

Morita Cohomology

Julian V. S. Holstein

St John's College

Department of Pure Mathematics and Mathematical Statistics

University of Cambridge

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. No part of this dissertation has been submitted for any other qualification.

Summary

This work constructs and compares different kinds of categorified cohomology of a locally contractible topological space X . Fix a commutative ring k of characteristic 0 and also denote by k the differential graded category with a single object and endomorphisms k . In the Morita model structure k is weakly equivalent to the category of perfect chain complexes over k .

We define and compute derived global sections of the constant presheaf k considered as a presheaf of dg-categories with the Morita model structure. If k is a field this is done by showing there exists a suitable local model structure on presheaves of dg-categories and explicitly sheafifying constant presheaves. We call this categorified Čech cohomology *Morita cohomology* and show that it can be computed as a homotopy limit over a good (hyper)cover of the space X .

We then prove a strictification result for dg-categories and deduce that under mild assumptions on X Morita cohomology is equivalent to the category of homotopy locally constant sheaves of k -complexes on X .

We also show categorified Čech cohomology is equivalent to a category of ∞ -local systems, which can be interpreted as categorified singular cohomology. We define this category in terms of the cotensor action of simplicial sets on the category of dg-categories. We then show ∞ -local systems are equivalent to the category of dg-representations of chains on the loop space of X and find an explicit method of computation if X is a CW complex. We conclude with a number of examples.

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1. Introduction

The aim of this thesis is to construct and compare categorifications of the cohomology of topological spaces by considering coefficients in the category of differential graded categories.

We begin with the calculation of $R\Gamma_{\text{Morita}}(X, \underline{k})$, for k a field, the cohomology of a locally contractible topological space X with coefficients in a constant sheaf. We categorify by considering the constant sheaf \underline{k} not as a sheaf of rings, but as a presheaf of dg-categories with one object, where we equip dg-categories with the Morita model structure. In this model structure $k \simeq \mathbf{Ch}_{pe}$, which is fundamental to our construction.

Hence we call this categorified Čech cohomology *Morita cohomology*. We write $\mathcal{H}^M(X)$ for $R\Gamma_{\text{Morita}}(X, \underline{k})$.

Let X be a locally contractible topological space. The following characterization follows once we establish a local model structure on presheaves of dg-categories.

Theorem 2.21. *Given a good hypercover $\{U_i\}_{i \in I}$ of X one can compute $\mathcal{H}^M(X) \simeq \text{holim}_{i \in I^{op}} \mathbf{Ch}_{pe}$.*

Remark 1.1. To prove existence of the local model structure we show that \mathbf{dgCat}_k is left proper if k is a field. This may be of independent interest.

Under some further assumptions on X we also identify Morita cohomology with a more intuitive category associated to X .

Theorem 3.15. *Let X have a bounded locally finite good hypercover. Then the dg-category $\mathcal{H}^M(X)$ is quasi-equivalent to the dg-category of homotopy locally constant sheaves of perfect complexes.*

Remark 1.2. The homotopy category of the category of homotopy locally constant sheaves can be considered as the correct derived category of local systems on X in the sense that it contains the abelian category of local systems but its Ext-groups are given by cohomology of X with locally constant coefficients rather than group cohomology of the fundamental group.

The proof uses strictification to write the homotopy limit as a category of homotopy cartesian sections of a constant Quillen presheaf. Homotopy cartesian sections are then identified with homotopy locally constant sheaves.

Next we define a category of ∞ -local systems on a simplicial set K by the action of simplicial sets on dg-categories, $K \mapsto \mathbf{Ch}_{dg}^K$. This construction is well-known to give a Quillen adjunction from \mathbf{sSet} to \mathbf{dgCat} . For a topological space X one considers $\mathcal{Y}(X) := \mathbf{Ch}_{dg}^{\mathrm{Sing}^* X}$, which can be considered as a categorification of singular cohomology. We prove the following comparison theorem:

Theorem 4.4. *The category $\mathcal{H}^M(X)$ is equivalent to the category of ∞ -local systems $(\mathbf{Ch}_{pe})^{\mathrm{Sing}^*(X)}$.*

Homotopy invariance and a Mayer–Vietoris theorem are easy to establish for ∞ -local systems and hence for Morita cohomology.

The category of ∞ -local systems is closely related to the loop space of X , as is shown by the next result:

Theorem 4.8. *If X is a pointed and connected topological space the category $\mathcal{H}^M(X)$ is furthermore equivalent to the category of fibrant cofibrant representations in perfect complexes of chains on the based loop space ΩX .*

We then establish a method of computing $\mathbf{Ch}_{pe}^{C_*(\Omega X)}$ for a CW complex that allows us to calculate Morita cohomology for a number of examples.

This characterization allows us to compute Hochschild homology of Morita cohomology in some cases. It follows for example from some results available in the literature that for a simply connected space $HH_*(\mathcal{H}^M(X)) \simeq H^*(\mathcal{L}X)$, where the right hand side is cohomology of the free loop space.

The results of Chapters 3 to 5 still hold if k is an arbitrary commutative ring of characteristic 0. We indicate in Section 2.5 how to relate this to Chapter 2.

Relation to other work

Here we collect some references to ideas and results in the literature which are related to our constructions. This is not meant to be an exhaustive list.

The analogous statement to Theorem 3.15 for perfect complexes of coherent sheaves appears for example as Theorem 2.8 in [41] referring back to [24] and as an assertion in [53].

Carlos Simpson discusses non-abelian cohomology with coefficients in a stack as an internal hom-space in geometric stacks, for example in [40]. This construction also appears in work by Pantev, Toën, Vaquié, Vezzosi [35] who construct interesting additional structures on these mapping stacks. In particular in the topological situation they mention the derived stack $\mathrm{Map}(M, R\mathrm{Perf})$, where M is a manifold considered as a constant stack and $R\mathrm{Perf}$ is the moduli stack of perfect complexes. (One can of course use more general topological spaces, the manifold condition provides extra structure.) One can consider the construction of ∞ -local systems in Chapter 4 as a non-geometric version of this, which is already somewhat interesting and more tractable than the mapping stack.

The comparison of homotopy locally constant sheaves and ∞ -local systems is a linear and stable version of results in [47] or [39], where the corresponding result for presheaves of simplicial sets is proved by going via the category of fibrations.

An A_∞ -category of ∞ -local systems on a simplicial set is constructed in [7], where the authors go on to prove a Riemann-Hilbert theorem. Their explicit formulae can be seen to be equivalent to our construction, see Proposition A.3.

Outline

After briefly recalling some technical results and definitions in 2.1 we proceed in 2.2 to show that the two model structures on dg-categories are cellular and left proper. This allows us to define a local model structure on presheaves of dg-categories and define their cohomology in 2.3. We then explicitly sheafify the constant presheaf in 2.4 and use this to define Morita cohomology $\mathcal{H}^M(X)$ and write down a formula as a homotopy limit of a constant diagram with fiber \mathbf{Ch}_{pe} indexed by the distinct open sets of a hypercover. Chapter 2 closes with some comments on the situation if k is not a field in 2.5

We explain how strictification allows to compute a homotopy limit as a category of homotopy cartesian sections in 3.1 and prove a strictification result for dg-categories in 3.2. We restrict this correspondence to sections with perfect fibers to obtain a formula for Morita cohomology in 3.3. We then use this formula in 3.3 to identify $\mathcal{H}^M(X)$ with the category of homotopy locally constant sheaves on X .

Chapter 4 takes a different approach to Morita cohomology. We define a category of ∞ -local systems from the cotensor action of simplicial sets on dg-categories, and show it is equivalent to $\mathcal{H}^M(X)$ in 4.1. We

use this to identify $\mathcal{H}^M(X)$ with representations of the loop space, which is another natural generalization of local systems, in 4.2. As a by-product we recover the well-known equivalence $R\text{End}_{C_*(\Omega X)}(k, k) \simeq C^*(X, k)$. Section 4.3 is then concerned with providing an explicit method for computing the category of ∞ -local systems. In 4.4 we collect some results about finiteness of $\mathcal{H}^M(X)$ and show how to compute Hochschild (co)homology in some cases.

Chapter 5 consists of some example computations. Finally, the appendix provides a combinatorial model category of dg-categories and a construction of explicit simplicial resolutions in \mathbf{dgCat} from which we deduce an explicit formula for \mathcal{C}^K .

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2. Morita cohomology

2.1. Preliminaries

In this section we fix our notation and conventions and recall some definitions and preliminary results. Note that Lemma 2.2 does not seem to be explicitly available in the literature.

2.1.1. Notation and conventions

We assume the reader is familiar with the theory of model categories, but will try to recall all the less well-known facts about them that we use.

In any model category we write Q for functorial cofibrant replacement and R for functorial fibrant replacement. We write mapping spaces (with values in \mathbf{sSet}) in a model category \mathcal{M} as $\mathrm{Map}_{\mathcal{M}}(X, Y)$. All other enriched hom-spaces in a category \mathcal{D} will be denoted as $\underline{\mathrm{Hom}}_{\mathcal{D}}(X, Y)$. In particular we use this notation for differential graded hom-spaces, internal hom-spaces and hom-spaces of diagrams enriched over the target category. It will always be clear from context which category we enrich in.

We will work over the (underived) commutative ground ring k . We assume characteristic 0 in order to freely use differential graded constructions.

Remark 2.1. If k is not a field, for example if we work over $k = \mathbb{Z}$, then the category of dg-categories is not automatically k -flat, i.e. hom-spaces might not be cofibrant as chain complexes (see below). This means some technical results in Chapter 2 are unavailable, see Section 2.5.

$\mathbf{Ch} = \mathbf{Ch}_k$ will denote the model category of chain complexes over the ring k equipped with the projective model structure where fibrations are the surjections and weak equivalences are the quasi-isomorphisms. Note that we are using homological grading convention, i.e. the differential decreases degree. The degree is indicated by a subscript or the inverse of a superscript, i.e. $C_i = C^{-i}$.

We write \mathbf{Ch}_{pe} for the subcategory of perfect complexes in \mathbf{Ch} .

\mathbf{Ch}_{dg} denotes the dg-category whose objects are fibrant and cofibrant objects of \mathbf{Ch} . Note that there is a natural identification of the subcategory of compact objects in \mathbf{Ch}_{dg} with \mathbf{Ch}_{pe} . This follows since an object X in \mathbf{Ch}_{dg} is *compact* if $\mathrm{Hom}_{\mathrm{Ho}(\mathbf{Ch}_{dg})}(A, -)$ commutes with arbitrary coproducts. Then compact objects are precisely perfect complexes, i.e. bounded complexes which are level-wise projective. But perfect complexes are automatically cofibrant in the projective model structure.

There is a natural smart truncation functor $\tau_{\geq 0}$ from \mathbf{Ch} to $\mathbf{Ch}_{\geq 0}$, the category of non-negatively graded chain complexes, which naturally has

the projective model structure. The functor $\tau_{\geq 0}$ is right Quillen with left adjoint the inclusion functor.

2.1.2. Differential graded categories

Basic references for dg-categories are [29] and [49]. Many technical details are proven in [44].

Let \mathbf{dgCat} denote the category of categories enriched in \mathbf{Ch} . Given $\mathcal{D} \in \mathbf{dgCat}$ we define the homotopy category $H_0(\mathcal{D})$ as the category with the same objects as \mathcal{D} and $\mathrm{Hom}_{H_0(\mathcal{D})}(A, B) = H_0 \underline{\mathrm{Hom}}_{\mathcal{D}}(A, B)$. If \mathcal{D} is a model category enriched in \mathbf{Ch} we define $L\mathcal{D}$ as its subcategory of fibrant cofibrant objects. We say \mathcal{D} is a dg-model category if the two structures are compatible, that is if they satisfy the pushout-product axiom, see for example the definitions in Section 3.1 of [49]. Then $H_0(\mathcal{D}) \simeq H_0(L\mathcal{D})$, where we take the homotopy category in the sense of model categories on the left and in the sense of dg-categories on the right. We mostly consider dg-categories with unbounded hom-spaces, but there is a natural truncation functor $\tau_{\geq 0} : \mathbf{dgCat} \rightarrow \mathbf{dgCat}_{\geq 0}$ that is just truncation on hom-spaces.

Recall that there are two model structures on \mathbf{dgCat} . Firstly there is the *Dwyer–Kan model structure*, denoted \mathbf{dgCat}_{DK} . Weak equivalences are *quasi-equivalences*, i.e. dg-functors that induce weak equivalences on hom-spaces and are essentially surjective on the homotopy category. Fibrations are those dg-functors F that are surjective on hom-spaces and have the property that every homotopy equivalence $F(a) \rightarrow b'$ in the

codomain of F lifts to a homotopy equivalence $a \rightarrow b$ with $F(b) = b'$. A set of generating cofibrations is given by the following.

- $\emptyset \rightarrow k$
- $\mathcal{S}(n) \rightarrow \mathcal{D}(n)$ for all $n \in \mathbb{Z}$.

Here $\mathcal{S}(n)$ is the linearization of the category $a \xrightarrow{g} b$ where g has degree n and is the only non-identity morphism. $\mathcal{D}(n)$ equals $\mathcal{S}(n)$ with additional morphisms $k.f$ in degree $n + 1$ such that $df = g$.

Recall the functor $\mathcal{D} \mapsto \mathcal{D}\text{-Mod}$ sending a dg-category to its model category of modules, i.e. $\mathcal{D}\text{-Mod}$ is the category of functors $\mathcal{D} \rightarrow \mathbf{Ch}$ and strict natural transformations. This is naturally a cofibrantly generated model category enriched in \mathbf{Ch} whose fibrations and weak equivalences are given levelwise. We usually consider its subcategory of fibrant and cofibrant objects, $L(\mathcal{D}\text{-Mod})$.

Remark 2.2. The construction of the model category $\mathcal{D}\text{-Mod}$ follows Chapter 11 of [23], but there are some changes since we are considering enriched diagrams. Let I and J denote the generating cofibrations and generating trivial cofibrations of \mathbf{Ch} . The generating (trivial) cofibrations of $\mathcal{D}\text{-Mod}$ are then of the form $h^X \otimes A \rightarrow h^X \otimes B$ for $A \rightarrow B \in I$ (resp. J), where h^X denotes the contravariant Yoneda embedding. As in Theorem 11.6.1 of [23] we transfer the model structure from $\mathbf{Ch}^{\text{discrete}(\mathcal{D})}$. This works since h^X is compact in $\mathcal{D}\text{-Mod}$ and so are its tensor products with the domains of I , ensuring condition (1) of Theorem 11.3.2 holds. For the second condition we have to check that relative $J \otimes h^X$ -cell complexes are weak equivalences. Pushouts are constructed levelwise. The generating trivial cofibrations of \mathbf{Ch} are of

the form $0 \rightarrow D(n)$. Since the pushout $U \leftarrow 0 \rightarrow D(n)$ is weakly equivalent to U we are done.

Note that cofibrations in this model category need not be levelwise cofibrations, unless hom-spaces in \mathcal{D} are cofibrant, in which case the map $\underline{\mathbf{Hom}}(\alpha, \beta) \otimes A \rightarrow \underline{\mathbf{Hom}}(\alpha, \beta) \otimes B$ is a cofibration and the proof goes through just like in [23]. In fact, if hom-spaces are cofibrant this works for categories of functors enriched in any monoidal model category \mathcal{V} .

Remark 2.3. In order to satisfy the smallness assumption we will always assume that all our dg-categories are small relative to some larger universe.

The homotopy category of the model category $L\mathcal{D}^{\text{op}}\text{-Mod}$ is called the derived category of \mathcal{D} and denoted $D(\mathcal{D})$.

Definition. We denote by $\mathbf{dgCat}_{\text{Mor}}$ the category of dg-categories with the *Morita model structure*, i.e. the Bousfield localization of \mathbf{dgCat}_{DK} along functors that induce equivalences of the derived categories, see Chapter 2 of [44].

Fibrant objects in $\mathbf{dgCat}_{\text{Mor}}$ are dg-categories \mathcal{A} such that the homotopy category of \mathcal{A} is equivalent (via Yoneda) to the subcategory of compact objects of $D(\mathcal{A})$ [29]. We can phrase this as: every compact object is quasi-representable. An object $X \in D(\mathcal{A})$ is called *compact* if $\text{Hom}_{D(\mathcal{A})}(X, -)$ commutes with arbitrary coproducts. We denote by $(\)_{pe}$ the subcategory of compact objects. Morita fibrant dg-categories are also called *triangulated* since their homotopy category is an (idempotent complete) triangulated category.

With these definitions $\mathcal{D} \mapsto L(\mathcal{D}^{\text{op}}\text{-Mod})_{pe}$, often denoted the *triangulated hull*, is a fibrant replacement, for example $k \mapsto \mathbf{Ch}_{pe}$.

The category \mathbf{dgCat} is symmetric monoidal with tensor product $\mathcal{D} \otimes \mathcal{E}$ given as follows. The objects are $\text{Ob}\mathcal{D} \times \text{Ob}\mathcal{E}$ and $\underline{\text{Hom}}_{\mathcal{D} \otimes \mathcal{E}}((D, E), (D', E')) := \underline{\text{Hom}}_{\mathcal{D}}(D, D') \otimes \underline{\text{Hom}}_{\mathcal{E}}(E, E')$. The unit is the one object category k , which is cofibrant in either model structure.

While \mathbf{dgCat} is not a monoidal model category there is a derived internal Hom space and the mapping spaces in \mathbf{dgCat}_{Mor} can be computed as follows [48]: Let $\underline{RHom}(\mathcal{C}, \mathcal{D})$ be the dg-category of right-quasirepresentable $\mathcal{C} \otimes \mathcal{D}^{\text{op}}$ -modules, i.e. functors $F : \mathcal{C} \otimes \mathcal{D}^{\text{op}} \rightarrow \mathbf{Ch}$ such that for any $c \in \mathcal{C}$ we have that $F(c, -)$ is isomorphic in the homotopy category to a representable object in $\mathcal{D}^{\text{op}}\text{-Mod}$ and moreover cofibrant. Then \underline{RHom} is right adjoint to the derived tensor product \otimes^L . Moreover $\text{Map}(\mathcal{C}, \mathcal{D})$ is weakly equivalent to the nerve of the subcategory of quasi-equivalences in $\underline{RHom}(\mathcal{C}, \mathcal{D})$. We will quote further properties of this construction as needed.

We will need the following lemma relating the two model structures. The definition of homotopy limits will be recalled in Section 2.1.4

Lemma 2.1. *Fibrant replacement as a functor from \mathbf{dgCat}_{Mor} to \mathbf{dgCat}_{DK} preserves homotopy limits.*

Proof. We know that \mathbf{dgCat}_{Mor} is a left Bousfield localization of \mathbf{dgCat}_{DK} , hence the identity is a right Quillen adjunction and its derived functor, given by fibrant replacement, preserves homotopy limits. \square

This means we can compute homotopy limits in \mathbf{dgCat}_{Mor} by computing the homotopy limit of a levelwise Morita-fibrant replacement in \mathbf{dgCat}_{DK} .

We will abuse notation and write R for the dg-algebra by the same name as well as for the 1-object dg-category with endomorphism space R concentrated in degree 0.

Recall that there is a model structure on differential graded algebras over k with unbounded underlying chain complexes, which can be considered as the subcategory of one-object-categories in \mathbf{dgCat}_{DK} .

Note that all objects of \mathbf{dgCat}_{DK} are fibrant and hence \mathbf{dgCat}_{DK} is right proper (i.e. pullbacks along fibrations preserve weak equivalences), while \mathbf{dgCat}_{Mor} as a left Bousfield localization need not be and in fact is not, cf. Example 4.10 of [46]. We will consider the question of left properness (i.e. whether the pushout along a cofibration preserves weak equivalences) in 2.5.

Recall also the category $\mathbf{sModCat}_k$ of categories enriched over simplicial k -modules and the natural Dold–Kan or Dold–Puppe functor $DK : \mathbf{dgCat}_{\geq 0} \rightarrow \mathbf{sModCat}$ that is defined hom-wise. DK and its left adjoint, normalization, form a Quillen equivalence between non-negatively graded dg-categories and $\mathbf{sModCat}$. For details see section 2.2 of [41] or [45].

Remark 2.4. While we are working with differential graded categories we are facing some technical difficulties for lack of good internal hom-spaces. It would be interesting to know if another model of stable linear $(\infty, 1)$ -categories could simplify our treatment.

2.1.3. Model \mathcal{V} -categories and simplicial resolutions

Model categories are naturally models for ∞ -categories and in fact have a notion of mapping spaces. Even if a model category is not enriched in \mathbf{sSet} one can define mapping spaces in $Ho(\mathbf{sSet})$. One way to do this is by defining simplicial resolutions, which we will make extended use of.

Let Δ be the simplex category and consider the constant diagram functor $c : \mathcal{M} \rightarrow \mathcal{M}^{\Delta^{op}}$. Then a *simplicial resolution* M_\bullet for $M \in \mathcal{M}$ is a fibrant replacement for cM in the Reedy model structure on $\mathcal{M}^{\Delta^{op}}$. (For a definition of the Reedy model structure see for example Chapter 15 of [23].)

For example, this construction allows us to compute mapping spaces: If $cB \rightarrow \tilde{B}$ is a simplicial framing in $\mathcal{M}^{\Delta^{op}}$ and QA a cofibrant replacement in \mathcal{M} then $\text{Map}(A, B) \simeq \text{Hom}^\bullet(QA, \tilde{B}) \simeq R(\text{Hom}^\bullet(-, c-))$, where the right-hand side uses the bifunctor $\text{Hom}^\bullet : \mathcal{M}^{op} \times \mathcal{M}^{\Delta^{op}} \rightarrow \mathbf{Set}^{\Delta^{op}}$ that is defined levelwise. The dual notion is a *cosimplicial resolution*.

