THE CLASSIFICATION OF MAPS BETWEEN THE CLASSIFYING SPACES OF LIE GROUPS

by
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Declaration: this dissertation is not substantially the same as any being submitted for a degree or any other qualification at any other university.

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration.

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CONTENTS.

Introduction.

Let C be the catagory of pairs (X,BX) where X is a C.W. complex and BX is a C.W. complex with $\Omega BX \simeq X$. A morphism between objects $(X,BX) \rightarrow (Y,BY)$ is a continuous function $BX \rightarrow BY$. We will assume that X is simply connected and take X to be a homotopy class rather than a single complex. The catagory C is meant to be a homotopy version of the catagory of Lie groups and Lie homomorphisms.

If G is a compact Lie group and BG a classification space as constructed in [14], then $(G,BG) \in C$. If $h:G \rightarrow H$ is a Lie homomorphism then $Bh:BG \rightarrow BH$ is a morphism in C. On the other hand, a map $f:BG \rightarrow BH$ corresponds to an A_∞-map $G \rightarrow H$, see [13]. There is a bijection $[G,H]_{A_\infty} \leftrightarrow [BG,BH]$, where the first set is the set of homotopy classes of A_∞-maps homotopic through A_∞-homotopies, [13].

Our object is to compare the sizes of the sets $Hom(G,H) \subset [G,H]_{A_\infty}$.

When G=H, a compact, connected, simply connected simple Lie group then for any map $f:BG \rightarrow BG$, $f_*:\pi_4(BG) \circlearrowleft$ is multiplication by an integer, m. If $G \neq G_2, F_4$, it is proved in [9] that $f^*:H^*(BG,Q) \circlearrowleft$ is determined by one integer (m). We discuss the possibility of generalisations of this result to maps $f:BG \rightarrow BH$.

In Chapter 1 the work in [9] is extended to cover G_2 and F_4.

In Chapter 2 the maps $BSp(1) \rightarrow BH$ are discussed.

In chapter 3, a cohomlogical description of the maps $BG \rightarrow BH$ is given in the light of some conjectures, while in chapter 4 these conjectures are discussed.

Chapter 1. G_2 and F_4.

The work presented here complements [9] in that we show that any map $BG \to BG$, $G = G_2, F_4$, is determined in cohomology by one integer. The proofs are by explicit computation.

G_2 and F_4 are respectively the exceptional compact simple (connected, simply-connected) Lie groups of rank 2 and 4. [See 11 page 84, and 19, page 268].

Before we give our cohomological description of maps $BG_2 \to BG_2$, we list some results which will be used later on.

Theorem D. (Dirichlet). Let a,b be relatively prime integers. Then the set $\{a + nb \mid n = 1,2, \ldots\}$ contains an infinite number of prime integers.

For the proof, see [18, vol. II, page 217].

Let $\left(\frac{b}{p}\right)$ denote the Legendre symbol for an integer b and a prime p.

Theorem R. If $\left(\frac{b}{p}\right) = 1$ for all but a finite number of primes p, then b is a square.

For the proof, see [18, vol. I, page 75], or [6].

Next a combinatorial result.

In the graded polynomial ring $Z[t_1, \ldots, t_n]$, let the t_i have equal dimension and consider the power sums $S_k = \sum_{1 \leq i \leq n} t_i^k$.

Let x_i be the i-th elementary symmetric function of the t_j.

If $E = (e_1, \ldots, e_n)$ is a sequence of non-negative integers, define $|E| = \Sigma e_i$, $||E|| = \Sigma i e_i$, $x^E = \Pi x_i^{e_i}$, and $E! = \Pi e_i!$, where both sums and products are over $1 \leq i \leq n$. Then, from [10, page 5], we find

<u>Theorem S.</u> $$S_k = \sum_{||E||=k} (-1)^{|E|+k} k[(|E|-1)!]\, x^E/E! .$$

This expresses S_k as a polynomial with integer coefficients in the x_i. Note that every monomial which could occur in S_k for dimensional reasons, does occur with non-zero coefficient. We will mostly use the Theorem reduced modulo a large prime p, and the form of k will be such that all these coefficients remain non-zero when reduced mod p.

Now let G be a compact, connected, simply connected simple Lie group, and $t(G)$ the set of primes for which $H_*(G,Z)$ has torsion. Thus $t(G) \subset \{2,3,5\}$, see [20,21]. Let R be any subring of the rationals in which one can invert each of the primes in $t(G)$. Let $i : Z \to R$ be the inclusion.

It is well known that BG is 3-connected and $\Pi_4(BG)$ is isomorphic to Z, [20]. Hence by the Hurewicz theorem, $H_4(BG,Z) \cong Z$. Also by the universal coefficient theorem [12, page 243], $H^4(BG,Z) \cong Z$ and $H^4(BG,R) \cong \mathrm{Hom}(Z,R) \cong R$, as abelian groups.

Let $\bar{x}$ be a generator of $H^4(BG,Z)$, as a Z-module.

Let x be a generator of $H^4(BG,R)$ as an R-module.

Lemma 1.1. Let $f : BG \to BG$ be a continuous function and $f^*x = ax$. Then a is an integer.

Proof. Let $f^*\bar{x} = b\bar{x}$. Then b is an integer.

Consider the exact sequence $0 \to Z \to R \to R/Z \to 0$ and the corresponding coefficient sequence [12, page 239]:

$$0 \to H^4(BG,Z) \xrightarrow{i_*} H^4(BG,R) \to H^4(BG,R/Z) \to \cdots .$$

We deduce that $i_*\bar{x} = \alpha x$, $\alpha \neq 0$ and since i_* is natural, $\alpha a x = b\alpha x$. Hence $a = b =$ integer.

Finally let $T \subset G$ be a maximal torus, and $j : BT \to BG$ the inclusion. The Weyl group of G acts on $H^*(BT,R)$. Let IG be the subring of Weyl group invariants. Then we easily deduce the following from Theorem 20.3, page 67 of [4].

Theorem W. There is a monomorphism $j^* : H^*(BG,R) \to H^*(BT,R)$, with image IG.

One can paraphrase this informally as "$H^*(BG,R)$ is the subring of Weyl group invariants in $H^*(BT,R)$".

Section 1 : G_2.

From [11, page 84] we find that we can choose a maximal torus $T \subset G_2$ with $H^*(BT,Z) \cong Z[t(1),t(2),t(3)]/I$ where $\dim t(i) = 2$ and I is the ideal generated by $t(1) + t(2) + t(3)$.

We use Theorem W to describe $H^*(BG_2)$, so since $t(G_2) = \{2\}$, let $R = Z[\frac{1}{2}]$.

The Weyl group of G_2 acts by permuting the $t(i)$ and the transformation $(t(1), t(2), t(3)) \to -(t(1), t(2), t(3))$. Let $y(i)$ be the ith elementary symmetric function in $t(j)^2$. Then $y(2) = \frac{1}{4} y(1)^2$ in $H^*(BT, R)$ and $H^*(BG_2, R)$ is the subring generated by $x(2) = y(1)$ and $x(6) = y(3)$.

For more information on G_2, see [5, section 18].

With this notation, we will prove the following :

<u>Proposition 1.2.</u> For any map $f : BG_2 \to BG_2$, there is an integer k such that either

$$\text{(i)} \quad f^*x(2) = k^2x(2) , \quad f^*x(6) = k^6x(6)$$

$$\text{or} \quad \text{(ii)} \quad f^*x(2) = 3k^2x(2) , \quad f^*x(6) = -27k^6x(6) + \tfrac{1}{2}k^6x(2)^3 .$$

Notice that in (ii) $f^*f^*x(2) = (3k^2)^2x(2)$, $f^*f^*x(6) = (3k^2)^6x(6)$.

We make a comment on whether a map satisfying (ii) can actually exist.

In [17, page 5.95], Sullivan conjectures that if p is a prime there does not exist a map $g : BSU(p) \to BSU(p)$ with

$$g^! = \psi^p : K^0(BSU(p)) \to K^0(BSU(p)) .$$

<u>Explanation.</u> Here K^0 is the complex K-theory functor and ψ^p the cohomology operation defined in : J. F. Adams, "Vector fields on spheres" Ann. Maths. 1962, vol. 75. The induced map in K^0 is denoted $g^!$; $SU(p)$ denotes the special unitary group.

A generalization of Sullivan's conjecture is the following:

Let W be the Weyl group of G, with G as above. Then if p divides the order of W, there does not exist a map $g : BG \to BG$ with

$g^{!} = \psi^p : K^0(BG) \to K^0(BG)$.

Recall from [11, page 84] that the Weyl group of G_2 has order 12 . By using [9] , we see that

$$f^{!} \circ f^{!} = \psi^{3k^2} : K^0(BG_2) \to K^0(BG_2) . \qquad (*) .$$

Thus the generalization of Sullivan's conjecture is relevant to the existence of a map with the property $(*)$.

We will prove 1.2 by writing down the condition that f^* commutes with P_p^1 the Steenrod reduced power [15] .

We therefore need to compute the action of $P^1 = P_p^1$ on $H^*(BG_2, Z_p)$, p an odd prime. By abuse of notation, we will also denote the generators of $H^*(BG_2, Z_p)$ by $x(2), x(6)$.

Let $S_j = \sum_{1 \leq i \leq 3} t(i)^{2j}$. By Theorem S , this can be expressed as a polynomial with integer coefficients in the $y(i)$, hence also as a polynomial in $x(2), x(6)$ with coefficients in R .

Lemma 1.3. (i) $P^1 S_j = 2j \sum_i t(i)^{2j+p-1}$ (ii) $P^1 x(6) = 2x(6)S_{\frac{1}{2}(p-1)}$.

Proof Clear from the fact that $P^1 t(i) = t(i)^p$ and the Cartan formula $P^1 xy = xP^1 y + yP^1 x$, see [15] .

We use this to identify some of the monomials in $P^1 x(i)$:

Cor. 1.4. If $p = 6t - 1$, then

(i) $\frac{1}{2} P^1 x(2) = 3x(6)^t + (t^2(2t-1)/4)x(2)^3 x(6)^{t-1} + \ldots$.

(ii) $\frac{1}{2} P^1 x(6) = x(6)[((3t-1)(2t-1)/4)x(2)^2 x(6)^{t-1} + \ldots]$.

Proof (i) By theorem S ,

$$\sum_{1\leqslant i\leqslant 3} t(i)^{6t} = 3y(3)^t + \tfrac{1}{2}t^2(t+1)y(1)^3y(3)^{t-1} - 3t^2y(2)y(1)y(3)^{t-1} + \dots$$

$$= 3x(6)^t + \tfrac{1}{4}t^2(2t-1)x(2)^3x(6)^{t-1} + \dots ,$$

since $y(2) = \tfrac{1}{4}y(1)^2$.

(i) now follows since $P^1x(1) = \Sigma P^1 t(i)^2 = 2\,\Sigma\, t(i)^{p+1}$.

(ii) Similar .

Lemma 1.5. If $p = 6t+1$, then $P^1x(6) = ix(6)^{t+1} + \dots,\ i \not\equiv 0$ mod p .

Proof From 1.3 (ii) $P^1x(6) = 2x(6)S_{3t}$

$$= 2x(6)(3y(3)^t + \dots) = 6x(6)^{t+1} + \dots .$$

To start the proof of Prop. 1.2, note that for dimensional reasons, $f^*x(2) = a(2)x(2)$, $f^*x(6) = a(6)x(6) + b(6)x(2)^3$.

Our task is to compute $a(2)$, $a(6)$ and $b(6)$. In the course of the proofs of lemma 1.6 and lemma 1.8 , we show that $a(2) = 0 \Rightarrow f^* = 0$.

Lemma 1.6. $a(6) = ja(2)^3$, $j = \pm 1$.

Proof In $P^1f^*x(2) = f^*P^1x(2)$, with $p = 6t-1$, equate coefficients of $x(6)^t$. Using the computation of $P^1x(2)$ in 1.4, this gives $a(6)^t = a(2)$ mod p , for infinitely many p , (by theorem D) .

Thus $a(2) = 0 \Rightarrow a(6) = 0$.

If $a(2) \not\equiv 0$, we have, $a(6)^{6t} = a(6)^2 = a(2)^6$ mod p , and so

$a(6)^2 = a(2)^6$ and $a(6) = \pm a(2)^3$.

By Lemma 1.1 at the begining of the Chapter, $a(2)$ is an integer.

<u>Lemma 1.7.</u> If $a(2) = 0$ then $f^* = 0$. If $a(2) \neq 0$, then

(i) $j = 1 \Rightarrow b(6) = 0$ and $a(2) = k^2$ for some integer k .

(ii) $j = -1 \Rightarrow a(2) = 3k^2$, $b(6) = \frac{1}{2} k^6$ for some integer k .

<u>Proof</u> With $p = 6t - 1$, in $P^1 f^* x(2) = f^* P^1 x(2)$ and $f^* P^1 x(6) = P^1 f^* x(6)$ respectively, equate coefficients of $x(2)^3 x(6)^{t-1}$ and $x(2)^2 x(6)^t$.

This gives

(1) $a(2)b = 3t(ja(2)^3)^{t-1} b(6) + ba(2)^3 (ja(2)^3)^{t-1}$, $b = \frac{1}{4} t^2 (2t - 1) = -\frac{1}{63}$

(2) $a.a(2)^2 (ja(2)^3)^t = ja(2)^3 a + 9b(6)$, $a = \frac{1}{4}(3t - 1)(2t - 1) = \frac{1}{12}$.

If $a(2) = 0$, (2) $\Rightarrow b(6) = 0$ and Lemma 1.6 $\Rightarrow a(6) = 0$.
Thus $f^* = 0$.

Assume henceforth that $a(2) \neq 0$.

(i) Putting $j = 1$ in (2) and (1) gives, after rearrangement and using the relation $\left(\frac{a(2)}{p}\right) = a(2)^{3t-1}$,

$$9b(6) = a(2)^3 a \left(\left(\frac{a(2)}{p}\right) - 1 \right) ,$$

$$b(6) = 2ba(2)^3 \left(\left(\frac{a(2)}{p}\right) - 1 \right) .$$

Hence $b(6) = 0$ and $\left(\frac{a(2)}{p}\right) = 1$, $p = 6t - 1$.

Now choose $p = 6t + 1$ and equate coefficients of $x(6)^{t+1}$ in $P^1 f^* x(6) = f^* P^1 x(6)$ to get

$$a(6) = a(6)^{t+1} \mod p \quad \text{i.e.} \quad \left(\frac{a(2)}{p}\right) = 1 .$$

Hence $\left(\frac{a(2)}{p}\right) = 1$ for $p > 3$, so $a(2) = k^2$.

(ii) Putting $j = -1$ in (1) and (2) gives, after rearrangement

$$-2a(2)^3 b\left((-1)^t\left(\frac{a(2)}{p}\right) + 1\right) = b(6)$$

$$a\, a(2)^3\left((-1)^t\left(\frac{a(2)}{p}\right) + 1\right) = 9b(6) .$$

If we use the values of a and b, we see that both these equations become

$$a(2)^3\left((-1)^t\left(\frac{a(2)}{p}\right) + 1\right) = 108\, b(6) .$$

If $\left(\frac{a(2)}{p}\right) = -(-1)^t$ for $p = 6t - 1$, then $b(6) = 0$. But if we equate coefficients of $x(6)^{t+1}$ and $x(2)^3 x(6)^t$ respectively in $P^1 f^* x(6) = f^* P^1 x(6)$ with $p = 6t + 1$, we get

$$a(6) = a(6)^{t+1} \qquad (3) ,$$

$$a(6)b' = b' a(2)^3 a(6)^t ,$$

Hence $-1 = 1$, contradiction.

So we must have $\left(\frac{a(2)}{p}\right) = (-1)^t$ with $p = 6t - 1$, and $a(2)^3 = 54b(6)$. Therefore $a(2) = 3b(2)$ and $b(2)^3 = 2b(6)$.

Now $\left(\frac{3}{p}\right) = (-1)^t$ if $p = 6t \pm 1$, hence

$$\left(\frac{b(2)}{p}\right) = 1 , \quad p = 6t - 1 , \text{ and from } (3) ,$$

$$\left(\frac{b(2)}{p}\right) = 1 , \quad p = 6t + 1 .$$

Thus $b(2) = k^2$ for some integer k.

<u>Proof of 1.2.</u> Taking $j = 1$ (resp. $j = -1$) in 1.6 and 1.7 shows that f^* has the form given in 1.2 (i) (resp. 1.2 (ii)).

It is still possible that if a morphism $h : H^*(BG_2) \to H^*(BG_2)$ has the form given in 1.2 (ii), then h may not commute with P^1 for all large primes p. We will prove in Chapter 4, Cor 4.12., that such an h does commute with P^1.

Section 2. F_4.

We show in this section that in cohomology, maps $f : BF_4 \to BF_4$, fall into two distinct types, just as for G_2. These cohomology classifications can be best understood in terms of our general conjecture on maps $BG \to BH$, formulated in Chapter 3. When $G = H = F_4$, this is proved as Corollary 1.22., below. The first step in this is

Theorem 1.8. For any map $f : BF_4 \to BF_4$, there is an integer k, such that either (i) $f^*x = k^{2n}x$, all $x \in H^{4n}(BF_4)$ or
(ii) $f^*f^*x = (2k^2)^{2n}x$, all $x \in H^{4n}(BF_4)$.

Before starting the proof of 1.8, we quote the following result from [9].

Lemma 1.9. [Hubbuck] If A is a polynomial algebra over the mod p Steenrod algebra, let $x \in A$ have dimension $2m$. Then there is a $y \in A$, with $\dim y = 2q$, $q + p - 1 = tm$, $t > 0$, such that if x and y are members of a basis for the indecomposables, so that the monomials in this basis form a Z_p - basis for A, then,

$$P^1 y = \alpha x^t + \ldots, \quad \alpha \neq 0 \bmod p.$$

To begin the proof of 1.8, we need to describe $H^*(BF_4)$. First note that since $t(F_4) = \{2, 3\}$, we will take coefficients in $Z[\frac{1}{2}, \frac{1}{3}] = R$.

Let $T \subset F_4$ be a maximal torus. Then $H^*(BT, R) \cong R[t(1), t(2), t(3), t(4)]$ and $H^*(BF_4, R)$ is the subring of Weyl group invariants. T will be chosen as in [5, page 534].

Let $y(i)$ be the i th elementary symmetric function in the $t(i)^2$, then the generators of $H^*(BF_4)$ are polynomials in the $y(i)$, from the form of the Weyl group.

We can choose generators $x(i)$ as follows:

$$x(1) = y(1) \ , \quad x(3) = y(3) - \frac{1}{6}\, y(1)y(2) \ ,$$

$$x(4) = y(4) + \frac{1}{12}y(2)^2 - \frac{1}{24}\, y(1)^2y(2) \ ,$$

and

$$x(6) = y(2)y(4) - \frac{1}{36}\, y(2)\Big(y(2)^2 - \frac{3}{2}\, y(1)^2y(2) + \frac{9}{16}\, y(1)^4\Big) .$$

The first three generators are taken from [5, section 19]. To see that $x(6)$ is invariant under the Weyl group, we know from [5], that we have to check that $x(6)$ is invariant under

(i) permutations of the $t(i)$ and sign changes $t(i) \to -t(i)$,

(ii) the map $t(i) \to t(i) - \frac{1}{2}\,(t(1) + t(2) + t(3) + t(4))$.

