Aspects of higher dimensional Einstein theory and M-theory

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Dissertation submitted for the Degree of
Doctor of Philosophy
at the University of Cambridge

Department of Applied Mathematics and Theoretical Physics
& St John's College
University of Cambridge

April 2012

هنگام سپیده دم خروس سحری　　　دانی که چرا کند همی نوحه گری
یعنی که نمودند در آئینه صبح　　　کز عمر شبی گذشت و تو بی خبری

Attributed to Omar Khayyàm (1048–1131)

Acknowledgements

I am extremely grateful to my Ph.D. supervisor Harvey Reall for his guidance and supervision of my research, but most importantly for teaching me how research is done. His many comments, suggestions and advice have been invaluable throughout my Ph.D. research.

I would like to thank my collaborators David Berman, Alan Coley, Mark Durkee, Hadi Godazgar, Sigbjorn Hervik and Chris Pope from whom I have learnt a lot. In particular, I am very grateful to Malcolm Perry for many helpful discussions and advice, and especially his contagious enthusiasm when confronted with any problem.

During the past four years, I have benefited from the friendship of and enjoyed useful discussions with many people including Anastasios Avgoustidis, Anirban Basu, Tae Joon Cho, Michele Cicoli, Sera Cremonini, Mihalis Dafermos, Anne Davis, Matt Dolan, Maciej Dunajski, Mathieu Ehrhardt, Pau Figueras, Gary Gibbons, Steffen Gielen, Michael Green, Alex Kaus, Jonathan Kommemi, David Kubiznak, Hari Kunduri, Sven Krippendorf, Joao Laia, Jock McOrist, Ricardo Monteiro, Keiju Murata, Hugh Osborn, Miguel Paulos, Donough Regan, Raquel Ribeiro, Norman Rink, Jacques Smulevici, Julian Sonner, David Tong, Paul Townsend, Linda Uruchurtu-Gomez, Claude Warnick, Scott Watson, Martin Wolf and Anna Zytkow. Special thanks are due to Oscar Dias, Anshuman Maharana and Jorge Santos whose uniquely fun characteristics have helped make my time at Cambridge during the past few years unforgettable.

My Ph.D. was supported by a studentship from the Engineering and Physical Sciences Research Council (EPSRC) and, in its final stages, by St John's College, Cambridge. I am also grateful to the Department of Applied Mathematics and Theoretical Physics (DAMTP), Dalhousie University, Institute of Physics, Cambridge Philosophical Society and George Mitchell and Sheridan Lorenz for their generosity.

This thesis is dedicated to my parents Hossein Godazgar and Masoumeh Velayati, my brother Hadi Godazgar and my sisters Faezeh and Mahdieh Godazgar for their continuous and never-ending support throughout my education.

Declaration

This dissertation is the result of my own work and includes nothing that is the outcome of work done in collaboration except where specifically indicated in the text. The research described in this dissertation was carried out in the Department of Applied Mathematics and Theoretical Physics at the University of Cambridge between October 2008 and April 2012. Except where reference is made to the work of others, all the results are original and based on the following works of mine:

- **Algebraically special axisymmetric solutions of the higher-dimensional vacuum Einstein equation.**
  (with H. S. Reall)
  Class. Quantum Grav., **26** *(2009) 165009* [1]

- **Spinor classification of the Weyl tensor in five dimensions.**
  Class. Quantum Grav., **27** *(2010) 245013* [2]

- **The Local symmetries of M-theory and their formulation in generalised geometry.**
  (with D. S. Berman, H. Godazgar and M. J. Perry)
  JHEP, **1201** *(2012) 012* [3]

- **The perturbation theory of higher dimensional spacetimes *á la* Teukolsky.**
  Class. Quantum Grav., **29** *(2012) 055008* [4]

- **Peeling of the Weyl tensor and gravitational radiation in higher dimensions.**
  (with H. S. Reall)
  Phys. Rev. D, **85** *(2012) 084021* [5]

None of the original works contained in this dissertation has been submitted by me for any other degree, diploma or similar qualification.

Signed: .... (Mohammad Mahdi Godazgar)
Date: ....

Summary

This thesis contains two main themes. The first is Einstein's theory of general relativity in higher dimensions, while the second is M-theory.

The first part of the thesis concerns the use of classification techniques based on the Weyl curvature in an attempt to systematically study higher dimensional general relativity and its solutions. After a review of the various classification schemes, the application of these schemes to the study of higher dimensional solutions is explained.

The first application of the tensor approach that is discussed is the systematic classification of higher dimensional axisymmetric solutions. A complete classification of all algebraically special axisymmetric solutions to the vacuum Einstein equation in higher dimensions is presented.

Next, the study of perturbations of higher dimensional solutions within this framework and the possibility of decoupling equations for black hole solutions of interest, as has been successfully done in four dimensions, is considered. In the case where such a decoupling of the perturbations is possible, a map for constructing solutions of the perturbation equation is presented and is applied to the Kerr/CFT correspondence.

Also, the property of gravitational radiation emitted from an isolated source in higher dimensions is considered and the tensor classification scheme is used to derive the peeling property of the Weyl tensor in higher dimensions. This is shown to be different to that which occurs in four dimensions.

Finally, after an in-depth exposition of the spinor classification scheme and its relation to the tensor approach, solutions belonging to the most special type in the spinor classification are classified. In addition, the classification of the black ring in this scheme is discussed.

The second part of the thesis explores the use of generalised geometry as a tool for better understanding M-theory. After briefly reviewing the curious phenomenon of M-theory dualities, it is explained how generalised geometry can be used to show that these symmetries are not exclusive to compactifications of the theory, but can be made manifest without recourse to compactification. Finally, results regarding the local symmetries of M-theory in the generalised geometry framework for a particular symmetry group are presented.

Contents

Part I

Introduction

Chapter 1

Introductory remarks

The job of a theoretical physicist is to study theories that usually rely on a certain amount of mathematics and which have, or will hopefully have, something to say about physical phenomena. The idea is to use an understanding of the mathematical structure of a theory to study its physical consequences, its strengths, but most importantly its weaknesses. It's usually problems with a certain theory that indicate a need to think differently and this is what leads to advances and new concepts.

One of the classic examples of such a theory is Einstein's theory of general relativity. This is a theory of gravity that at the time of its conception, just under one hundred years ago, used the most state of the art of mathematics: differential geometry to explain discrepancies between the Newtonian theory of gravity and observations, and furthermore successfully predict new phenomena such as the bending of light due to strong gravitational fields, which was famously verified by Arthur Eddington in his 1919 Principe expedition. Coincidentally, the study of general relativity led to many advances in differential geometry that were needed to properly understand the theory and inspired many mathematicians such as Cartan, Poincaré, Weyl and many others to consider problems that they may not have thought about otherwise. Furthermore, it has presented us with many challenges. The singularities in the theory are symptomatic of a problem with the theory and are yet to be properly understood. This too is a power that general relativity has in revealing its own limitations and boundaries. General relativity is the archetype of a successful theory in theoretical physics.

Despite what popular culture may often have us believe, Einstein's theory of general relativity did not emerge solely from the mind of a lone genius without any prior context or setting. It developed as part of a long line of ideas and problems going back to the work of Maxwell on electromagnetism, which can itself be traced back indefinitely. Thus, new ideas in theoretical physics, which are coherent enough to be recognised and celebrated as theories are not revolutions in our understanding of nature but merely milestones in an ever continuing and endless process that started with the very first question regarding our surroundings. New theories arise from an in-depth study and understanding of older theories and ideas as eluded to earlier.

For any given theory, one needs a variety of different techniques to be able to investigate its consequences and properties. Some techniques are standard. However, sometimes in order to be able to view a theory from a different perspective, new techniques need to be formulated. This dissertation concerns the development of new approaches for two different theories whose study is mainly motivated by the emergence of string theory as a viable candidate of quantum gravity.

The first theory is higher dimensional general relativity, which, as the name suggests, is the study of Einstein's theory of general relativity in higher than the usual four spacetime dimensions. To date, most studies of higher dimensional general relativity and its solutions have used four dimensional theorems and solutions as guide. If we are to learn about higher dimensional general relativity without the prejudice carried over from our knowledge of four dimensional general relativity, then we need to undertake a more systematic study of higher dimensional general relativity. One possibility is to develop classification schemes based on properties of the curvature of solutions.

The second theory is M-theory. This is a not-so-well understood theory that aims to unify all known string theories into one framework. There is a lot of evidence to suggest that the low energy limit of M-theory has a host of symmetries and there has been recent progress in addressing this issue using an approach that goes by the name of generalised geometry. Any geometry, for example a solution of the Einstein equation, has an associated local symmetry. One can see this by zooming enough into the geometry and finding that it looks a lot like flat space. In generalised geometry one would like to describe a geometry that does not have the same symmetry as the geometries that we are used to. That is to say, if we were to zoom into a generalised geometry, we would not find flat space, but something entirely different.

Part II

Higher dimensional general relativity

Chapter 2

Introduction

2.1 Motivation

There has been much growing interest in the study of higher dimensional gravity in recent years. The motivation has been provided, in large part, by the emergence of string theory as a viable candidate of quantum gravity, and particularly the rise of brane-world scenarios [6], in which the extra dimensions, needed for string theory to be consistent, can be large and need not be compactified. The gauge/gravity correspondence (the most well-understood example being the AdS_5/CFT_4 correspondence [7]) is also an important development that aims to relate gauge theories with gravitational theories in higher dimensions.

The study of higher dimensional gravity is also interesting in itself [8]. Understanding how gravity changes as the dimension, which is essentially an extra parameter in Einstein's theory, changes can shed further light on general relativity and help us understand why four dimensional gravity is special. In some naive sense, this is obvious since four is the first dimension in which general relativity is non-trivial and contains propagating degrees of freedom. But a relevant question is how the introduction of further degrees of freedom changes the theory. Indeed there are many differences between 4d and $d > 4$ general relativity, some of which will be explained in more detail later.

2.2 General relativity in four dimensions

Einstein's theory of gravitation [9], general relativity, is a geometric theory that ascribes gravitational effects in spacetime to the existence of curvature on a manifold that is identified with spacetime. Ultimately, the reason why this should be the case is due to two key assumptions or principles that general relativity relies on.

The first is the principle of general covariance, which is simply the statement that the result of a physical observation or experiment should not depend on a particular observer. That is, the existence of different frames and coordinates is an artefact of how we choose to observe a particular event rather than an essential feature of nature. All observers are equal and any measurement made by one observer is covariantly equivalent to that made by another. We mean covariant equivalence in the sense that they are equivalent up to a map that translates between the two frames. The mathematical consequence of this is that the theory be covariant under diffeomorphisms. For any physical statement to make sense, it must be expressible in terms of quantities that are covariant, i.e. tensors.

The second principle is that of local equivalence with special relativity. Thus, any observer working in a freely falling laboratory that is small compared to the length scale over which the gravitational field outside is important will not be able to distinguish this situation from one in which there are no gravitational forces. Mathematically, this means that locally one must satisfy Lorentz invariance.

Consider a Lorentzian manifold endowed with a metric g_{ab}, which measures the interval between two points. The metric is the fundamental quantity that can be used to find various geometric properties of the manifold. Suppose we would like to formulate a theory of gravity using an action defined solely from the metric and its derivatives up to second order, i.e.

$$S = \int \mathcal{L}(g_{ab}, \partial_c g_{ab}, \partial^2_{cd} g_{ab}) + S_{matter}, \tag{2.1}$$

where $\mathcal{L}$ is the gravitational Lagrangian density and S_{matter} is the action of some other non-gravitational matter fields. The Lagrangian density is equal to some scalar density times a Lagrangian, which is a scalar. A scalar density with the appropriate weight that can be formed from the metric is its determinant. Hence,

$$\mathcal{L} = \sqrt{-g} L. \tag{2.2}$$

The only possible up to second order in derivatives scalar constructed from the metric and its derivatives is the full contraction of its Riemann curvature tensor (i.e. its Ricci scalar curvature) and a constant term. Thus,

$$L = R + \lambda \tag{2.3}$$

for some constant λ. This is the gravitational Lagrangian up to an overall normalisation that is fixed by requiring consistency with the Newtonian limit. Defining the energy-momentum tensor to be

$$T^{ab} \propto \frac{1}{\sqrt{-g}} \frac{\delta S_{matter}}{\delta g_{ab}} \tag{2.4}$$

and varying the action given in (2.1) with respect to the metric gives the Einstein equation

$$R_{ab} - \tfrac{1}{2} R g_{ab} + \lambda g_{ab} = 8\pi G T_{ab} \tag{2.5}$$

that any metric in Einstein's theory must satisfy.