Recall \mathcal{V} is a symmetric monoidal model category if it is both symmetric monoidal and a model category and the structures are compatible, to be precise they satisfy the pushout-product axiom, see Definition 4.2.1 in [25]. This means in particular that tensor and internal Hom give rise to Quillen functors. We then call the adjunction of two variables satisfying the pushout-product axiom a *Quillen adjunction of two variables*.

Similarly a *model \mathcal{V} -category* \mathcal{M} is a model category \mathcal{M} that is tensored, cotensored and enriched over \mathcal{V} such that the pushout product axiom holds. We call a model **Ch**-category a *dg-model category*.

For example a model **sSet**-category, better known as a *simplicial model category*, \mathcal{M} consists of the data $(\mathcal{M}, \text{Map}, \otimes, \text{map})$ where the enrichment $\text{Map} : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathbf{sSet}$, the cotensor (or power) map $: \mathbf{sSet}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{M}$ and the tensor $\otimes : \mathbf{sSet} \times \mathcal{M} \rightarrow \mathcal{M}$ satisfy the obvious adjointness properties (in other words, they form an adjunction of two variables). The pushout-product axiom says that the natural map $f \square g : A \otimes L \amalg_{A \otimes K} B \otimes K \rightarrow B \otimes L$ is a cofibration if f and g are and is acyclic if f or g is moreover acyclic.

While not every model category is simplicial, every homotopy category of a model category is enriched, tensored and cotensored in $Ho(\mathbf{sSet})$. In fact, \mathcal{M} can be turned into a simplicial category in the sense that there is an enrichment Map and there are a tensor and cotensor which can be constructed from the simplicial and cosimplicial resolutions. Let a cosimplicial resolution $A^\bullet \in \mathcal{M}^\Delta$ and a simplicial set K be given. Consider ΔK , the category of simplices of K , with the natural map $u : \Delta K \rightarrow \Delta$ sending $\Delta[n] \mapsto K$ to $[n]$. We define $A^\bullet \otimes K = \text{colim}_{\Delta K} A^n$ to be the image of A^\bullet under $\text{colim} \circ u^* : \mathcal{C}^\Delta \rightarrow \mathcal{C}^{\Delta K} \rightarrow \mathcal{C}$. Similarly there is A^K which is the image of the simplicial resolution $A_\bullet \in \mathcal{M}^{\Delta^{\text{op}}}$ under $\text{lim} \circ v^*$, where $v : \Delta K^{\text{op}} \rightarrow \Delta^{\text{op}}$. This can also be written as $A^K = \text{lim}_n(\prod_{K_n} A_n)$.

If \mathcal{M} is a simplicial category one can use $(RA)^{\Delta^n}$ for A_n and $(QA) \otimes \Delta^n$ for A^n .

Remark 2.5. Note that A^K can also be written as a homotopy limit, $\operatorname{holim}_{\Delta K^{\text{op}}} A_n$. (The definition of a homotopy limit is recalled below.) This follows for example from Theorem 19.9.1 of [23], the conditions are satisfied by Propositions 15.10.4 and 16.3.12.

The functor $(A, K) \mapsto A^K$ is adjoint to the mapping space construction $A, B \mapsto \operatorname{Hom}(B, A_\bullet) \in \mathbf{sSet}$. Similarly $(B, K) \mapsto B \otimes K$ is adjoint to the mapping space construction $A, B \mapsto \operatorname{Hom}(B^\bullet, A) \in \mathbf{sSet}$, see Theorem 16.4.2 in [23]. Hence on the level of homotopy categories the two bifunctors together with Map give rise to an adjunction of two variables. This is of course not a Quillen adjunction, but it is sensitive enough to the model structure to allow for certain derived functors. We will quote further results about this construction as needed.

2.1.4. A very short introduction to homotopy limits

Ordinary limits in a model category are not very well behaved, in particular they are not invariant under weak equivalence. A much better notion is provided by *homotopy limits*.

Let I be a small category, \mathcal{M} a model category and $C : I \rightarrow \mathcal{M}$ a diagram. On the category of diagrams \mathcal{M}^I we can often define the *injective model structure* with levelwise weak equivalences and cofibrations. Limits are right adjoints of the constant diagram functor and with the injective model category structure on diagrams they become Quillen adjoints. Then homotopy limits are just their right derived functors.

in general, the injective model structure only exists if \mathcal{M} is combinatorial (or if the index category is direct) and the dual projective model structure still needs \mathcal{M} to be cofibrantly generated (or the index category to be inverse). But even if we only have \mathcal{M}^I as a category with weak equivalences, i.e. as a homotopical category, we can still define derived functors, see for example [15]. Simply put, the homotopy limit is the right adjoint to the constant functor $Ho(\mathcal{M}) \rightarrow Ho(\mathcal{M}^I)$ on the level of homotopy categories.

Note that the derived functors of right Quillen functors preserve homotopy limits, and so do all Quillen equivalences. (The reason for the latter is that Quillen equivalences induce equivalences of the homotopy categories of diagram categories.)

To compute homotopy limits explicitly there is a number of formulae available. Let us assume C_i is levelwise fibrant, and replace fibrantly if that is not the case. (Sometimes taking the homotopy limit of a levelwise fibrant replacement is called the *corrected homotopy limit*.) Then we will use the following, cf. e.g. Definition 19.1.4 in [23].

$$\operatorname{holim}_i C_i = \operatorname{eq} \left(\prod_{i \in I} C_i^{N(I \downarrow i)} \rightrightarrows \prod_{h \rightarrow j} C_j^{N(I \downarrow h)} \right)$$

Here we use a simplicial cotensor defined using a *simplicial frame* $(C_i)_\bullet$ for C_i . Since C_i is assumed fibrant this is just a simplicial resolution such that $C_i \rightarrow (C_i)_0$ is an isomorphism, and since the construction is invariant under weak equivalences between fibrant objects we can take any simplicial resolution.

By contrast, for a much more computable example, let us consider a homotopy pullback. It is provided by replacing the target fibrantly and both maps by fibrations before taking the limit, i.e.

$$\mathrm{holim}(A \rightarrow B \leftarrow C) \simeq \mathrm{lim}(\tilde{A} \rightarrow \tilde{B} \leftarrow \tilde{C})$$

where $\tilde{A} \rightarrow \tilde{B}$ and $\tilde{C} \rightarrow \tilde{B}$ are fibrations and $\tilde{A} \simeq A$ etc. If the model category is right proper, i.e. pushout along fibrations preserves weak equivalences, it suffices to replace one map by a fibration.

Similarly we construct homotopy ends of bifunctors. Recall that an end is a particular kind of limit. Let $\alpha(I)$ denote the *twisted arrow category* of I : Objects are arrows, $f : i \rightarrow j$, and morphisms are opposites of factorizations, i.e. $(f : i \rightarrow j) \Rightarrow (g : i' \rightarrow j')$ consists of maps $i' \rightarrow i$ and $j \rightarrow j'$ such that their obvious composition with f equals g . Then there are natural maps s and t (for source and target) from $\alpha(I)$ to I^{op} and I respectively. For a bifunctor $F : I^{\mathrm{op}} \times I \rightarrow \mathcal{C}$ one defines the end $\int_i F(i, i)$ to be $\mathrm{lim}_{\alpha(I)} (s \times t)^* F$. Then the homotopy end is:

$$\int_i^h F(i, i) := \mathrm{holim}_{\alpha(I)} (s \times t)^* F$$

Details on this view on homotopy ends can be found (dually) in [27].

The canonical example for an end is that natural transformations from F to G can be computed as $\int_A \mathrm{Hom}(FA, GA)$. A similar example of the use of homotopy ends is provided by the computation of mapping spaces in the diagram category of a model category $\mathrm{Map}(A_\bullet, B_\bullet) \simeq \int_i^h \mathrm{Map}(A_i, B_i)$. The case of simplicial sets is dealt with in [17].

In general, we have the following lemma. Assume \mathcal{M}^I exists with the injective model structure and let Q and R denote cofibrant and fibrant replacement in this model category.

Lemma 2.2. *Consider a right Quillen functor $\underline{H} : \mathcal{M}^{\text{op}} \times \mathcal{M} \rightarrow \mathcal{V}$. Then there is a natural Quillen functor $(F, G) \rightarrow \int_i \underline{H}(Fi, Gi)$ from $(\mathcal{M}^I)^{\text{op}} \times \mathcal{M}^I$ to \mathcal{V} whose derived functor is*

$$(F, G) \mapsto \int_i \underline{H}(QFi, RGi)$$

which is weakly equivalent to

$$(F, G) \mapsto \int_i^h R\underline{H}(Fi, Gi)$$

Proof. The \mathcal{V} -structure exists by standard results in [30]. We have a model \mathcal{V} -category by Lemma 2.3 below. Hence the derived functor is $(F, G) \mapsto \int_i \underline{H}(QFi, RGi)$.

On the other hand $\int_i \underline{H}(Fi, Gi)$ is the composition of levelwise hom-spaces with the limit,

$$\lim \circ \underline{H}^{\alpha(I)} \circ (s \times t)^* : (\mathcal{M}^I)^{\text{op}} \times \mathcal{M}^I \rightarrow (\mathcal{M}^{\text{op}} \times \mathcal{M})^{\alpha(I)} \rightarrow \mathcal{V}^{\alpha(I)} \rightarrow \mathcal{V}$$

But then the derived functor is the composition of derived functors, $\int_i^h R\underline{H}(Fi, Gi)$.

This is a little subtle, since we do not want to fibrantly replace at the level of diagram categories. However, levelwise $R\underline{H}$ from $(\mathcal{M}^{\text{op}})^I \times \mathcal{M}^I$ to $\mathcal{V}^{\alpha(I)}$ is a derived functor. This is the case since levelwise fibrant replacement gives a right deformation retract in the sense of 40.1 in [15]

since $(s \times t)^*$ preserves all weak equivalences and levelwise \underline{H} preserves weak equivalences between levelwise fibrant objects. \square

Remark 2.6. A slight modification of the lemma implies the formula for mapping spaces. We just have to replace \underline{H} by $\text{Hom}^\bullet : \mathcal{M}^{\text{op}} \times \mathcal{M}^{\Delta^{\text{op}}} \rightarrow \mathbf{sSet}$ and adjust the proof accordingly.

Lemma 2.3. \mathcal{M}^I is a model \mathcal{V} -category if \mathcal{M} is.

Proof. Given cofibrations $f : V \rightarrow W$ in \mathcal{V} and $g : A \rightarrow B$ in \mathcal{M}^I we have to show that whenever f and g are cofibrations then so is

$$f \square g : (V \otimes B) \amalg_{V \otimes A} (W \otimes A) \rightarrow W \otimes B$$

and $f \square g$ is an acyclic cofibration if f or g is.

Cofibrations and weak equivalences in \mathcal{M}^I are defined levelwise so it is enough to check $(f \square g)_i$. Colimits in diagram categories are also defined levelwise, so it is enough to check $f \square (g_i)$. But by assumption \mathcal{M} is a model \mathcal{V} -category. \square

These results generalise verbatim to categories of presections which will be defined in Section 3.1.

Homotopy colimits etc. can be defined entirely dually.

2.2. Further properties of \mathbf{dgCat}

In this section we will show that \mathbf{dgCat}_{Mor} is cellular and, if k is a field, left proper. This will be used to localize diagrams of dg-categories. It may also be of independent interest.

Proposition 2.4. *The categories \mathbf{dgCat}_{DK} and \mathbf{dgCat}_{Mor} and their small diagram categories are cellular.*

Proof. Recall that a model category is cellular if it is cofibrantly generated with generating cofibrations I and generating trivial cofibrations J such that the domains and codomains of the elements of I are compact, the domains of the elements of J are small relative to I and the cofibrations are effective monomorphisms. See Chapter 10 of [23] for precise definitions.

Left Bousfield localization preserves being cellular, and so does taking the category of diagrams indexed by a small category I with the projective model structure, see Theorem 4.1.1 and Proposition 12.1.5 of [23]. So it is enough to show \mathbf{dgCat}_{DK} is cellular.

The domains and codomains of elements of I are categories with at most two objects and perfect hom-spaces, so maps from these objects to relative I -complexes factor through small subcomplexes. So domains and codomains of I are compact.

Similarly the domains of the elements of J have two objects and perfect hom-spaces. Hence taking maps from a domain of J commutes with filtered colimits. So domains of J are small relative to I .

We are left to check that relative I -cell complexes, i.e. transfinite compositions of pushouts of generating cofibrations, are effective monomorphisms, i.e. any relative I -cell complex $f : X \rightarrow Y$ is the equalizer of $Y \rightrightarrows Y \amalg_X Y$. Note that we form the pushout along a generating cofibration by attaching maps freely. If we form \mathcal{C}' and \mathcal{C}'' from \mathcal{C} by attaching maps freely then the equalizer will have the same objects and the hom-spaces are given by considering morphisms of the pushout that are in the image of both \mathcal{C}' and \mathcal{C}'' . But these are precisely the hom-spaces of \mathcal{C} . \square

Proposition 2.5. *If k is a field the categories \mathbf{dgCat}_{DK} and \mathbf{dgCat}_{Mor} and their small diagram categories are left proper.¹*

Proof. Left Bousfield localization preserves left properness, see Proposition 3.4.4 of [23], and so does taking the category of diagrams indexed by a small category I (with the injective or projective model structure) since pushout and pullback are constructed levelwise. So it is enough to show \mathbf{dgCat}_{DK} is left proper.

The main work is showing that pushout along the generating cofibrations preserves quasi-equivalences.

To see this suffices note firstly that transfinite compositions are just filtered colimits, and filtered colimits preserve quasi-equivalences as follows: A filtered colimit of categories can be computed set-theoretically on objects and morphisms. Now filtered colimits preserve weak equivalences of simplicial sets (since S^n is compact) and hence

¹Thanks to Jon Pridham for helpful discussions about this result.

of mapping spaces. They also preserve the homotopy category since a filtered colimit of equivalences of categories is an equivalence of categories and taking the homotopy category commutes with filtered colimits.

Secondly, if pushout along some map preserves weak equivalences then so does pushout along a retract by functoriality of colimits. Since all cofibrations are retracts of transfinite compositions of generating cofibrations, it does indeed suffice to check generating cofibrations.

It is clear that pushout along $\emptyset \rightarrow k$ preserves quasi-equivalences.

So consider the generating cofibration $\mathcal{S}(n) \rightarrow \mathcal{D}(n)$ with a map $j : \mathcal{S}(n) \rightarrow \mathcal{C}$ and a quasi-equivalence $F : \mathcal{C} \rightarrow \mathcal{E}$. Let the objects of $\mathcal{S}(n)$ be denoted a, b and the non-identity morphism g . Then in forming the pushforward we adjoin a new map f with $df = j(g)$. We call the resulting category \mathcal{C}' . Then let \mathcal{E}' be the pushout of $\mathcal{S}(n) \rightarrow \mathcal{D}(n)$ along $F \circ j$.

The pushout along j has the same objects as \mathcal{C} . The morphism space is obtained by collecting maps from C to D , graded by how often they factor through $j(a) \rightarrow j(b)$. Write $\mathcal{C}(A, B)$ etc. for the enriched hom-spaces $\underline{\text{Hom}}_{\mathcal{C}}(A, B)$ etc. Then the hom-spaces in \mathcal{C}' are given as follows:

$$\mathcal{C}'(C, D) = \text{Tot}^{\oplus} (\mathcal{C}(C, D) \oplus (\mathcal{C}(j(b), D) \otimes k.f \otimes T \otimes \mathcal{C}(C, j(a)))) \quad (2.1)$$

Here $T = \sum_n (\mathcal{C}(j(b), j(a)) \otimes k.f)^{\otimes n}$ and we introduce a horizontal degree n with $\mathcal{C}(C, D)$ in degree -1 . The right hand side has a vertical differential given by the internal differential and a horizontal differential

given by $f \mapsto j(g) \in \text{Hom}(j(b), j(a))$ composed with the necessary compositions.

If the functor F is not the identity on objects from \mathcal{C} to \mathcal{E} we factor $F = Q \circ H : \mathcal{C} \rightarrow \mathcal{D} \rightarrow \mathcal{E}$ where \mathcal{D} has as objects the objects of \mathcal{C} but $\text{Hom}_{\mathcal{D}}(A, B) = \text{Hom}_{\mathcal{E}}(FA, FB)$. Then H is identity on objects and Q is an isomorphism on hom-spaces. We form the pushforward and obtain the factorization $F' = Q' \circ H'$ through \mathcal{D}' .

So it suffices to prove the following two lemmas. □

Lemma 2.6. *Q' defined as above is a quasi-isomorphism if Q is.*

Proof. Q' is quasi-essentially surjective if Q is since both $\mathcal{D} \rightarrow \mathcal{D}'$ and $\mathcal{E} \rightarrow \mathcal{E}'$ are essentially surjective as pushout along j does not change the set of objects.

We filter the formula 2.1 for hom-spaces in \mathcal{E}' by columns, i.e. by n . This filtration is bounded below and exhaustive for the direct sum total complex and hence the spectral sequence converges. Now we use the fact that Q induces isomorphisms on hom-spaces to obtain an isomorphism of spectral sequences, which in turn induces an isomorphism $\mathcal{D}'(C, D) \cong \mathcal{E}'(QC, QD)$, so Q' gives weak equivalences on mapping spaces.

Note that since $\mathcal{S}(n)$ maps to \mathcal{E} via \mathcal{D} all the hom-spaces involved in computing $\mathcal{E}'(QC, QD)$ are indeed images of hom-spaces in \mathcal{D} . □

Lemma 2.7. *H' defined as above is a quasi-isomorphism if H is.*

Proof. H' is quasi-essentially surjective if H is for the same reason that Q' is.

To consider the effect of H' on mapping spaces proceed as in the previous lemma.

Now H only induces weak equivalences on hom-spaces, but we know all hom-spaces are flat over k as k is a field. Hence the tensor product in 2.1 preserves weak equivalences. So the same spectral sequence argument applies. \square

Remark 2.7. Note that Dwyer and Kan prove left properness for simplicial categories on a fixed set of objects in [16].

Remark 2.8. If we have some non-flat hom-spaces then things go wrong even in the subcategory of dg-algebras. Consider Example 2.11 in [37]. In that paper the existence of a proper model for simplicial k -algebras is proven. A similar result for dg-categories may or may not be true, but is certainly beyond the scope of this work

Hence the DK -model category of dg-categories is only left proper if all dg-categories are k -flat, i.e. if and only if k has flat dimension 0, which is the case if and only if k has no nilpotents and Krull dimension 0. In practice we may as well assume k is a field. See Section 2.5 for how to interpret some of the results in the remainder of this chapter for more general k .

2.3. Cohomology of presheaves of model categories

Now we will define what we mean by cohomology of a sheaf with coefficients in a model category.

Let us assume \mathcal{M} is a cellular and left proper model category. The case we are interested in is $\mathcal{M} = \mathbf{dgCat}_{Mor}$.

We will consider the category \mathcal{M}^J of presheaves on a category J^{op} with values in a model category \mathcal{M} . We will denote by \mathcal{M}_{proj}^J the projective model structure on \mathcal{M}^J with levelwise weak equivalences and fibrations, and whose cofibrations are defined by the lifting property. If \mathcal{M} is cofibrantly generated this is well-known to be a model structure, which is cellular and left proper if \mathcal{M}^J is.

We are interested in enriching the model category \mathcal{M}_{proj}^J . Let us start by recalling the case where the construction is straightforward. Let \mathcal{V} be a monoidal model category and assume that it has a cofibrant unit. $\mathcal{V} = \mathbf{sSet}, \mathbf{Ch}$ are examples. Then if \mathcal{M} is a model \mathcal{V} -category, then so is \mathcal{M}^J . In particular if \mathcal{M} is monoidal then \mathcal{M}^J is a model \mathcal{M} -category. We can write $\underline{\mathbf{Hom}}$ for the enriched hom-spaces, and the functor $\mathbf{Hom} : (\mathcal{M}^J)^{op} \times \mathcal{M}^J \rightarrow \mathcal{M}$ is right Quillen and there is a pleasant derived functor $R\underline{\mathbf{Hom}}$ obtained by fibrant and cofibrant replacement, see Lemma 2.2

If \mathcal{M} is monoidal and a model category, but not a monoidal model category, then we can still construct an \mathcal{M} -enrichment of \mathcal{M}_{proj}^J as a plain category, which will of course not be a model category enrichment. We define $\underline{\mathbf{Hom}}_{\mathcal{M}^J}(A, B) = \int_j \underline{\mathbf{Hom}}(A(j), B(j))$, see [30].

Note that this enrichment is not in general derivable, i.e. weak equivalences between cofibrant and fibrant pairs of objects do not necessarily go to weak equivalences. So defining a suitable substitute for \underline{RHom} takes some care, see the proof of Lemma 2.9.

We have to consider this case since our example of interest is $\mathcal{M} = \mathbf{dgCat}$, which is a symmetric monoidal category and a model category, see [49], but not a symmetric monoidal model category. (The tensor product of two cofibrant objects need not be cofibrant.)

Now fix a locally contractible topological space X , for example a CW complex, and consider presheaves on $Op(X)$. We consider the Grothendieck topology induced by the usual topology on X and write the site as $(\mathbf{Set}^{Op(X)^{op}}, \tau)$. In other words τ is just the collection of maps represented by open covers. (We will not use any more general Grothendieck topologies or sites.) We let $J = Op(X)^{op}$. Our aim is to localize presheaves on $Op(X)$ with respect to covers in τ .

Recall that a *left Bousfield localization* of a model category \mathcal{N} at a set of maps S is a left Quillen functor $\mathcal{N} \rightarrow \mathcal{N}_S$ that is initial among left Quillen functors sending the elements of S to isomorphisms in the homotopy category. We need to know that left Bousfield localizations of \mathcal{M}^I exist. There are two general existence results: If \mathcal{M} is combinatorial and left proper or if \mathcal{M} is cellular and left proper. We have shown that diagrams in \mathbf{dgCat} satisfy the latter condition in Section 2.2.

