Vortices and Rossby-wave radiation on the beta-plane

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To my parents

‘The ruling power within, when it is in its natural state, is so related to outer circumstances that it easily changes to accord with what can be done and what is given it to do.’

Marcus Aurelius Antoninus, *Meditations, iv. 1*.

Contents

Summary

The Earth's atmosphere and oceans contain strongly swirling coherent structures. The sphericity of the Earth's surface, which may be modelled by the beta-effect, is responsible for the motion of these vortices, and also for the existence of Rossby waves.

This dissertation examines the evolution of distributed vortices on the beta-plane when the vortices are much stronger than the background beta-effect. There is a small nondimensional parameter ϵ, and a solution to the problem may be sought as an asymptotic expansion in ϵ. The near-field equation is then, to first order, a forced radial Rayleigh equation. For the nondivergent case, with no vortex stretching, a different dynamical balance holds in the far field.

Chapter 2 derives an exact mode-one solution for the radial Rayleigh operator and completely solves the classical inviscid instability initial-value problem for mode-one disturbances to a circular basic state. Disturbances tend to a steady-state solution if the basic state circulation is nonzero, and grow algebraically without bound otherwise. Chapter 3 examines properties of the causal Green's function for the far-field Rossby wave equation.

Chapter 4 calculates the first-order solution to the global problem in the nondivergent case by the method of Matched Asymptotic Expansions. For zero circulation, there is no far field. The resulting trajectory of vortices is computed; several ways of identifying the centre of the vortices are presented. Trajectories are calculated for the Rankine and Gaussian vortices. Chapter 5 calculates the asymptotic behaviour of the second-order solution, and examines the nonuniformity of the asymptotic expansion. In the case of nonzero circulation, the expansion loses validity for times $t = O(\epsilon^{-\frac{2}{3}})$ and spatial scales $r = O(\epsilon^{-\frac{1}{3}})$, rather than $t = O(\epsilon^{-1})$ and $r = O(1)$ respectively.

Chapter 6 solves the divergent problem numerically to first order, and also analytically for asymptotically large Rossby radii of deformation. The order of breakdown is then $t = O(\epsilon^{-1})$. Chapter 7 presents conclusions and suggestions for further research.

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This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. It is not the same as any work that I have submitted, or am in the process of submitting, for a degree or diploma or other qualification at any other university. Section 2·3 has been published as Llewellyn Smith (1995).

1·Introduction

Huracan was a 'God of Evil' for the Tainos, an ancient tribe of Central America. From their god has come the word hurricane, the tropical cyclone of the Atlantic and Eastern Pacific. In the western Pacific, tropical cyclones are typhoons, while in the Philippines they are bagiuos, after the city of Baguio which received 46 inches of rain in a 24 hour period in July 1911. The 80–100 tropical cyclones that occur each year cause an average number of 20,000 deaths and an economic loss of $6–7 billion (1982 values). Between 1964 and 1978, over 400,000 people were killed by tropical cyclones, including 300,000 in 1970 alone in Bangladesh, making tropical cyclones the most deadly natural disasters during that period (Anthes 1982, after Southern 1979). Naturally, understanding the behaviour of tropical cyclones is of great social importance as well as scientific interest. The book by Anthes (*op. cit.*) is a good reference on tropical cyclones, while the collection edited by Lighthill *et al.* (1993) comprises the proceedings of a conference on tropical cyclone disasters.

The Earth's atmosphere and oceans in fact contain an array of strongly swirling coherent structures – after all, fluid motion at high Reynolds number is characterised by the formation of coherent structures and eddies. The winter polar vortex, tropical cyclones, and tornadoes are examples of vortices in the atmosphere. Perhaps less well known is the fact, suspected since the 1930s, that the large-scale ocean circulation contains a large number of eddies: Gulf Stream rings, Kuroshio rings, meddies (Mediterranean eddies), and many others (Wunsch 1981). These have been likened to the 'weather' of the ocean. Many of these structures are generated in frontal regions, and are potentially of great importance in the horizontal transport of quantities such as heat and momentum, as well as biota. The atmospheres of other planets, too, contain strong vortices; for example Jupiter's Great Red Spot has been visible for over 300 years (Marcus 1993).

One of the striking features of these vortices is their persistence in the presence of effects such as adverse shear, vorticity gradients, dissipation and wave dispersion, which would tend to destroy them. They also move: the track of hurricanes is of great interest to the inhabitants of Bangladesh, the Philippines and Florida for example, while in the ocean, Gulf Stream rings and meddies have been followed for years at a stretch (Richardson 1993). As it happens, hurricanes are not just blown about by the prevailing winds. It has been recognised since Rossby (1949) that they

move on their own due to the sphericity of the Earth's surface. Similarly, rings are not simply advected by the waters around them.

There has been a wide variety of research on these eddies, examining their formation, evolution, propagation, and decay. The meteorological, oceanographic, and planetary science literatures have naturally concentrated on slightly different aspects of the structures, but there are strong similarities between the fields. The following brief review highlights aspects of the eddies' evolution that will be examined in this dissertation.

One strand of the meteorological literature has naturally concentrated on the formation of tropical cyclones and their subsequent motion, and has done so more recently than, and largely independently of, the classical Bergen school of extra-tropical, frontal, weather systems. Field observations of tropical storms are hard to carry out, even with the advent of satellite instruments. Theoretical investigations start with the work of Rossby (1949), although this is now generally recognised as not being entirely relevant to the tropical cyclone problem. The paper of Adem (1956) was probably the first attempt to look at a simple model of cyclone motion by solving the barotropic vorticity equation as a Taylor series in time. It is worthy of note that terms of order β^2 (the planetary vorticity squared) were neglected in the numerical computations. Since then, understanding the motion of cyclones in a prevailing wind, the 'steering current' problem, has possibly been the main focus of research (see, for example, Kasahara & Platzman 1963 and more recent investigations). This vein of work, concentrating in particular on the division between the cyclone and the environment, has continued ever since.

The use of numerical models has been very important in the study of these phenomena, with more sophisticated two- and three-dimensional models being progressively developed during the 1970s and 1980s. This subfield is reviewed in Anthes (*op. cit.*). Interestingly, this work has probably led to a revival in interest of theoretical work, to try and explain the results of numerical calculations using simple systems. Chan & Williams (1987) show very clearly how an intense vortex in a quiescent environment on the beta-plane (i.e. the simplest possible dynamical model of the sphericity of the Earth's surface) will decay into *Rossby waves* in the absence of nonlinearity, whereas it will propagate *coherently and to the Northwest* because of nonlinearity.

The more recent theoretical work foreshadowed in the previous paragraph has fitted in with the development of the field of geophysical fluid dynamics, and has looked extensively at the stability of vortices and at models of the propagation of

cyclones with simplified dynamics. The former problem involves solving a *radial Rayleigh equation*, and the corresponding eigenvalue problem has been examined by Gent & McWilliams (1986), Flatau & Stevens (1989), Peng & Williams (1991), Weber & Smith (1993) and Gill, Sneddon & Hosking (1993). The latter problem has been considered by, amongst others, Willoughby (1988), Smith & Ulrich (1990), Ross & Kurihara (1992) and Smith & Weber (1993). All of these have examined, in one form or another, the partition of the flow into background and cyclone contributions. One feature that has emerged from this work is the importance of the difference in magnitude between the vorticity gradient across a cyclone and the background vorticity gradient beta. The ratio of the latter to the former is small, and *expansions in this parameter* ϵ are considered in the last two of these papers. As may be seen from the above, incomplete, list of papers, this field is currently an area of intense research.

The study of these structures in oceanography dates back to the MODE (Mid-Ocean Dynamics Experiment) programme (MODE Group 1978). Theoretical work since has followed a great number of directions. Flierl (1977) showed theoretically that an eddy on the beta-plane *would disperse into Rossby waves* in linearised theory. Since then, a variety of linear and nonlinear models have been developed, in a fairly eclectic fashion, possibly reflecting the intrinsic difficulty of observing these structures, and also the variety of different phenomena encountered in the oceans. As in the meteorological case, numerical simulation has been an important tool in the study of coherent structures. McWilliams & Flierl (1979) and Mied & Lindemann (1979), showed that the eddies can *move and retain coherence on the beta-plane*.

More recently, an interesting overview of some lines of research is the review of Flierl (1987). This study concentrates on *isolated structures with weak far fields*. Flierl, Stern & Whitehead (1983) showed that steadily propagating isolated structures on the beta-plane cannot be simple, monopolar vortices with nonzero angular momentum. The evolution of nonisolated structures, they argued, generates long barotropic Rossby waves that alter the momentum balance. Sutyrin (1987,1989) considered truncated azimuthal modes models to examine vortex motion. The latter paper comes from Nihoul & Jamart (1989), an interesting collection of work on such problems. In a paper about the coastal North Atlantic circulation, Csanady (1988) examined the *generation of Rossby waves* by the pinching-off of Gulf Stream meanders which is a fundamentally nonlinear process. The propagation of Rossby waves in the ocean has long been recognised as important, and Veronis (1958) and Longuet-Higgins (1965) both studied the Rossby wave response of a beta-plane ocean to unsteady forcing.

There has also been a great deal of interest in the behaviour of rotating fluids in the field of geophysical fluid dynamics. This may be traced back to the work of pioneers such as Taylor, Ekman, Bjerknes and Richardson, but its rapid growth dates from after the Second World War and even more recently. In particular, it has been spurred on by new discoveries in the oceans and atmosphere. A very important contribution of the geophysical fluid dynamics community has been experimental work (which is after all of an observational nature). The motion of a monopolar vortex on a beta-plane has been studied in rotating tank experiments since Firing & Beardsley (1976). Carnevale, Kloosterziel & van Heijst (1991) provide a good overview of such work. Figure 1.1, adapted from this last paper, shows the *difference between the trajectory of a sink and that of a stirred vortex*. The former has nonzero circulation and the latter has zero. The trajectories appear noticeably different.

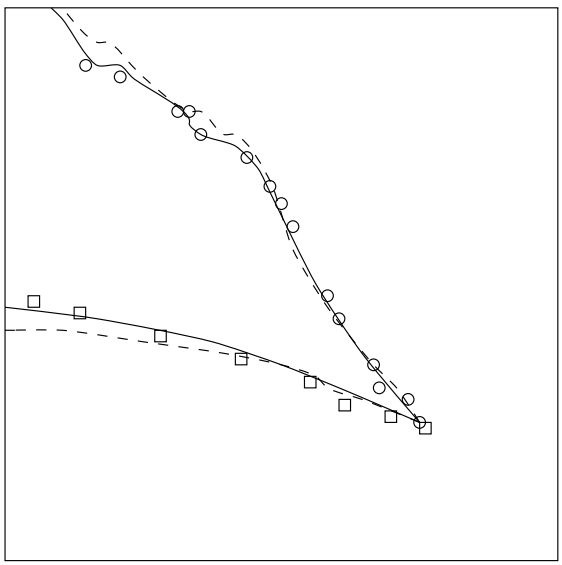

Figure 1.1: Trajectories of sink and stirred vortices on an inclined plane. Symbols represent the results of laboratory experiments: stirred vortices are indicated by squares, sink vortices by circles. The solid (dashed) lines correspond to viscous (inviscid) one-layer, quasigeostrophic, numerical simulations. Adapted from Carnevale *et al.* (1991).

The review article by Hopfinger & van Heijst (1993) provides a good overview of work on vortices in rotating fluids. The search for simple theoretical systems has led to the study of point vortex models, extensions of the classical notion of point vortices. Unfortunately, point vortices do not remain purely singular structures in the presence of the beta-effect and *generate a distributed vorticity field*. Reznik (1992) has looked at the behaviour of actual point vortices on the beta-plane, and the vorticity field they generate. There have been attempts to avoid this distributed vorticity field via modulated point vortex models (Zabusky & McWilliams 1982), but these are *ad hoc* rather than rational models.

The question of the stability of circular vortices in two-dimensional fluid flows has been the focus of much work. This topic goes back to the work of Rayleigh and Kelvin in the last century but

has been much extended by the meteorological and oceanographic communities. A number of numerical investigations in the meteorological literature have already been mentioned. Flierl (1988) and Kloosterziel & Carnevale (1992) have looked at theoretical aspects of the problem. In a general manner, the evolution of a circular vortex in the presence of some sufficiently (locally) weak distorting effect such as the beta-effect will be governed to a first approximation by a *radial Rayleigh operator* familiar in two-dimensional hydrodynamics.

Finally, another important line of work has been the general theory of waves and their interaction with mean flows. McIntyre (1983) provides a general overview. This work has led to the notion of 'potential vorticity thinking', where the dynamical evolution of the system is governed by the evolution of potential vorticity. The latter is a very powerful tool, as shown by the theorems of Haynes & McIntyre (1987) and more recent works.

The division between subjects in the brief review above is quite artificial, but serves to emphasise the different problems that have been considered and the different results obtained. To quote Smith & Weber (1993),'the last few years have seen a major upsurge in the study of tropical-cyclone motion with particular emphasis on the dynamical processes involved'. To a certain degree however, this diversity of work could be viewed as showing the lack of any general fluid-dynamical theory of tropical cyclones.

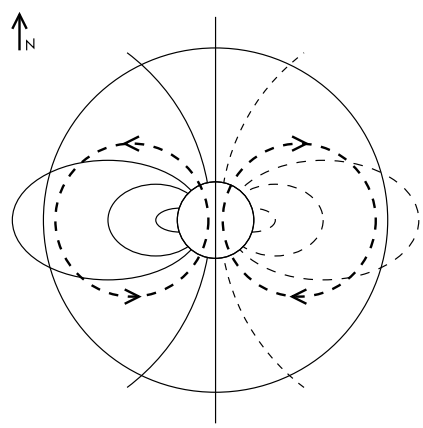

Figure 1.2: Field of $\partial\zeta/\partial t$, the rate of change of relative vorticity, centred on a cyclone. Heavy dashed lines show the induced secondary circulation as a result of the vorticity changes. Adapted from Holland (1983).

So how is cyclone motion explained anyway? The most convincing explanation of the motion of a cyclone on the beta-plane, the simplest possible system, is probably that of Holland (1983), indicating the basis for poleward drift. Figure 1.2 reproduces the relevant sketch. The conservation of absolute vorticity induces a positive relative vorticity anomaly to the East of the vortex, as lower values of background vorticity from South of the vortex are advected around it, causing an increase in the local relative vorticity. Similarly a negative relative vorticity anomaly is induced to the West of the vortex. The induced secondary circulation produces a flow to the North across the centre of the vortex. Therefore the vortex should *move to the North*.

According to Adem (1956), the motion to the North is a second-order effect; the first-order effect produces motion to the *West*. However, this ordering refers to terms in a Taylor series in time, which may converge for no values of time at all.

In fact, the above is far from a rigorous mathematical description. This dissertation starts from the issues that have been emphasised (italicised in fact) in the past pages. These are as follows: *the importance of the small quantity ϵ, the generation of Rossby waves, the role of the radial Rayleigh equation and the effect of the circulation of the vortices, all of these effects leading to coherent motion to the Northwest*. Therefore, the programme of this dissertation, as of November 1993, was to look at how an eddy evolves on the beta-plane, how it generates Rossby waves, how it evolves according to the radial Rayleigh equation, and which of these physical processes is relevant and where. This will be carried out here via an asymptotic expansion in ϵ, and the limits of the mathematical framework will also be investigated. This is the first rigorous investigation of the problem in that it considers a rational procedure of solving the governing equations in a physically plausible case.

Since November 1993, two very important papers have come out on the question, both using the same asymptotic approach in terms of the parameter ϵ. Sutyrin & Flierl (1994) examine the evolution of small disturbances to localised step-profile vorticity distributions in the presence of a weak beta-effect. This physical system is rather unusual, since the presence of Heaviside functions leads to coupled algebraic equations for each mode, with one equations per contour (jump in basic-state vorticity). In addition, the far-field is not treated separately, although it is recognised that in the nondivergent case*, the solution will break down in the far field when the basic-state circulation is nonzero. Thus the Rankine vortex falls outside the remit of the paper. Reznik & Dewar (1994) look at the evolution of isolated vortices in strictly two-dimensional flows. A variety of profiles were studied, but all with zero circulation and some with nonlocalised vorticity. The asymptotic expansion is recognised to become invalid in the far field at second order, and a nonrigorous patching is outlined. The work described here is more general, since both isolated and nonisolated distributed vortices are considered, in the divergent and nondivergent cases. The results are hence different from those of Sutyrin & Flierl (1994), since that paper considered only shifts in the positions of the basic-state jumps in vorticity. Here, however, the distributed correction vorticity field is indeed calculated. In addition, the validity of the expansion is investigated, and the far-field response

* 'Divergent' and 'nondivergent' will both be defined later.

correctly computed, for isolated and nonisolated vortices.

The evolution of a vortex with Gaussian streamfunction was considered by Korotaev & Fedotov (1994); a quasi-equilibrium regime was sought. It is not clear, however, how the claim that 'the vortex radiates only those waves that are quasi-stationary in the frame moving with it' fits in with the theorem of Flierl *et al.* (1983) outlined previously. In addition, a 'trap zone' is considered, inside which vorticity will be homogenised, although viscosity is never explicitly included in the model. While an asymptotic expansion is sought, the scalings used are not very clearly motivated. It is apparent, then, that this work does not solve the initial-value problem of the motion of an eddy on the beta-plane. Indeed, as the authors admit, '. . . this mathematical formulation does not provide an exact (even in a strict asymptotic sense) solution of the initial-value problem [. . .]; some fine-scale features of the solution are ignored.' Korotaev (1994*a*,1994*b*) are extended analyses with much the same characteristics. None of these approaches even aims to solve the initial-value problem of the evolution of a vortex on the beta-plane by a rational asymptotic expansion.

Since 1993, a few other more general papers have appeared. Flierl (1994) is a theoretical overview of rings in the ocean. In fact, the whole of that issue of *Chaos* is dedicated to coherent phenomena in planetary atmospheres and oceans, and provides an interesting overview of the subject.

The physical system

To begin with, then, it is important to identify the simplest model that will enable this programme to be carried out. Fluid motion on the surface of the Earth is governed by complicated nonlinear partial differential equations. However, the rotation of the Earth and the shallowness of the Earth's atmosphere and oceans mean that on large scales, simpler sets of equations may be used. The simplest such dynamical set are the quasigeostrophic equations. These are formally obtained by an asymptotic expansion in the Rossby number of the full equations of motion. The particular form adopted here is the barotropic vorticity equation on a mid-latitude beta-plane, in the meteorological terminology, or the one-layer quasigeostrophic equations on the beta-plane, to use a more fluid-dynamical language. This equation is

$$\frac{\partial}{\partial t}(\nabla^2 - R_\gamma^{-2})\psi + \mathcal{J}(\psi, \nabla^2\psi) + \beta\frac{\partial\psi}{\partial x} = 0, \tag{1.1}$$

where the x and y axes are eastward and northward respectively, ψ is the two-dimensional streamfunction, R_γ the Rossby radius of deformation, β the planetary

vorticity gradient and $\mathcal{J}$ the Jacobian. The Rossby radius of deformation,

$$R_\gamma = \frac{(gD)^{\frac{1}{2}}}{f}, \tag{1.2}$$

is a measure of the relative importance of vortex stretching and planetary vorticity. In this expression, L is a lengthscale characterising the motion, D is the depth of the fluid and f is the Coriolis parameter. The $R_\gamma^2\psi$ term approximates the vortex stretching due to small changes in fluid depth for quasigeostrophic flow.

A derivation of (1.1) may be found in Pedlosky (1987). The conditions for this approximate equation to be valid are as follows. The fluid must be shallow, which may be written (Pedlosky, Equation 3.2.1)

$$\delta = \frac{D}{L} \ll 1. \tag{1.3}$$

This is a reasonable approximation for hurricanes which may extend up to the tropopause, at around 10 km or so, and have a diameter of 500 km, and also a good approximation for the very shallow ring-like structures in the ocean. The Rossby number Ro, which must be small (Pedlosky, Equation 3.12.4), is

$$\mathrm{Ro} = \frac{V}{fL} \ll 1, \tag{1.4}$$

where V is a velocity scale associated with the fluid motion. This nondimensional parameter is a measure of the relative magnitude of the rotation associated with the fluid motion and the Earth's rotation. Ro is usually a small number for large-scale fluid motion on the Earth's surface, although it may reach $O(1)$ for intense disturbances (and this has been considered in some asymptotic theories of hurricane motion; see, for example, Shapiro & Montgomery 1993). Nevertheless, the quasi-geostrophic approximation is a reasonable one to make, especially since it reduces to the exact two-dimensional equations of motion for $R_\gamma^{-1} = 0$. In this nondivergent case, the equations are valid for any value of the Rossby number. The final approximation is the beta-plane approximation, which replaces the sphericity of the Earth's surface by a tangent plane approximation, by taking $f = f_0 + \beta y$, where y is the meridional coordinate. This is valid (Pedlosky, Equation 3.17.4) for

$$\mathrm{Ro}\tan\theta \ll 1, \tag{1.5}$$

where θ is the latitude at which this approximation is being made. Bearing in mind the comments already made about Ro, this is a good approximation to make here. At 12.5° North, the Coriolis parameter takes the value $3.157 \times 10^{-5}\,\mathrm{s}^{-1}$, and its

gradient, β, is $2.23 \times 10^{-11}\,\mathrm{m}^{-1}\mathrm{s}^{-1}$.

The equation (1.1) corresponds to the conservation of quasigeostrophic potential vorticity

$$q = (\nabla^2 - R_\gamma^{-2})\psi + f_0 + \beta y. \tag{1.6}$$

The quantity q is advected by the two-dimensional fluid velocity, which is given by

$$\mathbf{u} = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right), \tag{1.7}$$

following the usual geophysical convention whereby the relative vorticity ζ is given by $\zeta = \nabla^2\psi$. The crucial property of (1.1) as far as this dissertation is concerned, is that it supports Rossby waves as well as conserving q exactly, and is probably the simplest system to do so (especially when $R_\gamma^{-1} = 0$). The case $R_\gamma^{-1} = 0$ will be called nondivergent, following the meteorological usage. It is equivalent to strictly two-dimensional fluid flow in the presence of a background vorticity gradient β.

Finally, it is worth noting that the nondivergent barotropic vorticity equation is the simplest system which conserves potential, or rather absolute, vorticity, and which supports Rossby waves. Hence it is natural to consider it to investigate phenomena linking material conservation of vorticity (such as in a vortex) and wave emission. Admittedly, incompressible two-dimensional flows lead to strong flows at infinity. The divergent equations, on the other hand, provide a natural system in which to consider exponential shielding. While the equations are more complicated, the same basic mathematical framework may be adopted.

The nondimensional parameter ϵ

It is natural to nondimensionalise (1.1) to give

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\frac{\partial\psi}{\partial x} = 0, \tag{1.8}$$

using an appropriate space scale L and velocity scale V. Then the nondimensional parameter γ is

$$\gamma = \frac{R_\gamma}{L} \tag{1.9}$$

and ϵ is given by

$$\epsilon = \frac{\beta L^2}{V}. \tag{1.10}$$

The streamfunction scales as LV. The quantity ϵ is the main natural nondimensional quantity in the problem, and it is instructive to consider its physical meaning. For

a circular vortex of radius L (L could be the radius of maximum wind for example) and azimuthal velocity V at that radius, the change in vorticity from the centre to the edge of the vortex is $2V/L$, approximating the vorticity at radius L by twice the local angular velocity. The change in background vorticity is βL. Hence ϵ is approximately twice the ratio of the difference in background vorticity to the difference in relative vorticity across the vortex. A strong vortex (ϵ small) will remain coherent (Chan & Williams 1987), while a weak vortex will break up into Rossby waves (Flierl 1977).

It is hence natural to seek a solution to (1.8) as an asymptotic expansion in ϵ, with

$$\psi = \psi_0 + \epsilon\psi_1 + \cdots, \tag{1.11}$$

as in Sutyrin & Flierl (1994) and Reznik & Dewar (1994). The first paper considered both the nondivergent and divergent cases, but only for step profiles of vorticity, and noted that the resulting solutions were not valid in the far field in the divergent case. The second paper considered distributed vortices with zero circulation in the nondivergent case. In this dissertation, the general distributed case will be considered, in the divergent and nondivergent cases.

The expansion (1.11) will become disordered in the far field in the nondivergent case, as is recognised by Reznik & Dewar (1994, Section 8). For $\gamma = 0$, a rescaling of the equations of the motion according to $r = \epsilon^\beta R$ leads to

$$\frac{\partial}{\partial t}\epsilon^{-2\beta}\nabla_R^2\psi + \epsilon^{-2\beta}\mathcal{J}_R(\psi, \epsilon^{-2\beta}\nabla_R^2\psi) + \epsilon^{1-\beta}\frac{\partial\psi}{\partial X} = 0. \tag{1.12}$$

There is a second distinguished scaling beside the inner one (i.e. the one corresponding to (1.8)). The second scaling is an outer, or far-field, scaling, given by $\beta = -1$ and hence $\mathbf{R} = \epsilon\mathbf{r}$, so the far-field equation is

$$\frac{\partial}{\partial t}\nabla_R^2\phi + \frac{\partial\phi}{\partial X} + \epsilon^2\mathcal{J}_R(\phi, \nabla_R^2\phi) = 0, \tag{1.13}$$

where ϕ is the far-field streamfunction. If ϕ is also expanded in ϵ, the combined problem (1.8), (1.13) may be solved by the method of matched asymptotic expansions. The zeroth-order equation in the far field will be the linear Rossby wave equation, since the Jacobian term is two orders smaller. As is common in many systems supporting waves, the far-field behaviour of the system is dominated by linear dynamics.

In the nondivergent case, the simple scale analysis that led to (1.13) is naive, since first two terms balance in the far field. This leads to exponential decay of the

streamfunction (and of all quantities), and (1.8) remains valid over all space.

The expansion (1.11) will not remain valid for all time, and the order of breakdown of the validity of the expansion is of interest. This will presumably scale as some inverse power of ϵ. The dimensional counterpart is given by multiplying this quantity by L/V.

The radial Rayleigh equation

The leading-order near-field solution to (1.8) in the expansion (1.11) is the initial streamfunction, which satisfies the nonlinear vorticity equation

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi_0 + \mathcal{J}(\psi_0, \nabla^2\psi_0) = 0. \tag{1.14}$$

Given that the basic state leads to a vanishing Jacobian term, ψ_0 will be a steady solution of the above equation. This corresponds naturally to the vortex retaining its shape as it evolves, to the lowest order. The equation satisfied by the next order term is a forced radial Rayleigh equation, modified by a stretching term in γ, specifically

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi_1 + \Omega(r)\frac{\partial}{\partial \theta}\nabla^2\psi_1 - \frac{Q'}{r}\frac{\partial \psi_1}{\partial \theta} = -\frac{\partial \psi_0}{\partial x}, \tag{1.15}$$

where $\Omega(r)$ is the angular velocity of the basic flow, and $Q(r)$ its vorticity. The left-hand side is the radial Rayleigh operator, whose properties have been investigated since last century. Michalke & Timme (1967) found that the streamfunction

$$\psi_1 = r\Omega(r)\mathrm{e}^{\mathrm{i}\theta} \tag{1.16}$$

was a steady solution of the homogeneous, nondivergent counterpart of (1.15). They noted that 'it did not appear to be the limit of any normal mode of the equations'. This solution has been commented on in the meteorological literature, and has been viewed as evidence of inaccuracies in location of the centre of the vortex (Weber & Smith 1993). However, due to the linearity of (1.15), the convolution of (1.16) with arbitrary functions of time will be a homogeneous solution of the equation, whatever the value of γ. It is also clear that due to the form of the divergence term, namely preceded by a time-derivative, the above steady solution is also valid in the divergent case.

This solution corresponds almost to a change-of-frame term. An arbitrary axisymmetric function $\Psi(r)$ may be rewritten in a new coordinate system (x', y') related to the old one by

$$\mathbf{r} = \mathbf{r}' + \mathbf{X}_c. \tag{1.17}$$

The answer is simply $\Psi'(\mathbf{r}') = \Psi(|\mathbf{r}' + \mathbf{X}_c|)$, which is in general a difficult transformation to calculate explicitly. For $r' > X_c$, however, the usual Taylor expansion gives

$$\Psi'(\mathbf{r}') = \Psi(r') + \mathbf{X}_c . \nabla\Psi|_{r'} + \cdots \tag{1.18}$$

$$= \Psi(r') + (\mathbf{X}_c \cdot \hat{\mathbf{r}})r\Omega(r) + \cdots . \tag{1.19}$$

Hence, to first order, any change of frame (whether time-independent or not) can be represented by adding a term in $r\Omega(r)\mathrm{e}^{\mathrm{i}\theta}$ to the streamfunction. Conversely, the presence of such a term may be removed, to first order, by changing the origin of the coordinate system. It is important to note, however, that in the new coordinate system in which the $r\Omega\mathrm{e}^{\mathrm{i}\theta}$ term has vanished, there will be dynamically significant terms at higher order.

The Rossby wave equation

In the nondivergent case, the leading-order far-field solution ϕ_0 will satisfy a linear Rossby wave equation

$$\frac{\partial}{\partial t}\nabla_R^2\phi_0 + \frac{\partial\phi_0}{\partial X} = 0. \tag{1.20}$$

The study of solutions to this equation goes back to Rossby (1939), who incidentally first introduced the beta-plane approximation. The wave solutions to (1.20) of the form $\exp(\mathrm{i}\mathbf{k}.\mathbf{R} - \mathrm{i}\sigma t)$ satisfy the dispersion relation

$$\sigma = -\frac{l}{l^2 + m^2}, \tag{1.21}$$

where $\mathbf{k} = (l, m)$. This is an anisotropic dispersion relation, with the unusual property that long waves propagate faster than short waves. In fact, the speed of propagation increases without bound as the wavenumber decreases. The divergent counterpart of (1.20) has a finite maximal wave speed, but is not required in this dissertation. Rossby waves exist thanks to 'Rossby elasticity', the restoring effect of the background vorticity gradient. Their study goes back to the classical work by Kelvin on waves on the edge of a vortex patch (for further details, including nomenclature, see for example McIntyre 1993). Figure 1.3 is a schematic of the physical restoring mechanism.

The appropriate kind of solution to consider for the far-field problem here is the Green's function, since convolutions of the Green's function will yield functions satisfying (1.20) with arbitrary behaviour (including time-dependence) at the origin. Kamenkovich (1989) provides the relevant analysis, extending the work of Veronis

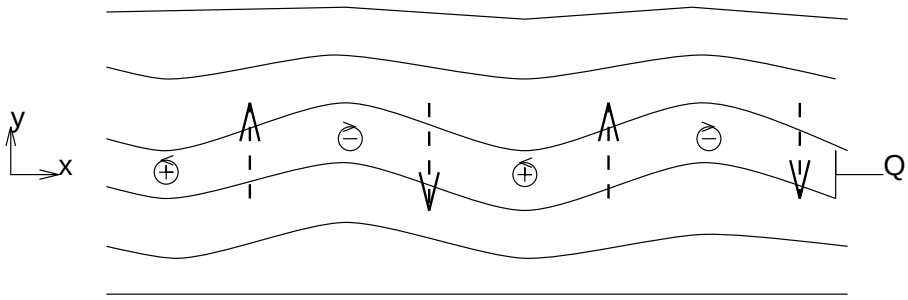

Figure 1.3: Sketch of the Q (absolute vorticity) contours, the Q-anomaly pattern given by the plus and minus signs, and the induced velocity field (dashed arrows) giving rise to the 'Rossby elasticity' in a Rossby wave. From McIntyre (1993) (figure and caption).

(1958) and Longuet-Higgins (1965).

Initial conditions and circulation

The initial condition of the flow will be taken here as given by a radially symmetric vorticity distribution $Q(r)$ which is *localised* in space, i.e. one for which all multipole moments $\int r^n Q(r)\,\mathrm{d}r$ converge. This corresponds to decay at infinity faster than any inverse power of r and will be denoted by the formal notation

$$Q(r) = O(r^{-\infty}). \tag{1.22}$$

Exponential decay of $Q(r)$, or compact support for $Q(r)$, are both cases of localised vorticity distributions. This excludes a number of azimuthal wind profiles that have been previously considered, for example those of Holland (1980) (except in a special case) and some of the profiles of Reznik & Dewar (1994), but is in keeping with the 'potential vorticity thinking' approach, where vorticity is the dynamical quantity of major importance. The latter paper also considered vortices with nonzero values of the gradient of vorticity Q' at the origin, but in this dissertation, the vorticity will be taken to be of the form

$$Q = 2\Omega_0 + 2r^2\Omega_0'' + O(r^3) \tag{1.23}$$

near the origin, without much loss of generality. The quantities Ω_0 and Ω_0'' are the angular velocity at the origin and its second derivative in r at the origin, respectively. The streamfunction then becomes

$$\Psi = \Psi_0 + \frac{\Omega_0}{2}r^2 + O(r^4), \tag{1.24}$$

where Ψ_0 is a constant. The most important implication for the rest of the dissertation is the result $\Omega_0' = 0$.

In the nondivergent case, the behaviour of the far-field streamfunction can be obtained for localised vorticity distributions. Poisson's equation (i.e. $\nabla^2\psi = \zeta$) for mode n[†] is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi_n}{\partial r}\right) - \frac{n^2}{r^2}\psi_n = q_n. \tag{1.25}$$

The regularity and decay conditions on ψ lead to the following Green's functions:

$$G(r, u) = u \ln r_> \tag{1.26}$$

for mode 0 and

$$G(r, u) = -\frac{u}{2n}\left(\frac{r_<}{r_>}\right)^n \tag{1.27}$$

for modes n other than 0, where $r_<$ and $r_>$ are the lesser and greater respectively of (r, u). For a radial (mode-zero) vorticity distribution $q_0(r)$, the solution for ψ_0 is then

$$\psi_0 = \ln r \int_0^r u q_0(u)\,\mathrm{d}u + \int_r^\infty u \ln u\, q(u)\,\mathrm{d}u. \tag{1.28}$$

For large r, the asymptotic behaviour of ψ_0 follows as

$$\psi_0 = \frac{\Gamma}{2\pi}\ln r + O(r^{-\infty}), \tag{1.29}$$

as can be shown by a formal expansion about the point $r = \infty$; Γ is the circulation:

$$\Gamma = \iint q(\mathbf{r})\,\mathrm{d}^2\mathbf{r} = 2\pi\int_0^\infty q_0(r) r\,\mathrm{d}r, \tag{1.30}$$

since $q(\mathbf{r}) = q_0(r)$ here.

The relevance of vortices with nonzero circulation is a delicate issue. Meteorologists use profiles which usually have zero or infinite circulation. From a fluid-mechanical perspective, one could well argue that the former case is hardly a vortex, while the latter is too much of a vortex. The work of Haynes & McIntyre (1987) indicates that a potential vorticity disturbance with nonzero circulation may only be formed, in an atmosphere initially at rest, by mass entrainment across isentropes or from infinity. While the strong updraughts in the cyclone eye might cause such a transfer,[‡] the prevailing view is that zero circulation is the relevant scenario. The experiments of Carnevale *et al.* (1991), however, show a difference between zero and

[†]Mode will be used to denote trigonometric mode, so mode n corresponds to $\mathrm{e}^{\mathrm{i}n\theta}$.

[‡]This is a speculative suggestion.

nonzero circulation vortices, as was shown in figure 1.1, and this merits investigation. This is to be expected, since a 'sink' vortex removes mass from the system, which is equivalent to a mass flux to infinity.

The terminology also merits some discussion. Flierl (1987) calls a vortex isolated if 'eddy flows vanish as $|x|$, $|y|$ becomes large'. This is vague, and in fact point vortex models are considered in that paper. Carnevale *et al.* (1991) call a vortex isolated if the velocity far from the vortex decays faster than r^{-1}. This corresponds to vortices with zero circulation. In fact, this is the mathematical condition required for the results of Flierl (1987). This definition will be adopted in this dissertation.

Outline of the dissertation

Given all this background, the work to be described in this dissertation has the following outline. First, Chapter 2 analyses the homogeneous radial Rayleigh equation, i.e. the left-hand side of (1.15) in the nondivergent case, relating the steady solution of Michalke & Timme (1967) to the full mode-one evolution. The governing equation is solved by quadrature, in anticipation of the full solution of (1.15) needed for the beta-plane problem. The analogous mode-zero and mode-two equations are also discussed, and the connection coefficient $\Delta(t)$ is introduced. It relates the small and large argument solutions of the radial Rayleigh equation and is needed in the second-order solution of the beta-plane problem.

Chapter 3 then solves (1.20), obtaining the limiting behaviour of the Green's function solution for small R in both the Laplace and time variable. The solution is valid for both zero and nonzero circulation, with the latter case being a subcase of the former. Appendix A considers the Rossby-wave radiation due to a steadily translating vortex on the beta-plane, and clarifies some previous work on the problem.

Chapter 4 solves the joint problem (1.8), (1.13) in the nondivergent case, to first order. Ways of defining the trajectory of the vortex are put forward, and discussed, and the trajectory of the Rankine and Gaussian vortices is calculated. The trajectories are to the Northwest, and are qualitatively in agreement with both experimental results and numerical calculations (even though these take place in a finite domain). Appendices B and C contain material relevant to Chapter 4 which would have unduly disrupted the flow of the text.

The second-order nondivergent problem is solved in Chapter 5 in an asymptotic sense. The full far-field solution is found, but only the far-field limit of the inner solution may be calculated. The implications for vortex motion are discussed, including the case of zero circulation. The behaviour of the nondivergent problem

is examined for different times and space scales, to understand the nonuniformity of the expansion in time.

Chapter 6 addresses the divergent problem. For $\gamma = O(1)$, the first-order near-field equation replacing the radial Rayleigh equation now has to be solved numerically, but there is no longer a far field. A simple numerical procedure is presented, and some trajectories calculated. The case of asymptotically small γ is also investigated.

Lastly, Chapter 7 briefly summarises the main results of the dissertation, and discusses possible future work. Appendix D outlines the modifications to the far-field problem that would be required to address one of the possible extensions discussed in Chapter 7.

2·The radial Rayleigh operator

2·1·Formulation

'Rayleigh's equation' may be used to denote the linearised equation governing the evolution of perturbation vorticity for two-dimensional constant-density flows, with a basic flow depending only on one spatial coordinate. Two-dimensional constant-density flows, also known as two-dimensional vortex dynamics, are equivalent to the nondivergent barotropic vorticity equation of Chapter 1, with no beta-effect. In this chapter, the effects of stretching will not be considered, and the inverse Rossby radius of deformation is zero. The vorticity (or Helmholtz) equation states that relative vorticity is materially conserved. That is,

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathcal{J}(\psi, \nabla^2\psi) = 0, \tag{2.1}$$

where ψ is the streamfunction defined according to the convention of Chapter 1, with the vorticity then becoming $\zeta = \nabla^2\psi$. The Jacobian $\mathcal{J}(a, b)$ of a and b is given by $\mathcal{J}(a, b) = a_x b_y - a_y b_x$ in Cartesian coordinates, and by $\mathcal{J}(a, b) = (a_r b_\theta - a_\theta b_r)/r$ in polar coordinates.

The evolution of small-amplitude perturbations ψ' to a basic state Ψ may be investigated by expanding (2.1). Then

$$\frac{\partial}{\partial t}\nabla^2\Psi + \mathcal{J}(\Psi, \nabla^2\Psi) = 0, \tag{2.2}$$

$$\frac{\partial}{\partial t}\nabla^2\psi' + \mathcal{J}(\Psi, \nabla^2\psi') + \mathcal{J}(\psi', \nabla^2\Psi) = 0, \tag{2.3}$$

on linearising. The interest of simple geometries lies in their giving simple steady state solutions to the basic-state equation (2.2). The classical Rayleigh equation arises from considering a basic state $\Psi(y)$. The first equation is automatically satisfied, while the second becomes

$$\frac{\partial}{\partial t}\nabla^2\psi' + U(y)\frac{\partial}{\partial x}\nabla^2\psi' + Q'(y)\frac{\partial\psi'}{\partial x} = 0, \tag{2.4}$$

where $U(y)$ is the basic-state velocity, and $Q'(y)$ the basic-state vorticity gradient. In the case of a radial basic state $\Psi(r)$, the disturbance equation is

$$\frac{\partial}{\partial t}\nabla^2\psi' + \Omega(r)\frac{\partial}{\partial\theta}\nabla^2\psi' - \frac{Q'(r)}{r}\frac{\partial\psi'}{\partial\theta} = 0. \tag{2.5}$$

In this equation, $\Omega(r)$ is the basic-state angular velocity given by $\Omega(r) = \Psi'(r)/r$, and $Q'(r)$ is the derivative with respect to r of the basic-state vorticity $Q(r) = \nabla^2\Psi$. The polar components of the velocity are related to the streamfunction by

$$u_r' = -\frac{1}{r}\frac{\partial\psi'}{\partial\theta} \tag{2.6}$$

and

$$u_\theta' = \frac{\partial\psi'}{\partial r}, \tag{2.7}$$

from (1.7).

The two Rayleigh equations (2.4) and (2.5) are linear in ψ', and the coefficients of ψ' do not depend on the variables x and θ respectively. Hence they may be decomposed by Fourier analysis into modes of the form e^{ikx} and $e^{in\theta}$. This shows the importance of the independence of the basic state from one geometric coordinate. The resulting equations are

$$\left(\frac{\partial}{\partial t} + ikU\right)\left[\frac{\partial^2\psi'}{\partial y^2} - k^2\psi'\right] + ikQ'\psi' = 0 \tag{2.8}$$

and

$$\left(\frac{\partial}{\partial t} + in\Omega\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi'}{\partial r}\right) - \frac{n^2}{r^2}\psi'\right] - \frac{inQ'}{r}\psi' = 0. \tag{2.9}$$

These may be further altered by Laplace or Fourier transforming in time. The designation 'Rayleigh's equation', properly speaking, probably refers to the Fourier transform of (2.8), but the designation will here be used interchangeably for any of (2.8), (2.9) or their integral transforms. However, as defined here, Rayleigh's equation governs an initial-value problem. Any search for normal modes (i.e. pure exponential solutions) will not address the question of initial conditions.

The perturbation equations (2.8) and (2.9) describe the behaviour of small-amplitude disturbances, which satisfy $|\psi'| \ll |\Psi|$. Significantly, the conditions for the validity of this inequality, or order relation, between Ψ and ψ' may break down at some time, when the disturbance has grown enough. This is for example the case for exponentially growing, or unstable, perturbations.

Rayleigh's equation has been a major area of research in the classical stability theory of fluid flow since the work of Rayleigh at the end of last century. However, the same linear operator applies to other physical situations of inviscid fluid flow containing additional physical phenomena. These problems may contain a natural expansion parameter corresponding in nondimensional terms to some nominally

small physical effect. The influence of this effect on otherwise steady solutions may then be investigated by solving a version of Rayleigh's equation with a different right-hand side. In the most general case, these situations may be represented by

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\mathcal{L}\psi = 0, \tag{2.10}$$

where $\mathcal{L}$ is some operator. The solution may then be expanded in the small parameter ϵ about some basic state Ψ that satisfies the vorticity equation. The perturbation equation is then

$$\frac{\partial}{\partial t}\nabla^2\psi' + \mathcal{J}(\Psi, \nabla^2\psi') + \mathcal{J}(\psi', \nabla^2\Psi) = -\mathcal{L}\Psi. \tag{2.11}$$

The new physical effect leads to a forced perturbation equation. This is no longer Rayleigh's equation, but the left-hand side is still the Rayleigh operator. Any results which depend only on the nature of the operator, and not the forcing term, will hold for the extended system. Hence solutions to the homogeneous equation will be relevant to all such systems.

From now on, only the radial equation will be considered, in keeping with the area of interest of this dissertation. The main body of research on the radial Rayleigh equation has been in the context of stability theory. For this the Fourier representation, with time dependence $\exp(-\mathrm{i}\omega t)$, is the most common. In this representation, the governing equation for mode n becomes

$$-(r\phi')' + \left[\frac{n^2}{r} + \frac{nQ'}{n\Omega - \omega}\right]\phi = 0, \tag{2.12}$$

with ϕ standing for the Fourier transform of ψ', and where the prime now denotes differentiation with respect to r, as it will in the rest of this chapter. This is an eigenvalue equation for normal modes, assuming the usual boundary condition of decaying velocity at infinity, and a regularity condition at the origin.

Various classical theorems relating to the stability of solutions of (2.12) exist (Gent & McWilliams 1986). Rayleigh's theorem states that a necessary criterion for instability, i.e. $c_i \neq 0$, is that Q' must be zero somewhere in the domain, where $c_i = \mathrm{Im}\,(\omega/n)$ is the growth rate of the disturbance. A refinement of this criterion is Fjørtoft's theorem, namely that $Q'(r)[\Omega(r) - \Omega(r_s)] < 0$ for all r and some value of r_s. Finally, the semicircle theorem shows that c must be located in a semicircle in the complex c-plane according to $[c_r - \frac{1}{2}(\Omega_M + \Omega_m)]^2 + c_i^2 \leq [\frac{1}{2}(\Omega_M - \Omega_m)]^2$, where Ω_M and Ω_m are the maximum and minimum of Ω respectively.

2·2·The mode-one solution

Particular solutions to Rayleigh's equation are known for special basic states, but there is no general solution. Such a solution would correspond to a complete knowledge of the linear evolution for all basic states. However, for (2.5), the particular geometry of polar coordinates leads to a remarkable special case. The treatment is easiest in the Laplace variable, in which (2.9) becomes

$$-(r\phi_n')' + \left[\frac{n^2}{r} + \frac{\mathrm{i}nQ'}{p + \mathrm{i}n\Omega}\right]\phi_n = -\frac{rq_n^i}{p + \mathrm{i}n\Omega}, \tag{2.13}$$

where q_n^i is the initial mode-n vorticity. The actual form of the right-hand side is irrelevant, and it may be called $H(r; p)$ for the purpose of this analysis. Consequently, this exact solution will be valid even when the right-hand side forcing (or initial-value) term is different.

Two homogeneous solutions may be found for the mode-one Rayleigh operator. This was shown in Smith & Rosenbluth (1993). Certain formulas of Reznik & Dewar (1994) are equivalent to this solution, but the exact solution is not mentioned explicitly. Both of these cases are of particular interest since they treat physical systems other than standard two-dimensional vortex dynamics: the first deals with plasmas, the second with flow on the beta-plane. In both cases, though, the Rayleigh operator is the same. The derivation reproduced below here was obtained independently of both the above papers (and concurrently with the second). It is perhaps worth emphasising that this solution is valid for all basic states.

As first noted in Michalke & Timme (1967), Rayleigh's equation has an exact, steady, solution for mode one, given by $\phi^s = r\Omega(r)$. This was the only mode-one solution found numerically for the eigenvalue problem (2.12) by Gent & McWilliams (1986). It is known (Pryce 1993) that for classical Sturm–Liouville problems, in which the eigenvalue appears linearly, there may be pseudo-eigenvalues in the singular case, which do not possess all the usual properties of eigenfunctions (in particular they may fail to be normalisable). The solution ϕ^s, however, has all the usual eigenfunction properties. Michalke & Timme (1967) pointed out that their neutrally stable solution did not appear to be the limit of any other stable or unstable solutions.

The existence of this particular solution suggests the substitution

$$\phi_1 = r(p + \mathrm{i}\Omega)f(r) \tag{2.14}$$

to try and cancel the denominator in (2.13). The mode-one Rayleigh equation

$$-(r\phi_1')' + \left[\frac{1}{r} + \frac{\mathrm{i}Q'}{p + \mathrm{i}\Omega}\right]\phi_1 = -\frac{rq_1^i}{p + \mathrm{i}\Omega} \tag{2.15}$$

then reduces to

$$-\frac{1}{r(p + \mathrm{i}\Omega)}[r^3(p + \mathrm{i}\Omega)^2 f']' = -\frac{rq_1}{p + \mathrm{i}\Omega}. \tag{2.16}$$

This is possible, essentially, because of the particular form of Q' in plane polar coordinates, which is given by $Q' = 3\Omega' + r\Omega''$. This change of variable hence gives a quadrature of the equations of motion, or equivalently, two linearly independent solutions to the homogeneous Rayleigh operator. The first is clearly

$$\phi_1^A = r(p + \mathrm{i}\Omega). \tag{2.17}$$

The second may be written as

$$\phi_1^B = r(p + \mathrm{i}\Omega)\int_r^\infty \frac{\mathrm{d}u}{u^3(p + \mathrm{i}\Omega(u))^2} \tag{2.18}$$

for nonzero p, in which case it decays for large r. For zero p, however, the integral does not converge, in which case the upper limit may be replaced by any value r_0 except ∞. Changing the upper limit of integration corresponds to adding a multiple of ϕ_1^A.

For nonzero p, it is apparent that $\phi_1^A \sim r$ and $\phi_1^B \sim r^{-1}$ near the origin. This may also be deduced from the form of the governing operator (noting that $\Omega(r)$ is bounded at the origin). Similarly, $\phi_1^A \sim r$ and $\phi_1^B \sim r^{-1}$ for large r, since $Q'(r)$ must decay faster than r^{-1} for large r in the case of a localised vortex. For $p = 0$, the behaviour of the homogeneous solutions changes: ϕ_1^A becomes an eigenfunction (the steady solution mentioned above), still with behaviour r near the origin, but now decaying at infinity like $r\Omega(r)$, i.e. like r^{-1} (if the basic-state circulation does not vanish) or faster. The second homogeneous function still behaves like r^{-1} near the origin and like r far from the origin. It is now badly behaved at both endpoints of the range (as could be verified from the Wronskian corresponding to the Rayleigh operator).

For nonzero p, it is hence possible to construct a Green's function for the problem. Of course, there is no Green's function in the case of an eigenvalue. The Green's function then, when it exists, will satisfy the equation

$$-(rG')' + \left[\frac{1}{r} + \frac{\mathrm{i}Q'}{p + \mathrm{i}\Omega}\right]G = \delta(r - \xi). \tag{2.19}$$

Combining the two homogeneous solutions appropriately then gives

$$G(r,\xi) = r(p + \mathrm{i}\Omega(r))\xi(p + \mathrm{i}\Omega(\xi)) \int_{r_>}^{\infty} \frac{du}{u^3(p + \mathrm{i}\Omega(u))^2}, \tag{2.20}$$

writing $r_>$ to mean the greater of r and ξ.