Here we are only interested in vacuum solutions to the Einstein equation, i.e. solutions for which $T_{ab} = 0$. The equation can be rewritten as

$$R_{ab} = \Lambda g_{ab}, \tag{2.6}$$

where Λ is proportional to λ. Let us consider solutions to the equation above. Of course, flat Minkowski spacetime is a solution to this equation with $\Lambda = 0$. This is a maximally symmetric solution to the Einstein equation in the sense that it has the largest number of isometries (or Killing directions) possible. Other maximally symmetric solutions are the de Sitter ($\Lambda > 0$) and Anti de Sitter ($\Lambda < 0$) solutions with metrics

$$ds^2 = -\left(1 - \frac{r^2}{\alpha^2}\right) dt^2 + \left(1 - \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)}, \tag{2.7}$$

$$ds^2 = -\left(1 + \frac{r^2}{\alpha^2}\right) dt^2 + \left(1 + \frac{r^2}{\alpha^2}\right)^{-1} dr^2 + r^2 d\Omega^2_{(2)}, \tag{2.8}$$

respectively, where $d\Omega^2_{(2)}$ is the round metric on a unit 2-sphere and $\alpha^2 \propto 1/|\Lambda|$, explaining the similarity between the two metrics. Note, also, that the α large (Λ small) limit recovers flat Minkowski spacetime. Particularly interesting solutions of the Einstein equation are black hole solutions such as the Schwarzschild and Kerr solutions.

Black holes

The Schwarzschild solution [10, 11] described in Schwarzschild coordinates (t, r, θ, ϕ) by the metric

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega_{(2)}^2 \tag{2.9}$$

was initially thought of as describing the gravitational field outside a spherical, uncharged, non-rotating object with mass M. Indeed, it can be used as the starting point for the calculations that experimentally verify general relativity in solar system tests such as the perihelion precession of Mercury or the deflection of light by the Sun. However, our modern understanding of this solution is that it is also a black hole solution with an event horizon at $r = 2GM$.

The term 'black hole' was coined in 1967 by John Wheeler in his public lecture "Our universe: the known and unknown" [12]. Black holes are a consequence of Einstein's theory of general relativity, and the Schwarzschild solution is one example of such a solution.[1] However, many have doubted their relevance to physical reality and it took a while for black holes to be accepted as *bona fide* consequences of general relativity. Karl Schwarzschild had realised that his metric is singular at $r = 2GM$, but argued that an infinite pressure gradient is needed to compress a star to this radius, and so the singularity has no physical significance [11].

Einstein also believed that the "Schwarzschild singularities do not exist in physical reality" [14]. In his 1939 paper, he sets out to show that black holes cannot exist. Today, however, there is a general consensus that not only is it possible for black holes to exist in nature, but that they are unavoidable. We will briefly outline key events that lead to an understanding of black holes and singularities in general relativity.

The first breakthrough came in 1930, when Subrahmanyan Chandrasekhar calculated the mass limit of white dwarfs [15]. He showed that no white dwarf can exist of mass greater than this limit, the Chandrasekhar limit, equalling 1.4 solar masses. Chandrasekhar did not give an explanation as to what happens to stars with mass greater than 1.4 solar masses, but his work was already controversial enough to attract severe criticism and opposition from Arthur Eddington [16]. Eddington did not like what was being implied by Chandrasekhar's limit and believed that some mechanism had to exist in order to prevent the indefinite collapse of a larger star.

[1] In fact, the idea of considering an object so dense that its escape velocity is greater than the speed of light was considered much earlier by John Michell in 1784 (see [13]).

It was Robert Oppenheimer and Hartland Snyder, who in 1939, provided an explanation of what happens to a star with mass greater than the Chandrasekhar limit, and in doing so provided a mechanism by which black holes can be formed [17]. They showed that a spherical homogeneous non-rotating star made of pressure-free fluid of mass greater than three solar masses will collapse to form a black hole. They realised that what happens during the collapse of the star depends on the observer: observers at rest with respect to the star move through the Schwarzschild radius with no significant events occurring, while for observers far away, the star tends to "close itself off from any communication" and appears to freeze at the Schwarzschild radius. Crucially their calculations displayed a physical singularity at $r = 0$.

Further research into black holes had to wait for around two decades until objects were beginning to be found with radii comparable to their Schwarzschild radii. This removed a major objection from critics that the Schwarzschild radius is too small for the body to be able to collapse beyond it. The Schwarzschild radius of the sun, for example, is only 3 km. But observations confirmed that this is no barrier in the formation of black holes and that gravitational forces can compress stars to such densities.

The next major development came in 1958, when Finkelstein introduced the concept of the horizon at the Schwarzschild radius by using the Eddington-Finkelstein coordinates that he had rediscovered to show that the surface at $r = 2GM$ "is not a singularity but acts as a perfect unidirectional membrane" [18]. Eddington had discovered the Eddington-Finkelstein coordinates 34 years earlier, but seems to have missed the significance of his discovery [19].

Roger Penrose became interested in the subject of black holes and singularities after attending a talk by Finkelstein in 1958 explaining his discovery concerning the Schwarzschild horizon. What troubled Penrose was the $r = 0$ singularity and whether this can be avoided. Oppenheimer and Snyder had only showed that a star collapses to a singularity under strict assumptions, the main one being spherical symmetry. Could it be that irregularities and asymmetries in physical stars prevent collapse to a singularity, and that the collapsing star simply passes "through a complicated central configuration to be flung outwards again" with no singularity being formed [20]. Penrose showed that this cannot be the case. In his singularity theorem [21], he proved under very general assumptions, including an appropriate energy condition, that any gravitational collapse to within a region similar to a Schwarzschild horizon produces a singularity. The genericity of singularities in general relativity remains an unsolved problem. It is hoped that a fully understood quantum theory of gravity will be able to explain these problems in general relativity.

A generalisation of the Schwarzschild metric to include rotation is given by the Kerr metric [22]. In Boyer-Lindquist coordinates (t, r, θ, ϕ), the Kerr metric is

$$\begin{aligned} ds^2 = &-\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\, dt^2 - 2a\sin^2\theta \frac{r^2 + a^2 - \Delta}{\Sigma}\, dt\, d\phi \\ &+ \frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2 + \frac{\Sigma}{\Delta}\, dr^2 + \Sigma\, d\theta^2, \end{aligned} \tag{2.10}$$

where $\Delta = r^2 - 2GMr + a^2$ and $\Sigma = r^2 + a^2\cos^2\theta$. Parameter a has the physical interpretation of angular momentum per unit mass and the solution is a black hole only for $M \geq |a|$. The solution is called extremal if $M = |a|$. Many astrophysical black holes are thought to be near-extremal [23]. Notice that this metric reduces to the Schwarzschild metric in Schwarzschild coordinates (2.7) for $a = 0$.

A set of uniqueness theorems establish these two black holes as the unique final states of the generic gravitational collapse of isolated matter. The Schwarzschild solution [11] is the unique regular, asymptotically flat, static black hole solution of Einstein's equations [24], while the Kerr solution [22] is the unique stationary black hole solution [25].

Furthermore, both black hole solutions are expected to be classically stable. The question of the classical stability of solutions to the Einstein equation is an important one. If a solution is thought to describe some physical process such as the end-state of collapsing matter as established above, then one would expect this to be stable under perturbations. The non-linear analysis of the stability of most solutions remains a challenge. Although, the non-linear stability of the Minkowski solution as the simplest solution of the Einstein equation has been established [26]. However, one can consider the question of the linear stability of solutions as a first step. The linear (mode) stability of the Schwarzschild solution has been established and relies on the Regge-Wheeler [27] and Zerilli [28] equations. In the case of the Kerr solution, its linear (mode) stability has been established [29, 30], thanks in large part, to the decoupling result of Teukolsky [31, 32].

Petrov Classification

The decoupling result of Teukolsky utilised a formalism due to Newman and Penrose [33] that is itself influenced by the Petrov classification [34, 35] of the Weyl tensor. The reason why we consider the Weyl tensor of the solution is because the Weyl tensor is the part of the curvature that is not constrained by the Einstein equation. The Riemann tensor can be decomposed into two parts

$$Riem = C + S \circ_{KN} g, \tag{2.11}$$

where C is the Weyl tensor and represents the tracefree part of the Riemann curvature, S is the Schouten tensor

$$S = \frac{1}{2}\left(Ric - R/6g\right),$$

and $\circ_{KN}$ represents the Kulkarni-Nomizu product that ensures that the symmetries of the Riemann tensor are respected. The Schouten tensor is fixed by the Einstein equation. Thus, it is the Weyl tensor that encodes the propagating degrees of freedom in a solution.

The Petrov classification is a method for classifying the Weyl tensor of a solution by investigating its algebraic structure. It has been an instrumental tool in studying solutions of Einstein's equations. It not only enables a systematic study of solutions and their properties, but has also been used to find some of the most well-known and fundamental solutions, including the Kerr solution. Appendix A contains a brief explanation of the tensor derivation of the Petrov classification. The spinor approach to the Petrov classification is reviewed in section 6.2.3.

One important result in the context of the Petrov classification is the Goldberg-Sachs theorem [36]. It states that a non-conformally flat vacuum spacetime is algebraically special if and only if it contains a shear-free geodesic null congruence. More specifically, a null vector field ℓ can be chosen such that $\Psi_0 = \Psi_1 = 0$ (but not all Weyl scalars vanishing) if and only if ℓ defines a geodesic, shear-free null congruence, which is equivalent to $\kappa = \sigma = 0$ (see table A.1).

It is very simple to see why this should be the case using the Geroch-Held-Penrose formalism.[2] Assume first that there exists a ℓ such that $\Psi_0 = \Psi_1 = 0$ and $\Psi_2 \neq 0$. Then Bianchi equations (A.17) and (A.20)$'$ reduce to

$$-3\kappa\Psi_2 = 3\sigma\Psi_2 = 0. \tag{2.12}$$

Since $\Psi_2 \neq 0$, by assumption, $\kappa = \sigma = 0$. The cases where $\Psi_2 = 0$, $\Psi_1' \neq 0$ and $\Psi_2 = \Psi_1' = 0$ must be considered separately. However, the argument is essentially the same, except using different Bianchi equations. Now assume that there exists a ℓ such that $\kappa = \sigma = 0$. Newman-Penrose equation (A.13) reduces to $\Psi_0 = 0$, while equations

[2] The GHP formalism [37], which is similar to the Newman-Penrose formalism, is reviewed in appendix A.

(A.11) and (A.14) reduce to

$$ð\rho = (\rho - \bar{\rho})\tau - \Psi_1, \tag{2.13}$$

$$þ\tau = (\tau - \bar{\tau}')\rho + \Psi_1, \tag{2.14}$$

respectively. Furthermore, equations (A.17) and (A.20)$'$ reduce to

$$þ\Psi_1 = 4\rho\Psi_1, \tag{2.15}$$

$$ð\Psi_1 = 4\tau\Psi_1, \tag{2.16}$$

respectively. Consider the combination ð (2.15) $-$ þ (2.16)

$$(ðþ - þð)\Psi_1 = 4(\rho\bar{\tau}' - \tau\bar{\rho})\Psi_1 - 8\Psi_1^2, \tag{2.17}$$

where we have used equations (2.13)–(2.16) to simplify the expression on the right hand side. Now using commutator equation (A.22) to simplify the left hand side gives $\Psi_1^2 = 0$, which establishes the Goldberg-Sachs theorem. Perhaps the most important application of the Goldberg-Sachs theorem has been Kinnersley's work on finding all type D vacuum ($\Lambda = 0$) solutions [38], which lead to the discovery of the rotating C-metric. This is a solution that describes two rotating black holes held apart by semi-infinite line masses or strings.

