Automorphisms of free products of groups

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James T. Griffin
Corpus Christi College

UNIVERSITY OF
CAMBRIDGE

**Department of Pure Mathematics
and Mathematical Statistics**

To my grandparents.

Abstract

The symmetric automorphism group of a free product is a group rich in algebraic structure and with strong links to geometric configuration spaces. In this thesis I describe in detail and for the first time the (co)homology of the symmetric automorphism groups.

To this end I construct a classifying space for the Fouxe-Rabinovitch automorphism group, a large normal subgroup of the symmetric automorphism group. This classifying space is a moduli space of 'cactus products', each of which has the homotopy type of a wedge product of spaces.

To study this space we build a combinatorial theory centred around 'diagonal complexes' which may be of independent interest. The diagonal complex associated to the cactus products consists of the set of forest posets, which in turn characterise the homology of the moduli spaces of cactus products. The machinery of diagonal complexes is then turned towards the symmetric automorphism groups of a graph product of groups.

I also show that symmetric automorphisms may be determined by their categorical properties and that they are in particular characteristic of the free product functor. This goes some way to explain their occurence in a range of situations.

The final chapter is devoted to a class of configuration spaces of Euclidean n-spheres embedded disjointly in $(n+2)$-space. When $n = 1$ this is the configuration space of unknotted, unlinked loops in 3-space, which has been well studied. We continue this work for higher n and find that the fundamental groups remain unchanged. We then consider the homology and the higher homotopy groups of the configuration spaces.

Our last contribution is an epilogue which discusses the place of these groups in the wider field of mathematics. It is the functoriality which is important here and using this new-found emphasis we argue that there should exist a generalised version of the material from the final chapter which would apply to a far wider range of configuration spaces.

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. No part of this dissertation is substantially the same as any that I have submitted or will be submitting for a degree or diploma or other qualification at this or any other University.

James Thomas Griffin
Cambridge, August 2011.

Parts of this work are presented in the following article,

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Contents

Introduction

The free product of a pair of groups G and H is easy to describe; one takes a presentation $\langle X \mid R \rangle$ for G and a presentation $\langle Y \mid S \rangle$ for H and then takes their union $\langle X \cup Y \mid R \cup S \rangle$ which is a presentation for the free product $G * H$. Equivalently the free product may be described using the language of category theory as the binary coproduct in the category of groups. For groups G and H the free product is the unique (up to isomorphism) group $G * H$ such that for any group K with maps $G \to K$ and $H \to K$ there exists a unique factorisation

$$\begin{tikzcd}
G \arrow[dr] \arrow[ddr, bend right] & & H \arrow[dl] \arrow[ddl, bend left] \\
& G * H \arrow[d, dotted] & \\
& K &
\end{tikzcd}$$

The n-ary free product of groups $G_1, \ldots, G_n$ is given by repeated applications of the binary free product which is associative. These objects occur naturally. Suppose one has a pair of connected, pointed spaces X and Y, then the wedge product is given by identifying their basepoints. The Seifert van-Kampen theorem tells us that the fundamental group of the wedge product is the free product of the fundamental groups of X and Y.

So free products are simple to describe and occur naturally. It turns out that they also have an interesting group of symmetries.

Automorphisms of free products

A presentation for the automorphism group of a free product of groups has been known for some time [21]. Suppose that we are given a group as a free product $G = G_1 * \ldots * G_n$; to give a presentation of the automorphism group, $\mathrm{Aut}(G)$ we would need to know

A. the automorphism group $\mathrm{Aut}(G_i)$ of each individual factor,

B. which of the pairs G_i, G_j are isomorphic,

C. which of the G_i are infinite cyclic, and finally

D. that each G_i is freely indecomposable.

If we knew A, B, C and D then we would be able to give the presentation. So what does each individual piece of information give us?

To write down the presentation we need to know A, B and C, then the final piece D would tell us that the group obtained is in fact the full automorphism group. If we were to find that D were false then the group written down would still act on the free product, but it would be a proper subgroup of $\mathrm{Aut}(G)$. Similarly if we know only a subset of A, B and C then we could still write down a partial presentation, however the group presented would be a subgroup of $\mathrm{Aut}(G)$.

The group obtained when only A and B are taken into account is called the *symmetric automorphism group*, $\Sigma\mathrm{Aut}(G)$ and it is this group which is the central object of study in this thesis. It is the full automorphism group when the answer to C is that no G_i is infinite cyclic and when D is known to be true. The main step in studying $\Sigma\mathrm{Aut}(G)$ is to first understand the group of automorphisms we may give when we know none of A, B, C or D; this group is called the *Fouxe-Rabinovitch group*, $\mathrm{FR}(G)$. The presentation of $\mathrm{FR}(G)$ contains only the relations from the factor groups G_i and certain commutators. Viewed from this perspective the Fouxe-Rabinovitch group does not depend on any of the properties of the factors and so it should come as no surprise that it is (a) functorial in the factor groups and (b) very well behaved.

History

As we have already mentioned, a presentation of $\mathrm{Aut}(G_1 * \ldots * G_n)$ where A, B, C and D are known was given by Fouxe-Rabinovitch [21]. This also gives a presentation of the symmetric automorphism group $\Sigma\mathrm{Aut}(G_1 * \ldots * G_n)$ for any G_i (we merely pretend that each G_i is freely indecomposable and not infinite cyclic). The group $\Sigma\mathrm{Aut}(\mathbb{Z}^{*n})$ appeared in the study of configuration spaces as the fundamental group of the configuration space of n unlinked, unknotted loops in 3-space [18]. A refined presentation of $\Sigma\mathrm{Aut}(\mathbb{Z}^{*n})$ was given by McCool [36]. It was not long after that significant progress was made on the cohomological properties of this group [14]. Collins described a version of Outer space consisting of graphs in a cactus shape; these may be seen as prototypical examples of the cactus products introduced in this thesis.

Attention next moved to the cohomology groups of $\Sigma\mathrm{Aut}(\mathbb{Z}^{*n})$; the integral cohomology of $\mathrm{FR}(\mathbb{Z}^{*3})$ was calculated in [9] and Brownstein and Lee conjectured the structure for general n. It should be stressed that up to this point most attention had been restricted to the case of $\mathbb{Z}^{*n}$ rather than general free products.

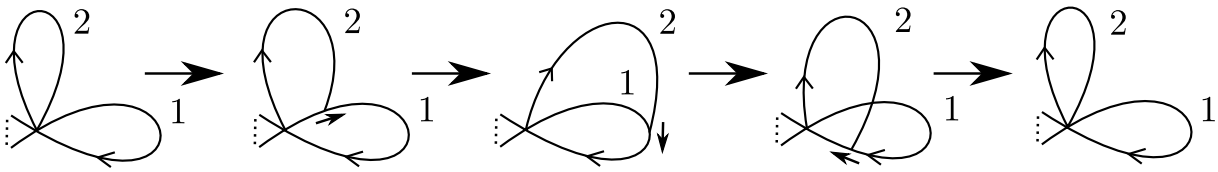
In [37], McCullough and Miller described a space on which $\Sigma\mathrm{Aut}(G_1 * \ldots * G_n)$ acts. In particular they showed that the space is contractible and that the action of $\mathrm{FR}(G)$ has stabilisers which are direct products of the G_i. This was however a departure from the space described by Collins [14]. Still the space allowed many results to be established [5], [35], [13] culminating in a proof of the Brownstein-Lee conjecture [27] for $\Sigma\mathrm{Aut}(\mathbb{Z}^{*n})$.

Since then calculations of the Euler characteristic [28] of $\Sigma\mathrm{Aut}(G)$ and partial calculations of the cohomology of $\Sigma\mathrm{Aut}(G)$ over a field have appeared [4], see Remark 3.4.10 for a discussion.

The Outer space of a free group

To understand the purpose of McCullough-Miller space it is best to first look at the motivation from the theory of automorphism groups of free groups. Let F_n be the free group on n letters, this is the group one obtains when given n generators and no relations. In one sense this is one of the simplest groups, for example to give a homomorphism from F_n to a group H it is enough to list the elements of H to which each generator of F_n is assigned; since there are no relations to check this is all that is required. Despite this the symmetries of the free group F_n are rather complicated and the study of the automorphism group $\mathrm{Aut}(F_n)$ has been long, there are many open problems and much is left to explore.

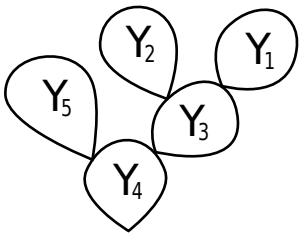
To begin with algebraic and combinatorial techniques were used to study $\mathrm{Aut}(F_n)$ and these yielded a presentation. In this thesis we will not interest ourselves with such methods, we will concentrate on the topological methods. The breakthrough in this area was made in [17] with the definition of the 'Outer space' of a free group, although the name came later. The basic idea is that since a rank n graph has fundamental group a free group F_n and since graphs can be continuously altered by adjusting edge lengths and merging and splitting vertices, a path in the 'moduli space of rank n graphs' will trace out an automorphism of the free group F_n.



Culler and Vogtmann made this intuition precise and by looking at 'marked graphs', which are graphs decorated with a basis of F_n, one may define a moduli space which holds an action of $\mathrm{Aut}(F_n)$. Furthermore they showed that this space was contractible and although the action by $\mathrm{Aut}(F_n)$ wasn't free, the simplex stabilisers are finite. So the motivation behind McCullough-Miller space is to provide a space on which $\mathrm{Aut}(G_1 * \ldots * G_n)$ acts. They were successful and defined a contractible space and although the stabiliser subgroups are not necessarily finite, they are well understood in terms of the factor groups $G_1, \ldots, G_n$. Outer space and McCullough-Miller space are combined in the Outer space of a free product defined in [23].

A different approach

The methods used in this thesis are different in that the object at the centre is not McCullough-Miller space, but we are still motivated by 'Outer space', we just take things in a slightly different direction. The approach is more in line with that of Collins [14]. The central construction is a functor that takes an n-tuple of pointed spaces $\mathbf{Y} = (Y_1, \ldots, Y_n)$ and gives a moduli space $\mathcal{M}\mathbf{Y}$ of objects which we name cactus products. An example of a single cactus product is pictured below.



A cactus product of pointed spaces is much like a wedge product, in fact it is homotopy equivalent to the wedge product. The space of cactus products is naturally functorial and there is an equivalence of functors

$$\pi_1(\mathcal{M}\mathbf{Y}) \cong \mathrm{FR}\big(\pi_1(Y_1) * \ldots * \pi_1(Y_n)\big),$$

which says that the fundamental group of the moduli space of cactus products is the Fouxe-Rabinovitch automorphism group of the free product $\pi_1(Y_1) * \ldots * \pi_1(Y_n)$.

A stronger result holds.

Theorem. *Suppose that $Y_1, \ldots, Y_n$ are classifying spaces for $G_1, \ldots, G_n$. Then $\mathcal{M}\mathbf{Y}$ is a classifying space for* $\mathrm{FR}(G_1 * \ldots * G_n)$.

This is the central theorem of this thesis and it is proved in Section 3.3.

Diagonal complexes

To prove this we require a strong handle on the moduli space and so an alternative description is required: we construct it as a *diagonal complex product*. When introducing the notion of diagonal complex we are inspired by graph products, indeed the diagonal complex products are generalisations of graph products. Recall that for each graph Γ with vertex set $[n]$ there is a product of groups $\mathbf{G} = (G_1, \ldots, G_n)$ which is generated by the groups G_i along with addition relations: for each edge $(i, j) \in E(\Gamma)$ we have the commutator relations $[g_i, g_j]$ for $g_i \in G_i$ and $g_j \in G_j$. Similarly for each diagonal complex there will be defined a product of groups, although the presentation for the products is necessarily more complicated.

We will see that there is a diagonal complex of *forest posets* which corresponds to the Fouxe-Rabinovitch automorphism groups. This diagonal complex will also define a product of pointed spaces which will give the moduli space of cactus products. It is the formalism of diagonal complexes which allows us to prove the theorem above using combinatorial properties of the forest posets and a variant of McCullough-Miller space from [13]. The diagonal complex also naturally describes the (co)homology of the moduli space and by extension the (co)homology of both the groups $\mathrm{FR}(G)$ and the symmetric automorphism groups $\Sigma\mathrm{Aut}(G)$.

Further applications

Other diagonal complexes describe other products of groups. There is a generalisation of a forest poset for each graph G which we call a G-admissable poset; a forest poset is G-admissable for the discrete graph (consisting only of vertices). The G-admissable posets also define diagonal complexes and we conjecture that the corresponding product of groups $\mathbf{H} = (H_1, \ldots, H_n)$ is the symmetric automorphism group of the graph product $\mathbf{H}G$.

One of the motivations for research into the symmetric automorphism group of a free group $F_n \cong \mathbb{Z}^{*n}$ was that it is the fundamental group of the configuration space of n unknotted, unlinked loops in $\mathbb{R}^3$. As such we investigate the configuration spaces of k-spheres in $\mathbb{R}^{k+2}$. In fact the fundamental groups are isomorphic to $\mathrm{FR}(\mathbb{Z}^{*k})$ for all $k \geq 1$.

Plan of the thesis

In Chapter 1 we review graph products of groups and their (co)homological properties. The graph product $\mathbf{G}\Gamma$ of factor groups $\mathbf{G} = (G_1, \ldots, G_n)$ over a graph Γ is functorial with respect to the factor groups and we proceed by defining classifying spaces for the $\mathbf{G}\Gamma$. Ofcourse we do this functorially obtaining spaces $\mathbf{Y}\Gamma$ for pointed spaces $\mathbf{Y} = (Y_1, \ldots, Y_n)$. The purpose of this chapter is to introduce the general methodology we use for the more complicated functors which follow.

So in Chapter 2 we introduce diagonal complexes, which are more versatile than graph products. We give formulae for the (co)homology of a diagonal complex product of pointed spaces as well as presentations of the diagonal complex products of groups. Unfortunately the extra versatility comes at a price and it is now harder to determine whether a space is a classifying space or not. The chapter finishes with two decomposition theorems which are aimed at making this task easier along with the definition of a coset complex which allows geometric techniques to be used.

It is in Chapter 3 that we turn to symmetric automorphisms. Cactus products and their moduli spaces are introduced and the geometric interpretation of partial conjugations

is described. The commutator relations between partial conjugations may be seen as tori embedded in the moduli space. The purpose of Chapters 1 and 2 was to give a gentle introduction to diagonal complexes and it is here that we give their first major application. We prove that the set of forest posets is in fact a diagonal complex and furthermore that it does give the moduli spaces of cactus products and the groups of partial conjugations that we wanted to study. Now that we have a good description of the moduli spaces we can prove that they are classifying spaces for the groups $\mathrm{FR}(G_1 * \ldots * G_n)$ when the factor spaces are classifying spaces for the G_i. The formulae for the (co)homology of diagonal complexes, established in Chapter 2 now give the (co)homology of the symmetric automorphism groups.

We then go on to discuss the relationship between Outer space and the moduli space of cactus products, in particular we define a hybrid space which we conjecture to be aspherical. If this conjecture were to be true it would be reasonable to call the space an Outer space of a free product, the advantage over the definition of [23] is that the stabiliser subgroups are far smaller. We also turn our attention to symmetric automorphisms of graph products of groups, which involves new diagonal complexes and significantly more complicated combinatorics.

The final Chapter 4 is devoted to configuration spaces. The configuration space of n unknotted, unlinked loops embedded in 3-space is well treated in the literature. We look at the analogue for k-spheres embedded in $(k+2)$-space and find that they have the same fundamental group $\mathrm{FR}(\mathbb{Z}^{*n})$. We then turn to the higher homotopy groups and conjecture that for $j \geq 2$ the homotopy groups π_j vanish for $k \geq j$. This would mean that when one takes the limit as $k \to \infty$ one gets a classifying space for $\mathrm{FR}(\mathbb{Z}^{*n})$. The main distinction between this classifying space and the moduli space of cactus graphs is that the action of the symmetric group is proper.

We finish the thesis with a short Epilogue. The purpose of this is to try to bring together the different strands that run through the thesis. In particular we emphasise the importance of the functoriality of our constructions. Ofcourse there is one area that lacks any functoriality at all, that of the configuration spaces in the final chapter. In the Epilogue we address this by making conjectures about more general configuration spaces and sketch how the theory may pan out.

Chapter 1

Preliminaries

The purpose of this chapter is to provide a gentle introduction to the ideas contained within the thesis. The diagonal complexes which are the subject of Chapter 2 offer a generalisation of graph products of groups. Since our techniques simplify when applied to graph products it makes sense to study this simplification before moving on to the general case.

In Section 1.2 we do just that; we give a gentle introduction to graph products of groups, we construct classifying spaces for these groups and then we calculate their (co)homology. If the reader is already familiar with graph products then they may skip this chapter without worry.

1.1 Algebraic topology

Algebraic topology is concerned with the properties and invariants of spaces *up to homotopy*. Let X be a path connected topological space with a chosen point $p \in X$. One of the basic homotopy invariants of this space is the fundamental group $\pi_1(X, p)$. But there are other homotopy invariants, for instance the higher homotopy groups and the homology and cohomology of the space.

We wish to study invariants of a group G. The general idea is to construct a classifying space; a space X with fundamental group G and trivial higher homotopy groups, and then to study the homotopy invariants of this space.

It is desirable to describe small models of classifying spaces for a group. There are a number of approaches to this. A common approach is to construct a contractible space equipped with a proper action of the group, often the space will carry some kind of geometry or Morse function allowing one to show that it is contractible. The quotient of the space by the group action is then a classifying space for the group.

Another method is to show that the classifying spaces are gluings of pairs of nice subspaces which are themselves classifying spaces of subgroups. One of the requirements for this is that the inclusion of the intersection of the subspaces into the subspaces themselves

must be injective on fundamental groups. The theorem required for this is the Seifert-van Kampen theorem which is reviewed below. It is this method that we will apply to the graph products of groups described later in this chapter.

For our central theorem in Chapter 3 we will actually use a combination of these two approaches.

1.1.1 The Seifert-van Kampen theorem

CW complexes

In this thesis all spaces will be assumed to be CW complexes and all maps between spaces will be maps of CW complexes. For definitions and a detailed treatment of their properties see [25]. The reason for considering CW complexes is that we avoid pathological examples of topological spaces whilst (for the purposes of algebraic topology) retaining full generality. In fact every topological space X can be approximated by a CW complex X' in such a way that their homotopy groups are identical, we say that X and X' are weakly homotopic. This means that all the algebraic invariants we use see X and X' as being identical.

We must be a little careful with our CW complexes. In the category of CW complexes the colimit of a diagram is homeomorphic to the colimit in the category of topological spaces, however the limit of a diagram in the category of CW complexes is not necessarily homeomorphic to the limit in topological spaces. The limits have isomorphic underlying sets but the topologies differ slightly, although (importantly) the two limits are weakly homotopic. In practice the only limit we take in this thesis is the product. This means that when we take the direct product of two CW complexes we don't endow it with the product topology, but with a slightly weaker topology, the CW product topology.

The important fact to take from the above comments is that if we start with CW complexes, all of our constructions will still be CW complexes, so we will be able to apply Theorem 1.1.1 and Proposition 1.1.2 without worry.

The Seifert-van Kampen theorem

The following theorem can be found in [7]. It may also be viewed in the context of Bass-Serre theory.

Theorem 1.1.1. *Suppose that X is a CW complex with subcomplexes A and B so that $X = A \cup B$. Suppose further that X, A, B and $C := A \cap B$ are connected. By picking a point in the intersection C these spaces may be pointed. Then the fundamental group is an amalgamation of the fundamental groups of the spaces A, B and C:*

$$\pi_1(X) \cong \pi_1(A) *_{\pi_1(C)} \pi_1(B). \tag{1.1.1.1}$$

Now suppose further that A, B and C are classifying spaces for $\pi_1(A), \pi_1(B)$ and $\pi(C)$ respectively. Suppose also that both $\pi_1(C) \to \pi_1(A)$ and $\pi_1(C) \to \pi_1(B)$ are injective. Then X is a classifying space for $\pi_1(X)$.

The theorem allows us to construct classifying spaces for amalgamations.

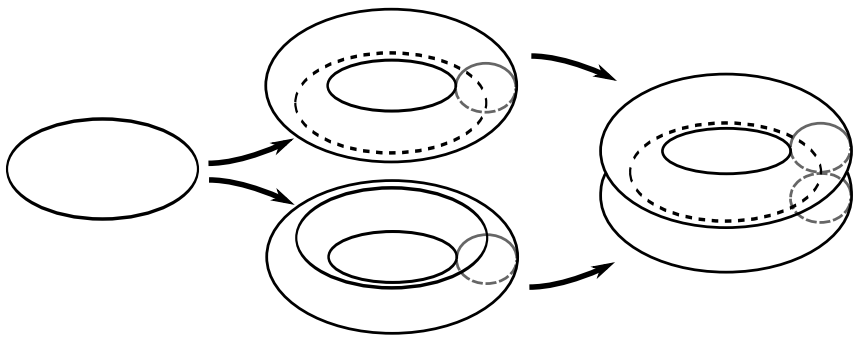
Example 1.1.1.1. Let G be the group with presentation

$$\langle a, b, c \mid [a, b], [b, c] \rangle . \tag{1.1.1.2}$$

The subgroups $\langle a, b\rangle$ and $\langle b, c\rangle$ are both isomorphic to $\mathbb{Z}^2$. Their intersection is $\langle b\rangle \cong \mathbb{Z}$ and G is equal to the amalgamation

$$\langle a, b\rangle *_{\langle b=b'\rangle} \langle b', c\rangle . \tag{1.1.1.3}$$

Let X be the space given by glueing two tori A, B together along radial circles.



(1.1.1.4)

Then applying the theorem shows that X is a classifying space for G. We may use this to calculate the homology of G with coefficients in any ring R.

$$H_i(G, R) = H_i(X, R) = \begin{cases} 0 & \text{if } i \geq 3, \\ R^2 & \text{if } i = 2, \\ R^3 & \text{if } i = 1 \text{ and} \\ R & \text{if } i = 0. \end{cases} \tag{1.1.1.5}$$

1.1.2 Colimits of CW complexes

One of the reasons for restricting ourselves to CW complexes, besides being a convenient setup for the Seifert-van Kampen theorem, is that a CW complex may be reconstructed from a covering of CW subcomplexes in the following sense.

Let X be a CW complex and let $\{X_i \mid i = 1, \ldots, n\}$ be CW subcomplexes of X. Then if each point $p \in X$ is in some X_j we say that $\{X_i\}$ is a *CW cover* of X. The intersection of two CW subcomplexes $X_i \cap X_j$ is still a CW subcomplex and hence so is $\bigcap_{i\in A} X_i$ for

each $A \subseteq [n]$. For each CW cover $\{X_i\}$ of a CW complex X there is a diagram D in CW complexes which has

- objects $\bigcap_{i \in A} X_i$ for each $A \subseteq [n]$ and
- morphisms the natural inclusions $\bigcap_{j \in B} X_j \hookrightarrow \bigcap_{i \in A} X_i$ for each $A \subseteq B \subseteq [n]$.

Proposition 1.1.2. *Let X be a CW complex and $\{X_i \mid i = 1, \ldots, n\}$ be a CW cover. Let D be the natural diagram associated to $\{X_i\}$. Then the colimit of D is isomorphic to X.*

See [25], Section 4.$\mathbb{G}$ for a discussion of this result.

1.1.3 Homotopy quotients

Let G be a group acting on a space Y. The homotopy quotient can be calculated as follows:

- take a contractible space E on which G acts properly,
- now consider the product $E \times Y$ on which G acts diagonally,
- we may take the quotient by the group action

$$E \times_G Y = \frac{E \times Y}{(e, y) \sim (g.e, g.y) \mid g \in G} \tag{1.1.3.1}$$

  and since G acts properly on $E \times Y$ this is the *homotopy quotient*.

We now treat the analogue for homology. Let M be a differentially graded (d.g.) RG-module, and suppose also that M is projective over R. The homotopy quotient, or homology of G with coefficients in M, is calculated as follows:

- take an RG-projective resolution of the trivially RG-module R

$$\ldots \to P_2 \to P_1 \to P_0 \to R \to 0. \tag{1.1.3.2}$$

  An example is given by $C_*(E)$, where E is as above.

- Take the tensor product $P_\bullet \otimes M$ on which RG acts diagonally,
- now take the quotient $(P_\bullet \otimes M)^G$.

The connection between the homotopy quotient of a G-space and the homological quotient is given by

$$H_*(E \times_G Y, R) \cong H_*\big((C_*(E) \otimes C_*(Y))^G, R\big). \tag{1.1.3.3}$$

So the homology of $E \times_G Y$ is the homology of $(P_\bullet \otimes M)^G$ when $M = C_*(Y)$ and $P_\bullet = C_*(E)$.

Finally we recall that if Y is a $K(H,1)$ with an action of G, then $E \times_G Y$ is a $K(H \rtimes G, 1)$. This means that the homology of $H \rtimes G$ with coefficients in R is the homology of the G-module $C_*(Y)$

$$H_*(G, C_*(Y)). \tag{1.1.3.4}$$

A note of warning, the homology of H, which is given by $H_*(C_*(Y))$ is a G-module, but there is not necessarily a G-module map to $C_*(Y)$. This means that the homology $H_*(G, H_*(H))$ does not necessarily compute the homology of $H \rtimes G$. It is the first page of a spectral sequence which computes $H \rtimes G$, however we will not need to use spectral sequences if we are careful about taking homology.

1.2 Graph products and right-angled Artin groups

The graph product of groups assigns to a graph $\Gamma = (V, E)$ and a V-labelled set of groups $\mathbf{G}$ the graph product $\mathbf{G}\Gamma$. When the graph has no edges the product is the free product of groups and when the graph is complete it is the direct product. So a graph product may be seen as an interpolation between the free and direct products. When each group is infinite cyclic the product $\mathbb{Z}\Gamma$ is called a right-angled Artin group (RAAG) and is often denoted A_Γ. The name 'right-angled Artin group' comes from the fact that A_Γ can be given as the Artin group of a Coxeter matrix (m_{ij}) where each m_{ij} is either 2 or ∞.

The RAAGs were studied before the graph products, an early reference is [3]. Much has been written since, for a summary with a long list of references see [10]. The general graph products were studied in the thesis [22], where a normal form for group elements was given.

1.2.1 Definitions

A *graph* $\Gamma = (V, E)$ consists of a set of *vertices* V and a set of *edges* E consisting of 2-elements subsets of V. If $e = \{v_1, v_2\} \in E$ then we say that v_1 and v_2 are the *ends* of the edge e.

Important examples are the *discrete graph* D_V on a vertex set V, where E is empty and the *complete graph* K_V on a vertex set V, where E is the set of all 2-elements subsets of V.

Definition 1.2.1. Let $\Gamma = (V, E)$ be a graph and let $\mathbf{G}$ be a V-tuple of groups, that is a V-indexed set of groups $\{G_v\}_{v \in V}$. Then the *graph product of* $\mathbf{G}$ *indexed by* Γ is the group

$$\mathbf{G}\Gamma := \langle G_v \mid R \rangle\,, \tag{1.2.1.1}$$

where v ranges over V and R is the set of relations

$$R := \bigcup_{e \in E} \big\{ [g,h] \mid g \in G_{v_1} \text{ and } h \in G_{v_2}, \text{ where } e = \{v_1, v_2\} \big\} \tag{1.2.1.2}$$

We will refer to the groups G_v as *vertex groups*. So a graph product is a group generated by the vertex groups, the additional relations assert that two elements of distinct vertex groups commute when the vertices are joined by an edge.

Example 1.2.1.1. Let D_V be the discrete graph on a vertex set V and let $\mathbf{G}$ be a V-tuple of groups. Then $\mathbf{G}D_V$ is the group generated by the vertex groups with no additional relations (because D_V has no edges). Hence $\mathbf{G}D_V$ is the free product of the vertex groups.

Example 1.2.1.2. Let K_V be the complete graph on a vertex set V and let $\mathbf{G}$ be a V-tuple of groups. Then $\mathbf{G}K_V$ is the group generated by the vertex groups where elements from distinct vertex groups commute (because K_V contains every possible edge). Hence $\mathbf{G}K_V$ is direct product of the vertex groups.

Example 1.2.1.3. Let Γ be the graph with $V = \{1,2,3,4\}$ and $E = \{12, 23, 34, 14\}$, where we write vw as a shorthand for the set $\{v, w\}$. This may be drawn as

Let $\mathbf{G} = (G_1, G_2, G_3, G_4)$ be a 4-tuple of groups. Then the elements G_1 in $\mathbf{G}\Gamma$ commute with those of G_2 and G_4, but not with the non-trivial elements of G_3. In fact we may check that $\mathbf{G}\Gamma$ is isomorphic to

$$(G_1 * G_3) \times (G_2 * G_4), \tag{1.2.1.3}$$

the direct product of the free products of opposite vertex groups.

It is important to note that not every graph product may be written as a combination of free and direct products. The smallest counterexample is given by:

Example 1.2.1.4. Let $V = \{1,2,3,4\}$ and let $E = \{12, 23, 13, 14\}$ and $E' = \{12, 23, 14\}$, which define two graphs drawn as

It is a fun and helpful exercise to check that the graph products indexed by these graphs can not be given by iterated free and direct products. It is also worth checking that these are the only such graphs with 4 vertices and that there are no such graphs with less than 4 vertices.

1.2.2 Right-angled Artin groups

Let $\Gamma = (V, E)$ be a graph and let $\mathbf{G}$ be the V-tuple with each $G_v = \mathbb{Z}$. Then $\mathbf{G}\Gamma$ is the *right-angled Artin group* (or RAAG) indexed by Γ. It is often denoted A_Γ. The presentation of A_Γ is relatively simple:

$$A_\Gamma := \langle g_v \text{ for } v \in V \mid [g_{v_1}, g_{v_2}] \text{ with } v_1, v_2 \in E \rangle . \tag{1.2.2.1}$$

Remark 1.2.2. A morphism of graphs is defined by a map f on the set of vertices for which each edge vw is either taken to an edge $f(v)f(w)$ or $f(v) = f(w)$. With this definition the RAAG construction becomes a functor $A_\bullet$ from the category of graphs to the category of groups. This functor is a left adjoint, where the right adjoint $\Gamma_\bullet$ is the following functor: for a group G let Γ_G be the graph with vertex set G and with edges the pairs gh for which g commutes with h. The fact that these functors form an adjoint pair means that for each graph H and each group G the two following homsets are naturally isomorphic:

$$\mathrm{Hom}_{\text{groups}}(A_H, G) \cong \mathrm{Hom}_{\text{graphs}}(H, \Gamma_G). \tag{1.2.2.2}$$

1.2.3 A topological approach

There is a similar product of pointed spaces. Rather than use graphs as indexing objects we use simplicial complexes. A *simplicial complex* $X = (V, S)$ consists of a set of *vertices* V and a set of *simplices* S consisting of (finite) subsets of V. The set S must satisfy another condition: if $A \in S$ and B is a subset of A, then $B \in S$.

An example is the simplicial complex $\Delta_V = (V, P_f V)$ consisting of the finite subsets of V.

Example 1.2.3.1. Let $\Gamma = (V, E)$ be a graph, then two natural constructions of simplicial complexes are given by:

- let $X_\Gamma = (V, V \cup E)$ be the simplicial complex consisting of the vertices and edges of Γ.

- let $\mathrm{Fl}_\Gamma = (V, S)$ be the simplicial complex where S consists of the subsets $A \subseteq V$ such that each 2-element subset of A is in E. Alternatively S is the set of subsets A such that the subgraph of Γ spanned by A is the complete graph K_A. The simplicial complex Fl_Γ is often called the *flag complex* of Γ.

One may also go the other way and construct a graph from any simplicial complex $X = (V, S)$. Let $W = \{v \in V \mid \{v\} \in S\}$ and let $E = S \cap P_2 W$ consist of the 2-element subsets in S. Then $\Gamma_X = (W, E)$ is the *1-skeleton graph* of X.

Remark 1.2.3. The sets of graphs and of simplicial complexes actually form categories. The morphisms are the maps on the underlying vertex sets which induce maps on the edge and simplex sets respectively. In the example above we are actually describing two functors X and Fl from the category of graphs to the category of simplicial complexes and we describe one functor Γ going the other way. In fact Γ is left adjoint to Fl and right adjoint to X.

Recall that a pointed space $(P, *)$ is a topological space P equipped with a chosen point $* \in P$.

Definition 1.2.4. Let $X = (V, S)$ be a simplicial complex and let $\mathbf{P}$ be a V-tuple of pointed spaces $\{(P_v, *_v)\}_{v \in V}$. If $p = (p_v)_{v \in V} \in \prod_{v \in V} P_v$ then the *support of* p is the set

$$\operatorname{supp}(p) := \{v \mid p_v \neq *_v\}. \tag{1.2.3.1}$$

The *simplicial complex product* of $\mathbf{P}$ indexed by X is the subset

$$\{p \mid \operatorname{supp}(p) \in S\} \subseteq \prod_{v \in V} P_v \tag{1.2.3.2}$$

and is denoted $\mathbf{P}X$. It is the set of points with support in S.

Example 1.2.3.2. Let V be a set and Δ_V be the simplicial complex defined above and let $\mathbf{P}$ be a V-tuple of pointed spaces. Then $\mathbf{P}\Delta_V$ is the subspace of the direct product $\prod_{v \in V} P_v$ consisting of the elements with finite support.

Example 1.2.3.3. Recall that we constructed a space by gluing together two tori in Example 1.1.1.1. This may be seen to be a simplicial complex product. Let $X = (V, S)$ be the simplicial complex with $V = \{1, 2, 3\}$ and $S = \{1, 2, 3, 12, 23\}$ which may be visualised as

$$1 \text{ —— } 2 \text{ —— } 3 \tag{1.2.3.3}$$

Then the space from (1.1.1.4) is isomorphic to the simplicial complex product $S^1 X$.

The simplicial complex product and graph product are closely linked. This is demonstrated by

Theorem 1.2.5. *Let $X = (V, S)$ be a simplicial complex and let $\mathbf{P}$ be a V-tuple of pointed CW complexes. Let $\mathbf{G} = \{\pi_1(P_v)\}$ be the V-tuple of their fundamental groups. Then the fundamental group of $\mathbf{P}X$ is isomorphic to the graph product $\mathbf{G}\Gamma_X$.*

This theorem follows from the following two lemmas.

Lemma 1.2.6. *Let $X = (V, S)$ be a simplicial complex and let $\mathbf{P}$ be a V-tuple of pointed CW complexes. Then $\mathbf{P}X$ is the colimit of the diagram which*

- *has objects the spaces* $\mathbf{P}_A := \prod_{v \in A} P_v$ *for* $A \in S$*, and*
- *has morphisms the inclusions*

$$\phi : \mathbf{P}_B \hookrightarrow \mathbf{P}_A \tag{1.2.3.4}$$

for each $B \subseteq A \in S$*, where* $\phi(p)_v = p_v$ *if* $v \in B$ *and* $\phi(p)_w = *_w$ *if* $w \notin B$*.*

Proof. Let $A \in S$, then the image of

$$\mathbf{P}_A = \prod_{v \in A} P_v \hookrightarrow \prod_{v \in V} P_v \tag{1.2.3.5}$$

consists of all the elements which have support in either A or one of its subsets. The simplicial complex product $\mathbf{P}X$ was defined as a subspace of the direct product $\prod_{v \in V} P_v$. It consisted of the points p with support set contained in S. Hence all of the maps of the form (1.2.3.5) form a cover of $\mathbf{P}X$. Since each P_i is a CW complex and so each $\mathbf{P}_A$ is a CW subcomplex we have that $\{\mathbf{P}_A \mid A \in P_f V\}$ is a CW cover of $\mathbf{P}X$ and so by Proposition 1.1.2 we are done. □

Lemma 1.2.7. *Let* $\Gamma = (V, E)$ *be a graph and let* $\mathbf{G}$ *be a* V*-tuple of groups. Then* $\mathbf{G}\Gamma$ *is the colimit of the diagram which*

- *has objects the groups* G_v *for each* $v \in V$ *and* $G_e := G_{v_1} \times G_{v_2}$ *for each* $e = \{v_1, v_2\} \in E$*, and*
- *has morphisms the inclusions*

$$\phi : G_v \hookrightarrow G_e \tag{1.2.3.6}$$

for each $v \in e = \{v, w\}$*, where* $\phi(g) = (g, e)$*.*

Proof. We proceed by induction on the number of edges e. When $e = 0$, the diagram is a set of objects $\{G_v \mid v \in V\}$ and so the colimit is the free product of the vertex groups as desired.

Now suppose the lemma holds for any graph with $e = m - 1$ edges and let $\Gamma = (V, E)$ be a graph with m edges. Choose an edge $\{v, w\} \in E$ and let Γ' be the graph with edge set $E - \{v, w\}$. Since Γ' has $m - 1$ edges we know by the assumption that $\mathbf{G}\Gamma' \cong \operatorname{colim} D'$ for the associated diagram of groups D'. The diagram D associated to Γ consists of D' along with the object $G_v \times G_w$ and morphisms $G_v \to G_v \times G_w \leftarrow G_w$. So the colimit of D is the colimit of

$$\begin{tikzcd} \operatorname{colim} D' & G_v \times G_w \\ G_v \arrow[u] \arrow[ur] & G_w. \arrow[u] \arrow[ul] \end{tikzcd} \tag{1.2.3.7}$$

But the colimit of this is

$$\frac{\mathbf{G}\Gamma'}{\langle [g, h] \mid g \in G_v, h \in G_w \rangle} \tag{1.2.3.8}$$

which is isomorphic to $\mathbf{G}\Gamma$. Thus by induction we are done. $\square$

The proof of the theorem then follows

Proof of Theorem 1.2.5. Lemma 1.2.6 realises $\mathbf{P}X$ as a colimit of a diagram $D^{\mathbf{P}}$ of pointed spaces. Meanwhile Lemma 1.2.7 realises $\mathbf{G}\Gamma_X$ as the colimit of a diagram $D^{\mathbf{G}}_{\Gamma}$. The fundamental group functor preserves colimits, so the fundamental group of $\mathbf{P}X$ is the colimit of $\pi_1(D^{\mathbf{P}})$ which we will denote $D^{\mathbf{G}}$. So we have reduced the Theorem to proving that the colimits of $D^{\mathbf{G}}_{\Gamma}$ and of $D^{\mathbf{G}}$ are isomorphic.

Comparing the two diagrams we find that $D^{\mathbf{G}}_{\Gamma}$ is the full subdiagram of $D^{\mathbf{G}}$ consisting of the binary products and the vertex groups themselves. For an object not in the subdiagram, $\mathbf{G}_A = \prod_{v\in A} G_v$ for $A \subseteq [n]$ with $|A| \geq 3$, there are natural maps $G_v \times G_w \to G_A$ for each pair $\{v, w\} \subset A$. And inside the group $\mathbf{G}\Gamma$ these groups $G_v \times G_w$ generate a subgroup isomorphic to $\mathbf{G}_A$. Hence there is a map from $\mathbf{G}_A$ to the colimit of $D^{\mathbf{G}}_{\Gamma}$ for every subset A. And so a unique extension from the colimit of $D^{\mathbf{G}}$ to that of $D^{\mathbf{G}}_{\Gamma}$. This implies that the two colimits are isomorphic. $\square$

1.2.4 Flag complexes and classifying spaces

We have seen that the fundamental group of a simplicial complex product of pointed spaces is equal to the graph product of the fundamental groups of the pointed spaces, where the graph is obtained from the simplicial complex by taking the 1-skeleton. A related problem is to go the other way. Given a graph product of groups, can one find a simplicial complex and pointed spaces such that the simplicial complex product is a classifying space for the graph product? The answer is that we can.

Recall that a flag complex is a simplicial complex which may be constructed from its 1-skeleton graph, in fact it is the maximal simplicial complex for a fixed graph.

Theorem 1.2.8. *Let $\Gamma = (V, E)$ be a finite graph, let $X = Fl_{\Gamma}$ be its corresponding flag complex and let $\mathbf{P}$ be a V-tuple of classifying spaces for $\mathbf{G}$ a V-tuple of groups. Then the simplicial complex product $\mathbf{P}X$ is a classifying space for $\mathbf{G}\Gamma$.*

This will be proved shortly. First we introduce some terminology.