2·3·The classical mode-one stability problem

2·3·1·Solution

The classical theory of flow instability has been of central importance in fluid dynamics for over a century. However, it was realised during the second half of this century that the classical approach of normal mode instability fails to completely describe the possible linear evolution of perturbations in unbounded inviscid flows. The continuous spectrum must also be taken into account. Maslowe (1985) gives a good overview of the ideas involved in the case of shear flows.

As mentioned in Section 2·1, Rayleigh derived a condition for the linear instability of a purely azimuthal flow, namely that the basic-state vorticity should have an extremum somewhere in the flow. The work of Michalke & Timme (1967) examined the stability of circular vortices produced by boundary layer breakdown. There has since also been a large body of related work in dynamical meteorology and oceanography, as observed structures in the atmosphere and oceans, such as tropical cyclones and warm-core rings, can be modelled by quasi-two-dimensional vortices (Flierl 1987, Hopfinger & van Heijst 1993).

Rayleigh's instability criterion for an azimuthal basic state is a condition for the existence of a growing normal mode perturbation. The related continuous spectrum seems to have attracted very little attention. For example, a well-known work on hydrodynamic instability, Drazin & Reid (1981), derives Rayleigh's condition for swirling flows, and mentions that a proper initial-value treatment could be developed, taking the continuous spectrum into account, but says no more. The following section completely solves the initial-value problem of the linear evolution of a mode-one perturbation. Previous results have been restricted to the normal modes of the system (Gent & McWilliams 1986).

The Laplace-transformed mode-one equation is

$$-(r\phi_1')' + \left[\frac{1}{r} + \frac{\mathrm{i}Q'}{p + \mathrm{i}\Omega}\right]\phi_1 = -\frac{rq_1^i}{p + \mathrm{i}\Omega}. \tag{2.21}$$

The solution may formally be derived by Green's functions, as in Case (1960);

this leads to a consistent solution to the initial-value problem, incorporating the continuous spectrum. In the parallel-flow case with lateral boundaries, considered by Case, the underlying Sturm–Liouville problem is regular, and a Green's function can always be constructed formally. In the geometry considered here, however, the interval is infinite, and the underlying Sturm–Liouville problem is no longer regular.

For definiteness, the basic-state vorticity will be taken to be bounded in magnitude, and localised in space, as in Chapter 1, and the physical domain under consideration will be taken to be unbounded – this is important, as will be seen later. The basic-state angular velocity will be taken to be monotonic decreasing and positive. This corresponds to an initial circular vortex monopole. Consequently, as shown in Chapter 1, the far-field angular velocity will be of the form

$$\Omega = \frac{\Gamma}{2\pi r^2} + O(r^{-\infty}), \tag{2.22}$$

where Γ is the circulation (which is positive). The basic-state vorticity will also be taken to be piecewise continuous or smoother, which permits distributions such as the Rankine vortex. The initial perturbation vorticity will be taken to be localised and continuous.

The governing equation may be solved using the Green's function derived previously, but it is easier to integrate it directly twice. The same substitution as before, $\phi_1 = r(p + \mathrm{i}\Omega)f$, leads to

$$\frac{1}{r(p+\mathrm{i}\Omega)}\left[r^3(p+\mathrm{i}\Omega)^2 f'\right]' = \frac{rq_1^i}{p+\mathrm{i}\Omega}. \tag{2.23}$$

The most general solution to this equation is

$$f = -\int_r^\infty \frac{\mathrm{d}v}{v^3(p+\mathrm{i}\Omega(v))^2}\int_0^v u^2 q_1^i(u)\,\mathrm{d}u + A(p) + B(p)\int_r^\infty \frac{\mathrm{d}v}{v^3(p+\mathrm{i}\Omega(v))^2}. \tag{2.24}$$

The boundary condition at infinity may be rewritten as $r(p + \mathrm{i}\Omega)f \to 0$; this is why the upper limit for the outer integral has been taken to be infinity. This boundary condition must be satisfied for all p, so A must be zero. Near the origin, the inner integral behaves like v^3, since the initial perturbation vorticity is bounded near the origin (as well as everywhere else over the domain). Hence B must be zero; otherwise f would behave like r^{-2}, which would be unacceptable. This leads to the solution

$$\phi_1 = -r(p+\mathrm{i}\Omega(r))\int_r^\infty \frac{m(v)}{(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v, \tag{2.25}$$

where

$$m(v) = \frac{1}{v^3}\int_0^v u^2 q_1^i(u)\,\mathrm{d}u = \frac{\partial}{\partial v}\left(\frac{\psi_1^i(v)}{v}\right). \tag{2.26}$$

The inverse Laplace transform of the solution is given by

$$\begin{aligned}
\psi_1 &= -r\int_r^\infty [1 + t(-\mathrm{i}\Omega(v) + \mathrm{i}\Omega(r))]\, m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v && (2.27)\\
&= -r\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\left[t\int_r^\infty m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v\right] && (2.28)\\
&= \psi_1^i(r)\mathrm{e}^{-\mathrm{i}\Omega(r)t} - \mathrm{i}r\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\left[t^2\int_r^\infty \frac{\psi_1^i(v)}{v}\Omega'(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v\right]. && (2.29)
\end{aligned}$$

It clearly satisfies the initial condition. The special case $m = \Omega'$ leads to the solution $\psi_1 = r\Omega$. This is to be expected, since that choice of m corresponds to taking $r\Omega$ as the initial condition, and this is known to be a steady solution.

The corresponding expression for the vorticity may be found from (2.13), using the fact that $\exp(-\mathrm{i}\Omega t)$ is annihilated by the differential operator $\partial_t + \mathrm{i}\Omega$. Hence

$$q_1 = -\mathrm{i}Q'\,t\int_r^\infty m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v + q_1^i(r)\mathrm{e}^{-\mathrm{i}\Omega(r)t}. \tag{2.30}$$

The vorticity is always localised in space, and the far-field behaviour of ψ_1 is hence unaltered.

2·3·2·Large-time behaviour

The solution for ψ_1 may be examined in the limit of large t. It is useful to rewrite the integral in (2.28) as

$$I = \int_0^{1/r} m\left(\frac{1}{s}\right)\mathrm{e}^{-\mathrm{i}\Omega(1/s)t}\,\frac{\mathrm{d}s}{s^2}. \tag{2.31}$$

The limit of large time corresponds to very rapid oscillation in the exponential function, while m varies comparatively slowly. The method of steepest descent may be used to obtain an asymptotic approximation to the integral, but care needs to be taken, since Ω may not have simple analytic behaviour at critical points. The behaviour of I will depend on the circulation of the basic flow, since that quantity determines the form of the flow at infinity, which is the critical point $s = 0$ of the integral I.

Localised circulation

For basic states whose vorticity vanishes beyond a certain radius r_M, the angular velocity beyond that distance is $\Gamma/2\pi r^2$. The angular velocity and m must both be

continuous at this radius, although the vorticity may have a jump there (as in the case of the Rankine vortex). Then I may be rewritten as

$$I = \int_0^{1/r_P} m\left(\frac{1}{s}\right) \mathrm{e}^{-\mathrm{i}\Gamma s^2 t/2\pi} \frac{\mathrm{d}s}{s^2} + \int_{1/r_P}^{1/r} m\left(\frac{1}{s}\right) \mathrm{e}^{-\mathrm{i}\Omega(1/s)t} \frac{\mathrm{d}s}{s^2}, \tag{2.32}$$

when r_P is the greater of r and r_M. If $r_M < r$, the second integral will be absent.

The first integral may be evaluated as an asymptotic series by the method of steepest descent, by treating $s = x + \mathrm{i}y$ as a complex variable. The path is transformed onto the steepest descent contour starting from the origin, going to $e^{-\mathrm{i}\pi/4}\infty$, and returning along the hyperbola $x^2 - y^2 = 1/r_P^2$. The behaviour of m is only required near the origin and the point $1/r_P$. The latter may be derived by an extension of the arguments applied to localised vortices in Chapter 1. From (1.27),

$$\psi_1 = -\frac{1}{2r}\int_0^r u^2 q(u)\,\mathrm{d}u + -\frac{r}{2}\int_r^\infty q(u)\,\mathrm{d}u, \tag{2.33}$$

so the far-field behaviour of the streamfunction is given by

$$\psi_1 = -\frac{M}{2r} + O(r^{-\infty}), \tag{2.34}$$

where

$$M = \int_0^\infty u^2 q_1(u)\,\mathrm{d}u \tag{2.35}$$

is related to the first moment in the multipole expansion of ψ. The definition of m then leads to

$$m = \frac{M}{r^3} + O(r^{-\infty}) \tag{2.36}$$

for large r.

Parametrising the contour from the origin by*

$$s = \left(\frac{2\pi}{\Gamma}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} u^{\frac{1}{2}} \tag{2.37}$$

*Llewellyn Smith (1995) used the local parametrisation

$$s = \left(\frac{2\pi}{\Gamma}\right)^{\frac{1}{2}} \mathrm{e}^{-\frac{\mathrm{i}\pi}{4}} u^{\frac{1}{2}} + O(u^{\frac{3}{2}}).$$

This resulted in a weaker bound in (2.39), namely $O(t^{-2})$, but made no difference to any other equation.

(where u is real, since it describes a constant phase contour) leads to a contribution to (2.32) of

$$I_1^0 = -\frac{2\mathrm{i}\pi M}{\Gamma}\int_0^\infty \mathrm{e}^{-ut}\left[u^{\frac{3}{2}} + O(u^\infty)\right]\frac{u^{-\frac{1}{2}}}{2u}\,\mathrm{d}u \tag{2.38}$$

$$= -\frac{\mathrm{i}\pi M}{\Gamma t} + O(t^{-\infty}). \tag{2.39}$$

The notation $\Omega'(r_P^+)$ will be used to highlight the discontinuity in the slope of Ω at r_P. The second path may be locally parametrised near the point $s = 1/r_P$ by

$$s = \frac{1}{r_P} + \frac{\mathrm{i}u}{r_P^2\Omega'(r_P^+)} + O(u^2); \tag{2.40}$$

this comes from expanding the equation $-\mathrm{i}\Omega(1/s) = -u - \mathrm{i}\Omega(r_P)$ in powers of u. The contribution from this path to the first integral is

$$I_1^1 = \mathrm{e}^{-\mathrm{i}\Omega(r_P)t}\int_\infty^0 \mathrm{e}^{-ut}\left[r_P^2 m(r_P) + O(u)\right]\left[\frac{\mathrm{i}}{r_P^2\Omega'(r_P^+)} + O(u)\right]\mathrm{d}u \tag{2.41}$$

$$= -\frac{\mathrm{i}m(r_P)}{\Omega'(r_P^+)t}\mathrm{e}^{-\mathrm{i}\Omega(r_P)t} + O\left(\frac{1}{t^2}\right). \tag{2.42}$$

The second integrand has no stationary points, and the integral can be evaluated asymptotically by repeated integration by parts. This is more easily done in the original variable. Thus

$$I_2 = \int_r^{r_P} m(v)\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v \tag{2.43}$$

$$= \frac{\mathrm{i}}{t}\left[\frac{m(r_P)\mathrm{e}^{-\mathrm{i}\Omega(r_P)t}}{\Omega'(r_P^-)} - \frac{m(r)\mathrm{e}^{-\mathrm{i}\Omega(r)t}}{\Omega'(r)}\right]_r^{r_P} - \frac{\mathrm{i}}{t}\int_r^{r_P}\left(\frac{m(v)}{\Omega'(v)}\right)'\mathrm{e}^{-\mathrm{i}\Omega(v)t}\,\mathrm{d}v. \tag{2.44}$$

This integration by parts may be repeated to give an asymptotic series in t^{-1}. There are now two cases, depending on whether Ω is smooth at r_M. If Ω is smooth there, i.e. if $\Omega'(r_M) = -\Gamma/\pi r_M^3$, then all the terms in r_P will cancel with the equivalent terms in I_1, and the leading-order behaviour of I is given by

$$I = -\frac{\mathrm{i}\pi M}{\Gamma t} - \frac{\mathrm{i}m(r)}{\Omega'(r)t}\mathrm{e}^{-\mathrm{i}\Omega(r)t} + O\left(\frac{1}{t^2}\right). \tag{2.45}$$

This is also the solution when the second integral is not present, and $r_P = r$. The second term is annihilated by the $\partial_t + \mathrm{i}\Omega$ operator in front of the integral, so the large time behaviour of the solution is given by

$$\psi_1 = -\frac{\pi M}{\Gamma}r\Omega(r) + O\left(\frac{1}{t}\right). \tag{2.46}$$

This proves that any smooth initial distribution with compact vorticity support, and nonzero circulation, tends to the steady solution found by Michalke and Timme.

If Ω is not smooth, then the exponential terms in r_P will not cancel, and the leading behaviour will be given by

$$\begin{aligned}\psi_1 &= -\frac{\pi M}{\Gamma} r\Omega(r) - r[\Omega(r) - \Omega(r_P)]\mathrm{e}^{-\mathrm{i}\Omega(r_P)t} m(r_P)\left[\frac{1}{\Omega'(r_P^+)} - \frac{1}{\Omega'(r_P^-)}\right] \\ &\quad + O\left(\frac{1}{t}\right). \end{aligned} \tag{2.47}$$

Any localised, discontinuous basic-state vorticity distribution will therefore tend to the steady solution in the region with zero vorticity. However, there will be an oscillatory term in the region inside the discontinuity. It is clear that this argument may be extended to any basic-state profile with jumps in Ω', by decomposing the integral into suitable portions. Each point of discontinuity will give an oscillating contribution exactly as above. There will be no oscillatory contribution beyond the point of discontinuity with the largest radius.

Nonvanishing circulation

The above argument may be modified to cover the case of a basic-state profile with nonzero circulation. Such a state has the following form for the angular velocity near the origin of the s-plane: $\Omega \sim \Gamma s^2/2\pi + O(s^\infty)$, i.e. the correction near the origin decays faster than any power of s. The correction term is flat to all algebraic orders near $s = 0$, and hence should not contribute, to any algebraic order, to the asymptotic behaviour of the solution for large t.

This can be formalised as follows. The integral may be split into two sub-ranges:

$$I = \left(\int_0^\delta + \int_\delta^{1/r}\right) \mathrm{e}^{-\mathrm{i}\Omega(1/s)t} m\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s^2}, \tag{2.48}$$

where $\delta \ll 1$. The first integral can then be approximated by the far-field Rankine vortex behaviour, since the basic-state profile is exponentially close to the Rankine vortex profile in the range of integration. Then

$$\int_0^\delta \mathrm{e}^{-\mathrm{i}\Omega(1/s)t} m\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s^2} = \int_0^\delta [1 + O(s^2 t)][Ms + O(s^\infty)]\,\mathrm{d}s \tag{2.49}$$

$$= \frac{M}{2}\delta^2 + O(\delta^4 t). \tag{2.50}$$

The second integral may be evaluated to yield

$$\int_\delta^{1/r} \mathrm{e}^{-\mathrm{i}\Omega(1/s)t} m\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s^2} = \frac{\mathrm{i}}{t}\left[\frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t} m(v)}{\Omega'(v)}\right]_r^{1/\delta} - \frac{\mathrm{i}}{t}\int_r^{1/\delta}\left(\frac{m(v)}{\Omega'(v)}\right)' \mathrm{e}^{\mathrm{i}\Omega(v)t}\,\mathrm{d}v$$

$$= -\frac{\mathrm{i}m(r)}{\Omega'(r)t}\mathrm{e}^{-\mathrm{i}\Omega(r)t} - \frac{\mathrm{i}\pi M}{\Gamma t} + O(\delta^{\infty})$$
$$-\frac{\mathrm{i}}{t}\int_r^{1/\delta}\left(\frac{m(v)}{\Omega'(v)}\right)'\mathrm{e}^{\mathrm{i}\Omega(v)t}\,\mathrm{d}v. \tag{2.51}$$

The first term is the normal contribution from the integration by parts. If the second term is integrated by parts again, the function multiplying the exponential will become unbounded for small δ. However, further integration is unnecessary, since the Riemann–Lebesgue lemma gives

$$\int_r^{1/\delta}\left(\frac{\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega'(v)t}\right)m'(v)\,\mathrm{d}v = o\left(\frac{1}{t}\right) \tag{2.52}$$

for large t. The $O(\delta^2)$ term can be removed by taking $\delta \ll t^{-1/2}$. Thus

$$I \sim -\frac{\mathrm{i}M\pi}{\Gamma t} - \frac{\mathrm{i}m(r)}{\Omega'(r)t}\mathrm{e}^{-\mathrm{i}\Omega(r)t} + o\left(\frac{1}{t}\right). \tag{2.53}$$

This is essentially the same result as in the previous case, i.e.

$$\psi_1 \sim -\frac{\pi M}{\Gamma}r\Omega(r) + o(1) \tag{2.54}$$

for large t. The argument clearly carries through if Ω is not smooth, and oscillatory terms will be forced exactly as before.

Zero circulation

For basic states with zero circulation, the Taylor series of the term $\mathrm{i}\Omega(1/s)$ in the exponential vanishes at the origin (i.e. its righthand derivatives all vanish at that point). This corresponds to an essential singularity in the complex s-plane. The method of steepest descent is no longer valid, and a different procedure must be followed to obtain the large time behaviour. The Riemann–Lebesgue lemma shows that $I(t)$ must vanish for large t. The failure of steepest descents shows that the decay must be slower than any inverse power of t, hence $tI(t)$ must be larger than $O(1)$ in the large t limit. This means that the solution grows without bound, albeit more slowly than t, since $I(t)$ decays in time.

As an example, consider the Gaussian streamfunction $\Psi = -\exp(-r^2)/2$, which has zero circulation. The associated basic-state angular velocity is $\Omega = \exp(-r^2)$, so the integral to be evaluated is

$$I = \int_0^{1/r}\exp\left(-\mathrm{i}\mathrm{e}^{-1/s^2}t\right)m\left(\frac{1}{s}\right)\frac{\mathrm{d}s}{s^2}. \tag{2.55}$$

This is very similar to an integral treated by Bender & Orszag (1978, Section 6.6,

Example 3). The change of variable $v = \mathrm{i}\exp(-1/s^2)$ leads to the integral

$$I = \int_0^{\mathrm{i}\exp(-r^2)} \mathrm{e}^{-vt} m\left(\frac{1}{s}\right)\frac{1}{s^2}\frac{\mathrm{d}s}{\mathrm{d}v}\,\mathrm{d}v, \tag{2.56}$$

where s is a function of v. The contours of stationary phase in the v-plane are the real axis, and the line $v = x + \mathrm{i}\exp(-r^2)$ (for $0 \le x < \infty$). The contribution from the point $\mathrm{i}\exp(-r^2)$ is exponentially small compared with that from the origin, so the leading-order behaviour is obtained by considering

$$I \sim \int_0^{\infty} \mathrm{e}^{-vt} m\left(\frac{1}{s}\right)\frac{1}{s^2}\frac{\mathrm{d}s}{\mathrm{d}v}\,\mathrm{d}v. \tag{2.57}$$

Integrating once by parts removes the apparent singularity at the origin and leads to

$$I \sim -t\int_0^{\infty} \mathrm{e}^{-vt}\psi_1^i\left(\frac{1}{s}\right) s\,\mathrm{d}v. \tag{2.58}$$

Following the steps in Bender & Orszag (1978), and using $\psi_1^i \sim -Ms/2 + O(s^{\infty})$, leads to

$$I = -\frac{M}{2\ln t}\left\{1 - \frac{\mathrm{i}\pi/2 + \gamma_E}{\ln t} + O\left(\frac{1}{(\ln t)^2}\right)\right\}, \tag{2.59}$$

where $\gamma_E = 0.5772\ldots$ is Euler's constant. An equality sign may be used due to the presence of an explicit order term. Hence the long-time behaviour of ψ_1 is given by†

$$\psi_1 = \frac{\mathrm{i}Mt}{2\ln t} r\Omega(r) + O\left(\frac{t}{(\ln t)^2}\right). \tag{2.60}$$

There is a phase shift of $\pi/2$ in the solution, since it is the imaginary part which dominates in the long-time limit. It is clear that angular velocities of the form $\exp(-r^a)$ will lead to growth rates of the form $(\ln t)^{-2/a}$.

2·3·3·Discussion

The exact solution to the mode-one linearised vorticity equation for an azimuthal basic state has been derived in this section. For basic states with nonzero circulation and continuous vorticity, the perturbation tends to the steady solution first described in Michalke & Timme (1967). This explains why the steady solution is not the limit of any other amplified or decaying normal mode of the system: there are none. This had already been shown by Reznik & Dewar (1994). Discontinuities in the basic-state vorticity lead in addition to oscillatory behaviour, with frequency given by the

†There are some typographical errors in this and the preceding equation in Llewellyn Smith (1995). The conclusions of that paper are unaffected.

angular velocity of the basic flow at the point of discontinuity. These oscillations may be interpreted as waves propagating on the basic-state vorticity discontinuity (i.e. Rossby waves), whose influence is felt over the disc bounded by the furthest radius of discontinuity of the vorticity.

Distributions with zero circulation eventually grow without bound as t increases, at a rate slower than t. The disturbances exhibit a $\pi/2$ phase shift in general for large time. Vortices with zero circulation have attracted a lot of attention in geophysical applications as was mentioned in Chapter 1. However, the preceding result shows that these isolated eddies are potentially unstable to any perturbation with a mode-one component.

The large-time limiting behaviour is not a uniform limit. The higher-order terms in (2.44) will not decay for large r, and neither will the higher-order terms in (2.60). However, these equations can only be expected to hold for $\Omega(r)t \gg 1$. In the nonzero circulation case, this corresponds to $\Gamma t \gg r^2$ for large r. For zero circulation, the corresponding condition does not take such a simple form.

The evolution of the initial perturbation is entirely associated with the continuous spectrum. Nonlinear effects are not considered. However, when the continuous spectrum can grow in time, it must be of importance in the appearance of nonlinear effects, even though the basic state is stable to normal mode-one disturbances. Of course, the nonlinear evolution of the system will involve mode coupling, but the possibility of nonlinear growth being triggered by the algebraic growth of the continuous spectrum component of the mode-one deformation is very intriguing, and shows the vital importance of the circulation of the basic state.

2·4·Higher modes – the connection problem

2·4·1·The connection problem

As mentioned previously, the Rayleigh operator governs the linearised behaviour of a variety of inviscid fluid systems, such as (2.11). In some, the 'small' term $\epsilon\mathcal{L}$ may not in fact be small everywhere in the domain under consideration. For the beta-plane problem, as shown in Chapter 1, this term becomes comparable in magnitude to the partial time-derivative term for large values of r (specifically for $\epsilon r = O(1)$). Thus, while only solutions that decay at infinity are of interest in the study of the Rayleigh equation, solutions to the full beta-plane problem, whose behaviour in the near field is governed by an inhomogeneous Rayleigh equation, may have different behaviour

in the far field.

Thus the Green's function derived in §2·2 is not necessarily the most interesting homogeneous solution of systems of the form

$$-(r\phi_n')' + \left[\frac{n^2}{r} + \frac{\mathrm{i}nQ'}{p + \mathrm{i}n\Omega}\right]\phi_n = 0 \tag{2.61}$$

(it is the unforced equation that is of interest here) – strictly speaking, it is not really a purely homogeneous solution at all, more a judicious combination of homogeneous solutions. The relevant homogeneous solution to consider is ϕ_n^A, which may be defined as the homogeneous solution that is well-behaved at the origin. For mode one, ϕ_1^A is known from (2.17). For higher modes, however, there is no general analytic expression for ϕ_n^A. The solution ϕ_n^A will not be bounded in the far field. Analysis of the governing equation shows that, in fact, $\phi_n^A \sim r^n$ in the far field. To be more precise, the asymptotic behaviour of ϕ_n^A is given by

$$\phi_n^A = \Delta r^n + \Lambda r^{-n} + O(r^{-\infty}), \tag{2.62}$$

owing to the rapid decay of Q' for large r. From the definition of ϕ_n^A, its behaviour near the origin is, with an appropriate normalisation that is consistent with (2.17),

$$\phi_n^A = (p + \mathrm{i}n\Omega_0)r^n + \frac{4\mathrm{i}n\Omega_0''}{2n+4}r^{n+2} + O(r^{n+3}). \tag{2.63}$$

The coefficient $\Delta = \Delta(p)$ may be called the 'connection coefficient', since it relates the behaviour of the solution near the regular singular point at the origin to the behaviour of the solution near the irregular singular point of rank infinity at infinity. It is related to the usual connection coefficient (cf. Olver 1974).

In particular, Δ will give the leading-order behaviour of the homogeneous solution ϕ_n^A in the far field, where it needs to be matched onto an outer solution. Thus the coefficient Δ is crucial to the calculation of a matched asymptotic expansion solution to the beta-plane problem for higher modes, since the inner field is given by an inhomogeneous Rayleigh equation. The inverse Laplace transform of $\Delta(p)$ will be denoted by $D(t)$. It will enter expressions as part of a convolution rather than as a product as is the case for $\Delta(p)$ in the Laplace variable. However, it is easier to represent than $\Delta(p)$, since it is a function of a real variable.

2·4·2·The mode-one solution

The value of the connection coefficient $\Delta(p)$ may easily be calculated from the knowledge of the mode-one homogeneous solutions ϕ_1^A. The ratio of the coefficient

of r of the large-r solution to the small-r solution gives, from (2.17),

$$\Delta_1 = \frac{p}{p + \mathrm{i}\Omega_0}. \tag{2.64}$$

This expression may be transformed back into the time variable. This leads to

$$D_1 = \delta(t) - \mathrm{i}\Omega_0 \mathrm{e}^{-\mathrm{i}\Omega_0 t}. \tag{2.65}$$

2·4·3·The exact solution for the Rankine vortex

For the Rankine vortex with vorticity $Q = 2\Omega_0$ inside a disc of radius d, the governing equation may be solved exactly. In this case, the vorticity gradient is given by a delta-function:

$$Q' = -2\Omega_0 \delta(r - d). \tag{2.66}$$

This corresponds to an angular velocity distribution

$$\Omega = \Omega_0 \left[1 - H(r - d) + H(r - d)\frac{d^2}{r^2} \right], \tag{2.67}$$

where H is the Heaviside step function. Then (2.13) becomes

$$f'' + \frac{1}{r}f' - \frac{n^2}{r^2}f + \frac{2\mathrm{i}n\Omega_0}{r(p + \mathrm{i}n\Omega_0)}\delta(r - d)f = 0. \tag{2.68}$$

Picking the solution that is well-behaved near the origin leads immediately to

$$f = r^n \tag{2.69}$$

for $r < d$ and

$$f = Ar^n + \frac{B}{r^n} \tag{2.70}$$

for $r > d$. The constants A and B may be determined by demanding continuity of the solution across the point $r = d$, and by imposing the jump condition obtained by integrating (2.68) across that point. The latter is

$$[f']_d + \frac{2\mathrm{i}n\Omega_0}{d(p + \mathrm{i}n\Omega_0)}f(d) = 0. \tag{2.71}$$

Then

$$d^n = Ad^n + \frac{B}{d^n} \tag{2.72}$$

and

$$nAd^{n-1} - \frac{nB}{d^{n+1}} - nd^{n-1} + \frac{2\mathrm{i}n\Omega_0}{p + \mathrm{i}n\Omega_0} d^{n-1} = 0. \tag{2.73}$$

Solving these two equations leads to

$$A = 1 - \frac{\mathrm{i}\Omega_0}{p + \mathrm{i}n\Omega_0}, \tag{2.74}$$

$$B = \frac{\mathrm{i}\Omega_0 d^{2n}}{p + \mathrm{i}n\Omega_0}. \tag{2.75}$$

The connection coefficient is in fact A, since that is the ratio between the powers of r^n and r^n in the dominant term at infinity and the well-behaved term at the origin respectively. This result agrees with the mode-one solution (2.64), noting that for the Rankine vortex considered above $\Gamma = 2\pi\Omega_0 d^2$. In the time variable, the connection coefficient is

$$D = \delta(t) - \mathrm{i}\Omega_0 \mathrm{e}^{-\mathrm{i}n\Omega_0 t}. \tag{2.76}$$

2·4·4·Numerical solution

The only general method available for solving the connection problem is numerical integration. However, the dominant solution at infinity grows without bound and it is hence more practical to calculate the function $g = f/r^n$, which satisfies the equation

$$g'' + \frac{2n+1}{r} g' - \frac{\mathrm{i}nQ'}{r(p + \mathrm{i}n\Omega)} g = 0. \tag{2.77}$$

The desired bounded solution has the behaviour $g = 1 + O(r^2)$ near the origin, and this may be used as its starting value in the integration. Its behaviour near infinity is also $O(1)$, while the other solution decays like r^{-2n}. Hence the limiting value of g will provide the desired connection coefficient $\Delta(p)$. The governing equation has a singularity at $p + \mathrm{i}\Omega(r) = 0$, which may lead to singularities in $\Delta(p)$ for $0 < \mathrm{Im}\, p < \Omega_M$. There is no singularity in $D(t)$, although it turns out to have delta-function behaviour.

To calculate Δ, the conditions $g_0 = 1$ and $g_0' = 0$ may be imposed at a starting point close to the origin and integrating to a large value leads to a limiting value $g_\infty = \Delta(p)$. The special solution for mode-one provides a check for the numerical procedure. Figure 2.1 shows the computed values of the real and imaginary parts of Δ for different arguments of p and $|p| \leq 6$ for a Gaussian vortex base profile $Q = \exp(-r^2)$, for which $\Omega_0 = 1/2$. The values match perfectly with the theoretical

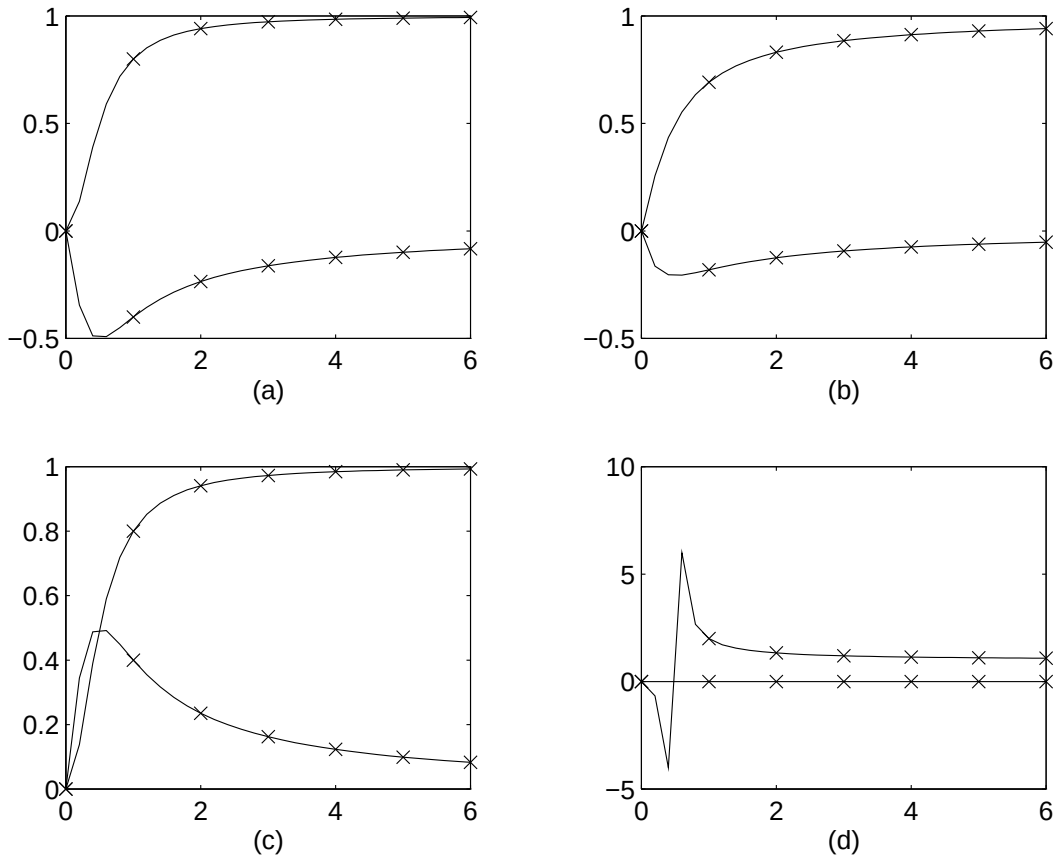

Figure 2.1: Computed values of Δ for mode one, plotted again $|p|$ for different values of $\arg p$. The basic state is the Gaussian vortex, for which $\Omega_0 = 1/2$. The cases correspond to different values of $\arg p$: (a) is 0, (b) $\frac{\pi}{4}$, (c) π and (d) $-\frac{\pi}{2}$. Crosses are the analytical mode-one coefficients.

result

$$\Delta = \frac{p}{p + \mathrm{i}\Omega_0}. \tag{2.78}$$

The connection coefficient has a simple pole at $p = -\mathrm{i}\Omega_0$, which is poorly resolved in (d).

The corresponding result for mode two is shown in figure 2.2. The crosses are the values of Δ for the Rankine vortex. They asymptote to the correct value for large $|p|$, but are wrong for smaller $|p|$. This shows that the structure of the vortex influences Δ. The asymptotic behaviour may however be enough to give a good approximation to the actual solution. The numerical procedure becomes very inaccurate near the singularity of $\Delta(p)$, which is somewhere in the range $-1.2 < \mathrm{Im}\, p < 0$. Presumably, it is actually at $p = -\mathrm{i}$, which would correspond to $p + \mathrm{i}m\Omega_0 = 0$ for mode two.

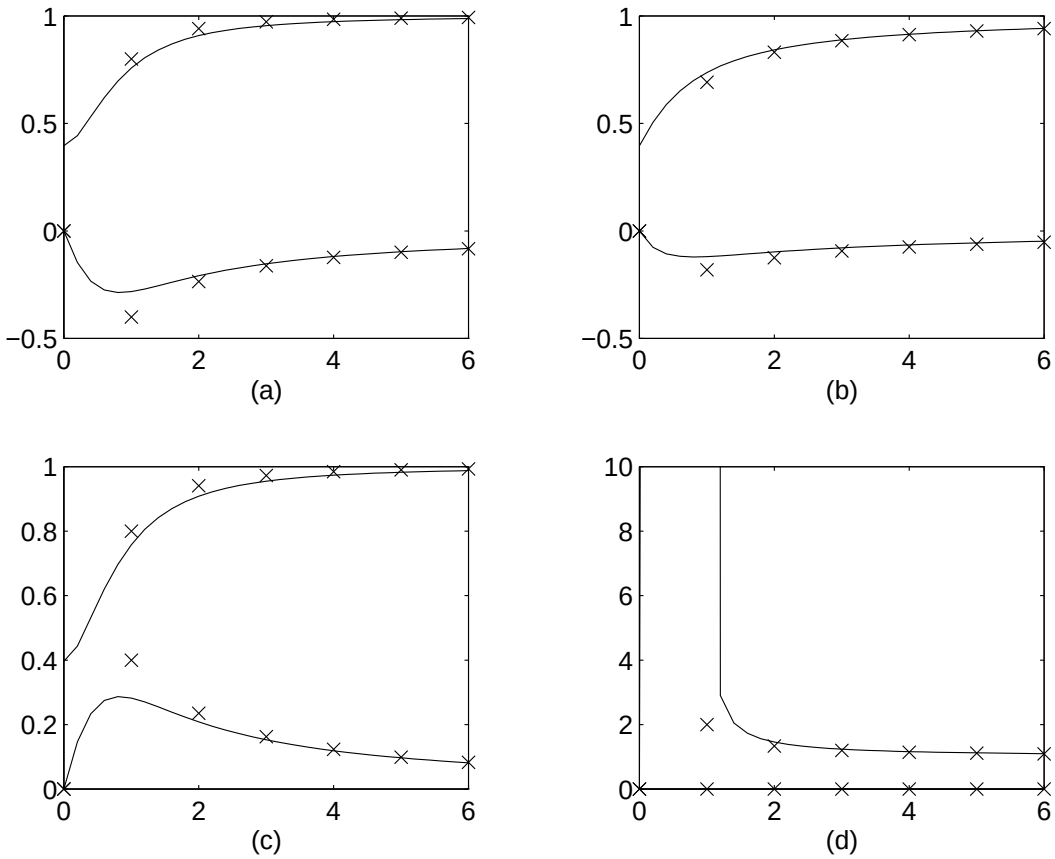

Figure 2.2: As figure 2.1, but for mode two. The numerical procedure is inaccurate in case (d) near the singularity of $\Delta(p)$.

The value of the real and imaginary parts of $D^r(t) = D(t) - \delta(t)$[‡] for modes one, two and ten is shown in figure 2.3 (again for the Gaussian vortex). The inverse Laplace transform is evaluated using Talbot's algorithm, which is described in Talbot (1979). It is apparent that $D^r(t)$ behaves like $\mathrm{e}^{-\mathrm{i}n\Omega_0 t}$ for small t, which shows that the pole at $p = -\mathrm{i}n\Omega_0$ governs the oscillatory behaviour of $D^r(t)$. For large t, however, there are no appropriate Tauberian theorems to give the decay rate.

2·5·Conclusions

This chapter has explored some aspects of the radial Rayleigh equation. Its exact solution for mode one, valid for all basic states, has been derived, and the large-time asymptotics of the solution have been investigated. For basic states with nonzero circulation, the mode-one solution to Rayleigh's equation tends to the steady mode-

[‡]The r stands for regular, i.e. no delta-function.

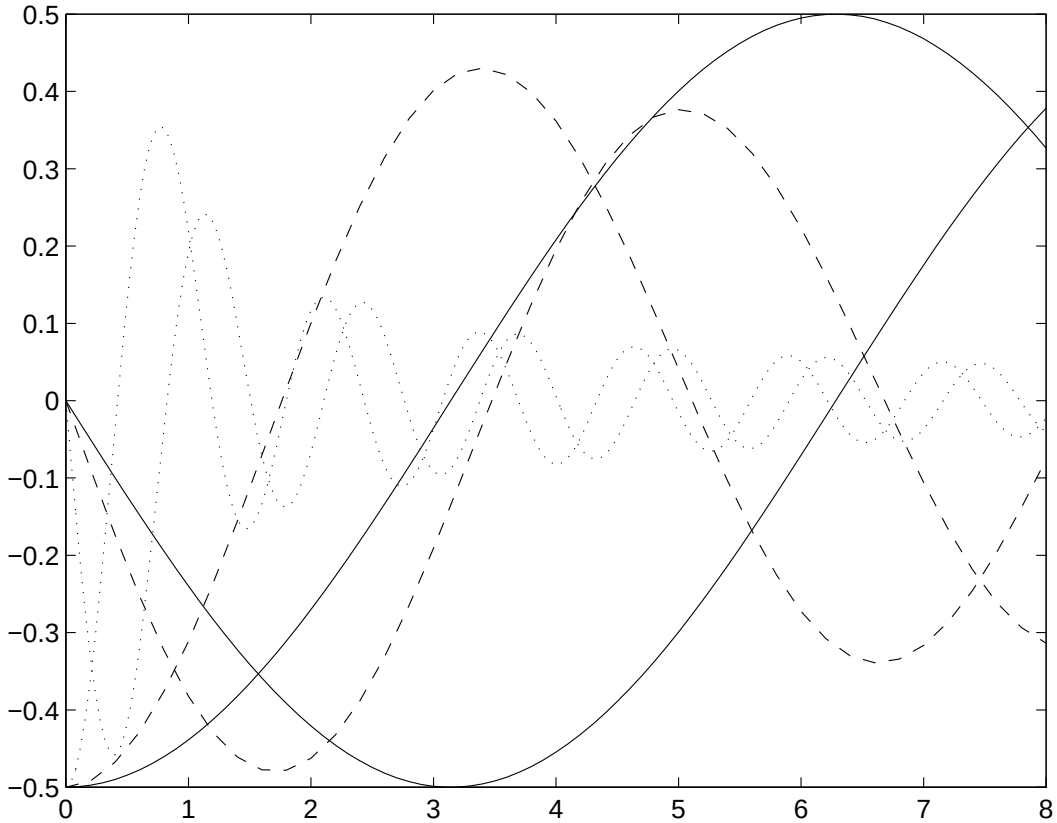

Figure 2.3: Computed values of $D^r(t) = D(t) - \delta(t)$ for modes one, two, and ten. The value of $D^r(t)$ at $t = 0$ is $-\mathrm{i}/2$. The basic state is the Gaussian vortex. The solid line is mode one, the dashed line mode two, and the dotted line mode ten.

one solution. For states with zero circulation, it grows without bound. Montgomery & Kallenbach (1995) found qualitatively the same large-time behaviour for the particular angular velocity profile $\Omega(r) = 2/(1 + r^2)$ (which does not have localised vorticity).

For modes other than one and zero (where the behaviour is trivial), solutions must be sought numerically. The function $\Delta(p)$ in the Laplace variable connects the near-field to the far-field behaviour, and is relevant to the matching when considering the full beta-plane problem at second order.

Numerical solutions of Rayleigh's equation have shown that many of the profiles used in tropical cyclone studies are stable at all modes. Naturally, all support the steady mode-one response. Gent & McWilliams (1986) showed that the Gaussian vorticity profile was stable, while Michalke & Timme (1967) showed that the Rankine vortex was stable. Each vortex profile, however, must be investigated separately.

Weber & Smith (1993) showed that the vortex profiles of Chan & Williams (1987)

and Willoughby (1988) were stable to all normal-mode disturbances, although the WA- and WB-profiles of the latter had a neutrally stable mode-one normal mode. In any case, the circulation is zero, so the initial-value analysis of this chapter for nonzero Γ is not relevant. It is not clear how the initial-value problem is related to the existence of normal modes. The former should be solved exactly for these profiles to see whether the normal modes found numerically are actually relevant to the evolution of the vortex. Weber & Smith (1993) also showed that one of the two profiles of Smith & Ulrich (1991) was unstable to mode-two perturbations. However, all the profiles they considered have zero circulation, and are hence unstable to the algebraic instability considered in this chapter, although it is not clear how this instability changes on the beta-plane. They do mention that the mode-one steady solution arises in numerical calculations of cyclone motion on the beta-plane, but attribute it to inaccuracies in calculating the vortex centre. It could be argued, however, that this response, is a fundamental aspect of the evolution of vortices, whether mode one is forced by the beta-effect or not, and not just a numerical artifact. In addition, numerical simulations are by necessity carried out on finite domains, where the response of the system is different (see Smith & Rosenbluth 1990). Nevertheless, their assertion that calculations of vortex-stability on an f-plane are relevant to tropical cyclone motion is probably correct.

In the situation considered in this dissertation, with a purely radial vortex evolving from rest on the beta-plane, barotropic instability for all modes other than one is irrelevant, since there is no initial disturbance to evolve. Numerical inaccuracies will certainly ensure that this is not the case for numerical computations. However, the nonisolated vortices considered in later chapters are stable. In addition, the timescale of the growth rate of the unstable vortex of Smith & Ulrich (1991) for example is $3.775 \times 10^{-5}\,\mathrm{s}^{-1}$, which is of the order of a day, and probably longer than the times on which the asymptotic solution to the problem considered in this dissertation remains valid. Hence barotropic instability will not be considered in the rest of this dissertation.

In the nondivergent case, ϕ^s is still a steady solution, but the evolution of the system must be considered numerically. The results of Chapter 6 imply that the vortices considered in that chapter are stable, since the mode-one forcing of the vortex by the beta-effect does not trigger any exponential instability.

3·The linear Rossby wave equation

3·1·The linear regime

Far from the vortex, where amplitudes are low, the governing equations of motion should be approximately linear, irrespective of the size of the beta-effect in comparison with the strength of the vortex. This is true for a large variety of nonlinear systems supporting waves, and has been used extensively in the aerodynamic theory of sound for example (Lighthill 1952). The aim of this chapter is not, however, to derive a theory of waves generated 'betaly', but to understand the aspects of the linear behaviour of the system appropriate to the situation examined in this dissertation.

The nondivergent linear vorticity equation on the beta-plane, recast in the variable $\mathbf{r}$, is

$$\frac{\partial}{\partial t}\nabla^2\phi + \epsilon\frac{\partial\phi}{\partial x} = 0. \tag{3.1}$$

The value of beta (albeit represented by ϵ) has been reintroduced by using $\mathbf{r}$. The equation (3.1) has wave solutions, known generally as Rossby waves. These are dispersive waves with dispersion relation

$$\sigma = -\frac{\epsilon l}{l^2 + m^2} \tag{3.2}$$

for a two-dimensional plane wave solution of the form $\exp(\mathrm{i}\mathbf{k}.\mathbf{r} - \mathrm{i}\sigma t)$, with $\mathbf{k} = (l, m)$. The wave speed increases without bound for large wavelengths, which is an inherent problem with the nondivergent model. The divergent case does not exhibit this feature.

The solution of the initial-value problem for the evolution of a distributed vortex on the beta-plane will require the solution of an initial-value problem in the far field with unknown inner boundary condition. The causal Green's function solution is hence the required solution to (3.1). The inhomogeneous version of (3.1), with some propagating forcing near the origin, is of some interest in understanding the effect of the wavefield, in particular with respect to the power radiated away by the waves. However, this last analysis is not directly relevant to the main goals of this dissertation. It may be found in Appendix A.

3·2·The causal Green's function

3·2·1·Derivation

The causal Green's function for the linear Rossby wave equation has been investigated in Kamenkovich (1989), or K89 hereafter. This section uses some of that work's results, but derives new representations for the function. The Green's function L is defined by

$$\frac{\partial}{\partial t}\nabla^2 L + \epsilon\frac{\partial L}{\partial x} = \delta(\mathbf{r})\delta(t), \tag{3.3}$$

with appropriate boundary conditions. The properties of the Green's function enable the solution of (3.1) to be written as a convolution of the Green's function with the forcing (or the initial value, which may be considered as a forcing with delta-function temporal dependence).

Laplace transforming (3.3) in time yields

$$p\nabla^2\overline{L} + \epsilon\frac{\partial\overline{L}}{\partial x} = \delta(\mathbf{r}). \tag{3.4}$$

The substitution $\overline{L} = l\exp(-\epsilon x/2p)$ removes the anisotropic term and leads to an inhomogeneous Helmholtz equation

$$\nabla^2 l + \frac{\epsilon^2}{4p^2}l = \frac{\delta(\mathbf{r})}{p}. \tag{3.5}$$

The solutions to the homogeneous equation are modified Bessel and MacDonald functions, and hence

$$\overline{L} = \exp\left(-\frac{\epsilon x}{2p}\right)\left[a(p)K_0\left(\frac{\epsilon r}{2p}\right) + b(p)I_0\left(\frac{\epsilon r}{2p}\right)\right], \tag{3.6}$$

where $r = |\mathbf{r}|$.

However, the desired solution $\overline{L}$ must have a bounded gradient at infinity and the appropriate singular behaviour near the origin. As a Laplace transform, $\overline{L}$ is certainly defined for $\mathrm{Re}(p) > 0$, and its behaviour for small and large r must be investigated for such p. The solution for real p has $b = 0$ and $a = -1/2\pi p$. This can be seen from the series expansion of I_0 and K_0 for small arguments: I_0 grows exponentially with r and is hence unacceptable, while K_0 has a logarithmic singularity which provides the appropriate delta-function behaviour near the origin, where the Laplacian is dominant. This solution is clearly valid throughout the right half complex p-plane. $I_0(z)$ has the same behaviour for all z, and $K_0(z)$ decays as $|z| \to \infty$ in the open right half complex z-plane. Since $1/p$ is in the right half-plane when p is, it follows that

$K_0(\epsilon r/2p)$ decays appropriately. The exponential term in x grows for $x < 0$, but this is cancelled by the decay of K_0. Hence

$$\overline{L} = -\frac{1}{2\pi p}\exp\left(-\frac{\epsilon x}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right). \tag{3.7}$$

The large $|p|$ behaviour of $\overline{L}$ governs the causal behaviour of the inverse Laplace transform. For example, the two dimensional wave equation Green's function has Laplace transform $\overline{G_w} = K_0(prc)$, and using the asymptotic expansion of K_0 for large arguments, it can be shown that $G_w = 0$ for $r > ct$. In the Rossby wave case, the asymptotic behaviour of the integrand of the inverse Laplace transformation is given by

$$\overline{L}\,\mathrm{e}^{pt} = -\frac{1}{2\pi p}\exp\left(-\frac{\epsilon x}{2p} + pt\right)K_0\left(\frac{\epsilon r}{2p}\right) \tag{3.8}$$

$$= -\frac{1}{2\pi p}\exp\left(-\frac{\epsilon x}{2p} + pt\right)\left[\ln p + O(1)\right]. \tag{3.9}$$

Writing $p = p_r + \mathrm{i}p_i$ shows that the real part of the argument of the exponential function is

$$\mathrm{Re}\left(-\frac{\epsilon x}{2p} + pt\right) = p_r\left(t - \frac{\epsilon x}{2|p|^2}\right). \tag{3.10}$$

There are thus two cases, depending on the sign of t:

- For $t < 0$, the contour may be closed in the right half-plane, since the quantity in brackets is always negative for large enough $|p|$. There are no singularities or cuts in the right half-plane, and L is zero. This shows that L has the appropriate causal behaviour.

- For $t > 0$, and for $|p|$ large enough, the real part of the argument of the exponential has the sign of p_r, so the contour must be closed in the left half-plane, and L is not zero. This is true for all values of r.

This immediate response over all space is due to the fact that the velocity of Rossby waves increases without bound as their wavelength increases. A spatial delta-function source contains all wavenumbers, and hence influence propagates through all space immediately. As mentioned previously, a divergent model, with nonzero Rossby deformation radius, will not have this feature.

