



Anti-self-dual fields and manifolds

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ABSTRACT

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In this thesis we study anti-self-duality equations in four and eight dimensions on manifolds of special Riemannian holonomy, among these hyper-Kähler, Quaternion-Kähler and $Spin(7)$ -manifolds.

We first consider the octonionic anti-self-duality equations on manifolds with holonomy $Spin(7)$. We construct explicit solutions to their symmetry reductions, the non-abelian Seiberg-Witten equations, with gauge group $SU(2)$. These solutions are singular for flat and Eguchi-Hanson backgrounds, however we find a solution on a co-homogeneity one hyper-Kähler metric with a domain wall, and the solution is regular away from the wall.

We then turn to Quaternion-Kähler four-manifolds, which are locally determined by one scalar function subject to Przanowski's equation. Using twistorial methods we construct a Lax Pair for Przanowski's equation, confirming its integrability. The Lee form of a compatible local complex structure gives rise to a conformally invariant differential operator, special cases of the associated generalised Laplace operator are the conformal Laplacian and the linearised Przanowski operator. Using recursion relations we construct a contour integral formula for perturbations of Przanowski's function. Finally, we construct an algorithm to retrieve Przanowski's function from twistor data.

At last, we investigate the relationship between anti-self-dual Einstein metrics with non-null symmetry in neutral signature and pseudo-, para- and null-Kähler metrics. We classify real-analytic anti-self-dual null-Kähler metrics with a Killing vector that are conformally Einstein. This allows us to formulate a neutral signature version of Tod's result, showing that around non-singular points all real-analytic anti-self-dual Einstein metrics with symmetry are conformally pseudo- or para-Kähler.

DECLARATION

This thesis:

- is my own work and contains nothing which is the outcome of work done in collaboration with others, except where specified in the text; and
- is not substantially the same as any that I have submitted for a degree or diploma or other qualification at any other university.
- It partly builds on previous results obtained in my Diploma Thesis [1] under the supervision of M. Dunajski and M. G. Schmidt. Such results are clearly specified in the text and appear only in chapter 3, in particular the content of sections 3.1 and 3.3.1 is taken from [1].

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INTRODUCTION

1.1 Introduction

The concept of anti-self-duality in four dimensions is very closely tied to the integrability of specific field equations and manifolds of special Riemannian holonomy. Indeed, interest in the subject arose with Penrose [2] showing that four-manifolds with anti-self-dual conformal structures can be linked via the twistor correspondence to three-dimensional complex twistor spaces with certain algebraic properties. Equipping this twistor space with a holomorphic fibration and a symplectic structure along the fibres yields anti-self-dual Ricci-flat four-manifolds, commonly referred to as hyper-Kähler, while a holomorphic contact structure on twistor space leads to anti-self-dual Einstein manifolds with non-zero cosmological constant [3], also known as Quaternion-Kähler. The significance of these correspondences lies in the fact that they translate differential equations into algebraic constraints, hence integrating the differential equation. One can also extract Lax Pairs for these differential equations from the geometry of the twistor space, thus providing another of the key features of integrable equations. In this geometric context the Lax Pair spans a distribution which is integrable if and only if the differential equation is satisfied.

From a physicist's point of view the differential equations leading to anti-self-dual, hyper-Kähler or Quaternion-Kähler manifolds are similar in kind to those of Einstein's general relativity, as they impose restrictions on the Riemannian curvature of four-manifolds. Turning to field equations, one of the goals of twistor geometry is to find solutions of the Yang-Mills equations. Again the concept of anti-self-

duality plays a prominent role, field equations amenable to twistor transforms are those that require the self-dual part of the curvature of a Lie algebra-valued connection on a principal bundle over an anti-self-dual manifold to vanish. These field equations have been named the anti-self-dual Yang–Mills equations, as their solutions automatically satisfy the ubiquitous Yang–Mills equations by virtue of the Bianchi identity. Ward [4] established a correspondence between solutions of the anti-self-dual Yang–Mills equations on conformally flat background and vector bundles over twistor space. This correspondence has subsequently been extended to cover anti-self-dual Yang–Mills equations on all anti-self-dual backgrounds [5].

A further source of integrable systems stems from the various symmetry reductions of the anti-self-duality equations to one, two or three dimensions. Such a dimensional reduction is possible if the four-dimensional system admits one or more Killing vectors, the features of the resulting symmetry-reduced model depend on the commutation relations of the Killing vectors and their properties: whether a Killing vector is conformal, pure or a homothety, whether it is null or not, whether it is hypersurface orthogonal or not, whether the action is free or not. One problematic issue with symmetry reductions is that the resulting twistor spaces are not necessarily Hausdorff if the spacetime on which the field equations are formulated is not geodesically convex [6, 7]. Lower-dimensional models that inherit their integrability from the anti-self-dual Yang–Mills equations are monopoles in three, vortices in two and kinks in one dimension [8, 9]. Examples of integrable equations include the dispersionless Kadomtsev–Petviashvili (dKP), $SU(\infty)$ Toda, Korteweg–de Vries, non-linear Schrödinger and Toda field equation as well as Painlevé’s and Nahm’s equations and many others, for an overview see [7] and references therein. In fact Ward conjectures [8] that many or perhaps all integrable differential equations may be obtained by symmetry reduction from the anti-self-dual Yang–Mills equations or some generalisation.

In this thesis we shall discuss in detail each of the three areas tied so closely to anti-self-duality: the octonionic instanton equation is a field equation on eight-manifolds modeled on the anti-self-dual Yang–Mills equation, Quaternion–Kähler manifolds are examples of Riemannian manifolds with special holonomy and we will also consider symmetry reductions of anti-self-dual Einstein manifolds in neutral signature by a pure, non-null Killing vector.

First we probe the limits of integrability by studying an extended version of anti-self-duality in higher dimensions. Inspired by the geometry underlying anti-self-

duality in four dimensions, there is a natural extension to higher-dimensional manifolds with special Riemannian holonomy. We explain this more general notion of anti-self-duality, and use it to introduce the octonionic instanton equation on manifolds with holonomy $Spin(7)$. Besides G_2 , this is one of the two exceptional Riemannian holonomy groups whose origin can be traced back to the existence of the octonions. The aim of this part of the thesis will be to find explicit solutions of the octonionic instanton equation. To make the equation somewhat tractable, we study a symmetry reduction from eight to four dimensions leading to the non-abelian Seiberg–Witten equation [10, 11]. Exploiting the symmetry of the background we make an Ansatz that mimics the behaviour of instantons in four dimensions. Since there is no known twistor construction for $Spin(7)$ -manifolds, we don't expect the octonionic instanton equation to be integrable. None the less, our Ansatz reduces the full equations to a second-order non-linear ordinary differential equation (ODE) for one scalar function. This is rather remarkable, as in intermediate stages we find highly-overdetermined coupled non-linear second-order partial differential equations (PDEs). While regular solutions on flat space are ruled out by scaling arguments, we do find explicit solutions on a gravitational instanton with a single-sided domain wall that are regular away from the wall. These can be viewed as solutions on a group-manifold with a hyper-Kähler metric where the singularity is present only in an overall conformal factor. We complete the discussion by numeric evaluations of the 2nd-order ODEs in question. The foundations of this work have been laid in previous work by the author [1] in collaboration with his supervisor and M. G. Schmidt, however the results on curved manifolds, in particular the solutions on Gibbons–Hawking background, are new.

The octonionic instanton equation illustrates rather nicely the effects of the lack of integrability that one encounters when leaving the territory of the twistor correspondence. The link between integrability and twistor constructions is very well understood for anti-self-dual Ricci-flat manifolds. The metric of an anti-self-dual Ricci-flat manifold is determined by the partial derivatives of one scalar function which is subject to a second-order partial differential equation, Plebanski's heavenly equation [12]. In [13] it is demonstrated that the heavenly equation is integrable using twistor methods. The authors derive a Lax Pair for the heavenly equation and relate the heavenly function to the geometry of twistor space. One can also find an contour integral formula for perturbations of the heavenly function and hence for deformations of the metric [14].

The aim of the second part of this thesis is to establish similar results for Quaternion–Kähler manifolds, which represent the second class of manifolds whose Riemannian holonomy is based on the existence of the quaternions. Four–dimensional Quaternion–Kähler manifolds are by definition anti–self–dual Einstein with non–zero scalar curvature, hence they also lie within the realm of the twistor correspondence. Without using twistor theory Przanowski [15] showed that anti–self–dual Einstein four–metrics can be written locally in terms of one scalar function subject to a second–order non–linear PDE. Hence, by the dogma of twistor theory, we would expect this equation to be integrable. And indeed, our first result here is to exhibit Przanowski’s equation as an integrable equation by providing a Lax Pair. Exploiting the integrability of Przanowski’s equation we go on to establish an integral formula for perturbations of solutions of Przanowski’s PDE. This formula links cohomology classes on twistor space to deformations of arbitrary Quaternion–Kähler four–manifolds, and hence extends previous results of [16], where only Quaternion–Kähler four–manifolds with isometries were considered. Finally we clarify the geometric origin of Przanowski’s function in the twistor correspondence using the double–fibration picture. To illustrate this, we discuss a number of explicit examples with positive and negative scalar curvature.

At last we want to elaborate on symmetry reductions in the context of integrable equations. To this end we study anti–self–dual Einstein metrics with a symmetry. In Euclidean signature the situation is well understood: As we have seen, the four–metric can be expressed in terms of a scalar function subject to Przanowski’s equation. In the presence of a Killing vector, the system is dimensionally reduced to three dimensions and we obtain an Einstein–Weyl structure which can also be expressed in terms of a scalar function [17], however this scalar function is subject to the $SU(\infty)$ Toda equation. So the symmetry reduction reduces Przanowski’s equation to the Toda $SU(\infty)$ equation.

In neutral signature new features arise, the main difference is the appearance of null–Kähler metrics linked to another integrable equation, namely the dispersionless Kadomtsev–Petviashvili (dKP) equation. The purpose of the last part of this thesis is to gain some insights into the symmetry reductions of ASD Einstein manifolds in neutral signature, at least in the real–analytic case. The main new result in this direction is the classification of real–analytic null–Kähler metrics with a Killing vector which are conformally equivalent to an anti–self–dual Einstein metric. With this information at our hands, we can show that away from singular points any real–analytic anti–self–dual Einstein metric with a non–null

Killing vector is conformally pseudo- or para-Kähler. As a corollary we obtain a useful classification of anti-self-dual conformal structures with a symmetry that admit a null-Kähler as well as a pseudo- or para-Kähler metric.

1.2 Outline

The remainder of this thesis is organised as follows. In chapter 2 we provide some mathematical background, starting with holonomy groups of Riemannian manifolds in section 2.1. Here we follow [18, 19] to introduce some concepts that will appear throughout this thesis. The next sections 2.2–2.5 are devoted to introducing the reader to spinorial notation, twistor theory, Plebanski’s heavenly equation and deformation theory following [20, 6, 7, 21, 13]. These are fundamental to the content of the following chapters. Having established the relevant basics, we proceed in chapter 3 to discuss the octonionic instanton equation. We start section 3.1 by explaining an extension of anti-self-duality to eight dimensions which is valid on any Riemannian manifold with holonomy $Spin(7)$, this leads us to the octonionic instanton equation. Pushing on, in section 3.2 we choose an explicit holonomy reduction of the background together with a symmetry reduction of the octonionic instanton equation, leading to a non-abelian version of the Seiberg–Witten equations. We present an Ansatz with gauge group $SU(2)$ in section 3.3 and deduce some exact solutions on flat and curved background. Also we discuss the singular or regular behaviour of these exact solutions and provide some further numeric solutions. The results of chapter 3 have been published in a joint paper [22] with M. Dunajski.

Chapter 4 is devoted to Quaternion-Kähler four-manifolds and Przanowski’s function. After introducing Przanowski’s form of a Quaternion-Kähler metric, we demonstrate in section 4.1 that a metric of this form is indeed anti-self-dual and Einstein. Furthermore, we construct a conformally invariant differential operator and consider the associated generalised Laplacian. In section 4.2 we construct the twistor space of a Quaternion-Kähler manifold and as a spin-off obtain a Lax Pair for Przanowski’s equation. We discuss recursion relations relating solutions of the generalised Laplace equation to cohomology classes on twistor space. At the end of this section, we focus on the linearised Przanowski operator as a special case of the generalised Laplacian and describe deformations of the holomorphic contact structure on twistor space generated by perturbations of Przanowski’s

function. In section 4.3 we provide an algorithm to obtain Przanowski's function from twistor data in the double-fibration picture by making a suitable choice of gauge. We then use section 4.4 to illustrate this procedure in a few examples: S^4 , H^4 , $\mathbb{C}P^2$ and $\widetilde{\mathbb{C}P^2}$, the non-compact version of $\mathbb{C}P^2$ with the Bergmann metric. The content of this chapter has appeared in [23].

The final chapter 5 is concerned with Quaternion-Kähler metrics in neutral signature with a non-null symmetry. In section 5.1 we review Einstein-Weyl structures and the Jones-Tod construction for neutral signature metrics with a non-null symmetry. Furthermore we recall that the $SU(\infty)$ Toda equation leads to scalar-flat pseudo- and para-Kähler and the dKP equation to anti-self-dual null-Kähler metrics. The next section 5.2 is concerned with the classification of anti-self-dual Einstein metrics within the conformal class of a null-Kähler metric with a Killing vector. With this result at our disposal, we proceed in section 5.3 to derive the general form of a real-analytic anti-self-dual Einstein metric with non-null symmetry in neutral signature, away from singular points. We finish with the classification of the overlap between null-Kähler and pseudo- or para-Kähler metrics.

1.3 Notation

We denote frames of the co-tangent bundle and more generally one-forms by e^a or $e^{AA'}$ and the dual vector fields by ∂_a or $\partial_{AA'}$, where $a, b, \dots = 0, 1, 2, 3$ and $A, B, \dots = 0, 1$. For a coordinate-induced vector field we replace the index by the coordinate, e.g. if (w, z) are coordinates then $\partial_w = \frac{\partial}{\partial w}$ and $\partial_z = \frac{\partial}{\partial z}$. We abbreviate partial derivatives of a function $K(w, z)$ by $K_w = \partial_w K = \frac{\partial K}{\partial w}$. A comprehensive list of symbols can be found in the Appendix.

MATHEMATICAL BACKGROUND

2.1 Riemannian Holonomy

In this section we will introduce the concept of holonomy on Riemannian manifolds and give an overview of the classification of Riemannian holonomy groups based on [18, 19]. Details of selected cases will be discussed in the following sections. Suppose (M, g) is a connected n -dimensional Riemannian manifold, then the Levi-Civita connection ∇ associated with the metric g induces parallel transport of vectors along curves $\gamma : \mathbb{R} \mapsto M$. Hence we have a map

$$P : T_{\gamma(0)}M \mapsto T_{\gamma(1)}M. \quad (2.1)$$

For every closed loop based at $m \in M$ the map P induces an automorphism of T_mM . The set of endomorphisms induced from all possible closed loops in M has the structure of a Lie group of endomorphisms of T_mM , the holonomy group Hol_m associated to $m \in M$. Note that the holonomy group $Hol_{\tilde{m}}$ of a different point $\tilde{m} \in M$ is given by $Hol_{\tilde{m}} = P_\gamma Hol_m P_\gamma^{-1}$, where γ is a path connecting m and \tilde{m} . Hence up to conjugation the Levi-Civita connection associates a holonomy group Hol to the metric g . By definition the Levi-Civita connection preserves lengths and angles and hence $Hol \subseteq O(n)$. Note that $O(n)$ is precisely the stabiliser of the metric g : all orthonormal frames, i.e. frames in which g corresponds to the identity matrix $\mathbb{1}_4$ are related by an $O(n)$ -transformation. The holonomy group can be a proper subset of $O(n)$, hence it is possible to classify Riemannian manifolds by their holonomy groups. One large class of this classification are the Riemannian symmetric spaces, which reduce to quotients of Lie groups and are

characterised by a covariantly constant curvature tensor. The non-symmetric Riemannian manifolds can locally be decomposed into products of irreducible components, the possible holonomy groups of these irreducible non-symmetric Riemannian manifolds have been classified by Berger [24]. Their origin can be traced back to the existence of the four division algebras \mathbb{R} , \mathbb{C} , \mathbb{H} and \mathbb{O} . Most of these holonomy groups will appear at some point of this thesis, we use this section to present all of them in one place. We shall see that each of the possible holonomy groups is the stabiliser of one or more globally-defined, covariantly constant tensors in addition to the metric.

We start with the Riemannian holonomy groups $O(n)$ and $SO(n)$ corresponding to automorphisms of \mathbb{R}^n . As explained above, every Riemannian manifold with the Levi-Civita connection has holonomy $O(n)$ with covariantly constant metric g . If we add an orientation this reduces to holonomy $SO(n)$ and beyond the metric g there exists a globally-defined covariantly constant volume form vol_n .

All other holonomy groups are considered 'special', as they impose substantial restrictions on the curvature of the metric. Consider first the holonomy groups associated with automorphisms of \mathbb{C}^n , which are $U(m)$ and $SU(m)$ where $n = 2m$. The unitary group $U(m)$ characterises Kähler manifolds, i.e. Riemannian manifolds with a compatible and integrable complex structure I and a covariantly constant Kähler form $\Sigma = g(I(\cdot), \cdot)$. The special unitary group $SU(m)$ is the holonomy group of Calabi-Yau manifolds, which are also Kähler but furthermore have a 'complex orientation', namely a covariantly closed holomorphic volume form $\nu \in \Lambda^{(m,0)}M$. Calabi-Yau manifolds are always Ricci-flat. Manifolds with holonomy $U(m)$ or $SU(m)$ lie in the overlap of Riemannian and complex geometry and have been extensively studied, they are amenable to methods of differential as well as algebraic geometry.

The quaternionic holonomies corresponding to automorphisms of \mathbb{H}^n divide into Quaternion-Kähler manifolds with holonomy¹ $Sp(k) \cdot Sp(1)$ and hyper-Kähler manifolds with holonomy $Sp(k)$, where $n = 4k$ for $k > 1$. Here $Sp(k)$ denotes the compact symplectic group. Hyper-Kähler manifolds admit a two-sphere worth of complex structures compatible with the metric. We can parametrise these by $aI_1 + bI_2 + cI_3$, where I_1, I_2 and I_3 are three anti-commuting complex structures and $a^2 + b^2 + c^2 = 1$. The metric is Kähler with respect to all of these complex structures, hence we have a basis of three covariantly constant self-dual Kähler

¹Here $Sp(k) \cdot Sp(1) = (Sp(k) \times Sp(1)) / \mathbb{Z}_2$.

forms Σ_1, Σ_2 and Σ_3 . All hyper-Kähler manifolds are Ricci-flat. The case $k = 1$ is special: we have the isomorphism $Sp(1) = SU(2)$ and hence hyper-Kähler and Calabi-Yau four-manifolds coincide.

Quaternion-Kähler manifolds in turn are not necessarily complex, however in every point they admit a two-sphere of local complex structures which are compatible with the metric. The metric need not be Kähler with respect to any of these local complex structures and so the fundamental two-forms Σ_i need not be closed. None the less the four-form

$$\Delta := \Sigma_1 \wedge \Sigma_1 + \Sigma_2 \wedge \Sigma_2 + \Sigma_3 \wedge \Sigma_3 \quad (2.2)$$

is covariantly constant. Quaternion-Kähler manifolds are not Ricci-flat, but Einstein with non-zero scalar curvature. Again the case $k = 1$ is special: $SO(4) \cong Sp(1) \cdot Sp(1)$ and so every Riemannian four-manifold has holonomy group $Sp(1) \cdot Sp(1)$. Hence one defines a Quaternion-Kähler four-manifold to be anti-self-dual Einstein, for details see section 2.3.

Manifolds with quaternionic holonomies can also be studied with algebro-geometric tools via the twistor transform, this will be discussed in section 2.2 and is the object of study in chapter 4 for Quaternion-Kähler manifolds.

Finally we come to the exceptional holonomies G_2 and $Spin(7)$, which are related to the non-associative octonions \mathbb{O} . In contrast to the six infinite families of holonomy groups we have encountered so far, the exceptional holonomy groups arise only for one particular dimension each.

G_2 is the group of automorphisms of the seven-dimensional space of imaginary octonions. G_2 is 14-dimensional and is a subgroup of $SO(7)$, manifolds with holonomy G_2 are seven-dimensional and come equipped with a four-form Θ that is covariantly constant, as is its Hodge-dual $*_7 \Theta$.

$Spin(7)$ is the group of automorphisms of $\mathbb{O} \cong \mathbb{R}^8$ preserving some part of the multiplicative structure. As we will work with $Spin(7)$ -manifolds in chapter 3, we characterise $Spin(7)$ in more detail: It is the 21-dimensional subgroup of $SO(8)$ preserving a self-dual four-form. Manifolds with holonomy $Spin(7)$ are eight-dimensional. Set $e^{\mu\nu\rho\sigma} = e^\mu \wedge e^\nu \wedge e^\rho \wedge e^\sigma$. On any $Spin(7)$ -manifold there exists an orthonormal frame in which the metric and the four-form are given by

$$\begin{aligned} g_8 &= (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2 + (e^4)^2 + (e^5)^2 + (e^6)^2 + (e^7)^2 + (e^8)^2, \\ \Xi &= e^{0123} - e^{0145} - e^{0167} - e^{0246} + e^{0257} - e^{0347} - e^{0356} \\ &\quad - e^{1247} - e^{1256} + e^{1346} - e^{1357} - e^{2345} - e^{2367} + e^{4567}, \end{aligned} \quad (2.3)$$

furthermore Ξ is parallel and self-dual with respect to g_8 and the Levi-Civita connection. Both G_2 - and $Spin(7)$ -manifolds have vanishing Ricci tensor. As neither complex geometry nor twistor methods can get a hold on manifolds of exceptional holonomy, a lot less is known about them. Local existence was first demonstrated in [25] and the first complete, non-compact examples were given in [26, 27]. Joyce [28] constructed compact Riemannian manifolds with holonomy $Spin(7)$, and many further explicit non-compact $Spin(7)$ -metrics have appeared in the literature since then [29, 30, 31].

It is worth noting that the differential forms $\Sigma_1, \Sigma_2, \Sigma_3, \Delta, \Theta, *_7 \Theta$ and Ξ on manifolds with quaternionic or exceptional holonomy are in fact covariantly constant if and only if they are closed. This follows from a representation-theoretic decomposition of their covariant derivatives, which are all completely anti-symmetric [18].

2.2 Spinor formalism

In order to discuss hyper-Kähler and Quaternion-Kähler four-manifolds in more detail, we first introduce spinor formalism in four dimensions following [7, 21]. Following Penrose's ideas [2] the starting point is a holomorphic Riemannian four-manifold (\mathbb{M}, g) , i.e. a complex² four-dimensional manifold \mathbb{M} with a holomorphic, symmetric metric g . Under the group isomorphism $SO(4, \mathbb{C}) \cong SL(2, \mathbb{C}) \times SL(2, \mathbb{C})$ the tangent bundle $T\mathbb{M}$ of (\mathbb{M}, g) can locally be regarded as a tensor product $T\mathbb{M} = \mathbb{S} \otimes \mathbb{S}'$ of two rank 2 spin bundles \mathbb{S} and \mathbb{S}' . We choose a null tetrad $e^{AA'}$ of $T^*\mathbb{M}$, in which

$$g = \varepsilon_{AB}\varepsilon_{A'B'} \otimes e^{AA'} e^{BB'} = 2 \left(e^{00'} e^{11'} - e^{01'} e^{10'} \right), \quad (2.4)$$

with dual vector fields $\partial_{AA'}$. Primed and unprimed indices will always run from 0 to 1. Equation (2.4) amounts to choosing a basis (o^A, ρ^A) of \mathbb{S} with dual basis (o_A, ρ_A) of \mathbb{S}^* and a basis $(o^{A'}, \rho^{A'})$ of \mathbb{S}' with dual basis $(o_{A'}, \rho_{A'})$ of \mathbb{S}'^* over every point of \mathbb{M} and setting

$$\begin{aligned} e^{00'} &= o_A o_{A'} e^{AA'}, & e^{01'} &= o_A \rho_{A'} e^{AA'}, \\ e^{10'} &= \rho_A o_{A'} e^{AA'}, & e^{11'} &= \rho_A \rho_{A'} e^{AA'}. \end{aligned} \quad (2.5)$$

²The relation to real Riemannian manifolds, which we are ultimately interested in, will become clear in the next section.

The metric thus induces symplectic structures ϵ_{AB} on \mathbb{S} , ϵ^{AB} on \mathbb{S}^* , $\epsilon_{A'B'}$ on \mathbb{S}' and $\epsilon^{A'B'}$ on \mathbb{S}'^* which in the bases (o, ρ) and (o', ρ') are simply given by the Levi–Civita symbols,

$$\epsilon^{AB} = \epsilon^{A'B'} = \epsilon_{AB} = \epsilon_{A'B'} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.6)$$

We can use the ϵ -symbols to raise and lower indices of primed and unprimed spinors. Our conventions are

$$\alpha_B = \alpha^A \epsilon_{AB}, \quad \alpha^A = \epsilon^{AB} \alpha_B, \quad (2.7)$$

and equally for primed spinors. Since a symplectic structure is anti-symmetric we have $\alpha_A \beta^A = -\alpha^A \beta_A$, so it is important to maintain the order of indices. The null tetrad (2.4) identifies a vector V with a 2×2 -matrix

$$V^{AA'} = \begin{pmatrix} V^{00'} & V^{01'} \\ V^{10'} & V^{11'} \end{pmatrix}, \quad (2.8)$$

the square of the length of the vector is now proportional to the determinant of the matrix, $|V|^2 = 2 \det(V^{AA'})$. A null vector corresponds to a degenerate matrix and thus can be written as $V^{AA'} = \beta^A \alpha^{A'}$, where $\alpha^{A'} \in \mathbb{S}'$ and $\beta^A \in \mathbb{S}$. A totally null plane is a two-dimensional plane spanned by two null vectors V, W which are orthogonal, $V_{AA'} V^{AA'} = W_{AA'} W^{AA'} = V_{AA'} W^{AA'} = 0$. Choosing a constant primed spinor $\alpha^{A'}$ and varying β^A sweeps out a two-dimensional plane of null vectors $V^{AA'} = \beta^A \alpha^{A'}$ which are all orthogonal, we call such a totally null plane an α -plane. Conversely, an α -plane determines a primed spinor up to scale by $\alpha_{A'} V^{AA'} = 0$ for all vectors in the plane. If instead we choose a constant unprimed spinor and vary the primed spinor, we call the totally null plane a β -plane. It is easy to see that all totally null planes are either α - or β -planes and are in one-to-one correspondence with non-zero primed and unprimed spinors up to scale. We call a surface an α - or β -surface if all its tangent planes are α - or β -planes.

Now consider a two-form F given by

$$F = \frac{1}{2} F_{AA'BB'} e^{AA'} \wedge e^{BB'}. \quad (2.9)$$

Since $F_{AA'BB'}$ is anti-symmetric in the index-pairs AA' and BB' , we must have a decomposition

$$F_{AA'BB'} = F_{A'B'}^+ \epsilon_{AB} + F_{AB}^- \epsilon_{A'B'}, \quad (2.10)$$

with two symmetric spinors $F_{A'B'}^+$ and F_{AB}^- . We now fix an orientation by choosing a volume form

$$\text{vol}_g := \frac{1}{4!} \epsilon_{AA'BB'CC'DD'} e^{AA'} \wedge e^{BB'} \wedge e^{CC'} \wedge e^{DD'} = e^{00'} \wedge e^{10'} \wedge e^{01'} \wedge e^{11'}, \quad (2.11)$$

where

$$\epsilon_{AA'BB'CC'DD'} := (\epsilon_{AC}\epsilon_{BD}\epsilon_{A'D'}\epsilon_{B'C'} - \epsilon_{AD}\epsilon_{BC}\epsilon_{A'C'}\epsilon_{B'D'}). \quad (2.12)$$

Then

$$F^+ := \frac{1}{2} F_{A'B'}^+ \epsilon_{AB} e^{AA'} \wedge e^{BB'}, \quad F^- := \frac{1}{2} F_{AB}^- \epsilon_{A'B'} e^{AA'} \wedge e^{BB'} \quad (2.13)$$

correspond to the self-dual and anti-self-dual parts of F respectively,

$$*_g F^+ = F^+, \quad *_g F^- = -F^-. \quad (2.14)$$

Hence we have a splitting

$$\Lambda^2 \mathbb{M} = \Lambda_+^2 \mathbb{M} \oplus \Lambda_-^2 \mathbb{M}, \quad (2.15)$$

and we can define a basis $\Sigma^{A'B'}$ of the self-dual (SD) two-forms $\Lambda_+^2 \mathbb{M}$ as well as a basis Σ^{AB} of the anti-self-dual (ASD) two-forms $\Lambda_-^2 \mathbb{M}$ by

$$\Sigma^{A'B'} := \frac{1}{2} \epsilon_{AB} e^{AA'} \wedge e^{BB'}, \quad \Sigma^{AB} := \frac{1}{2} \epsilon_{A'B'} e^{AA'} \wedge e^{BB'}. \quad (2.16)$$

These satisfy identities

$$\Sigma^{A'B'} \wedge \Sigma^{C'D'} = \frac{1}{4} \epsilon_{AB} \epsilon_{CD} e^{AA'} \wedge e^{BB'} \wedge e^{CC'} \wedge e^{DD'}, \quad (2.17)$$

and similarly for Σ^{AB} , while all other wedge products vanish. Note that every α -plane associated to a primed spinor α determines a SD two-form $\Sigma = \alpha_{A'} \alpha_{B'} \Sigma^{A'B'}$. Since every SD two-form is of that form for some primed spinor, the converse also holds and similarly we have a one-to-one correspondence between unprimed spinors and ASD two-forms up to scale.

On the Lie algebra level, we have an induced isomorphism of $so(4, \mathbb{C}) \cong sl(2, \mathbb{C}) \oplus sl(2, \mathbb{C})$ which leads to a splitting of the Levi-Civita connection Γ . Taking Cartan's first structural equation³

$$de^{AA'} = \Gamma^{AA'}_{BB'} e^{BB'} \quad (2.18)$$

³We are suppressing the one-form index, $\Gamma^{AA'}_{BB'} = \Gamma^{AA'}_{BB'CC'} e^{CC'}$.

as the definition of the connection coefficients the decomposition

$$\Gamma_{AA'BB'} = \epsilon_{AB}\Gamma_{A'B'} + \epsilon_{A'B'}\Gamma_{AB}, \quad (2.19)$$

of Γ into a symmetric unprimed connection Γ_{AB} on \mathbb{S} and a symmetric primed connection $\Gamma_{A'B'}$ on \mathbb{S}' preserves the splitting of the tangent bundle. Similarly, the curvature of \mathbb{M} splits up into the primed curvature $R^{A'}_{B'}$ of \mathbb{S}' and the unprimed curvature R^A_B of \mathbb{S} , where

$$R^A_B = d\Gamma^A_B + \Gamma^A_C \wedge \Gamma^C_B, \quad R^{A'}_{B'} = d\Gamma^{A'}_{B'} + \Gamma^{A'}_{C'} \wedge \Gamma^{C'}_{B'}. \quad (2.20)$$

Using the SD and ASD two-forms to decompose the primed and unprimed curvature spinors, we find

$$\begin{aligned} R^{A'}_{B'} &= \frac{1}{12}R\Sigma^{A'}_{B'} + W^{A'}_{B'C'D'}\Sigma^{C'D'} + \varrho^{A'}_{B'CD}\Sigma^{CD}, \\ R^A_B &= \frac{1}{12}R\Sigma^A_B + W^A_{BCD}\Sigma^{CD} + \varrho^A_{BC'D'}\Sigma^{C'D'}. \end{aligned} \quad (2.21)$$

Here W^A_{BCD} and $W^{A'}_{B'C'D'}$ are the anti-self-dual and self-dual Weyl spinors, $\varrho^A_{BC'D'}$ is the trace-free Ricci spinor and $R = 12\Lambda$ is the scalar curvature. Regarding the curvature $\mathcal{R} = R^A_B + R^{A'}_{B'}$ as a map

$$\mathcal{R} : \Lambda^2\mathbb{M} \mapsto \Lambda^2\mathbb{M} \quad (2.22)$$

then under the splitting (2.15) this map becomes

$$\mathcal{R} = \left(\begin{array}{c|c} W_+ + \Lambda & \varrho \\ \hline \varrho & W_- + \Lambda \end{array} \right), \quad (2.23)$$

with short-hand notation W_{\pm} for the SD and ASD Weyl spinor and ϱ for the trace-free Ricci spinor. There are various curvature restrictions one can impose:

- The metric is ASD if and only if $W_+ = 0$. Since W is conformally invariant, it suffices to consider ASD conformal structures.
- The metric is scalar-flat if and only if $\Lambda = 0$.
- The metric is Einstein if and only if $\varrho = 0$. In this case g satisfies Einstein's vacuum field equations with cosmological constant Λ .