Links and stars

Definition 1.2.9. Let $X = (V, S)$ be a simplicial complex and v be a vertex in V. The *star at v* is defined to be

$$\{A \in S \mid A \cup \{v\} \in S\}. \tag{1.2.4.1}$$

Then the *link at v* is defined to be the simplicial complex

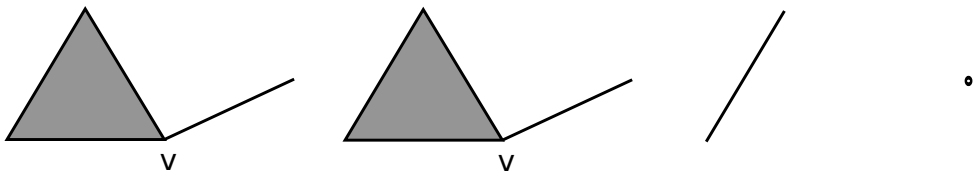
$$\{A \in S \mid v \notin A \text{ and } A \cup \{v\} \in S\}. \tag{1.2.4.2}$$

The *cone of* X is defined to be the simplicial complex $C(X) = (V \cup \{0\}, S')$ where

$$S' = S \cup \{A \cup \{0\} \mid A \in S\}. \tag{1.2.4.3}$$

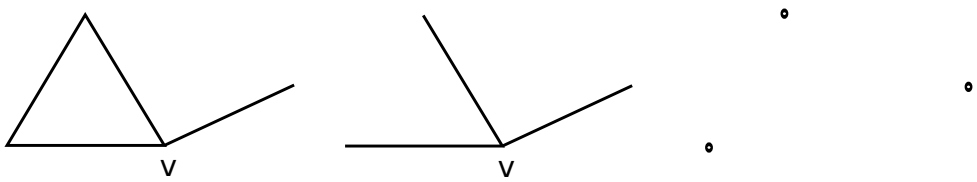
It is immediate from the definition that $\mathrm{st}(v)$ is the cone of $\mathrm{lk}(v)$ where v takes the role of the point 0.

Example 1.2.4.1. Consider the unique simplicial complex with 4 vertices, 4 edges and a single triangle (or 2-simplex). We draw below the simplicial complex, followed by the star at v, followed by the link at v.



(1.2.4.4)

The next example is the same but with the triangle missing. We again show the simplicial complex, followed by the star at v, followed by the link at v.



(1.2.4.5)

Lemma 1.2.10. *Let X be a flag complex and v be a vertex in V. Then both the link and star at v are full subcomplexes of X and hence themselves flag complexes.*

Proof. Let A be a simplex in X such that each $w \in A$ is contained in $\mathrm{lk}(v)$. To prove that $\mathrm{lk}(v)$ is a full subcomplex of X we need only prove that A is also a simplex of $\mathrm{lk}(v)$.

By the definition of the link at v we have that for each $w \in \mathrm{lk}(v)$ the pair $\{v, w\}$ is contained in X. But then $A \cup \{v\}$ spans a complete graph in the 1-skeleton of X, hence by the flag condition $A \cup \{v\}$ is a simplex of X. Therefore A is contained in the link at v, which is what we needed to prove.

The argument for $\mathrm{st}(v)$ is identical. The fact that a full subcomplex of a flag complex is itself a flag complex is immediate from the definition. □

Proof of Theorem 1.2.8. We proceed by induction on the number of vertices n of Γ. When $n = 1$ the theorem is trivially true. Now assume that it is true for $m < n$ and let Γ be a graph with n vertices. Let v be a vertex of Γ. Define $X_{\overline{v}}$ to be the simplicial complex consisting of the simplices of X which do not contain v. This is the flag complex corresponding to the graph $\Gamma_{\overline{v}}$ obtained by removing the vertex v and all adjacent edges $\{v, w\}$. Of course $\Gamma_{\overline{v}}$ has $n - 1$ vertices and so by the assumption the space $\mathbf{P}X_{\overline{v}}$ is a

classifying space for $\mathbf{G}\Gamma_{\overline{v}}$. Using Lemma 1.2.10 we may also apply the assumption to the link at v denoted $\mathrm{lk}(v)$ which also does not contain v so has $n-1$ or fewer vertices: we get that $\mathbf{P}\,\mathrm{lk}\,x$ is a classifying space for the group $\mathbf{G}\Gamma_{\mathrm{lk}(v)}$ where $\Gamma_{\mathrm{lk}(v)}$ is the subgraph of Γ spanned by vertices which are joined by an edge to v.

The star at v denoted $\mathrm{st}(v)$ may have n vertices so we can not apply the induction hypothesis. We may however note that since the star is the cone of the link we have that

$$\mathbf{P}\,\mathrm{st}(v) \cong \mathbf{P}\,\mathrm{lk}(v) \times P_v. \tag{1.2.4.6}$$

Hence $\mathbf{P}\,\mathrm{st}(v)$ is also a classifying space of $\mathbf{G}\,\mathrm{st}(v)$.

Both $\mathbf{P}\,\mathrm{st}(v)$ and $\mathbf{P}X_{\overline{v}}$ are CW subspaces of $\mathbf{P}X$ and in fact they cover $\mathbf{P}X$ and have intersection $\mathbf{P}\,\mathrm{lk}(v)$, so we may apply Proposition 1.1.2 to get that $\mathbf{P}X$ is the colimit of the diagram

$$\mathbf{P}X_{\overline{v}} \longleftarrow \mathbf{P}\,\mathrm{lk}(v) \longrightarrow \mathbf{P}\,\mathrm{st}(v) \tag{1.2.4.7}$$

and so on fundamental groups we have that $\mathbf{G}\Gamma$ is the amalgamation of

$$\mathbf{G}\Gamma_{\overline{v}} \longleftarrow \mathbf{G}\Gamma_{\mathrm{lk}(v)} \longrightarrow \mathbf{G}\Gamma_{\mathrm{lk}(v)} \times G_v, \tag{1.2.4.8}$$

using (1.2.4.6) to decompose $\pi_1(\mathbf{P}\,\mathrm{st}(v))$ as a direct product. The right-hand arrow is clearly injective, to see that the left-hand arrow is also injective we may just note that there is a one-sided inverse given by sending any vertex group element corresponding to a vertex not in $\Gamma_{\mathrm{lk}(v)}$ to the identity. We have now satisfied all of the conditions of Theorem 1.1.1 and so $\mathbf{P}X$ is a classifying space for $\mathbf{G}\Gamma$ and we are done by induction. □

Now that we can construct classifying spaces for graph products of groups we can compute invariants such as the homology of the groups.

Corollary 1.2.11. *Let Γ be a finite graph and $\mathbf{G}$ be a V-tuple of groups. Then for any commutative ring R*

$$H_*(\mathbf{G}\Gamma, R) = R \oplus \bigoplus_K H_*\Big(\bigotimes_{v\in K} \widehat{C}_*(G_v, R)\Big), \tag{1.2.4.9}$$

where K ranges over the complete subgraphs of Γ and $\widehat{C}_*(G_v, R)$ is an R-projective complex computing the reduced homology of G_v.

The (co)homology of the classifying space $\mathbf{P}\mathrm{Fl}_\Gamma$ can be calculated using Theorem 2.3.2. Since the proof for simplicial complexes does not significantly simplify from the general case of diagonal complexes we will not rewrite it here.

1.2.5 Summary

The following diagram summarises this introductory section. This is analagous to our approach to diagonal complexes in the next chapter, so it will be very helpful if the reader understands the arguments presented in Section 1.2.

$$\begin{tikzcd}
 & \text{Subspace of } \prod_v Y_v \arrow[d] & \\
 & \text{Covering by product spaces} \arrow[d] & \\
 & \text{Colimit diagram} \arrow[dl] \arrow[d] \arrow[dr, "X \text{ a flag complex}"] & \\
\pi_1 & H_* \text{ and } H^* & \text{decomposition into smaller pieces} \arrow[d, "Y_v \text{ aspherical}"] \\
 & & \text{Proof of asphericity.}
\end{tikzcd} \tag{1.2.5.1}$$

Chapter 2

Diagonal complexes

In the introductory chapter we studied the graph product of groups and set up a theory to compute their homology. We now proceed to a more general class of products of groups. Whereas graphs describe RAAGs, *diagonal complexes* describe 'diagonal right-angled Artin groups', or DRAAGs. The draw of this new class of groups is present for a number of reasons:

1. groups that we already know well are included in their number,
2. most of the structures used to study RAAGs carry over in a simple manner,
3. the potential structures of the groups are more intricate than the RAAGs and the methods used, such as CAT(0) geometry, do not carry over to the DRAAGs.

I hope that the DRAAGs provide fellow group theorists with interesting new examples to explore. In particular we uncover examples of groups with simple, naturally defined classifying spaces built out of tori, but which do not appear to have naturally defined non-positively curved metrics.

However the original motivation for defining diagonal complexes came from a particular example. We wished to provide a solid combinatorial framework to

1. describe moduli spaces of cactus products (defined in Chapter 3) and then to
2. prove the asphericity of the moduli spaces and to compute their homology.

It soon became clear that the notion arrived at was a good generalization of the graph product of groups and that the theory encompassed a wide variety of groups sharing some properties of graph products, but not all.

Since many interesting spaces (including the moduli spaces) can be embedded into product spaces as unions of diagonal maps we will first describe the poset of partial partitions under partial coarsening. In Theorem 2.1.2 we show that every union of diagonal maps in a product space is uniquely described as a subposet closed from beneath. However

this notion is too general for our purposes; the fundamental groups are tricky to describe and their homological properties hard to compute.

The choice of definition of diagonal complexes restricts the spaces we are able to describe, on the other hand the spaces we get are more pleasant to work with. Also the posets involved are subsets of a power set, rather than subsets of the set of all partial partitions. As such we need only work with 'relatively small' combinatorial structures.

2.1 Definitions

2.1.1 Diagonal maps

Let Y be a set, topological space or group. In each case there is a diagonal map[1]

$$\begin{aligned} \Delta_Y : Y &\to Y \times Y \\ y &\mapsto (y, y) \end{aligned} \tag{2.1.1.1}$$

from Y to the direct product $Y \times Y$. We will denote by Δ_Y^{n-1} the map from Y to the n-fold direct product $Y^{\times n}$,

$$\Delta_Y^{n-1}(y) = (y, \ldots, y). \tag{2.1.1.2}$$

In particular when $n = 1$ then Δ_Y^0 is the identity on Y.

Now suppose that Y is pointed, that is there is a chosen map

$$p : \{1\} \mapsto Y. \tag{2.1.1.3}$$

In the case where Y is a group there is one choice of map which is the unique map from the trivial group.

We are interested in the natural maps which can be built between $Y^{\times k}$ and $Y^{\times n}$ from the diagonal map Δ_Y and the point map p. Since both generating maps increase the number of copies of Y by 1 we require that $k \leq n$. Such maps can be written as a product

$$\Delta_Y^{n_1-1} \times \Delta_Y^{n_2-1} \times \ldots \times \Delta_Y^{n_k-1} \times p^m, \qquad \text{where } n_1 + \ldots + n_k + m = n, \tag{2.1.1.4}$$

composed with a permutation $\sigma \in \mathfrak{S}_n$ which permutes the factors of Y^n. These maps can be denoted by an ordered partition $(U_1, \ldots, U_k, B)$ of $[n]$. To see this apply the permutation $\sigma \in \mathfrak{S}_n$ to the partition

$$\big(\{1, \ldots, n_1\}, \{n_1{+}1, \ldots, n_1{+}n_2\}, \ldots, \{n_1{+}\ldots{+}n_{k-1}{+}1, \ldots, n_1{+}\ldots{+}n_k\}, \\ \{n{-}m{+}1, \ldots, n\}\big) \tag{2.1.1.5}$$

[1] In fact in any category with binary products there exist such diagonal maps.

Conversely given an ordered partition $(U_1, \ldots, U_k, B)$ choose a permutation $\sigma \in \mathfrak{S}_n$ which sends it to the form (2.1.1.5) where $n_i = |U_i|$. The corresponding map is then (2.1.1.4) followed by σ^{-1} permuting the factors of Y^n. Since Δ_Y is symmetric the choice of σ does not affect the given map. We will denote the map by

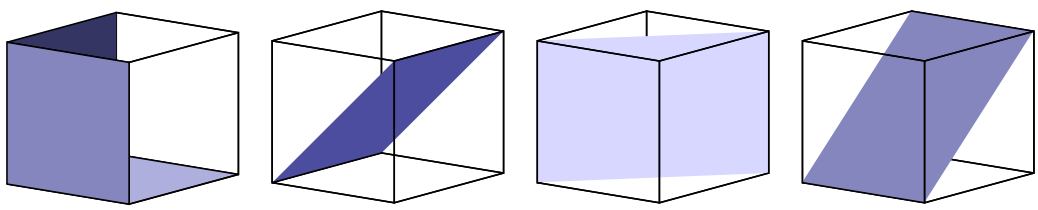
$$D_Y^{(U_1,\ldots,U_k)} \tag{2.1.1.6}$$

In practice we use partial partitions $\{U_1, \ldots, U_k\}$ of $[n]$ because B can be recovered by taking the complement of the union of the U_i's in $[n]$. We choose to forget the ordering because although the ordered $(U_1, \ldots, U_k)$ determines the map $Y^{\times k} \to Y^{\times n}$ we are interested in the image of the maps which depends only on the unordered partial partitions. We call the image of such a map a *diagonal subspace* (or diagonal subset, or diagonal subgroup depending on what type of object we are interested in). The number k is referred to as the *rank* of the diagonal subset.

We may illustrate this with an example; the following is a list of all six partial partitions of $[3] = \{1, 2, 3\}$ consisting of two subsets, along with a choice of image of (x, y) in $Y^{\times 3}$.

$$\begin{array}{rl} 1 \mid 2 & (x, y, 1) \\ 1 \mid 3 & (x, 1, y) \\ 2 \mid 3 & (1, x, y) \\ 12 \mid 3 & (x, x, y) \\ 13 \mid 2 & (x, y, x) \\ 23 \mid 1 & (y, x, x) \end{array} \tag{2.1.1.7}$$

So these six partial partitions correspond to the six copies of $Y^{\times 2}$ inside $Y^{\times 3}$ naturally given by diagonal and point maps or in other words, the diagonal subspaces of rank 2. These are pictured for $Y = I$ the unit interval below. The first three 'orthogonal' subspaces are pictured in the first cube, the remaining three are pictured in the other cubes in no particular order.



(2.1.1.8)

2.1.2 The poset of partial partitions

We saw above that the diagonal subspaces of a product Y^n were given by partial partitions of $[n]$. The aim is to construct a theory to study these subspaces and their unions. The

union of two diagonal subspaces is not a diagonal subspace unless one is contained in the other, but their intersection is. We devote this section to understanding this meet-semilattice[2] and it turns out that we need not look further than the set of partial partitions and a partial order on this set.

Denote the set of all subsets of a set X by PX and the set of all finite subsets by P_fX. Recall that a *partition* of X is a set of subsets $\{U_i \subseteq X \mid i \in I\}$ which are pairwise disjoint and whose union is X. A *partial partition* of X is a partition of a subset of X. Alternatively a partial partition is a set $\{U_i\}$ of pairwise disjoint subsets. In this thesis all partial partitions will be partitions of finite subsets of X. Denote by PP_fX the set of partial partitions whose union is a finite set. We will refer to a subset $\{U_j \mid j \in J\}$ of a partial partition $\{U_i \mid i \in I\}$ for $J \subseteq I$ as a *subpartition of* $\{U_i\}$. Let

$$D_Y^{\{U_i\}} : Y^I \to Y^X \tag{2.1.2.1}$$

represent the diagonal subspace, then precomposing this with the natural inclusion

$$\mathrm{id}_Y^J \times p_Y^{I-J} : Y^J \to Y^I \tag{2.1.2.2}$$

one gets a representative of the diagonal subspace corresponding to $\{U_j \mid j \in J\}$. So being a subpartition implies the inclusion of diagonal subspaces, however the converse does not hold so we require a finer ordering on PP_fX to fully represent the diagonal subspaces.

Suppose that $\{U_i\}$ and $\{V_j\}$ are two elements of PP_fX. Then we say that $\{U_i\}$ is a *coarsening* of $\{V_j\}$ (or that $\{V_j\}$ is a refinement of $\{U_i\}$) if the unions are equal and if for each V_j there is a U_i containing V_j. If $\{U_i\}$ is a coarsening of a subpartition of $\{V_j\}$ then we say that $\{U_i\}$ is a *partial coarsening* of $\{V_j\}$ and write $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$.

Lemma 2.1.1. *The pair* $(PP_fX, \leq_{pc})$ *forms a poset.*

Proof. The reflexivity condition is clear, $\{U_i\}$ is a coarsening of $\{U_i\}$. Now for antisymmetry, suppose that both $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$ and $\{V_j\} \leq_{\mathrm{pc}} \{U_i\}$, then both their unions are contained in one another and so must be equal, hence they are coarsenings of each other. Now take some $U_i \in \{U_i\}$, since V_j is a coarsening of U_i there exists a $V_j \in \{V_j\}$ containing U_i, but $\{U_i\}$ is a coarsening of $\{V_j\}$ so there exists a U_k containing V_j and hence U_i. But the subsets in $\{U_i\}$ are pairwise disjoint, so $U_i = U_k$ and since V_j is sandwiched between them $V_j = U_i$. This is true for each U_i so $\{U_i\} = \{V_j\}$.

Finally transitivity, suppose that $\{U_i\} \leq_{\mathrm{pc}} \{V_j\} \leq_{\mathrm{pc}} \{W_k\}$ and pick a $W_k \in \{W_k\}$. Either W_k does not intersect $\{V_j\}$ in which case it does not intersect $\{U_i\}$ or there is some V_j containing it, suppose the latter. Now either V_j does not intersect $\{U_i\}$ in which case W_k does not intersect $\{U_i\}$ or there is some U_i containing V_j and hence also W_k. We are

[2] a meet-semilattice is a poset which has a meet (a greatest lower bound) for any non-empty finite subset.

left with the possibility either that W_k does not intersect $\{U_i\}$ or that there is some U_i containing W_k, hence $\{U_i\} \leq_{\mathrm{pc}} \{W_k\}$. $\square$

We will now examine the diagonal subspaces again. Let $\{U_1, \ldots, U_k\}$ be a partial partition of $[n]$ then we have a diagonal subspace the image of the map

$$D_Y^{\{U_i\}} : Y^k \to Y^n. \tag{2.1.2.3}$$

Recall that this may be given by

$$D_Y^{\{U_i\}}(y_1, \ldots, y_k) = (x_1, \ldots, x_n), \tag{2.1.2.4}$$

where

$$x_i = \begin{cases} y_j & \text{if } i \in U_j, \\ * & \text{if } i \notin U_j \text{ for each } j = 1, \ldots, k. \end{cases}. \tag{2.1.2.5}$$

The image may be characterised by

$$\left\{(y_i)_{i\in[n]} \mid y_i = y_j \text{ for } i, j \in U_i \text{ and } y_l = * \text{ for } l \notin \bigcup_i U_i\right\}. \tag{2.1.2.6}$$

Theorem 2.1.2. *Let Y be a pointed space containing some point $y \neq *$. Then the poset $(DS, \subseteq)$ consisting of the diagonal subspaces of $Y^{\times n}$ ordered by inclusion is isomorphic to the poset $(PP_f[n], \leq_{pc})$. Furthermore the intersection of two diagonal subspaces is itself a diagonal subspace and so meets are defined in $(PP_f[n], \leq_{pc})$.*

Proof. A diagonal subspace is the image of a direct sum of diagonal and point maps which as discussed in Section 2.1.1 is given by a partial partition. Showing that two partial partitions give different diagonal subspaces will tell us that the sets $PP_f[n]$ and DS are equal. We then show that these two sets are also isomorphic as posets. So let $\{U_i\}$ and $\{V_j\}$ be distinct partial partitions. If the unions are distinct then we may assume that $\bigcup_i U_i \not\subseteq \bigcup_j V_j$, then the element $(y_x)_{x\in X}$ with

$$y_x = \begin{cases} y & \text{if } x \in \bigcup_i U_i, \\ * & \text{otherwise} \end{cases} \tag{2.1.2.7}$$

is in $\mathrm{Im} D_Y^{\{U_i\}}$ but not in $\mathrm{Im} D_Y^{\{V_j\}}$, which can be seen by examining (2.1.2.6). Suppose now that the unions are equal. Since $\{U_i\}$ and $\{V_j\}$ are distinct we may find U_i and V_j such that $U_i \cap V_j \neq \emptyset$ and $U_i \neq V_j$. Suppose that neither is contained in the other then $(y_x)_{x\in X}$ with

$$y_x = \begin{cases} y & \text{if } x \in U_i, \\ * & \text{otherwise} \end{cases} \tag{2.1.2.8}$$

is in $\mathrm{Im}D_Y^{\{U_i\}}$ but not in $\mathrm{Im}D_Y^{\{V_j\}}$. We are left with the case that one is contained in the other, so assume that $U_i \subset V_j$, then the same element (2.1.2.8) serves to distinguish $\mathrm{Im}D_Y^{\{U_i\}}$ and $\mathrm{Im}D_Y^{\{V_j\}}$.

We have now shown that the set of diagonal subspaces is equal to the set of partial partitions. If $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$ then by (2.1.2.6) we have the containment $\mathrm{Im}D_Y^{\{U_i\}} \subseteq \mathrm{Im}D_Y^{\{V_j\}}$. Now for the converse, suppose that $\{U_i\}, \{V_j\} \in PP_f[n]$ and $\mathrm{Im}D_Y^{\{U_i\}} \subseteq \mathrm{Im}D_Y^{\{V_j\}}$, we need to show that $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$.

If the union $\bigcup_i U_i$ was not contained in $\bigcup_j V_j$ then the element defined in (2.1.2.7) would contradict the inclusion. Finally for any U_i consider the element of $\mathrm{Im}D_Y^{\{U_i\}}$ defined by (2.1.2.8). Since this element is in $\mathrm{Im}D_Y^{\{V_j\}}$, the set U_i must be a union of some of the $\{V_j\}$. We have shown the equality of the two posets.

For the second part of the theorem we will describe the intersection of two subspaces $\mathrm{Im}D_Y^{\{U_i\}}$ and $\mathrm{Im}D_Y^{\{V_j\}}$ and then remark that it is given by the meet of $\{U_i\}$ and $\{V_j\}$ in $(PP_f[n], \leq_{\mathrm{pc}})$. Let $(y_x)_{x\in X} \in \mathrm{Im}D_Y^{\{U_i\}} \cap \mathrm{Im}D_Y^{\{V_j\}}$, then if either $a, b \in U_i$ or $a, b \in V_j$, then $y_a = y_b$. So define an equivalence relation on $[n] \cup \{0\}$ as follows, write $a \sim b$ if either $a, b \in U_i$ or $a, b \in V_j$ for some U_i or V_j. Also write $a \sim 0$ if either $a \notin \bigcup_i U_i$ or $a \notin \bigcup_j V_j$. Extend $\sim$ to an equivalence relation by transitivity. If $a \sim b$ in $([n] \cup \{0\}, \sim)$ then $y_a = y_b$ in $(y_x)_{x\in X}$.

Define $\{W_k\}$ to be the equivalence classes of $\sim$ not containing 0. Then $(y_x)_{x\in X}$ is in $\mathrm{Im}D_Y^{\{W_k\}}$, also any element of $\mathrm{Im}D_Y^{\{W_k\}}$ is certainly in $\mathrm{Im}D_Y^{\{U_i\}}$ and $\mathrm{Im}D_Y^{\{V_j\}}$. The partial partition $\{W_k\}$ is the meet of $\{U_i\}$ and $\{V_j\}$ in $(PP_f[n], \leq_{\mathrm{pc}})$ so we have

$$\mathrm{Im}D_Y^{\{U_i\}} \cap \mathrm{Im}D_Y^{\{V_j\}} = \mathrm{Im}D_Y^{\{U_i\}\wedge\{V_j\}}. \tag{2.1.2.9}$$

□

We set out to describe the spaces which can be built as unions of diagonal subspaces of a product space. Using the previous theorem we have our answer, such spaces are described by the subposets of $(PP_f[n], \leq_{\mathrm{pc}})$ which are closed in the sense that if $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$ and $\{V_j\}$ is in the poset, then $\{U_i\}$ is also in the poset.

2.1.3 Diagonal complexes

Theorem 2.1.2 above described any space built as the union of diagonal subspaces. We now turn our attention to a smaller class of such unions, those given by diagonal complexes. The main body of this chapter is devoted to describing their homotopical invariants.

Let X be a set and write X^+ for $\left\{x^+ = \{x\} \mid x \in X\right\}$ the set of singleton subsets of X.

Definition 2.1.3. A *diagonal complex on* X consists of a pair (Γ, γ), with $\Gamma \subseteq P_f X$ and $\gamma : \Gamma \to P_f\Gamma$ such that

1. $X^+ \subseteq \Gamma$,

2. for each $U \in \Gamma$ the set $\gamma(U)$ is a partition of U. So the image of γ is contained in PP_fX. Furthermore if U is not a singleton then $\gamma(U)$ is a proper partition. That is, if $U \in \Gamma - X^+$ then $|\gamma(U)| > 1$,

3. (*simplicial condition*) for $U \in \Gamma$ we write $\gamma(U) = \{U_1, \ldots, U_k\}$. For each $A \subseteq [k]$ we require that $U_A \in \Gamma$, where
$$U_A := \bigcup_{i \in A} U_i. \tag{2.1.3.1}$$
We also require that $\gamma(U_A)$ is either $\{U_i \mid i \in A\}$ or a refinement of $\{U_i \mid i \in A\}$. We call the U_A the *faces* of U.

The *dimension* of $U \in \Gamma$ is defined to be $|\gamma(U)|$. Note that the dimension of a face of U may be greater than the dimension of U.

Consider the poset on Γ transitively generated by $V \leq U$ if V is a face of U. In this poset if $V \leq U$ we say that V is a *descendant of* U. For example a face of a face of U is a descendant, but not necessarily a face of U. If a diagonal complex (Γ, γ) satisfies

4. the descendance order agrees with the ordering by inclusion.

then we say that (Γ, γ) is *proper*.

Remark 2.1.4. Condition 1. in the above definition is not necessary for many of the results in this thesis. However asking for it gives the diagonal complex product of groups a minimal set of generators.

Condition 2. says that γ has image in the intersection of $P_f\Gamma$ and PP_fX, in fact it implies that γ is a setwise section of the map of posets $(PP_fX, \leq_{\mathrm{pc}}) \to (P_fX, \subseteq)$ given by taking the union of a partial partition.

A diagonal complex which is proper is a convenient object to work with because by the proposition below the inclusion ordering on Γ determines the map γ. With this in mind we will sometimes just write Γ for (Γ, γ) when it is proper.

Proposition 2.1.5. *Let (Γ, γ) be a diagonal complex on a set X. Then (Γ, γ) is proper if and only if for each $U \in \Gamma$*
$$\gamma(U) = \{U - M_1, \ldots, U - M_k\}, \tag{2.1.3.2}$$
where $M_1, \ldots, M_k$ are the set of maximal subsets under U in the inclusion order on Γ.

Proof. Let $U \in \Gamma$, then $\gamma(U) = \{U_1, \ldots, U_k\}$ is a partition of U. Recall that the unions of the U_i are called the faces and so the maximal faces are the unions
$$\bigcup_{i \neq j} U_i \tag{2.1.3.3}$$

for each j. But this is just $U - U_j$ and by the definition of the descendance order these are maximal under U in the descendance order.

Now suppose that (Γ, γ) is proper, then the inclusion order agrees with the descendance order and the sets $U - U_j$ are maximal subsets of U.

Conversely, if (2.1.3.2) holds then $U - U_j$ are both maximal in the descendents order and as subsets of U for every $U \in \Gamma$. Such a property of finite posets implies that the two orderings are equal. □

2.1.4 The geometric realisation

A diagonal complex (Γ, γ) over a set X may be encoded as a diagram called the *geometric realisation* consisting of a subset of the simplex Δ^X with vertex set X. In fact the realisation is a sub-simplicial complex of the barycentric subdivision of Δ^X. Recall that the barycentric subdivision of a simplex with vertex set X has vertex set $P_f X$, the set of non-empty finite subsets of X. Then a simplex in the subdivision is a set of subsets which form a total order under the partial order by inclusion:

$$A_1 \subset A_2 \subset \ldots \subset A_k. \tag{2.1.4.1}$$

Now suppose that $\{U_i\}$ is a partial partition of X. Then each U_i corresponds to a vertex of the barycentric subdivision. If Δ^X is embedded in $\mathbb{R}^X$ as the convex hull of the vectors $\{e_x \mid x \in X\}$ then the vertex corresponding to U_i is given by

$$e_{U_i} = \frac{1}{|U_i|} \sum_{x \in U_i} e_x \tag{2.1.4.2}$$

We may associate a subspace to $\{U_i\}$ by taking the convex hull of the e_{U_i}, call this simplex $S(\{U_i\})$. This is infact spanned by simplices in the barycentric subdivision of Δ^X, the vertices are all possible unions of the U_i's.

Definition 2.1.6. To (Γ, γ) associate a subset of Δ^X:

$$|(\Gamma, \gamma)| = \bigcup_{U \in \Gamma} S(\gamma(U)). \tag{2.1.4.3}$$

We call this the *geometric realisation of* (Γ, γ).

By the discussion above this realisation can also be viewed as a realisation of some sub-simplicial complex of the barycentric subdivision of Δ^X.

2.1.5 Examples

The following list of examples should help to build up some intuition for the kind of structures we can build.

Example 2.1.5.1. Let $A = (X, S)$ be a simplicial complex with vertex set X. Let $\Gamma = S$ and define $\gamma(U) = U^+ = \big\{x^+ = \{x\} \mid x \in U\big\}$. Then (Γ, γ) is a proper diagonal complex. It's geometric realisation is isomorphic to the geometric realisation of A.

Example 2.1.5.2. Let $X = [3]$, we define Γ to be the set $\big\{\{1,2,3\} = X, \{1,2\}\big\} \cup X^+$ and γ is defined by $\gamma(\{1,2,3\}) = \big\{\{1,2\}, 3^+\big\}$ and $\gamma(\{1,2\}) = \{1^+, 2^+\}$. We may represent (Γ, γ) by the geometric realisation:

$$\begin{array}{ccc} & 3 & \\ & | & \\ 1 & \text{---} & 2 \end{array} \tag{2.1.5.1}$$

The descendance order agrees with the inclusion order and so Γ is proper.

Example 2.1.5.3. An example of a diagonal complex which is not proper is given by adding the element $\{1,3\}$ onto the previous example. The picture we now get is

$$\begin{array}{ccc} & 3 & \\ / & | & \\ 1 & \text{---} & 2 \end{array} \tag{2.1.5.2}$$

Note that $\{1,3\}$ is a subset of $\{1,2,3\}$ but it is not a descendant and so this diagonal complex is not proper.

Example 2.1.5.4. The set X can be infinite, for example let $X = \{1, 2, \ldots\}$ be the strictly positive natural numbers. Define Γ to be

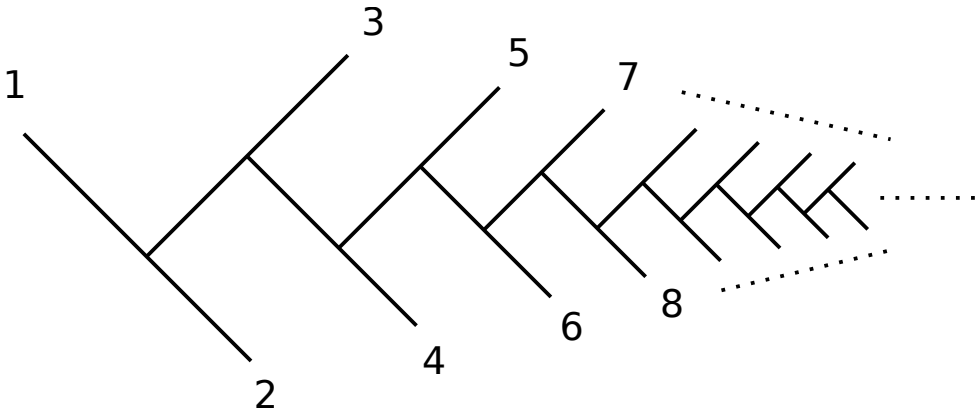
$$\big\{\{1, \ldots, n\} \mid n = 1, 2, \ldots\big\} \cup X^+ \tag{2.1.5.3}$$

and γ on the non-singleton sets by

$$\gamma\big(\{1, \ldots, n\}\big) = \big\{\{1, \ldots, n-1\}, \{n\}\big\}. \tag{2.1.5.4}$$

So the diagonal complex (Γ, γ) consists of subsets of unbounded cardinality, however the dimension of the subsets is bounded by two. It can be represented by the following

diagram



(2.1.5.5)

2.1.6 Morphisms of diagonal complexes

The morphisms between diagonal complexes are important because with a good definition of a morphism the diagonal complex product (to be defined in the next section) will be functorial and so morphisms of diagonal complexes will induce maps between diagonal complex products.

So let (Γ_1, γ_1) and (Γ_2, γ_2) be diagonal complexes over sets X_1 and X_2 respectively. Then a morphism

$$f : (\Gamma_1, \gamma_1) \to (\Gamma_2, \gamma_2) \tag{2.1.6.1}$$

is given by a map of sets also denoted f

$$f : X_1 \to \Gamma_2 \tag{2.1.6.2}$$

which may be extended to a function denoted F

$$F : \Gamma_1 \to \Gamma_2 \tag{2.1.6.3}$$

by taking unions. This must make the following diagram commute

$$\begin{tikzcd} \Gamma_1 \arrow[r, "F"] \arrow[d, "\gamma_1"] & \Gamma_2 \arrow[d, "\gamma_2"] \\ P_f\Gamma_1 \arrow[r, "P_f F"] & P_f\Gamma_2. \end{tikzcd} \tag{2.1.6.4}$$

The composition of morphisms $f : (\Gamma_1, \gamma_1) \to (\Gamma_2, \gamma_2)$ and $g : (\Gamma_2, \gamma_2) \to (\Gamma_3, \gamma_3)$ is given by the composition

$$\begin{tikzcd} X_1 \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \Gamma_2 \arrow[r, "G"] & \Gamma_3 \end{tikzcd} \tag{2.1.6.5}$$

Lemma 2.1.7. *The map from Γ_1 to Γ_3 induced from $g \circ f$ is precisely $G \circ F$ and this*

map satisfies the diagram (2.1.6.4).

It is now a simple task to check that we have defined a category of diagonal complexes.

Now suppose that $f : (\Gamma_1, \gamma_1) \to (\Gamma_2, \gamma_2)$ may be given by an inclusion of vertex sets like so

$$X_1 \hookrightarrow X_2 \cong X_2^+ \hookrightarrow \Gamma_2. \tag{2.1.6.6}$$

For such f's we say that (Γ_1, γ_1) is a *subcomplex of* (Γ_2, γ_2). Suppose now that $Y \subseteq X$ is a subset of X and that (Γ, γ) is defined over X. Let $\Gamma_Y = \Gamma \cap P_f Y$, then we call $(\Gamma_Y, \gamma|_{\Gamma_Y})$ *the full diagonal subcomplex on* Y.

Example 2.1.6.1. Let (Γ, γ) be the diagonal complex defined in Example 2.1.5.2 with diagram

$$\begin{array}{ccc} & 3 & \\ & | & \\ 1 & \text{---} & 2 \end{array} \tag{2.1.6.7}$$

Then up to isomorphism there are two injective maps from the line $\Gamma_L = \{\{s\}, \{t\}, \{s, t\}\}$. One sends s to $\{1\}$ and t to $\{2\}$ and realises Γ_L as the full diagonal subcomplex on $\{1, 2\}$. The other sends s to $\{3\}$ and t to $\{1, 2\}$.

Levels of a diagonal complex

Let (Γ, γ) be a diagonal complex on a set X. We will define the *level* $\mathrm{lev} : \Gamma \to \mathbb{N}$ inductively as follows:

- the level is zero if $U \in X^+$, otherwise
- the level is defined to be the maximum of the levels of the maximal subfaces of U, plus one:

$$\mathrm{lev}(U) = \sup \{\mathrm{lev}(U - V) \mid V \in \gamma(U)\} + 1. \tag{2.1.6.8}$$

The level is well-defined because for $U \in \Gamma - X^+$, the cardinalities of the elements of $\gamma(U)$ are strictly less than the cardinality of U. The level will be used in inductive arguments.

For $n \in \mathbb{N}$ we define (Γ_n, γ_n) to be the diagonal subcomplex given by elements of level n or below. This defines a filtration of (Γ, γ):

$$(X^+ = \Gamma_0, \gamma_0) \leq (\Gamma_1, \gamma_1) \leq (\Gamma_2, \gamma_2) \leq \dots. \tag{2.1.6.9}$$

The union is the whole of (Γ, γ).

Remark 2.1.8. There is also a *coarse level* $\mathrm{lev}_c : \Gamma \to \mathbb{N}$ which is similarly defined with $\mathrm{lev}_c(X^+) = \{0\}$ and

$$\mathrm{lev}_c(U) = \sup \{\mathrm{lev}_c(V) \mid V \in \gamma(U)\} + 1. \tag{2.1.6.10}$$

This defines the *coarse filtration* (Γ_i^c, γ_i^c). It is worth noting that the zeroth term of both filtrations consists of just the points of X, whilst in the coarse case the first term comes from a simplicial complex Γ_1^c and in the regular case the first term Γ_1 is the 1-skeleton of Γ_1^c. The coarse level will not be used in the sequel although it would be sufficient in some arguments.

Example 2.1.6.2. The diagonal complex (Γ, γ) of Example 2.1.5.4 defined over $X = \{1, 2, \ldots\}$ has the levelwise filtration given by

$$\Gamma_n = X^+ \cup \big\{\{1, \ldots, m\} \mid m = 2, \ldots, n+1\big\} \tag{2.1.6.11}$$

2.1.7 Products

The very reason for defining diagonal complexes is to study the diagonal complex products they index and which we define in this section. We will begin by constructing the products for pointed spaces and then by considering this case expand the definition to other categories.

Labelled diagonal complexes

Before defining products of spaces we must consider the spaces we are taking the product over. For this we require a system for labelling the factor spaces.

Definition 2.1.9. Let (Γ, γ) be a diagonal complex defined over a set X and let $l : X \to Z$ be a map of sets with codomain Z. Then (Γ, γ, l) is a *Z-labelled diagonal complex* if

- for each $U \in \Gamma$ and each $V \in \gamma(U)$, the map l is constant on V.

We call l the *labelling* of (Γ, γ).

Example 2.1.7.1.

1. Let $Z = \{1\}$ then any diagonal complex is Z-labelled with the constant map $X \to \{1\}$.

2. Let Γ be the proper diagonal complex defined by a simplicial complex $S_\bullet$ over a set X. See Example 2.1.5.1 for details. Then any labelling $l : X \to Z$ for any Z makes (Γ, γ, l) into a Z-labelled diagonal complex.

3. Let Γ be the proper diagonal complex given in Example 2.1.5.2. Let $Z = \{\bullet, \circ\}$ and define l to be
$$l(1) = \bullet, \quad l(2) = \bullet \quad \text{and} \quad l(3) = \circ. \tag{2.1.7.1}$$
Then (Γ, γ, l) is a $\{\bullet, \circ\}$-labelled complex.

In fact each diagonal complex (Γ, γ) defined over X has a universal labelling $l_0 : X \to Z_0$ with the property that if $l : X \to Z$ is another valid labelling then there exists a factorisation

$$\begin{tikzcd} X \arrow[r, "l_0"] \arrow[rr, bend right, "l"'] & Z_0 \arrow[r] & Z. \end{tikzcd} \tag{2.1.7.2}$$

It may be defined as follows, consider the subsets $V \subseteq X$ which occur as $V \in \gamma(U)$ for some $U \in \Gamma$. Now define an equivalence relation on X by saying that x and y are equivalent if they both lie in such a V and extending by transitivity. Now let Z_0 be the set of equivalence classes with $l_0 : X \to Z_0$ the natural map. It is not hard to see that this enjoys the above universal property.

Let $l : X \to Z$ be a map of sets and $\mathbf{Y} = (Y_i \mid i \in Z)$ a Z-tuple of pointed spaces. Let $\{U_i \mid i = 1, \ldots, k\}$ be a partial partition of X such that l is constant on each U_i, we write the value as $l(U_i)$. There is now a diagonal map

$$D_{\mathbf{Y}}^{\{U_i\}} : \prod_{i=1}^{k} Y_{l(U_i)} \to \prod_{i \in X} Y_{l(i)}. \tag{2.1.7.3}$$

Suppose that we are given two partial partitions valid with respect to a labelling. The next lemma says that the meet of these partial partitions is also valid with respect to the labelling. This should come as no surprise because in light of Theorem 2.1.2 this is the intersection of the two diagonal subspaces of the form (2.1.7.3).