There are several possible integral representations of L. All the following results should be understood as incorporating a Heaviside function of time $H(t)$. This

just states that there is no response for negative t. Following K89, an integral representation may be derived for L using the identity

$$K_0(z) = 2\mathrm{e}^{-z} \int_0^\infty \frac{\exp(-2u^2 z)}{(u^2+1)^{\frac{1}{2}}}\,\mathrm{d}u. \tag{3.11}$$

This leads to the representation

$$L = -\frac{1}{\pi} \int_0^\infty \frac{\mathcal{J}_0(2[\epsilon r t(u^2+c^2)]^{\frac{1}{2}})}{(u^2+1)^{\frac{1}{2}}}\,\mathrm{d}u, \tag{3.12}$$

where $c = \cos(\theta/2)$. This representation in plane polar coordinates shows that the only way ϵ, t, and r can enter L is through the combination $\epsilon r t$. The integral in (3.12) exists for all r and c, but its partial derivatives are not defined. This simply means that such an integral representation does not exist for those derivatives. Particular expressions of L may be found on the x-axis:

- for $\theta = 0$,

$$L = \frac{1}{2}\mathcal{J}_0\left([\epsilon r t]^{\frac{1}{2}}\right)\mathcal{Y}_0\left([\epsilon r t]^{\frac{1}{2}}\right) \tag{3.13}$$

- for $\theta = \pi$,

$$L = -\frac{1}{\pi} I_0\left([\epsilon r t]^{\frac{1}{2}}\right) K_0\left([\epsilon r t]^{\frac{1}{2}}\right). \tag{3.14}$$

The convolution theorem may be used to deduce new integral representations for L. The convolution theorem states

$$\overline{f * g} = \overline{f} \times \overline{g}, \tag{3.15}$$

where

$$f * g = \int_0^t f(\tau) g(t-\tau)\,d\tau. \tag{3.16}$$

Now $\overline{L}$ may be written in two different ways:

$$\overline{L} = -\frac{1}{2\pi}\left\{\frac{1}{p^{\frac{1}{2}}}\exp\left(\frac{\epsilon r}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right)\right\} \times \left\{\frac{1}{p^{\frac{1}{2}}}\exp\left(-\frac{\epsilon r c^2}{p}\right)\right\} \tag{3.17}$$

and

$$\overline{L} = -\frac{1}{2\pi}\left\{\left[\exp\left(-\frac{\epsilon r c^2}{p}\right) - 1\right] + 1\right\} \times \left\{\frac{1}{p}\exp\left(\frac{\epsilon r}{2p}\right)K_0\left(\frac{\epsilon r}{2p}\right)\right\}. \tag{3.18}$$

Use of the convolution theorem then gives

$$L = -\frac{1}{\pi^2} t^{-\frac{1}{2}} K_0\left(2[\epsilon rt]^{\frac{1}{2}}\right) * t^{-\frac{1}{2}} \cos\left(2c[\epsilon rt]^{\frac{1}{2}}\right) \tag{3.19}$$

$$= -\frac{2}{\pi^2} \int_0^{\pi/2} K_0\left(2[\epsilon rt]^{\frac{1}{2}} \sin\phi\right) \cos\left(2c[\epsilon rt]^{\frac{1}{2}} \cos\phi\right) \mathrm{d}\phi \tag{3.20}$$

and

$$L = -\frac{1}{2\pi} \left\{ -\frac{c(\epsilon r)^{\frac{1}{2}}}{t^{\frac{1}{2}}} \mathcal{J}_1\left(2c[\epsilon rt]^{\frac{1}{2}}\right) + \delta(t) \right\} * 2I_0\left([\epsilon rt]^{\frac{1}{2}}\right) K_0\left([\epsilon rt]^{\frac{1}{2}}\right) \tag{3.21}$$

$$\begin{aligned} = & \frac{2c}{\pi} (\epsilon rt)^{\frac{1}{2}} \int_0^{\pi/2} \mathcal{J}_1\left(2c[\epsilon rt]^{\frac{1}{2}} \cos\phi\right) \times \\ & \times I_0\left([\epsilon rt]^{\frac{1}{2}} \cos\phi\right) K_0\left([\epsilon rt]^{\frac{1}{2}} \cos\phi\right) \sin\phi \, \mathrm{d}\phi \\ & -\frac{1}{\pi} I_0\left([\epsilon rt]^{\frac{1}{2}}\right) K_0\left([\epsilon rt]^{\frac{1}{2}}\right) \end{aligned} \tag{3.22}$$

respectively ($c = \cos\theta/2$ is positive). These expressions are computationally more efficient than (3.12) since they are integrals over a finite range.

Evaluating (3.22) on the negative x-axis ($c = 0$) leads to (3.14), while identifying (3.20) on the negative x-axis with (3.14) leads to

$$\int_0^{\pi/2} K_0(2z \sin\theta) \, \mathrm{d}\theta = \frac{\pi}{2} I_0(z) K_0(z), \tag{3.23}$$

with $z = (\epsilon rt)^{\frac{1}{2}}$. This is equation 6.681.4 of Gradshteyn & Ryzhik (1980). Proceeding analogously on the positive x-axis ($c = 1$) leads to two definite integrals:

$$\begin{aligned} & \int_0^{\pi/2} \mathcal{J}_1(2z\cos\theta) I_0(z\sin\theta) K_0(\sin\theta) \sin\theta \, \mathrm{d}\theta \\ = & \frac{\pi}{2z} \left(\frac{1}{2} \mathcal{J}_0(z) \mathcal{Y}_0(z) + \frac{1}{\pi} I_0(z) K_0(z) \right). \end{aligned} \tag{3.24}$$

and

$$\int_0^{\pi/2} K_0(2z\sin\theta) \cos(2z\cos\theta) \, \mathrm{d}\theta = -\frac{\pi^2}{4} \mathcal{J}_0(z) \mathcal{Y}_0(z) \tag{3.25}$$

These two integrals do not seem to appear in Gradshteyn & Ryzhik (1980) or in Luke (1962).

3·2·2·Expansion for small ϵ

The expansions for the exponential and MacDonald functions may be used to obtain a series expansion for $\overline{L}$, valid for small $\epsilon r/p$:

$$\overline{L} = -\frac{1}{2\pi p} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma_E \right] + \frac{\epsilon x}{4\pi p^2} \left[\ln p - \ln \frac{\epsilon r}{4} - \gamma_E \right]$$

$$-\frac{\epsilon^2 x^2}{16\pi p^3}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] - \frac{\epsilon^2 r^2}{32\pi p^3}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E + 1\right]$$
$$+O\left(\frac{\epsilon^3 r^3}{p^4}\right). \tag{3.26}$$

This series may then be inverted using the inverse Laplace transform identity

$$\frac{\ln p}{p^k} \longrightarrow \frac{t^{k-1}}{\Gamma(k)}[\psi(k) - \ln t], \tag{3.27}$$

where $\psi(k)$ is the Euler digamma function. There are only integral powers of p in the expression, so the particular values $\Gamma(1) = 1$, $\Gamma(2) = 1$, $\Gamma(3) = 2$, ..., and $\psi(1) = -\gamma_E$, $\psi(2) = 1 - \gamma_E$, $\psi(3) = 1 + \frac{1}{2} - \gamma_E$, ..., may be used, to give

$$\begin{aligned} L &= \frac{1}{2\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma_E\right] - \frac{\epsilon xt}{4\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma_E - 1\right] \\ &\quad + \frac{\epsilon^2 t^2}{128\pi}\left[(2x^2 + r^2)(2\ln\frac{\epsilon rt}{4} + 4\gamma_E - 3) - 2r^2\right] + O(\epsilon^3 r^3 t^3). \end{aligned} \tag{3.28}$$

This series is clearly nonuniform in time. When ϵrt is $O(1)$, the asymptotic ordering breaks down.

3·2·3·Fourier transform

With the definition

$$\tilde{f}(l, m) = \frac{1}{(2\pi)^2}\iint f(x,y)\exp(-\mathrm{i}lx - \mathrm{i}my)\,\mathrm{d}x\mathrm{d}y, \tag{3.29}$$

the Fourier transform of (3.3) gives

$$\frac{\partial}{\partial t}(-k^2\tilde{L}) + \mathrm{i}\epsilon l\tilde{L} = \frac{\delta(t)}{(2\pi)^2}, \tag{3.30}$$

where $k^2 = l^2 + m^2$. The solution to this equation is

$$\tilde{L} = -\frac{H(t)}{(2\pi k)^2}\exp\left(\frac{\mathrm{i}\epsilon lt}{k^2}\right). \tag{3.31}$$

Then inverse Fourier transforming yields

$$L = -\frac{H(t)}{(2\pi)^2}\iint\frac{1}{k^2}\exp\left(\frac{\mathrm{i}\epsilon lt}{k^2}\right)\exp(\mathrm{i}lx + \mathrm{i}my)\,\mathrm{d}l\mathrm{d}m \tag{3.32}$$

$$= -\frac{H(t)}{(2\pi)^2}\int_0^\infty\int_0^{2\pi}$$
$$\exp\left\{\mathrm{i}\sin(\phi - \chi)\left[\left(\frac{\epsilon t}{k} + kx\right)^2 + k^2y^2\right]^{\frac{1}{2}}\right\}\mathrm{d}\phi\frac{\mathrm{d}k}{k} \tag{3.33}$$

$$= \quad -\frac{H(t)}{2\pi}\int_0^\infty \mathcal{J}_0\left(\left[\left(\frac{\epsilon t}{k}+kx\right)^2+k^2y^2\right]^{\frac{1}{2}}\right)\frac{\mathrm{d}k}{k}, \tag{3.34}$$

where χ is an angle whose value is not actually required. This is equivalent to the solution described in Flierl (1977), after a deconvolution.

The two integral representations (3.12) and (3.34) derived above can be shown to be equivalent simply by changing variable. Writing

$$4\epsilon rt(u^2+c^2) = \left[\left(kx+\frac{\epsilon t}{k}\right)^2+k^2y^2\right] \tag{3.35}$$

leads to

$$u = \pm\frac{1}{2(\epsilon rt)^{\frac{1}{2}}}\left(kr-\frac{\epsilon t}{k}\right). \tag{3.36}$$

There is therefore a multivalued relation between u and k. The integration range in the Laplace integral means that u is positive, so all dependence on sign and range will be taken into account by splitting the k integral into two parts:

$$\int_0^\infty dk \quad = \quad \left(\int_0^{k^*}+\int_{k^*}^\infty\right)\mathrm{d}k \tag{3.37}$$

$$= \quad 2\int_0^\infty \left|\frac{\mathrm{d}k}{\mathrm{d}u}\right|\mathrm{d}u. \tag{3.38}$$

The cut-off point is $k^* = (\epsilon rt)^{\frac{1}{2}}$. Noting that $4\epsilon rt(u^2+1) = (kr+\epsilon t/k)^2$ and using

$$\frac{\mathrm{d}u}{\mathrm{d}k} = \pm\frac{1}{2(\epsilon rt)^{\frac{1}{2}}}\left[r+\frac{\epsilon t}{k^2}\right] \tag{3.39}$$

leads to

$$\frac{|\mathrm{d}u|}{(u^2+1)^{\frac{1}{2}}} = \frac{|\mathrm{d}k|}{k}. \tag{3.40}$$

Hence the two integral representations are equal.

3·2·4·The general solution to the unforced initial-value problem

The unforced initial-value problem alone will be considered. The Laplace transform of (3.1) gives

$$p\nabla^2\overline{\psi}+\frac{\partial\overline{\psi}}{\partial x} = \nabla^2\Psi, \tag{3.41}$$

where Ψ is the initial streamfunction. Rewriting the right-hand side of this equation as the initial vorticity Q and using the property of the Green's function gives

$$\overline{\psi} = \overline{L} * Q \tag{3.42}$$

The inverse Laplace transform of this equation clearly gives

$$\psi = L * Q, \tag{3.43}$$

as expected. The solution to the forced problem is more complicated, but not required here.

Like any field governed by a Laplace-like operator, the streamfunction may be expressed as a multipole expansion in the form

$$\psi \sim L \iint Q \, \mathrm{d}^2 s + \frac{\partial L}{\partial x_i} \iint s_i Q \, \mathrm{d}^2 s + \cdots \tag{3.44}$$

for large values of r, provided that the vorticity is localised, and hence that all the integrals exist.

4·The nondivergent first-order solution

4·1·Asymptotic solution

The governing equation to be solved, corresponding to the conservation of absolute vorticity, is

$$\frac{\partial}{\partial t}\nabla^2\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\frac{\partial\psi}{\partial x} = 0, \tag{4.1}$$

with initial condition $\psi = \Psi(r)$ at $t = 0$. This is an initial-value problem, with appropriate decay conditions on the streamfunction at infinity. The initial vorticity will be taken to be localised, so $Q = \nabla^2\Psi$ decays faster than any power of r at infinity. This corresponds to

$$Q = O(r^{-\infty}) \tag{4.2}$$

in the notation being employed. This means that all moments of the form $\int u^n Q(u)\,\mathrm{d}u$ will converge. In addition, the circulation of the vortex, that is to say, its net integrated vorticity, is Γ. Then, as shown in Chapter 1,

$$\Psi = \frac{\Gamma}{2\pi}\ln r + O(r^{-\infty}). \tag{4.3}$$

The angular velocity $\Omega(r)$ will be taken positive and decreasing; then Γ is positive, which corresponds to a cyclone.

As discussed in Chapter 1, the nondimensional parameter ϵ is small for strong vortices. Then (4.1) may be solved as an asymptotic series in ϵ, with

$$\psi = \psi_0 + \epsilon\psi_1 + \cdots. \tag{4.4}$$

This inner problem was considered by Reznik & Dewar (1994), hereafter RD94. However, the dominant balance in the equation for large values of r will not be between the rate-of-change of relative vorticity and the Jacobian, but between the former and the beta term. Thus there is a far field, in which the natural variable is $\mathbf{R} = \epsilon\mathbf{r}$, and in which the governing equation of motion is

$$\frac{\partial}{\partial t}\nabla_R^2\phi + \frac{\partial\phi}{\partial X} + \epsilon^2\mathcal{J}_R(\phi, \nabla_R^2\phi) = 0. \tag{4.5}$$

In the far field, the asymptotic expansion is

$$\phi = \phi_0 + \epsilon\phi_1 + \cdots. \tag{4.6}$$

The two expansions must be related via some matching procedure. Van Dyke's rule (Van Dyke 1975) is an appropriate device. It states:

$$\psi^{(n,m)} = \phi^{(m,n)}, \tag{4.7}$$

where $\psi^{(n,m)}$ is the inner solution truncated to order n in the inner variable r, and then subsequently reexpressed in the outer variable R and truncated to order m in that variable. The right-hand side of the equality corresponds to the same procedure (swapping r and R) applied to the outer solution. For this rule to give correct results, terms of logarithmic order must be included in the expansion truncated at algebraic order (e.g., any $O(\epsilon \ln \epsilon)$ terms must be included in the truncations $\psi^{(1,.)}$ and $\phi^{(1,.)}$). In expressions of this sort in this chapter, order terms will refer to the variable used in the terms on the same side of the equation; that spatial variable and t will be considered to be constant.

Neither expansion in ϵ will in general be uniform in time. For long times, the structure of the velocity field becomes very complicated with the formation of a wake behind the vortex (e.g., Sutyrin *et al.* 1994). The asymptotic solution here is appropriate for times of $O(1)$, and should hold up to asymptotically larger times. The precise order of breakdown is not immediately apparent, though, and will be investigated in detail in Chapter 5.

4·2·Zeroth-order solution

4·2·1·Inner solution

Derivation

The zeroth-order equation in the near field is

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + \mathcal{J}(\psi_0, \nabla^2\psi_0) = 0, \tag{4.8}$$

with initial condition $\psi_0 = \Psi$ at $t = 0$. The boundary conditions at infinity come from the matching with the outer solution.

However, $\psi_0 = \Psi$ is clearly a steady solution to the equation, ignoring the boundary condition. The eigenfunctions of the Laplace operator are also solutions of the equation which do not necessarily satisfy any boundary conditions; they correspond to irrotational flow. A simple form for the inner field, regular at $r = 0$, is then

$$\psi_0 = \Psi(r) + \sum a_n(t) r^n \mathrm{e}^{\mathrm{i}n\theta}, \tag{4.9}$$

with integer n, where the a_n are all initially zero for $n \geq 1$. The constant function of space $a_0(t)$ is physically irrelevant, and hence satisfies no initial condition. Complex notation will be used for this type of trigonometric sum; the real part is to be understood. This sum includes all the necessary terms for a complete zeroth-order solution to the inner problem, as will be shown by the matching.

Far-field behaviour

The far-field streamfunction corresponds to a localised vorticity distribution, so using (4.3), the inner solution may be rewritten as

$$\psi^{(0,,)} = \frac{\Gamma}{2\pi} \ln\left(\frac{R}{\epsilon}\right) + \sum a_n \left(\frac{R}{\epsilon}\right)^n \mathrm{e}^{\mathrm{i}n\theta} + O\left(\frac{\epsilon^\infty}{R^\infty}\right) \tag{4.10}$$

in the far-field coordinate.

The order-infinity term cannot appear at any order of the matching procedure. In addition, there can be no terms in the above sum with n greater than zero, since these would have to match onto terms of the far-field solution containing negative powers of ϵ, whereas the perturbation expansion in the outer field cannot be large as ϵ becomes very small. Hence all the a_n are zero except for $n = 0$, and the appropriate truncation for Van Dyke's rule is

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi}(\ln R - \ln \epsilon) + a_0(t). \tag{4.11}$$

This shows that there must be terms of the form $\psi_0' \ln \epsilon$ in the inner expansion. However, such terms cannot be dynamically significant, since the dominant motion is at zeroth order (i.e. smaller than $\ln \epsilon$). Hence the only possible term is constant in space, which corresponds to a streamfunction of the form

$$\psi_0' = a'(t). \tag{4.12}$$

Therefore the correct truncated expansion to use, including logarithmic terms, is

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi}(\ln R - \ln \epsilon) + a_0(t) + a'(t) \ln \epsilon. \tag{4.13}$$

Primes will be used to denote functions of logarithmic order in ϵ.

4·2·2·Outer solution

Derivation

The zeroth-order equation in the far field is

$$\frac{\partial}{\partial t} \nabla_R^2 \phi_0 + \frac{\partial \phi_0}{\partial X} = 0. \tag{4.14}$$

The boundary condition at the origin comes from the matching. The initial condition may be written as $\phi_0 = \Phi(R)$, where $\Phi(R)$ is the zeroth-order term in the expansion of $\Psi(R/\epsilon)$. However, the initial vorticity $q(r)$ is assumed to be localised, and hence decays faster than any inverse power of r, and hence of R also, in the far field. Expanding its far-field counterpart $Q(R)$ to any power in ϵ gives nothing, since all terms will decay too fast. Hence the initial condition for the vorticity is $Q_0(R) = 0$, and the solution to (4.14) can be taken from Chapter 3 as

$$\phi_0 = \sum A_{i\ldots j} \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{R}, t). \tag{4.15}$$

Tensorial notation is used for the multipole sum, and the number of indices on A is equal to n. The sum corresponds to solutions with some unknown forcing at the origin.

Near-field behaviour

The behaviour of the Green's function and its derivatives near the origin can be copied from (3.28), using the appropriate far-field variable:

$$L = \frac{1}{2\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E\right] - \frac{Xt}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right] + O(R^2t^2). \tag{4.16}$$

Hence

$$\nabla_R L = \frac{1}{2\pi}\frac{\mathbf{R}}{R^2} - \frac{t}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right]\mathbf{i} - \frac{Xt}{4\pi}\frac{\mathbf{R}}{R^2} + O(Rt). \tag{4.17}$$

As expected, each differentiation raises the degree of singularity of L. Changing to the inner variable by

$$\frac{\partial}{\partial X_i} = \frac{1}{\epsilon}\frac{\partial}{\partial x_i} \tag{4.18}$$

leads to a new expression for the sum:

$$\begin{aligned}\sum_n A_{i\ldots j}\frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{R}, t) \quad = \quad & A_0\left\{\frac{1}{2\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma_E\right]\right. \\ & \left. -\frac{\epsilon xt}{4\pi}\left[\ln\frac{\epsilon rt}{4} + 2\gamma_E - 1\right] + O(\epsilon^2)\right\} \\ & + \mathbf{A}_1 O(\epsilon^{-1}) + \cdots,\end{aligned} \tag{4.19}$$

for fixed r and t. This shows that all the $A_{i\ldots j}$ must be zero for $n \geq 1$, since any higher A's would have to match onto terms of negative order in ϵ in the inner expansion, and the leading order behaviour of the inner solution is of order zero.

4·2·3·Matching

The truncated expansions for the inner and outer solutions are now

$$\psi^{(0,0)} = \frac{\Gamma}{2\pi} \ln R + a_0 + \ln \epsilon \left(a' - \frac{\Gamma}{2\pi} \right) \tag{4.20}$$

and

$$\phi^{(0,0)} = \frac{A_0}{2\pi} \left[\ln \frac{\epsilon r t}{4} + 2\gamma_E \right] \tag{4.21}$$

respectively. The outer expansion may be rewritten in terms of R as

$$\phi^{(0,0)} = \frac{A_0}{2\pi} \left[\ln \frac{R t}{4} + 2\gamma_E \right], \tag{4.22}$$

and Van Dyke's rule then leads to the following three relations:

$$A_0 = \Gamma, \tag{4.23}$$

$$a_0 = \frac{\Gamma}{2\pi} \left[\ln \frac{t}{4} + 2\gamma_E \right], \tag{4.24}$$

$$a' = \frac{\Gamma}{2\pi}. \tag{4.25}$$

If there had been a logarithmic term in the far field (ϕ_0' say), it would have had to satisfy (4.14), and hence been proportional to L. The three equations (4.23), (4.24) and (4.25) could only then have been satisfied with ϕ_0' identically zero.

The complete zeroth-order solution, including logarithmic terms, is thus

$$\psi_0 = \Psi(r) + \frac{\Gamma}{2\pi} \left[\ln \frac{t}{4} + 2\gamma_E \right] \tag{4.26}$$

and

$$\psi_0' = \frac{\Gamma}{2\pi} \tag{4.27}$$

for the near field, and

$$\phi_0 = \Gamma L(\mathbf{R}, t) \tag{4.28}$$

for the far field. In the Laplace coordinate, these expressions become

$$\overline{\psi_0} = \frac{\Psi(r)}{p} - \frac{\Gamma}{2\pi} \left[\ln p - \ln \frac{1}{4} - \gamma_E \right] \tag{4.29}$$

in the near field and

$$\overline{\psi_0'} = \frac{\Gamma}{2\pi p}; \tag{4.30}$$

and

$$\overline{\phi_0} = \Gamma \overline{L}(\mathbf{R}, p), \tag{4.31}$$

in the far field

The only dynamically significant part of the inner solution is the initial streamfunction. The other terms are just functions of time that match onto the outer solution. The Green's function term corresponds to the response to a vortex of circulation Γ at the origin. To the far field, the only 'visible' property of the vortex at zeroth order is its circulation. This illustrates the important difference between isolated and nonisolated vortices.

4·3·First order solution

4·3·1·Inner solution

Derivation

The governing equation for the first order solution in the inner field is the linearised inhomogeneous Euler equation

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right) \nabla^2 \psi_1 - \frac{Q'}{r} \frac{\partial \psi_1}{\partial \theta} + \Omega r \cos\theta = 0, \tag{4.32}$$

with zero initial condition. The forcing term comes from the beta-effect acting on the radial order-zero streamfunction. Laplace transforming in time, and decomposing into radial modes, leads to

$$(p + \mathrm{i}l\Omega) \left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} r \frac{\mathrm{d}}{\mathrm{d}r} - \frac{l^2}{r^2}\right] \overline{\psi_1^l} - \frac{Q'}{r} \mathrm{i}l \overline{\psi_1^l} = -\frac{\Omega r \delta_{1l}}{p}, \tag{4.33}$$

where the real part of this equation is to be understood, and where

$$\psi_1 = \sum \psi_1^l \mathrm{e}^{\mathrm{i}l\theta}. \tag{4.34}$$

Each mode ψ_1^l has zero initial condition, and (4.33) is homogeneous for all modes except $l = 1$. Hence ψ_1^l is identically zero for all l except $l = 1$. Dropping the subscript l, the governing equation for mode one may be rewritten as

$$-\frac{\mathrm{d}}{\mathrm{d}r}\left(r \frac{\mathrm{d}\overline{\psi_1}}{\mathrm{d}r}\right) + \left[\frac{1}{r} + \frac{\mathrm{i}Q'}{p + \mathrm{i}\Omega}\right] \overline{\psi_1} = \frac{1}{p + \mathrm{i}\Omega} \frac{\Omega r^2}{p}. \tag{4.35}$$

As in Chapter 2, the change of variable $\overline{\psi_1} = r(p + \mathrm{i}\Omega)f$ leads to the equation

$$-\frac{1}{r(p+\mathrm{i}\Omega)}\frac{\mathrm{d}}{\mathrm{d}r}\left[r^3(p+\mathrm{i}\Omega)^2\frac{\mathrm{d}f}{\mathrm{d}r}\right] = \frac{1}{p+\mathrm{i}\Omega}\frac{\Omega r^2}{p}. \tag{4.36}$$

The solution to this equation is

$$\overline{\psi_1} = -\frac{r(p+\mathrm{i}\Omega(r))}{p}\int_{B(p)}^{r}\frac{h(v)-h(A(p))}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v, \tag{4.37}$$

where h is defined by

$$h(v) = \int_0^v \Omega(u)u^3\,\mathrm{d}u \tag{4.38}$$

and A and B are undetermined functions of p. Changing A and B corresponds to adding multiples of the homogeneous solutions. This working reproduces that of Chapter 2, and is essentially the solution presented in RD94, albeit with different (as yet unspecified) large-r boundary conditions. The far-field behaviour of (4.37) is quite different here.

The function $h(v)$ is the basic-state relative angular momentum within a disc of radius v. The behaviour of the function $h(v)$ is given by

$$h(v) = \frac{\Omega_0}{4}v^4 + O(v^6) \tag{4.39}$$

for small v, and by

$$h(v) = \frac{\Gamma v^2}{4\pi} + H + O(v^{-\infty}) \tag{4.40}$$

for large v, where

$$H = \int_0^\infty \left[\Omega(u) - \frac{\Gamma}{4\pi u^2}\right]u^3\,\mathrm{d}u. \tag{4.41}$$

This last quantity will in general be nonzero, even for vortices with zero circulation, although there will be vortices for which it vanishes. For vortices with zero circulation, it has been called the Relative Angular Momentum (RAM) in the tropical cyclone literature (Willoughby 1988).

Denoting by $\overline{f_l}$ the mode-l solution to (4.33) that is well-behaved at the origin, ($\overline{f_l}$ corresponds to ϕ_l^A of Chapter 2), a complete solution to the inner problem is given by

$$\overline{\psi_1} = -\frac{r(p+\mathrm{i}\Omega(r))}{p}\mathrm{e}^{\mathrm{i}\theta}\int_0^r\frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v + \sum_l \overline{b_l}(p)\overline{f_l}(r,p)\mathrm{e}^{\mathrm{i}l\theta}. \tag{4.42}$$

The homogeneous solutions are multiplied by functions of p which remain to be determined. The functions A and B have been fixed to ensure convergence of the integral. This freedom is due to the presence of the unspecified function $\overline{b_1}$. The homogeneous solutions for the lowest two modes that are well-behaved at the origin may be written down explicitly as

$$\overline{f_0} = 1 \tag{4.43}$$

and

$$\overline{f_1} = r(p + \mathrm{i}\Omega). \tag{4.44}$$

Far-field behaviour

Truncating the complete inner solution to first order gives

$$\overline{\psi^{(1,\cdot)}} = \overline{\psi_0} - \frac{\epsilon r(p + \mathrm{i}\Omega(r))}{p}\mathrm{e}^{\mathrm{i}\theta}\int_0^r \frac{h(v)}{v^3(p + \mathrm{i}\Omega(v))^2}\,\mathrm{d}v + \epsilon\sum_l \overline{b_l}(p)\overline{f_l}(r,p)\mathrm{e}^{\mathrm{i}l\theta}. \tag{4.45}$$

The integral in (4.45) is

$$I = \int_0^{R/\epsilon} \frac{h(v)}{v^3(p + \mathrm{i}\Omega(v))^2}\,\mathrm{d}v. \tag{4.46}$$

It clearly behaves logarithmically for small ϵ, but it is the order-zero and order-one terms that are of interest. The singular part of the integral may be subtracted off by writing

$$I = \int_0^{R/\epsilon} \frac{h(v)}{v^3}\left[\frac{1}{(p + \mathrm{i}\Omega(v))^2} - \frac{1}{p^2}\right]\mathrm{d}v + \int_0^{R/\epsilon}\frac{h(v)}{p^2v^3}\,\mathrm{d}v. \tag{4.47}$$

The first integral in (4.47) exists for $\epsilon = 0$ and may be expressed as a Taylor series in ϵ. The far-field behaviour of the integrand is given by

$$-\frac{\mathrm{i}\Gamma^2\epsilon^3}{4\pi p^3R^3} + O\left(\frac{\epsilon^5}{R^5}\right) \tag{4.48}$$

for large R/ϵ. Using the chain rule,

$$\frac{dI_1}{d\epsilon} = -\frac{R}{\epsilon^2}\left[-\frac{\mathrm{i}\Gamma^2\epsilon^3}{4\pi p^3R^3} + O\left(\frac{\epsilon^5}{R^5}\right)\right]. \tag{4.49}$$

This leads to

$$I_1 = \frac{1}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p + \mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v + O\left(\frac{\epsilon^2}{R^2}\right). \tag{4.50}$$

The second integral in (4.47) may be rewritten as

$$I_2 = \frac{1}{p^2}\int_0^{R/\epsilon}\int_0^v \frac{u^3\Omega(u)}{v^3}\,\mathrm{d}u\,\mathrm{d}v, \tag{4.51}$$

which may be transformed into

$$I_2 = \frac{\Psi(R/\epsilon)}{2p^2} - \frac{\Psi_0}{2p^2} - \frac{\epsilon^2 h(R/\epsilon)}{2p^2R^2}. \tag{4.52}$$

The behaviour of this expression for small ϵ is

$$I_2 = \frac{\Gamma}{4\pi p^2}\ln\frac{R}{\epsilon} - \frac{\Psi_0}{2p^2} - \frac{\Gamma}{8\pi p^2} - \frac{\epsilon^2 H}{2p^2R^2} + O(\epsilon^\infty). \tag{4.53}$$

The first term is exactly the logarithmic term required by the matching, while the second and third are constants which may not be discarded. Their appearance is due to the boundary condition at infinity which ensures that no constant term appears in the expansion of Ψ for large R/ϵ. Combining the two expressions above gives

$$\begin{aligned} I \;=\; & \frac{\Gamma}{4\pi p^2}\ln\frac{R}{\epsilon} + \frac{1}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v \\ & -\frac{\Psi_0}{2p^2} - \frac{\Gamma}{8\pi p^2} + O(\epsilon^2) \end{aligned} \tag{4.54}$$

for small ϵ and fixed R.

The leading order behaviour of the integral for small ϵ and $R = O(1)$ is given by $\Gamma\ln(R/\epsilon)/4\pi p^2$. The contribution of the integral term in (4.45), in the far-field variable, is

$$\begin{aligned} & -\frac{\Gamma X}{4\pi p^2}\ln\frac{R}{\epsilon} - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v \\ & +\frac{\Gamma X}{8\pi p^2} + \frac{\Psi_0 X}{2p^2} + O(\epsilon^2), \end{aligned} \tag{4.55}$$

using (4.54) and the far-field behaviour of Ω. The expression $R\mathrm{e}^{\mathrm{i}\theta}$ can be replaced by X when it is multiplying a real quantity.

The asymptotic form of the integral term suggests that mode-zero and mode-one homogeneous terms will be needed in the first-order solution. The bounded mode-zero solution is

$$\overline{f_0} = 1, \tag{4.56}$$

while the appropriate mode-one solution is

$$\overline{f_1} = r(p + \mathrm{i}\Omega) = \frac{Rp}{\epsilon} + O(\epsilon) \tag{4.57}$$

in the far-field variable. Owing to the factor ϵ multiplying it in (4.45), the mode-zero term is of first order in the far-field variable expansion. Discarding all modes other than zero and one gives the contribution

$$\overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} p + \epsilon \overline{b_0} + O(\epsilon^2) \tag{4.58}$$

to (4.45) from the homogeneous terms.

The $O(\epsilon \ln \epsilon)$ terms

The form of the integral term shows that a logarithmic term will also be required in the expansion. The inner solution at $O(\ln \epsilon)$ has already been found in (4.12) and is dynamically insignificant. The governing equation for the inner $O(\epsilon \ln \epsilon)$ term is

$$\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right) \nabla^2 \psi_1' - \frac{Q'}{r} \frac{\partial \psi_1'}{\partial \theta} = 0, \tag{4.59}$$

which is the homogeneous counterpart of (4.32). This has the solution

$$\psi_1' = \sum \overline{b_l'}(p) \overline{f_l}(r, p) \mathrm{e}^{\mathrm{i}\theta}, \tag{4.60}$$

i.e. the homogeneous term of (4.42). The form of the logarithmic term in (4.55) shows that the only term actually required is the mode-one response. Its far-field behaviour is given by (4.57).

Putting these results together leads to

$$\begin{aligned} \overline{\psi^{(1,.)}} &= \overline{\psi^{(0,1)}} - \frac{\Gamma X}{4\pi p^2} \ln \frac{R}{\epsilon} - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p + \mathrm{i}\Omega(v))^2} (\Omega(v) - 2\mathrm{i}p)\, \mathrm{d}v \\ &\quad + \frac{\Gamma X}{8\pi p^2} + \frac{\Psi_0 X}{2p^2} + \overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} p + \epsilon \overline{b_0} + \overline{b_1'} R\mathrm{e}^{\mathrm{i}\theta} p \ln \epsilon + O(\epsilon^2). \end{aligned} \tag{4.61}$$

The zeroth- and first-order truncations may now be easily computed.

4·3·2·Outer solution

Derivation

The governing equation is again (4.14), since nonlinearity only enters at second order in the far-field. The solution with zero initial condition, and unspecified behaviour at the origin, is

$$\phi_1 = \sum_n B_{i..j} \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{R}, t). \tag{4.62}$$

It is not necessary to compute terms of logarithmic order in the far-field. If this were done, they would be identically zero through the matching process. At higher

orders, however, such terms will be required, as will be seen in Chapter 5.

Near-field behaviour

The limiting behaviour of such a sum has already been investigated in section 4·2·2. Working in the Laplace coordinate, the first-order truncation of the outer solution is

$$\begin{aligned}\overline{\phi_1^{(1,,)}} &= \Gamma\overline{L}(\mathbf{R},p)v + \epsilon\sum_n B_{i\ldots j}\frac{\partial^n}{\partial X_i\ldots\partial X_j}\overline{L}(\mathbf{R},p)\\ &= \Gamma\left\{-\frac{1}{2\pi p}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] + \frac{\epsilon x}{4\pi p^2}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] + O(\epsilon^2)\right\}\\ &\quad +\epsilon\sum_n B_{i\ldots j}\frac{\partial^n}{\epsilon^n\partial x_i\ldots\partial x_j}\left\{-\frac{1}{2\pi p}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right]\right.\\ &\quad \left.+\frac{\epsilon x}{4\pi p^2}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] + O(\epsilon^2)\right\},\end{aligned} \tag{4.63}$$

when rewritten in the inner coordinate. The order term in the second equality refers to the expansion as written in the inner variable (i.e. for small ϵ and fixed r). As before, the B must be zero, except for B_0 and B_i, since there is no ϵ^{-1} term in the inner expansion. Hence, truncating to the appropriate order gives

$$\begin{aligned}\overline{\phi_1^{(1,1)}} &= -\frac{\Gamma}{2\pi p}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] + \frac{\mathbf{B}_1}{2\pi p}\cdot\frac{\mathbf{r}}{r^2}\\ &\quad +\epsilon\left\{\frac{1}{\pi}\left(\frac{\Gamma x}{4p^2} - \frac{B_0}{2p}\right)\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right]\right.\\ &\quad \left.+\frac{\mathbf{B}_1}{4\pi p^2}\cdot\left(\mathbf{i}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] - x\frac{\mathbf{r}}{r^2}\right)\right\},\end{aligned} \tag{4.64}$$

where $\mathbf{B}_1$ is the vector form of B_i. The truncation of $\overline{\phi^{(1,,)}}$ to zeroth order may be written down immediately from the previous line as

$$\overline{\phi_1^{(1,0)}} = -\frac{\Gamma}{2\pi p}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right] + \frac{\mathbf{B}_1}{2\pi p}\cdot\frac{\mathbf{r}}{r^2}. \tag{4.65}$$

The 'off-diagonal' element $\psi^{(1,0)} = \phi^{(0,1)}$ of Van Dyke's rule requires the first-order truncation of the zeroth-order outer solution. This is

$$\overline{\phi^{(0,1)}} = \overline{\phi^{(0,0)}} + \frac{\epsilon\Gamma x}{4\pi p^2}\left[\ln p - \ln\frac{\epsilon r}{4} - \gamma_E\right]. \tag{4.66}$$

4·3·3·Matching

The $\overline{\psi^{(0,1)}} = \overline{\phi^{(1,0)}}$ matching

All Van Dyke truncations of the zeroth-order inner solution are the same, since the only powers of ϵ present when expressed in the far-field coordinate are zero

(including the logarithmic term) and infinity. Hence

$$\overline{\psi^{(0,1)}} = \overline{\psi^{(0,0)}} \tag{4.67}$$

and no further calculation is required. The off-diagonal element of Van Dyke's rule gives

$$\overline{\phi^{(1,0)}} = \overline{\phi^{(0,0)}} + \frac{\mathbf{B}_1}{2\pi p} \cdot \frac{\mathbf{R}}{\epsilon R^2} = \overline{\psi^{(0,1)}}. \tag{4.68}$$

Using the result of the zeroth-order matching gives $\mathbf{B}_1 = \mathbf{0}$.

THE $\overline{\psi^{(1,0)}} = \overline{\phi^{(0,1)}}$ MATCHING

Truncating (4.61) at zeroth order, and equating it to (4.66) leads to the equation

$$\begin{aligned} &\overline{\psi^{(0,1)}} - \frac{\Gamma X}{4\pi p^2} \ln \frac{R}{\epsilon} - \frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v \\ &+\frac{\Gamma X}{8\pi p^2} + \frac{\Psi_0 X}{2p^2} + \overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} p + \overline{b_1'}(p) R\mathrm{e}^{\mathrm{i}\theta} p \ln \epsilon \\ = \;& \overline{\phi^{(0,0)}} + \frac{\Gamma X}{4\pi p^2}\left[\ln p - \ln \frac{R}{4} - \gamma_E\right], \end{aligned} \tag{4.69}$$

written in the far-field variable. Again, the two terms formally truncated at zeroth order cancel from the zeroth-order matching (using the preceding paragraph). The logarithmic terms in R cancel also, and the equation decouples into two, since the $\ln \epsilon$ terms must be taken into account separately. The two resulting equations are

$$\begin{aligned} &-\frac{R\mathrm{e}^{\mathrm{i}\theta}}{p^2} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v + \frac{\Gamma X}{8\pi p^2} + \frac{\Psi_0 X}{2p^2} + \overline{b_1} R\mathrm{e}^{\mathrm{i}\theta} \\ = \;& \frac{\Gamma X}{4\pi p^2}\left[\ln p - \ln \frac{1}{4} - \gamma_E\right] \end{aligned} \tag{4.70}$$

and

$$\frac{\Gamma X}{4\pi p^2} + \overline{b_1'}(p) R\mathrm{e}^{\mathrm{i}\theta} p = 0. \tag{4.71}$$

The real part of both equations is understood. However, both decouple into a real and an imaginary equation, and solving these is equivalent to solving the original complex equations for $\overline{b_1}$ and $\overline{b_1'}$, replacing X by its complex counterpart $R\mathrm{e}^{\mathrm{i}\theta}$. This leads to

$$\begin{aligned} \overline{b_1} \;=\; & \frac{\Gamma}{4\pi p^3}\left[\ln p - \ln \frac{1}{4} - \gamma_E - \frac{1}{2}\right] - \frac{\Psi_0}{2p^3} \\ & +\frac{1}{p^3} \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v \end{aligned} \tag{4.72}$$

and

$$\overline{b_1'} = -\frac{\Gamma}{4\pi p^3}. \tag{4.73}$$

THE $\overline{\psi^{(1,1)}} = \overline{\phi^{(1,1)}}$ MATCHING

Truncating (4.61) at first order now gives

$$\overline{\psi^{(1,1)}} = \overline{\psi^{(1,0)}} + \epsilon\overline{b_0}(p), \tag{4.74}$$

since the integral does not contribute an order-one term to the expansion. The necessary truncation of the outer expansion has already been calculated, and may be rewritten as

$$\overline{\phi^{(1,1)}} = \overline{\phi^{(0,1)}} - \frac{\epsilon B_0}{2p}\left[\ln p - \ln\frac{R}{4} - \gamma_E\right]. \tag{4.75}$$

Van Dyke's rule gives

$$\overline{b_0} = -\frac{B_0}{2p}\left[\ln p - \ln\frac{R}{4} - \gamma_E\right]. \tag{4.76}$$

The only possible solution to this equation is $\overline{b_0} = B_0 = 0$; any other choice leaves a logarithmic term that cannot be matched. Hence there is no first-order disturbance in the far field (this could have been guessed from the appearance of the Jacobian at $O(\epsilon^2)$ in the far-field equation).

The complete solution to the first-order problem is thus given by the outer solution

$$\phi_1 = 0 \tag{4.77}$$

and the inner solution

$$\begin{aligned}\overline{\psi_1} &= -\frac{r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{p}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v + r(p+\mathrm{i}\Omega(r))\overline{b_1} && (4.78)\\ &= -\frac{r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{p}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v \\ &\quad + \frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\frac{1}{4} - \gamma_E - \frac{1}{2}\right] - \frac{\Psi_0}{2}\right. \\ &\quad \left. + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\}, && (4.79)\end{aligned}$$

including the logarithmic term

$$\overline{\psi_1'} = r(p+\mathrm{i}\Omega(r))\overline{b_1'} = -\frac{\Gamma r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{4\pi p^3}. \tag{4.80}$$

An equivalent expression for the $O(\epsilon)$ solution, which highlights its behaviour in the matching region, is

$$\begin{aligned}\overline{\psi_1} &= \frac{r(p+\mathrm{i}\Omega(r))\mathrm{e}^{\mathrm{i}\theta}}{p^3}\left\{-\frac{\Psi(r)}{2}+\frac{h(r)}{2r^2}+\frac{\Gamma}{4\pi}\left[\ln p-\ln\frac{1}{4}-\gamma_E-\frac{1}{2}\right]\right.\\ &\quad\left.+\int_r^\infty \frac{h(v)\Omega(v)}{v^3}\frac{\Omega(v)-2\mathrm{i}p}{(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v.\right\}. \qquad (4.81)\end{aligned}$$

In the time variable, these expressions become

$$\begin{aligned}\psi_1 &= -r\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial t}+\mathrm{i}\Omega(r)\right)\int_0^r\frac{h(v)}{v^3}\left[\frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)}+\frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t}-1}{\Omega(v)^2}\right]\mathrm{d}v\\ &\quad+r\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial t}+\mathrm{i}\Omega(r)\right)\left\{\frac{\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4}-2\gamma_E+1\right]-\frac{\Psi_0 t^2}{4}\right.\\ &\quad\left.+\int_0^\infty\frac{h(v)}{v^3}\left[-\frac{t^2}{2}+\frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)}+\frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t}-1}{\Omega(v)^2}\right]\mathrm{d}v\right\} \qquad (4.82)\end{aligned}$$

and

$$\psi_1' = -\frac{\Gamma r\mathrm{e}^{\mathrm{i}\theta}}{4\pi}\left[t+\mathrm{i}\Omega(r)\frac{t^2}{2}\right]. \qquad (4.83)$$

While the last expression seems to suggest that the asymptotic expansion must lose validity for $t = O(\epsilon^{-\frac{1}{2}})$, when ψ_1' will be of order ϵ^{-1}, and hence become a zeroth-order term (actually an $O(\ln\epsilon)$ term), this is erroneous. In fact, the spatial dependence of the solution must be taken into account as well. This will be further investigated later.

4·4·Vortex trajectory

4·4·1·Moving coordinate system

Working in a coordinate system moving with the vortex will be advantageous in this asymptotic framework. Assuming that the velocity of this new frame with respect to the old is (U, V), (4.1) becomes

$$\frac{\partial}{\partial t}\nabla^2\psi+\mathcal{J}(\psi,\nabla^2\psi)+\epsilon\frac{\partial\psi}{\partial x}-\left(U\frac{\partial}{\partial x}+V\frac{\partial}{\partial y}\right)\nabla^2\psi=0, \qquad (4.84)$$

in the inner region, and

$$\frac{\partial}{\partial t}\nabla_R^2\phi+\frac{\partial\phi}{\partial X}+\epsilon^2\mathcal{J}_R(\phi,\nabla_R^2\phi)-\epsilon\left(U\frac{\partial}{\partial X}+V\frac{\partial}{\partial Y}\right)\nabla_R^2\phi=0 \qquad (4.85)$$

in the outer region. However, the relative velocity between the frames is of order ϵ (or $\epsilon \ln \epsilon$), since it cannot appear in the absence of the beta-effect. Hence the previous analysis is valid to order zero in the near field, and order one in the far field. The full new equations have to be solved at higher orders to find a solution to the equation of motion in the new frame.

In the inner region, the first-order equation for ψ_1 becomes

$$\begin{aligned} & \frac{\partial}{\partial t}\nabla^2\psi_1 + \mathcal{J}(\psi_0, \nabla^2\psi_1) + \mathcal{J}(\psi_1, \nabla^2\psi_0) \\ = \quad & -\frac{\partial \psi_0}{\partial x} + \left(U_1 \frac{\partial}{\partial x} + V_1 \frac{\partial}{\partial y} \right) \nabla^2 \psi_0, \end{aligned} \tag{4.86}$$

where $U = \epsilon U_1 + \epsilon \ln \epsilon \, U_1' + \cdots$, and similarly for V. This is just equation (4.32) with an extra forcing term due to the relative motion of the frame. Since (4.86) is a linear equation, the solution can be decomposed into a part due to the beta-effect, which has already been calculated, and a part due to the new forcing term, provided both satisfy appropriate boundary and initial conditions.

The first-order part of the streamfunction ψ_1^f due to the change of frame may be written as

$$\psi_1^f = \int_0^t \left(U_1(\tau) \frac{\partial \Psi}{\partial x} + V_1(\tau) \frac{\partial \Psi}{\partial y} \right) \mathrm{d}\tau \tag{4.87}$$

$$= X_1(t) \frac{\partial \Psi}{\partial x} + Y_1(t) \frac{\partial \Psi}{\partial y} \tag{4.88}$$

$$= (X_1(t) \cos\theta + Y_1(t) \sin\theta) \Psi', \tag{4.89}$$

where $\Psi(r)$ is the original streamfunction and $\mathbf{X}(t) = \epsilon \mathbf{X}_1 + \cdots$ is the location of the origin in the new frame, as seen from the old one. This solution may be written in complex form as

$$\psi_1^f = \mathcal{Z}_1^*(t) \mathrm{e}^{\mathrm{i}\theta} r \Omega, \tag{4.90}$$

where the asterisk denotes complex conjugation. The function Ψ may be taken instead of ψ_0, since the two differ only by a constant function of space. The function $r\Omega \mathrm{e}^{\mathrm{i}\theta}$ is actually a steady mode-one solution to the linearised vorticity equation (in the absence of beta), as was discussed in Chapter 2. The corresponding vorticity is easily calculated to be

$$\nabla^2 \psi_1^f = X_1 \frac{\partial Q}{\partial x} + Y_1 \frac{\partial Q}{\partial y} = \mathcal{Z}_1^* \mathrm{e}^{\mathrm{i}\theta} Q', \tag{4.91}$$

which leads to the following two Jacobian terms in the equation of motion:

$$\mathcal{J}(\Psi, \nabla^2\psi_1^f) = \frac{\Psi'}{r}\mathcal{Z}_1^* \mathrm{i}\mathrm{e}^{\mathrm{i}\theta} Q' \tag{4.92}$$

and

$$\mathcal{J}(\psi_1^f, Q) = -\frac{\mathcal{Z}_1^* \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}\Psi'}{r} Q'. \tag{4.93}$$

Hence (4.86) is satisfied. This part of the streamfunction decays at infinity, and clearly vanishes at $t = 0$. Its Laplace transform is simply obtained by replacing $\mathcal{Z}_1$ by $\overline{\mathcal{Z}_1}$.

The behaviour of $r\Omega$ at infinity is given by

$$r\Omega(r) = \frac{\Gamma}{2\pi r} + O\left(\frac{1}{r^\infty}\right). \tag{4.94}$$

Therefore, expressed in terms of the far-field space variable and of the Laplace transform variable, ψ_1^f takes the form

$$\overline{\psi^{f(1,,)}} = \epsilon\overline{\mathcal{Z}_1^*}\mathrm{e}^{\mathrm{i}\theta}\left[\frac{\Gamma\epsilon}{2\pi R} + O\left(\frac{\epsilon^\infty}{R^\infty}\right)\right]. \tag{4.95}$$

This is formally a second-order quantity, so its truncations to zeroth and first order will vanish. Hence this contribution to the streamfunction from the change of frame does not affect the far-field expansion to zeroth and first order (via matching).

The governing equation at $O(\epsilon \ln \epsilon)$ is

$$\frac{\partial}{\partial t}\nabla^2\psi_1' + \mathcal{J}(\psi_0, \nabla^2\psi_1') + \mathcal{J}(\psi_1', \nabla^2\psi_0) = -\left(U_1'\frac{\partial}{\partial x} + V_1'\frac{\partial}{\partial y}\right)\nabla^2\psi_0. \tag{4.96}$$

There is no forcing due to the beta-effect and no advection of $O(\ln \epsilon)$ vorticity. The solution to the homogeneous problem was found in (4.80), while the solution to the drift problem is clearly formally of the same form as above, i.e.

$$\psi_1^{f\,\prime} = \mathcal{Z}_1^{*\prime}(t)\mathrm{e}^{\mathrm{i}\theta} r\Omega. \tag{4.97}$$

The analysis of the previous paragraph shows that this term does not require matching in the far field until second order in the logarithmic terms.

4·4·2·Origin of the coordinate system

It is logical to centre the moving coordinate system in the middle of the vortex. This condition will determine X_1 and Y_1 and hence close the set of equations. This is easier than solving an implicit set of equations in the original frame. However, the centre of the vortex must be specified somehow. A number of possibilities for identifying

the centre of an initially monopolar distribution exist. Four are presented here: the vorticity maximum, the streamfunction maximum, and the location of the particle initially at the origin, which are all described in RD94, and also a pseudo-secularity condition.