Now $x(6)$ is clearly invariant under (i), whilst under (ii)

$$y(1) \to y(1) \ , \quad y(2) \to \frac{1}{8}\left(3y(1)^2 - 4y(2) + 24\, X\right)$$

$$y(4) \to \frac{1}{256}\left(4y(2) + 8X - y(1)^2\right)^2 \ , \quad X = t(1)t(2)t(3)t(4) .$$

Thus one checks that $x(6)$ is invariant under (ii) .

To prove 1.8, we will compute f^* in terms of these generators. For dimensional reasons, f^* has the following form

$$f^*x(1) = a(1)x(1) \quad , \qquad f^*x(3) = a(3)x(3) + b(3)x(1)^3$$

$$f^*x(4) = a(4)x(4) + b(4)x(1)^4 + c(4)x(1)x(3) \; .$$

$$f^*x(6) = a(6)x(6) + b(6)x(1)^6 + c(6)x(3)^2 + d(6)x(4)x(1)^2 + e(6)x(3)x(1)^3 \; .$$

We will assume $a(1) \neq 0$, otherwise it follows from the arguments below that $f^* = 0$.

Using Lemma 1.1, we see that $a(1)$ is an integer.

Our task now is to compute the coefficients in f^*.

Lemma 1.10 $\qquad a(i) = \alpha(i)a(1)^i , \quad \alpha(i) = \pm 1$.

Proof Choose $p = 12t - 1$, and in $P^1f^*x(1) = f^*P^1x(1)$, equate coefficients of $x(6)^t$:

$$P^1x(1) = \beta x(6)^t + \dots , \quad \beta \neq 0 , \text{ by Lemma 1.9, and}$$

$$P^1f^*x(1) = a(1)(\beta x(6)^t + \dots)$$

$$f^*P^1x(1) = \beta(a(6)x(6) + \dots)^t + \dots \; .$$

Hence $a(1) = a(6)^t \mod p$, so $a(6)^2 = a(1)^{12}$, since the congruence is true for infinitely many p . Similarly $a(4)^2 = a(1)^8$.

Next choose $p = 8t + 3$, and equate coefficients of $x(8)^{t+1}$ in $P^1f^*x(3) = f^*P^1x(3)$, to get $a(3) = a(4)^{t+1} \mod p$, for infinitely many p . Hence $a(3) = \alpha(4)^{t+1}a(1)^{4t+4} = \alpha(4)^{t+1}a(1)^3a(1)^{\frac{1}{2}(p-1)}$. Thus $a(3)^2 = a(1)^6$.

Lemma 1.11.

$$\left(\frac{a(1)}{p}\right) = \alpha(4)^t \mod p \qquad p = 8t-1 \qquad (1)$$
$$= \alpha(3)\alpha(4)^t \qquad = 8t-5 \qquad (2)$$
$$= \alpha(4)^{t-1} \qquad = 8t-7 \qquad (3)$$
$$= \alpha(6)\alpha(4)^t \qquad = 8t-11 \qquad (4)$$
$$= \alpha(6)^t \qquad = 12t-1 \qquad (5)$$
$$= \alpha(3)\alpha(6)^t \qquad = 12t-5 \qquad (6)$$
$$= \alpha(4)\alpha(6)^t \qquad = 12t-7 \qquad (7)$$
$$= \alpha(6)^{t-1} \qquad = 12t-11 \qquad (8)$$

Proof If $p = 12t - 1$, we know from the proof of 1.10 that $a(1) = a(6)^t \mod p$; also $a(6) = \alpha(6)a(1)^6$. Hence $\alpha(6)^t a(1)^{6t} = a(1) \mod p$ and since $\left(\frac{a(1)}{p}\right) = a(1)^{\frac{1}{2}(p-1)}$, (5) is proved. The rest of relations are similarly derived.

Lemma 1.12. $\alpha(4) = 1$, $\alpha(6) = \alpha(3)$.

Proof Put $t = 4s + 2$ in (5) and $t = 6s + 3$ in (1). This gives $\left(\frac{a(1)}{p}\right) = 1 = \alpha(4)^{6s+3} \mod p$, $p = 48s + 23$. Hence $\alpha(4) = 1$.

In (5) choose $t = 4s + 1$, and in (2), $t = 6s + 2$. This gives $\alpha(6) = \alpha(3) \mod p$, $p = 48s + 11$. Hence $\alpha(6) = \alpha(3)$.

We can now prove 1.8.

Proof of 1.8.

(i) If $\alpha(3) = 1$, then all the $\alpha(i)$ are 1 , and using the techniques of [9] one can easily prove 1.8 (i) .

(ii) Assume that $\alpha(3) = -1$. Lemma 1.11 then gives

$$\begin{aligned} \left(\frac{a(1)}{p}\right) &= 1 && \text{mod } p \qquad p = 8t-1 \\ &= -1 && \qquad\qquad p = 8t-5 \\ &= 1 && \qquad\qquad p = 8t-7 \\ &= -1 && \qquad\qquad p = 8t-11 \end{aligned}$$

To "solve" this system for $a(1)$, we need

Lemma 1.13. If $\alpha(3) = -1$, then $a(1) = 2k^2$ for some integer k .

Proof We have $\left(\frac{a(1)}{p}\right) = 1$ (resp. -1) for $p = 1, 7$ (resp. $3, 5$) mod 8 . Hence $\left(\frac{2a(1)}{p}\right) = \left(\frac{2}{p}\right)\left(\frac{a(1)}{p}\right) = 1$ for all primes $p > 3$.

This implies that $2a(i) = j^2$ for some even j . Hence $a(1) = 2(\frac{1}{2} j)^2 = 2k^2$. The proof of 1.8 (ii) can now be completed using [9] by noting that $f^* f^* x(i) = (2k^2)^{2i} x(i)$, mod decomposables.

To get a better idea of the form of f^* in this case, we need detailed computations of the action of P^1 on $H^*(BF_4)$.

Proposition 1.14. If $\alpha(3) = -1$, then $a(1) = 2k^2$, and

(i) $c(4) = -4k^8$, $d(6) = 40k^{12}$, $c(6) = 24k^{12}$

(ii) $b(3) = -k^6/3$

(iii) $e(6) = -4k^{12}$, $b(4) = -\frac{1}{12}k^8$.

This is the main computation of the section.

Lemma 1.15. If $p = 8t + 3$, then

$$P^1 x(1) = 2(-1)^t (2t+1)[(t+1)x(1)^2 x(4)^t - tx(3)^2 x(4)^{t-1} - 2x(6)x(4)^{t-1} + \dots].$$

Proof $P^1 x(1) = 2 \Sigma t(i)^{4t+2}$ = a polynomial in the $y(i)$ by Theorem S.
The coefficient of $y(4)^t y(1)^2$ is $2(-1)^t (2t+1)(t+1)$ and
$y(4)^t y(1)^2$ can come only from $x(1)^2 x(4)^t$.
The coefficient of $y(3)^2 y(4)^{t-1}$ is $-2(-1)^t (2t+1)t$ and
$y(3)^2 y(4)^{t-1}$ can come only from $x(3)^2 x(4)^{t-1}$.
The coefficient of $y(2)y(4)^t$ is $-2(-1)^t (4t+2)$ and can come only from
$x(6)x(4)^{t-1}$.

Lemma 1.16. If $p = 8t - 1$, then

$$P^1 x(1) = 8(-1)^t (x(4)^t - t^2 x(1)x(3)x(4)^{t-1} + \dots)$$

Proof As in 1.15.

Lemma 1.17. If $p = 8t + 5$, and $\alpha = \frac{1}{6}(t+1)(t+2)$ then

$$P^1 x(1) = 2(-1)^t (4t+3)[\alpha\, x(1)^3 x(4)^t + x(3)x(4)^t - (t + \tfrac{5}{6})x(1)x(6)x(4)^{t-1} + \dots].$$

Proof As in 1.15, but one has to be careful to note that the term $y(1)y(2)y(4)^t$ occurs in $x(3)x(4)^t$ and in $x(1)x(6)x(4)^{t-1}$, when computing the coefficient of $x(1)x(6)x(4)^{t-1}$.

Armed with these computations, we can prove 1.14 (i).

Proposition 1.18. $c(4) = -4k^8$, $d(6) = 40k^{12}$, $c(6) = 24k^{12}$.

Proof To find $c(4)$, equate coefficients of $x(1)x(3)x(4)^{t-1}$ in $P^1f^*x(1) = f^*P^1x(1)$, with $p = 8t - 1$ in 1.16 :

$$-a(1)t^2 = tc(4)a(4)^{t-1} - t^2a(1)a(3)a(4)^{t-1} .$$

So, $-a(1)a(4)t = a(1)(c(4) - ta(1)a(3))$, since $a(1)^{4t-1} = 1$, mod p , and hence $-16tk^8 = c(4) + 16t\,k^8$.

$$\therefore\ c(4) = -4k^8 , \text{ since } 8t = 1 \mod p .$$

To find $d(6)$ and $c(6)$, using 1.15, with $p = 8t + 3$, equate coefficients of $x(1)^2x(4)^t$ and $x(3)^2x(4)^{t-1}$, respectively in

$$P^1f^*x(1) = f^*P^1x(1) .$$

So, for example, for $d(6)$,

$$(t + 1)a(1)^2a(1)^{4t} - 2d(6)a(1)^{4t-4} = (t + 1)a(1) ,$$

$$-((t + 1)a(1)^6 - 2d(6)) = (t + 1)a(1)^6 , \text{ since } \left(\frac{2}{p}\right) = -1 .$$

Therefore $d(6) = (t + 1)a(1)^6 = 40k^{12}$.

Proof of 1.14 (ii)

In $P^1f^*x(1) = f^*P^1x(1)$, with $p = 8t + 5$, using 1.17., equate coefficients of $x(1)^3x(4)^t$:

$$\frac{1}{6}a(1)(t+1)(t+2) = \frac{1}{6}(t+1)(t+2)a(1)^3a(4)^t + b(3)a(4)^t$$

$$- (t + \tfrac{5}{6})d(6)a(1)a(4)^{t-1} .$$

Now note that $\left(\frac{2}{p}\right) = -1$, $d(6) = 40k^{12}$, from 1.18 . Hence we get

$$b(3) = -\frac{1}{3}k^6 .$$

For 1.14 (iii) we need

Lemma 1.19.

(i) $16k^6 b(4) + c(4)b(3) = 0$,

(ii) $d(6)c(4) - 2a(1)^4 e(6) - 4k^2 c(6)b(3) = 0$.

Proof Using [9] , we know that $f^* f^* x(4) = a(1)^8 x(4)$ and $f^* f^* x(6) = a(1)^{12} x(6)$. So to prove (ii) for instance, we equate coefficients of $x(1)^3 x(3)$ in the latter equation :

$a(6)e(6) - 2c(6)b(3)a(1)^3 + d(6)c(4)a(1)^2 - a(1)^6 e(6) = 0$, which simplifies to 1.19 (ii) .

Proof of 1.14 (iii) Substitute the values of $c(4)$, $b(3)$, $d(6)$ and $c(6)$ in 1.19 (i) and 1.19 (ii) . This gives $b(4)$ and $e(6)$.

Notice now that in f^* only $b(6)$ remains to be determined. To finish the determination of f^*, we change tack.

Define a ring homomorphism

$$f_0^* : H^*(BT) \to H^*(BT) \quad \text{as follows :}$$

$f_0^*(t(1), t(2), t(3), t(4)) = k(t(1) + t(2), t(1) - t(2), t(3) - t(4), t(3) + t(4))$, with k an integer.

Lemma 1.20.

(i) f_0^* commutes with P^1 .

(ii) $f_0^* H^*(BF_4) \subset H^*(BF_4)$.

Proof (i) Clear.

(ii) By obvious computation,

$f_0^* \, y(1) = 2k^2 y(1)$

$f_0^* \, y(2) = k^4[(t(1)^2 - t(2)^2)^2 + (t(3)^2 - t(4)^2)^2 + 4(t(1)^2 + t(2)^2)(t(3)^2 + t(4)^2)]$

$f_0^* \, y(3) = 2k^6[(t(1)^2 - t(2)^2)^2(t(3)^2 + t(4)^2) + (t(1)^2 + t(2)^2)$

$(t(3)^2 - t(4)^2)^2]$.

$f_0^* \, y(4) = k^8[(t(1)^2 - t(2)^2)(t(3)^2 - t(4)^2)\,]^2$.

If one computes $f_0^* \, x(i)$, one finds

$f_0^* \, x(1) = 2k^2 x(1)$

$f_0^* \, x(3) = -\, 2^3 k^6 x(3) - \frac{1}{3} k^6 x(1)^3$

$f_0^* \, x(4) = k^8(2^4 x(4) - 4x(1)x(3) - \frac{1}{12} x(1)^4)$ and

$f_0^* \, x(6) = k^{12}(-2^6 x(6) - \frac{1}{9} x(1)^6 + 24\, x(3)^2 + 40\, x(1)^2 x(4) - 4\, x(1)^3 x(3))$.

So $f_0^* \, H^*(BF_4) \subset H^*(BF_4)$.

We use this lemma to complete our determination of f^* .

Lemma 1.21. In case (ii) of Theorem 1.11, f^* has $b(6) = -\frac{1}{9} k^{12}$.

Proof Note that $(f^* - f_0^*)\, x(i) = 0$, $i \neq 6$ and

$(f^* - f_0^*)x(6) = (b(6) + \frac{k^{12}}{9})\, x(1)^6$.

Also $f^* P^1 \, x(1) = P^1 f^* x(1) = 2k^2 x(1)$

$f_0^* P^1 x(1) = P^1 f_0^* \, x(1) = 2k^2 x(1)$.

Hence $f^* P^1 \, x(1) = f_0^* P^1 \, x(1)$ (*)

For $p = 8t + 3$, look at the computation of $P^1x(1)$ in 1.15. Using that, equate coefficients of $x(1)^6x(4)^{t-1}$ in (*). From our previous determinations of f_0^* and f^*, we see that this leads to

$$-2b(6)a(4)^{t-1} + \beta = -2(-\frac{1}{9}k^{12})a(4)^{t-1} + \beta,$$

where β does not involve $b(6)$

$$\therefore \quad b(6) = -\frac{1}{9}k^{12}.$$

This lemma enables us to give a cohomological description of maps $f : BF_4 \to BF_4$, namely

<u>Cor. 1.22.</u> For any map $f : BF_4 \to BF_4$, there exists

$$f_0^* : H^*(BT) \to H^*(BT)$$

such that

$$f^* = f_0^*|H^*(BF_4).$$

<u>Proof</u> In Theorem 1.8., we divided the maps f^* into two cases. We find an f_0^* for each of these cases:

(i) Clearly we take $f_0^*(t(1), t(2), t(3), t(4)) = k(t(1), t(2), t(3), t(4))$.

(ii) Again from Lemma 1.20, 1.21, we find that we can take
$f_0^*(t(1), t(2), t(3), t(4)) = k(t(1) + t(2), t(1) - t(2), t(3) - t(4), t(3) + t(4))$.

Chapter 2 Maps $BSp(1) \to BH$.

Let H be a simple, connected, simply connected, compact Lie group, and $f : BSp(1) \to BH$ a continuous function. We abbreviate "continuous function" to "map" . In this chapter we give a cohomological classification of the maps f . Our method requires that we deal with each group H individually and with specific generators for $H^*(BH)$. We will give the details of the classification when $H = Sp(n)$, $SU(n)$, $Spin\ (n)$ or G_2 .

For a precise statement of the classification for $H = Sp(n)$, we need more notation. Let $T \subset Sp(n)$ be the standard maximal torus of the symplectic group $Sp(n)$. Then $H^*(BT) \cong Z[t(1),\ldots, t(n)]$ since T has rank n , and from [11, page 82] we deduce that the Weyl group will act by permuting the $t(i)$ and changing signs. Hence by Theorem W of chapter 1 , $H^*(BSp(n)) \cong Z[x(1),\ldots, x(n)]$, as a graded ring, where $x(i)$ is the i th elementary symmetric function in the $t(i)^2$. Notice that since the dimension of $t(i)$ is 2 , $\dim x(i) = 4i$.

From above $H^*(BSp(1)) \cong Z[x(1)]$. Put $x = x(1)$.

Abbreviation $e_j(Z(i))$ = j th elementary symmetric function of the variables $Z(i)$. Put $e_j(Z(i)) = e_j$ when the $Z(i)$ are understood.

So for example, $x(j) = e_j(t(i)^2)$.

Let $f : BSp(1) \to BSp(n)$ be a map . If $f^*: H^*(BSp(n)) \to H^*(BSp(1))$ is the induced homomorphism, assume that there exist integers $m(1),\ldots, m(n)$, such that $f^*x(j) = a(j)x^j$, $a(j) = e_j(m(i)^2)$.

Call $\{m(1),\ldots, m(n)\}$, the degree of f .

The main result of Section 1 is

Theorem 2.1. Any map $f : BSp(1) \to BSp(n)$ has a degree.

In Section 3, we will use Sullivan's construction of maps $BSp(1) \to BSp(1)$ to construct a map $f : BSp(1) \to BSp(n)$ of degree $\{m(i)\}$, where each $m(i)$, $i = 1, \ldots, n$, is odd. We also compute the degrees of some maps induced from representations $Sp(1) \to Sp(n)$.

In Section 2 we use symplectic K-theory to put mod 2 restrictions on the possible degrees of maps.

In Section 4 we will give the analogue of 2.1. for $H = SU(n)$, Spin (n) or G_2. In Chapter 3 we make a conjecture on what the analogue should be for maps $BG \to BH$, where G is any compact, connected simply connected simple Lie group.

Section 2.1. Homomorphisms $H^*(BSp(n)) \to H^*(BSp(1))$.

We need the following result from number theory : "If a polynomial in one variable with integer coefficients factors into linear factors modulo every large prime p then it factors into linear factors over the integers." See [6].

If the polynomial is of degree 2, this result is Theorem R from the introduction to Chapter 1.

We will prove Theorem 2.1 by giving necessary and sufficient conditions for graded ring homomorphisms $H^*(BSp(n)) \to H^*(BSp(1))$ to commute with P^1 for all large primes p. The idea behind the proof is that mod large p, f^* has the stated form when we take coefficients in some extension of Z_p. The naturality of P^1 then tells us that f^* has the stated form with Z_p coefficients. A use of the above number-theory result knits this mod p information together to give f^* the stated form over Z.

Theorem 2.1. is a corollary of

Proposition 2.2 A graded ring homomorphism $h : H^*(BSp(n)) \to H^*(BSp(1))$ commutes with P^1 for all large p iff there exist integers $m(1), \ldots, m(n)$, such that $hx(j) = a(j)x^j$, $a(j) = e_j(m(i)^2)$.