Lemma 2.1.10. *Let X be a set with a labelling $l : X \to Z$. Suppose that $\{U_i\}$ and $\{V_j\}$ are two partial partitions with the property that l is constant on any W in $\{U_i\}$ or $\{V_j\}$. Then the meet $\{W_k\} = \{U_i\} \wedge \{V_j\}$ in the poset $(PP_f X, \leq_{pc})$ also has the property that l is constant for any $W \in \{W_k\}$.*

Proof. Recall that $\{W_k\}$ may be constructed by defining an equivalence relation of $X \cup \{0\}$ with generators given by the sets U_i and V_j and also the complements of the partial partitions inside $X \cup \{0\}$. Since l is constant on the generating sets of the form U_i and V_j, l must be constant on the equivalence classes which do not include 0. But these equivalence classes are precisely the elements of $\{W_k\}$. □

Preliminary construction

Let $\mathbf{Y} = (Y_j)_{j \in Z}$ be a Z-tuple of pointed spaces and let (Γ, γ, l) be a Z-labelled diagonal complex on X, then define $\mathbf{Y}^X$ to be

$$\big\{(y_i)_{i \in X} \mid y_i \neq *_{l(i)} \text{ for finitely many } i \in X\big\} \subseteq \prod_{i \in X} Y_{l(i)}. \tag{2.1.7.4}$$

Recall that diagonal subspaces were defined by partial partitions of X. For each $U \in \Gamma$ we are given a partial partition $\gamma(U)$ of X and this defines a map

$$D_{\mathbf{Y}}^{\gamma(U)} : \mathbf{Y}^{\gamma(U)} := \prod_{U' \in \gamma(U)} Y_{l(U')} \to \mathbf{Y}^X. \tag{2.1.7.5}$$

The space $\mathbf{Y}'(\Gamma, \gamma)$ is defined as

$$\bigcup_{U \in \Gamma} \operatorname{Im} D_{\mathbf{Y}}^{\gamma(U)}. \tag{2.1.7.6}$$

This will turn out to be isomorphic to the diagonal complex product of $(Y_i)_{i \in Z}$ indexed by (Γ, γ, l) as proved in Proposition 2.1.12.

Definition as a colimit

Let (Γ, γ, l) be a Z-labelled diagonal complex defined over a set X. Recall that γ has codomain $PP_f X$. Let P_Γ be the subposet of $(PP_f, \leq_{\mathrm{pc}})$ containing $\gamma(\Gamma)$ and closed under meets.

Let $\mathcal{C}$ be a category with finite products. Then there is an initial object, call it k. Along with the binary product $\otimes$ this makes $(\mathcal{C}, k, \otimes)$ into a symmetric monoidal category. Furthermore for each object C in $\mathcal{C}$ there is a natural diagonal map $\Delta_C : C \to C \otimes C$. Let $\mathbf{C} = \{C_i \mid i \in Z\}$ be a Z-tuple of objects in $\mathcal{C}$.

Now viewing P_Γ as a category we define a functor $F_{(\mathbf{C},\Gamma)}$ as follows:

- An object $\{U_i\}$ is taken to $\bigotimes_i C_{l(U_i)}$,
- the image of a morphism $\{U_i\} \leq_{\mathrm{pc}} \{V_j\}$ is the product of maps

$$\bigotimes_{U_i = \amalg_{k \in I} V_k} \Delta_{C_{l(U_i)}}^{|I|-1} \otimes \bigotimes_{V_k \not\subseteq \cup_i U_i} p_{C_{l(V_k)}} \tag{2.1.7.7}$$

  consisting of a diagonal map for each U_i and an initial morphism for each V_k not contained in a U_i.

In the category of pointed spaces or groups the maps are those that take (y_i) to (y'_j), where if $V_j \subseteq U_i$ for some i then $y'_j = y_i \in Y_{l(U_i)} = Y_{l(V_j)}$, otherwise $y'_j = * \in Y_{l(V_j)}$.

Definition 2.1.11. If the colimit of $F_{(\mathbf{C},\Gamma)}$ exists then we call it the *diagonal complex product* $\mathbf{C}(\Gamma, \gamma)$ of $\mathbf{C}$ indexed by (Γ, γ).

Recall (2.1.7.6) the definition of $\mathbf{Y}'(\Gamma, \gamma)$.

Proposition 2.1.12. *Let $\mathcal{C}$ be the category of pointed spaces and $\mathbf{Y} = \mathbf{C}$ and (Γ, γ, l) be as in the definition above. Then the diagonal complex product $\mathbf{Y}(\Gamma, \gamma)$ is isomorphic to $\mathbf{Y}'(\Gamma, \gamma)$.*

Proof. The product $\mathbf{Y}'(\Gamma,\gamma)$ is defined to be the union of the inclusions

$$i_U = D_{\mathbf{Y}}^{\gamma(U)} : \mathbf{Y}^U \to \mathbf{Y}^X. \tag{2.1.7.8}$$

So in particular the images of i_U cover the space $\mathbf{Y}'(\Gamma,\gamma)$. For $U_1,\ldots,U_k \in \Gamma$ the intersection $\operatorname{Im} i_{U_1} \cap \ldots \cap \operatorname{Im} i_{U_m}$ is given by $\operatorname{Im} D_{\mathbf{Y}}^{\{W_k\}}$ where $\{W_k\}$ is the meet of the $\gamma(U_j)$ for $j = 1,\ldots,m$.

Hence the $\operatorname{Im} i_U$ form a cover of $\mathbf{Y}'(\Gamma,\gamma)$ and the functor $F_{(\mathbf{Y},\Gamma)} : P_\Gamma \to \{\text{Pointed Spaces}\}$ is the diagram consisting of the spaces $\operatorname{Im} i_U$ and their intersections in $\mathbf{Y}'(\Gamma,\gamma)$. In the situation that all the spaces and maps are CW complexes this implies that $\mathbf{Y}'(\Gamma,\gamma) \cong \operatorname{colim} F_{(\mathbf{Y},\Gamma)} = \mathbf{Y}(\Gamma,\gamma)$ by Proposition 1.1.2, see also [25] Section 4.$\mathbb{G}$. $\square$

Example 2.1.7.2. We defined a diagonal complex (Γ,γ) in Example 2.1.5.2 and gave a labelling in Example 2.1.7.1. The category P_Γ is

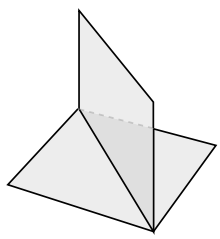
$$\begin{tikzcd}
 & \bullet \arrow[d] \\
 & \bullet \\
 & \bullet \arrow[u] \arrow[d] \\
\bullet \arrow[r] & \bullet & \bullet \arrow[l]
\end{tikzcd} \tag{2.1.7.9}$$

and the corresponding diagram for objects C_1 and C_2 is given by

$$\begin{tikzcd}
 & C_2 \arrow[d, "p\otimes 1"] & \\
 & C_1 \otimes C_2 & \\
 & C_1 \arrow[u, "1\otimes p"'] \arrow[d, "\Delta"] & \\
C_1 \arrow[r, "1\otimes p"] & C_1 \otimes C_1 & C_1. \arrow[l, "p\otimes 1"']
\end{tikzcd} \tag{2.1.7.10}$$

where the morphisms from the initial object are denoted p and the identity morphisms are denoted 1.

Now suppose that each C_i is a line segment $I = [0,1]$ with basepoint 0 in the category of pointed spaces. Then the product $I\Gamma$ looks like:



(2.1.7.11)

Another definition of the geometric realisation

There are a number of different ways to define the geometric realisation of a diagonal complex, originally defined in Section 2.1.4. Here is a method using the product associated to a diagonal complex. Let (Γ, γ) be a diagonal complex over a set X and let $I = [0, 1]$ be the unit interval with basepoint 0. Then the geometric realisation is equal to the intersection

$$I(\Gamma, \gamma) \cap \left\{(t_i)_{i \in X} \mid \text{finitely many } t_i \text{ are non-zero and } \sum_i t_i = 1\right\} \tag{2.1.7.12}$$

of the diagonal product of I indexed by (Γ, γ) seen as a subspace of I^X and the full simplicial complex on X embedded in I^X. It may be a helpful exercise to examine the figure above in Example 2.1.7.2 in order to see that the geometric realisation in this case is a 'T' shape.

In fact $I(\Gamma, \gamma)$ is a simplicial cone over the basepoint (0). The geometric realisation is isomorphic to the base of this cone. In fact this gives the realisation as a subcomplex of the barycentric subdivision of the full simplicial complex on the set X.

Spherical realisation

Now let $\mathbb{R}$ be the real line with basepoint chosen to be 0. For a diagonal complex (Γ, γ) the diagonal complex product $\mathbb{R}(\Gamma, \gamma)$ is a subspace of

$$\mathbb{R}^X = \{(x_i)_{i \in X} \mid \text{ finitely many } x_i \text{ are non-zero}\}. \tag{2.1.7.13}$$

The finite support condition means that $\mathbb{R}^X$ may be given the standard Euclidean metric, i.e. that for two points (x_i) and (y_i) the following is well-defined

$$\sqrt{\sum_{i \in X} (x_i - y_i)^2}. \tag{2.1.7.14}$$

Definition 2.1.13. The *spherical realisation* of a diagonal complex (Γ, γ) is the intersection

$$\mathbb{R}(\Gamma, \gamma) \cap \left\{(x_i) \mid \sum_i x_i^2 = 1\right\} \tag{2.1.7.15}$$

of the unit sphere and the diagonal complex product $\mathbb{R}(\Gamma, \gamma)$ embedded in $\mathbb{R}^X$.

The spherical resolution is of interest because it occurs as the link of the vertex in the diagonal complex product of copies of the circle. This means it is of interest in the study of possible geometries on diagonal complex products.

Levelwise construction of products of pointed spaces

We previously defined the level of a $U \in \Gamma$ and saw that there was a filtration

$$(X^+ = \Gamma_0, \gamma_0) \leq (\Gamma_1, \gamma_1) \leq (\Gamma_2, \gamma_2) \leq \dots. \tag{2.1.7.16}$$

Our aim now is to describe how the diagonal complex product $\mathbf{Y}(\Gamma_n, \gamma_n)$ of pointed spaces $\mathbf{Y}$ may be built from that of $\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1})$. Let $U \in (\Gamma_n, \gamma_n)$ be of level n and write $\gamma(U) = \{U_1, \dots, U_k\}$. Then the maximal faces $U - U_i$ are all of level $n-1$ or less. For each proper subset $A \subsetneq [k]$ there is an inclusion

$$\prod_{i \in A} Y_{l(U_i)} \hookrightarrow \prod_{i \in [k]} Y_{l(U_i)} = \mathbf{Y}^U, \tag{2.1.7.17}$$

we call this subspace $\mathbf{Y}^U_A$ and the union of such subspaces we will denote $\delta \mathbf{Y}^U$. This is the subspace consisting of elements of $\mathbf{Y}^U$ where at least one of the coordinates is equal to $*$. From the definition of a diagonal complex the set $U_A = \bigcup_{i \in A} U_i$ is in $(\Gamma_{n-1}, \gamma_{n-1})$ and $\gamma(U_A)$ is a refinement of $\{U_i\}_{i \in A}$. Therefore there is a map

$$\mathbf{Y}^U_A \to \mathbf{Y}^{U_A} \to \mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1}), \tag{2.1.7.18}$$

which realises $\mathbf{Y}^U_A$ as the intersection of $\mathbf{Y}^U$ and $\mathbf{Y}^{U_A}$ in $\mathbf{Y}(\Gamma_n, \gamma_n)$. Taking the union of the $\mathbf{Y}^U_A$ we get that the intersection of $\mathbf{Y}^U$ and $\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1})$ is $\delta \mathbf{Y}^U$. So we have a diagram

$$\mathbf{Y}^U \leftarrow \delta \mathbf{Y}^U \to \mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1}) \tag{2.1.7.19}$$

whose colimit attaches $\mathbf{Y}^U$ onto $\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1})$. In this way $\mathbf{Y}(\Gamma_n, \gamma_n)$ is given by attaching each U of level n.

2.2 Products of groups

2.2.1 As a fundamental group

Suppose that $\mathcal{C}$ and $\mathcal{D}$ are categories with finite products and that $G : \mathcal{C} \to \mathcal{D}$ is a functor preserving both finite products and colimits. Now let (Γ, γ, l) be a Z-labelled diagonal complex and $\mathbf{C}$ be a Z-tuple of objects of $\mathcal{C}$. Recall that the diagonal complex $\mathbf{C}(\Gamma, \gamma, l)$ is defined if the colimit of a certain functor $F_{(\mathbf{C},\Gamma)}$ exists. Now since G preserves finite products we have that

$$G \circ F_{(\mathbf{C},\Gamma)} = F_{(G(\mathbf{C}),\Gamma)}. \tag{2.2.1.1}$$

Since G also preserves colimits we have further that

$$G(\mathbf{C}(\Gamma,\gamma,l)) \cong G(\mathbf{C})(\Gamma,\gamma,l). \tag{2.2.1.2}$$

Now let $\mathcal{C}$ be the category of pointed, connected spaces and $\mathcal{D}$ be the category of groups and $G = \pi_1$ be the fundamental group functor, which preserves finite products and colimits. Let (Γ,γ,l) be a Z-labelled diagonal complex as before and let $\mathbf{Y}$ be a Z-tuple of pointed spaces. Also denote the Z-tuple of groups $(\pi_1(Y_i))$ by $\mathbf{G}$. Then Equation (2.2.1.2) becomes

$$\pi_1(\mathbf{Y}(\Gamma,\gamma,l)) \cong \mathbf{G}(\Gamma,\gamma,l). \tag{2.2.1.3}$$

In words this is '*the fundamental group of a diagonal complex product of pointed spaces is the diagonal complex product of the fundamental groups of the spaces*'.

2.2.2 An explicit presentation

We will now construct a presentation for a diagonal complex product of groups in terms of the diagonal complex and of the factor groups themselves. This is summarised as

Theorem 2.2.1. *Let (Γ,γ,l) be a Z-labelled diagonal complex and let $\mathbf{G} = (G_i)_{i\in Z}$ be a Z-tuple of groups. Then $\mathbf{G}(\Gamma,\gamma)$ is generated by elements*

$$g_U = (g,U) \tag{2.2.2.1}$$

where $U \in \Gamma$ is such that l is constant on U and $g \in G_{l(U)}$. These are subject to the relations

$$g_U.h_U = (gh)_U \qquad \text{for any } g_U, h_U, \tag{2.2.2.2}$$

$$e_U = e \qquad \text{for each } U, \tag{2.2.2.3}$$

$$[g_U, h_V] = e \qquad \text{for } U,V \in \gamma(W) \text{ and } W \in \Gamma, \tag{2.2.2.4}$$

$$g_U = g_{U_1}\dots g_{U_k} \qquad \text{where } \gamma(U) = \{U_1,\dots,U_k\}. \tag{2.2.2.5}$$

These relations suffice to present $\mathbf{G}(\Gamma,\gamma)$. The element g_U is given by

$$g_U = D_{\mathbf{G}}^{\gamma(U)} \circ \Delta_{G_{l(U)}}^{\dim U - 1}(g). \tag{2.2.2.6}$$

Proof. To give a presentation of $\mathbf{G}(\Gamma,\gamma)$ we look to the levelwise construction. The group $\mathbf{G}(\Gamma,\gamma)$ is the colimit of the diagram.

$$\mathbf{G}(\Gamma_0,\gamma_0) \to \mathbf{G}(\Gamma_1,\gamma_1) \to \mathbf{G}(\Gamma_2,\gamma_2) \to \dots. \tag{2.2.2.7}$$

So it will suffice to prove the theorem for each level, which we will do by induction. Note that $\mathbf{G}(\Gamma_0, \gamma_0)$ is the free product $*_{i\in X} G_i$ and that $\mathbf{G}(\Gamma_1, \gamma_1)$ is a graph product of groups. So for the case $n = 0$ the theorem can easily be seen to be true. As with the levelwise construction of spaces, we construct each successive group by amalgamations,

$$\mathbf{G}^U \leftarrow \delta\mathbf{G}^U \rightarrow \mathbf{G}(\Gamma_{n-1}, \gamma_{n-1}) \tag{2.2.2.8}$$

where U is of level n, $\gamma(U) = \{U_1, \ldots, U_k\}$, the group $\mathbf{G}^U$ is $\prod_{i\in[k]} G_{l(U_i)}$ and $\delta\mathbf{G}^U$ is the colimit of the diagram consisting of all the groups $\prod_{i\in A} G_{l(U_i)}$ for $A \in \delta\Delta_k$. However in the category of groups, if the dimension of U is greater than two, then $\delta\mathbf{G}^U \cong \mathbf{G}^U$. Only in the case when the dimension of U is two, when

$$\delta\mathbf{G}^U = G_{l(U_1)} * G_{l(U_2)} \tag{2.2.2.9}$$

and so the diagram looks like

$$G_{l(U_1)} \times G_{l(U_2)} \leftarrow G_{l(U_1)} * G_{l(U_2)} \rightarrow \mathbf{G}(\Gamma_{n-1}, \gamma_{n-1}). \tag{2.2.2.10}$$

does taking the colimit of the diagram have any effect. The effect in question is that of adding commutation relations. Even though the amalgamation may not change the group it is still a good time to prove that the relations (2.2.2.2)-(2.2.2.5) hold for U. The group $\mathbf{G}^U$ is a copy of $\prod_{U'\in\gamma(U)} G_{l(U')}$ and each of the factors are seen to be one of the $F(\{U'\})$, which map diagonally into $F(\gamma(U'))$. Hence by (2.2.2.6) each of the factors consists of the elements $g_{U'}$ for $g \in G_{l(U')}$, and so relation (2.2.2.4) is seen to hold. Now assume that $l(U) \neq 0$. Then g_U for each g is given by the diagonal $G_{l(U)} \rightarrow \mathbf{G}^U$, hence $g_U = \prod_{U'\in\gamma(U)} g_{U'}$ giving relation (2.2.2.5). Finally relations (2.2.2.2) and (2.2.2.3) are also given by the inclusion $G_{l(U)} \rightarrow \mathbf{G}^U$. □

Note that the set $\{g_U \mid U \in X^+\}$ generates the group and that the remaining g_U are defined for notational convenience.

It will be helpful to examine the presentation in the case of a simple example.

Example 2.2.2.1. Let (Γ, γ, l) be the 'T' shaped labelled diagonal complex from Example 2.1.7.1 and let $G_\bullet = \mathbb{Z}/(m)$ and $G_\circ = \mathbb{Z}/(n)$ be two cyclic groups. Then $\mathbf{G}(\Gamma, \gamma, l)$ is the group generated by

$$g_1, g_2, g_3 \text{ and } g_{12} \tag{2.2.2.11}$$

with relations given by

$$g_1^m = g_2^m = g_{12}^m = e \text{ and } g_3^n = e, \tag{2.2.2.12}$$

coming from the factor groups, the commutator brackets

$$[g_1, g_2] = e \text{ and } [g_{12}, g_3] = e, \tag{2.2.2.13}$$

coming from the elements $\{1,2\}$ and $\{1,2,3\}$ of Γ respectively and finally

$$g_{12} = g_1 g_2. \tag{2.2.2.14}$$

Remark 2.2.2. In fact the group just described is isomorphic to the graph product given by the graph

$$1 \text{——} 2 \text{——} 3 \tag{2.2.2.15}$$

with $G_1 = G_2 = \mathbb{Z}/(m)$ and $G_3 = \mathbb{Z}/(m)$. If we let the natural generators of this graph product be x_1, x_2 and x_3 then we may give the isomorphism by sending x_1 to g_1, x_2 to $g_{12} = g_1.g_2$ and x_3 to g_3. This isomorphism may be given whenever $G_\bullet$ is abelian.

Unfortunately the higher homotopy group functors π_i for $i \geq 2$ do not preserve colimits and so computing the higher homotopy groups of a diagonal complex product is significantly harder.

2.3 (Co)homology of products

Fortunately computing the homology and cohomology of a diagonal complex is not very difficult and there is a simple formula. Once this is derived we will then study the ring structure of the cohomology and finally will look at what happens when there is an action of a group of automorphisms on the diagonal complex.

2.3.1 (Co)homology

Let Y be a pointed space and let $C_*(Y)$ be the chain complex of the space Y_i over a ring R. Then $C_*(Y_i)$ splits as $R \oplus \widehat{C}_*(Y_i)$, where $\widehat{C}_*(Y_i)$ is the reduced chain complex. Recall that there is a decomposition of the homology of a product of pointed spaces:

Proposition 2.3.1. *Let Y_i be pointed spaces for $i = 1, \ldots, n$. Then the homology of the product of the pointed spaces decomposes as*

$$H_*\left(\prod_{i=1}^{n} Y_i\right) = R \oplus \bigoplus_{A \in \Delta_n} H_*\left(\bigotimes_{i \in A} \widehat{C}_*(Y_i)\right), \tag{2.3.1.1}$$

where $\Delta_n = P_f[n]$ is the n-simplex.

Proof. Using the definition of the homology and a Künneth formula we have

$$\begin{aligned} H_*\left(\prod_{i=1}^{n} Y_i\right) &\cong H_*\left(C_*\left(\prod_{i=1}^{n} Y_i\right)\right) \\ &\cong H_*\left(\bigotimes_{i=1}^{n} C_*(Y_i)\right) \end{aligned}$$

which uses the fact that $C_*(Y_i)$ is a free R-module. By using the decomposition $C_*(Y_i) \cong R \oplus \widehat{C}_*(Y_i)$ we have

$$\begin{aligned}\bigotimes_{i=1}^{n} C_*(Y_i) &\cong \bigotimes_{i=1}^{n} (R \oplus \widehat{C}_*(Y_i)) \\ &\cong \bigoplus_{A \subseteq [n]} \bigotimes_{i \in A} \widehat{C}_*(Y_i).\end{aligned}$$

We are done because H_* commutes with $\oplus$ and the R term comes from $A = \emptyset \subset [n]$. $\square$

We get a similar decomposition of the homology of a diagonal complex product.

Theorem 2.3.2. *Let (Γ, γ) be a Z-labelled diagonal complex on a set X and $\mathbf{Y} = (Y_i)_{i \in Z}$ be pointed spaces. The homology of the diagonal complex product $\mathbf{Y}(\Gamma, \gamma)$ splits as*

$$R \oplus \bigoplus_{U \in \Gamma} H_*(U), \tag{2.3.1.2}$$

where $H_*(U)$ is given by

$$H_*\Big(\bigotimes_{U' \in \gamma(U)} \widehat{C}_*(Y_{l(U')}) \Big). \tag{2.3.1.3}$$

Proof. We proceed by induction on the level. For the case of $n = 0$, the product is a wedge product of the spaces Y_i and so the theorem holds. Now suppose that (2.3.1.2) holds for level $n-1$, that is for the diagonal complex $(\Gamma_{n-1}, \gamma_{n-1})$. In Section 2.1.7 the space $\mathbf{Y}(\Gamma_n, \gamma_n)$ was formed by gluing spaces $\mathbf{Y}^U$ onto $\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1})$ using diagrams of the form

$$\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1}) \leftarrow \delta \mathbf{Y}^U \rightarrow \mathbf{Y}^U. \tag{2.3.1.4}$$

Write $\gamma(U) = \{U_1, \ldots, U_k\}$. Using Proposition 2.3.1 we are able to write

$$C_*(\delta \mathbf{Y}^U) \rightarrow C_*(\mathbf{Y}^U) \tag{2.3.1.5}$$

as the split monomorphism

$$R \oplus \bigoplus_{A \in \delta \Delta_k} \bigotimes_{i \in A} \widehat{C}_*(Y_{l(U_i)}) \hookrightarrow R \oplus \bigoplus_{A \in \Delta_k} \bigotimes_{i \in A} \widehat{C}_*(Y_{l(U_i)}).$$

For each U_i there is a map $Y_{l(U_i)} \rightarrow \mathbf{Y}^{U_i} \cong Y_{l(U_i)}^{\dim U_i}$ given by the diagonal map, and this induces the inclusion $\delta \mathbf{Y}^U \hookrightarrow \mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1})$. On chains the map $Y_{l(U_i)} \rightarrow \mathbf{Y}^{U_i}$ induces an injection

$$C_*(Y_{l(U_i)}) \rightarrow C_*(\mathbf{Y}^{U_i}) \tag{2.3.1.6}$$

and so the map $C_*(\delta \mathbf{Y}^U) \rightarrow C_*(\mathbf{Y}(\Gamma_{n-1}, \gamma_{n-1}))$ is injective. Taking the colimit of (2.3.1.4)

has the effect of adding on a term

$$\bigotimes_{i=1}^{k} \widehat{C}_*(Y_{l(U_i)}), \tag{2.3.1.7}$$

which gives the term $H_*(U)$ in (2.3.1.2). Thus we have proved the theorem. □

Remark 2.3.3 (cohomological version of Theorem 2.3.2). The cohomological version is similar, although because taking cochains is contravariant, the colimit is replaced by a limit and so the direct sum should be replaced by the direct product. In the case that Γ and X are finite the direct product is isomorphic to the direct sum and so we have that the cohomology of the diagonal complex product $\mathbf{Y}(\Gamma, \gamma)$ splits as

$$R \oplus \bigoplus_{U \in \Gamma} H^*(U), \tag{2.3.1.8}$$

where $H^*(U)$ is given by

$$H^*\Big(\bigotimes_{U' \in \gamma(U)} \widehat{C}^*(Y_{l(U')}) \Big) \tag{2.3.1.9}$$

and

$$C^*(Y_{l(U')}, R) \cong R \oplus \widehat{C}^*(Y_{l(U')}). \tag{2.3.1.10}$$

Remark 2.3.4. It is worth noting that the proof of the theorem above offers more than a calculation of the homology of diagonal complex products, there is also a natural quasi-isomorphism realising this equivalence

$$R \oplus \bigoplus_{U \in \Gamma} \Big(\bigotimes_{U' \in \gamma(U)} \widehat{C}_*(Y_{l(U')}) \Big) \to C_*\big(\mathbf{Y}(\Gamma, \gamma)\big). \tag{2.3.1.11}$$

The importance of this will become clear in Theorem 2.3.9. There is also a cohomological version, where of course the arrow is reversed

$$R \oplus \bigoplus_{U \in \Gamma} \Big(\bigotimes_{U' \in \gamma(U)} \widehat{C}^*(Y_{l(U')}) \Big) \leftarrow C^*\big(\mathbf{Y}(\Gamma, \gamma)\big). \tag{2.3.1.12}$$

Hilbert-Poincaré Series

Let (Γ, γ, l) be a Z-labelled diagonal complex over a finite set X. To each simplex $U \in \Gamma$ we assign a monomial in the elements of $Z = \{z_1, \ldots, z_k\}$:

$$m(U) = \prod_{U' \in \gamma(U)} l(U'). \tag{2.3.1.13}$$

The Hilbert-Poincaré series of (Γ, γ) in the polynomial ring $\mathbb{Z}[Z]$ is

$$h_{(\Gamma,\gamma)}(z_1, \ldots, z_k) = \sum_{U \in \Gamma} m(U). \tag{2.3.1.14}$$

Example 2.3.1.1. The Hilbert-Poincaré series of Γ with the labelling by $\{\bullet, \circ\}$ from Example 2.1.7.1 is

$$h_\Gamma(z_\bullet, z_\circ) = 2z_\bullet + z_\circ + z_\bullet^2 + z_\bullet z_\circ. \tag{2.3.1.15}$$

Let $\mathcal{R}[\![t]\!]$ be the complete $\mathbb{N}$-graded ring generated by the indecomposable modules of R with the product Tor^R. We use the variable t to keep track of the grading. For example if $R = \mathbb{Z}$ then the indecomposable modules are of the form $\mathbb{Z}$ and $\mathbb{Z}/(p^i)$. The module $\mathbb{Z}$ is the unit of $\mathrm{Tor}^{\mathbb{Z}}$ and we will denote $\mathbb{Z}/(p^i)$ by x_{p^i}. Then

$$x_{p^i}.x_{q^j} = \begin{cases} (1+t)x_{p^i} & \text{if } p = q \text{ and } i \leq j, \\ 0 & \text{if } p \neq q \end{cases} \tag{2.3.1.16}$$

is used to denote the fact that

$$\mathrm{Tor}_k^{\mathbb{Z}}\left(\mathbb{Z}/(p^i), \mathbb{Z}/(q^j)\right) \cong \begin{cases} 0 & \text{if } k \geq 2, \\ \mathbb{Z}/(p^i) & \text{if } p = q,\ i \leq j \text{ and } k = 0, 1, \\ 0 & \text{if } p \neq q \text{ and } k = 0, 1. \end{cases} \tag{2.3.1.17}$$

Now let $y_i(t)$ and $y_i'(t)$ in $\mathcal{R}[\![t]\!]$ be the Hilbert-Poincaré series of $H_*(Y_i)$ and $H^*(Y_i)$ respectively. From Theorem 2.3.2 we get the following

Corollary 2.3.5. *The Hilbert-Poincaré series of the homology and the cohomology of the diagonal complex product $\mathbf{Y}(\Gamma, \gamma)$ are respectively*

$$1 + h_{(\Gamma,\gamma)}(y_1 - 1, \ldots, y_k - 1) \quad \textit{and} \quad 1 + h_{(\Gamma,\gamma)}(y_1' - 1, \ldots, y_k' - 1).$$

2.3.2 Structure of the cohomology ring

We may use the inclusion $\mathbf{Y}(\Gamma, \gamma) \hookrightarrow \mathbf{Y}^X$ to calculate the cup product on cohomology.

Lemma 2.3.6. *Let (Γ, γ, l) be a Z-labelled diagonal complex defined over a finite set X. Then the map $H^*(\mathbf{Y}^X, R) \to H^*(\mathbf{Y}(\Gamma, \gamma), R)$ is surjective.*

Proof. By the remarks following Theorem 2.3.2 the cohomology $H^*(\mathbf{Y}(\Gamma, \gamma), R)$ decomposes as

$$R \oplus \bigoplus_{U \in \Gamma} H^*\left(\widehat{C}^*(U)\right). \tag{2.3.2.1}$$

And by a cohomological version of Proposition 2.3.1 the complex $H^*(\mathbf{Y}^X, R)$ decomposes as

$$R \oplus \bigoplus_{A \in \Delta_X} H^*\Big(\bigotimes_{i \in A} \widehat{C}^*(Y_{l(i)})\Big). \tag{2.3.2.2}$$

The inclusion of $\mathbf{Y}^U = \prod_{U' \in \gamma(U)} Y_{l(U')}$ into $\prod_{x \in U} Y_{l(x)}$ by diagonal maps induces a map

$$H^*\Big(\bigotimes_{x \in U} C^*(Y_{l(x)})\Big) \to H^*\Big(\bigotimes_{U' \in \gamma(U)} C^*(Y_{l(U')})\Big) \to H^*\left(\widehat{C}^*(U)\right). \tag{2.3.2.3}$$

So each summand in (2.3.2.1) is mapped to from a summand in (2.3.2.2). It now only remains to note that since there is a projection from $\mathbf{Y}^X$ to $\mathbf{Y}^U$ providing a one-sided inverse to the inclusion, the maps (2.3.2.3) on the individual summands are surjective. □

The product can now be calculated by considering the diagram

$$\begin{tikzcd} H^*(\mathbf{Y}(\Gamma,\gamma),R) \otimes H^*(\mathbf{Y}(\Gamma,\gamma),R) \arrow[d] & H^*(\mathbf{Y}^X,R) \otimes H^*(\mathbf{Y}^X,R) \arrow[l, two heads] \arrow[d] \\ H^*(\mathbf{Y}(\Gamma,\gamma),R) & H^*(\mathbf{Y}^X,R) \arrow[l, two heads] \end{tikzcd} \tag{2.3.2.4}$$

Given two cocycles in $H^*(\mathbf{Y}(\Gamma,\gamma),R)$ to multiply, first lift along the surjective map to cocycles in $H^*(\mathbf{Y}^X,R)$, multiply them there, then map back to $H^*(\mathbf{Y}(\Gamma,\gamma),R)$.

2.3.3 Extensions by symmetry groups

Let (Γ,γ) be a diagonal complex on a set X, then by inspecting the definition of a morphism of diagonal complexes we may see that an automorphism of (Γ,γ) must act by permuting the set X. Now suppose that $l : X \to Z$ is a labelling of (Γ,γ), then an automorphism of the labelled diagonal complex is a permutation σ of X with $l \circ \sigma = l$.

Let G be a group of automorphisms of a Z-labelled diagonal complex (Γ,γ,l) and let $\mathbf{Y}$ be any Z-tuple of pointed spaces. Then G acts on $\mathbf{Y}(\Gamma,\gamma)$ and fixes the basepoint and hence G also acts on

$$\pi_1(\mathbf{Y}(\Gamma,\gamma)) \cong (\pi_1(\mathbf{Y}))(\Gamma,\gamma). \tag{2.3.3.1}$$

The purpose of this section is to calculate the homology of the homotopy quotient of the action of G on the spaces $\mathbf{Y}(\Gamma,\gamma)$. The homotopy quotient was reviewed in Section 1.1.3.

Homology of the homotopy quotient of a diagonal complex

Proposition 2.3.7. *Let G act on the Z-labelled diagonal complex (Γ, γ, l) and let $\mathbf{Y}$ be a Z-tuple of pointed spaces. Then*

$$H_*\big(E \times_G \mathbf{Y}(\Gamma, \gamma), R\big) \cong H_*(G, R) \oplus \bigoplus_{U \in \Gamma/G} H_*\big(\mathrm{Stab}(U), \widehat{C}_*(U)\big), \tag{2.3.3.2}$$

where

$$\widehat{C}_*(U) = \bigotimes_{U' \in \gamma(U)} \widehat{C}_*(Y_{l(U')}) \tag{2.3.3.3}$$

with the action inherited from the action of $\mathrm{Stab}(U)$ *on* $\gamma(U)$.

Proof. The homology of $E \times_G \mathbf{Y}(\Gamma, \gamma)$ may be computed as the homology of

$$\big(C_*(E) \otimes C_*(\mathbf{Y}(\Gamma, \gamma))\big)^G. \tag{2.3.3.4}$$

Recall from Theorem 2.3.2 and Remark 2.3.4 that $C_*(\mathbf{Y}(\Gamma, \gamma))$ is quasi-isomorphic to a complex which splits into a direct sum indexed by elements $U \in \Gamma$. The action of G is inherited from the action of G on Γ, so as a G-module it splits as

$$R \oplus \bigoplus_{[U] \in \Gamma/G} \Big(\bigoplus_{U' \in [U]} \bigotimes_{i \in U'} \widehat{C}_*(Y_{l(i)}) \Big). \tag{2.3.3.5}$$

This gives the desired splitting. We now need to show that

$$H_*\Big(G, \bigoplus_{U' \in [U]} \bigotimes_{i \in U'} \widehat{C}_*(Y_{l(i)})\Big) \cong H_*(\mathrm{Stab}(U), \widehat{C}_*(U)). \tag{2.3.3.6}$$

To see this note that

$$\bigoplus_{U' \in [U]} \bigotimes_{i \in U'} \widehat{C}_*(Y_{l(i)}) \tag{2.3.3.7}$$

is isomorphic to the $\mathrm{Stab}(U)$-module $\widehat{C}_*(U)$ induced up to G. Now Shapiro's lemma gives the desired result. □

2.3.4 Homotopy quotients by automorphism groups

Let (Γ, γ, l) be a Z-labelled diagonal complex and let $\mathbf{Y}$ be a Z-tuple of pointed spaces. Let H_i be a basepoint fixing group acting on Y_i for each $i \in Z$. Since the diagonal complex product is functorial with respect to the Z-tuple of factor groups there is an action of

$$H := \prod_{i \in Z} H_i \tag{2.3.4.1}$$

on $\mathbf{Y}(\Gamma,\gamma)$.

Proposition 2.3.8. *Let (Γ,γ,l) and $\mathbf{Y}$ be as above and let R be a ring. Then*

$$H_*\big(E\times_H \mathbf{Y}(\Gamma,\gamma),R\big)\cong H_*(H,R)\oplus\bigoplus_{U\in\Gamma}H_*\Big(\bigotimes_{i\in Z}C_*\big(H_i,\widehat{C}_*(Y_i)^{\otimes U_i}\big)\Big),\tag{2.3.4.2}$$

where U_i is the (perhaps empty) subset of i-coloured elements of U and the action of H_i is via the diagonal action. When U_i is empty we take $\widehat{C}_*(Y_i)^{\otimes U_i}$ to be the trivial module R.

Proof. The homology of the homotopy quotient is given be the homology of H with coefficients in the module $C_*(\mathbf{Y}(\Gamma,\gamma))$. By Remark 2.3.4 there is a quasi-isomorphism relating $C_*(\mathbf{Y}(\Gamma,\gamma))$ to a split complex (2.3.1.11). With the action of $h\in H_i$ on the summand $\widehat{C}_*(U)$ given by

$$h.a_{U'}=\begin{cases}ha_{U'} & \text{if } l(U')=i \text{ and}\\ a_{U'} & \text{otherwise,}\end{cases}\tag{2.3.4.3}$$

the quasi-isomorphism (2.3.1.11) is a morphism of H-modules. Hence the homology splits as

$$H_*\big(E\times_H \mathbf{Y}(\Gamma,\gamma),R\big)\cong H_*(H,R)\oplus\bigoplus_{U\in\Gamma}H_*\big(H,\widehat{C}_*(U)\big),\tag{2.3.4.4}$$

We remark that for a G_1-module M and a G_2-module N, the $G_1\times G_2$-coinvariants of $M\otimes N$ are isomorphic to the G_1-coinvariants of M tensored with the G_2-coinvariants of N. Since H is a direct product of groups and $\widehat{C}_*(U)$ a tensor product of modules for those groups we get the tensor product in (2.3.4.2). □

Suppose now that G is a group of automorphisms of (Γ,γ,l) and $\mathbf{Y}$ is a Z-tuple of pointed spaces. Suppose also that for each $z\in Z$ the discrete group H_z acts on Y_z.

Then $\mathbf{Y}(\Gamma,\gamma)$ has both an action of G and of H. In fact together they form a semidirect product $H\rtimes G$ acting on $\mathbf{Y}(\Gamma,\gamma)$.

Combining Propositions 2.3.7 and 2.3.8 we get the main theorem.

Theorem 2.3.9. *Let (Γ,γ,l), $\mathbf{Y}$, H and G be as above. Then the homology of the homotopy quotient of $\mathbf{Y}(\Gamma,\gamma)$ by $H\rtimes G$ with coefficients in R splits as*

$$H_*(H\rtimes G,R)\oplus\bigoplus_{[U]\in\Gamma/G}H_*\big(H\rtimes Stab_G(U),\widehat{C}_*(U)\big),\tag{2.3.4.5}$$

where

$$\widehat{C}_*(U):=\bigotimes_{U'\in\gamma(U)}\widehat{C}_*(Y_{l(U')})\tag{2.3.4.6}$$

has the permutation action by $Stab_G(U)$. The action of H on $C_*\big(\mathbf{Y}(\Gamma,\gamma)\big)$ restricts to $\widehat{C}_*(U)$.

Proof. The action of G on Γ permutes the $U \in \Gamma$ and so permutes the terms in the direct sum (2.3.1.11), thus making the quasi-isomorphism (2.3.1.11) into a morphism of G-modules. Since the action of H on $C_*\big(\mathbf{Y}(\Gamma,\gamma)\big)$ restricts to each term

$$\bigotimes_{U'\in\gamma(U)} \widehat{C}_*(Y_{l(U')}), \tag{2.3.4.7}$$

the morphism from (2.3.1.11) is in fact a quasi-isomorphism of $(H \rtimes G)$-modules. The homology of the homotopy quotient of $\mathbf{Y}(\Gamma,\gamma)$ may be calculated by the homology of $H \rtimes G$ with coefficients in $C_*\big(\mathbf{Y}(\Gamma,\gamma)\big)$ so we can compute it by calculating the homology of $H \rtimes G$ with coefficients in the decomposition

$$R \oplus \bigoplus_{U\in\Gamma}\Big(\bigotimes_{U'\in\gamma(U)} \widehat{C}_*(Y_{l(U')})\Big). \tag{2.3.4.8}$$

The direct sum

$$\bigoplus_{U\in\Gamma} \tag{2.3.4.9}$$

factors into

$$\bigoplus_{[U]\in\Gamma/G}\ \bigoplus_{V\in[U]} \tag{2.3.4.10}$$

and so the $(H \rtimes G)$-module (2.3.4.8) decomposes over the sum

$$\bigoplus_{[U]\in\Gamma/G}. \tag{2.3.4.11}$$

Now we may concentrate on individual terms in this decomposition. A term

$$\bigoplus_{V\in[U]}\ \bigotimes_{U'\in\gamma(U)} \widehat{C}_*(Y_{l(U')}) \tag{2.3.4.12}$$

may be given by inducing the $H \rtimes \mathrm{Stab}_G(U)$-module $\widehat{C}_*(U)$ up to $H \rtimes G$. So by the Shapiro lemma the homology of the corresponding term is given by the homology of $H \rtimes \mathrm{Stab}_G(U)$ with coefficients in $\widehat{C}_*(U)$. This completes the proof. □

Example 2.3.4.1. A special case of this theorem is when the diagonal complex comes from a full simplicial complex $\Delta_n = P_f\,[n]$ and so the diagonal complex product is the direct product. With the trivial labelling this has automorphism group $\mathfrak{S}_n$. So for a space Y with H the trivial group, the homotopy quotient is $Y^n \times_{\mathfrak{S}_n} E_{\mathfrak{S}_n}$. When Y is a classifying space for a group G, this homotopy quotient is a classifying space for the wreath product of G with $\mathfrak{S}_n$. The homology of these spaces was studied in [32].