Relative vorticity maximum

One way of defining the centre of the vortex is to look at the maximum in relative vorticity. Presumably, this approach will not work if there is no such point (e.g. for the Rankine vortex). The position of the origin is then determined by the condition

$$\nabla(\nabla^2\psi)|_O = \mathbf{0}. \tag{4.98}$$

Clearly, the inner solution is the relevant one to employ here. The results of Appendix B must hold, since the relative vorticity is a well-defined quantity for the vortices considered here. So expanding in ϵ, (4.98) becomes

$$\frac{\partial}{\partial r}\nabla^2(\psi_0 + \ln\epsilon\,\psi_0' + \epsilon\psi_1 + \epsilon\psi_1^f + \epsilon\ln\epsilon\,\psi_1' + \epsilon\ln\epsilon\,\psi_1^{f\prime} + \cdots)\bigg|_{r=0} = 0. \tag{4.99}$$

The $O(1)$ term of (4.99) vanishes at the origin, since Ψ has a maximum there, and the $O(\ln\epsilon)$ term is dynamically insignificant. As for the other terms, it may be seen that

$$\nabla^2 r(p + \mathrm{i}\Omega(r)) = (3\Omega'(r) + r\Omega''(r)), \tag{4.100}$$

with the obvious special case $p = 0$; in addition

$$\begin{aligned} &-\frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\int_0^r \frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v \\ &= \left[-\frac{r(p+\mathrm{i}\Omega_0)}{p} + O(r^2)\right]\int_0^r\left[\frac{\Omega_0 v}{4} + O(1)\right]\left[\frac{1}{(p+\mathrm{i}\Omega_0)^2} + O(v)\right]\mathrm{d}v \qquad (4.101)\\ &= -\frac{\Omega_0 r^3\mathrm{e}^{\mathrm{i}\theta}}{8p(p+\mathrm{i}\Omega_0)} + O(r^4\mathrm{e}^{\mathrm{i}\theta}), \qquad (4.102)\end{aligned}$$

for small r, which leads to

$$\nabla^2\left[-\frac{r(p+\mathrm{i}\Omega(r))}{p^3}\mathrm{e}^{\mathrm{i}\theta}\int_0^r\frac{h(v)}{v^3(p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v\right] = -\frac{\Omega_0 r\mathrm{e}^{\mathrm{i}\theta}}{p(p+\mathrm{i}\Omega_0)} + O(r^2\mathrm{e}^{\mathrm{i}\theta}). \tag{4.103}$$

Hence (4.99) becomes the two equations

$$4\Omega_0''(\overline{\mathcal{Z}_1^*} + \mathrm{i}\overline{b_1}) - \frac{\Omega_0}{p(p+\mathrm{i}\Omega_0)} = 0 \tag{4.104}$$

and

$$4\Omega_0''(\overline{\mathcal{Z}_1^*} + \mathrm{i}\overline{b_1'}) = 0. \tag{4.105}$$

Clearly, if $\Omega_0'' = 0$, the first equation is inconsistent and the second one useless. This happens for vorticity profiles with an inflection point at the origin. In this special case, taking the r-derivative of (4.98) at the origin will lead to an answer. A higher number of derivatives may need to be taken for profiles that are very flat at the origin, such as $Q(r) = \mathrm{e}^{-r^6}$. However, if the vorticity profile is actually constant about the origin, as is the case for the Rankine vortex, the technique will not work at all.

When $\Omega_0'' \neq 0$, the resulting equations for the motion of the vortex are

$$\begin{aligned}\overline{\mathcal{Z}_1^*} &= -\frac{\mathrm{i}}{p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\frac{1}{4} - \gamma_E - \frac{1}{2}\right] - \frac{\Psi_0}{2}\right.\\ &\quad\left. + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\} + \frac{\Omega_0}{4\Omega_0''p(p+\mathrm{i}\Omega_0)},\end{aligned} \tag{4.106}$$

and

$$\overline{\mathcal{Z}_1^{*\prime}} = \frac{\mathrm{i}\Gamma}{4\pi p^3}. \tag{4.107}$$

Inverse Laplace transforming these expressions and taking their complex conjugates gives

$$\begin{aligned}\mathcal{Z}_1^{rv} &= \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4}\\ &\quad + \mathrm{i}\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} - \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v\\ &\quad + \frac{\mathrm{i}}{4\Omega_0''}(1 - \mathrm{e}^{\mathrm{i}\Omega_0 t})\end{aligned} \tag{4.108}$$

and

$$\mathcal{Z}_1^{rv\prime} = -\frac{\mathrm{i}\Gamma t^2}{8\pi}. \tag{4.109}$$

It is interesting to note that the first-order trajectory has an oscillatory component, forced at what might be termed the vortex frequency Ω_0. This oscillatory term is governed by the local behaviour of the basic state near the origin, while the other terms depend on the circulation and other global properties of the vortex.

The small-time behaviour of the displacement of the centre is given by

$$\mathcal{Z}_1^{rv} = \frac{\Omega_0 t}{4\Omega_0''} + \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} + O(t^3) \tag{4.110}$$

and by (4.109). The initial displacement of the vortex is zonal and to the West, since $\Omega_0 > 0$ and $\Omega_0'' < 0$. It comes from the oscillatory component of the displacement, and hence depends on the local behaviour of the basic-state vorticity near the origin. The next order term is meridional and to the North; it contains a logarithmic contribution. These results agree qualitatively with experimental observations of the motion of cyclones on the beta-plane (cf. Carnevale *et al.* 1991).

The large time behaviour of the trajectory can also be derived. For large t, the result is, from Appendix C,

$$\tilde{z}_1^{rv} = \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] + \frac{\mathrm{i}\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma_E + \frac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] O(t). \quad (4.111)$$

This result is unphysical, as can be seen by comparing it to computed vortex trajectories. It predicts a displacement to the South, which is clearly erroneous. However, the asymptotic expansion used breaks down for asymptotically large time. Either the current expansion must match onto some other expansion valid for these larger times, or there is no expansion possible for large times.

Streamfunction maximum

The position of the streamfunction maximum is determined by the condition

$$\nabla\psi|_O = \mathbf{0}. \quad (4.112)$$

The working is as above, replacing $\nabla^2\psi$ by ψ. Rewriting the two first-order equations leads to

$$\overline{b_1}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{\tilde{z}_1^*}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = 0 \quad (4.113)$$

and

$$\overline{b_1'}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{\tilde{z}_1^{*\prime}}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = 0. \quad (4.114)$$

Substituting the appropriate expressions gives

$$\begin{aligned}\overline{\tilde{z}_1^*} \;=\; & -\frac{p + \mathrm{i}\Omega_0}{\Omega_0 p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\frac{1}{4} - \gamma_E - \frac{1}{2}\right] - \frac{\Psi_0}{2}\right. \\ & \left. + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p + \mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\}, \end{aligned} \quad (4.115)$$

and

$$\overline{\tilde{z}_1^{*\prime}} = \frac{\Gamma(p + \mathrm{i}\Omega_0)}{4\pi\Omega_0 p^3}. \quad (4.116)$$

In the time variable, these expressions become

$$\mathcal{Z}_1^{sf} = \left(1 + \frac{\mathrm{i}}{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\right)\left[\mathcal{Z}_1^{rv} - \frac{\mathrm{i}}{4\Omega_0''}(1 - \mathrm{e}^{\mathrm{i}\Omega_0 t})\right] \tag{4.117}$$

or

$$\begin{aligned}\mathcal{Z}_1^{sf} &= \left(\mathrm{i}t^2 - \frac{2t}{\Omega_0}\right)\left\{\frac{\Gamma}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\Psi_0}{4}\right\} + \frac{\Gamma t}{8\pi\Omega_0} \\ &\quad + \int_0^\infty \frac{h(v)}{v^3}\left[-\frac{\mathrm{i}t^2}{2} + \left(\frac{t}{\Omega_0} - \frac{\mathrm{i}}{\Omega(v)^2}\right)\left(1 - \mathrm{e}^{\mathrm{i}\Omega(v)t}\right) + \frac{t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)}\right]\mathrm{d}v, \end{aligned} \tag{4.118}$$

and

$$\mathcal{Z}_1^{sf\prime} = \left(1 + \frac{\mathrm{i}}{\Omega_0}\frac{\mathrm{d}}{\mathrm{d}t}\right)\mathcal{Z}_1^{rv\prime} = \frac{\Gamma}{4\pi}\left[\frac{t}{\Omega_0} - \frac{\mathrm{i}t^2}{2}\right]. \tag{4.119}$$

The small time behaviour of the trajectory may be obtained from (4.110):

$$\begin{aligned}\mathcal{Z}_1^{sf} &= -\frac{\Gamma t}{4\pi\Omega_0}\left[-\ln\frac{t}{4} - 2\gamma_E + \frac{1}{2}\right] + \frac{\Psi_0 t}{2\Omega_0} + \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] \\ &\quad -\frac{\mathrm{i}\Psi_0 t^2}{4} - \frac{\mathrm{i}t^2}{\Omega_0}\int_0^\infty \frac{h(v)\Omega(v)^2}{v^3}\,\mathrm{d}v + O(t^3). \end{aligned} \tag{4.120}$$

The initial displacement is again to the West thanks to the $t\ln t$ term. The large time behaviour can similarly be obtained, and is given by

$$\begin{aligned}\mathcal{Z}_1^{sf} &= \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] + \frac{\mathrm{i}\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma_E + \frac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] \\ &\quad + O(t\ln t). \end{aligned} \tag{4.121}$$

This is almost the same as (4.111).

Motion of the origin

The particle initially at the origin of space moves with the flow, and its motion is described by the equations

$$\left.\frac{\partial\psi}{\partial y}\right|_{r=0} = -U \tag{4.122}$$

and

$$\left.\frac{\partial\psi}{\partial x}\right|_{r=0} = V. \tag{4.123}$$

Transforming to the Laplace variable, and working in polar coordinates, these two equations correspond to

$$\left.\frac{\partial\overline{\psi}}{\partial r}\right|_{r=0} = \mathrm{i}\mathrm{e}^{\mathrm{i}\theta}(\overline{U} - \mathrm{i}\overline{V}) = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{Z}^*}. \tag{4.124}$$

This equation may be expanded in powers of ϵ, which corresponds to adding the appropriate truncation of the right-hand side of (4.124) to (4.113) and (4.114). This leads to

$$\overline{b_1}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{\mathcal{Z}_1^*}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{Z}_1^*} \tag{4.125}$$

and

$$\overline{b_1'}(p + \mathrm{i}\Omega_0)\mathrm{e}^{\mathrm{i}\theta} + \overline{\mathcal{Z}_1^{*\prime}}\mathrm{e}^{\mathrm{i}\theta}\Omega_0 = \mathrm{i}p\mathrm{e}^{\mathrm{i}\theta}\overline{\mathcal{Z}_1^{*\prime}}. \tag{4.126}$$

Solving these equations leads to

$$\begin{aligned}\mathcal{Z}_1^o &= \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} \\ &\quad + \mathrm{i}\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} - \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v\end{aligned} \tag{4.127}$$

and

$$\mathcal{Z}_1^{o\prime} = -\frac{\mathrm{i}\Gamma t^2}{8\pi}. \tag{4.128}$$

These equations are almost the same as for the relative vorticity maximum technique. The only difference is the bounded and oscillatory term $\mathrm{i}(1-\mathrm{e}^{-\mathrm{i}\Omega_0 t})/4\Omega_0''$. The large-time behaviour of this technique will this be given by (4.111), while the small-time behaviour will be given by

$$\begin{aligned}\mathcal{Z}_1^o &= \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4} - \frac{t^3}{3}\int_0^\infty \frac{h(v)\Omega(v)^2}{v^3}\,\mathrm{d}v \\ &\quad + O(t^4).\end{aligned} \tag{4.129}$$

With this technique, the vortex initially moves to the North, which is different from the previous methods. It also moves more slowly initially.

Pseudo-secularity condition

The three conditions derived above make crucial use of the fact that the extremum of a mode-one quantity has a natural expression in terms of the derivative of that quantity near the origin. This was used to cancel off the mode-one component $\mathcal{Z}_1^* r\Omega$ which corresponds, to first order, to a change of frame, as was seen in Chapter 1. From Appendix B, it turns out that in fact any higher-mode terms in the inner solution ψ_1 do not play any part in the matching. The relevant proof was derived under the assumptions governing $\Psi(r)$ set forward in Chapter 1, but could almost certainly be generalised.

The discussion in Chapter 1 about the effect of a change in origin on a function

of space, however, suggests another way of picking the centre of the frame, namely to choose $\mathcal{Z}$ such that the streamfunction has no $r\Omega$ term.* This is a well-defined criterion, and corresponds to picking the reference frame in which the dynamical contribution to the streamfunction does not contain any contribution from the special change-of-frame solution.

This condition is easy to apply in the Laplace variable; it leads to

$$\overline{\mathcal{Z}_1^*} + \mathrm{i}\overline{b_1} = 0 \tag{4.130}$$

and

$$\overline{\mathcal{Z}_1^{*\prime}} + \mathrm{i}\overline{b_1'} = 0. \tag{4.131}$$

Thus

$$\begin{aligned}\overline{\mathcal{Z}_1^*} \;=\; & -\frac{\mathrm{i}}{p^3}\left\{\frac{\Gamma}{4\pi}\left[\ln p - \ln\frac{1}{4} - \gamma_E - \frac{1}{2}\right] - \frac{\Psi_0}{2}\right.\\ & \left. + \int_0^\infty \frac{h(v)\Omega(v)}{v^3(p+\mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v\right\},\end{aligned} \tag{4.132}$$

which becomes

$$\begin{aligned}\mathcal{Z}_1^s \;=\; & \frac{\mathrm{i}\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Psi_0 t^2}{4}\\ & +\mathrm{i}\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} + \frac{\mathrm{i}t\mathrm{e}^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v.\end{aligned} \tag{4.133}$$

The expression for the logarithmic displacement is also very simple and takes the form

$$\overline{\mathcal{Z}_1^{*\prime}} = \frac{\mathrm{i}\Gamma}{4\pi p^3}, \tag{4.134}$$

that is

$$\mathcal{Z}_1^{s\prime} = -\frac{\mathrm{i}\Gamma t^2}{8\pi}. \tag{4.135}$$

Interestingly enough, this recovers the result of the displacement of the origin. In a sense, this is because this technique is concerned with minimising the first-order difference between frames, and the Lagrangian technique of tracking the origin corresponds very naturally to picking a frame. However, this technique will behave very differently for higher orders. It may be called a 'pseudo-secularity' condition, since it does not really remove the ultimate breakdown in the asymptotic expansion.

*I am grateful to Peter Haynes for suggesting this approach.

Nevertheless, it fully removes all extraneous order-one contributions that may be ascribed merely to a change of frame. In effect, it corresponds to the transformation that most fully keeps the vortex symmetric. The overall physical effect is not just a change of frame though, since such a transformation would have higher-order terms too. For higher-order terms, this technique may also be used, although it cannot remove asymmetries in modes other than mode one.

The behaviour of the trajectory is already known from previous results. It is clear that while this technique removes unwanted changes of frame, the predicted trajectory will eventually reverse to the South. Hence the expansion cannot remain valid longer than with any other technique.

4·5·Examples

4·5·1·Rankine vortex

The Rankine profile is a distribution with vorticity $2\Omega_0$ within a radius d say, and zero vorticity outside. Then

$$\Omega(r) = \Omega_0 \tag{4.136}$$

inside the disc, and

$$\Omega(r) = \frac{\Omega_0 d^2}{r^2} \tag{4.137}$$

outside it. The corresponding streamfunction is

$$\Psi(r) = \frac{1}{2}\Omega_0(r^2 - d^2) + \Omega_0 d^2 \ln d \tag{4.138}$$

inside the disc, and

$$\Psi(r) = \Omega_0 d^2 \ln r \tag{4.139}$$

outside it. This corresponds to a circulation of $2\pi\Omega_0 d^2$, and a value of the streamfunction of $\Psi_0 = \Omega_0 d^2 \left(\ln d - \frac{1}{2}\right)$ at the origin. For the numerical calculations of this chapter, the values $\Omega_0 = \frac{1}{2}$ and $d = 1$ have been chosen, leading to $\Gamma = \pi$.

The integral in (4.127) can be evaluated analytically. The position of the centre of the vortex is given by

$$\begin{aligned} \mathcal{Z}_1^o &= \frac{\mathrm{i}\Omega_0 d^2 t^2}{4}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\mathrm{i}\Omega_0 d^2 t^2(\ln d - 1/2)}{4} \\ &\quad + \frac{\mathrm{i}\Omega_0 d^2 t^2}{8}\left[-\ln\Omega_0 t - \gamma_E + \frac{\mathrm{i}\pi}{2} + \frac{1}{2} - E_1\left(-\mathrm{i}\Omega_0 t\right)\right] + \frac{d^2 t}{8} \end{aligned}$$

$$+\frac{\mathrm{i}d^2}{8\Omega_0}\left(\mathrm{e}^{\mathrm{i}\Omega_0 t}-1\right), \tag{4.140}$$

where E_1 is the exponential integral (Abramowitz & Stegun 1969), and by

$$z_1^{0\prime} = -\frac{\mathrm{i}\Omega_0 d^2 t^2}{4}. \tag{4.141}$$

However, the large-time asymptotic behaviour of the vortex is not given by (4.111). This is due to the assumption about the strict monotonicity of Ω employed in the derivation of the large-time behaviour, which is not appropriate in the case of the Rankine vortex. The oscillatory behaviour of the trajectory comes from the exponential integral term, which leads to a leading-order oscillatory term $t\mathrm{e}^{\mathrm{i}\Omega_0 t}/8$ for large times. These oscillations are hence of lower order than the actual displacement of the vortex, which behaves like t^2.

The trajectory of the streamfunction maximum may also be calculated. This results in

$$\begin{aligned} z_1^{sf} \;=\; & \left(t^2+\frac{2\mathrm{i}t}{\Omega_0}\right)\left\{\frac{\mathrm{i}\Omega_0 d^2}{4}\left[-\ln\frac{t}{4}-2\gamma_E+1\right]-\frac{\mathrm{i}\Omega_0 d^2(\ln d-1/2)}{4}\right\} \\ & +\left(t^2+\frac{2\mathrm{i}t}{\Omega_0}\right)\frac{\mathrm{i}\Omega_0 d^2}{8}\left[-\ln\Omega_0 t-\gamma_E+\frac{\mathrm{i}\pi}{2}-E_1\left(-\mathrm{i}\Omega_0 t\right)+\frac{1}{2}\right] \\ & -\frac{d^2 t}{8}\left(\mathrm{e}^{\mathrm{i}\Omega_0 t}-4\right) \end{aligned} \tag{4.142}$$

and

$$z_1^{sf\prime} = \frac{d^2 t}{2}-\frac{\mathrm{i}\Omega_0 d^2 t^2}{4}. \tag{4.143}$$

Figure 4.1 shows the trajectory of the vortex, identified by the two different techniques, until the time at which the trajectory starts to curve back down to the South. The motion of the centre of the vortex is oscillatory, but for the streamfunction maximum, the oscillations are suppressed for this trajectory, since the $-d^2 t\mathrm{e}^{\mathrm{i}\Omega_0 t}/8$ term cancels with the leading-order contribution from the exponential integral. There are oscillations, but they are at a higher order, and hence cannot be seen on the plot. The difference between the two methods is most marked for large ϵ.

The time of validity of the expansion may be evaluated from the maximum y-position of the trajectory, for the particle-tracking and for the streamfunction-maximum techniques. Figure 4.2 shows logarithmic plots of time against ϵ for both methods. A straight line fit to this curve, using values of ϵ between 10^{-6} and 0.1 gives a slope of -0.6642 for the particle technique, and -0.7092 for the streamfunction method. This suggests that the expansion breaks down for times

of order $\epsilon^{-\frac{2}{3}}$. These values become -0.6064 and -0.6956, respectively, when ϵ is restricted to the range 0.005 to 0.1, slightly under two decades. This dependence on $\epsilon^{-\frac{2}{3}}$ may be understood as follows. The maximum value of y must occur for large time (or else the expansion is useless), and hence the dominant contribution to the time-derivative of the imaginary part of $\mathcal{Z}_1^o + \ln\epsilon\mathcal{Z}_1^{o\prime}$ is

$$-\frac{\Gamma t \ln t}{4\pi} - \frac{\Gamma t \ln t}{8\pi} - \frac{\Gamma t \ln \epsilon}{4\pi}. \tag{4.144}$$

This vanishes asymptotically for $t = O(\epsilon^{-\mathcal{N}})$ when $\mathcal{N}+\mathcal{N}/2-1 = 0$, i.e. for $t = O(\epsilon^{-\frac{2}{3}})$. The same argument holds for the streamfunction method, since the two behave identically for large time.

4·5·2·Gaussian vortex

The vorticity profile for the Gaussian vortex will be taken to be

$$Q(r) = \mathrm{e}^{-r^2}. \tag{4.145}$$

This corresponds to angular velocity

$$\Omega(r) = \frac{1-\mathrm{e}^{-r^2}}{2r^2} \tag{4.146}$$

and streamfunction

$$\Psi(r) = \frac{1}{2}\ln r + \frac{1}{4}E_1(r^2). \tag{4.147}$$

The values of the streamfunction and angular velocity at the origin are given by $\Psi_0 = -\gamma_E/4$ and $\Omega_0 = \frac{1}{2}$. The circulation of the vortex is equal to π. The integrals in the preceding section cannot be evaluated in closed form, but may be calculated numerically. This was done using the NAG routine D01AMF.

Figure 4.3 shows the trajectory of the vortex, using the path of the origin and the streamfunction-maximum techniques. Again, the trajectory is truncated when it turns South. The path of the maximum in relative vorticity could be plotted, but it is almost identical to figure 4.3 except for large ϵ, where high resolution is needed for small time.

Again, the time of validity of the expansion may be estimated by considering the y-position reached. Figure 4.4 shows the maximum time values as functions of ϵ. The slopes of the lines are now -0.688 and -0.679. The same argument as in the preceding section explains the $-2/3$ slope for both methods.

4·6·Comparison with numerical simulation

This section presents a comparison of numerical simulations with the analytical predictions of the present theory. The numerical model is inviscid and spectral, and is that used in Carnevale & Kloosterziel (1994). Gaussian vortices were the initial condition in a square domain. Different values of ϵ were used, and the figures show the trajectory of the relative vorticity maximum. In the numerical case, this was calculated by bilinear interpolation between grid points (the numerical model having a grid on which to calculate the nonlinear terms). This approach results in a smooth trajectory, but the resulting velocity has discontinuities in it, resulting from the centre of the vortex jumping from grid point to grid point.

The values of ϵ used are 0.0039 for figure 4.5, 0.0155 for figure 4.6, 0.062 for figure 4.7 and 0.2482 for figure 4.7.

The results are quite good, considering that the numerical calculations took place on a finite domain, and the numerical vortices simulated had their circulation removed. The former problem may be addressed by re-performing the analysis on a finite domain. The latter is more problematic. The whole analysis of this chapter carries through with $\Gamma = 0$ so that is not the problem. However, the simulations are performed in a finite domain, and the total circulation is subtracted from the vorticity field. This is impossible to do in an unbounded domain since the resultant velocity field grows like r at infinity. Hence it is not possible to produce a circulation-free initial vorticity profile to use for comparison purposes. For the moment at least, the question of the effect of circulation is not properly addressed by these numerical simulations. The questions of the effect of circulation and domain size will be returned to in Chapter 5.

4·7·Conclusions

The initial-value problem for the evolution of a circular vortex on the beta-plane has been solved to first order in an expansion in ϵ, the nondimensional beta-effect. To zeroth order, the near-field response is just the initial condition: an intense vortex is unaffected by the beta-induced perturbation. The far-field zeroth-order response is the Green's function of the linear Rossby wave equation, with amplitude equal to the circulation of the initial vortex. This response only exists therefore for nonisolated vortices.

The first-order response in the near field is a time-dependent dipole, which

corresponds to the beta-gyres (Sutyrin & Flierl 1994). There is an $O(\epsilon \ln \epsilon)$ response proportional to the circulation; this means that the trajectory is different for different values of ϵ. The first-order far-field response is identically zero. The solution (4.82) reduces to that of RD94 in the case of zero circulation, in which case the logarithmic response vanishes.

The equation of motion may be solved in a reference frame centred in the vortex, and the steady mode-one solution ϕ^s to the radial Rayleigh equation may be generalised by convolution to enable the problem to be solved in the new frame.

The fundamental issue then becomes locating the centre of the vortex. Four possibilities are set out (following RD94): the maximum in relative vorticity, the maximum in streamfunction, the position of the particle initially at the origin of space, and a pseudo-secularity condition. The first of these techniques fails when the basic-state vorticity is constant at the origin, as in the case of the Rankine vortex. The second is not Galilean invariant. The third turns out to be almost identical with the first. This is to be expected, since the difference between the relative vorticity of the point initially at the origin and its value at the instant $t = 0$ is essentially ϵy (y in the original coordinate frame), which is an asymptotically small quantity.

The final technique is probably the most interesting. Mathematically, it leads to the same result as the Lagrangian origin-following approach to first order. However, it is obtained by removing all terms of the $r\Omega$, or convolutions of the trajectory with the steady mode-one solution.

The large-time behaviour of all these methods gives motion to the South, which disagrees with experiments. In fact, the time at which the paths curve back down to the South is $O(\epsilon^{-\frac{2}{3}})$, which suggests that naive estimation of the breakdown of the expansion as occurring at $O(\epsilon^{-1})$ is simplistic.

The trajectory of the Rankine vortex is computed analytically, and that of the Gaussian vortex numerically. Both these profiles have nonzero circulation. The resulting trajectories are plotted. The case of the Gaussian vortex is compared to numerical experiments. Agreement is good for small times, but it is hard to compare the situations exactly, since the numerical calculations took place on a finite domain, with vortices whose circulation was removed.

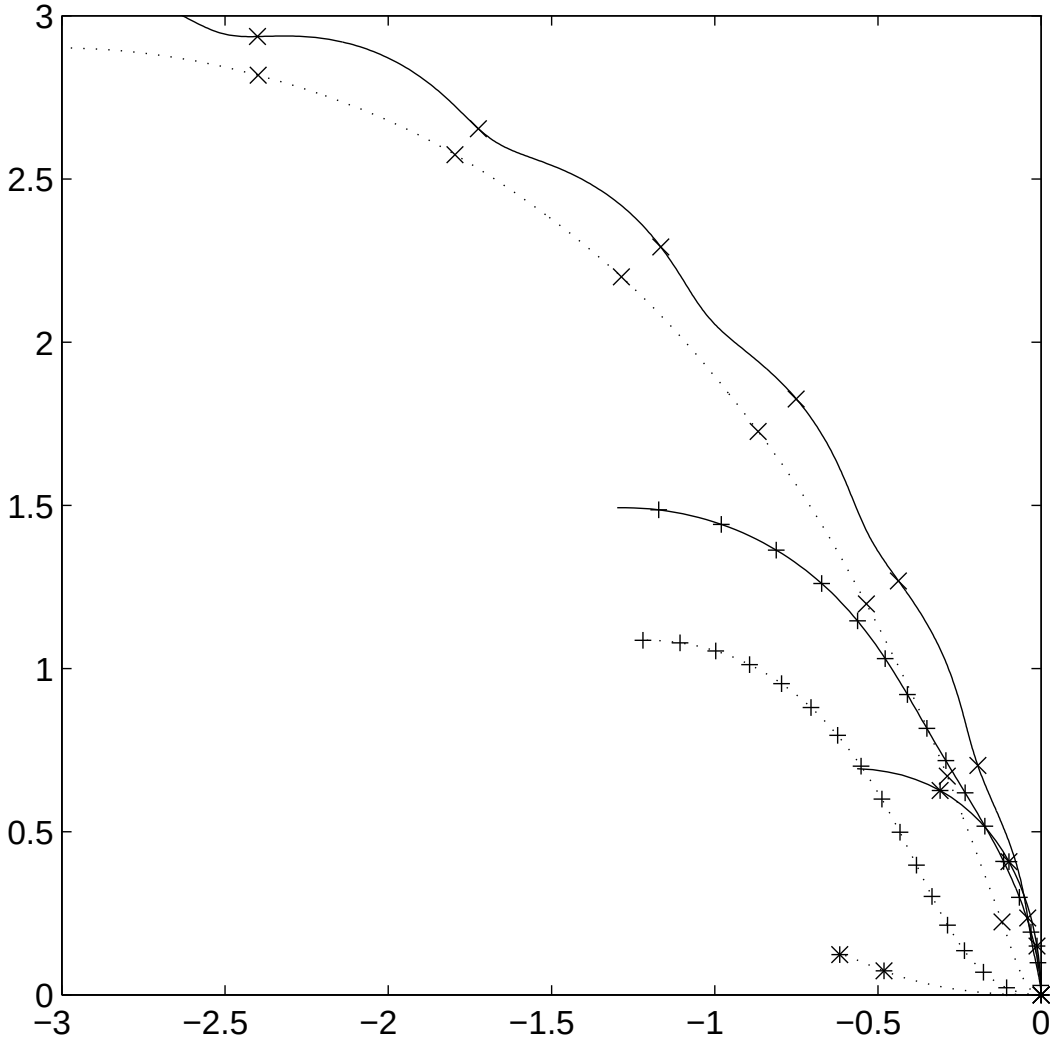

Figure 4.1: Trajectory of a Rankine vortex stopping at the maximum y-position. Full lines correspond to the relative vorticity maximum (v), dotted lines to the streamfunction maximum (s). For each technique, there is a lower line, corresponding to $\epsilon = 0.5$, with stars at unit time intervals and a maximum time of 3.75 for v and 2 for s, a middle line with $\epsilon = 0.05$, crosses at unit time intervals and maximum time 16.625 for v and 16 for s, and an upper line, with $\epsilon = 0.005$, crosses every 10 time units and maximum time 78.75 for v and 79.5 for s.

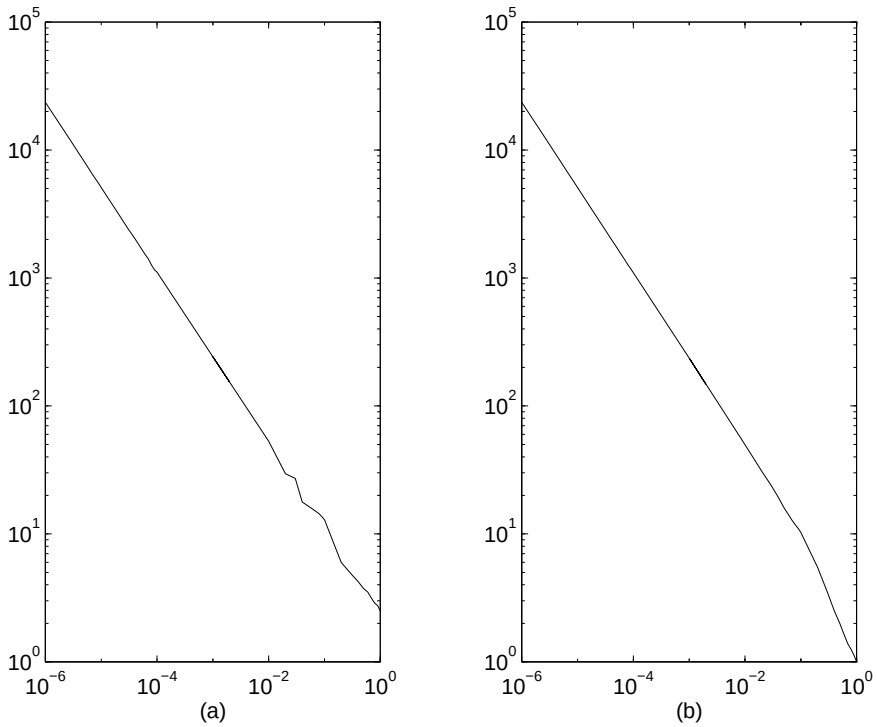

Figure 4.2: Log-log plot of approximate time of breakdown of expansion versus ϵ for the Rankine vortex; (a) corresponds to the motion of the origin technique, (b) to the streamfunction maximum. The slope of the line gives the power of ϵ for breakdown.

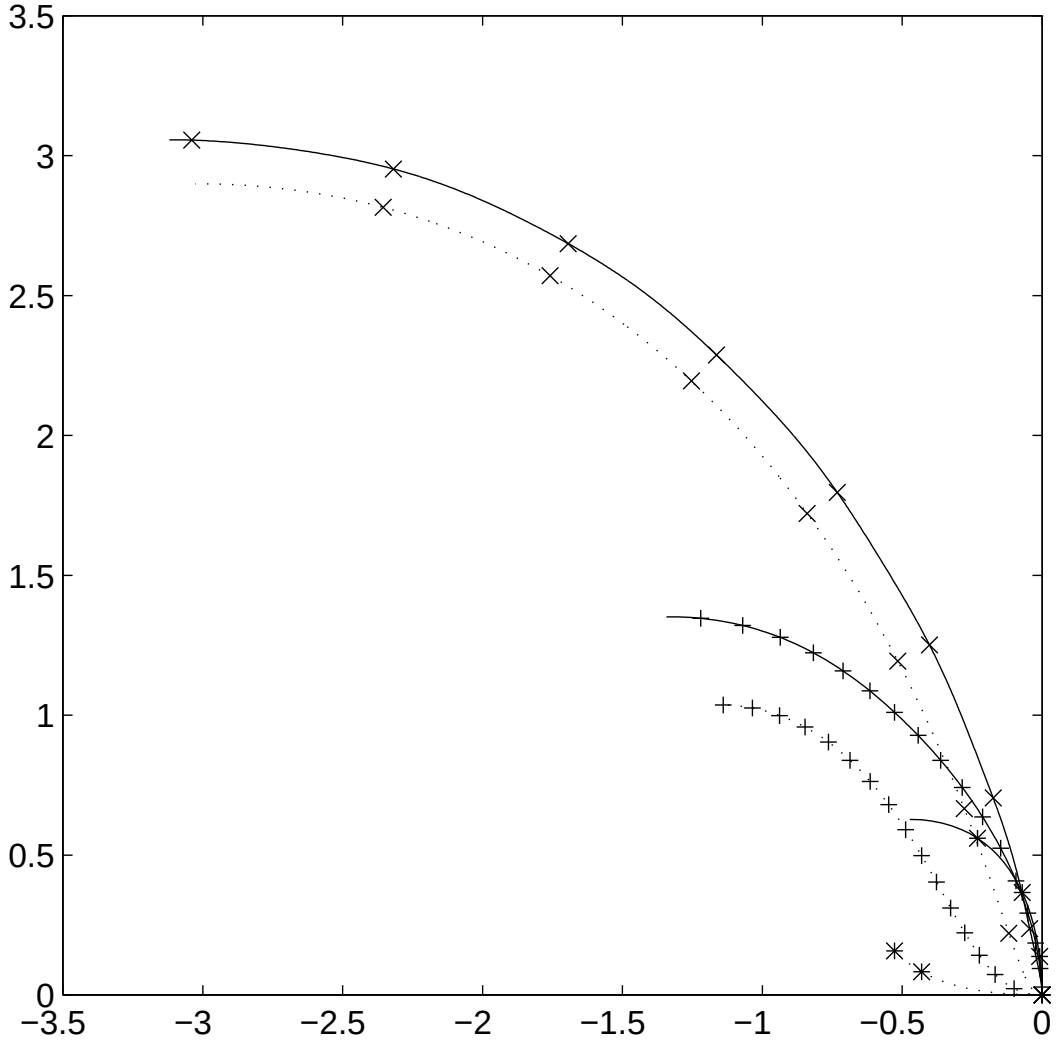

Figure 4.3: Trajectory of a Gaussian vortex stopping at the maximum y-position. Full lines correspond to the relative vorticity maximum (v), dotted lines to the streamfunction maximum (s). For each technique, there is a lower line, corresponding to $\epsilon = 0.5$, with stars at unit time intervals and a maximum time of 3.875 for v and 2.25 for s, a middle line with $\epsilon = 0.05$, crosses at unit time intervals and maximum time 17.75 for v and 16.125 for s, and an upper line, with $\epsilon = 0.005$, crosses every 10 time units and maximum time 81 for v and 79.75 for s.

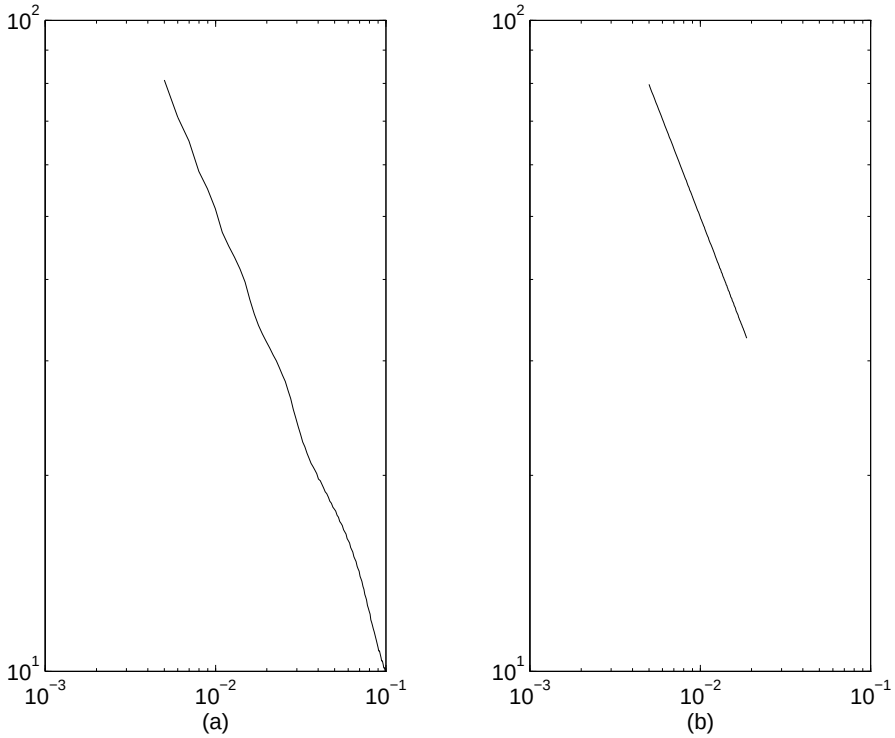

Figure 4.4: Log-log plot of approximate time of breakdown of expansion versus ϵ for the Gaussian vortex; (a) corresponds to the motion of the origin technique, (b) to the streamfunction maximum. The slope of the line gives the power of ϵ for breakdown.

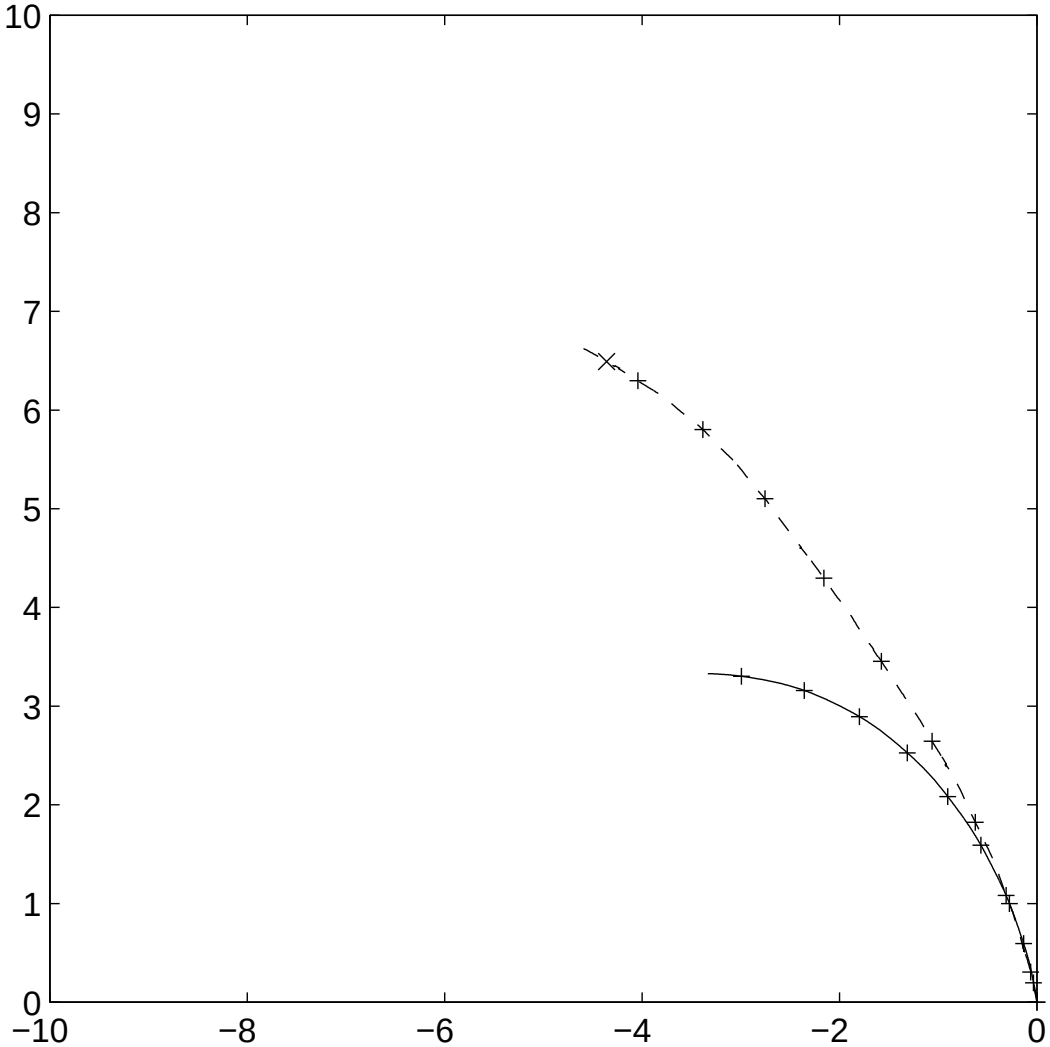

Figure 4.5: Comparison of numerical and analytical trajectories for the Gaussian vortex. The numerical trajectory is dashed. The nondimensional beta-parameter is $\epsilon = 0.0039$. The time of breakdown of the analytical solution is taken to be when the centre of the vortex reaches its maximum y-displacement, 95.375 here. Plus crosses on the trajectories are every 10 time units. The times cross shows the position on the numerical trajectory when the analytical one breaks down.

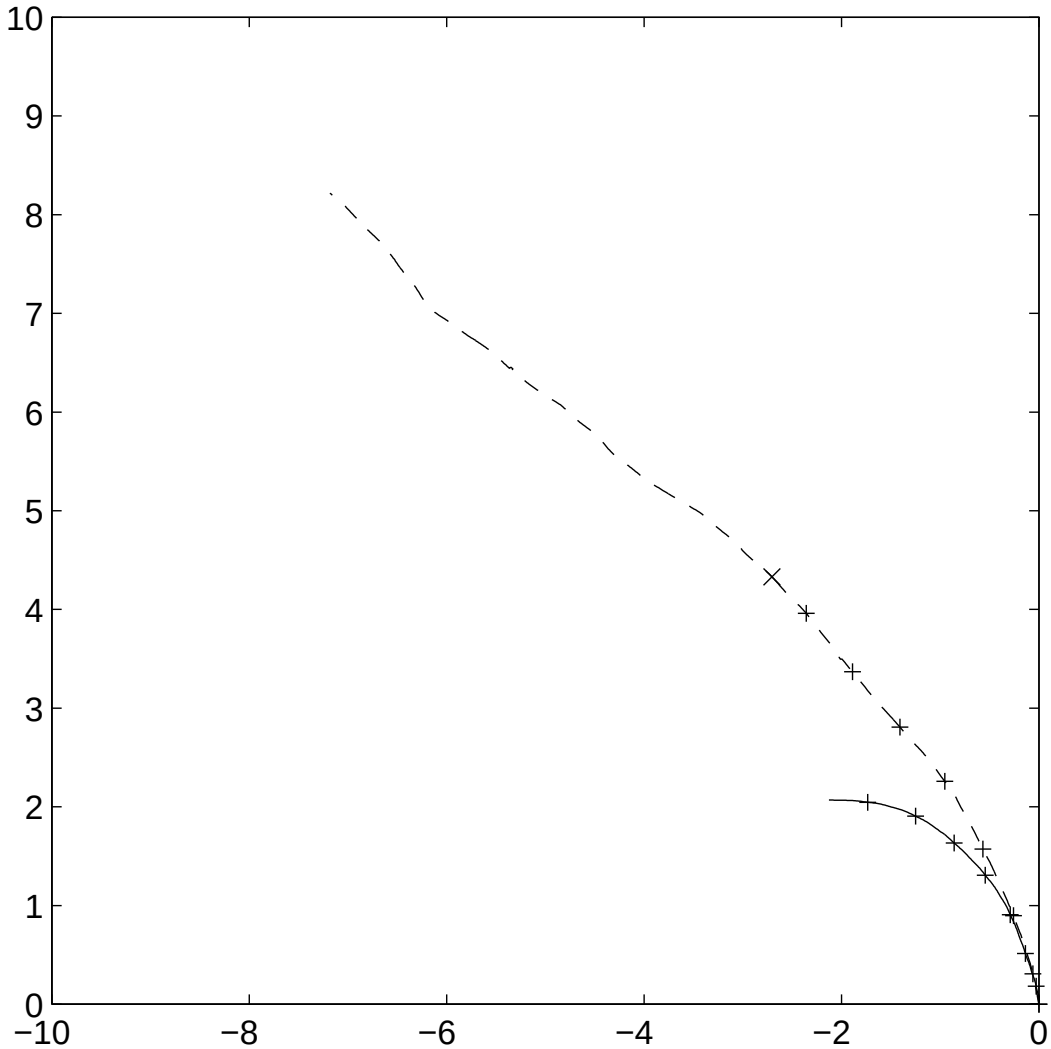

Figure 4.6: Comparison of numerical and analytical trajectories for the Gaussian vortex. The numerical trajectory is dashed. The nondimensional beta-parameter is $\epsilon = 0.0155$. The time of breakdown of the analytical solution is taken to be when the centre of the vortex reaches its maximum y-displacement, 38.25 here. Plus crosses on the trajectories are every 5 time units. The times cross shows the position on the numerical trajectory when the analytical one breaks down.

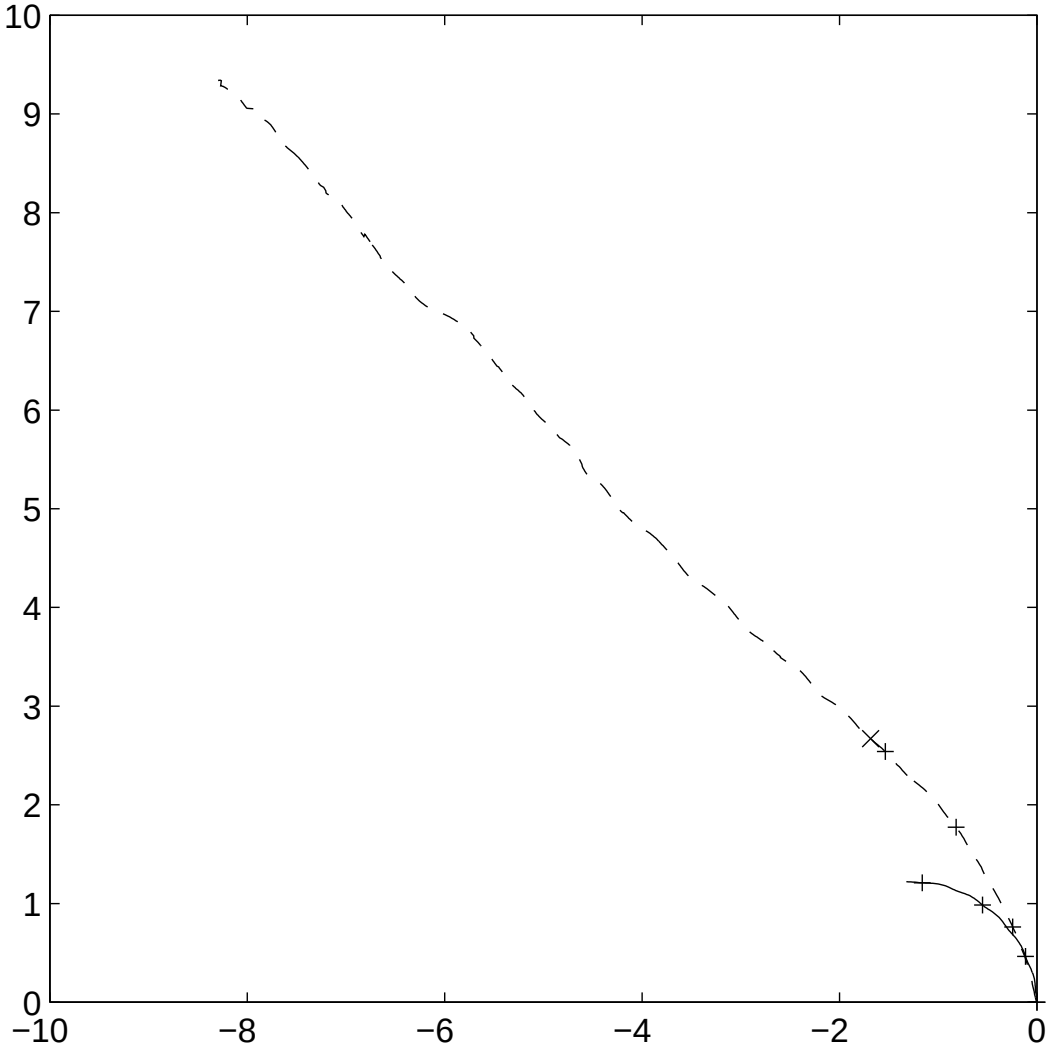

Figure 4.7: Comparison of numerical and analytical trajectories for the Gaussian vortex. The numerical trajectory is dashed. The nondimensional beta-parameter is $\epsilon = 0.062$. The time of breakdown of the analytical solution is taken to be when the centre of the vortex reaches its maximum y-displacement, 15.5 here. Plus crosses on the trajectories are every 5 time units. The times cross shows the position on the numerical trajectory when the analytical one breaks down.