Note The proof of 2.1. follows by taking $h = f^*$.

We prove 2.2 by computing the action of P^1 on $H^*(BSp(n))$ and writing down the condition that h commutes with P^1.

To begin the proof, we introduce some notation.

(i) $S_k = \sum_{1\leq i\leq n} t(i)^{2k}$ (ii) $2s = p + 1$.

From one of the axioms for the Steenrod algebra, we have $P^1 t(i) = t(i)^p$ since $\dim t(i) = 2$. There is also the Cartan formula : $P^1 uv = uP^1 v + vP^1 u$, for $u, v \in H^*(BSp(n))$. Thus $\frac{1}{2}P^1 t(i)^2 = t(i)^{p+1}$.

Proposition 2.3. (i) $\frac{1}{2}P^1 S_k = kS_{s+k-1}$

(ii)
$$\frac{1}{2}P^1 x(i) = \sum_{1\leq j\leq i} (-1)^{j-1} x(i-j) S_{s+j-1}$$
$$= \sum_{1\leq j\leq n-i+1} (-1)^{j-1} x(i+j-1) S_{s-j}, \quad \text{where } x(0)=1.$$

Proof (i) The Cartan formula and linearity give

$$P^1 \sum_{1\leq i\leq n} t(i)^{2k} = 2k \sum t(i)^{2k-1+p}.$$

This is equivalent to $\frac{1}{2}P^1 S_k = kS_{s+k-1}$.

(ii) The Newton relation

$$S_r - x(1)S_{r-1} + \ldots + (-1)^i x(i) S_{r-i} + \ldots + (-1)^r r x(r) = 0, \quad S_0 = n, \qquad (2.4),$$

shows that the two given expressions are equivalent. We prove the first one by induction on i. It is true for $i = 1$, since $x(1) = S_1$.

Assume that it is true for i . Then

$$\tfrac{1}{2}P^1 t(1)^2 \ldots t(i+1)^2 = \sum_{1 \leq j \leq i+1} t(1)^2 \ldots t(j)^{2s} \ldots t(i+1)^2$$

Hence,

$$x(i)S_s = \tfrac{1}{2}P^1 x(i+1) + \sum \sum_j t(k_1)^2 \ldots t(k_j)^{p+3} \ldots t(k_i)^2 ,$$

where the first summation is over all sets $\{k_j\}$ with $1 \leq k_1 < \ldots < k_i \leq n$.

So by the induction hypothesis,

$$x(i)S_s = \tfrac{1}{2}P^1 x(i+1) + \sum_{1 \leq j \leq i} (-1)^{j-1} x(i-j) S_{s+j} .$$

This completes the induction.

If h is as in Proposition 2.2., then for dimensional reasons, $hx(i) = b(i)x^i$, $i = 1, \ldots, n$, where the $b(i)$ are integers and $x \in H^4(BSp(1),Z)$ is the generator.

If h commutes with P^1 ,

$$P^1 hx(i) = hP^1 x(i) , \quad i = 1, \ldots, n , \qquad (2.5) .$$

We lose no generality if we assume that $b(n) \neq 0$, for otherwise it is clear from the proof of 2.2. below, that we can work with the largest m such that $b(m) \neq 0$, and we would then be dealing essentially with a homomorphism $H^*(BSp(m)) \to H^*(BSp(1))$.

Assume henceforth that $\rho > \max \{ |b(n)|, 2, n! \}$.

When $i = n$, by using 2.3 , we see that (2.5) becomes,

$$nb(n)x^n S_{s-1} = b(n)x^n hS_{s-1} .$$

Hence $hS_{s-1} = nx^{s-1}$.

Recall from Chapter 1 that S_k is a polynomial, with integer coefficients, in the $x(i)$. By abuse of notation we also denote

$S_k(b(1), \ldots, b(n))$ by S_k . Then with this notation, we have proved that

$$S_{s-1} = n .$$

Lemma 2.6. $S_{s-1+k} = S_k \mod p$, $S_0 = n$.

Proof We have proved the lemma for $k = 0$. Assume as an induction hypothesis that $S_{s-1+j} = S_j$ for $0 \leqslant j \leqslant k$.

Then

$$hx(k+1) = b(k+1)x^{k+1} .$$

Hence from 2.3. and 2.5.,

$$(k+1)b(k+1)x^{k+s} = h(x(k)S_s - x(k-1)S_{s+1} + \ldots) .$$

Using the induction hypothesis we get

$$(k+1)b(k+1) = b(k)S_1 - b(k-1)S_2 + \ldots + (-1)^{k-1}b(1)S_k + (-1)^k S_{s+k} .$$

However, we have the "Newton" relation, (2.4) :

$$S_{k+1} - b(1)S_k + \ldots + (-1)^{k+1}(k+1)b(k+1) = 0 .$$

The lemma follows.

We have to "solve" the system of equations $S_{s-1+k} = S_k$ for $b(i)$. For this purpose, choose a finite extension K of Z_p in which

$$\underset{\sim}{x}^{2n} - b(1)\underset{\sim}{x}^{2n-2} + \ldots + (-1)^i b(i)\underset{\sim}{x}^{2n-2i} + \ldots + (-1)^n b(n) = 0 \quad (2.7) ,$$

has $2n$ roots, namely let K be the splitting field of (2.7) over Z_p .

In particular,

$$z^n - b(1).z^{n-1} + \ldots + (-1)^n b(n) \quad (2.8) ,$$

is a product of linear factors in $K[z]$, if we consider $b(i)$ as being reduced mod p .

If the roots of (2.8) are $r(i)$, $i = 1, \ldots, n$, and those of (2.7) $m(j)$, $j = 1, \ldots, 2n$, then by renumbering if necessary, we can arrange that

$r(i) = m(i)^2$, $i = 1, \ldots, n$. Then we have

$$b(j) = e_j(m(i)^2) , \quad m(i) \in K .$$

We show that (2.6) => $m(i) \in Z_p$.

Proposition 2.9. For each $i = 1, \ldots, n$, $m(i) \in Z_p$.

Proof Our assumption that $p > |b(n)|$ ensures that $m(i) \neq 0$ for $i = 1, \ldots, n$. By elementary Galois theory, [see for instance : "Algebra" by S. Lang, page 205] we know that

$$m(i) \in Z_p \text{ iff } m(i)^{p-1} = 1 .$$

Put $m(i)^{p-1} = 1 + u(i)$, $u(i) \in K$. Then

$$\sum_{1 \leq i \leq n} m(i)^{2k+p-1} = \sum m(i)^{2k}(u(i) + 1) , \quad k = 0, \ldots, (n-1) . \qquad (2.10)$$

But (2.6) says that $\sum m(i)^{2k+p-1} = \sum m(i)^{2k}$. Hence (2.10) gives

$$\sum_{1 \leq i \leq n} m(i)^{2k}u(i) = 0 , \quad k = 0, \ldots, (n-1) . \qquad (2.11)$$

Lemma 2.12. For each $i = 1, \ldots, n$, $u(i) = 0$.

Proof By induction on n . If $n = 1$, (2.11) becomes $m(1)^{2k}u(1) = 0$, $k = 0$. Hence $u(1) = 0$.

As an induction hypothesis assume that

$$\left\{ \sum_{1 \leq i \leq n-1} m(i)^{2k}w_i = 0 , \quad k = 0, \ldots, (n-2) , \quad m(i)^{p-1} = 1 + w_i , \quad w_i \in K \right\}$$

implies that $w_i = 0$ for $i = 1, \ldots, (n-1)$.

Treat (2.11) as a system of linear equations for $u(i)$. If one of the $u(i)$ is zero, we use the induction hypothesis to prove that the remaining $u(i)$ are also 0 .

Assume therefore that no $u(i)$ is 0 .

For a fixed n , consider the following statement:
{at least r of the $m(i)^2$ are equal} . (*). This (*) is true for $r = 1$.

Assume (*) true for r . Without loss of generality, we can in fact assume that the last r of the $m(i)^2$ are equal: $m(n-r+1)^2 = \dots = m(n)^2$.

Now put $v_n = ru(n)$ and $v_j = u(j)$, $j \leqslant n - r$. [This is where we need the assumption that $p > n!$]

Note that $m(i)^2 = m(j)^2 \Rightarrow m(i)^{p-1} = m(j)^{p-1}$, hence $u(i) = u(j)$, and so we have $u(n-r+1) = u(n-r+2) = \dots = u(n)$.

Hence (2.11) gives

$$\sum_{1 \leqslant i \leqslant n-r+1} m(i)^{2k} v_i = 0 \text{ , } k = 0, \dots, (n-r) \text{ .}$$

Since not all the v_j are zero, we must have $\det A = 0$, where $A_{ij} = m(i)^{2j}$, and this is a Vandermonde determinant:

$$\det A = \prod_{i<j} (m(i)^2 - m(j)^2) = 0 \text{ .}$$

Hence there exist distinct i and j with $m(i)^2 = m(j)^2$, so that $(r+1)$ of the $m(i)^2$ are equal . Hence by induction $m(1)^2 = m(2)^2 = \dots = m(n)^2$, consequently $u(1) = \dots = u(n)$, which implies that $n.u(1) = 0$, so <u>all</u> the $u(i)$ are 0 .

This completes the proof of 2.12., hence of 2.9.

Theorem 2.13. The polynomial (2.7) factors into linear factors over the integers.

Proof Proposition 2.9. tells us that (2.7) factors into linear factors mod p , for all large p . Hence 2.13 follows from the number theory result mentioned at the begining of the section: see [6, page 22 9] .

Proof of Proposition 2.2. From 2.13 we know that given an h , there are integers $m(i)$ such that

$$hx(j) = a(j)x^j, \quad a(j) = e_j(m(i)^2) .$$

Conversely, it is obvious from (2.6) that such a homomorphism commutes with P^1 for all odd primes p .

Section 2.2. Homomorphisms $KSp(BSp(n)) \to KSp(BSp(1))$.

Let $f : BSp(1) \to BSp(n)$ be a map . Then $f^* : H^*(BSp(n)) \to H^*(BSp(1))$ must have the form described in Proposition 2.2.

Recall the ψ^k operations in complex K-theory $KU^0(X)$. The fact that $f^! : KU^0(BSp(n)) \to KU^0(BSp(1))$ must commute with ψ^k for all k , gives no further restrictions on the possible form of f^* , but since we shall not use this result, we omit the proof .

To obtain further information on the integers $m(i)$, we use the representation theory of $Sp(n)$.

First we describe $KU^0(BSp(n))$.

Let $T \subset Sp(n)$ be the maximal torus. Then $KU^0(BT) \cong Z[[s(1) ,..., s(n)]]$, see [2, theorem 4.8] , and $KU^0(BSp(n))$ is isomorphic to the subring of Weyl group invariants, [2, Theorem 4.8. and Theorem 4.4] .

Put $Z(i) = 1 + s(i)$ so that $Z(i)$ is the canonical (virtual) line bundle over BS^1 , where S^1 is the group of complex numbers of unit modulus. The action of the Weyl group of $Sp(n)$ is to permute the $Z(i)$ and to invert: $Z(i) \to Z(i)^{-1}$. Hence

$$KU^0(BSp(n)) \cong Z[[y(1), \ldots, y(n)]], \quad y(j) = e_j(Z(i) + Z(i)^{-1} - 2) .$$

All this follows from the Atiyah-Hirzebruch results in [2] relating the complex representation ring, $R(G)$, of a compact connected Lie group G to $KU^0(BG)$.

$KU^0(BSp(1) \cong Z[[y(1)]]$ from above . Put $y = y(1)$.

Let $Ch : KU^0(\ ) \to \prod_m H^{2m}(\ , Q)$ be the Chern character, [2, section 1.10], and let Ch_{2m} be the m th component . Ch is a natural ring homomorphism .

If $\{x(i)\}$ is the set of generators of $H^*(BSp(n))$ defined in Section 2.1. then $Ch_{4i}y(i) = x(i)$. This is shown for instance in the proof of 2.17. below. The first non-zero component of $Ch\,y(i)$, $Ch : KU^0(BSp(n)) \to \prod_m H^{2m}(BSp(n), Q)$, is $Ch_{4i}y(i)$.

Recall from Proposition 2.2. that $f^*x(i) = a(i)x^i$.

<u>Lemma 2.14.</u> With the above notation for generators,

$$f^!y(i) = a(i)y^i + y^{i+1}Y \quad \text{for some } Y \in KU^0(BSp(1)) .$$

<u>Proof</u> Clearly $f^!$ has the form

$f^!y(i) = a(i)'y^i + y^{i+1}Y$, $a(i)'$ an integer and $Y \in KU^0(BSp(1))$.

By the naturality of Ch , $Chf^!y(i) = f^*Ch\ y(i)$. Equate coefficients of x^i in this equation :

$$a(i)' x^i = f^* x(i) = a(i) x^i . \quad \text{Hence } a'(i) = a(i) .$$

Note 2.15. General references for the structure of $KU^0(BG)$ are [3] and [2] .

To get restrictions on the $m(i)$, we will need to compute $f^! y(1)$, and for this purpose we describe the relation between $R(G)$ and $KU^0(BG)$ in more detail .

In [2], page 29, an isomorphism is given: $\hat{\alpha} : \widehat{R(G)} \to KU^0(BG)$, where $\widehat{R(G)}$ is the completion of $R(G)$ under the augmentation topology. Again, in section 4.7. of [2] , there is a monomorphism $R(G) \to \widehat{R(G)}$ and a monomorphism $\alpha : R(G) \to KU^0(BG)$.

If Sp and U are the "big" symplectic and unitary groups, the standard inclusion $Sp \subset U$ defines a transformation $i : KSp^*(\ ) \to KU^*(\ )$ of group valued functors, where $KSp^*(\ )$ is the symplectic K - theory functor. An element of $KU^0(BSp(n))$ is called symplectic if it is in the image of i .

Now $y(1)$ is in the image of a symplectic representation under α , and so is symplectic. Consequently $f^! y(1)$ is symplectic . Our restrictions on the $m(i)$ arise from this fact.

Lemma 2.16. The subgroup of symplectic elements in $KU^0(BSp(1))$ is generated by $1, y, 2y^2, \ldots, y^{2i-1}, 2y^{2i}, \ldots$.

Proof Since an element of $KU^0(BSp(1))$ is a (formal) power series in y , we have to decide which monomials in y are symplectic.

Since y is symplectic, so is y^{2i-1} for $i \geq 1$. Since y^{2i} is self conjugate, $2y^{2i}$ is symplectic. Finally we observe that y^{2i} is not symplectic. A proof of this fact can be based on page 144 of [11] .

So if $f^! y(1) = \sum_{r\geq 1} \alpha(r) y^r$, $\alpha(2r)$ must be even. We note that by 2.14., $\alpha(1) = a(1) = \sum_{1\leq i\leq n} m(i)^2$.

<u>Theorem 2.17</u> $\alpha(r) = \sum_{1\leq i\leq n} \frac{m(i)}{r}\binom{m(i) + r - 1}{2r - 1}$, where ( ) is the binomial coefficient.

<u>Cor 2.18</u> $\sum_{1\leq i\leq n} \frac{m(i)}{2r}\binom{m(i) + 2r - 1}{4r - 1}$ is even.

<u>Proof</u> This is just the condition that $\alpha(2r)$ is even, and puts 2-primary restrictions on the $m(i)$ as we shall see below.

The proof of 2.17 requires the

<u>Proposition 2.19.</u> Theorem 2.17 is true for $n = 1$ i.e.

$$\alpha(r) = \frac{m}{r}\binom{m + r - 1}{2r - 1} \qquad m = m(1) .$$

<u>Proof</u> We have $f : BSp(1) \to BSp(1)$ with $f^! y = m^2 y + \sum \alpha(r) y^r$. Now $\psi^2 : KU^0(BSp(1)) \to KU^0(BSp(1))$ is easily computed, since $\psi^2 Z(i) = Z(i)^2$ and so $\psi^2 y = 4y + y^2$.

The naturality, $\psi^2 f^! y = f^! \psi^2 y$, of ψ^2 gives

$$m^2(4y + y^2) + \sum_{i\geq 2} \alpha(i)\,(4y + y^2)^i = 4f^! y + (f^! y)^2 \qquad (*) .$$

One can calculate the $\alpha(i)$ inductively by equating coefficients in $(*)$.

We know a priori that $f^{!} = \psi^m$, so computing the $\alpha(r)$ amounts to writing $Z^m + Z^{-m} - 2$ as a polynomial in $Z + Z^{-1} - 2$.

<u>Proof of 2.17.</u> Consider the Chern character Ch. On $KU^0(BT)$ this can be defined on generators by $\mathrm{Ch}\, Z(i) = e^{t(i)}$ where $t(i)$ is a generator of $H^2(BT)$ and $e^{t(i)} = 1 + t(i) + \dots + \frac{t(i)^j}{j!} + \dots$.

Since Ch is a ring homomorphism, we can make the following calculations.

$$\mathrm{Ch}\, y(1) = \mathrm{Ch} \sum (Z(i) + Z(i)^{-1} - 2)$$
$$= \sum_{1 \le i \le n} (e^{t(i)} + e^{-t(i)} - 2) \text{ , so}$$

$$\mathrm{Ch}\, y(1) = 2 \sum_r \frac{Sr}{(2r)!} \text{ , where } S_r = \sum_{1 \le i \le n} t(i)^{2r} .$$

It is easy to see that $f^* S_r = A(r) x^r$ where $A(r) = \sum_i m(i)^{2r}$ and $x \in H^*(BSp(1))$ is the usual generator.

Hence $f^* \mathrm{Ch}\, y(1) = 2 \sum_{r \ge 1} \frac{A(r) x^r}{(2r)!}$ and

$$\mathrm{Ch}\, f^{!}_{\circ}\, y(1) = \sum_{r \ge 1} \alpha(r) (e^t + e^{-t} - 2)^r \text{ where } t^2 = x .$$

From the special case $n = 1$ in 2.19, we have

$$2 \sum_r m(i)^{2r} \frac{x^r}{(2r)!} = \sum_r \frac{m(i)}{r} \binom{m(i) + r - 1}{2r - 1} (e^t + e^{-t} - 2)^r$$

for $i = 1, 2, \dots, n$.

If we combine this with $f^* \mathrm{Ch}\, y(1) = \mathrm{Ch}\, f^! y(1)$ we get

$$2 \sum_r \frac{A(r)x^r}{(2r)!} = \sum_r \left[\sum_i \frac{m(i)}{r} \binom{m(i)+r-1}{2r-1} \right] (e^t + e^{-t} - 2)^r$$

$$= \sum_r \alpha(r)\,(e^t + e^{-t} - 2)^r .$$

Hence $$\alpha(r) = \sum_{1 \leqslant i \leqslant n} \frac{m(i)}{r} \binom{m(i)+r-1}{2r-1} .$$

We now come to exactly what restrictions the condition that $\alpha(2r)$ must be even puts on the $m(i)$. First a lemma and definitions.