Of course one can combine these curvature restrictions to obtain Ricci-flat metrics, ASD Einstein metrics and so forth.

2.3 Twistor theory

Having at hand the spinor formalism for four-manifolds, we turn to the twistor correspondence [20, 7, 6, 21], which will be a valuable tool throughout this thesis. Consider the primed spin bundle without the zero section, $\mathfrak{F} = \mathbb{S}' \setminus \{\xi^{A'} = 0\}$, where $\xi^{A'}$ are coordinates on the fibres of \mathbb{S}' . For every section of \mathfrak{F} , we obtain a distribution of α -planes in $T\mathbb{M}$ spanned by two vector fields $\xi^{A'} \partial_{AA'}$. Multiplying a spinor section $\xi^{A'}$ by a non-vanishing function on \mathbb{M} leaves the null planes unchanged, to eliminate this redundancy we need to projectivise the fibres of \mathfrak{F} and we arrive at the correspondence space F . The fibres are now no longer copies of \mathbb{C}^2 without the origin but copies of $\mathbb{C}P^1$. The space \mathfrak{F} can be understood as a holomorphic line bundle over F , the points in the fibre representing different multiples of a given null plane. When restricted to a fibre of F over \mathbb{M} , this line bundle is just the tautological bundle $\mathbb{C}^\times \hookrightarrow \mathbb{C}^2 \setminus \{0, 0\} \rightarrow \mathbb{C}P^1$.

Parallel transport with respect to the Levi-Civita connection maps null planes to null planes, giving rise to a one-form τ homogeneous of degree two on \mathfrak{F} ,

$$\tau := \xi_{A'} \left(d\xi^{A'} + \xi^{B'} \Gamma^{A'}_{B'} \right). \quad (2.24)$$

Using τ , we can lift the vector fields $\xi^{A'} \partial_{AA'}$ to \mathfrak{F} to obtain⁴

$$d_A := \xi^{A'} \partial_{AA'} - \xi^{A'} \xi^{B'} \Gamma_{AA'B'}^{C'} \partial_{\xi^{C'}}. \quad (2.25)$$

Since the Euler vector field

$$\Upsilon := \xi^{A'} \partial_{\xi^{A'}} \quad (2.26)$$

lies in the kernel of the one-form τ , the vector fields d_A are only determined up to the addition of terms proportional to Υ . Since by definition $d_A \lrcorner \tau = 0$, the vector fields d_A form a distribution on \mathfrak{F} that lies within the kernel of τ , called the twistor distribution. It is well-known [2] that this twistor distribution is integrable if and only if (\mathbb{M}, g) is ASD. In general the following identity

$$[d_0, d_1] = \xi^{A'} \Gamma_{AA'}^{AB} d_B \quad (2.27)$$

holds for manifolds with vanishing self-dual Weyl spinor. The leaves of this integrable distribution are the α -surfaces of \mathbb{M} . From this we can construct

⁴Here AA' are one-form indices, so $\Gamma_{B'C'} = \Gamma_{AA'B'C'} e^{AA'}$.

the following spaces: take the quotient space $\mathfrak{T} = \mathfrak{F} / \langle d_0, d_1 \rangle$, this four-dimensional complex manifold is the non-projective twistor space \mathfrak{T} . The vector fields d_A project to non-zero vector fields l_A on F , so we can also consider the three-dimensional complex manifold $T = F / \langle l_0, l_1 \rangle$, the projective twistor space. Now a point $p \in T$ corresponds to an integral surface α of the twistor distribution in F . We can restrict the line bundle \mathfrak{F} to this integral surface to obtain a line bundle \mathfrak{F}_α . However this line bundle has to be trivial, since we can find a global trivialisation over α using the leaves of the distribution $\langle d_0, d_1 \rangle$. Thus by construction \mathfrak{T} is a line bundle over the twistor space T and if we pull \mathfrak{T} back to the correspondence space F we recover \mathfrak{F} .

The fibres of \mathfrak{F} over \mathbb{M} project to a four-parameter family of copies of $\mathbb{C}^2 \setminus \{0\}$ in \mathfrak{T} , correspondingly the $\mathbb{C}P^1$ -fibres of F project to the twistor lines in T . Again, \mathfrak{T} restricted to such a twistor line is the tautological bundle over $\mathbb{C}P^1$. So we have the following double fibration:

$$\begin{array}{ccccc}
 \mathfrak{F} & \xrightarrow{\pi_{\mathfrak{T}}} & \mathfrak{T} & & \\
 \pi_F \downarrow & & \downarrow \pi_T & & \\
 \mathbb{M} & \xleftarrow{\pi_1} & F & \xrightarrow{\pi_2} & T
 \end{array} \tag{2.28}$$

Taking the pre-image of a point $m \in \mathbb{M}$ and mapping it to T and vice versa, we obtain the following correspondence: The point $m \in \mathbb{M}$ corresponds to the set of all α -surfaces which contain m , this is a $\mathbb{C}P^1$ or twistor line \hat{m} in T . Conversely, a point in T corresponds to an α -surface in \mathbb{M} . Two points $m_1, m_2 \in \mathbb{M}$ are connected by a null geodesic if and only if they lie on a common α -surface, in this case their twistor lines intersect.

The last step in this correspondence is to identify the normal bundle $N_{\hat{m}}$ of a twistor line \hat{m} , denoting the tangent bundle of twistor space by TT , this is defined as the quotient

$$N_{\hat{m}} = TT \Big|_{\hat{m}} / T\hat{m}. \tag{2.29}$$

So elements of the normal bundle are tangent vectors of the twistor space modulo tangent vectors of the twistor line \hat{m} . Suppose the integral surface α corresponding to a point $p \in T$ contains the point $m \in \mathbb{M}$, a vector $Y \in T_p T$ corresponds to a tangent vector $X \in T_m \mathbb{M}$ plus a variation of the integral surface α going through m , where X is determined only up to the addition of an element in $T\alpha$. Therefore a fibre of the normal bundle can be identified with the quotient of $T_m \mathbb{M}$ by $T\alpha$. This quotient is given by $X^{AA'} \mapsto X^{AA'} \xi_{A'}$, which is linear in $\xi_{A'}$ and thus

$N = \mathcal{O}(1) \oplus \mathcal{O}(1)$. More rigorously we obtain the same result from the following exact sequence, where $\langle d_0, d_1 \rangle$ denotes the twistor distribution in F ,

$$\begin{aligned} 0 \longrightarrow \langle d_0, d_1 \rangle \longrightarrow \pi_1^* TM \longrightarrow \pi_2^* N_{\hat{m}} \longrightarrow 0. \\ \mu^A \longmapsto \mu^A \xi^{A'} \\ X^{AA'} \longmapsto X^{AA'} \xi_{A'} \end{aligned} \quad (2.30)$$

The converse is the statement of

Theorem 2.3.1 [2] *There is a one-to-one correspondence between complex ASD conformal structures and three-dimensional complex manifolds containing a four-parameter family of rational curves with normal bundle $N = \mathcal{O}(1) \oplus \mathcal{O}(1)$.*

What makes this theorem so useful is the fact that it turns the differential problem of finding ASD metrics into an algebraic question of determining complex manifolds with certain properties. The theorem shows that the ASD equations $W_+ = 0$ are integrable in the sense that they have a Lax Pair given by the vector fields spanning the twistor distribution. Imposing further constraints on the metric will result in additional properties of the twistor space. We will discuss two examples, ASD Ricci-flat and ASD Einstein metrics. For an exhaustive list and more details see [20, 7, 6, 21], as well as [32, 33] for similar curvature restrictions in supergravity theories.

First we turn to ASD Ricci-flat metrics with $W_+ = \varrho = R = 0$. Then the only non-vanishing component of the curvature is W_- and from (2.21) we see that $R^{A'}_{B'} = 0$. Therefore we can choose a null tetrad such that $\Gamma_{A'B'} = 0$ and consequently

$$\tau = \xi_{A'} d\xi^{A'}, \quad d_A = \xi^{A'} \partial_{AA'}. \quad (2.31)$$

It's easy to see that $\mathcal{L}_{d_A} \tau = 0$, so τ descends to a well-defined one-form on \mathfrak{T} or an $\mathcal{O}(2)$ -valued one-form τ_T on T , as it is quadratic in the primed indices. Furthermore on \mathfrak{T} it satisfies

$$d\tau \wedge \tau = 0, \quad (2.32)$$

and defines an integrable distribution with two-dimensional leaves. The points on the leaves correspond to parallel, non-intersecting α -surfaces and the leaves are parametrised by all α -planes going through an arbitrary base point, i.e. they

are parametrised by a twistor line. Then \mathfrak{T} has the structure of a fibre bundle over $\mathbb{C}^2 \setminus \{0\}$, the twistor lines of \mathfrak{T} are sections of this fibre bundle. On top of this, we have a non-degenerate two-form $\Sigma(\xi) = \xi_{A'} \xi_{B'} \Sigma^{A'B'}$ on the fibres of this fibration. To see this, first note that on an ASD Ricci-flat manifold we always have $d\Sigma^{A'B'} = 0$ as a consequence of (2.18) and (2.19) since the curvature of \mathbb{S}' vanishes. Then the Lie derivative $\mathcal{L}_{d_A} \Sigma(\xi)$ vanishes since $d_A \lrcorner \Sigma(\xi) = 0$ and so $\Sigma(\xi)$ descends to \mathfrak{T} as it is constant along the twistor distribution. Along the fibres of \mathfrak{T} over $\mathbb{C}^2 \setminus \{0\}$ this two-form can be regarded as a symplectic structure, as it is non-degenerate and closed when treating $\xi_{A'}$ as a parameter. Projecting to T we arrive at the following characterisation of ASD Ricci-flat manifolds

Theorem 2.3.2 [2, 34] *There is a one-to-one correspondence between complex ASD Ricci-flat metrics and three-dimensional complex manifolds T with*

- a projection $\mu : T \rightarrow \mathbb{C}P^1$
- a four-parameter family of sections of μ with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$
- a non-degenerate two-form Σ on the fibres of μ , with values in the pull-back of $\mathcal{O}(2)$ from $\mathbb{C}P^1$.

Remark 1: Starting from a basis of $\Lambda_{\pm}^2 \mathbb{M}$ consisting of three covariantly constant self-dual two-forms $\Sigma^{A'B'}$ we construct three non-degenerate closed two-forms

$$\Sigma_1 := 2i\Sigma^{0'1'}, \quad \Sigma_2 := i\left(\Sigma^{0'0'} - \Sigma^{1'1'}\right), \quad \Sigma_3 := -\left(\Sigma^{0'0'} + \Sigma^{1'1'}\right). \quad (2.33)$$

Using the metric these give rise to three complex structures J_i via $\Sigma_i(X, Y) = g(J_i X, Y)$. All of the J_i are compatible with the metric with respective Kähler form Σ_i . As the curvature of \mathbb{S}' vanishes, we have a full S^2 of Kähler structures and hence an ASD Ricci-flat metric is hyper-Kähler and vice versa.

Remark 2: Note that we can use $\xi^{A'}$ as coordinates on the base manifold $\mathbb{C}^2 \setminus \{0\}$ of the fibre bundle \mathfrak{T} , as $\xi^{A'}$ are annihilated by d_A in (2.31).

Finally consider ASD Einstein metrics with $W_- = \varrho = 0$ and $R = 12\Lambda$, where now $\Lambda \neq 0$. We still have $\mathcal{L}_{d_A} \tau = 0$, as this only requires $\varrho = 0$, so τ as defined in (2.24) descends to a well-defined one-form on \mathfrak{T} . However, τ now satisfies $d\tau \wedge d\tau = 4\Lambda\rho$ and so defines a symplectic structure on \mathfrak{T} . Here

$\rho = d\xi^{A'} \wedge d\xi_{A'} \wedge \xi^{B'} \xi^{C'} \Sigma_{B'C'}$ serves only to encode the cosmological constant. LeBrun [35] shows that symplectic structures on the total space of holomorphic line bundles are in one-to-one correspondence with contact structures on the base manifold of the bundle. This works as follows: The pull-back of a contact form $\tilde{\tau}$ from the base manifold to the total space of a line bundle is equal to the contraction of the symplectic structure ω with the Euler vector field \mathcal{Y} ,

$$\pi^* \tilde{\tau} = \mathcal{Y} \lrcorner \omega. \quad (2.34)$$

Here π is the projection from the total space of the line bundle to the base manifold. A short computation shows that in our setting $\pi_T^* \tilde{\tau}$ is simply τ and the contact form $\tilde{\tau}$ on T satisfies $\tilde{\tau} \wedge d\tilde{\tau} = 4\Lambda\tilde{\rho}$, where $\pi_T^* \tilde{\rho} = 2\mathcal{Y} \lrcorner \rho$. Hence we arrive at the following⁵

Theorem 2.3.3 [34, 3] *There is a one-to-one correspondence between complex ASD Einstein metrics and three-dimensional complex manifolds T with*

- *a four-parameter family of rational curves with normal bundle $\mathcal{O}(1) \oplus \mathcal{O}(1)$*
- *a holomorphic contact structure τ_T*
- *a volume form ρ such that $\tau_T \wedge d\tau_T = 4\Lambda\rho$.*

Remark 3: As mentioned earlier, by definition four-dimensional Quaternion-Kähler manifolds are ASD Einstein. Hence the curvature restrictions we are investigating in this section are precisely the two cases of special quaternionic Riemannian holonomies.

A final step of the twistor correspondence is to relate all these results to real Riemannian manifolds. Some crucial input at this stage is provided by Atiyah, Hitchin and Singer [5]: an anti-self-dual real Riemannian four-manifold (M, g) in Euclidean signature is always real-analytic. Hence we can complexify such a manifold M by regarding the coordinates as complex and making the transition functions holomorphic, call the resulting complex four-manifold \mathbb{M} . The metric g is then also holomorphic and symmetric. We can now apply the spinor formalism and the results of twistor theory as presented above to the complex ASD Riemannian four-manifold (\mathbb{M}, g) .

⁵We now drop the tilde over $\tilde{\tau}$ and $\tilde{\rho}$, and denote them by the same symbol as their pullbacks.

The converse task is to recover the underlying real manifold M from the complexified manifold \mathbb{M} . To achieve this we restrict the metric to a real slice M of \mathbb{M} . The real structure on M is encoded in the twistor picture by an anti-holomorphic involution ι on twistor space T , inherited from complex conjugation on \mathbb{M} . The points of M are real and hence invariant under complex conjugation, the corresponding twistor lines are invariant under ι when acting on T . However the involution has no fixed points on T but acts as the antipodal map on such a twistor line. We can recover M by restricting \mathbb{M} to twistor lines on which ι acts in this way. The details of this procedure will be illuminated in the context of ASD Einstein manifolds in chapter 4.

While the concept of anti-self-duality is trivial for Lorentzian signature, it is meaningful to extend it to neutral signature. The twistor methods as introduced in this section only apply to real-analytic metrics, and not all ASD metrics in neutral signature are real-analytic. However twistor approaches can be extended to include all neutral signature metrics [36] when using the single-fibration picture established by [5]. We will consider ASD neutral signature metrics in chapter 5, but without resorting to twistor constructions explicitly and therefore are content with referring to the literature.

2.4 Heavenly equation

As mentioned in the introduction, the virtue of the twistor correspondences is to turn differential equations into algebraic ones, thus effectively integrating the initial equations. This has been studied in detail for various different curvature restrictions [37, 13, 38, 39, 40], as an example of the interplay between ASD structures and integrable equations we will now discuss the connection between ASD Ricci-flat metrics and the integrability of Plebanski's [12] heavenly equation. In chapter 4 we will extend much of the following to ASD Einstein manifolds. As we pointed out in Remark 1 in section 2.3, ASD Ricci-flat metrics admit a basis of closed self-dual two-forms $\Sigma^{A'B'}$. From the identities (2.17) we read off that $\Sigma^{0'0'}$ and $\Sigma^{1'1'}$ have rank 2, whereas $\Sigma^{0'1'}$ has rank 4 and furthermore that $\Sigma^{0'0'} \wedge \Sigma^{1'1'} \neq 0$. Then Darboux' theorem implies that we can choose coordinates (w, z, \bar{w}, \bar{z}) on \mathbb{M} such that

$$\Sigma^{0'0'} = dw \wedge dz, \quad \Sigma^{1'1'} = d\bar{w} \wedge d\bar{z}. \quad (2.35)$$

Here (w, z, \bar{w}, \bar{z}) are four independent holomorphic coordinates on the complex manifold \mathbb{M} , when returning to an underlying real ASD Ricci–flat manifold M we would find that (\bar{w}, \bar{z}) are complex conjugates of (w, z) . Now $\Sigma^{0'1'} \wedge \Sigma^{0'0'} = 0$ constrains the form of $\Sigma^{0'1'}$ to be

$$\Sigma^{0'1'} = \frac{1}{2} (dw \wedge dx - dz \wedge dy), \quad (2.36)$$

for two functions x, y , whereas $\Sigma^{0'1'} \wedge \Sigma^{1'1'} = 0$ enforces

$$x_z + y_w = 0, \quad (2.37)$$

which is the integrability condition for the existence of a function $H(w, z, \bar{w}, \bar{z})$ such that

$$x = H_w, \quad y = -H_z. \quad (2.38)$$

Then

$$\Sigma^{0'1'} = \frac{1}{2} (H_{w\bar{w}} dw \wedge d\bar{w} + H_{w\bar{z}} dw \wedge d\bar{z} + H_{z\bar{w}} dz \wedge d\bar{w} + H_{z\bar{z}} dz \wedge d\bar{z}), \quad (2.39)$$

and $\Sigma^{0'1'} \wedge \Sigma^{0'1'} = -2\Sigma^{0'0'} \wedge \Sigma^{1'1'}$ yields Plebanski’s heavenly equation,

$$H_{w\bar{w}} H_{z\bar{z}} - H_{w\bar{z}} H_{z\bar{w}} = 1. \quad (2.40)$$

Out of the S^2 worth of complex structures that exist on any hyper–Kähler manifold, we have picked a preferred one. It comes with holomorphic coordinates (w, z) and Kähler form $-2i\Sigma^{0'1'}$. The heavenly function H acts as Kähler potential for the metric,

$$g = 2 (H_{w\bar{w}} dw d\bar{w} + H_{w\bar{z}} dw d\bar{z} + H_{z\bar{w}} dz d\bar{w} + H_{z\bar{z}} dz d\bar{z}). \quad (2.41)$$

Choosing a null tetrad adapted to the previous discussion,

$$\begin{aligned} e^{00'} &= dw, & e^{01'} &= -H_{z\bar{w}} d\bar{w} - H_{z\bar{z}} d\bar{z}, \\ e^{10'} &= dz, & e^{11'} &= H_{w\bar{w}} d\bar{w} + H_{w\bar{z}} d\bar{z}, \end{aligned} \quad (2.42)$$

we obtain for the vector fields spanning the twistor distribution

$$\begin{aligned} d_0 &= \xi^{0'} \partial_w + \xi^{1'} (H_{w\bar{z}} \partial_{\bar{w}} - H_{w\bar{w}} \partial_{\bar{z}}), \\ d_1 &= \xi^{0'} \partial_z + \xi^{1'} (H_{z\bar{z}} \partial_{\bar{w}} - H_{z\bar{w}} \partial_{\bar{z}}), \end{aligned} \quad (2.43)$$

subject to (2.40). The vector fields d_0 and d_1 commute if and only if (2.40) is satisfied, and so constitute a Lax Pair for the heavenly equation, which arises as the compatibility condition of an over-determined system

$$d_0\Theta = 0, \quad d_1\Theta = 0, \quad (2.44)$$

for some function $\Theta(w, z, \bar{w}, \bar{z}, \xi)$. The existence of a such a Lax Pair is a characteristic feature, if not the defining property, of an integrable equation.

2.5 Deformation theory

A further aspect of the twistor correspondence is the fact that it paves the way for a deformation theory of ASD metrics. Again we will illuminate this using the example of ASD Ricci-flat metrics. Let us first approach deformations starting from the heavenly equation. Perturbations of the metric (2.41) are governed by a perturbation δH of the heavenly function, where this perturbation has to satisfy the linearisation of (2.40),

$$(H_{w\bar{w}}\partial_{z\bar{z}} + H_{z\bar{z}}\partial_{w\bar{w}} - H_{w\bar{z}}\partial_{z\bar{w}} - H_{z\bar{w}}\partial_{w\bar{z}})\delta H = 0. \quad (2.45)$$

This simply requires δH to lie in the kernel of the background-coupled wave operator,

$$\square_H \delta H := \epsilon^{AB}\epsilon^{A'B'}\partial_{AA'}\partial_{BB'}(\delta H) = 0. \quad (2.46)$$

On the other hand, coming from the twistor space picture, such a deformation is generated by a complex deformation of twistor space T together with a deformation of the symplectic structure along the fibres [2]. However we have to check that this deformation preserves the properties of T as a twistor space. A discussion of deformation theory by Kodaira [41] reveals that $\mathbb{C}P^1$ is rigid and cannot be deformed, so a deformation can only affect the fibres of T over $\mathbb{C}P^1$. Furthermore a theorem of Kodaira [42] ensures that a sufficiently small deformation preserves the existence of a four-parameter family of twistor lines with the appropriate normal bundle. The third property of a twistor space of ASD Ricci-flat metrics is the existence of a symplectic structure along the fibres of T . Hence the transformation has to be canonical. While a general complex deformation is determined by an element

$$\theta = f^A\partial_{\omega^A} + f^{A'}\partial_{\xi^{A'}}, \quad (2.47)$$

of the first cohomology group $H^1(T, \Theta(0))$ with values in the tangent sheaf $\Theta(0)$ of $\mathcal{O}(0)$ -valued holomorphic vector fields, rigidity of $\mathbb{C}P^1$ restricts the form of θ by $f^{A'} = 0$. Here we are using homogeneous coordinates on T , where we have supplemented the coordinates $\xi^{A'}$ on the base of \mathfrak{T} by two coordinates ω^A along the fibres. The transformation generated by θ is canonical if θ is a Hamiltonian vector field,

$$\Sigma(\xi)(\theta, \cdot) = d\Psi, \quad (2.48)$$

for some Hamiltonian function $\Psi(\omega^A, \xi^{A'})$. Since $\Sigma(\xi)$ is $\mathcal{O}(2)$ -valued, we require $\Psi \in H^1(T, \mathcal{O}(2))$. So in the twistor picture deformations are encoded in elements of the first cohomology group of $\mathcal{O}(2)$ -valued functions on T .

We now want to relate this to the perturbation δH of the heavenly function. To this end note that the pull-back⁶ of Ψ satisfies

$$d_A \Psi = 0. \quad (2.49)$$

We can always write $\Psi = \xi^{0'} \xi^{0'} \sum_{n=-\infty}^{\infty} \psi_n \xi^n$, where $\xi = \frac{\xi^{1'}}{\xi^{0'}}$. Then dividing (2.49) by the third power of $\xi^{0'}$ we obtain

$$\begin{aligned} \partial_{00'} \psi_{n+1} &= -\partial_{01'} \psi_n, \\ \partial_{10'} \psi_{n+1} &= -\partial_{11'} \psi_n, \end{aligned} \quad (2.50)$$

for all $n \in \mathbb{N}$. Now since $[\partial_{00'}, \partial_{10'}] = [\partial_{01'}, \partial_{11'}] = 0$ we can cross-differentiate to find

$$\square_H \psi_n = 0, \quad \forall n \in \mathbb{N}, \quad (2.51)$$

where \square_H is defined in (2.46). Therefore every term of the power series of Ψ satisfies the background-coupled wave equation, which is the linearised heavenly equation. In this way we obtain an infinite number of perturbations δH from every $\Psi \in H^1(T, \mathcal{O}(2))$. The reverse reasoning goes as follows: Since δH is a solution of the linearised heavenly equation, we define $\psi_0 = \delta H$. Then ψ_0 satisfies the integrability condition (2.51) and we can use (2.50) to define ψ_1 , which will automatically lie in the kernel of the background-coupled wave operator. Thus ψ_1 satisfies the necessary integrability condition so that we can use (2.50) again to define ψ_2 and so on. In theory this allows us to construct $\Psi \in H^1(T, \mathcal{O}(2))$

⁶For convenience also denoted by Ψ .

from a single solution of (2.45).

Remark: Since every term in the power series expansion of Ψ satisfies the linearised heavenly equation (2.45), one can use (2.50) to define a recursion operator that generates new solutions of (2.45) from old ones: Start with a known solution δH , use (2.50) to generate the next term in the power series of Ψ , which provides the new solution $\widetilde{\delta H}$.

OCTONIONIC INSTANTONS

Gauge theory in dimension higher than four has been investigated in both theoretical physics [43, 44, 45, 46, 47] and pure mathematics [11, 48] contexts. While the solutions to the full second-order Yang–Mills equations seem to be out of reach, the first-order higher-dimensional analogues of four-dimensional anti-self-duality equations admit some explicit solutions. Such equations can be written down on any n -dimensional Riemannian manifold M_n , once a differential form Ξ of degree $(n - 4)$ has been chosen. The generalised self-duality equations state that the curvature two-form \mathbb{F} of a Yang–Mills connection takes its values in one of the eigenspaces of the linear operator $T : \Lambda^2 M_n \rightarrow \Lambda^2 M_n$ given by $T(\mathbb{F}) = *(\Xi \wedge \mathbb{F})$. The full Yang–Mills equations are then implied by the Bianchi identity if Ξ is closed. If $n = 4$, and the zero-form $\Xi = 1$ is canonically given by the orientation, the eigenspaces of T are both two-dimensional, and are interchanged by reversing the orientation. In general the eigenspaces corresponding to different eigenvalues have different dimensions. For the construction to work, one of these eigenspaces must have dimension equal to $\frac{1}{2}(n - 1)(n - 2)$, as only then the number of equations matches the number of unknowns modulo gauge. Any Riemannian manifold with special holonomy $Hol \subset SO(n)$ admits a preferred parallel $(n - 4)$ -form, and the eigenspace conditions above can be equivalently stated as $\mathbb{F} \in \mathfrak{hol}$, where we have identified the Lie algebra \mathfrak{hol} of the holonomy group with a subspace of $\Lambda^2 M_n \cong \mathfrak{so}(n)$. One of the most interesting cases corresponds to eight-dimensional manifolds with holonomy $Spin(7)$. The only currently known explicit solution on $M_8 = \mathbb{R}^8$ with its flat metric has a

gauge group $Spin(7)$, while we will construct explicit solutions to the system

$$*_8(\mathbb{F} \wedge \Xi) = -\mathbb{F}, \quad (3.1)$$

with gauge group $SU(2)$. This will be achieved by exploiting the embedding $SU(2) \times SU(2) \subset Spin(7)$. This holonomy reduction allows a canonical symmetry reduction to the Yang–Mills–Higgs system in four dimensions– a non–abelian analogue of the Seiberg–Witten equations involving four Higgs fields [11, 46, 49]. The explicit $SU(2)$ solutions arise from a t’Hooft–like ansatz which turns out to be consistent despite a vast overdeterminancy of the equations. The resulting solutions on \mathbb{R}^8 fall into two classes, both of which are singular along a hypersurface. To overcome this, and to evade Derrick’s theorem prohibiting finite action solutions in dimensions higher than four we shall consider the case of curved backgrounds of the form $M_8 = M_4 \times \mathbb{R}^4$, where M_4 is hyper–Kähler. The gauge fields on the Eguchi–Hanson gravitational instanton are still singular, but if M_4 is taken to be a Bianchi II gravitational instanton representing a domain wall [50], then the Yang–Mills curvature is regular away from the wall. The following theorem is the main result of this chapter. The relevant notation will be developed in detail in section 3.2, while the proof will follow from section 3.3.2.

Theorem 3.0.1 *Let \mathbb{H} denote the quaternions and \mathcal{H} be the simply–connected Lie group whose left–invariant one–forms satisfy the Maurer–Cartan relations*

$$d\sigma_0 = 2\sigma_0 \wedge \sigma_3 - \sigma_1 \wedge \sigma_2, \quad d\sigma_1 = \sigma_1 \wedge \sigma_3, \quad d\sigma_2 = \sigma_2 \wedge \sigma_3, \quad d\sigma_3 = 0,$$

with the regular and left–invariant metric

$$\hat{g} = \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2.$$

- *The metric $g = e^{3\rho}\hat{g}$, where $d\rho = \sigma_3$, is hyper–Kähler.*
- *The $\mathfrak{su}(2)$ –valued one–form and $\mathfrak{su}(2) \otimes \mathbb{H}$ –valued Higgs field*

$$\mathcal{A} = \frac{3}{4}(\sigma_2 \otimes T_1 - \sigma_1 \otimes T_2 + \sigma_0 \otimes T_3), \quad \Phi = -\frac{\sqrt{21}}{4}e^{-\frac{3}{2}\rho}(iT_1 + jT_2 + kT_3)$$

with $[T_1, T_2] = T_3$, $[T_3, T_1] = T_2$, $[T_2, T_3] = T_1$ satisfy the non–abelian Seiberg–Witten equations

$$\mathcal{F}_+^{\mathcal{A}} - \frac{1}{4}[\Phi, \bar{\Phi}]_{\mathfrak{J}} = 0, \quad \mathcal{D}^{\mathcal{A}}\Phi = 0,$$

where $\mathcal{F}_+^{\mathcal{A}}$ is the self–dual part of $\mathcal{F}^{\mathcal{A}} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$ with respect to g , $\bar{\Phi}$ is the quaternionic conjugate of Φ , \mathfrak{J} a map $\mathfrak{Im}\mathbb{H} \mapsto \Lambda_+^2 \mathcal{H}$ and $\mathcal{D}^{\mathcal{A}}$ the Dirac operator coupled to \mathcal{A} acting on Φ under the identification $\mathbb{H} \cong \mathbb{R}^4$.

Finally we should mention that there are other candidates for 'self-duality' equations in higher dimensions. One possibility in dimension eight, exploited by Polchinski in the context of heterotic string theory [51], is to consider the system

$$*\mathbb{F} \wedge \mathbb{F} = \pm \mathbb{F} \wedge \mathbb{F}. \quad (3.2)$$

These equations are conformally invariant, and thus the finite action solutions compactify \mathbb{R}^8 to the eight-dimensional sphere, but unlike the system (3.1) considered in this thesis they do not imply the Yang-Mills equations.