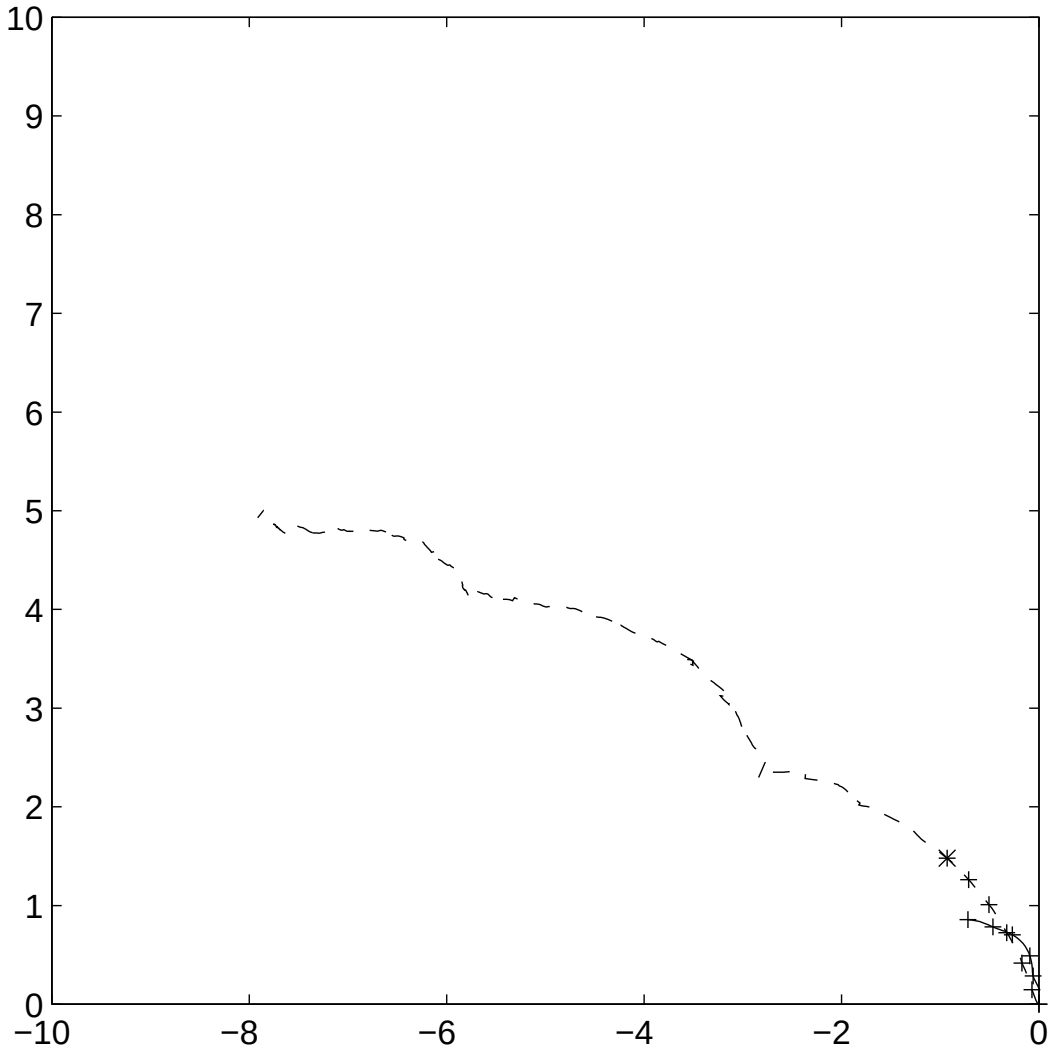

Figure 4.8: Comparison of numerical and analytical trajectories for the Gaussian vortex. The numerical trajectory is dashed. The nondimensional beta-parameter is $\epsilon = 0.2482$. The time of breakdown of the analytical solution is taken to be when the centre of the vortex reaches its maximum y-displacement, 5.75 here. Plus crosses on the trajectories are every time unit. The times cross shows the position on the numerical trajectory when the analytical one breaks down.

5·Nondivergent second-order solution and breakdown

5·1·Preliminary results

Calculating the second-order solution is clearly important in order to better understand the nature of the asymptotic expansion. Including an extra order in the solution should not only give better predictions for the cyclone trajectory, at least if ϵ is small enough, but also give an insight into the breakdown of the expansion. Since closed-form solutions are not in general available for the differential operators present in the second-order equations, the only way to proceed analytically is by an asymptotic expansion in r. The main motivation of the second-order analysis is to understand the asymptotic behaviour of the solution in the matching region, and of the far field. The nature of the expansion means that the complete behaviour of the second-order solution in the far field may be obtained.

The behaviour of the zeroth- and first-order solutions is needed to higher order than was previously considered. Since the second-order equations have nonlinear forcing terms, it is necessary to work in the time variable to avoid convolution operators in the Laplace variable.

5·1·1·Large-r behaviour of previous inner solutions

The zeroth-order streamfunction is given for large r by

$$\psi_0 = \frac{\Gamma}{2\pi}\left[\ln\frac{rt}{4} + 2\gamma_E\right] + O(r^{-\infty}); \tag{5.1}$$

there are no higher-order terms to consider. The first-order streamfunction is given by

$$\psi_1 = F(r,t)\mathrm{e}^{\mathrm{i}\theta} + \mathcal{Z}_1^*(t)r\Omega\mathrm{e}^{\mathrm{i}\theta}, \tag{5.2}$$

where

$$\begin{aligned} F(r,t) \;=\; & -r\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\int_0^r \frac{h(v)}{v^3}\left[\frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t} - 1}{\Omega(v)^2}\right]\mathrm{d}v \\ & + r\mathrm{e}^{\mathrm{i}\theta}\left(\frac{\partial}{\partial t} + \mathrm{i}\Omega(r)\right)\left\{\frac{\Gamma t^2}{8\pi}\left[-\ln\frac{t}{4} - 2\gamma_E + 1\right] - \frac{\Psi_0 t^2}{4}\right. \end{aligned}$$

$$+\int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2}+\frac{\mathrm{i}t\mathrm{e}^{-\mathrm{i}\Omega(v)t}}{\Omega(v)}+\frac{\mathrm{e}^{-\mathrm{i}\Omega(v)t}-1}{\Omega(v)^2}\right]\mathrm{d}v\Bigg\}. \tag{5.3}$$

The large-r behaviour of this expression may most easily be found using (4.81). This yields

$$\begin{aligned}\overline{F} &= \frac{r}{p^3}\left[p+\frac{\mathrm{i}\Gamma}{2\pi r^2}+O(r^{-\infty})\right]\Bigg\{-\frac{\Gamma}{4\pi}\ln r+\frac{\Gamma}{8\pi}+\frac{H}{2r^2}\\ &\quad+\frac{\Gamma}{4\pi}\left[\ln p-\ln\frac{1}{4}-\gamma_E-\frac{1}{2}\right]+\int_r^\infty\left[-\frac{\mathrm{i}\Gamma^2}{4\pi p v^3}+O\left(\frac{1}{v^4}\right)\right]\mathrm{d}v\Bigg\} \qquad (5.4)\\ &= -\frac{\Gamma}{4\pi p^2}r\ln r+\frac{\Gamma}{4\pi p^2}\left[\ln p-\ln\frac{1}{4}-\gamma_E\right]r-\frac{\mathrm{i}\Gamma^2}{8\pi^2p^3}\frac{\ln r}{r}\\ &\quad+\left\{\frac{\mathrm{i}\Gamma}{8\pi^2p^3}\left[\ln p-\ln\frac{1}{4}-\gamma_E\right]+\frac{H}{2p^2}-\frac{\mathrm{i}\Gamma^2}{8\pi^2p^3}\right\}\frac{1}{r}+O\left(\frac{1}{r^3}\right). \qquad (5.5)\end{aligned}$$

Inverse Laplace transforming in time gives

$$\psi_1=\left[a(t)r\ln r+b(t)r+d(t)\frac{\ln r}{r}+e(t)\frac{1}{r}+O\left(\frac{1}{r^3}\right)\right]\mathrm{e}^{\mathrm{i}\theta}, \tag{5.6}$$

with

$$a(t) = -\frac{\Gamma t}{4\pi}, \tag{5.7}$$

$$b(t) = -\frac{\Gamma t}{4\pi}\left[\ln\frac{t}{4}+2\gamma_E-1\right], \tag{5.8}$$

$$d(t) = -\frac{\mathrm{i}\Gamma^2t^2}{16\pi^2}, \tag{5.9}$$

$$e(t) = -\frac{\mathrm{i}\Gamma^2t^2}{16\pi^2}\left[\ln\frac{t}{4}+2\gamma_E-\frac{1}{2}\right]+\frac{Ht}{2}+\frac{\Gamma\mathcal{Z}_1^*}{2\pi}. \tag{5.10}$$

with H defined as in (4.41). The expressions for a and b may be checked by comparing them to the mode-one terms in ϕ_0.

The $O(\epsilon\ln\epsilon)$ solution is

$$\psi_1' = -\frac{\Gamma r\mathrm{e}^{\mathrm{i}\theta}}{4\pi}\left[t+\mathrm{i}\Omega\frac{t^2}{2}\right]+\mathcal{Z}_1^{*\prime}r\Omega\mathrm{e}^{\mathrm{i}\theta} \tag{5.11}$$

$$= \left[-\frac{\Gamma t}{4\pi}r+\frac{1}{r}\left(-\frac{\mathrm{i}\Gamma^2t^2}{16\pi^2}+\frac{\Gamma\mathcal{Z}_1^{*\prime}}{2\pi}\right)\right]\mathrm{e}^{\mathrm{i}\theta}+O(r^{-\infty}), \tag{5.12}$$

over the whole domain, and for large r, respectively. Its Laplacian is

$$\nabla^2\psi_1' = -\frac{\mathrm{i}\Gamma t^2\mathrm{e}^{\mathrm{i}\theta}}{8\pi}\mathcal{Q}'+\mathcal{Z}_1^{*\prime}\mathcal{Q}'\mathrm{e}^{\mathrm{i}\theta}. \tag{5.13}$$

This decays faster than any power of r for large r. Hence any terms containing $\nabla^2\psi_1'$ cannot contribute to the matching.

The behaviour of the Laplacian operator acting on these expressions is given by

$$\nabla^2\psi_0 = O(r^{-\infty}) \tag{5.14}$$

since the basic-state vorticity is localised, by

$$\nabla^2\psi_1 = \frac{2a}{r}\mathrm{e}^{\mathrm{i}\theta} + O\left(\frac{1}{r^3}\right), \tag{5.15}$$

for the first-order term, and by

$$\nabla^2\psi_1' = O(r^{-\infty}) \tag{5.16}$$

for the $O(\epsilon \ln \epsilon)$ term.

The behaviour of the truncation of the inner solution in the outer variable at second order is given by

$$\psi^{(1,2)} = \psi^{(1,1)} + \epsilon^2\frac{d\ln R/\epsilon}{R} + \epsilon^2\frac{e}{R} + \frac{\epsilon^2\ln\epsilon}{R}\left[-\frac{\mathrm{i}\Gamma^2t^2}{16\pi^2} + \frac{\Gamma\mathcal{Z}_1^{*\prime}}{2\pi}\right]. \tag{5.17}$$

5·1·2·Small-R behaviour of previous outer solutions

The only outer solution is the zeroth-order response $\phi_0 = \Gamma L(\mathbf{R}, t)$. The inner limit of this function has already been computed in Chapter 3, and is reproduced here for completeness. It is given by

$$\begin{aligned}\phi_0 &= \frac{\Gamma}{2\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E\right] - \frac{\Gamma Xt}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right] \\ &\quad + \frac{\Gamma t^2}{128\pi}\left[(2X^2 + R^2)\left(2\ln\frac{Rt}{4} + 4\gamma_E - 3\right) - 2R^2\right] + O(R^3).\end{aligned} \tag{5.18}$$

Rewritten in complex form to highlight the behaviour of different trigonometric modes, this is

$$\begin{aligned}\phi_0 &= \frac{\Gamma}{2\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E\right] - \frac{\Gamma R\mathrm{e}^{\mathrm{i}\theta}t}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right] \\ &\quad + \frac{\Gamma R^2t^2}{128\pi}\left[(2 + \mathrm{e}^{2\mathrm{i}\theta})\left(2\ln\frac{Rt}{4} + 4\gamma_E - 3\right) - 2\right] + O(R^3).\end{aligned} \tag{5.19}$$

The Laplacian of (5.18) is

$$\nabla^2\phi_0 = -\frac{\Gamma Xt}{2\pi R^2} + \frac{\Gamma t^2}{16\pi}\left[2\ln\frac{Rt}{4} + 4\gamma_E - 2 + \cos 2\theta\right] + O(R) \tag{5.20}$$

$$= -\frac{\Gamma\mathrm{e}^{\mathrm{i}\theta}t}{2\pi} + \frac{\Gamma t^2}{16\pi}\left[2\ln\frac{Rt}{4} + 4\gamma_E - 2 + \mathrm{e}^{2\mathrm{i}\theta}\right] + O(R) \tag{5.21}$$

$$\tag{5.22}$$

for small R.

The order-zero outer solution is just Γ times the Green's function for the linear outer wave equation L. The behaviour of the gradient of this function is needed for the matching problem. This is recapitulated here:

$$\nabla L = \frac{1}{2\pi}\frac{\mathbf{R}}{R^2} - \frac{t}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right]\mathbf{i} - \frac{Xt}{4\pi}\frac{\mathbf{R}}{R^2} + O(R). \tag{5.23}$$

Terms of order R or higher will not appear in the matching, since they behave like ϵ^3 (or higher) in the inner variable. The double-derivative is also needed. This comes out as

$$\begin{aligned}\frac{\partial^2 L}{\partial X_i \partial X_j} &= \frac{1}{2\pi}\left[\frac{\delta_{ij}}{R^2} - \frac{2X_iX_j}{R^4}\right] - \frac{t}{4\pi}\frac{\delta_{i1}X_j + \delta_{j1}X_i}{R^2} \\ &\quad -\frac{Xt}{4\pi}\left[\frac{\delta_{ij}}{R^2} - \frac{2X_iX_j}{R^4}\right] + O(1).\end{aligned} \tag{5.24}$$

Some of the $O(R^{-1})$ terms in this expansion, which will appear in the second-order matching, are mode-three terms, but do not have the right behaviour to match onto homogeneous mode-three inner solutions. Hence there cannot be any such term in the outer-region solution, and the outer solution, to second order, may only contain terms proportional to L and ∇L.

5·1·3·Complex representation of forcing terms

The forcing terms in the second-order equations are Jacobians, x-derivatives, and terms corresponding to changes of frame due to the first-order solution, i.e. due to a mode-one function. Hence they contain zeroth- and second-order contributions, from the nonlinear interaction and from the derivative. These may be represented in a convenient complex form, which however needs to be derived from the original real equation with care.

The beta-effect, when applied to a mode-one term, gives the following result:

$$\begin{aligned}\frac{\partial}{\partial x} g e^{\mathrm{i}\theta} &= \left[\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right][g_r\cos\theta - g_i\sin\theta] \\ &= \frac{1}{2}\left(g_r' + \frac{g_r}{r}\right) + \frac{\cos 2\theta}{2}\left(g_r' - \frac{g_r}{r}\right) + \frac{\sin 2\theta}{2}\left(-g_i' + \frac{g_i}{r}\right) \\ &= \frac{1}{2r}(rg)' + \frac{re^{2\mathrm{i}\theta}}{2}\left(\frac{g}{r}\right)'.\end{aligned} \tag{5.25}$$

The change-of-frame term takes the following form:

$$\mathbf{U}_1.\nabla g e^{\mathrm{i}\theta} = \left(U_1\frac{\partial}{\partial x} + V_1\frac{\partial}{\partial y}\right)[g_r\cos\theta - g_i\sin\theta]$$

$$
\begin{aligned}
&= \frac{U_1}{2}\left(g_r' + \frac{g_r}{r}\right) + \frac{V_1}{2}\left(-g_i' - \frac{g_i}{r}\right) \\
&\quad + \frac{U_1}{2}\left[\cos 2\theta\left(g_r' - \frac{g_r}{r}\right) + \sin 2\theta\left(-g_i' + \frac{g_i}{r}\right)\right] \\
&\quad + \frac{V_1}{2}\left[\cos 2\theta\left(g_i' - \frac{g_i}{r}\right) + \sin 2\theta\left(g_r' - \frac{g_r}{r}\right)\right] \\
&= \frac{W_1}{2r}(rg)' + \frac{r\mathrm{e}^{2\mathrm{i}\theta}W_1^*}{2}\left(\frac{g}{r}\right)', \qquad (5.26)
\end{aligned}
$$

where W_1 is the complex representation of the velocity,

$$W_1 = U_1 + \mathrm{i}V_1. \qquad (5.27)$$

Finally, the Jacobian term takes the more complicated form

$$
\begin{aligned}
\mathcal{J}\left(f\mathrm{e}^{\mathrm{i}\theta}, g\mathrm{e}^{\mathrm{i}\theta}\right) &= \mathcal{J}(f_r\cos\theta - f_i\sin\theta, g_r\cos\theta - g_i\sin\theta) \\
&= \frac{1}{2r}\left[-f_r'g_i + f_ig_r' + f_i'g_r - f_rg_i'\right] \\
&\quad + \frac{\cos 2\theta}{2r}\left[-f_r'g_i + f_ig_r' - f_i'g_r + f_rg_i'\right] \\
&\quad + \frac{\sin 2\theta}{2r}\left[-f_rg_r' + f_i'g_i + f_rg_r' - f_ig_i'\right] \\
&= -\frac{\mathrm{i}}{2r}\left[fg^*\right]' + \frac{\mathrm{i}\mathrm{e}^{2\mathrm{i}\theta}}{2r}\left(f'g - g'f\right). \qquad (5.28)
\end{aligned}
$$

The real parts of (5.25), (5.26) and (5.28) are to be understood on the left-hand side of the equalities, and in the last right-hand side expressions. Primes denote differentiation with respect to r.

5·1·4·Complex representation of differential operators

The non-exponentially-decaying linearised Rayleigh operator is

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)\nabla^2\psi = \mathcal{L}\psi. \qquad (5.29)$$

It may be decomposed into its respective mode-zero,-one and -two components as

$$\mathcal{L}\psi = \mathcal{L}_0\psi^0 + \mathcal{L}_1\psi^1\mathrm{e}^{\mathrm{i}\theta} + \mathcal{L}_2\psi^2\mathrm{e}^{\mathrm{i}\theta}, \qquad (5.30)$$

where the ψ^i are the respective mode-zero, -one and -two components of the function ψ.

5·2·The second-order near-field solution

The second-order equation is

$$\left(\frac{\partial}{\partial t}+\Omega\frac{\partial}{\partial\theta}\right)\nabla^2\psi_2-\frac{Q'}{r}\frac{\partial\psi_2}{\partial\theta}-\mathbf{U}_2.\nabla\left(\nabla^2\psi_0\right)$$
$$=\quad -\frac{\partial\psi_1}{\partial x}+\mathbf{U}_1.\nabla\left(\nabla^2\psi_1\right)-\mathcal{J}\left(\psi_1,\nabla^2\psi_1\right). \tag{5.31}$$

This is a linear equation, so the solution may be decomposed into the parts forced by the various terms on the right, as well as complementary functions (i.e. homogeneous solutions). For the purposes of the matching procedure, terms up to $O(\epsilon^2)$ in the Van Dyke expansion of ψ need to be retained; this corresponds to $O(1)$ terms in ψ_2, and terms up to $O(r^{-2})$ in the forcing terms.

The $\mathcal{Z}_2^* r\Omega$ term corresponding to the second-order change of frame does not enter the matching process at second order (being of order ϵ^3/R in the far-field variable), just as the $\mathcal{Z}_1^* r\Omega$ term did not appear in the first-order matching. Hence it will be ignored in the rest of this section.

5·2·1·The $O(\epsilon^2)$ solution

The beta-effect

This corresponds to the subproblem

$$\mathcal{L}\psi_2^\beta=-\frac{\partial\psi_1}{\partial x}. \tag{5.32}$$

Mode zero The mode-zero equation is given by

$$\mathcal{L}_0\psi_2^{\beta 0}=-\frac{1}{2r}\left[rF+\mathcal{Z}_1^* r^2\Omega\right]'. \tag{5.33}$$

The right-hand side becomes

$$-a\left(\ln r+\frac{1}{2}\right)-b-\frac{d}{2r^2}+O\left(\frac{1}{r^4}\right) \tag{5.34}$$

for large r. However, d is pure imaginary, and can hence be ignored completely in the solution of the forced equation (since only real parts are physically relevant and mode zero has no $\mathrm{e}^{\mathrm{i}n\theta}$ term multiplying it). The solution is then of the form $\psi_2^{\beta 0}=Ar^2\ln r+Br^2+O(r^{-2})$. The left-hand side of (5.33) then becomes

$$4\frac{\mathrm{d}A}{\mathrm{d}t}\ln r+4\frac{\mathrm{d}A}{\mathrm{d}t}+4\frac{\mathrm{d}B}{\mathrm{d}t}+O\left(\frac{1}{r^4}\right). \tag{5.35}$$

Writing down the two equations that are satisfied up to $O(r^{-3})$ gives

$$4\frac{\mathrm{d}A}{\mathrm{d}t} = -a, \tag{5.36}$$

$$4\frac{\mathrm{d}A}{\mathrm{d}t} + 4\frac{\mathrm{d}B}{\mathrm{d}t} = -\frac{a}{2} - b. \tag{5.37}$$

These equations may be solved to give

$$\psi_2^{\beta 0} = -r^2 \ln r \int_0^t \frac{a}{4}\,\mathrm{d}\tau + r^2 \int_0^t \left[\frac{a}{8} - \frac{b}{4}\right]\,\mathrm{d}\tau + O\left(\frac{1}{r^3}\right). \tag{5.38}$$

The mode-zero equation (5.33) could be solved exactly, but it is the asymptotic behaviour of the solution, given by (5.38), that is required.

Mode two The mode-two equation does not have a simple closed-form solution. Its asymptotic behaviour for large r is given by

$$\begin{aligned}\mathcal{L}_2\psi_2^{\beta 2} &= -\frac{r}{2}\left[a\ln r + b + \frac{d\ln r}{r^2} + \frac{e}{r^2} + O\left(\frac{1}{r^4}\right)\right]' \\ &= -\frac{a}{2} - \frac{d(1-2\ln r)}{2r^2} + \frac{e}{r^2} + O\left(\frac{1}{r^4}\right).\end{aligned} \tag{5.39}$$

This must have a solution of the form $Ar^2\ln r + B\ln r + C$, which leads to the following left-hand side of (5.31):

$$\left[\frac{\partial}{\partial t} + \frac{\mathrm{i}\Gamma}{\pi r^2}\right]\left[4A - 4\frac{\ln r}{r^2}B - \frac{4C}{r^2}\right] + O(r^{-\infty}). \tag{5.40}$$

The unknown functions A, B and C hence satisfy

$$4\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{a}{2}, \tag{5.41}$$

$$-4\frac{\mathrm{d}B}{\mathrm{d}t} = d, \tag{5.42}$$

$$-4\frac{\mathrm{d}C}{\mathrm{d}t} + \frac{4\mathrm{i}\Gamma}{\pi}A = -\frac{d}{2} + e. \tag{5.43}$$

The solution to this set of equations is

$$\begin{aligned}\psi_2^{\beta 2} &= -r^2\ln r\int_0^t \frac{a}{8}\,\mathrm{d}\tau - \ln r\int_0^t \frac{d}{4} - \frac{\mathrm{i}\Gamma}{8\pi}\int_0^t (t-\tau)a\,\mathrm{d}\tau \\ &\quad + \int_0^t \left[\frac{d}{8} - \frac{e}{4}\right]\,\mathrm{d}\tau + O\left(\frac{1}{r^2}\right).\end{aligned} \tag{5.44}$$

Change of frame

This corresponds to the solution of

$$\mathcal{L}\psi_2^f = \mathbf{U}_1.\,\nabla(\nabla^2\psi_1). \tag{5.45}$$

Mode zero The mode-zero forcing is given by

$$\frac{W_1}{2r}\left[r\left(\frac{2a}{r}+O\left(\frac{1}{r^3}\right)+\mathcal{Z}_1^*Q'\right)\right]'=O\left(\frac{1}{r^4}\right). \tag{5.46}$$

So to the order required, there is no response; that is

$$\psi_2^{f0}=O\left(\frac{1}{r^2}\right). \tag{5.47}$$

Mode two The mode-two forcing takes the form

$$\frac{-2W_1^*a}{r^2}+O\left(\frac{1}{r^4}\right), \tag{5.48}$$

and so the appropriate solution is

$$\psi_2^{f2}=\frac{1}{2}\int_0^t W_1^*a\,\mathrm{d}\tau+O\left(\frac{1}{r^2}\right). \tag{5.49}$$

THE JACOBIAN TERM

The last part of the forced response is

$$\mathcal{L}\psi_2^{\mathcal{J}}=-\mathcal{J}(\psi_1,\nabla^2\psi_1). \tag{5.50}$$

Mode zero The forcing term may be expressed for large r as

$$\begin{aligned}&\frac{\mathrm{i}}{2r}\left\{\left[ar\ln r+br+O\left(\frac{1}{r}\right)\right]\left[\frac{2a^*}{r}+O\left(\frac{1}{r^3}\right)\right]\right\}'\\ =\;&\frac{\mathrm{i}|a|^2}{r^2}+O\left(\frac{1}{r^4}\right).\end{aligned} \tag{5.51}$$

To leading order, this is pure imaginary, and can hence be ignored in the complex notation that is being used. Hence

$$\psi_2^{\mathcal{J}0}=O\left(\frac{1}{r^2}\right). \tag{5.52}$$

Mode two The mode-two forcing term is

$$\begin{aligned}&-\frac{\mathrm{i}}{2r}\left\{\left[a(\ln r+1)+b+O\left(\frac{1}{r^2}\right)\right]\left[\frac{2a}{r}+O\left(\frac{1}{r^3}\right)\right]\right.\\ &\left.-\left[ar\ln r+br+O\left(\frac{1}{r}\right)\right]\left[-\frac{2a}{r^2}+O\left(\frac{1}{r^4}\right)\right]\right\}\\ =\;&-\frac{\mathrm{i}}{2}\left[\frac{4a^2\ln r}{r^2}+\frac{2a^2}{r^2}+\frac{4ab}{r^2}+O\left(\frac{1}{r^4}\right)\right].\end{aligned} \tag{5.53}$$

The solution to this equation is

$$\psi_2^{\mathcal{J}2}=\mathrm{i}\int_0^t\left[\frac{a^2\ln r}{2}+\frac{a^2}{4}+\frac{ab}{2}\right]\mathrm{d}\tau+O\left(\frac{1}{r^2}\right) \tag{5.54}$$

in the far field.

THE HOMOGENEOUS SOLUTION

The homogeneous solutions to the left-hand side operator are the usual f_n, defined so as to be well-behaved near the origin. The only modes that will be required are modes zero, one and two. Hence the homogeneous solution may be written as

$$c_0 * f_0 + c_1 * f_1 + c_2 * f_2. \tag{5.55}$$

The convolution operators are necessary in the time variable. In Chapter 4, the homogeneous solution was only considered in the Laplace variable, and hence products appeared rather than convolutions.

5·2·2·The $O(\epsilon^2 \ln \epsilon)$ solution

At this order, the governing equation takes the form

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + \Omega \frac{\partial}{\partial \theta}\right) \nabla^2 \psi_2' - \frac{Q'}{r} \frac{\partial \psi_2'}{\partial \theta} - \mathbf{U}_2' . \nabla \left(\nabla^2 \psi_0\right) \\ = \quad & -\frac{\partial \psi_1'}{\partial x} + \mathbf{U}_1 . \nabla \left(\nabla^2 \psi_1'\right) + \mathbf{U}_1' . \nabla \left(\nabla^2 \psi_1\right) \\ & -\mathcal{J}\left(\psi_1, \nabla^2 \psi_1'\right) - \mathcal{J}\left(\psi_1', \nabla^2 \psi_1\right), \end{aligned} \tag{5.56}$$

which appears more complicated than (5.31). However, the function ψ_1' is simpler in form than ψ_1, which leads to rather different results.

THE BETA-EFFECT

This comes from solving

$$\mathcal{L} \psi_1^{\beta\prime} = -\frac{\partial \psi_1'}{\partial x}. \tag{5.57}$$

Mode zero The forcing term is now given by

$$-\frac{1}{2r}\left[-\frac{\Gamma r^2 t}{4\pi} - \frac{\mathrm{i}\Gamma t^2}{8\pi} r^2 \Omega + \mathcal{Z}_1^{*\prime} r^2 \Omega\right]'. \tag{5.58}$$

For large r, this behaves like

$$\frac{\Gamma t}{4\pi} + O(r^{-\infty}). \tag{5.59}$$

The far-field behaviour of the solution is hence very simple, and is given by

$$\psi_2^{\beta 0\prime} = \frac{\Gamma r^2 t^2}{32\pi} + O(r^{-\infty}). \tag{5.60}$$

Mode two The forcing term now takes the form

$$-\frac{\mathrm{i}\Gamma^2 t^2}{16\pi^2 r^2} + \frac{\Gamma \mathcal{Z}_1^{*\prime}}{2\pi r^2} + O(r^{-\infty}). \tag{5.61}$$

Solving for the mode-two operator gives

$$\psi_2^{\beta 2\prime} = \frac{\mathrm{i}\Gamma^2 t^3}{192\pi^2} - \frac{\Gamma}{8\pi}\int_0^t \mathcal{Z}_1^{*\prime}\,\mathrm{d}\tau + O\left(\frac{1}{r^\infty}\right). \tag{5.62}$$

Change of frame

There are two forcing terms from the change of frame, since this portion of the second-order solution comes from solving

$$\mathcal{L}\psi_2^{f\prime} = \mathbf{U}_1\nabla(\nabla^2\psi_1') + \mathbf{U}_1'.\,\nabla(\nabla^2\psi_1). \tag{5.63}$$

The first contains the Laplacian of ψ_1', which is exponentially small in the far field, and hence cannot contribute to the asymptotic expansion of this term for large r (or rather can only contribute an $O(r^{-\infty})$ amount). The second term, however, is exactly the same as in the previous section, with the substitution of $W_1^{*\prime}$ for W_1^*. Then (5.47) and (5.49) immediately lead to

$$\psi_2^{f0\prime} = O\left(\frac{1}{r^2}\right) \tag{5.64}$$

and

$$\psi_2^{f2\prime} = \frac{1}{2}\int_0^\infty W_1^{*\prime} a\,\mathrm{d}\tau + O\left(\frac{1}{r^2}\right). \tag{5.65}$$

The Jacobian term

The Jacobian forcing term comes from products of derivatives of the Laplacians mentioned above, and the appropriate equation is

$$\mathcal{L}\psi_2^{\mathcal{J}\prime} = -\mathcal{J}(\psi_1, \nabla^2\psi_1') - \mathcal{J}(\psi_1', \nabla^2\psi_1). \tag{5.66}$$

Again, the first term contains $\nabla^2\psi_1'$ and can only contribute an exponentially small amount (as well as homogeneous solutions). The second term needs to be calculated explicitly.

Mode zero The far-field governing equation is now given by

$$\mathcal{L}_0\psi_2^{\mathcal{J}0\prime} = \frac{\mathrm{i}}{2r}\left\{\left[-\frac{\Gamma r t}{4\pi} + O\left(\frac{1}{r}\right)\right]\left[\frac{2a}{r} + O\left(\frac{1}{r^3}\right)\right]\right\}' \tag{5.67}$$

$$= O\left(\frac{1}{r^4}\right). \tag{5.68}$$

This is at a higher order than is necessary for the matching, so the solution may be written

5·3·The second-order far-field solution

$$\psi_2^{\mathcal{J}0\prime} = O\left(\frac{1}{r^2}\right). \tag{5.69}$$

Mode two This is the more complex term. The equation to be solved becomes

$$\begin{aligned}\mathcal{L}_2\psi_2^{\mathcal{J}2\prime} &= -\frac{\mathrm{i}}{2r}\left\{\left[-\frac{\Gamma t}{4\pi} + O\left(\frac{1}{r^2}\right)\right]\left[\frac{2a}{r} + O\left(\frac{1}{r^3}\right)\right]\right. \\ &\quad \left. - \left[-\frac{\Gamma rt}{4\pi} + O\left(\frac{1}{r}\right)\right]\left[-\frac{2a}{r^2} + O\left(\frac{1}{r^4}\right)\right]\right\} \qquad (5.70) \\ &= \frac{\mathrm{i}\Gamma at}{2\pi r^2} + O\left(\frac{1}{r^4}\right). \qquad (5.71)\end{aligned}$$

Integrating this, and taking homogeneous solutions into account, gives

$$\psi_2^{\mathcal{J}2\prime} = -\frac{\mathrm{i}\Gamma}{8\pi}\int_0^t a\tau\,\mathrm{d}\tau + O\left(\frac{1}{r^2}\right). \tag{5.72}$$

THE HOMOGENEOUS SOLUTION

The homogeneous solutions to the left-hand side operator are the same as in the previous section, so the form of homogeneous solution to take is

$$c_0' * f_0 + c_1' * f_1 + c_2' * f_2. \tag{5.73}$$

5·2·3·The $O\left(\epsilon^2 \ln^2\epsilon\right)$ solution

The governing equation at this order is forced entirely by the Jacobian term due to the self-interaction of the $O(\epsilon \ln \epsilon)$ terms. It is thus

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)\nabla^2\psi_2'' - \frac{Q'}{r}\frac{\partial\psi_2''}{\partial\theta} - \mathbf{U}_2''.\nabla\left(\nabla^2\psi_0\right) = -\mathcal{J}\left(\psi_1', \nabla^2\psi_1'\right). \tag{5.74}$$

All forcing terms are exponentially small in the far field, and hence the response at this order is entirely homogeneous. In the far field, it is thus given by

$$\psi_2'' = c_0'' * f_0 + c_1'' * f_1 + c_2'' * f_2. \tag{5.75}$$

5·3·The second-order far-field solution

The far-field equation at second order is, from (4.85),

$$\frac{\partial}{\partial t}\nabla^2\phi_2 + \frac{\partial\phi_2}{\partial X} = \mathbf{U}_1.\nabla\left(\nabla^2\phi_0\right) - \mathcal{J}(\phi_0, \nabla^2\phi_0), \tag{5.76}$$

where the Laplacian operator is in the variable R. This is again a linear equation, and the Green's function associated with the operator on the left-hand side is known: it is L. Hence the solution may be formally written as

$$\phi_2 = L * * \left[\mathbf{U}_1 . \nabla \left(\nabla^2 \phi_0\right) - \mathcal{J}(\phi_0, \nabla^2 \phi_0)\right], \tag{5.77}$$

where $**$ denotes convolution in time and space. However, this expression is hard to work with. For the purpose of the matching, it is sufficient to have expressions for the right-hand side and the solution for small R. Since the equation is linear, the response may be decomposed into two forced contributions, and the homogeneous response, which comprises derivatives of L.

5·3·1·The $O(\epsilon^2)$ solution

Change of frame

This is the solution to

$$\frac{\partial}{\partial t}\nabla^2 \phi_2^f + \frac{\partial \phi_2^f}{\partial X} = \mathbf{U}_1 . \nabla(\nabla^2 \phi_0). \tag{5.78}$$

The forcing term may be expanded by (5.25) to give

$$\left[U_1 \frac{\partial}{\partial X} + V_1 \frac{\partial}{\partial Y}\right] \nabla^2 \phi_0 = \frac{\Gamma W_1^* t}{2\pi R^2} \mathrm{e}^{2\mathrm{i}\theta} + O\left(\frac{1}{R}\right). \tag{5.79}$$

The asymptotic solution to this is

$$\phi_2^f = -\frac{\Gamma \mathrm{e}^{2\mathrm{i}\theta}}{8\pi} \int_0^t \tau W_1^* \, \mathrm{d}\tau + O(R). \tag{5.80}$$

The Jacobian term

This is the solution to

$$\frac{\partial}{\partial t}\nabla^2 \phi_2^{\mathcal{J}} + \frac{\partial \phi_2^{\mathcal{J}}}{\partial X} = -\mathcal{J})\phi_0, \nabla^2 \phi_0). \tag{5.81}$$

The forcing produced by the nonlinear interaction of the zeroth-order far-field solution with itself is given by

$$\begin{aligned} -\mathcal{J}(\phi_0, \nabla^2 \phi_0) \quad = \quad & -\frac{\Gamma^2}{R}\left\{\frac{1}{2\pi R} - \frac{t\cos\theta}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E\right] + O(R)\right\} \\ & \times \left\{\frac{t\sin\theta}{2\pi R} - \frac{t^2 \sin 2\theta}{8\pi} + O(R)\right\} \\ & + \frac{\Gamma^2}{R}\left\{\frac{Rt\sin\theta}{4\pi}\left[\ln\frac{Rt}{4} + 2\gamma_E - 1\right] + O(R^2)\right\} \end{aligned}$$

$$\times \left\{ \frac{t\cos\theta}{2\pi R^2} + O\left(\frac{1}{R}\right) \right\} \tag{5.82}$$

$$= -\frac{\Gamma^2 t \sin\theta}{4\pi^2 R^3} + \frac{\Gamma^2 t^2 \sin 2\theta}{8\pi^2 R^2}\left[\ln\frac{Rt}{4} + 2\gamma_E\right] + O\left(\frac{1}{R}\right). \tag{5.83}$$

The solution to this equation will be of the form

$$\phi_2^J = A(t)\sin\theta\frac{\ln R}{R} + B(t)\sin 2\theta \ln R + C(t)\sin 2\theta + O(R). \tag{5.84}$$

Then the left-hand side of (5.81) becomes

$$\begin{aligned} &-2\frac{\mathrm{d}A}{\mathrm{d}t}\frac{\sin\theta}{R^3} + A\frac{\sin 2\theta}{2R^2}(1 - 2\ln R) \\ &-4\frac{\mathrm{d}B}{\mathrm{d}t}\sin 2\theta\frac{\ln R}{R^2} - 4\frac{\mathrm{d}C}{\mathrm{d}t}\frac{\sin 2\theta}{R^2} + O\left(\frac{1}{R}\right) \end{aligned} \tag{5.85}$$

Substituting and equating coefficients leads to

$$-2\frac{\mathrm{d}A}{\mathrm{d}t} = -\frac{\Gamma^2 t}{4\pi^2}, \tag{5.86}$$

$$-A - 4\frac{\mathrm{d}B}{\mathrm{d}t} = \frac{\Gamma^2 t^2}{8\pi^2}, \tag{5.87}$$

$$\frac{A}{2} - 4\frac{\mathrm{d}C}{\mathrm{d}t} = \frac{\Gamma^2 t^2}{8\pi^2}\left[\ln\frac{t}{4} + 2\gamma_E\right]. \tag{5.88}$$

The final solution is given by

$$\begin{aligned} \phi_2^J = &\frac{\Gamma^2 t^2}{16\pi^2}\sin\theta\frac{\ln R}{R} - \frac{\Gamma^2 t^3}{64\pi^2}\sin 2\theta \ln R \\ &-\frac{\Gamma^2 t^3}{96\pi^2}\sin 2\theta\left[\ln\frac{t}{4} + 2\gamma_E - \frac{7}{12}\right] + O(R). \end{aligned} \tag{5.89}$$

None of these terms will match onto homogeneous mode-one and mode-two solutions; they will correspond to forced responses.

The homogeneous solution

The homogeneous solution at this order will be of the usual form:

$$\phi_2^h = \sum C_{i..j} * \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{R}, t). \tag{5.90}$$

Only the first two Cs can be nonzero, by the argument following (5.24). The truncation of this solution in the near-field variable is needed to second order. It is given by

$$\psi_2^{(h,2)} = \frac{\epsilon^2 C_0}{2\pi} * \left[\ln\frac{\epsilon r t}{4} + 2\gamma_E\right]$$

$$+\epsilon^2 C_i * \left(\frac{x_i}{2\pi\epsilon r^2} - \frac{\delta_{i1} t}{4\pi} \left[\ln \frac{\epsilon r t}{4} + 2\gamma_E - 1 \right] - \frac{x x_i t}{4\pi r^2} \right) \tag{5.91}$$

$$= \frac{\epsilon^2 C_0}{2\pi} * \left[\ln \frac{\epsilon r t}{4} + 2\gamma_E \right] - \frac{\epsilon^2 C_x}{4\pi} * t \left[\ln \frac{\epsilon r t}{4} + 2\gamma_E - \frac{1}{2} \right]$$
$$+ \frac{\epsilon \mathrm{e}^{\mathrm{i}\theta} C_1^* * 1}{2\pi r} - \frac{\epsilon^2 \mathrm{e}^{2\mathrm{i}\theta} C_1^* * t}{8\pi}, \tag{5.92}$$

where C_1^* is the complex conjugate of the complex representation of C_i, $C_1 = C_x + \mathrm{i} C_y$.

5·3·2·The $O(\epsilon^2 \ln \epsilon)$ solution

Unlike the first-order problem, a logarithmic term is required in the far field at second order. It can easily be verified that adding in an $\epsilon \ln \epsilon$ term does not affect the previous matching, since its coefficient has to be zero. At second order, however, a logarithmic term might be necessary. The only forcing term at this order comes from the change of frame, and may be immediately deduced from the previous section. It takes the form

$$\phi_2^{f\prime} = -\frac{\Gamma \mathrm{e}^{2\mathrm{i}\theta}}{8\pi} \int_0^t \tau W_1^{*\prime} \, \mathrm{d}\tau + O(R). \tag{5.93}$$

There is also the usual homogeneous response

$$\phi_2' = \sum C_{i..j}' * \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{R}, t). \tag{5.94}$$

Again, only the first two C's may be nonzero. Expanding in the near-field variable gives the same expansion as (5.92), with C_0' replacing C_0 and C_1' replacing C_1.

5·3·3·The $O(\epsilon^2 \ln^2 \epsilon)$ solution

At this order, there are no forced terms, so the only response is the homogeneous solution

$$\phi_2'' = \sum C_{i..j}'' * \frac{\partial^n}{\partial X_i \ldots \partial X_j} L(\mathbf{r}, t). \tag{5.95}$$

For the same reasons as above, only the first two C''s may be nonzero. The same sort of expansion as (5.92) is again obtained.

5·4·Matching

5·4·1·Working

The matching procedure is quite complicated. To simplify the algebra, the various modes will be considered separately, which is possible in this problem. In each of the following sections, the subscripts 0, 1 and 2 should be understood in the expressions for the inner solution, for each mode in turn. The presence of convolutions in (5.55), (5.73), (5.75), (5.90), (5.94) and (5.95) leads to solutions expressed in terms of convolutions.

Mode one

The mode-one component of the inner function, expressed in terms of the far-field variable, becomes

$$\begin{aligned}\psi^{(2,.)} &= \psi^{(1,1)} + \epsilon^2 \frac{d \ln R/\epsilon}{R} + \frac{\epsilon^2 e}{R} + \epsilon R[c_1 + \ln\epsilon\, c_1' + \ln^2\epsilon\, c_1''] * 1 \\ &\quad + \frac{\epsilon^2 \ln\epsilon}{R}\left[-\frac{\mathrm{i}\Gamma^2 t^2}{16\pi^2} + \frac{\Gamma \mathcal{Z}_1^{*\prime}}{2\pi}\right] + O(\epsilon^3), \end{aligned} \tag{5.96}$$

omitting the explicit $\mathrm{e}^{\mathrm{i}\theta}$ factor, and where $\psi^{(1,1)}$ is in fact the mode-one component of that term. There are no terms from ψ_1. In terms of the inner variable, the far-field mode-one term is

$$\phi^{(2,.)} = \phi^{(1,1)} + \frac{\epsilon\Gamma^2 t^2}{16\pi^2}\sin\theta\frac{\ln\epsilon r}{r} + \frac{\epsilon \mathrm{e}^{\mathrm{i}\theta}(C_1^* + \ln\epsilon\, C_1^{*\prime}) * 1}{2\pi r} + O(\epsilon^3 r), \tag{5.97}$$

using (5.89), 5.92), (5.94) and (5.95). There is no contribution from $\phi^{(1,2)}$: all the terms come from ϕ_2. The above equation is in mixed notation, since the third term on the right-hand side is in fact a real part. It is immediately apparent that $c_1 = c_1' = c_1'' = 0$. The absence of any $\ln^2\epsilon$ terms in (5.96) immediately gives $C_1'' = 0$, and hence this term has not been included in the far-field expansion. The $\psi^{(1,1)}$ and $\phi^{(1,1)}$ terms are equal, and may be ignored from now on, if one considers only the (2,2) matching equation. Reinserting the trigonometric terms in the inner-region truncation leads to

$$\mathrm{Re}\left\{-\frac{\mathrm{i}\Gamma^2 t^2}{16\pi^2}\mathrm{e}^{\mathrm{i}\theta}\right\} = \frac{\Gamma^2 t^2}{16\pi^2}\sin\theta, \tag{5.98}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{C_1^*}{2\pi}, \tag{5.99}$$

$$-\frac{\mathrm{i}\Gamma^2 t}{8\pi^2} + \frac{\Gamma W_1^{*\prime}}{2\pi} = \frac{C_1^{*\prime}}{2\pi} - \frac{\mathrm{i}\Gamma^2 t}{8\pi^2}, \tag{5.100}$$

for the $(\epsilon \ln r)/r$, ϵ/r and $(\epsilon \ln \epsilon)/r$ terms respectively. The second and third equations have been differentiated in time to remove the convolutions. The first of these is an identity, while the second and third give the coefficients C_1 and C_1' multiplying the homogeneous ∇L terms. The solutions to these two equations are

$$C_1 = \frac{\mathrm{i}\Gamma^2 t}{4\pi}\left[\ln\frac{t}{4} + 2\gamma_E\right] + \pi H + \Gamma W_1, \tag{5.101}$$

and

$$C_1' = \Gamma W_1'. \tag{5.102}$$

The coefficient of the inner $\epsilon^2 \ln \epsilon$ homogeneous term is essentially the velocity of the origin (however the latter has been selected).

Mode zero

In terms of the far-field variable, the mode-zero component of the inner solution is

$$\begin{aligned}\psi^{(2,.)} &= \psi^{(1,1)} + \epsilon^2 c_0 + \epsilon^2 \ln\epsilon\, c_0' + \epsilon^2 \ln^2\epsilon\, c_0'' \\ &\quad + \int_0^t \left[-\frac{a}{4}R^2 \ln(R/\epsilon) + \left(\frac{a}{8} - \frac{b}{4}\right)R^2\right] \mathrm{d}\tau \\ &\quad + \ln\epsilon \frac{\Gamma t^2 R^2}{32\pi} + O(\epsilon^3),\end{aligned} \tag{5.103}$$

where $\psi^{(1,1)}$ is actually its mode-zero truncation. The outer-solution mode-zero component is given by

$$\begin{aligned}\phi^{(2,.)} &= \phi^{(1,1)} + \frac{\epsilon^2 \Gamma r^2 t^2}{32\pi}\left[\ln\frac{\epsilon r t}{4} + 2\gamma_E - 2\right] \\ &\quad + \frac{\epsilon^2 (C_0 + \ln\epsilon\, C_0')}{2\pi} * \left[\ln\frac{\epsilon r t}{4} + 2\gamma_E\right] \\ &\quad - \frac{\epsilon^2 (C_x + \ln\epsilon\, C_x')}{4\pi} * t\left[\ln\frac{\epsilon r t}{4} + 2\gamma_E - \frac{1}{2}\right] + O(\epsilon^3).\end{aligned} \tag{5.104}$$

There is no $O(\epsilon^2 \ln^2 \epsilon)$ term in (5.103), so $C_0'' = 0$. Ignoring the previously matched terms, and writing down the (2,2) matching gives a large number of equations. The ϵ^2, $\epsilon^2 \ln\epsilon$ and $\epsilon^2 \ln^2 \epsilon$ equations give information on c_0, c_0' and c_0'', which has no dynamical significance and will hence be ignored.

The $\epsilon^2 \ln r$ equation gives

$$0 = \frac{C_0 * 1}{2\pi} - \frac{C_x * t}{4\pi}, \tag{5.105}$$

which may be simplified to

$$C_0 = \frac{1}{2}\int_0^t C_x \,\mathrm{d}\tau = \frac{\pi H t}{2} + \frac{\Gamma X_1}{2}. \tag{5.106}$$

The $\epsilon^2 \ln\epsilon \ln r$ equation becomes, similarly,

$$C_0' = \frac{1}{2}\int_0^t C_x' \,\mathrm{d}\tau = \frac{\Gamma X_1'}{2}. \tag{5.107}$$

The remaining four equations (corresponding to $\epsilon^2 r^2 \ln r$, $\epsilon^2 r^2$, $\epsilon^2 \ln^2 r$ and $\epsilon^2 r^2 \ln\epsilon$ terms respectively) are completely separate and reduce to

$$-\frac{1}{4}\int_0^t a \,\mathrm{d}\tau = \frac{\Gamma t^2}{32\pi}, \tag{5.108}$$

$$\int_0^t \left[\frac{a}{8} - \frac{b}{4}\right] \mathrm{d}\tau = \frac{\Gamma t^2}{32\pi}\left[\ln\frac{t}{4} + 2\gamma_E - 2\right], \tag{5.109}$$

$$\frac{\Gamma t^2}{32\pi} = \frac{\Gamma t^2}{32\pi}. \tag{5.110}$$

These three equations are indeed satisfied.