The type D condition is known to be equivalent to the existence of hidden symmetries for vacuum ($\Lambda = 0$) solutions [39, 40]. More precisely, the statement is that a vacuum ($\Lambda = 0$) solution is type D if and only if there exists a non-degenerate conformal Killing-Yano 2-form k on the background, satisfying

$$\nabla_{(a}k_{b)c} = -1/3(g_{ab}\nabla^d k_{cd} - g_{c(a}\nabla^d k_{b)d}). \tag{2.18}$$

It is not too hard to see why type D solutions should admit such a structure, which is half of the statement above. In [40], this is shown with the use of spinor calculus methods. However, the proof is in fact easier using the GHP formalism. On a type D background the only non-vanishing Weyl scalar is Ψ_2. Therefore, take as our starting ansatz

$$k = f(\bar{\Psi}_2\Psi_2)\ell^\flat \wedge n^\flat + ig(\bar{\Psi}_2\Psi_2)m^\flat \wedge \bar{m}^\flat, \tag{2.19}$$

where $\ell^\flat$, $n^\flat$, $m^\flat$, $\bar{m}^\flat$ are the 1-form duals of the basis vectors $(\ell, n, m, \bar{m})$. Equation (2.18) reduces to the following set of equations in this basis

$$þ(f+g)+\rho(f+g)=0, \quad þ'(f+g)+\rho'(f+g)=0, \quad ð(f+g)+\tau(f+g)=0, \tag{2.20}$$
$$þ(f-g)+\bar{\rho}(f-g)=0, \quad þ'(f-g)+\bar{\rho}'(f-g)=0, \quad ð(f-g)+\bar{\tau}'(f-g)=0. \tag{2.21}$$

Using the Goldberg-Sachs theorem, Bianchi equations (A.18) and (A.19) reduce to

$$þ\Psi_2 = 3\rho\Psi_2, \quad ð\Psi_2 = 3\tau\Psi_2, \tag{2.22}$$

respectively. Thus, one particular solution to the above equations is

$$f+g=\Psi_2^{-1/3}, \quad f-g=\bar{\Psi}_2^{-1/3}. \tag{2.23}$$

Notice that equations (2.20) and (2.21) are invariant under the interchange of f and g. This means that there is another solution to equation (2.18) with f and g swapped in the ansatz. This new solution is the dual of the original. The existence of hidden symmetries is very important. For example, the separability of the Hamilton-Jacobi equation for the geodesic problem or the Teukolsky decoupled equation on the Kerr background occur precisely because of this result [41].

Another important result is the peeling of the Weyl tensor [42, 43, 44, 45]. Radiation from an isolated source displays a special property in that however complicated the radiative source is, as one moves away from the source towards asymptotic infinity, various components of the Weyl tensor decay away leaving only the most special types. This means that the radiation observed far away from the source is much simpler than the complicated dynamics at play within the source. The peeling result is covered in greater detail in chapter 5.

2.3 General relativity in higher dimensions

Higher dimensional general relativity is the study of possible solutions to the (vacuum) Einstein equation (2.6) and their properties for spacetime dimensions greater than four. There are certainly some similarities between the 4d and $d > 4$ cases. For example, the Schwarzschild and Kerr solutions can be generalised to higher dimensions, albeit with some new features.

The higher dimensional generalisation of the Schwarzschild solution, known as the

Schwarzschild-Tangherlini solution, is given by the metric

$$ds^2 = -\left[1 - \left(\frac{a}{r}\right)^{d-3}\right] dt^2 + \left[1 - \left(\frac{a}{r}\right)^{d-3}\right]^{-1} dr^2 + r^2 d\Omega^2_{(d-2)}. \tag{2.24}$$

Like the 4d solution, the higher dimensional generalisation is known to be the unique asymptotically flat static black hole solution [46]. Furthermore, the linear (mode) stability of the Schwarzschild-Tangherlini solution has been demonstrated [47].

However, there are also some interesting differences. Consider circular orbits (in the equatorial plane) on this background. The effective potential for a free particle with mass m and angular momentum h on this background is

$$V_{eff}(r) = \frac{1}{2}\left(m + \frac{h^2}{mr^2}\right)\left(1 - \left(\frac{a}{r}\right)^{d-3}\right). \tag{2.25}$$

Analysing the potential, it is very simple to show that a local minimum is only possible for the 4d case. An example of a local minimum for the 4d case is given on the left hand side of figure 2.1, while the right hand side curve demonstrates a typical potential in higher dimensions.

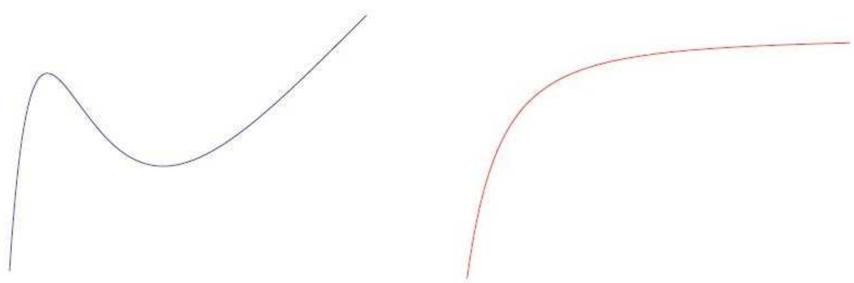

Fig. 2.1: An example of typical potentials in four and higher dimensions on the Schwarzschild background.

The generalisation of the Kerr solution in higher dimensions is the Myers-Perry solution [48], except that in higher dimensions, there are a greater number of planes of rotation. In fact the number of planes of rotation in $d-$dimensions is equal to the integer part of $(d-1)/2$. Another new feature in higher dimensions is the existence of ultraspinning regimes for $d \geq 6$. Recall that for the Kerr solution, the angular momentum parameter a is constrained to be less than the mass parameter for the solution to remain a black hole solution. For $d \geq 6$ Myers-Perry solutions the angular momenta can be made arbitrarily large.

The Myers-Perry solution is given by the metric

$$ds^2 = -dt^2 + \frac{U}{V-2M}dr^2 + \frac{2M}{U}\left(dt + \sum_{i=1}^{n} a_i{\mu_i}^2 d\phi_i\right)^2$$
$$+ \sum_{i=1}^{n}(r^2 + {a_i}^2)({\mu_i}^2 {d\phi_i}^2 + {d\mu_i}^2) + \varepsilon r^2 d\mu_{n+\varepsilon}^2, \tag{2.26}$$

where

$$V = r^{\varepsilon-2}\prod_{i=1}^{n}(r^2 + {a_i}^2), \qquad U = V\left(1 - \sum_{i=1}^{n}\frac{{a_i}^2{\mu_i}^2}{r^2+{a_i}^2}\right), \tag{2.27}$$

with the constraint

$$\sum_{i=1}^{n}{\mu_i}^2 + \varepsilon\mu_{n+\varepsilon}^2 = 1. \tag{2.28}$$

n is equal to the integer part of $(d-1)/2$ and ε is equal to 1 for even spacetime dimensions d and 0 for odd.

A similar result to that discussed above, regarding the existence of stable circular orbits holds true for rotating black holes too. There exist stable circular orbits in the Kerr spacetime, but it can be shown that none exist on the 5d Myers-Perry background [49].

The status of the uniqueness and stability of Myers-Perry solutions is much more interesting. The existence of a rotating black ring solution with horizon topology $S^1 \times S^2$ in five dimensions [50, 51] (see section 6.3.3) provides a dramatic counterexample to the uniqueness of the Myers-Perry solution in higher dimensions. It is expected that there are many other solutions in higher dimensions with varying properties [52, 53, 54].

Although, the linear (mode) stability of the Schwarzschild-Tangherlini solution has been demonstrated, the situation for other black objects is very different. For example, it is known that black strings and membranes are Gregory-Laflamme unstable [55]. And, it has been known for some time, by analogy with Gregory-Laflamme (GL) type instabilities, that one would expect instabilities to occur in certain regimes of the Myers-Perry and black ring families. For example, one would expect thin black rings to suffer from the same type of GL instability that occurs for a black string [50]. Similarly, one would expect ultraspinning ($d \geq 6$) Myers-Perry black holes to suffer from the same kind of GL type instability that one finds for p-branes [56]. There has been a great deal of recent progress in using numerical methods to tackle and confirm these conjectures, at least, as far as the Myers-Perry solution is concerned [57]. Such numerical investigations have hinted at a relation between certain unstable modes and the branching of new solutions providing a relation between the stability and the uniqueness of black hole solutions, which we discussed above.

Most higher dimensional solutions found to date, have been direct generalisations of four dimensional solutions. One way to investigate higher dimensional gravity independent of the 4d case is to attempt a systematic study of $d > 4$ general relativity. As discussed before, in four dimensions, the Petrov classification is a very important tool in studying solutions of Einstein's equations.

Given the success of a systematic approach to the study of 4d GR, it is natural to extend such ideas to higher dimensions, in which such considerations have been hitherto lacking. Higher dimensional gravitation will be studied with this focus in mind in this part of the dissertation.

2.4 Classification schemes in higher dimensions

The emphasis here will be on the study of solutions of the higher dimensional vacuum Einstein equation using techniques employed and developed from classification schemes based on the curvature properties of solutions. In four dimensions, there are several different, but equivalent, ways in which one can formulate the Petrov classification of the Weyl tensor. However, when extended to higher dimensions, these different methods become inequivalent.

In this section, we briefly review the main methods used to classify the Weyl tensor in higher dimensions. We begin by describing the tensorial approach due to Coley, Milson, Pravda and Pravdová [58, 59] (henceforth abbreviated to CMPP) in section 2.4.1. The CMPP classification relies on the existence of special null directions adapted to the Weyl tensor. These are known as *Weyl aligned null directions* or WANDs. Any d-dimensional solution can be classified in this scheme. The CMPP classification scheme has received the most attention and has been successfully applied to studying many important properties of higher dimensional gravity (see [60] and references therein), including the possibility of finding new solutions. These include a study of higher dimensional axisymmetric solutions [1] (see chapter 3); the study of solutions admitting a hyper-surface orthogonal WAND [61] and the study of Kundt [62] and Robinson-Trautman [63] solutions. Unfortunately, these investigations have not lead to the discovery of interesting solutions such as the higher dimensional C-metric. However, the possibility of finding new solutions using these methods has not been exhausted. The approach taken in Ref. [1], where a class of solutions with a certain symmetry are considered with the extra assumption of algebraic specialness may yield interesting results for other symmetry groups.

Other applications of the CMPP classification include a partial generalisation of the Goldberg-Sachs theorem to higher dimensions [64, 65]; the asymptotic properties of higher dimensional spacetimes [66]; the perturbation theory of higher dimensional solutions [67, 68, 4] (see chapter 4); a study of the instabilities of near-extremal cohomogeneity-1 Myers-Perry solutions [69] and a study of the peeling property of the Weyl tensor in higher dimensions [5] (see chapter 5).

In section 2.4.2, we review the spinorial De Smet classification [70], which is definable only in five dimensions. The relation between the CMPP and De Smet classification is not straightforward and they certainly do not share any equivalent notions such as the existence of a preferred null direction (see section 6.3.2) [2]. The possibility of finding new solutions using this classification scheme has been investigated [70, 71, 2]. In Refs. [70, 71], axisymmetric solutions within the context of this classification scheme were studied (see section 3.1), while in Ref. [2], solutions belonging to the most special type in this scheme were classified (see chapter 6).

Finally, in section 2.4.3, we briefly review the extension of the bivector approach of classifying the Weyl tensor to higher dimensions [72]. The classification of black hole solutions using this approach has been investigated in Ref. [73].

There are other approaches to the classification of higher dimensional solutions that do not directly involve properties of the Weyl tensor, such as the existence of hidden symmetries [74, 75] or the existence of a certain optical structure [76, 77].