3.1 Anti-self-duality in eight dimensions

Let (M_8, g_8) be an eight-dimensional oriented Riemannian manifold with holonomy $Spin(7)$ and associated four-form Ξ as in (2.3). Let $T : \Lambda^2 M_8 \rightarrow \Lambda^2 M_8$ be a self-adjoint operator given by $\mathbb{F} \rightarrow *_8(\Xi \wedge \mathbb{F})$, where $*_8$ is the Hodge operator of g_8 corresponding to the orientation $\Xi \wedge \Xi$. The 28-dimensional space of two-forms in eight dimensions splits into $\Lambda_{21}^2 M_8 \oplus \Lambda_+^2 M_8$, where $\Lambda_{21}^2 M_8$ and $\Lambda_+^2 M_8$ are eigenspaces of T with eigenvalues -1 and 3 respectively. The 21-dimensional space Λ_{21}^2 can be identified with the Lie algebra $\mathfrak{spin}(7) \subset \mathfrak{so}(8) \cong \Lambda^2 M_8$. Let \mathbb{A} be a one-form on M_8 with values in a Lie algebra \mathfrak{g} of a gauge group G . The $Spin(7)$ anti-self-duality condition states that the curvature two-form

$$\mathbb{F} := d\mathbb{A} + \mathbb{A} \wedge \mathbb{A} \quad (3.3)$$

takes its values in $\Lambda_{21}^2 M_8$. This leads to a system of seven first order equations

$$*_8(\mathbb{F} \wedge \Xi) = -\mathbb{F}, \quad (3.4)$$

which we call the octonionic instanton equation. Explicitly the components are given by

$$\begin{aligned} \mathbb{F}_{01} + \mathbb{F}_{23} - \mathbb{F}_{45} - \mathbb{F}_{67} &= 0, \\ \mathbb{F}_{02} - \mathbb{F}_{13} - \mathbb{F}_{46} + \mathbb{F}_{57} &= 0, \\ \mathbb{F}_{03} + \mathbb{F}_{12} - \mathbb{F}_{47} - \mathbb{F}_{56} &= 0, \\ \mathbb{F}_{04} + \mathbb{F}_{15} + \mathbb{F}_{26} + \mathbb{F}_{37} &= 0, \\ \mathbb{F}_{05} - \mathbb{F}_{14} - \mathbb{F}_{27} + \mathbb{F}_{36} &= 0, \\ \mathbb{F}_{06} + \mathbb{F}_{17} - \mathbb{F}_{24} - \mathbb{F}_{35} &= 0, \\ \mathbb{F}_{07} - \mathbb{F}_{16} + \mathbb{F}_{25} - \mathbb{F}_{34} &= 0. \end{aligned} \quad (3.5)$$

This is a determined system of PDEs as one of the eight components of \mathbb{A} can be set to zero by a gauge transformation

$$\mathbb{A} \longrightarrow \rho \mathbb{A} \rho^{-1} - d\rho \rho^{-1}, \quad \text{where } \rho \in \text{Map}(M_8, G). \quad (3.6)$$

Equations (3.1) were first investigated in [43], and some solutions were found in [52, 44] for the gauge group $Spin(7)$. If \mathbb{A} is a solution to (3.1), then it is a Yang–Mills connection because

$$D *_8 \mathbb{F} = -D\mathbb{F} \wedge \Xi = 0, \quad \text{where } D = d + [\mathbb{A}, \dots] \quad (3.7)$$

by the Bianchi identities. The Derrick scaling argument (see e.g. [21, 9]) shows there are no nontrivial finite action solutions to the pure Yang–Mills equations on \mathbb{R}^8 . This obstruction can be overcome if some dimensions are compactified. If (M_8, g_8) is a compact manifold with holonomy $Spin(7)$, then the Yang–Mills connections which satisfy (3.1) are absolute minima of the Yang–Mills functional

$$E(\mathbb{A}) := -\frac{1}{4\pi} \int_{M_8} \text{tr}(\mathbb{F} \wedge \mathbb{F}). \quad (3.8)$$

To see this write $\mathbb{F} = \mathbb{F}_+ + \mathbb{F}_-$, where $\mathbb{F}_+ \in \Lambda_+^2 M_8 \otimes \mathfrak{g}$ and $\mathbb{F}_- \in \Lambda_-^2 M_8 \otimes \mathfrak{g}$, then verify that

$$\mathbb{F} \wedge *_8 \mathbb{F} = \mathbb{F}_+ \wedge *_8 \mathbb{F}_+ + \Xi \wedge \mathbb{F} \wedge \mathbb{F}. \quad (3.9)$$

The integral of the trace of the second term on the right–hand side is independent of \mathbb{A} .

3.2 Non–abelian Seiberg–Witten equations

Holonomy reduction

We shall consider the special case of product manifolds with holonomy equal to or contained in $SU(2) \times SU(2) \subset Spin(7)$, namely

$$M_8 = M_4 \times \widetilde{M}_4, \quad g_8 = g_4 + \widetilde{g}_4, \quad (3.10)$$

where M_4 and \widetilde{M}_4 are hyper–Kähler manifolds. This is one of the possible holonomy reductions of a $Spin(7)$ –manifold [19]. Let Σ_i^\pm span the spaces $\Lambda_+^2 M_4$ and $\Lambda_-^2 M_4$ of self–dual and anti–self–dual two–forms respectively. Thus

$$g_4 = (e^0)^2 + (e^1)^2 + (e^2)^2 + (e^3)^2, \quad \text{and } \Sigma_i^\pm := e^0 \wedge e^i \pm \frac{1}{2} \varepsilon^i_{jk} e^j \wedge e^k, \quad (3.11)$$

where $i, j, \dots = 1, 2, 3$ with analogous expressions for \tilde{g}_4 and $\tilde{\Sigma}_i^\pm$. The self-dual $Spin(7)$ four-form (2.3) is then given by

$$\Xi := \text{vol}_4 + \widetilde{\text{vol}}_4 - \sum_{i=1}^3 \Sigma_i^+ \wedge \tilde{\Sigma}_i^+, \quad (3.12)$$

where $\text{vol}_4, \widetilde{\text{vol}}_4$ are volume forms on M_4 and \widetilde{M}_4 respectively. Ξ is closed if Σ_i^+ and $\tilde{\Sigma}_i^+$ are, which can always be achieved by parallel transport along the flat bundles $\Lambda_+^2 M_4$ and $\Lambda_+^2 \widetilde{M}_4$. Hence we have a global parallelism of self-dual two-forms on M_4 and \widetilde{M}_4 . Furthermore the $Spin(7)$ -structure on the product manifold M_8 determined by Ξ induces an isomorphism between Σ_i^+ on M_4 and $\tilde{\Sigma}_i^+$ on \widetilde{M}_4 and so we can also identify the fibres of $\Lambda_+^2 M_4$ and $\Lambda_+^2 \widetilde{M}_4$ with each other. By contraction with the metric the self-dual two-forms Σ_i^+ and $\tilde{\Sigma}_i^+$ correspond to two algebras of endomorphisms (I_1, I_2, I_3) of $\Lambda^1 M_4$ and $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ of $\Lambda^1 \widetilde{M}_4$, each of which is isomorphic to the imaginary quaternions $\mathfrak{Im}(\mathbb{H})$. Here the S^2 worth of complex structures on the two hyper-Kähler manifolds corresponds to imaginary quaternions of unit length. We choose to identify (I_1, I_2, I_3) with (i, j, k) and hence every fibre of $\Lambda_+^2 M_4$ with $\mathfrak{Im}\mathbb{H}$ using a map

$$\mathfrak{J} : \mathfrak{Im}\mathbb{H} \mapsto \Lambda_+^2 M_4, \quad \mathfrak{J}(i) = \Sigma_1^+, \quad \mathfrak{J}(j) = \Sigma_2^+, \quad \mathfrak{J}(k) = \Sigma_3^+, \quad (3.13)$$

and similarly for $\Lambda_+^2 \widetilde{M}_4$. The map (3.13) is unique up to global (i.e. constant) $Sp(1)$ gauge transformations.

Symmetry reduction

We shall now consider the anti-self-duality equations (3.1) for a \mathfrak{g} -valued connection \mathbb{A} over an eight-manifold M_8 of the form (3.10), where M_4 is an arbitrary hyper-Kähler four-manifold, and $\widetilde{M}_4 = \widetilde{\mathbb{R}}^4$ is flat. We shall look for solutions \mathbb{A} that admit a four-dimensional symmetry group generated by the translations on $\widetilde{\mathbb{R}}^4$. The product structure of M_8 and the translational symmetry along the second factor \widetilde{M}_4 imply a decomposition of \mathbb{A} into

$$\mathbb{A} = \mathcal{A} + \Phi, \quad (3.14)$$

where $\mathcal{A} \in \Gamma(M_4, \Lambda^1 M_4 \otimes \mathfrak{g})$ and $\Phi \in \Gamma(M_4, \Lambda^1 \widetilde{M}_4 \otimes \mathfrak{g}) \cong \Gamma(M_4, \mathbb{R}^4 \otimes \mathfrak{g})$. If we require admissible gauge transformations to obey the translational symmetry then \mathcal{A} can be regarded as a \mathfrak{g} -valued connection on M_4 but Φ corresponds to

four \mathfrak{g} -valued Higgs fields. On the level of the curvature \mathbb{F} of \mathbb{A} , the product structure of M_8 enforces a further decomposition into $\mathbb{F} = \mathbb{F}_{(2,0)} + \mathbb{F}_{(1,1)} + \mathbb{F}_{(0,2)}$ since

$$\Lambda^2 M_8 = \Lambda^2 M_4 \oplus \left(\Lambda^1 M_4 \otimes \Lambda^1 \widetilde{M}_4 \right) \oplus \Lambda^2 \widetilde{M}_4. \quad (3.15)$$

Using (3.14) and the definition of the curvature, $\mathbb{F} = d\mathbb{A} + \frac{1}{2}[\mathbb{A}, \mathbb{A}]$, we find the components of this decomposition,

$$\mathbb{F} = d\mathcal{A} + \frac{1}{2}[\mathcal{A}, \mathcal{A}] + d\Phi + [\mathcal{A}, \Phi] + \frac{1}{2}[\Phi, \Phi] = \mathcal{F}^{\mathcal{A}} + D^{\mathcal{A}}\Phi + \frac{1}{2}[\Phi, \Phi], \quad (3.16)$$

where $\mathcal{F}^{\mathcal{A}}$ is the curvature of the connection \mathcal{A} and $D^{\mathcal{A}}\Phi$ the covariant derivative of Φ under the adjoint action. The octonionic instanton equation (3.1) imposes constraints on these components, the first three equations of (3.5) translate into

$$\mathcal{F}_+^{\mathcal{A}} - \frac{1}{2}[\Phi, \Phi]_+ = 0. \quad (3.17)$$

This is well-defined due to the isomorphism of $\Lambda_+^2 M_4$ and $\Lambda_+^2 \widetilde{M}_4$. The conditions on $D^{\mathcal{A}}\Phi$ are harder to interpret geometrically. First note that the map \mathcal{J} allows us to identify the fibres of $\Lambda^1 M_4$ and $\Lambda^1 \widetilde{M}_4$ with the quaternions: Choose frames e^a of T^*M_4 and \tilde{e}^a of $T^*\widetilde{M}_4$ in which Σ_i^+ and $\tilde{\Sigma}_i^+$ are given by (3.11) and define

$$\begin{aligned} \mathcal{A} = \mathcal{A}_a e^a &\longmapsto \mathcal{A} = \mathcal{A}_0 + i\mathcal{A}_1 + j\mathcal{A}_2 + k\mathcal{A}_3, \\ \Phi = \Phi_a \tilde{e}^a &\longmapsto \Phi = \Phi_4 + i\Phi_5 + j\Phi_6 + k\Phi_7, \end{aligned} \quad (3.18)$$

then the action of (I_1, I_2, I_3) on $\Lambda^1 M_4$ and $(\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$ on $\Lambda^1 \widetilde{M}_4$ is simply given by quaternionic multiplication with (i, j, k) on the left. Note that (3.18) is defined up to $SU(2)$ gauge transformations of \mathbb{S} and $\tilde{\mathbb{S}}$, the unprimed spin bundles of M_4 and \widetilde{M}_4 . These gauge transformations are part of the gauge freedom via $SU(2) \times SU(2) \subset Spin(7)$. Under the identification (3.18) the octonionic instanton equation (3.1) is a set of equations for a \mathfrak{g} -valued connection \mathcal{A} and a $\mathbb{H} \otimes \mathfrak{g}$ -valued Higgs field Φ on M_4 , given by

$$\mathcal{F}_+^{\mathcal{A}} - \frac{1}{4}[\Phi, \bar{\Phi}]_{\mathcal{J}} = 0, \quad \mathcal{D}^{\mathcal{A}}\Phi = 0. \quad (3.19)$$

Here $\bar{\Phi}$ and $\mathcal{D}^{\mathcal{A}} = D_0^{\mathcal{A}} - iD_1^{\mathcal{A}} - jD_2^{\mathcal{A}} - kD_3^{\mathcal{A}}$ are the quaternionic conjugates of Φ and the quaternion-valued covariant derivative $D^{\mathcal{A}}$ coupled to \mathcal{A} . The bracket in the first equation is a combination of the Lie bracket and point-wise identification of

$\mathfrak{Im}\mathbb{H}$ with self-dual two-forms on M_4 using the map \mathfrak{J} . The freedom of choosing \mathfrak{J} in (3.13) corresponds to a global $SU(2)$ -rotation of $\Lambda_+^2 M_4$ and $\Lambda_+^2 \widetilde{M}_4$. In the identification of $\Lambda^1 M_4$ and $\Lambda^1 \widetilde{M}_4$ with \mathbb{H} this comes down to multiplying Φ by a unit quaternion and $\bar{\Phi}$, \mathcal{D}^A by its quaternionic conjugate. Expanding equations (3.19) into real and imaginary parts shows the equivalence with (3.5). We call (3.19) the non-abelian Seiberg–Witten equations.

To motivate the choice of this name as well as the symbol \mathcal{D}^A for the differential operator in (3.19) note the following: M_4 is hyper-Kähler and hence its primed spinor bundle is flat. Thus we can identify the product space $M_8 = M_4 \times \widetilde{\mathbb{R}}^4$ with the primed spinor bundle of M_4 . Then \mathcal{A} is a \mathfrak{g} -valued connection as before, but Φ in this setting is a \mathfrak{g} -valued section of the primed spinor bundle. In this context, \mathcal{D}^A is precisely the Dirac operator coupled to \mathcal{A} acting on \mathbb{S} and \mathbb{S}' and mapping them onto each other. Hence the system (3.19) can indeed be regarded as a non-abelian version [46, 11, 53, 54, 49] of the equations found by Seiberg and Witten [55].

3.3 Ansatz for $SU(2)$ solutions

To find explicit solutions to (3.19) and (3.1) we specialise to the gauge group $G = SU(2)$. We shall proceed with an analogy to the t'Hooft ansatz for the self-dual Yang–Mills equations on \mathbb{R}^4 . Let T_i , ($i = 1, 2, 3$) denote a basis of $\mathfrak{su}(2)$ with commutation relations $[T_i, T_j] = \epsilon_{ijk} T_k$ and $T_i T^i := T_i T_j \delta^{ij} = -\frac{3}{4} \mathbf{1}_2$. We can then define two $\mathfrak{su}(2)$ -valued two-forms σ and $\tilde{\sigma}$ such that $*_4 \sigma = \sigma$ and $*_4 \tilde{\sigma} = -\tilde{\sigma}$ by

$$\sigma := \frac{1}{2} \sigma_{ab} e^a \wedge e^b = \sum_i T_i \Sigma_i^+, \quad \tilde{\sigma} := \frac{1}{2} \tilde{\sigma}_{ab} e^a \wedge e^b = \sum_i T_i \Sigma_i^-, \quad (3.20)$$

where Σ_i^\pm are given by (3.11). Thus the forms σ_{ab} select the three-dimensional space of SD two forms $\Lambda_+^2 M_4$ from the six-dimensional space $\Lambda^2 M_4$ and project it onto the three-dimensional subspace $\mathfrak{su}(2)$ of $\mathfrak{so}(4)$. An analogous isomorphism between $\Lambda_-^2 M_4$ and another copy of $\mathfrak{su}(2)$ is provided by $\tilde{\sigma}$. The following identities [21] hold

$$\tilde{\sigma}_{ab} \sigma^{ab} = 0, \quad \sigma_{ab} \sigma^b{}_c = \frac{3}{4} \mathbf{1}_2 \delta_{ac} + \sigma_{ac}, \quad \sigma_{ab} \sigma^{ab} = -3 \mathbf{1}_2. \quad (3.21)$$

We now return to equations (3.19) and, identifying the $\mathfrak{su}(2) \otimes \mathbb{H}$ -valued Higgs field $\Phi = \Phi_0 + i\Phi_1 + j\Phi_2 + k\Phi_3$ with $\Phi = \Phi_a e^a$, we make the following ansatz for

two $\mathfrak{su}(2)$ -valued one-forms,

$$\mathcal{A} := *_4(\sigma \wedge dG) = \sigma_{ab} \partial^b G e^a, \quad \Phi := *_4(\sigma \wedge dH) = \sigma_{ab} \partial^b H e^a, \quad (3.22)$$

where $G, H : M_4 \rightarrow \mathbb{R}$ are functions on M_4 and ∂_a are the vector fields dual to e^a . This Ansatz for \mathcal{A} was first suggested by t'Hooft, it leads to instantons in four dimensions if and only if G is harmonic. Let $\square = *d*d + d*d*$ be the Laplacian and d be the exterior derivative on M_4 , and let $d(e^a) = C^a{}_{bc} e^b \wedge e^c$. The following Proposition will be proved in Appendix A:

Proposition 3.3.1 [1] *The non-abelian Seiberg–Witten equations (3.19) are satisfied by Ansatz (3.22) if and only if G and H satisfy the following system of coupled partial differential equations:*

$$\square G + |dG|^2 - |dH|^2 = 0, \quad (3.23a)$$

$$(\epsilon_{ea}{}^{bc} C^a{}_{bc} \sigma^{ed} - \sigma^{ab} C^d{}_{ab}) \partial_d G = 0, \quad (3.23b)$$

$$\tilde{\sigma}_{ac} \sigma^c{}_b (\partial^a \partial^b H - 2\partial^a G \partial^b H) = 0, \quad (3.23c)$$

$$\sigma_{ab} (\partial^a \partial^b H - 2\partial^a G \partial^b H) = 0. \quad (3.23d)$$

Note that equation (3.23d) is equivalent to the anti-self-duality of the anti-symmetric part of

$$\partial^a \partial^b H - 2\partial^a H \partial^b G. \quad (3.24)$$

A similar interpretation of equation (3.23c) is given by the following

Lemma 3.3.2 [1] *Let Ψ_{ab} be an arbitrary tensor. Then*

$$\tilde{\sigma}^{ab} \sigma^c{}_b \Psi_{ac} = 0 \quad \Leftrightarrow \quad \Psi_{(ac)} = \frac{1}{4} \Psi_b{}^b \delta_{ac}. \quad (3.25)$$

Proof. Starting from the left hand side we first define a two-form $(\Psi\sigma) := \sigma^c{}_{[b} \Psi_{a]c} e^a \wedge e^b$. Therefore

$$\tilde{\sigma}^{ab} \sigma^c{}_b \Psi_{ac} = \tilde{\sigma}^{ab} \sigma^c{}_{[b} \Psi_{a]c} = *[\tilde{\sigma} \wedge (\Psi\sigma)] = 0, \quad (3.26)$$

and so $(\Psi\sigma)$ is self-dual, i.e.

$$(\Psi\sigma)_{01} = (\Psi\sigma)_{23}, \quad (\Psi\sigma)_{02} = -(\Psi\sigma)_{13}, \quad (\Psi\sigma)_{03} = (\Psi\sigma)_{12}. \quad (3.27)$$

Using the definition (3.20) of σ_{ab} in terms of the generators of $\mathfrak{su}(2)$ this is equivalent to a system of nine linear equations for the components of Ψ_{ac} : six of them

set off-diagonal terms to zero, three more equate the four diagonal terms of Ψ_{ac} . Solving this system is straightforward: the only solution is $\Psi_{(ac)} = \Psi\delta_{ac}$ for some scalar function Ψ . \square

Thus equations (3.23c) and (3.23d) together imply that $\partial^a\partial^b H - 2\partial^a H\partial^b G$ is the sum of a (symmetric) pure-trace term and an (anti-symmetric) ASD term. To continue with the analysis of (3.23) we need to distinguish between flat and curved background spaces.

3.3.1 Flat background

Our first choice for M_4 is the flat space \mathbb{R}^4 with $e^a = dx^a$ for Cartesian coordinates x^a . Since the one-forms e^a are closed we have $C^a{}_{bc} = 0$ and the dual vector fields ∂_a commute. This implies that (3.23b) is identically satisfied. Equation (3.23d) implies that the simple two-form $dG \wedge dH$ is ASD. Therefore this form is equal to zero, since there are no real simple ASD two-forms in Euclidean signature and thus H and G are functionally dependent. Therefore we can set $H = H(G)$. Thus the tensor $\Psi_{ab} = \partial_a\partial_b H - 2\partial_a H\partial_b G$ is symmetric. Next, we turn our attention to (3.23c). Applying lemma 3.3.2 we deduce that Ψ_{ab} is pure trace. Defining a one-form $f := \exp(-2G)dH$ we find that

$$\partial_a f_c = \Psi e^{-2G} \delta_{ac} \tag{3.28}$$

for some Ψ . Equating the off-diagonal components of (3.28) to zero shows that f_c depends on x^c only, and the remaining four equations yield

$$dH = e^{2G} dw, \tag{3.29}$$

where $w := \frac{1}{2}\gamma x_a x^a + \kappa_a x^a$, for some constants γ, κ_a . Thus G also depends only on w and, defining $g(w) := \exp G(w)$, equation (3.23a) yields

$$g''(2\gamma w + \kappa^2) + 4\gamma g' - g^5(2\gamma w + \kappa^2) = 0. \tag{3.30}$$

There are two cases to consider

- Assume that $\gamma = 0$, in which case

$$g' = \pm \sqrt{\frac{1}{3}g^6 + \gamma_1}. \tag{3.31}$$

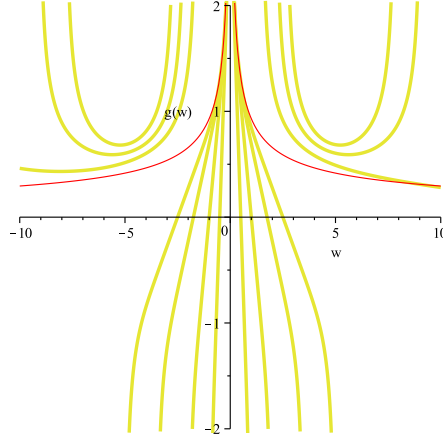


Figure 3.1: Numerical plot of solutions to $g'' = g^5$

To obtain an explicit solution we set the constant $\gamma_1 = 0$. Using the translational invariance of (3.19) we can always put $w = x^3$. Reabsorbing the constant of integration and rescaling yields

$$G = -\frac{1}{2} \ln |x^3|, \quad H = \frac{\sqrt{3}}{2} \ln |x^3|. \quad (3.32)$$

Using these functions in the ansatz (3.22) we obtain

$$\mathcal{A} = \frac{1}{2x^3} (e^2 \otimes T_1 - e^1 \otimes T_2 + e^0 \otimes T_3), \quad \Phi = -\frac{\sqrt{3}}{2x^3} (T_3 - iT_2 + jT_1), \quad (3.33)$$

with curvature

$$\mathcal{F} = \frac{-1}{8(x^3)^2} (3\sigma + \tilde{\sigma}). \quad (3.34)$$

Note that the connection \mathcal{A} is singular along a hyperplane in \mathbb{R}^4 and thus \mathbb{A} is also singular along a hyperplane in \mathbb{R}^8 because of the translational symmetry. The curvature for this solution is singular along a hyper-plane with normal κ_a , and blows up like $|x^3|^{-2}$. A numerical plot of solutions of (3.31) for different γ_1 is displayed in Figure 3.1. Since the equation is autonomous, one can obtain the general solution by translating any curve in the x^3 -direction. The red line corresponds to (3.32). Note that all other curves have two vertical asymptotes and do not extend to the whole range of x^3 .

- We will now present a second, radially symmetric solution. If $\gamma \neq 0$ we translate the independent variable by $w \rightarrow w - \frac{\kappa^2}{2\gamma}$, then (3.30) is

$$g''w + 2g' - g^5w = 0. \quad (3.35)$$

Figures 3.2 and 3.3 contain numerical plots for two different sets of initial conditions obtained using the computer algebra system MAPLE. An explicit analytic solution is given by

$$g(w) = \frac{1}{\sqrt{\frac{1}{3}w^2 - 1}}. \quad (3.36)$$

If we define the radial coordinate $r := \left| \sqrt{\frac{\gamma}{2\sqrt{3}}} \left(x_a + \frac{\kappa_a}{\gamma} \right) \right|$, then $w = \sqrt{3}r^2$ and

$$G(r) = -\frac{1}{2} \ln(r^4 - 1), \quad H(r) = \frac{\sqrt{3}}{2} \ln \left[\frac{r^2 - 1}{r^2 + 1} \right], \quad (3.37)$$

leading to

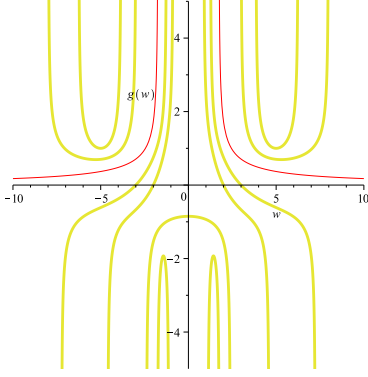
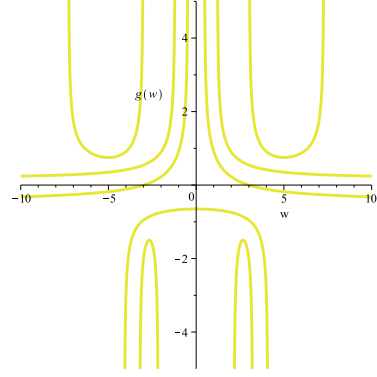
$$\mathcal{A} = \frac{-r^2}{r^4 - 1} x^a \sigma_{ab} e^b, \quad \Phi = \frac{\sqrt{3}}{r^4 - 1} (iq \otimes T_1 + jq \otimes T_2 + kq \otimes T_3), \quad (3.38)$$

using the quaternionic coordinate $q := x^0 + ix^1 + jx^2 + kx^3$. Both \mathcal{A} and Φ are singular on the sphere $r = 1$ in \mathbb{R}^4 . In \mathbb{R}^8 this corresponds to cylinders of a hypersurface type. The curvature is given by

$$\mathcal{F} = \frac{2}{(r^4 - 1)^2} (-r^2 \sigma + x^c x_{[a} \sigma_{b]c} e^a \wedge e^b), \quad (3.39)$$

with the same singularity. The numerical results suggest that there are no regular solutions to (3.35) and most solution curves do not even extend to the full range of r .

This concludes the process of solving the initial system of coupled partial differential equations (3.23) for vanishing $C^a{}_{bc}$. We have shown that the most general solution to this system is given by two functions of one variable, $G(w)$ and $H(w)$ with $w := \frac{1}{2}\gamma x_a x^a + \kappa_a x^a$, which are determined by an ordinary differential equation. We presented two classes of solutions on \mathbb{R}^8 in closed form.


Figure 3.2: Solutions of ODE (3.35) I

Figure 3.3: Solutions of ODE (3.35) II

3.3.2 Curved backgrounds

The solutions we found in the last section 3.3.1 have extended singularities resulting in an unbounded curvature and infinite action. While we could argue that the former is an artifact resulting from the form of our ansatz, there is no hope to cure the latter. The existence of the finite action solutions to pure Yang–Mills theory on \mathbb{R}^8 or to Yang–Mills–Higgs theory on \mathbb{R}^4 is ruled out by the Derrick scaling argument [21]. To evade Derrick’s argument we shall now look at curved hyper–Kähler manifolds M_4 in place of \mathbb{R}^4 . The one–forms e^a in the orthonormal frame (3.11) are no longer closed and the vector fields ∂_a do not commute, as $C_{ab}^c \neq 0$. The equations (3.23c) and (3.23d) imply that $\partial_a \partial_b H - 2\partial_a G \partial_b H$ is a sum of a pure–trace term and an ASD term, but examining the integrability conditions shows that the trace term vanishes unless the metric g_4 is flat. Thus

$$\partial_a H = \delta_a e^{2G}, \quad (3.40)$$

where δ_a are some constants of integration. We shall analyse two specific examples of M_4 . The first class of solutions on the Eguchi–Hanson manifold generalises the spherically symmetric solutions (3.37), which were singular at $r = 1$. In the Eguchi–Hanson case the parameter in the metric can be chosen so that $r = 1$ does not belong to the manifold. The second class of solutions on the domain wall backgrounds generalises the solutions (3.32).

Eguchi–Hanson background

Consider (M_4, g_4) to be the Eguchi–Hanson manifold [56], with the metric

$$g_4 = \left(1 - \frac{a^4}{r^4}\right)^{-1} dr^2 + \frac{1}{4}r^2 \left(1 - \frac{a^4}{r^4}\right) \sigma_3^2 + \frac{1}{4}r^2(\sigma_1^2 + \sigma_2^2). \quad (3.41)$$

Here $\sigma_i, i = 1, 2, 3$ are the left-invariant one-forms on $SU(2)$

$$\sigma_1 + i\sigma_2 = e^{-i\psi}(d\theta + i \sin \theta d\phi), \quad \sigma_3 = d\psi + \cos \theta d\phi \quad (3.42)$$

and to obtain the regular metric we take the ranges

$$r > a, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \phi \leq 2\pi, \quad 0 \leq \psi \leq 2\pi. \quad (3.43)$$

Choose an orthonormal frame

$$e^0 = \frac{1}{\sqrt{1 - \frac{a^4}{r^4}}} dr, \quad e^1 = \frac{r}{2} \sqrt{1 - \frac{a^4}{r^4}} \sigma_3, \quad e^2 = \frac{r}{2} \sigma_2, \quad e^3 = \frac{r}{2} \sigma_1. \quad (3.44)$$

Computing the exterior derivatives $d(e^a)$ explicitly we can evaluate (3.23b) and find that it is trivially zero. Furthermore, we know that equations (3.23c) and (3.23d) are equivalent to (3.40). The integrability conditions $d^2H = 0$ imply

$$df = 2f \wedge dG, \quad \text{where} \quad f = \delta_a e^a \quad (3.45)$$

The condition $dG \neq 0$ implies $\delta_i = 0$. Then

$$f = \frac{\delta_0 dr}{\sqrt{1 - \frac{a^4}{r^4}}}, \quad (3.46)$$

and $df = 0$. Thus $f \wedge dr = dH \wedge dr = dH \wedge dG = 0$ and consequently H and G depend on r only and satisfy the following relation:

$$\frac{dH}{dr} = \frac{\delta_0 e^{2G}}{\sqrt{1 - \frac{a^4}{r^4}}}. \quad (3.47)$$

Using this in equation (3.23a) and substituting $g := \frac{e^G}{\sqrt{\delta_0}}$ yields

$$\left(1 - \frac{a^4}{r^4}\right) g'' + \frac{1}{r} \left(3 + \frac{a^4}{r^4}\right) g' - g^5 = 0. \quad (3.48)$$

The numerical results (Figures 3.4 and 3.5, where $a = 1$) indicate that yet again there are no regular functions among the solutions. Analysing the limit $r \rightarrow a$ we

find that the solution curves have to satisfy $g'(a) = \frac{a}{4}g(a)^5$ if they intersect with the line $r = a$ in the (r, g) plane for finite g . All other curves necessarily blow up for $r \rightarrow a$. However, numerical evidence strongly suggests that all curves that do intersect with the line $r = a$ for finite g are monotonically in- or decreasing and escape to infinity in finite time.