Remark 2.9. In the appendix we prove that there is an equivalent subcategory of \mathbf{dgCat}_{DK} that is combinatorial, see Section A.1. Then

the existence of Bousfield localizations follows from Jeff Smith's theorem, which is for example proven as Theorem 2.11 in [1].

Lemma 2.8. *Assume \mathcal{N} is a cellular and left proper model category and let S be a set of maps. Then \mathcal{N}_S exists. The cofibrations are equal to projective cofibrations, weak equivalences between are S -local weak equivalences and fibrant objects are S -local objects.*

Proof. This is Theorem 4.1.1 of [23]. □

Recall for future reference that an object P is S -local if it is fibrant in \mathcal{N} and every $f : A \rightarrow B \in S$ induces a weak equivalence $\text{Map}_{\mathcal{N}}(B, P) \simeq \text{Map}_{\mathcal{N}}(A, P)$. A map $g : C \rightarrow D$ is an S -local weak equivalence if it induces a weak equivalence $\text{Map}_{\mathcal{N}}(D, P) \simeq \text{Map}_{\mathcal{N}}(C, P)$ for every S -local P .

Given a set N we write $N \cdot M := \coprod_N M \in \mathcal{M}$ for the tensor over **Set** and extend this notation to presheaves.

Definition. Let $\mathcal{M}_\tau^J := (\mathcal{M}_{proj}^J)_{H_\tau}$ denote the left Bousfield localization of \mathcal{M}_{proj}^J with respect to

$$H_\tau = \{S \cdot \mathbf{1}_{\mathcal{M}} \rightarrow h_W \cdot \mathbf{1}_{\mathcal{M}} \mid S \rightarrow h_W \in \tau\}$$

Here h_- denotes the covariant Yoneda embedding $X \mapsto \text{Hom}(-, X)$.

We have assumed \mathcal{M} and hence \mathcal{M}^J is cellular and left proper. Since H_τ is a set the localization \mathcal{M}_τ^J exists.

We have now localized with respect to Čech covers. We are interested in the *local model structure* which is obtained by localizing at all hypercovers.

Remark 2.10. By way of motivation see [12] for the reasons that localizing at hypercovers gives the local model structure on simplicial presheaves, i.e. weak equivalences are precisely stalk-wise weak equivalences.

Definition. A *hypercov*er of an open set $W \subset X$ is a simplicial presheaf U_* on the topological space W such that:

1. For all $n \geq 0$ the sheaf U_n is isomorphic to a disjoint union of a small family of presheaves representable by open subsets of W . We can write $U_n = \coprod_{i \in I_n} h_{U_n^{(i)}}$ for a set I_n where the $U_n^{(i)} \subset W$ are open.
2. The map $U_0 \rightarrow *$ lives in τ , i.e. the $U_0^{(i)}$ form an open cover of W .
3. For every $n \geq 0$ the map $U_{n+1} \rightarrow (\text{cosk}_n U_*)_{n+1}$ lives in τ . Here $(\text{cosk}_n U)_{n+1} = M_n^W U$ is the n -th matching object computed in simplicial presheaves over W .

Intuitively, the spaces occurring in U_1 form a cover for the intersections of the $U_0^{(i)}$, the spaces in U_2 form a cover for the triple intersections of the $U_1^{(i)}$ etc. To every Čech cover one naturally associates a hypercover in which all $U_{n+1} \rightarrow (\text{cosk}_n U_*)_{n+1}$ are isomorphisms.

Note that despite the notation U_n is not an open set but a presheaf on open sets that is a coproduct of representables.

We denote by $I = \cup I_n$ the category indexing the representables making up the hypercover.

Associated to any hypercover of a topological space is the simplicial space $n \mapsto \prod_{i \in I_n} U_n^i$ which is also sometimes called a hypercover.

Hypercovers are naturally simplicial presheaves. We work with presheaves with values in a more general model category. The obvious way to associate to a simplicial object in a model category a plain object is to take the homotopy colimit.

Definition. Let the set of *hypercovers in \mathcal{M}^J* be defined as

$$\check{H}_\tau = \{ \text{hocolim}_I (U_* \cdot \mathbf{1}_{\mathcal{M}}) \rightarrow h_W \cdot \mathbf{1}_{\mathcal{M}} \mid U_* \rightarrow h_W \text{ a hypercover} \}$$

where we take the levelwise tensor and the homotopy colimit in \mathcal{M}^J with the projective model structure. Since disjoint union commutes with cofibrant replacement we could equivalently take the limit of U_n over Δ^{op} , the opposite of the simplex category.

Remark 2.11. Note that the homotopy colimit does not change if instead we use the localised model structure \mathcal{M}_τ^J . Left Bousfield localization is left Quillen and hence preserves homotopy colimits.

Definition. Let the left Bousfield localization of \mathcal{M}_τ^J at the hypercovers \check{H}_τ be denoted by $\mathcal{M}_\check{\tau}^J$ and call it the *local model structure*.

The localization exists just as before. The fibrant objects are the \check{H}_τ -local objects of \mathcal{M}^J .

Note that $\underline{\text{Hom}}(h_W, \mathcal{F}) \simeq \mathcal{F}(W)$ if the model structure on \mathcal{M}^J is enriched over \mathcal{M} . So we sometimes write hypercovers as if they

are open sets. For example given a hypercover U_* and a presheaf $\mathcal{F} \in \mathcal{M}^J$ we write $\mathcal{F}(U_n)$ for $\underline{\text{Hom}}(U_n, \mathcal{F})$ etc. In particular $\mathcal{F}(U_n) = \mathcal{F}(\coprod_i U_n^{(i)}) := \prod_i \mathcal{F}(U_n^{(i)})$.

Definition. We call a presheaf \mathcal{F} a *sheaf* (sometimes called *hypersheaf*) if it satisfies

$$\mathcal{F}(W) \simeq \text{holim}_{I^{\text{op}}} \mathcal{F}(U_*) \text{ for every hypercover } U_* \text{ of every open } W \subset X \quad (2.2)$$

The limit is over $I^{\text{op}} = \cup I_n$; we could write it $\text{holim}_n \text{holim}_{i \in I_n} \mathcal{F}(U_n^{(i)})$ which can be considered as $\text{holim}_{n \in \Delta} \mathcal{F}(U_n)$ using the convention above. This condition is also called *descent* with respect to hypercovers.

Remark 2.12. If we give, say, the category of abelian groups the trivial model structure (only identities are weak equivalences and all maps are fibrations and cofibrations) we recover the usual notion of sheaf.

Note that in general being a hypersheaf is a stronger condition than being a sheaf of underived objects.

For the next Lemma we need \mathcal{M} to have a certain homotopy enrichment over itself. For simplicity we specialise to $\mathcal{M} = \mathbf{dgCat}_{\text{Mor}}$.

Lemma 2.9. *Levelwise fibrant sheaves are fibrant in the above model structure.*

Proof. We need to show that for a levelwise fibrant presheaf \mathcal{F} the sheaf condition on \mathcal{F} implies that \mathcal{F} is \check{H}_τ -local, i.e. that whenever $\epsilon : \text{hocolim}(U_* \cdot \mathbf{1}) \rightarrow h_W \cdot \mathbf{1}$ is in \check{H}_τ we have $\text{Map}(\text{hocolim}(U_* \cdot \mathbf{1}), \mathcal{F}) \simeq$

$\text{Map}(h_W \cdot \mathbf{1}, \mathcal{F})$. We will show that both sides are weakly equivalent to $\text{Map}_{\mathbf{dgCat}_{Mor}}(\mathbf{1}, \mathcal{F}(W))$.

We need a suitable derived hom-space between sheaves of dg-categories with values in dg-categories. We define $\underline{RHom}'(A_\bullet, B_\bullet) := \int_V^h \underline{RHom}(A_V, B_V)$, where \underline{RHom} is Toën's internal derived Hom of dg-categories.

First note that

$$\underline{RHom}'(h_W \cdot \mathbf{1}, \mathcal{F}) \simeq \int_{V \subset W}^h \underline{RHom}(\mathbf{1}, \mathcal{F}(V)) \simeq \text{holim}_{V \subset W} \mathcal{F}(V)$$

The first weak equivalence holds since $h_W(V)$ is just the indicator function for $V \subset W$ and the second since the homotopy end over a bifunctor that is constant in the first variable degenerates to a homotopy limit, by comparing the diagrams. Then we observe $\text{holim}_{V \subset W} \mathcal{F}(V) \simeq \mathcal{F}(W)$ if \mathcal{F} satisfies the sheaf condition.

We claim that this implies $\text{Map}(h_W \cdot \mathbf{1}, \mathcal{F}) \simeq \text{Map}(\mathbf{1}, \mathcal{F}(W))$. Note that in \mathbf{dgCat} we have $\text{Map}(A, B) \simeq \text{Map}(\mathbf{1}, \underline{RHom}(A, B))$, see Corollary 6.4 of [48]. Moreover the mapping space in diagram categories is given by a homotopy end, see Lemma 2.2.

Putting these together we see $\text{Map}(A_\bullet, B_\bullet) = \int_V^h \text{Map}(\mathbf{1}, \underline{RHom}(A_V, B_V))$. Then the claim follows since $\text{Map}(\mathbf{1}, -)$ commutes with homotopy limits and hence homotopy ends.

Similarly we have

$$\begin{aligned} \text{Map}(\text{hocolim}_i(U_i \cdot \mathbf{1}), \mathcal{F}) &\simeq \text{holim}_i \text{Map}(\mathbf{1}, \underline{RHom}'(U_i \cdot \mathbf{1}, \mathcal{F})) \\ &\simeq \text{Map}(\mathbf{1}, \text{holim}_i \text{holim}_{V \subset U_i} \mathcal{F}(V)) \end{aligned}$$

which is $\text{Map}(\mathbf{1}, \mathcal{F}(W))$ again by applying the sheaf condition twice.

□

Remark 2.13. The theory of enriched Bousfield localizations from [1] says that in the right setting \mathcal{M}_τ^J is an enriched model category and fibrant objects are precisely levelwise fibrant sheaves. However, this theory requires that we work with a category \mathcal{M} that is tractable, left proper and a symmetric monoidal model category with cofibrant unit. The characterization of fibrant objects in particular depends on the enriched hom-space being a Quillen bifunctor. While \mathbf{dgCat}_{Mor} is left proper by Proposition 2.5 and equivalent to a combinatorial and tractable subcategory by Section A.1, it is well-known \mathbf{dgCat}_{Mor} is not symmetric monoidal. Tabuada's equivalent category Lp of localizing pairs has a derivable internal Hom object, but is not a monoidal model category either. In fact, tensor product with a cofibrant object is not left Quillen. Recall the dg-category $\mathcal{S}(0)$ that is the linearization of $a \rightarrow b$. The example $\mathcal{S}(0) \otimes \mathcal{S}(0)$ in \mathbf{dgCat} gives rise to

$$(\emptyset \subset \mathcal{S}(0)) \otimes (\emptyset \subset \mathcal{S}(0)) \simeq (\emptyset \subset \mathcal{S}(0) \otimes \mathcal{S}(0))$$

which is again a tensor product of cofibrant objects that is not cofibrant. Then of course $\underline{\text{Hom}}(\mathcal{S}(0), -)$ cannot be Quillen either.

Lemma 2.10. *Let $\mathcal{M} = \mathbf{dgCat}$. Assume that for two presheaves \mathcal{F} and \mathcal{F}' there is a hypercover V_* on which \mathcal{F} and \mathcal{F}' agree and which restricts to a hypercover of W for every open W . Then \mathcal{F} and \mathcal{F}' are weakly equivalent in \mathcal{M}_τ^J .*

Proof. We need to show that there is a \check{H}_τ -local equivalence between \mathcal{F} and \mathcal{F}' , i.e. $\text{Map}_{\mathcal{M}^J}(\mathcal{F}, \mathcal{G}) \simeq \text{Map}_{\mathcal{M}^J}(\mathcal{F}', \mathcal{G})$ for any fibrant G .

Specifically, we consider sets V in the hypercover of agreement contained in W . Then we know $\text{Map}(\mathcal{F}(V), \mathcal{G}(V)) \simeq \text{Map}(\mathcal{F}'(V), \mathcal{G}(V))$. To compute $\text{Map}(\mathcal{F}, \mathcal{G}) = \int_W^h \text{Map}(\mathcal{F}(W), \mathcal{G}(W))$ note that the homotopy end can be computed as follows:

$$\int_W^h \text{Map}(\mathcal{F}(W), \mathcal{G}(W)) \simeq \int \text{Hom}((Q^\bullet \mathcal{F})(W), R\mathcal{G}(W))$$

Here we use fibrant replacement and a cosimplicial frame in \mathcal{M}^J . But now $\text{holim}_V \mathcal{G}(V) = \lim_V R\mathcal{G}(V)$ by fibrancy of the diagram $R\mathcal{G}$. So it suffices to consider $\int_W \text{Hom}(Q^\bullet \mathcal{F}(W), R\mathcal{G}(W))$ where $R\mathcal{G}(W) = \lim_{V \subset W} R\mathcal{G}(V)$. But an end is just given by the collection of all compatible maps, and every map from $Q^i \mathcal{F}(W)$ to $R\mathcal{G}(W)$ is determined by the maps from $Q^i \mathcal{F}(W)$ to $R\mathcal{G}(V)$, which factor through $Q^i \mathcal{F}(V)$. So the end over the V is the same as the end over all W and

$$\begin{aligned} \text{Map}(\mathcal{F}, \mathcal{G}) &\simeq \int_V \text{Hom}(Q^\bullet \mathcal{F}(V), R\mathcal{G}(V)) \\ &\simeq \int_V \text{Hom}(Q^\bullet \mathcal{F}'(V), R\mathcal{G}(V)) \simeq \text{Map}(\mathcal{F}', \mathcal{G}) \end{aligned}$$

This completes the proof. \square

Remark 2.14. If \mathcal{M} is a symmetric monoidal model category then by Remark 2.13 fibrant objects are precisely levelwise fibrant sheaves and are again determined on a hypercover and Lemma 2.10 holds again.

To compute cohomology we need to compute the derived functor of global sections. First we need to know that pushforward is right Quillen.

Lemma 2.11. *Consider a map $r : C \rightarrow D$ of diagrams and a model category M . Then there is a Quillen adjunction $r_! : M_{proj}^C \rightleftarrows M_{proj}^D : r^*$.*

Proof. We define r^* by precomposition. Then $r_!$ exists as a Kan extension. Clearly r^* preserves levelwise weak equivalences and fibrations. \square

Lemma 2.12. *Given any map $r : (C, \tau) \rightarrow (D, \sigma)$ that preserves covers and hypercovers we get a Quillen adjunction $r_! : M_\tau^C \rightleftarrows M_\sigma^D : r^*$. The same adjunction works if we only localize with respect to Čech covers.*

Proof. To prove the result for the localization with respect to covers we use the universal property of localization applied to the map $M^C \rightarrow M^D \rightarrow M_\sigma^D$ which is left Quillen and sends hypercovers to weak equivalences and hence must factor through $M^C \rightarrow M_\tau^C$ in the category of left Quillen functors, giving rise to $r_! \vdash r^*$.

To prove the result for the localization at hypercovers we repeat the same argument for $M_\tau^C \rightarrow M_\tau^D$ etc. \square

The arguments in the proofs of these two lemmas are Propositions 1.22 and 3.37 in [1].

Consider now locally contractible topological spaces X and Y with sites of open sets $(Op(X), \tau)$ and $(Op(Y), \sigma)$. Given a map $f : X \rightarrow Y$ consider $f^{-1} : (Op(X), \tau) \rightarrow (Op(Y), \sigma)$. Then $f_* := (f^{-1})^*$ and by the above it is a right Quillen functor. (This is the usual definition of pushforward: $f_* \mathcal{F}(U) := \mathcal{F}(f^{-1}u)$.)

As usual we write Γ or $\Gamma(X, -)$ for $(\pi_X)_*$ where $\pi_X : X \rightarrow *$.

Definition. Let \mathcal{C} be a presheaf with values in a model category \mathcal{M} and let $\mathcal{C}^\#$ be a fibrant replacement for \mathcal{C} in the local model category $\mathcal{M}_{\tilde{\tau}}^J$ defined above. Then we define global sections as

$$R\Gamma(X, \mathcal{C}) = \mathcal{C}^\#(X)$$

In Section 2.4 we will compute $\mathcal{C}^\#$ if \mathcal{C} is constant.

Since a sheaf satisfies $\mathcal{F}(X) = \operatorname{holim}_i \mathcal{F}(U_i)$ for some cover $\{U_i\}$ we can also think of global section as a suitable homotopy limit. A concise formulation of this is Theorem 2.21.

Definition. Consider the presheaf \underline{k} that is constant with value $k \in \mathbf{dgCat}$ and let $\underline{k}^\#$ be a fibrant replacement for \underline{k} . Then we define *Morita cohomology* as

$$R\Gamma_{\text{Morita}}(X, \underline{k}) := R\Gamma(X, \underline{\mathbf{Ch}}_{pe}) = \underline{k}^\#(X)$$

in $\operatorname{Ho}(\mathbf{dgCat}_{\text{Mor}})$. We write $\mathcal{H}^M(X) := R\Gamma_{\text{Morita}}(X, \underline{k})$.

We will also consider the version with unbounded fibers, $R\Gamma_{\text{Morita}}(X, \underline{\mathbf{Ch}})$.

Remark 2.15. As usual $R\Gamma(\emptyset, \underline{k}) \simeq 0$, the terminal object of \mathbf{dgCat} .

Remark 2.16. The term cohomology is slightly misleading as our construction corresponds to the underlying complex and not the cohomology groups. The closest analogue to taking cohomology is probably semi-orthogonal decomposition, see for example [8].

2.4. Sheafification of constant presheaves

Our aim now is to compute a sheafification of the constant presheaf with values in a model category.

We assume X is a locally contractible topological space. Fix a model category \mathcal{M} that is cellular and left proper and that is moreover homotopy enriched over itself and has a cofibrant unit. We will also need that the derived internal hom-space commutes with homotopy colimits.

The example we care about is $\mathcal{M} = \mathbf{dgCat}_{Mor}$. The fact that $\mathrm{holim} \underline{RHom}(A_i, B) \simeq \underline{RHom}(\mathrm{hocolim} A_i, B)$ in \mathbf{dgCat} follows from Corollary 6.5 of [48]. The one object dg-category k is a cofibrant unit.

We write \underline{P} for the constant presheaf with fiber $P \in \mathcal{M}$.

First we will need two lemmas about comparing homotopy limits.

Given a functor $\iota : I \rightarrow J$, recall the natural map $e_j : (j \downarrow \iota) \rightarrow J$ from the undercategory, sending $(i, j \rightarrow \iota(i))$ to $\iota(i)$.

Lemma 2.13. *Let $\iota : I \rightarrow J$ be functor between small categories such that for every $j \in J$ the overcategory $(\iota \downarrow j)$ is nonempty with a contractible nerve and let $X : J \rightarrow \mathcal{M}$ be a diagram. Then the map $\mathrm{holim}_J X \rightarrow \mathrm{holim}_I \iota^* X$ is a weak equivalence.*

Lemma 2.14. *Let $\iota : I \rightarrow J$ be a functor between small categories and let $X : J \rightarrow \mathcal{M}$ a diagram with values in a model category. Suppose that the composition*

$$X_j \rightarrow \lim_{(j \downarrow \iota)} e_j^*(X) \rightarrow \mathrm{holim}_{(j \downarrow \iota)} e_j^*(X)$$

is a weak equivalence for every j . Then the natural map $\text{holim}_j X \rightarrow \text{holim}_j t^* X$ is a weak equivalence.

Proofs. For topological spaces these are Theorems 6.12 and 6.14 of [11] and the proofs (in section 9.6) do not depend on the choice of model category. \square

We will also rely on the following results of [13]. The first statement is Theorem 1.3 and the second is a corollary of Proposition 4.6 as any basis is a complete open cover.

Proposition 2.15. *Consider a hypercover $U_* \rightarrow X$ of a topological space as a simplicial space. Then the maps $\text{hocolim } U_* \rightarrow |U_*| \rightarrow X$ are weak equivalences in **Top**.*

The colimit here is over the category Δ^{op} , but recall that $\text{hocolim}_{\Delta^{\text{op}}} U_n \simeq \text{hocolim}_I U_n^i$.

Proposition 2.16. *Consider a basis \mathfrak{U} of a topological space X as a simplicial space. Then the map $\text{hocolim}_{U \in \mathfrak{U}} U \rightarrow X$ is a weak equivalence in **Top**.*

Let X be locally contractible. Then we can define the (nonempty) set $\{\mathfrak{U}^s\}_{s \in S}$ of all bases of contractible sets for X .

Definition. Fix a basis of contractible sets \mathfrak{U}^s for X . Let \underline{P} be a constant presheaf with fiber $P \in \mathcal{M}$ and define a presheaf $\mathcal{L}_{\underline{P}}^s$ by

$$\mathcal{L}_{\underline{P}}^s(U) = \text{holim}_{V \subset U, V \in \mathfrak{U}^s} \underline{RP}(V)$$

where $P \rightarrow RP$ is a fibrant replacement in \mathcal{M} . Denote the natural map by $\lambda : \underline{P} \rightarrow \mathcal{L}_P^s$. The restriction maps are induced by inclusion of diagrams.

This construction proceeds via constructing rather large limits, so even the value of \mathcal{L}^s on a contractible set is hard to make explicit.

We will mainly be interested in $\mathcal{L}_k^s \simeq \mathcal{L}_{\mathbf{Ch}_{pe}}^s$.

The following lemma is the first step towards showing that our construction does indeed give a sheaf.

Lemma 2.17. *Consider a constant presheaf \underline{P} with fibrant fiber $P \in \mathcal{M}$ on $Op(X)$. Then on any contractible set $U \subset Op(X)$ we have $\mathcal{L}_P^s(U) \simeq P$.*

Proof. Consider \mathfrak{U} as a category. We need to show that $\text{holim}_{\mathfrak{U}^{op}} \underline{P} \simeq P$. The crucial input is that the weak equivalences $V \rightarrow *$ give rise to $U \simeq \text{hocolim}_{V \subset \mathfrak{U}} V \simeq \text{hocolim}_{\mathfrak{U}} *$ via Proposition 2.16.