Mode two

For this mode, the inner-solution expansion has the form

$$\begin{aligned}
\psi^{(2,.)} = {} & \psi^{(1,1)} + D(t) * (c_2 + \ln\epsilon\, c_2' + \ln^2\epsilon\, c_2'')R^2 \\
& -R^2 \ln(R/\epsilon)\int_0^t \frac{a}{8}\,\mathrm{d}\tau - \epsilon^2 \ln(R/\epsilon)\int_0^t \frac{d}{4}\,\mathrm{d}\tau \\
& -\frac{\mathrm{i}\epsilon^2\Gamma}{8\pi}\int_0^t (t-\tau)a\,\mathrm{d}\tau + \epsilon^2\int_0^t \left[\frac{d}{8} - \frac{e}{4}\right]\mathrm{d}\tau + \frac{\epsilon^2}{2}\int_0^t W_1^* a\,\mathrm{d}\tau \\
& +\mathrm{i}\epsilon^2\int_0^t \left[\frac{a^2}{2}\ln(R/\epsilon) + \frac{a^2}{4} + \frac{ab}{2}\right]\mathrm{d}\tau \\
& +\epsilon^2\ln\epsilon\frac{\mathrm{i}\Gamma^2 t^3}{192\pi^2} - \epsilon^2\ln\epsilon\frac{\Gamma}{8\pi}\int_0^t \mathcal{Z}_1^{*\prime}\,\mathrm{d}\tau + \frac{\epsilon^2\ln\epsilon}{2}\int_0^t W_1^{*\prime} a\,\mathrm{d}\tau \\
& -\epsilon^2\ln\epsilon\frac{\mathrm{i}\Gamma}{8\pi}\int_0^t a\tau\,\mathrm{d}\tau + O(\epsilon^4),
\end{aligned} \tag{5.111}$$

where $D(t)$ is the connection factor for the second-order homogeneous solutions between the origin and infinity (see Chapter 2). The far-field expansion, written in the inner variable, becomes

$$\begin{aligned}
\phi^{(2,.)} = {} & \phi^{(1,1)} + \frac{\Gamma\epsilon^2 r^2 \mathrm{e}^{2\mathrm{i}\theta} t^2}{128\pi}\left[2\ln\frac{\epsilon r t}{4} + 4\gamma_E - 3\right] \\
& -\frac{\epsilon^2\Gamma\mathrm{e}^{2\mathrm{i}\theta}}{8\pi}\int_0^t \tau W_1^*\,\mathrm{d}\tau - \frac{\epsilon^2\ln\epsilon\,\Gamma\mathrm{e}^{2\mathrm{i}\theta}}{8\pi}\int_0^t \tau W_1^{*\prime}\,\mathrm{d}\tau \\
& -\frac{\epsilon^2\Gamma^2 t^3}{64\pi^2}\sin 2\theta \ln\epsilon r - \frac{\epsilon^2\Gamma^2 t^3}{96\pi^2}\sin 2\theta\left[\ln\frac{t}{4} + 2\gamma_E - \frac{7}{12}\right]
\end{aligned}$$

$$-\frac{\epsilon^2 e^{2i\theta}(C_1^* + \ln\epsilon\, C_1^{*\prime}) * t}{8\pi} + O(\epsilon^3) \tag{5.112}$$

in mixed notation (real parts are required for the complex exponentials). Again, solving the $\epsilon^2 r^2$, $\epsilon^2 r^2 \ln\epsilon$ and $\epsilon^2 r^2 \ln^2\epsilon$ equations leads to

$$c_2 = \frac{\Gamma}{128\pi D(t)} * t^2 \left[2\ln\frac{t}{4} + 4\gamma_E - 3\right], \tag{5.113}$$

$$c_2' = \frac{\Gamma}{64\pi D(t)} * t^2, \tag{5.114}$$

$$c_2'' = 0. \tag{5.115}$$

The other equations resulting from the matching are

$$-\int_0^t \frac{a}{8}\,\mathrm{d}\tau = \frac{\Gamma t^2}{64\pi}, \tag{5.116}$$

$$-\int_0^t \frac{d}{4}\,\mathrm{d}\tau + \mathrm{i}\int_0^t \frac{a^2}{2}\,\mathrm{d}\tau = \frac{\mathrm{i}\Gamma^2 t^3}{64\pi^2}, \tag{5.117}$$

from $\epsilon^2 r^2 \ln r$ and $\epsilon^2 \ln r$ respectively. These two equations are identities. The last two equations are

$$\begin{aligned} &-\frac{\mathrm{i}\Gamma}{8\pi}\int_0^t (t-\tau)a\,\mathrm{d}\tau + \int_0^t \left[\frac{d}{8} - \frac{e}{4}\right]\mathrm{d}\tau \\ &+\frac{1}{2}\int_0^t W_1^* a\,\mathrm{d}\tau + \frac{\mathrm{i}}{2}\int_0^t \left[\frac{a^2}{2} + ab\right]\mathrm{d}\tau \\ =\ &\frac{\mathrm{i}\Gamma^2 t^3}{96\pi^2}\left[\ln\frac{t}{4} + 2\gamma_E - \frac{7}{12}\right] - \frac{C_1^* * t}{8\pi} - \frac{\Gamma}{8\pi}\int_0^t W_1^*\tau\,\mathrm{d}\tau \end{aligned} \tag{5.118}$$

from ϵ^2, and

$$\begin{aligned} &\frac{\mathrm{i}\Gamma^2 t^3}{192\pi^2} - \frac{\Gamma}{8\pi}\int_0^t \mathcal{Z}_1^{*\prime}\,\mathrm{d}\tau + \frac{1}{2}\int_0^t W_1^{*\prime} a\,\mathrm{d}\tau - \frac{\mathrm{i}\Gamma}{8\pi}\int_0^t a\tau\,\mathrm{d}\tau \\ =\ &\frac{\mathrm{i}\Gamma^2 t^3}{64\pi^2} - \frac{C_1^{*\prime} * t}{8\pi} - \frac{\Gamma}{8\pi}\int_0^t W_1^{*\prime}\tau\,\mathrm{d}\tau, \end{aligned} \tag{5.119}$$

from $\epsilon^2 \ln\epsilon$, which are rather more complicated. They are both identities though, and the matching is consistent.

5·4·2·Summary

The second-order matching has been carried out using an asymptotic expansion, corresponding to the far field of the inner solution, and to the near field of the outer solution. The quantities of interest resulting from the matching are the coefficients of the homogeneous solutions for both inner and outer solutions.

The inner solution, at second order, consists of

$$\psi_2 = \psi_2^\beta + \psi_2^f + \psi_2^{\mathcal{J}} + c_0 + \frac{\Gamma}{128\pi\Delta(t)} * t^2 \left[2\ln\frac{t}{4} + 4\gamma_E - 3\right] * f_2 \tag{5.120}$$

at $O(\epsilon^2)$,

$$\psi_2' = \psi_2^{\beta\prime} + \psi_2^{f\prime} + \psi_2^{\mathcal{J}\prime} + c_0' + \frac{\Gamma}{64\pi\Delta(t)} * t^2 * f_2 \tag{5.121}$$

at $O(\epsilon^2 \ln\epsilon)$, and

$$\psi_2'' = \psi_2^{\mathcal{J}\prime\prime} + c_0'' \tag{5.122}$$

at $O(\epsilon^2 \ln^2\epsilon)$.

The outer solution is

$$\begin{aligned}\phi_2 &= \frac{\pi H t + \Gamma X_1(t)}{2} * L \\ &\quad + \left(\pi H, -\frac{\Gamma^2 t}{4\pi}\left[\ln\frac{t}{4} + 2\gamma_E\right]\right)' * \nabla L + \Gamma \mathbf{U}_1 .* \nabla L\end{aligned} \tag{5.123}$$

at $O(\epsilon^2)$. The notation $.*$ corresponds to an inner product combined with a convolution. At $O(\epsilon^2 \ln\epsilon)$, the solution is

$$\phi_2' = \frac{\Gamma X_1'}{2} * L + \Gamma \mathbf{U}_1' .* \nabla L. \tag{5.124}$$

5·5·The trajectory at second order

All methods of calculating the second-order trajectory essentially reduce to solving the equation

$$\mathcal{Z}_2^* = -c_1 * G(t) \tag{5.125}$$

for some function of time $G(t)$ depending on the method. The function c_1 is the coefficient convolved with the homogeneous solution f_1. Here, however, this coefficient vanishes, as do its counterparts at logarithmic order. As a result, the trajectory correction is identically zero at second order.

5·6·The solution for zero circulation

The solution calculated in the previous chapter considered the case of nonzero circulation, for which the outer solution was an impulsive response $\Gamma L(\mathbf{r}, t)$. For zero circulation, there is no zeroth-order outer solution. However, the inner response still

exists. This is the situation considered by Reznik & Dewar (1994). It is clear from the results of this chapter, though, that in general there will be a nonzero response in the far field at second order. This results from the H term in ϕ_2. The outer solution is given by

$$\phi_2 = \frac{\pi H t}{2} * L + \pi H * \frac{\partial L}{\partial x}. \tag{5.126}$$

The inner solution, at second order, is more complicated. However, the first-order solution may easily be calculated, since all that is required is to replace Γ by zero in all the results of the previous chapter. Hence those results are also valid for zero circulation too.

This shows that the analysis of Reznik & Dewar (1994) is deficient in that it does not address the presence of a far field at higher order (as is indeed acknowledged in §8 of that work).

5·7·Nonuniformity of the asymptotic solution

5·7·1·Scaling behaviour

The question of the validity of the asymptotic expansion has already been raised previously. The expansion cannot *a priori* be expected to remain valid for all time. If it is indeed nonuniform in time, then understanding the regimes in which it holds is an important aspect of the problem.

Distinguished scalings revisited

Taking a completely general rescaling of space and time $r = \epsilon^{\beta} r^*$ and $t = \epsilon^{\alpha} t^*$ leads to the governing equation

$$\frac{\partial}{\partial t^*}\nabla_*^2\psi + \epsilon^{\alpha-2\beta}\mathcal{J}_*(\psi, \nabla_*^2\psi) + \epsilon^{1+\alpha+\beta}\frac{\partial\psi}{\partial x_*} = 0. \tag{5.127}$$

The terms may be denoted 1, 2 and 3. The magnitude of ψ has not been rescaled, but due to the nonlinearity of the equation, it is implicitly given by $\epsilon^{-\alpha+2\beta}$. The dynamical scalings for $\alpha = 0$, i.e. corresponding to $O(1)$ timescales, have already been calculated. These are motions with time variations of the same order as the basic state.

A general analysis of the above equation reveals seven different scaling regimes, not all of which are dynamically significant.

I $1 \sim 2$: this is essentially the regime of vorticity advection. The condition on the time and space scales is $\alpha = 2\beta, \alpha > -\frac{2}{3}$. This is a two-term dynamical balance, which hence results in an asymptotic expansion in $\epsilon^{1+3\beta}$. The basic-state spatial and temporal scales are part of this scaling, so it includes the inner region (and time implicitly) already investigated.

II $2 \sim 3$: this is also a two-term balance, but the time-derivative is now recessive. Thus it can only be a dynamically significant balance as a quasi-steady-state. It cannot arise from other initial conditions. In particular, since it corresponds to $\beta = -\frac{1}{3}$ and $\alpha < -\frac{2}{3}$, i.e. large time and space scales, it cannot be reached from the initial conditions. An asymptotic expansion in $\epsilon^{\frac{1}{3}}$ is however possible.

III $1 \sim 3$: this is the final two-term balance, with the tendency term and the beta term dominating. Then $1 + \alpha + \beta = 0$, and $\alpha < -\frac{2}{3}$. This scaling is always on larger scales than case I. It contains the outer scaling previously considered, which has $\alpha = 0$, i.e. has the same timescale as the basic state. The asymptotic expansion in $\epsilon^{3\alpha+2}$ is then possible. This underlines the previous result given by (4.77) that the first-order solution in the far-field vanishes, $\phi_1 = 0$, for the natural timescale $\alpha = 0$.

IV $1 \sim 2 \sim 3$: this is the only three-term balance. All the terms have the same magnitude, and no asymptotic expansion is possible. The initial conditions for such an evolution, however, depend on the previous behaviour of the system, as discussed below.

V $1 \gg 2, 3$: the tendency term is dominant in this part of the plane. This is not a dynamical balance (since it is not a balance at all). An expansion in ϵ corresponds to a Taylor series expansion of the solution in time. In essence, the solution is showing that the wrong timescale has been chosen.

VI $2 \gg 1, 3$: the Jacobian term dominates. This is not a balance.

VII $3 \gg 1, 2$: the beta term dominates. This is not a balance.

Figure 5.1 shows the different regions of the (α, β) plane. In certain ways, it is analogous to figure 1 in Crighton (1983), albeit in a very different context since what it represents is regions of validity of an expansion in different regions of a two-dimensional parameter space.

For all regimes with $\alpha < 0$, the question as to whether they can be reached starting from the order-zero initial conditions depends on the actual details of the

time evolution. There is no way of knowing whether the spatial and temporal scales of the solution will change without an actual investigation of the appropriate expressions. Some nonlinear physical systems, e.g. soliton equations, have solutions whose spatial and temporal scales do not change in their evolution, even though other dynamical balances might be possible. In such systems, however, non-soliton-like initial conditions will evolve towards soliton states which then persist (Drazin & Johnson 1989), and may be calculated via the inverse scattering transform. In this respect, there might also exist here such persistent states on the $(-\frac{2}{3}, -\frac{1}{3})$ scale.

The dynamical evolution of the system must be along the lines I and III, since these are the only possible dynamical scalings attainable starting from a timescale of order unity. It is not possible in advance to tell whether the timescale of the dynamics of the system does in fact evolve. However, the behaviour of the previously calculated solutions ψ_0, ψ_1, ϕ_0 and ϕ_1 along different rays of the (α, β) plane should give expressions valid for the subsequent evolution of the solution from its behaviour for $\alpha = 0$. Only the upper portion of the plane should be investigated, as it makes no sense to continue the inner and outer solutions onto and across the lines III and I respectively. This means that the two conditions $1 + \alpha + \beta > 0$ and $\alpha > 2\beta$ will be met in the subsequent analysis. From now on, the asterisks will be dropped.

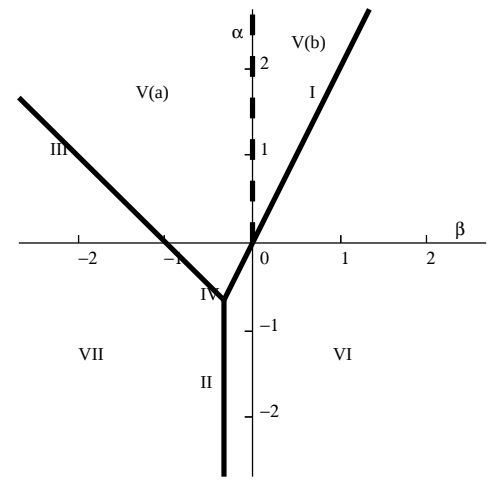

Figure 5.1: Different regimes of balance of the equation. The dashed line corresponds to the inner-region spatial scaling.

Scaling limits of the outer solution

The outer solution ϕ_0 is given by $\phi_0 = \Gamma L(\mathbf{R}, t)$. Using the integral representation of (3.12), with $c = \cos\theta/2$, gives

$$L = -\frac{1}{\pi}\int_0^\infty \frac{\mathcal{J}_0(2[\epsilon^{1+\alpha+\beta} rt(u^2 + c^2)]^{\frac{1}{2}})}{(u^2 + 1)^{\frac{1}{2}}}\,\mathrm{d}u. \tag{5.128}$$

So in regions I, V and part of VI, which are the regions of interest, the near-field expansion previously used may be rewritten as

$$\begin{aligned} L &= \frac{1}{2\pi}\left[\ln\frac{\epsilon^{1+\alpha+\beta}rt}{4}+2\gamma_E\right] \\ &\quad -\frac{\epsilon^{1+\alpha+\beta}xt}{4\pi}\left[\ln\frac{\epsilon^{1+\alpha+\beta}rt}{4}+2\gamma_E-1\right]+O(\epsilon^{2(1+\alpha+\beta)}). \end{aligned} \tag{5.129}$$

The outer solution along III keeps the same form, despite the different individual scalings of space and time. The appropriate combination of the r and t is of order one, and hence the solution cannot be expanded in ϵ. The first-order outer solution is identically zero, as are all terms of logarithmic order (until $\epsilon^2\ln\epsilon$).

Scaling limits of the inner solution

Order zero In the scaled variables, the order-zero solution, including logarithmic terms, is

$$\psi^{(0,)} = \Psi(\epsilon^\beta r)+\frac{\Gamma}{2\pi}\left[\ln\frac{\epsilon^\alpha t}{4}+2\gamma_E\right]+\frac{\Gamma}{2\pi}\ln\epsilon. \tag{5.130}$$

There are three different expansions for this expression, depending on the value of β. For reasons that will become apparent, only the $\beta<0$ case will be considered. Then the rescaled variable is a far-field variable, and the expansion becomes

$$\psi^{(0,)} = \frac{\Gamma}{2\pi}\left[(1+\alpha+\beta)\ln\epsilon+\ln\frac{rt}{4}+2\gamma_E\right]+O(\epsilon^\infty). \tag{5.131}$$

The fact that the form of this solution depends on the value of β, and not on one of the two special linear combinations of α and β giving a distinguished scaling, suggests that there will be another constraint on the scaling regimes that may be reached from both $O(\epsilon)$ solutions.

Order one It is convenient to work in the Laplace variable for the $O(\epsilon)$ response. It should be noted that the rescaling of time used in this section corresponds to changing a function $\overline{f(p)}$ of the Laplace variable to $\epsilon^{-\alpha}\overline{f(\epsilon^{-\alpha}p)}$.

The $O(\epsilon)$ solution limit The appropriate form of the first-order solution to use is

$$\begin{aligned} \overline{\psi_1} &= \epsilon^{2\alpha}\frac{r(\epsilon^{-\alpha}p+\mathrm{i}\Omega(\epsilon^\beta r))\mathrm{e}^{\mathrm{i}\theta}}{p^3}\left\{-\frac{\Psi(\epsilon^\beta r)}{2}+\frac{h(\epsilon^\beta r)}{2\epsilon^{2\beta}r^2}\right. \\ &\quad +\frac{\Gamma}{4\pi}\left[\ln\epsilon^{-\alpha}p-\ln\frac{1}{4}-\gamma_E-\frac{1}{2}\right] \\ &\quad \left.+\int_{\epsilon^\beta r}^\infty\frac{h(v)\Omega(v)}{v^3}\frac{\Omega(v)-2\mathrm{i}\epsilon^{-\alpha}p}{(\epsilon^{-\alpha}p+\mathrm{i}\Omega(v))^2}\,\mathrm{d}v\right\}. \end{aligned} \tag{5.132}$$

The $\epsilon^{-\alpha}$ term in the second fraction dominates the Ω term over the portion of the plane $\alpha > 2\beta$. Therefore the second fraction may be expanded, and the resulting integral is

$$I = \int_{\epsilon^{\beta}r}^{\infty} \frac{h(v)\Omega(v)}{v^3}\left[-\frac{2\mathrm{i}\epsilon^{\alpha}}{p} + O(\epsilon^{2\alpha}\Omega(v))\right] \mathrm{d}v. \tag{5.133}$$

There are then two cases, depending on the sign of β. For $\beta < 0$, the result is

$$I = O(\epsilon^{\alpha-2\beta}), \tag{5.134}$$

while for $\beta \geq 0$, the result is

$$I = O(\epsilon^{\alpha}). \tag{5.135}$$

These results may be combined with appropriate expansions for the rest of (5.132), which leads to

$$\overline{\psi_1} = \frac{\epsilon^{\alpha}\Gamma r}{4\pi p^2}\mathrm{e}^{\mathrm{i}\theta}\left[\ln \epsilon^{-\alpha} r - \ln\frac{1}{4} - \gamma_E +\right] + O(\epsilon^{2\alpha-2\beta}). \tag{5.136}$$

for $\beta < 0$. For $\beta \geq 0$, however, it is clear that functions of space will be present, albeit in Taylor-series form in the case of $\beta > 0$. It will be impossible to match these functions to anything in (5.129). Therefore, the region of overlap of the two expansions must also satisfy the constraint $\beta < 0$.

The $O(\epsilon \ln \epsilon)$ solution limit The solution at this order is

$$\psi_1' = -\epsilon^{\alpha+\beta}\ln\epsilon\frac{\Gamma r\mathrm{e}^{\mathrm{i}\theta}}{4\pi}\left[t + \mathrm{i}\Omega(\epsilon^{\beta}r)\epsilon^{\alpha}\frac{t^2}{2}\right]. \tag{5.137}$$

There are hence three regions of validity, depending on the sign of β. Only the negative β expansion is needed. This is

$$\psi_1' = -\epsilon^{\alpha+\beta}\frac{\Gamma r\mathrm{e}^{\mathrm{i}\theta}}{4\pi}\left[t + \epsilon^{\alpha-2\beta}\frac{\mathrm{i}\Gamma t^2}{4\pi r^2} + O(\epsilon^{\infty})\right]. \tag{5.138}$$

The dominant term is t since $\alpha - 2\beta > 0$.

Combined limits

It is imperative that any limit reached along a ray in the (α, β) plane from the inner solution $(0, 0)$ be the same as that reached along any other ray from the outer solution $(0, 1)$ in the region between the two expansions. Only the contributions from the first two orders will be considered. Any breakdown of the solution will be manifest in these orders. The region of the plane in which the two expansions have to agree is V(a). Reaching the remainder of the plane from lines I and III entails crossing

over a critical scaling, and is hence not relevant to the validity of the expansion. In particular, the whole purpose of matched asymptotic expansions is to relate two solutions in different regions of space (of the (α, β) plane here) to each other by using a region where they are both valid and 'overlap', as has been done on the β-axis. Hence there is no sense in investigating the behaviour of an asymptotic solution past the region of validity of the other solution.

In the whole of V, the outer solution is given by

$$\begin{aligned}\phi &= \frac{\Gamma}{2\pi}\left[\ln\frac{\epsilon^{1+\alpha+\beta}rt}{4}+2\gamma_E\right]-\frac{\epsilon^{1+\alpha+\beta}\Gamma xt}{4\pi}\left[\ln\frac{\epsilon^{1+\alpha+\beta}rt}{4}+2\gamma_E-1\right]\\ &\quad +O(\epsilon^{2(1+\alpha+\beta)}).\end{aligned} \tag{5.139}$$

In V(a), the order relations useful to the categorisation of the behaviour of ψ are $\beta < 0$ and $\alpha > 2\beta$. Then the inner solution is

$$\begin{aligned}\psi &= \frac{\Gamma}{2\pi}\left[(1+\alpha+\beta)\ln\epsilon+\ln\frac{rt}{4}+2\gamma_E\right]+O(\epsilon^{\infty})\\ &\quad +\epsilon^{1+\alpha}\frac{\Gamma t}{4\pi}re^{i\theta}\left[-(\alpha+\beta)\ln\frac{\epsilon rt}{4}-2\gamma_E+1\right]\\ &\quad -\epsilon^{1+\alpha+\beta}\ln\epsilon\frac{\Gamma t}{4\pi}re^{i\theta}t+O(\epsilon^{1+\alpha-2\beta}).\end{aligned} \tag{5.140}$$

The matching clearly works in this region, as can be seen by identifying terms in the above equation.

5·7·2·Approach to the breakdown scaling

The critical point IV corresponds to the dynamical scalings of the inner and outer regions becoming identical. It is therefore a universal scaling regime, where a single evolution governs the ultimate fate of all disturbances. However, it would appear that the inner and outer solutions tend to different limits as they approach this point.

This is because the asymptotic scalings of the inner and outer solutions become disordered. Their respective expansion parameters are $\epsilon^{1+3\beta}$ and $\epsilon^{3\alpha+2}$, both of which become order-zero quantities in region IV. Hence it is not possible to construct a universal initial condition to start off the evolution on this scale, since the asymptotic expansions have broken down. The only possible approach would be to resum the expansions, but this is obviously impossible. Hence there is no way of studying the universal behaviour of the original governing equation without solving it numerically.

5·8·Conclusions

The second-order solution has been calculated in this chapter. The far-field expansion is found exactly, while the inner-field expansion is given as an expansion for large r. The second-order contribution to the motion of the vortex is found to be *identically zero*. However, there is a non-zero response in the far field at this order, unless the RAM of the vortex vanishes.

The order of breakdown of the solution is examined, and it is found that the expansion becomes invalid on timescales of order $t = O(\epsilon^{-\frac{2}{3}})$ and on lengthscales of order $t = O(\epsilon^{-\frac{1}{3}})$. Hence the statements often made about the validity of such asymptotic expansions in ϵ, namely that they hold up to $t = O(\epsilon^{-1})$, are incorrect, at least for nonzero circulation. The inner and outer expansions are shown to hold in a region of validity of time and space given by the region V(a) of figure 5.1.

6·Divergent models

6·1·Nonzero Rossby radius dynamics

The nondivergent model investigated so far has the advantage of relative analytic simplicity. One of the major drawbacks of this model, however, is the unbounded growth of wave speed with wavelength. The divergent case does not suffer from this drawback. The dispersion relation for Rossby waves is then

$$\sigma = -\frac{\epsilon l}{l^2 + m^2 + \gamma^2}, \tag{6.1}$$

where γ is the inverse nondimensionalised Rossby radius of deformation. The phase velocity of waves asymptotes to $(-\epsilon/\gamma^2, -\epsilon l/m\gamma^2)$, which is finite for very large wavelengths.

The conserved quantity is now the quasigeostrophic potential vorticity

$$q = (\nabla^2 - \gamma^2)\psi + \epsilon y. \tag{6.2}$$

As a result, the governing equation becomes

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\frac{\partial \psi}{\partial x} = 0. \tag{6.3}$$

The only difference in the equation is the replacement of the Laplacian operator ∇^2 by the modified Helmholtz operator $(\nabla^2 - \gamma^2)$. This will lead to effects such as shielding and exponential decay and will greatly alter the dynamics.

Unlike the nondivergent case, there is no far field. The operator $(\nabla^2 - \gamma^2)$ leads to far-field decay of the form $\exp(-\gamma r)/r^{1/2}$, and so for large r the two terms in the operator scale the same way. There is hence no far-field distinguished scaling.

However, the governing equation (6.3) may still be solved as an asymptotic expansion in ϵ. The zeroth-order equation is

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi_0 + \mathcal{J}(\psi_0, \nabla^2\psi_0) = 0. \tag{6.4}$$

The initial radially symmetric streamfunction $\Psi(r)$ is a steady solution of this equation, as for the nondivergent case.

The first-order equation is now

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi_1 + \mathcal{J}(\psi_0, \nabla^2\psi_1) + \mathcal{J}(\psi_1, \nabla^2\psi_0) + \frac{\partial \psi_0}{\partial x} = 0. \tag{6.5}$$

This equation is no longer solvable by quadratures, but the exponential decay of solutions to the equations makes calculating a numerical solution quite straightforward. The trajectory of the vortex may hence be investigated as before.

6·2·The basic-state streamfunction

A localised, radially symmetric profile $Q(r)$ of potential vorticity $(\nabla^2 - \gamma^2)\Psi$ (without an ϵ term naturally) will be considered. Then the basic-state streamfunction $\Psi(r)$ satisfies

$$\Psi'' + \frac{\Psi'}{r} - \gamma^2 \Psi = Q \tag{6.6}$$

The solution to this equation that decays at infinity and is well-behaved at the origin is

$$\Psi = -I_0(\gamma r) \int_r^\infty K_0(\gamma u) Q(u) u \, \mathrm{d}u - K_0(\gamma r) \int_0^r I_0(\gamma r) Q(u) u \, \mathrm{d}u. \tag{6.7}$$

The far-field behaviour of the potential vorticity will be taken to be given by $Q \sim r^\eta \exp(-\delta r)$. The actual value of η is irrelevant. If the vorticity decays faster than exponentially, for example if Q has compact support, then δ may be considered to be infinite.

Under these assumptions, the far-field behaviour of the streamfunction depends on the relative magnitudes of γ and δ (both positive quantities). For a relatively slowly decaying vorticity profile, where $\delta < \gamma$, the streamfunction behaves like $r^\eta \exp(-\delta r)$. For a more rapidly decaying vorticity distribution, the behaviour of the streamfunction is given by $r^{-\frac{1}{2}} \exp(-\gamma r)$. In the special case $\gamma = \delta$, the result is $f(r) \exp(-\gamma r)$, where f is an algebraic or logarithmic function of r. These results may be put together to give

$$\Psi \sim f(r) \exp(-\delta_c r) \tag{6.8}$$

for large r, with

$$\delta_c = \min[\delta, \gamma] \tag{6.9}$$

and $f(r)$ as above.

The simplest example is the divergent Rankine vortex, where $Q(r) = H(d - r)$. The resulting streamfunction is

$$\Psi = \begin{cases} I_0(\gamma r)K_1(\gamma d)d/\gamma - 1/\gamma^2 & \text{for } r < d, \\ -K_0(\gamma r)I_1(\gamma d)d/\gamma & \text{for } r > d. \end{cases} \tag{6.10}$$

For more complicated potential vorticity distributions, the streamfunction must be found numerically.

The second-order ordinary differential equation (6.6) may be discretised using finite differences (see, e.g., Smith 1983) on a finite interval $(0, M)$. The streamfunction will be calculated at the points $r_n = nh$, where n ranges from 0 to m, with $M = mh$. Similarly $Q(r_n)$, which is known, is denoted by Q_n. Centred differences are used for the two differential operators. The second derivative at r_n is approximated by

$$\Psi_n'' = \frac{\Psi_{n+1} - 2\Psi_n + \Psi_{n-1}}{h^2} + O(h^2), \tag{6.11}$$

while the first derivative there is

$$\Psi_n' = \frac{\Psi_{n+1} - \Psi_{n-1}}{2h} + O(h^2). \tag{6.12}$$

The differential equation then becomes the difference equation

$$\left(\frac{1}{h^2} - \frac{1}{2hr_n}\right)\Psi_{n-1} - \left(\frac{2}{h^2} + \gamma^2\right)\Psi_n + \left(\frac{1}{h^2} + \frac{1}{2hr_n}\right)\Psi_{n+1} = Q_n \tag{6.13}$$

for $n = 1, \ldots, m - 1$. At the endpoints of the interval, other forms of the equations need to be found.

The differential operator in (6.6) is singular at the origin, but l'Hôpital's rule gives

$$2\Psi''(0) - \gamma^2\Psi(0) = Q(0) = Q_0 \tag{6.14}$$

there. Hence the appropriate discretisation to use at the origin is

$$2\frac{\Psi_1 - 2\Psi_0 + \Psi_{-1}}{h^2} - \gamma^2\Psi_0 = Q_0. \tag{6.15}$$

The value Ψ_{-1} is not a physical one, but the boundary condition $\Psi'(0) = 0$ may be discretised to $O(h^2)$ to give $\Psi_{-1} = \Psi_1$. The discretised form of the equation at the origin is thus

$$-\left(\frac{4}{h^2} + \gamma^2\right)\Psi_0 + \frac{4}{h^2}\Psi_1 = Q_0. \tag{6.16}$$

The main purpose in using a finite-difference numerical method, rather than a preexisting numerical package, is to enable the decay condition for large r to be treated simply. If the endpoint of the interval M is taken to be large enough (10 say), the behaviour of the streamfunction is given by

$$\Psi \sim f(M)\exp(-\delta_c M)[1 + O(M^{-1})]. \tag{6.17}$$

The proportionality constant is unknown, but may be eliminated from the difference equations by considering $\Psi'(M)$. The finite-difference approximation to this value is

$$\Psi'_m = \frac{\Psi_{m+1} - \Psi_{m-1}}{2h} + O(h^2). \tag{6.18}$$

But

$$\Psi'_m = -\delta_c f(M)\exp(-\delta_c M)[1 + O(M^{-1}] \tag{6.19}$$

$$= -\delta_c \Psi_m + O\left(f(M)M^{-1}\exp(-\delta_c M)\right). \tag{6.20}$$

Combining these two expressions gives

$$\Psi_{m+1} = \Psi_{m-1} - 2h\delta_c\Psi_m + O\left(hf(M)M^{-1}\exp(-\delta_c M), h^3\right). \tag{6.21}$$

The governing equation (6.13), which holds at the point M, may then have this value of Ψ_{m+1} substituted into it to give

$$\frac{2}{h^2}\Psi_{m-1} - \left[\frac{2}{h^2} + \gamma^2 + 2h\delta_c\left(\frac{1}{h^2} + \frac{1}{2hM}\right)\right]\Psi_m = Q_m. \tag{6.22}$$

The system of equations (6.13), (6.16) and (6.22) is a set of $m+1$ linear equations that may be solved to give a finite-difference approximation to Ψ on $(0, M)$. Figure 6.1 shows the numerically-calculated approximation to the Rankine vortex streamfunction on the interval (0, 10). The numerical result is close to (6.10), with the error being less than 1.5% over most of the range (certainly for $r < 8$) for $m = 1000$. This could be decreased, formally, by taking $M = O(h^{-1})$, but this leads to more expensive computations. In practice, no significant increase in accuracy was found.

The angular velocity Ω_n is obtained numerically to the same accuracy from the basic-state streamfunction by

$$\Omega_n = \frac{\Psi_{n+1} - \Psi_{n-1}}{2hr_n}, \tag{6.23}$$

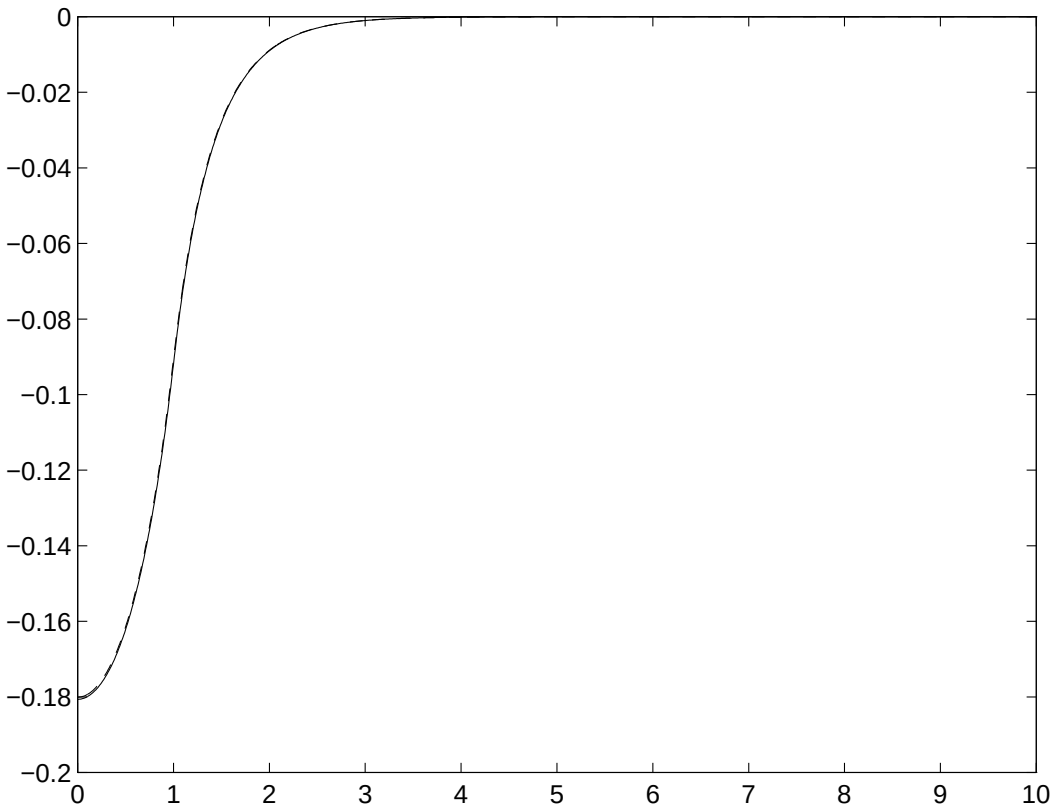

Figure 6.1: Exact (solid line) and numerical (dashed line) solutions to (6.6). In this calculation, $m = 1000$, and $M = 10$ giving a distance between grid points of $h = 0.01$. The radius of the vortex is $d = 1$, while $\gamma = 2$.

except for Ω_0 and Ω_m. The former is given by

$$\Omega_0 = 2\frac{\Psi_2 - \Psi_1}{h^2} \tag{6.24}$$

to $O(h^2)$, the latter by

$$\Omega_m = -\delta_c \Psi_m / M, \tag{6.25}$$

with an error of the same order as the discretisation.

The numerical discretisation outlined above is very simple but is sufficient to calculate the basic-state streamfunction to sufficient accuracy. While an off-the-shelf package might have provided more efficiency and accuracy, the simplicity of this approach will become particularly valuable when dealing with the first-order problem, where such systems of equations have to be solved repeatedly.

6·3·The first-order solution

The equation governing the evolution of ψ_1 may be obtained as in the nondivergent case. The governing equation for the Laplace-transformed, mode-one, part of ψ_1, denoted ϕ here, is

$$\phi'' + \frac{\phi'}{r} - \left[\frac{1}{r^2} + \frac{\mathrm{i}Q' + \gamma^2 pr}{r(p + \mathrm{i}\Omega)}\right]\phi = -\frac{\Omega r}{p(p + \mathrm{i}\Omega)}. \tag{6.26}$$

This equation may be discretised in a similar way to (6.5). The resulting difference equation is

$$\begin{aligned} &\left(\frac{1}{h^2} - \frac{1}{2hr_n}\right)\phi_{n-1} - \left(\frac{2}{h^2} + \frac{1}{r_n^2} + \frac{\mathrm{i}Q'_n + \gamma^2 pr_n}{r_n(p + \mathrm{i}\Omega_n)}\right)\phi_n \\ &+ \left(\frac{1}{h^2} + \frac{1}{2hr_n}\right)\phi_{n+1} \\ = \quad & -\frac{\Omega_n r_n}{p(p + \mathrm{i}\Omega_n)} \end{aligned} \tag{6.27}$$

for n in the range $1, \ldots, m$. The gradient of Q may just be calculated from $Q'_n = (Q_{n+1} - Q_{n-1})/2h + O(h^2)$. The boundary condition at the origin is very simple, since $\phi_0 = 0$. That at infinity is slightly different from its previous representation, although the basic idea is the same. The solutions of the homogeneous version of (6.26) behave like $\exp(-\gamma r)$, so the behaviour of the solutions to (6.26) for large r is given by

$$\phi \sim g(r)\exp(-\delta_c)r, \tag{6.28}$$

again, where g has some algebraic or logarithmic dependence. The equation at M then becomes

$$\begin{aligned} & \frac{2}{h^2}\phi_{m-1} - \left[\frac{2}{h^2} + \frac{1}{M^2} + \frac{\mathrm{i}Q_m + \gamma^2 M}{M(p+\mathrm{i}\Omega_m)} + 2h\delta_c\left(\frac{1}{h^2} + \frac{1}{2hM}\right)\right]\phi_m \\ = & -\frac{\Omega_m r_m}{p(p+\mathrm{i}\Omega_m)}. \end{aligned} \tag{6.29}$$

As before, this is a closed set of equations, and may be solved for any value of p. Talbot's algorithm (Talbot 1979) may be used to calculate the mode-one solution for any value of time.

6·4·The motion of the vortex

It is again instructive to work in a coordinate system centred in the vortex. The velocity of the new frame, (U, V), scales like ϵ, so the governing equation for the first-order perturbation is, analogously to Chapter 4,

$$\begin{aligned} & \frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi_1 + \mathcal{J}(\psi_0, \nabla^2\psi_1) + \mathcal{J}(\psi_1, \nabla^2\psi_0) \\ = & -\frac{\partial\psi_0}{\partial x} + \left(U_1\frac{\partial}{\partial x} + V_1\frac{\partial}{\partial y}\right)(\nabla^2 - \gamma^2)\psi_0, \end{aligned} \tag{6.30}$$

where $(U, V) = \epsilon(U_1, V_1) + \cdots$.

The function $\phi(r, p)$ calculated in section 6·3 is the Laplace transform of the mode-one component of the solution of (6.30) without the U_1 and V_1 terms. However, (6.30) is linear, and so as previously, its solution may be decomposed into two parts, each satisfying appropriate initial and boundary conditions.

Yet again, the function $\psi_1^f = r\Omega\mathrm{e}^{\mathrm{i}\theta}$ is a steady solution of (6.30) with only the change-of-frame terms on the right-hand side. The proof is the same as in Chapter 4. Physically, this is because ψ_1^f again corresponds to the first-order term in the expansion of a change-of-frame transformation.

The motion of the vortex may then be identified in the same ways as previously. It is again convenient to work in the Laplace variable. Then the total mode-one streamfunction is

$$\overline{\psi_1} = \phi\mathrm{e}^{\mathrm{i}\theta} + \overline{\zeta_1^*}r\Omega\mathrm{e}^{\mathrm{i}\theta}. \tag{6.31}$$

The condition for the centre to be the maximum in relative vorticity is

$$\frac{4}{3}\phi_0''' + 4\Omega_0''\overline{\zeta_1^*} = 0. \tag{6.32}$$

The value of the triple-derivative of ϕ_1 at the origin may be removed using the governing equation. The $O(r)$ term in the Rayleigh equation is equivalent to

$$\frac{4}{3}\phi_0''' - \frac{4\mathrm{i}\Omega_0'' + \gamma^2 p}{p + \mathrm{i}\Omega_0}\phi_0' = -\frac{\Omega_0}{p(p + \mathrm{i}\Omega_0}. \tag{6.33}$$

This leads to

$$\overline{\mathcal{Z}_1^{rv*}} = -\frac{\mathrm{i}\phi_0'}{p + \mathrm{i}\Omega_0} - \frac{\gamma^2 p\phi_0'}{4\Omega_0''(p + \mathrm{i}\Omega_0)} + \frac{\Omega_0}{4\Omega_0'' p(p + \mathrm{i}\Omega_0)}. \tag{6.34}$$

The last term in this equation is analogous to the oscillatory term in the equivalent nondivergent expression. The second term is a new, divergent, contribution, which vanishes for zero γ.

The condition for the maximum in streamfunction becomes

$$\phi_0' + \overline{\mathcal{Z}_1^*}\Omega_0 = 0, \tag{6.35}$$

or

$$\overline{\mathcal{Z}_1^{sf*}} = -\frac{\phi_0'}{\Omega_0}, \tag{6.36}$$

where ϕ_0' is the value of the derivative of ϕ at the origin. In this case, it is irrelevant whether the inverse Laplace transform is carried out on ϕ or on $\overline{\mathcal{Z}_1^*}$.

Finally, the condition for tracking the particle initially at the origin, and hence for the pseudo-secularity condition, becomes

$$\phi_0' + \overline{\mathcal{Z}_1^*}\Omega_0 = \mathrm{i}p\overline{\mathcal{Z}_1^*}, \tag{6.37}$$

or

$$\overline{\mathcal{Z}_1^{o*}} = -\frac{\mathrm{i}\phi_0'}{p + \mathrm{i}\Omega_0}. \tag{6.38}$$

In this case, the Laplace transform should be carried out on $\overline{\mathcal{Z}_1^*}$ to avoid an unnecessary integration in the time variable.

There is no longer a logarithmic term ψ_1', so the trajectory is given just by the $O(\epsilon)$ quantities (6.36) and (6.38). Hence the trajectories calculated hold for all values for ϵ, unlike the nondivergent case where the actual value of ϵ led to different trajectories. Of course, γ is now a physical parameter which may be varied in these calculations.

Figure 6.2 shows the trajectory of the Rankine vortex $Q = H(r - 1)$, again for $\gamma = 2$. Figure 6.3 shows the trajectory of the Gaussian vortex $Q = \exp(-r^2)$ for $\gamma = 2$. The path of the relative vorticity maximum has oscillations around the trajectory of the origin, which is to be expected. The actual curves plotted are splines

fitted through the crosses which are at every time unit – this is for computational efficiency. Higher accuracy in time is not in itself difficult. In both cases, the vortex starts off travelling to the Northwest, but then changes direction and moves to the East. The Rankine vortex even goes around the origin! For moderately small values of ϵ (say around 0.1 or so), the asymptotic expansion is probably breaking down at about the location where the vortex changes direction, which is not unreasonable. For smaller values of ϵ, however, the vortex seems to eventually travel to the East at times where the expansion is still valid. There are thus two possibilities, which cannot be resolved from the expansion alone. Either the physical process of vortex stretching permits travel to the East while the expansion is valid, or the expansion breaks down earlier in time, when the vortex starts moving East. The results of the next section suggest that the former is more probable. It is hard, however, to come up with a sketch like that of Chapter 1, to understand the dynamics of the system. For very small ϵ, γ is a large quantity, and the stretching is the dominant part of the dynamics. Hence it is not surprising that the previous arguments about vorticity anomaly do not hold anymore.

Figure 6.4 shows the trajectory of the Rankine vortex for $\gamma = 1$, while figure 6.6 shows it for $\gamma = 0.2$ The former is more zonal than that of figure 6.2, and the latter even more so, and more similar to the nondivergent trajectories. This is to be expected with smaller values of γ. The particle initially at the origin circles around the streamfunction maxima in a cycloidal manner, since it is caught inside a predominantly swirling flow and is hence being swept around the streamfunction maxima as it moves. However, it is probably the discontinuity in the basic-state potential vorticity that is responsible for the major part of the oscillations. Figure 6.5 shows the trajectory of the Gaussian vortex for $\gamma = 1$, and 6.7 for $\gamma = 0.2$. These two figures reinforce the previous conclusions. The trajectories are less oscillatory, however, which is presumably due to the smoothness of the initial profile.

It is necessary to multiply the trajectories in figures 6.2 and 6.3 by ϵ to obtain the physical displacement of the vortex. The fact that the predicted trajectories depend linearly on ϵ is a prediction that could be tested in numerical experiments. The exponential decay of the streamfunction also means that such experiments could be performed in finite-size domains without the conclusions being affected.

It is clear from comparing figures 6.2 and 6.3 to figures 4.1 and 4.3 that vortices will move much more slowly in the divergent case, and in particular more slowly the larger γ. This can be related to the change in the far-field response as evidenced by the analysis of the next section.

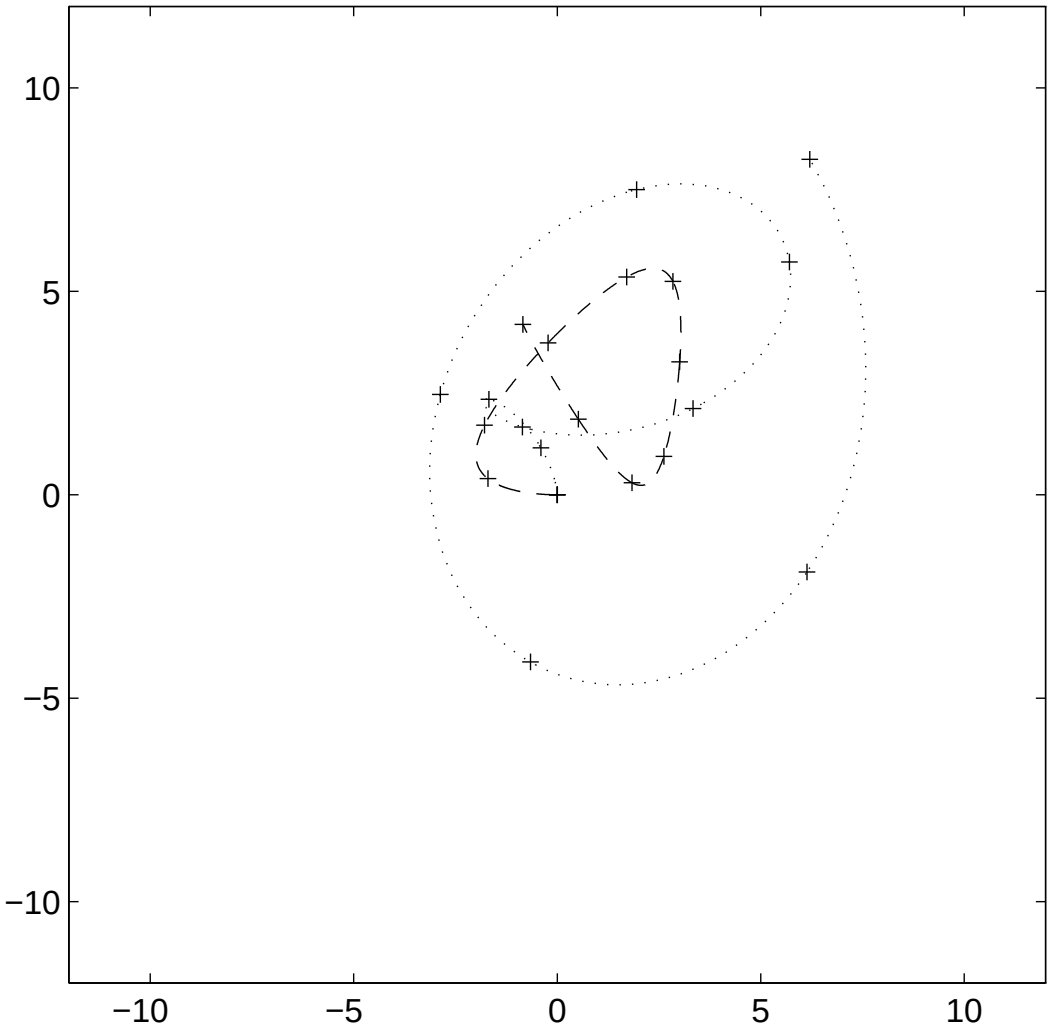

Figure 6.2: Trajectories for the Rankine vortex in the divergent case for $\gamma = 2$. The trajectory of the particle at the origin is dotted, while that of the streamfunction maximum is dashed. Crosses mark every 10 times units.