2.4.1 CMPP classification

The CMPP classification relies on the existence of independent null vectors ℓ and n, such that in the null frame $m_0 \equiv \ell$, $m_1 \equiv n$, m_i $(i = 2, \dots, d-1)$, where m_i are a set of orthonormal spacelike vectors orthogonal to ℓ and n, certain components of the Weyl tensor vanish, implying the solution to be of a certain type.

Given a null frame (ℓ, n, m_i), continuous Lorentz transformations are generated by: null rotations about ℓ and n, spins (rotating the spacelike basis vectors), and boosts, given by

$$\ell' = \lambda \ell, \qquad n' = \lambda^{-1} n, \qquad m_i' = m_i, \tag{2.29}$$

where $\lambda \neq 0$. Under a boost, a particular component of a p-rank tensor T in the null frame transforms as

$$T_{a_1 \ldots a_p} \longrightarrow \lambda^b T_{a_1 \ldots a_p}, \tag{2.30}$$

where b is the *boost weight* of $T_{a_1 \ldots a_p}$, and is equal to the number of a_i that are 0 minus the number that are 1.

For the Weyl tensor, the possible boost weights lie in the range $-2 \leq b \leq 2$. For example, boost weight 2 components of the Weyl tensor are C_{0i0j}.

The solution is said to be of type G at a point p if there does not exist a ℓ such that $C_{0i0j} = 0$ at p, i.e. $C_{0i0j} \neq 0$ at p for any choice of ℓ. On the other hand, the solution is of given type

- I $\iff$ $C_{0i0j} = 0$,
- II $\iff$ $C_{0i0j} = C_{0ijk} = 0$,
- D $\iff$ $C_{0i0j} = C_{0ijk} = C_{1ijk} = C_{1i1j} = 0$,
- III $\iff$ $C_{0i0j} = C_{0ijk} = C_{01ij} = C_{ijkl} = 0$,
- N $\iff$ $C_{0i0j} = C_{0ijk} = C_{01ij} = C_{ijkl} = C_{1ijk} = 0$,
- O $\iff$ $C_{abcd} = 0$.

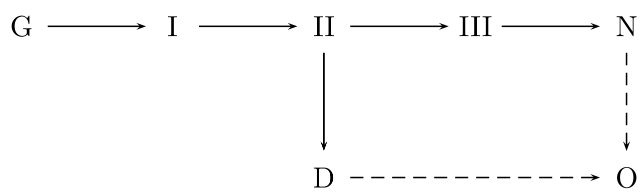

Fig. 2.2: Penrose diagram of the CMPP classification

The algebraic type of the solution is defined to be the type of its most algebraically general point. If the solution is type I, then ℓ for which $C_{0i0j} = 0$ is said to be a *Weyl aligned null direction* or *WAND*. A given WAND need not be unique. Indeed there may be infinitely many WANDs in a spacetime. For example, consider $dS_{d-2} \times S^2$. Any null vector in dS_{d-2} is a WAND.

We define a solution to be *algebraically special* if it is type II or more special. The reason for this is that there are many examples where the solution is of type G in one open subset and type I in another [1], including Gross-Perry solitons and the black ring.

Note that the definition of type D solutions depends on n being *Weyl aligned* as well, i.e. $C_{1i1j} = 0$. Thus, the type D definition requires a secondary classification, in which n is chosen such that as many trailing Weyl tensor components as is possible can be set to zero, that is, of course, once a WAND ℓ has been found such that as many leading Weyl tensor components as is possible have been set to zero. For example one defines type I_i solutions to be those for which a ℓ and n can be found such that $C_{0i0j} = C_{1i1j} = 0$. We shall not utilise the secondary classification scheme here, except in the definition of type D solutions.

2.4.2 De Smet classification

The De Smet classification is a spinorial classification of the Weyl tensor of 5d solutions. We start with a representation of the Clifford algebra such that

$$\Gamma^{ab}{}_{AB} = (C\Gamma^{[a}\Gamma^{b]})_{AB} \tag{2.31}$$

is symmetric and where the charge conjugation matrix C is antisymmetric. Indices $A, B \ldots$ refer to Dirac spinor indices.

We define the *Weyl spinor*, associated with the Weyl tensor, to be

$$C_{ABCD} = C_{abcd}\Gamma^{ab}{}_{AB}\Gamma^{cd}{}_{CD}. \tag{2.32}$$

It can be shown that this spinor quantity is totally symmetric (see section 6.2.2). Hence, we can treat it as a homogeneous quartic polynomial in four complex variables[3] and ask how this polynomial factorises into lower degree polynomials. We then classify the Weyl tensor according to the degree and multiplicity of the factors. Of course, it could be that the polynomial cannot be factorised, in which case we denote the Weyl type to be 4 (since we have a single quartic or fourth order polynomial) and call the corresponding solution *algebraically general*. If the polynomial does factorise, then the corresponding solution is *algebraically special*.

The Weyl tensor has 35 real independent components in five dimensions whereas the Weyl spinor C_{ABCD} has 35 complex components. Therefore, C_{ABCD} must satisfy some reality condition, which constrains the possible ways in which it can factorise [2].

In all there are eight different possibilities for how the polynomial can factorise and

[3] Since C_{ABCD} is totally symmetric, we can form such a polynomial by fully contracting it with a set of Dirac spinors ψ^A, i.e. $C(\psi) = C_{ABCD}\psi^A\psi^B\psi^C\psi^D$.

so there are eight different types. These are depicted in figure 2.3.

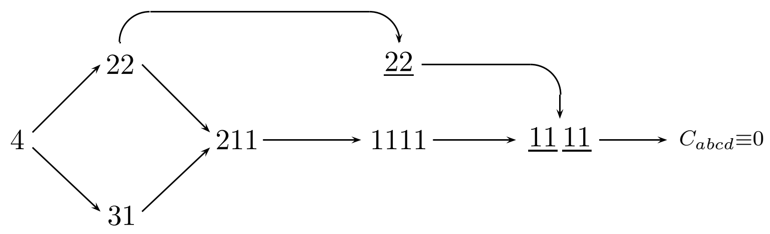

Fig. 2.3: Figure showing the 8 different algebraic types in the spinor classification.

The notation is such made that a number represents the degree of the polynomial factor and an underline represents its multiplicity. For example, type 22 corresponds to the case where the Weyl polynomial factorises into two quadratic factors that are proportional to one another and cannot be further factorised.

2.4.3 Bivector approach

The bivector scheme relies on the fact that the symmetries of the Weyl tensor are such that it can be treated as an operator from the space of 2-forms (bivectors) to itself

$$X_{ab} \mapsto X'_{ab} = \tfrac{1}{2} C_{ab}{}^{cd} X_{cd}. \tag{2.33}$$

Denoting a basis for bivectors with capital letters this implies that

$$X'_A = C_A{}^B X_B, \quad X' = C(X). \tag{2.34}$$

The classification scheme involves a study of the eigenvalue problem of operator C. The eigenvector structure of this operator was studied in [72] assuming certain CMPP types.

2.5 Scope of part II

In chapter 3, algebraically special, axisymmetric solutions of the higher dimensional vacuum Einstein equation (with cosmological constant) are investigated using the tensor approach due to Coley, Milson, Pravda and Pravdová. Necessary and sufficient conditions for static axisymmetric solutions to belong to different algebraic classes are presented. Then general (possibly time-dependent) axisymmetric solutions are discussed. All axisymmetric solutions of algebraic types II, D, III and N are obtained.

In chapter 4, we consider the possibility of deriving a decoupled equation for gravitational perturbations of the Schwarzschild-Tangherlini solution using the higher dimensional GHP formalism [67]. We find that a particular gauge invariant component of the Weyl tensor does decouple and expect this to correspond to the vector modes of Ishibashi and Kodama [47]. Also, we construct a Hertz potential map for solutions of the electromagnetic and gravitational perturbation equations of a higher dimensional Kundt background using the decoupled equation of Durkee and Reall [68]. Motivated by recent work of Guica and Strominger [78], we use this to construct the asymptotic behaviour of metric perturbations of the near horizon geometry of the 5d cohomogeneity-1 Myers-Perry black hole.

The peeling behaviour of the Weyl tensor near null infinity is determined for an asymptotically flat higher dimensional spacetime in chapter 5. The result is qualitatively different from the peeling property in 4d. To leading order, the Weyl tensor is type N. The first subleading term is type II. The next term is algebraically general in 6 or more dimensions but in 5 dimensions another type N term appears before the algebraically general term. The Bondi energy flux is written in terms of "Newman-Penrose" Weyl components.

We investigate the spinor classification of the Weyl tensor in five dimensions due to De Smet in chapter 6. We show that a previously overlooked reality condition reduces the number of possible types in the classification. We classify all vacuum solutions belonging to the most special algebraic type. The connection between this spinor and the tensor classification is investigated and the relation between most of the types in each of the classifications is given. We show that the black ring is algebraically general in the spinor classification.

Chapter 3

Algebraically special axisymmetric solutions

3.1 Introduction

3.1.1 Background

A d-dimensional spacetime is "axisymmetric" if it possesses an $SO(d-2)$ isometry group whose orbits are $(d-3)$-spheres. There are several motivations for studying axisymmetric solutions of the higher-dimensional vacuum Einstein equation (with cosmological constant)

$$R_{\mu\nu} = \Lambda g_{\mu\nu}. \tag{3.1}$$

These include the problem of finding an exact solution describing a black hole bound to a 3+1 dimensional brane in the (single brane) Randall-Sundrum model [6], and determining the phase structure of General Relativity with a compactified dimension [79].

In $d = 4$ dimensions, all static axisymmetric solutions of the vacuum Einstein equation (with $\Lambda = 0$) were obtained by Weyl, who showed that they are characterized by a single axisymmetric harmonic function in R^3 (see e.g. [80]). Weyl's result has been generalized to higher dimensions: the class of solutions of the d-dimensional vacuum Einstein equation (with $\Lambda = 0$) admitting $d-2$ commuting, orthogonal, non-null Killing fields is specified by $d-3$ axisymmetric harmonic functions in R^3 [81]. If one of the Killing fields generates time translations and the others generate rotations then these solutions have isometry group $R \times U(1)^{d-3}$, generalizing the $R \times U(1)$ symmetry of Weyl's solutions.

However, these solutions are not axisymmetric for $d > 4$.

It is desirable to know the general static axisymmetric solution in $d > 4$ dimensions but, unfortunately, the Einstein equation cannot be solved analytically for $d > 4$ (or even for $d = 4$ with $\Lambda \neq 0$). The impediment arises from the curvature of S^{d-3} [82]. Note that S^{d-3} is flat if $d = 4$, which is why the Einstein equation can be solved for $d = 4$.

The goal of this chapter is to determine whether the Einstein equation can be solved analytically if one makes the additional assumption that the spacetime admits a WAND. In $d = 4$, an algebraically special static, axisymmetric spacetime must be type D (or O). For $\Lambda = 0$, the only such solutions are, in the terminology of Ehlers and Kundt [83], the A-metrics, the B-metrics, and the C-metric [35]. The A-metrics are labelled by the parameters $k \in \{-1, 0, 1\}$ and $M \neq 0$. The metric takes the generalized Schwarzschild form

$$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega_k^2, \tag{3.2}$$

where $U(r) = k - 2M/r$, and $d\Omega_k^2$ is the metric on a space of constant curvature with sign k. The B-metrics are analytic continuations of the A-metrics in which the time coordinate t is Wick rotated to a spatial coordinate ϕ and $d\Omega_k^2$ to a Lorentzian metric of constant curvature $d\Sigma_k^2$:

$$ds^2 = U(r)d\phi^2 + \frac{dr^2}{U(r)} + r^2 d\Sigma_k^2. \tag{3.3}$$

The C-metric describes a pair of black holes being accelerated apart by a conical singularity [84].

We shall follow the algebraic classification of Coley, Milson, Pravda and Pravdova (CMPP) [58], which applies for general d.