It is interesting to note that for $r \gg a$ the ODE (3.48) simplifies to

$$g'' + \frac{3}{r}g' - g^5 = 0, \quad (3.49)$$

which is the same as the equation we obtain from (3.48) when considering the flat limit $a = 0$. The reason is that far away from the lump of curvature in the middle of the Eguchi–Hanson manifold, the gravitational instanton approximates flat-space. The equation differs from ODEs (3.31) and (3.35) since we are using a non-integrable coordinate frame. We would expect the same equation when starting with Ansatz (3.22) on \mathbb{R}^4 using the frame (3.44) with $a = 0$.

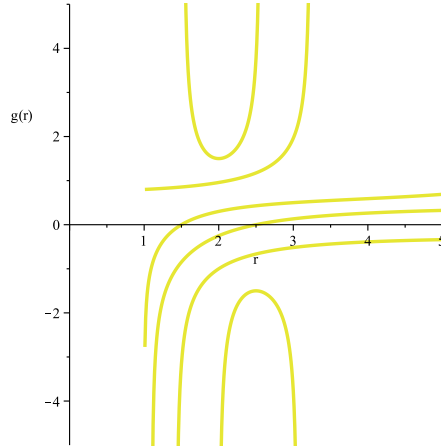
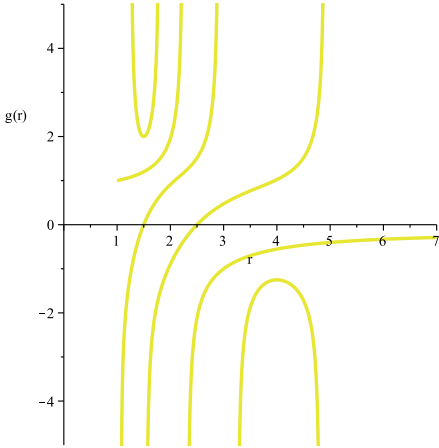


Figure 3.4: Solutions of ODE (3.48) I

Figure 3.5: Solutions of ODE (3.48) II

Non-abelian Seiberg–Witten equations on Bianchi II domain wall

In this Section we shall prove theorem 3.0.1. Consider the Gibbons–Hawking [57] class of hyper-Kähler metrics characterised by the existence of a tri-holomorphic isometry. The metric is given by

$$g_4 = V \left((dx^1)^2 + (dx^2)^2 + (dx^3)^2 \right) + V^{-1} (dx^0 + \alpha)^2. \quad (3.50)$$

The function V and the one-form $\alpha = \alpha_i dx^i$ depend on x^j and satisfy

$$*_3 dV = -d\alpha, \quad (3.51)$$

where $*_3$ is the Hodge operator on \mathbb{R}^3 . Thus the function V is harmonic. Chose the orthonormal frame

$$e^0 = \frac{1}{\sqrt{V}}(dx^0 + \alpha), \quad e^i = \sqrt{V}dx^i, \quad (3.52)$$

and the dual vector fields ∂_0 and ∂_i . In comparison to the Eguchi–Hanson background, for the Gibbons–Hawking case the equation (3.23b) is no longer trivially satisfied. It only holds if $dG \wedge dV = 0$. Thus, in particular $\partial_0 G = 0$. The equations (3.23c) and (3.23d) are equivalent to (3.40). The integrability conditions force $\delta_0 = 0$. Setting $w := \delta_i x^i$, we can determine H from the relation

$$dH = \sqrt{V}e^{2G}dw. \quad (3.53)$$

Thus H and $\sqrt{V}e^{2G}$ are functions of w only. We claim that $\sqrt{V}e^{2G} \neq C$ for any constant¹ C . Therefore $dV \wedge dw = dG \wedge dw = 0$, since $dV \wedge dG = 0$, and we must have $V := V(w)$, $G := G(w)$. Furthermore $V(w)$ is harmonic, so the potential must be linear in w , i.e. without loss of generality

$$V = x^3, \quad \alpha = x^2 dx^1. \quad (3.55)$$

The resulting metric admits a Bianchi II (also called *Nil*) group of isometries generated by the vector fields

$$X_0 := \frac{\partial}{\partial x^0}, \quad X_1 := \frac{\partial}{\partial x^1}, \quad X_2 := \frac{\partial}{\partial x^2} - x^1 \frac{\partial}{\partial x^0} \quad (3.56)$$

with the Heisenberg Lie algebra structure

$$[X_0, X_1] = 0, \quad [X_0, X_2] = 0, \quad [X_2, X_1] = X_0. \quad (3.57)$$

There is also a homothety generated by

$$D := 2x^0 \frac{\partial}{\partial x^0} + x^1 \frac{\partial}{\partial x^1} + x^2 \frac{\partial}{\partial x^2} + x^3 \frac{\partial}{\partial x^3}, \quad (3.58)$$

¹Suppose the opposite. Using $V = C^2 e^{-4G}$ in (3.23a) we find $\partial_{x_i} \partial^{x_i} G + \partial_{x_i} G \partial^{x_i} G = C^2 \delta_i \delta^i$. The Laplace equation on V implies $\partial_{x_i} \partial^{x_i} G = 4 \partial_{x_i} G \partial^{x_i} G$, and

$$\partial_{x_i} \partial^{x_i} G = 4c^2, \quad \partial_{x_i} G \partial^{x_i} G = c^2, \quad \text{where } c^2 := \frac{C^2 \delta_i \delta^i}{5}. \quad (3.54)$$

Differentiation of the first relation reveals that all derivatives of G are harmonic. Two partial differentiations of the second relation and contracting the indices then yields $|\partial_{x_i} \partial_{x_j} G|^2 = 0$. This implies $c = 0$ and thus $\partial_{x_i} G = 0$, which rules out this special case.

such that

$$\mathcal{L}_D g_4 = 3g_4. \quad (3.59)$$

The conformally rescaled metric $\hat{g} = (x^3)^{-3}g_4$ admits D as a proper Killing vector. Using the Bianchi classification [58, 59] of 3-dimensional real Lie algebras we note that the vector fields $\{X_0, X_1, X_2\}$ span the Bianchi II algebra of isometries of \hat{g} and $\{X_0, X_1, D\}$ span the Bianchi V algebra of isometries of \hat{g} . Setting $x^3 := \exp(\rho)$ puts g_4 in the form

$$g_4 = e^{3\rho}(d\rho^2 + e^{-2\rho}((dx^1)^2 + (dx^2)^2) + e^{-4\rho}(dx^0 + x^2 dx^1)^2). \quad (3.60)$$

This metric is singular at $\rho \rightarrow \pm\infty$ but we claim that this singularity is only present in an overall conformal factor, and g_4 is a conformal rescaling of a regular homogeneous metric on a four-dimensional Lie group with the underlying manifold $\mathcal{H} = Nil \times \mathbb{R}^+$ generated by the right-invariant vector fields $\{X_0, X_1, X_2, D\}$. To see it, set

$$\sigma_0 := e^{-2\rho}(dx^0 + x^2 dx^1), \quad \sigma_1 := e^{-\rho} dx^1, \quad \sigma_2 := e^{-\rho} dx^2, \quad \sigma_3 := d\rho. \quad (3.61)$$

Then

$$g_4 = e^{3\rho} \hat{g} \quad \text{where} \quad \hat{g} = \sigma_0^2 + \sigma_1^2 + \sigma_2^2 + \sigma_3^2, \quad (3.62)$$

and the left-invariant one-forms satisfy

$$d\sigma_0 = 2\sigma_0 \wedge \sigma_3 - \sigma_1 \wedge \sigma_2, \quad d\sigma_1 = \sigma_1 \wedge \sigma_3, \quad d\sigma_2 = \sigma_2 \wedge \sigma_3, \quad d\sigma_3 = 0. \quad (3.63)$$

Thus the metric \hat{g} is left-invariant and hence complete [60], i.e. regular. In [50] the singularity of g_4 at $\rho = -\infty$ has been interpreted as a single side domain wall in the space-time

$$M_4 \times \mathbb{R}^{p-3,1} \quad (3.64)$$

with its product metric. This domain wall is a p -brane: either a nine-brane of 11D super gravity if $p = 9$ or a three-brane of the $4 + 1$ dimensional space-time $g_4 - dt^2$. In all cases the direction ρ is transverse to the wall. In the approach of [50] the regions $x^3 > 0$ and $x^3 < 0$ are identified. In this reference it is argued that (M_4, g_4) with such identification is the approximate form of a regular metric constructed in [61] on a complement of a smooth cubic curve in $\mathbb{C}P^2$.

Using this linear potential $V = w = x^3$ in (3.23a) and setting $g(w) := e^{G(w)}$ yields

$$g'' - wg^5 = 0. \quad (3.65)$$

This equation changes its character as w changes from positive to negative sign, we find infinitely many singularities for $G(w)$ for $w < 0$. We thus focus on the region $w > 0$, which is in agreement with the identification of these two regions proposed by [50]. Numerical plots for solutions of this equation corresponding to two one-parameter families of initial conditions are given in Figures 3.6 and 3.7. One explicit solution is given by

$$g(w) = \pm \frac{1}{2} \sqrt[4]{21} w^{-\frac{3}{4}}. \quad (3.66)$$

If we choose $w = x^3$, the curvature for this solution blows up like $(x^3)^{-3}$, this is singular only on the domain wall. Alternatively, the curvature is regular on \mathcal{H} , however the metric g_4 restricted to \mathcal{H} , while conformal to a regular metric \hat{g} , is not complete.

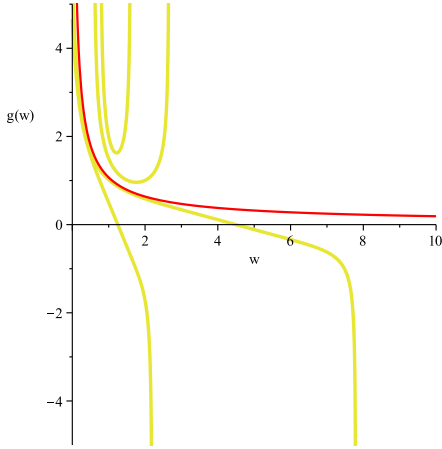


Figure 3.6: Solutions of ODE (3.65) I

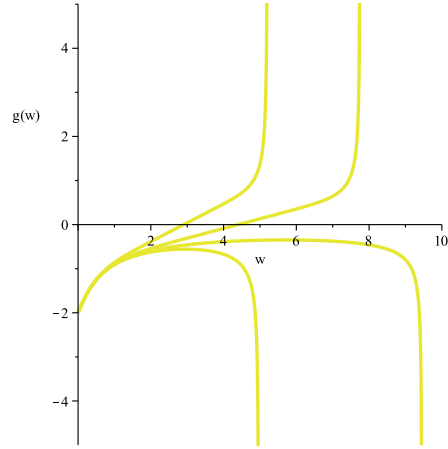


Figure 3.7: Solutions of ODE (3.65) II

Explicitly, the solution (3.66) gives

$$G = -\frac{3}{4}\rho + \frac{1}{4} \ln 21 - \ln 2, \quad H = -\frac{\sqrt{21}}{3}G. \quad (3.67)$$

and

$$\begin{aligned}
 \mathcal{A} &= \frac{3}{4}(\sigma_2 \otimes T_1 - \sigma_1 \otimes T_2 + \sigma_0 \otimes T_3), & \Phi &= -\frac{\sqrt{21}}{4}e^{-\frac{3}{2}\rho}(iT_1 + jT_2 + kT_3), \\
 \mathcal{F} &= \left(\frac{9}{16}\sigma_0 \wedge \sigma_1 + \frac{3}{4}\sigma_2 \wedge \sigma_3\right) \otimes T_1 + \left(\frac{9}{16}\sigma_0 \wedge \sigma_2 - \frac{3}{4}\sigma_1 \wedge \sigma_3\right) \otimes T_2 \\
 &+ \left(\frac{3}{2}\sigma_0 \wedge \sigma_3 - \frac{3}{16}\sigma_1 \wedge \sigma_2\right) \otimes T_3. & & (3.68)
 \end{aligned}$$

Here we exploited the gauge freedom in (3.18) by multiplying Φ with $k \in \Im\mathbf{m}\mathbb{H}$ on the left.

QUATERNION–KÄHLER FOUR–MANIFOLDS

Quaternion–Kähler four–manifolds can be characterised in three rather different ways. Firstly, with motivation coming from higher–dimensional Quaternion–Kähler manifolds which are Riemannian manifolds with holonomy $Sp(n) \cdot Sp(1)$, one can define a four–dimensional Quaternion–Kähler manifold to be anti–self–dual Einstein with non–vanishing cosmological constant [62]. Secondly, as explained in section 2.3, they can be described by means of their twistor space, a three–dimensional complex manifold with a four–parameter family of holomorphic curves and a contact structure [3, 6, 7, 2]. Finally, these manifolds are locally determined by one scalar function, known in the literature as Przanowski’s function, which is subject to a second–order partial differential equation [15].

While this last description is only local in nature, it appears to be very useful in applications, as explicit expressions for the metric in local coordinates are easily obtained. In particular the hyper–multiplet moduli space in string theory is an example of a Quaternion–Kähler four–manifold, and Przanowski’s function has been used in that context [16, 63, 64].

The purpose of this chapter is to introduce Przanowski’s function and the associated partial differential equation as well as its linearisation and clarify their geometric origin in the twistor construction. In particular, we construct a Lax Pair for Przanowski’s Equation and exhibit its linearisation as the generalised Laplacian associated to a natural, conformally invariant differential operator. We relate solutions of the generalised Laplace equation to twistor cohomology

using recursion relations, leading to a contour integral formula for perturbations of Przanowski's function. Eventually, we provide an algorithm that extracts Przanowski's function from twistor data in the double-fibration picture, extending work of [16]. All considerations and computations in this chapter will be local in nature.

4.1 Invariant differential operators

In this section, we will introduce Przanowski's form of a Quaternion-Kähler metric on a four-manifold (M, g) . Locally on M one can always [18] find a complex structure compatible with g with complex coordinates (w, z) and complex conjugates (\bar{w}, \bar{z}) . Of course this need not be true globally, as a counter-example consider S^4 which is anti-self-dual Einstein with the round metric, but does not admit a global complex structure. With respect to such a local complex structure the metric can be written in Hermitian form [15]

$$g = 2 \left(K_{w\bar{w}} dw d\bar{w} + K_{w\bar{z}} dw d\bar{z} + K_{z\bar{w}} dz d\bar{w} + \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right) dz d\bar{z} \right), \quad (4.1)$$

where $\Lambda \neq 0$ is the cosmological constant and $K(w, \bar{w}, z, \bar{z})$ is a real function on M . The metric g is ASD Einstein if and only if K satisfies Przanowski's equation

$$K_{z\bar{w}} K_{w\bar{z}} - K_{w\bar{w}} \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right) + K_w K_{\bar{w}} e^{\Lambda K} = 0. \quad (4.2)$$

as shown in [15]. We will see that equation (4.2) is sufficient at the end of this section, while the necessity will become clear when recovering Przanowski's formulation from the twistor description of a Quaternion-Kähler manifold. Locally we can always find a null tetrad adapted to the complex structure so that $e^{A0'} \in \Lambda^{(1,0)}M$ while $e^{A1'} \in \Lambda^{(0,1)}M$. This reduces the gauge freedom from $SO(4, \mathbb{C})$ to $GL(2, \mathbb{C}) \cong SL(2, \mathbb{C}) \times \mathbb{C}^\times$. Here $SL(2, \mathbb{C})$ acts on \mathbb{S} while \mathbb{C}^\times is a subgroup of $SL(2, \mathbb{C})$ acting on \mathbb{S}' via¹ $e^{A0'} \mapsto e^\Theta e^{A0'}$ and $e^{A1'} \mapsto e^{-\Theta} e^{A1'}$. We can fix the gauge freedom by choosing

$$\begin{aligned} e^{00'} &= dw, & e^{01'} &= -K_{z\bar{w}} d\bar{w} - \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right) d\bar{z}, \\ e^{10'} &= dz, & e^{11'} &= K_{w\bar{w}} d\bar{w} + K_{w\bar{z}} d\bar{z}. \end{aligned} \quad (4.3)$$

¹This corresponds to a transformation $o^{A'} \mapsto e^\Theta o^{A'}$ and $\rho^{A'} \mapsto e^{-\Theta} \rho^{A'}$ in (2.5).

Using the abbreviation

$$\tilde{K} := K_{w\bar{z}}K_{z\bar{w}} - K_{w\bar{w}} \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right), \quad (4.4)$$

we obtain for the self-dual two-forms (2.16) on M

$$\Sigma^{0'0'} = dw \wedge dz, \quad \Sigma^{1'1'} = -\tilde{K} d\bar{w} \wedge d\bar{z}, \quad (4.5)$$

as well as

$$2i\Sigma^{0'1'} = i\partial\bar{\partial}K + \frac{2i}{\Lambda} e^{\Lambda K} dz \wedge d\bar{z}, \quad (4.6)$$

which is Hermitian. Here $d = \partial + \bar{\partial}$ is the splitting of the exterior derivative induced by the complex structure. Again, note the Dolbeault types of these forms: $\Sigma^{0'0'} \in \Lambda^{(2,0)}M$, $\Sigma^{0'1'} \in \Lambda^{(1,1)}M$ and $\Sigma^{1'1'} \in \Lambda^{(0,2)}M$. The Hermitian two-form depends only on the choice of complex structure, while the other self-dual two-forms transform with weight ± 2 under the \mathbb{C}^\times -action, e.g. $\Sigma^{0'0'} \mapsto e^{2\Theta}\Sigma^{0'0'}$. Using the vector fields

$$\begin{aligned} \partial_{00'} &:= \partial_w, & \partial_{11'} &:= \frac{1}{\tilde{K}} \left[- \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right) \partial_{\bar{w}} + K_{z\bar{w}} \partial_{\bar{z}} \right], \\ \partial_{10'} &:= \partial_z, & \partial_{01'} &:= \frac{1}{\tilde{K}} [-K_{w\bar{z}} \partial_{\bar{w}} + K_{w\bar{w}} \partial_{\bar{z}}], \end{aligned} \quad (4.7)$$

which are dual to the null tetrad, we obtain for the primed connection

$$\begin{aligned} \Gamma_{0'0'} &= -\partial_{A0'}(\ln K_{\bar{w}})e^{A1'}, & \Gamma_{1'1'} &= \partial_{A1'}(\ln K_w)e^{A0'}, \\ \Gamma_{0'1'} &= \frac{1}{2} \left[\partial_{A0'}(\ln \tilde{K} - \ln K_{\bar{w}})e^{A0'} + \partial_{A1'}(\ln K_w)e^{A1'} \right]. \end{aligned} \quad (4.8)$$

To simplify these expressions we used Przanowski's equation. Our choice of adapted null tetrad (4.3) leads to a particularly simple form of the primed connection:

$$\begin{aligned} \Gamma_{0'0'} &\in \Lambda^{(0,1)}M, & \Gamma_{1'1'} &\in \Lambda^{(1,0)}M, & d\Gamma_{0'1'} &\in \Lambda^{(1,1)}M, \\ &\text{with } \Gamma_{0'0'} \wedge d\Gamma_{0'0'} &= \Gamma_{1'1'} \wedge d\Gamma_{1'1'} &= 0. \end{aligned} \quad (4.9)$$

At this point we can check directly that the metric (4.1) is ASD and Einstein. To do this, we compute the primed curvature spinor $R_{A'B'}$ and upon substituting (4.2) find

$$R_{A'B'} = \Lambda \Sigma_{A'B'}. \quad (4.10)$$

Thus the self-dual Weyl spinor and the trace-free Ricci spinor vanish as claimed. The converse was already shown by Przanowski almost 30 years ago [15]. We will obtain it from the twistor picture in section 4.3. Using the exterior derivative of (4.10) we define the Lee form B according to

$$d\Sigma^{0'1'} =: B \wedge \Sigma^{0'1'}, \quad (4.11)$$

and a second one-form A by

$$d\Sigma^{0'0'} =: (B - A) \wedge \Sigma^{0'0'}, \quad d\Sigma^{1'1'} =: (B + A) \wedge \Sigma^{1'1'}. \quad (4.12)$$

The Lee form B only depends on the choice of complex structure, while A transforms as $A \mapsto A - 2d\Theta$ under the \mathbb{C}^\times -action. Using these definitions, the one-forms A and B are related to the connection coefficients by

$$A = \Gamma_{A(A'0'1')} e^{AA'}, \quad B = -2\Gamma_{A1'0'0'} e^{A0'} + 2\Gamma_{A0'1'1'} e^{A1'}, \quad (4.13)$$

or in terms of K , they are given by

$$A = \partial \left(\ln \tilde{K} - 2 \ln K_{\bar{w}} \right) + \bar{\partial} (2 \ln K_w), \quad B = \partial (\ln K_w) + \bar{\partial} (\ln K_{\bar{w}}). \quad (4.14)$$

Remark: When $K_w = K_{\bar{w}}$ the Lee form B is exact, so in this case (M, g) is locally conformally Kähler. However, $K_w = K_{\bar{w}}$ implies that (M, g) has an isometry [65, 66]. Hence this is an example of the more general correspondence proved in [67] that an ASD Einstein four-manifold is conformally Kähler if and only if it has an isometry.

More generally, under conformal rescalings where $g \mapsto e^{2\Omega}g$ the Lee form transforms as $B \mapsto B + 2d\Omega$, we also have $A \mapsto A + 2d\Omega$ if we keep $e^{A0'}$ invariant. We can now consider the line bundle

$$L^{l,m} = (\Lambda^{(2,0)}M)^{\frac{l}{2}} \otimes (\Lambda^{(2,2)}M)^{\frac{m}{4}}, \quad (4.15)$$

which is locally well-defined. A local trivialisation of $L^{l,m}$ is determined by the complex structure as well as a choice of conformal scale and the \mathbb{C}^\times -gauge, namely we can choose the volume form as basis of $\Lambda^{(2,2)}M$ and the element $\Sigma^{0'0'}$ as basis of $\Lambda^{(2,0)}M$. A section $f_{l,m}$ of $L^{l,m}$ transforms as $f_{l,m} \mapsto e^{l\Theta+m\Omega}f_{l,m}$ under the \mathbb{C}^\times -action and a change of conformal scale. Defining a differential operator D acting on sections $f_{l,m}$ of $L^{l,m}$ by

$$Df_{l,m} := \left(d + \frac{l}{2}A - \frac{l+m}{2}B \right) f_{l,m}, \quad (4.16)$$

we find that $Df_{l,m}$ is an $L^{l,m}$ -valued one-form and so D is a well-defined first-order differential operator on $L^{l,m}$ that depends only on the conformal class of the ASD Einstein metric and the choice of a compatible complex structure. The origin of D lies in twistorial methods and will be explained in section 4.2.2, however the relevance of D can be understood without making use of twistor theory: Taking care to transform the conformal weight of an $L^{l,m}$ -valued one-form appropriately under the action of the Hodge star operator we note that $*Df_{l,m} \in L^{l,m+2} \otimes \Lambda^3 M$ and we can consider the Laplacian $*D * D$. Evaluating this Laplacian explicitly using the definition (4.14) of B in terms of K , we find that when acting on sections of $L^{0,-1}$ it reproduces the conformal Laplacian

$$*D * D = *d * d - \frac{1}{6}R, \quad (4.17)$$

since first-order derivatives cancel and $*d * B - \frac{1}{2} * (B \wedge *B) = -\frac{1}{3}\Lambda$ using Przanowski's equation (4.2). For sections of $L^{0,1}$ in turn we find

$$*D * D = *d * d + 2 * (*B \wedge d) + \frac{3}{4} * (B \wedge *B) - \frac{1}{2} * d * B, \quad (4.18)$$

which is equivalent to the linearised Przanowski operator defined by

$$\begin{aligned} D_{Prz} := & K_{z\bar{w}}\partial_{w\bar{z}} + K_{w\bar{z}}\partial_{z\bar{w}} - K_{w\bar{w}}\partial_{z\bar{z}} - \left(K_{z\bar{z}} + \frac{2}{\Lambda}e^{\Lambda K} \right) \partial_{w\bar{w}} \\ & + e^{\Lambda K} (K_w\partial_{\bar{w}} + K_{\bar{w}}\partial_w + \Lambda K_w K_{\bar{w}} - 2K_{w\bar{w}}), \end{aligned} \quad (4.19)$$

when expressed in terms of K and the vector fields $\partial_{AA'}$. Solutions $\delta K \in \Gamma(M, L^{0,1})$ to the differential equation $*D * D \delta K = 0$ are infinitesimal perturbations of the Przanowski function K and thus correspond to deformations of the underlying Quaternion-Kähler manifold. Having established the linearised Przanowski operator as a Laplacian acting on sections of a line bundle $L^{0,1}$ over M is the first step towards extracting solutions to the linearised Przanowski equation from cohomology classes on twistor space, which we will achieve in section 4.2.2.

4.2 Twistor theory and Przanowski's function

As was explained in section 2.3, from any four-dimensional Quaternion-Kähler manifold one can construct an associated three-dimensional complex twistor space with a four-parameter family of holomorphic curves called twistor lines

and a contact structure [3, 6, 7, 2, 21, 5]. We will first use the more general correspondence for anti-self-dual manifolds to construct a Lax Pair for Przanowski's equation as well as a recursion relation relating solutions of a generalised Laplace equation to cohomology classes $H^1(T, \mathcal{O}(k))$. Using the recursion relation we provide a contour integral for perturbations δK of Przanowski's function.

We will work with the double-fibration picture, so we need to complexify the underlying manifold M , which we can assume to be real-analytic [5]. We thus promote (w, \bar{w}, z, \bar{z}) to four independent complex variables $(w, \tilde{w}, z, \tilde{z})$ and denote the resulting complex four-manifold by \mathbb{M} . From the complex conjugation of the underlying real manifold M we inherit an anti-holomorphic involution

$$\iota_{\mathbb{M}} : \mathbb{M} \longrightarrow \mathbb{M}, \quad (w, \tilde{w}, z, \tilde{z}) \longmapsto (\bar{\tilde{w}}, \bar{w}, \bar{\tilde{z}}, \bar{z}). \quad (4.20)$$

The fixed points of this map allow us to retrieve M , corresponding to reality conditions $\bar{w} = \tilde{w}$, $\bar{z} = \tilde{z}$. To make notation more convenient we use four independent holomorphic coordinates (w, \bar{w}, z, \bar{z}) on \mathbb{M} , we retrieve M when (\bar{w}, \bar{z}) are complex conjugates of (w, z) . Note that the action of the involution $\iota_{\mathbb{M}}$ can be extended to an involution ι on F . $\iota_{\mathbb{M}}^*$ pulls back $(2, 0)$ -forms to $(0, 2)$ -forms and therefore ι , while leaving the fibres over real points in \mathbb{M} invariant, acts as the antipodal map on such a fibre.

4.2.1 Lax Pair

While the integrability of the twistor distribution $\langle d_0, d_1 \rangle$ is equivalent to the anti-self-duality of (\mathbb{M}, g) , the fact that K satisfies Przanowski's equation (4.2) is sufficient but not strictly necessary for this. To obtain a Lax Pair consider the modified vector fields

$$\tilde{\partial}_{A0'} := \partial_{A0'}, \quad \tilde{\partial}_{01'} := -\frac{\tilde{K}}{e^{\Lambda K} K_w K_{\bar{w}}} \partial_{01'}, \quad \tilde{\partial}_{11'} = -\frac{\tilde{K}}{e^{\Lambda K} K_w K_{\bar{w}}} \partial_{11'}, \quad (4.21)$$

which reduce to (4.7) if and only if (4.2) is satisfied. Similarly

$$\tilde{d}_A := \xi^{A'} \partial_{AA'} - \xi^{A'} \left(\xi^{B'} \tilde{\partial}_{AB'} \lrcorner \Gamma^{C'}_{A'} \right) \partial_{\xi^{C'}}, \quad (4.22)$$

reduces to d_A as defined in (2.25) if and only if (4.2) holds. We now introduce a trivialisation of \mathfrak{F} over F based on the standard trivialisation of the tautological line bundle over $\mathbb{C}P^1$: Consider

$$U_0 := \left\{ x \in \mathfrak{F} \mid \xi^{0'} \neq 0 \right\}, \quad U_1 := \left\{ x \in \mathfrak{F} \mid \xi^{1'} \neq 0 \right\}. \quad (4.23)$$

On U_0 , define $\xi = \frac{\xi^{1'}}{\xi^{0'}}$ and on U_1 , define² $\eta = \frac{\xi^{0'}}{\xi^{1'}}$. Holomorphic functions on U_0 homogeneous of degree zero, which can be regarded as functions on F holomorphic in ξ , are annihilated by the Euler vector field Υ and thus $\partial_{\xi^{A'}}$ acts as $\frac{\xi^{A'}}{(\xi^{0'})^2} \partial_{\xi}$. So the projection of \tilde{d}_A to TF is

$$\tilde{l}_A = \xi^{A'} \partial_{AA'} + (\xi^{0'})^{-2} \xi^{A'} \xi^{B'} \xi^{C'} \left(\tilde{\partial}_{AA'} \lrcorner \Gamma_{B'C'} \right) \partial_{\xi}, \quad (4.24)$$

or in terms of K we find in inhomogeneous form the vector fields

$$\begin{aligned} l_0 &= \partial_w - \xi \frac{K_{w\bar{z}}}{\tilde{K}} \partial_{\bar{w}} + \xi \frac{K_{w\bar{w}}}{\tilde{K}} \partial_z + \left(\frac{\tilde{K}_w + e^{\Lambda K} K_w K_{w\bar{w}}}{\tilde{K}} - \frac{K_{w\bar{w}}}{K_{\bar{w}}} \right) \xi \partial_{\xi}, \\ l_1 &= \partial_z - \frac{\xi}{\tilde{K}} \left(K_{z\bar{z}} + \frac{2}{\Lambda} e^{\Lambda K} \right) \partial_{\bar{w}} + \xi \frac{K_{z\bar{w}}}{\tilde{K}} \partial_z + \\ &\quad + \left(\frac{\tilde{K}_z + e^{\Lambda K} K_w K_{z\bar{w}} - e^{\Lambda K} K_{\bar{w}} \xi}{\tilde{K}} - \frac{K_{z\bar{w}}}{K_{\bar{w}}} + \frac{\xi}{K_w} \right) \xi \partial_{\xi}. \end{aligned} \quad (4.25)$$

Note that ∂_w and ∂_z commute and hence l_0 and l_1 form an integrable distribution if and only if they commute. Przanowski's equation (4.2) is a sufficient and necessary condition for $[l_0, l_1] = 0$. To see this, write the various components of the commutator as

$$[l_0, l_1] = A \partial_{01'} + B \partial_{11'} + (C_1 \xi + C_2 \xi^2 + C_3 \xi^3) \partial_{\xi}, \quad (4.26)$$

we find

$$\begin{aligned} A &= \frac{\xi K_w K_{z\bar{w}} - \xi^2 K_{\bar{w}}}{\tilde{K} K_w K_{\bar{w}}} \cdot Prz_K, & B &= -\frac{\xi K_{w\bar{w}}}{\tilde{K} K_w} \cdot Prz_K, \\ C_1 &= -\frac{1}{K_{\bar{w}}} (K_{w\bar{w}} \partial_z - K_{z\bar{w}} \partial_w) \left(\frac{Prz_K}{\tilde{K}} \right), & C_3 &= -\frac{1}{K_w} \partial_{01'} \left(\frac{Prz_K}{\tilde{K}} \right), \\ C_2 &= \left(\partial_{00'} \partial_{11'} - \partial_{10'} \partial_{01'} + \frac{e^{\Lambda K} K_{\bar{w}}}{\tilde{K}} \partial_w + \frac{e^{\Lambda K} K_w}{\tilde{K}} \partial_{\bar{w}} + \frac{2e^{\Lambda K} K_{w\bar{w}}}{\tilde{K}} \right) \left(\frac{Prz_K}{e^{\Lambda K} K_w K_{\bar{w}}} \right), \end{aligned} \quad (4.27)$$

where

$$Prz_K := \tilde{K} + K_w K_{\bar{w}} e^{\Lambda K} \quad (4.28)$$

and (4.2) is equivalent to

$$Prz_K = 0. \quad (4.29)$$

²Whenever we use this trivialisation of \mathfrak{F} over F , we will work in the patch U_0 . The formulae valid over U_1 can be easily inferred.