2.4 Decompositions

We are now well equipped to calculate a number of homotopical invariants of diagonal complex products; we have a presentation for their fundamental groups and a formula for their homology. However we know little about the higher homotopy groups, in particular we would like to know when diagonal complex products are aspherical, that is, all their higher homotopy groups vanish. Via the maps

$$\begin{tikzcd} Y_i \arrow[r] & \mathbf{Y}(\Gamma,\gamma) \arrow[r, "i"] & \mathbf{Y}^X \arrow[l, bend left, "\pi_i"] \end{tikzcd} \tag{2.4.0.13}$$

we can see that if any Y_i has a higher homotopy group then so does $\mathbf{Y}(\Gamma,\gamma)$. We call (Γ,γ) *aspherical* if for every labelling $l : X \to Z$ and every Z-tuple of aspherical pointed spaces $\mathbf{Y} = (Y_i)_{i\in Z}$, the diagonal complex product $\mathbf{Y}(\Gamma,\gamma)$ is aspherical.

We would like a combinatorial condition which would tell us which diagonal complexes are aspherical, however without one we can still make progress. The following two decompositions are tools which allow us to prove that certain diagonal complexes are aspherical.

2.4.1 Orthogonal decomposition

This is the simpler of the decomposition theorems. Let $X = A \cup B$ for proper subsets A and B of X, let (Γ,γ,l) be a Z-labelled diagonal complex on X and let $\mathbf{Y} = (Y_i)_{i\in Z}$ be a Z-tuple of pointed spaces. Suppose that each $U \in \Gamma$ is contained in at least one of A or B. Then we say that (Γ,γ) *decomposes orthogonally.*

Theorem 2.4.1. *Suppose that (Γ,γ) decomposes orthogonally over a set $X = A \cup B$. Then*

$$\mathbf{Y}(\Gamma,\gamma) = \mathbf{Y}(\Gamma_A,\gamma_A) \cup_{\mathbf{Y}(\Gamma_{A\cap B},\gamma_{A\cap B})} \mathbf{Y}(\Gamma_B,\gamma_B), \tag{2.4.1.1}$$

where Γ_A, Γ_B and $\Gamma_{A\cap B}$ are the full diagonal subcomplexes corresponding to A, B and $A \cap B$ respectively.

Proof. We may characterise $\mathbf{Y}(\Gamma_A,\gamma_A) \subseteq \mathbf{Y}(\Gamma,\gamma)$ as

$$\{(y_i)_{i\in X} \in \mathbf{Y}(\Gamma,\gamma) \mid \text{ if } y_i \neq * \text{ then } i \in A\}. \tag{2.4.1.2}$$

Then it is clear that $\mathbf{Y}(\Gamma_A,\gamma_A) \cap \mathbf{Y}(\Gamma_B,\gamma_B) = \mathbf{Y}(\Gamma_{A\cap B},\gamma_{A\cap B})$ and it remains to show that $\mathbf{Y}(\Gamma_A,\gamma_A)$ and $\mathbf{Y}(\Gamma_B,\gamma_B)$ cover $\mathbf{Y}(\Gamma,\gamma)$. By Proposition 2.1.12, the product $\mathbf{Y}(\Gamma,\gamma)$ is the subspace of $\mathbf{Y}^X$ covered by maps i_U for $U \in \Gamma$. Since $U \in \Gamma$ is either contained in A or B, this means that i_U lands in either of $\mathbf{Y}(\Gamma_A,\gamma_A)$ or $\mathbf{Y}(\Gamma_B,\gamma_B)$. $\square$

Remark 2.4.2. The reason for decomposing diagonal complex products in this way is that one may hope to apply the Seifert van-Kampen theorem. However the natural inclusions of the intersection into the two subspaces is not necessarily injective on fundamental groups. Examples where the inclusions are injective are supplied by any orthogonal decomposition of a diagonal complex coming from a simplial complex. We will see in the next chapter in Section 3.3.3 an example where the inclusions are not injective.

2.4.2 Conical decomposition

Let $x \in X$ and Γ be a proper diagonal complex on X. Suppose that for each $U \in \Gamma$ containing x

$$x^+ = \{x\} \in \gamma(U). \tag{2.4.2.1}$$

Then we say that Γ *decomposes conically at* x.

Theorem 2.4.3. *Suppose that* Γ *defined over a set* X *decomposes conically at* x*. Then there is a diagonal complex* $(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)})$ *and a decomposition*

$$\mathbf{Y}\Gamma = \mathbf{Y}\Gamma_{X-x} \cup_{\mathbf{Y}(\Gamma_{\mathrm{lk}(x)},\gamma_{\mathrm{lk}(x)})} \left(\mathbf{Y}(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)}) \times Y_{l(x)}\right). \tag{2.4.2.2}$$

If $U \in \Gamma_{X-x}$ and $U \cup x^+ \in \Gamma$ then we call U a *link simplex*. The *link dimension* of a link simplex U is defined to be equal to the dimension of $U \cup x^+$ subtract one. The following is the set of link simplices of link dimension one.

$$X_{\mathrm{lk}(x)} = \left\{U \in \Gamma_{X-x} \mid U \cup x^+ \in \Gamma \text{ and } \gamma(U \cup x^+) = \left\{U, x^+\right\}\right\} \tag{2.4.2.3}$$

Write Γ_l for the set of link simplices and define γ_l by

$$\gamma_l(U) = \gamma(U \cup x^+) - x^+ \tag{2.4.2.4}$$

The pair (Γ_l, γ_l) is not necessarily a diagonal complex on X because there may exist $U \in X_{\mathrm{lk}(x)}$ but not in X^+.

Applying γ_l to U gives a partition, applying it again to the elements of $\gamma_l(U)$ gives a still finer partition. We repeatedly apply γ_l until we have reached a partition consisting of elements of $X_{\mathrm{lk}(x)}$ at which point applying γ_l does not refine the partition further. Define $\widehat{U}$ to be the subset of $X_{\mathrm{lk}(x)}$ obtained in this manner. The set $\Gamma_{\mathrm{lk}(x)} \subseteq P_f X_{\mathrm{lk}(x)}$ is defined to be

$$\left\{\widehat{U} \mid U \in \Gamma_l\right\}. \tag{2.4.2.5}$$

The function $\gamma_{\mathrm{lk}(x)} : \Gamma_{\mathrm{lk}(x)} \to P_f \Gamma_{\mathrm{lk}(x)}$ is defined as

$$\gamma_{\mathrm{lk}(x)}(\widehat{U}) = \widehat{\gamma_l(U)} = \left\{\widehat{U'} \mid U' \in \gamma_l(U)\right\}. \tag{2.4.2.6}$$

Lemma 2.4.4. *1. If $V \subseteq U$ are link simplices and $U \in X_{\mathrm{lk}(x)}$ then $V = U$.*

2. If $V \subseteq U$ are link simplices then $\widehat{V} \subseteq \widehat{U}$.

3. The pair $(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)})$ is a diagonal complex on $X_{\mathrm{lk}(x)}$, which we call the link at x.

Proof. 1. V is a subset of U, so $V \cup x^+$ is a subset of $U \cup x^+$ and since Γ is proper $V \cup x^+$ is a descendant of $U \cup x^+$. Hence $\gamma(V \cup x^+)$ is a partial refinement of $\gamma(U \cup x^+) = \{U, x^+\}$. So $V = U$.

2. Since Γ is proper and $V \subseteq U$ we have that $V \cup x^+$ is a descendant of $U \cup x^+$. Hence $\gamma_l(V)$ is a partial refinement of $\gamma_l(U)$. Applying this argument again to the $W \in \gamma_l(V)$ and $W' \in \gamma_l(U)$ with $W \subseteq W'$ we have that $\gamma_l^2(V)$ is a partial refinement of $\gamma_l^2(U)$. We repeat this until we have that $\gamma_l^n(U) = \widehat{U}$. Then since $\gamma_l^n(V)$ is a partial refinement we may apply part (1) to get that it is a subset of $\widehat{U}$ and hence that $\widehat{V} \subseteq \widehat{U}$.

3. Using (2) each of the required properties may be lifted from those of Γ. For example, the simplicial condition: if $\widehat{V}$ is a face of $\widehat{U}$ then V is a face of U and so $\gamma_l(V)$ is a partial refinement of $\gamma_l(U)$. So using (2), $\gamma_{\mathrm{lk}(x)}(V) = \widehat{\gamma_l(V)}$ is a partial refinement of $\gamma_{\mathrm{lk}(x)}(U)$.

□

Let $(\Gamma_{\mathrm{st}(x)}, \gamma_{\mathrm{st}(x)})$ be the diagonal complex on $X_{\mathrm{lk}(x)} \cup x^+$ defined to consist of both U or $U \cup x^+$, where U are the link simplices. Then $\gamma_{\mathrm{st}(x)}(U) = \gamma_{lk(x)}(U)$ and $\gamma_{\mathrm{st}(x)}(U \cup x^+) = \gamma_{lk(x)}(U) \cup \{x^+\}$. This is an example of a direct product of diagonal complexes and $\mathbf{Y}(\Gamma_{\mathrm{st}(x)}, \gamma_{\mathrm{st}(x)}) \cong \mathbf{Y}(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)}) \times Y_{l(x)}$.

Proof of Theorem 2.4.3. The subspace $\mathbf{Y}\Gamma_{X-x} \subseteq \mathbf{Y}\Gamma$ is characterised by

$$\{(y_i)_{i \in X} \in \mathbf{Y}\Gamma \mid y_x = *\}. \tag{2.4.2.7}$$

Meanwhile the subspace $\mathbf{Y}\Gamma_{\mathrm{st}(x)} \subseteq \mathbf{Y}\Gamma$ is characterised by

$$\{(y_i)_{i \in X} \in \mathbf{Y}\Gamma \mid \text{ the set } \{i \mid y_i \neq *\} - x \text{ is a link simplex}\}. \tag{2.4.2.8}$$

And so their intersection consists of those elements whose support is a link simplex. So by the definition of $(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)})$ the intersection is $\mathbf{Y}(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)})$. To prove the amalgamation we just need to note that any $U \in \Gamma$ either does not contain x or $U - x$ is a link simplex, so the two spaces do cover $\mathbf{Y}(\Gamma, \gamma)$.

Finally we have that

$$\mathbf{Y}(\Gamma_{\mathrm{st}(x)}, \gamma_{\mathrm{st}(x)}) = \mathbf{Y}(\Gamma_{\mathrm{lk}(x)}, \gamma_{\mathrm{lk}(x)}) \times Y_{l(x)}. \tag{2.4.2.9}$$

□

Example 2.4.2.1. Let Γ be the infinite two dimensional diagonal complex from Example 2.1.5.4. We saw in Example 2.1.6.2 that Γ is the colimit of a filtration by Γ_n, which is given in Equation (2.1.6.11). The diagonal complex Γ_n is the disjoint union of an infinite number of points $\{n+2, \ldots\}$ and a finite diagonal complex on $\{1, \ldots, n+1\}$ which we denote by Γ'_n. The filtration

$$\Gamma'_1 < \Gamma'_2 < \ldots \tag{2.4.2.10}$$

has colimit Γ and is injective on fundamental group, so to show that Γ is aspherical it is enough to show that Γ'_n is aspherical for each n. We do this by induction using the conical decomposition, the diagonal complex Γ'_1 is just a line so is aspherical, now assume that Γ'_{n-1} is aspherical for some $n \geq 2$. The diagonal complex Γ'_n decomposes conically at $n+1$; the star at $n+1$ is a line and the link is a point. The inclusion of the link space is injective on fundamental group so the conical decomposition allows us to apply 1.1.1 to see that Γ'_n is aspherical. Now by induction Γ'_n is aspherical for every n and hence so is Γ.

2.5 Coset complexes

In the previous section we introduced two decompositions as a means to study diagonal complexes as iterated pushouts. In this section we introduce a different technique which applies to diagonal complex products of groups. The approach is to consider certain coset complexes associated to natural subgroups. The notion of coset complex we use was defined in [1] although this name isn't used. The coset complex of a finite group is studied in [8] as the simplicial complex associated to the *coset poset*.

Recall that the diagonal complex product of groups is defined as the colimit from a certain category P_Γ, see Section 2.1.7. Each object $\{U_i\}$ of P_Γ determines a subgroup

$$\prod_{U_j \in \{U_i\}} G_{l(U_j)} \hookrightarrow \mathbf{G}(\Gamma, \gamma), \tag{2.5.0.11}$$

the fact that this morphism is injective can be seen by taking the composition

$$\prod_{U_j \in \{U_i\}} G_{l(U_j)} \to \mathbf{G}(\Gamma, \gamma) \to \prod_{x \in X} G_{l(x)}, \tag{2.5.0.12}$$

which is a diagonal map and hence injective. The category P_Γ was defined as a subposet closed under meets of the partial partitions of X under partial coarsening. The meet in this poset corresponds to the intersection of subgroups in $\mathbf{G}(\Gamma, \gamma)$, so the collection of subgroups parametrised by P_Γ is closed under intersections.

We now recall some material from the paper [1] on which our treatment of the coset complex is based. Let G be a group with a finite family of subgroups $\mathcal{H} = \{H_j \mid j \in J\}$

closed under intersection. For such a group let $\mathfrak{H}$ be the set of cosets

$$\coprod_{j\in J} G/H_j. \tag{2.5.0.13}$$

Since $\mathcal{H}$ is closed under taking intersections and since a non-empty intersection of cosets $g_1H_1 \cap g_2H_2$ is a coset of the intersection of the respective subgroups $u(H_1 \cap H_2)$, the cosets $\mathfrak{H}$ are closed under taking non-empty intersections. The set $\mathfrak{H}$ may be viewed as a cover of G.

Let X be a set and $\mathfrak{U}$ be a covering of that set, we will assume that $\mathfrak{U}$ is closed under taking non-empty intersections. Under inclusion $\mathfrak{U}$ forms a poset $(\mathfrak{U}, \subseteq)$, which has a nerve $N(\mathfrak{U}, \subseteq)$ which is a simplicial set where the k-simplices are chains

$$U_0 \subseteq U_1 \subseteq \ldots \subseteq U_k. \tag{2.5.0.14}$$

We also define another simplicial set $X^{\text{simp}}(\mathfrak{U})$ where the k-simplices are $(k+1)$-tuples $(x_0, \ldots, x_k)$ of X such that there is a U in $\mathfrak{U}$ containing each x_i for $i = 1, \ldots, k$. Results from Section 1.6 and Theorem 1.4 from [1] imply that $X^{\text{simp}}(\mathfrak{U})$ and $N(\mathfrak{U}, \subseteq)$ are homotopic.

Definition 2.5.1. Let G be a group with a finite family of subgroups $\mathcal{H} = \{H_j \mid j \in J\}$ closed under intersection and let $\mathfrak{H}$ be the associated set of cosets which may be viewed as a cover of G. Then the nerve $N(\mathfrak{H}, \subseteq)$ is called the *coset complex*.

The functor B assigns to a group G the standard simplicial set $B(G)$, a classifying space for G. Recall that the set of k-simplices of $B(G)$ is the set G^k.

Theorem 2.5.2. *Let G be a group with a finite family of subgroups $\mathcal{H} = \{H_j \mid j \in J\}$ closed under intersection. Let $(\mathfrak{H}, \subseteq)$ be the poset of cosets as above. Then the colimit $\operatorname{colim}_{H\in\mathcal{H}} B(H)$ is a classifying space for G if and only if the coset complex $N(\mathfrak{H}, \subseteq)$ is contractible.*

Proof. For a group H, define $X^{\text{simp}}(H)$ to be the simplicial set with k-simplices the $(k+1)$-tuples of H. The face maps are given by forgetting a coordinate and the degeneracy maps given by duplicating coordinates. This has a free H-action given by

$$h.(h_0, \ldots, h_k) = (hh_0, \ldots, hh_k). \tag{2.5.0.15}$$

This construction is functorial so if H is a subgroup of G then there is an inclusion of simplicial sets $X^{\text{simp}}(H) \hookrightarrow X^{\text{simp}}(G)$. Define $X^{\text{simp}}(H) \uparrow^G$ to be the G-orbit of the image of $X^{\text{simp}}(H)$ in $X^{\text{simp}}(G)$. The induction notation is appropriate because both H and G actions are free. Explicitly the k-simplices of $X^{\text{simp}}(H) \uparrow^G$ consist of those $(k+1)$-tuples

which lie in a single coset gH. Note that if $H \leq H'$ then $X^{\text{simp}}(H) \uparrow^G \subseteq X^{\text{simp}}(H') \uparrow^G$ and furthermore

$$X^{\text{simp}}(H_1) \uparrow^G \cap X^{\text{simp}}(H_2) \uparrow^G = X^{\text{simp}}(H_1 \cap H_2) \uparrow^G . \tag{2.5.0.16}$$

So the colimit $\text{colim}_{H \in \mathcal{H}} X^{\text{simp}}(H) \uparrow^G$ is a subspace of $X^{\text{simp}}(G)$. This subspace is the span of all of the inclusions of $X^{\text{simp}}(H) \uparrow^G$ for $H \in \mathcal{H}$, so explicitly it consists of $(k+1)$-tuples $(g_0, \ldots, g_k)$ for which there exists a coset gH containing each g_i for $i = 0, \ldots, k$. But this is precisely the space $X^{\text{simp}}(\mathfrak{H})$. The space $X^{\text{simp}}(\mathfrak{H})$ has a free G-action and we now take the quotient:

$$X^{\text{simp}}(\mathfrak{H})/G = \text{colim}_{H \in \mathcal{H}} X^{\text{simp}}(H) \uparrow^G /G \cong \text{colim}_{H \in \mathcal{H}} X^{\text{simp}}(H)/H. \tag{2.5.0.17}$$

Each $X^{\text{simp}}(H)$ is a contractible space with a free H-action and $X^{\text{simp}}(H)/H \cong B(H)$, this means that

$$X^{\text{simp}}(\mathfrak{H})/G \cong \text{colim}_{H \in \mathcal{H}} B(H). \tag{2.5.0.18}$$

So $\text{colim}_{\mathcal{H}} B(H)$ is a classifying space for G if and only if $X^{\text{simp}}(\mathfrak{H})$ is contractible. However $X^{\text{simp}}(\mathfrak{H})$ is homotopic to $N(\mathfrak{H}, \subseteq)$. We are done. □

This may be applied to a diagonal complex product as follows. Let (Γ, γ) be a Z-labelled diagonal complex and let $\mathbf{G}$ be a Z-tuple of groups. As discussed above the category P_Γ parametrises a family of subgroups of $\mathbf{G}(\Gamma, \gamma)$ which are closed under intersections. This satisfies the conditions of Theorem 2.5.2 and we write $\text{CC}_\Gamma(\mathbf{G})$ for the coset complex of this family.

2.6 Questions and discussion

We now take stock of our progress in the study of diagonal complexes. Each Z-labelled diagonal complex supplies a product, which can be seen as a functor from $\mathcal{C}^Z$ to $\mathcal{C}$ where $\mathcal{C}$ is a category with finite products and colimits. The diagonal complex product is defined as the colimit of a diagram and in the case of pointed spaces this diagram comes from a covering of the product by subspaces.

Every diagonal complex comes with a levelwise filtration and to construct a diagonal complex product one may glue on pieces level by level. This is used to compute the diagonal complex product of groups and the (co)homology of the diagonal complex product of pointed spaces. Recall that the fundamental group of a product of pointed spaces is the product of the fundamental groups.

After Sections 2.2 and 2.3 we have a good description of the fundamental group and (co)homology of a diagonal complex product of pointed spaces. The major gap in our understanding is the higher homotopy groups of diagonal complex products. In particular

we would like to know when a diagonal complex is aspherical, recall that this means that the associated product of aspherical spaces is itself aspherical. The problem with higher homotopy groups is that general colimits are very difficult to calculate and using the level-wise construction is not helpful. However in Section 2.4 we describe two decompositions, the main point of these is to write a complicated product as a pushout of three simpler products. Unfortunately we can not assume that the inclusions from the intersection are injective on fundamental groups but often we are in a position to check the injectivity. This is important because although colimits are in general hard to compute, the pushout of aspherical spaces with injective maps is itself aspherical.

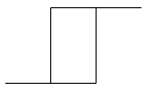
In the next chapter we will use the decompositions to show that particular examples of diagonal complexes are aspherical, however we will also see an example where the injectivity fails. In this example we use the coset complex, which is an alternative method for proving that diagonal complexes are aspherical.

Our aim should be to have a combinatorial condition which would tell us which diagonal complexes are aspherical. The following case would be an important step towards the general case.

Question A. *Let Γ be a proper diagonal complex on a set X. Suppose that $X \in \Gamma$, then is Γ aspherical?*

When working with right-angled Artin groups, which are the graph products of the infinite cyclic group $\mathbb{Z}$, one uses CAT(0) geometry to show that the graph product of the circle S^1 is aspherical. The following diagonal complex is an example where the product of copies of S^1 is not CAT(0), but is aspherical.

$$\left\{\begin{matrix} \{1,2,3\}, & \{2,3,4\}, \\ \{1,2\}, & \{3,4\} \end{matrix}\right\} \cup X^+ \qquad (2.6.0.19)$$



Even when the spaces are not CAT(0) we may hope that geometrical techniques such as Morse theory could be helpful. However these apply only directly to products of S^1 and so an answer to the next question would mean they are helpful for general products.

Question B. *Let Γ be a proper diagonal complex on a set X and let Y_1 be the unit circle. If $\mathbf{Y}(\Gamma, \gamma)$ is aspherical, does this imply that Γ is aspherical?*

Chapter 3

Symmetric automorphisms of free products

We now get onto the main topic of this thesis; automorphisms of free products. In Section 3.1 the different types of automorphism we are interested in are introduced. At the center of our methodology is a classifying space for a group of partial conjugations (introduced shortly). This space is most naturally described geometrically as a moduli space of certain diagrams which we name *cactus products*, see Section 3.2. The paths in this moduli space give a geometric interpretation for partial conjugations.

The previous chapter was devoted to diagonal complexes, which we will put to use in Section 3.3. We show that the moduli space of cactus products may be written as a diagonal complex product, associated to a particular diagonal complex of forest posets. Using this description we may prove the main theorems of this chapter and so compute the (co)homology of the symmetric automorphism groups in Section 3.4.

There is another interpretation of the partial conjugations; this time as the automorphisms which are extendable with respect to the free product functor. We will say what this means in Section 3.5. Then in Section 3.6 the relationship with Outer space will be discussed. Finally in Section 3.7 we give generalisations of the diagonal complex of forest posets, then conjecture that the associated groups describe the symmetric automorphisms of graph products of groups.

3.1 Introduction

Let $\mathbf{G}$ be an n-tuple of groups and let G be their free product. We are interested in automorphisms of this G. Example automorphisms include:

Example 3.1.0.2. 1. Let $\phi \in \mathrm{Aut}(G_1)$ be an automorphism of G_1. Then

$$G_1 * G_2 * \ldots * G_n \xrightarrow{\phi * 1_{G_2} * \ldots * 1_{G_n}} G_1 * G_2 * \ldots * G_n \tag{3.1.0.1}$$

is an automorphism of G which we call a *factor automorphism*. The original automorphism could have been of any G_i.

2. Suppose that $\varphi : G_1 \to G_2$ is an isomorphism. Let τ_{12} be the map which switches G_1 and G_2 like so

$$G_1 * G_2 * \ldots * G_n \xrightarrow{\tau_{12}} G_2 * G_1 * \ldots * G_n, \tag{3.1.0.2}$$

then composing this with

$$G_2 * G_1 * \ldots * G_n \xrightarrow{\varphi^{-1} * \varphi * \ldots * 1_{G_n}} G_1 * G_2 * \ldots * G_n, \tag{3.1.0.3}$$

gives an automorphism of G. There is ofcourse an analogue for each pair $\{i, j\}$.

3. Let $i \neq j$ and h be an element of G_j. Define an automorphism of G by

$$\alpha_i^h : g_k \mapsto \begin{cases} h^{-1} g_k h & \text{if } k = i \text{ and } g_k \in G_k \\ g_k & \text{if } k \neq i \text{ and } g_k \in G_k. \end{cases} \tag{3.1.0.4}$$

This is called a *partial conjugation* of G.

Remark 3.1.1. If we let $i = j$ then Equation 3.1.0.4 defines a factor automorphism induced by the inner automorphism $\mathrm{inn}(h)$ of G_i.

Let $\mathrm{FR}(G)$ be the subgroup of $\mathrm{Aut}(G)$ generated by the partial conjugations. Then it is simple to check that the following relations hold:

$$\alpha_i^h . \alpha_i^g = \alpha_i^{gh} \qquad \text{for } g, h \in G_j, \tag{3.1.0.5}$$

$$\alpha_i^{e_j} = e \qquad \text{for } e_j \text{ the identity in } G_j, \tag{3.1.0.6}$$

$$\alpha_i^h . \alpha_k^{h'} = \alpha_k^{h'} . \alpha_i^h, \tag{3.1.0.7}$$

where $h \in G_j$ and $h' \in G_{j'}$ with $i \neq k$ and $j, j' \notin \{i, k\}$. The last relation says that if automorphisms 'act on different factors' then they commute, note that j could be equal to j'. The next relation

$$\alpha_i^h . \alpha_k^h . \alpha_i^t = \alpha_i^t . \alpha_i^h . \alpha_k^h, \tag{3.1.0.8}$$

where $h \in G_j$ and i, j and k are distinct and $t \in G_k$, requires checking: first the left hand side

$$\alpha_i^h . \alpha_k^h . \alpha_i^t (g_i) = \alpha_i^h . \alpha_k^h (g_i^t) = \alpha_i^h (g_i^{h^{-1}th}) = g_i^{th} \tag{3.1.0.9}$$

$$(g_k) = \alpha_i^h \alpha_k^h (g_k) = g_k^h \tag{3.1.0.10}$$

$$(g_l) = g_l \qquad \text{for } l \neq i, k. \tag{3.1.0.11}$$

Now for the right hand side:

$$\alpha_i^t.\alpha_i^h.\alpha_k^h(g_i) = \alpha_i^t.\alpha_i^h(g_i) = \alpha_i^t(g_i^h) = g_i^{th} \tag{3.1.0.12}$$

$$(g_k) = \alpha_i^t\alpha_i^t(g_k^h) = g_k^h \tag{3.1.0.13}$$

$$(g_l) = g_l \qquad \text{for } l \neq i, k. \tag{3.1.0.14}$$

So the relation holds. We will give a geometric interpretation of these automorphisms in Section 3.2. The following proposition is taken from the work of Collins and Zieshang [15], [16]. It generalises the presentation of the symmetric automorphism group given by McCool [36].

Proposition 3.1.2. *The relations* (3.1.0.5), (3.1.0.6), (3.1.0.7) *and* (3.1.0.8) *along with the generators* α_i^h *give a presentation for* $\mathrm{FR}(G)$.

Recall from Example 3.1.0.2 that each automorphism group $\mathrm{Aut}(G_i)$ acts on the free product G. If $i \neq j$ then the automorphism groups act on different factors and so the actions commute. Hence we have a subgroup

$$\mathbf{Aut(G)} = \prod_{i=1}^{n} \mathrm{Aut}(G_i) \tag{3.1.0.15}$$

of the automorphism group $\mathrm{Aut}(G)$. Now let $h \in G_j$ and α_i^h be a generating partial conjugation. Let $\phi \in \mathrm{Aut}(G_k)$. If $k \neq j$ then ϕ commutes with α_i^h. Now suppose that $k = j$ then

$$\phi\alpha_i^h\phi^{-1} = \alpha_i^{\phi(h)} \tag{3.1.0.16}$$

as can be checked by looking at the action of the left hand side on the factors of G. Hence as a subgroup of $\mathrm{Aut}(G)$, the partial conjugations and the factor automorphisms generate a semi-direct product

$$\mathrm{FR}(G) \rtimes \mathbf{Aut(G)}. \tag{3.1.0.17}$$

We denote this by $P\textsc{Aut}(\mathbf{G})$ and call it the *pure symmetric automorphism group of* G. Now let $\mathbf{Inn(G)}$ be the subgroup

$$\prod_{i=1}^{n} \mathrm{Inn}(G_i) \tag{3.1.0.18}$$

of $\mathbf{Aut(G)}$ consisting of the automorphisms of G induced by inner automorphisms of the factors. Note that this subgroup is not necessarily inner itself, and does not usually contain the inner automorphisms of G either. The subgroup of $\mathrm{Aut}(G)$ generated by $\mathbf{Inn(G)}$ and $\mathrm{FR}(G)$ is called the *Whitehead group of* G and is denoted by

$$\mathrm{WH}(G) = \mathrm{FR}(G) \rtimes \mathbf{Inn(G)}. \tag{3.1.0.19}$$

Consider the n-tuple of groups $\mathbf{G} = (G_i)$ and divide it into equivalence classes under isomorphism. For each equivalence class $(G_i)_{i\in A}$ choose a representative G_i and for each $j \in A - \{i\}$ choose an isomorphism $\varphi_j : G_i \to G_j$, also let φ_i be the identity. Now each pair of isomorphic groups G_j, G_k may be identified uniquely by

$$G_j \xrightarrow{\varphi_j^{-1}} G_i \xrightarrow{\varphi_k} G_k. \tag{3.1.0.20}$$

Using this isomorphism in 2) of Example 3.1.0.2 we get an automorphism of G. All such automorphisms generate a subgroup

$$\mathfrak{S}_{\mathbf{G}} \cong \mathfrak{S}_{n_1} \times \ldots \times \mathfrak{S}_{n_k}, \tag{3.1.0.21}$$

where n_i are the sizes of the equivalence classes. Note that different choices for the isomorphisms ϕ_j would have given a different embedding of (3.1.0.21) into $\mathrm{Aut}(G)$.

Let $\sigma \in \mathfrak{S}_{\mathbf{G}}$ and $h \in G_j$ with α_i^h an associated partial conjugation. Then

$$\sigma\alpha_i^h\sigma^{-1} = \alpha_{\sigma(i)}^{\varphi_{\sigma(j)}\varphi_j^{-1}(h)}. \tag{3.1.0.22}$$

For convenience we will write h for $\varphi_{\sigma(j)}\varphi_j^{-1}(h)$ using $\varphi_{\sigma(j)}\varphi_j^{-1}$ to identify the sets G_i and $G_{\sigma(i)}$. The subgroup generated by $\mathfrak{S}_{\mathbf{G}}$ and $\mathrm{FR}(G)$ is a semidirect product

$$\Sigma\mathrm{FR}(G) = \mathrm{FR}(G) \rtimes \mathfrak{S}_{\mathbf{G}} \tag{3.1.0.23}$$

and is called the *symmetric Fouxe-Rabinovitch group*. The group $\mathfrak{S}_{\mathbf{G}}$ also acts by conjugation on $\mathbf{Aut}(\mathbf{G})$ and $\mathbf{Inn}(\mathbf{G})$, so we may define subgroups

$$\Sigma\mathrm{WH}(G) = \mathrm{WH}(G) \rtimes \mathfrak{S}_{\mathbf{G}} \text{ and} \tag{3.1.0.24}$$

$$\Sigma\mathrm{Aut}(G) = P\mathrm{Aut}(G) \rtimes \mathfrak{S}_{\mathbf{G}}, \tag{3.1.0.25}$$

which are called the *symmetric Whitehead group of* G and the *symmetric automorphism group of* G respectively. The following proposition is derived from the Kurosh subgroup theorem.

Proposition 3.1.3. *Let* $\mathbf{G}$ *be an* n*-tuple of groups such that each* G_i *is freely indecomposable and not isomorphic to the infinite cyclic group. Then*

$$\Sigma\mathrm{Aut}(G) \cong \mathrm{Aut}(G). \tag{3.1.0.26}$$

Summarising we have a diagram of subgroups of $\mathrm{Aut}(G)$:

$$\begin{tikzcd} \Sigma\mathrm{FR}(G) \arrow[r] & \Sigma\mathrm{WH}(G) \arrow[r] & \Sigma\mathrm{AUT}(G) \\ \mathrm{FR}(G) \arrow[r] \arrow[u] & \mathrm{WH}(G) \arrow[r] \arrow[u] & P\mathrm{AUT}(G). \arrow[u] \end{tikzcd} \tag{3.1.0.27}$$

Every arrow is a normal embedding as are their composites. Factoring out by the normal subgroup $\mathrm{FR}(G)$ gives the corresponding diagram.

$$\begin{tikzcd} \mathfrak{S}_{\mathbf{G}} \arrow[r] & \mathbf{Inn(G)} \rtimes \mathfrak{S}_{\mathbf{G}} \arrow[r] & \mathbf{Aut(G)} \rtimes \mathfrak{S}_{\mathbf{G}} \\ I \arrow[r] \arrow[u] & \mathbf{Inn(G)} \arrow[r] \arrow[u] & \mathbf{Aut(G)}. \arrow[u] \end{tikzcd} \tag{3.1.0.28}$$

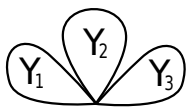
Most importantly we have that

$$\Sigma\mathrm{AUT}(G) \cong \mathrm{FR}(G) \rtimes \mathbf{Aut(G)} \rtimes \mathfrak{S}_{\mathbf{G}} \tag{3.1.0.29}$$

We have omitted the bracketing deliberately, either semi-direct product, $\rtimes$ may be evaluated first.

3.2 Moduli spaces of cactus products

In the category of pointed spaces the analogue of the n-fold free product of groups is the n-fold wedge sum; this takes n pointed spaces and identifies their basepoints.



(3.2.0.30)

The fundamental group of a wedge sum is the free product of the fundamental groups of the summands. The assumption that the spaces have the homotopy type of CW complexes means that changing the basepoint does not alter the homotopy type[1], so forming the wedge sum with a different choice of basepoints gives the same homotopy type. The moduli space of cactus products describes a space of possible ways to 'wedge' n spaces together.

3.2.1 Cactus products

Let t be a tree with vertex set $[n] = \{1, \ldots, n\}$. A tree t with a chosen vertex i is called a *rooted tree*. The edges of a rooted tree may be oriented by pointing them towards the root,

[1] we are assuming that all our spaces are connected

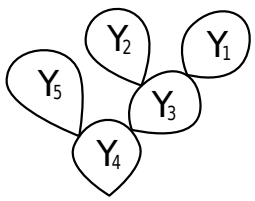
for an edge e the source vertex is denoted $s(e)$ and the target vertex $t(e)$. An example of a rooted tree with root 4:

$$\begin{tikzcd} & 2 \arrow[dr] & & 1 \arrow[dl] \\ 5 \arrow[dr] & & 3 \arrow[dl] & \\ & 4 & & \end{tikzcd} \tag{3.2.1.1}$$

Now let $(Y_i, *_i)$ be pointed spaces for $i = 1, \ldots, n$. A *cactus diagram* T over rooted tree t consists of an edge labelling of t: to each edge e there is a label $y_e \in Y_{t(e)}$. A cactus diagram T gives a *cactus product space* Y_T:

$$\frac{Y_1 \amalg \ldots \amalg Y_n}{*_{s(e)} \sim y_e \mid e \in t}. \tag{3.2.1.2}$$

An example corresponding to the rooted tree (3.2.1.1) above:



(3.2.1.3)

If each pointed space is homotopy equivalent to a CW complex then the cactus product space is homotopic to the wedge product $Y_1 \vee \ldots \vee Y_n$. A cactus product space has a canonical choice of basepoint given by the basepoint of the space corresponding to the root of the tree. For each i there is a natural inclusion $Y_i \hookrightarrow Y_T$, but note that this inclusion does not respect the basepoint. The *cactus product* consists of the cactus product space in the context of the diagram

$$\begin{tikzcd} & Y_T & \\ Y_1 \arrow[ur] & \cdots\cdots & Y_n. \arrow[ul] \end{tikzcd} \tag{3.2.1.4}$$

3.2.2 The moduli space of cactus products

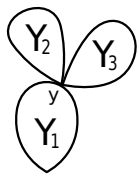
Let T and T' be two cactus products with the same number of spaces. Then we say that T and T' are *congruent* if there is a map $Y_T \to Y_{T'}$ making the diagram

$$\begin{tikzcd} Y_T \arrow[rr] & & Y_{T'} \\ & Y_i \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.2.2.1}$$

commute for each i. For example the two decorated trees

$$\begin{tikzcd} 2 \arrow[dr, "y"'] & & 3 \arrow[dl, "y"] \\ & 1 & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} 3 \arrow[d, "*"] \\ 2 \arrow[d, "y"] \\ 1 \end{tikzcd} \tag{3.2.2.2}$$

give the same space:



(3.2.2.3)

Lemma 3.2.1. *For a fixed tree t, a cactus product Y_T determines the labelling T. And so the set of cactus products over t is naturally identified with*

$$\prod_e Y_{t(e)}. \tag{3.2.2.4}$$

Proof. Given an edge e the intersection

$$Y_{s(e)} \cap Y_{t(e)} \hookrightarrow Y_{t(e)}. \tag{3.2.2.5}$$

consists of a single point, which is the label $y_e \in Y_{t(e)}$. □

The *moduli space of cactus products* $\mathcal{M}\mathbf{Y}$ is defined to be the set of cactus products modulo congruence, with the topology inherited from the CW product topology of (3.2.2.4). It has a canonical basepoint given by the wedge product, which is realised by any tree with every label a basepoint.

3.2.3 An embedding into a direct product

Proposition 3.2.2. *The moduli space of cactus graphs $\mathcal{M}\mathbf{Y}$ may be embedded into*

$$Y_1^{n-1} \times \ldots \times Y_n^{n-1}. \tag{3.2.3.1}$$

Proof. A coordinate in the product space is denoted

$$(y_{ij})_{i \neq j}, \tag{3.2.3.2}$$

where $i, j = 1, \ldots, n$ and y_{ij} is an element of Y_i. Let T be a cactus diagram over a tree t. For each pair (i, j) choose a path within the cactus product Y_T from the basepoint of Y_j to the basepoint of Y_i. Let y_{ij} be the element in Y_i first reached by the path.

Combinatorially this may be defined by taking the unique undirected path in t from j to i. If each edge of the path points towards the root, then define y_{ij} to be y_e where e is the last edge in the path (and so has end vertex i). If the path goes through the root then let y_{ij} be the basepoint $* \in Y_i$.

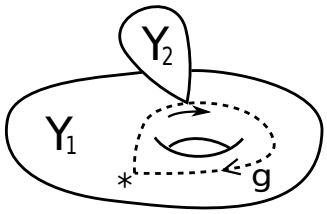
This map is both well-defined and injective. □

3.2.4 Geometric interpretation of partial conjugations

A based loop in the moduli space of cactus products of pointed spaces $Y_1, \ldots, Y_n$ can be seen to act on the free product of groups

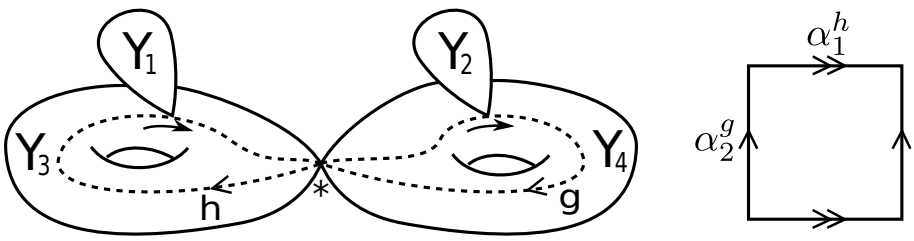
$$\pi_1 Y_1 * \ldots * \pi_1 Y_n. \tag{3.2.4.1}$$

This is illustrated by the following diagram which shows a path representing the automorphism α_2^g.