Lemma 2.20. For any integers m and n, and a prime p let $m = \sum a_i p^i$, $n = \sum b_j p^j$ be their p-adic expansions with $0 \leqslant a_i, b_j \leqslant p-1$. Then

$$\binom{m}{n} = \prod_i \binom{a_i}{b_i} \mod p .$$

Proof See [15, page 5]. We need the lemma only for the case $p = 2$.

Definition 2.21. (i) For any integer m, write $m = 2^s n'$, n' odd and define $\beta(m) = s$.

(ii) Divide the set $\{m(1), \dots, m(n)\}$ into disjoint subsets $I_0, I_1, \dots$, such that if $a, b \in I_s$ then $\beta(a) = \beta(b) = s$.

(iii) In the factorisation of (2.8) consider the factor $(z - m(i')^2)^{d(i')}$, $d(i') \geqslant 1$. Assume that I_s contains the distinct elements $m(j_1), \dots, m(j_i)$ and define Card I_s to be $d(j_1) + \dots + d(j_i)$. Note that under this definition, $\sum_s \mathrm{Card}\, I_s = n$.

(iv) Write $C_i(r) = \frac{m(i)}{r}\binom{m(i)+r-1}{2r-1}$.

<u>Proposition 2.22.</u> (i) $C_i(r) = \frac{2m(i)}{m(i)+r}\binom{m(i)+r}{2r}$.

(ii) If $\beta(r) = s$ and $\beta(m(i)) \neq s+1$, then $C_i(2r)$ is even .

<u>Proof</u> (i) $C_i(r) = \frac{m(i)}{r}\frac{(m(i)+r-1)!}{(2r-1)!\,(m(i)-r)!} = \frac{2m(i)}{(m(i)+r)}\frac{(m(i)+r)!}{(2r)!\,(m(i)-r)!}$.

(ii) Note that $\beta(mn) = \beta(m) + \beta(n)$. Hence

$$\begin{aligned}\beta(m(i)/(m(i)+2r)) &= \beta(m(i)) - \beta(m(i)+2r)\\ &= \beta(m(i)) - (s+1) \quad \geqslant 0 \text{ if } \beta(m(i)) > s+1\\ &= \beta(m(i)) - \beta(m(i)) \quad \geqslant 0 \quad .. \quad .. \quad < s+1 .\end{aligned}$$

Combining this with (i) gives (ii) .

We can now state what restrictions symplectic K-theory puts on the $m(i)$. With the above notation,

<u>Theorem 2.23.</u> (i) If I_s is not empty, then $s > 0$ implies that Card I_s is even .

(ii) Again let $s > 0$, and let the elements of I_s for which d( ) is odd be the first $2t$ of the m's , [there has to be an even number of such m's by (i)] $m(1), \ldots, m(2t)$. Then there exist integers w_i and C_i with $C_i = 0$ or 1 such that

$$\begin{aligned}m(2i) &= 2^s(1 + 4w_i + 2C_{2i})\\ m(2i-1) &= 2^s(1 + 4w_i + 2C_{2i-1}) , \quad \text{for } i = 1, \ldots, t .\end{aligned}$$

Cor 2.24. If all the $m(i)^2$ are equal to m^2 say, then

(i) n odd implies that m is odd or zero

(ii) n even implies that $\alpha(2r)$ is even.

Proof (i) We are given that $m \in I_s$ for some $s \geqslant 0$ and Card $I_s = n$.
If $s > 0$, 2.23 (i) tells us that n is even.
(ii) This is obvious since $C_i(2r) = C(2r)$, say, and $\alpha(2r) = n\,C(2r)$.

Notes (a) When $n = 1$, part (i) of the corollary has been known for several years. See "Proceeding of a conference on algebraic topology", University of Illinois at Chicago circle, 1968, page 293, conjecture 38.
(b) It is clear from 2.24 (ii) that in Theorem 2.23 (ii), we cannot get any information on those $m(i)$ for which $d(i)$ is even.
(c) There is a precise formula for C_i and w_i given below in terms of the 2-adic expansions of the $m(i)$.
(d) With 2.23. and 2.24., we have a necessary and sufficient condition for $\alpha(2r)$ to be even.

Proof of 2.23. (i) First, we may assume that the distinct elements in I_s are the first t' out of $m(1), m(2), \ldots, m(n)$.

Write $m(i)$ as

$$m(i) = \sum_{u \geqslant 0} a_{iu} 2^{u+s}, \quad a_{i0} = 1, \quad a_{iu} = 0 \text{ or } 1 \text{ and } 1 \leqslant i \leqslant t', \text{ so that}$$

$m(i) \in I_s$.

Let $r = 2^{s-1} + b(1)2^s + b(2)2^{s+1} + \ldots$. Then 2.22. implies that $C_j(2r)$ is even if $m(j) \notin I_s$, and hence 2.18 becomes

$$\sum_{1 \leqslant i \leqslant t'} d(i) C_i(2r) = 0 \mod 2.$$

Since $\beta(m(i)/2r) = 0$, we see that

$$\sum_i d(i)\binom{m(i) + 2r - 1}{4r - 1} = 0 \mod 2 .$$

From lemma 2.20., we have

$$\binom{m(i) + 2r-1}{4r - 1} = \prod_{j\geq 2} \binom{b(j) + a_{ij}}{b(j - 1)} \mod 2 .$$

If we choose $b(j) = 0$ for each j , all the binomial coefficients in the above line become 1 , hence

$$\sum_i d(i) = 0 \mod 2 .$$

This proves (i) , since the left hand side is Card I_s .

(ii). Again we assume for the sake of notational simplicity that the $m(i)$ are the first $2t$ out of $m(1) ,\ldots, m(n)$.

From the proof of (i) , it is clear that the information we have is

$$\sum_{1\leq i\leq 2t} d(i)a_{ik_1} \ldots a_{ik_r} = 0 , \quad r > 0 , \quad k_1 < k_2 < \ldots < k_r .$$

Since we are assuming that the $d(i)$ are odd, this becomes

$$\sum_{1\leq i\leq 2t} a_{ik_1} \ldots a_{ik_r} = 0 \mod 2 , \quad 2 \leq k_1 < \ldots < k_r , \quad r \geq 1 . \qquad (**)$$

Notice that this does not involve a_{i1} . When $t = 1$, take $r = 1$ in (**) to get, $a_{1u} = a_{2u}$ for all $u > 1$.

Define $w_1 = \sum_{u\geq 2} a_{1u}2^{u+s}$, and $C_1 = a_{11}, C_2 = a_{21}$, and we have

$$m_1 = 2^s(1 + 2C_1 + 4w_1), \; m_2 = 2^s(1 + 2C_2 + 4w_1) .$$

In general, to solve the system (**), we need the following:

Lemma.

Consider the following system of equations over Z_2 :

$$\sum_{1\leq i\leq 2t} a_{i,k_1} \cdots a_{i,k_r} = 0 \quad (**) \quad 2 \leq k_1 < \dots < k_r , \quad r \geq 1 .$$

This system is satisfied iff the $a_{,k}$ are equal in pairs i.e. for each i , $1 \leq i \leq 2t$, there is an i' , $i' \neq i$, such that

$$a_{i,k} = a_{i',k} \text{ for all } k \geq 2 .$$

Proof (i). Obviously, if $a_{i,k} = a_{i',k}$, the system is satisfied.
(ii). We solve (**) by induction on t . The system has been solved when $t = 1$.

Assume that the solution has the stated form for systems,

$$\sum_{1\leq i\leq 2t''} a'_{i,k_1} \cdots a'_{i,k_r} = 0 , \quad t'' < t , \quad 2 \leq k_1 < \dots < k_r .$$

If in (**) the a's are all 0 or all 1 , we are finished. Assume therefore that not all the $a_{i,2}$, for instance, are equal . Without loss of generality, we can assume in fact that

$$a_{1,2} = \dots = a_{2q,2} = 1 , \quad a_{2q+1,2} = 0 = \dots = a_{2t,2} = 0 , \quad \text{for some } q \leq t-1 .$$

In (**) if we take $k_1 = 2$, we get

$$\sum_{1\leq i\leq 2q} a_{i,k_2} \cdots a_{i,k_r} = 0 , \quad 3 \leq k_2 < \dots < k_r .$$

By the induction hypothesis, for each i , there is an i', $\neq i$, with $1 \leq i, i' \leq 2q$, such that

$$a_{i,k} = a_{i',k} , \text{ for all } k \geq 3 .$$

Putting this information into (**) reduces the system to,

$\sum_{2q+1 \leq i \leq 2t} a_{i,k_2} \dots a_{i,k_r} = 0$, $3 \leq k_2 < \dots < k_r$ and again by the induction hypothesis applied to this system, we get: for each i , there is an i', $\neq i$, $2q + 1 \leq i , i' \leq 2t$ such that

$$a_{i,k} = a_{i',k} \text{ for each } k \geq 3 .$$

So finally, for each i , there is an i' , $1 \leq i , i' \leq 2t$, such that $a_{i,k} = a_{i',k}$ for all $k \geq 2$. This completes the proof of the lemma .

To complete the proof of 2.23 (ii), using the lemma just proved, we can renumber the $m(i)$ so that for each i , $1 \leq i \leq t$,

$$a_{2i,k} = a_{2i-1,k} \text{ for all } k \geq 2 .$$

We can define C_i to be a_{i1} , $1 \leq i \leq 2t$ and

$$w_j \text{ to be } \sum_{u \geq 2} a_{2j-1,u} 2^{u+s} , \quad 1 \leq j \leq t .$$

Note It is easy to see that

$\frac{m}{r} \binom{m + r - 1}{2r - 1} = 2m^2(m^2 - 1) \dots (m^2 - (r - 1)^2)/(2r)!$, so we don't need to worry about the signs of the $m(i)$.

Section 2.3. Construction of maps $BSp(1) \to BSp(n)$.

In this section we realise geometrically those maps whose degrees contain only odd integers, and also compute the degrees of some maps induced from representations. First, some notation.

Let H denote the quaternions and $M_n(H)$ the ring of $n \times n$ matrices, with entries in H. If $A \in M_n(H)$ denote by $\bar{A}$ the quaternion conjugate i.e. $(\bar{A})_{ij} = \bar{A}_{ij}$, where the second bar denotes quaternion conjugation. With this notation, $Sp(n) = \{A \in M_n(H) \mid A^t \bar{A} = I\}$, where I is the $n \times n$ identity matrix, and A^t the tranpose of A.

Let $T \subset Sp(n)$ be a quaternionic torus i.e.

$T = \{A \in M_n(H) \mid A_{ij} = 0\ ,\ i \neq j\ ,\ A_{ii}\bar{A}_{ii} = 1\}$, so that $T = Sp(1) \times \dots \times Sp(1)$, n factors.

<u>Theorem 2.25.</u> If $\{m(1), \dots, m(n)\}$ is a sequence of odd integers, there is a map $f : BSp(1) \to BSp(n)$ of degree $\{m(1), \dots, m(n)\}$.

<u>Proof</u> Let $f(m) : BSp(1) \to BSp(1)$ be a map of odd degree m (m^2 in Sullivan's sense) as constructed in [17, Corollary 5.10].

By [14], $BT \simeq BSp(1) \times \dots \times BSp(1)$. Hence we can define

$$\prod = \prod_{1 \leq i \leq n} f(m(i)) : BT \to BT$$ to be the cartesian product.

Define $f : BSp(1) \to BSp(n)$ to be the composite,

$BSp(1) \xrightarrow{\Delta} BT \xrightarrow{\Pi} BT \xrightarrow{Bi} BSp(n)$, where Δ is the diagonal and i the [illegible] the inclusion $T \to Sp(n)$.

It is clear that f has degree $\{m(1), \dots, m(n)\}$.

We now come to maps induced, by Lie group homomorphisms, $Sp(1) \to Sp(n)$, in cohomology.

The only Lie group maps $Sp(1) \to Sp(1)$ are isomorphisms, or constants.

To describe representations of $Sp(n)$, it is useful to have the following alternative description of $Sp(n)$:

$$Sp(n) = \left\{ A \in GL(2n, \mathbb{C}) \,\middle|\, \bar{A}A^t = I \;,\; A^t \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} A = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \right\},$$ see [7, page 21].

The (virtual) complex representation ring of $Sp(1)$ is $RSp(1) \cong \mathbb{Z}[\alpha]$, where $\alpha : Sp(1) \to Sp(1)$ is the identity, see [1], last chapter. The tensor power α^{2r+1} is symplectic and we want to determine the action of $B\alpha^{2r+1}$ in cohomology. For $r \geq 0$, α^{2r+1} is a homomorphism $Sp(1) \to Sp(2^{2r})$.

<u>Proposition 2.26.</u> Let A be the diagonal matrix $\mathrm{diag}\,(z,\bar{z})$ in $Sp(1)$, $z\bar{z} = 1$, so that A is in a maximal (complex) torus. Then

$$\alpha^{2r+1}A = \mathrm{diag}(z^{2r+1}, z^{2r-1}, \ldots, z^{2r-3}, \ldots, \bar{z}^{2r-3}, \ldots, \bar{z}^{2r-1}, \ldots, \bar{z}^{2r+1}),$$

where the number of entries $z^{2(r-i)+1}$ (or $\bar{z}^{2(r-i)+1}$, since there are equal numbers of them) is $\binom{2r+1}{i}$, $0 \leq i \leq r$.

<u>Proof</u> $\alpha^{2r+1}A$ is the $(2r+1)$ - st tensor power of A, call it A_{2r+1}. This is defined inductively by :

$$A_1 = A \;, \quad A_r = \begin{bmatrix} zA_{r-1} & 0 \\ 0 & \bar{z}A_{r-1} \end{bmatrix}, \quad r \geq 2 .$$

The number of entries of the form $z^{2(r-i)+1}$ is easily calculated: one uses the relation $(1+1)^{2r+1} = 2 \sum_{0\leq i\leq r} \binom{2r+1}{i}$.

<u>Proposition 2.27.</u> The integers $m(i)$ corresponding to $(B\alpha^{2r+1})^* : H^*(BSp(2^{2r})) \to H^*(BSp(1))$ are as follows:

$$m(k) = 2i + 1 \ , \ \sum_{0 \leqslant j < i} \binom{2r+1}{r-j} < k \leqslant \sum_{0 \leqslant j \leqslant i} \binom{2r+1}{r-j} \ , \quad 1 \leqslant i \leqslant r$$

$$m(k) = 1 \ , \quad 1 \leqslant k \leqslant \binom{2r+1}{r} \ .$$

Proof In 2.26., we computed the action of α^{2r+1} in the maximal torus of $Sp(1)$. The integers $m(i)$ are the exponents of Z.

To determine the action of a sum of representations, note that if α, β are two representations of $Sp(1)$, then

$$(\alpha + \beta)g = \begin{bmatrix} \alpha(g) & 0 \\ 0 & \beta(g) \end{bmatrix} \quad \text{for} \quad g \in Sp(1) \ .$$

Hence we can state,

Proposition 2.28. If $n\alpha$ is the sum of n copies of the identity representation of $Sp(1)$, then each integer in the degree of $Bn\alpha$ is 1.

Proof Under $n\alpha$, $\text{diag}(Z,\bar{Z})$ in $Sp(1)$ goes to $\text{diag}(Z,\bar{Z},\ldots,Z,\bar{Z})$ in $Sp(n)$ so the proposition follows.

From 2.28. and 2.27. we can compute the effect in cohomology, of any polynomial in α with non-negative integer coefficients.

It is interesting to note that all the maps we have constructed have only odd integers in their degrees. In the light of this, we make the following conjecture. Let $f : BSp(1) \to BSp(n)$ be a map and $x(n) \in H^{4n}(BSp(n))$ the usual generator. Then if $f^*x(n) \neq 0$, each integer in the degree of f is odd.

The requirement that $f^*x(n) \neq 0$ is essential, otherwise the degree may have even integers in it : see 3.19. in Chapter 3 .

Note that this cannot be proved using symplectic K - theory, with the methods we have used. See the note (b) after Cor. 2.24.

Section 2.4. Homomorphisms $H^*(BH) \to H^*(BSp(1))$.

In this section we give a cohomological classification of maps $BSp(1) \to BH$ for some groups H other than $Sp(n)$. So let $f : BSp(1) \to BH$ be a map and consider the following particular cases for H

(a) $H = SU(m)$, the special unitary group.

First we describe the cohomology of $BSU(m)$.

Let $T \subset SU(m)$ be the standard maximal torus, so that

$$H^*(BT, Z) \cong Z[t(1), \ldots, t(m)] \text{ where } \sum_{1 \leqslant i \leqslant m} t(i) = 0 .$$

Hence by Theorem W of Chapter 1 , $H^*(BSU(m)) \subset H^*(BT, Z)$ is the subring of Weyl group invariants. Since this group acts by permuting the $t(i)$, ([11 page 7 9], [4, last chapter])

$$H^*(BSU(m), Z) \cong Z[x(1), \ldots, x(m)], \quad x(i) = e_i(t(j)) , \quad x(1) = 0 .$$

Since the dimension of $x(m)$ is $2m$, $f^*x(m) = 0$, if m is odd, so we may as well assume that m is even, $= 2n$ and $f^*x(m) \neq 0$.

When $H = SU(m)$, it is convenient to regard $Sp(1)$ as $SU(2)$ and choose the generator of $H^4(BSU(2))$ accordingly i.e. if $S \subset SU(2)$ is the maximal torus, then $H^*(BS) \cong Z[s_1, s_2]$, $s_1 + s_2 = 0$ and

$$H (BSU(2), Z) \cong Z[x] , \quad x = s_1 s_2 .$$

For dimensional reasons, $f^*x(2i) = a(i)x^i$, and $f^* x(2i + 1) = 0$.

We have to determine the $a(i)$. Just as in Section 1, we first have to compute P^1 and then write down the condition that f^* should commute with P^1.

Proposition 2.29. $P^1x(i) = \sum_{j>0} (-1)^{j-1}x(i-j)S_{p+j-1}$, where $x(0) = 1$, $x(1) = 0$ and $S_k = \sum_{1\leq i\leq m} t(i)^k$.

Proof We need only know the Cartan formula $P^1xy = xP^1y + yP^1x$ for $x,y \in H^*(BSU(m))$ and the fact that $P^1t(i) = t(i)^p$

We can now state what form the $a(i)$ take.

Theorem 2.30. If $f : BSU(2) \to BSU(2n)$ is a map and $f^*x(2i) = a(i)x^i$, then there exist integers $m(1), \ldots, m(n)$, such that $a(i) = e_i(m(j)^2)$.