Now consider any $N \in \mathcal{M}$ and a cosimplicial resolution N^\bullet . Then we have the functor $K \mapsto N \otimes K$ defined in the introduction which is left Quillen, as is shown in Corollary 5.4.4 of [25]. Hence it preserves homotopy colimits and we have:

$$N = N \otimes \text{hocolim}_{\mathfrak{U}} * \simeq \text{hocolim}_{\mathfrak{U}} (N \otimes *) = \text{hocolim}_{\mathfrak{U}} \underline{N}$$

Finally, we use the fact that \mathcal{M} has internal hom-spaces. Replace N above by the cofibrant unit. Then we conclude:

$$\begin{aligned} \operatorname{holim}_{U \in \mathfrak{U}^{\text{op}}} \underline{P}(U) &\simeq \operatorname{holim}_{\mathfrak{U}^{\text{op}}} \underline{\operatorname{RHom}}(\mathbf{1}, \underline{P}(U)) \\ &\simeq \operatorname{holim}_{\mathfrak{U}^{\text{op}}} \underline{\operatorname{RHom}}(\mathbf{1}(U), P) \simeq \underline{\operatorname{RHom}}(\operatorname{hocolim}_{\mathfrak{U}} \mathbf{1}, P) \\ &\simeq \underline{\operatorname{RHom}}(\mathbf{1}, P) \simeq P \end{aligned}$$

In the second line we use the fact that $\underline{\operatorname{RHom}}(-, P)$ sends homotopy colimits to homotopy limits. \square

Proposition 2.18. *For two choices \mathfrak{U}^t and \mathfrak{U}^s there is a chain of quasi-isomorphisms between $\mathcal{L}_{\underline{P}}^t$ and $\mathcal{L}_{\underline{P}}^s$. Hence there is a presheaf $\mathcal{L}_{\underline{P}}$ well defined in the homotopy category.*

Proof. By considering the union of \mathfrak{U}^s and \mathfrak{U}^t it suffices to show the result if \mathfrak{U}^t is a subcover of \mathfrak{U}^s . By Lemma 2.14 it then suffices to fix $U_i \in \mathfrak{U}^t$ and check that $\operatorname{holim}_{i/\iota} \underline{P} \simeq P$ where ι is the natural inclusion map. But the arrow category stands for the opposite of the category of all the elements of \mathfrak{U}^s contained in U_i . These form a basis and hence the homotopy limit is given by Lemma 2.17. \square

Proposition 2.19. *For any choice of \mathfrak{U}^s the presheaf $\mathcal{L}_{\underline{P}}^s$ is fibrant, i.e. it is \check{H} -local.*

Proof. By Lemma 2.9, it is enough to show $\mathcal{L}_{\underline{P}}^s$ is levelwise fibrant (immediate from definition) and satisfies the sheaf condition.

Given a hypercover $\{W_i\}_{i \in I}$ of U we may assume that any element of \mathfrak{U}^s is a subset of one of the W_i . Then we consider for every i the basis

of contractibles $\mathfrak{U}^{s(i)}$ for W_i of elements of \mathfrak{U}^s that are contained in W_i . Then we obtain the following:

$$\operatorname{holim}_{I^{\text{op}}} \mathcal{L}_{\underline{P}}^s(W_i) \simeq \operatorname{holim}_{i \in I^{\text{op}}} \operatorname{holim}_{U \in \mathfrak{U}^{s(i)\text{op}}} \underline{P}(U) \leftarrow \operatorname{holim}_{U \in \mathfrak{U}^{s\text{op}}} \underline{P}(U)$$

And our aim is to show the arrow on the right is a weak equivalence.

By considering $\underline{R}\underline{\text{Hom}}(\mathbf{1} \otimes \operatorname{hocolim} *, P)$ as in the proof of Lemma 2.17 it suffices to show $\operatorname{hocolim}_{i \in I} \operatorname{hocolim}_{V \in \mathfrak{U}^{s(i)}} V \rightarrow \operatorname{hocolim}_{U \in \mathfrak{U}^s} U$ is a weak equivalence. But if we apply Proposition 2.16 this is weakly equivalent to $\operatorname{hocolim}_{i \in I} W_i \rightarrow X$, which is a weak equivalence by Proposition 2.15. \square

If X is locally contractible then it has a basis of contractible open sets. Moreover one can associate a hypercover to any basis. For details on the construction see Section 4 of [13] and note that a basis is a complete cover.

Proposition 2.20. *If \mathcal{P} is constant then the natural map $\mathcal{P} \rightarrow \mathcal{L}_{\mathcal{P}}$ is a weak equivalence of presheaves.*

Proof. To show that \mathcal{L} resolves \mathcal{P} it is enough to observe that $\mathcal{L}_{\mathcal{P}}(U) \simeq \mathcal{P}$ for contractible U by Lemma 2.17. Now the contractible opens give rise to a hypercover on which \mathcal{P} and $\mathcal{L}_{\mathcal{P}}$ agree and that restricts to a hypercover on every open set. By Lemma 2.10 that suffices to prove the proposition. \square

Remark 2.17. An object \mathcal{F} in $\mathcal{M}_{\mathbb{Z}}^J$ is called a *homotopy locally constant presheaf* if there is a hypercover U_* such that all restrictions $\mathcal{F}|_{U_n^{(i)}}$ are weakly equivalent to constant presheaves.

If \mathcal{P} is only homotopy locally constant we still have the construction of $\mathcal{L}_{\mathcal{P}}$ and Proposition 2.20 holds as well as Lemma 2.17 for small enough contractible open sets. We expect that $\mathcal{P} \rightarrow \mathcal{L}_{\mathcal{P}}$ will still be a fibrant replacement. However, the proof of Proposition 2.19 relies explicitly on the fact that we are considering constant presheaves, so it does not readily adapt to the more general case.

With Proposition 2.20 we can compute $R\Gamma(X, \underline{P})$ as $\mathcal{L}_{\underline{P}}(X)$. Note that since we have not used functorial factorization this is not a functor on the level of model categories but only on the level of homotopy categories.

Definition. We will call a cover in $(\mathbf{Set}^{Op(X)^{op}}, \tau)$ a *good cover* if all its elements and all their finite intersections are contractible. Correspondingly a *good hypercover* is a hypercover such that all its open sets $U_n^{(i)}$ are contractible.

We will now consider a good hypercover $\{U_i\}_{i \in I}$. For computations it is easier not to consider the full simplicial presheaf given by open sets in the cover but only the semi-simplicial diagram of nondegenerate open sets, i.e. leaving out identity inclusions. This becomes particularly relevant when we consider locally finite covers in the Chapter 3.

Theorem 2.21. *Let $U_* \rightarrow h_X$ be a good hypercover of a topological space X . Let \underline{P} be a constant presheaf on X . Then $R\Gamma(X, \underline{P}) = \text{holim}_{I_0^{op}} P$ where I_0 indexes the distinct contractible sets of U_* .*

Proof. We consider a fibrant replacement $\mathcal{L}_{\underline{P}}$ as in Definition 2.4. Let I index the connected open sets of U_* . Then we have:

$$\begin{aligned} R\Gamma(X, \underline{P}) &\simeq \mathcal{L}_{\underline{P}}(X) \simeq \operatorname{holim} \mathcal{L}_{\underline{P}}(U_*) \\ &\simeq \operatorname{holim}_{I^{\text{op}}} \mathcal{L}_{\underline{P}}(U_n^{(i)}) \simeq \operatorname{holim}_{I^{\text{op}}} P \end{aligned}$$

Here we use Lemma 2.17 to identify $\mathcal{L}_{\underline{P}}(U_n^{(i)})$ and P . Now consider $\iota : I_0^{\text{op}} \subset I^{\text{op}}$ and note that all the overcategories $\iota \downarrow i$ are trivial (any $i \in I$ is isomorphic to some $j \in I_0$) so by Lemma 2.13 we have

$$R\Gamma(X, \underline{U}) \simeq \operatorname{holim}_{I_0^{\text{op}}} P \quad \square$$

Remark 2.18. Note that we can of course take the hypercover associated to a Čech cover in this theorem. In fact, since we are concerned with locally constant presheaves it makes very little difference whether we use the Čech or local model structure for computations. Considering hypercovers simplifies the theory and Čech covers make for simpler examples.

We conclude this section with some results on functoriality.

Lemma 2.22. *Let $f : X \rightarrow Y$ be a continuous map and let $\underline{P}_{X \text{ or } Y}$ denote the constant presheaf with fiber P on X or Y . Then $R\Gamma(X, \underline{P}) \simeq R\Gamma(Y, Rf_*(\underline{P}))$.*

Proof. The fact that $R\Gamma \circ Rf_* \simeq R\Gamma$ follows immediately from $\pi_{Y,*} \circ f_* = \pi_{X,*}$ and the fact that all these maps preserve fibrations. \square

Lemma 2.23. *In the setting of the previous lemma there is a functor $R\Gamma(Y, \underline{P}_Y) \rightarrow R\Gamma(X, \underline{P}_X)$.*

Proof. Γ is a covariant functor. From Lemma 2.22 we have a natural weak equivalence $R\Gamma(Y, Rf_*(\underline{P}_X)) \rightarrow R\Gamma(X, \underline{P}_X)$.

Let $\underline{P}_\bullet \rightarrow \mathcal{P}_\bullet$ be a fibrant replacement. It is then enough to construct a map $f^\bullet : \mathcal{P}_Y \rightarrow Rf_*(\mathcal{P}_X)$ of sheaves on Y . On any open set U this is given by $\underline{P}_Y(U) = P \rightarrow f_*\underline{P}_X(U) \rightarrow f_*\mathcal{P}_X(U)$. \square

Remark 2.19. With $P = \mathbf{Ch}_{pe}$ this gives functoriality for Morita cohomology if we use functorial factorizations.

Note that our computation using good covers is not functorial unless we pick compatible covers. However, if X and Y have bases of contractible sets which are suitably compatible we can just write down the comparison map between homotopy limits.

2.5. Morita cohomology over general rings

Everything we have done since Section 2.2 was built on the assumption that \mathbf{dgCat} is left proper, which is only the case if k is of flat dimension zero, see Remark 2.8.

Nevertheless, we can consider the question of what Morita cohomology should be over other ground rings. The obvious way out is to use $\Gamma(X, \mathcal{L}_{\mathbf{Ch}_{pe}})$ as our *definition* of Morita cohomology if k has positive flat dimension. All pertinent results in the remainder of the chapter then still apply, in particular Theorem 2.21, and we can prove equivalence with the category of homotopy locally constant sheaves in Chapter 3 and with ∞ -local systems in Section 4.1.

3. Homotopy locally constant sheaves

3.1. Strictification and computation of homotopy limits

In this chapter we will show that Morita cohomology of X is equivalent to the category of homotopy locally constant sheaves of perfect complexes on X .

We begin in this section with generalities on strictification and the computation of homotopy limits.

But first we set up the situation with which we will be concerned. Fix throughout this chapter a topological space X and assume that X has a good hypercover $\mathfrak{U} = \{U_i\}_{i \in I}$ satisfying certain finiteness conditions.

Specifically we assume that \mathfrak{U} satisfies the following two conditions, which we sum up by saying \mathfrak{U} is *bounded locally finite*.

- \mathfrak{U} is locally finite. (Every point has a neighbourhood meeting only finitely many elements of \mathfrak{U} .)
- There is some positive integer n such that no chain of distinct open sets in \mathfrak{U} has length greater than n .

Remark 3.1. If X is a finite-dimensional CW complex it has a bounded locally finite cover. One can show this by induction on the n -skeleta using *collaring*, see Lemma 1.1.7 in [20], to extend a bounded locally finite hypercover on X_n to one on a neighbourhood in X_n in X_{n+1} . Then one extends over the $n + 1$ -cells.

Note that by Theorem 2.21 we can compute cohomology as the homotopy limit of a diagram indexed by $I_0 \subset I$, the category of non-degenerate objects. In the next section we will use strictification to compute this small homotopy limit explicitly as a category of homotopy cartesian sections.

Let us consider the fiber \mathbf{Ch} instead of \mathbf{Ch}_{pe} at first, which has the advantage of being a model category. Model categories are often a convenient model to do computations with ∞ -categories. However, as the category of model categories is not itself a model category there exist no homotopy limits of model categories. Instead one can compute categories of homotopy cartesian sections and strictification results compare them to homotopy limits of the ∞ -categories associated with the model categories in question.

Generally speaking, using strictification to compute a homotopy limit proceeds as follows. As ingredients we need some localization functor $L : \mathbf{MC} \rightarrow \infty\mathbf{Cat}$ from model categories to some model of $(\infty, 1)$ -categories and a method, call it hsect , of computing homotopy cartesian sections of a Quillen presheaf, i.e. of a suitable diagram of model categories. Then given a diagram (\mathcal{M}_i) of model categories indexed by I one proves $\text{holim}_{i \in I^{\text{op}}} L\mathcal{M}_i \simeq L\text{hsect}(I, \mathcal{M}_i)$.

We will proceed by adapting the strictification result for inverse diagrams of simplicial categories from Spitzweck [42] to dg-categories. J is an *inverse category* if one can associate to every element a non-negative integer, called the *degree*, and every non-identity morphism lowers degree. This is certainly the case for I_0 if \mathcal{U} is bounded locally finite.

We then have to restrict to compact objects in the fibers to compute $R\Gamma(X, \underline{\mathbf{Ch}}_{pe})$ rather than $R\Gamma(X, \underline{\mathbf{Ch}})$.

Remark 3.2. There is a wide range of strictification results in the literature: For simplicial sets [18, 52], simplicial categories [42], Segal categories (Theorem 18.6 of [24]) and complete Segal spaces [4, 5].

Most of the above results make fewer assumptions on the index category, for example Theorem 18.6 of [24] proves strictification of Segal categories with general Reedy index categories, and a generalization to arbitrary small simplicial index categories is mentioned in Theorem 4.2.1 of [51]. But since it is unclear how to adapt this proof to the dg-setting and since the existence of a bounded locally finite good hypercover for X does not seem a very restrictive assumption we stay with it.

We will deal with model categories that are already enriched in some symmetric monoidal model category \mathcal{V} and our ∞ -categories will be \mathcal{V} -categories. (Think $\mathcal{V} = \mathbf{sSet}$ or \mathbf{Ch} .)

Definition. Denote by L the localization functor $L : \mathcal{V}\mathbf{MC} \rightarrow \mathcal{V}\mathbf{Cat}$ that sends M to M^{cf} , the subcategory of fibrant cofibrant objects of M .

The fibrant cofibrant replacement is necessary to ensure that the \mathcal{V} -hom spaces are invariant under weak equivalences. In the case $\mathcal{V} = \mathbf{sSet}$ compare the homotopy equivalence between LM and the Dwyer–Kan localization of M .

Let us set up the machinery:

Definition. A *left Quillen presheaf* on a small category I is a contravariant functor $M_\bullet : I \rightarrow \mathbf{Cat}$, written as $i \mapsto M_i$ such that for every $i \in \text{Ob}(I)$ the category M_i is a model category and for every map $f : i \rightarrow j$ in I the map $f^* : M_j \rightarrow M_i$ is left Quillen. (One can similarly define right Quillen presheaves.)

Definition. The *constant left Quillen presheaf with fiber M* , denoted as \underline{M} is the Quillen presheaf with $\underline{M}_i = M$ for all i and $f^* = \mathbf{1}_M$ for all f .

Remark 3.3. One can define Quillen presheaves in terms of pseudofunctors instead of functors, see [42]. The complicated definition is in [25]. One then rectifies the pseudofunctor to turn it into a suitable functor, i.e. into a left Quillen presheaf as defined above.

Definition. Let M_\bullet be a left Quillen presheaf of model categories. We define a *left section* to be a tuple consisting of (X_i, ϕ_f) for $i \in \text{Ob}(I)$ and $f \in \text{Mor}(I)$ where $X_i \in M_i$ and $\phi_f : f^*X_j \rightarrow X_i$, satisfies $\phi_g \circ (g^*\phi_f) = \phi_{f \circ g} : (f \circ g)^*X_k \rightarrow X_i$ for composable pairs $g : X_i \rightarrow X_j$, $f : X_j \rightarrow X_k$.

A *morphism of sections* consists of $m_i : X_i \rightarrow Y_i$ in M_i making the obvious diagrams commute. We write the category of sections of

M_\bullet as $\text{psect}(I, M_\bullet)$. The levelwise weak equivalences make it into a homotopical category.

Definition. A *homotopy cartesian section* is a section for which all the comparison maps $\phi_f : Rf^*X_j \rightarrow X_i$ are isomorphisms in $Ho(M_i)$. We write the category of homotopy cartesian sections of M as $\text{hsect}(I, M_\bullet)$.

If I is an inverse category or M is combinatorial then the category of left sections $\text{psect}(I, M)$ has an injective model structure, just like a diagram category, in which the weak equivalences and cofibrations are defined levelwise, cf. Theorem 1.32 of [1].

We write $L\text{hsect}(I, M_\bullet)$ for the subcategory of homotopy coherent sections whose objects are moreover fibrant and cofibrant.

Note that $\text{hsect}(I, M_\bullet)$ is not itself a model category since it is not in general closed under limits.

Remark 3.4. One would like homotopy cartesian sections to be the fibrant cofibrant objects in a suitable model structure. If we are working with the projective model structure of right sections then (under reasonable conditions) there exists a Bousfield localization, the so-called homotopy limit structure (cf. Theorem 2.44 of [1]). The objects of $L\text{hsect}_R(I, M)$ (which are projective fibrant) are precisely the fibrant cofibrant objects of $(\text{psect}_R)_{\text{holim}}(I, M)$.

The homotopy limit structure on left sections is subtler. It is the subject matter of [3]. Assuming the category of left sections is a right proper model category Bergner constructs a right Bousfield localization where the cofibrant objects are the homotopy cartesian ones in Theorem 3.2

of [5]. Without the hard properness assumption the right Bousfield localization only exists as a right semimodel category, cf. [2].

Note that we will still use model category theory, all we are losing is a conceptually elegant characterization of the subcategory we are interested in.

In the remainder of this section we recall the construction of enrichments of presections and presheaves that will be used in the proof of strictification.

Assume that \mathcal{V} is a symmetric monoidal model category and that we are given a left Quillen presheaf such that all the M_i are model \mathcal{V} -categories. Note that \mathcal{V} will be the category **Ch** in our application.

Lemma 3.1. *If M_\bullet is as above and the comparison functors are \mathcal{V} -functors then $\text{psect}(I, M_\bullet)$ is a model \mathcal{V} -category.*

Proof. Tensor and cotensor can be defined levelwise.

We define $\underline{\text{Hom}}_{\text{psect}}(X_\bullet, Y_\bullet)$ as the end $\int_i \underline{\text{Hom}}(X_i, Y_i)$. The same reasoning as in diagram categories applies, see the discussion before Lemma 2.2.

Since cofibrations and weak equivalences in $\text{psect}(I, M_\bullet)$ are defined levelwise the pushout product axiom holds, cf. Lemma 2.3, and we have a model \mathcal{V} -structure. \square

It follows that the derived internal hom-spaces can be computed by cofibrantly and fibrantly replacing source and target, $R\underline{\text{Hom}}_{\text{psect}}(X_\bullet, Y_\bullet) = \int_i \underline{\text{Hom}}((QX)_i, (RY)_i)$, cf. Lemma 2.2.

In particular if all M_i are dg-model categories then $\text{psect}(I, M)$ is a dg-model category.

Definition. If M is enriched in \mathcal{V} let $\mathcal{V}Psh(M)$ be the category of \mathcal{V} -functors from M to \mathcal{V} , i.e. functors such that the induced map on hom-spaces is a morphism in \mathcal{V} .

$\mathcal{V}Psh(M)$ is a model category if $\mathcal{V} = \mathbf{Ch}$ or if \mathcal{V} has cofibrant hom-spaces, see Remark 2.2.

Lemma 3.2. *$\mathcal{V}Psh(M)$ is enriched, tensored and cotensored over \mathcal{V} .*

Proof. We can tensor and cotensor levelwise. For the enrichment we have to define an object in \mathcal{V} of \mathcal{V} -natural transformations between two \mathcal{V} -functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$. Recall that a \mathcal{V} -natural transformation is, for every object $A \in \mathcal{A}$ a morphism $\mathbf{1}_{\mathcal{V}} \rightarrow \mathcal{B}(FA, GA)$. And morphism spaces in \mathcal{V} live themselves in \mathcal{V} so $\text{Nat}(F, G)$ is a limit (to be specific, an end) in \mathcal{V} . This is of course entirely standard, see Chapter 1 of [30]. \square

Lemma 3.3. *There is an enriched Yoneda embedding $M \rightarrow \mathcal{V}Psh(M)$. If \mathcal{V} has a cofibrant unit and fibrant hom-spaces then the Yoneda embedding factors through the subcategory of fibrant cofibrant objects.*

Proof. For the existence of the embedding see 2.35 in [30]. It is clear from the projective model structure that the objects in the image are fibrant. To see the image consists of cofibrations, we just note that the maps $0 \rightarrow h^X \otimes \mathbf{1}$ are generating cofibrations. \square

The conditions of the lemma are satisfied in \mathbf{Ch} .

3.2. Strictification for dg-categories

Our goal now is to prove the following theorem:

Theorem 3.4. *Let I be a direct category. Let M_i be a presheaf of model categories enriched in \mathbf{Ch} . Then $L\text{hsect}(I, M_\bullet) \cong \text{holim}_{i \in I^{\text{op}}} LM_i$ in $\text{Ho}(\mathbf{dgCat}_{DK})$.*

This theorem allows us to characterize Morita cohomology of X . From Theorem 2.21 we immediately obtain the following:

Corollary 3.5. *Let $\{V_i\}_{i \in I}$ be a locally finite good hypercover of X . Then $R\Gamma(X, \mathbf{Ch}_{dg}) \simeq L\text{hsect}(I_0, \mathbf{Ch})$.*

We are mainly interested in restricting attention to \mathbf{Ch}_{pe} . In this case we have to be careful about defining the right hand side. We will consider this situation in Section 3.3.