3.1.2 Summary of results

In this chapter, we start (in section 3.2) by considering static, axisymmetric solutions and determine the condition for them to admit a WAND. It was shown in Ref. [85] that a static solution must be of algebraic type G, I, D or O. We derive simple necessary and sufficient conditions for a solution to belong to the various algebraic types. We also show that many analytic solutions are type G in one open subset of the spacetime and type I in another. This suggests that distinguishing between type G and type I solutions is not very useful in practice, and that the type I condition alone will not be much help in finding new solutions.

In the rest of the chapter, we relax the condition of staticity and consider general (possibly time-dependent) axisymmetric solutions admitting a WAND. The starting point of our analysis is the observation that, for $d > 4$, the action of $SO(d-2)$ on S^{d-3} must be *orthogonally transitive* [35], i.e., spacetime is locally a product $M_3 \times S^{d-3}$ with warped product metric

$$ds^2 = g_{ab}(x)dx^a dx^b + E(x)^2 d\Omega^2, \tag{3.4}$$

for some 3-metric g_{ab} and function $E(x)$ on M_3, where $d\Omega^2$ is the metric on S^{d-3} normalized to unit radius. The analysis naturally divides into two cases depending on whether or not the WAND is axisymmetric, i.e., invariant under $SO(d-2)$.

In section 3.3, we consider the case in which the WAND is axisymmetric. The axisymmetry implies that the null geodesic[1] congruence tangent to the WAND has vanishing rotation, hence it is hypersurface-orthogonal. We determine all solutions with an axisymmetric geodesic WAND without assuming a particular algebraic type. The solutions are all type II or more special.[2] There are several classes.

- Type O (conformally flat) solutions. Irrespective of axisymmetry, the only such solutions are Minkowski, de Sitter, and anti-de Sitter spacetimes.

- The Schwarzschild solution, generalized to allow for flat or hyperbolic slices (i.e., higher-dimensional analogues of the A-metrics) and a cosmological constant. The metric is given by equation (3.55). The solution is type D. The null congruence tangent to the WAND has vanishing shear and non-vanishing expansion, so these solutions are a subset of the higher-dimensional Robinson-Trautman family of solutions (defined to be solutions admitting a null geodesic congruence with vanishing shear and rotation and non-vanishing expansion) obtained in Ref. [86].

- "Black string" solutions obtained, for $\Lambda = 0$, by foliating Minkowski spacetime with $(d-1)$-dimensional Minkowski, or de Sitter, slices and replacing the slices with a Schwarzschild, or Schwarzschild-de Sitter, metric respectively. In the former case, this gives the familiar Schwarzschild black string solution. There is an analogous construction for $\Lambda > 0$ based on a de Sitter foliation of de Sitter spacetime, and for $\Lambda < 0$ based on Minkowski, de Sitter, or anti-de Sitter foliations of anti-de Sitter spacetime. The latter includes the anti-de Sitter black string of Ref. [87]. The

[1] Any algebraically special solution admits a geodesic multiple WAND [65].

[2] For $\Lambda = 0$, this implies that these solutions belong to the class of spacetimes discussed in Ref. [61], i.e., those admitting a hypersurface orthogonal multiple WAND. The dependence on the affine parameter along the geodesics was determined in that paper.

metric of these solutions is given by equation (3.49). They are all type D. The null congruence associated with the WAND has non-vanishing expansion and shear.

- For $\Lambda > 0$, $dS_3 \times S^{d-3}$ is type D. A *general* multiple WAND of this spacetime is non-geodesic. However, any null geodesic congruence in dS_3 defines a geodesic multiple WAND. Such a congruence may be expanding and shearing or non-expanding and non-shearing (in the latter case the solution is a special case of the Kundt solutions discussed next).

- Axisymmetric Kundt solutions (section 3.3.1). A Kundt spacetime is a spacetime admitting a null geodesic congruence with vanishing expansion, rotation and shear [35]. Such solutions are type II, or more special, for any $d \geq 4$ [88]. In general they involve arbitrary functions of time. In our axisymmetric case, these solutions are expressed in terms of solutions of certain ODEs that cannot be solved analytically in general. We show that some of these solutions are type D or N (but not III). The type N solutions can be obtained analytically. They describe gravitational waves in Minkowski (eq. (3.90)), de Sitter (eq. (3.96)) or anti-de Sitter (eq. (3.95)) spacetime. The general type D solution is cohomogeneity-1 with surfaces of homogeneity $M_2 \times S^{d-3}$ where M_2 is 2d Minkowski or (anti-) de Sitter spacetime:

$$ds^2 = dz^2 + A(z)^2 d\Sigma^2 + R(z)^2 d\Omega^2, \tag{3.5}$$

where $d\Sigma^2$ is the metric on M_2. The functions $A(z)$ and $R(z)$ can be determined analytically only in special cases e.g. the product spaces $dS_3 \times S^{d-3}$, $dS_2 \times S^{d-2}$ and $AdS_2 \times H^{d-2}$, or for flat M_2 with $\Lambda = 0$. Various solutions of this form have been discussed previously in the literature. For positively curved M_2, if one analytically continues to Riemannian signature (so that M_2 becomes S^2) then these are metrics of the form discussed by Böhm. He proved that, for low enough $d > 4$, and $\Lambda > 0$, there exist infinitely many Einstein metrics on spheres and products of spheres of the form (3.5) [89]. For $\Lambda \leq 0$ he constructed complete, non-compact metrics of the form (3.5) [90]. The Lorentzian interpretation of some of the Böhm solutions has been discussed in Ref. [91]. Some singular solutions with flat M_2 and $\Lambda = 0$ were discussed in Ref. [92], analogous solutions with non-flat M_2 were discussed in Ref. [93]. A regular solution with $d = 5$, flat M_2 and $\Lambda < 0$ describes the metric dual to the ground state of $\mathcal{N} = 4$ super Yang-Mills theory on $R \times S^1 \times S^2$ (with fermions periodic on S^1) [94]. A solution with $d = 5$, negatively curved M_2 and $\Lambda < 0$ describes the bulk near-horizon geometry of an extremal charged Randall-Sundrum black hole, or the metric dual to the ground state of $\mathcal{N} = 4$ SYM in $AdS_2 \times S^2$

[95]. More generally, solutions of the form (3.5) with $\Lambda < 0$ that are asymptotically locally anti-de Sitter presumably describe the metrics dual to the ground state of a CFT in $M_2 \times S^{d-3}$.

For $d = 4$, it was proposed in Ref. [96] that some type II Kundt solutions describe gravitational waves propagating in a "background" spacetime described by a type D Kundt solution. The same is true for $d > 4$: given a type D background of the form (3.5), one can construct explicitly axisymmetric type II Kundt solutions which describe gravitational waves propagating along the space M_2 of this background. (Some particular examples of such solutions were obtained in Refs. [97, 98].) For $\Lambda < 0$, some of these solutions will be asymptotically locally AdS, and will describe metrics dual to certain CFT states in $M_2 \times S^{d-3}$ for which there is a null energy-momentum flux along M_2.

The second case to consider is when the WAND is not axisymmetric. Acting with $SO(d-2)$ generates a continuously infinite family of WANDs, which suggests that the solution should have an enhanced symmetry. This is indeed the case: assuming that the solution admits a multiple WAND, we are able to show that $SO(d-2)$ is enhanced to the de Sitter symmetry $SO(1, d-2)$, with dS_{d-2} orbits, and the only non-trivial (i.e. not type O) solutions are:

- For any Λ, a Kaluza-Klein bubble solution [99] (i.e. a higher-dimensional analogue of the B-metrics) obtained by analytic continuation of the Schwarzschild solution (so that $S^{d-2} \to dS_{d-2}$). The metric is given by equation (3.113). This is type D.
- For positive Λ there is a $dS_{d-2} \times S^2$ solution. This is also type D.

For both solutions, any null vector field tangential to the dS_{d-2} orbits of $SO(1, d-2)$ is a multiple WAND. These solutions are examples of type D vacuum solutions admitting non-geodesic multiple WANDs. In fact, it has been proved in [65] that the only five dimensional solutions that admit non-geodesic multiple WANDs are the examples presented above, and their $\Lambda \leq 0$ generalisations.

Combining these results, we conclude that any axisymmetric solution admitting a WAND that is *not* encompassed by our analysis must be type I and such that every WAND is either not invariant under $SO(d-2)$ or is invariant under $SO(d-2)$ but is not geodesic. In particular, any axisymmetric solution admitting a multiple WAND is one of the solutions listed above. It is convenient to summarize our results according to algebraic type:

- Type O: the only type O Einstein solutions are Minkowski or (anti)-de Sitter spacetime.
- Type N: the only axisymmetric solutions are the type N axisymmetric Kundt solutions.
- Type III: there are no axisymmetric type III solutions.
- Type D: all axisymmetric solutions are contained in the following list: Kaluza-Klein bubble; $dS_{d-2} \times S^2$; generalized Schwarzschild; generalized black string; solutions of the form (3.5).
- Type II: the only axisymmetric solutions are the type II axisymmetric Kundt solutions.
- Type I: if a WAND of an axisymmetric type I solution is axisymmetric then it is non-geodesic.

Note that the type D solutions all have isometry groups larger than the $SO(d-2)$ that was assumed initially.

We can compare these results to those of De Smet, who classified *static*, axisymmetric $d = 5$ spacetimes belonging to classes 22 and $\underline{22}$ in his classification scheme for $\Lambda = 0$ [70] and $\Lambda \neq 0$ [71]. For $d = 5$, our list of type D solutions is very similar to the set of the solutions that he found.[3] One significant difference is for $\Lambda = 0$, where he found a solution that is not on our list (eq. 4.6 of Ref. [70], a "homogeneous wrapped object"). The results of section 3.2 below show that this solution is type G in the CMPP classification. Curiously, no analogous solution with $\Lambda \neq 0$ was obtained in Ref. [71]. Some of the generalized "black string" solutions that we found (eq. (3.49)) do not appear in De Smet's results. The connection between the De Smet scheme and the CMPP scheme is studied in section 6.3.2.

It is also interesting to compare our results with results for $d = 4$. For $d = 4$, axisymmetry is much less restrictive than for $d > 4$. This is because, the action of a 1-dimensional group such as $SO(2)$ need not be orthogonally transitive, so the orthogonal decomposition (3.4) is not always possible, e.g. it does not apply for the Kerr solution. The natural $d = 4$ analogues of $d > 4$ axisymmetric spacetimes are spacetimes with a spacelike, hypersurface-orthogonal Killing vector field. Coordinates can then be chosen so

[3] We are taking results from the arXiv version of Ref. [70], which differs significantly from the published version. De Smet worked in Euclidean signature and hence could not distinguish between the Schwarzschild solution and a KK bubble.

that the Killing field is $\partial/\partial\phi$ and there is a discrete isometry $\phi \to -\phi$. The metric then can be written in the form (3.4) with $d\Omega^2 = d\phi^2$. So this is the class of $d = 4$ spacetimes analogous to our spacetimes. Algebraically special $d = 4$ vacuum solutions (with $\Lambda = 0$) with these symmetries were first classified by Kramer and Neugebauer [100], so we shall refer to them as KN solutions.