Proof First note that from 2.29, if $2s = p + 1$,

$$P^1x^i = 2i(-1)^{s-1}\,x^{s+i-1} \quad \text{and if } m = 2n,$$

$$P^1x(m) = x(m)S_{p-1}$$

so that

$$f^*P^1x(m) = a(n)x^nf^*S_{p-1},$$

$$P^1f^*x(m) = 2na(n)x^{s+n-1}(-1)^{s-1}.$$

Hence,

$$f^*S_{p-1} = 2n(-x)^{s-1}. \tag{2.31}$$

Now by Theorem C of Chapter 1 ,

$$S_{p-1+2k} = \sum_{||E|| = p-1+2k} (-1)^{|E|} \frac{(p-1+2k)(|E|-1)!}{E!} x^E ,$$

where if

$$E = (e_1, \dots, e_m), \; x^E = x(1)^{e_1} \dots x(m)^{e_m} .$$

Hence,

$$f^* S_{p-1+2k} = x^{s-1+k} \sum_{e_2+2e_4+\dots = s-1+k} \Big[(-1)^{|E|} \frac{(p-1+2k)(e_2+e_4+\dots+e_m-1)!}{e_2! \dots e_m!} a(1)^{e_2} a(2)^{e_4} \dots \Big] , \qquad (*) ,$$

since

$$f^* x(2i+1) = 0 .$$

Let

$$S'_{s-1+k} = \sum_{||F||=s-1+k} (-1)^{f_2+f_4+\dots} \frac{(s-1+k)(|F|-1)!}{F!} a^F . \qquad (**)$$

Now, $\Sigma f_i + \Sigma f_{2j} = s - 1 + k \mod 2$.

Hence by comparing $(*)$ and $(**)$, we have

$$f^* S_{p-1+2k} = 2(-x)^{s-1+k} S'_{s-1+k} .$$

In particular, if we take $k = 0$ and look at 2.31, we get $S'_{s-1} = n$.

Similarly, we prove that

$$S'_{s-1+k} = S'_k , \; k = 0, 1, \dots, (n-1) , \text{ with } S'_0 = n .$$

We are now formally in the same position as in Section 2.1. Namely we factor,

$$Z^{2n} - a(1)Z^{2n-2} + a(2)Z^{2n-4} + \dots + (-1)^n a(n)$$

in some extension of Z_p , then show the factoring to be in Z_p and finally, show that the polynomial factors over Z . The $m(i)$ are the roots of this polynomial.

Note that we have not used the fact that f^* comes from a geometric map, but merely that it commutes with P^1 .

We have made the central theme of this work, the conjecture to be formulated in Chapter 3 . For maps $BSU(2) \to BSU(m)$, the conjecture amounts to the following:

Corollary 2.32. Let S, T be maximal tori in $SU(2)$ and $SU(m)$, and let $w \in H^2(BS)$ be the generator so that $w = s_1 = - s_2$ in our previous notation. Then if $h' : H^*(BSU(m)) \to H^*(BSU(2))$ is a graded ring homomorphism which commutes with P^1 for all large p , there is an extension $h : H^*(BT) \to H^*(BS)$. In fact,

$$ht(2i - 1) = m(i)w \ , \quad ht(2i) = - m(i)w \ , \quad i = 1 , \ldots , n .$$

Proof We have to check that (i) $hx(2i + 1) = 0$ and (ii) $hx(2i) = a(i)x^i$. For (i) note that $hS_{2i+1} = \sum_j h\, t(j)^{2i+1} = 0$, so $hx(2i + 1) = 0$, for each $i \geq 0$.

The proof of (ii) again involves manipulating symmetric functions. Assume by induction that $hx(2k) = a(k)x^k$ for $k \leq i$. To start the induction note that $hS_2 = 2 \sum_{1 \leq j \leq n} m(j)^2 w^2 = 2a(1)w^2$.

Since $x^i = (-1)^i w^{2i}$ and $S_2 = - 2x(2)$, we have $hx(2) = a(1)x$.

For the inductive step we have $hS_{2i+2} = 2w^{2i+2} \sum_j m(j)^{2i+2}$,

and h applied to the Newton relation gives

$$hS_{2i+2} + a(1)xhS_{2i} + \dots + a(i)x^i hS_2 + (2i + 2)hx(2i + 2) = 0 .$$

Hence, if we define $S_i'' = \sum_j m(j)^{2i}$, we have

$$2(-1)^{i+1}x^{i+1}S''_{i+1} + 2a(1)x^{i+1}(-1)^i S_i'' + \dots + (2i+2)hx(2i + 2) = 0 .$$

Using a Newton relation again, this gives

$$\begin{aligned}(2i + 2)hx(2i + 2) &= 2(-1)^i x^{i+1}(S''_{i+1} - a(1)S_i'' + a(2)S''_{i-1} + \dots + \dots) \\ &= 2(-1)^i x^{i+1}[(a(1)S_i'' - \dots + (i+1)(-1)^i a(i+1)) - \\ &\quad a(1)S_i'' + a(2)S''_{i-1} + \dots] \\ &= 2(i + 1)a(i + 1)x^{i+1} .\end{aligned}$$

Hence $hx(2i + 2) = a(i + 1)x^{i+1}$. This completes the inductive step .

Next we consider the case $H = \mathrm{Spin}\ (m)$.

(b) $H = \mathrm{Spin}\ (m)$.

$H^*(B\mathrm{Spin}\ (m), Z)$ has only 2 - torsion, see [16, page 290] .

So $H^*(B\mathrm{Spin}\ (2n + 1),R) \cong H^*(BSp(n),R)$, if $R = Z[\frac{1}{2}]$.

Hence the classification of maps is the same as for $BSp(n)$, except that the $m(i)$ are in R .

If $m = 2n$, let $T \subset \mathrm{Spin}\ (2n)$ be a maximal torus, then $H^*(B\mathrm{Spin}\ (m), R)$ is isomorphic to the subring of Weyl group invariants in

$H^*(BT,R) = R[t(1),\dots,t(n)]$, by Theorem W .

Hence for the standard T,

$H^*(B\ Spin\ (2n),R) \cong R[x(1),\dots,x(n-1),z]$, $x(i) = e_i(t(j)^2)$

and

$$z = t(1)t(2)\dots t(n) .$$

To classify induced homomorphisms, note first that $P^1 z = zS_{s-1}$ where $S_k = \Sigma\, t(i)^{2k}$, $2s = p + 1$, and $P^1 x(i)$ is the same as for $Sp(n)$. Then we can state

Proposition 2.33. For any map $f : BSp(1) \to B\ Spin\ (2n)$, there exist elements $m(1),\dots,m(n) \in R$, such that if $f^* : H^*(B\ Spin\ (2n),R) \to H^*(BSp(1),R)$ then $f^* x(i) = a(i)x^i$, $f^* z = ax^{\frac{1}{2}n}$ where $a(i) = e_i(m(j)^2)$ and $a^2 = e_n(m(j)^2)$. [a = 0 if n is odd] .

Proof We have described the action of P^1 on $H^*(B\ Spin\ (2n),Z_p)$. From this we notice that we can copy the proof from the case $H = Sp(n)$.

If n is odd, a will have to be zero, and some $m(i) = 0$. The $m(i)$ are in R since $f^* x(i) = b(i)x^i$, $f^* z = bx^{\frac{n}{2}}$ for some $b, b(i) \in R$ and the $m(i)$ are roots of $t^{2n} - b(1)t^{2n-2} + \dots + (-1)^n b(n) = 0$, $b(n) = b^2$.

Finally we classify maps $f^* : H^*(BG_2,R) \to H^*(BSp(1),R)$ where G_2 is the exceptional group of rank 2 , and $R = \mathbb{Z}[\frac{1}{2}]$, ($H_*(G_2, \mathbb{Z})$ has 2 - torsion), so if $T \subset G_2$ is the maximal torus, $H^*(BG_2,R)$ will be isomorphic to the Weyl group invariants in $H^*(BT,R) \cong R[t(1),t(2),t(3)]$, $t(1) + t(2) + t(3) = 0$. [See Chapter 1.]

So, $H^*(BG_2,R) = R[y(1),y(2),y(3)]$, $y(j) = e_j(t(i)^2)$ and there is a relation $y(2) = \frac{1}{4} y(1)^2$ corresponding to $t(1) + t(2) + t(3) = 0$. For more information on G_2, see [5].

With this notation we can state

Proposition 2.34. Let $f : BSp(1) \to BG_2$ be a map. Then there are elements $m(1), m(2), m(3) \in R$, satisfying $m(1) \pm m(2) \pm m(3) = 0$ (for some choice of signs) such that $f^*: H^*(BG_2,R) \to H^*(BSp(1),R)$ has the form, $f^*y(i) = a(i)x^i$, $i = 1,2,3$, $a(i) = e_i(m(j)^2)$.

Proof The action of P^1 in $H^*(BG_2, \mathbb{Z}_p)$ is the same as in $H^*(BSp(3), Z_p)$ apart from the relation $y(2) = \frac{1}{4} y(1)^2$. Hence $f^*y(i) = a(i)x^i$ and we can find the elements $m(i) \in R$ with the stated properties, from the work on $Sp(3)$. The relation $y(2) = \frac{1}{4} y(1)^2$ gives, $a(2) = \frac{1}{4} a(1)^2$ and this is equivalent to $0 = m(1)^4 + m(2)^4 + m(3)^4 - 2a(2)$. The latter equals

$$(m(1) + m(2) + m(3))(m(1) + m(2) - m(3))(m(1) - m(2) + m(3))(m(1) - m(2) - m(3)).$$

By now it is clear that we seem to be getting the same sort of classification for maps $BSp(1) \to BH$. Using our methods, we have to work with a specific set of generators and this entails a separate calculation for each group. A technique which deals with all groups at once, is required.

We make conjectures on the cohomological classification of maps $BG \to BH$ in Chapter 3.

Chapter 3 . The maps $BG \to BH$.

In this chapter we discuss a cohomological description of the maps $BG \to BH$. Henceforth G and H will be compact, connected, simply connected simple Lie groups.

We first formulate the conjecture alluded to at the end of the last chapter.

Let $A(p)$ be the mod p Steenrod algebra.

Choose maximal tori T, S in G and H respectively.

Conjecture A' . Given any morphism $f : H^*(BH,Zp) \to H^*(BG,Zp)$ of graded rings and $A(p)$-modules, for p sufficiently large, then there is a morphism f' to make the following diagram commute :

$$\begin{array}{ccc} H^*(BS,Zp) & \xrightarrow{f'} & H^*(BT,Zp) \\ (Bi)^*\uparrow & & \uparrow(Bi)^* \\ H^*(BH,Zp) & \xrightarrow{f} & H^*(BG,Zp) \end{array}$$

where i is the appropriate inclusion.

Remarks (i) One would hope to be able to knit together the mod p information as in section 2.1.

(ii) Proposition 2.9 and the results in the last section of chapter 2 prove the conjecture when $G = Sp(1)$, and $H \neq F_4, E_6, E_7, E_8$, the exceptional groups.

We will illustrate the implications of the conjecture A' by discussing the maps $H^*(BSp(n)) \to H^*(BSp(r))$ in detail. For this purpose it is convenient to give another formulation of A' , which is equivalent to A' when $G = Sp(r)$, $H = Sp(n)$.

So let T,S be maximal tori in Sp(r) and Sp(n) respectively and let $\{y(i)\}$ (resp. $\{x(i)\}$) be the corresponding set of generators of $H^*(BSp(r))$ (resp. $H^*(BSp(n))$ defined in Chapter 2 . For brevity put $C_n = H^*(BSp(n),Z)$. We shall abuse notation by using the same symbol to denote mod p cohomology where convenient.

Let $f : C_n \to C_r$ be any morphism of graded rings and define g(i) by the formula

$$fx(i) = g(i)(y(1) ,\ldots, y(r)) .$$

Choose a transcendental, t over C_r and form the polynomial $F(t) = 1 - g(1)t + \ldots + (-1)^i g(i)t^i + \ldots + (-1)^n g(n)t^n$. Thus $F(t) \in C_r[t] \subset Z[t(1) ,\ldots, t(r), t]$.

<u>Assumption A.</u> If f is a morphism of graded rings and A(p) modules for a sufficiently large prime p , then F(t) factors into linear factors, over $\bar{Z}_p[t(1) ,\ldots, t(r)]$, as

$\prod_{1\leq i\leq n} (1 - th(i))$ where $\bar{Z}_p$ is the algebraic closure of Z_p .

With this particular set of generators for C_n and C_r , if A' is true, so is A since the linear factors of F(t) will be $(1 - f's(i)^2 t)$ where s(i) is a generator of $H^2(BS)$. Also A implies A' : see 3.9 below , and 3.10.

At the moment we are unable to prove A' in complete generality, but we will give the proof in special cases, essentially when G and H are "about the same size" . For a fuller statement of what can be proved, see the end of Chapter 4 .

We also make the following conjecture, which we take to be a homotopy version of A' .

<u>Conjecture B</u>. For any map $g : BG \to BH$, there is a map $\bar{g} : BT \to BS$ to make the following diagram homotopy commute:

$$\begin{array}{ccc} BG & \xrightarrow{g} & BH \\ \uparrow & & \uparrow \\ BT & \xrightarrow{\bar{g}} & BS \end{array} .$$

A summary of the chapter follows.

In section 3.1, we shall construct many maps between cohomology rings which commute with P^1 for all large p . The statement of A amounts to saying that these are <u>all</u> the maps which will commute with P^1 .

In section 3.2, we realize some cohomology maps geometrically, and in section 3.3 we prove that if $f : H^*(BH) \to H^*(BG)$ is a morphism commuting with P^1 for all large p and G is "bigger" than H , then $f = 0$. The term "bigger" is explained there.

<u>Section 3.1</u>. Morphisms $C_n \to C_r$.

Our programme will be to obtain a complete list of morphisms $C_n \to C_r$ which commute with P^1 , under the assumption A . The generalization to other groups is mentioned in Chapter 4.

Recall that for a graded ring morphism $f : C_n \to C_r$ we defined

$fx(i) = g(i)(y(1), \ldots, y(r))$. If f satisfies assumption A , we can identify some terms in the polynomials $g(i)$. For example,

Lemma 3.1. If we take integer coefficients and the coefficient of the monomial $y(1)^i$ in $g(i)$ is $a(i)$ then there exist integers $m(j)$ such that $a(i) = e_i(m(j)^2)$.

Proof We have $g(i) = a(i)y(1)^i + \ldots$. In $P^1fx(i) = fP^1x(i)$ equate coefficients of $y(1)^{s-1+i}$. We are then essentially dealing with morphisms, $C_n \to C_1$. Now use Proposition 2.2.

Assume henceforth that $fx(n) \neq 0$, otherwise it will be clear that we could work with the largest n' such that $fx(n') \neq 0$. Assume also that $p > \max\{m(i)^2\}$.

With $\bar{Z}_p$ coefficients, we have

$$F(t) = \prod_{1 \leq i \leq n} (1 - h(i)t) . \qquad (3.2)$$

Hence $g(i)(t(1)^2, \ldots, t(r)^2) = e_i(h(j))$, a symmetric function in the $t(i)^2$ of degree i (in the $t(i)^2$) .

The fact that f commutes with P^1 enables us to prove that Assumption A => Conjecture A' , see lemma 3.7. To begin the proof of this lemma we need,

Lemma 3.3. Each $h(i)$ is a quadratic form over $\bar{Z}_p$ in the $t(i)$.

Proof First, $\sum_i h(i)^j$ is for each j a polynomial in the

$e_k(h(i))$, hence a polynomial in the $g(i)$, so homogeneous of degree $2k$ in the $t(i)$, i.e.

$$\sum_{1\leq i\leq n} h(i)^j = \text{homogeneous polynomial in the } t(i)^2 \text{, of degree } 2j . \qquad (*)$$

<u>Step 1</u>. Let $h(i) = k(i) + \ell(i,1)t(1) + \dots + \ell(i,r)t(r)$ + higher degree terms, where the k's and ℓ's are in $\bar{Z}_p$.

Equate constants in $(*)$:

$$\sum_i k(i)^j = 0 \quad \text{for } j = 1,2,\dots, n .$$

Hence $k(i) = 0$ for all i. One way to see this is to note that each elementary symmetric function of the $k(i)$ must be 0. Hence the $k(i)$ are roots of the polynomial with all but the leading coefficient zero.

Next, equate coefficients of $t(q)^j$ in $(*)$:

$$\sum_i \ell(i,q)^j = 0 \quad \text{for all } j .$$

Hence $\ell(i,q) = 0$ for each i and q.

We prove that $h(i)$ contains no terms of degree three or higher.

<u>Step 2</u>. Write all monomials in the t's in the form $t(r)^{e_r} \dots t(1)^{e_1}$. Order them as follows:

$$t(r)^{e_r} \dots t(1)^{e_1} > t(r)^{f_r} \dots t(1)^{f_1}, \text{ if } e_r = f_r ,\dots, e_i = f_i$$

and $e_{i-1} > f_{i-1}$ for some $i \geq 2$.

For the sake of notational simplicity, drop references to the index i

for the moment. Let W' be the largest monomial of degree ≥ 3, which occurs in any h. Thus

$W'=Wt(j_1)^{e_1}\dots t(j_s)^{e_s}$, $j_1>\dots>j_s$, where if $W\neq 1$, all the t's in W are larger than $t(j_1)$, so that if any monomial, M, of degree equal to $\deg W'$ is divisible by $t(j_1)^{f_1}$, then $f_1<e_1$, unless $M=W'$.

We show that the coefficient of W' in $h(i)$ is 0.

Look at the coefficient of $W^j t(j_1)^{je_1}\dots t(j_s)^{je_s}$ in $h(i)^j$ as i varies. Such a coefficient can arise only from $(W')^j$ since in any case it comes only from monomials of degree equal to $\deg W'$ and all these except W' have $f_1<e_1$. Hence they cannot contribute the factor $W^j t(j_1)^{je_1}$.

Let the coefficient of W' in $h(i)$ be β_i.

Then equating coefficients of $(W')^j$ in (*) gives

$$\sum_{1\leq i\leq n}\beta_i^j=0 \quad \text{for } j=1,\dots,n.$$

Hence $\beta_i=0$ for each $i=1,\dots,n$.

We assumed that $\beta_i\neq 0$ for some i. This contradiction shows that there is no monomial of degree ≥ 3 in any $h(i)$.

This completes the proof of lemma 3.3.

We will prove that each $h(j)$ is a square, in lemma 3.7. Note the following:

(i) By the factoring of $F(t)$, $f's(j)^2$ is defined and $fS_i=\sum_{j=1}^{n} h(j)^i$.

(ii) The conclusion (and the proof) of lemma 3.11a is valid if Z is replaced by $\bar{Z}_p$, namely if $w\in W(r)$, and $1\leq i\leq n$, then $wh(i)=h(j)$ for some j.

(iii) Since $fx(n)\neq 0$, no $h(j)$ is zero.

(3.4). Assume that in each $h(j)$ the coefficient of some $t(k)^2$ is not 0.

We prove (3.4) in the course of proving 3.7. It then follows from lemma 3.11a, by the above remark, that given k the coefficient of $t(k)^2$ in some $h(j)$ is $\neq 0$.

Our aim is to show that $h(i) = (a(i1)t(1) + \dots + a(ir)t(r))^2$, and since this requires the assumption that f commutes with P^1 we first indicate how this information is to be used.