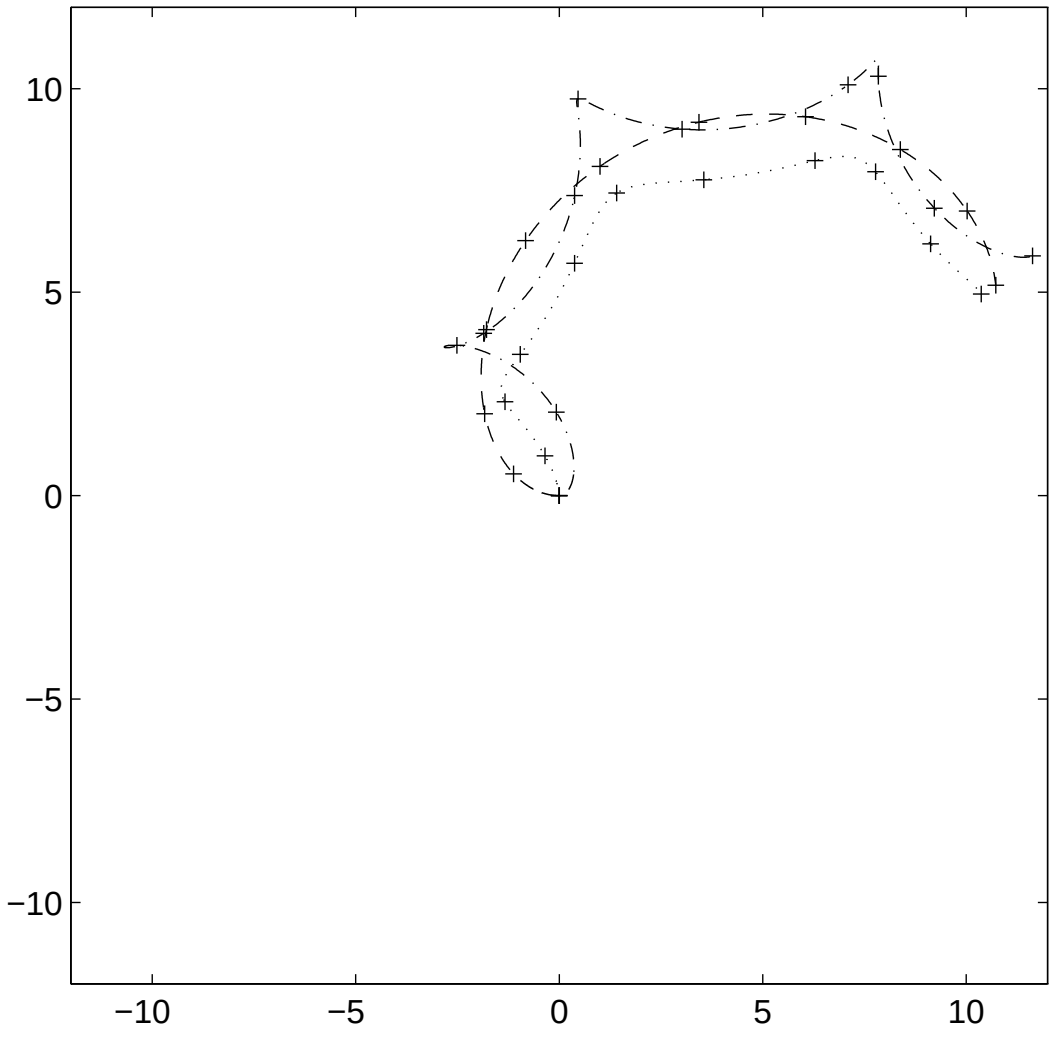

Figure 6.3: Trajectories for the Gaussian vortex in the divergent case for $\gamma = 2$. The trajectory of the particle at the origin is dotted, that of the streamfunction maximum is dashed and the path of the streamfunction is dash-dotted. Crosses mark every 10 times units.

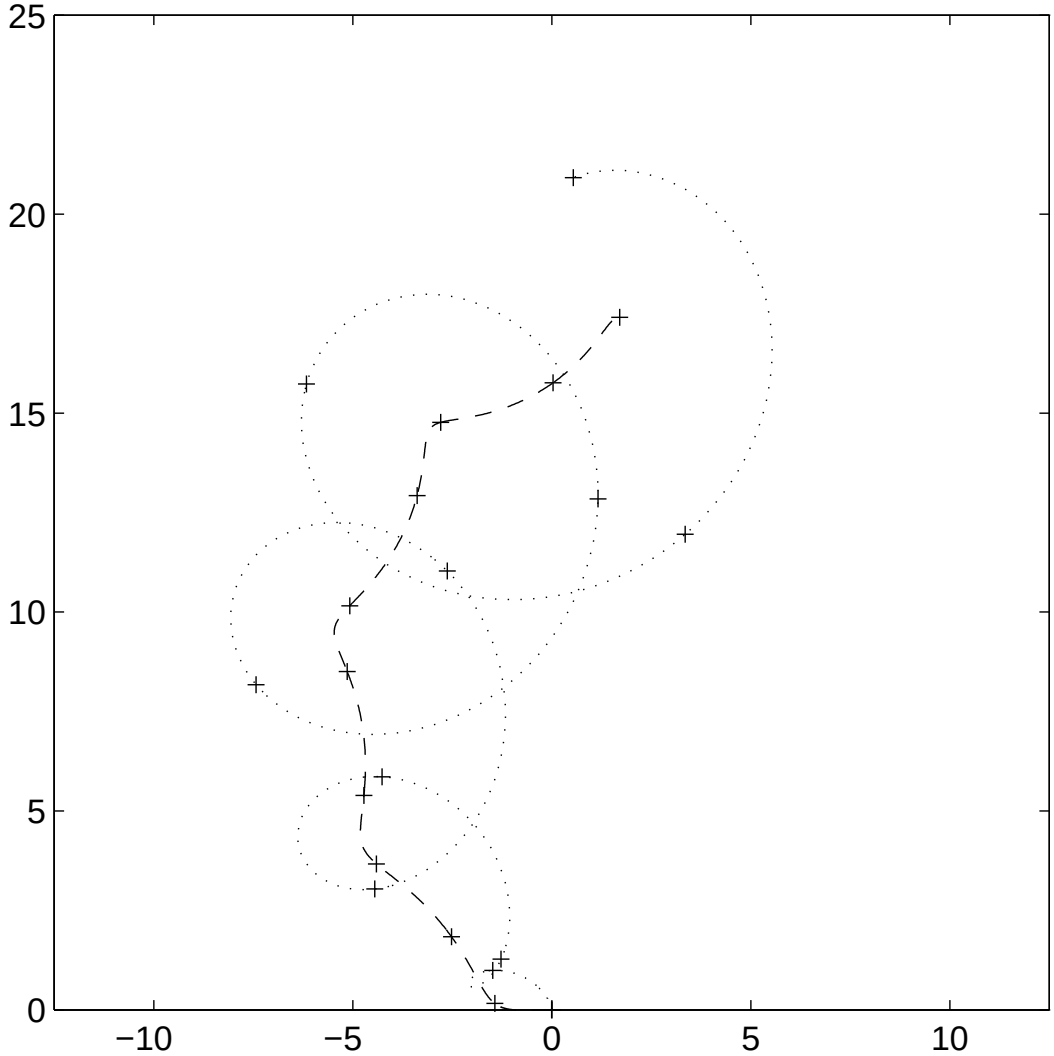

Figure 6.4: Trajectories for the Rankine vortex in the divergent case for $\gamma = 1$. The trajectory of the particle at the origin is dotted, while that of the streamfunction maximum is dashed. Crosses mark every 10 times units.

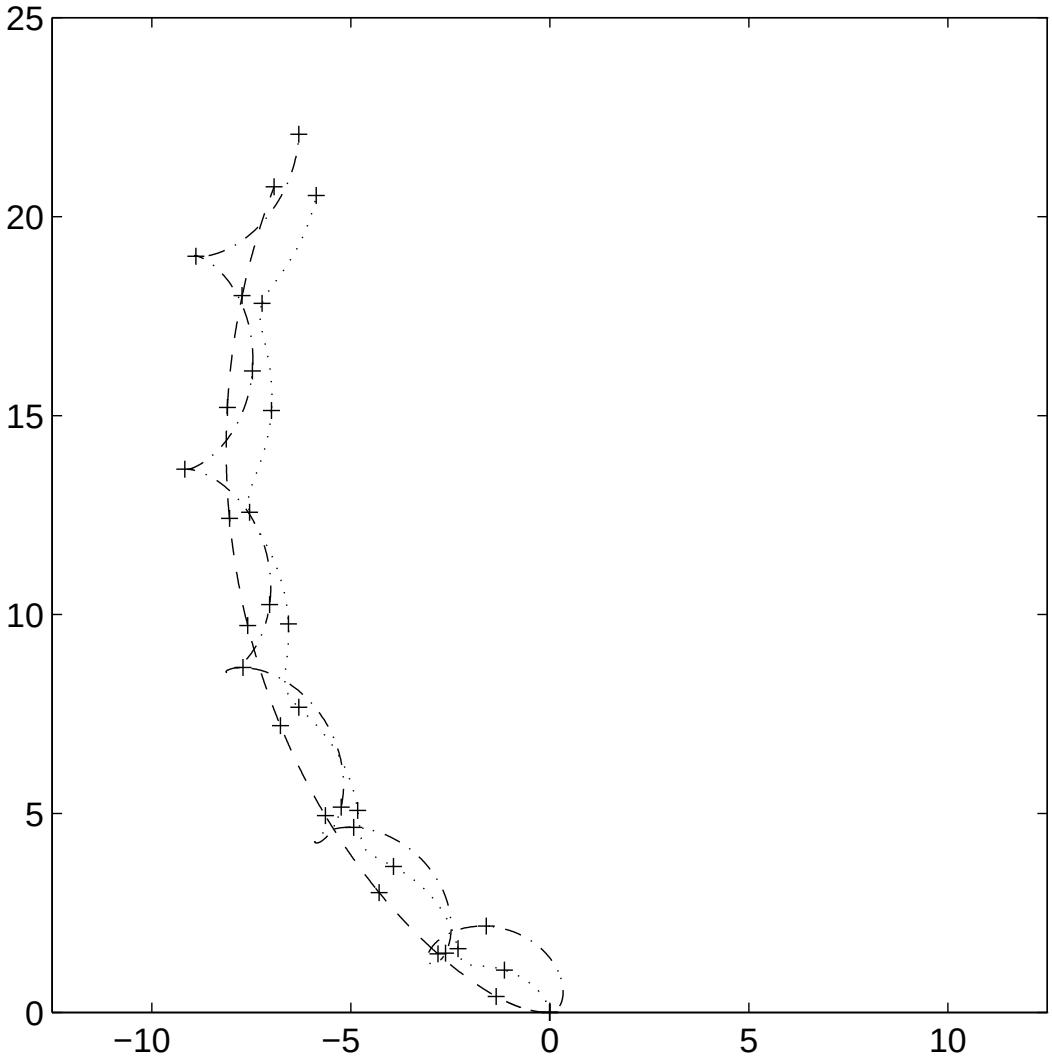

Figure 6.5: Trajectories for the Gaussian vortex in the divergent case for $\gamma = 1$. The trajectory of the particle at the origin is dotted, that of the streamfunction maximum is dashed and the path of the streamfunction is dash-dotted. Crosses mark every 10 times units.

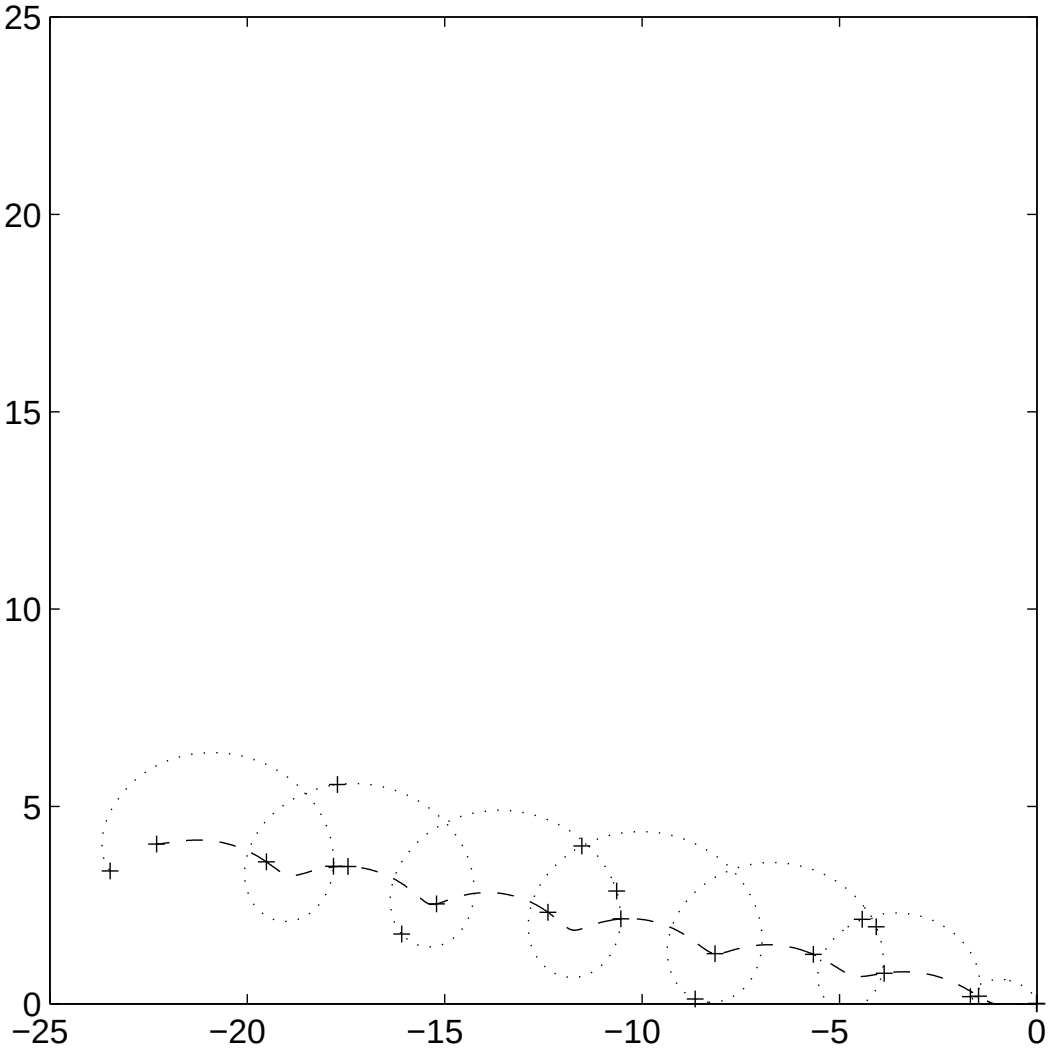

Figure 6.6: Trajectories for the Rankine vortex in the divergent case for $\gamma = 0.2$. The trajectory of the particle at the origin is dotted, while that of the streamfunction maximum is dashed. Crosses mark every 10 times units.

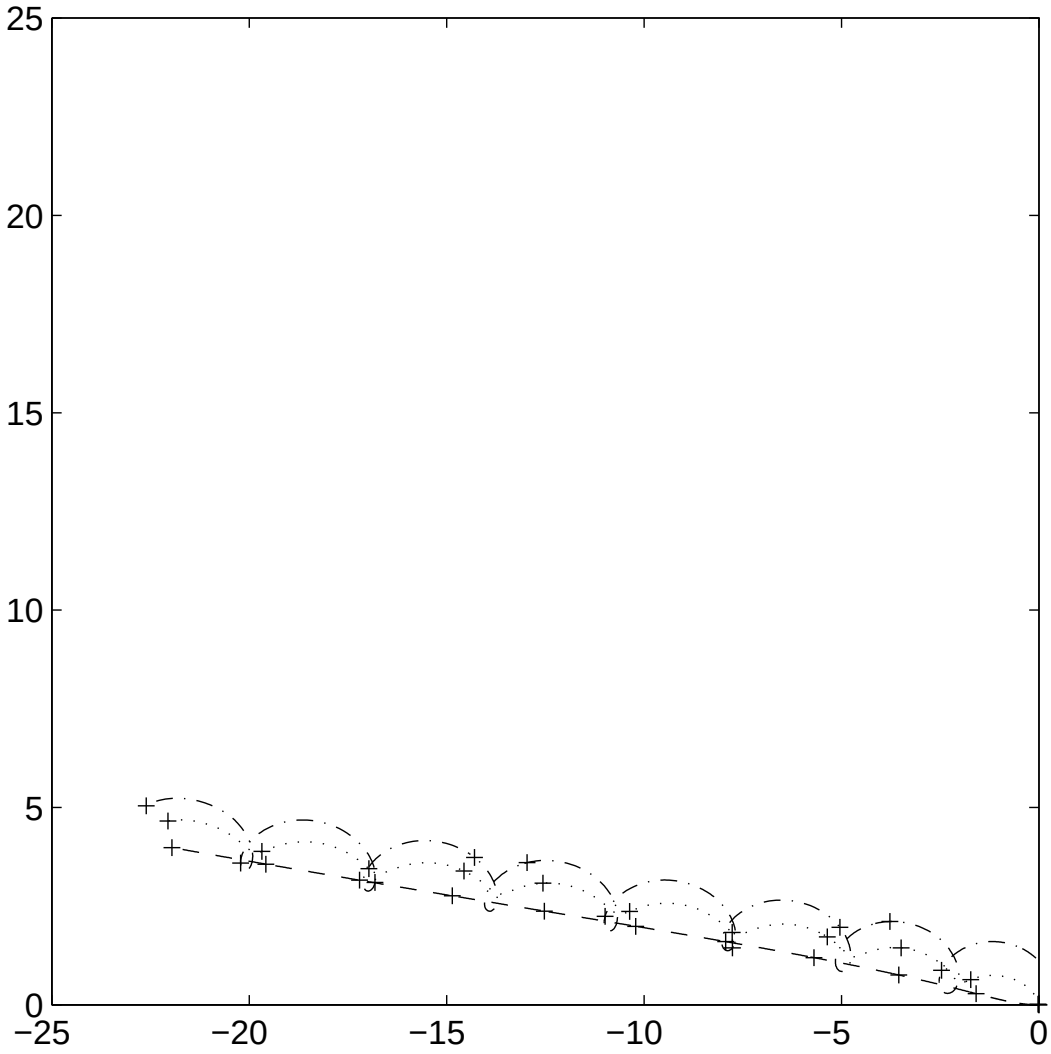

Figure 6.7: Trajectories for the Gaussian vortex in the divergent case for $\gamma = 0.2$. The trajectory of the particle at the origin is dotted, that of the streamfunction maximum is dashed and the path of the streamfunction is dash-dotted. Crosses mark every 10 times units.

6·5·Breakdown of the expansion

The scaling analysis of Chapter 5 cannot be immediately applied to the divergent case, essentially because there is no division between the near and far fields. The general rescaling of space and time of (5.127) is hence no longer appropriate, essentially because $\nabla^2\psi$ is of the same order as ψ for all regions of space.

The analysis needs to be done order by order. The basic state and the first-order solution both behave like $O(1)$ in the near field and $O(\mathrm{e}^{-\delta_c r})$ in the far field for fixed values of the Laplace variable p. However, for large time (i.e. for p small), the real part of $\gamma/p^{\frac{1}{2}}$ becomes very large, and so the behaviour in the far field is again $O(\mathrm{e}^{-\delta_c r})$. Hence the breakdown must occur over all space at the same time.

The growth in time of ψ_1 can be obtained from (6.26). For small p, this equation takes two forms, depending on the magnitude of r, and must hence be solved by matched asymptotic expansions. There are two expressions in the equation which contain functions of p and functions of r. The first is $\mathrm{i}Q' + \gamma^2 pr$: for $r = O(1)$, the first term is dominant, while for a scaling of $r = O(\epsilon^{-\beta})$ for *any* positive β, the second term dominates. The second expression is $\mathrm{i}\Omega + p$, which behaves in the same way. Hence for $r = O(1)$, the equation is

$$\phi'' + \frac{\phi'}{r} - \left[\frac{1}{r^2} + \frac{Q'}{r\Omega}\right]\phi = \frac{\mathrm{i}r}{p}. \tag{6.39}$$

For larger r, it becomes

$$\phi'' + \frac{\phi'}{r} - \left[\frac{1}{r^2} + \gamma^2\right]\phi = -\frac{\Omega r}{p^2}. \tag{6.40}$$

The second equation holds for all values of r such that $r^{-1}f(r)\exp(-\delta_c r) \gg p$, which will hold for almost all values of r as soon as p is relatively small. It shows that ϕ must behave like $1/p^2$ for large r (although r does not have to be at all large to satisfy the previous inequality). This is not a rigorous analysis, since the equations have not been solved, but is enough to show that ψ_1 grows linearly with time. This approach was not possible in Chapter 5, since both the far and near field need to be considered, but then rescaling time and space leads to the breakdown time and space scales.

Numerically, the first-order solution ψ_1 grows linearly in time over all space. Figure 6.8 shows the absolute value of the mode-one component of ψ_1 plotted as a function of time for the Gaussian vortex. Figure 6.8 confirms that (6.39) holds for essentially no values of r, and that the growth is well predicted by (6.40).

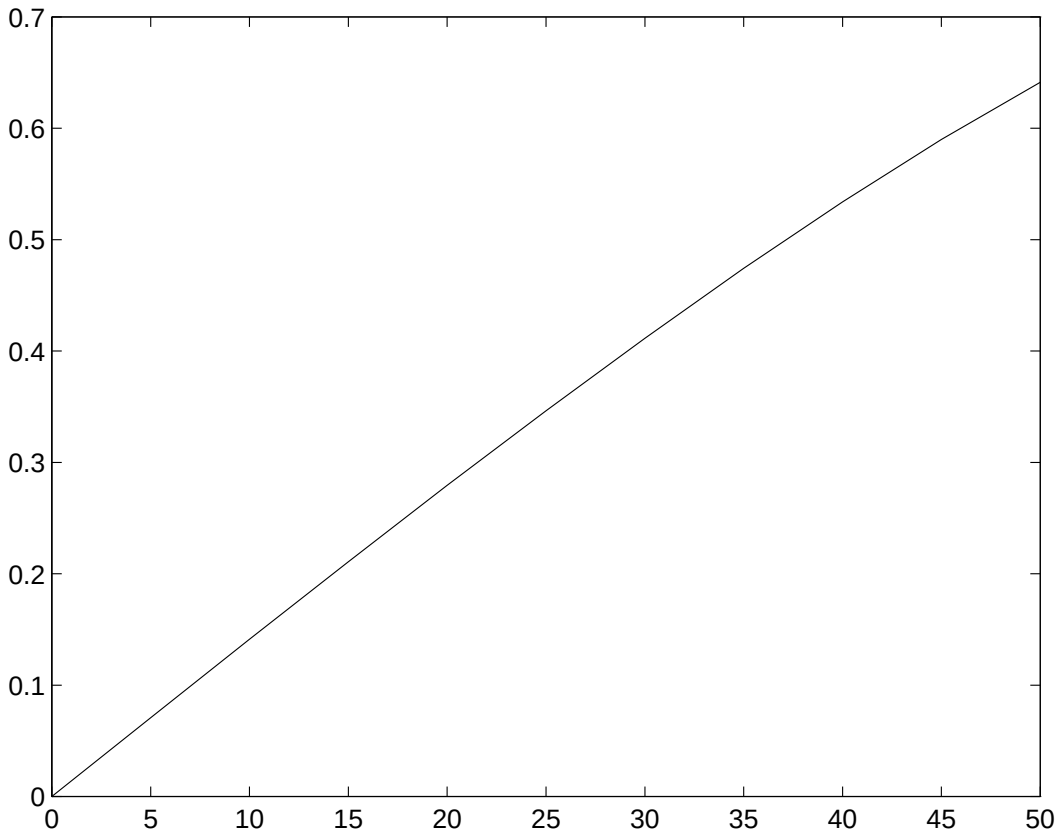

Figure 6.8: The mode-one disturbance amplitude plotted again time. The growth is essentially linear for all time.

This indicates that the asymptotic expansion formally breaks down at times of order ϵ^{-1}. Of course, like all asymptotic expansions, it may produce accurate results for larger times. This, however, may only be checked numerically.

6·6·Asymptotically large radius of deformation

6·6·1·Scaling analysis

The previous analysis has investigated the case of an $O(1)$ Rossby radius of deformation, which corresponds to $\gamma = O(1)$ in the present notation. In the atmosphere, however, the Rossby radius is likely to be much larger than the inner lengthscale of the problem as previously nondimensionalised, so the possibility of asymptotically small γ deserves to be considered. This also links back to the purely nondivergent analysis of Chapters 4 and 5.

For $\gamma = O(\epsilon^{\delta})$, where $\delta > 0$, the equation of motion becomes

$$\frac{\partial}{\partial t}(\nabla^2 - \epsilon^{2\delta}\gamma^2)\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\frac{\partial \psi}{\partial x} = 0. \tag{6.41}$$

A general rescaling of space $r = \epsilon^{\beta} r^*$ leads to

$$\frac{\partial}{\partial t}(\nabla_*^2 - \epsilon^{2(\delta+\beta)}\gamma^2)\psi + \epsilon^{-2\beta}\mathcal{J}_*(\psi, \nabla_*^2\psi) + \epsilon^{1+\beta}\frac{\partial \psi}{\partial x_*} = 0. \tag{6.42}$$

The possible dynamical balances (i.e. those involving the operator ∂_t) are

1. $\beta = 0$: this is the inner scaling, where the stretching term and beta effect are small. This is a leading-order balance for $\delta > 0$, i.e. for small stretching.

2. $\beta = -1$: this is the previous outer scaling, where the rate-of-change and beta terms balance. This is a leading-order balance for $\delta > 1$, i.e. when the stretching is weaker than the beta-effect.

3. $\beta = -\delta$: this is a new scaling, where the operator $(\nabla^2 - \gamma^2)$ is dominant. It is a leading-order balance for $0 < \delta < 1$, i.e. when the stretching is stronger than the beta-effect.

The case $\delta = 1$ leads to a three-term balance when the far fields due to the beta-effect and to the stretching are the same. Whatever the value of δ satisfying $0 < \delta < 1$, one of the balances 2 or 3 will apply and determine the far-field balance.

6·6·2·Beta-effect stronger than stretching

In the case of very large Rossby radius, with $\delta > 1$, the matching procedure, to first order, leads to exactly the same results as in Chapter 4. The Rossby waves can propagate far enough in an $O(1)$ time for the problem to look essentially nondivergent in the near field.

6·6·3·Equal magnitudes of the beta-effect and stretching

The governing equation in the far field is now

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\phi_0 + \frac{\partial \phi_0}{\partial X} = 0. \tag{6.43}$$

This equation is investigated in Kamenkovich (1989). Its solution in the Laplace variable is*

$$\overline{L_\gamma} = -\frac{1}{2\pi p}\exp\left(-\frac{X}{2p}\right)K_0\left(\left[\frac{1}{4p^2} + \gamma^2\right]^{\frac{1}{2}} R\right). \tag{6.44}$$

As in Chapter 4, the appropriate solution to the outer problem is some combination of derivatives of the new Green's function L_γ. However, only the term with no spatial derivative can match onto the inner solution appropriately.

The zeroth-order inner equation is

$$\frac{\partial}{\partial t}\nabla^2\psi_0 + \mathcal{J}(\psi_0, \nabla^2\psi_0) = 0, \tag{6.45}$$

whose solution is $\psi_0 = \Psi + F(t)$, where Ψ is the basic-state streamfunction. Since the stretching is small, the $O(1)$ relative vorticity is $Q = \nabla^2\Psi$. For large r, the behaviour of ψ_0 is again $(\Gamma/2\pi)\ln r$.

The truncation of the outer solution to zeroth order in the inner variable gives

$$\overline{\phi_0^{(0,.)}} = \frac{A}{2\pi p}\left[\ln\left(\frac{1}{4p^2} + \gamma^2\right)^{\frac{1}{2}} + \ln r + \gamma_E + \ln\epsilon\right], \tag{6.46}$$

where Euler's constant is now denoted by γ_E to avoid confusion. The matching condition gives $A = \Gamma$ again. Logarithmic terms will again be forced and must be the same as in Chapter 4. The $O(1)$ logarithmic terms are still dynamically irrelevant.

The next-order term in the expansion is $O(\epsilon)$, since $\delta = 1$. The inner equation is the same as in Chapter 4, since the stretching term is $O(\epsilon^2)$, while the outer equation

*The derivation of (6.44) is analogous to that of $\overline{L}$ in Chapter 3. In fact, the only difference is the replacement of $1/2p$ by $(1/4p^2 + \gamma^2)^{\frac{1}{2}}$ in the argument of K_0.

is

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\phi_1 + \frac{\partial \phi_1}{\partial X} = 0. \tag{6.47}$$

The solutions to both these equations are known, but the matching must be repeated, since the near-field behaviour of (6.44) is different from the purely nondivergent case.

The appropriate matching equations may be obtained by repeating the analysis of section 4·3·3 but modifying the coefficients in the expansion of $\overline{\phi^{(1,)}}$ in the inner variable. In fact, $\overline{\phi^{(1,1)}}$ may be obtained merely by replacing the expression $\ln p$ by

$$-\ln\left(\frac{1}{p^2} + 4\gamma^2\right)^{\frac{1}{2}}, \tag{6.48}$$

or equivalently, adding the expression

$$-\frac{1}{2}\ln(1 + 4\gamma^2 p^2) \tag{6.49}$$

to $\overline{\psi_1}$. At higher order, extra correction terms would be required. To first order though, the results of chapter 4 carry through with this substitution, and the solution to first order is given by the appropriate modification of (4.79). The only difference comes from the inverse Laplace transform of (6.49). In particular, the form of the $\ln \epsilon$ term is unchanged.

In essence, the problem has not changed except for this one feature. This may be understood by considering the properties of Rossby waves in this scaling regime. The maximum phase velocity of these waves is γ^2/ϵ, so the waves excited when the system is 'switched on' can reach the far field in an $O(1)$ time and hence lead to coupling between the near and far fields. As a result, the qualitative features of the first-order motion of the vortex, which occurs in the near-field, do not change. Of course, there will be differences at higher orders.

The asymptotic scaling of the time of breakdown of the expansion does not change since the large-time behaviour of the vortex trajectories is still dominated by the $t^2 \ln t$ contribution from the integral I of (4.82). In combination with the t^2 behaviour of the $\ln \epsilon$ terms, the breakdown must again be at $O(\epsilon^{-\frac{2}{3}})$.

6·6·4·Stretching stronger than beta-effect

For the case $0 < \delta < 1$, the stretching effect should be more important than the beta-effect, and hence the results should be similar to those of section 6·3. In addition, the analysis is complicated by the relative orders of the beta-effect and of the stretching in the near-field.

The new far-field scaling is $R_\delta = \epsilon^\delta r$. In this scaling, the zeroth-order far-field equation is

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\phi_0 = 0. \tag{6.50}$$

The appropriate solution is

$$\overline{\phi_0} = \frac{A}{2\pi} K_0(\gamma R_\delta) \tag{6.51}$$

in the Laplace variable, which is incidentally the limit of (6.44) for small δ. This is a term whose spatial structure is independent of time. Hence wave effects are not possible at this order. The order-zero near-field equation is the usual one, with solution $\Psi + F(t)$. The large-r behaviour yields $A = \Gamma$ again. The same logarithmic terms are again needed in the inner field.

The next-order equation in the near field depends on the value of δ. For $\delta > \frac{1}{2}$, it is (4.32), driven by the beta-effect. The same solution holds as in Chapter 4. In this case, the far-field equation at first order will not contain a Jacobian term, and will again have a K_0 solution. As a result, the appropriate truncation of the far-field solution is

$$\phi^{(1,1)} = \frac{\Gamma}{2\pi p}\left[\ln\frac{\epsilon^\delta r}{2} + \gamma_E\right] + \frac{\epsilon B_0}{2\pi p}\left[\ln\frac{\epsilon^\delta r}{2} + \gamma_E\right] + \frac{\mathbf{B}_1}{2\pi p}\cdot\frac{\mathbf{r}}{r^2}. \tag{6.52}$$

Euler's constant is again denoted by γ_E. Repeating the analysis of section 4·3·3 leads to $\mathbf{B}_1 = \mathbf{0}$ and $B_0 = \overline{b_0} = 0$ as before, as well as to

$$\overline{b_1'} = -\frac{\Gamma}{4\pi p^3}, \tag{6.53}$$

but also to the following new equation:

$$\overline{b_1} = -\frac{\Gamma}{8\pi p^3} - \frac{\Psi_0}{2p^3} + \frac{1}{p^3}\int_0^\infty \frac{h(v)\Omega(v)}{v^3(p + \mathrm{i}\Omega(v))^2}(\Omega(v) - 2\mathrm{i}p)\,\mathrm{d}v. \tag{6.54}$$

The inner-field is killing its mode-one component for large δ. The mode-one terms cancel to give the two above equations. Hence the motion of the vortex in this asymptotic regime differs from the nondivergent case by the absence of the term $\Gamma[\ln p - \ln 1/4 - \gamma]/4\pi p^3$ (in the Laplace variable).

In the case $\delta = \frac{1}{2}$, the first-order disturbance equation in the near field is

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial\theta}\right)\nabla^2\psi_1 - \frac{Q'}{r}\frac{\partial\psi_1}{\partial\theta} - \gamma^2\psi_0 + \Omega r\cos\theta = 0. \tag{6.55}$$

The mode-one component of this equation is the same as for the nondivergent case,

but the mode-zero component becomes

$$\frac{\partial}{\partial t}\nabla^2\psi_1^0 = \gamma^2\psi_0. \tag{6.56}$$

This equation is easy to solve and shows very clearly that the expansion is not uniformly valid, since its solution grows linearly in time. However, it does not lead to any new features in the motion of the vortex, as the mode-zero component of the solution does not affect the various methods of following the centre of the frame. Although ψ_1^0 will grow like t, the expansion is already known to break down for $t = O(\epsilon^{-\frac{2}{3}})$, and that conclusion remains unaffected by the addition of the stretching term.

If $\delta < \frac{1}{2}$, the sequence of equations becomes complicated and rebarbative. There are an unlimited number of possibilities depending on the size of δ, and the analysis would be unilluminating.

6·7·Conclusions

The effect of a nonzero Rossby radius of deformation has been included in the model of vortex propagation on the beta-plane. For asymptotically small stretching, the analysis is very similar to the purely nondivergent case for Rossby radii greater than $O(\epsilon^{-\frac{1}{2}})$. In fact, the motion of the vortex is the same as for the nondivergent case, to first order, for radii greater than $O(\epsilon^{-1})$. The far-field response is no longer wavelike for radii smaller than $O(\epsilon^{-1})$. The $O(\epsilon^{-1})$ radius falls in between the two previous cases. The motion of the vortex may be identified in much the same way too, since the particular solution $r\Omega e^{i\theta}$ remains valid in this situation. In particular, a logarithmic response is generated, and the vortex trajectory has a logarithmic dependence on ϵ. The asymptotic expansion must break down at $O(\epsilon^{-\frac{2}{3}})$ in time. The actual details of the vortex trajectory depend on the magnitude of the stretching term.

The qualitative features of the vortex motion remain similar because Rossby waves can propagate into the far field on $O(1)$ timescales, and hence the far-field and near-field coupling affects the vortex motion. The actual behaviour in the distant far field is, however, very different from the nondivergent case, and exhibits exponential shielding.

For Rossby radii smaller than $O(\epsilon^{-\frac{1}{2}})$, the analysis has not been performed. For $O(1)$ Rossby radii, the distinction between outer and inner problems vanishes. The resulting $O(1)$ problem may be solved analytically, but the $O(\epsilon)$ problem must be

solved numerically. This is relatively straightforward, given the exponential decay of the response in the far field. A quick and moderately accurate procedure has been implemented, using the Talbot algorithm for inversion of the Laplace transform. This may be combined with the usual change-of-frame function $r\Omega e^{i\theta}$ to follow the motion of the cyclone. The numerical calculations gives trajectories that resemble the nondivergent case, qualitatively, for small γ. For larger γ, however, the trajectories are highly oscillatory, and the dominant physical balances are no longer those of relative vorticity advection with small beta. Rather, the stretching terms are now important, and the motion is more zonal. It may even reverse and go to the East for large enough γ. The asymptotic expansion lose validity for $t = O(\epsilon^{-1})$.

7·Conclusion

7·1·Discussion

The physical mechanisms outlined in Chapter 1 appear to give a good description of the essential dynamics associated with cyclone evolution. However, the actual details, and in particular such things as the logarithmic response at $O(\epsilon \ln \epsilon)$, revealed by the analysis are of some complexity.

It is perhaps appropriate at this point to consider what values ϵ may take, and what the timescales and lengthscales of the beta-plane response might be. The work of Adem (1956) leads to a value of $\epsilon = 0.57$, which is big, and to a time unit of $T = 3 \times 10^4$ s, or approximately 8 hours. With such a large value of ϵ, ϵ^{-1} and $\epsilon^{-\frac{2}{3}}$ are very similar, and give a value of perhaps 10 hours for the validity of the expansion. This is long, but with ϵ being so large, it is not clear how good the expansion is. There is a slight problem, which arises from the fact that $\epsilon = \beta L^2/V$. Hence large values of V, i.e. intense vortices, will lead to small values of ϵ, but also to small values of L/V, the natural timescale of the problem.

Smith (1993) has used the numerical results of Chan & Williams (1987) to calculate the dependence of the steady drift speed V_d of a vortex on the beta-plane on the value of beta through dimensional analysis. This leads to

$$V_d = 0.9 r_m (V_m \beta)^{\frac{1}{2}}, \tag{7.1}$$

where V_m is the maximum wind and r_m the radius at which it occurs. This is an interesting result, but cannot be directly compared to the predictions of this dissertation, because it considers steady beta-drift, and hence much longer timescales. In the asymptotic scheme considered in this work, the velocity scales as ϵ in nondimensional terms, or βL^2 in dimensional terms. It is interesting to note that V does not enter directly into this scaling law, which may hence be viewed as linear, since it corresponds to the scaling of linear Rossby wave dispersion. However, the logarithmic term in the asymptotic expansion will correspond to a scaling of $\beta L^2 \ln(\beta L^2/V)$, which does depend on the magnitude of V. And similarly, higher-order terms in the expansion will scale like $\beta^2 L^4/V$ and so on.*

*Intriguingly, this depends on V^{-1}.

7·2·Possible future work

Ekman, or Rayleigh friction, can be treated quite simply in the framework of this dissertation. Physically, this corresponds to a linear drag at the bottom of a container in experimental situations, or to Ekman pumping at the top of the layer of fluid. The governing equation becomes

$$\frac{\partial}{\partial t}(\nabla^2 - \gamma^2)\psi + \mathcal{J}(\psi, \nabla^2\psi) + \epsilon\frac{\partial\psi}{\partial x} = \lambda\nabla^2\psi. \tag{7.2}$$

The coefficient of friction is nondimensionalised appropriately as $\lambda = \lambda^* L/V$, where λ^* is the original dimensional coefficient. Clearly, depending on the magnitude of λ with respect to ϵ, various asymptotic regimes may exist and be investigated. However, the whole formalism of the preceding chapters, if applied to this extended case, leads to a series of linear equations, with a tendency term in vorticity on the left-hand side, and a friction term on the right-hand side. It is crucial that the basic-state nonlinear interaction (its Jacobian) vanish identically for this conclusion to hold. Hence, taking λ to be of order zero, all the equations are identically satisfied by multiplying the solutions by $\mathrm{e}^{-\lambda t}$. Thus the only effect of Ekman friction is uniform decay in time of the solutions.

The effect of viscosity has not been investigated here. There is, however, a long tradition, supported by scale analysis, of investigating inviscid systems in geophysical fluid dynamics. Nonetheless, there are two ways in which viscosity might be expected to manifest itself. Firstly, the wrapping-up of material contours around the centre of the vortex to give spaghetti-like structures (cf. Sutyrin & Flierl 1994, Figure 2) will be smeared out by viscous diffusion. Secondly, secular phenomena of vortex evolution due to viscosity such as those examined by Moffatt (1995) might take place. The latter of these two, certainly, should be a formidable problem.

The Earth's oceans and atmosphere contain other kinds of coherent structure beside vortices and rings, in particular dipoles such as modons. Flierl & Haines (1994) considered the decay of a modon by the emission of Rossby waves. The scalings are very different, and it would be interesting to further investigate the links between the two problems.

The possibility of arbitrarily-shaped domains is a very important one, especially given that computational experiments must perforce be carried out on finite domains. If these domains are large enough, as is natural when considering real-world applications, the only change in the working is to have an outer field with boundaries. Hence only L changes (see Appendix D). However, this leads to potential problems,

as may be seen by calculating

$$\Gamma = \oint_C \mathbf{U}.\ \mathbf{ds}, \tag{7.3}$$

the circulation in the domain. This is acceptable in an inviscid fluid, but the natural doubly-periodic domain geometry of computational experiments cannot support such jumps in the streamfunction in the boundary – at least not without substantial attention being paid to the Laplacian inversion operator if the model is spectral. In addition, any numerical viscosity will lead to very strong shear layers developing on the boundary that are quite unphysical. The approach adopted in numerical simulations to avoid the problem of circulation is to subtract off a constant level of vorticity to cancel the circulation. This, however, is dynamically inconsistent. It is therefore clear that some work remains to be done before theoretical and numerical results may be compared in detail.

Grimshaw *et al.* (1994) considered the related problem of the motion of an eddy over a topographic slope. The governing equations are very similar, and a theoretical approach analogous to that of Smith & Ulrich (1990) was considered. It would be interesting to consider this problem further in the light of the results of this dissertation and the discussion of the previous paragraph, and to try and relate it to the semi-infinite beta-plane case.

There have been recent extensions of the point-vortex and step-profile models of vortex propagation on the beta-plane to the case of multiple layers. A two-layer point-vortex is outlined in Reznik & Grimshaw (1996). However, it only concerns a purely baroclinic scenario, where there is no far field. Sutyrin & Morel (1996) extend the model of Sutyrin & Flierl (1994) to several layers. Both of these more recent approaches suffer from the same limitations as the earlier works on which they are based. It should be possible, though, to extend the work of this dissertation to more than one layer. There would then be both a barotropic and a baroclinic response, and the results of Chapters 4 and 6 would need to be synthesised.

It should also be possible to add far-field background flows. They would then appear only as X and Y terms in the inner-field Rayleigh equations (as they would be expanded in r), although the far-field equation would become much more complicated. The results could then be compared to the results of Smith & Weber (1993) and Williams & Chan (1994) for example.

Finally, we should stress that even if the models investigated in this dissertation are mathematical and might be criticised as toy-like, understanding the underlying phenomena in the atmosphere and ocean is an important task. The relevance of

global warming to tropical cyclones, for example, is addressed in Lighthill *et al.* (1994) – to be sure, the answer appears to be negative, but that and many other problems remain open.

A·Moving sources on the beta-plane

A·1·Far-field radiation from moving sources

The solution to the initial-value problem given by (3.43) is in general a complicated time-dependent function. However, the behaviour of the far field should be linked in some way to the gross properties of the forcing if the latter is localised. In particular, an understanding of the relation of the near-field forcing to the far-field response, which is predominantly linear and wave-like, should help elucidate the fully nonlinear problem. A simple model of the behaviour of the full equation of motion in the far field is the linearised equation of motion (3.1) with some uniformly propagating forcing. A simple framework exists for such problems.

The asymptotic behaviour of forced linear wave systems has been extensively studied since the classical work of Kelvin, Airy, Sommerfeld and others. Lighthill (1978) provides a comprehensive and physically enlightening presentation of waves in fluids, and in particular includes a presentation of the behaviour of linear systems for large times and distances. This work complements and synthesises previous papers (Lighthill 1960, 1965, 1967, but see also Lighthill 1990). The 1967 work, in particular, addresses the same physical system, referred to as the 'beta-plane ocean', as the linear Rossby wave equation. The notation here follows this last paper (while Lighthill 1978 uses (k, l, m) in place of (l, m, n), B and ω instead of P and σ, and $-\mathbf{U}$ rather than $\mathbf{U}$), although ϵ will be used in place of β.

An unforced linear partial differential equation with constant coefficients may be written as

$$P\left(\mathrm{i}\frac{\partial}{\partial t}, -\mathrm{i}\frac{\partial}{\partial x}, -\mathrm{i}\frac{\partial}{\partial y}, -\mathrm{i}\frac{\partial}{\partial z}\right)\phi = 0. \tag{A.1}$$

The existence of a plane-wave solution $\phi = \phi_0 \exp(\mathrm{i}\mathbf{k}.\mathbf{r} - \mathrm{i}\sigma t)$ is equivalent to the existence of a solution to the dispersion relation

$$P(\sigma, l, m, n) = 0, \tag{A.2}$$

where $\mathbf{k} = (l, m, n)$. The corresponding causal solution to the forced problem, with localised forcing term $f(\mathbf{r} - \mathbf{U}t)$, may be expressed as a Fourier integral

$$\phi = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{F(\mathbf{k})\exp\{\mathrm{i}\mathbf{k}.(\mathbf{r} - \mathbf{U}t)\}}{P(\mathbf{U}.\mathbf{k}, l, m, n)}\,\mathrm{d}l\mathrm{d}m\mathrm{d}n, \tag{A.3}$$

where $F(\mathbf{k})$ is the Fourier transform of the forcing function f (using the same definition as in Chapter 2, albeit in three dimensions). This expression is valid for three spatial dimensions; its counterpart for fewer dimensions is obtained merely by removing the n or the (m, n) dependence and appropriately modifying the definition of F.

Lighthill's method gives an expression for the behaviour of ϕ in its far field. There, the amplitude of ϕ is asymptotically

$$\frac{4\pi^2}{|K|^{\frac{1}{2}}R}\frac{F(\mathbf{k})}{|\nabla P(\mathbf{U}.\mathbf{k}, l, m, n)|}, \tag{A.4}$$

where $R = |\mathbf{r} - \mathbf{U}t|$, ∇ is the gradient operator in $\mathbf{k}$ space, and K is the principal curvature of the surface

$$P(\mathbf{U}.\mathbf{k}, l, m, n) = 0. \tag{A.5}$$

This result is valid in three dimensions. The two-dimensional equivalent is obtained by ignoring the coordinate m, while the first factor in (A.4) becomes

$$\frac{(2\pi)^{\frac{3}{2}}}{|\kappa|^{\frac{1}{2}}r^{\frac{1}{2}}}. \tag{A.6}$$

This corresponds to the characteristic inverse-square-root decay of two-dimensional wavefields. The term κ is now the curvature of the wavenumber curve (A.5). A plane portion of the wavenumber curve generates waves with no attenuation: the factor then becomes simply 2π.

The wavenumber of waves found in any direction of physical space is given by the solution $\mathbf{k}$ of (A.5) when the oriented normal to P, i.e. ∇P in $\mathbf{k}$ space, is in the physical direction away from the origin. Some directions may not have any waves propagating along them away from the origin. Some may correspond to more than one point of (A.5). In that case, the amplitude of the various contributions must be evaluated separately from (A.4).

For the two-dimensional linear Rossby wave system, the governing polynomial is

$$P(\sigma, l, m) = \sigma(l^2 + m^2) + \epsilon l, \tag{A.7}$$

corresponding to the forced linear equation of the particular form

$$\frac{\partial}{\partial t}\nabla^2\psi + \epsilon\frac{\partial\psi}{\partial x} = \frac{\partial}{\partial t}q(\mathbf{r}, t). \tag{A.8}$$

For a radially symmetric structure propagating on the beta-plane, the right-hand

side forcing term is

$$\frac{\partial}{\partial t} q(|\mathbf{r} - \mathbf{U}t|), \tag{A.9}$$

where $q(r)$ describes the radially symmetric vorticity disturbance. The velocity may be taken as $\mathbf{U} = U(\cos\alpha, \sin\alpha)$. Then the appropriate Fourier transform of this forcing term is

$$F(\mathbf{k}) = -\mathrm{i}\mathbf{U}.\mathbf{k}Q(k) = -\mathrm{i}Uk\cos(\varphi - \alpha)Q(k), \tag{A.10}$$

where $k^2 = l^2 + m^2$. The function $Q(k)$ is now the Fourier transform of the spatial vorticity distribution. It now remains to calculate the geometric factors in the expression. These come from the wavenumber curve $P(\mathbf{U}.\mathbf{k}, l, m)$. The curve $P = 0$ corresponds to the equation

$$k[Uk^2\cos(\varphi - \alpha) + \epsilon\cos\varphi] = 0, \tag{A.11}$$

where

$$\mathbf{k} = k(\cos\varphi, \sin\varphi), \tag{A.12}$$

which is equivalent to

$$k = \left(-\frac{\epsilon\cos\varphi}{U\cos(\varphi - \alpha)}\right)^{\frac{1}{2}} \tag{A.13}$$

when

$$\cos\varphi\cos(\varphi - \alpha) < 0. \tag{A.14}$$

These are the points on the wavenumber curve that contribute to the far-field radiation. The point $k = 0$ does not generate any radiation. The wavenumber curve $P = 0$ is shown in figure A.1 for different values of α.