By the Goldberg-Sachs theorem, the null congruence tangent to the repeated PND is geodesic and shear-free. For KN solutions it can be shown that it is also rotation-free, i.e., hypersurface-orthogonal [100]. Hence a KN solution belongs to the Robinson-Trautman (RT) or Kundt family of solutions depending on whether the congruence associated to the repeated PND is expanding or not. In 4d, the general vacuum solution belonging to either of these classes involves arbitrary functions of time, and cannot be written down in closed form [35]. However, with the KN symmetries, the general Kundt (but not RT) solution can be obtained in closed form [100]. Special cases of RT solutions with the KN symmetries are the A-metrics and the C-metric. The Kundt family of solutions contains the B-metrics.[4]

The main differences between these $d = 4$ results and our results are (i) the absence of time-dependent axisymmetric $d > 4$ RT solutions; (ii) the absence of a $d > 4$ analogue of the C-metric. The first difference extends beyond axisymmetry: the general class of $d > 4$ RT solutions was investigated in Ref. [86] and found to be considerably simpler than the $d = 4$ class. In particular, the only $d > 4$ RT solutions with non-vanishing "mass function" are simple static generalizations of the Schwarzschild solution, in contrast with the $d = 4$ case where such solutions are generically time-dependent. Concerning point (ii), to explain what we mean by "analogue", we note that the main interest in constructing such a solution is to obtain an exact solution describing a black hole on a Randall-Sundrum brane, as explained in Ref. [101]. Such a solution would have $d = 5$, $\Lambda < 0$ and would be axisymmetric (if the black hole were spherically symmetric on the brane) and describe an object with an event horizon accelerating along the axis of symmetry. The $d = 4$ C-metric belongs to the Weyl class, the RT class and the class of type D metrics. However, no $d > 4$ analogue was found in the generalized Weyl class (for $\Lambda = 0$) [81] or, as we have just discussed, the $d > 4$ RT class [86]. Our results demonstrate that this negative conclusion extends to the $d > 4$ type D class too. However, we note that a type D metric of the form (3.5) does describe the *near-horizon geometry* of an extremal charged Randall-Sundrum

[4]We have ignored a special case that arises in the KN analysis, which occurs when the repeated PND is not invariant under the discrete symmetry (and must therefore be mapped to another repeated PND so the spacetime is type D). KN showed that the only such solution is the $k = 0$ B-metric. However, this admits a second hypersurface-orthogonal spacelike Killing field, and the associated discrete symmetry *does* preserve the repeated PNDs. This implies that the solution also belongs to the Kundt class.

black hole [95].

We are interested in solutions of the vacuum Einstein equation (3.1). For such spacetimes, the WAND conditions can be reformulated in terms of the Riemann tensor:

$$R_{0i0j} = 0 \Leftrightarrow \text{WAND}, \qquad R_{0i0j} = R_{0ijk} = 0 \Leftrightarrow \text{multiple WAND}. \tag{3.6}$$

We shall make use of several general results for warped product spacetimes. A warped product is a spacetime of the form

$$ds^2 = A(y)^2 g_{AB}(x) dx^A dx^B + B(x)^2 g_{IJ}(y) dy^I dy^J, \tag{3.7}$$

where g_{AB} is Lorentzian and g_{IJ} is Riemannian. Such a spacetime is type D or O if the Lorentzian factor is (i) two-dimensional; (ii) a three-dimensional Einstein space; (iii) a type D Einstein space [85].

We now summarize our index conventions in this chapter. As described above, indices $i, j, k, \ldots$ refer to the spacelike basis vectors, and take values from 2 to $d-1$. Later, we shall choose $d-3$ of the basis vectors to like tangent to the S^{d-3} orbits of $SO(d-2)$. Indices $\alpha, \beta, \gamma, \ldots$ ranging from 3 to $d-1$ will refer to these basis vectors.

3.2 Static, axisymmetric, solutions

In this section we consider higher-dimensional solutions that are static and axisymmetric, i.e., they admit a hypersurface orthogonal timelike Killing vector field that commutes with the generators of $SO(d-2)$. Introduce coordinates adapted to the isometries:

$$ds^2 = -A(r,z)^2 dt^2 + B(r,z)^2 (dr^2 + dz^2) + C(r,z)^2 d\Omega^2. \tag{3.8}$$

The components of the Einstein equation (3.1) for this metric are given in Ref. [102]. Define a complex coordinate $w \equiv (r + iz)/\sqrt{2}$. Consider $R^r_r - R^z_z + 2iR^r_z = 0$. This gives

$$\frac{\partial^2 A}{A} - 2\frac{\partial A \partial B}{AB} + (d-3)\left(\frac{\partial^2 C}{C} - 2\frac{\partial B \partial C}{BC}\right) = 0, \tag{3.9}$$

where $\partial \equiv \partial/\partial w$. This implies

$$\frac{\partial B}{B} = \frac{C^{d-3}\partial^2 A + (d-3)AC^{d-4}\partial^2 C}{2\partial(AC^{d-3})}. \tag{3.10}$$

We must consider the denominator since it could vanish identically, i.e., AC^{d-3} might be constant. The equation $R_t^t + (d-3)R_{\theta_1}^{\theta_1} = (d-2)\Lambda$ implies[5]

$$\frac{\Delta(AC^{d-3})}{AC^{d-3}} = \frac{B^2}{C^2}\left[(d-3)(d-4) - (d-2)\Lambda C^2\right]. \tag{3.11}$$

Constancy of $\partial(AC^{d-3})$ implies the RHS must vanish hence C is a constant and Λ is positive. But then A must also be constant so the spacetime has a flat time direction. This is incompatible with positive Λ. Hence AC^{d-3} cannot be constant.

If A and C are known then equation (3.10) determines B. Furthermore, $R_r^r + R_z^z - 2R_t^t = 0$ implies

$$\frac{(\nabla B)^2}{B^2} - \frac{\Delta B}{B} = \frac{(d-3)\Delta C}{2C} - \frac{\Delta A}{2A} - (d-3)\frac{\nabla A \cdot \nabla C}{AC}. \tag{3.12}$$

It can be checked that this is compatible with equation (3.10).

We assume that the spacetime is type I or more special, so it admits a WAND ℓ. We shall assume for now that the WAND is axisymmetric. Assuming $d > 4$, this implies that it is orthogonal to S^{d-3}. By rescaling ℓ we can arrange that

$$\ell = \frac{\partial}{\partial t} + \frac{A}{B}\left(\cos\alpha(r,z)\frac{\partial}{\partial r} + \sin\alpha(r,z)\frac{\partial}{\partial z}\right), \tag{3.13}$$

for some function $\alpha(r,z)$. Staticity implies that

$$n = \frac{\partial}{\partial t} - \frac{A}{B}\left(\cos\alpha(r,z)\frac{\partial}{\partial r} + \sin\alpha(r,z)\frac{\partial}{\partial z}\right), \tag{3.14}$$

is also a WAND[6], i.e., WANDs come in pairs, which implies that the algebraic type must be I, D or O [85]. Choose

$$m^2 = \frac{1}{B}\left(-\sin\alpha(r,z)\frac{\partial}{\partial r} + \cos\alpha(r,z)\frac{\partial}{\partial z}\right), \tag{3.15}$$

and

$$m^\alpha = \frac{1}{C}\hat{e}^\alpha, \qquad \alpha = 3, \ldots, d-1 \tag{3.16}$$

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[5]We have defined $\nabla = (\partial_r, \partial_z)$, $\Delta = \nabla^2$, and indices are raised with the flat metric $dr^2 + dz^2$.

[6]These null vectors don't obey $\ell \cdot n = 1$ but this can arranged by rescaling them, which doesn't affect anything below.

where $\hat{e}^\alpha$ is a vielbein for S^{d-3}. We find that the WAND condition (3.6) reduces to

$$\mathrm{Re}\left(e^{2i\alpha}W\right) = X, \tag{3.17}$$

where

$$W = \frac{\partial^2 A}{A} - \frac{\partial^2 C}{C} - 2\frac{\partial A \partial B}{AB} + 2\frac{\partial B \partial C}{BC} = (d-2)\frac{C^{d-4}\left(\partial^2 A \partial C - \partial A \partial^2 C\right)}{\partial (AC^{d-3})}, \tag{3.18}$$

$$X = \frac{\Delta A}{2A} - \frac{\Delta B}{B} + \frac{\Delta C}{2C} - \frac{\nabla A \cdot \nabla C}{AC} + \frac{(\nabla B)^2}{B^2} = \frac{(d-2)A^2}{2C}\nabla \cdot (\frac{\nabla C}{A^2}), \tag{3.19}$$

where the second equality in each case follows from equations (3.10) and (3.12). The spacetime is type I or more special if, and only if, there exists a real solution α of the WAND condition. Hence

$$|W| \geq |X| \qquad \Leftrightarrow \qquad \text{admits WAND} \tag{3.20}$$

Now consider the additional condition required for a multiple WAND (equation (3.6)). This gives the single equation $\mathrm{Im}\left(e^{2i\alpha}W\right) = 0$. Combining with the type I condition gives

$$e^{2i\alpha}W = X. \tag{3.21}$$

We conclude that, assuming an axisymmetric WAND

$$\begin{array}{rclcl} |W| & < & |X| & \Leftrightarrow & \text{Type G} \\ |W| & > & |X| & \Leftrightarrow & \text{Type I} \\ |W| & = & |X| & \Leftrightarrow & \text{Type D or O} \end{array} \tag{3.22}$$

We shall now comment on our assumption that the WAND is axisymmetric. In general, this need not be true. However, for odd d, S^{d-3} is even dimensional so the projection of ℓ onto the sphere must vanish somewhere. Working at such a point, we can argue as above to arrive at equations (3.22) that depend only on r and z and must therefore hold *everywhere* on the sphere, which implies the existence of an axisymmetric WAND. Hence, for odd d, there is no loss of generality in restricting to an axisymmetric WAND. For even d, this argument does not work. However, in section 3.4, we shall consider axisymmetric spacetimes with a non-axisymmetric *multiple* WAND, and show that no such spacetime is static and axisymmetric.[7] Therefore the multiple WAND of a static axisymmetric type D

[7] Actually, the spacetimes we find there are "static" and "axisymmetric" but they are not "static

spacetime must be axisymmetric. However, it is possible that some spacetimes with even $d > 4$ and $|W| < |X|$ may be type I with a non-axisymmetric WAND.

To illustrate these conditions, consider the z-independent $d = 5$, $\Lambda = 0$ solution of Ref. [103], written in the form given in Ref. [104]

$$\begin{aligned} ds^2 &= -\left(\frac{1-m/R}{1+m/R}\right)^{2/\alpha} dt^2 + \left(\frac{1-m/R}{1+m/R}\right)^{2\beta/\alpha} dz^2 \\ &+ \left(1+\frac{m}{R}\right)^4 \left(\frac{1-m/R}{1+m/R}\right)^{2(\alpha-\beta-1)/\alpha} \left(dR^2 + R^2 d\Omega^2\right), \end{aligned} \tag{3.23}$$

where $\alpha = \sqrt{\beta^2+\beta+1}$. Assume $m \neq 0$ (so the spacetime is not flat). Then a calculation reveals that $|W| = |X|$ if, and only if, $\beta = 0$ or $\beta = 1$. The first possibility gives the Schwarzschild black string. The second possibility gives a boost invariant singular spacetime discussed in Ref. [92]. This spacetime is of the form (3.5) discussed in the introduction.

Another interesting example is the static Kaluza-Klein bubble (the product of a flat time direction with the Euclidean Schwarzschild solution). This can be obtained by taking the limit $\beta \to \infty$ of the above metric. This spacetime has $W = 0$, $X \neq 0$ hence it is type G.

Since type G is distinguished from type I only by an inequality, it is possible that there exist (connected) analytic spacetimes that are type G in some open subset of spacetime and type I in some other open subset. Indeed, if we choose $m > 0$ and $\beta = 1/2$ in the above metric then it is type G for $R \sim m$ but type I for $R \gg m$.[8] As discussed in the introduction, this kind of behaviour suggests that the type I condition alone will not be much help in solving the Einstein equation.

It would be nice to use the type D condition obtained above to solve the Einstein equation. However, we have not made progress using the coordinates employed here. (Even in $d = 4$, this approach would not work for $\Lambda \neq 0$.) However, in subsequent sections we shall see that all static axisymmetric type D solutions *can* be found, indeed we shall relax the condition of stationarity and determine all axisymmetric type D solutions.

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and axisymmetric" because the generator of time translations does not commute with the generators of axisymmetry.

[8]The general behaviour appears to be that, for $0 < \beta < 1$ and $m > 0$, the solution is type G near $R = m$ and type I for $R \gg m$. For (finite) $\beta > 1$ and $m > 0$, it is type I near $R = m$ and type G for $R \gg m$.

3.3 Axisymmetric solutions with an axisymmetric WAND

3.3.1 Introducing coordinates

In this section we shall consider general (possibly time-dependent) axisymmetric spacetimes with an axisymmetric WAND. First we shall introduce coordinates adapted to the WAND. Consider the metric in the form (3.4), where spacetime is locally a warped product $M_3 \times S^{d-3}$. Axisymmetry implies that the WAND is tangential to M_3.