To show Theorem 3.4 we closely follow the method of proof in [42], replacing enrichments in simplicial sets by enrichment in chain complexes wherever appropriate. For easier reference we write in terms of \mathcal{V} -categories, where $\mathcal{V} = \mathbf{Ch}$ for our purposes and $\mathcal{V} = \mathbf{sSet}$ in [42].

One simplification is that we are assuming the model categories we start with are already enriched in \mathbf{Ch} , so that we can use restriction to fibrant cofibrant objects instead of Dwyer–Kan localization as the localization functor.

There are two times two steps to the proof: First we define homotopy embeddings ρ_1 and ρ_2 of the two sides into $L\text{psect}(I, \mathcal{V}PS\text{h}(RLM_\bullet))$.

We then show that their images are given by homotopy cartesian section whose objects are in the image of M_i . The first pair of steps are quite formal. The second pair is given by explicit constructions using induction along the degree of the index category.

The proof of the strictification result depends on setting up a comparison between the limit construction and presections. Since the fibrant replacement of LM_\bullet is not a Quillen presheaf we have to embed everything into a presheaf of enriched model categories. This is achieved by using the Yoneda embedding.

We write RLM_\bullet for $i \mapsto (RLM)_i$, where R stands for fibrant replacement in the injective model structure on diagrams of \mathcal{V} -categories and L is taking fibrant cofibrant objects of every M_i .

Lemma 3.6. *Let D_i be an I^{op} -diagram of \mathcal{V} -categories. We have a canonical full \mathcal{V} -embedding:*

$$\rho_2 : \text{holim } D_\bullet = \lim RD_\bullet \hookrightarrow L \text{psect}(I, \mathcal{V}PS h(RD_\bullet))$$

Proof. The map to $\text{psect}(I, \mathcal{V}Psh(D_\bullet^f))$ is obtained by composing the Yoneda embedding with the map of \mathcal{V} -categories $\lim_i C_i \rightarrow \text{psect}(I, C_\bullet)$ that sends a to $\{\pi_i(a)\}$ if $\pi_i : \lim_j C_j \rightarrow C_i$ are the universal maps. (C_\bullet is not a model category, but we can still take psect with the obvious meaning, the comparison maps are identities by definition.)

Recall that $\text{Ob}(\lim C_i)$ consists of collections $\{c_i \in \text{Ob}C_i\}$ such that $C(f)(c_j) = c_i$ for $f : i \rightarrow j$ and $\text{Mor}(\lim C_i)$ consist of collections $\{g_i \in \text{Mor}(C_i)\}$ such that $C(f) \circ g_i = g_j \circ C(f)$, i.e. the hom-space

between $\{c_i\}$ and $\{d_i\}$ is given by $\int_i \text{Hom}(c_i, d_i)$. If the C_i are enriched then $\underline{\text{Hom}}_{\lim c_i}(a, b) = \int \underline{\text{Hom}}(\pi_i(a), \pi_i(b))$ in \mathcal{V} .

The universal property is clear. Hence the objects are a subset of the objects of presections, namely the ones with identities for comparison maps.

By definition of the morphisms in $\text{psect}(I, D_\bullet)$ we have the following isomorphisms of hom-spaces:

$$\begin{aligned} \underline{\text{Hom}}_{\text{psect}}(c_\bullet, d_\bullet) &\cong \int_i \underline{\text{Hom}}_{\mathcal{V}Psh(RD_\bullet)_i}(c_i, d_i) \\ &\cong \int_i \underline{\text{Hom}}_{RD_i}(c_i, d_i) \\ &\cong \underline{\text{Hom}}_{\lim RD_\bullet}(\{c_i\}, \{d_i\}) \end{aligned}$$

Here we use that the enriched Yoneda embedding is indeed an enriched embedding. We abusively write c_i for the image of c_i in $\mathcal{V}Psh(RD_i)$. This proves there is an embedding of the homotopy limit into $\text{psect}(I, \bullet)$.

To show this embedding factors through fibrant cofibrant objects note first that cofibrations are defined levelwise. For fibrations one uses the fibrancy of RLM_\bullet , this is Lemma 6.3 of [42]. \square

It follows from this embedding that homotopy equivalences in the homotopy limit are determined levelwise since in $L\text{psect}$ homotopy equivalences are weak equivalences and weak equivalences are defined levelwise. This is Corollary 6.5 in [42].

From now on we will write ρ_2 for the case $D_i = LM_i$.

We also have the following:

Lemma 3.7. *There is a natural homotopy \mathcal{V} -embedding*

$$\rho_1 : L\text{hsect } M_\bullet \hookrightarrow L\text{psect}(I, \mathcal{V}PSh(RLM_\bullet))$$

Proof. We have an embedding $\text{hsect} \hookrightarrow \text{psect}$ and homotopy embeddings $M_i \hookrightarrow \mathcal{V}Psh(RLM_i)$ which give a homotopy embedding when we apply $L\text{psect}(I, -)$ since the hom-spaces of presections between fibrant cofibrant objects are given by homotopy ends, which are invariant under levelwise weak equivalence. \square

Remark 3.5. Note that the situation is a little more intricate in [42] where simplicial localization and restriction to fibrant cofibrant objects are a priori distinct and need to be compared through another embedding.

Next we have to identify the images of ρ_1 and ρ_2 . The explicit computation is done in Lemma 6.6 of [42]. The only use of special properties of the category \mathbf{sCat} made in this lemma (and the results needed for it) is the characterization of fibrations in terms of lifting homotopy equivalences. But this characterization is also valid for fibrations in \mathbf{dgCat}_{DK} .

We provide the argument here for convenience and future reference.

Lemma 3.8. *The image of ρ_1 consists of homotopy cartesian sections $X_\bullet \in L\text{psect}(I, \mathcal{V}Psh(RLM_\bullet))$ such that all X_i are in the image of M_i .*

Proof. We proceed by induction on the degree of the indexing category. Let X_\bullet be given as in the statement and assume by induction we have a

levelwise equivalence $Y_{<n} \simeq X_{<n}$ where $Y_{<n} \in L \text{hsect}(I_{<n}, M_\bullet)$. Here we write $()_{<n}$ for the obvious restriction to the index category with objects of degree less than n .

Let i be an element of degree n in I and consider the embedding $\iota : M_i \rightarrow \mathcal{V}Psh(RLM_i)$. So there is a cofibrant Y'_i with $\iota(Y'_i) \simeq X_i$.

Next note that ι commutes with homotopy limits and M_i in a fibrant diagram can be written as a homotopy limit of the diagram $\{f_*X_j\}$ indexed by I_i , the category of non-identity maps in I/i . Now levelwise f_* from $\text{psect}(I_i, X_\bullet)$ to $\text{psect}(I_i, \underline{X}_i) = X_i^{I_i}$ is a right adjoint by definition, and hence commutes with the Yoneda embedding. (This is Lemma 5.3 in [42], which is entirely formal.)

Since ι commutes with matching objects we can take homotopy limits and find that $\iota(M_i Y_{<n}) \simeq M_i X_\bullet$. Then $p' : Y'_i \rightarrow M_i Y_{<n}$ can be factored as a trivial cofibration followed by a fibration $Y_i \rightarrow M_i Y_{<n}$. Then the map $f : \iota(Y_i) \rightarrow X_i$ can be chosen such that it commutes with the existing maps, completing the induction step. (Details on the last step, which is valid in any model category, can be found in [42]). \square

Lemma 3.9. *The image of ρ_2 consists of homotopy cartesian sections $X_\bullet \in L \text{psect}(I, \mathcal{V}Psh(RLM_\bullet))$ such that all X_i are in the image of M_i .*

Proof. We proceed by induction on the degree of the indexing category. Assume we have a section X_\bullet as in the statement of the lemma. We will abuse notation and use the same name for images of the same object in different categories under the natural embeddings.

By induction we assume there is $Y_{<n} \simeq X_{<n}$ in RLM_{\bullet} and $Y_{<n}$ is the image of an element of $\lim RLM_{\bullet}$. Fix an object i of degree n and consider $Y_{<i} \in M_i RLM_{\bullet}$. By assumption $p : RLM_i \rightarrow M_i RLM_{\bullet}$ is a fibration. Since X is homotopy coherent and homotopy equivalences are levelwise $p(X_i) \simeq X_{<i} \simeq Y_{<i}$. Putting this together with the fact that p is a fibration we find $Y_i \in RLM_i$ with $p(Y_i) = Y_{<i}$. This completes the induction step. \square

Putting this together we obtain a zig-zag of quasi-essentially surjective maps between $L \text{psect}(I, M_{\bullet})$ and $\text{holim}_I LM$, showing the two categories are isomorphic in $Ho(\mathbf{dgCat}_{DK})$.

3.3. Restriction to perfect complexes

In this section we restrict the equivalence obtained by strictification to sections with compact fibers.

The compact objects in \mathbf{Ch} form the subcategory \mathbf{Ch}_{pe} consisting of compact complexes which are automatically fibrant and cofibrant. Note that \mathbf{Ch}_{pe} is not a model category, so in the next lemma we extend strictification to subcategories.

Proposition 3.10. *The dg-category $\text{holim}_I \mathbf{Ch}_{pe}$ is quasi-equivalent to the dg-category $\text{hsect}(I, \mathbf{Ch}_{pe})$, defined to be the subcategory of $\text{hsect}(I, \mathbf{Ch})$ consisting of sections X_{\bullet} such that every X_i is in \mathbf{Ch}_{pe} .*

Remark 3.6. Note that this is not the subcategory of compact objects in $\text{hsect}(I, \mathbf{Ch})$.

Proof. Considering the proof of strictification we aim to show that $\text{hsect}(I, \mathbf{Ch}_{pe})$ and $\text{holim}_I \mathbf{Ch}_{pe}$ can be identified with the subcategory of objects $X_\bullet \in L\text{hsect}(I, \mathcal{V}Psh(\underline{RLCh}_\bullet))$ such that every X_i is in the image of \mathbf{Ch}_{pe} .

For $\text{hsect}(I, \mathbf{Ch}_{pe})$ this is immediate from the proof of Lemma 3.8: We inductively pick $Y'_i \simeq X_i \in \text{Im}(\mathbf{Ch}_{pe})$ and replace them by $Y_i \simeq Y'_i$, ensuring every Y_i is the image of a compact object.

Now we consider the construction of an object Y_\bullet in the homotopy limit. There is a natural map between fibrant diagrams $\underline{RLCh}_{pe} \rightarrow \underline{RLCh}$ through which $\mathbf{Ch}_{pe} \rightarrow \underline{RLCh}$ factors by functoriality of fibrant replacement. So we can inductively construct Y_\bullet such that the Y_i live in $(\underline{RLCh}_{pe})_i$. \square

Theorem 3.11. $\mathcal{H}^M(X) \simeq \text{hsect}(I_0, \mathbf{Ch}_{pe})$ for any locally finite good hypercover $\{U_i\}_{i \in I}$ of X where $I_0 \subset I$ is the subcategory of non-degenerate objects.

Proof. We apply Proposition 3.10 to Theorem 3.4 and recall Theorem 2.21. \square

3.4. Homotopy locally constant sheaves

Theorem 3.11 is just a precise way of saying that an object of $\mathcal{H}^M(X)$ is given by a collection of chain complexes, one for every open set in the cover, with quasi-isomorphic transition function. We will now turn this

into an equivalence with the dg-category of homotopy locally constant sheaves of perfect chain complexes.

To define homotopy locally constant sheaves we put the local model structure (as in as Section 2.3) on presheaves of chain complexes on X ,

Definition. We call *homotopy locally constant* a presheaf \mathcal{F} such that there is a cover U_i such that all the restrictions $\mathcal{F}|_{U_i}$ are weakly equivalent to constant sheaves. (In particular the transition functions between $\mathcal{F}(U_i)|_{U_{ij}}$ and $\mathcal{F}(U_j)|_{U_{ij}}$ are weak equivalences.)

Then we denote by $LC_H(X)$ the subcategory of homotopy locally constant sheaves of perfect chain complexes. We note that this category consists of fibrant and cofibrant homotopy locally constant presheaves. This is a dg-category and the hom-spaces are derived hom-spaces of complexes of sheaves.

Remark 3.7. Note that the homology sheaves of a homotopy locally constant sheaf are finite dimensional vector bundles which have isomorphisms as transition functions with respect to the above cover, i.e. they are local systems.

Proposition 3.12. *Let X be a topological space with a locally finite good hypercover \mathfrak{U} and let I_0 index the nondegenerate connected open sets. There is a restriction functor from $LC_H(X)$ to $\text{hsect}(I_0, \mathbf{Ch}_{pe})$ that is quasi-essentially surjective.*

Proof. There is an obvious functor $r : LC_H(X) \rightarrow \text{hsect}(I_0, \mathbf{Ch}_{pe})$ sending a sheaf \mathcal{F} to $i \mapsto \mathcal{F}(U_i)$. (If \mathcal{F} is fibrant cofibrant in the local

model structure it is fibrant cofibrant in the injective model structure.) We show that r is quasi-essentially surjective by producing a left inverse in the homotopy category.

Let \mathfrak{U} also denote the category of all connected open sets making up the hypercover \mathfrak{U} . Pick a basis \mathfrak{B} of contractible sets for the topology of X and assume it is subordinate to \mathfrak{U} in the sense that any $B \in \mathfrak{B}$ is contained in any $U \in \mathfrak{U}$ it intersects. This is possible since \mathfrak{U} is locally finite. Consider the presheaf $S^{\mathfrak{B}}(A)$ on \mathfrak{B} that sends B to A_U where U is minimal containing B , such U exists by our assumptions. Extend $S^{\mathfrak{B}}(A)$ to a presheaf $S^p(A)$ on X by $S^p(A)(W) = \text{holim}_{C \subset W} S^{\mathfrak{B}}(A)(C)$. Let $S(A)$ denote a functorial fibrant and cofibrant replacement of $S^p(A)$ (in particular it is a sheafification). Now if we restrict $S^p(A)$ to $U \in \mathfrak{U}$ there is an obvious weak equivalence with the constant presheaf A_U , via $S^p(B) \simeq A_{U'} \simeq A_U$ if $B \subset U' \subset U$. Hence $S(A)$ is a homotopy locally constant sheaf.

To show that $S(A)(U) \simeq A_U$ we can take homology and since the homology sheaves are constant on U the canonical map to the stalk at any point of U is a weak equivalence. (The value at the stalk is weakly equivalent to the limit of the constant diagram A_U .)

Hence $r \circ S \simeq \mathbf{1}$ and r is indeed quasi-essentially surjective. \square

The following lemma is well-known. We sketch a proof for lack of a reference.

Lemma 3.13. *Let X_\bullet be a cosimplicial diagram of chain complexes. Then $\text{holim } X_\bullet \simeq \text{Tot}^\Pi X_\bullet$.*

Proof. \mathbf{Ch} is an abelian category so there is an equivalence of categories between cosimplicial objects in \mathbf{Ch} and nonpositive chain complexes in \mathbf{Ch} , we can write this as $\mathbf{Ch}^\Delta \cong \mathbf{Ch}^{\mathbb{N}}$. Now note that the Dold–Kan correspondence respects levelwise quasi-isomorphisms. (To see the normalized chain functor preserves quasi-isomorphisms consider the splitting of the Moore complex $M(A) = N(A) \oplus D(A)$.)

It follows that the associated categories with weak equivalences and hence the homotopy categories $Ho(\mathbf{Ch}^\Delta)$ and $Ho(\mathbf{Ch}^{\mathbb{N}})$ are equivalent and the homotopy limit of the cosimplicial diagram is the homotopy limit of the corresponding \mathbb{N} -diagram.

But taking the homotopy limit of a complex of chain complexes is just taking the product total complex. If the complex is concentrated in two degrees this is the well-known cone construction which generalizes in the obvious way. \square

Remark 3.8. It is worth pointing out that while this is an ad-hoc construction there is a complete Dold–Kan theorem for stable $(\infty, 1)$ -categories in Section 1.2 of [33].

Proposition 3.14. *In the setting of the previous proposition, for $A_\bullet, B_\bullet \in \text{hsect}(I, \mathbf{Ch}_{pe})$ we have:*

$$\underline{Hom}_{\text{hsect}(I, \mathbf{Ch}_{pe})}(A_\bullet, B_\bullet) \simeq \underline{Hom}_{LC_H(X)}(S(A), S(B))$$

In particular the cohomology groups of $\underline{Hom}(A_\bullet, B_\bullet)$ are the Ext groups of $S(A)$ and $S(B)$.

Proof. We know that the right hand side can be computed as a Čech complex of the good hypercover. (See for examples the section

Hypercovers in [43].) It remains to show that the left-hand side is quasi-isomorphic to $\check{C}_{\mathbb{N}}^*(\text{Hom}(S(A), S(B)))$. The Čech complex is the total complex, and hence by Lemma 3.13 the homotopy limit, of the cosimplicial diagram

$$n \mapsto \underline{\text{Hom}}(A_{U_n}, B_{U_n}) := \prod_{i \in I_n} \underline{\text{Hom}}(A_{U_n^i}, B_{U_n^i})$$

which is in turn equal to the homotopy limit of the diagram $i \mapsto \underline{\text{Hom}}(A_{U_n^i}, B_{U_n^i})$. Here we can replace $\underline{\text{Hom}}_{\mathcal{O}}(S(A)(U_n^i), S(B)(U_n^i))$ by $\underline{\text{Hom}}(A_{U_n^i}, B_{U_n^i})$ as the U_n^i are contractible.

By adapting Lemma 2.2 to presections one sees that the derived functor of $\int \underline{\text{Hom}}(A_{\bullet}, B_{\bullet})$ is given by $\underline{\text{Hom}}_{\text{hsect}}$ between a cofibrant replacement of A_{\bullet} and a fibrant replacement of B_{\bullet} in the injective model category structure. In other words, since all objects in hsect are assumed fibrant and cofibrant, $\underline{\text{Hom}}_{\text{hsect}}(A_{\bullet}, B_{\bullet})$ is already the derived functor of $\int \underline{\text{Hom}}(A_{\bullet}, B_{\bullet})$.

Now, adapting Lemma 3.1 of [42] to dg-model categories, we can also compute hom-spaces in hsect as a homotopy limit of the diagram $i \mapsto \underline{\text{Hom}}_{\text{hsect}(I/i, \mathbf{Ch}_{pe})}(A|_{(I/i)}, B|_{(I/i)})$ where the comparison maps are induced by the inclusion of diagrams. The underived version follows from a diagram chase comparing the end and the limit of ends, and both sides give fibrant diagrams since A_{\bullet} and B_{\bullet} are fibrant cofibrant.

By 3.1 of [42] again $\underline{\text{Hom}}_{\text{hsect}(I/i, \mathbf{Ch}_{pe})}(A|_{(I/i)}, B|_{(I/i)})$ is weakly equivalent to $\underline{\text{Hom}}_{\mathbf{Ch}_{pe}}(A_{U_n^i}, B_{U_n^i})$. So the objects in the diagrams on the left-hand side and the right-hand side agree.

It remains to show that the comparison maps on the left-hand side correspond to the restriction map of sheaf homs on the right-hand side. Note that giving a sheaf Hom from $S(A)(U)$ to $S(B)(U)$ corresponds to giving morphisms $S(A)(W) \rightarrow S(B)(W)$ for all $W \subset U$, so giving a morphisms of presections in the overcategory $Op(X)^{op}/U$. But when applying the weak equivalence with A_U the only non-identity restrictions come from the fixed cover \mathfrak{U} and we can take the limit over \mathfrak{U}^{op}/U and obtain the same expression we have on the left-hand side.

Hence the two homotopy limits agree and the enriched hom-space is weakly equivalent to the Čech complex. \square

Summing up we have proven:

Theorem 3.15. *Let X be a topological space with a bounded locally finite good hypercover. Then $\mathcal{H}^M(X)$ is quasi-equivalent to the dg-category $LC_H(X)$.*

The corresponding results also hold if the fiber is **Ch**.

Remark 3.9. With this interpretation of Morita cohomology the pullback f^* is the map induced by pull-back of complexes of sheaves.

Since pushforwards of homotopy locally constant sheaves are not homotopy locally constant it is clear that we do not in general expect a map f_* or $f_!$ going in the other direction.

Remark 3.10. If we use the Dwyer–Kan model structure rather than the Morita model structure on **dgCat** we find that $R\Gamma_{DK}(X, k)$ is the category of topological line bundles on X .

4. Infinity-local systems

4.1. Infinity-local systems

We will now consider another approach to $\mathcal{H}^M(X)$ which begins with the definition of ∞ -local systems.

Recall from 2.1.3 that while the model categories on \mathbf{dgCat} are not simplicial there is a bifunctor $\mathbf{sSet}^{\mathrm{op}} \times \mathbf{dgCat}_{DK} \rightarrow \mathbf{dgCat}_{DK}$ that induces a natural $Ho(\mathbf{sSet})$ cotensor action on $Ho(\mathbf{dgCat}_{DK \text{ or } Mor})$. We write this as $(K, \mathcal{D}) \mapsto \mathcal{D}^K$.

Definition. We define the *dg-category of ∞ -local systems* on a simplicial set K as \mathbf{Ch}_{pe}^K . We write $\mathcal{Y}(K)$ for \mathbf{Ch}_{pe}^K . For a topological space X we recall the (unpointed) singular simplicial set $\mathrm{Sing}_*(X)$ and define $\mathcal{Y}(X) := \mathcal{Y}(\mathrm{Sing}_*(X))$. We also define $\mathcal{Y}^u(K) = \mathbf{Ch}_{dg}^K$.