(3.2.4.2)

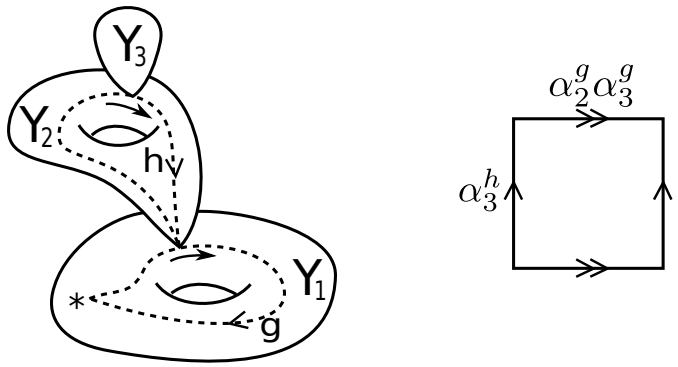
Any loop γ contained in Y_2 is taken to a loop which first follows g in Y_1 and then follows γ before returning around g in the opposite direction. The commutator relations between partial conjugations can be viewed as embedded tori inside the moduli space, for example the following represents such a torus:



(3.2.4.3)

The factor space Y_1 may pass around loop h independently to space Y_2 passing around loop g. The torus pictured on the right records a homotopy between paths representing

$\alpha_1^h\alpha_2^g$ and $\alpha_2^g\alpha_1^h$. This next diagram represents a different type of torus:



(3.2.4.4)

One edge of the torus has space Y_3 passing around loop h, which represents the automorphism α_3^h. The other edge has both spaces Y_2 and Y_3 passing around loop g, this represents the automorphism $\alpha_2^g\alpha_3^g$. Therefore this torus represents a homotopy between paths representing $\alpha_2^g\alpha_3^g$ and α_3^h.

3.3 The diagonal complex of forest posets

The moduli space of cactus products has an intuitive description, but when it comes to working hands on with the space; decomposing it into smaller pieces, calculating its fundamental group and calculating its homology, an alternative description using diagonal complexes is far more powerful. The diagonal complex that does this consists of a set of different partial orders on a fixed set $[n]$. The partial orders of interest are those which give a Hasse diagram of planted forests, hence the name *forest posets*.

3.3.1 The construction of $\Gamma_{\mathcal{F}_n}$

Let $(P, \leq)$ be a finite poset. The *Hasse diagram* of $(P, \leq)$ is the directed graph with vertex set P and an edge $\overrightarrow{ij}$ when $i < j$ are adjacent, that is, when $i \leq k \leq j$ implies that $k = i$ or j. Conversely a directed graph without directed loops defines a poset given by setting $i < j$ when $\overrightarrow{ij}$ is an edge and then taking the transitive closure. A directed graph is called a *planted forest* if for each vertex the number of incoming edges is not greater than one, if there are no cycles and if the edge set is non-empty. Under the transitive closure the planted forests are taken to posets $(P, \leq)$ with the *underset condition*:

$$\text{for all } x \in P, \{y \mid y \leq x\} \text{ is a total order.} \tag{3.3.1.1}$$

In fact the correspondence is a bijection between the set of planted forests and the set of finite non-trivial posets satisfying (3.3.1.1). Let P_n be the set $[n]$. A poset $(P_n, \leq)$ defines

a set

$$\{(i,j) \mid i < j\}, \tag{3.3.1.2}$$

which is a subset of $X_n = P_n \times P_n - \Delta P_n$. We write $\Gamma_{\mathcal{F}_n} \subseteq P_f X_n$ for the set of subsets of X_n given by forest posets $(P_n, \leq)$. An uppercase letter such as U will be used to denote the poset $(P_n, \leq)$, the subset of X_n and the corresponding planted forest. This should not cause confusion.

We will next define a map $\gamma_{\mathcal{F}_n} : \Gamma_{\mathcal{F}_n} \to P_f \Gamma_{\mathcal{F}_n}$. For a forest poset U take $i < j$, then choose a maximal path from i to j

$$i = i_0 < i_1 < \ldots < i_m = j \tag{3.3.1.3}$$

The pair (i, i_1) necessarily gives an edge $\overrightarrow{ii_1} \in U$, write $\mu_U(i,j) = \overrightarrow{ii_1}$. Then μ_U is a map from the set U to $E(U)$, the edge set of the planted forest. Since the path is within the set $\{k \mid k \leq j\}$ which is a total order, the path is unique and so μ_U is well-defined. For an edge $\overrightarrow{ij}$ we have $\mu_U(i,j) = \overrightarrow{ij}$, so μ_U is surjective. The map μ_U defines a partition of U, each subset is given in the form $\mu_U^{-1}(\overrightarrow{ij})$ for some edge of U, and this defines $\gamma(U)$. For example

$$\gamma_{\mathcal{F}_n}\left(\begin{array}{ccc} 1 & & \\ \uparrow & & \\ 2 & & 4 \\ \uparrow & \nearrow & \\ 3 & & \end{array}\right) = \left\{\begin{array}{ccccc} 1 & 1 & & 2 & 3 \\ \uparrow, & \uparrow & \nearrow & , & \uparrow \\ 2 & 3 & & & 4 \end{array}\right\}. \tag{3.3.1.4}$$

Proposition 3.3.1. *The pair $(\Gamma_{\mathcal{F}_n}, \gamma_{\mathcal{F}_n})$ is a diagonal complex on X_n. Furthermore this diagonal complex is proper.*

Proof. For each element (i,j) of $X_n = P_n \times P_n - \Delta P_n$, the tree $i \to j$ gives the poset given by $\{(i,j)\}$ and so condition (1) in the definition of a diagonal complex is satisfied. That $\gamma(U)$ is a partition of U is immediate from the definition. If $\gamma(U) = \{U\}$ then U is a forest with only one edge and so is contained in X_n^+, hence condition (2) is satisfied.

Next we must show that $\Gamma_{\mathcal{F}_n}$ contains the faces of each $U \in \Gamma_{\mathcal{F}_n}$. Let $\{U_i\}_{i \in E(U)} = \gamma(U)$ and suppose $Z \subseteq E(U)$. We write U_Z for $\bigcup_{i \in Z} U_i$. Then U_Z is the subset of U consisting of $i <_Z j$ when the maximal path in U

$$i = i_0 < i_1 < \ldots < i_m = j \tag{3.3.1.5}$$

has $\overrightarrow{ii_0} \in Z$. To check that this is a poset we need only check that it is transitively closed. This is immediate because for $i < j < k$ the maximal path from i to j is a subpath of the maximal path from i to k, so $i <_Z j$ implies that $i <_Z k$. We now need to check that each underset in U_Z is a total order. So suppose that $i \leq_Z k$ and $j \leq_Z k$. Since both $i < k$ and $j < k$ in U and U satisfies the underset condition, we may assume that $i < j$. But since the maximal path joining i and j is a subpath of the path joining i and k, then $i <_Z j$

and so U_Z satisfies the underset condition and so is in $\Gamma_{\mathcal{F}_n}$.

For the remainder of condition (3) it is enough to show that each $V \in \gamma_{\mathcal{F}_n}(U_Z)$ is contained in some $U' \in \gamma_{\mathcal{F}_n}(U)$. The set V corresponds to some edge $\overrightarrow{ij} \in E(U_Z)$, let $\overrightarrow{ik}$ be the edge $\mu_U(i < j)$ and let $U' = \mu_U^{-1}(\overrightarrow{ik})$. For $i < l$ in V, we have $j \leq l$ and so the maximal path joining i and j in U travels through j and hence the maximal path joining i and j is a subpath. So the maximal path joining i and l in U starts with $\overrightarrow{ik}$ and hence $\mu_U(i, l) = \overrightarrow{ik}$ and so $(i, l) \in U'$. We have shown that $V \subseteq U'$ and have completed the proof that $(\Gamma_{\mathcal{F}_n}, \gamma_{\mathcal{F}_n})$ is a diagonal complex.

Finally to see that $(\Gamma_{\mathcal{F}_n}, \gamma_{\mathcal{F}_n})$ is proper we must show that for each $U \in \Gamma_{\mathcal{F}_n}$ the maximal faces $U - \mu_U^{-1}(\overrightarrow{ij})$ are maximal subsets of U in $\Gamma_{\mathcal{F}_n}$. So suppose that

$$U - \mu_U^{-1}(\overrightarrow{ij}) < V \leq U \tag{3.3.1.6}$$

in $\Gamma_{\mathcal{F}_n}$. Let (i, k) be an element in $V \cap \mu_U^{-1}(\overrightarrow{ij})$. Unless $j = k$ (and so $\overrightarrow{ij} \in V$), we have that

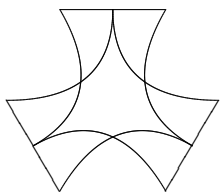
$$(j, k) \in U - \mu_U^{-1}(\overrightarrow{ij}) < V \tag{3.3.1.7}$$

and so by the underset condition

$$(i, k), (j, k) \in V \in \Gamma_{\mathcal{F}_n} \Rightarrow (i, j) \in V. \tag{3.3.1.8}$$

Therefore we must have $V = U$ by transitivity and so $U - \mu_U^{-1}(\overrightarrow{ij})$ is maximal. So by Proposition 2.1.5 the diagonal complex $\Gamma_{\mathcal{F}_n}$ is proper. □

We may draw the diagonal complex when $n = 3$ as follows:



(3.3.1.9)

To assign an $[n]$-labelling to $\Gamma_{\mathcal{F}_n}$ we give the pair (i, j) the label $i \in [n]$. By the definition of $\gamma_{\mathcal{F}_n}$ the U' in $\gamma_{\mathcal{F}_n}(U)$ are corollas with base i and so are labelled by i. Hence this labelling of X_n is compatible with the diagonal complex structure of $\Gamma_{\mathcal{F}_n}$, in fact it is the universal labelling.

3.3.2 Properties of $\mathbf{Y}\Gamma_{\mathcal{F}_n}$

Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an n-tuple of pointed spaces. We are now able to prove the following assertion.

Theorem 3.3.2. *The fundamental group $\pi_1(\mathbf{Y}\Gamma_{\mathcal{F}_n})$ is isomorphic to the Fouxe-Rabinovitch group* $\mathrm{FR}(\pi_1(Y_1) * \ldots * \pi_1(Y_n))$.

Proof. Using Theorem 2.2.1 the relations of a diagonal complex product of groups all come from either the summand groups or from the $U \in \Gamma_{\mathcal{F}_n}$ of dimension 2. In $\Gamma_{\mathcal{F}_n}$ these are

$$\begin{tikzcd} k & j \\ i \arrow[u] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} k & j \\ l \arrow[u] & i \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} j \\ i \arrow[u] \\ k \arrow[u] \end{tikzcd}. \tag{3.3.2.1}$$

Let $G_i = \pi_1(Y_i)$, then a presentation for $\mathbf{G}\Gamma_{\mathcal{F}_n}$ has generators

$$g_{ij} \tag{3.3.2.2}$$

for each $(i, j) \in X_n$ and $g \in G_i$. The relations consist of

$$g_{ij}h_{ij} = (gh)_{ij} \tag{3.3.2.3}$$

coming from the generating groups and

$$[g_{ik}, g_{ij}], \quad [g_{lk}, g_{ij}] \quad \text{and} \quad [g_{ki}g_{kj}, g_{ij}], \tag{3.3.2.4}$$

corresponding respectively to the forests in (3.3.2.1). Equating g_{ij} with $\alpha_j^{g_i^{-1}} \in \mathrm{FR}(G_1 * \ldots * G_n)$ gives an equivalence between this presentation and the presentation in Proposition 3.1.2. □

Theorem 3.3.3. *The moduli space of cactus products $\mathcal{M}\mathbf{Y}$ is isomorphic to $\mathbf{Y}\Gamma_{\mathcal{F}_n}$.*

Proof. By Proposition 3.2.2, the moduli space $\mathcal{M}\mathbf{Y}$ embeds into

$$Y_1^{n-1} \times \ldots \times Y_n^{n-1}. \tag{3.3.2.5}$$

We will now prove that this coincides with the embedding of $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ into $\mathbf{Y}^{X_n}$, see Proposition 2.1.12. For each tree t, there is a planted forest U which is isomorphic to t. The spaces $\mathbf{Y}^U$ and

$$t(\mathbf{Y}) := \{y(T) \mid T \text{ is a cactus diagram over } t\} \tag{3.3.2.6}$$

have the same image in $\mathbf{Y}^X$. Hence the moduli space is contained in $\mathbf{Y}\Gamma_{\mathcal{F}_n}$. Now for a planted forest $U \in \Gamma_{\mathcal{F}_n}$, there exists a maximal, connected planted forest V containing U. Each edge of U is contained in V and so each $U' \in \gamma_{\mathcal{F}_n}(U)$ is contained in some $V' \in \gamma_{\mathcal{F}_n}(V)$. Therefore $\mathbf{Y}^U$ is contained in $\mathbf{Y}^V$ and since V is a tree t, this shows that $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ is contained in the moduli space. □

3.3.3 Decompositions of $\Gamma_{\mathcal{F}_n}$

We have so far shown that the diagonal complex of forest posets serves to describe the moduli space of cactus products. Using the theory built up in the previous chapter we have shown that the fundamental group of the moduli space is the Fouxe-Rabinovitch group of the factor spaces. Our eventual aim is to show that the moduli spaces provide classifying spaces for the Fouxe-Rabinovitch groups. The technique of decomposing diagonal complexes does not quite suffice to prove this, however the decompositions are still of interest and we do manage a proof of an analogous result for some subgroups.

In the following we will write Γ_V for the full diagonal subcomplex of $\Gamma_{\mathcal{F}_n}$ spanned by a subset $V \subseteq X_n$.

Lemma 3.3.4. *For each $V \subseteq X_n$, either $V \in \Gamma_{\mathcal{F}_n}$, or there exist (i,j) and $(k,l) \in V$ such that no $U \in \Gamma_V$ contains both (i,j) and (k,l).*

Proof. Suppose that there exist $i \neq j$ such that V contains both (i,j) and (j,i). Then choose this pair, no poset could contain both. Suppose now that V does not contain such a pair. If V were not transitively closed then there would exist (i,j) and (j,k) in V, but with $(i,k) \notin V$. Then we would pick this pair, no poset can contain the pair without containing (i,k). So now we may assume that V is transitively closed and does not contain both (i,j) and (j,i) for any i and j. So V is a poset. Now either V satisfies the property that the under poset of any element is a total order in which case $V \in \Gamma_{\mathcal{F}_n}$, or the Hasse diagram of V contains the following sub diagram

$$\begin{tikzcd} & k \\ i \arrow[ur] & j \arrow[u] \end{tikzcd} \tag{3.3.3.1}$$

for some i, j, k. But this means that neither (i,j) nor (j,i) is contained in V. So no $U \in \Gamma_V$ may contain both (i,k) and (j,k). □

This means that if $V \notin \Gamma_{\mathcal{F}_n}$ then Γ_V decomposes orthogonally with $A = V - (i,j)$ and $B = V - (k,l)$.

Lemma 3.3.5. *Suppose j is maximal in a poset $U \in \Gamma_{\mathcal{F}_n}$, that is j has one ingoing edge $\overrightarrow{ij}$ and no outgoing edges in the planted forest U. Then for each $W \in \Gamma_U$ with $(i,j) \in W$ the singleton $\{(i,j)\}$ is contained in $\gamma_{\mathcal{F}_n}(W)$.*

Proof. For any $W \in \Gamma_U$ containing (i,j), j will be maximal. Also since W is a subposet of U, the edge $\overrightarrow{ij}$ is in the Hasse diagram of W. So

$$\mu_U^{-1}(\overrightarrow{ij}) = \{(i,j)\}. \tag{3.3.3.2}$$

□

This means that if $U \in \Gamma_{\mathcal{F}_n}$ then Γ_U decomposes conically at (i,j). We now turn our attention to identifying the link at (i,j). Supposing that $U \neq \{(i,j)\}$, let $U_{\hat{j}}$ be the poset given by removing j from the poset U.

Lemma 3.3.6. *As diagonal complexes $\Gamma_{\mathrm{lk}(i,j)}$ is isomorphic to $\Gamma_{U_{\hat{j}}}$.*

Proof. Suppose that $W \in \Gamma_U$ contains (i,j) and that $\gamma_{\mathcal{F}_n}(W) = \{\{(i,j)\}, W - (i,j)\}$. Then W has two edges and so is of the form

$$\begin{tikzcd} k & j \\ i \arrow[u] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} k & j \\ l \arrow[u] & i \arrow[u] \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} j \\ i \arrow[u] \\ k \arrow[u] \end{tikzcd}. \tag{3.3.3.3}$$

So $X_{\mathrm{lk}(i,j)}$ consists of sets

$$\begin{cases} \{(k,l)\} & \text{with } k \text{ and } l \text{ distinct from } i \text{ and } j, \\ \{(i,k)\} & \text{with } k \text{ distinct from } i \text{ or} \\ \{(k,i),(k,j)\} & \text{with } k \text{ distinct from } i \text{ and } j. \end{cases} \tag{3.3.3.4}$$

The map from $\Gamma_{U_{\hat{j}}}$ to $\Gamma_{\mathrm{lk}(i,j)}$ is defined by

$$(k,l) \mapsto \begin{cases} \{(k,l)\} & \text{if } k,l \text{ are distinct from } i, \\ \{(i,l)\} & \text{if } k = i \\ \{(k,i),(k,j)\} & \text{if } l = i. \end{cases} \tag{3.3.3.5}$$

That these give isomorphic diagonal complexes can be seen by observing that the map takes a planted forest $W \in \Gamma_{U_{\hat{j}}}$ and adds j with an edge $\overrightarrow{ij}$. The inverse just deletes the edge $\overrightarrow{ij}$. □

The remaining lemma regards the inclusion of $\Gamma_{\mathrm{lk}(i,j)}$ into Γ_U.

Lemma 3.3.7. *The map $\Gamma_{\mathrm{lk}(i,j)} \to \Gamma_U$ gives an injection*

$$\pi_1 \mathbf{Y}\Gamma_{\mathrm{lk}(i,j)} \to \pi_1 \mathbf{Y}\Gamma_U \tag{3.3.3.6}$$

for any n-tuple of pointed spaces $\mathbf{Y} = (Y_1, \ldots, Y_n)$.

Proof. Since the fundamental group of a diagonal complex product of spaces is the diagonal complex product of the fundamental groups, we need to show that for groups $G_1, \ldots, G_n$ the map

$$\mathbf{G}\Gamma_{U_{\hat{j}}} \cong \mathbf{G}\Gamma_{\mathrm{lk}(i,j)} \to \mathbf{G}\Gamma_{U-(i,j)} \tag{3.3.3.7}$$

is injective. By (3.3.3.5) the map is defined by

$$g_{kl} \mapsto \begin{cases} g_{kl} & \text{if } k, l \text{ distinct from } i, \\ g_{il} & \text{if } k = i \\ g_{ki} g_{kj} & \text{if } l = i. \end{cases} \tag{3.3.3.8}$$

By showing that the one-sided inverse $\mathbf{G}\Gamma_{U-(i,j)} \to \mathbf{G}\Gamma_{U_{\widehat{j}}}$

$$g_{kl} \mapsto \begin{cases} g_{kl} & \text{if } k, l \text{ distinct from } i, j, \\ g_{il} & \text{if } k = i \\ g_{ki} & \text{if } l = i \\ e & \text{if } l = j \end{cases} \tag{3.3.3.9}$$

is well-defined we will prove the lemma. The relations that need checking are the ones containing g_{kj} for some k. Relations of this type are given by the forests

$$\begin{tikzcd} j & l \\ k \arrow[u] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} j & m \\ k \arrow[u] & l \arrow[u] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} j \\ k \arrow[u] \\ l \arrow[u] \end{tikzcd}. \tag{3.3.3.10}$$

Written out they are

$$[g_{kj}, h_{kl}], \quad [g_{kj}, h_{lm}] \quad \text{and} \quad [g_{kj}, h_{lk} h_{lj}] \tag{3.3.3.11}$$

for $g \in G_k$ and $h \in G_k$ for the first and $h \in G_l$ for the second and third relations. When mapped to $\mathbf{G}\Gamma_{U_{\widehat{j}}}$ these become

$$[e_{kj}, h_{kl}], \quad [e_{kj}, h_{lm}] \quad \text{and} \quad [e_{kj}, h_{lk} e_{lj}], \tag{3.3.3.12}$$

so each is taken to the identity and the map is well-defined. □

We have shown that both types of decompositions occur for the diagonal complex of forest posets. In the second case the decompositions involve injective maps on fundamental groups allowing the Seifert van-Kampen theorem to be applied. This allows the following theorem to be proved.

Theorem 3.3.8. *Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be aspherical pointed spaces and let $T_n \in \Gamma_{\mathcal{F}_n}$ be the total order $1 < 2 < \ldots < n$. Then the diagonal complex product $\mathbf{Y}\Gamma_{T_n}$ is also aspherical.*

Proof. The proof is by induction on elements of Γ_{T_n}; we wish to prove that for each forest

poset $V \subseteq T_n$, the space $\mathbf{Y}\Gamma_V$ is aspherical. We use induction on the cardinality of the sets. If $|V|= 1$ then $\mathbf{Y}\Gamma_V$ is just one of the spaces Y_i and so we are done. Suppose now that $|V|= m$. The diagonal complex Γ_V decomposes conically at some (i, j) by Lemma 3.3.5. The set $V - (i, j)$ is of lesser cardinality so $\mathbf{Y}\Gamma_{V-(i,j)}$ may be assumed to be aspherical. The diagonal complex $\Gamma_{\mathrm{lk}(i,j)}$ is isomorphic to $\Gamma_{U_{\hat{j}}}$ and $V - \{(k, j) \mid k < j\}$ has cardinality less than V, so $\mathbf{Y}\Gamma_{\mathrm{lk}(i,j)}$ may be assumed to be aspherical. Finally

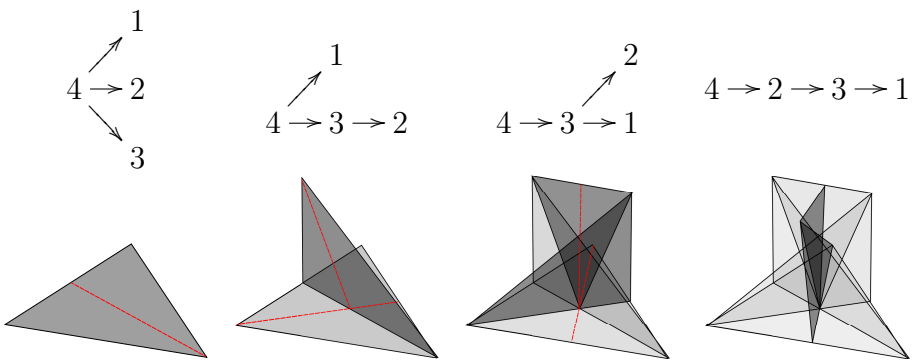
$$\mathbf{Y}\Gamma_{\mathrm{st}(i,j)} \cong \mathbf{Y}\Gamma_{\mathrm{lk}(i,j)} \times Y_i, \tag{3.3.3.13}$$

so is aspherical. By Lemma 3.3.7 the map

$$\mathbf{Y}\Gamma_{\mathrm{lk}(i,j)} \to \mathbf{Y}\Gamma_{U-(i,j)} \tag{3.3.3.14}$$

is injective on fundamental groups. So by Theorem 2.4.3 the space $\mathbf{Y}\Gamma_V$ may be written as the pushout of aspherical spaces with maps injective on fundamental groups and so is itself aspherical. □

This following series of diagrams shows a diagonal complex Γ_U being constructed by adding vertices successively. The dotted lines represent where the next parts will be glued, while the darker triangles are those that have most recently been added.



(3.3.3.15)

So we have now constructed a classifying space for groups $\mathbf{G}\Gamma_{T_n}$ using the conical decomposition to prove that it is aspherical. It may be hoped that the orthogonal decomposition can be used to complete the proof that $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ is a classifying space for the Fouxe-Rabinovitch group $\mathbf{G}\Gamma_{\mathcal{F}_n}$ when $\mathbf{Y} = (Y_1, \ldots, Y_n)$ are classifying spaces for $\mathbf{G}$.

However we will now show that this approach fails because arrows in the orthogonal decomposition given by Lemma 3.3.4 are not necessarily injective. To see this we will use the results of [41]. In this paper the group $P\Sigma_3 \cong \mathrm{FR}(\mathbb{Z}^{*3})$ is considered. This group has a presentation with generating set $A = \{\alpha_{ij}\}$ for $i \neq j \in \{1, 2, 3\}$. The relations are certain commutator brackets, a diagram representing these is pictured in (3.3.3.18).

There is a homomorphism

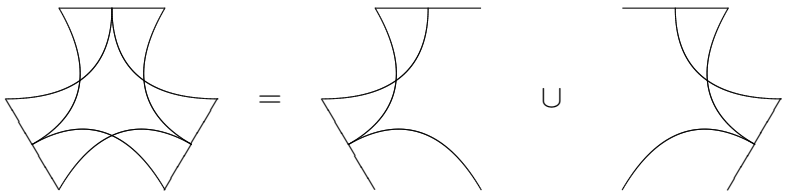
$$P\Sigma_2 \to F_2 = \langle a, b \rangle \,, \tag{3.3.3.16}$$

which takes α_{23} to a, α_{32} to b and the remaining generators α_{1i} and α_{i1} to the identity. The kernel K_3 is generated as a subgroup by the generators α_{1i} and α_{i1}. In [41] it is shown that the kernel is not finitely presented using a short homological argument.

However in Lemma 3.3.4 it is shown that $\Gamma_{\mathcal{F}_3}$ decomposes orthogonally over the subsets $A - \alpha_{23}$ and $A - \alpha_{32}$. Hence $P\Sigma_3 \cong \mathbb{Z}\Gamma_{\mathcal{F}_3}$ is the pushout of

$$\mathbb{Z}\Gamma_{A-\alpha_{32}} \leftarrow \mathbb{Z}\Gamma_{A-\{\alpha_{32},\alpha_{23}\}} \to \mathbb{Z}\Gamma_{A-\alpha_{23}}. \tag{3.3.3.17}$$

We may write this diagrammatically as



(3.3.3.18)

with intersection

(3.3.3.19)

• •

Were the two arrows injections this would mean that the map from $P\Sigma_3$ to F_2 had kernel the central group in the diagram above. However this group has finite presentation given by generators $\{\alpha_{12}, \alpha_{13}, \alpha_{21}, \alpha_{31}\}$ and the single relation $[\alpha_{31}, \alpha_{21}]$. This contadicts the result that K_3 is not finitely presented, hence the arrows in (3.3.3.17) can not both be injective.

3.3.4 McCullough-Miller space and coset complexes

To prove the asphericity of the diagonal complex of forest posets and so show that the moduli spaces of cactus products of classifying spaces are themselves classifying spaces we will use a variant of McCullough-Miller space introduced in [13] and the concept of a coset complex for a diagonal complex product of groups as described in Section 2.5 of the previous chapter.

Bipartite planted forests

The diagonal complex $\Gamma_{\mathcal{F}_n}$ defines a category $P_{\Gamma_{\mathcal{F}_n}}$, see Section 2.1.7; we will shorten the notation to $P_{\mathcal{F}_n}$. Recall from Section 2.1.2 that the set PP_fX_n of partial partitions of X_n forms a poset $(PP_fX_n, \leq_{pc})$ with the partial coarsening relation. The category $P_{\mathcal{F}_n}$

is defined to be the minimal subposet of $PP_f X_n$ containing $\gamma_{\mathcal{F}_n}(\Gamma_{\mathcal{F}_n})$ and closed under taking meets. So to describe $P_{\mathcal{F}_n}$ we need to understand not only the forest posets but also their meets in $(PP_f X_n, \leq_{pc})$.

A *bipartite planted forest* on $[n]$ is a planted forest f with vertex set $[n] \cup N$ such that

- all leaves and isolated vertices of f (the vertices of valence 1 and 0 respectively), are in $[n]$,
- each edge has one end in $[n]$ and one end in N and
- the root vertices are in $[n]$.

The second condition states that the underlying graph is bipartite, whilst the first says that each extremal vertex is in $[n]$. A planted forest is naturally directed by setting each edge to be directed away from the root of the tree in which it is contained. As such each internal vertex v has a unique *parent* vertex $p(v)$ such that $\overrightarrow{p(v)v}$ is an edge. Since each vertex in N is internal this means that every such vertex has a unique parent, so p restricts to a map $N \to [n]$.

For each $v \in N$ define a subset $U_v \subset X_n$ by

$$U_v = \{(p(v), w) \mid \text{ there exists a directed path from } p(v) \text{ to } w \text{ in } f\}. \tag{3.3.4.1}$$

Then $\{U_v\}_{v \in N}$ is a partial partition of X_n associated to the bipartite planted forest f.

Each planted forest f defines a bipartite planted forest by setting $N = E$, the edge set of f. The edges of the new forest are then given by $\overrightarrow{ve}$ and $\overrightarrow{ew}$ for each edge $e = \overrightarrow{vw} \in E$. This is the barycentric subdivision of the forest. The associated partial partition is equal to $\gamma_{\mathcal{F}_n}(f)$.

The partial partitions have an ordering by partial coarsening; $\{U_i\} \leq_{pc} \{V_j\}$ if for each $U_k \in \{U_i\}$ there exists a subset of $\{V_j\}$ whose union is U_k. In this ordering the minimal elements above a partial partition $\{U_i\}$ are given (a) by taking the union of two of the subsets, or (b) by removing one of the subsets. There is an induced ordering of the bipartite planted forests. Let f be a bipartite planted forest and let

$$\begin{tikzcd} i & & & j \\ & x \arrow[ul] & y \arrow[ur] & \\ & & k \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.3.4.2}$$

be a subtree, where $i, j, k \in [n]$ and $x, y \in N$. Now define a new forest f_{xy} by identifying

x and y to obtain z, the subtree now becomes

$$\begin{tikzcd} i & & j \\ & z \arrow[ul] \arrow[ur] & \\ & k \arrow[u] & \end{tikzcd} . \tag{3.3.4.3}$$

Let us now consider the associated partial partitions. If $w \in N$ is not x or y then U_w is left unchanged by the operation of identifying x and y. This accounts for all subsets associated to f_{xy} except U_z, this is equal to $U_x \cup U_y$. We call the operation of identifying vertices x and y a *horizontal folding*. Now suppose that we instead have the following subtree

$$\begin{tikzcd} i \\ x \arrow[u] \\ j \arrow[u] \\ y \arrow[u] \\ k \arrow[u] \end{tikzcd} . \tag{3.3.4.4}$$

We again identify x and y to obtain z and the subtree looks like the Y-shaped tree (3.3.4.3). However this time the effect on partial partitions is to remove U_x. We call this a *vertical folding*. For two bipartite planted forests f and f' we write $f' \leq f$ if f' is obtained by a chain of foldings from f to f'. We then observe that the associated poset is the same poset structure as the one induced from the poset structure from the partial partitions.

Proposition 3.3.9. *The set of partial partitions of X_n associated to the bipartite planted forests is equal to the object set of the category $P_{\mathcal{F}_n}$.*

Proof. To show this we need only show that every bipartite planted forest corresponds to the meet in $PP_f X_n$ of partial partitions coming from subdivisions of forest posets. So let f be a bipartite planted forest with vertex set $[n] \cup N$ and corresponding partial partition $\{U_v\}_{v \in N}$. Every vertex in N of valence greater than two may be unfolded in a variety of ways. Repeating this will give a forest in which each vertex in N is bivalent and so is the barycentric subdivision of a planted forest with vertices in $[n]$. We claim that $\{U_v\}$ is the meet of all the partial partitions corresponding to maximal unfoldings, this would complete the proof.

Let $\{V_j\}$ be the meet of all the maximal unfoldings of f. So $\{V_j\}$ is the greatest partial partition which is less than each unfolding, since $\{U_v\}$ is less than each unfolding it must be less than (or equal to) $\{V_j\}$. Now consider the maximal f_h given by horizontal unfoldings, the partial partition $\{W_k\}$ of this has the same union as $\{U_v\}$, the partial partition of f. Since we have $\{U_v\} \leq_{pc} \{V_j\} \leq_{pc} \{W_k\}$, the union of $\{V_j\}$ must be the same as that of $\{U_v\}$.

Now pick a maximal vertical unfolding f_v. The partial partition $\{W_k\}$ associated to f_v contains $\{U_v\}_{v\in N}$ as a subpartition. Since $\{U_v\} \leq_{pc} \{V_j\} \leq_{pc} \{W_k\}$ this means that $\{V_j\}$ also contains $\{U_v\}$ as a subpartition. However their unions are the same, so they must be equal.

Therefore $\{U_v\}_{v\in N}$ is the meet and we are done. □

So we now have a combinatorial description of the category $P_{\mathcal{F}_n}$ in terms of bipartite planted forests and foldings. To see how they give the diagonal complex product associated to $\Gamma_{\mathcal{F}_n}$ we need only describe the associated functor. So let $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ be an n-tuple of objects equipped with point and (commutative) diagonal maps in a symmetric monoidal category $(\mathcal{C}, \otimes, k)$. The diagonal maps $\Delta_i : Y_i \to Y_i \otimes Y_i$ should be coassociative and symmetric in the sense that $\Delta_i^{\mathrm{op}} = \Delta_i$ and the point maps are morphisms from the unit of the symmetric monoidal category, $p_i : k \to Y_i$. We assign to a bipartite planted forest f the product

$$\mathbf{Y}_f := \bigotimes_{x\in N} Y_{p(x)} \tag{3.3.4.5}$$

Suppose that f' is obtained by a horizontal folding identifying x and y in N to give a vertex z. Then $p(x) = p(y) = p(z)$ and so there is a map $\mathbf{Y}_{f'} \to \mathbf{Y}_f$ induced by the coalgebra map $Y_{p(z)} \to Y_{p(z)} \otimes Y_{p(z)} \cong Y_{p(x)} \otimes Y_{p(y)}$.

Now suppose that f' is obtained by a vertical folding which identifies x and y to z where $x < y$. Then $p(z) = p(x)$ and the map $\mathbf{Y}_{f'} \to \mathbf{Y}_f$ is given by the unit map $Y_{p(z)} \otimes k \to Y_{p(x)} \otimes Y_{p(y)}$. The diagonal complex product $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ is the colimit of this functor $F_{\mathbf{Y}} : P_{\mathcal{F}_n} \to \mathcal{C}$.

3.3.5 Based McCullough-Miller space

In [37] for each free product $G = G_1 * \ldots * G_n$ a space MM_G on which $\mathrm{OWH}(G) := \mathrm{WH}(G)/\mathrm{Inn}(G)$ acts was described. Their main result was that MM_G was contractible and that the simplex stabilisers were of the form $\prod_{j=1}^{k} G_{i_j}$ for some i_j's. The equivariant spectral sequence of this action has been used in papers such as [27] and [4] to study the homology of $\mathrm{OWH}(G)$ and hence of $\mathrm{WH}(G)$. The spaces MM_G were defined using bipartite trees with an ordering by folding, along with a 'marking' which consists of a basis of G.

In [13] a variant of McCullough-Miller space was studied. In this variant the bipartite trees had a chosen basepoint $*$. The space was denoted $L(G)$ and had an action of $\mathrm{FR}(G)$ rather than the outer version. In was shown that the methods of [37] could be applied to show that $L(G)$ is contractible. Ofcourse rooted trees are equivalent to planted forests, one just removes the base vertex and the neighbouring vertices describe the planting. So we may describe $L(G)$ using the bipartite planted forests which we described above.

The following definition is a restatement of the definition of $L(G)$ from [13].

Definition 3.3.10. An automorphism is *carried by a bipartite planted forest* f if it is in the subgroup generated by elements $\alpha^g_{U_x} := \alpha^g_{i_1} \dots \alpha^g_{i_k}$ where $x \in N$, $g \in G_{p(x)}$ and $U_x = \{(p(x), i_j)\}$. This subgroup is denoted $\mathbf{G}_f$ and is isomorphic to the term given by (3.3.4.5).

A marked bipartite planted forest is a pair $([\alpha], f)$ where $[\alpha]$ is a left coset in $\mathrm{FR}(G)/\mathbf{G}_f$. The marked bipartite planted forests form a poset where $([\alpha], f) \leq ([\beta], f')$ if $[\alpha] \subseteq [\beta]$ and f is given by a chain of foldings of f'. We denote this poset $M(G)$. There is an action by $\mathrm{FR}(G)$ given by multiplication of the cosets on the left. The space $L(G)$ is the geometric realisation of $M(G)$, the poset of marked bipartite planted forests.

In the paper [13] it is shown that

Theorem 3.3.11 (3.1 from [13])**.** *The space $L(G)$ is contractible.*

Note that although [13] only considers finite groups G_i, this is only applied from Section 4 onwards, so Theorem 3.1 is valid for all groups G_i.

Note also that in the definition above the poset $M(G)$ is precisely the poset of cosets in the family $\{\mathbf{G}_f\}$ indexed by $P_{\mathcal{F}_n}$. Therefore $L(G)$ is the coset complex $\mathrm{CC}_{\Gamma_{\mathcal{F}_n}}(\mathbf{G})$ associated to the diagonal complex $\Gamma_{\mathcal{F}_n}$. Therefore by Theorem 3.3.11 and Theorem 2.5.2 we have our main theorem.

Theorem 3.3.12. *Let $\mathbf{Y} = (Y_1, \dots, Y_n)$ be aspherical pointed spaces. Then the space $\mathbf{Y}\Gamma_{\mathcal{F}_n} \cong \mathcal{M}\mathbf{Y}$ is also aspherical.*

3.4 (Co)homology of the symmetric automorphism groups

The position we are now in is that we have for any n-tuple of groups $\mathbf{G} = (G_1, \dots, G_n)$ with classifying spaces $\mathbf{Y} = (Y_1, \dots, Y_n)$ a classifying space $\mathcal{M}\mathbf{Y} \cong \mathbf{Y}\Gamma_{\mathcal{F}_n}$ for the Fouxe-Rabinovitch automorphism group $\mathrm{FR}(G_1 * \dots * G_n) \cong \mathbf{G}\Gamma_{\mathcal{F}_n}$. So we will first use the methods of Chapter 2 to calculate the homology of the space $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ and hence of the group $\mathrm{FR}(G)$. Then we will use the results of Section 2.3.3 to calculate the homology of the symmetric automorphism group,

$$\Sigma\mathrm{Aut}(G) \cong \mathrm{FR}(G) \rtimes \mathbf{Aut}(\mathbf{G}) \rtimes \mathfrak{S}_{\mathbf{G}}. \tag{3.4.0.1}$$

3.4.1 The Fouxe-Rabinovitch groups

We start with the homology of the Fouxe-Rabinovitch groups, which by Theorems 3.3.2 and 3.3.12 is given by $H_*(\mathbf{Y}\Gamma_{\mathcal{F}_n}, R)$. Let U be a planted forest. The monomial, see (2.3.1.13), attached to U is

$$x_1^{\mathrm{OUT}(1)} \dots x_n^{\mathrm{OUT}(n)}, \tag{3.4.1.1}$$

where $\mathrm{OUT}(i)$ is the number of outgoing edges of U from i. The following is a restatement of Theorem 5.3.4 of [44].

Proposition 3.4.1. *The Hilbert-Poincaré series of* $\Gamma_{\mathcal{F}_n}$ *is*

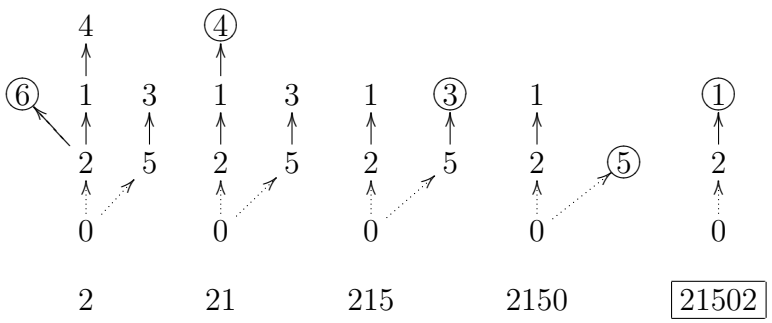
$$h_{\mathcal{F}_n}(x_1, \ldots, x_n) = (1 + x_1 + \ldots + x_n)^{n-1}. \tag{3.4.1.2}$$

We refer to [44] for the full proof, but in the interests of self-containment we sketch part of the proof below.