We shall choose the local coordinates x^a on M_3 as follows. Pick a 2-surface in M_3 transverse to the WAND ℓ and let $x^{\hat{a}}$ be coordinates on this surface, where $\hat{a} = 1, 2$. Now carry these coordinates to the rest of spacetime along the integral curves of ℓ, and let $\hat{r}$ be the parameter distance along these curves. Now use $(x^{\hat{a}}, \hat{r})$ as coordinates on M_3, so $\ell = \partial/\partial\hat{r}$. The metric takes the form

$$ds^2 = 2g_{\hat{r}\hat{a}} d\hat{r} dx^{\hat{a}} + g_{\hat{a}\hat{b}} dx^{\hat{a}} dx^{\hat{b}} + E^2 d\Omega^2, \tag{3.24}$$

where $g_{\hat{r}\hat{a}} \neq 0$ for some $\hat{a}$. Without loss of generality we may assume $g_{\hat{r}1} \neq 0$. Now let $r = \int g_{\hat{r}1}(\hat{r}, x^{\hat{a}}) d\hat{r}$, $v = x^1$, $z = x^2$ and use coordinates (v, r, z). In this chart, the metric takes the form

$$\begin{aligned} ds^2 &= 2\left[dv + B(v,r,z)dz\right]\left[dr - \frac{1}{2}U(v,r,z)\left(dv + B(v,r,z)dz\right) + C(v,r,z)dz\right] \\ &+ D(v,r,z)^2 dz^2 + E(v,r,z)^2 d\Omega^2, \end{aligned} \tag{3.25}$$

for some functions U, B, C, D, E. The WAND is proportional to $\partial/\partial r$ so we can rescale it so that $\ell = \partial/\partial r$. It is convenient to complete this to a null basis as follows:

$$\begin{aligned} \ell_a dx^a &= dv + B dz, \qquad n_a dx^a = dr - \frac{1}{2}U \ell_a dx^a + C dz, \\ m^2 &= D dz, \qquad m^\alpha = E \hat{e}^\alpha, \end{aligned} \tag{3.26}$$

where $\hat{e}^\alpha$ $(\alpha = 3 \ldots d-1)$ is an orthonormal basis of 1-forms on S^{d-3} with no dependence on v, r, z. We shall denote the spacelike basis 1-forms collectively as m^i, $i = 2 \ldots d-1$.

Now consider the null congruence associated with the WAND. This is geodesic if, and only if, $\partial_r B \equiv 0$. The "expansion matrix" of the congruence is

$$S_{ij} \equiv m_i^\mu m_j^\nu \nabla_{(\mu} \ell_{\nu)} = \text{diag}\left(\frac{\partial_r D}{D}, \frac{\partial_r E}{E}, \ldots, \frac{\partial_r E}{E}\right). \tag{3.27}$$

The expansion of the congruence is the trace of this matrix and the shear tensor is the traceless part. The rotation matrix of the congruence vanishes: this is a consequence of axisymmetry.

The results of Ref. [64, 65] establish that an axisymmetric multiple WAND in a vacuum solution can be assumed to be geodesic. In this section we shall determine all solutions with an axisymmetric geodesic WAND. We shall not assume that the WAND is a *multiple* WAND (so *a priori* the solution might be type I but we shall see that this does not happen).

We have $\partial_r B = 0$ because the WAND is geodesic. We can now introduce new coordinates v' and r' such that $v = v' + F(v', z)$, $r = r'G(v', z)$ for some functions F, G that can be chosen to bring the metric to the same form as before but with $B \equiv 0$. Dropping the primes on the coordinates, we have

$$ds^2 = -U(v, r, z)dv^2 + 2dvdr + 2C(v, r, z)dvdz + D(v, r, z)^2dz^2 + E(v, r, z)^2d\Omega^2. \quad (3.28)$$

By rescaling, the WAND can be taken to be $\ell = \partial/\partial r$. We saw above that the null congruence associated with the WAND has vanishing rotation. Since it is geodesic, this implies that it is hypersurface orthogonal. In the above coordinates, it is orthogonal to hypersurfaces of constant v. Furthermore, r is an affine parameter along the null geodesics.[9] There is some coordinate freedom remaining: the form of the metric is invariant under the transformations

$$\begin{aligned} v &\rightarrow V(v), \qquad r \rightarrow r/\partial_v V, \\ r &\rightarrow r - F(v, z), \\ z &\rightarrow z(v, z). \end{aligned} \quad (3.29)$$

All of this is well-known in the context of 4d solutions with a hypersurface orthogonal null geodesic congruence [35].

We shall employ the same null basis as before (i.e., (3.26) with $B \equiv 0$). The Riemann tensor of the above metric in this basis is given in Appendix 3.A. The WAND condition (3.6) reduces to

$$\partial_r^2 D = \partial_r^2 E = 0, \quad (3.30)$$

[9] If we assumed that ℓ is a *multiple* WAND then the r-dependence of the metric could be read off from Ref. [61]. However we shall not make this assumption.

Hence

$$D(v,r,z) = D_0(v,z) + rD_1(v,z), \qquad E(v,r,z) = E_0(v,z) + rE_1(v,z), \tag{3.31}$$

for some functions D_0, D_1, E_0, E_1. The 00 component of the Einstein equation is now automatically satisfied. Axisymmetry implies that the 0α component is trivial. The 02 component reduces to an equation linear in C:

$$\partial_r^2 C - \left(\frac{\partial_r D}{D} - (d-3)\frac{\partial_r E}{E}\right)\partial_r C - 2(d-3)\frac{\partial_r D \partial_r E}{DE} C = 2(d-3)\left(\frac{\partial_r \partial_z E}{E} - \frac{\partial_r D \partial_z E}{DE}\right). \tag{3.32}$$

This will determine the r-dependence of C. There are several different cases to consider.

$E_1 \neq 0,\ D_1 \equiv 0$

We can use the residual freedom in r and z (equation (3.29)) to set $E_0 \equiv 0$ and $D_0 \equiv 1$, i.e. $D \equiv 1$. Then (3.32) reduces to

$$\partial_r^2 C + \frac{d-3}{r}\partial_r C = 2\frac{d-3}{r}\frac{\partial_z E_1}{E_1}, \tag{3.33}$$

which can be solved to give

$$C(v,r,z) = C_0(v,z) + \frac{C_1(v,z)}{(d-4)r^{d-4}} + \frac{2\partial_z E_1}{E_1} r, \tag{3.34}$$

for arbitrary functions C_0 and C_1. The r-dependence of $U(v,r,z)$ is determined by the 01 component of Einstein's equation:

$$\begin{aligned} U(v,r,z) = &-\frac{C_1(v,z)^2}{2(d-4)^2 r^{2(d-4)}} - \frac{U_1(v,z)}{(d-4)r^{d-4}} + U_0(v,z) - \frac{2r}{E_1}\left(\partial_v E_1 + C_0 \partial_z E_1\right) \\ &+ \frac{r^2}{d-2}\left(\frac{\partial_z^2 E_1}{E_1} - d\frac{(\partial_z E_1)^2}{E_1^2} - \Lambda\right) - \chi(r)\left(\partial_z C_1 + \frac{(d^2-9d+22)}{d-4} C_1 \frac{\partial_z E_1}{E_1}\right), \end{aligned} \tag{3.35}$$

where U_0 and U_1 are arbitrary functions, and

$$\chi(r) = \begin{cases} \log(r)\,, & d = 5 \\ -\dfrac{1}{(d-5)r^{d-5}}\,, & d > 5 \end{cases} . \tag{3.36}$$

The r-dependence of the metric is now fully determined. Comparing coefficients of terms with different r dependence in the remaining components of the Einstein equation can be used to restrict the arbitrary functions above. The $\alpha\beta$ components of Einstein's equation give

$$C_1 = 0, \qquad \partial_z C_0 = 0, \tag{3.37}$$

$$U_0(v,z) = \frac{1}{E_1(v,z)^2} - C_0(v)^2, \tag{3.38}$$

The residual coordinate freedom $z \to z - f(v)$ can be used to set

$$C_0 = 0. \tag{3.39}$$

The 22 component of Einstein's equation gives

$$\frac{\partial_z^2 E_1}{E_1} - 2\frac{(\partial_z E_1)^2}{E_1^2} - \frac{\Lambda}{d-1} = 0. \tag{3.40}$$

Now the 12 component of Einstein's equation implies

$$\partial_z U_1 + 2(d-3)\frac{\partial_z E_1}{E_1} U_1 = 0, \tag{3.41}$$

which in turn implies

$$U_1(v,z) = \frac{m(v)}{E_1(v,z)^{2(d-3)}}, \tag{3.42}$$

for some arbitrary function $m(v)$. The 11 component of Einstein's equation then implies

$$\partial_v m(v) = (d-4)m(v)\frac{\partial_v E_1(v,z)}{E_1(v,z)}. \tag{3.43}$$

$m(v) = 0$ implies that the spacetime is conformally flat (type O), so assume that $m(v) \neq 0$. Then (3.43) implies

$$E_1(v,z) = \frac{|m(v)|^{1/(d-4)}}{g(z)}, \tag{3.44}$$

for some positive function $g(z)$. Inserting this into equation (3.40) gives the linear equation

$$g''(z) + \frac{\epsilon}{L^2} g(z) = 0, \tag{3.45}$$

where L is defined by

$$\Lambda = \frac{(d-1)\epsilon}{L^2}, \qquad \epsilon \in \{-1, 0, 1\}. \tag{3.46}$$

Define a positive constant μ via the first integral

$$g'(z)^2 + \frac{\epsilon}{L^2} g(z)^2 = \eta\mu^2, \tag{3.47}$$

where $\eta \in \{-1, 0, 1\}$. Using the freedom to shift z by a constant (and $z \to -z$) we have $g(z) = \mu G(z)$, where $G(z)$ is given by

$G(z)$	$\eta = 1$	$\eta = 0$	$\eta = -1$
$\epsilon = 1$	$L\sin(z/L)$		
$\epsilon = 0$	z	α	
$\epsilon = -1$	$L\sinh(z/L)$	$\alpha e^{\pm z/L}$	$L\cosh(z/L)$

where α is a positive constant. Defining new coordinates (V, ρ) by

$$dV = \mu \frac{dv}{|m(v)|^{1/(d-4)}}, \qquad \rho = \frac{r|m(v)|^{1/(d-4)}}{\mu G(z)^2}, \tag{3.48}$$

the metric becomes

$$ds^2 = dz^2 + G(z)^2 \left[-\left(1 - \frac{M}{\rho^{d-4}} - \eta\rho^2\right) dV^2 + 2dV d\rho + \rho^2 d\Omega^2 \right], \tag{3.49}$$

where $M = \text{sign}(m(v))\mu^{d-4}$. This is the warped product of a line, parametrized by z, with the $(d-1)$-dimensional Schwarzschild (anti)-de Sitter metric. The $\epsilon = \eta = 0$ case is the Schwarzschild-Tangherlini black string. The $\epsilon = 0$, $\eta = 1$ case corresponds to taking a de Sitter slicing of Minkowski spacetime and replacing the de Sitter slices with the Schwarzschild-de Sitter metric. The $\epsilon = \eta = 1$ case corresponds to doing the same thing for de Sitter spacetime. The $\epsilon = -1$ cases correspond to the same idea for slicings of anti-de Sitter space (In $d = 5$, $\epsilon = -1$, $\eta = 0$ is the AdS black string of Ref. [87]). A warped product whose Lorentzian factor is a type D Einstein space is also type D [85]. Hence these solutions are all type D.

