we shall choose as our standard basis the spheres A_i given by straightline paths joining adjacent critical points $i-1$ and i in Figure 12.1. When included into our total space, these all determine nonzero homology classes in E, but now with the relation $\sum A_i = 0$. We shall therefore choose to identify $H^2(E;\mathbb{R})$ with the n-dimensional vector space $V = \{v \in \mathbb{R}^{n+1} : \sum v_i = 0\}$.

Pick some vector $\mathbf{v} = (v_1, \ldots, v_n, v_{n+1}) \in V$. By the same process as in Chapter 7, we can construct a deformation of the symplectic structure on M_{n+1}, by adding on 2-forms in the regions between the critical point weighted according to the components. The condition on $\mathbf{v}$ means the that the homological obstruction to the matching paths above defining matching cycles vanishes, so we can once more build the corresponding deformation of (X_n^j, ω_j). We are interested in what choices of j and $\mathbf{v}$ mean that (X_n^j, ω_j) contains a Lagrangian sphere after the deformation coresponding to $\mathbf{v}$.

We first observe that, as in Chapter 7, we shall get a Lagrangian sphere in X_n^j when we can "lift" V_j over the critical points and onto V_0. For this to be true, we need

$$\sum_r^k v_r \neq 0 \text{ for all } k \leq j.$$

In this case we shall get a Lagrangian sphere in X_j^n once we perturb in the direction of $\mathbf{v}$. We shall now show that in all other cases we do not get such a sphere.

Fix some direction $\mathbf{v} \in V$. In what follows, we shall as before denote by HF_t^* the Floer cohomology group computed with respect to the time-t deformation of ω in the direction of $\mathbf{v}$. For the same reasons as already discussed, all these groups are well-defined (perhaps after rescaling $\mathbf{v}$).

Suppose that there is a homologically essential Lagrangian sphere $L \subset (X_n^j, \omega_t)$. Then, as in Chapter 9, we must have $L \cdot \Delta_j \neq 0$, which implies that $HW_t^*(\Delta_j, \Delta_j) \neq 0$. This wrapped Floer group fits in an exact triangle as before.

$$\begin{array}{ccc} HF_t^*(\Delta_j, \Delta_j^{\epsilon}) & \longrightarrow & HF_t^*(\Delta_j, \Delta_j^{2\pi+\epsilon}) \\ & \nwarrow & \downarrow \\ & & HF_t^*(\mu(V_j), V_j). \end{array} \tag{12.1}$$

where the bottom group is calculated in the fibre E_z. Here μ is, up to isotopy, $\tau_{V_0} \circ \tau_{V_j}$, so we shall need to consider the group $HF_t^*(\tau_{V_0} V_j, V_j)$.

The argument in this chapter largely follows that found in [27], from where we reproduce the following basic observation.

Lemma 12.1. *If we have an exact triangle of graded vector spaces*

$$\begin{array}{ccc} K & \xrightarrow{F} & L \\ & {\scriptstyle [1]}\nwarrow & \downarrow \\ & & M, \end{array}$$

then $\mathrm{rank}(M) = \mathrm{rank}(K) + \mathrm{rank}(L) - 2\,\mathrm{rank}(\mathrm{im}(F))$.

We shall consider this lemma applied to the following triangle coming from the long exact sequence in [38].

$$\begin{array}{ccc} HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) & \longrightarrow & HF_t^*(V_j, V_j) \\ & \nwarrow & \downarrow \\ & & HF_t^*(\tau_{V_0} V_j, V_j). \end{array}$$

Remark 12.2. *To apply Seidel's long exact sequence in this nonexact setting, we can no longer filter the Floer cochain groups by the symplectic action, as discussed in Chapter 9.3. However, we can introduce a filtration by powers of our formal Novikov parameter q. This will give us an appropriate $\mathbb{Z}$-filtration as the energy spectrum of the (unperturbed) holomorphic curves u will form a discrete set.*

Consider now the Lagrangian balls B_i associated to the dotted paths in Figure 12.1 and suppose there is an i such that $HF_t^*(V_j, B_i)$ is nonzero. Then the product

$$HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) \to HF_t^*(V_j, V_j) \cong H^*(S^2)$$

does not contain the identity in its image, because if it did, then the composite

$$HF_t^*(V_j, B_i) \otimes HF_t^*(V_0, V_j) \otimes HF_t^*(V_j, V_0) \to HF_t^*(V_j, B_i)$$

would hit the identity despite factoring through $HF_t^*(V_0, B_i)$ which is zero as these Lagrangians are disjoint. Here we use the fact that the product structure on Floer cohomology is associative. However, the fundamental class of $H^2(S^2)$ is in the image, by Poincaré duality for Floer cohomology.

So, when we consider the ranks of the groups in the above triangle, we see that

Lemma 12.3. *If* $HF_t^*(V_j, B_i) \neq 0$ *for any* i*, then* $\operatorname{rank} HF_t^*(\tau_{V_0} V_j, V_j) = 4$.

We now consider the triangle (12.1) relating the first few terms in the system of groups computing $HW_t^*(\Delta_j, \Delta_j)$. Again, by computing ranks we see that, if $\operatorname{rank} HF_t^*(\tau_{V_0} V_j, V_j) = 4$, then the rank of the image of the horizontal map must be zero, and therefore take 1 to 0, which in turn forces $HW_t^*(\Delta_j, \Delta_j) = 0$. We conclude

Lemma 12.4. *If* $HF_t^*(V_j, B_i) \neq 0$ *for any* i*, then there exists no homologically essential Lagrangian sphere.*

For $i > j$, V_j and B_i are disjoint so $HF_t^*(V_j, B_i) = 0$ is automatic. For $i \leq j$, the criterion that $HF_t^*(V_j, B_i)$ be nonzero corresponds to

$$\sum_r^k v_r \neq 0 \text{ for all } k \leq i$$

since, in the fibre where the paths defining V_j and B_i intersect we either get disjoint circles or instead two copies of some circle C whose self-Floer cohomology $HF_t^*(C, C) \cong H^*(C)$ is nonzero.

Remark 12.5. *In particular, the above argument shows that, in the undeformed case,* $HW^*(\Delta_j, \Delta_j) = 0$*. A similar argument also shows that* $HW^*(\Delta_0, \Delta_0) = 0$*, which, by Lemma 11.1, proves that, for our undeformed exact symplectic manifolds* $SH^*(X_n^i) = 0$ *for all* i*.*

Therefore, if we consider the $(n-1)$-Grassmanian invariant $\Gamma_{n-1}(X_n^j)$, we see that the planes for which we get no Lagrangians appearing are, in our choice of basis, precisely those $(n-1)$-planes defined by any one of the equations

$$\sum_r^k v_r = 0 \text{ for some } k \leq j,$$

so that $\Gamma_{n-1}(X_n^j)$ is a set consisting of j points.

We now have, for $1 \leq i \leq n$, exact symplectic manifolds such X_n^i is not symplectomorphic to X_n^j for $i \neq j$, even though neither contains any exact Lagrangian submanifolds. Our final manifold $(X_n^{n+1}, \omega_{n+1})$ simply comes from adding n handles to some exotic Maydanskiy-Seidel example, just as in Figure 9.1. The same argument as in Chapter 9 will show that $\Gamma_{n-1}(X_n^{n+1}, \omega_{n+1}) = Gr_{n-1}(\mathbb{R}^n)$, so X_n^{n+1} cannot be symplectomorphic to any of the X_n^i for $i \leq n$. This completes the proof of Theorem 1.3.

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