Sketch of Proof. A proof of this proposition involves the theory of *Prüfer codes*. The set of planted forests with vertex set $[n]$ is shown to be in bijection with the set of words of length $n-1$ in the letters $\{x_0, x_1, \ldots, x_n\}$. To give the word of a planted forest, attach a new vertex 0 onto the forest by adding the edge $\overrightarrow{0i}$ for each of the roots i. A *leaf* is a vertex which is attached to only one other vertex. A word $w = s_1 \ldots s_{n-1}$ is produced for each tree t as follows:

1. Let $t_1 = t$.
2. Suppose that t_i is defined. Take the leaf v of maximal value in t_i and define s_i to be the value of its unique adjacent vertex.
3. If $i = n-1$ then stop.
4. Now define t_{i+1} to be the tree created by removing v and its unique adjoining edge from t_i.
5. Go back to step (2).

The following diagram illustrates an example; the forest on the left is turned into the word $x_2x_1x_5x_0x_2$.



(3.4.1.3)

This algorithm may be reversed in order to build a tree from a word and this produces the bijection. The element

$$(x_0 + x_1 + \ldots + x_n)^{n-1} \tag{3.4.1.4}$$

in the free commutative ring on $\{x_0, \ldots, x_n\}$ is the sum over all words of length $n-1$. The word associated with a planted forest may be turned into a monomial by viewing it

in the free commutative monoid on $\{x_1, \ldots, x_n\}$ and setting $x_0 = 1$. Since the number of x_i's in a word is the number of outgoing edges of i, this monomial is the same as the monomial defined in (3.4.1.1). Hence setting $x_0 = 1$ and viewing (3.4.1.4) in the free commutative ring gives the Hilbert-Poincaré series of the planted forests. □

The series for the planted forests is identical to the Hilbert-Poincaré series of the diagonal complex describing $(G_1 * \ldots * G_n)^{\times n-1}$ and so by Corollary 2.3.5:

Theorem 3.4.2. *Let* $\mathbf{G}$ *be an* n*-tuple of groups and* G *be the* n*-fold free product of these groups. Then*

$$H_*(\mathrm{FR}(G), R) \cong H_*(G^{n-1}, R), \tag{3.4.1.5}$$

where G^{n-1} *is the* $(n-1)$*-fold direct product of* G*. This also holds for the cohomology.*

The Euler characteristic of $\mathrm{WH}(\mathbb{Z}^{*n}) \cong \mathrm{FR}(\mathbb{Z}^{*n})$, was computed in [35], and the homology was calculated in [27]. Recalling that the Hilbert-Poincaré series of $\mathbb{Z}$ is $1 + t$ we may reobtain these results.

Corollary 3.4.3. *The Hilbert-Poincaré series of* $H_*\big(\mathrm{WH}(F_n), \mathbb{Z}\big)$ *is*

$$(1 + tn)^{n-1} \tag{3.4.1.6}$$

and so the Euler characteristic is

$$(1 - n)^{n-1}. \tag{3.4.1.7}$$

We may take this further by calculating the homology of $\mathrm{WH}\big(\mathbb{Z}/(p)^{*n}\big)$. The question of the cohomology of this group was posed by Jensen in [39].

Corollary 3.4.4. *The Hilbert-Poincaré series of* $H_*\big(\mathrm{WH}(\mathbb{Z}/(p)^{*n}), \mathbb{Z}\big)$ *is*

$$1 + y\frac{1}{1+t}\left[\left(1 + \frac{nt}{1-t}\right)^{n-1} - 1\right], \tag{3.4.1.8}$$

where the y *coefficient encodes the number of* $\mathbb{Z}/(p)$*-summands and the constant term encodes the number of* $\mathbb{Z}$*-summands.*

Proof. The homology of $\mathbb{Z}/(p)$ is given by

$$H_i(\mathbb{Z}/(p), \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } i = 0, \\ \mathbb{Z}/(p) & \text{if } i = 2k-1 \text{ for } k \geq 1 \text{ and} \\ 0 & \text{if } i = 2k \text{ for } k \geq 1. \end{cases} \tag{3.4.1.9}$$

As a Hilbert-Poincaré series this is

$$1 + yt + yt^3 + yt^5 + \ldots = 1 + \frac{ty}{1-t^2}. \tag{3.4.1.10}$$

Putting this into the Hilbert-Poincaré series for $\Gamma_{\mathcal{F}_n}$ gives

$$\left(1+\frac{nyt}{1-t^2}\right)^{n-1} = 1+\sum_{k=1}^{n-1}\binom{n-1}{k}\left(\frac{nyt}{1-t^2}\right)^k. \tag{3.4.1.11}$$

Using the identity $y^2 = y(1+t)$ from (2.3.1.16) to give $y^k = y(1+t)^{k-1}$ and simplifying we get the desired series (3.4.1.8). □

3.4.2 The pure automorphism groups

Recall that the pure automorphism group, $P\mathrm{AUT}(G)$ consists of those automorphisms that take an element $g \in G_i$ and give a conjugate $(g')^h$ of an element $g' \in G_i$ inside $G = G_1 * \ldots * G_n$. It is a semidirect product of $\mathrm{FR}(G)$ and $\mathbf{Aut}(\mathbf{G}) \cong \prod_i \mathrm{Aut}(G_i)$. The action of $\mathbf{Aut}(\mathbf{G})$ on $\mathrm{FR}(G)$ may be described as follows. Let $\phi \in \mathrm{Aut}(G_i)$, then

$$(\alpha_k^{g_j})^\phi = \begin{cases} \alpha_k^{g_j} & \text{if } j \neq i \text{ and} \\ \alpha_k^{\phi(g_j)} & \text{if } j = i. \end{cases} \tag{3.4.2.1}$$

The group $\mathbf{Aut}(\mathbf{G})$ has an action on the free product G and hence on G^{n-1} by the diagonal action, so we may form the group

$$G^{n-1} \rtimes \mathbf{Aut}(\mathbf{G}). \tag{3.4.2.2}$$

Theorem 3.4.5. *The homology of $P\mathrm{AUT}(G)$ decomposes as*

$$H_*\big(\boldsymbol{Aut}(\mathbf{G}), R\big) \oplus \bigoplus_{F\in\Gamma_{\mathcal{F}_n}} H_*\big(\boldsymbol{Aut}(\mathbf{G}), \widehat{C}_*(F)\big), \tag{3.4.2.3}$$

where

$$\widehat{C}_*(F) = \bigotimes_{\overrightarrow{ij}\in E(F)} \widehat{C}_*(Y_i) \tag{3.4.2.4}$$

and Y_i is a classifying space for G_i for each $i = 1, \ldots, n$.

Therefore

$$H_*\big(P\mathrm{AUT}(G), R\big) \cong H_*\big(G^{n-1} \rtimes \boldsymbol{Aut}(\mathbf{G}), R\big). \tag{3.4.2.5}$$

Proof. The diagonal complex product $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ is a classifying space for $\mathrm{FR}(G)$ and has an action of $\mathbf{Aut}(\mathbf{G})$. The homotopy quotient is then a classifying space for the pure automorphism group, $P\mathrm{AUT}(G) = \mathrm{FR}(G) \rtimes \mathbf{Aut}(\mathbf{G})$.

Let $\Gamma_{\mathcal{F}_n}$ have the labelling by $[n]$ given by $l(i,j) = i$. For each $i \in [n]$ let $H_i = \mathrm{Aut}(G_i)$ be the automorphism group of G. The direct product, H of the H_i acts on $\mathbf{Y}\Gamma_{\mathcal{F}_n}$ see Section 2.3.4. This direct product is isomorphic to $\mathbf{Aut}(\mathbf{G})$ and has the same action on

$\mathbf{Y}\Gamma_{\mathcal{F}_n}$. Letting $\mathfrak{S}$ be the trivial group, then we may apply Theorem 2.3.9 to calculate the homology of $P\mathrm{Aut}(G)$, which gives the decomposition.

For the second part observe that for any planted forest $F \in \Gamma_{\mathcal{F}_n}$ with the same monomial as a term U in the homology of G^{n-1}, the automorphism groups always act diagonally and so $\widehat{C}_*(F) \cong \widehat{C}_*(U)$ as H-modules. Since the Hilbert-Poincaré polynomials are identical the corresponding sums of H-modules are isomorphic. Hence applying Theorem 2.3.9 for both groups $P\mathrm{Aut}(G)$ and $G^{n-1} \rtimes \mathbf{Aut}(\mathbf{G})$ we get the same homology. □

3.4.3 The symmetric automorphism groups

Let $n_1, \ldots, n_k$ be the sizes of the isomorphism classes of $\mathbf{G} = (G_1, \ldots, G_n)$. Then write $\mathfrak{S}_{\mathbf{G}}$ for the group

$$\mathfrak{S}_{n_1} \times \ldots \times \mathfrak{S}_{n_k} \leq \mathfrak{S}_n \tag{3.4.3.1}$$

of symmetries of $\mathbf{G}$. The symmetric automorphism group $\Sigma\mathrm{Aut}(G)$ is the semidirect product of the pure automorphism group $P\mathrm{Aut}(G)$ and $\mathfrak{S}_{\mathbf{G}}$.

Remember that the diagonal complex $\Gamma_{\mathcal{F}_n}$ has a universal labelling $X_n \to [n]$. So a map $L : [n] \to Z = [k]$ induces a labelling $l : X_n \to Z$. Let n_i be the size of the set $L^{-1}(i)$ for $i \in Z$. We will call a planted forest Z-coloured if there is a vertex colouring by Z and if for each $i \in Z$ there are n_i vertices coloured i.

Lemma 3.4.6. *The $\{1\}$-labelled diagonal complex $\Gamma_{\mathcal{F}_n}$ carries an action of $\mathfrak{S}_n$. The orbits are given by the isomorphism types of unlabelled planted forests.*

The Z-labelled diagonal complex $\Gamma_{\mathcal{F}_n}$ carries an action of $\mathfrak{S}_{\mathbf{G}} = \mathfrak{S}_{n_1} \times \ldots \times \mathfrak{S}_{n_k}$. The orbits are given by the isomorphism types of Z-coloured planted forests.

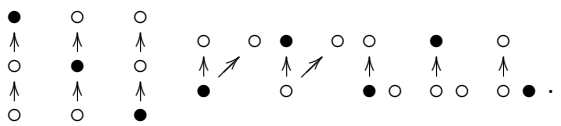
Proof. The symmetric group $\mathfrak{S}_n$ has the permutation action on $[n]$. This gives an action on $X_n = \{(i,j) \mid i \neq j\}$. The induced action on $P_f X_n$ restricts to $\Gamma_{\mathcal{F}_n}$: an element $\sigma \in \mathfrak{S}_n$ takes the edge $\overrightarrow{ij}$ of $U \in \Gamma_{\mathcal{F}_n}$ to $\overrightarrow{\sigma(i)\sigma(j)}$ of $\sigma(U)$.

The action of $\mathfrak{S}_{\mathbf{G}}$ on $[n]$ fixes the morphism $L : [n] \to Z$ and hence the action on $\Gamma_{\mathcal{F}_n}$ satisfies

$$l(\sigma(U)) = l(U), \tag{3.4.3.2}$$

for all $\sigma \in \mathfrak{S}_{\mathbf{G}}$ and $U \in \Gamma_{\mathcal{F}_n}$. □

Example 3.4.3.1. Let $L : \{1,2,3\} \to \{\circ, \bullet\}$ be defined by $l(1) = l(2) = \circ$ and $l(3) = \bullet$. Then the orbits are enumerated by the Z-coloured planted forests as follows:



(3.4.3.3)

Theorem 3.4.7. *Let $G = G_1 * \ldots * G_n$ where G_i is neither freely decomposable nor infinite cyclic. Then the homology of the automorphism group of G is given by*

$$H_*\big(\mathrm{Aut}(G), R\big) \cong H_*\big(\boldsymbol{Aut}(\mathbf{G}) \rtimes \mathfrak{S}_{\mathbf{G}}, R\big) \oplus \bigoplus_{f \in \textsc{Forests}_Z} H_*\left(\boldsymbol{Aut}(\mathbf{G}) \rtimes \mathrm{Aut}(f), \widehat{C}_*(f)\right), \tag{3.4.3.4}$$

where $\textsc{Forests}_Z$ *is the set of Z-coloured planted forests and $\widehat{C}_*(f)$ is given by*

$$\bigotimes_{i \in [n]} \widehat{C}_*(Y_{l(i)}, R)^{\otimes \mathrm{OUT}(i)}, \tag{3.4.3.5}$$

where Y_i is a classifying space for G_i for each $i = 1, \ldots, n$.

Proof. There are two labellings of $\Gamma_{\mathcal{F}_n}$, the labelling by Z coming from the isomorphism type of the G_i and the universal labelling by $[n]$. The group $\mathfrak{S}_{\mathbf{G}}$ acts on $\Gamma_{\mathcal{F}_n}$ preserving the first labelling. Thus we may apply Theorem 2.3.9 to compute the homology of the homotopy quotient of the action of $\mathbf{Aut}(\mathbf{G}) \rtimes \mathfrak{S}_{\mathbf{G}}$ on $\mathbf{Y}\Gamma_{\mathcal{F}_n}$. This homotopy quotient is a classifying space for $\Sigma\textsc{Aut}(G)$.

By Lemma 3.4.6 the orbits of $\Gamma_{\mathcal{F}_n}$ are given by $\textsc{Forests}_Z$ and the stabiliser of a Z-coloured forest f is $\mathrm{Aut}(f)$. □

Remark 3.4.8 (cohomological version of Theorem 3.4.7). Precisely the same arguments may be used to prove that given the hypotheses of Theorem 3.4.7 the cohomology of $\mathrm{Aut}(G)$ is given by

$$H^*\big(\mathrm{Aut}(G), R\big) \cong H^*\big(\mathbf{Aut}(\mathbf{G}) \rtimes \mathfrak{S}_{\mathbf{G}}, R\big) \oplus \bigoplus_{f \in \textsc{Forests}_Z} H^*\left(\mathbf{Aut}(\mathbf{G}) \rtimes \mathrm{Aut}(f), \widehat{C}^*(f)\right), \tag{3.4.3.6}$$

where $\widehat{C}^*(f)$ is given by

$$\bigotimes_{i \in [n]} \widehat{C}^*(Y_{l(i)})^{\otimes \mathrm{OUT}(i)}. \tag{3.4.3.7}$$

Remark 3.4.9. Let $G = G_1^{*n_1} * \ldots * G_k^{*n_k}$ with each G_i neither freely decomposable nor $\mathbb{Z}$, then $\mathrm{Aut}(G) \cong P\textsc{Aut}(G) \rtimes \mathfrak{S}_{\mathbf{G}}$, so we have that the Euler characteristic of $\mathrm{Aut}(G)$ is

$$\frac{1}{(n_1)! \ldots (n_k)!}\big(1 + n_1(\chi(G_1) - 1) + \ldots + n_k(\chi(G_k) - 1)\big)^{n-1} \prod_{i=1}^{k} \chi\big(\mathrm{Aut}(G_i)\big)^{-n_i}, \tag{3.4.3.8}$$

which is the same as that of $(G^{n-1} \rtimes \mathbf{Aut}(\mathbf{G})) \rtimes \mathfrak{S}_{\mathbf{G}}$. However their integral homologies

are not necessarily the same due to the presence of the groups $\mathrm{Aut}(f)$. Hence the pattern

$$H_*(\mathrm{FR}(G), R) \cong H_*(G^{n-1}, R), \tag{3.4.3.9}$$

$$H_*(\mathrm{WH}(G), R) \cong H_*(G^{n-1} \rtimes \mathbf{Inn}(\mathbf{G}), R), \tag{3.4.3.10}$$

$$H_*(P\textsc{Aut}(G), R) \cong H_*(G^{n-1} \rtimes \mathbf{Aut}(\mathbf{G}), R) \tag{3.4.3.11}$$

is broken when non-pure symmetric automorphisms are present.

Remark 3.4.10. There is a discrepancy between the three isomorphisms we obtain in (3.4.3.9), (3.4.3.10) and (3.4.3.11) and the results of [28] and [4]. Let $G = G_1 * \ldots * G_n$ be a free product with n factors. Calculating the Euler characteristic we find that

$$\chi(\mathrm{FR}(G)) = \chi(G^{n-1}), \tag{3.4.3.12}$$

whereas in [28] it is calculated to be

$$\chi(G^{n-1}). \prod_i \chi(\mathrm{Inn}(G_i))^{-1}. \tag{3.4.3.13}$$

Across all of the results there is a factor of $\chi(\mathbf{Inn}(\mathbf{G}))$ difference. There is an analogous difference with the results of [4]. We attribute this to a misquoting of Proposition 5.1 of [37] in each of the papers.

Example 3.4.3.2. Let $G_i = \mathbb{Z}$ and consider the group $\Sigma\mathrm{FR}(F_n) \cong \mathrm{FR}(F_n) \rtimes \mathfrak{S}_n$. Using Theorem 3.4.7 we may calculate the homology as

$$H_*\big(\Sigma\mathrm{FR}(F_n), R\big) = H_*(\mathfrak{S}_n, R) \oplus \bigoplus_{f \in \textsc{Forests}} H_*\big(\mathrm{Aut}(f), \widehat{C}_*(f)\big) \tag{3.4.3.14}$$

$$\cong H_*(\mathfrak{S}_n, R) \oplus \bigoplus_{f \in \textsc{Forests}} t^{|E(f)|}\, H_*\big(\mathrm{Aut}(f), R_f\big), \tag{3.4.3.15}$$

where t is of degree one and R_f is the one-dimensional 'determinant' $\mathrm{Aut}(f)$-module: a factor of -1 is introduced every time two edges of f are swapped.

3.5 Categorical interpretation of symmetric automorphisms

The partial conjugations play an important role in the theory of automorphisms of free products. In this section we aim to show that they can be viewed as generalisations of inner automorphisms in a sense which can be made precise in the language of category theory. In fact we may show that the Whitehead automorphism group $\mathrm{WH}(G)$ occurs naturally as automorphisms which can be extended with respect to the free product

functor

$$*^{n-1} : \textsc{Gps}^{\times n} \to \textsc{Gps}. \tag{3.5.0.16}$$

In comparison, the inner automorphisms of a group are those automorphisms which extend with respect to the identity functor

$$I_{\textsc{Gps}} : \textsc{Gps} \to \textsc{Gps}. \tag{3.5.0.17}$$

3.5.1 Extendable automorphisms

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

Definition 3.5.1. Let C be an object of $\mathcal{C}$ and $\alpha : FC \to FC$ be an automorphism of FC in $\mathcal{D}$. Then we say that the pair (C, α) is *extendable with respect to* F if for every morphism $f : C \to B$ with source C there exists an automorphism β of FB making the square

$$\begin{tikzcd} FC \arrow[r, "\alpha"] \arrow[d, "Ff"'] & FC \arrow[d, "Ff"'] \\ FB \arrow[r, "\beta"] & FB \end{tikzcd} \tag{3.5.1.1}$$

commute. The extendable automorphisms $\{(C, \alpha)\}$ of FC form a group, which we denote $E_F C$.

Although it is not necessarily the case, in many examples the extendable automorphisms form a functor $E_F : \mathcal{C} \to \textsc{Gps}$ to the category of groups, where the commuting squares take the form

$$\begin{tikzcd} FC \arrow[r, "\alpha \in E_F C"] \arrow[d, "Ff"'] & FC \arrow[d, "Ff"'] \\ FB \arrow[r, "(E_F f)(\alpha)"] & FB. \end{tikzcd} \tag{3.5.1.2}$$

Example 3.5.1.1. Let F be the identity functor on the category of groups, $I : \textsc{Gps} \to \textsc{Gps}$. Then by a theorem of Schupp [43] the extendable automorphisms consist of the inner automorphisms.

Theorem 3.5.2 (due to Schupp [43])**.** *Let G be a group and α an automorphism of G. The automorphism α is an inner automorphism of G if and only if α has the property that whenever G is embedded in a group H then α extends to some automorphism of H.*

Proof. A subgroup $K \leq H$ is called *malnormal in* H if $hKh^{-1} \cap K = \{1\}$ for all $h \in H \backslash K$. Schupp noted that it was enough to prove that any group G is embeddable as a malnormal subgroup of a complete group H, which he then went on to prove. This is enough because any automorphism of H is inner by the completeness of H and it restricts to G only if the conjugating element is in G by the malnormality of the embedding. □

We record Schupp's intermediate result as the following lemma, referring back to [43] for the proof.

Lemma 3.5.3 (due to Schupp [43])**.** *Any group G is embeddable as a malnormal subgroup of a complete group H.*

Remark 3.5.4. The inner automorphism group is normal inside the full automorphism group. It might be hoped that the extendable automorphism group $E_F C$ of an object FC is normal inside the full automorphism group $\mathrm{Aut}(FC)$, however this is not necessarily the case. An example will be given by the Whitehead automorphism group $\mathrm{WH}(F_n)$ which is not normal inside $\mathrm{Aut}(F_n)$. A weaker result holds.

Proposition 3.5.5. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor and C an object of $\mathcal{C}$. Then the subgroup $F\big(\mathrm{Aut}(C)\big) \leq \mathrm{Aut}(FC)$ normalises the subgroup $E_F C$ of extendable automorphisms of F.*

Proof. Let α be an extendable automorphism of FC and let $\gamma \in \mathrm{Aut}(C)$. We need to show that we may complete the square

$$\begin{tikzcd} FC \arrow[r, "F\gamma\alpha F\gamma^{-1}"] \arrow[d, "Ff"'] & FC \arrow[d, "Ff"'] \\ FB & FB \end{tikzcd} \tag{3.5.1.3}$$

for any $f \in \mathrm{Hom}(C, B)$. Redraw the above diagram as

$$\begin{tikzcd} FC \arrow[r, "F\gamma^{-1}"] \arrow[dr, "Ff"'] & FC \arrow[r, "\alpha"] \arrow[d, "F(f\gamma)"] & FC \arrow[r, "F\gamma"] \arrow[d, "F(f\gamma)"'] & FC \arrow[dl, "Ff"] \\ & FB & FB & \end{tikzcd} \tag{3.5.1.4}$$

Looking at the central square in this diagram, α may be extended along $F(f\gamma)$ in order to fill in the lower edge. The same extension serves to extend $F\gamma\alpha F\gamma^{-1}$ along Ff. Hence $F\gamma\alpha F\gamma^{-1}$ is extendable with respect to F. □

3.5.2 A characterisation of $WH(G)$

The direct product $\textsc{Gps}^{\times n}$ of n copies of the category of groups has as objects n-tuples of groups $\mathbf{G} = (G_1, \ldots, G_n)$ and the morphisms are n-tuples of group homomorphisms $\mathbf{f} = (f_1, \ldots, f_n)$. The free product functor

$$*^{n-1} : \textsc{Gps}^{\times n} \to \textsc{Gps} \tag{3.5.2.1}$$

takes an n-tuple of groups and gives their free product $G_1 * \ldots * G_n$. So an extendable automorphism α of the free product $G_1 * \ldots * G_n$ with respect to the functor $*^{n-1}$ is an

automorphism α for which each square

$$\begin{tikzcd} G_1 * \ldots * G_n \arrow[r, "\alpha"] \arrow[d, "f_1*\ldots*f_n"'] & G_1 * \ldots * G_n \arrow[d, "f_1*\ldots*f_n"'] \\ H_1 * \ldots * H_n \arrow[r, dotted] & H_1 * \ldots * H_n \end{tikzcd} \tag{3.5.2.2}$$

may be completed for each n-tuple of group morphisms

$$(f_1, \ldots, f_n) : (G_1, \ldots, G_n) \to (H_1, \ldots, H_n)\,. \tag{3.5.2.3}$$

Recall that the Whitehead automorphism group of $G_1 * \ldots * G_n$ is generated by partial conjugations $\alpha_i^{g_j}$ for $i \neq j$ and $g_j \in G_j$ and by inner factor automorphisms $\mathrm{inn}(g_i)$ for $g_i \in G_i$. Both kinds of generators are extendable; indeed for $(f_1, \ldots, f_n)$, the generator $\alpha_i^{g_j}$ extends to $\alpha_i^{f_j(g_j)}$ and the generator $\mathrm{inn}(g_i)$ extends to $\mathrm{inn}(f_i(g_i))$. The purpose of this section is to show the converse, that every extendable automorphism is contained in the Whitehead automorphism group.

Theorem 3.5.6. *Let* $\mathbf{G} = (G_1, \ldots, G_n)$ *be an* n*-tuple of groups and* α *be an automorphism of the free product* $G = G_1 * \ldots * G_n$*. Then* α *is a Whitehead automorphism if and only if it extends with respect to the free product functor* $*^{n-1}$*. That is for every morphism*

$$f_1 * \ldots * f_n : G_1 * \ldots * G_n \to H_1 * \ldots * H_n, \tag{3.5.2.4}$$

there exists an automorphism β *of* $H_1 * \ldots * H_n$ *such that the following diagram commutes.*

$$\begin{tikzcd} G_1 * \ldots * G_n \arrow[r, "\alpha"] \arrow[d, "f_1*\ldots*f_n"'] & G_1 * \ldots * G_n \arrow[d, "f_1*\ldots*f_n"'] \\ H_1 * \ldots * H_n \arrow[r, "\beta"] & H_1 * \ldots * H_n. \end{tikzcd} \tag{3.5.2.5}$$

Proof. We have already seen that every Whitehead automorphism extends. For the opposite implication, suppose that α is an automorphism of G which satisfies the extending condition. By Lemma 3.5.3 each G_i is embeddable as a malnormal subgroup of a complete group H_i. Since each H_i is complete, it's true that each H_i is neither freely decomposable nor isomorphic to $\mathbb{Z}$. And so the automorphism group $\mathrm{Aut}(H_1 * \ldots * H_n)$ is $\Sigma\mathrm{WH}(H_1 * \ldots * H_n)$.

We now show that any $\gamma \in \Sigma\mathrm{WH}(H_1 * \ldots * H_n)$ restricting to $G_1 * \ldots * G_n$ must restrict to an element of $\Sigma\mathrm{WH}(G_1 * \ldots * G_n)$. For $g \in G_i$, the action of γ is given by

$$\gamma(g) = (g')^{h_i}, \tag{3.5.2.6}$$

where $g' = g \in G_j$ for some isomorphic factor $G_j \cong G_i$ and for some fixed $h_i \in H_1 * \ldots * H_n$.

Reducing $(g')^{h_i}$ to a reduced word in the letters $H_1, \ldots, H_n$ we get

$$\gamma(g) = ((g')^{h'_i})^w, \tag{3.5.2.7}$$

where $h'_i \in H_j$ and w is a reduced word with its first letter not in H_j. Since $\gamma(g) \in G_1 * \ldots * G_n$ this implies that $g^{h'_i}$ is in G_i and w is in $G_1 * \ldots * G_n$. And since G_i is malnormally embedded in H_i, the element h'_i must be in G_i, hence $h_i = h'_i w$ is in $G_1 * \ldots * G_n$. Therefore γ does restrict to an an element of $\Sigma\mathrm{WH}(G_1 * \ldots * G_n)$.

The automorphism α extends to an automorphism β of $H_1 * \ldots * H_n$, which by the above must be in $\Sigma\mathrm{WH}(H_1 * \ldots * H_n)$. But α is the restriction of β to $G_1 * \ldots * G_n$ and so must itself by in $\Sigma\mathrm{WH}(G_1 * \ldots * G_n)$.

We have proved that α is in the symmetric Whitehead group, it remains to show that it is a pure Whitehead automorphism. We may embed the G_i into pairwise non-isomorphic groups, K_i, none of which are free decomposable or $\mathbb{Z}$. So α extends to $\kappa \in \mathrm{Aut}(K_1 * \ldots * K_n)$. Since no two factors K_i are isomorphic the automorphism group of their free product is generated by factor automorphisms and partial conjugations. So κ must take $g \in G_i \leq K_i$ to a conjugate of an element of G_i, but since $\kappa(g) = \alpha(g)$, this must be a conjugate of g. Hence α is a pure Whitehead automorphism. □

3.6 Relationship with Outer space

The moduli space of cactus products is pictured as a 'space of spaces' each of which has fundamental group a free product of groups. Likewise Outer space is a 'space of graphs' each of which has fundamental group the free group. So it is natural to compare both concepts to look for both similarities and differences. But it is also natural to ask if the two concepts can be united into one space which would act as an Outer space for a general free product in which the integer factors are taken into account.

3.6.1 Background on Outer space

Outer space was defined in [17] in order to study the automorphism group of a free group. In that paper the authors show that $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ have virtual cohomological dimensions $2n-2$ and $2n-3$ respectively by showing that they act with finite stabilisers on contractible spaces of those respective dimensions. The spaces are called *Auter space* and *Outer space*[2] and are both spaces of marked metric graphs, the difference being that the graphs in Auter space have a basepoint.

A graph here will be a one dimensional finite CW-complex of fixed rank r. A metric graph is such a graph where each edge is assigned a length. A graph G is called marked

[2]to be precise these spaces have greater dimensions but they equivariantly retract to spaces (the *spines*) which do have the required dimensions

when it possesses a homotopy class of maps

$$m : \bigvee_{i=1}^{r} S^1 \to G. \tag{3.6.1.1}$$

The marking m must represent a homotopy equivalence. The main result of [17] is that the spaces of such graphs are contractible and that the natural action by $\mathrm{Aut}(F_n)$ and $\mathrm{Out}(F_n)$ respectively have finite stabilisers.

3.6.2 Cactus graphs

Let $Y = S^1$ be the circle with a chosen basepoint. Then a cactus product is a glueing of circles and in particular is a graph. We call the graphs obtained in this manner *cactus graphs*.

A cactus graph has a basepoint and at each vertex the valence is even, also every vertex is a separating vertex. The moduli space of cactus graphs is a classifying space for its fundamental group $\mathrm{FR}(\mathbb{Z}^{*n})$. Its universal cover E may be described by taking the space of pairs, each consisting of a cactus graph G and a marking (3.6.1.1). Assigning the usual metric to S^1 we have a moduli space of marked metric pointed graphs and so E maps into Auter space. The property of being a cactus graph is quite restrictive, indeed the dimension of the moduli space of cactus graphs is $(n-1)$, half that of Auter space.

Cactus graphs without basepoint were considered by Collins in [14], although they are referred to as symmetric graphs. Their moduli space is also contractible and this was shown using 'symmetric analogues' of the techniques used by Culler and Vogtmann in [17].

3.6.3 Outer space with boundaries

An Outer space with boundaries was studied in [29]. The *boundaries* are circles embedded inside the marked metric graphs, the corresponding groups are subgroups of $\mathrm{Aut}(F_n)$ which fix certain conjugacy classes. When the total rank of the graphs is contributed solely by the boundaries we reobtain the space of symmetric graphs (or marked unpointed cactus graphs).

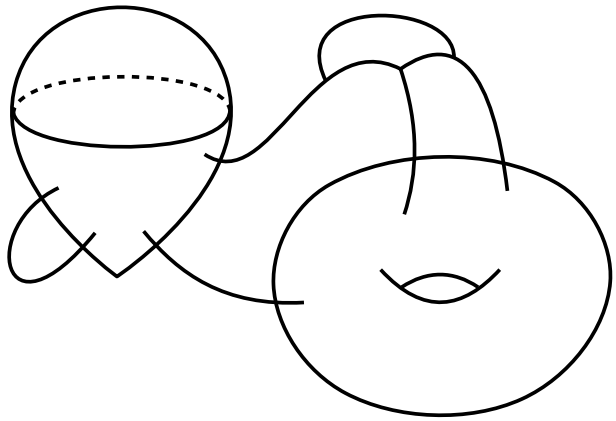
A similar space, this time called relative Outer space was defined in [38]. This is similar to Outer space with boundaries, however now the embedded circles may be replaced with embedded wedges of circles. This time when the total rank of the graph is contributed by the embedded wedges the space obtained may be compared to the unpointed cactus products of wedges of circles.

Definition 3.6.1. These two constructions motivate a common generalisation of Outer space and cactus products. Let $\mathbf{Y} = (Y_1, \ldots, Y_n)$ be an n-tuple of pointed spaces. Then

a $\mathbf{Y}$-graph is a space A

- containing each Y_i as a subspace
- with $Y_i \cap Y_j$ either empty or a single point for distinct i, j and
- where $A/\mathbf{Y}$, the space given by contracting each subspace Y_i to a point in turn, is a graph.

For example if $Y_1 = S^2$ and $Y_2 = S^1 \times S^1$ then a $\mathbf{Y}$-graph could look like



(3.6.3.1)

The rank of A is defined to be the rank of $A/\mathbf{Y}$. The fundamental group of A is

$$\pi_1(Y_1) * \ldots * \pi_1(Y_n) * F_r, \tag{3.6.3.2}$$

where r is the rank of $A/\mathbf{Y}$.

A *marked* $\mathbf{Y}$*-graph* (A, m) is a $\mathbf{Y}$-graph with a homotopy class of maps

$$m : \bigvee_{i=1}^{n} Y_i \vee \bigvee_{i=1}^{r} S^1 \to A, \tag{3.6.3.3}$$

we also ask that m is a homotopy equivalence and that for each Y_i, m preserves the conjugacy class of the image of $\pi_1(Y_i)$.

By placing a metric on the edge components of A we may consider *marked metric* $\mathbf{Y}$*-graphs*. Define $\mathrm{OS}(Y)$ to be the space of marked metric $\mathbf{Y}$-graphs. This may be topologised in the standard way, but also using the topology of the Y_i.

Conjecture 3.6.1. *Let* $\mathbf{Y}$ *be an* n*-tuple of pointed spaces and suppose that each* Y_i *is aspherical. Then* $OS(\mathbf{Y})$ *is connected and contractible.*

Remark 3.6.2. This moduli space of marked metric $\mathbf{Y}$-graphs can be directly compared with the Outer space of a free group. It is tempting to call this space the Outer space of a free product. However there is already a notion of the Outer space of a free product

as introduced in [23]. The definition in that paper is a hydrid of the McCullough-Miller space of a free product and the Outer space of a free group.

Remark 3.6.3. An approach to a proof of this conjecture is to view the Outer space of Guirardel and Levitt as an analogue of McCullough-Miller space and to attempt to show that the contractibility of the moduli space defined above is equivalent to the contractibility of the Outer space, which was proved in [23].

A corollary of this conjecture would be a formula for the virtual cohomological dimension of $\mathrm{Aut}(G)$ for any finitely generated G, in terms of the factors of the free product decomposition of G.

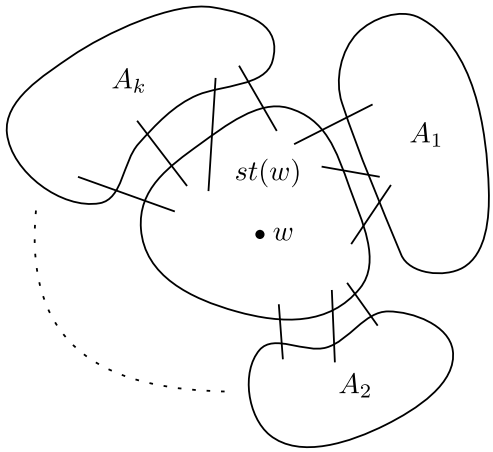
3.7 Automorphisms of graph products

In this section we will use diagonal complexes to construct groups which act on graph products of groups. Let $G = (V, E)$ be a graph, then for each V-tuple of groups $\mathbf{H}$ there is a graph product $\mathbf{H}G$. We are interested in automorphisms which act by conjugation on the vertex subgroups.

We will proceed by defining a diagonal complex Γ_G which depends only on the graph G. Next we will show that the diagonal complex products $\mathbf{H}\Gamma_G$ act on $\mathbf{H}G$ by symmetric automorphisms. This will be followed by a discussion of the literature on automorphisms of graph products, with the outcome that we do not know whether the group of partial conjugations of $\mathbf{H}G$ is isomorphic to $\mathbf{H}\Gamma_G$ or not. Finally we will approach the question of whether the diagonal complex Γ_G is aspherical[3] by defining an analogue of the McCullough-Miller complex. The asphericity of Γ_G is equivalent to the contractibility of these complexes.

3.7.1 The diagonal complex Γ_G

Let w be a vertex of G and write G_w for the full subgraph of G with vertices in the complement of the star of w. So this is the graph given by taking away from G the vertex w and any other vertices joined by a single edge to w. We say that a subset $A \subseteq V$ is *admissable with respect to* w if it is the vertex set of a connected component of G_w. The set of admissable sets with respect to w is denoted $\mathrm{adm}(w)$. This may be represented by the diagram



(3.7.1.1)

where $\mathrm{adm}(w) = \{A_1, \ldots, A_k\}$.

Definition 3.7.1. For a graph $G = (V, E)$ let X_G be the set of pairs (w, A) with $w \in V$ and $A \in \mathrm{adm}(w)$. We call such a pair (w, A) a *G-admissable pair*. A *G-admissable poset* $(V, \leq)$ has vertex set V, is non-trivial and satisfies the following two conditions:

[3]Recall that a diagonal complex is aspherical if the diagonal complex products of aspherical spaces are themselves aspherical.

1. (the 'overset condition') For every $w \in V$, the set $\{v \mid w < v\}$ is the union of sets admissable with respect to w.

2. (the 'underset condition') For every $w \in V$, any two elements x, y in $\{v \mid v < w\}$ satisfy either

   - $x \leq y$,
   - $y \leq x$ or
   - $(x, y) \in E$ is an edge in G.

We denote by Γ_G the set of G-admissable posets.

Since the overset condition implies that $\{(v, w) \mid v < w\}$ is the union of sets $(w, A) \in X_G$ we may view any G-admissable poset as a subset of X_G and so Γ_G as a subset of $P_f X_G$. When $v < w$ in a G-admissable poset we write $[v < w]$ for the element $(v, A) \in X_G$ where A is the v-admissable subset containing w.

Example 3.7.1.1. Let G be the graph

$$\begin{tikzcd}[column sep=small] & 6 \arrow[d, dash] & \\ 5 \arrow[dr, dash] \arrow[d, dash] & 3 \arrow[d, dash] & \\ 1 \arrow[ur, dash] & 2 \arrow[r, dash] & 4 \end{tikzcd} \tag{3.7.1.2}$$

and let P be the poset

$$\begin{tikzcd}[column sep=small] 2 & & 4 \\ & 6 \arrow[ul] \arrow[ur] & \\ 1 \arrow[ur] & & 5 \arrow[ul] \end{tikzcd} \tag{3.7.1.3}$$

Then P is a G-admissable poset, it is the union of the elements $(1, 24), (1, 6), (5, 24), (5, 6)$ and $(6, 24)$ of X_G.

Let $P \in \Gamma_G$ and $v < w$ in P. Choose a maximal path[4]

$$v = v_0 < v_1 < \ldots < v_r = w \tag{3.7.1.4}$$

between v and w. Let A be the v-admissable subset containing v_1, we claim that A does not depend on the choice of maximal path and so we may define a map $\gamma_G^P : \{v < w\} \to X_G$ taking $v < w$ to (v, A). Indeed, choose another maximal path this time with v_1' the first step, then by the underset condition applied to w the vertices v_1 and v_1' are joined by an edge and therefore v_1 and v_1' are contained in the same connected component of G_v.

[4]a maximal path has the property that if $v_i < v' < v_{i+1}$ then v' is equal to either v_i or v_{i+1}.

Lemma 3.7.2. *Let P be a G-admissable poset.*

1. *Let $v < w < w' \in P$, then $\gamma_G^P(v < w) = \gamma_G^P(v < w')$.*

2. *Let v be a vertex of G and let w and w' be two vertices in the same v-admissable subset, then $\gamma_G^P(v < w) = \gamma_G^P(v < w')$.*

Proof. The concatenation of maximal paths from v to w and from w to w' is a maximal path from v to w'. The first step from the path from v to w is the same as the first step of the composite path from v to w', hence $\gamma_G^P(v < w) = \gamma_G^P(v < w')$.

Now for the second part. The vertices w and w' are in the same connected component of G_v and so it is enough to assume that w and w' are connected by an edge as equality under γ_G^P in this case would imply equality for the whole connected component. Let

$$v = v_0 < v_1 < \ldots < v_{r-1} < v_r = w \tag{3.7.1.5}$$

and

$$v = v'_0 < v'_1 < \ldots < v'_{s-1} < v'_s = w' \tag{3.7.1.6}$$

be maximal paths in P. Suppose that $r = s = 1$, then $\gamma_G^P(v < w) = [v < w]$ and $\gamma_G^P(v < w') = [v < w']$, which by assumption are the same class so in this case we are done.