Remark 4.1. We are using the Dwyer–Kan model structure for simplicity, but of course we think of \mathbf{Ch}_{pe} as a Morita fibrant replacement of k and one can show that $\mathcal{Y}(K)$ is weakly equivalent to k^K as constructed in \mathbf{dgCat}_{Mor} , cf. the proof of Theorem 4.4.

As we will mainly consider topological spaces via the functor Sing_* in this chapter we restrict attention to compactly generated Hausdorff spaces so that Sing_* is part of a Quillen equivalence.

Remark 4.2. We can think of this definition as categorified singular cohomology.

Lemma 4.1. $K \mapsto \mathcal{Y}(K)$ is a left Quillen functor from \mathbf{sSet} to $\mathbf{dgCat}_{DK}^{\text{op}}$ with right adjoint given by $\text{Map}(-, \mathbf{Ch}_{pe})$,

Proof. This follows for example from Theorems 16.4.2 and 16.5.7 in [23]. \square

As all simplicial sets are cofibrant we obtain the following corollaries:

Corollary 4.2. $K \mapsto \mathcal{Y}(K)$ preserves weak equivalences.

Corollary 4.3. The functor $K \mapsto \mathcal{Y}(X)$ sends homotopy colimits to homotopy limits.

Since Sing_* sends cofibrations of topological spaces to cofibrations in \mathbf{sSet} the lemma also holds for $\mathcal{Y} : \mathbf{CGHaus} \rightarrow \mathbf{dgCat}_{DK}^{\text{op}}$. Moreover, as Sing_* is a Quillen equivalence it preserves weak equivalences and homotopy colimits. Then the last result can be interpreted as a Mayer–Vietoris theorem:

$$\mathcal{Y}(U \cup V) \simeq \mathcal{Y}(U) \times_{\mathcal{Y}(U \cap V)} \mathcal{Y}(V)$$

This definition of ∞ -local systems looks a little indirect. But note that an ∞ -local systems does provide us with an object of $(\mathbf{Ch}_{pe})_n$ for every n -simplex of K . The explicit simplicial resolution \mathcal{C}_\bullet in the appendix shows that this is the data one would expect. Cf. also Proposition A.3 and the following remark.

Section 4.2 will provide a more explicit way of looking at ∞ -local systems, but first we show that ∞ -local systems are equivalent to Morita cohomology.

Fix a topological space X with a good hypercover $\{U_i\}_{i \in I}$.

Theorem 4.4. $\mathcal{H}^M(X)$ and $\mathcal{Y}(X)$ are isomorphic in $\text{Ho}(\mathbf{dgCat}_{DK})$.

Proof. By Proposition 2.15 there is a weak equivalence $\text{hocolim } U_* \simeq X$. Let $I = \cup I_n$ by the indexing category. Then we can consider the data of the category I as a simplicial set $n \mapsto I_n$ with the induced face and degeneracy maps, or in fact as a simplicial space where every I_n is considered as a discrete space. Then we can consider the comparison map from U_n to $\amalg_{I_n} *$ sending every connected open to a distinct point to get $\text{hocolim}_\Delta U_n \simeq \text{hocolim}_\Delta I_n$ where we take homotopy colimits of simplicial spaces. Then I_* considered as a simplicial space has free degeneracies in the sense of Definition A.4 in [13]. Hence we can apply Theorem 1.2 of [13] and find $|I_*| \simeq \text{hocolim}_\Delta I_n$. So the simplicial set I_* is weakly equivalent to $\text{Sing}_* X$ and it suffices to analyse $\mathbf{Ch}_{pe}^{I_*}$.

Hence by Remark 2.5 and Theorem 2.21 we are left to compare $\text{holim}_{I^{\text{op}}} \mathbf{Ch}_{pe}$ and $\text{holim}_{\Delta I_*^{\text{op}}} (\mathbf{Ch}_{pe})_n$. But the category I is exactly the category of simplices of the simplicial set I_* and the weak equivalences $\mathbf{Ch}_{pe} \rightarrow (\mathbf{Ch}_{pe})_n$ induce a weak equivalence of homotopy limits. \square

Remark 4.3. This argument still applies if we replace \mathbf{Ch}_{pe} by any other dg-category P . Hence we know that $R\Gamma_{\text{Morita}}(X, \underline{P}) \simeq P^{\text{Sing}_* K}$. For example $R\Gamma_{\text{Morita}}(X, \mathbf{Ch}_{dg}) \simeq \mathcal{Y}^u(X)$.

Corollary 4.5. $\mathcal{H}^M(X)$ is homotopy invariant and sends homotopy colimits to homotopy limits.

Proof. This is immediate from Theorem 4.4 and the topological versions of Lemma 4.1 and its corollaries. \square

Definition. With this equivalence in mind we can define the *Morita homology* $\mathcal{H}_M(K)$ of a simplicial set K as $\mathbf{Ch}_{pe} \otimes K$.

Note, however, that computing this involves a cosimplicial resolution in dg-categories which looks difficult to produce.

4.2. Loop space representations

If we consider the fiber of a locally constant sheaf on X we obtain an action of the fundamental group of X . Similarly the fiber of a homotopy locally constant sheaf acquires an action of the loop group of X .

In V.5 of [21] explicit looping and delooping functors for simplicial sets and simplicial groupoids are constructed. For arbitrary simplicial sets there is a functor $G : \mathbf{sSet} \rightarrow \mathbf{sGpd}$ with right adjoint \overline{W} . Together they form a Quillen equivalence. The obvious composition with the normalization functor $NkG : \mathbf{sSet} \rightarrow \mathbf{dgCat}_{DK}$ is left Quillen. Essentially this lets us consider a simplicial set as a dg-category. The restriction of G to simplicial sets with a single vertex is a Quillen equivalence with simplicial groups.

As in the introduction, for any dg-category \mathcal{D} we consider $L(\mathcal{D}\text{-Mod})$, where L just restricts to fibrant cofibrant objects. Let us write this as $\mathbf{Ch}^{\mathcal{D}}$. This is quasi-equivalent to $\underline{RHom}(\mathcal{D}, \mathbf{Ch}_{dg})$. Similarly write $\mathbf{Ch}_{pe}^{\mathcal{D}}$ for the subcategory of $\mathbf{Ch}^{\mathcal{D}}$ with underlying complexes perfect over k .

Notation. If X is a topological space we write $N\Omega X$ for $N(kG \text{ Sing}_*(X))$.

Definition. The category of *loop space representations* of X is $\mathbf{Ch}_{pe}^{N\Omega X}$.

Theorem 4.6. *For a simplicial set K the dg-categories \mathbf{Ch}_{pe}^K and \mathbf{Ch}_{pe}^{NkGK} are quasi-equivalent, as are \mathbf{Ch}_{dg}^K and \mathbf{Ch}_{dg}^{NkGK} .*

Proof. The proofs for $\mathbf{Ch}_{dg}^{(-)}$ and $\mathbf{Ch}_{pe}^{(-)}$ are identical, so let us write the proof for \mathbf{Ch}_{pe} .

By the Yoneda embedding it is enough to prove

$$\text{Map}_{\text{dgCat}_{DK}}(\mathcal{D}, \mathbf{Ch}_{pe}^K) \simeq \text{Map}_{\text{dgCat}_{DK}}(\mathcal{D}, \mathbf{Ch}_{pe}^{NkGK})$$

for arbitrary dg-categories \mathcal{D} . (In fact an isomorphism of connected components of the mapping space would be enough.)

The left-hand side is $\text{Map}_{\text{sSet}}(K, \text{Map}_{\text{dgCat}}(\mathcal{D}, \mathbf{Ch}_{pe}))$ by the usual adjunction. Meanwhile, for the right-hand side we have the following computation. We use the adjunctions $\otimes^L \dashv \underline{RHom}$, $\iota \dashv \tau_{\geq 0}$ (inclusion and truncation), $N \dashv DK$ (Dold–Kan), $k \dashv U$ (free and forgetful) and $G \dashv \overline{W}$ (looping and delooping). For legibility we contract $DK \circ \tau_{\geq 0}$ to

DK and suppress ι and U .

$$\begin{aligned}
\text{Map}(\mathcal{D}, \mathbf{Ch}_{pe}^{NkGK}) &\simeq \text{Map}_{\text{dgCat}_{DK}}(NkGK, \underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe})) \\
&\simeq \text{Map}_{\text{dgCat}_{DK}}(NkGK, \tau_{\geq 0}(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) && \text{as LHS} \subset \text{Im}(\tau_{\geq 0}) \\
&\simeq \text{Map}_{\text{sModCat}}(kGK, DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) \\
&\simeq \text{Map}_{\text{sCat}}(GK, DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) \\
&\simeq \text{Map}_{\text{sGpd}}(GK, DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) && \text{as LHS is a groupoid} \\
&\simeq \text{Map}_{\text{sSet}}(K, \overline{W}(DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))))
\end{aligned}$$

Hence it suffices to show that $\overline{W}(DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe})))$ is weakly equivalent to $\text{Map}(\mathcal{D}, \mathbf{Ch}_{pe}) = \text{Map}(\mathbf{1}, \underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))$. Since any simplicial set K is weakly equivalent to $\text{Map}(*, K)$ we consider the following.

$$\begin{aligned}
\text{Map}_{\text{sSet}}(*, \overline{W}(DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe})))) &\simeq \text{Map}_{\text{sGpd}}(*, DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) \\
&\simeq \text{Map}_{\text{sModCat}}(\mathbf{1}, DK(\underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))) \\
&\simeq \text{Map}_{\text{dgCat}_{DK}}(\mathbf{1}, \underline{\text{RHom}}(\mathcal{D}, \mathbf{Ch}_{pe}))
\end{aligned}$$

Here we use some of the same observations as before and note moreover that $G* \simeq *$, the trivial simplicial groupoid. Here the unit $\mathbf{1}$ is the one object category with morphism space $DK(k)$ respectively k . \square

We can restrict from the dg-category $N(\Omega X)$ to a more familiar dg-algebra if X is connected and pointed. Let ΩX denote the topological group of based Moore loops on X . Then $C_*(\Omega X)$ is a dg-algebra.

Lemma 4.7. *Let X be a pointed and connected topological space. $C_*(\Omega X)$ considered as a dg-category with one object is quasi-equivalent to $N(kG(\text{Sing}_* X))$.*

Proof. $\text{Sing}_* X$ is a connected simplicial set and by the existence of minimal Kan complexes has a reduced model K , i.e. there is a weakly equivalent simplicial set with a single vertex. Then we have $G \text{Sing}_* X \simeq GK$ as simplicial groupoids and thus as simplicial categories. It follows that $N(kG \text{Sing}_* X) \simeq NkGK$. Finally, there is a weak equivalence of simplicial groups between GK and $\text{Sing}_* \Omega X$. \square

We have the following corollary:

Theorem 4.8. *The dg-categories $\mathbf{Ch}_{pe}^{C_*(\Omega X)}$ and $\mathcal{Y}(X)$ are Morita equivalent, as are $\mathbf{Ch}_{dg}^{C_*(\Omega X)}$ and $\mathcal{Y}^u(X)$.*

We can sum this up as a slogan: Morita cohomology is controlled by chains on the loop space.

If ΩX is formal we can study representations of $H_*(\Omega X)$ instead of $C_*(\Omega X)$. This is for example the case for $X = S^n$. But note that we will construct explicit models for $C_*(\Omega X)$ in Theorem 4.13.

Example 4.1. The category of loop space representations of S^2 is the category of bounded chain complexes with a degree 1 endomorphism which are fibrant and cofibrant. Recall that here the model structure has fibrations and weak equivalences determined by the forgetful map to \mathbf{Ch} . This follows since the homology algebra of ΩS^2 is equivalent to a polynomial algebra on a single generator in degree 1. We will provide details of the computation in Example 5.2.

Recall that all bounded objects are fibrant and cofibrant objects are determined by the lifting condition with respect to trivial fibrations.

Remark 4.4. There is a beautiful duality between $C^*(X)$ and $C_*(\Omega X)$, cf. [14]. It is well known that $R\text{Hom}_{C_*\Omega X}(k, k) \simeq C^*(X, k)$. We can interpret this as saying that the cohomology of k as a $C_*(\Omega X)$ -representation and as a constant sheaf on X agree, and in fact this is a direct consequence of our results characterizing $\mathcal{H}^M(X)$ as homotopy locally constant sheaves and as $C_*(\Omega X)$ -representations. (Recall that Morita cohomology is equivalent to the category of fibrant and cofibrant objects in homotopy locally constant sheaves, so the hom-spaces are automatically derived.) Conversely if X is simply connected, k is a field and all homology groups are finite dimensional over k it is true that $R\text{Hom}_{C^*(X, k)}(k, k) \simeq C_*(\Omega X)$. It would be very interesting to have a similar interpretation of $C^*(X)$ -modules where it is clear that endomorphisms of k are given by $C_*(\Omega X)$.

4.3. Cellular computations

The previous computations correspond to computing Čech cohomology and singular cohomology of topological spaces. This is often not the most effective way of computing, and it becomes even more cumbersome when we deal with categories rather than vector spaces.

In this section we will write down a simpler way of computing a model for $\mathbf{Ch}_{pe}^{C_*(\Omega X)}$ if X is a CW-complex. This model will be given by representations of an algebra $\mathcal{B}(X)$ with a generator in degree $e - 1$ for every e -cell (with an inverse if $e = 1$). We can think of this as

categorified cellular cohomology. The case for $\mathbf{Ch}_{dg}^{C_*(\Omega X)}$ works exactly in the same manner and for simplicity write $\mathbf{Ch}^{(-)}$ for both cases.

Note that if X has no 1-cells and k is a field we prove a stronger result and construct $\mathcal{B}(X)$ as a cofibrant dg-algebra weakly equivalent to $C_*(\Omega X)$.

Remark 4.5. The shift in the degree of generators corresponds to the correspondence of loop space and shift. To see another incarnation of this recall for example that the loop groupoid GK we used in the previous section looks as follows if K is reduced: GK_n is the free group on the set $K_{n+1} \setminus s_0 K_n$. So representations of GK consist, roughly, of a morphism in degree $n - 1$ for every element of K_n .

For later reference we note:

Lemma 4.9. $\mathcal{D} \mapsto \mathbf{Ch}^{\mathcal{D}}$ sends colimits to limits.

Proof. The construction $\mathcal{D} \mapsto \mathcal{D}\text{-Mod}$ is the naive category of dg-functors and is adjoint to the tensor product $- \otimes \mathbf{Ch}$. All objects are fibrant so we are left to compare cofibrants in $(\text{colim}_i \mathcal{A}_i)\text{-Mod}$ with the limit of the categories of cofibrants in $\mathcal{A}_i\text{-Mod}$. But since acyclic fibrations agree, the left lifting property gives the same conditions on both sides. \square

Next we compute an explicit model for $\mathbf{Ch}^{N\Omega(X)}$. The plan is to proceed by induction on the cells of X . To perform this we first need good models for the cofibrations $N\Omega S^{n-1} \hookrightarrow N\Omega B^n$.

Let $D(n)$ be the differential graded algebra $k[x_{n-1}, x_n \mid dx_n = x_{n-1}]$. Let $S(n) = k[x_n \mid dx_n = 0]$. Then $k \rightarrow S(n)$ and $S(n-1) \rightarrow D(n)$ are the generating cofibrations for the model structure on dg-algebras.

First we observe that $S(n-1) \simeq N\Omega S^n$ if $n > 1$. In other words $S(n-1)$ provides a model for singular chains on ΩS^n equipped with the Pontryagin product. This is of course well-known: $H_*(\Omega X)$ is a polynomial algebra on a generator in degree $n-1$. A computation via the James construction is in 3.C and 4.J of [22]. So there is a natural map $S(n-1) \cong H_*(\Omega S^n) \rightarrow N\Omega S^n$ which is a quasi-isomorphism. (We will see in Example 5.2 how to show directly that $S(n-1) \simeq N\Omega S^n$.)

We also need to know that there is a map $D(n) \rightarrow N\Omega B^n$ compatible with $S(n-1) \rightarrow D(n)$. This follows by the lifting property of the cofibration $S(n-1) \rightarrow D(n)$ with respect to the trivial fibration $N\Omega B^n \rightarrow *$.

These are the building blocks needed to associate to any connected CW-complex X without 1-cells a dg-algebra $\mathcal{B}(X)$ quasi-isomorphic to $C_*(\Omega X)$ that approximates the way X is glued from cells.

Lemma 4.10. *The inclusion of dg-algebras as dg-categories with one object preserves pushouts.*

Proof. Constructing the pushout of dg-categories with one object we obtain a dg-category with a single object. Furthermore maps from a one-object dg-category to other dg-categories are just maps from the dg-algebra of endomorphism to the dg-algebra of endomorphisms of the image. \square

In the proof of the next theorem we need to compute some homotopy pushouts. We assume k is a field so that the model category \mathbf{dgAlg}_k is proper and we can compute homotopy pushouts as pushouts whenever one of the constituent maps is a cofibration.

Theorem 4.11. *Assume k is a field. Associated to every connected CW complex X with cells in dimension ≥ 2 there is a cofibrant dg-algebra $\mathcal{B}(X)$ with one generator in degree $n - 1$ for every n -cell, that is quasi-equivalent to $N(\Omega X)$. In particular $\mathcal{Y}(X) \simeq \mathbf{Ch}^{\mathcal{B}(X)}$.*

Proof. We proceed by induction on the cells, let α be the index. We begin with the 2-skeleton of X which by assumption is a wedge of s spheres. Since the derived versions of N , $k \otimes -$, G and Sing_* all preserve homotopy colimits, so does $N\Omega$ on cofibrant spaces in \mathbf{CGHaus} . This gives $N\Omega(\bigvee_s S^2) \simeq \otimes_s k[x_1]$, i.e. a free dg-algebra on s generators in degree 1. We define $\mathcal{B}(X_2) := \otimes_s k[x_1]$.

From here on we can compute $N(\Omega X)$ inductively by forming, for every pushout $X_{\leq \alpha} = \text{colim}(e_\alpha^{n+1} \leftarrow S^n \rightarrow X_{< \alpha})$, the diagram of dg-categories $N(\Omega B^{n+1}) \leftarrow N(\Omega S^n) \rightarrow N(\Omega X_{< \alpha})$. Our aim is to show the pushout of the second diagram is weakly equivalent to $N(\Omega X_{\leq \alpha})$. Since \mathbf{CGHaus} and \mathbf{dgCat}_{DK} are proper the pushouts of both diagrams are homotopy pushouts and hence matched up by $N\Omega$.

Hence $N(\Omega X_{\leq \alpha})$ is determined by the map $N(\Omega S^n) \rightarrow N(\Omega X_{< \alpha})$. Since the left-hand side is weakly equivalent to a free dg-algebra on a single generator in degree $n - 1$ it suffices to specify its image, which is a homology class on the right-hand side, i.e. an element of $H_{n-1}(\Omega X_{< \alpha})$.

Inductively assume there is a weak equivalence $\mathcal{B}(X_{<\alpha}) \rightarrow N(\Omega X_{<\alpha})$. In particular there is an isomorphism on H_{n-1} and hence there is a map $S(n-1) \rightarrow \mathcal{B}(X_{<\alpha})$ corresponding to the attachment map $S^n \rightarrow X_{<\alpha}$. This gives a weak equivalence between pushout diagrams. We define $\mathcal{B}(X_{\leq\alpha})$ to be $\text{colim}(D(n) \leftarrow S(n-1) \rightarrow \mathcal{B}(X_{<\alpha}))$. By the previous lemma the pushout computed in **dgAlg** is the same as the one in **dgCat**. Since the pushouts are homotopy pushouts of diagrams which are levelwise weakly equivalent they agree up to homotopy and we have $\mathcal{B}(X_{\leq\alpha}) \simeq N(\Omega X_{\leq\alpha})$.

To extend to infinite CW-complexes we have to check the same argument goes through for filtered colimits. Since the maps $X_{<\alpha} \rightarrow X_{\leq\alpha}$ are cofibrations the filtered colimit is a homotopy colimit and commutes with $N\Omega$. So $N\Omega X_{\leq\lambda} \simeq \text{hocolim}_{\alpha<\lambda} N\Omega X_{\alpha}$ and we can define $\mathcal{B}(X_{\leq\lambda})$ as $\text{colim}_{\alpha<\lambda} \mathcal{B}(X_{\leq\alpha})$. \square

Proposition 4.12. *Let k be a commutative ring of characteristic 0. Assume the CW complex X as above is such that all attachment maps are cofibrations. Then $\mathcal{B}(X)$ constructed as above is weakly equivalent to $N\Omega(X)$.*

Proof. The only place we used that k is a field was in asserting that pushouts are homotopy pushouts if one of the maps is a cofibration. With our new assumption both $N(\Omega B^n) \rightarrow N(\Omega S^n)$ and $N(\Omega S^n) \rightarrow N(\Omega X_{\alpha})$ are cofibrations, so the pushout is a homotopy pushout without assuming properness. \square

Remark 4.6. To use this computation in practice we need to identify the degree $n - 1$ element y of $\mathcal{B}(X_{<a})$ that corresponds to the image of S^{n-1} . Then we adjoin a new generator x with $dx = y$. This can of course be quite non-trivial. There are some examples in the next section.

Next we deal with the case of 1-cells.

Remark 4.7. The main difficulty in considering \mathbf{Ch}^{S^1} arises as follows. It is clear that $C_*(\Omega S^1) \simeq k[\mathbb{Z}]$. However, $k[\mathbb{Z}]$ is not a cofibrant dg-algebra, so cannot be used for computing homotopy pushouts.

Theorem 4.13. *Associated to every connected CW complex X there is a dg-algebra $\mathcal{B}(X)$ with one generator in degree $n - 1$ for every n -cell with $n \geq 2$, and with two inverse generators in degree 0 for every 1-cell, such that $\mathcal{Y}(X) \simeq \mathbf{Ch}^{\mathcal{B}(X)}$.*

Proof. Let us define $S^*(0) = k[a, a^{-1}]$ and $D^*(1) = k[a, a^{-1}, b \mapsto a - 1]$ and consider the cofibration $S^*(0) \hookrightarrow D^*(0)$. Of course $D^*(0) \simeq k$.