Recall the following identity :

$$\log(1 - x(1)t + \dots + (-1)^i x(i)t^i + \dots) = - \sum_i S_i t^i/i \qquad (3.4)$$

where $S_i = \sum_j t(j)^{2i}$.

This identity can be proved by noting that the left hand side is

$$\log \prod_j (1 - t(j)^2 t) = - \sum_j \sum_{i\geq 1} t(j)^{2i} t^i/i$$
$$= - \sum_i S_i t^i/i .$$

Apply f to (3.4) :

$$- \sum_{i\geq 1} fS_i t^i/i = \log(1 - fx(1)t + \dots)$$
$$= \log \prod_j (1 - h(j)t)$$
$$= \sum_j \log(1 - h(j)t) .$$

$$\text{Hence } fS_i = \sum_{1\leq j\leq n} h(j)^i .$$

Since $P^1 S_i = 2iS_{i+s-1}$, $2s = p + 1$, the equation $P^1 fS_i = fP^1 S_i$ gives

$$P^1 fS_i = 2ifS_{i+s-1} = 2i \sum_j h(j)^{i+s-1} . \qquad (3.5)$$

Now if p is large enough, for each i , $1 \leq i \leq n$, we can express fS_i as a polynomial in the S_j , so that

$$fS_i = \alpha(i)S_1^i + \text{other monomials, for some } \alpha(i) \in Z_p .$$

Hence

$P^1 fS_i = 2i\alpha(i)S_1^{i-1}S_s + \dots$, and we arrive at

$$2i\alpha(i)S_1^{i-1}S_s + \dots + \dots = 2i \sum_{1 \leqslant j \leqslant n} h(j)^{i+s-1} . \qquad (3.6)$$

Lemma 3.7. For each $j = 1,2, \dots, n$, $h(j) = (a(j1)t(1) + \dots + a(jr)t(r))^2$.

Proof. By induction on r : true for $r = 1$.

Assume true for $r - 1$, when $1 \leqslant i \leqslant n$. If we work modulo the ideal generated by $t(r)$, the induction hypothesis gives

$$h(j) = (a(j1)t(1) + \dots + a(j,r-1)t(r-1))^2 + t(r) \sum_{1 \leqslant k \leqslant r} \beta(k)t(k)$$

$$= (a(j1)t(1) + \dots + a(jr)t(r))^2 + (b(j2)t(2) + \dots + b(jr)t(r))^2 + \dots .$$

By looking at the coefficient of $t(i)t(r)$ and $t(i)^2$, $1 \leqslant i < r$, in this we see that

$$h(j) = (a(j1)t(1) + \dots + a(jr)t(r))^2 + \gamma(j)t(r)^2 \text{ for some } \gamma(j) .$$

So combining this with (3.6) we get,

$$\sum_{j=1}^{n} h(j)^{s+i-1}$$

$$= \sum [(a(j1)t(1) + \dots + a(jr)t(r))^{p-1+2i} + (s+i-1)(a(j1)t(1) + \dots$$

$$\dots + a(jr)t(r))^{p-3+2i} \gamma(j)t(r)^2 + \dots]$$

$$= \alpha(i)S_1^{i-1}S_s + \dots . \qquad (3.8)$$

Let ξ_j be the smallest integer with $a(j, \xi_j) \neq 0$.

Let $j \in M$.

In (3.8) equate coefficients of $t(r)^{2i} t(1)^{p-1}$ for $i = 1, \ldots, m$. This gives $\sum_{j \in M} \gamma(j)^i = 0$ for $1 \leq i \leq m$. Hence $\gamma(j) = 0$ for $\xi_j = 1$.

By induction on ξ_j, we prove that $\gamma(j) = 0$, $j = 1, 2, \ldots, n$. This completes the induction and the proof of the lemma.

We can now prove that Assumption A => Conjecture A' .

<u>Cor. 3.9.</u> There is an "extension" of f defined by

$f's(i) = a(i1)t(1) + \ldots + a(ir)t(r)$, for $1 \leq i \leq n$ [Z_p coefficients.]

<u>Proof</u> We know from the factoring of $F(t)$ that $fs(i)^2 = h(i)$, renumbering the $s(i)$ if necessary. The corollary follows since each $h(i)$ is a square.

Next, we identify the $a(ij)^2$ in terms of the integers $m(i)$ defined in Lemma 3.1.

<u>Lemma 3.10.</u> The sets $\{a(ij)^2 | i = 1, \ldots, n\}$ and $\{m(i)^2 | i = 1, \ldots, n\}$ are equal for each $j = 1, \ldots, r$.

<u>Proof</u> We have $fx(i) = e_i(h(k))$. Equating coefficients of $t(j)^{2i}$ gives:

$$a(i) = e_i(a(kj)^2) \quad \text{for } 1 \leq j \leq r. \quad \text{But}$$

$$a(i) = e_i(m(k)^2) \quad \text{from 3.1.} \quad \text{The lemma follows.}$$

From 3.10. we see that in fact $h(i) \in Z_p[t(1), \ldots, t(r)]$ for each i and $F(t)$ factors over the integers i.e. the $h(i)$ are mod p reductions of elements in $Z[t(1), \ldots, t(r)]$. The factorisation of

$F(t)$ puts even more restrictions on the $a(ij)$. In particular, $e_i(h(j)) = fx(i)$ must be invariant under the Weyl group of $Sp(r)$.

We are now ready for the main part of the chapter.

Recall that for any graded ring morphism $f : C_n \to C_r$, we always assume that $fx(n) \neq 0$. This involves no loss of generality.

Under the assumption A, we are able to give a complete list of those f which will commute with P^1 for all large p. First some definitions.

Definition 3.11. Choose an integer U, $1 \leqslant U \leqslant r$. Let $P = \{u_1, \ldots, u_\beta\}$ be a partition of U so that $U = \sum_{1 \leqslant i \leqslant \beta} u_i$, with $1 \leqslant u_i \leqslant U$.

Given P choose a set, B, of non-zero integers $\{b(1), \ldots, b(\beta)\}$ with the $b(i)^2$ distinct. P and B depend on U. Now define

$$I(U,P,B) = \prod \left[1 - t(b(1)(t(i_1) \pm \cdots \pm t(i_{u_1})) \pm \cdots \pm b(\beta)(t(i_{U-u_\beta+1}) \pm \cdots \pm t(i_U)))^2\right]$$

where the product is taken over all possible signs $\pm$, over all subsets of $\{1, 2, \ldots, r\}$ containing U elements and all partitions $\{i_1, \ldots, i_{u_1}\}, \ldots, \{i_{U-u_\beta+1}, \ldots, i_U\}$ of these subsets into β parts containing $u_1, \ldots, u_\beta$ elements.

The number of factors in $I(U,P,B)$ is therefore

$n(U,P,B) = 2^{U-1}\binom{r}{U} \dfrac{U!}{u_1! \cdots u_\beta!}$: the 2^{U-1} enumerates the signs,

$\binom{r}{U}$ the subsets of $\{1, \ldots, r\}$ and the remaining factor the partitions of these subsets.

Notice that $I(U,P,B)$ is uniquely defined by U,P and B. We will abbreviate $I(U,P,B)$ to I when the U,P and B are understood.

We show that I represents an "irreducible" morphism into C_r which commutes with P^1 for all odd primes p. For a precise statement see Proposition 3.12. below. In preparation for this, we need the following discussion.

Let $W(r)$ be the Weyl group of $Sp(r)$; it acts on C_r by permuting the $t(i)$ and changing their signs.

For a graded ring morphism $f : C_n \to C_r$, let $F(t)$ factor as $F(t) = \prod_{1 \leq i \leq n} (1 - th(i))$ with $h(i) \in Z\,[t(1), \ldots, t(r)]$. Denote this latter ring by $Z(r)$. Then if $w \in W(r)$, $wh(i) \in Z(r)$.

Since Z is a unique factorisation domain, so is $Z(r)[t]$.

<u>Lemma</u> 3.11a. For any $w \in W(r)$ and $i \in \{1, \ldots, r\}$, $wh(i) = h(j)$ for some $j \in \{1, \ldots, r\}$.

<u>Proof</u> It is easy to see from the definition of $F(t)$ that $wF(t) = F(t)$.

By renumbering the h's assume that $i = 1$. Then

$$(1 - wh(1)t) \ldots (1 - wh(n)t) = (1 - h(1)t) \ldots (1 - h(n)t) .$$

Clearly $(1 - wh(1)t) \in Z(r)[t]$ is an irreducible polynomial, hence prime since $Z(r)[t]$ is a unique factorisation domain. Therefore

$(1 - wh(1)t)$ divides some $(1 - h(j)t)$ which is irreducible. Since the only units in $Z(r)[t]$ are ± 1 this can only mean that $1 - wh(1)t = 1 - h(j)t$, which proves our lemma.

We paraphrase this lemma as follows : Given $f : C_n \to C_r$, form $F(t) = \prod (1 - h(j)t)$. Then $\{h(j)\}$ is invariant under $W(r)$.

If $G(t) = \prod_{1 \leq i \leq n} (1 - k(i)t) \in Z(r)[t]$, and $G(t) \in C_r[t]$, we define a morphism $g : C_n \to C_r$ by sending $x(j)$ to $e_j(k(i))$. We say that $G(t)$ corresponds to g.

We are now ready to state

Proposition 3.12. (a) If I factors as $G(t).H(t)$, then neither $G(t)$ nor $H(t)$ corresponds to a morphism $C_n \to C_r$ unless $G(t) = 1$ or $H(t) = 1$.

(b) I corresponds to a morphism

$$f : C_{n(U,P,B)} \to C_r ,$$ which commutes with P^1 for odd primes p.

Proof (a) If some product $\Pi(1 - h(i)t)$ corresponds to a morphism into C_r, we have proved in the above lemma that the set $\{h(i)\}$ is invariant under $W(r)$. I was defined so that it contained exactly the factors needed to make it invariant: if any factor is omitted, it won't be.

(b) If we take any linear factor of $I \in Z(r)[t]$, and apply $W(r)$ to it, we find that it goes into another factor of I. Hence $W(r)I \subset C_r[t] \subset Z(r)[t]$, so we have a morphism $f : C_n \to C_r$.

We know that $fs(j)^2$ is defined and equals $h(j)$, say.

Also [c.f. 3.9. and 3.5.],

$$fP^1 s_i = 2i \sum_j h(j)^{i+s-1} ,$$ and

$$P^1 fS_i = P^1 \sum fs(j)^{2i} = P^1 \sum h(j)^i$$

$$= i \sum h^{i-1}_{(j)} P^1 h(j) .$$

Now, $$P^1 h(j) = P^1 (\sum a(jk)t(k))^2 = 2(\sum a(jk)t(k))\; P^1 \sum a(jk)t(k)$$

$$= 2(\cdot\ \cdot) \sum a(jk)t(k)^p$$

$$= 2(\cdot\ \cdot)(\sum a(jk)t(k))^p$$

$$= 2h(j)^s .$$

So, $$P^1 fS_i = 2i \sum h(j)^{s+i-1} \quad \text{and}$$

$$P^1 fS_i = fP^1 S_i .$$

This implies that $P^1 fx = fP^1 x$ for all $x \in C_n$. Thus f commutes with P^1 for all odd p.

Remark 3.13. In view of 3.12(a), we may say that $I(U,P,B)$ corresponds to an "irreducible" morphism $C_n \to C_r$.

We are now ready for the main result of the chapter.

Notice that we have established the following : if $f : C_n \to C_r$ satisfies Assumption A, then there is a 1 - 1 correspondence

$$f \leftrightarrow F(t) = \Pi(1 - h(i)t) \in Z(r)[t] .$$

Theorem 3.14. (i) If $f : C_n \to C_r$ satisfies assumption A, and $fx(n) \neq 0$, then $F(t) \in Z(r)[t]$ factors as follows:

There exist for each $U \in \{1,2,\ldots, r\}$,

(a) Sets, P of positive integers $u_1, \ldots, u_\beta$ with

$$U = \sum_i u_i ;$$

(b) for each U and P, some sets, B of non-zero integers $\{b(1), \ldots, b(\beta)\}$ with the $b(i)^2$ distinct ;

(c) for each U,P,B a unique integer $\alpha(U,P,B) \geq 0$, such that

$$F(t) = \prod_{U,P,B,\alpha(U,P,B)} (I(U,P,B))^{\alpha(U,P,B)} .$$

(ii) Conversely each such F(t) defines a morphism $C_n \to C_r$ commuting with P^1 for all odd primes.

Let us use the abbreviation $\alpha(U) = \sum_{P,B,\alpha(U,P,B)} \frac{U!\ \alpha(U,P,B)}{u_1! \ldots u_\beta!}$

with each $\alpha(U,P,B) > 0$.

Cor. 3.15.
$$n = \sum_{1 \leq U \leq r} 2^{U-1} \binom{r}{U} \alpha(U) .$$

Proof With the usual notation, $fx(n) = \prod_{1 \leq i \leq n} h(i)$. Since $fx(n) \neq 0$, no h(i) is zero so n is the degree of $F(t) \in Z(r)[t]$. This degree is the number of factors $(1 - th(i))$. The corollary now follows by counting these factors.

Cor 3.16. (Hubbuck).

Let $f : C_n \to C_r$ satisfy Assumption A (and $fx(n) \neq 0$). Then

(i) if $r = n \neq 2$, there is an integer k such that for any

$$x \in H^{4m}(BSp(n)) \ , \quad fx = k^{2m}x \ ;$$

(ii) if $r = n = 2$, there is a k with $fx = k^{2m}x$ or

$$fx(1) = 2k^2x(1)$$

$$fx(2) = -4k^4x(2) + k^4x(1)^2 \ .$$

<u>Proof (i)</u> Put $r = n$ in 3.15 :

$n = \alpha(1)n + 2\alpha(2)\binom{n}{2} + \ldots + \alpha(n)2^{n-1}$. Hence $\alpha(U) = 0$, $1 < U < n$,

since in this range $\binom{n}{U} > n$. So $n = \alpha(1)n + \alpha(n)2^{n-1}$.

If $n > 2$, $2^{n-1} > n$ hence $\alpha(n) = 0$. So , $\alpha(1) = 1$ and

$\alpha(U) = 0$, $1 < U \leq n$.

Therefore $1 = \sum_{P,B,\alpha(U,P,B)} \dfrac{U!\ \alpha(U,P,B)}{u_1! \ldots u_\beta!}$ and there is only one

set P , only one integer $b(i) = k$ say, and only one $\alpha(U,P,B)$ which is

$\neq 0$: it is 1 .

Hence $F(t) = \prod_{1\leq i\leq n} (1 - t(kt(i))^2)$ and $fx(i) = k^{2i}x(i)$.

If $n = 1 = r$, $1 = \alpha(1).1$ so $\alpha(1) = 1$ and again $F(t) = 1 - tk^2t(1)^2$.
This completes the proof of (i) .

For the <u>proof of (ii)</u> , we have $n = r = 2$ and $2 = 2\alpha(1) + 2\alpha(2)$.
Thus

$$\alpha(1) = 1 \ , \quad \alpha(2) = 0$$

$$\text{or} \quad \alpha(1) = 0 \quad \alpha(2) = 1 \ .$$

If one constructs the corresponding $F(t)$, one gets the stated result.

As a final corollary, we list the possible $h(i)$ for a morphism $C_n \to C_2$.

Cor. 3.17. For a morphism $C_n \to C_2$ the possible $h(i)$ have the form

(i) $a^2 t(i)^2$, $i = 1,2$

(ii) $b^2(t(1) \pm t(2))^2$ (iii) $(ct(1) \pm dt(2))^2$ $c^2 \neq d^2$.

Proof The $h(i)$ are squares of homogeneous polynomials of degree 1 , so must have the given form. The significant fact is that each of the three types will give us a morphism $C_n \to C_2$.

With regard to these corollaries, it should be noted that we will not prove Conjecture A' here . The corollaries are meant to illustrate the usefulness of the conjecture, (if true) .

We now come to the

Proof of 3.14. (i) Take a particular $h(j)$ from $F(t)$. Under the stated assumptions, we have already proved that $h(j)$ is a square in $Z(r)$:

$$h(j) = (b(j1)(t(i_1) + \dots t(i_{u_1})) + \dots + b(j\beta)(t(i_{U-u_\beta+1}) + \dots + t(i_U)))^2 .$$

This determines an integer $U \in \{1 ,\dots, r\}$, a set $\{u_1,\dots, u_\beta\}$ with $\sum_i u_i = U$ and integers $b(ji) \neq 0$.

Apply $W(r)$ the Weyl group of $Sp(r)$ to $h(j)$: the h's which arise as images of $h(j)$ under $W(r)$ form a unique $I(U,P,B)$ which must be a factor of $F(t)$.

If this exhausts all the h's , stop . If not, then $h(j)$ may still be one of the remaining factors of $F(t)$ and will give another copy of I(U,P,B) . Continuing in this way, we break off $\alpha(U,P,B)$ copies of I(U,P,B) from F(t) . If this exhausts the h's in F(t) , stop . If not, take an h(k) not in I(U,P,B) and form another I , etc.

Since F(t) has only a finite number of factors, this process stops. Each h in F(t) must be in some I , since F(t) is invariant under W(r) : there are no h's left over.

(ii) This follows from 3.12.

Section 3.2. Construction of maps $BSp(r) \to BSp(n)$.

We show that some of the morphisms $C_n \to C_r$ listed in 3.14. are induced from maps $BSp(r) \to BSp(n)$. For this purpose we compute the induced homomorphisms of some representations.

Example 3.18. There is a map $\phi : BSp(r) \to BSp(rm)$ such that if $\phi^* = f : C_{rm} \to C_r$, then

$$F(t) = \prod_{1 \leq j \leq r} (1 - t\, t(j)^2)^m , \text{ where } r , m \geq 1 .$$

Proof Let $BSp(r)^m$ be the m-fold cartesian product $BSp(r) \times \dots \times BSp(r)$, and $J : BSp(r)^m \to BSp(r)^m$ the identity. Then take ϕ to be the composite

$$BSp(r) \xrightarrow{\Delta} BSp(r)^m \xrightarrow{J} BSp(r)^m \xrightarrow{i} BSp(rm) ,$$

where Δ is the diagonal and i the inclusion .

Next we compute the induced morphism of a particular representation

$Sp(3) \to Sp(\frac{1}{2}\, 6^3)$, using the (alternative) description of $Sp(n)$ given in section 2.3.

Lemma 3.19. Let $\alpha : Sp(3) \to Sp(3)$ be the identity and α^3 the third tensor power. Then the $h(i)$ corresponding to $(B\alpha^3)^* : C_n \to C_3$ $(n = \frac{1}{2}\, 6^3)$ are of the form:

$t(j)^2,\ 9t(j)^2\ ,\ (2t(j) \pm t(k))^2\ ,\ (t(1) \pm t(2) \pm t(3))^2\ ,\ 1 \leq j,\ k \leq 3\ ,\ j \neq k$.