First consider the coefficient A , it vanishes only if Przanowski's equation (4.2) holds. Conversely, if (4.2) is satisfied all coefficients vanish, so (4.25) are a Lax Pair for Przanowski's equation, showing that it is another example of an integrable equation coming from anti-self-duality equations in four dimensions. While we used twistorial methods to derive this Lax Pair, the advantage of having a Lax Pair for (4.2) is that it makes many of the usual properties of integrable systems manifest without resorting to the twistor construction explicitly.

4.2.2 Recursion relations

We now want to explain how one can construct solutions to generalised Laplace equations from elements of $H^1(\mathfrak{T}, \mathcal{O}(k))$, or conversely construct those cohomology classes from solutions of the generalised Laplace equation using a recursion relation. So suppose we have constructed the twistor space \mathfrak{T} from \mathfrak{F} by taking the quotient along the twistor distribution. Furthermore we have an involution ι , a holomorphic contact structure τ and the volume form ρ on non-projective twistor space as defined above theorem 2.3.3. Starting with $\Psi \in H^1(\mathfrak{T}, \mathcal{O}(k))$, we can pull Ψ back to \mathfrak{F} to obtain $\Psi \in H^1(\mathfrak{F}, \mathcal{O}(k))$ satisfying

$$d_A \Psi = 0, \tag{4.30}$$

where d_A span the twistor distribution in \mathfrak{F} and were defined in (2.25). Since Ψ is an element of the first cohomology group, its domain can be assumed to be $U_0 \cap U_1$ where $\xi^{0'} \neq 0$, so we can trivialise and write $\Psi = \sum_n (\xi^{0'})^{(k-n)} (\xi^{1'})^n \psi_n$ where ψ_n are functions on M . Recall that the \mathbb{C}^\times -action scales the null tetrad according to $e^{A0'} \mapsto e^\Theta e^{A0'}$ and $e^{A1'} \mapsto e^{-\Theta} e^{A1'}$, hence we need to scale the basis of the twistor lines by $\xi^{0'} \mapsto e^\Theta \xi^{0'}$ and $\xi^{1'} \mapsto e^{-\Theta} \xi^{1'}$ since the vector fields (2.25) spanning the twistor distribution have to be invariant under any change of frame. Note that this leaves the volume form ρ invariant. Similarly under a conformal transformation $g \mapsto e^{2\Omega} g$ we scale the null tetrad by $e^{A0'} \mapsto e^{A0'}$ and $e^{A1'} \mapsto e^{2\Omega} e^{A1'}$, since this keeps the trivialisation of $\Lambda^{(2,0)} M$ invariant. Keeping (2.25) invariant requires $\xi^{0'} \mapsto \xi^{0'}$ and $\xi^{1'} \mapsto e^{2\Omega} \xi^{1'}$, which would also scale the volume form by $\rho \mapsto e^{6\Omega} \rho$. However we still have the freedom to change the normalisation of $\xi^{A'}$, we choose to keep ρ invariant and hence compensate the transformation of ρ by scaling the entire basis of the twistor lines simultaneously by a factor of $e^{-\frac{3}{2}\Omega}$ and hence obtain $\xi^{0'} \mapsto e^{-\frac{3}{2}\Omega} \xi^{0'}$ and $\xi^{1'} \mapsto e^{\frac{1}{2}\Omega} \xi^{1'}$. Consequently the functions ψ_n transform as $\psi_n \mapsto e^{(2n-k)\Theta + (\frac{3}{2}k - 2n)\Omega} \psi_n$ under a change of \mathbb{C}^\times -gauge and conformal scale.

Hence we see that ψ_n are sections of the line bundle $L^{(2n-k, \frac{3}{2}k-2n)}$ defined in (4.15). Now we look at the action of d_A when acting on Ψ , expanding in powers of $\xi^{A'}$ and using (4.13) we have

$$d_A \Psi = \sum_{n=-\infty}^{\infty} (\xi^{0'})^{(k-n)} (\xi^{1'})^n \xi^{A'} \left(\partial_{AA'} + \left(n - \frac{k}{2} \right) A_{AA'} - \frac{k}{4} B_{AA'} \right) \psi_n, \quad (4.31)$$

where we recognise the bracket on the right side of the equation³ as the linear first-order operator⁴ $D^{(n,k)}$ acting on $L^{(2n-k, \frac{3}{2}k-2n)}$ defined in (4.16). From (4.30) we obtain the recursion relations

$$D_{A0'}^{(n+1,k)} \psi_{n+1} = -D_{A1'}^{(n,k)} \psi_n. \quad (4.32)$$

For (4.32) to be consistent, ψ_n has to satisfy an integrability condition. Since (2.27) implies

$$\epsilon^{AB} \xi^{A'} \xi^{B'} \left[D_{AA'}^{(n,k)} D_{BB'}^{(n,k)} - \left(\Gamma_{BA'B'}^{C'} + \Gamma_{CB'B}^C \epsilon_{A'}^{C'} \right) D_{AC'}^{(n,k)} \right] = 0, \quad (4.33)$$

cross-differentiating (4.32) and imposing (4.33) requires

$$\left(D_{AA'}^{(n,k)} + \Gamma_{AB'A'}^{B'} + \Gamma_{BA'A}^B + B_{AA'} \right) D^{(n,k) AA'} \psi_n = 0. \quad (4.34)$$

Rewriting this expression covariantly, we find

$$*D * D \psi_n = 0. \quad (4.35)$$

So starting with ψ_n satisfying this integrability condition we can use the recursion relations (4.32) to determine ψ_{n+1} , where $*D * D \psi_{n+1} = 0$ is automatically guaranteed, again using (4.33). This allows us to use the recursion relations to define ψ_{n+2} , and so forth. Thus a single coefficient ψ_n satisfying (4.35) is sufficient to determine $\Psi \in H^1(T, \mathcal{O}(k))$. Conversely, given such a Ψ , each coefficient will satisfy a second-order integrability condition.

We will now show that one can use the correspondence above to construct coordinates on T from solutions to (4.35). Consider solutions of (4.35) with weight $(0, 0)$, one can check that holomorphic functions $f(w, z)$ on M as well as anti-holomorphic functions $f(\bar{w}, \bar{z})$ on M are examples. If we choose ψ_0 to be an anti-holomorphic function on M , then the recursion relations (4.32) imply that $\psi_n = 0$ for all negative coefficients and so Ψ will in fact be an element of $H^0(U_0, \mathcal{O}(0))$

³The one-forms A and B were defined in (4.11) and (4.12).

⁴To avoid confusion we denote the indices (n, k) of D explicitly.

that descends to T . Therefore we can recursively construct coordinates of T on the image of U_0 under the canonical projection to T by setting ψ_0 equal to \bar{w} , \bar{z} or a constant. Similarly, on the image of U_1 we can start with ψ_0 equal to w , z or a constant.

4.2.3 Perturbations

The twistor space with its twistor lines only encodes the conformal structure of \mathbb{M} , the information necessary to retrieve an Einstein metric within the conformal class is contained in the contact structure on T . Essentially all that is needed to fix the metric within the conformal class is a scale, which is specified uniquely by the symplectic structures ϵ_{AB} and $\epsilon_{A'B'}$ of \mathbb{S} and \mathbb{S}' . Since the basis $\xi^{A'}$ of \mathbb{S}' is normalised such that $\epsilon_{0'1'}\xi^{0'}\xi^{1'} = 1$ and similarly ϵ_{AB} is contained within the definition of $\Gamma_{A'B'}$, all this information is stored in the one-form τ on \mathfrak{F} , given by (2.24). It corresponds to a one-form τ_F quadratic in ξ on F , where

$$\tau_F = d\xi - \Gamma_{0'0'} - 2\xi\Gamma_{0'1'} - \xi^2\Gamma_{1'1'}. \quad (4.36)$$

As explained in section 2.3, the Lie derivative of τ along the twistor distribution vanishes and hence τ descends to a holomorphic one-form homogeneous of degree two on \mathfrak{F} . In inhomogeneous form, this yields an $\mathcal{O}(2)$ -valued contact form τ_T on the twistor space T . According to Darboux' theorem, one can always choose canonical coordinates on T such that $\tau_T = dx - ydt$. Comparing with the illustration (2.28), τ gives rise to a contact structure on the projective and non-projective twistor and correspondence space. However, the pull-back of τ_T from T to F is proportional but not in general equal to τ_F . This proportionality factor will prove important when retrieving Przanowski's function from the twistor picture.

Now recall that for $(l, m) = (0, 1)$ or equivalently $(n, k) = (1, 2)$ equation (4.35) is the linearised Przanowski equation. Thus by definition the coefficient ψ_1 of every element $\Psi \in H^1(T, \mathcal{O}(2))$ is a solution $\delta K \in L^{0,1}$ of (4.35). Indeed perturbations of Quaternion-Kähler metrics are known to be generated by elements of $H^1(T, \mathcal{O}(2))$ [35]. To see this, regard a representative Ψ of this cohomology class as a Hamiltonian of a one-parameter family of symplectic transformations. So $d\Psi = d\tau_T(\theta, \cdot)$ where $\theta \in H^1(T, \Theta(0))$ is an element of the first cohomology group with values in the sheaf of holomorphic vector fields. Therefore θ encodes

a deformation of the holomorphic symplectic structure of \mathfrak{T} and consequently a deformation of the metric of M . For details on complex deformations see [41]. We can obtain δK from $Psi \in H^1(F, \mathcal{O}(2))$, Cauchy's integral formula yields

$$\delta K = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\psi_1}{\xi} d\xi = \frac{1}{2\pi i} \oint_{\Gamma} \frac{\Psi \xi_{A'} d\xi^{A'}}{(\xi^{A'} o_{A'})^2 (\xi^{A'} \rho_{A'})^2}. \quad (4.37)$$

The constant spinors $o_{A'} = (1, 0)$ and $\rho_{A'} = (0, 1)$ are determined by the choice of complex structure on M and Γ is any contour around the equator of $\mathbb{C}P^1$. This is similar to [68] but for non-zero cosmological constant, and extends results of [69, 16] to Quaternion-Kähler four-manifolds with no isometries.

4.3 Przanowski's function from Twistor data

We now want to explain how to derive the existence of Przanowski's function as well as the second-order partial differential equation (4.2) it satisfies from the description of a Quaternion-Kähler four-manifold by its twistor space. From this, we will obtain an algorithm to extract Przanowski's function and a compatible complex structure from twistor data.

A similar procedure has been established in [16] in the single-fibration picture: While in this thesis we employ the twistor correspondence in the double-fibration picture (2.28) introduced by Penrose [2, 20], there is an equivalent version established by Atiyah, Hitchin and Singer [5] called the single-fibration picture. The starting point of the double-fibration picture are complex anti-self-dual four-manifolds \mathbb{M} , since real anti-self-dual four-manifolds are always real-analytic this captures the general case. The correspondence space F is then a five-dimensional complex manifold, taking the quotient by the two-dimensional twistor distribution we obtain a three-dimensional complex twistor space T . The single-fibration picture instead starts with real anti-self-dual four-manifolds M . Their correspondence space Z is a six-dimensional real manifold with a two-dimensional real integrable twistor distribution. Rather than take a quotient one extends the twistor distribution to a three-dimensional integrable distribution which one defines to be $T^{(0,1)}M$. Atiyah, Hitchin and Singer show that this defines an integrable complex structure on Z , which is hence a three-dimensional complex manifold. In this context Z is usually referred to as twistor space of M , and we have a fibration of Z over M explaining the term single-fibration picture. The advantage of the single-fibration picture is that it extends easily to Quaternion-Kähler

manifolds of higher dimensions, while the double-fibration picture is restricted to four dimensions. Furthermore, Swann [70] showed that Z admits a Kähler Einstein metric if M is ASD Einstein. The Kähler potential of this metric has been identified in [16] as the origin of Przanowski's function: Given the twistor data in the single-fibration picture of an ASD Einstein four-manifold, one can read off Przanowski's function from the Kähler potential subject to the choice of complex structure.

In the double-fibration picture a similar algorithm achieves the same, however we don't require any information about a metric or Kähler structure on twistor space. We thus assume we have at our disposal the complete twistor data that describes a real ASD Einstein manifold M : a three-dimensional complex twistor space T with a four-parameter family \mathbb{M} of twistor lines, furthermore a holomorphic contact structure on T determined by a one-form τ_T homogeneous of degree two such that $\tau_T(Q) \neq 0$ for any non-zero vector Q tangent to one of the twistor lines. And finally let $R = 12\Lambda$ be the scalar curvature and cosmological constant of the Quaternion-Kähler manifold respectively. The real structure of the underlying manifold M is encoded in an involution ι on T . The steps of the algorithm to extract K from twistor data are as follows:

- Find canonical coordinates (x, y, t) for the contact form $\tau_T = dx - ydt$ on twistor space T and pull them back⁵ to correspondence space F .
- M inherits a complex structure from the complex surface $S = \{p \in F \mid t = 0\}$ with holomorphic coordinates $\bar{w} = y|_S$, $\bar{z} = x|_S$ and complex conjugates $w = \iota(y)|_{\iota(S)}$, $z = \iota(x)|_{\iota(S)}$, where ι is the anti-holomorphic involution on twistor space preserving the real lines. The asymmetry of w and z in (4.2) is reflected in the fact that $\tau_F|_S = d\bar{z}$.
- Extend these coordinates to a coordinates system $(w, \bar{w}, z, \bar{z}, \xi)$ on correspondence space such that $\xi|_S = 0$ and $(\xi)^{-1}|_{\iota(S)} = 0$. In a neighbourhood of S , the restriction of the contact form to the twistor lines \hat{m} for $m \in M$ is of the form $\tau_F|_{\hat{m}} = e^\Phi d\xi$, defining a 'contact potential' Φ on correspondence space. Similarly we obtain around $\iota(S)$ the contact potential $\tilde{\Phi}$ and find the Przanowski function to be $K = -\frac{1}{\Lambda} \left(\Phi|_S + \tilde{\Phi}|_{\iota(S)} \right)$.

⁵We denote the pull-backs by (x, y, t) and τ_T as well.

Not all choices in this procedure are unique, the resulting freedom will be seen to be a gauge freedom of the Przanowski gauge.

Remark 1: In this procedure the coordinates (w, z, \bar{w}, \bar{z}) are determined only up to holomorphic coordinate transformations given by

$$w \rightarrow w'(w, z), \quad z \rightarrow z'(z), \quad \bar{w} \rightarrow \bar{w}'(\bar{w}, \bar{z}), \quad \bar{z} \rightarrow \bar{z}'(\bar{z}). \quad (4.38)$$

Under such a change of coordinates, K will transform as

$$K(w, z, \bar{w}, \bar{z}) \rightarrow K(w', z', \bar{w}', \bar{z}') - \frac{1}{\Lambda} \ln(\partial_z z') - \frac{1}{\Lambda} \ln(\partial_{\bar{z}} \bar{z}'). \quad (4.39)$$

One can check that this gives rise to the same metric.

Remark 2: The Przanowski function K determines $d\Gamma_{0'1'}$, a non-degenerate closed two-form. This symplectic form, which is neither compatible with the metric nor covariant, admits both $(w, z, -K_w, -K_z)$ as well as $(\bar{w}, \bar{z}, K_{\bar{w}}, K_{\bar{z}})$ as canonical coordinates,

$$d\Gamma_{0'1'} = dK_{\bar{w}} \wedge d\bar{w} + dK_{\bar{z}} \wedge d\bar{z} = -dK_w \wedge dw - dK_z \wedge dz. \quad (4.40)$$

Thus $K(w, z, \bar{w}, \bar{z})$ can be regarded as the generating function for the symplectic transformation that maps 'initial positions' (w, z) to 'final positions' (\bar{w}, \bar{z}) . This is a remnant of the interpretation of the heavenly function as a transition function on Hyper-Kähler manifolds [13].

We now give the details of the construction. Suppose that x, y, t are local holomorphic coordinates on T and $\tau_T = dx - ydt$. To obtain a local complex structure, choose a complex surface S'_1 in the twistor space T transversal to the twistor lines. A Quaternion-Kähler manifold does not in general carry a global complex structure, hence this may not be possible for all lines, the complex structure is not defined for points in M whose twistor lines are tangent to S'_1 , we may wish to exclude these points from M . For instance we can choose $S'_1 = \{p \in T \mid t = 0\}$. The pre-image of this surface in the correspondence space F is a four-dimensional holomorphic surface S_1 which is also a section s of the $\mathbb{C}P^1$ -bundle over the base manifold \mathbb{M} . We can use x, y as coordinates on S'_1 , pulled back to F the one-forms dx and dy annihilate the twistor distribution. We define $\bar{z} = x|_{S_1}$, $\bar{w} = y|_{S_1}$ and

$$\Lambda^{(0,1)}M := \langle d\bar{w}, d\bar{z} \rangle. \quad (4.41)$$

From $S'_2 = \iota(S'_1)$ with pre-image S_2 in F we obtain two more coordinates $z = \iota(x)|_{S_2}$ and $w = \iota(y)|_{S_2}$. By construction these will be complex conjugates of (\bar{w}, \bar{z}) on the underlying real manifold M . We use them to define $\Lambda^{(1,0)}M := \langle dw, dz \rangle$. As any two non-intersecting totally null planes through a point span the entire tangent space, the functions (w, z, \bar{w}, \bar{z}) will be independent on M . Locally this defines an integrable complex structure compatible with the metric [18].

Remark: Infinitely many other choices for S'_1 are possible, each corresponding to a different local complex structure compatible with the metric. Since a contact structure on a three-dimensional manifold has no integral sub-manifolds of dimension higher than one, the one-form τ_T is non-zero when restricted to any two-dimensional surface S'_1 . Darboux' theorem ensures that we can always choose coordinates (w, z, \bar{w}, \bar{z}) on M such that $\tau_T = f'_1 d\bar{z}$ on S_1 and $\tau_T = f'_2 dz$ on S_2 for some functions f'_1 and f'_2 on M . This step is well-defined up to transformations of the form (4.38) and is the origin of the asymmetry between w and z in the Przanowski equation.

By construction the metric g of M is Hermitian with respect to the coordinates (w, z, \bar{w}, \bar{z}) , choosing an adapted tetrad with $e^{A0'} \in \Lambda^{(1,0)}M$ and $e^{A1'} \in \Lambda^{(0,1)}M$ reduces the gauge freedom to $GL(2, \mathbb{C})$. In the trivialisation (4.23) the pre-images of the hypersurfaces S'_1 and S'_2 are then given by $S_1 = \{p \in F \mid \xi = 0\}$ and $S_2 = \{p \in F \mid \eta = 0\}$. For convenience we choose $e^{00'} = dw$, $e^{10'} = dz$ to fix the frame uniquely, to find K we need the explicit expressions of $e^{A1'}$. Using the primed connection one-forms they can be obtained from $\Sigma^{0'1'}$, which is proportional to the Hermitian two-form compatible with the metric g and the complex structure.

To see how Przanowski's function arises, we show that the primed connection one-forms are of the form

$$\Gamma_{0'0'} = f_1 d\bar{z}, \quad \Gamma_{1'1'} = f_2 dz, \quad d\Gamma_{0'1'} = \frac{\Lambda}{2} \bar{\partial} \partial K, \quad (4.42)$$

for some complex-valued functions f_1, f_2 and K on \mathbb{M} with $K = \frac{1}{\Lambda} \ln f_1 f_2$. Under the induced real structure on M , K is a real function and we will identify it with Przanowski's function. It may be instructive to compare (4.42) with (4.8). We start by classifying $\Gamma_{A'B'}$ according to their Dolbeault-type. By definition,

$$de^{AA'} = (\Gamma^{AA'}_{CC'})_{BB'} e^{BB'} \wedge e^{CC'}. \quad (4.43)$$

Now integrability of the complex structure means that

$$\Gamma^{A1'}_{[00'10']} = 0, \quad \Gamma^{A0'}_{[01'11']} = 0. \quad (4.44)$$

Therefore, using anti-symmetry in the Lie algebra indices and considering separately the cases $A = 0$ and $A = 1$,

$$\Gamma_{0'0'} \in \Lambda^{(0,1)}M, \quad \Gamma_{1'1'} \in \Lambda^{(1,0)}M. \quad (4.45)$$

Now recall the components of the ASD Einstein equation,

$$\begin{aligned} d\Gamma_{0'0'} + 2\Gamma_{0'0'} \wedge \Gamma_{0'1'} &= \Lambda\Sigma_{0'0'}, \\ d\Gamma_{0'1'} + \Gamma_{0'0'} \wedge \Gamma_{1'1'} &= \Lambda\Sigma_{0'1'}, \\ d\Gamma_{1'1'} + 2\Gamma_{0'1'} \wedge \Gamma_{1'1'} &= \Lambda\Sigma_{1'1'}. \end{aligned} \quad (4.46)$$

Denoting the component of $\Gamma_{0'1'}$ in $\Lambda^{(a,b)}M$ by $\Gamma_{0'1'}^{(a,b)}$, the first of equations (4.46) splits up into

$$d\Gamma_{0'0'}^{(0,2)} + 2\Gamma_{0'0'}^{(0,1)} \wedge \Gamma_{0'1'}^{(0,1)} = \Lambda\Sigma_{0'0'}, \quad d\Gamma_{0'0'}^{(1,1)} + 2\Gamma_{0'0'}^{(0,1)} \wedge \Gamma_{0'1'}^{(1,0)} = 0. \quad (4.47)$$

Note that $\Gamma_{0'0'} = \Gamma_{0'0'}^{(0,1)}$, hence $d\Gamma_{0'0'}^{(0,2)} \wedge \Gamma_{0'0'} = d\Gamma_{0'0'}^{(1,1)} \wedge \Gamma_{0'0'} = 0$, and therefore

$$d\Gamma_{0'0'} \wedge \Gamma_{0'0'} = 0. \quad (4.48)$$

In fact all identities in (4.9) follow from this analysis. Now recall that on S'_1 we have $\tau_T = d\bar{z}$ and on S'_2 we find $\tau_T = dz$. Similarly $\tau_F = -\Gamma_{0'0'}$ on S_1 and $\tau_F = -\Gamma_{1'1'}$ on S_2 . But the contact structure on T is induced from the one on F , so the pull-back of τ_T to F is proportional to τ_F , consequently

$$\Gamma_{0'0'} = f_1 d\bar{z}, \quad \Gamma_{1'1'} = f_2 dz, \quad (4.49)$$

for some complex-valued functions f_1 and f_2 . Furthermore, since $d\Gamma_{0'1'}$ is a closed (1,1)-form, it can be written as

$$d\Gamma_{0'1'} = \frac{\Lambda}{2} \bar{\partial}\partial K \quad (4.50)$$

for some complex-valued function K . So far we have established (4.42), it remains to show that K is indeed the Przanowski function and real.

From (4.50) K is determined only up to the addition of two functions $c(w, z)$ and

$\tilde{c}(\bar{w}, \bar{z})$. Using equations (4.46) it is easy to show that one can choose c and \tilde{c} such that

$$\Gamma_{0'0'} = e^{\Lambda K} K_w d\bar{z}, \quad \Gamma_{1'1'} = \frac{1}{K_w} dz, \quad \Gamma_{0'1'} = \frac{1}{2} (d(\ln(K_w)) + \Lambda \partial K), \quad (4.51)$$

together with the self-dual two-forms $\Sigma^{A'B'}$ as in (4.5) and (4.6). Then $\Sigma^{0'0'} \wedge \Sigma^{1'1'} = -2\Sigma^{0'1'} \wedge \Sigma^{0'1'}$, which follows from (2.17), is equivalent to Przanowski's equation (4.2). We saw earlier that $2i\Sigma^{0'1'}$ is the Hermitian two-form with respect to the complex structure and metric g on M , so g must be given by (4.1). Thus K in (4.50) is indeed Przanowski's function and real. To determine K explicitly, observe that (4.51) implies

$$K = \frac{1}{\Lambda} \ln f_1 f_2. \quad (4.52)$$

Evaluating the restriction of τ to the sections $(\xi^{0'}, \xi^{1'}) = (1, 0)$ and $(0, 1)$ of \mathfrak{F} provides f_1 and f_2 from (4.49) and thus yields Przanowski's function using (4.52). This however requires knowledge of the symplectic structure on \mathfrak{F} , to obtain Przanowski's function from the associated contact structure on T note that the pull-back of τ_T to F is a scalar multiple of τ_F as they are contact forms of the same contact structure. This scalar function, which depends on the choice of complex structure and associated coordinates on M , is referred to as the 'contact potential' in [16]. Evaluating the contact potential along S_1 and S_2 yields f_1 and f_2 . Hence $f_1(m)$ and $f_2(m)$ for $m \in M$ can be obtained from the restriction of τ_T to the twistor line \hat{m} , since on the intersection of \hat{m} and S_1 this restriction of τ_T satisfies

$$\tau_T \Big|_{\hat{m}} = f_1(m) d\xi, \quad (4.53)$$

while in the intersection of \hat{m} with S_2 we have

$$\tau_T \Big|_{\hat{m}} = f_2(m) d\eta. \quad (4.54)$$

Both f_1 and f_2 are functions on M and yield Przanowski's function using (4.52).

4.4 Examples

We will demonstrate the procedure of writing a Quaternion-Kähler metric in Przanowski's form explicitly for a few examples: S^4 , H^4 , $\mathbb{C}P^2$ with the Fubini-

Study metric and $\widetilde{\mathbb{C}P^2}$ with the Bergmann metric. The first two cases are conformally flat with negative and positive scalar curvature respectively⁶ and are treated in [16]. The other two examples instead are non-trivial, the Fubini–Study metric has negative scalar curvature and the Bergmann metric positive scalar curvature. The twistor data for the Fubini–Study metric is given in [5, 3] and can be easily adapted to accommodate for the Bergmann metric⁷.

4.4.1 S^4 and H^4

S^4 and H^4 are conformally flat, the only difference in their twistor data arises in the contact structure. However, it is convenient to use slightly different parametrisations of the twistor lines. Defining ϵ to be the sign of the cosmological constant, $\Lambda = \epsilon|\Lambda|$, we can treat both cases simultaneously by including ϵ as a parameter. We will initially normalise Λ to ± 1 and return to the general case at the end. To obtain S^4 , set $\epsilon = -1$, to obtain H^4 , set $\epsilon = 1$. The twistor space is $\mathbb{C}P^3$ for S^4 and an open subset thereof for H^4 . Parametrising $\mathbb{C}P^3$ by homogeneous coordinates (u_0, u_1, v_0, v_1) , the twistor lines are given by [16]

$$\begin{aligned} u_0 &= \frac{\xi^{0'}}{\sqrt{1 - \epsilon|w|^2(1 + |z|^2)}}, & v_0 &= \frac{w\xi^{0'} + \bar{w}\bar{z}\xi^{1'}}{\sqrt{1 - \epsilon|w|^2(1 + |z|^2)}}, \\ u_1 &= \frac{\xi^{1'}}{\sqrt{1 - \epsilon|w|^2(1 + |z|^2)}}, & v_1 &= \frac{wz\xi^{0'} - \bar{w}\xi^{1'}}{\sqrt{1 - \epsilon|w|^2(1 + |z|^2)}}. \end{aligned} \quad (4.55)$$

The incidence relations are obtained from this by eliminating $(\xi^{0'}, \xi^{1'})$,

$$v_0 = wu_0 + \bar{w}\bar{z}u_1, \quad v_1 = wz u_0 - \bar{w}u_1. \quad (4.56)$$

Here (w, z, \bar{w}, \bar{z}) are coordinates on \mathbb{M} , the four-parameter family of twistor lines, and $(\xi^{0'}, \xi^{1'})$ are homogeneous coordinates along such a line. The twistor lines are invariant under the involution ι if (\bar{w}, \bar{z}) are complex conjugates of (w, z) . We specify a contact structure by

$$\tau = \varepsilon^{AB} (u_A du_B + \epsilon v_A dv_B). \quad (4.57)$$

The parametrisation of the twistor lines is chosen so that when restricted to a line the contact form is $\tau \Big|_{\hat{m}} = \varepsilon_{A'B'} \xi^{A'} d\xi^{B'}$ so $(\xi^{0'}, \xi^{1'})$ is a normalised basis of \mathbb{S}' .

⁶According to our conventions S^4 has negative scalar curvature, following [3, 21].

⁷See [16] for a description of the latter twistor space with Przanowski’s function in a different gauge.

On $U_0 = \{u \in \mathbb{C}P^3 \mid u_0 \neq 0\}$ we can introduce inhomogeneous coordinates

$$\left(\frac{u_1}{u_0}, \frac{v_0}{u_0}, \frac{v_1}{u_0} \right) := (\xi, w + \bar{w}\bar{z}\xi, wz - \bar{w}\xi), \quad (4.58)$$

and choose a holomorphic surface in T by setting $S_1 = \{p \in T \mid \xi = 0\}$. On $U_1 = \{u \in \mathbb{C}P^3 \mid u_1 \neq 0\}$ we use coordinates

$$\left(\frac{u_0}{u_1}, \frac{v_0}{u_1}, \frac{v_1}{u_1} \right) := (\eta, \bar{w}\bar{z} + \eta w, -\bar{w} + \eta wz), \quad (4.59)$$

and find $S_2 = \iota(S_1) = \{p \in T \mid \eta = 0\}$. This yields a complex structure on M with holomorphic coordinates w, z induced from

$$w := \frac{v_0}{u_0} \Big|_{S_1}, \quad wz := \frac{v_1}{u_0} \Big|_{S_1}. \quad (4.60)$$

As complex conjugates we obtain \bar{w}, \bar{z} from S_2 . Now observe that

$$\tau \Big|_{S_1} = \frac{\epsilon w^2 dz}{1 - \epsilon |w|^2 (1 + |z|^2)}, \quad \tau \Big|_{S_2} = \frac{\epsilon \bar{w}^2 d\bar{z}}{1 - \epsilon |w|^2 (1 + |z|^2)}, \quad (4.61)$$

so we can use (4.52) to find Przanowski's function,

$$K = \frac{2}{\Lambda} \ln \left[\frac{|w|^2}{1 - \epsilon |w|^2 (1 + |z|^2)} \right], \quad (4.62)$$

where we have now included the cosmological constant as a free parameter.

4.4.2 $\mathbb{C}P^2$ and $\widetilde{\mathbb{C}P^2}$

As a non-trivial example we now consider $\mathbb{C}P^2 = SU(3)/U(2)$ with the Fubini-Study metric, which has negative scalar curvature, and its non-compact version $\widetilde{\mathbb{C}P^2} = SU(2,1)/U(2)$ with the Bergmann metric, which has positive scalar curvature. Recall that $\mathbb{C}P^2$ is the space of lines through the origin in \mathbb{C}^3 , the Fubini-Study metric is induced from a Hermitian form with signature $(+++)$. In contrast, for $\widetilde{\mathbb{C}P^2}$ consider \mathbb{C}^3 equipped with a Hermitian form with signature $(++-)$. Then $\widetilde{\mathbb{C}P^2}$ is the space of time-like lines and the Hermitian form induces the Bergmann metric. Although not conformally equivalent, we can again treat both cases simultaneously by introducing a parameter ϵ where $\Lambda = \epsilon|\Lambda|$, alternatively ϵ is the negative of the third eigenvalue of the Hermitian form. We

initially assume $\Lambda = \pm 1$. The twistor space T is the flag manifold F_{12} of \mathbb{C}^3 , so every point of T consists of a pair (l, p) where p is a plane in \mathbb{C}^3 and l is a line in p , both containing the origin. For $\widetilde{\mathbb{C}P^2}$ we furthermore require that l be space-like and that p contain a time-like direction. Using homogeneous coordinates, we can write any point in T as a pair (l^j, p_j) where $j = 0, 1, 2$ and $p_j l^j = 0$.