The gradient of the locus $P = 0$ is

$$\nabla P = \begin{pmatrix} U(l^2 + m^2)\cos\alpha + 2Ul(l\cos\alpha + m\sin\alpha) + \epsilon \\ U(l^2 + m^2)\sin\alpha + 2Um(l\cos\alpha + m\sin\alpha) \end{pmatrix}. \tag{A.15}$$

This may be rewritten in a more geometric form as

$$\nabla P = \begin{pmatrix} Uk^2\cos\alpha - \epsilon\cos 2\varphi \\ Uk^2\sin\alpha - \epsilon\sin 2\varphi \end{pmatrix}, \tag{A.16}$$

which shows that the group velocity (which is parallel to ∇P) can be viewed as the vector sum of a velocity in the direction of motion of the vortex and a velocity

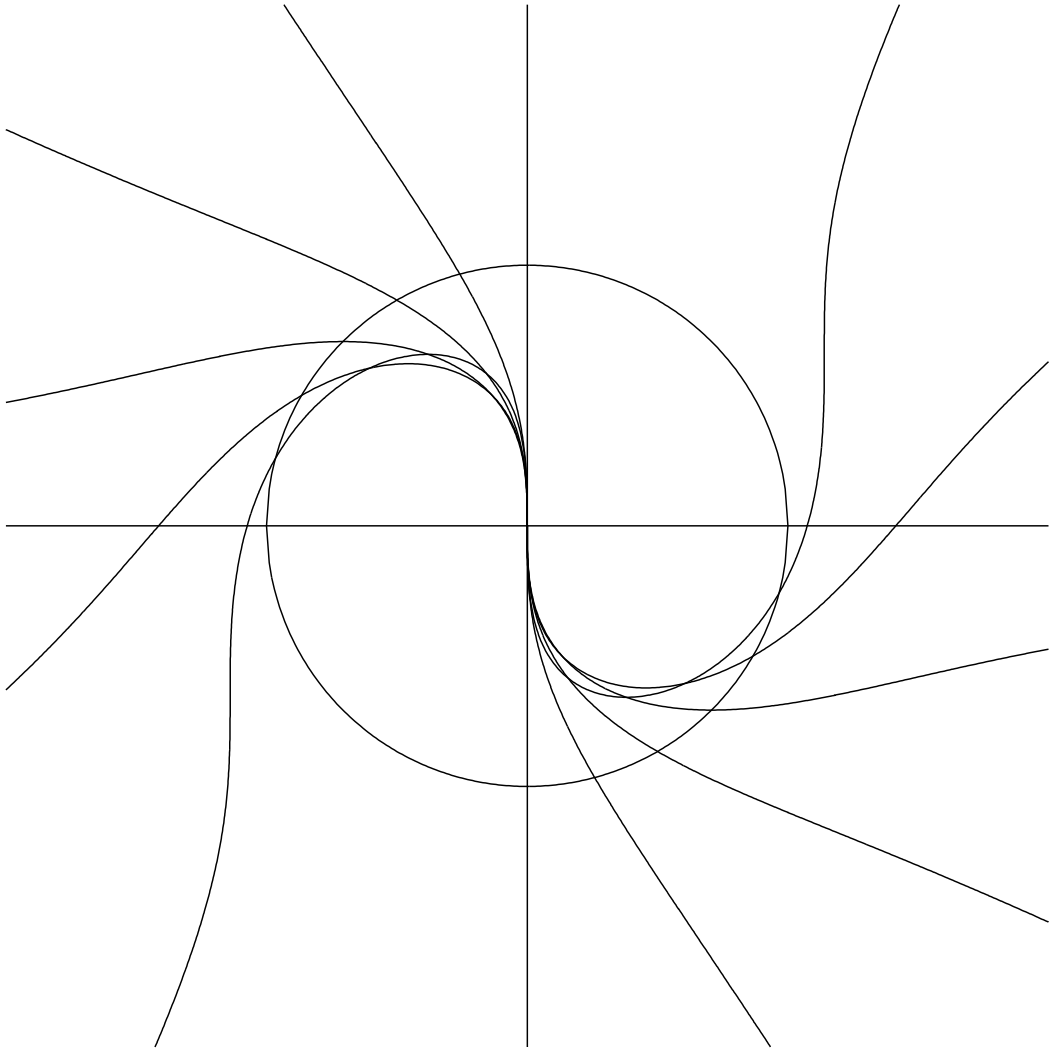

Figure A.1: Wavenumber curves generated by steadily propagating disturbances.

oriented at twice the angle between **k** and the x-axis. However, the most convenient form for subsequent calculation is

$$\nabla P = -\frac{\epsilon}{\cos(\varphi - \alpha)} \begin{pmatrix} \cos\varphi\cos\alpha + \cos 2\varphi \cos(\varphi - \alpha) \\ \cos\varphi\sin\alpha + \sin 2\varphi \cos(\varphi - \alpha) \end{pmatrix}. \tag{A.17}$$

The modulus of $|\nabla P|$ is then

$$\begin{aligned} |\nabla P| &= \frac{\epsilon}{|\cos(\varphi - \alpha)|} \times \\ &\quad \times [\cos^2\varphi + \cos^2(\varphi - \alpha) \\ &\quad + 2\cos\varphi\cos(\varphi - \alpha)\cos(2\varphi - \alpha)]^{\frac{1}{2}} \end{aligned} \tag{A.18}$$

$$= \frac{\epsilon\Phi(\varphi,\alpha)^{\frac{1}{2}}}{|\cos(\varphi-\alpha)|}. \tag{A.19}$$

The only other term that remains to be calculated is the curvature of P. A general expression for the curvature is

$$\kappa = \frac{1}{|\nabla P|^3}[P_{ll}P_m^2 + P_{mm}P_l^2 - 2P_{lm}P_lP_m]. \tag{A.20}$$

The higher derivatives of P may be written as follows:

$$P_{ll} = 2Uk\cos(\varphi-\alpha) + 4Uk\cos\alpha\cos\varphi, \tag{A.21}$$

$$P_{mm} = 2Uk\cos(\varphi-\alpha) + 4Uk\sin\alpha\sin\varphi, \tag{A.22}$$

$$P_{lm} = 2Uk\sin(\alpha+\varphi). \tag{A.23}$$

Therefore, the curvature is given by

$$\begin{aligned}\kappa &= \frac{U^{\frac{1}{2}}}{\epsilon^{\frac{1}{2}}}\frac{\cos\varphi-\alpha)}{\Phi(\varphi,\alpha)^{\frac{3}{2}}}[-\cos\varphi\cos(\varphi-\alpha)]^{\frac{1}{2}}\times \\ &\quad\times[3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)].\end{aligned} \tag{A.24}$$

For a symmetric disturbance, as considered here, the wavefield amplitude in the far field is given by

$$A = \frac{(2\pi)^{\frac{3}{2}}U^{\frac{1}{4}}\Phi^{\frac{1}{4}}}{\epsilon^{\frac{1}{4}}r^{\frac{1}{2}}}\frac{[-\cos\varphi\cos^3(\varphi-\alpha)]^{-\frac{1}{4}}|Q(k)|}{|3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)|^{\frac{1}{2}}} \tag{A.25}$$

This quantity is a positive real number. The phase information, which is required to construct the actual far-field behaviour, is ignored here.

For delta-function forcing, the Fourier transform Q becomes $1/(2\pi)^2$ in the convention used here. Figure A.2 is a plot of the resultant wavefield amplitude for $\alpha = 2\pi/3$ as a function of φ, the angle of the wave vector. The function $A(\varphi)$ has singularities at locations where the curvature κ vanishes. These are not physical singularities, but correspond rather to caustics where the assumptions used to derive (A.25) are invalid. A more detailed investigation reveals an Airy function transition across the caustic.

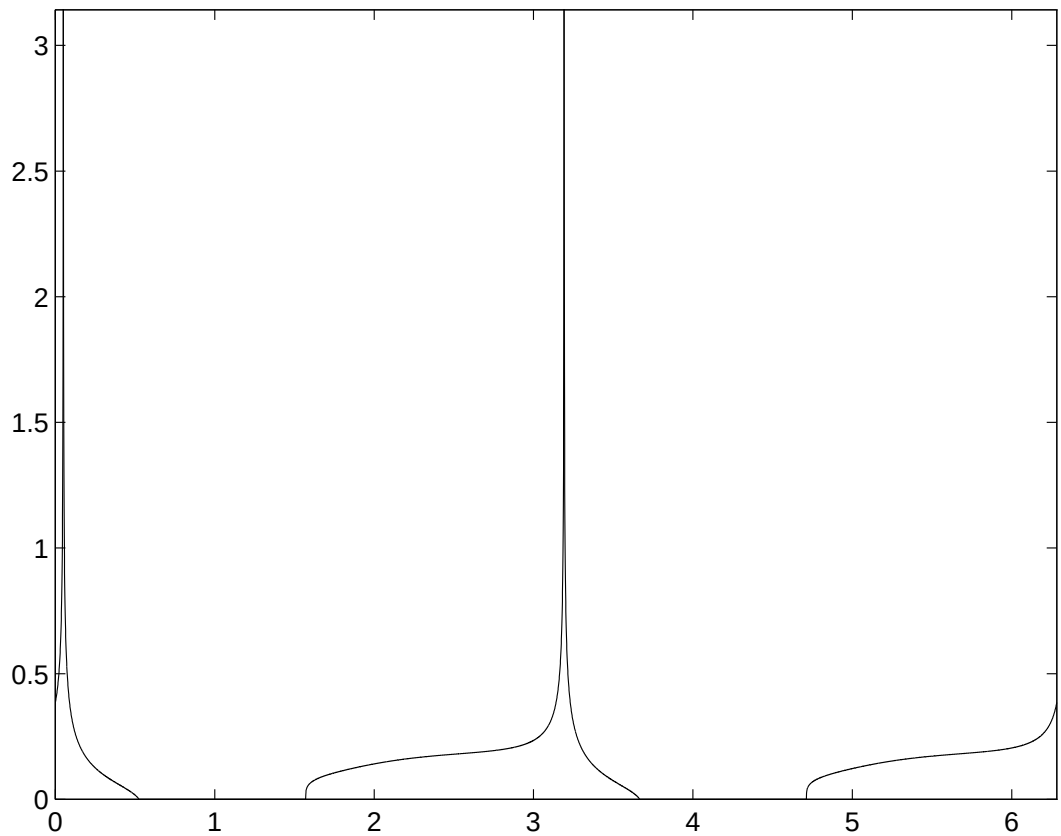

Figure A.2: Plot of the nondimensionalised far-field wave amplitude $(\epsilon/U)^{\frac{1}{4}} r^{\frac{1}{2}} A$ against ϕ for $\alpha = 2\pi/3$. The singularities in A correspond to caustics in the wavefield proper.

A·2·Energy balances

A·2·1·Wave-energy flux

Lighthill (1978) provides a general methodology for calculating the power radiated from a source in an anisotropic wave system. The following discussion is restricted to two-dimensional systems. The flux of wave-energy density is then (Lighthill 1978, Chapter 4, equations 431 and 434)

$$W^{(0)} |F|^2 \left| \frac{\partial P}{\partial \sigma} \right|^{-1} \left| \frac{\partial P}{\partial n} \right|^{-1} \left(\frac{4\pi^3}{\kappa r} \right), \tag{A.26}$$

where κ is the curvature of the wavenumber surface P. The quantity $W^{(0)}$ is the function required to obtain the appropriate wave-energy density from the far-field amplitude. A factor of $\frac{1}{2}$ has already been included in this function, from integrating

over a wave period. Hence the appropriate $W^{(0)}$ for the energy, for example, will be ρk^2 and not $\rho k^2/2$. Integrating (A.26) around a curve gives the power

$$P_W = 4\pi^3 \int W^{(0)} |F|^2 \left|\frac{\partial P}{\partial \sigma}\right|^{-1} \left|\frac{\partial P}{\partial n}\right|^{-1} \mathrm{d}s, \tag{A.27}$$

since the curvature factor cancels on integration. This expression has no singularities at caustics, and hence no detailed analysis is required near the caustics.

In the case of Rossby waves, the first partial derivative is

$$\frac{\partial P}{\partial \sigma} = k^2, \tag{A.28}$$

while for uniformly-translating forcing,

$$F = -\mathrm{i}Uk\cos(\varphi - \alpha)\mathcal{Q} \tag{A.29}$$

where $\mathcal{Q}$ is the Fourier transform of the forcing function. The variable of integration can be changed to φ, using the relations

$$\mathrm{d}s^2 = \mathrm{d}k^2 + k^2\mathrm{d}\varphi^2 = \mathrm{d}\varphi^2 \left[k^2 + \left(\frac{\mathrm{d}k}{\mathrm{d}\varphi}\right)^2\right] \tag{A.30}$$

and

$$\frac{\mathrm{d}k}{\mathrm{d}\varphi} = -\frac{\partial P/\partial \varphi}{\partial P/\partial k} \tag{A.31}$$

on the wavenumber curve $P = 0$. Then

$$\mathrm{d}s^2 = k^2\mathrm{d}\varphi^2 \left(\frac{\partial P}{\partial k}\right)^{-2} \left[\left(\frac{\partial P}{\partial k}\right)^2 + \frac{1}{k^2}\left(\frac{\partial P}{\partial \varphi}\right)^2\right] \tag{A.32}$$

$$= k^2\mathrm{d}\varphi^2 \left(\frac{\partial P}{\partial k}\right)^{-2} \left(\frac{\partial P}{\partial n}\right)^2. \tag{A.33}$$

Thus, integrating over the appropriate angular range, the wave-energy flux is

$$P_W = 8\pi^3 U^2 \int_{\pi/2}^{\pi/2+\alpha} W^{(0)} \cos^2(\varphi - \alpha)|\mathcal{Q}|^2 \left|\frac{\partial P}{\partial k}\right|^{-1} k\,\mathrm{d}\varphi, \tag{A.34}$$

where the factor of 2 comes from halving the range of integration by symmetry. The partial derivative in the integrand is given by

$$\frac{\partial P}{\partial \kappa} = -2\epsilon\cos\varphi \tag{A.35}$$

(φ is held fixed in this partial derivative), so the integral is actually

$$P_W = \frac{4\pi^3 U^{\frac{3}{2}}}{\epsilon^{\frac{1}{2}}} \int_{\pi/2}^{\pi/2+\alpha} W^{(0)} |Q|^2 \left(-\frac{\cos^3(\varphi - \alpha)}{\cos\varphi} \right)^{\frac{1}{2}} \mathrm{d}\varphi. \tag{A.36}$$

A·2·2·Examples

The function $W^{(0)}$ depends on the quantity whose radiated flux is being estimated. For the wave-energy density, it is ρk^2 as mentioned above. Then

$$P_W = 4\pi^3 \rho (\epsilon U)^{\frac{1}{2}} \int_{\pi/2}^{\pi/2+\alpha} |Q|^2 [-\cos(\varphi - \alpha)\cos\varphi]^{\frac{1}{2}} \mathrm{d}\varphi. \tag{A.37}$$

Thus for point forcing, where $Q = \Gamma/(2\pi)^2$, the wave power is

$$P_W = \frac{\rho \Gamma^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \int_{\pi/2}^{\pi/2+\alpha} [-\cos(\varphi - \alpha)\cos\varphi]^{\frac{1}{2}} \mathrm{d}\varphi \tag{A.38}$$

$$= \frac{\rho \Gamma^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} [2E(\sin^2 \alpha/2) - (1 + \cos\alpha) K(\sin^2 \alpha/2)], \tag{A.39}$$

where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, with parameter m. This is the result obtained by Korotaev (1988) using very complicated algebra. However, in the Russian notation, the argument of the elliptic integral is its modulus k, which is related to m by $m = k^2$.

The wave power can also be calculated for other forcing functions. For the Gaussian forcing $q = q_0 \exp(-r^2/a^2)$, the quantity $d = a(\epsilon/U)^{\frac{1}{2}}$ is a nondimensional measure of the spatial extent of the forcing. The wave power is then

$$P_W = \frac{\rho \Gamma^2 (\epsilon U)^{\frac{1}{2}}}{4\pi} \times \times \int_{\pi/2}^{\pi/2+\alpha} \exp\left(\frac{d^2 \cos(\varphi - \alpha)}{4\cos\varphi} \right) [-\cos(\varphi - \alpha)\cos\varphi]^{\frac{1}{2}} \mathrm{d}\varphi, \tag{A.40}$$

where $\Gamma = \pi q_0 a^2$ is the appropriate measure of the forcing in the Fourier transform variable.

Finally, the power can also be calculated for the Rankine forcing $q = q_0 H(a - r)$, where H is the Heaviside function. The Fourier transform of this forcing is given by $2\Gamma \mathcal{J}_1(ka)/ka$, where $\Gamma = \pi q_0 a^2$. Thus, using the same definition of d as before, the power flux is

$$P_W = \frac{\rho \Gamma^2 (\epsilon U)^{\frac{1}{2}}}{\pi d^2} \int_{\pi/2}^{\pi/2+\alpha} \mathcal{J}_1 \left(\left[-\frac{\cos\varphi}{\cos(\varphi - \alpha)} \right]^{\frac{1}{2}} d \right)^2 \times$$

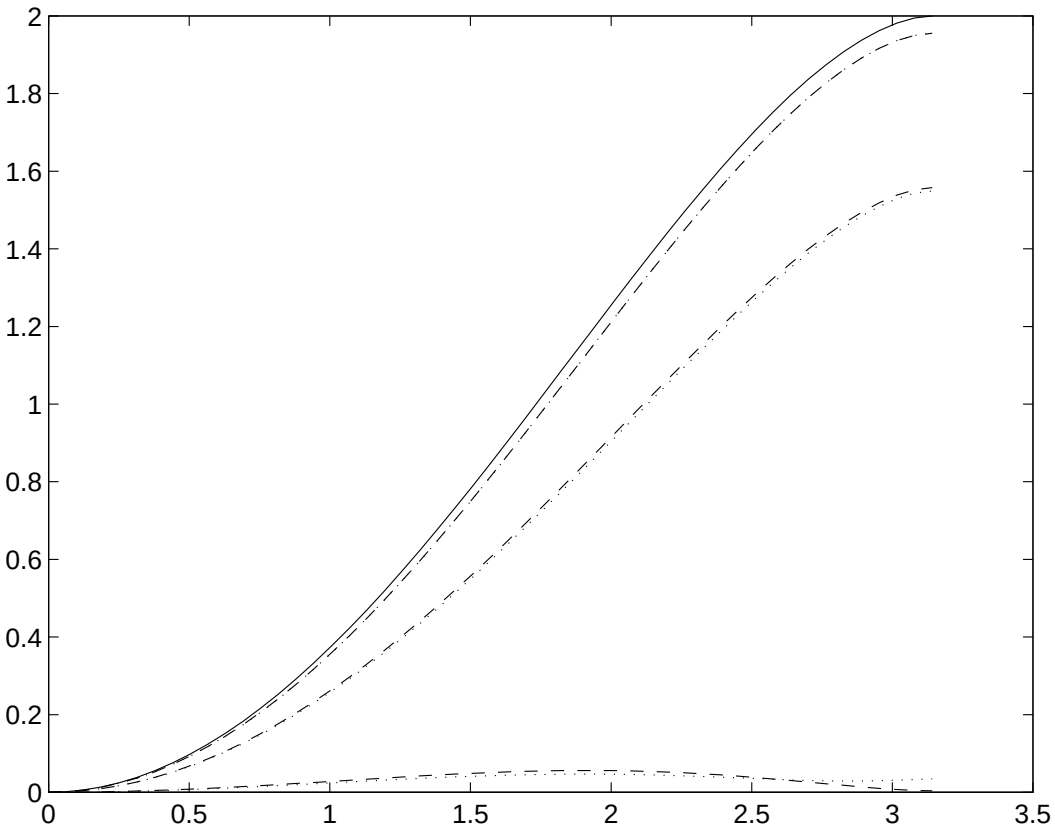

Figure A.3: Plot of the nondimensionalised power $4\pi P_W/\rho\Gamma^2(\epsilon U)^{\frac{1}{2}}$ against the angle α of the trajectory of the vortex for different kinds of forcing. The solid curve corresponds to point forcing, the dashed curves to Gaussian forcing, and the dotted lines to Rankine forcing. For the Gaussian and Rankine forcing, the upper line corresponds to $d = 0.3$, the middle to $d = 1$ and the lower to $d = 5$.

$$\times \left(-\frac{\cos^3(\varphi - \alpha)}{\cos\varphi} \right)^{\frac{1}{2}} \mathrm{d}\varphi. \tag{A.41}$$

The three different powers are shown in figure A.3. For large d with Gaussian forcing and small d with Rankine forcing, the curves are almost identical to the point vortex, as must be the case with the scaling adopted for Γ. The power becomes very small in the opposite limit, corresponding to extensively distributed forcing in physical space.

A·3·Wave momentum flux and force balances

Energy is a quadratic quantity in ϕ, and the linearised governing equation of motion on the beta-plane (which has not actually been required so far),

$$\frac{\partial \mathbf{u}}{\partial t} + (f + \epsilon y)\mathbf{k} \times \mathbf{u} = -\nabla p, \tag{A.42}$$

leads immediately to the flux conservation law

$$\frac{\partial W}{\partial t} = -\nabla.(p\mathbf{u}) = -\nabla.\mathbf{I}, \tag{A.43}$$

where the energy density is $W = |\mathbf{u}|^2/2$ and the wave flux $\mathbf{I} = p\mathbf{u}$. There is hence an easily identifiable physical expression for the wave-energy flux $\mathbf{I}$, although there is no simple expression for it in terms of ψ alone. However, the definition of W leads to the relation $W^{(0)} = k^2$ used above. The power radiated by the vortex could be related to the force due to the waves on the vortex. Power is force multiplied by velocity, and the velocity of the vortex is U; of course this only gives the component of the force parallel to the displacement of the vortex. Hence the three preceding expressions (A.39), (A.40) and (A.41) for the power give the corresponding force on division by U. This is the procedure followed by Korotaev (1988).

The wave enstrophy density $\mathcal{Z}$ could also be calculated using $W^{(0)} = k^4$. In fact, the conservation equation for wave enstrophy is simpler than that for energy, and is given by

$$\frac{\partial \mathcal{Z}}{\partial t} = -\nabla.\left[\frac{\epsilon}{2}(\psi_x^2 - \psi_y^2, \psi_x\psi_y)\right], \tag{A.44}$$

where $\mathcal{Z} = (\nabla^2\psi)^2/2$. However, there is no direct connection between enstrophy flux and force. The procedure analogous to that above would involve dividing the resultant flux by the quantity ϵ. This would give a function of α, with dimensions $\Gamma^2(U/\epsilon)^{\frac{1}{2}}$, as in the previous section, but without any physical meaning.

Stepanyants & Fabrikant (1992) (hereafter SF92) attempt to use similar ideas to compute the force on a steadily translating vortex, but their working is flawed. They define the quasiparticle number per unit volume $\mathcal{N}$ as $\rho|\psi|^2$ and the quasi-particle-number-density-flux by

$$S(\varphi) = \mathcal{N}|\mathbf{c_g}|, \tag{A.45}$$

where $\mathbf{c_g}$ is the group velocity. They claim to derive this function $S(\varphi)$ from Korotaev (1988) and obtain

$$S(\varphi) = \frac{\rho\Gamma^2}{4\pi r}. \tag{A.46}$$

This is clearly wrong, since the results of the previous two sections give

$$\begin{aligned} S(\varphi) &= \frac{\rho U^{\frac{3}{2}}}{2\pi r\epsilon^{\frac{3}{2}}}\left(-\frac{\cos(\varphi-\alpha)}{\cos\varphi}\right)^{\frac{1}{2}} \times \\ &\quad \times \frac{\Phi(\varphi,\alpha)}{|3+4\cos(2\varphi-2\alpha)+\cos(4\varphi-2\alpha)|}, \end{aligned} \tag{A.47}$$

using

$$\mathbf{c_g} = -\frac{U\cos(\varphi-\alpha)}{\epsilon\cos\varphi}\nabla P. \tag{A.48}$$

It is clear from this expression that S is not a constant function of its argument.

However SF92 recovers the correct relation for the power loss due to the waves. First, it is important to note that the quantity whose flux SF92 computes is the pseudomomentum. The correct version of the argument then goes as follows. For a linearised system, the pseudomomentum may be written (McIntyre 1981)

$$\mathbf{q} = \frac{E\mathbf{k}}{\sigma}, \tag{A.49}$$

where E is the wave-energy and σ the Doppler-shifted frequency. Taking components parallel and perpendicular to the trajectory of the vortex gives

$$\mathbf{q} = \frac{E}{U}(1, \tan(\varphi-\alpha)). \tag{A.50}$$

Hence the rate of change of pseudomomentum (obtained by integrating its flux) parallel to the trajectory of the vortex is $P_{\|} = P_W/U$, as found by SF92. The rate of change perpendicular to the trajectory of the vortex is

$$P_{\perp} = 4\pi^3\rho(\epsilon/U)^{\frac{1}{2}}\int_{\pi/2}^{\pi/2+\alpha}|\underset{\sim}{Q}|^2\left(-\frac{\sin^2(\varphi-\alpha)\cos\varphi}{\cos(\varphi-\alpha)}\right)^{\frac{1}{2}}\mathrm{d}\varphi. \tag{A.51}$$

For point forcing, this becomes

$$P_{\perp} = \frac{\rho\Gamma^2(\epsilon/U)^{\frac{1}{2}}}{4\pi}K(\sin^2\alpha/2)\sin\alpha. \tag{A.52}$$

The rate of change of pseudomomentum could also be found for Gaussian and Rankine forcing by computing the appropriate integrals.

However, the physical relevance of this calculation is not immediately apparent. The 'photon analogy' (McIntyre 1993) says that 'the rate at which momentum is transported from location A to location B, when a wave packet is generated at A, and dissipated at B, is the same as if (a) the fluid were absent, and (b) the wave packet had momentum equal to its pseudomomentum'. However, this is just an analogy, and the actual details need to be checked for different physical systems. These calculations usually require considerations of the $O(a^2)$ behaviour of the system, which has not been attempted here. Hence, while it is undeniable that the waves radiate energy, it is not clear from the preceding calculation that it is possible to speak of a force on the vortex.

There are two other simple objections to the preceding calculation. Firstly, there is no real vortex in the system, just a moving forcing. While this approach has been used, for example, in internal gravity wave problems (e.g., Gorodtsov 1994), its validity is again suspect. Secondly, even if this forcing is considered to be a vortex, there is no physical object to support pressure, and hence feel a force.

Dolina & Ostrovsky (1990) considered the instability of an eddy due to radiation. The analysis bears some resemblance to the results of this Appendix. However, the eddy is explicitly modelled as a cylinder containing a line vortex, which can support pressure forces.

B·Identifying the vortex centre

The question of identifying the centre of the vortex to first order is addressed in Section 4·4. Locating the centre at the extremum of some flow quantity $f(\mathbf{r})$ corresponds to solving the equation

$$\nabla f|_{r=0} = \mathbf{0}, \tag{B.1}$$

where in general f is made up of all trigonometric components: $f = \sum f^n e^{in\theta}$. Writing the components of the gradient operator in polar coordinates gives

$$\frac{\partial}{\partial r} = \cos\theta\frac{\partial}{\partial x} + \sin\theta\frac{\partial}{\partial y}, \tag{B.2}$$

$$\frac{1}{r}\frac{\partial}{\partial\theta} = -\sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}. \tag{B.3}$$

Hence (B.1), which must hold for all values of θ, becomes

$$\left.\frac{\partial f}{\partial r}\right|_{r=0} = \left.\frac{1}{r}\frac{\partial f}{\partial\theta}\right|_{r=0} = 0. \tag{B.4}$$

Unless f vanishes at the origin for modes other than zero, the gradient of f is undefined at the origin. When the gradient is defined, l'Hôpital's rule may be used to give the single condition on f

$$\left.\frac{\partial f}{\partial r}\right|_{r=0} = 0. \tag{B.5}$$

The components of the near-field streamfunction in this dissertation satisfy one of two generic equations. The mode-zero term obeys a nonlinear vorticity equation. Since only profiles with $\Psi'(0) = 0$ are being considered, (B.5) is automatically satisfied for the streamfunction-maximum technique where $f = \psi_0$. The same remark applies for the relative-vorticity-maximum, since $Q'(0) = 0$. Hence (B.5) holds with $f = \nabla^2\psi_0$. The origin-following method also works for the same reason. The pseudo-secularity condition does not in fact require these conditions on Ψ, and (B.5) will hold with $f = \psi_0$ for profiles more general than those considered in this dissertation.

The higher-order near-field solutions obey a forced Rayleigh equation of the form

$$\left(\frac{\partial}{\partial t} + \Omega\frac{\partial}{\partial \theta}\right)\nabla^2\psi_n - \frac{Q'}{r}\frac{\partial \psi_n}{\partial \theta} - \mathbf{U}_n.\,\nabla(\nabla^2\psi_0) \tag{B.6}$$

$$= \quad -\frac{\partial \psi_{n-1}}{\partial x} + \sum_{k=0}^{n-1}\mathbf{U}_k.\,\nabla(\nabla^2\psi_{n-k}) - \sum_{k=1}^{n-1}\mathcal{J}(\psi_k, \nabla^2\psi_{n-k}), \tag{B.7}$$

with obvious modifications for the logarithmic terms. This is again a linear equation, and the change-of-frame term $\psi_n^f = \mathcal{Z}_n^* r\Omega e^{i\theta}$ is still a mode-one solution. The mode-m equation is

$$\left(\frac{\partial}{\partial t} + i\Omega\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_n^m}{\partial r}\right) - \frac{m^2}{r^2}\psi_n^m\right] - \frac{imQ'}{r}\psi_n^m$$

$$= \quad \left[-\frac{\partial \psi_{n-1}}{\partial x} + \sum_{k=0}^{n-1}\mathbf{U}_k.\,\nabla(\nabla^2\psi_{n-k}) - \sum_{k=1}^{n-1}\mathcal{J}(\psi_k, \nabla^2\psi_{n-k})\right]^{(m)}, \tag{B.8}$$

where the square bracket on the right-hand side corresponds to extracting mode m of the bracket's contents.

The behaviour of the left-hand side for small r is

$$\left(\frac{\partial}{\partial t} + i\Omega_0 + O(r^2)\right)\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi_n^m}{\partial r}\right) - \frac{m^2}{r^2}\psi_n^m\right] - in[2\Omega_0 + O(r^2)]\psi_n^m. \tag{B.9}$$

The solution to (B.8) near the origin is therefore of the form

$$\psi_n^m = c_n^m r^m + d_n^m r^{m+1} + e_n^m r^{m+2} + O(r^{m+3}), \tag{B.10}$$

where c_n^m is given by the matching procedure, and its Laplacian is

$$\nabla^2\psi_n^m = (2m+1)d_n^m r^{m-1} + (4m+4)e_n^m r^m + O(r^{m+1}). \tag{B.11}$$

Thus for the streamfunction-maximum technique, (B.5) will hold for all modes except possibly one and zero. The mode-one condition can be satisfied by using the change-of-frame term, with

$$\mathcal{Z}_n = -c_n^{m*}. \tag{B.12}$$

Mode zero will satisfy the appropriate condition if $d_n^0 = 0$. For the relative-vorticity-maximum technique, (B.5) will hold for all modes except possibly two, one and zero. Mode-one will give a change-of-frame term

$$\mathcal{Z}_n = -(4m+4)e_n^{m*}. \tag{B.13}$$

For the vorticity to remain nonsingular, the condition $d_n^0 = 0$ is required. For mode two to vanish appropriately in the Laplacian requires $d_n^2 = 0$.

Substituting the expansion for ψ_n^m into (B.9) leads to the following equation in the Laplace variable:

$$(p + \mathrm{i}\Omega_0)(2m+1)\overline{d_n^m} r^{m-1} = [(\ldots)r^{m-1}]^{(m)}. \tag{B.14}$$

The complex representation of Section 5·1·3 may be generalised to cover the right-hand side of (B.8). The contribution from the beta-effect term due to ψ_{n-1}^q is given by

$$-\frac{\mathrm{e}^{\mathrm{i}(q-1)\theta}}{2r^q}(r^q\psi_{n-1}^q)' - \frac{r^q\mathrm{e}^{\mathrm{i}(q+1)\theta}}{2}\left(\frac{\psi_{n-1}^q}{r^q}\right)'. \tag{B.15}$$

Substituting in the expansion for ψ_{n-1}^q, the only way a 'dangerous' r^{m-1} term can appear for mode m is from the term proportional to d_{n-1}^m. The change-of-frame terms lead to a sum of terms of the form

$$-\frac{W_k\mathrm{e}^{\mathrm{i}(q-1)\theta}}{2r^q}(r^q\psi_{n-k}^q)' - \frac{W_k^* r^q\mathrm{e}^{\mathrm{i}(q+1)\theta}}{2}\left(\frac{\psi_{n-k}^q}{r^q}\right)'. \tag{B.16}$$

Again, expanding near the origin leads to the same sort of equation, with d_n^m proportional to a sum of d_{n-k}^m terms. The Jacobian term leads to the following right-hand side (summed over q, s and k):

$$\frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(q-s)}}{2r}[q\psi_k^{q\prime}\nabla^2\psi_{n-k}^{s*} + s\psi_k^q\nabla^2\psi_{n-k}^{s*\prime}] - \frac{\mathrm{i}\mathrm{e}^{\mathrm{i}(q+s)}}{2r}[p\psi_k^{q\prime}\nabla^2\psi_{n-k}^s - n\psi_k^q\nabla^2\psi_{n-k}^{s\prime}]. \tag{B.17}$$

Substituting in

$$\psi_k^q = c_k^q r^q + d_k^q r^{q+1} + \cdots, \tag{B.18}$$

$$\nabla^2\psi_{n-k}^s = (2s+1)d_{n-k}^s r^{s-1} + \cdots \tag{B.19}$$

leads to the following set of terms which are possibly of the form $r^{m-1}\mathrm{e}^{\mathrm{i}m\theta}$. First

$$c_k^q d_{n-k}^{s*} r^{q+s-2}\mathrm{e}^{\mathrm{i}(q-s)\theta} \tag{B.20}$$

if $q + s - 2 = q - s - 1$, but this is impossible for integer s. Then

$$pd_k^q d_{n-k}^{s*} r^{q+s-1}, pe_k^q e_{n-k}^{s*} r^{q+s-1} \tag{B.21}$$

for $q + s - 1 = q - s - 1$, i.e. for $s = 0$. But then in fact the terms vanish. All higher terms lead to negative values for s which are impossible. The other set of terms contains terms like r^{q+s-j} for $j > 2$; it is therefore impossible to create forcing terms

of the appropriate form. The results of this paragraph lead, by strong induction on n, to the conclusion $d_n^m = 0$ for all n and m, since $d_0^m = 0$ for all m. Therefore the relative-vorticity and streamfunction conditions work to all orders. As a result, so does the origin-following technique. Once again, the pseudo-secularity condition does not depend on any of the smoothness conditions used here. While the trajectory is defined at higher orders, it is only the mode-one term that contributes directly to the motion of the vortex.

For the profiles considered in this dissertation, the relative vorticity is finite everywhere, and since absolute vorticity is preserved, and planetary vorticity is finite, it could be argued that no singularities can be generated. Then $d_n^0 = 0$, and (B.5) is satisfied for mode zero and the streamfunction technique is valid to all orders. This argument is not, however, totally rigorous however.

C·Large-time behaviour of I

The integral

$$I = \int_0^\infty \frac{h(v)}{v^3}\left[-\frac{t^2}{2} - \frac{\mathrm{i}te^{\mathrm{i}\Omega(v)t}}{\Omega(v)} + \frac{e^{\mathrm{i}\Omega(v)t}-1}{\Omega(v)^2}\right]\,\mathrm{d}v \tag{C.1}$$

governs the behaviour of the motion of the vortex. Differentiating in time gives

$$\frac{dI}{dt} = t\mathcal{J}(t) = t\int_0^\infty \frac{h(v)}{v^3}\left(e^{\mathrm{i}\Omega(v)t}-1\right)\,\mathrm{d}v. \tag{C.2}$$

The change of variable $u = \Omega(v)$ (a single-valued transformation, since Ω is monotonic as discussed in Chapter 1) leads to

$$\mathcal{J}(t) = \int_0^{\Omega_0} g(u)\left(e^{\mathrm{i}ut}-1\right)\,\mathrm{d}u, \tag{C.3}$$

where

$$g(u) = -\frac{h(v)}{v^3\Omega'(v)}. \tag{C.4}$$

The behaviour of g at the points 0 and Ω_0 can be deduced from the behaviour of $\Omega(v)$ and $h(v)$ as $v \to \infty$ and $v \to 0$. For small v,

$$\Omega'(v) = \Omega_0''v + O(v^2), \tag{C.5}$$

$$h(v) = \frac{\Omega_0}{4}v^4 + O(v^6). \tag{C.6}$$

For large v,

$$\Omega'(v) = -\frac{\Gamma}{\pi v^3} + O(v^{-\infty}), \tag{C.7}$$

$$h(v) = \frac{\Gamma v^2}{4\pi} + O(1). \tag{C.8}$$

This leads in turn to

$$g(u) = \frac{\Gamma}{8\pi u} + O(1), \tag{C.9}$$

$$g(u) = -\frac{\Omega_0}{4\Omega_0''} + O\left((\Omega_0 - u)^{\frac{1}{2}}\right), \tag{C.10}$$

for u near 0 and Ω_0 respectively.

Applying the method of stationary phase to $\mathcal{J}$ shows that the dominant contribution to the integral comes from the origin. This suggests subtracting off the singular

behaviour of g from the integrand, and hence rewriting $\mathcal{J}$ as

$$\mathcal{J} = \int_0^{\Omega_0} \left[g(u) - \frac{\Gamma}{8\pi u}\right] (\mathrm{e}^{\mathrm{i}ut} - 1)\ \mathrm{d}u + \frac{\Gamma}{8\pi} \int_0^{\Omega_0} \frac{\mathrm{e}^{\mathrm{i}ut} - 1}{u}\, \mathrm{d}u \tag{C.11}$$

$$= \mathcal{J}_1(t) + \frac{\Gamma}{8\pi}\left[-\ln \Omega_0 t - \gamma_E + \frac{\mathrm{i}\pi}{2} - E_1(-\mathrm{i}\Omega_0 t)\right], \tag{C.12}$$

where E_1 is the exponential integral (Abramowitz & Stegun 1969). The behaviour of the derivative of $\mathcal{J}_1$ can be obtained by integration by parts, thanks to the earlier change of variable. The result

$$\frac{d\mathcal{J}_1}{dt} = \mathrm{i} \int_0^{\Omega_0} u \left[g(u) - \frac{\Gamma}{8\pi u}\right] \mathrm{e}^{\mathrm{i}ut}\, \mathrm{d}u \tag{C.13}$$

leads to

$$\frac{d\mathcal{J}_1}{dt} = \left[\Omega_0 g(\Omega_0) - \frac{\Gamma}{8\pi}\right] \frac{\mathrm{e}^{\mathrm{i}\Omega_0 t}}{t} + O\left(\frac{1}{t^2}\right).\,, \tag{C.14}$$

essentially because $ug(u) = O(1)$ for small u. If $\mathcal{J}_1$ is rewritten in the form

$$\mathcal{J}_1 = \int_0^{\Omega_0} \left[g(u) - \frac{\Gamma}{8\pi u}\right] \mathrm{e}^{\mathrm{i}ut}\, \mathrm{d}u - \int_0^{\Omega_0} \left[g(u) - \frac{\Gamma}{8\pi u}\right]\, \mathrm{d}u, \tag{C.15}$$

the Riemann–Lebesgue lemma shows that the first integral must be $o(1)$ for large t. The actual behaviour of $\mathcal{J}_1$ is known up to a constant term (by integrating C.14), and hence this term must come entirely from the second integral in (C.15). This integral may be rewritten in the original variable as

$$\mathcal{J}_2 = -\lim_{R\to\infty} \int_0^R \left[\frac{h(v)}{v^3} + \frac{\Gamma}{8\pi} (\ln \Omega(v))'\right]\, \mathrm{d}v. \tag{C.16}$$

Then

$$\mathcal{J}_2 \quad -\lim_{R\to\infty} \left\{\frac{\Psi(R)}{2} - \frac{\Psi_0}{2} - \frac{h(R)}{2R^2} + \frac{\Gamma}{8\pi} \ln \frac{\Omega(R)}{\Omega_0}\right\} \tag{C.17}$$

$$= -\lim_{R\to\infty} \left\{\frac{\Gamma}{4\pi} \ln R - \frac{\Psi_0}{2} - \frac{\Gamma}{8\pi} + O\left(\frac{1}{R^2}\right) + \frac{\Gamma}{8\pi} \ln \left[\frac{\Gamma}{2\pi\Omega_0 R^2} + O\left(\frac{1}{R^\infty}\right)\right]\right\} \tag{C.18}$$

$$= \frac{\Psi_0}{2} + \frac{\Gamma}{8\pi} - \frac{\Gamma}{8\pi} \ln \frac{\Gamma}{2\pi\Omega_0}, \tag{C.19}$$

using (4.52)

Combining the above results leads to

$$\frac{dI}{dt} = \frac{\Gamma t}{8\pi}\left[-\ln \frac{\Gamma t}{2\pi} - \gamma_E + 1 + \frac{\mathrm{i}\pi}{2} - E_1(-\mathrm{i}\Omega_0 t)\right] + \frac{\Psi_0 t}{2} + \mathrm{i}\left[\frac{\Gamma}{8\pi} - \Omega_0 g(\Omega_0)\right] \mathrm{e}^{\mathrm{i}\Omega_0 t} + O\left(\frac{1}{t}\right). \tag{C.20}$$

Integrating once and using appropriate order relations gives

$$I = \frac{\Gamma t^2}{16\pi}\left[-\ln\frac{\Gamma t}{2\pi} - \gamma_E + \frac{3}{2} + \frac{\mathrm{i}\pi}{2}\right] + \frac{\Psi_0 t^2}{4} + O(t). \tag{C.21}$$

D·The Rossby-wave equation in other domains

The analysis of Chapter 4 examined an infinite domain. On this domain, the Green's function for the linear far-field response was easy to calculate, and its inner behaviour could be matched onto the small-r region where vorticity advection was dominant. Solving the inner zeroth-order equation depended crucially on having a radially symmetric basic state, which automatically satisfied the vorticity equation. However, the behaviour of this basic state did not matter for large r, where it matched onto the appropriate inner limit of the outer solution.

Hence the whole of the preceding analysis may be applied to other domain configurations, provided these are of the size of the far field. Then the boundary conditions only enter the far-field problem. Then the whole inner problem is exactly as previously, with the only difference coming from the matching conditions. This is clearly relevant to numerical calculations, since only finite domains can be considered by a computer.

For a rectangular domain of size L, M (two far-field quantities), the function of interest is the appropriate Green's function, since it is this impulsive response that matches back onto the near field. The periodic initial-value Green's function for the Rossby wave equation is obtained by solving

$$\frac{\partial}{\partial t}\nabla^2\psi + \frac{\partial\psi}{\partial x} = \delta(t)\left[\delta(\mathbf{r}) - \frac{1}{\mathcal{A}}\right] \tag{D.1}$$

in a domain of area $\mathcal{A}$ with periodic boundary conditions. The presence of the constant area term is necessary to ensure existence of the solution. It vanishes for infinite domains.

This kind of problem has been extensively treated in the low-Reynolds-number and solid-state-physics literature. The natural method of solution is by expansion in terms of Fourier modes, and techniques exist to accelerate the slow convergence of the resulting sums, going back to Ewald (1921). A very clear treatment for the case of Stokes flow is given by Hasimoto (1959), whose notation will be followed. Nijboer & De Wette (1957) give essentially the same technique in the context of crystal lattices.

Treating the problem as a lattice sum will enable the use of such techniques. Considering a rectangle with sides L and M corresponds to working on a lattice in

physical space with basic vectors $\mathbf{a}^{(1)} = (L, 0)$ and $\mathbf{a}^{(2)} = (0, M)$. Then any point of the lattice may be written as $\mathbf{r_n} = n_1\mathbf{a}^{(1)} + n_2\mathbf{a}^{(2)}$, where the n_i are integers. The corresponding reciprocal lattice is given by $\mathbf{b}^{(1)} = (0, 1/M)$ and $\mathbf{b}^{(2)} = (1/L, 0)$, and then $\mathbf{k} = m_1\mathbf{b}^{(1)} + m_2\mathbf{b}^{(2)}$.

Writing the streamfunction as

$$\psi = \sum_{\mathbf{k}} \psi_k \mathrm{e}^{-2\pi\mathrm{i}(\mathbf{k}.\mathbf{r})} \tag{D.2}$$

leads to the equations

$$-4\pi^2k^2\frac{\partial\psi_k}{\partial t} - 2\pi\mathrm{i}l\psi_k = \frac{\delta(t)}{\mathcal{A}}[1 - \delta_{l0}\delta_{m0}], \tag{D.3}$$

where the components of $\mathbf{k}$ are l and m. The area $\mathcal{A}$ of the rectangle is LM. Hence the solution may be written as

$$\psi = -\sum_{\mathbf{k}}{}' \frac{\mathrm{e}^{-\mathrm{i}lt/2\pi k^2}}{4\pi^2k^2\mathcal{A}}\mathrm{e}^{-2\pi\mathrm{i}(\mathbf{k}.\mathbf{r})}. \tag{D.4}$$

The convergence of this sum may be radically accelerated by writing it in the form

$$\psi = \sum_{\mathbf{k}}{}' \frac{1 - \mathrm{e}^{-\mathrm{i}lt/2\pi k^2}}{4\pi^2k^2\mathcal{A}}\mathrm{e}^{-2\pi\mathrm{i}(\mathbf{k}.\mathbf{r})} - \sum_{\mathbf{k}}{}' \frac{\mathrm{e}^{-2\pi\mathrm{i}(\mathbf{k}.\mathbf{r})}}{4\pi^2k^2\mathcal{A}}. \tag{D.5}$$

The first sum converges very fast, as may be seen from the behaviour of its terms for large k, while the second one has a simple representation. It is in fact the solution ϕ to

$$\nabla^2\phi = \sum_{\mathbf{n}} \delta(\mathbf{r} - \mathbf{r_n}) - \frac{1}{\mathcal{A}}. \tag{D.6}$$

Inside the basic cell, the solution to this equation may be written as

$$\phi = \frac{1}{2\pi}\ln r - \frac{r^2}{4\mathcal{A}} + d. \tag{D.7}$$

The streamfunction then satisfies the equation

$$\psi - \frac{1}{2\pi}\ln r = \sum_{\mathbf{k}}{}' \frac{1 - \mathrm{e}^{-\mathrm{i}lt/2\pi k^2}}{4\pi^2k^2\mathcal{A}}\mathrm{e}^{-2\pi\mathrm{i}(\mathbf{k}.\mathbf{r})} - \frac{r^2}{4\mathcal{A}} + d. \tag{D.8}$$

Hence the leading-order behaviour of ψ is logarithmic, as in the infinite-domain case.

The behaviour of this expression near the origin is required. By symmetry, the y-derivative of the streamfunction vanishes at the origin. Calling the left-hand side

of the above equation ψ' leads to

$$\frac{\partial \psi'}{\partial x} = \sum_{\mathbf{k}}{}' \frac{-\mathrm{i}l(1 - \mathrm{e}^{-\mathrm{i}lt/2\pi k^2})}{2\pi k^2 \mathcal{A}} \mathrm{e}^{-2\pi \mathrm{i}(\mathbf{k}.\mathbf{r})} - \frac{x}{2\mathcal{A}}. \tag{D.9}$$

The last term will obviously not contribute to the behaviour near the origin to first order, while the sum may again be decomposed to enhance its convergence. As a result,

$$\begin{aligned} \frac{\partial \psi'}{\partial x} &= \sum_{\mathbf{k}}{}' \left\{ \frac{-\mathrm{i}l(1 - \mathrm{e}^{-\mathrm{i}lt/2\pi k^2})}{2\pi k^2 \mathcal{A}} - \frac{l^2 t}{4\pi^2 k^4 \mathcal{A}} \right\} \mathrm{e}^{-2\pi \mathrm{i}(\mathbf{k}.\mathbf{r})} \\ &\quad + \sum_{\mathbf{k}}{}' \frac{l^2 t}{4\pi^2 k^4 \mathcal{A}} \mathrm{e}^{-2\pi \mathrm{i}(\mathbf{k}.\mathbf{r})} + O(r). \end{aligned} \tag{D.10}$$

The first sum is well-behaved for small r, while the second may be conventionally written as

$$\frac{t}{4\pi} \frac{\partial^2 S_2}{\partial x^2}, \tag{D.11}$$

whose behaviour on the x-axis is given by

$$\frac{t}{4\pi} \left\{ -\ln r - \frac{1}{2} \ln \pi - 1 - \frac{\gamma_E}{2} + \Sigma_1 + O(r^2) \right\}. \tag{D.12}$$

The dominant behaviour is $\ln r/4\pi$, as in the infinite-domain case. The number Σ_1 may be calculated very efficiently from equation (6.3') of Hasimoto (1959), using Ewald's method; it depends on L and M. However, for square domains, it can be shown that the above behaviour becomes

$$\frac{t}{4\pi} \left\{ -\ln r + \ln L - 1.3205 + O(r^2) \right\}. \tag{D.13}$$

From now on, only the case of square domains will be considered, although all the results may be generalised to rectangular domains. With this restriction, the behaviour of the streamfunction near the origin is

$$\psi = \frac{1}{2\pi} \ln r + C - \frac{x}{4\pi} \left[t \ln \frac{r}{L} + S(t) + 1.3205t \right] + O(r^2), \tag{D.14}$$

where

$$S(t) = \sum_{\mathbf{k}}{}' \left\{ \frac{\mathrm{i}l(1 - \mathrm{e}^{-\mathrm{i}lt/2\pi k^2})}{2\pi k^2 \mathcal{A}} + \frac{l^2 t}{4\pi^2 k^4 \mathcal{A}} \right\} \tag{D.15}$$

Of course the function $S(t)$ depends on the lattice. Figure D.1 shows the behaviour of $S(t)$ for some values of L. It grows in time more slowly than the other terms in the square bracket: like t^2 presumably.

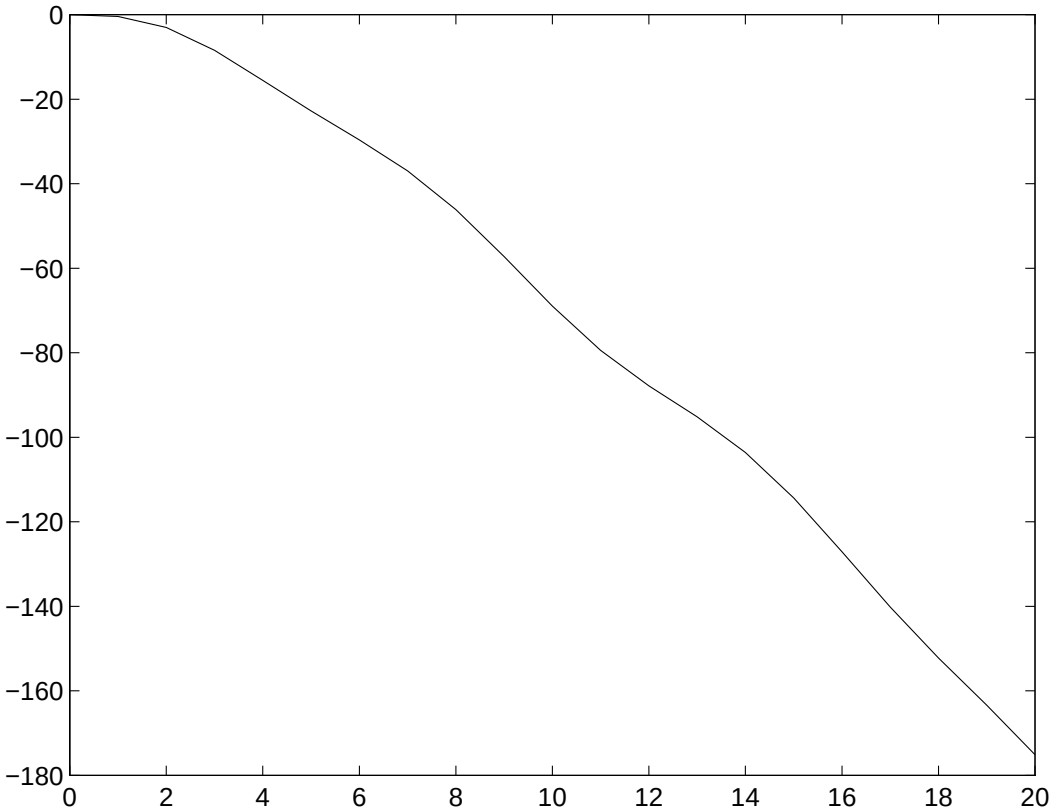

Figure D.1: Plot of $S(t)$ against time, with $L = 1$.

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