Proof Take the diagonal matrix $\mathrm{diag}(Z_1, \bar{Z}_1, Z_2, \bar{Z}_2, Z_3, \bar{Z}_3) = D$, in the maximal torus of $Sp(3)$. $\alpha^3 D = D^3$, the third tensor power. We defined tensor powers of such matrices in the proof of Proposition 2.26.

So , $D^2 = \mathrm{diag}(Z_1^2, 1, Z_1Z_2, Z_1\bar{Z}_2, Z_1Z_3, Z_1\bar{Z}_3, 1, \bar{Z}_1^2, \bar{Z}_1Z_2\ , \ldots,\ Z_2Z_3\ , \ldots)$,

and

$D^3 = \mathrm{diag}(Z_1^3, Z_1, Z_1^2Z_2, Z_1^2\bar{Z}_2,\ Z_1^2Z_3, Z_1^2\bar{Z}_3, Z_1, \bar{Z}_1, Z_2\ , \ldots,\ Z_1Z_2Z_3, \ldots)$

with 6^3 entries on the diagonal.

From the exponents of the Z's in D^3 we see that the $h(i)$ must have the stated form .

Note. Let $i_1 : Sp(1) \to Sp(1)^3 = T$ be inclusion into the first factor and $j : T \to Sp(3)$ the diagonal inclusion of section 2.3. If we take $ji_1 : Sp(1) \to Sp(3)$ and follow by α^3 , we construct a map $BSp(1) \to BSp(\frac{1}{2}\, 6^3)$ with an even integer (namely 2) in its degree. The "2" arises from $h(i) = (2t(j) \pm t(k))^2$ in the notation of 3.19.

This does not affect our conjecture on degrees of maps $f : BSp(1) \to BSp(n)$ since in this case $f^*x(n) = 0$.

Next we clarify our notion of a map $BSp(r) \to BSp(n)$ being irreducible by an example.

Take again $\alpha : Sp(3) \to Sp(3)$ and consider its exterior power $\Lambda^3\alpha : Sp(3) \to Sp(10)$.

Lemma 3.20. The $h(i)$ corresponding to $B\Lambda^3\alpha : BSp(3) \to BSp(10)$ are of the form $t(i)^2$ and $(t(1) \pm t(2) \pm t(3))^2$, $1 \leq i \leq 3$.

Proof Again we calculate $\Lambda^3\alpha$ on the maximal torus

$$\{\mathrm{diag}(z_1,\bar{z}_1,z_2,\bar{z}_2,z_3,\bar{z}_3) \mid |z_i| = 1\}.$$

Take $e_1, \ldots, e_6$ as a basis for $\mathbb{C}^6$. Then $e_i \wedge e_j \wedge e_k$, $1 \leq i < j < k \leq 6$ is a basis for $\Lambda^3\mathbb{C}^6 \cong \mathbb{C}^{20}$, and the action of $\Lambda^3\alpha$ on this basis is

$$\Lambda^3\alpha(g)(e_i \wedge e_j \wedge e_k) = \alpha(g)e_i \wedge \alpha(g)e_j \wedge \alpha(g)e_k \quad \text{for } g \in Sp(3).$$

If g is in the maximal torus then

$$\Lambda^3\alpha(g)e_1 \wedge e_2 \wedge e_3 = z_1e_1 \wedge \bar{z}_1e_2 \wedge z_2e_3 = z_2e_1 \wedge e_2 \wedge e_3 \quad \text{and}$$

$$\Lambda^3\alpha(g)e_1 \wedge e_3 \wedge e_5 = z_1e_1 \wedge z_2e_3 \wedge z_3e_5 = z_1z_2z_3e_1 \wedge e_3 \wedge e_5 \quad \text{etc.}$$

Thus one calculates $\Lambda^3\alpha$ on the maximal torus. The $h(i)$ follow from this calculation. For example, for the two calculations just made, the corresponding $h(i)$ would be $t(2)^2$ and $(t(1) + t(2) + t(3))^2$.

Remark $\mathbb{C}^{20}$ with the given action of $Sp(3)$ is reducible : see [8, page 23.3].

Now in cohomology, we have a commutative diagram

$$\begin{array}{ccc} C_{10} & \xrightarrow{(B\Lambda^3\alpha)^*} & C_3 \\ (Bi)^* \downarrow & \nearrow (Bi)^* \times (Bi)^* \times f & \\ C_3 \times C_3 \times C_4 & & \end{array}$$

where the i's are inclusions and $f : C_4 \to C_3$ is the morphism whose h's are $(t(1) \pm t(2) \pm t(3))^2$.

So $(B\Lambda^3\alpha)^*$ is reducible.

<u>Question 3.21.</u> Can one make a homotopy commutative diagram

$$\begin{array}{ccc} Sp(3) & \xrightarrow{\Lambda^3\alpha} & Sp(10) \\ \downarrow & \nearrow\swarrow & \\ Sp(3) \times Sp(3) \times Sp(4) & & \end{array}$$

of A_∞-maps and spaces?

<u>Section 3.3.</u> Maps $BG \to BH$.

We take care not to use Assumption A in this section, and prove that if G is "bigger than" H , then for any map $f : BG \to BH$, $f^* = 0$. The phrase "bigger than" is clarified below : see 3.22. and 3.25.

Let R be a subring of the rationals in which one can invert each of the primes for which $H^*(G,Z)$ has torsion. Then $H^*(BG,R) \cong R[y(1),\ldots,y(n)]$ by theorem W . Similarly,

$$H^*(BH,R') \cong R'[x(1),\ldots,x(n')] .$$

We will use the same notation $\{x(i)\}$, $\{y(j)\}$ for mod p generators, if p is a large prime and assume that $\dim x(1) = \dim y(1) = 4$.

Let $\alpha(G)$ be the set of half dimensions of the generators of $H^*(BG,Q)$. For example

$$\alpha(Sp(n)) = \{2,4,\ldots,2n\} .$$

Having established notation we can begin. Let $f : BG \to BH$ be a map.

Lemma 3.22. (i) If there is a generator $y \in H^*(BG)$ such that $\frac{1}{2}\dim y \notin \alpha(H)$, $\frac{1}{2}(\dim y + 4) \notin \alpha(G)$, $\dim y > \max \alpha(H) \cup \alpha(G)$, then $f^*x(1) = 0$.

(ii) If in addition there is a map $i : BH \to BG$ with $i^* : H^*(BG) \to H^*(BH)$ surjective, then $f^* = 0$.

Remark The conditions in (i) are designed to ensure that for a priori dimensional reasons, no power of y can occur in the image of f^* .

Proof of 3.22. (i) If $f^*x(1) = a(1)y(1)$ for some $a(1)$, then by the naturality of P^1 , $a(1)P^1y(1) = f^*P^1x(1)$. (*)

Since $\frac{1}{2}(\dim y + 4) \notin \alpha(G)$ and $\dim y > \max \alpha(G)$, lemma 1.9 gives us the following: for infinitely many primes,

$$P^1y(1) = by^t + \ldots , \quad b \not\equiv 0 \mod p .$$

Since no power of y is in the image of f^* , equating coefficients in (*) gives $a(1)b = 0$. Hence $a(1) = 0$ modulo infinitely many primes, so $a(1) = 0$.

(ii). We have $if : BG \to BG$, with $f^*i^*y(1) = 0$. Hence from [9] and Chapter 1 , $f^*i^* = 0$ and since i^* is epi, we must have $f^* = 0$.

Recall that the groups G were classified by Cartan. See [8] for this classification.

<u>Cor. 3.23.</u> If G and H are in the same class in Cartan's classification and rank $G >$ rank H, then for any map $f : BG \to BH$, $f^* = 0$.

<u>Proof</u> We can take $i : BH \to BG$ to be the inclusion induced from $H \subset G$, The hypotheses of 3.22. are then satisfied since for y we can take the generator of maximum dimension in $H^*(BG)$, and i^* is epi.

Assume now that G and H are classical groups i.e. $SU(n)$, $Sp(n)$ or $Spin(n)$. Then we can strengthen Lemma 3.22 by showing that if G is "bigger" than H, then any map $BG \to BH$ is zero in cohomology. We have found no simple condition to define the term "bigger". The condition is neither "rank $G >$ rank H" nor "dimension $G >$ dimension H", as we shall see. Of course, if G and H belong to the same Cartan class, then the condition <u>is</u> "rank $G >$ rank H" and then 3.23. is the best possible in the sense that if rank G = rank H, then $G = H$ and the identity map $BG \to BH$ is not zero in cohomology.

<u>Remark 3.24.</u> There follows a rather motley collection of results. The idea behind the proof in each case is to find conditions on G and H so that there is a generator $y \in H^*(BG)$, no power of which can occur in the image of f^*. We defined generators for $H^*(BG)$, when G is a classical group, in Chapter 2. We will always use those generators.

So for maps $f : BG \to BH$, we discuss various cases.

(a) $G = SU(m)$, the special unitary group, $m \geq 2$.

<u>Proposition 3.25.</u> If $f : BSU(m) \to BSp(n)$ is a map with m odd and $n \geq 3$, $m \geq n + 1$, then $f^* = 0$.

<u>Remark</u> Note that rank $SU(m) = m - 1$, rank $Sp(n) = n$.

Take generators $\{x(i)\}$ and $\{y(j)\}$ for $H^*(BSp(n))$ and $H^*(BSU(m))$ as in Chapter 2 .

In the proof of 3.25., we will need

<u>Lemma 3.26.</u> For any integer $m \geq 2$, let $Y \in H^*(BSU(m))$ be a monomial. Then there is a large prime, $p = 1 + mt$, such that

$$P^1 Y = BY\, y(m) + \ldots , \quad B \neq 0 .$$

<u>Proof</u> By 2.29., for any r ,

$$P^1 y(r) = y(r-1)S_p + \ldots + (-1)^{r-1} S_{p-1+r} , \quad p = 1 + mt .$$

By the Cartan formula, for any monomial $X \in H^*(BSU(m))$,

$$P^1 X y(r)^a = y(r)^a P^1 X + aXy(r)^{a-1}(y(r-1)S_p + \ldots + (-1)^{r+1} S_{p-1+r}) .$$

If $r \neq m$, the coefficient of $y(r)y(m)^t$ in S_{p-1+r} is $(-1)^{t+r+1}(p - 1 + r)$ by Theorem C . Now let $Y = y(m)^{a_m} y(r_1)^{a_1} y(r_2)^{a_2} \ldots$. Then the coefficient of $Y\, y(m)^t$ in $P^1 Y$ is $B = (-1)^t (ma_m + a_1(r_1 - 1) + a_2(r_2 - 1) + \ldots)$. This is non-zero mod p , if p is large enough.

Before we begin the proof of 3.25. note that our choice of m ensures that no power of $y(m)$ can occur in the image of f^* : $y(m)$ can't occur because m is odd and $y(m)^i$, $i \geq 2$, because $m \geq n + 1$.

<u>Lemma 3.27.</u> f^* is zero mod decomposables.

<u>Proof</u> Let $f^* x(i) = \alpha(i) y(2i) + \ldots$.

Since $\dim x(i) = 4i$ and $\dim y(j) = 2j$, $\alpha(i) = 0$ if $i > \frac{1}{2}m$, so we need only prove that $\alpha(j) = 0$ if $\frac{1}{2}m > j \geq 1$.

Choose a large prime $p = 1 + mt$. Then $P^1 y(2i) = \beta y(2i)y(m)^t + \dots$, $\beta \neq 0$, by 3.26.

By naturality,

$$\begin{aligned} f^* P^1 x(i) &= P^1 f^* x(i) \\ &= P^1(\alpha(i)y(2i) + \text{decomposables}) \\ &= \alpha(i)(\beta y(2i)y(m)^t + \dots) + \dots . \end{aligned}$$

Equating coefficients of $y(2i)y(m)^t$ gives $\alpha(i)\beta = 0$. Hence $\alpha(i) = 0$.

To proceed further, we need some more notation.

Definition 3.28. The length of a monomial $y(j_1)^{e_1} y(j_2)^{e_2} \dots$, is $\sum_i e_i$.

Order the monomials as follows: write all monomials as $y(j_1)^{e_1} \dots y(j_s)^{e_s}$, $j_1 > j_2 > \dots > j_s$. Then $y(j_1)^{e_1} \dots y(j_s)^{e_s} > y(j_1)^{f_1} \dots y(j_s)^{f_s}$ if $e_1 = f_1, \dots, e_i = f_i$ and $e_{i+1} > f_{i+1}$, for some i , $0 \leq i \leq s - 1$.

Lemma 3.29. Each term in f^* has length ≥ 3 .

Proof Let $f^* x(i) = (\beta(m)y(m)y(2i - m) + \beta(m - 1)y(m - 1)y(2i - m + 1)$

$$+ \dots) + v(i) ,$$

where each term in $v(i)$ has length ≥ 3.

By 3.26.,

$$P^1 y(m)y(2i - m) = ay(2i - m)y(m)^{t+1} + \dots , \quad a \neq 0 .$$

The coefficient of $y(2i - m)y(m)^{t+1}$ in $P^1 f^* x(i)$ is $a\beta(m)$, but in $f^* P^1 x(i)$ it is zero. Hence $\beta(m) = 0$.

Assume by induction that $\beta(m), \dots, \beta(m - j) = 0$. Then the coefficient of $y(m)^t y(m - j - 1)y(2i - m + j + 1)$ in $P^1 f^* x(i)$ is $a'\beta(m - j + 1)$, for some $a' \neq 0$, but in $f^* P^1 x(i)$ it is zero. Hence $\beta(m - j + 1) = 0$. This finishes the inductive step.

The proofs of the previous two lemmas are meant to motivate!

<u>Lemma 3.30.</u> Let $W_1 = W_3 y(j_1)^{e_1} \dots y(j_s)^{e_s}$ and $W_2 = W_3 y(j_1)^{f_1} \dots y(j_s)^{f_s}$ be monomials of equal degree with $e_1 > f_1$. Assume that if $y(i)$ divides W_3, then $i > j_1$. Then the coefficient of $W_1 y(m)^t$ in $P^1 W_2$ (with $p = 1 + mt$) is zero.

<u>Proof</u> Assume that this coefficient is not zero. Then by the Cartan formula applied to $P^1 W_2$, we see that except for possibly one e (say e_r), we must have $f_i \leq e_i$ (and $f_r \leq e_r + 1$).

Now equate dimensions of the W_i:

$$j_1 e_1 + \dots + j_r e_r + \dots + j_s e_s = j_1 f_1 + \dots + j_r f_r + \dots + j_s f_s , \quad \text{so} ,$$

$$j_1(e_1 - f_1) = j_2(f_2 - e_2) + \dots + j_r(f_r - e_r) + \dots + j_s(f_s - e_s) \leq j_r(f_r - e_r) .$$

If $r = 1$, this gives $e_1 - f_1 \leq f_1 - e_1$ i.e. $e_1 \leq f_1$, contrary to assumption.

If $r > 1$, we have $j_1(e_1 - f_1) \leq j_r(f_r - e_r) < j_1$. Hence $e_1 - f_1 < 1$, contrary to assumption.

The coefficient of $W_1 y(m)^t$ in $P^1 W_2$ must therefore be zero .

<u>Proof of 3.25.</u> If $f^* x(i) \neq 0$, let W be the largest monomial in $f^* x(i)$ with a non-zero coefficient: $f^* x(i) = aW + \dots$, $a \neq 0$.

Now it is clear that if $p = 1 + mt$, t large, no monomial $Wy(m)^t$ can occur in $f^* P^1 x(i)$.

By 3.26.,

$$P^1 W = \beta W y(m)^t + \dots , \quad \beta \neq 0 .$$

So by 3.30., the coefficient of $Wy(m)^t$ in $P^1 f^* x(i)$ is βa . This coefficient is zero in $f^* P^1 x(i)$, hence $a = 0$, This contradiction shows that $f^* x(i) = 0$.

<u>Cor. 3.31.</u> If $f : BSU(m) \to BSp(n)$ is a map and m is even with $m \geq n + 2$, then $f^* = 0$.

<u>Proof</u> Let $f^* x(\frac{1}{2}m) = \alpha y(m) + \dots$.

There are infinitely many primes satisfying $p - 1 = (m - 1)t$, and $P^1 y(m) = y(m) S_{p-1} = \beta y(m) y(m - 1)^t + \dots$, $\beta \neq 0$, by Theorem C of Chapter 1 .

The coefficient of $y(m) y(m - 1)^t$ in $P^1 f^* x(\frac{1}{2} m)$ is $\alpha\beta$.

Since $4(m - 1) > 4n$, no power of $y(m - 1)$ can occur in the image of f^* and hence the coefficient of $y(m)y(m-1)^t$ in $f^*P^1x(\frac{1}{2}m)$ is 0. Therefore $\alpha = 0$.

We conclude that no power of $y(m)$ appears in f^*. The proof now proceeds as for 3.25.

Returning to the situation of maps $f : BSU(m) \to BH$, if $n \geq 2$, $H = Spin\ (2n+1)$, the proof of 3.25. applies to give

Cor. 3.32. If $f : BSU(m) \to BSpin\ (2n+1)$ is a map with m even (resp. odd) and $m \geq n+1$ (resp. $m \geq n+2$), then $f^* = 0$.

Now let $H = Spin\ (2n)$, $n \geq 4$. Then

Lemma 3.33. If $f : BSU(m) \to BSpin\ (2n)$ with m odd (resp. m even) and $m \geq n+1$ (resp. $m \geq n+2$), then $f^* = 0$.

Proof $H^*(BSpin\ (2n), Zp) = Zp\ [x(1), \ldots, x(n-1), z]$, $\dim z = 2n$, $\dim x(i) = 4i$, where the generators are as in 2.31.

When m is odd, we are assured that there is no power of $y(m)$ in the image of f^* because it can't be in f^*z, since $m \neq n$, and it can't be in $f^*x(i)$, because $2.2\,m > 4(n-1)$.

When m is even $y(m)$ can't occur in f^*z because $m \neq n$.

If $f^*x(\frac{1}{2}m) = \alpha y(m) + \ldots$, we can prove that $\alpha = 0$ as in Corollary 3.31.

We can now use the proof of 3.25. to prove 3.33.

(b) For maps $BG \to BH$, $G = Sp(\ )$, $Spin\ (2m)$, $Spin\ (2m+1)$, we will give less details. First, we need an analogue of Lemma 3.26.

Lemma 3.34. For any monomial $X \in H^*(BSpin\,(2m))$, there is a large prime $p = 1 + 2mt$, such that the coefficient of Xz^{2t} in P^1X is non-zero .

Proof $\frac{1}{2} P^1x(i) = \sum_{1 \leq j \leq i} (-1)^{j-1}x(i-j)S_{s+j-1}$, $2s = p+1$

and $P^1z = zS_{s-1}$, $S_k = \sum_j t(j)^{2k}$.

The coefficient of $x(i)z^{2t}$ in $\frac{1}{2}P^1x(i)$ is $(-1)^{s-1+t}(i+mt) = (-1)^{s-1+t}(i-\frac{1}{2})$.
The coefficient of $z.z^{2t}$ in P^1z is $m(-1)^{s-1+t}$.