Then we have compatible quasi-isomorphisms $N\Omega S^1 \rightarrow S^*(0)$ and $N\Omega B^2 \rightarrow D^*(1)$. The first is induced by projection to connected components $G \text{Sing}_* S^1 \rightarrow \mathbb{Z}$, the second map exists since $D^*(1) \rightarrow 0$ is a trivial fibration and $N\Omega S^1 \rightarrow N\Omega B^2$ is a cofibration.

Let X_1 be the 1-skeleton of X and define $\mathcal{B}(X_1) = \mathcal{B}(\bigvee_s S^1) := \otimes_s S^*(0)$ which is weakly equivalent to $C_*(\Omega(\bigvee_s S^1))$. There is an obvious map from $S^*(0)$ to $\mathcal{B}(X_1)$ for any attachment map $S^1 \rightarrow X_1$. Assume first that X is obtained from X_1 by attaching a 2-cell. Then we define

$$\mathcal{B}(X) = \text{colim}(D^*(1) \leftarrow S^*(0) \rightarrow \mathcal{B}(X_1))$$

Now $\mathcal{Y}(X)$ is the homotopy pullback of $\mathcal{Y}(B^2) \leftarrow \mathcal{Y}(S^1) \rightarrow \mathcal{Y}(X_1)$. But this diagram is weakly equivalent to $\mathbf{Ch}^{N\Omega B^2} \rightarrow \mathbf{Ch}^{N\Omega S^1} \leftarrow \mathbf{Ch}^{N\Omega X_1}$ which is in turn weakly equivalent to $\mathbf{Ch}^{D^*(1)} \rightarrow \mathbf{Ch}^{S^*(0)} \leftarrow \mathbf{Ch}^{\mathcal{B}(X_1)}$.

These are all pullback diagrams of fibrant objects with one map a fibration, hence they are homotopy pullbacks as \mathbf{dgCat}_{DK} is right proper. Since the diagrams are levelwise quasi-equivalent their pullbacks are quasi-equivalent, and thus also isomorphic in $Ho(\mathbf{dgCat}_{Mor})$. But since $\mathcal{D} \mapsto \mathbf{Ch}^{\mathcal{D}}$ sends colimits to limits it also follows that

$$\begin{aligned} \mathcal{Y}(X) &\simeq \text{holim} \left(\mathcal{Y}(B^2) \rightarrow \mathcal{Y}(S^1) \leftarrow \mathcal{Y}(X_1) \right) \\ &\simeq \text{holim} \left(\mathbf{Ch}^{D^*(1)} \rightarrow \mathbf{Ch}^{S^*(0)} \leftarrow \mathbf{Ch}^{\mathcal{B}(X_1)} \right) \\ &\simeq \lim \left(\mathbf{Ch}^{D^*(1)} \rightarrow \mathbf{Ch}^{S^*(0)} \leftarrow \mathbf{Ch}^{\mathcal{B}(X_1)} \right) \\ &\simeq \mathbf{Ch}^{\text{colim}(D^*(1) \leftarrow S^*(0) \rightarrow \mathcal{B}(X_1))} \end{aligned}$$

The colimit in the exponent is how we have defined $\mathcal{B}(X)$.

Now consider the general case. First to obtain $\mathcal{B}(X_2)$ note that any attachment map from S^1 factors through X_1 , so we can repeat the previous step as often as required. Attachment of higher-dimensional cells works in exactly the same manner, we just have to replace $S^*(0)$ by $S(n-1)$ and $D^*(1)$ by $D(n)$.

The argument extends to filtered colimits just like in the proof of Theorem 4.11. \square

Remark 4.8. By construction $\mathcal{B}(X)$ is Morita-equivalent to $N\Omega X$, but it does not follow from the construction whether the two dg-algebras are isomorphic in $Ho(\mathbf{dgAlg})$.

4.4. Finiteness and Hochschild homology

In this section we consider conditions for Morita cohomology to satisfy various finiteness properties, and determine Hochschild homology in some cases by quoting relevant results from the literature.

Let us first make some definitions. Here R denotes fibrant replacement in \mathbf{dgCat}_{Mor} . Specifically, $RB = L(B^{op}\text{-Mod})_{pe}$.

We say a dg-category \mathcal{D} is *locally proper* if the hom-space between any two objects is a perfect complex. \mathcal{D} is *proper* if moreover the triangulated category $H_0(R\mathcal{D})$ has a compact generator, i.e. a compact object which detects all objects.

Recall an object X in a model category is *homotopically finitely presented* if $\text{Map}(X, -)$ commutes with filtered colimits. \mathcal{D} is *smooth* if it is homotopically finitely presented as a $\mathcal{D}^{op} \otimes \mathcal{D}$ -module. \mathcal{D} is *saturated* if it is smooth, proper and Morita fibrant.

\mathcal{D} is *of finite type* if there is a homotopically finitely presented dg-algebra B such that $R\mathcal{D} \simeq R(B^{op})$.

These definitions are Morita-invariant (except for the condition of being Morita fibrant). Toën shows in Lemma 2.6 of [50] that a dg-category has a compact generator if and only if $R\mathcal{D} \simeq RB^{op}$ for some dg-algebra B and is moreover proper if and only if the underlying complex of B is perfect. Moreover any dg-category of finite type is smooth (Proposition 2.14 of [50]).

Remark 4.9. One reason to be interested in these finiteness conditions is that if a dg-category is saturated there is a nice moduli stack of objects, this is the main result of [50].

Proposition 4.14. *$\mathcal{Y}^u(X)$ is triangulated and has a compact generator. If X is a finite CW-complex without 1-cells then \mathcal{Y}^u is smooth. If moreover $H_*(\Omega X)$ is of finite type then $\mathcal{Y}^u(X)$ is saturated.*

Proof. Note first that as a homotopy limit $\mathcal{Y}^u(X)$ is fibrant and the compact generator is given by $C_*(\Omega X)$.

Theorem 4.13 implies that in the absence of 1-cells the dg-algebra $\mathcal{B}(X)$ is homotopically finitely presented. So the category $\mathcal{Y}^u(X)$ is of finite type and hence smooth. If $H_*(\Omega X)$ is of finite type, then $\mathcal{B}(X)$ is a perfect complex over k , and \mathcal{Y}^u is moreover proper and we find that $\mathcal{Y}^u(X)$ is saturated. \square

By contrast if X is an infinite CW-complex then $\mathcal{B}(X)$ is usually not homotopically finitely presented. For example consider $\mathcal{B}(\mathbb{C}P^\infty) \simeq k[x_1]/(x_1^2)$. Any cofibrant replacement has infinitely many generators and hence the identity does not factor through any subobject of finite type.

Next we consider properness for $\mathcal{Y}(X)$. The category $\mathcal{H}^M(X)$ is locally proper if all cohomology groups of X with coefficients in local systems are finite dimensional and concentrated in finitely many degree. This is for example the case if X has a finite good cover. Then the hom-spaces are finite limits of perfect chain complexes.

This is in contrast to Ext-groups of local systems which can be large even if X is very well behaved, for example if X is a smooth projective variety [10].

The example $X = S^1$ shows that we cannot expect $\mathcal{Y}(X)$ to be proper in general. \mathbf{Ch}^{S^1} is the category of complexes of \mathbb{Z} -representations, with infinitely many connected components, see Example 5.1.

Proposition 4.15. *If $\pi_1(X)$ has only finitely many irreducible representations which are all finite dimensional then there exists a compact generator A and $\mathcal{Y}(X) \simeq L(\text{End}(A)^{\text{op}}\text{-Mod})_{pe}$. $\mathcal{Y}(X)$ is proper if $C^*(X, \text{End}(A))$ is a perfect complex.*

Proof. We define A to be the sum of all the irreducibles. Then A maps to the lowest nontrivial homology group of any object in $\mathcal{Y}(X)$ and hence generates the dg-category since objects with trivial homology are quasi-isomorphic to 0.

By Lemma 2.6 of [50] $L(\mathcal{Y}(X)^{\text{op}}\text{-Mod}) \simeq L(\text{End}_{\mathcal{Y}(X)}(A)^{\text{op}}\text{-Mod})$. Since $\mathcal{Y}(X) \simeq L(\mathcal{Y}(X)^{\text{op}}\text{-Mod})_{pe}$ we deduce that $\mathcal{Y}(X)$ is the subcategory of compact objects in $\text{End}(A)\text{-Mod}$.

The second statement is clear. □

The proposition applies for example if the fundamental group is finite. Then we can take A to be the group ring.

Example 4.2. Let X be simply connected. Then we can take $A = k$ and find $\text{End}(A) \simeq R\text{Hom}_{\Omega X}(k, k) \simeq C^*(X, k)$ by earlier results. In particular $\mathcal{Y}(X) \simeq C^*(X, k)$ in \mathbf{dgCat}_{Mor} . Then $\mathcal{Y}(X)$ is proper if and only

if $C^*(X, k)$ is a perfect complex. If $C^*(X, k)$ is homotopically finitely presented then $\mathcal{Y}(X)$ is moreover smooth and saturated.

If $\mathcal{Y}(X)$ has a compact generator it becomes much easier to compute secondary invariants. In particular we can compute Hochschild homology and cohomology. For definitions and a summary of results see [29]. Since Hochschild homology and cohomology are Morita-invariant we can compute them on a generator of a dg-category if there is one.

Example 4.2 implies the following proposition. Here HH stands for either HH^* or HH_* .

Proposition 4.16. *Let X be simply connected then $HH(\mathcal{Y}(X)) \cong HH(C^*(X))$.*

So we can compute Hochschild (co)homology of Morita cohomology from minimal models (in the sense of Sullivan).

Proposition 4.17. $HH(\mathcal{Y}^u(X)) \cong HH(C_*(\Omega X)) \cong HH(\mathcal{B}(X))$.

Proof. The second isomorphism follows since Hochschild (co)homology is Morita-invariant. \square

The following applications follows from results readily available in the literature.

Proposition 4.18. *Let X be simply connected then $HH_*(\mathcal{Y}(X)) \cong H^*(\mathcal{L}X)$. If M is a simply connected closed oriented manifold of*

dimension d then $HH^*(\mathcal{Y}(M)) \cong H_{*+d}(\mathcal{L}M)$ as graded algebras with the Chas-Sullivan product on the right hand side.

Proof. If X is simply connected it is well known (see [32]) that $HH_*(C^*(X, k)) \cong H^*(\mathcal{L}X)$ where $\mathcal{L}X$ is the free loop space.

The second part follows since the Hochschild cohomology ring of singular cochains on M (with the cup product) is isomorphic to its loop homology with the Chas-Sullivan product, cf. [9]. \square

Proposition 4.19. $HH_*(\mathcal{Y}^u(X)) \cong H_*(\mathcal{L}X)$. If X is simply connected $HH^*(\mathcal{Y}^u(K)) \cong H^*(\mathcal{L}X)$ as graded algebras.

Proof. We find $HH_*(\mathcal{Y}^u(X)) \cong HH_*\Omega(X) \cong H_*(\mathcal{L}X)$ from 7.3.14 in [31].

For a the result that $HH^* \text{Sing}_* \Omega X \cong H^*(\mathcal{L}X)$ (as graded algebras) for a simply connected CW-complex X , see [34]. \square

Note that we do not expect Hochschild homology of $\mathcal{Y}(X)$ to be particularly tractable if X is not simply connected. For example $\mathcal{Y}(S^1)$ has $|k^*|$ simple objects with no morphisms between them, cf. Example 5.1. Hence it follows from the explicit definition in [29] that Hochschild homology consists of $|k^*|$ copies of $HH_*(k[y])$ where y lives in degree 1 and has square 0.

5. Computation and Examples

In this chapter we compute some examples of Morita cohomology. In the following whenever an element has a subscript, this will denote its degree.

For simplicity we will often not mention the restriction to fibrant cofibrant objects in representations of $C_*(\Omega X)$ or $\mathcal{B}(X)$.

5.1. Spheres

Example 5.1. We begin with the case $X = S^1$. Clearly $\mathcal{H}^M(S^1)$ is equivalent to the category of representations of $\mathbb{Z} \simeq \Omega S^1$.

We can also view this as the category of bounded chain complexes of local systems on S^1 since S^1 is $K(\mathbb{Z}, 1)$ and the cohomological dimension of \mathbb{Z} is 1, so that all non-trivial extensions in chain complexes of \mathbb{Z} -modules are extensions in the abelian category of \mathbb{Z} -modules.

We can also characterize $\mathcal{H}^M(S^1)$ as the explicit limit

$$(\mathbf{Ch}_{pe})^I \times_{\mathbf{Ch}_{pe} \times \mathbf{Ch}_{pe}} \mathbf{Ch}_{pe}$$

Here \mathbf{Ch}_{pe}^I is the path object in dg-categories, which is $(\mathbf{Ch}_{pe})_1$ in the simplicial resolution constructed in the appendix, see Example A.2.

The limit then comes out as the category of pairs $(M, \phi \in \text{Aut}(M))$ with morphisms $(f, g, h) : (M, \phi) \rightarrow (N, \psi)$ in $\underline{\text{Hom}}(M, N)^{\oplus 2} \oplus \underline{\text{Hom}}(M, N)[-1]$ with differential

$$(f, g, h) \mapsto (df, dg, dh - (-1)^{|g|}g\phi + \psi f)$$

In particular $\underline{\text{Hom}}^*(k, k) \cong k \oplus k[1]$, which is exactly cohomology of S^1 , as predicted.

We can also compute $\mathcal{H}^M(S^1)$ with a Čech cover. Using a good cover by three open sets and their intersections and an explicit model for \mathbf{Ch}_1 , see above, we find that an object is given by three chain complexes with weak equivalences between them. We can use two weak equivalences to identify the complexes and are left with a single homotopy invertible map.

As we have stated before, the category $\mathcal{H}^M(S^1)$ is highly disconnected, in fact isomorphism classes of simple objects are naturally in bijection with k^* . Of course k^* has a geometric structure, and one way of interpreting large sets of isomorphism classes of objects is to consider a moduli stack of objects of $\mathcal{H}^M(X)$. We will not follow this direction here.

Example 5.2. If $n > 1$ then $\mathcal{H}^M(S^n) \simeq \mathbf{Ch}_{pe}^{S(n)}$, i.e. the category of perfect chain complexes with an endomorphism in degree $n - 1$ that are fibrant and cofibrant as such modules.

Proof 1. This is immediate from the quasi-isomorphism $S(n) \rightarrow N(\Omega \text{Sing}_* S^n)$ which was mentioned in the last section. \square

Proof 2. We can also compute $\mathcal{B}(S^2)$ using the method of Theorem 4.13 by gluing two copies of B^2 along S^1 . The resulting dg-algebra has one invertible generator with two trivialising homotopies, which is quasi-isomorphic to $k[x_1] = S(1)$.

Once we know the case $n = 2$ we can inductively compute $S^n = D^n \amalg_{S^{n-1}} D^n$ and note that $S(n) \simeq D(n) \otimes_{S(n)}^L D(n)$.

Note that we can use this construction of $\mathcal{B}(S^n)$ in the proof of Theorem 4.13. There is no circularity as we only need a model for spheres in smaller dimensions to compute $\mathcal{B}(S^n)$. \square

Example 5.3. Next consider some more detail for $n = 2$. Since k is a generator and $R\underline{\text{End}}_{C_*(\Omega S^2)} \simeq C^*(S^2) \simeq k[x_2, x_3 \xrightarrow{d} x_2^2] =: A$ we can characterize $\mathcal{Y}(S^2)$ as compact objects in $A\text{-Mod}$.

An example of an object of $R\Gamma_{\text{Morita}}(S^2, k)$ is the chain complex associated to the Hopf fibration $p : S^3 \rightarrow S^2$. As a homotopy locally constant sheaf we can consider this as $Rp_* \text{Sing}_*(S^3)$. As a representation of ΩS^2 this can be written as $k \oplus k[-1]$ with the canonical map of degree 1.

Since $\pi_1(S^2)$ is trivial, we can also view $R\Gamma_{\text{Mor}}(S^2, \underline{k})$ as generated by the trivial local system and the information $H^*(S^2, -)$ provides about (iterated) extensions. This provides a slightly different viewpoint on Morita cohomology.

Specifically, consider the forgetful map $\mathcal{D} \rightarrow \mathbf{Ch}$. The objects in the fibre over $M \cong \bigoplus M^i[-i]$ are all the homotopy locally constant sheaves with homology M . They can be determined

iteratively. For example, over $M^0 \oplus M^1[-1]$ the fiber is parametrized by $C^*(X, \text{Hom}^1(M^1[-1], M^0))$.

5.2. Other topological spaces

Example 5.4. $\mathcal{H}^M(BG)$ is just the dg-category of perfect complexes with an action of G .

Example 5.5. $R\Gamma(\mathbb{R}P^2, \mathbf{Ch}_{pe})$ is given by representations of $\mathcal{B}(\mathbb{R}P^2)$ on perfect complexes, and $\mathcal{B}(\mathbb{R}P^2)$ has generators a_0, a_0^{-1}, b_1 such that $db_1 = a_0 \circ a_0 - 1$. This follows from Theorem 4.13. The identification $db_1 = a_0 \circ a_0 - 1$ is induced by the attaching map from the boundary of the 2-cell to $\mathbb{R}P^1$.

If we are working over the field \mathbb{Q} Morita cohomology has certain similarities to rational homotopy theory, cf. the duality between $C_*(\Omega X)$ and $C^*(X)$ in the simply connected case. On the other hand we see that $\mathbb{R}P^2$ has trivial minimal model, but its Morita cohomology is a dg-category with two simple objects corresponding to the irreducible reps of $\mathbb{Z}/2$.

We can obtain $\mathcal{B}(\mathbb{R}P^3)$ from $\mathcal{B}(\mathbb{R}P^2)$ by adding c_2 with $dc_2 = 0$.

Example 5.6. Next we compute the map $p^* : \mathcal{H}^M(S^2) \rightarrow \mathcal{H}^M(S^3)$ induced by the Hopf fibration. On the level of loop spaces we see that the map is induced by $\Omega p_* : H_*(\Omega S^3) \rightarrow H_*(\Omega S^2)$ which is given by $x_2 \mapsto y_1^2$ on the generators.

With this in mind we can work out $\mathcal{H}^M(\mathbb{C}P^2)$ explicitly by considering the following diagram:

$$\mathcal{H}^M(B^4) \xrightarrow{i^*} \mathcal{H}^M(S^3) \xleftarrow{p^*} \mathcal{H}^M(\mathbb{C}P^1)$$

On the level of dg-algebras we have

$$D(3) \xleftarrow{i_*} S(3) \xrightarrow{p_*} \mathcal{B}(S^2) \cong S(2)$$

The attaching map p_* , is induced by the Hopf fibration. As we have just seen it corresponds to the map $H_*(\Omega S^3) \rightarrow H_*(\Omega S^2)$ given by sending $x_2 \mapsto y_1^2$. Hence we find:

$$\mathcal{B}(\mathbb{C}P^2) \simeq k[\alpha_1, \alpha_3 \mid d\alpha_3 = \alpha_1^2]$$

Example 5.7. We can generalise this to $\mathbb{C}P^n$, every extension over a $2i$ -cell corresponding to another map α_{2i-1} in degree $2i - 1$. We find $d : \alpha_3 \mapsto \alpha_1^2; \alpha_5 \mapsto \alpha_3\alpha_1 + \alpha_1\alpha_3; \alpha_7 \mapsto \alpha_5\alpha_1 + \alpha_3^2 + \alpha_1\alpha_5$ etc.

Let us compare this with a computation of $H_*(\Omega\mathbb{C}P^n)$, which is done e.g. in [38]. The fibration $\Omega S^{2n+1} \rightarrow \Omega\mathbb{C}P^n \rightarrow S^1$ is a direct product. Hence $H_*(\Omega\mathbb{C}P^n) \cong \Lambda(y_1) \otimes k[y_{2n}]$ as a Hopf algebra, in particular the Pontryagin products agree. $C_*(\Omega\mathbb{C}P^n)$ is moreover formal since $C_*(\Omega S^{2n+1})$ and $C_*(\Omega S^1)$ are.

To relate this to the above description identify $y_{2n} = \alpha_{2n-1}\alpha_1 + \cdots + \alpha_1\alpha_{2n-1}$. The dg-algebra $\mathcal{B}(X)$ is larger since it is quasi-free (i.e. the underlying graded associative algebra is free), while $H_*(\Omega\mathbb{C}P^n)$ is only quasi-free as a commutative dg-algebra.

Example 5.8. Taking the limit we find $\mathcal{B}(\mathbb{C}P^\infty)$.

Of course the homology algebra of $\Omega\mathbb{C}P^\infty$ is just that of S^1 . Indeed $k[\alpha_1, \alpha_3, \dots]$ is a quasi-free model for $k[z_1]$.

We conclude with a few examples that give some insight into what Morita cohomology (doesn't) tell us about a space.

From [47] we know that a CW-complex X is determined up to weak equivalence by the ∞ -category of homotopy locally constant sheaves of spaces with its fiber functor. We can think of this category as $R\Gamma(X, \mathbf{sSet})$ and it is natural to compare with $R\Gamma(X, \mathbf{Ch}_{pe})$. The main differences are linearization, stabilization and restriction to compact fibers.

Example 5.9. An example of a space with trivial Morita cohomology is provided by the classifying space of Higman's 4-group H . This is known to be a finite CW complex and H is an acyclic group without non-trivial finite dimensional representations. For references and other examples see e.g. [6].

To show that the Morita cohomology of BH is trivial we have to show it is Morita equivalent to \mathbf{Ch} . Now given an object M of $\mathcal{H}^M(BH)$ we can take homology. As H has no non-trivial finite-dimensional representations this is a direct sum of shifted trivial representations. We can now show by induction that M must be a trivial extension, as there is no cohomology $H^{-0}(H, k)$. Since quasi-isomorphisms are detected on the underlying complex an object with trivial homology is quasi-isomorphic to 0. It follows that $\mathcal{H}^M(BH) \cong \mathbf{Ch}$ in $Ho(\mathbf{dgCat}_{Mor})$.