Next we need the twistor lines, these are of the following form [5]: let P be a plane in \mathbb{C}^3 and L a line in \mathbb{C}^3 *not* in P . For $\widetilde{\mathbb{C}P^2}$ we need L to be time-like while P must be spanned by two space-like vectors. Then a twistor line in T is given by all pairs (l, p) where p contains L and where the two planes p and P intersect in l . According to this description the equation for the twistor line is $P_j l^j = p_j l^j = p_j L^j = 0$ using homogeneous coordinates for P and L . If we write $P_j = (W, Z, 1)$ and $L^j = (\tilde{W}, \tilde{Z}, 1)$ we can use $(W, Z, \tilde{W}, \tilde{Z})$ as coordinates⁸ on \mathbb{M} . One can check [3] that

$$\begin{aligned} l^j &= \left(-(1 + Z\tilde{Z})\xi^{1'}, \frac{\xi^{0'}}{1 + Z\tilde{Z}} + W\tilde{Z}\xi^{1'}, \frac{-Z\xi^{0'}}{1 + Z\tilde{Z}} + W\xi^{1'} \right), \\ p_j &= \left(\frac{\xi^{0'}}{1 + W\tilde{W} + Z\tilde{Z}}, \frac{-\tilde{W}Z\xi^{0'}}{(1 + Z\tilde{Z})(1 + W\tilde{W} + Z\tilde{Z})} + \xi^{1'}, \right. \\ &\quad \left. \frac{-\tilde{W}\xi^{0'}}{(1 + Z\tilde{Z})(1 + W\tilde{W} + Z\tilde{Z})} - \tilde{Z}\xi^{1'} \right), \end{aligned} \quad (4.63)$$

satisfy the defining equations of a twistor line. To fix a metric within the conformal structure, we chose a contact form

$$\tau = \frac{1}{2} (p_j dl^j - l^j dp_j). \quad (4.64)$$

The parametrisation (4.63) of the twistor lines has been chosen to ensure that the restriction of the contact form to the twistor lines is in canonical form,

$$\tau_F \Big|_{\hat{m}} = \xi^{0'} d\xi^{1'} - \xi^{1'} d\xi^{0'}. \quad (4.65)$$

A further difference between the Fubini–Study metric and the Bergmann metric on the level of their twistor description arises when we describe the involution on T . This involution ι is induced from the Hermitian form on \mathbb{C}^3 which defines an anti-linear map from \mathbb{C}^3 to the dual space, and thus an anti-holomorphic map from T to itself. Under this map ι a pair (l, p) is mapped to (\bar{p}, \bar{l}) , pairs invariant

⁸These serve as coordinates on all of $\widetilde{\mathbb{C}P^2}$, but only on a coordinate patch of $\mathbb{C}P^2$.

under this map correspond to real twistor lines. Applied to a twistor line (L^j, P_j) we obtain the reality conditions

$$\bar{W} = -\epsilon\tilde{W}, \quad \bar{Z} = -\epsilon\tilde{Z}. \quad (4.66)$$

For the Bergmann metric, the condition that L be time-like and P space-like together with the reality conditions implies $W\bar{W} + Z\bar{Z} < 1$. This gives a complete description of the two metrics in terms of twistor data. We can now use this information to deduce a complex structure and Przanowski's function in both cases.

We set $l^0 = 0$ to select a holomorphic surface in T , from (4.63) we see that this amounts to choosing the complex structure induced from $S_1 = \{(l, p) \in T \mid (\xi^{0'}, \xi^{1'}) = (1, 0)\}$. The twistor lines restricted to S_1 are

$$l^j = (0, 1, -Z), \quad p_j = \left(1 + Z\tilde{Z}, -\tilde{W}Z, -\tilde{W}\right), \quad (4.67)$$

so we can choose holomorphic coordinates $z := Z$ and $w := \frac{(1+Z\tilde{Z})}{W\tilde{W}}W$. Note that the contact form restricted to S_1 is indeed proportional to dz :

$$\tau \Big|_{S_1} = \frac{\tilde{W}}{\left(1 + W\tilde{W} + Z\tilde{Z}\right) \left(1 + Z\tilde{Z}\right)^2} dZ. \quad (4.68)$$

The parametrisation (4.63) is chosen to ensure that $\iota(S_1) = S_2$ where $S_2 = \{(l, p) \in T \mid (\xi^{0'}, \xi^{1'}) = (0, 1)\}$ for both reality conditions. On S_2 we have

$$l^j = \left(-1 + Z\tilde{Z}, W\tilde{Z}, W\right), \quad p_j = \left(0, 1, -\tilde{Z}\right), \quad (4.69)$$

so we can choose anti-holomorphic coordinates $\bar{z} := -\epsilon\tilde{Z}$ and $\bar{w} := -\epsilon\frac{(1+Z\tilde{Z})}{W\tilde{W}}\tilde{W}$. Again

$$\tau \Big|_{S_2} = Wd\tilde{Z} \quad (4.70)$$

as required. To retrieve the Przanowski function we need only use (4.52),

$$K = \frac{1}{\Lambda} \ln \left[\frac{W\tilde{W}}{\left(1 + W\tilde{W} + Z\tilde{Z}\right) \left(1 + Z\tilde{Z}\right)^2} \right], \quad (4.71)$$

which is valid for arbitrary cosmological constant. In terms of the coordinates (w, z, \bar{w}, \bar{z}) and taking account of reality conditions we have

$$K = -\frac{1}{\Lambda} \ln [(1 - \epsilon w\bar{w} - \epsilon z\bar{z})(z\bar{z} - \epsilon)]. \quad (4.72)$$

ASD EINSTEIN METRICS WITH SYMMETRY

Having dealt with the general case of ASD Einstein metrics in chapter 4, we now specialise to ASD Einstein metrics with Killing vectors. Let us first consider Euclidean signature: Generic ASD Einstein metrics in Euclidean signature are generated by solutions of Przanowski's equation, as we have seen in the previous chapter. Przanowski [66] and Tod [65, 17] considered symmetry reductions of this system and showed that all ASD Einstein metrics with at least one Killing vector can be derived from solutions of the $SU(\infty)$ Toda field equation. Conversely every solution of the $SU(\infty)$ Toda field equation leads to an ASD Einstein manifold with a Killing vector. The underlying reason for the appearance of Toda's equation in this setting is the fact that ASD Einstein metrics with a Killing vector in Euclidean signature are always conformal to scalar-flat Kähler metrics, and these are known to be generated by the $SU(\infty)$ Toda equation [71]. The crucial difference between Toda's and Przanowski's equation lies in the number of independent variables: four for Przanowski's equation, but only three for Toda's equation. This dimensional reduction is possible because of the additional symmetry in the system.

We now turn to neutral signature, as this is the only other signature where the concept of anti-self-duality is non-trivial. A couple of new features arise in neutral signature: besides pseudo-Kähler metrics, which are the equivalent of Kähler metrics in Euclidean signature, we can introduce para-Kähler and null-Kähler metrics. Whereas a pseudo-Kähler structure is a triple (g, J, Σ) of a neutral-

signature metric g Hermitian with respect to a parallel, complex structure J such that the associated self-dual fundamental form Σ is closed and hence Kähler, we define a para-Kähler structure to be a triple (g, S, Σ) of a neutral-signature metric g anti-Hermitian with respect to a parallel involutive structure S such that the associated self-dual fundamental form Σ is closed. A null-Kähler structure in turn is a triple (g, N, Σ) consisting of a neutral-signature metric g compatible with a parallel, nilpotent structure N in the sense that

$$g(N(X), Y) + g(X, N(Y)) = 0, \quad (5.1)$$

for two vector fields X, Y on M . Furthermore we require the associated self-dual two-form $\Sigma = g(N(\cdot), \cdot)$ to be closed. By construction $\Sigma \wedge \Sigma = 0$ and hence the name null-Kähler. Examples of pseudo-Kähler, para-Kähler and null-Kähler structures will be presented in section 5.1.2.

The aim of this chapter is to understand better the relationship between ASD Einstein four-metrics with a non-null symmetry in neutral signature and pseudo-Kähler, para-Kähler and null-Kähler metrics. One of our main results is the general form of an ASD Einstein metric that is conformally equivalent to a real-analytic null-Kähler metric with a Killing vector. Such metrics are type N gravitational waves in a conformally flat background parametrised by one free scalar function of one variable. We then turn to pseudo- and para-Kähler metrics and find that away from singular points all real-analytic ASD Einstein metrics with a non-null Killing vector are generated by solutions of the $SU(\infty)$ Toda equation. This is the neutral signature analogon of the results in [65, 17]. Finally, we classify ASD conformal structures with symmetry that contain both a real-analytic null-Kähler as well as a scalar-flat pseudo- or para-Kähler metric.

5.1 Einstein–Weyl structures

Throughout this last chapter of the thesis we consider ASD Einstein metrics g with a symmetry, where by a symmetry we mean the existence of a conformal Killing vector K such that

$$\mathcal{L}_K g = cg, \quad (5.2)$$

where c is some function. For constant c we call K a homothety, if $c = 0$ we have a (pure) Killing vector K . Throughout this first section 5.1 we will drop the

Einstein condition, so it suffices to consider ASD conformal structures $[g]$ since anti-self-duality is a conformally invariant concept. A conformal structure $[g]$ has a symmetry K if one and hence every $\hat{g} \in [g]$ satisfies (5.2). The conformal Killing vector K can be non-null or null with respect to $[g]$, we will restrict attention to non-null symmetries, null symmetries are related to projective structures and have been discussed extensively in [72]. The study of ASD conformal structures with a non-null symmetry can be dimensionally reduced to the study of three-dimensional Einstein–Weyl (EW) spaces. Guided by [40], we will now introduce EW spaces and continue with the Jones–Tod correspondence [73] relating EW spaces to ASD conformal structures.

Before continuing to do so, we should mention that there is also a twistor correspondence for EW spaces leading to mini-twistor spaces. Their geometry was first explored by Hitchin [34]. The relation between the twistor correspondence for four-dimensional ASD manifolds and the mini-twistor correspondence for EW spaces is presented in great detail in [73]. In fact, just as the EW space arises as the symmetry reduction of a four-dimensional ASD manifold, the mini-twistor space arises by a similar reduction out of the twistor space of that four-manifold. However, as mentioned earlier, there are some subtleties when working with twistor theory in neutral signature. For the rest of this chapter we will only make use of the Jones–Tod correspondence without resorting to twistor theory and so will not go into any further details.

5.1.1 Einstein–Weyl geometry

Let \mathcal{W} be a three-dimensional manifold with a conformal structure $[h]$ of signature $(2, 1)$ and a torsion-free connection D that preserves $[h]$, i.e.

$$Dh = \omega \otimes h, \tag{5.3}$$

for some $h \in [h]$ and some 1-form ω , which depends on the choice of $h \in [h]$. Then we call the triple $(\mathcal{W}, [h], D)$ a Weyl space, for convenience denoted by (h, ω) . Condition (5.3) is weaker than compatibility of D with h , the Levi–Civita connection of h will satisfy (5.3), but so will any connection D under which null geodesics of $[h]$ remain geodesic. If ω is exact, then D is the Levi–Civita connection of some $h \in [h]$.

Both the curvature W^i_{jkl} of D and its Ricci tensor W_{ij} , which is not necessarily symmetric, depend only on the connection D . However, to compute a scalar

curvature $W = W_{ij}h^{ij}$, we need to choose a scale within the conformal structure $[h]$. Under conformal transformations $h \rightarrow e^{2\Omega}h$ we have $W \rightarrow e^{-2\Omega}W$ and thus W is of conformal weight -2 . This allows us to write down a conformally invariant version of the Einstein equations for D and $[h]$,

$$W_{(ij)} - \frac{1}{3}Wh_{ij} = 0. \quad (5.4)$$

Note that the left-hand side is indeed conformally invariant, as the weights of W and h_{ij} cancel. We call a Weyl space satisfying (5.4) an Einstein–Weyl (EW) structure. The Jones–Tod construction relates Einstein–Weyl structures to ASD conformal structures with a symmetry.

Theorem 5.1.1 [73, 72] *Let $(M, [g])$ be a neutral signature ASD four–manifold with a non–null conformal Killing vector K . An Einstein–Weyl structure on the space \mathcal{W} of trajectories of K is defined by*

$$h := |K|^{-2}g - |K|^{-4}\mathbf{K} \odot \mathbf{K}, \quad \omega := 2|K|^{-2} *_g(\mathbf{K} \wedge d\mathbf{K}), \quad (5.5)$$

where $|K|^2 := g(K, K)$, $\mathbf{K} := g(K, \cdot)$ and $*_g$ is the Hodge star operator with respect to g . All EW structures arise in this way. Conversely, there is a pair (V, α) consisting of a function V of weight -1 and a one–form α on W satisfying the generalised monopole equation

$$*_h \left(dV + \frac{1}{2}\omega V \right) = d\alpha, \quad (5.6)$$

such that $[g]$ with conformal Killing vector $K = \partial_z$ is determined by

$$g = Vh - \frac{1}{V}(dz + \alpha)^2. \quad (5.7)$$

Note that the Jones–Tod construction is conformally invariant, although we used explicit representatives of the conformal structures $[g]$ and $[h]$ in the theorem. Also, as a consequence of (5.6), the function V has to satisfy the integrability condition $d *_h (d + \frac{1}{2}\omega) V = 0$.

5.1.2 Examples

In this section we want to present three examples of EW structures and describe the corresponding ASD conformal structures. We start with a neutral signature version of LeBrun’s [71] scalar–flat Kähler metrics.

Scalar–flat pseudo–Kähler metrics

The first example are scalar–flat pseudo–Kähler metrics arising from the $SU(\infty)$ Toda equation. One can check that

$$h := e^U (dx^2 + dy^2) - dt^2, \quad \omega := 2U_t dt, \quad (5.8)$$

is an EW structure if and only if the function $U(x, y, t)$ satisfies the $SU(\infty)$ Toda equation¹,

$$(e^U)_{tt} - U_{xx} - U_{yy} = 0. \quad (5.9)$$

By theorem 5.1.1 the conformal class determined by

$$g = V (e^U (dx^2 + dy^2) - dt^2) - \frac{1}{V} (dz + \alpha)^2 \quad (5.10)$$

is ASD and $\mathcal{L}_K g = 0$ where $K = \partial_z$ if V and α obey the generalised monopole equation (5.6). In particular the integrability condition implies that V is a solution of the linearised $SU(\infty)$ Toda equation,

$$(Ve^U)_{tt} - V_{xx} - V_{yy} = 0. \quad (5.11)$$

Furthermore, and this goes beyond theorem 5.1.1, the particular element g of the conformal class $[g]$ given by (5.10) is scalar–flat and pseudo–Kähler. The vanishing of the scalar curvature is easy to confirm by direct computation, to see the latter we define an almost complex structure J on M by

$$dt \mapsto V^{-1} (dz + \alpha), \quad dx \mapsto dy, \quad (5.12)$$

such that $J^2 = -1$. Note that g is Hermitian with respect to this complex structure, $g(JX, JY) = g(X, Y)$. Integrability of J follows from the generalised monopole equation (5.6). The pseudo–Kähler form Σ associated to J and g is

$$\Sigma := (dz + \alpha) \wedge dt + Ve^U dx \wedge dy, \quad (5.13)$$

which again is closed by virtue of (5.6). LeBrun [71] also shows the converse², every scalar–flat pseudo–Kähler metric with a Killing vector is ASD and locally of the form (5.10) subject to the $SU(\infty)$ Toda equation, its linearisation and the generalised monopole equation.

¹This equation will appear with different signs, we always refer to it as $SU(\infty)$ Toda equation.

²LeBrun’s theorem deals with Euclidean signature, but carries over to neutral signature [72].

Scalar–flat para–Kähler metrics

In neutral signature there is some freedom in choosing space–like and time–like directions, a slight modification of (5.8) yields the Weyl space given by

$$h := e^U (dx^2 - dy^2) + dt^2, \quad \omega := 2U_t dt, \quad (5.14)$$

which is an EW structure if and only if

$$-(e^U)_{tt} - U_{xx} + U_{yy} = 0. \quad (5.15)$$

Again, the conformal class determined by this EW structure is ASD and has a Killing vector ∂_z if V and α satisfy (5.6), but the integrability condition of (5.6) is now

$$-(Ve^U)_{tt} - V_{xx} + V_{yy} = 0. \quad (5.16)$$

We define an involution S by (5.12) with the difference that $S^2 = 1$, the eigenvalues of S are ± 1 and the corresponding two–dimensional eigenspaces form an integrable distribution by virtue of the generalised monopole equation (5.6). The specific element of $[g]$ given by

$$g = V (e^U (dx^2 - dy^2) + dt^2) - \frac{1}{V} (dz + \alpha)^2, \quad (5.17)$$

is anti–Hermitian with respect to the involution, $g(SX, SY) = -g(X, Y)$, and scalar–flat. The associated para–Kähler form $\Sigma = g(S(\cdot), \cdot)$ is given by

$$\Sigma := -(dz + \alpha) \wedge dt + Ve^U dx \wedge dy, \quad (5.18)$$

and closed by virtue of (5.6). Again, following LeBrun’s proof with $S^2 = 1$, we find that every scalar–flat para–Kähler metric with a Killing vector is ASD and locally of the form (5.17). So we see that in neutral signature the $SU(\infty)$ Toda equation gives rise to scalar–flat pseudo– and para–Kähler metrics, we will refer to the EW structures (5.8) and (5.14) as EW structures in Toda–form.

Anti–self–dual null–Kähler metrics

A further example of an EW structure comes from the dispersionless Kadomtsev–Petviashvili (dKP) equation [40], indeed the Ansatz

$$h := dy^2 - 4dt(dx + Udt), \quad \omega := -4U_x dt \quad (5.19)$$

satisfies equation (5.4) if

$$(U_t - UU_x)_x - U_{yy} = 0, \quad (5.20)$$

which is the dKP equation for U . It has been shown in [74] that an EW structure is locally of dKP-form (5.19) whenever there is a covariantly constant vector l of weight $-\frac{1}{2}$,

$$Dl + \frac{1}{4}\omega \otimes l = 0. \quad (5.21)$$

Here D is the connection of the EW space.

Remark 1: [74] There is some freedom in the choice of coordinates, the coordinate transformation

$$(\tilde{x}, \tilde{y}, \tilde{t}) := (x + f'y + 2ff' + k, y + 2f, t), \quad (5.22)$$

where $f := f(t)$, $k := k(t)$ are arbitrary functions and $f' := f_t$, leaves (5.19) invariant if U transforms according to

$$\tilde{U}(\tilde{x}, \tilde{y}, \tilde{t}) := U(\tilde{x} - f'\tilde{y} - k, \tilde{y} - 2f, \tilde{t}) - \tilde{y}f'' - f'^2 - k'. \quad (5.23)$$

The new dKP function $\tilde{U}(\tilde{x}, \tilde{y}, \tilde{t})$ then satisfies the dKP equation in the new coordinates.

Remark 2: [74] We can also map $t \mapsto \hat{t}$, where $\hat{t} := c(t)$ is an arbitrary function with first derivative c' etc. The transformation

$$(\hat{x}, \hat{y}, \hat{t}) := \left(c'^{\frac{1}{3}}x + \frac{c''}{6c'^{\frac{2}{3}}}y^2, c'^{\frac{2}{3}}y, c(t) \right), \quad (5.24)$$

and a redefinition of U along

$$\hat{U}(\hat{x}, \hat{y}, \hat{t}) := c'^{-\frac{2}{3}}U\left(c'^{-\frac{1}{3}}\hat{x} - \frac{c''\hat{y}^2}{6c'^{\frac{2}{3}}}, c'^{-\frac{2}{3}}\hat{y}, t\right) - \frac{\hat{x}c''}{3c'^2} + \frac{\hat{y}^2}{18c'^3}\left(\frac{5c''}{c'} - 3c'''\right) \quad (5.25)$$

can be compensated by a change of the conformal scale of h . If we define $\hat{h} := e^{2\Omega}h$ where $\Omega := \frac{2}{3}\ln c'$ and change ω accordingly, then \hat{h} is of the form (5.19) with hats over all coordinates and the dKP function \hat{U} , which accordingly satisfies the dKP equation (5.20) in hatted coordinates.

Now we use the correspondence between EW structures and ASD conformal structures in four dimensions: If V is a solution of the linearised dKP equation,

$$V_{xt} - (UV)_{xx} - V_{yy} = 0, \quad (5.26)$$

and α is determined by (5.6), then

$$g = V(dy^2 - 4dxdt - 4Udt^2) - \frac{1}{V}(dz + \alpha)^2 \quad (5.27)$$

represents an ASD conformal structure with a Killing vector $K = \partial_z$. Again, we can say more about g than what the Jones–Tod construction tells us about the conformal class $[g]$. Namely, g is an example of an ASD null–Kähler metric, which we define now.

Definition 5.1.2 [40] *A null–Kähler structure on a four–manifold consists of a metric g of neutral signature and a real spinor field $\iota \in \Gamma(S')$ parallel with respect to the Levi–Civita connection. A null–Kähler metric is ASD if the self–dual part of the Weyl tensor vanishes.*

Null–Kähler metrics are always scalar–flat [40]. The relation to the equivalent but less technical definition given at the start of this chapter is the following: The isomorphism $\Lambda_+^2 M \cong S' \odot S'$ between the bundle of self–dual two–forms and the symmetric tensor product of S' with itself implies that the real self–dual two–form $\Sigma := \iota_{A'}\iota_{B'}\Sigma^{A'B'}$ is covariantly constant and null, i.e. $\Sigma \wedge \Sigma = 0$. Associated to the null–Kähler metric is an endomorphism N using the relation (5.1) between the metric and the null–Kähler form. By construction N is parallel and nilpotent [72].

Similar to LeBrun’s characterisation of scalar–flat Kähler metrics we have the following theorem by Dunajski, which provides a very useful explicit form of real–analytic ASD null–Kähler metrics:

Theorem 5.1.3 [40] *Let $H := H(x, y, t)$ and $W := W(x, y, t)$ be smooth, real–valued functions on an open set $\mathcal{W} \subset \mathbb{R}^3$ which satisfy*

$$H_{yy} - H_{xt} + H_x H_{xx} = 0, \quad (5.28)$$

$$W_{yy} - W_{xt} + (H_x W_{xx})_x = 0. \quad (5.29)$$

Then

$$g = W_x(dy^2 - 4dxdt - 4H_x dt^2) - W_x^{-1}(dz - W_x dy - 2W_y dt)^2 \quad (5.30)$$

is an ASD null-Kähler metric on a circle bundle $M \rightarrow \mathcal{W}$. All real-analytic ASD null-Kähler metrics with a Killing vector preserving the parallel spinor arise from this construction.

Defining $U := H_x$ and $V := W_x$ and differentiating (5.28) and (5.29) with respect to x yields the dKP equation and its linearisation. The advantage of using the potential forms (5.28), (5.29) is that one can solve the monopole equation and determine α in (5.27) explicitly so that g in (5.30) depends on two functions H and W only. If we pick a null tetrad for g by defining

$$\begin{aligned} e^{00'} &= \frac{dz - 2W_y dt}{2W_x}, & e^{01'} &= dx + H_x dt, \\ e^{10'} &= 2W_x dt, & e^{11'} &= -dz + 2W_x dy + 2W_y dt, \end{aligned} \tag{5.31}$$

then the two-forms $\Sigma^{0'0'}$ and $\Sigma^{0'1'}$ are closed. Here $\Sigma^{0'0'} = dz \wedge dt$ is the null-Kähler form associated to the parallel spinor $\iota_{A'} := (1, 0)$, note that $\mathcal{L}_K \Sigma^{0'0'} = 0$ and hence $\mathcal{L}_K \iota = 0$ as claimed. Despite the closure of $\Sigma^{0'1'}$ the metric g is not necessarily pseudo-Kähler: While the ideal spanned by $e^{A0'}$ is closed, the ideal spanned by $e^{A1'}$ is not unless g is pseudo hyper-Kähler [40]. Therefore in general the almost-complex structure associated with g and $\Sigma^{0'1'}$ is not integrable.

5.2 Anti-self-dual Einstein and null-Kähler metrics

We now return to ASD Einstein metrics with a non-null symmetry, thus we consider a particular representative g of an ASD conformal class, where g satisfies the Einstein equation and has a conformal non-null Killing vector. As mentioned above, in Euclidean signature there is a well-known result by Tod [65] which establishes that every solution of the $SU(\infty)$ Toda field equation with a suitable potential and conformal factor gives rise to an ASD Einstein manifold with a Killing vector. In this section we ask a similar question in neutral signature with regard to the dKP equation: Are there ASD Einstein metrics with a non-null symmetry that project to an EW structure in dKP-form (5.19)? Equivalently, starting with a metric g as in (5.30), is there a conformal factor Ω and a monopole $V = W_x$ such that the metric $\hat{g} = e^{2\Omega}g$ is Einstein with non-zero cosmological constant? The answer to this question will be very useful in section 5.3, where we extend Tod's result to neutral signature. It is given by the following

Theorem 5.2.1 *Let g be a real-analytic ASD null-Kähler metric with a non-null Killing vector K and a parallel spinor ι whose Lie derivative $\mathcal{L}_K \iota$ vanishes. If g admits an Einstein metric \hat{g} within its conformal class then \hat{g} is a type N gravitational wave embedded in a conformally flat background, it is given by*

$$\hat{g} = \frac{4}{(w - \Lambda t)^2} \left[dydz - dt dw + \left(f(w) - \frac{1}{2} \dot{f}(w) (w - \Lambda t) \right) dz^2 \right], \quad (5.32)$$

where $f(w) \neq 0$ is an arbitrary function with derivative $\dot{f}(w) := f_w(w)$ and the Killing vector is $K = \partial_z$.

Note that in the limiting case $f = 0$ the Killing vector K becomes null. The only non-trivial parts of the curvature of the metric (5.32) are the cosmological constant Λ and the anti-self-dual Weyl spinor. Indeed, in the frame

$$e^{00'} = \frac{dz}{w - \Lambda t}, \quad e^{10'} = \frac{2dt}{w - \Lambda t}, \quad e^{01'} = \frac{dw}{w - \Lambda t}, \quad e^{11'} = \frac{2dy - 2F(w, t)dz}{w - \Lambda t}, \quad (5.33)$$

where

$$F(w, t) := \frac{1}{2} \dot{f}(w) (w - \Lambda t) - f(w), \quad (5.34)$$

we find that the only non-vanishing component of the Weyl spinor is

$$W_{0000} = (w - \Lambda t)^3 \ddot{f}. \quad (5.35)$$

So we obtain a conformally flat metric for $f(w)$ quadratic in w . The metric (5.32) is of type N according to the Penrose-Petrov classification [20]. Direct inspection of the metric shows it consists of a conformally flat background metric³ of neutral signature and an additional term proportional to dz^2 , representing a gravitational wave propagating with the speed of light along the null wave vector ∂_y . Hence the solution is a neutral signature version of the Kerr-Schild spacetimes explored in [75], where the authors discuss gravitational waves in de-Sitter background. The rest of this section is devoted to proving this theorem. To do so, we first establish two lemmas.

Lemma 5.2.2 *Let g be a real-analytic ASD null-Kähler metric with a non-null Killing vector K such that $\mathcal{L}_K \iota = 0$. If g admits an Einstein metric \hat{g} within its conformal class, then g has a second Killing vector K_0 which is null.*

³The neutral signature analogons of S^4 and H^4 coincide up to an overall sign of the metric.

Proof: Suppose g is real-analytic ASD null-Kähler and consider a conformal rescaling $\hat{g} := e^{2\Omega}g$, we choose to scale the symplectic structures on \mathbb{S} and \mathbb{S}' by

$$\hat{\epsilon}_{AB} := e^{\Omega}\epsilon_{AB}, \quad \hat{\epsilon}_{A'B'} := e^{\Omega}\epsilon_{A'B'}. \quad (5.36)$$

Let $\iota \in \mathbb{S}'$ be the parallel spinor associated to the ASD null-Kähler metric g , it transforms with weight 1 under changes of the conformal scale, $\hat{\iota}_{A'} := e^{\Omega}\iota_{A'}$. This is accompanied by a change of the Levi-Civita connection given by [20, 6]

$$\hat{\nabla}_{AA'}\chi_{B'} = \nabla_{AA'}\chi_{B'} - \Upsilon_{AB'}\chi_{A'}, \quad (5.37)$$

for any primed spinor $\chi_{B'}$, where $\Upsilon_{AA'} := \nabla_{AA'}\Omega$. By definition $\nabla\iota = 0$, this is no longer true for the conformally rescaled metric, instead we have

$$\hat{\nabla}_{AA'}\hat{\iota}_{B'} = \hat{\alpha}_A\hat{\epsilon}_{A'B'}, \quad (5.38)$$

for $\hat{\alpha}_A := \iota^{A'}\Upsilon_{AA'}$. Note that neither $\hat{\alpha}_{A'}$ nor $\Upsilon_{AA'}$ are conformally invariant, we denote $\hat{\alpha}$ with a hat to indicate that we use $\hat{\epsilon}^{AB}$ to raise the index. Taking second derivatives and using the Ricci spinor identities [20]

$$\square_{AB\iota_{B'}} = \varrho_{ABA'B'}\iota^{A'}, \quad (5.39)$$

$$\square_{A'B'\iota_{C'}} = \Psi_{A'B'C'D'}\iota^{D'} - \frac{R}{12}\iota_{(A'}\epsilon_{B')C'}, \quad (5.40)$$

where

$$\square_{AB} := \nabla_{A'(A}\nabla_{B)}^{A'}, \quad \square_{A'B'} := \nabla_{A(A'}\nabla^{A}{}_{B')}, \quad (5.41)$$

we find

$$\hat{\nabla}_{AA'}\hat{\alpha}_B = -\hat{\varrho}_{ABA'B'}\hat{\iota}^{B'} - \frac{\hat{R}}{24}\hat{\iota}_{A'}\hat{\epsilon}_{AB}. \quad (5.42)$$

So if $\hat{\varrho}_{ABA'B'}\hat{\iota}^{B'} = 0$, which is weaker than the Einstein condition, then

$$\hat{\nabla}_{AA'}(\hat{\alpha}_A\hat{\iota}_{A'}) = \hat{\alpha}_A\hat{\alpha}_B\hat{\epsilon}_{A'B'} - \frac{\hat{R}}{24}\hat{\iota}_{A'}\hat{\iota}_{B'}\hat{\epsilon}_{AB} \quad (5.43)$$

is antisymmetric, so $K_0 := \hat{\alpha}^A\hat{\iota}^{A'}\hat{\partial}_{AA'}$ is a null Killing vector of \hat{g} . Now let $e^{AA'}$ be the null tetrad of g defined in (5.31) with dual vector fields $\partial_{AA'}$, then $\hat{e}^{AA'} = e^{\Omega}e^{AA'}$ is a null tetrad of \hat{g} with dual vector fields $\hat{\partial}_{AA'}$. Now we can determine the null Killing vector,

$$K_0 = e^{-\Omega}\partial^A{}_{1'}\Omega\partial_{A1'} = \frac{e^{-\Omega}}{2W_x}(\Omega_y\partial_x - \Omega_x\partial_y). \quad (5.44)$$

Since K_0 annihilates Ω , it is also a null Killing vector of the null-Kähler metric g . \square

While the Lie-derivatives of both g and \hat{g} with respect to K_0 vanish, the vector K is a pure Killing vector for the real-analytic null-Kähler metric g but in general only a conformal Killing vector of the Einstein metric \hat{g} . If K happens to be a pure Killing vector of \hat{g} , then the null Killing vector K_0 descends to the associated EW structure:

Lemma 5.2.3 *Let g be a real-analytic ASD null-Kähler metric with a non-null Killing vector K and a parallel spinor ι such that $\mathcal{L}_K \iota = 0$. If g admits an Einstein metric \hat{g} with the same Killing vector K within its conformal class, then g and \hat{g} have a null Killing vector K_0 that descends to the EW structure (h, ω) . Two possibilities arise:*

- the Killing vector K_0 is space-like with respect to h , then $K_0 = \partial_y$ and

$$\hat{g} = \frac{1}{(W_0(s) + W_1(r))^2} \left[V (dy^2 - 4(1 - rF(s)) drds) - \frac{1}{V} (dz - V dy)^2 \right], \quad (5.45)$$

with $V := \frac{\partial_s W_0(s)}{1 - rF(s)}$ for some functions $W_0(s)$, $W_1(r)$ and $F(s)$,

- the Killing vector K_0 is null with respect to h , then $K_0 = \partial_s$ and

$$\hat{g} = \frac{-1}{(W_0(r)y + W_1(r))^2} \left[W_0(r) (dy^2 - 4drds) - \frac{1}{W_0(r)} (dz - W_0(r)dy)^2 \right], \quad (5.46)$$

for some functions $W_0(r)$ and $W_1(r)$.