Now suppose that $r \neq 1$, then $v_1 < w$ so w is not in the star of v_1. Since w' is connected by an edge to w either w' is in the same connected component of G_{v_1} so $v < v_1 < w'$ or w' is in the star of v_1 and hence is joined by an edge to v_1. In the first case $\gamma_G^P(v < w') = [v < v_1] = \gamma_G^P(v < w)$ using part 1) of the lemma. In the second case we may replace w by v_1 because $\gamma_G^P(v < v_1) = \gamma_G^P(v < w)$ and v_1 is joined by an edge to w'. Our 'new' r is 1 and s remains unchanged. Hence we may assume that $r = 1$. The same argument applies to s meaning we may assume that $s = 1$ also and we have already covered the case $r = s = 1$. □

Definition 3.7.3. The previous lemma established that γ_G^P is constant on subsets $\{v < w\}$ where w ranges across a v-admissable subset. Hence γ_G^P can be lifted to P considered as a subset of X_G. We define γ_G to be the fibres of this lifting, explicitly this is

$$\gamma_G(P) = \left\{(\gamma_G^P)^{-1}(v, A) \mid (v, A) \in \mathrm{Im}(\gamma_G^P)\right\}. \tag{3.7.1.7}$$

Example 3.7.1.2. Recall the graph G and G-admissable poset P from Example 3.7.1.1. The image of γ_G^P is

$$(1, 6), (5, 6) \text{ and } (6, 24). \tag{3.7.1.8}$$

By taking the fibres we find that

$$\gamma_G(P) = \left\{ \begin{tikzcd} 2 & 4 \arrow[d] & 6 \\ & 1 \arrow[ul] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} 2 & 4 \arrow[d] & 6 \\ & 5 \arrow[ul] \arrow[ur] & \end{tikzcd}, \quad \begin{tikzcd} 2 & & 4 \\ & 6 \arrow[ul] \arrow[ur] & \end{tikzcd} \right\}. \tag{3.7.1.9}$$

Proposition 3.7.4. *The pair* (Γ_G, γ_G) *is a diagonal complex.*

Proof. We must show that (1)-(3) from Definition 2.1.3 hold. Throughout this proof P will any G-admissable poset.

Condition (1) states that each (v, A) is a G-admissable poset, this is trivially true. That $\gamma_G(P)$ is a partition of P is clear because $\gamma_G(P)$ consists of the fibres of the map $P \to X_G$. To complete condition (2) we need only remark that γ_G^P is constant only if $P \in X_G^+$. Now onto the first part of condition (3); we must show that any union of a subset of $\gamma_G(P)$ is itself a G-admissable subset. Let $A \subseteq \gamma_G(P)$ and let

$$P_A = \bigcup_{U \in A} U \tag{3.7.1.10}$$

be the union. To see that P_A is a poset we need only show transitivity, let $v_1 < v_2$ and $v_2 < v_3$ in P_A. Since

$$\gamma_G^P(v_1 < v_2) = \gamma_G^P(v_1 < v_3) \tag{3.7.1.11}$$

we have $v_1 < v_3$ in P_A. Next to see that P_A has the overset condition we need only note that by definition it is a union of elements of X_G. For the underset condition suppose that $v < w$ and $v' < w$ in P_A. Since $P_A \subseteq P$ the underset condition for P implies that either $v \leq v'$, $v' \leq v$ in P or (v, v') is an edge of G. In the case that $v < v'$ we have that $\gamma_G^P(v < v') = \gamma_G^P(v < w)$ so $v < v'$ in P_A. Thus the underset condition holds for P_A.

Now it only remains to show that $\gamma_G(P_A)$ is a partial refinement of $\gamma_G(P)$. This follows from

$$\gamma_G^P|_{P_A} = \gamma_G^P \circ \gamma_G^{P_A} \tag{3.7.1.12}$$

which implies that the fibres of $\gamma_G^{P_A}$ are a partial refinement of the fibres of γ_G^P. To show (3.7.1.12), let $v < w$ in P_A and let

$$v = v_0 < v_1 < \ldots < v_{r-1} < v_r = w \tag{3.7.1.13}$$

be a maximal path from v to w in P_A, then $\gamma_G^{P_A}(v < w) = [v < v_1]$. But $v < v_1 < w$ in P so $\gamma_G^P(v < w) = \gamma_G^P(v < v_1)$, which establishes (3.7.1.12). □

Example 3.7.1.3. Still examining the graph G and G-admissable poset P from Exam-

ple 3.7.1.1 we may give the maximal faces of P as

$$\begin{tikzcd}[column sep=small] & 2 & & 4 & & 6 \\ & & 1 \arrow[ul] \arrow[ur] \arrow[urr] & & 5 \arrow[ull] \arrow[ur] \arrow[ull] & \end{tikzcd}, \quad \begin{tikzcd}[column sep=small] 2 & & 4 \\ & 6 \arrow[ul] \arrow[ur] & \\ & 1 \arrow[u] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] 2 & & 4 \\ & 6 \arrow[ul] \arrow[ur] & \\ & 5 \arrow[u] & \end{tikzcd}. \tag{3.7.1.14}$$

Proposition 3.7.5. *The diagonal complex* (Γ_G, γ_G) *is proper. In addition with the map* $l : X_G \to V$ *given by* $l(v, A) = v$ *it is a* V*-labelled diagonal complex.*

Proof. First that the diagonal complex is proper. Let P be a G-admissable poset and let $(v, A) \in \gamma_G^P(P)$. It is enough to show that

$$P' = P - (\gamma_G^P)^{-1}(v, A) \tag{3.7.1.15}$$

is a maximal proper G-admissable poset inside P. Let $P' \subset P'' \subseteq P$ be an intermediate element of Γ_G and suppose that $v < w$ in P'' but not in P'. Let v_1 be the first step of a path from v to w inside P, then $v_1 \in A$. Now inside P'' we have $v, v_1 \leq w$ and so by the underset condition either $v \leq v_1$, $v_1 \leq v$ or (v, v_1) is an edge, but since P'' is contained in P it must be that $v \leq v_1$. Hence $(v, A) = [v < v_1] \in P''$. For any $v < w''$ in $(\gamma_G^P)^{-1}(v, A)$ we may find an intermediate $w' \in A$ such that $v < w' < w''$ in P, since $w < w''$is in P' and $v < w'$ in P'' this implies that $v < w''$ in P'' and hence $P = P''$. Hence the maximal faces of P are the maximal G-admissable posets in the inclusion order, so (Γ_G, γ_G) is proper.

That Γ_G is V-labelled is a simple observation; any $U \in \gamma_G(P)$ is a fibre of the map γ_G^P:

$$(\gamma_G^P)^{-1}(v, A) \tag{3.7.1.16}$$

which consists of pairs (v, B) hence l is constant on elements of $\gamma_G(P)$. □

3.7.2 Action on the graph products

The automorphism groups of right-angled Artin groups have similar properties and a similar 'flavour' to the automorphism groups of the free groups. Ofcourse a right-angled Artin group based on a discrete graph is a free group. At the other extreme the RAAG associated to a complete graph is a free abelian group and the automorphism group is $\mathrm{GL}_n(\mathbb{Z})$. So the automorphism groups of RAAGs may be considered to interpolate between $\mathrm{Aut}(F_n)$ and $\mathrm{GL}_n(\mathbb{Z})$. The relations of a right-angled Artin group provide restrictions to its possible automorphisms, however the automorphism groups are still large and the added complexity of the relations means that the automorphism groups are more complicated and trickier to study. In particular the presentation of the automorphism group depends on the graph.

The majority of work on automorphism groups of graph products has focussed on

the right-angled Artin group case, see for instance [19] and [12]. However there are also treatments of more general cases, for graph products of abelian groups see [24] and for results for a restricted class of graphs see [11].

Let $G = (V, E)$ be a graph and $\mathbf{H}$ be a V-tuple of groups. Recall from Chapter 1 that $\mathbf{H}G$ is the group with presentation

$$\frac{\coprod_{v \in V} H_v}{\langle [H_v, H_w] \mid vw \in E \rangle}. \tag{3.7.2.1}$$

Let (v, A) be a G-admissable pair and $h \in H_v$. Then $\alpha^h_{(v,A)}$ is an element of $\mathbf{H}\Gamma_G$ and by Theorem 2.2.1 the set of all such $\alpha^h_{(v,A)}$ generates $\mathbf{H}\Gamma_G$. The action of $\alpha^h_{(v,A)}$ on $\mathbf{H}G$ is defined as follows

$$\alpha^h_{(v,A)}(g) = \begin{cases} g^h & \text{if } g \in H_w \text{ for } w \in A, \\ g & \text{if } g \in H_w \text{ for } w \notin A. \end{cases} \tag{3.7.2.2}$$

To check that this action of $\alpha^h_{(v,A)}$ on $\mathbf{H}G$ is well-defined we must check that $\alpha^h_{(v,A)}$ preserves the relation of $\mathbf{H}G$; so let $e \in E$ be an edge joining w and w' in G, then for $g_w \in H_w$ and $g_{w'} \in H_{w'}$ there is a relation

$$[g_w, g_{w'}]. \tag{3.7.2.3}$$

Suppose that both $w, w' \in A$, then

$$\alpha^h_{(v,A)}([g_w, g_{w'}]) = \left[g_w^h, g_{w'}^h\right] = [g_w, g_{w'}]^h. \tag{3.7.2.4}$$

For $w, w' \notin A$, then

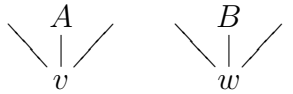
$$\alpha^h_{(v,A)}([g_w, g_{w'}]) = [g_w, g_{w'}]. \tag{3.7.2.5}$$

Now without loss of generality we are left with the case that $w \in A$ and $w' \notin A$. Since (v, A) is an admissable pair, the set A is a connected component of $G_v = G - \mathrm{st}(v)$. But e is an edge with one end w in A and the other end w' not in A, therefore w' must be in the star of v, hence $g_{w'}$ commutes with h and in particular $g_{w'} = g_{w'}^h$. Therefore

$$\alpha^h_{(v,A)}([g_w, g_{w'}]) = \left[g_w^h, g_{w'}\right] = \left[g_w^h, g_{w'}^h\right] = [g_w, g_{w'}]^h. \tag{3.7.2.6}$$

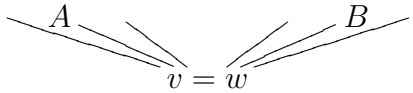
This concludes the check that each $\alpha^h_{(v,A)}$ defines an automorphism of $\mathbf{H}G$, we will denote this automorphism $\beta^h_{(v,A)}$ to distinguish it from the element of $\mathbf{H}\Gamma_G$. Our next task is to check that the relations of $\mathbf{H}\Gamma_G$ coming from the diagonal complex Γ_G hold for the automorphisms $\beta^h_{(v,A)}$. Relations for Γ_G come from the two dimensional elements, first

there are the elements of the form $\{(v, A), (w, B)\}$, the associated posets take the forms

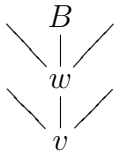


, (3.7.2.7)

, (3.7.2.8)



, (3.7.2.9)



. (3.7.2.10)

In the final tree (3.7.2.10) there is the inclusion $\{w\} \cup B \subseteq A$. The other elements are of the form

$$\{(v, A), (w, B), (v, C_1), \ldots, (v, C_k)\}, \tag{3.7.2.11}$$

where $\bigcup_{i=1,\ldots,k} C_i \subseteq B$, $w \in A$ and $B \subseteq A \cup \bigcup_{i=1,\ldots,k} C_i$. This forms a tree of height three just as in (3.7.2.10). The relations for (3.7.2.7)-(3.7.2.10) are of the form $\left[\beta^h_{(v,A)}, \beta^g_{(w,B)}\right]$, whereas the relation for (3.7.2.11) is

$$\left[\beta^g_{(w,B)}, \beta^h_{(v,A)}\beta^h_{(v,C_1)} \ldots \beta^h_{(v,C_k)}\right]. \tag{3.7.2.12}$$

This final relation may be simplified; the elements $\beta^h_{(v,A)}, \beta^h_{(v,C_1)}, \ldots, \beta^h_{(v,C_k)}$ commute by the relation (3.7.2.9) and if we let $A' = A \cup C_1 \cup \ldots \cup C_k$ be their union and write

$$\beta^h_{(v,A')} = \beta^h_{(v,A)}\beta^h_{(v,C_1)} \ldots \beta^h_{(v,C_k)}, \tag{3.7.2.13}$$

then (3.7.2.12) becomes $\left[\beta^g_{(w,B)}, \beta^h_{(v,A')}\right]$ and $w \cup B \subseteq A'$ and so this relation may be grouped with (3.7.2.10).

Checking these relations is little more complicated than for the automorphisms of free products; for (3.7.2.7) and (3.7.2.9), the sets A and B are disjoint and so the associated automorphisms act non-trivially on different subsets of the generators and hence commute. For the tree (3.7.2.8) the sets A and B do overlap, however by the underset condition v and w must be joined by an edge in G and so in $\mathbf{H}$ the elements g and h commute, so the relation holds.

We are left with the case that $w \cup B \subseteq A$, where A may be a union of v-admissable

sets. Let $w' \in B$ and $g' \in H_{w'}$ then

$$\beta^g_{(w,B)}\beta^h_{(v,A)}(g') = \beta^g_{(w,B)}(g'^h) = (g'^g)^h = g'^{gh} \text{ and} \tag{3.7.2.14}$$

$$\beta^h_{(v,A)}\beta^g_{(w,B)}(g') = \beta^g_{(w,B)}(g'^g) = (g'^h)^{g^h} = g'^{gh}. \tag{3.7.2.15}$$

The remaining cases where $w' \notin B$ are simpler.

We have just shown the following.

Proposition 3.7.6. *There is a map*

$$\mathbf{H}\Gamma_G \to \mathrm{Aut}(\mathbf{H}G). \tag{3.7.2.16}$$

Remark 3.7.7. The functoriality of $\mathbf{H}\Gamma_G$ in $\mathbf{H}$ means that those automorphisms are extendable in the sense of Section 3.5.1.

Question A. *Is the action of* $\mathbf{H}\Gamma_G$ *on* $\mathbf{H}G$ *is faithful? Also do all partial conjugations arise in this way?*

To answer this question we would need different methods, perhaps peak reduction, which we have not covered in this thesis. The next question to ask is

Question B. *For any graph* G*, is the diagonal complex* Γ_G *aspherical?*

This question is a direct analogue of Theorem 3.3.12 and in fact when G is the discrete graph reduces to the theorem. The most direct method of investigating this question appears to by using the associated coset complex for each n-tuple of groups $\mathbf{G}$. This coset complex may be seen as a candidate for an analogue of McCullough-Miller space and it suffices to prove that it is contractible.

A positive answer to both of these questions would allow one to compute the homology of the symmetric automorphism group of a graph product. The answer would be written in terms of direct sums indexed by G-graphs.

Chapter 4

Configuration spaces

Much of the initial interest in symmetric automorphism groups was concentrated on the symmetric automorphism group of a free group, $\mathrm{FR}(\mathbb{Z}^{*k})$ and one of the main motivations for studying this group was a result contained in the thesis of Dahm [18]. He showed that the fundamental group of a configuration space of k oriented loops with an unknotted, unlinked embedding in $\mathbb{R}^3$ was $\mathrm{FR}(\mathbb{Z}^{*k})$.

These spaces are of interest in physics, for instance the equations which govern the dynamics of an inviscid fluid in three dimensions force any vorticity (the curl of the velocity field) to lie along embedded paths, elsewhere it is zero. Furthermore the vorticity is a constant of any solution of the equations; when the fluid moves the vorticity moves with it and furthermore its strength at that point is conserved. The strength of the vorticity is constant along the line and the line must either be infinite or it joins up with itself to form a vortex ring. As such solutions to these dynamical equations have a topological order.

These results are due to Hermann von Helmholtz [26]. William Thomson (later to become Lord Kelvin) was so inspired that he championed [30] a "Vortex theory of Atoms" suggesting that ring vortices occuring in the 'ether' could account for the fundamental particles of nature. The motivation for this idea was the permanence of the vortex rings, they could neither be created nor destroyed, and also by being knotted or linked they could provide enough variety to account for all possible particles. However the theory made no successful predictions and was subsequently abandoned. Furthermore Kelvin was frustrated by the fact that he could not construct stable solutions of the hydrodynamical equations except in the simplest of examples. The theory of vortex rings is still an active area of study today, for example [2] and [31]. For a good summary of the history and current state of vortex dynamics see [40].

There has been a large overlap between work on $\mathrm{FR}(\mathbb{Z}^{*k})$ and on these configuration spaces. Relevant works include the thesis of Dahm [18] and papers [36], [14], [9], [5] and [27]. The roles of these papers were covered in the introduction to the thesis. Results of particular importance to configuration spaces of loops include the result of Dahm that

the fundamental groups correspond to the groups of partial conjugations $\mathrm{FR}(\mathbb{Z}^{*k})$ of the free group, in [42] there is discussion of fibrations involving these spaces and more recently in [6] it is proved that the configuration space of smooth embeddings is homotopy equivalent to a space of Euclidean embeddings.

It is the Euclidean embeddings which we consider in this chapter, however the particular example of configuration spaces of loops is extended to a family $\mathcal{L}_n(k)$ of configuration spaces of k copies of the n-sphere embedded into $\mathbb{R}^{n+2}$, we assume that $n \geq 1$. Since we consider Euclidean embeddings the configuration spaces are manifolds and our first task is to describe the manifold strata associated to them. This allows us to prove Theorem 4.1.3 which states that the fundamental group $\pi_1(\mathcal{L}_n(k))$ is $\mathrm{FR}(\mathbb{Z}^{*k})$ and so is independent of n. So in terms of the fundamental groups we find the same result for n-spheres as we do for loops. The proof is a direct translation from Proposition 3.4 of [6].

The next section considers other homotopy invariants of the spaces $\mathcal{L}_n(k)$; we consider the integral homology, the higher homotopy groups and the homotopy type of the suspension. These are significantly more challenging to compute however conjectures are made; these include a precise characterisation of the integral homology and a connectivity result concerning the higher homotopy groups. We show that the first conjecture would imply that the suspension of $\mathcal{L}_n(k)$ is homotopy equivalent to a bouquet of spheres. For $k = 2$ direct calculations are simple and so the conjectures are checked for this value. We finish the section by considering the colimit of the spaces $\mathcal{L}_n(k)$ as n ranges over $\mathbb{N}$. This is denoted $\mathcal{L}_\infty(k)$ and it would follow from the connectivity conjecture that it is a classifying space for $\mathrm{FR}(\mathbb{Z}^{*k})$. The advantage over the classifying space constructed in the previous chapter is that the action of $\mathfrak{S}_k$ on $\mathcal{L}_\infty(k)$ given by permuting the embedded spheres is proper and so the quotient by $\mathfrak{S}_k$ is a classifying space for

$$\Sigma\mathrm{FR}(\mathbb{Z}^{*k}) \cong \mathrm{FR}(\mathbb{Z}^{*k}) \rtimes \mathfrak{S}_k. \tag{4.0.2.1}$$

We finish the chapter by sketching a possible proof of the homology conjecture.

4.1 The spaces of codimension 2 spheres

There are a number of different ways to produce a space of embeddings of n-spheres into $\mathbb{R}^{n+2}$, one could take the space of smooth embeddings, or spaces of Euclidean embeddings. These are not necessarily equivalent and our choice of Euclidean embeddings via translations and scalings is chosen to give a space with elegant behaviour but without too complicated geometry.

4.1.1 Some definitions

Let $S^n \subseteq \mathbb{R}^{n+2}$ be the n-sphere with the following embedding

$$\left\{ (x_1, \ldots, x_{n+1}, 0) \in \mathbb{R}^{n+2} \mid \sum_{i=1}^{n+1} x_i^2 = 1 \right\}. \tag{4.1.1.1}$$

For example in $\mathbb{R}^3$, the 1-sphere S^1 is the unit circle lying in the xy-plane with centre the origin. The n-sphere is of codimension 2, nevertheless we refer to D^{n+1}

$$\left\{ (x_1, \ldots, x_{n+1}, 0) \in \mathbb{R}^{n+2} \mid \sum_{i=1}^{n+1} x_i^2 < 1 \right\}. \tag{4.1.1.2}$$

as the *interior* of S^n. Let $TS_{n+2} = \mathbb{R}^{n+2} \rtimes \mathbb{R}_{>0}$ be the group of translations and scalings acting on $\mathbb{R}^{n+2}$, so

$$(y_1, \ldots, y_{n+2}, t)\,(x_1, \ldots, x_{n+2}) = (y_1 + tx_1, \ldots, y_{n+2} + tx_{n+2})\,. \tag{4.1.1.3}$$

Then the stabiliser $\mathrm{Stab}_{TS_{n+2}}(S^n) = \{e\}$ and so we identify the *space of embeddings of the n-sphere* with TS_{n+2}. The sphere s associated to an element $g = (y_1, \ldots, y_{n+2}, t) \in TS_{n+2}$ is

$$g.S^n = \left\{ (x_1, \ldots, x_{n+1}, y_{n+2}) \mid \sum_{i=1}^{n+1} (x_i - y_i)^2 = t^2 \right\}. \tag{4.1.1.4}$$

Let $\mathbf{g} = (g_1, \ldots, g_k) \in TS_{n+2}^k$, then we write s_i for the sphere $g_i.S^n$. We say that s_i and s_j are disjoint if $s_i \cap s_j = \emptyset$.

Definition 4.1.1. Let $\mathcal{L}_n(k)$ be the subspace of TS_{n+2}^k consisting of the $\mathbf{g}$ with pairwise disjoint s_i. We call this the *space of n-spheres in* $\mathbb{R}^{n+2}$, points of $\mathcal{L}_n(k)$ will be called *configurations.*

The support of a configuration

Let $\mathbf{g} \in \mathcal{L}_n(k)$ be a configuration, we define a partial ordering $\leq$ on $[k] = \{1, \ldots, k\}$, by writing $s_i \leq s_j$ if $g_j D^{n+1} \subseteq g_i D^{n+1}$, that is the interior of s_i contains the interior of s_j. We call $([k], \leq)$ the *support*, $\mathrm{supp}(\mathbf{g})$ of $\mathbf{g}$.

Recall that a forest poset F is a non-empty poset which satisfies the underset condition (3.3.1.1):

$$\text{for all } x \in F,\ \{y \mid y \leq x\} \text{ is a total order.} \tag{4.1.1.5}$$

Lemma 4.1.2. *Let $\mathbf{g} \in \mathcal{L}_n(k)$, then the support* $\mathrm{supp}(\mathbf{g})$ *is either the empty poset or a forest poset.*

Proof. Let $l \in [k]$ be a point of $\mathrm{supp}(\mathbf{g})$ and suppose that $s_i < s_l$ and $s_j < s_l$. Then both of the interiors of s_i and s_j contain the interior of s_l and hence the interiors of s_i and

s_j have non-empty intersection. But this implies that one must be contained within the other, hence either $s_i \leq s_j$ or $s_j \leq s_i$. $\square$

4.1.2 The structure of the configuration space

Recall that by viewing a poset as a subset of $\{(i,j) \in [k] \times [k] \mid i \neq j\}$, we may form the poset of posets on $[k]$ ordered under inclusion. Let F be a forest poset and $\mathcal{L}_n^F$ be the subspace of $\mathcal{L}_n(k)$ consisting of those configurations $\mathbf{g}$ with $\mathrm{supp}(\mathbf{g}) \subseteq F$. Let $\mathcal{L}_n^0$ be the subspace of $\mathcal{L}_n(k)$ of configurations with empty support. We denote by $\mathcal{L}_n^{(F)}$ the subspace of configurations with support equal to F, hence as sets

$$\mathcal{L}_n^F = \mathcal{L}_n^0 \amalg \coprod_{F' \subseteq F} \mathcal{L}_n^{(F')}. \tag{4.1.2.1}$$

Of course

$$\mathcal{L}_n(k) = \mathcal{L}_n^0 \amalg \coprod_{F} \mathcal{L}_n^{(F)}. \tag{4.1.2.2}$$

Let F be a forest poset and let $j_1, \ldots, j_r$ be the roots of the trees within the Hasse diagram of F, equivalently these are the minimal elements in F. Let $\mathbf{g} \in \mathcal{L}_n^{(F)}$ be a configuration with support F. Since each j_s is minimal the sphere s_{j_s} can not be contained in any other spheres. Also if $i \neq j_s$ for any $s = 1, \ldots, r$, then s_i must be contained in one of the s_{j_s}. So there is a projection

$$P : \mathcal{L}_n^{(F)} \to X_{n+2}^r \tag{4.1.2.3}$$

where X_{n+2}^r is the space of disjoint embeddings (via elements of TS_{n+2}) of r copies of D^{n+1} into $\mathbb{R}^{n+2}$. This space is homotopic to the configuration space $\mathrm{Conf}_r(\mathbb{R}^{n+2})$ of r points in $\mathbb{R}^{n+2}$, which can be seen by sending the discs D^{n+1} to their centre points. The map going the other way is given by sending each point to the disc with centre that point and radius a quarter of the minimal distance between points, m say, the homotopy is then given by first shrinking the discs with radius larger than m and then enlarging those discs with smaller radii.

Now consider any element i of F, suppose that i has t outgoing edges. This means that for each configuration the sphere s_i contains within it t other spheres corresponding to the other ends of the outgoing edges. So let Y_{n+1}^t be the space of t disjoint Euclidean $n+1$ discs embedded into a larger $n+1$ disc. Then there is a projection

$$p_i : \mathcal{L}_n^{(F)} \to Y_{n+1}^t \tag{4.1.2.4}$$

which sends the configuration $\mathbf{g}$ to the configuration of t spheres embedded in the sphere s_i. Using the same argument as above, the space Y_{n+1}^t is homotopic to the configuration space of t distinct points in D^{n+1} and since D^{n+1} is open and hence homeomorphic to

$\mathbb{R}^{n+1}$, the space Y^t_{n+1} is homotopic to $\mathrm{Conf}_t(\mathbb{R}^{n+1})$.

The projections P and p_i determine a homeomorphism

$$P \times \prod_{i\in[k]} p_i : \mathcal{L}_n^{(F)} \to X_{n+2}^{\text{no. of trees}} \times \prod_{i\in[k]} Y_{n+1}^{|\mathrm{out}(i)|}. \tag{4.1.2.5}$$

Since each Y^t_{n+1} is n-connected and each X^r_{n+2} is $(n+1)$-connected we now know that $\mathcal{L}_n^{(F)}$ is n-connected. Applying the above arguments to $\mathcal{L}_n^0$ we find that it is homotopic to $\mathrm{Conf}_k(\mathbb{R}^{n+2})$ and so is $(n+1)$-connected.

The manifold strata

We will now show that there is a good stratification of $\mathcal{L}_n(k)$ which will allow us to use general position arguments in the proof of Theorem 4.1.3.

The space of n-spheres in $\mathbb{R}^{n+2}$ is an open subset of $TS^k_{n+2} \cong \mathbb{R}^{k(n+3)}$, making it a $k(n+3)$-manifold. Using (4.1.2.5) we may see that for a forest poset F, the space $\mathcal{L}_n^{(F)}$ is of dimension

$$(n+3)(\text{no. of minimal elements}) + (n+2)\sum_{i\in[k]}(\text{no. of outgoing edges from } i), \tag{4.1.2.6}$$

which by some simple combinatorics is $k(n+3)-|E(F)|$, so $\mathcal{L}_n^{(F)}$ is a codimension $|E(F)|$ submanifold of $\mathcal{L}_n(k)$. In particular $\mathcal{L}_n(k)$ is the union of $\mathcal{L}_n^0$ and strictly positive codimension strata, hence $\mathcal{L}_n^0$ is dense and open in $\mathcal{L}_n(k)$. We define $\mathcal{L}_n^1$ to be the union of the codimension 1 strata

$$\mathcal{L}_n^1 = \bigcup_{(i,j)} \mathcal{L}_n^{\overrightarrow{ij}} \tag{4.1.2.7}$$

where $\mathcal{L}_n^{\overrightarrow{ij}}$ is $\mathcal{L}_n^{(F)}$ for F the forest with a single edge $\overrightarrow{ij}$. Also we write $\mathcal{L}_n^2$ for the codimension 2 strata

$$\mathcal{L}_n^2 = \bigcup_{i,j,k,l \text{ distinct}} \mathcal{L}_n^{(\overrightarrow{ji},\overrightarrow{lk})} \cup \bigcup_{i,j,k \text{ distinct}} \mathcal{L}_n^{(\overrightarrow{ki},\overrightarrow{kj})} \cup \bigcup_{i,j,k \text{ distinct}} \mathcal{L}_n^{(\overrightarrow{kj},\overrightarrow{ji})}. \tag{4.1.2.8}$$

Here the respective terms correspond to $\mathcal{L}_n^{(F)}$ for forests F

$$\begin{tikzcd} i & k \\ j \arrow[u] & l, \arrow[u] \end{tikzcd} \qquad \begin{tikzcd} i & j \arrow[dl] \\ k \arrow[u] & \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} i \\ j \arrow[u] \\ k \arrow[u] \end{tikzcd} \tag{4.1.2.9}$$

respectively.

4.1.3 The fundamental group

The following is a straight generalisation of Proposition 3.4 of [6].

Theorem 4.1.3. *Let $n \geq 1$, then the fundamental group of $\mathcal{L}_n(k)$ is isomorphic to* $\mathrm{FR}(\mathbb{Z}^{*n})$.

Proof. We proceed via a general position argument. Any path in $\mathcal{L}_n(k)$ can be moved infinitesimally so that it passes through the submanifolds $\mathcal{L}_n^{(F)}$ transversally. This means that any path $\gamma : S^1 \to \mathcal{L}_n(k)$ may be perturbed to some smooth $\widehat{\gamma}$ so that it lies in $\mathcal{L}_n^0$ and the codimension 1 strata $\mathcal{L}_n^1$. Hence

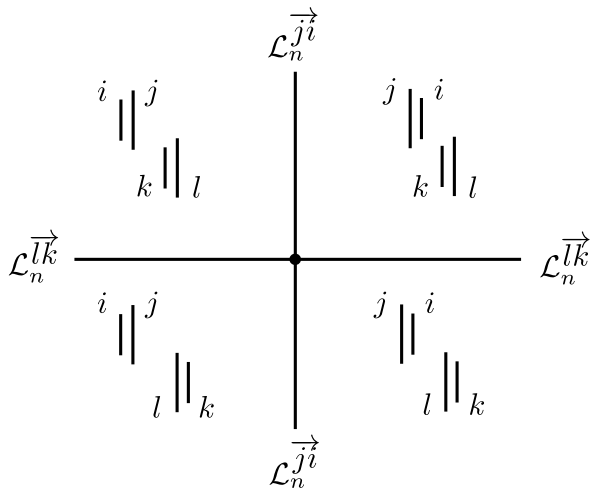
$$\pi_1(\mathcal{L}_n^0 \cup \mathcal{L}_n^1) \to \pi_1(\mathcal{L}_n(k)) \tag{4.1.3.1}$$

is surjective. Furthermore the transversality implies that for any $p \in S^1$ with $\widehat{\gamma}(p) \in \mathcal{L}_n^1$ there is a neighbourhood U such that $(\widehat{\gamma}\,|_U)^{-1}(\mathcal{L}_n^1) = \{p\}$. Since both $\mathcal{L}_n^0$ and each $\mathcal{L}_n^{\overrightarrow{ij}}$ are simply connected and since $\mathcal{L}_n^{\overrightarrow{ij}}$ has trivial normal bundle we have that $\pi_1(\mathcal{L}_n^0 \cup \mathcal{L}_n^1)$ is freely generated by the paths α_j^i which pass only once through $\mathcal{L}_n^1$ in the component $\mathcal{L}_n^{\overrightarrow{ij}}$.

Now suppose that $H : [0,1]^2 \to \mathcal{L}_n(k)$ is a homotopy between transverse paths γ_1, γ_2. Since $[0,1]^2$ is a 2 dimensional manifold with boundary we may perturb it to $\widehat{H}$ so that it lies transverse to any submanifold $\mathcal{L}_n^F$. This means that the homotopy lies in $\mathcal{L}_n^0 \cup \mathcal{L}_n^1 \cup \mathcal{L}_n^2$ and hence

$$\pi_1\big(\mathcal{L}_n^0 \cup \mathcal{L}_n^1 \cup \mathcal{L}_n^2\big) \to \pi_1\big(\mathcal{L}_n(k)\big) \tag{4.1.3.2}$$

is an isomorphism. The transverse homotopy $\widehat{H}$ intersects $\mathcal{L}_n^2$ locally in isolated points. Since each $\mathcal{L}_n^{(F)}$ is connected to find the relations for $\pi_1(\mathcal{L}_n(k))$ it suffices to look at the possible behaviour near each codimension 2 component. So we look case by case at the forests from (4.1.2.9). The first forest F consists of two disjoint edges and so a point of $\mathcal{L}_n^{(F)}$ is a configuration of spheres $\mathbf{g}$ in which sphere i is contained in sphere j and sphere k is contained in sphere l and there are no other nested spheres. Such a point p is represented by the centre point in the diagram (4.1.3.3) below

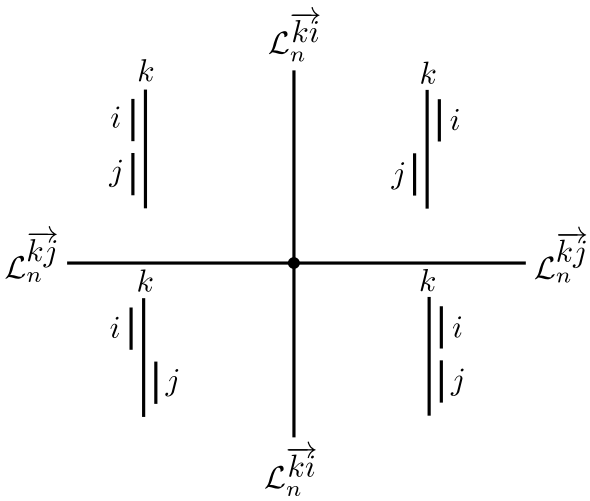


(4.1.3.3)

The remainder of the diagram represents the structure of a local intersection with the homotopy $\widehat{H}$. The four quadrants are areas intersecting with $\mathcal{L}_n^0$ and the figure in each quadrant represents a generic point in a neighbourhood of p. The four lines are labelled by the codimension 1 strata they are contained in. The homotopy asserts that the two possible paths travelling around the central point are homotopic and so we obtain the relation

$$\alpha_i^j.\alpha_k^l = \alpha_k^l.\alpha_i^j. \tag{4.1.3.4}$$

The next forest consists of a single tree with a root k and two leaves with end vertices i and j and a point p of $\mathcal{L}_n^{(F)}$ is a configuration in which spheres i and j are contained in sphere k. The next diagram (4.1.3.5) is analogous to diagram (4.1.3.3) above.



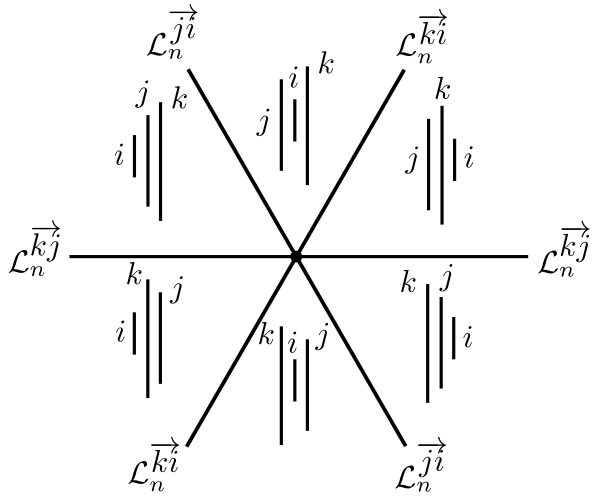
(4.1.3.5)

This time it represents the relation

$$\alpha_i^k.\alpha_j^k = \alpha_j^k.\alpha_i^k. \tag{4.1.3.6}$$

The final case is a forest of one two-edged tree with root k which is joined to j which is in turn joined to i. The point p this time is a configuration where sphere k contains sphere j which in turn contains sphere i. The intersection with the homotopy this time is more complex as there are six types of configurations in $\mathcal{L}_n^0$ occuring, each corresponding to an

ordering of i, j and k as pictured in (4.1.3.7).



(4.1.3.7)

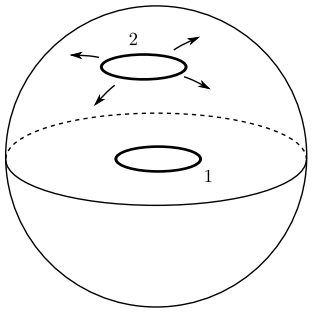
The corresponding relation thus has three generators on each side and is given by

$$\alpha_i^j.\alpha_i^k.\alpha_j^k = \alpha_j^k.\alpha_i^k.\alpha_i^j. \tag{4.1.3.8}$$

These three classes of relations along with the generators α_i^j give a defining presentation of $\pi_1(\mathcal{L}_n(k))$ and this is the same as the defining presentation for $\mathrm{FR}(\mathbb{Z}^{*n})$, see Proposition 3.1.2. □

4.2 Other homotopy invariants

Now that we know that the fundamental group of each of the spaces $\mathcal{L}_n(k)$ is $\mathrm{FR}(\mathbb{Z}^{*k})$ we should consider other homotopy invariants of $\mathcal{L}_n(k)$. There are the homology groups $H_i(\mathcal{L}_n(k), \mathbb{Z})$, the higher homotopy groups $\pi_i(\mathcal{L}_n(k))$ for $i \geq 2$ and also the homotopy type of the suspension $\Sigma\mathcal{L}_n(k)$. It is easy to see that the higher homotopy groups are non-trivial and that the homology groups differ from those of $\mathrm{FR}(\mathbb{Z}^{*k})$; consider the following diagram to see a 2-sphere in $\mathcal{L}_1(2)$.



(4.2.0.9)

We start this section with a conjecture which states precisely the homology of $\mathcal{L}_n(k)$; whereas the homology of $\mathrm{FR}(G_1 * \ldots * G_n)$ is given by forests with edges labelled by

homology classes from $H_*(G_i, \mathbb{Z})$, the 'homology conjecture' states that the homology of $\mathcal{L}_n(k)$ is given by the same forests but now with vertex labelling, this time by elements from the homology of the configuration space of points in $\mathbb{R}^{n+2}$. Next we discuss the suspension of $\mathcal{L}_n(k)$ and assuming that the homology conjecture holds we show that $\Sigma\mathcal{L}_n(k)$ is homotopic to a bouquet of spheres. This should come as no surprise because it is known [27] that the suspension of a $K(\mathrm{FR}(\mathbb{Z}^{*k}), 1)$ is a bouquet of spheres, as is the suspension of the configuration space of points in $\mathbb{R}^{n+2}$.

We then make another conjecture that $\mathcal{L}_n(k)$ is highly connected in the sense that $\pi_i(\mathcal{L}_n(k)) = 0$ for $1 < i \leq n$; we refer to this as the 'homotopy conjecture'. Our final result shows that the colimit $\mathcal{L}_\infty(k)$ of the $\mathcal{L}_n(k)$ as n tends to infinity is a $K(\mathrm{FR}(\mathbb{Z}^{*k}), 1)$ with a proper S_k-action, we show that this follows by either the homology or homotopy conjecture.

4.2.1 The homology groups of $\mathcal{L}_n(k)$

We know the homology of $\mathrm{FR}(\mathbb{Z}^{*k})$ from Theorem 3.4.2; recall that $H_i(\mathrm{FR}(\mathbb{Z}^{*k}), \mathbb{Z})$ is the free graded $\mathbb{Z}$-module on a basis of vertex labelled forests with i edges. By filling in higher spheres there is a map

$$\mathcal{L}_n(k) \to X, \tag{4.2.1.1}$$

where X is a $K(\mathrm{FR}(\mathbb{Z}^{*k}), 1)$ so there is necessarily a map on homology

$$H_*(\mathcal{L}_n(k), \mathbb{Z}) \to H_*(X, \mathbb{Z}) = H_*(\mathrm{FR}(\mathbb{Z}^{*k}), \mathbb{Z}). \tag{4.2.1.2}$$

It is likely that this map is both split on homology and that the homology of $\mathcal{L}_n(k)$ splits over a direct sum;

$$\bigoplus_{F \in \mathcal{F}_k}, \tag{4.2.1.3}$$

where $\mathcal{F}_k$ is the set of forest posets, just as the homology of $\mathrm{FR}(G_1 * \ldots * G_k)$ does. I will actually go further and give a conjectural precise description of the integral homology of $\mathcal{L}_n(k)$. For this we will need some notation; the homology of the configuration space of k labelled points in $\mathbb{R}^n$ is a free $\mathbb{Z}$-module which I denote $C_n(k)$. A combinatorial basis involving terms like

$$1.\,[2,4]\,.\,[5,[3,6]] \tag{4.2.1.4}$$

is possible but we will not go into details of that here. Let P be a forest poset with Hasse diagram F, this diagram is a rooted forest, for example

$$\begin{tikzcd}
1 & & 2 & & \\
8 & 6 \arrow[ul] \arrow[ur] & & 5 & \\
 & 4 \arrow[ul] \arrow[u] & & 3 \arrow[u] & 7.
\end{tikzcd} \tag{4.2.1.5}$$

We write $\textsc{out}(i)$ for the number of outgoing edges from the vertex i and define $\textsc{out}(0)$ to be the number of minimal elements of P or equivalently the number of roots of F. For instance in the forest above, $\textsc{out}(6) = \textsc{out}(4) = 2$ and $\textsc{out}(0) = 3$.