Example 5.10. Whenever a group G and its algebraic completion G^{alg} have the same cohomology with coefficients in their finite-dimensional representations then they have the same Morita cohomology. By definition this is true for algebraically good groups, see [28].

Example 5.11. Next we consider an example of information in the category of simplicial fibrations with fiber of finite type that is not detected by Morita cohomology.

Consider the outer automorphism group of the free group on four generators, $\text{Out}(F_4)$. This group is known to be finitely presented (since $\text{Aut}(F_4)$ is), and nonlinear [19]. It is also isomorphic to $\pi_0\text{Map}(F, F)$ if $F = \bigvee_4 S^1$. Hence there is a homotopy locally constant sheaf of simplicial sets of finite type (with fiber F) on $B\text{Out}(F_4)$ that does not descend to any BG for G a quotient of $\text{Out}(F_4)$. On the other hand the associated complex of local systems must descend to the classifying space of the image of $\text{Out}(F_4)$ in its linearization.

Remark 5.1. Contrast this with $\mathcal{S}^u(X)$ which loses little information about the weak homotopy type of X . Since ΩX is an H -space it is nilpotent and there is a Whitehead theorem for integral homology for nilpotent spaces. So if $N\Omega X$ and $N\Omega Y$ are quasi-equivalent (rather than just Morita equivalent) then $X \simeq Y$.

A. Some technical results on dg-categories

A.1. A combinatorial model for dg-categories

For some technical questions it is convenient to work with combinatorial model categories. (For example it is easier to prove the existence of localizations and injective model structures.) One does not expect \mathbf{dgCat}_{Mor} to be combinatorial as it is too large, but we show in this section that it is equivalent to a subcategory that is combinatorial.

Definition. A model category is *combinatorial* if the underlying category is locally presentable.

Being locally presentable is a finiteness condition.

Definition. Let λ be a regular cardinal. An object A in a category \mathcal{D} is λ -*presentable* if it is small with respect to λ -filtered colimits, i.e. if for every λ -filtered colimit $\operatorname{colim} B_i$ the map $\operatorname{colim} \operatorname{Hom}(A, B_i) \rightarrow \operatorname{Hom}(A, \operatorname{colim} B_i)$ is an isomorphism. We say A is *presentable* if it is λ -presentable for some λ . A cocomplete category is *locally presentable* if for some regular cardinal λ it has a set S of λ -presentable objects such that every object is a λ -directed colimit of objects in S .

Proposition A.1. *The categories \mathbf{dgCat}_{DK} and \mathbf{dgCat}_{Mor} are Quillen equivalent to combinatorial subcategories.*

Proof. This follows immediately from the proof of the main theorem of [36]. Let \mathcal{D} denote either of the two model structures. Let S be the collection of objects that are domains or codomains of the generating cofibrations and generating trivial cofibrations. Clearly S is a set. Let \mathcal{S} denote the full subcategory of \mathcal{D} with objects S . Define $\eta_S(X)$ to be the colimit of the forgetful diagram $(s \rightarrow A) \mapsto s$ indexed by the overcategory $\mathcal{S} \downarrow A$. Then an object $A \in \mathcal{D}$ is *S-generated* if it is isomorphic to $\eta_S(X)$.

Now by the proof of Theorem 1.1 in [36] the subcategory of S -generated objects of \mathcal{D} is a model category \mathcal{D}_S which is Quillen equivalent to the original one. Moreover, by Proposition 3.1 of [36], \mathcal{D}_S is locally presentable if every object in S is presentable. But this is clear since they have finitely many objects and generating morphisms. \square

Remark A.1. Note that Vopenka's principle is not needed here since the objects of S are presentable.

We get another finiteness condition for free. A model category is called *tractable* if the generating cofibrations and generating trivial cofibrations can be chosen to have cofibrant domains.

Corollary A.2. *The categories \mathcal{D}_S are tractable.*

Proof. By Corollary 1.12 in [1] it is enough to check all the generating cofibrations can be chosen to have cofibrant domains. This is immediate in our example. \square

A.2. Simplicial resolutions of dg-categories

In this section we will construct explicit simplicial resolutions $\mathcal{C} \mapsto \mathcal{C}_\bullet$ in \mathbf{dgCat}_{DK} to improve our understanding of the homotopy theory of \mathbf{dgCat}_{DK} . Such a resolution can be used for explicit, if unwieldy, computations. The resolutions we construct will be simplicial frames for a fibrant replacement.

Our construction is directly motivated by Simpson’s construction of global sections of a presheaf of dg-categories as a dg-category of Maurer–Cartan elements, cf. section 5.4 of [41].

Remark A.2. In fact, the construction of \mathcal{C}_n below corresponds to considering the constant presheaf of dg-categories on a covering of $|\Delta^n|$ by $n + 1$ open sets (corresponding to leaving out one of the faces).

Define \mathcal{C}_n as follows. We think of this as \mathcal{C}^{Δ^n} , but recall that the precise definition of \mathcal{C}^{Δ^n} differs by a limit over degenerate simplices.

Definition. Assume \mathcal{C} is fibrant, replace fibrantly otherwise. Then \mathcal{C}_n is a dg-category with objects given by pairs (E, η) where E is a collection $E_0, \dots, E_n \in \text{Ob}\mathcal{C}$ and η is a collection of $\eta_I = \eta(I) \in \underline{\text{Hom}}_{k-1}(E_{i_0}, E_{i_k})$ for all multi-indices $I = (i_0, \dots, i_k)$ with $1 \leq k \leq n$. The case $k = 0$ is subsumed by the differential on E . (We interpret $\eta(i) = 0$ where it

comes up in computation.) These pairs must satisfy the Maurer–Cartan condition: $\delta\eta + \eta^2 = 0$, explained below. We also demand that all $\eta_i \in \underline{\mathbf{Hom}}(E_i, E_i)$ are weak equivalences in \mathcal{C} .

Let us spell out the Maurer–Cartan condition. Intuitively, η provides all the comparison maps as well as homotopies between the different compositions. We define the differential

$$(\delta\eta)(i_0, \dots, i_k) := d(\eta(i_0, \dots, i_k)) + (-1)^{|\eta|} \sum_{j=1}^{k-1} (-1)^j \eta(i_0, \dots, \widehat{i_j}, \dots, i_k)$$

which lives in $\underline{\mathbf{Hom}}_k(E_{i_0}, E_{i_k})$. We write $\delta = d + \Delta$. Here we define $|\eta| = 1$. The product is:

$$(\phi \circ \eta)(i_0, \dots, i_k) := \sum_{j=0}^k (-1)^{|\phi|j} \phi(i_j, \dots, i_k) \circ \eta(i_0, \dots, i_j)$$

Both definitions follow section 5.2 of [41], with some corrections to the signs. We leave out the terms in $\delta\eta$ corresponding to leaving out i_0 and i_k as they do not live in the correct hom-spaces.

One can now check that $\Delta d = -d\Delta$ (and hence $\delta^2 = 0$) and we have the following Leibniz rule:

$$\delta(\phi \circ \eta) = (-1)^{|\eta|} (\delta\phi) \circ \eta + \phi \circ (\delta\eta)$$

The same equation holds for the summands d and Δ . (The unusual sign appears because of the backward notation for compositions.)

Example A.1. For $n = 1$ we have $(\delta\eta + \eta^2)_{01} = d(\eta_{01}) + 0$, the expected cycle condition. For $n = 2$ we have for example

$$(\delta\eta + \eta^2)_{012} = d(\eta_{012}) + \eta_{02} - \eta_{12} \circ \eta_{01} \in \underline{\mathbf{Hom}}_1(E_0, E_2)$$

So an element of \mathcal{D}_2 is of the form (E, η) where $E = (E_0, E_1, E_2)$ and $\eta = (\eta_{01}, \eta_{02}, \eta_{12}; \eta_{013})$ satisfies $d\eta + \eta^2 = 0$, which comes out to $d\eta_{ij} = 0$ and $d\eta_{012} = -\eta_{02} + \eta_{12} \circ \eta_{01}$. This agrees with our intuition that η_{012} is a homotopy from $\eta_{12} \circ \eta_{01}$ to η_{02} .

Morphisms from (E, η) to (F, ϕ) are as follows.

$$\underline{\text{Hom}}_{\mathcal{C}_n}^{-m}((E, \eta), (F, \phi)) = \{a(i_0, \dots, i_k)\}$$

where $a(i_0, \dots, i_k) \in \underline{\text{Hom}}_{m-k}(E_{i_0}, F_{i_k})$. (Here $C^{-m} = C_m$.) We write $m = |a|$ for the degree of a morphism. We have a differential $d_{\eta, \phi}$ defined by

$$(d_{\eta, \phi}(a))(i_0, \dots, i_k) = \delta(a) + \phi \circ a - (-1)^{|a|} a \circ \eta$$

where composition and differential are defined as above. The Maurer–Cartan condition on η and ϕ together with the Leibniz rule ensures $(d_{\eta, \phi})^2 = 0$.

Example A.2. For example \mathcal{C}_1 agrees with the path object in \mathbf{dgCat} as constructed in Section 3 of [46]. Indeed, objects are homotopy invertible morphisms $\eta : A \rightarrow B$ and morphisms from η to ϕ are given by triples (a_0, a_1, a_{01}) with differential

$$\delta : (a_0, a_1, a_{01}) \mapsto (da_0, da_1, da_{01} + \phi \circ a_0 - (-1)^{|a_0|} a_0 \circ \eta)$$

Notation. Given an object or morphism α and a positive integer k we write $\alpha_{[k]}$ for the collection of all $\alpha_{i_0 \dots i_k}$.

Before we embark on the somewhat technical proof that \mathcal{C}_\bullet is a simplicial resolution, we note the following application. We can extend

the definitions of the differentials and composition to functions defined on general simplices. (That is, we replace “leaving out the i -th term” by the map induced by ∂_i etc.)

Proposition A.3. *Given a simplicial set K we can construct \mathcal{C}^K as the dg-category with objects (E, η) where $E \in (\text{Ob } \mathcal{C})^{K_0}$ and η assigns to every k -simplex in $K_{\geq 1}$ a map in $\underline{\text{Hom}}_{k-1}(E(\partial_0^k \sigma), E(\partial_{\max}^k \sigma))$ satisfying the Maurer–Cartan equations. Hom-spaces are defined similarly to hom-spaces in \mathcal{C}_\bullet .*

Proof. This follows from the construction of $\mathcal{C}^K = \lim_{\Delta K^{\text{op}}} \mathcal{C}_\bullet$. \square

Remark A.3. This shows that the construction of ∞ -local systems as $\mathcal{C} \mapsto \mathcal{C}^K$ corresponds to the ∞ -local systems of [7], or the A_∞ -functor of [26] if K is the nerve of a category, or to the Čech globalization in [41] if K is the nerve of an open covering.

Proposition A.4. *The inclusion from the constant simplicial dg-category $c\mathcal{C}$ to \mathcal{C}_\bullet is a levelwise weak equivalence.*

Proof. We have to check the inclusion map $\iota : c\mathcal{C} \rightarrow \mathcal{C}_n$ is a quasi-equivalence.

Let us first show that ι induces weak equivalences on hom-complexes. We have to show that $\underline{\text{Hom}}_{\mathcal{C}^{\Delta^n}}((E, \eta), (F, \phi)) \simeq \underline{\text{Hom}}_{\mathcal{C}}(E, F)$ when both η and ϕ are of the form $(\mathbf{1}, 0)$, i.e. the constituent morphisms in degree 0 are the identity and all others are 0.

Write $(H, d_H) := \underline{\text{Hom}}(E, F)$ and note that from the definitions we can write

$$\underline{\text{Hom}}((E, 0), (F, 0)) \simeq (H[1] \otimes \bigwedge \langle e_0, \dots, e_n \rangle, D)$$

Here the e_i all have degree 1 and we identify $H.e_{i_0} \wedge \dots \wedge e_{i_k}$ with the $a(i_0, \dots, i_k)$. The differential D is $d_H + \iota_{\sum e_i}$ where the second term denotes contraction. This complex is a resolution of (H, d_H) .

Next we show ι is quasi-essentially surjective, i.e. show that any object (E, η) is equivalent to an object $(\underline{F}_0, (\mathbf{1}, 0))$ where \underline{F}_0 is of the form (F_0, \dots, F_0) .

We can deduce this if we can show that every (E, η) is equivalent to some (F, ϕ) such that all compositions which agree up to homotopy by $\delta\phi + \phi^2 = 0$ agree strictly, i.e. $\phi = (\phi_{[0]}, 0)$, and that any such $(F, (\phi_{[0]}, 0))$ is equivalent to $(\underline{F}_0, (\mathbf{1}, 0))$. The second part of this is immediate: We define a map from $(\underline{F}_0, (\mathbf{1}, 0))$ to $(F, (\phi_{[0]}, 0))$ by sending F_0 to F_i via $\phi(0, i) = \phi(i-1, i) \dots \phi(0, 1)$. Since all $\phi(j, j+1)$ are homotopy invertible there is a homotopy inverse.

We will now show that any (E, η) is equivalent to (F, ϕ) where ϕ has no higher homotopies. Let $F = E$ and let $\phi(i, j) = \eta(j-1, j) \dots \eta(i, i+1)$. We may assume by induction on n that all $\eta(i_0, \dots, i_k)$ with $i_k < n$ are 0.

We define the homotopy equivalence $H : (E, \eta) \rightarrow (E, \phi)$ as follows:

$$\begin{aligned} H(i) &= \mathbf{1} \\ H(i_0, \dots, i_k) &= (-1)^{k-1} \eta(i_0, \dots, i_{k-1}, n-1, n) \quad \text{if } i_k = n \text{ and } i_{n-1} \neq n-1 \\ H(i_0, \dots, i_k) &= 0 \quad \text{otherwise} \end{aligned}$$

And define H^- to be equal to H in degree 0 and $-H$ in degree > 0 , i.e. the sign of $H(i_0, \dots, i_k)$ is always $(-1)^k$.

Then it is clear that H and H^- are inverses. Since $H(i_0, \dots, i_n)$ is zero unless $i_n = n$ there are no nontrivial compositions and the composition $\mathbf{1} \circ H(\dots)$ and $H^-(\dots) \circ \mathbf{1}$ cancel in degrees greater than 0.

So it remains to show that $dH = dH^- = 0$ to show we have a genuine homotopy equivalence.

We consider H first. Putting together our definitions we find the following. Let us first assume $i_{k-1} \neq n-1$ and $i_k = n$. To obtain the correct signs recall that $|H| = 0$ and $|\eta| = |\phi| = 1$.

$$\begin{aligned}
(dH)(i_0, \dots, i_k) &= d(H(i_0, \dots, i_k)) + \sum_{j=1}^{k-1} (-1)^j H(i_0, \dots, \hat{i}_j, \dots, i_n) \\
&\quad + \sum_{j=0}^k (-1)^j \phi(i_j \dots i_n) \circ H(i_0, \dots, i_j) \\
&\quad - \sum_{j=0}^k H(i_j, \dots, i_k) \circ \eta(i_0, \dots, i_j) \\
&= (-1)^{k-1} d\eta(i_0, \dots, n-1, n) \\
&\quad + (-1)^{k-2} \sum_j (-1)^j \eta(i_0, \dots, \hat{i}_j, \dots, n-1, n) \\
&\quad + 0 - (-1)^{k-2} \eta(i_1, \dots, n-1, n) \circ \eta(i_0, i_1) - \mathbf{1} \circ \eta(i_0, \dots, i_k) \\
&= 0
\end{aligned}$$

The last equality holds since the penultimate term is of the form

$$(-1)^{k-1} (\delta\eta + \eta^2)(i_0, \dots, i_{k-1}, n-1, n)$$

This becomes clear if we write $\eta(i_0, \dots, i_k) = \eta(i_0, \dots, \widehat{n-1}, n)$ and observe that all the other terms we expect in $\delta\eta + \eta^2$ are 0.

The other cases are easier. If $i_k \neq n$ all terms in the differential are 0 and if $i_{k-1} = n-1$ and $i_k = n$ there are only two nonzero terms, which cancel.

When we consider dH^- the sign of the term $\eta(i_0, \dots, i_k)$ changes, as it now comes from $\eta \circ H$ and not $H \circ \eta$. This cancels the effect of the sign of $H(i)$ also changing by a factor of -1 . There are no other occurrences of the sign of $H(i)$ unless $k = 1$ when all but the last two terms are zero and the last two terms cancel. \square

Proposition A.5. \mathcal{C}_\bullet is Reedy fibrant.

Proof. Write

$$\eta_{<n} := (\eta_0, \dots, \widehat{\eta_{0\dots n}}) = (\eta_{[0]}, \dots, \eta_{[n-1]})$$

Then $M_n(\mathcal{C})$ is a subcategory of \mathcal{C}_n whose objects are of the form $(E, \eta_{<n})$. In particular note that the Maurer–Cartan condition holds on all indexing sets except on $(0, \dots, n)$. Similarly, morphisms are of the form $s_{<n}$ where s is a morphism in \mathcal{C}_n . This is easily seen to be the correct limit, see Proposition A.3. We write $\pi : \mathcal{C}_n \rightarrow M_n\mathcal{C}$ for the functor forgetting $\eta_{[n]}$.

It is immediate from the definition that there is a surjection on hom-spaces. So it remains to check the lifting property for homotopy invertible maps. We first reduce to lifting contractions, as is done in the case of path objects in Section 3 of [46].

Note that by assumption the dg-category \mathcal{C} is fibrant and hence has cones, cf. Section 2 of [46]. Then to see if a map h is homotopy invertible it suffices to see if $\text{cone}(h)$ is contractible. So assume $h : (E, \eta_{<n}) \rightarrow (F, \phi_{<n})$ is homotopy invertible in $M_n\mathcal{C}$ with homotopy inverse g and that $(E, \eta_{<n}) \in \text{Im}(\pi)$. Next we need to check that $(F, \phi_{<n})$ is also in the image of \mathcal{C}_n . It is enough to find $\phi_{[n]}$ such that $\delta\phi + \phi^2 = 0$ while we know that $\delta\phi_{<n} + \phi_{<n}^2 = 0$. In other words we are looking for $\phi_{[n]}$ such that $d\phi_{[n]} = (\Delta\phi + \phi^2)_{[n]}$.

We will first consider $g(n) \cdot (\Delta\phi + \phi^2)(0 \dots n)$. Define $\rho \circ' \sigma$ to be $\rho \circ \sigma$ minus the term $\rho(n) \cdot \sigma(\dots n)$. Then $- \circ' \sigma = - \circ \sigma$ if σ is η or ϕ . Note that d and Δ are compatible with \circ' just as with the usual product.

Then $g(n) \cdot \phi(i \dots n) = (-g \circ' \phi + \eta \circ g - \delta g)(i \dots n)$ and we can perform the following computation, where we deduce the Maurer–Cartan condition in degree n from the Maurer–Cartan conditions in lower degrees.

$$\begin{aligned}
g(n) \cdot (\Delta\phi + \phi \circ \phi) &= -\Delta(g \circ' \phi) + \Delta(\eta \circ g) - \Delta\delta g \\
&\quad + (-g \circ' \phi + \eta \circ g - \delta g) \circ \phi \\
&= -g \circ' (\Delta\phi + \phi \circ \phi) + \eta \circ (\Delta g + g \circ \phi) \\
&\quad - dg \circ' \phi + \Delta\eta \circ g + d\Delta g \\
&\simeq g \circ' d\phi - dg \circ' \phi + \eta \circ \eta \circ g - \eta \circ dg + \Delta\eta \circ g \\
&= d(g \circ' \phi) - d\eta \circ g - \eta \circ dg \\
&\simeq -d(\eta \circ g) \\
&\simeq 0
\end{aligned}$$

Since $dh(n) = 0$ we deduce that $h(n)g(n)(\Delta\phi + \phi^2) \simeq 0$ and it suffices to show $(h(n)g(n) - \mathbf{1}) \cdot (\Delta\phi + \phi^2) \simeq 0$. We know there exists K with $dK = h(n)g(n) - \mathbf{1}$ so the desired homotopy follows if we can show that $d(\Delta\phi + \phi^2) = 0$. One may check explicitly that $d(\Delta\phi) = -\Delta\phi \circ \phi + \phi \circ \Delta\phi$, using that $d\phi = -\Delta\phi - \phi^2$ in degree less than n . Then we can use Maurer–Cartan in lower degrees again to deduce:

$$\begin{aligned} d(\Delta\phi + \phi^2) &= d(\Delta\phi) - (-\Delta\phi - \phi^2) \circ \phi + \phi \circ (-\Delta\phi - \phi^2) \\ &= 0 \end{aligned}$$

Once we know the domain and codomain of h are in the image we can use surjectivity of hom-spaces to write $h = \pi(H)$. Now if the contraction of h lifts we find that H is also contractible and the preimages of $(E, \eta_{<n})$ and $(F, \phi_{<n})$ are homotopy equivalent.

So let us assume we are given a contraction $s_{<n}$ of $\text{cone}(h) = (G, \gamma_{<n})$, we have to find a contraction s of (G, γ) . By assumption we can write $d_\gamma(s_{<n}) = (\mathbf{1}, 0, \dots, 0, t_{[n]})$ for some $t_{[n]}$. Now consider $0 = d_\gamma d_\gamma(s_{<n}) = (0, \dots, 0, dt_{[n]} + 0)$. This forces $\delta t_{[n]} = dt_{[n]} = 0$. But now we know that $ds_{[0]} = \mathbf{1}$ and hence $d : s_{[0]}t_{[n]} \mapsto t_{[n]}$ and $(s_{[0]}, \dots, s_{[n-1]}, s_{[0]}t_{[n]})$ is a contraction of (G, γ) . \square

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