Proof: Using the same notation as above, (5.31) is a null tetrad for g , the Einstein metric is $\hat{g} = e^{2\Omega}g$ and $K = \partial_z$. Now lemma 5.2.2 implies that g and \hat{g} have a null Killing vector K_0 of the form (5.44). Since K is a pure Killing vector of g and \hat{g} , the conformal factor Ω cannot depend on z . As W does not depend on z either, K_0 commutes with K and so descends to a Killing vector of the Einstein-Weyl structure (h, ω) . Note that while K_0 is null with respect to g , it is only null with respect to h if $\Omega_x = 0$. We consider independently the cases where K_0 is null with respect to h , and the case $\Omega_x \neq 0$ where K_0 is space-like with respect to h . First assume $\Omega_x \neq 0$ and impose $\mathcal{L}_{K_0} \hat{g} = 0$, as explained above this implies

$\mathcal{L}_{K_0} h = \mathcal{L}_{K_0} \omega = 0$. These two equations are equivalent to⁴

$$\left(\partial_y - \left(\frac{ya_t}{2a} + b_t \right) \partial_x \right) \begin{pmatrix} \Omega \\ W_x \\ H_x \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{aa_{tt} - 2(a_t)^2}{2a^2} y + b_t - \frac{a_t b}{a} \end{pmatrix}, \quad (5.47)$$

and

$$ae^{-\Omega} + W - W_2(y, t) = 0. \quad (5.48)$$

Here $a(t), b(t)$ and $W_2(y, t)$ are arbitrary functions arising as constants of integration. The functions H and W are furthermore subject to equations (5.28) and (5.29). Exploiting the symmetries [40] of the dKP equation we can set $a = 1$ and $b = 0$. To achieve this, we use a combination of (5.22) and (5.24) and introduce coordinates

$$\begin{aligned} \hat{x} &:= \sqrt{a} \left(x + \frac{y^2 a_t}{4a} + by \right), \\ \hat{y} &:= ay - 2F, \\ \hat{t} &:= \hat{t}(t), \end{aligned} \quad (5.49)$$

where $dF = -abd t$ and $d\hat{t} = a^{\frac{3}{2}} dt$ and an adapted dKP function \hat{U} and monopole $\hat{W}_{\hat{x}}$ by

$$\begin{aligned} \hat{U} &:= \frac{1}{a} \left[U - \frac{xa_t}{2a} - \frac{y^2 a_{tt}}{4a} + \frac{3y^2}{8} \left(\frac{a_t}{a} \right)^2 \right] + y \frac{ba_t - 2ab_t}{2a^2} + \frac{b^2}{a^2}, \\ \hat{W} &:= \frac{W}{\sqrt{a}}. \end{aligned} \quad (5.50)$$

Then the equations (5.47) reduce to

$$\partial_{\hat{y}} \begin{pmatrix} \Omega \\ \hat{W}_{\hat{x}} \\ \hat{U} \end{pmatrix} = 0. \quad (5.51)$$

Here \hat{W} satisfies the Monopole equation (5.29) in the hatted coordinates and we have

$$\hat{W} = -\sqrt{a} e^{-\Omega} + \hat{W}_2(\hat{y}, \hat{t}). \quad (5.52)$$

⁴The individual components of the two equations can be found in Appendix B.

The metric h now reads $h = \frac{1}{a^2}\hat{h}$, where

$$\hat{h} = d\hat{y}^2 - 4d\hat{t} \left(d\hat{x} + \hat{U}d\hat{t} \right), \quad (5.53)$$

with Killing vector $K_0 = \partial_{\hat{y}}$ and \hat{U} satisfies the dKP equation (5.20) in the hatted coordinates. So our change of coordinates corresponds to a change of conformal scale in the Einstein–Weyl structure. Note that $V = a\hat{V}$, where $\hat{V} = \hat{W}_{\hat{x}}$, and so a can be absorbed into the conformal factor by defining $\hat{\Omega} = \Omega - \frac{1}{2} \ln a$. Since the one-form transforms trivially,

$$dz - W_x dy - 2W_y dt = dz - \hat{W}_{\hat{x}} d\hat{y} - 2\hat{W}_{\hat{y}} d\hat{t}, \quad (5.54)$$

we then have

$$\hat{g} = e^{2\hat{\Omega}} \left[\hat{V}\hat{h} - \frac{1}{\hat{V}} \left(dz - \hat{W}_{\hat{x}} d\hat{y} - 2\hat{W}_{\hat{y}} d\hat{t} \right)^2 \right] \quad (5.55)$$

as well as

$$\hat{W} = -e^{-\hat{\Omega}} + \hat{W}_2(\hat{y}, \hat{t}). \quad (5.56)$$

This reduces the general situation to the case where $a = 1$ and $b = 0$ and therefore we can now drop the hats and ignore a and b in equations (5.47) and (5.48).

Next, we show that we can also set $W_2(y, t) = 0$. We have so far imposed $\mathcal{L}_{K_0}h = 0$, which already implies $\mathcal{L}_{K_0}W_x = 0$. Then we are left with

$$\mathcal{L}_{K_0}(dz - 2W_x dy - 2W_y dt) = 0, \quad (5.57)$$

which is equivalent to $W_{yy} = 0$. Thus W is linear in y and W_y a function of t only. But then the term $2W_y dt$ can be absorbed by a change of the coordinate z , which will not effect any other part of the metric. This effectively sets $W_2(y, t) = 0$, so we now have

$$\Omega_y = W_y = U_y = 0, \quad \text{and} \quad W = -e^{-\Omega}. \quad (5.58)$$

These relations greatly simplify the dKP and monopole equation, so that we can now solve them. The dKP equation reads

$$U_t - UU_x = \alpha(t), \quad (5.59)$$

where again $\alpha(t)$ is a constant of integration. The method of characteristics implies that the general solution is of the form

$$U = f(x + tU + g_1(t)) + g_2(t), \quad (5.60)$$

where $g_1(t)$ and $g_2(t)$ are functions of t satisfying

$$\partial_t g_1(t) = -t\alpha(t), \quad \partial_t g_2(t) = \alpha(t), \quad (5.61)$$

and f is an arbitrary function of $s := x + tU + g_1(t)$. We can now make a hodograph [74, 76, 77] transform $(x, y, t) \rightarrow (s, y, r)$, where $r := t$ and

$$\partial_s = (1 - t\partial_s f(s)) \partial_x, \quad \partial_r = \partial_t - U\partial_x. \quad (5.62)$$

Using

$$U_x = \frac{\partial_s f(s)}{1 - t\partial_s f(s)}, \quad U_t = \frac{U\partial_s f(s)}{1 - t\partial_s f(s)} + \alpha(t), \quad (5.63)$$

we have

$$ds = \frac{dx + Udt}{1 - t\partial_s f(s)}, \quad (5.64)$$

and thus

$$h = dy^2 - 4(1 - t\partial_s f(s)) dr ds. \quad (5.65)$$

We now turn to the monopole equation which, using $W_y = 0$, is

$$(W_t - UW_x)_x = 0, \quad (5.66)$$

or, after the hodograph transform, simply $W_{rs} = 0$. So we find

$$W = -W_0(s) - W_1(r), \quad (5.67)$$

and consequently, defining $F(s) = \partial_s f(s)$,

$$\hat{g} = \frac{1}{(W_0(s) + W_1(r))^2} \left[V (dy^2 - 4(1 - tF(s)) dt ds) - \frac{1}{V} (dz - Vdy)^2 \right] \quad (5.68)$$

as claimed. We now turn to the second case, where $\Omega_x = 0$ and K_0 is null. Again we impose $\mathcal{L}_{K_0} h = \mathcal{L}_{K_0} \omega = 0$, this implies⁵

$$\begin{aligned} W_{xx} = W_{xy} = H_{xxx} = H_{xxy} = 0, \\ e^\Omega = \frac{1}{a(t)y + b(t)}, \quad \partial_t \ln \left(\frac{W_x}{a(t)} \right) - H_{xx} = 0. \end{aligned} \quad (5.69)$$

⁵See Appendix B for the individual components of these two equations.

Here $a(t)$ and $b(t)$ are constants of integration. In combination with the dKP equation, these relations imply that the dKP function is of the form

$$U(x, y, t) = U_0(t)x + \frac{1}{2}y^2 (\partial_t U_0(t) - U_0(t)^2) + U_1(t)y + U_2(t). \quad (5.70)$$

Such a dKP function leads to a conformally flat Einstein–Weyl structure. To see this, we again use the symmetries of the dKP equation and consider the change of coordinates combining (5.22) and (5.24),

$$\begin{aligned} \hat{x} &:= c'^{\frac{1}{3}} \left(x + \frac{y^2}{6} (\ln c')' \right) - f' c'^{\frac{2}{3}} y - g, \\ \hat{y} &:= c'^{\frac{2}{3}} y - 2f, \\ \hat{t} &:= c(t) \end{aligned} \quad (5.71)$$

where $c' := \partial_t c(t)$ and $(\ln c')' := 3U_0(t)$, $f'' := -c'^{-\frac{4}{3}}U_2(t)$, $g' := -c'^{-\frac{2}{3}}U_3(t) - f'^2$. One can check that under this coordinate transformation we have

$$g = e^{2\hat{\Omega}} \left[\hat{W}_{\hat{x}} (d\hat{y}^2 - d\hat{x}d\hat{t}) - \frac{1}{\hat{W}_{\hat{x}}} \left(dz - \hat{W}_{\hat{x}}d\hat{y} - 2\hat{W}_{\hat{y}}d\hat{t} \right)^2 \right], \quad (5.72)$$

for $\hat{W} := c'^{-\frac{1}{3}}W$, $\hat{\Omega} := \Omega + \frac{1}{3} \ln c'$ with $\hat{W}_{\hat{x}\hat{y}} = \hat{W}_{\hat{x}\hat{x}} = 0$ and $e^{-\hat{\Omega}} = \hat{W}_{\hat{x}}\hat{y} + W_1(t)$. Furthermore \hat{W} satisfies the Monopole Equation in hatted coordinates. So we can drop the hats and work with (5.69) using $H = 0$. The Monopole equation then leads to

$$W = -\frac{1}{2}W_0'(t)y^2 - xW_0(t) + yW_2(t) + W_3(t), \quad (5.73)$$

where $W_0(t)$, $W_2(t)$ and $W_3(t)$ are arbitrary functions. The last term $W_3(t)$ is irrelevant for the metric and can be dropped, while the term linear in y can be absorbed into the definition of the coordinate z . But then the final form of the metric arises from the transformation $(t, x) \rightarrow (r, s)$,

$$\hat{g} = \frac{-1}{(W_0(r)y + W_1(r))^2} \left[W_0(r) (dy^2 - 4drds) - \frac{1}{W_0(r)} (dz - W_0(r)dy)^2 \right]. \quad (5.74)$$

□

We are now in a position to prove the theorem established on the first page of this section:

Proof of Theorem 5.2.1: By definition the Einstein metric \hat{g} has a conformal Killing vector K . Now a well-known result which goes back to Brinkmann [78], see also Corollary 2.10 in [79], states that any four-dimensional Einstein manifold with non-zero cosmological constant that admits a proper conformal Killing vector (i.e. a conformal Killing vector which is neither a homothety nor a Killing vector) is conformally flat. Conformally flat Einstein metrics are the special case of (5.32) when $\ddot{f} = 0$. Hence we continue with Killing vectors K that are not proper conformal, i.e. we assume K is either a homothety or a Killing vector. If K is a homothety then it is necessarily a scalar curvature collineation [80] and hence

$$L_K g = cg, \quad L_K \Lambda = -c\Lambda, \quad (5.75)$$

for some constant c . But $\Lambda \neq 0$ implies $c = 0$ and so K must be (pure) Killing vector. So now the assumptions of lemma 5.2.3 are satisfied and \hat{g} is of the form (5.45) or (5.46). Imposing the Einstein equations amounts to the conditions $\hat{\varrho}_{AB0'0'} = 0$ and $\hat{R} = 12\Lambda$ in the frame given by (5.31). The case where K_0 is null with respect to the EW structure is trivial: Imposing constant scalar curvature enforces $W_0(r) = C\Lambda$, but then the Weyl spinor vanishes identically so this case is reduced to conformally flat spacetimes as well. More interesting is the case where K_0 is space-like with respect to the EW structure, here the constant scalar curvature condition is simply

$$\partial_r W_1 = -\Lambda, \quad (5.76)$$

so $W_1(r) = -\Lambda r$. Then all that remains of $\hat{\varrho}_{AB0'0'} = 0$ is one single ODE for $f(s)$ and $W_0(s)$:

$$\partial_s f W_0 \partial_{ss} W_0 + \partial_s f (\partial_s W_0)^2 - W_0 \partial_s W_0 \partial_{ss} f - \Lambda \partial_{ss} W_0 = 0. \quad (5.77)$$

This equation determines $f(s)$ once $W_0(s)$ has been chosen, and can be solved for $f(s)$,

$$f(s) = -\Lambda \int W_0 \partial_s W_0 \left(\int \frac{\partial_{ss} W_0}{(W_0 \partial_s W_0)^2} ds + C \right) ds. \quad (5.78)$$

However this leaves us with a quadrature that cannot be performed in general entering the final form of the metric. A way to circumvent this is to interchange dependent and independent variables. We thus consider the function $s(w)$ where

$w = W_0$ replaces the coordinate s and $f(s(w))$ is now also a function of w . Equation (5.77) now reads

$$\dot{f} - w\ddot{f} + \Lambda\dot{s} = 0, \quad (5.79)$$

which is readily integrated to give

$$2f - w\dot{f} + \Lambda\dot{s} - k = 0, \quad (5.80)$$

Here k is a constant of integration and $\dot{f} = \partial_w f$. Note that we cannot have $2f = k$, as then either $\Lambda = 0$ or $\dot{s} = 0$, implying $V = 0$ in (5.45). Redefining $f \rightarrow \Lambda f + \frac{k}{2}$, absorbing a factor of 2 into the coordinate y and using (5.80) to replace \dot{s} in \hat{g} yields the final form of the metric given by (5.32). \square

5.3 Einstein metrics and the $SU(\infty)$ Toda equation

In section 5.1.2 we mentioned three sources of examples of EW structures, leading to scalar-flat pseudo-Kähler, scalar-flat para-Kähler and anti-self-dual null-Kähler metrics respectively. In the previous section we discussed ASD Einstein metrics arising from solutions of the dKP equation. These metrics are conformally equivalent to a null-Kähler metric. Now we turn to the relation of ASD Einstein metrics and pseudo-Kähler or para-Kähler metrics generated by solutions of the $SU(\infty)$ Toda equation. We shall see that locally all real-analytic ASD Einstein metrics with a non-null Killing vector are conformally pseudo- or para-Kähler.

5.3.1 From Einstein metrics to the $SU(\infty)$ Toda equation

Tod [65, 17] showed that ASD Einstein metrics in Euclidean signature with a Killing vector are always conformally scalar-flat Kähler. Using LeBrun's characterisation of Euclidean scalar-flat Kähler metrics, this boils down to EW structures in Toda-form. In neutral signature, the same construction is possible and certainly leads to examples of ASD Einstein metrics. However, we need to take into account the existence of para-Kähler and null-Kähler metrics. We restrict attention to the real-analytic category, as we need to make use of theorem 5.1.3.

Theorem 5.3.1 *Let g be a real-analytic ASD Einstein metric in neutral signature with a non-null Killing vector K , where $d\mathbf{K}_+$ is the SD part of the exterior derivative of the one-form \mathbf{K} dual to K . Then $*_g(d\mathbf{K}_+ \wedge d\mathbf{K}_+) \neq 0$ in any neighbourhood, if $c := *_g(d\mathbf{K}_+ \wedge d\mathbf{K}_+) \neq 0$ in a point, then locally g is conformally scalar-flat pseudo-Kähler ($c > 0$) or para-Kähler ($c < 0$) and of the form*

$$g = \frac{V}{t^2} (e^U (dx^2 \pm dy^2) \mp dt^2) - \frac{1}{Vt^2} (dz + \alpha)^2, \quad (5.81)$$

with Killing vector $K = \partial_z$ and potential

$$V(x, y, t) := \pm \frac{tU_t - 2}{2\Lambda}. \quad (5.82)$$

Proof: The fact that $*_g(d\mathbf{K}_+ \wedge d\mathbf{K}_+)$ is not identically zero follows from theorem 5.2.1 using [40]. To prove the second half of the statement we follow [65, 17], translating it to neutral signature.

Assume that g is ASD Einstein in neutral signature with a non-null Killing vector K . The Killing equation implies

$$\nabla_a K_b = \phi_{AB} \epsilon_{A'B'} + \psi_{A'B'} \epsilon_{AB}, \quad (5.83)$$

where ϕ_{AB} and $\psi_{A'B'}$ are symmetric spinors. Using the identity

$$\nabla_a \nabla_b K_c = R_{bcad} K^d, \quad (5.84)$$

which holds for every Killing vector, we obtain from (5.83)

$$\nabla_{AA'} \phi_{BC} = -W_{BCAD}^- K^D{}_{A'} - \Lambda \epsilon_{A(B} K_{C)A'}, \quad (5.85)$$

$$\nabla_{AA'} \psi_{B'C'} = -\Lambda \epsilon_{A'(B'} K_{C')A}. \quad (5.86)$$

The second of these equations shows that $\psi_{A'B'}$ satisfies the twistor equation [20],

$$\nabla_{A(A'} \psi_{B'C')} = 0. \quad (5.87)$$

Note that $d\mathbf{K}_+ = \frac{1}{2} \psi_{A'B'} e_B{}^{A'} \wedge e^{BB'}$, so

$$c := *_g(d\mathbf{K}_+ \wedge d\mathbf{K}_+) = \psi_{A'B'} \psi^{A'B'}. \quad (5.88)$$

First we want to rule out that this wedge product is identically zero, so assume the converse. Note that $d\mathbf{K}$ anti-self-dual leads to $\Lambda = 0$ using (5.86), so $c \equiv 0$

implies $d\mathbf{K}_+$ is null. Then we must have $\psi_{A'B'} = \iota_{A'}\iota_{B'}$ for some constant spinor $\iota_{A'} \in \mathbb{S}'$. The twistor equation (5.87) implies

$$\nabla_{AA'}(\iota_{B'}\iota_{C'}) = \epsilon_{A'(B'}\iota_{C')}\nabla_{AD'}\iota^{D'} - \iota^{D'}\nabla_{AD'}(\iota_{(B'})\epsilon_{C')A'}, \quad (5.89)$$

using the Leibniz rule on the left-hand side we get

$$\nabla_{AA'}(\iota_{B'}\iota_{C'}) = \iota_{C'}\nabla_{AA'}\iota_{B'} + \iota_{B'}\nabla_{AA'}\iota_{C'}. \quad (5.90)$$

Now choose a second constant spinor $o_{A'} \in \mathbb{S}'$ such that $o_{A'}\iota^{A'} = 1$. Then contracting both sides of (5.89) with $o^{A'}o^{B'}o^{C'}$, $o^{A'}\iota^{B'}\iota^{C'}$ and $o^{A'}o^{B'}\iota^{C'} + \iota^{A'}o^{B'}o^{C'}$ respectively we find

$$\nabla_{A(A'}\iota_{B')} = 0. \quad (5.91)$$

But this equation is conformally invariant and implies that g is conformally null-Kähler [40]. We will now deduce that the conformally equivalent null-Kähler metric \hat{g} has the same Killing vector as g . From (5.86) we have

$$0 = -\frac{1}{2\Lambda}\nabla_{AA'}(\psi_{B'C'}\psi^{B'C'}) = -\frac{1}{\Lambda}\psi^{B'C'}\nabla_{AA'}\psi_{B'C'} = \psi_{A'B'}K_A^{B'}, \quad (5.92)$$

since $d\mathbf{K}_+$ is null. If instead we contract (5.83) with the Killing vector K , we find

$$\begin{aligned} K \lrcorner d\mathbf{K} &= \phi_{AB}K^B{}_{A'} + \psi_{A'B'}K_A^{B'}, \\ K \lrcorner *_g(d\mathbf{K}) &= -\phi_{AB}K^B{}_{A'} + \psi_{A'B'}K_A^{B'}, \end{aligned} \quad (5.93)$$

and thus, using (5.92),

$$K \lrcorner d\mathbf{K}_+ = 0. \quad (5.94)$$

Now $\mathcal{L}_K g = 0$ implies $\mathcal{L}_K d\mathbf{K}_+ = 0$, and so $K \lrcorner d(d\mathbf{K}_+) = 0$. Suppose the null-Kähler metric is $\hat{g} = e^{2\Omega}g$, then the null-Kähler form $\Sigma = e^{3\Omega}d\mathbf{K}_+$ is closed. Therefore

$$0 = K \lrcorner d\Sigma = K \lrcorner (d(e^{3\Omega}) \wedge d\mathbf{K}_+ + e^{3\Omega}d(d\mathbf{K}_+)) = 3e^{3\Omega}(K \lrcorner d\Omega)d\mathbf{K}_+. \quad (5.95)$$

Consequently the conformal factor does not depend on z and thus g and \hat{g} have the same Killing vector K . Note that furthermore $\mathcal{L}_K \Sigma = 0$ and thus $\mathcal{L}_K \iota = 0$. As we are working the real-analytic category, g is conformally equivalent to a real-analytic ASD null-Kähler metric with the same Killing vector K . However g is also Einstein and such metrics we classified in theorem 5.2.1. So g is of the

form (5.32) with $*_g(d\mathbf{K}_+ \wedge d\mathbf{K}_+) = 0$. A short computation reveals⁶ that this happens only if $f(w) = 0$, which is excluded. So c is not identically zero. Note that this does not exclude the possibility of c vanishing at a point or even along a hypersurface.

Let p be a point where c is non-vanishing, then there is a neighbourhood U of p where $c \neq 0$. On U we define a scalar $2\psi^2 := |c|$ and an endomorphism J by

$$J_a^b := \psi^{-1} \psi_{A'}^{B'} \epsilon_A^B, \quad (5.96)$$

so that $J^2 = \mp 1$ and thus J is an almost-complex structure or an involution depending on the sign⁷ of c . It follows from the twistor equation (5.87) that J is integrable [81, 67], thus the metric g is (anti)-Hermitian. From the definition of ψ and equation (5.86) we find

$$\pm 2\psi \nabla_{AA'} \psi = \psi^{B'C'} \nabla_{AA'} \psi_{B'C'} = -\Lambda \psi_{A'B'} K_A^{B'}. \quad (5.97)$$

Thus J maps \mathbf{K} to an exact form, if we define the coordinate $t := \frac{1}{2}\Lambda\psi^{-1}$ we have

$$J(\mathbf{K}) = \mp \frac{dt}{t^2}. \quad (5.98)$$

Next, we introduce a second coordinate z by $dz(K) = 1$ and a function V related to the norm of the Killing vector,

$$g(K, K) = -(Vt^2)^{-1}. \quad (5.99)$$

If $c > 0$ then g is Hermitian with respect to J , if $c < 0$ then it is anti-Hermitian, so

$$g\left(\frac{dt}{t^2}, \frac{dt}{t^2}\right) = \pm g(K, K) = \mp (Vt^2)^{-1}. \quad (5.100)$$

Since J is integrable, we can now introduce two further coordinates x, y such that dx and dy annihilate both K and $J(K)$. The metric is then of the form

$$g = \frac{V}{t^2} (e^U (dx^2 \pm dy^2) \mp dt^2) - \frac{1}{Vt^2} (dz + \alpha)^2, \quad (5.101)$$

for some function U and one-form α . Comparison with (5.8) and (5.14) shows that g is conformal to a scalar-flat pseudo- or para-Kähler metric with conformal

⁶For details we refer the reader to the next section 5.3.2.

⁷Throughout this proof the upper sign corresponds to $c > 0$ whereas the lower sign corresponds to $c < 0$.

factor t^{-2} if U , V and α satisfy the necessary conditions. We start with α , we have

$$K \lrcorner d\mathbf{K} = d(g(K, K)), \quad K \lrcorner d\mathbf{K}_+ = -\psi J(\mathbf{K}) = \pm \frac{\Lambda dt}{2t^3}. \quad (5.102)$$

Using these two relations as well as (5.101) to compute $K \lrcorner *(d\mathbf{K})$ leads to

$$d\alpha = V_x dy \wedge dt \pm V_y dt \wedge dx \mp e^U \left(V_t + \frac{2V}{t} (1 \pm \Lambda V) \right) dt \wedge dy. \quad (5.103)$$

This has an integrability condition that we need to return to. To find conditions on U and V we need to impose the curvature restriction that g be ASD Einstein. This yields the $SU(\infty)$ Toda equation for U and determines the form of the monopole,

$$\pm 2\Lambda V = tU_t - 2. \quad (5.104)$$

Using (5.104) the equation for α simplifies to

$$d\alpha = V_x dy \wedge dt \pm V_y dt \wedge dx \mp (e^U V)_t dx \wedge dy, \quad (5.105)$$

with integrability condition

$$-V_{xx} \mp V_{yy} \pm (e^U V)_{tt} = 0, \quad (5.106)$$

which is satisfied by V from (5.104). So away from singular points ASD Einstein metrics with a Killing vector are scalar-flat pseudo- or para-Kähler with a monopole given by (5.104), as claimed. \square

Remark: Some more insight into the underlying reasons for this division into pseudo-Kähler and para-Kähler metrics can be gained from a closer look at $\mathbb{S}' \odot \mathbb{S}'$, the space of self-dual two-forms. In Euclidean signature every point on a unit two-sphere corresponds to a self-dual two-form which, contracting it with the metric, gives rise to an endomorphism which squares to -1 , an almost complex structure. Not so in split signature, if we define three non-degenerate self-dual two-forms by

$$S := \Sigma^{0'0'} - \Sigma^{1'1'}, \quad I := \Sigma^{0'0'} + \Sigma^{1'1'}, \quad T := \Sigma^{0'1'}, \quad (5.107)$$

then we have

$$I \wedge I = -S \wedge S = -T \wedge T = \text{vol}_4. \quad (5.108)$$

Consequently the endomorphisms obtained by contraction with the metric, which we will also denote by I , S and T , satisfy

$$I^2 = -S^2 = -T^2 = -1, \quad IST = 1, \quad (5.109)$$

and

$$g(X, Y) = g(IX, IY) = -g(SX, SY) = -g(TX, TY), \quad (5.110)$$

for any real vectors X and Y . The almost complex structures $aI + bS + cT$ are thus parametrised by the two-sheeted hyperboloid $a^2 - b^2 - c^2 = 1$, while we obtain an involution for any point on $a^2 - b^2 - c^2 = -1$. The behaviour of an endomorphism under the action of a Killing vector is determined by $d\mathbf{K}$, its self-dual component corresponds to the action on \mathcal{S}' . Anti-self-dual $d\mathbf{K}$ implies $\Lambda = 0$ via (5.86), so this can happen at most on a hyper-surface. Excluding such points, we have a non-trivial action on the two hyperboloids of almost-complex structures and involutions with a fixed point given by $d\mathbf{K}_+$. Depending on whether this fixed point lies on the one-sheeted or two-sheeted hyperboloid, we obtain an invariant involution or almost-complex structure. Appropriate rescaling will make this endomorphism integrable, and so we have a conformally scalar-flat pseudo- or para-Kähler metric.

5.3.2 Einstein, dKP and $SU(\infty)$ Toda

Having reduced ASD Einstein metrics with a Killing vector to EW structures in Toda-form, we now take a closer look at the class of Einstein metrics (5.32) arising from the dKP equation and reduce them to the form (5.81) following theorem 5.3.1. So let \hat{g} be an Einstein metric given by (5.32), the non-null Killing vector $K = \partial_z$ has the dual one-form

$$\mathbf{K} = \frac{2}{(w - \Lambda t)^2} (dy - 2F(w, t)dz), \quad (5.111)$$

where $F(w, t)$ is defined in (5.34). Thus the self-dual part of the exterior derivative is

$$d\mathbf{K}_+ = \frac{4}{(w - \Lambda t)^3} (\Lambda f(w)dt \wedge dz + dy \wedge dw + F(w, t)dw \wedge dz), \quad (5.112)$$

which satisfies

$$*_{\hat{g}}(d\mathbf{K}_+ \wedge d\mathbf{K}_+) = \frac{8\Lambda f(w)}{(w - \Lambda t)^2}. \quad (5.113)$$

The wedge product is non-zero since $f(w) \neq 0$ as claimed. According to theorem 5.3.1 the metric \hat{g} is conformally equivalent to a scalar-flat pseudo- or para-Kähler metric g of the form (5.81). Indeed, if we define $g := e^{2\Omega}\hat{g}$ with conformal factor

$$\Omega := \ln(w - \Lambda t) - \frac{1}{2} \ln(4|\Lambda f(w)|), \quad (5.114)$$

then $\Sigma := e^{3\Omega}d\mathbf{K}_+$ is closed and satisfies

$$*_g(\Sigma \wedge \Sigma) = \pm 2, \quad (5.115)$$

depending on the sign of $|\Lambda f(w)| = \pm \Lambda f(w)$. With the pseudo- or para-Kähler form

$$\Sigma = \frac{1}{2|\Lambda f(w)|^{\frac{3}{2}}} (\Lambda f(w)dt \wedge dz - dw \wedge (dy - F(w, t)dz)), \quad (5.116)$$

and the metric g at hand, we compute the endomorphism J and find

$$J : dz \mapsto \frac{1}{\sqrt{|\Lambda f(w)|}} dw, \quad dt \mapsto \frac{1}{\sqrt{|\Lambda f(w)|}} (dy - F(w, t)dz), \quad (5.117)$$

with $J^2 = \mp 1$. One can check that J is integrable using the definition of $F(w, t)$. Now according to the algorithm from the proof of theorem 5.3.1 we need to use two coordinates (T, Z) with $dZ(K) = 1$ and $T = \Lambda e^\Omega$. The remaining coordinates must be annihilated by K and $J(K)$, where

$$J(K) = \frac{1}{|\Lambda f(w)|} (-\Lambda f(w)\partial_w - F(w, t)\partial_t), \quad (5.118)$$

one such function is y , another one is determined by

$$dX := \frac{\Lambda^2}{2\sqrt{|\Lambda f(w)|}} (\Lambda f(w)dt - F(w, t)dw). \quad (5.119)$$

This is locally well-defined since the right-hand side is closed. In summary we have a set of new coordinates given by $(T, X, Y, Z) = (\Lambda e^\Omega, X, \frac{1}{2}\Lambda^2 y, z)$, the metric \hat{g} is given by

$$\hat{g} = \frac{1}{T^2} \frac{|\Lambda f|}{\Lambda^2 F} \left[\frac{1}{\Lambda^2 f^2} (\pm dX^2 + dY^2) \mp dT^2 \right] - \frac{1}{T^2} \frac{\Lambda^2 F}{|\Lambda f|} \left(dZ - \frac{1}{2\Lambda^2 F} dY \right)^2. \quad (5.120)$$

Here $f = f(w(X, T))$ and $F = F(w(X, T), t(X, T))$ are functions of two variables X, T determined by the free function $f(w)$ and $F(w, t)$ in (5.34). Comparison with (5.81) shows that (5.120) is in Toda-form if we choose $U = -2 \ln |\Lambda f|$ and the potential $V = \frac{|\Lambda f|}{\Lambda^2 F}$. And indeed, U satisfies the $SU(\infty)$ Toda equation as can be checked implicitly using the definition of the coordinates X and T . Furthermore, the potential is of the form $\pm 2\Lambda V = T\partial_T U - 2$ as required, and the generalised monopole equation is satisfied.