To each F associate the graded $\mathbb{Z}$-module

$$C_n(F) := C_{n+2}\big(\textsc{out}(0)\big) \otimes \bigotimes_{i=1,\dots,k} C_{n+1}\big(\textsc{out}(i)\big) \tag{4.2.1.6}$$

We may now state the conjecture.

Conjecture 4.2.1. *The homology of $\mathcal{L}_n(k)$ is given by*

$$\bigoplus_{F\in\mathcal{F}_k} C_n(F)\,[|E(F)|]\,. \tag{4.2.1.7}$$

The notation $[|E(F)|]$ means that the graded $\mathbb{Z}$-module is suspended by the number of edges of a forest F.

4.2.2 The configuration space when $k = 2$

For this low value it is possible to compute the homotopy type using elementary methods. For a point $(\mathbf{g}_1, \mathbf{g}_2) \in \mathcal{L}_n(2)$ let z_1 and z_2 be the respective $(n+2)$th coordinates. Let $A = \{(\mathbf{g}_1, \mathbf{g}_2) \mid z_1 \le z_2\}$ and $B = \{(\mathbf{g}_1, \mathbf{g}_2) \mid z_1 \ge z_2\}$, then $\mathcal{L}_n(2) = A \cup B$ and the intersection $A \cap B$ is the space

$$\{(\mathbf{g}_1, \mathbf{g}_2) \mid z_1 = z_2\}. \tag{4.2.2.1}$$

Both A and B are contractable so we see that $\mathcal{L}_n(2)$ is equivalent to the suspension of $A \cap B$. This intersection is the disjoint union of three connected components. These are determined by their support; either $s_1 \le s_2$, $s_2 \le s_1$ or the support poset is empty. Both the components with non-trivial support are contractable. The remaining component is equivalent to the configuration space of two points in $\mathbb{R}^{n+1}$ and so equivalent to an n-sphere. So

$$H_i(A \cap B, \mathbb{Z}) = \begin{cases} 0 & \text{if } n < i, \\ \mathbb{Z} & \text{if } i = n, \\ 0 & \text{if } 0 < i < n \text{ and} \\ \mathbb{Z}^3 & \text{if } i = 0. \end{cases} \tag{4.2.2.2}$$

The space $\mathcal{L}_n(2)$ is equivalent to the suspension so

$$H_i(\mathcal{L}_n(2), \mathbb{Z}) = \begin{cases} 0 & \text{if } n+1 < i, \\ \mathbb{Z} & \text{if } i = n+1, \\ 0 & \text{if } 1 < i < n+1, \\ \mathbb{Z}^2 & \text{if } i = 1 \text{ and} \\ \mathbb{Z} & \text{if } i = 0. \end{cases} \tag{4.2.2.3}$$

This agrees with the conjecture. The two components in degree 1 correspond to the forests with a single edge, whilst the sum $\mathbb{Z} \oplus \mathbb{Z}[n+1]$ corresponds to the term of the forest with no edges.

With this realisation as a suspension it is easy to compute the homotopy type of $\mathcal{L}_n(2)$: it is a bouquet of spheres of dimensions 1, 1 and $n+1$. Of course we know that for higher values of k that $\mathcal{L}_n(k)$ is not a bouquet of spheres, afterall the fundamental groups are not free.

The projection p

We may use what we have learnt from $\mathcal{L}_n(2)$ to learn more about the general case $\mathcal{L}_n(k)$. For a pair $1 \leq i < j \leq k$, define the projection

$$p_{ij} : \mathcal{L}_n(k) \to \mathcal{L}_n(2) \tag{4.2.2.4}$$

to be the map which forgets each sphere except for spheres s_i and s_j. The full projection p is given by applying each map p_{ij} to get

$$p : \mathcal{L}_n(k) \to \prod_{1 \leq i < j \leq k} \mathcal{L}_n(2). \tag{4.2.2.5}$$

This map is reminiscent of an abelianisation map, albeit just a partial abelianisation on the fundamental groups

$$\pi_1(p) : \mathrm{FR}(\mathbb{Z}^{*k}) \to \prod_{1 \leq i < j \leq k} F_2, \tag{4.2.2.6}$$

where $p(\alpha_i^j)$ and $p(\alpha_j^i)$ pick out a pair of generators in the same factor F_2, the free group on two generators.

4.2.3 The suspension $\Sigma\mathcal{L}_n(k)$

The space $\mathcal{L}_n(2)$ is homotopy equivalent to a bouquet of spheres and hence so is the suspension $\Sigma\mathcal{L}_n(2)$. This section is devoted to the idea that the suspension of $\mathcal{L}_n(k)$ is always weakly equivalent to a bouquet of spheres, even though $\mathcal{L}_n(k)$ is not for $k > 2$.

In fact we will show that it is a corollary of the homology conjecture 4.2.1. We will use the map p defined above to show this; first we deal with the target of p. Since $\mathcal{L}_n(2)$ is a bouquet of spheres, and since the suspension of a direct product of spheres is also a bouquet of spheres (the most familiar example being the suspension of a torus), the suspension

$$\Sigma\left(\prod_{1\leq i<j\leq k}\mathcal{L}_n(2)\right) \tag{4.2.3.1}$$

is also a bouquet of spheres. In particular the homology is a free $\mathbb{Z}$-module. The map p induces a map on homology

$$H_*(p) : H_*(\mathcal{L}_n(k),\mathbb{Z}) \to \bigotimes_{1\leq i<j\leq k} H_*(\mathcal{L}_n(2),\mathbb{Z}). \tag{4.2.3.2}$$

Now suppose that Conjecture 4.2.1 holds, then $H_*(p)$ is a split monomorphism. This is not immediately obvious, however can be shown either using a careful analysis of the two homology groups each of which would be known, or it can be shown using techniques which would likely come from a proof of the conjecture.

Since $H_*(p)$ is a split monomorphism, so is $\Sigma H_*(p) = H_*(\Sigma p)$. But the target of Σp is a bouquet of spheres, so by picking a bouquet $S_{\mathcal{L}_n(k)}$ of representative spheres of the image of $H_*(\Sigma p)$ and collapsing the remaining spheres one has a map

$$\Sigma\mathcal{L}_n(k) \to \Sigma \prod_{1\leq i<j\leq k}\mathcal{L}_n(2) \to S_{\mathcal{L}_n(k)}, \tag{4.2.3.3}$$

which is an isomorphism on homology. But since the spaces are simply connected this implies that the spaces $\Sigma\mathcal{L}_n(k)$ and $S_{\mathcal{L}_n(k)}$ are weakly homotopic, so $\Sigma\mathcal{L}_n(k)$ does have the weak homotopy type of a bouquet of spheres.

Remark 4.2.1. This argument should be compared to the proof that $\Sigma B\left(\mathrm{FR}(\mathbb{Z}^{*k})\right)$ is a bouquet of spheres which was Corollary 6.9 of [27]. The proof was due to Fred Cohen and rather than use the map p used an abelianisation map.

4.2.4 Remarks on the higher homotopy groups

An explicit calculation of the higher homotopy groups of $\mathcal{L}_n(k)$ would be too much to hope for, however in this short section we may conjecture the following result.

Conjecture 4.2.2. *The higher homotopy groups $\pi_i(\mathcal{L}_n(k))$ are zero for $2 \leq i \leq n$.*

We may see that this conjecture holds for $k = 2$ using Section 4.2.2. It would be convenient if the map forgeting the kth sphere

$$\mathcal{L}_n(k) \to \mathcal{L}_n(k-1) \tag{4.2.4.1}$$

was a fibration, however we find that the fibres are not homeomorphic or even homotopic. But this is because we have chosen a very restrictive Euclidean geometry to work with. In [42], it was shown that for $n = 1$ the corresponding map for smooth embeddings is a fibration and that the kernel on fundamental groups was the group

$$\left\langle \alpha_i^n, \alpha_n^i \mid i = 1, \ldots, n-1 \right\rangle / \left(\left[\alpha_i^n, \alpha_j^n \right] \mid i, j = 1, \ldots, n-1 \right). \tag{4.2.4.2}$$

And in [6] it is shown that the smooth version of $\mathcal{L}_1(k)$ is homotopic to the Euclidean version where the restriction that the spheres lie in the same plane is lifted. The fibres of the fibration are given by the space of smooth embeddings of k loops but where the first $k-1$ are fixed. It would be enough to show that the conjecture held for these spaces, then the long exact sequence of homotopy groups would allow the full conjecture to be proved by induction on k.

4.2.5 Another classifying space for $\mathbf{FR}(F_k)$

In Chapter 3 we defined the moduli space of cactus products built from an k-tuple $\mathbf{Y} = (Y_1, \ldots, Y_k)$ and in Theorem 3.3.12 showed that if the pointed spaces Y_i were classifying spaces for G_i then the moduli space of cactus products is a classifying space for $\mathrm{FR}(G_1 * \ldots * G_k)$. This provides a good geometric model of the partial conjugations of a free product. When each space is the unit circle $Y_i = S^1$ the cactus products are called cactus graphs and provide a model for the group $\mathrm{FR}(F_k)$, we will now construct an alternative model using the configuration spaces of spheres.

In the previous sections we have made two conjectures on the homology and homotopy groups of $\mathcal{L}_n(k)$ respectively.

Proposition 4.2.2. *There are inclusions*

$$\mathcal{L}_1(k) \hookrightarrow \mathcal{L}_2(k) \hookrightarrow \mathcal{L}_3(k) \hookrightarrow \ldots, \tag{4.2.5.1}$$

which induce isomorphisms on the underlying fundamental groups.

Proof. Define a map

$$\varphi_m : \mathcal{L}_m(k) \to \mathcal{L}_{m+1}(k) \tag{4.2.5.2}$$

by considering the inclusion $TS_{m+2} \to TS_{m+3}$ given by

$$(y_1, \ldots, y_{m+2}, t) \mapsto (0, y_1, \ldots, y_{m+2}, t) \tag{4.2.5.3}$$

Recall that $\mathcal{L}_m(k)$ was defined as a subspace of ${TS_{m+2}}^k$, the induced map into ${TS_{m+3}}^k$ has image in $\mathcal{L}_{m+1}(k)$ and this defines our map φ_m. The map has the effect of taking an m-sphere with centre $(x_1, \ldots, x_{m+2})$ and radius r to an $(m+1)$-sphere with centre

$(0, x_1, \ldots, x_{m+2})$ and radius r.

The support of a configuration is also preserved, because if m-sphere s_i is contained in m-sphere s_j then $(m+1)$-sphere $\varphi_m(s_i)$ is contained in $(m+1)$-sphere $\varphi_m(s_j)$. Hence the map φ_m preserves the manifold strata defined using the support, in particular taking codimension 1 and 2 submanifolds to codimension 1 and 2 submanifolds respectively. Therefore $\pi_1(\varphi_m)$ is an isomorphism. □

Taking the union of the spaces (or colimit of the diagram) we get a space

$$\mathcal{L}_\infty(k) := \bigcup_{n\geq 1} \mathcal{L}_n(k). \tag{4.2.5.4}$$

Theorem 4.2.3. *(assuming Conjecture 4.2.2) The space $\mathcal{L}_\infty(k)$ is aspherical and so a classifying space for*

$$\mathrm{FR}(F_k) \cong \mathbb{Z}\Gamma_{\mathcal{F}_k}. \tag{4.2.5.5}$$

Proof. Suppose that the connectivity conjecture holds. Any based map from S^n for $n \geq 2$ is null-homotopic in $\mathcal{L}_m(k)$ for $m \geq n$ and hence also in $\mathcal{L}_\infty(k)$. Therefore all higher homotopy groups vanish and $\mathcal{L}_\infty(k)$ is a classifying space for $\mathrm{FR}(\mathbb{Z}^{*k})$. □

Each space $\mathcal{L}_n(k)$ has a proper action of $\mathfrak{S}_k$ given by relabelling the spheres and hence so does $\mathcal{L}_\infty(k)$.

Corollary 4.2.4. *(requiring Conjecture 4.2.2) The quotient $\mathcal{L}_\infty(k)/\mathfrak{S}_k$ is a classifying space for*

$$\Sigma\mathrm{FR}(F_k) \cong \mathbb{Z}\Gamma_{\mathcal{F}_k} \rtimes \mathfrak{S}_k. \tag{4.2.5.6}$$

4.2.6 Sketch of a possible proof of Conjecture 4.2.1

Before considering the space $\mathcal{L}_n(k)$ we will consider a simpler example, $\mathbb{R}^k$. Let $f : [k] \to [m]$ be a surjective map of sets, where $[k] = \{1, \ldots, k\}$. This defines a subset of $\mathbb{R}^k$; let A_f be

$$\{(z_1, \ldots, z_k) \mid z_i \leq z_j \text{ if } f(i) \leq f(j)\}. \tag{4.2.6.1}$$

So if $f(i) = f(j)$ then $z_i = z_j$. The dimension of A_f is easily seen to be k. The action of $(\mathbb{R}, +)$ on $\mathbb{R}^k$ by addition on each factor preserves the subspaces A_f, so we will consider the projection to the subspace

$$\{(z_1, \ldots, z_k) \mid z_1 + \ldots + z_k = 0\}. \tag{4.2.6.2}$$

The decomposition is also invariant with respect to multiplication by elements of $\mathbb{R}_{>0}$ so we will contract further to the $(k-1)$-disc

$$\big\{(z_1, \ldots, z_k) \mid z_1 + \ldots + z_k = 0 \text{ and } z_1^2 + \ldots + z_k^2 \leq 1\big\}. \tag{4.2.6.3}$$

What we have now is a kind of polyhedron. We may take its dual. This now has only one $(k-1)$-cell which is the whole of the $(k-1)$-disc and is the dual of the space A_c, where $c : [k] \to [1]$ is the constant map. This dual complex is in fact a filled-in version of the Coxeter complex of type A_{k-1}. The action of $\mathfrak{S}_k$ corresponds to the action of $\mathfrak{S}_k$ on $\mathbb{R}^k$.

There is a geometric interpretation of the dual. Define a function m on $\mathbb{R}^k$ by

$$m(z_1, \ldots, z_k) = 1 - \sum_{1 \le i < j \le k} (z_i - z_j)^2. \tag{4.2.6.4}$$

This is a Morse function on the $(k–1)$-disc and the descending cells correspond to the dual polyhedron. One may use the Morse complex MC_* to compute the integral homology of the disc. The stationary points contributing to MC_i are parametrised by the surjections $f : [k] \to [k-i]$. Viewing f as a partition of k into $k-i$ pieces, the differential of MC_* takes f to the sum with signs over all possible ways to refine partition f into $k-i+1$ pieces. For example

$$d(13 \mid 24) = 1 \mid 3 \mid 24 - 3 \mid 1 \mid 24 - 13 \mid 2 \mid 4 + 13 \mid 4 \mid 2. \tag{4.2.6.5}$$

The sign convention defines $\mid$ to have degree -1 and each number to have degree 1, passing d past either induces a sign change. Computing the homology of the complex MC_* gives the homology of $\mathbb{R}^k$.

Configurations of spheres

Note that there is a map $\pi : \mathcal{L}_n(k) \to \mathbb{R}^k$ given by taking the $(n+2)$-coordinate of each of the k spheres. Both the actions of $(\mathbb{R}, +)$ and $(\mathbb{R}_{>0}, .)$ by translation and scaling respectively apply to $\mathcal{L}_n(k)$ and the map π respects them. Furthermore the Morse function m is also defined on $\mathcal{L}_n(k)$. However the stationary points are not isolated, instead we have stationary sets again parametrised by the functions $f : [k] \to [k-i]$. It is now possible to argue that the Morse function decomposes $\mathcal{L}_n(k)$ into descending 'cells', however the 'cells' are not i-cells, they are the direct products of i-cells with the stationary sets. There is a further complication in that the descending sets do not totally cover $\mathcal{L}_n(k)$, however this may be resolved because the Morse function gives an explicit deformation retract onto the space which is covered.

We obtain a Morse complex $MC_*(\mathcal{L}_n(k); f)$ with the form

$$\bigoplus_{i=1,\ldots,k-1} \Big(\bigoplus_{f:[k]\to[k-i]} C_*(f) \Big)[i], \tag{4.2.6.6}$$

where $C_*(f)$ is the chain complex of the stationary set corresponding to f. The stationary sets are easy to describe. Let $\mathcal{N}\mathcal{L}_n(l)$ be the configuration space of n-spheres embedded in $\mathbb{R}^{n+1}$. Since the spheres may be nested (hence the $\mathcal{N}$), this space is not connected.

View $f : [k] \to [k-i]$ as an ordered partition, for each set $f^{-1}(a)$ there is $\mathcal{NL}_n(f^{-1}(a))$, the configuration space of n-spheres labelled by the elements of $f^{-1}(a)$ inside $\mathbb{R}^{n+1}$. The stationary set is the product of these configuration spaces:

$$\prod_{a \in [k-i]} \mathcal{NL}_n(f^{-1}(a)) \tag{4.2.6.7}$$

Using the ideas from Section 4.1.2 one may see that this space is homotopy equivalent to a disjoint union of products of configuration spaces of points in $\mathbb{R}^{n+1}$. The disjoint union is parametrised by forests and the homology of configuration spaces of labelled points is free as a $\mathbb{Z}$-module so we may compute the complexes $C_*(f)$ explicitly.

We are now left with the differential of $MC_*(\mathcal{L}_n(k), f)$ to compute. These are induced from the maps

$$\mathcal{NL}_n(A \amalg B) \to \mathcal{NL}_n(A) \times \mathcal{NL}_n(B), \tag{4.2.6.8}$$

which split the configurations of spheres into two. Looking at the projection

$$\mathcal{NL}_n(A \amalg B) \to \mathcal{NL}_n(A), \tag{4.2.6.9}$$

the map is defined by forgetting the positions of the spheres labelled by points in B. The differential on $C_*(f)$ is the sum of such maps over all refinements of f.

The hardest step in the "proof" is the last, one must compute the homology of the complex $MC_*(\mathcal{L}_n(k); f)$. Fortunately there is a theory of Gröbner basis for operads [20] which gives an explicit basis for the homology of the configuration space of points in $\mathbb{R}^{n+1}$. This allows us to construct explicit maps linking the Morse complex and the homology complex from the conjecture. The precise detail of this homotopy equivalence is the final piece of the proof.

Epilogue

This thesis contains results on and related to families of groups occuring as automorphism groups of free products. The exposition aims to prove these results in an efficient but most of all clear manner. However the methods for constructing the spaces and proving the results have been arrived at after a number of iterations. In this epilogue the author would like to discuss some of the ideas which lie behind the thesis. Just because they are no longer necessary does not mean that they are no longer relevant, indeed some of them will be returned to in future work.

The first section will discuss some algebraic structure which led the author to first investigate the automorphism groups of free products. The second section goes on to discuss the influence of the functoriality of the Fouxe-Rabinovitch groups. It is the functoriality which is the inspiration behind the final section which sketches a theory which aims to extend the objects of Chapter 4 to a whole functor of similar objects.

Additional algebraic structure

The author was first introduced to the groups $\mathrm{FR}(\mathbb{Z}^{*n})$ by Vladimir Dotsenko, who had a suspicion that they may possess some additional algebraic structure. He thought this because the Hilbert-Poincaré series of the homology groups $\mathrm{FR}(\mathbb{Z}^{*n})$ behaves as if such structure were present. Before a more detailed explanation is offered we will discuss some more familiar groups.

Let g be a pure braid on n strands. The group of all such braids is denoted PB_n and there are many accounts of the theory of braids, see for instance [33]. Now let h be a second pure braid this time on m strands. Choosing the ith strand of g there is a composition $g \circ_i h$ defined by replacing strand i of g with h; a way to imagine this is to replace strand i with a pipe, then the composition is given by simply threading the pure braid h through the pipe before removing the pipe entirely. This composition is a group morphism $\circ_i : PB_n \times PB_m \to PB_{n+m-1}$. The family of groups PB_n for $n = 2, 3, \ldots$ along with the compositions just described may be viewed as one algebraic object, a *non-symmetric operad*. We will not discuss the theory of operads any further, the reader only needs to know that they are pleasant algebraic objects with a rich theory and with plenty of interesting examples (at least in the categories of sets and vector spaces). A

good reference is [34].

And so it turns out that the groups $\mathrm{FR}(\mathbb{Z}^{*n})$ also possess an operad structure. This may seem to be a frivolous result only of interest to those who enjoy playing with exotic algebraic objects, however it was this that first allowed the moduli space of cactus products to be constructed. By finding a presentation for the homology of the groups as an operad and then interpreting this presentation in the category of pointed spaces the classifying spaces are revealed. It is somewhat unfortunate that this approach does not help one to prove that the relevant spaces actually are classifying spaces, to do so it is simplest to strip out all reference to operads and use other methods to describe the space. However the details of the operadic picture will be communicated in a future publication.

The importance of functors

The fact that the groups $\mathrm{FR}(G_1 * \ldots * G_n)$ are functorial in the factor groups was not directly used in the proof of any main theorem. However it is an observation whose importance should not be underestimated. One reason is that it is usually much easier to prove a theorem for a family of groups indexed by a functor because the possible techniques are constrained to those that can be made functorial.

In this instance the author was originally interested in the moduli space of cactus graphs and the obvious method to prove that this space is aspherical is to prove that it is locally CAT(0). However it is not locally CAT(0) as a check on the link will show; one may try other methods inspired by geometry to try to show that this space is aspherical and I'm sure that it's possible that one may work. On the other hand if presented with the whole functor of moduli spaces of cactus products one must give up on this approach immediately. Afterall the geometry would have to cope with all manner of exotic spaces because the properties of pointed spaces are reflected in the image of the functor $\mathcal{M}\mathbf{Y}$. This is a typical example of a situation where trying to prove a statement for a class of examples is easier than trying to prove it for just one example.

It is the observations of Section 3.5 that are most intriguing. They suggest that the Whitehead automorphisms of a free product are special in the same sense that the inner automorphisms of a general group are special. An analogous result for the moduli space of cactus graphs is conspicuously missing, perhaps the correct statement for this would give a completely new proof that the moduli spaces give classifying spaces. The correct language and formalism most likely lies in the emerging field of higher category theory. The exciting prospect is that a general theory could give the moduli space of cactus products in one case, the Outer space of a free group in another, new spaces for the graph products of groups and a new Outer space of a right-angled Artin group.

The categorical perspective may point the way to other possibilities. The configuration spaces of the final chapter are currently 'one-off' examples which perhaps should belong

to a functor of similar objects. The next section sketches a possible theory.

Mapping class groups

In Chapter 3 we calculated the homology of the group $\mathrm{FR}(G_1 * \ldots * G_m)$ which extended results from [27] which only applies to $\mathrm{FR}(\mathbb{Z} * \ldots * \mathbb{Z})$. Results about a single object have been extended to a whole functor. In chapter 4 we studied configuration spaces of spheres and showed that these have fundamental groups $\mathrm{FR}(\mathbb{Z} * \ldots * \mathbb{Z})$. This one object case is suggestive of a theory which covers a wider range of the functor $\mathrm{FR}(-)$. To end this thesis we sketch how our approach and our philosophy can be applied to mapping class groups.

The groups $\mathrm{FR}(G_1 * \ldots * G_m)$ act on the free product $G_1 * \ldots * G_m$. Such free products are images of the n-fold free product $*^{m-1}$ which is a functor from $(\textsc{Gps})^m$, m copies of the category of groups, to $\textsc{Gps}$. The plan is to replace the category of groups with a different category n-$\textsc{Mfld}_{S^{n-1}}$ which we describe below. The wedge product $*^{m-1}$ will be replaced by a different functor, the m-ary connected sum functor, denoted W^{m-1}. Finally a functor of automorphisms which act on the image objects of W^{m-1} is defined using the theory of extendable automorphisms developed in Section 3.5.1. Analogues of Dehn twists are used to construct extendable automorphisms explicitly however we make no attempt at a full classification.

Manifolds with a chosen boundary component

In the following we will not fix a particular category of manifolds, although a default choice could be a category of smooth manifolds of fixed dimension. However the morphisms between manifolds will always be embeddings. The reason for not fixing a particular type of manifold is that the theory should be well defined whatever choice is made.

Let M be an n-dimensional manifold with a chosen boundary component isomorphic to the $(n-1)$-sphere

$$S^{n-1} \longrightarrow M. \tag{E.1.1}$$

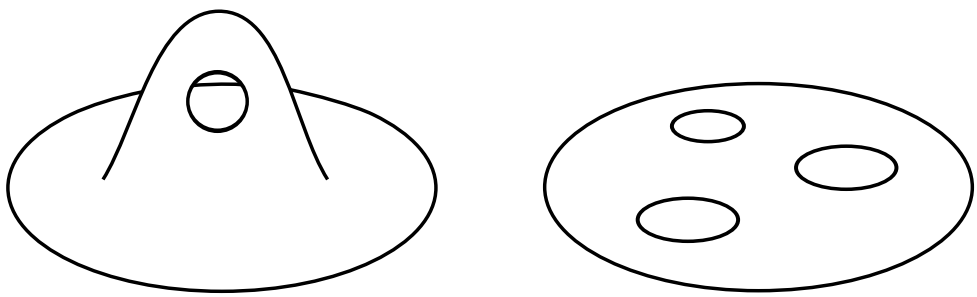
A morphism of such manifolds will be a triangle

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[dr] & M \arrow[d, "f"] \\ & N \end{tikzcd} \tag{E.1.2}$$

where f is an embedding which does not necessarily need to preserve the whole boundary. We denote this category n-$\textsc{Mfld}_{S^{n-1}}$. Let M be any n-manifold perhaps with boundary and p be a point in its interior. Then by removing an open n-disc around p we get an

object of n-$\textsc{Mfld}_{S^{n-1}}$ and any object can be obtained in this way.

Example E.1.1. Some objects of 2-$\textsc{Mfld}_{S^1}$:



(E.1.3)

The first is a disc with one handle attached and the only boundary component is the chosen one. This example could have been obtained by removing a disc from a two dimensional torus. The second is a disc with three discs removed from its interior, as such it has four boundary components; the chosen one and the boundaries of the three removed discs.

Example E.1.2. Examples of 3-manifolds are given by knots embedded in D^3. Let $K : S^1 \to D^3$ be such a knot, then define $M = D^3 - K(S^1)$. Since $K(S^1)$ is closed within D^3 the manifold M only has boundary the chosen component S^2. Replacing $K(S^1)$ with a tubular open neighbourhood N_K, we obtain $N = D^3 - N_K$ which is a manifold with boundary a union of S^2 and the torus $S^1 \times S^1$.

There is a functor given by the fundamental group functor

$$\pi_1 : n\text{-}\textsc{Mfld}_{S^{n-1}} \to \textsc{Gps} \tag{E.1.4}$$

whenever $n \geq 3$. This is because the chosen boundary sphere is simply connected and so may be used as a 'basepoint'. If $n = 2$ then boundary component is a circle and there is no canonical choice of basepoint, if we supplied one then the target of the functor would be the category of groups with a chosen element,

$$\mathbb{Z} \to \pi_1(M). \tag{E.1.5}$$

In the following we will restrict n to be strictly larger than two, although a theory would still exist for surfaces.

The m-ary connected sum, W^{m-1}

The *connected sum* of objects of n-$\textsc{Mfld}_{S^{n-1}}$ is given by identifying the two chosen boundary components. We now describe the connected sum of m objects of n-$\textsc{Mfld}_{S^{n-1}}$.

Choose a configuration of m closed n-discs inside a large n-disc

$$\begin{tikzcd} D^n \arrow[r, bend left, "b_1"] \arrow[r, bend right, "b_m"'] & D^n. \end{tikzcd} \tag{E.2.6}$$

These satisfy $b_i(D^n) \cap b_j(D^n) = \emptyset$ if $i \neq j$. Now take the manifold L^n_m which has the interiors of these discs taken away so has boundary components

$$\begin{tikzcd} S^{n-1} \arrow[r, bend left, "s_1"] \arrow[r, bend right, "s_m"'] & L^n_m & S^{n-1}. \arrow[l, "t"'] \end{tikzcd} \tag{E.2.7}$$

The map t gives the boundary of the large disc and with this t is in the category n-$\textsc{Mfld}_{S^{n-1}}$. We use this manifold to define a functor

$$W^{m-1} : (n\text{-}\textsc{Mfld}_{S^{n-1}})^m \to n\text{-}\textsc{Mfld}_{S^{n-1}} \tag{E.2.8}$$

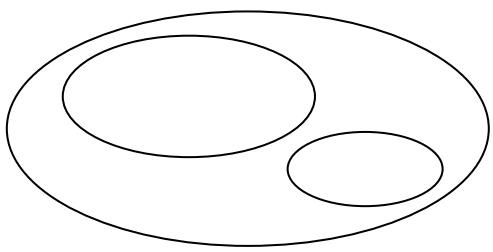
as follows: take the n-manifold L^n_m just defined and for each $i = 1, \ldots, m$ glue L^n_m to M_i along s_i:

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d, "s_i"'] & M_i \\ L^n_m. \end{tikzcd} \tag{E.2.9}$$

This results in an n-manifold $W^{m-1}(M_1, \ldots, M_m)$ with chosen boundary component t. Clearly this gluing also carries to morphisms $f : M_i \to N_i$ for $i = 1, \ldots, m$, so we do have a functor W^{m-1}. If $n \geq 2$ then there is a commuting diagram of functors

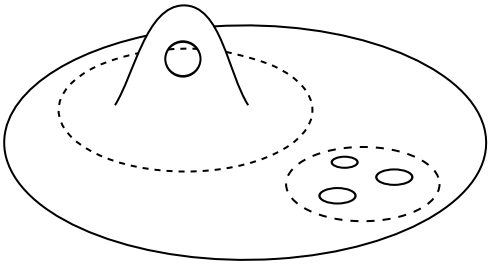
$$\begin{tikzcd} (n\text{-}\textsc{Mfld}_{S^{n-1}})^m \arrow[r, "W^{m-1}"] \arrow[d, "(\pi_1)^m"'] & n\text{-}\textsc{Mfld}_{S^{n-1}} \arrow[d, "\pi_1"] \\ (\textsc{Gps})^m \arrow[r, "*^{m-1}"] & \textsc{Gps} \end{tikzcd} \tag{E.2.10}$$

Example E.2.3. Taking the objects from Example E.1.1 and a configuration of discs



(E.2.11)

the functor W^1 gives the manifold



(E.2.12)

Remark E.1. In order to take the connected sum of m objects we had to choose a configurations of n-discs inside a large n-disc. So in fact there is a space of possible connected sums, although since the space of configurations is connected any two choices will give two functors connected by a natural transformation of homeomorphisms.

The fact that there is a space of products means that we must be careful with our definitions of monoidal categories; n-$\textsc{Mfld}_{S^{n-1}}$ isn't a strict monoidal category with our definitions, but it is weakly monoidal. In particular it is not symmetric monoidal, instead it is "E_n-monoidal". When $n = 2$ this definition means that the category is weakly braided monoidal.

Automorphisms extendable with respect to m-ary connected sums

Let M be an object of n-$\textsc{Mfld}_{S^{n-1}}$. The automorphisms of M naturally form a topological group fixing the boundary sphere S^{n-1}. The homotopy classes of automorphisms $\pi_0 \operatorname{Aut}(M)$ form the *mapping class group*. The definition of an extendable automorphism is contained in Section 3.5.1.

Definition E.2. Let $\mathrm{MFR}(M_1, \ldots, M_m)$ be the topological group of extendable automorphisms of $W^{m-1}(M_1, \ldots, M_m)$.

Remark E.3. Recall that the extendability of an automorphism α of $W^{m-1}(M_1, \ldots, M_m)$ says that for every m-tuple of morphisms $f_i : M_i \to N_i$ the diagram

$$\begin{tikzcd} W^{m-1}(M_1, \ldots, M_m) \arrow[r, "\alpha"] \arrow[d, "W^{m-1}(f_1,\ldots,f_m)"'] & W^{m-1}(M_1, \ldots, M_m) \arrow[d, "W^{m-1}(f_1,\ldots,f_m)"] \\ W^{m-1}(N_1, \ldots, N_m) \arrow[r, dotted, "\beta"'] & W^{m-1}(N_1, \ldots, N_m) \end{tikzcd} \tag{E.3.13}$$

may be filled in to a commuting square by some automorphism β of $W^{m-1}(N_1, \ldots, N_m)$.

Such extendable automorphisms in $\mathrm{MFR}(M_1, \ldots, M_m)$ may be constructed by choosing a method to build automorphisms by using only a minimal amount of information attached to $W^{m-1}(M_1, \ldots, M_m)$. The information will be mapped by any m-tuple of

morphisms $f_i : M_i \to N_i$ to $W^{m-1}(N_1, \ldots, N_m)$ where it may be used to construct an automorphism in $\mathrm{MFR}(N_1, \ldots, N_m)$ which extends the original one.

Twists through tubes

In the following a *tube* T_n will be a copy of $D^{n-1} \times S^1$ whose points may be parametrised by

$$(\mathbf{n}, r, t)\,, \tag{E.4.14}$$

where $\mathbf{n} \in S^{n-2}$, $r \in [0,1]$ and $t \in S^1 \cong [0, 2\pi]\,/(0 \sim 2\pi)$. The map

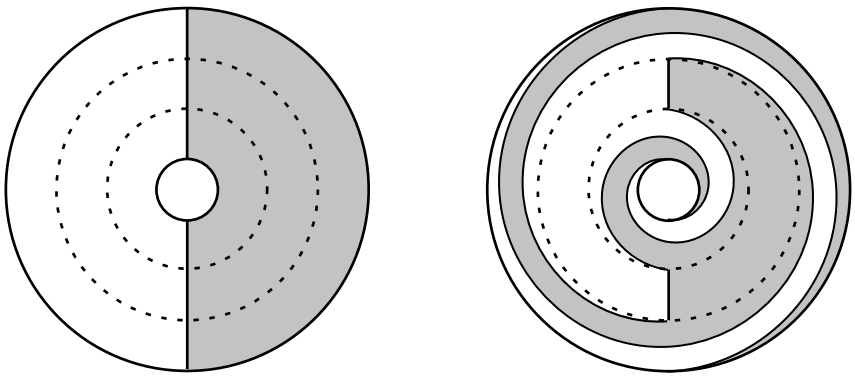
$$S^{n-2} \times [0,1] \to D^{n-1} \cong \left\{\mathbf{x} \in \mathbb{R}^{n-1} \mid |x| \leq 1\right\} \tag{E.4.15}$$

is defined by $(\mathbf{n}, r) \mapsto r.\mathbf{n}$. The *core* C_n of T_n is the set of points with $r \leq \frac{1}{2}$. We now define a twisting map $\phi : T_n \to T_n$ which fixes the core C_n and the boundary of T_n, but 'twists' the remaining part of the tube as follows

$$\varphi(\mathbf{n}, r, t) = \begin{cases} (\mathbf{n}, r, t) & \text{if } |r| \leq \frac{1}{2}\text{, that is a point in the core,} \\ (\mathbf{n}, r, t + 2\pi(2r-1)) & \text{otherwise.} \end{cases} \tag{E.4.16}$$

This is a homeomorphism, the inverse can be given just by changing a sign.

Example E.4.4. When $n = 2$ the tube T_n is an annulus and the twisting map φ is represented below. The vertical lines in the left diagram are taken to the spiralling lines in the right.



(E.4.17)

Note that if one restricts attention to the core which is bounded by the dashed lines, one sees that the regions coloured grey are identical, this is because φ is the identity on the core.

Definition E.4. A *tube with support on* $M_2, \ldots, M_m$ is an embedding E of T_n into $W^{m-1}(D^n, M_2, \ldots, M_m)$ such that the copy of D^n in the first position is contained within the core $E(C_n)$.

The automorphism $\alpha_E \in \mathrm{Aut}(W^{m-1}(M_1, \ldots, M_m))$ associated to E is defined using the tube twist φ as follows

$$\alpha_E(y) = \begin{cases} y & \text{if } y \in M_1, \\ y & \text{if } y \notin E(T_n), \\ E(\varphi(y)) & \text{if } y = E(x). \end{cases} \tag{E.4.18}$$

Proposition E.5. *Let $E(T_n)$ be a tube with support on $M_2, \ldots, M_m$. Then α_E is extendable with respect to W^{m-1}.*

Proof. Suppose (f_i) is an m-tuple of morphisms $f_i : M_i \to N_i$ in n-$\mathrm{MFLD}_{S^{n-1}}$. Then $W^{m-1}(I_{D^n}, f_2, \ldots, f_m)$ is an embedding of $W^{m-1}(D^n, M_2, \ldots, M_m)$ into $W^{m-1}(D^n, N_2, \ldots, N_m)$. Therefore $W^{m-1}(I_{D^n}, f_2, \ldots, f_m) \circ E$ is an embedding of a tube into $W^{m-1}(D^n, N_2, \ldots, N_m)$. Since D^n is fixed by $W^{m-1}(I_{D^n}, f_2, \ldots, f_m)$ it is also contained in the image of the tube embedding so $W^{m-1}(I_{D^n}, f_2, \ldots, f_m) \circ E$ gives a tube with support $N_2, \ldots, N_m$, we will denote the embedding $f(E)$. It is an easy check that the square

$$\begin{tikzcd} W^{m-1}(M_1, \ldots, M_m) \arrow[r, "\alpha_E"] \arrow[d, "W^{m-1}(f_1,\ldots,f_m)"'] & W^{m-1}(M_1, \ldots, M_m) \arrow[d, "W^{m-1}(f_1,\ldots,f_m)"] \\ W^{m-1}(N_1, \ldots, N_m) \arrow[r, "\alpha_{f(E)}"] & W^{m-1}(N_1, \ldots, N_m) \end{tikzcd} \tag{E.4.19}$$

commutes, so α_E is extendable. □

The automorphisms α_E are analogues of partial conjugations: for $n > 2$ the maps $\pi_1(\alpha_E)$ are partial conjugations of the groups $\pi_1(W^{m-1}(M_1, \ldots, M_m))$, which are free products of the groups $\pi_1(M_i; S^{n-1})$.

It is worth noting that the functor π_1 takes automorphisms to automorphisms, however there is no guarantee that the image of an extendable automorphism is itself extendable. So we can not conclude that every extendable automorphism induces a Whitehead automorphism.

Remark E.6. Suppose we were to restrict the morphisms in the category n-$\mathrm{MFLD}_{S^{n-1}}$ to be homeomorphisms. Then every automorphism of $W^{m-1}(M_1, \ldots, M_m)$ would be extendable because the morphisms they would extend along would be automorphisms and hence the extension is given by conjugation. This should serve as a warning; if M_i is a compact manifold where the whole boundary is S^{n-1} then the embeddings out of M_i are isomorphisms, hence M_i supplies no interesting morphisms to extend along and so the extendability problem is different from that of the theory for groups.

A further remark: we were able to show that the extendable morphisms with respect to the free product functor were precisely the Whitehead automorphisms. The proof relied on a lemma of Schupp 3.5.3 from [43] which says that every group is embeddable

into a malnormal subgroup of a complete group. We do not yet know an analogue of this Lemma in the category n-$\textsc{Mfld}_{S^{n-1}}$.

The classifying space of $\mathbf{MFR}(M_1, \ldots, M_m)$

Just as the group $\mathrm{FR}(G_1 * \ldots * G_m)$ has a classifying space given by the moduli space of cactus products, a space of gluings of manifolds should provide a classifying space for $\mathrm{MFR}(M_1 * \ldots * M_m)$. An example is given in the first half of the previous chapter, where the manifolds considered are closed n-discs with an unknotted $(n-2)$-sphere removed, see Example E.1.2. We studied spaces of configurations of these $(n-2)$-spheres in $\mathbb{R}^n$ (note that we have changed our dimension conventions slightly) and these configuration spaces should be related to the classifying space of $\mathrm{MFR}(D^n - S^{n-2}, \ldots, D^n - S^{n-2})$. This inspires the following conjecture.

Conjecture. *The topological group* $\mathrm{MFR}(M_1, \ldots, M_m)$ *has a classifying space with the homotopy type of a finite dimensional CW complex. Furthermore the homology*

$$H_*(B\mathrm{MFR}(M_1, \ldots, M_m), \mathbb{Z}) \tag{E.5.20}$$

splits into a direct sum over forest posets

$$\bigoplus_{\textit{forest poset } F} . \tag{E.5.21}$$

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