Hence, if $X = z^a x(m-1)^{e_{m-1}} \dots x(1)^{e_1}$, the coefficient of Xz^{2t} in P^1X is $(-1)^{s-1+t}(e_1(2-1) + e_2(4-1) + \dots + e_{m-1}(2m-2-1) + a.m)$, which is non-zero mod p if p is large .

Similar results can be proved for $Sp(m)$ and $Spin\,(2m+1)$.

Using Lemma 3.34., we can prove

Lemma 3.35. If $f : BSpin\,(2m) \to BSU(n)$ is any map and $2m \geq n+3$, then $f^* = 0$.

Proof We want z to play the role of $y(m)$ in the case $G = SU(\ )$. But we could have $f^*y(m) = \alpha_1 z + \alpha_2 x(\frac{1}{2}m)$ + decomposables.

We prove that α_1 , $\alpha_2 = 0$.

Now, $\frac{1}{2}P^1x(\frac{1}{2}m) = x(\frac{1}{2}m - 1)\,S_s + \dots \pm S_{s+\frac{1}{2}m-1}$, $2s = p+1$.

If $s + \frac{1}{2}m - 1 = (m-1)t + \frac{1}{2}m$, then $p = 2(m-1)t + 1$, for infinitely many t .

The coefficient of $x(\frac{1}{2}m)x(m-1)^t$ in $P^1x(\frac{1}{2}m)$ is $\beta \neq 0$, by Theorem C.

Since $2m \geq n+3$, no power of $x(m-1)$ can appear in the image of f^*. Hence the coefficient of $x(\frac{1}{2}m)x(m-1)^t$ in $f^*P^1y(m)$ is 0.

The coefficient of $x(\frac{1}{2}m)x(m-1)^t$ in $P^1f^*y(m)$ is $\alpha_2\beta = 0$,

$\therefore \alpha_2 = 0$.

So,

$$f^*y(m) = \alpha_1 z + \text{decomposables.}$$

By Theorem C, $P^1z = \beta_1 zx(m-1)^t + \dots$, $\beta_1 \neq 0$, for $p = 2(m-1)t+1$, sufficiently large.

The coefficient of $zx(m-1)^t$ in $P^1f^*y(m)$ is $\alpha_1\beta_1$.

The coefficient of $zx(m-1)^t$ in $f^*P^1y(m)$ is 0. Hence $\alpha_1 = 0$.

Thus we have established that no power of z can appear in the image of f^*.

We comment on the analogue of 3.20. for BSpin (2m) : order the monomials so that $z > x(i) > x(j)$ if $i > j$. Let $M_1 = z^{a+1}x(\frac{1}{2}m)^b$, $M_2 = z^a x(\frac{1}{2}m)^{b+1}$, $a,b \geq 1$. Then $M_1 > M_2$, and $M_1z^{t'}$ could appear in P^1M_2, but our condition $2m \geq n+3$ ensures that M_i can't appear in $f^*y(k)$.

Lemma 3.35. can therefore be proved by using z in place of $y(m)$ in the case G = SU( ).

One can prove similar vanishing results for maps $BG \to BH$ for all other pairs of (classical) groups.

Remark 3.36. Lemmas 3.35. and 3.25. show that the concept of "size of G" needed in this context is neither dimension nor rank.

Notice that we did not use the fact that f^* was induced from a geometric map, but only that it was a map of $A(p)$ - modules.

We believe that the proper statement to prove is

Conjecture 3.37. If y,x are the 4-dimensional generators for $H^*(BG)$, $H^*(BH)$ respectively and $h : H^*(BH) \to H^*(BG)$ a morphism of graded rings and $A(p)$ - modules for all large p , then $h = 0$ iff $hx = 0$.

A decent proof of this conjecture would we believe, require use of Assumption A .

One would use 3.37. (if true) as follows: first use Lemma 3.22 (i) to prove $hx = 0$ and then use 3.37. to conclude $h = 0$.

Chapter 4. The Assumption A .

We give evidence for "A" in this chapter and discuss when it can be proved.

Section 1. Morphisms $C_4 \to C_3$.

Choose the usual generators $\{x(i)\}$, $\{y(j)\}$ for C_4 and C_3 respectively, with the notation of Chapter 3, Section 1. Then for any morphism of graded rings,

$f : C_4 \to C_3$, we have

$$fx(1) = a(1)y(1) , \quad fx(2) = a(2)y(1)^2 + b(2)y(2) ,$$
$$fx(3) = a(3)y(1)^3 + c(3)y(1)y(2) + b(3)y(3) ,$$
$$fx(4) = a(4)y(1)^4 + b(4)y(2)^2 + c(4)y(1)^2y(2) + \alpha y(1)y(3) .$$

Assume further that f commutes with P^1 for all large p , and that $a(1) \neq 0$. (Otherwise it will be clear from the arguments below that $f = 0$.)

Lemma 4.1. $\alpha = 0$.

Proof Choose $p = 1 + 6t$. Then the coefficient of $y(3)^t$ in $P^1y(1)$ is $\beta \neq 0$. The coefficient of $y(3)^{t+1}$ in $P^1fx(4)$ is $\alpha\beta$.

$P^1x(4) = x(4)S_{s-1}$. Hence the coefficient of $y(3)^{t+1}$ in $fP^1x(4)$ is zero.

Hence $\alpha\beta = 0$ and $\alpha = 0$.

We will show that all other coefficients in f can be non-zero.

<u>Theorem 4.2.</u> "A" is true for morphisms $f : C_4 \to C_3$.

The proof consists of a series of lemmas. We will actually prove that the conclusion of Theorem 3.14. holds, without assuming "A" .

Most of what follows involves computing coefficients using Theorem C. For example

<u>Lemma 4.3.</u> $$S_{3t} = 3x(3)^t - 3t^2x(1)x(2)x(3)^{t-1} + \frac{1}{2}t^2(t+1)x(1)^3x(3)^{t-1} - \frac{1}{2}t^2(t-1)x(2)^3x(3)^{t-2} + 3t(t-1)x(2)x(4)x(3)^{t-2} + \dots .$$

<u>Proof</u> Just use the formula in Theorem C .

<u>Proposition 4.4.</u> If $p + 1 = 2s = 6t$, then

(i) $b(3)^t = a(1)$, (ii) $6c(3) = a(1)b(2) - b(3)$,

(iii) $6^3a(3) - 6^2a(1)a(2) + 7a(1)^3 = 7b(3)$.

<u>Cor. 4.5.</u> $a(1)^3 = \pm b(3)$.

<u>Proof</u> From 4.4. (i) , we have $b(3)^{6t} = a(1)^6$ i.e. $b(3)^2 = a(1)^6$.

<u>Proof of 4.4.</u> $\frac{1}{2}P^1x(1) = S_{3t}$. In $P^1fx(1) = fP^1x(1)$, equate coefficients of $y(3)^t$, $y(1)y(2)y(3)^{t-1}$ and $y(1)^3y(3)^{t-1}$ respectively. This gives

(i) $a(1) = b(3)^t$

(ii)' $-ta(1) = b(3)^{t-1}(c(3) - ta(1)b(2))$ and

(iii)' $\frac{1}{2}t(t+1)a(1) = b(3)^{t-1}(3a(3) - 3ta(1)a(2) + \frac{1}{2}t(t+1)a(1)^3)$.

(ii)' and (iii)' give

(ii) $-b(3) = 6c(3) - a(1)b(2)$ and

(iii) $7b(3) = 6^3 a(3) - 6^2 a(1)a(2) + 7a(1)^3$.

Note that strictly speaking, some of the equations in 4.4. should be over Z_p e.g. (i) and some over Z . But since we are working modulo a large prime, any equation not explicitly involving p can be taken over Z .

Lemma 4.6. (i) $b(2) = a(1)^2 - 4a(2)$, (ii) $8a(4) = -c(4)$, (iii) $2c(4) = -b(4)$.

Proof With $p + 1 = 2s = 6t$, $\frac{1}{2}P^1x(2) = x(1)S_s - S_{s+1} = \frac{3}{2}x(1)x(3)^t + \ldots$, and $\frac{1}{3}P^1x(4) = x(4)S_{s-1}$.

(i) Equate coefficients of $y(1)y(3)^t$ in $P^1fx(2) = fP^1x(2)$.

For (ii) and (iii), equate coefficients of $y(1)^3y(3)^t$ (resp. $y(1)y(2)y(3)^t$) in $P^1fx(4) = fP^1x(4)$.

Lemma 4.7. If $p = 6t + 1$ then (i) $1 = b(3)^t$,

(ii) $-2b(3) = 6c(3) + a(1)b(2)$ and

(iii) $22b(3) = 6^3a(3) - 5a(1)^3 + 6^2a(1)a(2)$.

Proof We assume that $a(4) \neq 0$. Otherwise the arguments below show that $f = 0$.

If $p = 6t + 1$, by using 4.3. and 4.6. ((ii) and (iii)) we see that $P^1fx(4) = fP^1x(4)$ gives

$$4(2S_{3t+2} - y(1)S_{3t+1}) =$$
$$(y(1)^2 - 4y(2))f(3x(3)^t - 3t^2x(1)x(2)x(3)^{t-1} + \tfrac{1}{2}t^2(t+1)x(1)^3x(3)^{t-1} + \ldots) .$$

Equating coefficients of $y(2)y(3)^t$, $y(1)y(2)^2y(3)^{t-1}$ and $y(1)^5y(3)^{t-1}$ in this gives

(i) $1 = b(3)^t$

(ii) $4t = -4(3tc(3)b(3)^{t-1} - 3t^2a(1)b(2)b(3)^{t-1})$ and

(iii) $t(t+1)(t+2) = 5b(3)^{t-1}(3ta(3) + \frac{1}{2}t^2(t+1)a(1)^3 - 3t^2a(1)a(2))$.

The lemma follows from these relations.

We collect together the information needed for the next lemma

$$-b(3) = 6c(3) - a(1)b(2) \qquad (1)$$

$$7b(3) = 6^3a(3) - 6^2a(1)a(2) + 7a(1)^3 \qquad (2)$$

$$b(2) = a(1)^2 - 4a(2) \qquad (3)$$

$$22b(3) = 6^3a(3) - 5a(1)^3 + 6^2a(1)a(2) \qquad (4)$$

$$-2b(3) = 6c(3) + a(1)b(2) \qquad (5)$$

These come respectively from 4.4. ((ii) and (iii)) , 4.6. (i) , 4.7. ((iii) and (ii)) .

<u>Lemma 4.8.</u> $b(3) = a(1)^3$.

<u>Proof</u> (1) and (5) above give $2a(1)b(2) = -b(3)$ and (3) with this gives $2a(1)(a(1)^2 - 4a(2)) = -b(3)$ (6) .

(2) and (4) give $4a(1)^3 - 24a(1)a(2) = -5b(3)$ (7) .

(6) and (7) give $3a(1)^3 = 8a(1)a(2)$ (8) .

By 4.5., $b(3) = \pm a(1)^3$. If $b(3) = -a(1)^3$, (6) becomes $a(1)^3 = 8a(1)a(2)$, which contradicts (8) , so $b(3) = a(1)^3$.

Lemma 4.9. There is an integer k such that

(i) $a(1) = 4k^2$ (ii) $a(2) = 6k^4$ (iii) $a(3) = 4k^6$.

Proof $b(3) = a(1)^3$. With 4.4(i) and 4.7. (i) this gives, $\left(\frac{a(1)}{p}\right) = 1$ for all primes $p = 6t \pm 1$. Hence $a(1) = k_1{}^2$ for some integer k_1 .

From (8) above, we see that $a(1)$ is even, so $a(1) = 4k^2$ for some k .

Parts (ii) and (iii) follow from (8) and (2) above.

Lemma 4.10. If k is as in 4.9., then $a(4) = k^8$.

Proof With $p + 1 = 6t$, equate coefficients of $y(2)^3y(3)^{t-2}$ in $fP^1x(1) = P^1fx(1)$. After simplification, this gives $-b(3)^2 = b(2)(36b(4) - b(2)^2)$, from which we get $b(4) = 4^2k^8$, since $b(3)$, and $b(2)$ are known in terms of $a(1)$ and $a(2)$.

Hence , from 4.6 (ii) and (iii) , $a(4) = k^8$.

We are now ready to prove 4.2.

Proof of 4.2. This is completed with 4.10., since, we have found that,

$$fx(1) = 4k^2y(1) , \qquad fx(2) = 6k^4\, y(1)^2 - 8k^4y(2) ,$$

$$fx(3) = 4k^6y(1)^3 - 16k^6y(1)y(2) + 4^3k^6y(3) ,$$

$fx(4) = k^8(y(1)^2 - 4y(2))^2$, which is exactly what 3.14. gives.

Remark Notice that the only monomial which doesn't appear in f , namely $y(1)y(3)$, is the one excluded by 4.1. The form of f also explains why we could assume $a(1), a(4) \neq 0$.

We now comment on when conjecture A' can be proved.

Let $p \geq 7$ and $T \subset G$ a maximal torus.

Recall that $A(p)$, the mod p Steenrod algebra, is generated by the P^i and β , together with the Adem relations.

If $t_j \in H^2(BT, Zp)$ is a generator, then

$$\begin{aligned} P^i t_j &= 0 \qquad i \geq 2 \\ &= t_j^p \qquad i = 1 \quad \text{and} \\ \beta t_j &= 0 \qquad\qquad (*) \end{aligned}$$

If $j : BT \to BG$ is the inclusion, then $j^* : H^*(BG, Z_p) \to H^*(BT, Z_p)$ is injective, onto the Weyl group invariants by [4, Theorem 20.3] , since $H_*(G,Z)$ has no p-torsion.

Thus the action of $A(p)$ on $H^*(BG, Zp)$ is completely determined by the conditions (*) and the Cartan formula. In particular this action is determined by the action of P^1 .

After these preliminary remarks, we make the following observations on the Conjecture A' .

(i) The above rather tedious method for morphisms $C_4 \to C_3$ will generalize to the case $C_n \to C_r$, $2r > n \geq r$, and probably to any situation $H^*(BH) \to H^*(BG)$, when

$$2 \max \alpha(G) > \max \alpha(H) \geq \max \alpha(G) ,$$

G and H classical groups or G_2 .

But obviously one needs to look for a more efficient method which doesn't waste effort on needless computation.

(ii) The work in [9] and Chapter 1 proves A' when $G = H$. For G_2 the cohomology map which is not a ψ^k does satisfy A' . As in the case $C_n \to C_r$, the "extension" f' can be described in terms of a polynomial $F(t)$.

<u>Lemma 4.11.</u> Let $f^* : H^*(BG_2) \to H^*(BG_2)$ be the morphism of Proposition 1.2. (ii) . Then the corresponding $F^*(t)$ is the following

$$F^*(t) = (1 - tk^2(2t(1) + t(2))^2)(1 - tk^2(t(1) - t(2))^2)(1 - tk^2(t(1) + 2t(2))^2))$$

<u>Proof</u> Just expand $F^*(t)$ and look at the coefficients of $-t$ and $-t^3$. For example, the coefficient of $-t$ is

$$k^2((t(1) - t(2))^2 + (2t(1) + t(2))^2 + (t(1) + 2t(2))^2)$$
$$= 6k^2(t(1)^2 + t(2)^2 + t(1)t(2)) = 3k^2x(2) .$$

<u>Cor. 4.12.</u> The f^* in Proposition 1.2. (ii) commutes with P^1 for all primes > 3 .

<u>Proof</u> Clear : compare the proof of 3.12. (b) .

(iii) Chapter 2 proves A' when $G = Sp(1)$, $H \neq F_4, E_6, E_7, E_8$, although the method could probably be extended to cover these remaining cases.

(iv) The method used to prove 4.2. won't generalize to the case $C_{2r} \to C_r$. Nothing simple emerges from equating coefficents, and one realizes that one must try something different.

We feel that A' could be proved for $C_n \to C_r$ (any n,r) by factoring $F(t)$ in a very large extension of $Z_p[t(1), \ldots, t(r)]$. The restriction that f commutes with P^1 should then be enough to ensure that the factoring is in $Z_p[t(1), \ldots, t(r)]$.

Finally, the concept of maximal symplectic torus makes sense : e.g. a maximal symplectic torus in $Sp(n)$ is $Sp(1)^n$.

One explanation of our inability to construct maps $BSp(1) \to BSp(n)$ with even degrees might be the following (where $G = Sp(1)$, $H = Sp(n)$.)

<u>Conjecture</u> If T,S are maximal symplectic tori in G and H, then for any map $g : BG \to BH$ with $g^* x(n) \neq 0$, there is a map $g' : BT \to BS$ such that the following diagram homotopy commutes:

$$\begin{array}{ccc} BG & \xrightarrow{g} & BH \\ \uparrow & & \uparrow \\ BT & \xrightarrow{g'} & BS \end{array},$$

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References.

1) J.F.Adams,Lectures on Lie groups,(1969),Benjamin.

2) M.F.Atiyah and F.Hirzebruch,Vector bundles and homogeneous spaces, Proc.of Symposia in Pure Maths 3,Differential Geometry A.M.S.,(1961),7-38.

3) andG.Segal ,Equivariant K-theory and completions, J.of Diff.Geo. 3(1969),1-18.

4) A.Borel ,Topics in the homology theory of fibre bundles, Lecture notes in mathematics ,36,Springer-Verlag.

5) and F.Hirzebruch,Characteristic classes and homogeneous spaces I,Amer.J.Maths.80(1958), 459-538.

6) J.Cassels and A.Frohlich,eds.,Algebraic number theory, Academic Press (1967),p.229.

7) C.Chevalley,Theory of Lie groups,Princeton U Press (1946).

8) H.Freudenthal and H.de Vries ,Linear Lie groups,Academic Press (1969).

9) J.R.Hubbuck,Mapping degrees for classifying spaces,I,II, (to appear).

10) P.A.Macmahon,Combinatory analysis,2 vols,Chelsea publishing company,New York,(1960).

11) H.Samelson, Notes on Lie algebras,Van Nostrand Reinhold,(1969).

12) E.Spanier, Algebraic topology,McGraw-Hill,New York (1966).

13).J.Stasheff, Lecture notes in mathematics,161,Springer-Verlag.

14).N.Steenrod, Milgram's classifying space of a topological group,Topology,vol.7,pp. 349-368.

15) and D.Epstein,Cohomlogy operations ,ANN.MATHS.STUDIES No.50,Princeton U Press,1962.

16) R.Stong, Notes on cobordism theory,1968,Princeton U Press, (Notes in mathematics.).

17) D.Sullivan ,Geometric Topology,part 1,Notes from M.I.T.(1971).

18)W.J.Le Veque,Topics in number theory,(2 vols),Addison-Wesley, 1961.

19)J.A.Wolf,Spaces of constant curvature,McGraw-Hill,(1967).

20) A.Borel,Topology of Lie groups and characteristic classes, Bull.A.M.S. ,61 (1955).

21) B.Harris,Torsion in Lie groups,Topology,vol 5,pp.347-354.