Since every EW structure in Toda-form lifts to an Einstein metric for a suitable choice of potential and we have classified all EW structures of dKP-form (5.19) that lift to an Einstein metric, we have singled out all EW structures that lie in the overlap of dKP and Toda.

Theorem 5.3.2 *Any EW structure $(\mathcal{W}, [h], D)$ that admits a metric $h_1 \in [h]$ in Toda-form (5.8) or (5.14) as well as a metric $h_2 \in [h]$ in dKP-form (5.19) is given by*

$$h = dy^2 - 8F(w, t)dt dw, \quad \omega = -\frac{2\Lambda w \dot{f}(w)}{F(w, t)} dt, \quad (5.121)$$

in some local coordinates (t, w, y) on \mathcal{W} , where $f(w) \neq 0$ is a free function that uniquely determines $F(w, t) = \frac{1}{2} \dot{f}(w) (w - \Lambda t) - f(w)$.

Equivalently, this is a classification of all ASD conformal structures with a non-null symmetry that admit both a real-analytic null-Kähler as well as a scalar-flat pseudo- or para-Kähler metric each with a Killing vector, such conformal structures hence contain an element of the form (5.32).

CONCLUSIONS AND OUTLOOK

In this thesis we studied anti-self-duality equations in four and eight dimensions. All examples were related to manifolds of special Riemannian holonomy, among these hyper-Kähler, Quaternion-Kähler and $Spin(7)$ -manifolds.

In chapter 3 we introduced the octonionic instanton equation, an 'anti-self-duality' field equation on background with exceptional holonomy $Spin(7)$. We used the identification of \mathbb{R}^8 with $\mathbb{R}^4 \times \mathbb{R}^4$, or the curved analogue when one of the \mathbb{R}^4 factors is replaced by a hyper-Kähler four-manifold (M_4, g_4) to construct explicit solutions of the 'anti-self-duality' equations in eight dimensions with gauge group $SU(2)$. The solutions all admit a four-dimensional symmetry group along the \mathbb{R}^4 factor, and thus they give rise to solutions of the non-abelian Seiberg-Witten equations on M_4 . Due to the restrictions imposed by this symmetry group, the holonomy of the background is reduced to hyper-Kähler. We have analysed three cases, where M_4 is \mathbb{R}^4 with the flat metric, the Eguchi-Hanson gravitational instanton, and finally the co-homogeneity one hyper-Kähler metric with Bianchi II group acting isometrically with three-dimensional orbits. In this last case the gauge fields are regular away from a domain wall in the five-dimensional space-time with the metric $g_4 - dt^2$. Alternatively, the background is a Lie group-manifold with a hyper-Kähler metric conformal to the homogeneous left-invariant metric and the singularity is only present in the conformal factor. While some of the contents of chapter 3 are previous work of the author in collaboration with his supervisor, the results on curved background are new. The symmetry reduction to four dimensions was based on the holonomy reduction

$SU(2) \times SU(2) \subset Spin(7)$. An analogous reduction from \mathbb{R}^8 with split signature metrics may provide a source of Lorentz invariant gauged solitons in $3 + 1$ dimensions. Moreover, there are other special realisations of $Spin(7)$ in terms of the Lie groups G_2 , $SU(3)$ and $SU(4)$. Each realisation leads to some symmetry reduction [82, 83], and picks a preferred gauge group, where an ansatz analogous to (3.22) can be made.

Witten [10] considered a complex-valued connection $\mathcal{A} = A + i\Phi$ on bundles over four-manifolds of the form $M_4 = \mathbb{R} \times M_3$ with the product metric $g_4 = dw^2 + g_3$, where (M_3, g_3) is a three-dimensional Riemannian manifold. He showed that the gradient flow equation

$$\frac{d\mathcal{A}}{dw} = - *_3 \frac{\delta \mathcal{I}}{\delta \bar{\mathcal{A}}} \quad (6.1)$$

for the holomorphic Chern–Simons functional \mathcal{I} yields two equations corresponding to the imaginary parts of (3.19). In this setup neither A nor Φ have a dw component.

The example (3.32) fits into this framework: g_3 is the flat metric on \mathbb{R}^3 , and the corresponding ODE is the reduction of the gradient flow equations. In all other examples in our paper the underlying four manifold is also of the form $M_4 = \mathbb{R} \times M_3$, where M_3 is a three-dimensional Lie group with left-invariant one-forms σ_i . Moreover in all cases there exists a gauge such that neither A nor Φ have components in the \mathbb{R} -direction orthogonal to the group orbits. However the Riemannian metric $g_4 = dw^2 + h_{ij}(w)\sigma_i\sigma_j$ on M_4 is not a product metric unless h_{ij} does not depend on w . It remains to be seen whether the gradient flow formulation of the non-abelian Seiberg–Witten equations can be achieved in this more general setup.

The dogma of twistor theory suggests that all symmetry reductions of the four-dimensional anti-self-duality equations should be integrable, as this is true for the non-reduced equations via the twistor correspondence. No such correspondence is known for the octonionic instanton equation, and therefore one can only hope but not expect to find explicit solutions for symmetry reductions of it. A more comprehensive approach to the octonionic instanton equation would be the attempt to come up with a construction for $Spin(7)$ -manifolds similar to the twistor correspondence. While many of the properties of ASD four-manifolds are tied to the existence of α -planes, the same can be said for $Spin(7)$ -manifolds and Cayley planes [84, 85]. The moduli space of Cayley planes through an arbitrary point of

a $Spin(7)$ -manifold M is given by the homogeneous space¹ $G = Spin(7)/Sp(1)^3$, thus the analogon of the correspondence space of the twistor construction is a G -bundle F over M . As it turns out the homogeneous space G , which is isomorphic to $SO(7)/SU(2)^3$, is a Quaternion-Kähler space appearing on Wolf's list [86, 62]. Now in the twistor correspondence of ASD Einstein manifolds the fibre $\mathbb{C}P^1$ has a complex structure, and the twistor space as a whole is in fact Kähler [70] with respect to a canonical metric. Salamon [87] introduced the notion of a quaternionic structure on a $4n$ -manifold, which in some sense is the quaternionic analogon of a complex structure. In terms of G -structures a quaternionic structure is a $GL(n, \mathbb{H}) \cdot \mathbb{H}$ -structure admitting a torsion-free connection. As a Quaternion-Kähler space, the 12-dimensional homogeneous space G has such a quaternionic structure and so do Cayley planes. It is possible to patch these structures together to an almost quaternionic structure on the 20-dimensional space of Cayley planes F . This part of the construction is based merely on linear algebra and the dimensions work out as naive counting confirms. The interesting and hard part is to investigate the intrinsic torsion of this almost quaternionic structure to see if or under what conditions it vanishes. The crucial point is that quaternionic manifolds admit twistor spaces themselves, translating the whole construction into algebraic terms. Depending on the results of these studies one could then aim to interpret the lift of the octonionic instanton equation to F .

In chapter 4 we considered Quaternion-Kähler four-manifolds, which by definition are anti-self-dual Einstein. We introduced their local description by Przanowski's function K and showed that metrics of this form are anti-self-dual Einstein provided K satisfies Przanowski's equation (4.2).

We continued with twistorial techniques to construct a Lax Pair, i.e. two vector fields l_A that commute if and only if Przanowski's equation is satisfied. The existence of this Lax Pair confirms that Przanowski's equation is integrable, as one would expect from an equation arising as a special case of anti-self-duality equations in four dimensions.

Furthermore, we encountered a conformally invariant differential operator acting on the line bundle $L^{l,m}$ as well as recursion relations relating solutions of the associated Laplace equation to cohomology classes on twistor space. Special cases are the conformal Laplacian and the linearised Przanowski operator. The

¹Here $Sp(1)^3 = Sp(1) \times Sp(1) \times Sp(1)/\mathbb{Z}_2$.

latter annihilates perturbations δK of Przanowski's function and thus describes deformations of the underlying manifold. We explained how the corresponding deformation of the twistor data is determined by the associated cohomology class $H^1(T, \mathcal{O}(2))$. We also constructed a contour integral formula for δK in terms of this cohomology class. If desired, it would be straight-forward to write down a contour integral for all other values of (l, m) . The Lax Pair as well as the differential operator and its relation to deformation theory with the resulting integral formula are new results obtained by the author and have not appeared in the literature before.

The next section was dedicated to the procedure of recovering a complex structure and Przanowski's function K with the associated choice of holomorphic coordinates from twistor data. We illustrated the necessary steps explicitly using a number of examples including the non-trivial cases of $\mathbb{C}P^2$ with the Fubini-Study and $\widehat{\mathbb{C}P^2}$ with the Bergmann metric. The latter is an interesting starting point for deformations, as $\mathbb{C}P^2$ with the Fubini-Study metric is rigid. A similar algorithm to obtain Przanowski's function from twistor data has been employed in the single-fibration picture in [16], while the version appearing in this thesis is adapted to the double-fibration picture.

Comparison of the results in chapter 4 with sections 2.4 and 2.5 reveals that we have extended many of the well-known results [13, 37, 14] valid for the heavenly function on ASD Ricci-flat manifolds to Przanowski's function on a background with non-zero cosmological constant. This includes in particular a Lax Pair for Przanowski's function, an integral formula producing perturbations from cohomology classes and an algorithm to deduce the explicit form of the ASD Einstein metric from twistor data. However, one feature that is missing is the recursion operator. In the twistor correspondence for ASD Ricci-flat metrics every coefficient of a power series of an element of $H^1(T, \mathcal{O}(2))$ serves as a perturbation of the heavenly function, hence one can retrieve new solutions from old ones². In the twistor correspondence for ASD Einstein metrics, only a single coefficient of a similar power series satisfies the linearised Przanowski equation, hence the recursion relation will recover the entire power series – and hence the cohomology class – from a single coefficient, but not provide new perturbations of Przanowski's function. It should be interesting to see whether one can somehow cure this problem and provide a recursion operator for perturbations of Przanowski's function.

²See the remark in section 2.5

Another useful extension of the present work would be to provide Przanowski's function for the family of quaternionic Taub–Nut metrics [88]. These are the family of four–dimensional ASD Einstein metrics that have as conformal infinity the Berger sphere, i.e. the squashed 3–sphere with metric

$$ds^2 = \sigma_1^2 + \sigma_2^2 + \lambda\sigma_3^2, \tag{6.2}$$

where $(\sigma_1, \sigma_2, \sigma_3)$ is a basis of left invariant one–forms satisfying $d\sigma_i = \sum_{j,k} \epsilon_{ijk} \sigma_j \wedge \sigma_k$ and λ is a parameter. Quaternionic Taub–Nut metrics are parametrised by two free parameters, the cosmological constant Λ and the squashing parameter λ in the metric at conformal infinity [89]. Note that for $\lambda = 1$ the Berger sphere reduces to the round metric on S^3 , the four–metric corresponding to this case is simply H^4 . Taking the limit where $\lambda \rightarrow \infty$ leads to a degenerate metric at conformal infinity, in this limit the four–metric reduces to $\widetilde{\mathbb{C}P^2}$. So both of the examples with positive scalar curvature that we treated in section 4.4 appear within this family. Hence Przanowski's function for a quaternionic Taub–Nut metric should interpolate between (4.72) and (4.62). Note that in comparison S^4 and $\mathbb{C}P^2$ are rigid in the sense that any deformation is no longer ASD Einstein, hence they cannot be part of any continuous family of ASD Einstein metrics.

Section 5.2 suggests a closer investigation of Przanowski's function for neutral signature ASD Einstein metrics. In the holomorphic category one would expect that restricting to a different real slice is all that is necessary. However, Przanowski's formulation for ASD Einstein metrics might well extend to non–analytic metrics in neutral signature, which are not captured by a twistor treatment. For further insights into that matter it might prove helpful to study Przanowski's original work [15, 90], which made no use of twistor theory.

Looking beyond the four–dimensional case, it would be interesting to see how much of the local description of a Quaternion–Kähler metric by a scalar function with one associated second–order partial differential equation remains valid in higher dimensions. Some comments in this direction have been made in [16] and some rigorous claims appear in [91], however in a much more physical setup. This generalisation to higher dimensions is more easily approached in the single fibration picture [5], which works for Quaternion–Kähler manifolds of arbitrary dimension. The work of Swann [70] shows that one can always construct a Kähler structure on the non–projective twistor space, hence providing a symplectic structure. Using LeBrun's [35] one–to–one correspondence between symplectic and contact structures we have a contact structure on the projective twistor space,

which we can use to find canonical coordinates and deduce the existence of potentials necessary to express the metric via an associated Hermitian form. One would certainly expect the number of PDEs to rise with the dimension of the Quaternion–Kähler manifold, but could hope that the spirit of the Przanowski construction carries through.

In chapter 5 we turned to ASD Einstein metrics with a symmetry. In Euclidean signature these metrics have been thoroughly investigated [65, 17, 66], here we focused on neutral signature metrics with a non–null symmetry. The symmetry can be exploited to reduce the four–metric to a conformal structure on a three–dimensional space, an Einstein–Weyl structure. Examples of EW spaces can be constructed from integrable equations such as the $SU(\infty)$ Toda equation and the dispersionless Kadomtsev–Petviashvili (dKP) equation. Toda’s equation leads to scalar–flat Kähler metrics in Euclidean signature and scalar–flat pseudo– and para–Kähler metrics in neutral signature. Furthermore we obtain anti–self–dual null–Kähler metrics in neutral signature from the dKP equation.

In Euclidean signature all ASD Einstein metrics with a Killing vector are conformally scalar–flat Kähler and hence project to EW structures that arise from Toda’s equation. Conversely, every solution of Toda’s equation leads to an ASD Einstein metric with a symmetry. To investigate the relationship between ASD Einstein metrics with non–null symmetry in neutral signature and pseudo–, para– and null–Kähler metrics in four dimensions was the aim of this last chapter of the thesis.

The first result was the classification of all ASD Einstein metrics that admit a real–analytic null–Kähler metric with a Killing vector within the same conformal class. It follows from the classification that in this case both metrics must have the same Killing vector unless they are conformally flat. This classification is a new result that has not appeared elsewhere.

We then continued to formulate a neutral signature version of Tod’s result, showing that around non–singular points all real–analytic ASD Einstein metrics with a Killing vector are conformally pseudo– or para–Kähler. This includes all metrics of the classification above, which hence contain an ASD Einstein metric, a null–Kähler metric and a pseudo– or para–Kähler metric within the same conformal class. So as a corollary we could clarify and answer affirmatively the open question of whether there are anti–self–dual null–Kähler metrics with a Killing vector that are conformally scalar–flat pseudo– or para–Kähler. Metrics of this type are

precisely the ones classified at the start of section (5.2), the corresponding EW structures arise from the $SU(\infty)$ Toda as well as the dKP equation.

Three nearby questions that this discussion leaves open present themselves readily: It would be preferable to eliminate twistorial arguments from the proofs of the theorems underlying the discussion, so as to be able to extend the results to all ASD Einstein metrics in neutral signature with non-null symmetry and drop the analyticity condition. In fact the proof of theorem 5.1.3 is the only one that uses twistor theory, hence an alternative proof based on purely geometrical and spinorial arguments would be desirable.

Secondly it would be interesting to study ASD Einstein metrics with symmetry around what we called singular points. These are points in which

$$(d\mathbf{K})_+ \wedge (d\mathbf{K})_+ = 0, \tag{6.3}$$

where $(d\mathbf{K})_+$ is the self-dual part of the exterior derivative of the one-form \mathbf{K} dual to the Killing vector K . This equation cannot hold in a full neighbourhood of a point, as theorem 5.3.1 states. However, the theorem doesn't rule out that (6.3) is satisfied in isolated points or even on entire hypersurfaces of \mathbb{M} . It would be interesting to see whether this can actually occur, and if so, what is the form of the metric in such singular points? Note that metric (5.32) satisfies (6.3) along a hypersurface if $f(w)$ has a simple zero and changes sign along that hypersurface. However, in this example the Killing vector K is null on the hypersurface, in fact the hypersurface marks the transition of K from a space-like to a time-like vector. The question we are raising here concerns the existence of metrics whose Killing vector is non-null everywhere and yet equation (6.3) holds.

Thirdly, as it stands theorem 5.2.1 only works in one direction: start with a real-analytic ASD null-Kähler metric with a non-null Killing vector, then every Einstein metric conformal to it is of the form specified by the theorem. If instead we start with an ASD Einstein metric with a non-null Killing vector, then what is the form of a conformal null-Kähler metric? Note that in general this null-Kähler metric only has a conformal Killing vector and hence need not be in the form given by theorem 5.1.3. In fact this amounts to extending theorem 5.2.1 to real-analytic ASD null-Kähler metrics with a conformal Killing vector.

APPENDIX A

Proof of Proposition 3.3.1. The non-abelian Seiberg–Witten equations (3.19) for a \mathfrak{g} -valued connection A and a $\mathfrak{g} \otimes \mathbb{H}$ -valued Higgs field $\Phi = \Phi_0 + i\Phi_1 + j\Phi_2 + k\Phi_3$ expanded in real and imaginary parts become

$$\sigma^{ab} (F_{ab} - \Phi_a \wedge \Phi_b) = 0, \quad (.0A4)$$

$$-\tilde{\sigma}^{ab} D_a \Phi_b = 0, \quad (.0A5)$$

$$D^a \Phi_a = 0. \quad (.0A6)$$

Here $D_a \Phi_b = \partial_a \Phi_b + [A_a, \Phi_b]$. Now, substituting (3.22) and using (3.21) in equation (.0A4) yields

$$\frac{3}{4} \partial_a \partial^a G + \sigma_{ac} \partial^a \partial^c G + \sigma_{cd} \partial^d G \sigma^{ab} d(e^c)_{ab} + \frac{3}{4} |\partial G|^2 - \frac{3}{4} |\partial H|^2 = 0. \quad (.0A7)$$

The term $\sigma_{cd} \partial^d G \sigma^{ab} d(e^c)_{ab}$ decomposes as

$$\sigma_{cd} \partial^d G \sigma^{ab} d(e^c)_{ab} = \frac{1}{4} [C^a{}_{da} + \epsilon_{da}{}^{bc} C^a{}_{bc}] \partial^d G \mathbf{1}_2 + \epsilon_{ea}{}^{bc} C^a{}_{bc} \partial^d G \sigma^e{}_d. \quad (.0A8)$$

The closure condition $d\sigma = 0$ yields

$$\sigma_{a[b} C^a{}_{cd]} = 0, \quad (.0A9)$$

which is a system of 12 linear equations. These equations imply the four relations

$$\epsilon_{da}{}^{bc} C^a{}_{bc} = 2C^a{}_{da}. \quad (.0A10)$$

Then the identity-valued part of (.0A4) becomes

$$\frac{3}{4} \partial_a \partial^a G + \frac{3}{4} C^a{}_{ba} \partial^b G + \frac{3}{4} |\partial G|^2 - \frac{3}{4} |\partial H|^2 = 0. \quad (.0A11)$$

The first two terms of these combine to give $\square G$, as can be seen by computing

$$\square G = *d * dG = *d \left(\frac{1}{3!} \epsilon_{abcd} \partial_a G e^b \wedge e^c \wedge e^d \right) = (\partial_a \partial^a + C^b{}_{ab} \partial^a) G. \quad (.0A12)$$

The other components of (.0A4) are given by

$$(\epsilon_{ea}{}^{bc} C^a{}_{bc} \sigma^{\epsilon d} - \sigma^{ab} C^d{}_{ab}) \partial_d G = 0. \quad (.0A13)$$

Using the spinor decomposition [21]

$$C^a{}_{bc} = \varepsilon^{A'}{}_{B'} \Gamma^A{}_{BCC'} + \varepsilon^A{}_B \Gamma^{A'}{}_{B'CC'} \quad (.0A14)$$

with the anti-self-duality conditions $d\sigma = 0$ equivalent to

$$\Gamma^{A'}{}_{B'CC'} = 0 \quad (.0A15)$$

gives

$$\Gamma^{AB}{}_{AC'} \sigma^{C'B'} \partial_{BB'} G = 0, \quad (.0A16)$$

where $\sigma^{A'B'} = \sigma^{(A'B')}$ and $\sigma^{ab} = \sigma^{A'B'} \varepsilon^{AB}$. Thus the three-dimensional distribution spanned by $\Gamma^{AB}{}_{A(C'} \partial_{B')} B$ is integrable and G is in its kernel. We now move to equation (.0A5), using the ansatz (3.22) for A and Φ we find

$$\tilde{\sigma}_{ab} \sigma^{bc} \partial^a \partial_c H + 2 \tilde{\sigma}_{ab} \sigma^{ad} \sigma^{bc} \partial_{(c} G \partial_{d)} H = \tilde{\sigma}_{ab} \sigma^b{}_c (\partial^a \partial^c H - 2 \partial^a H \partial^c G) = 0. \quad (.0A17)$$

Here we had to explicitly evaluate and symmetrise a product of three σ -matrices to obtain the last line. And finally, for equation (.0A6) we obtain from (3.21)

$$\partial_a (\sigma^{ab} \partial_b H) + \sigma_{ab} \sigma^a{}_c \partial^b G \partial^c H - \sigma_{ac} \sigma^a{}_b \partial^b G \partial^c H = \sigma_{ab} (\partial^a \partial^b H - 2 \partial^a G \partial^b H) = 0. \quad (.0A18)$$

□

APPENDIX B

In the notation of lemma 5.2.3 we provide the components of $\mathcal{L}_{K_0}h = \mathcal{L}_{K_0}\omega = 0$, where K_0 is defined in (5.44) and (h, ω) is an EW structure that lifts to a real-analytic null-Kähler metric g . They are

$$\begin{aligned}
W_x\Omega_x^2 + W_{xx}\Omega_x - W_x\Omega_{xx} &= 0, \\
W_x\Omega_x\Omega_y + W_{xx}\Omega_y - W_x\Omega_{xy} &= 0, \\
W_x\Omega_x\Omega_y + W_{xy}\Omega_x - W_x\Omega_{xy} &= 0, \quad (.0A19) \\
W_x\Omega_x\Omega_t + W_{xt}\Omega_x - W_x\Omega_{xt} + 2\Omega_y^2W_x + 2W_{xy}\Omega_y - 2W_x\Omega_{yy} &= 0, \\
W_x\Omega_y\Omega_t + W_{xt}\Omega_y - W_x\Omega_{yt} - W_x\Omega_yH_{xx} + W_x\Omega_xH_{xy} &= 0,
\end{aligned}$$

and

$$\Omega_xH_{xxy} - \Omega_yH_{xxx} = 0, \quad (.0A20)$$

which is already implicitly contained in the system above. Furthermore, the vanishing of the Lie derivative $\mathcal{L}_{K_0}\alpha$ of the one-form $\alpha = dz - W_xdy - 2W_ydt$ arising from the generalised monopole equation (5.6) in conjunction with the potential leading to the null-Kähler metric g is equivalent to

$$W_x\Omega_y^2 - W_x\Omega_{yy} + \Omega_xW_{yy} = 0. \quad (.0A21)$$

As one would expect, we obtain a linear combination of the same six equations when setting

$$\hat{\varrho}_{ABA'B'}\iota^{B'} = 0, \quad (.0A22)$$

for a metric $\hat{g} = e^{2\Omega}g$.

LIST OF RELATED PUBLICATIONS

The following list comprises the scientific work of the author that was published before or throughout the development of this thesis.

Moritz F. Högner

Quaternion-Kähler four-manifolds and Przanowski's function

J. Math. Phys. **53**, (2012), hep-th/1205.3977

Quaternion-Kähler four-manifolds, or equivalently anti-self-dual Einstein manifolds, are locally determined by one scalar function subject to Przanowski's equation. Using twistorial methods we construct a Lax Pair for Przanowski's equation, confirming its integrability. The Lee form of a compatible local complex structure, which one can always find, gives rise to a conformally invariant differential operator acting on sections of a line bundle. Special cases of the associated generalised Laplace operator are the conformal Laplacian and the linearised Przanowski operator. We provide recursion relations that allow us to construct cohomology classes on twistor space from solutions of the generalised Laplace equation. Conversely, we can extract such solutions from twistor cohomology, leading to a contour integral formula for perturbations of Przanowski's function. Finally, we illuminate the relationship between Przanowski's function and the twistor description, in particular we construct an algorithm to retrieve Przanowski's function from twistor data in the double-fibration picture. Using a number of examples, we demonstrate this procedure explicitly.

Maciej Dunajski, Moritz F. Högner

SU(2) solutions to self-duality equations in eight dimensions

J. Math. Phys. **53**, (2012), hep-th/1205.3977

We consider the octonionic self-duality equations on eight-dimensional manifolds of the form $M_8 = M_4 \times \mathbb{R}^4$, where M_4 is a hyper-Kähler four-manifold. We construct explicit solutions to these equations and their symmetry reductions to the non-abelian Seiberg-Witten equations on M_4 in the case when the gauge group is $SU(2)$. These solutions are singular for flat and Eguchi-Hanson backgrounds. For $M_4 = \mathbb{R} \times \mathcal{G}$ with a co-homogeneity one hyper-Kähler metric, where \mathcal{G} is a nilpotent (Bianchi II) Lie group, we find a solution which is singular only on a single-sided domain wall. This gives rise to a regular solution of the non-abelian Seiberg-Witten equations on a four-dimensional nilpotent Lie group which carries a regular conformally hyper-Kähler metric.

Moritz F. Högner

Anti-self-duality over eight-manifolds

Diploma Thesis (2010), Ruprecht-Karls-Universität Heidelberg

We consider Yang-Mills theory with gauge group $SU(2)$ over eight-manifolds. Using an extension of anti-self-duality to eight-manifolds with $Spin(7)$ -structure, we attempt to find explicit solutions for the Yang-Mills equations. Our first example is flat-space: Imposing a four-parameter translational symmetry, we reduce the anti-self-dual Yang-Mills equations for a specific choice of Ansatz to a second-order ODE for a scalar function. From this equation we obtain two explicit solutions, neither of which is regular. Then we consider the corresponding symmetry reduction of the Yang-Mills action to four dimensions and investigate the topological properties of this model. A scaling argument shows that no solutions with finite action exist. This leads us to consider a curved manifold as our second example: $M_{EH} \times \mathbb{R}^4$, where M_{EH} is the Eguchi-Hanson gravitational instanton. Similar to the first instance, we choose an Ansatz and impose translational symmetry. Again we succeed in reducing the anti-self-dual Yang-Mills equations to a second-order ODE. We cannot find any explicit solutions to this ODE.

LIST OF SYMBOLS

$\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$	real and complex numbers, quaternions and octonions
M, M_n	(pseudo-)Riemannian manifold (of dimension n)
\mathbb{M}	complex Riemannian manifold
$(w, \bar{w}, z, \bar{z}), (s, t, x, y)$	coordinates on 4-manifolds
U_0, U_1	neighbourhoods on M
H, K	Heavenly and Przanowski's function on M
d	exterior derivative
$\partial, \bar{\partial}$	Dolbeault operators
K, K_0, K_1	vector fields
\mathcal{L}_K	Lie derivative wrt vector field K
TM	tangent bundle of M
$\Lambda^n M, \Lambda^{(m,n)} M$	bundle of differential n - or (m, n) -forms on M
$L^{(l,m)}$	line bundle on M
ω, \mathbf{K}	one-forms on M
$K \lrcorner \omega$	contraction of one-form ω with vector field K
$\langle \cdot, \cdot, \cdot, \dots \rangle$	span of vector fields or one-forms
g, h	metrics on M or \mathbb{M}
∇	Levi-Civita connection
Γ	Christoffel symbols of ∇
P	parallel transport wrt ∇
$*$	Hodge star operator
\mathcal{R}	Riemannian curvature of ∇
W_{\pm}	self-dual and anti-self-dual Weyl curvature
ϱ	trace-free Ricci curvature
Λ	cosmological constant
e^{Ω}	conformal factor

LIST OF SYMBOLS

I, J	complex structures on M
S, T	involutive structures on M
N	nilpotent structure on M
$\Sigma, \Sigma_i, \Sigma_{AA'}$	self-dual two-forms on M or \mathbb{M}
B	Lee form wrt a complex structure
Δ, Θ, Ξ	global parallel differential four-forms on M
G	Lie group
\mathfrak{g}	Lie algebra
T_1, T_2, T_3	basis of $\mathfrak{su}(2)$
$\sigma_0, \sigma_1, \dots$	Maurer–Cartan one-forms on G
\mathbb{A}, \mathcal{A}	\mathfrak{g} -valued connections on M_8 and M_4
Φ	\mathfrak{g} -valued Higgs field on M_4
\mathbb{F}, \mathcal{F}	\mathfrak{g} -valued curvature of \mathbb{A} and \mathcal{A}
$e^{AA'}$	null tetrad on the complexified cotangent bundle of M
\mathbb{S}', \mathbb{S}	primed and unprimed spin bundle over M
$o^{A'}, o^A$	sections of \mathbb{S}' and \mathbb{S}
$\epsilon_{A'B'}, \epsilon_{AB}$	symplectic structures on \mathbb{S}' and \mathbb{S}
F	projective correspondence space, $\dim_{\mathbb{C}} F = 5$
\mathfrak{F}	non-projective correspondence space, $\dim_{\mathbb{C}} \mathfrak{F} = 6$
T	projective twistor space, $\dim_{\mathbb{C}} T = 3$
\mathfrak{T}	non-projective twistor space, $\dim_{\mathbb{C}} \mathfrak{T} = 4$
$\mathbb{C}P^1$	Riemann sphere
$\mathcal{O}(-1)$	tautological bundle over $\mathbb{C}P^1$
$\xi^{A'}, \xi$	homogeneous and inhomogeneous coordinates on $\mathbb{C}P^1$
ι	involution on $\mathbb{C}P^1$
u_0, u_1, v_0, v_1	homogeneous coordinates on $\mathbb{C}P^3$
Υ	Euler vector field on $\mathcal{O}(-1)$
$H^n(T, \mathcal{O}(m))$	n -th cohomology group with values in $\mathcal{O}(m)$
Ψ	cohomology class in $H^n(T, \mathcal{O}(m))$
τ	homogeneous one-form on \mathfrak{F}
τ_F, τ_T	the one-form τ written in inhomogeneous coordinates on F and T
ρ	volume form on \mathfrak{T}
d_A, l_A	vector fields spanning the twistor distribution in \mathfrak{F} and F
S_1, S_2	hypersurfaces in F
S'_1, S'_2	hypersurfaces in T
ι_A	parallel unprimed spinor on M

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