$E_1 \neq 0,\ D_1 \neq 0,\ D_0 \equiv 0$: Robinson-Trautman solutions

The coordinate freedom (3.29) can be used to set $E_0 \equiv 0$. From (3.27), these solutions have vanishing shear and non-vanishing expansion. Therefore they belong to the class of higher-dimensional Robinson-Trautman solutions [86]. To give a self-contained presentation, we shall rederive these solutions here (with the additional restriction of axisymmetry). Using a transformation $z \to z'(z, v)$, we can set $D_1 \equiv 1$. The general solution to equation (3.32) is

$$C(v,r,z) = C_1(v,z)r^2 + \frac{C_2(v,z)}{r^{d-3}}, \tag{3.50}$$

where C_1 and C_2 are arbitrary functions. But now $R_{22} - R_{\alpha\alpha}$ (no sum on α) is independent of U and, by the Einstein equation, must vanish. Equating coefficients of terms with different r-dependence gives $C_2 \equiv 0$ and

$$\frac{\partial_z^2 E_1}{E_1} - \frac{(\partial_z E_1)^2}{E_1^2} + \frac{1}{E_1^2} = 0, \tag{3.51}$$

$$\partial_z C_1 - \frac{\partial_z E_1}{E_1} C_1 + \frac{\partial_v E_1}{E_1} = 0. \tag{3.52}$$

The 22 component of the Einstein equation can then be solved to determine U:

$$U(v,r,z) = \frac{U_1(v,z)}{r^{d-3}} - \frac{\partial_z^2 E_1}{E_1} + 2r\partial_z C_1 - r^2 \left(C_1^2 + \frac{\Lambda}{d-1} \right), \tag{3.53}$$

where U_1 is an arbitrary function. Now examining the 12 component of the Einstein equation gives $\partial_z U_1 = 0$. The 11 component of the Einstein equation reduces to

$$E_1 \partial_v U_1 + (d-1)\left(U_1 \partial_v E_1 - C_1 U_1 \partial_z E_1 \right) = 0. \tag{3.54}$$

If $U_1 \equiv 0$ then it can be shown that the above equations imply that the Weyl tensor vanishes hence the solution is type O. If $U_1 \neq 0$ then we can use the gauge freedom $v \to V(v)$ and $r \to r/\partial_v V$ to set $U_1 \equiv \mu$ for some non-zero constant μ. Then (3.54) gives $\partial_v E_1 = C_1 \partial_z E_1$. From (3.52) we then learn that $\partial_z C_1 = 0$. This implies that C_1 can be gauged away by a shift $z \to z - f(v)$. In the new gauge we have $\partial_v E_1 = 0$. The solutions of (3.51) are $E_1 = R\sin(z/R)$, z or $R\sinh(z/R)$ (using $z \to z - \text{const}$ and $z \to -z$ to simplify) where R is a positive constant. R can be set to one by rescaling z, v and r. The solution takes the final form

$$ds^2 = -\left(k - \frac{M}{r^{d-3}} - \frac{\Lambda}{d-1} r^2 \right) dv^2 + 2dvdr + r^2 d\Sigma_k^2, \tag{3.55}$$

where M is a non-zero constant, and $d\Sigma_k^2$ is the metric on a $d-2$ dimensional space of unit constant curvature of sign k. This generalized Schwarzschild metric is of type D [86].

$E_1 \neq 0,\ D_1 \neq 0,\ D_0 \neq 0$

We use the transformations (3.29) to set $D_1 \equiv 1$ and $E_0 \equiv 0$. The general solution to equation (3.32) is

$$C(v,r,z) = (D_0(v,z) + r)^2 \left(C_1(v,z) + C_2(v,z) \int \frac{dr}{r^{d-3}(D_0+r)^3} \right) - \frac{D_0 \partial_z E_1}{E_1}, \quad (3.56)$$

where C_1 and C_2 are arbitrary functions. Now we consider the 22 and $\alpha\alpha$ components of the Einstein equation. These equations are linear in U and $\partial_r U$ and can be solved algebraically to determine U and $\partial_r U$. The r-dependence is completely determined hence consistency of the solutions for U and $\partial_r U$ gives an equation whose r dependence is fully determined. Equating coefficients of terms with the same r dependence then gives $C_2 \equiv 0$ together with

$$\begin{aligned} \partial_v D_0 &= -\frac{1}{E_1^2} + \partial_z(D_0 C_1) + \frac{(\partial_z E_1)^2}{E_1^2} - \frac{\partial_z^2 E_1}{E_1}, \\ \partial_v E_1 &= \frac{\Lambda}{d-1} D_0 E_1 - E_1 \partial_z C_1 + C_1 \partial_z E_1. \end{aligned} \quad (3.57)$$

The solution for U is then

$$\begin{aligned} U(v,r,z) &= \frac{1}{E_1^2} - C_1^2 D_0^2 + 2C_1 D_0 \frac{\partial_z E_1}{E_1} - \frac{(\partial_z E_1)^2}{E_1^2} \\ &+ \left(2\partial_z C_1 - \frac{2\Lambda}{d-1} D_0 - 2C_1^2 D_0 \right) r - \left(\frac{\Lambda}{d-1} + C_1^2 \right) r^2. \end{aligned} \quad (3.58)$$

These results imply that the Weyl tensor vanishes. Hence these solutions are type O, i.e., Minkowski or (anti)-de Sitter spacetime.

$E_1 \equiv 0$: Kundt solutions

Solving (3.32) gives

$$C(v,r,z) = C_0(v,z) + \left(C_*(v,z) D_0(v,z) + 2(d-3) \frac{\partial_z E_0}{E_0} \right) r + \frac{1}{2} C_*(v,z) D_1(v,z) r^2, \quad (3.59)$$

where $C_0(v,z)$ and $C_*(v,z)$ are arbitrary functions. The $\alpha\alpha$ component of the Einstein equation does not involve U so its r dependence is completely determined. Equating coefficients of terms with different dependence on r gives $D_1^3[(d-4) - \Lambda E_0^2] = 0$. Hence either $D_1 \equiv 0$ or $E_0^2 \equiv (d-4)/\Lambda$. The latter implies that spacetime is a direct product

$M_3 \times S^{d-3}$, which requires $\Lambda > 0$ and the spacetime must then be locally $dS_3 \times S^{d-3}$ which is of algebraic type D.

Assume instead that $D_1 \equiv 0$. We now have $\partial_r D = \partial_r E = 0$ so the geodesic congruence is free of expansion and shear as well as twist. Spacetimes with vanishing expansion, shear and twist are referred to as Kundt spacetimes [35, 60]. All vacuum Kundt solutions are type II or more special for any $d \geq 4$ [88]. General d-dimensional Kundt spacetimes have been discussed recently [62]. The general solution cannot be obtained in closed form. We shall now analyze such solutions assuming axisymmetry, which enables further progress to be made.

With $D_1 \equiv 0$, we can use the transformation $z \to z'(v,z)$ to set $D_0 \equiv 1$. Write the solution for C as $C(v,r,z) = C_0(v,z) + rC_1(v,z)$. The shift $r \to r - F(v,z)$ has the effect $C_0 \to C_0 - \partial_z F - C_1 F$. Hence we can choose $F(v,z)$ to set $C_0 \equiv 0$. To summarize, we have brought the metric to the form

$$ds^2 = -U(v,r,z)dv^2 + 2dvdr + 2rC_1(v,z)dvdz + dz^2 + E_0(v,z)^2 d\Omega^2. \tag{3.60}$$

Some gauge freedom remains. The transformations of the form (3.29) that preserve this form of the metric are

$$v = v', \qquad z = z' + f(v'), \qquad r = r' + g(v',z'), \qquad \partial_{z'} g + C_1 g = -\partial_{v'} f, \tag{3.61}$$

$$v = V(v'), \qquad r = \frac{r'}{\partial_{v'} V}, \qquad z = z'. \tag{3.62}$$

The $\alpha\alpha$ component of the Einstein equation reduces to

$$\frac{\partial_z^2 E_0}{E_0} + (d-4)\frac{(\partial_z E_0)^2}{E_0^2} - C_1 \frac{\partial_z E_0}{E_0} - \frac{d-4}{E_0^2} + \Lambda = 0, \tag{3.63}$$

and the 22 component of the Einstein equation reduces to

$$\partial_z C_1 - \frac{1}{2}C_1^2 - (d-3)\frac{\partial_z^2 E_0}{E_0} - \Lambda = 0. \tag{3.64}$$

The 01 component of the Einstein equation is satisfied if, and only if,

$$U(v,r,z) = U_0(v,z) + rU_1(v,z) + r^2 U_2(v,z), \tag{3.65}$$

where U_0 and U_1 are arbitrary functions, and

$$U_2(v,z) = \frac{1}{2}\partial_z C_1 + \frac{d-3}{2} C_1 \frac{\partial_z E_0}{E_0} - \frac{1}{2} C_1^2 - \Lambda. \tag{3.66}$$

The 12 Einstein equation reduces to

$$\partial_z U_1 = -\partial_v C_1 - (d-3) C_1 \frac{\partial_v E_0}{E_0} - 2(d-3)\frac{\partial_v \partial_z E_0}{E_0}. \tag{3.67}$$

Finally, using the above equations, the 11 Einstein equation reduces to

$$\partial_z^2 U_0 + \left(C_1 + (d-3)\frac{\partial_z E_0}{E_0} \right) \partial_z U_0 \\ + \left(\partial_z C_1 + (d-3) C_1 \frac{\partial_z E_0}{E_0} \right) U_0 = (d-3)\left(2\frac{\partial_v^2 E_0}{E_0} - U_1 \frac{\partial_v E_0}{E_0} \right). \tag{3.68}$$

As is familiar for Kundt solutions, the equations of motion separate into the "background" equations (3.63) and (3.64), which must be solved to determine E_0 and C_1. Given a solution of these equations, the other equations can be integrated to determine U_0 and U_1. The second step is trivial because the equations are linear. Hence solving the background equations is the non-trivial step that remains. However, the general solution to the background equations is not known analytically.

Since the background equations do not involve v-derivatives, solving these equations is equivalent to solving the corresponding equations assuming that E_0 and C_1 are independent of v and $U_0 = U_1 = 0$. But in this case, the metric is static. In fact, we shall see below that the general type D axisymmetric Kundt metric is of this form. The background equations can only be solved in special cases e.g. the general solution with $d = 4$ can be determined, and the general solution with $d = 5$, $\Lambda = 0$ and $U_2 = 0$ can also be obtained [92]. Some time-dependent solutions based on the latter solution of the background equations were obtained in Refs [97, 98].

It is convenient to define a positive function $W(v,z)$ by

$$W(v,z) = W_0(v)\exp(-\int^z C_1(v,z')dz'), \tag{3.69}$$

where $W_0(v)$ is an arbitrary positive function, so

$$C_1 = -\frac{\partial_z W}{W}. \tag{3.70}$$

The background equations become

$$\frac{\partial_z^2 E_0}{E_0} + (d-4)\frac{(\partial_z E_0)^2}{E_0^2} + \frac{\partial_z W}{W}\frac{\partial_z E_0}{E_0} - \frac{d-4}{E_0^2} + \Lambda = 0, \tag{3.71}$$

$$\frac{1}{2}\frac{(\partial_z W)^2}{W^2} - \frac{\partial_z^2 W}{W} - (d-3)\frac{\partial_z^2 E_0}{E_0} - \Lambda = 0. \tag{3.72}$$

Equation (3.68) becomes

$$U_2 = -\frac{\partial_z^2 W}{2W} - \frac{(d-3)\partial_z W \partial_z E_0}{2W E_0} - \Lambda. \tag{3.73}$$

These equations imply that

$$\partial_z (W U_2) = 0, \tag{3.74}$$

hence we can always choose $W_0(v)$ so that

$$U_2(v,z) = -\frac{k}{W(v,z)}, \tag{3.75}$$

with $k \in \{1, 0, -1\}$. We can now define a new coordinate R by

$$r = W(v,z)R. \tag{3.76}$$

The metric becomes

$$ds^2 = W(v,z)\left\{-\left[\frac{U_0(v,z)}{W(v,z)} + R\left(U_1(v,z) - 2\frac{\partial_v W}{W}\right) - kR^2\right]dv^2 + 2dvdR\right\} \\ +dz^2 + E_0(v,z)^2 d\Omega^2, \tag{3.77}$$

We now consider a further classification of the Kundt solutions according to their algebraic type. Using the above equations to simplify the Weyl tensor, we find that the only independent nonzero components are:

$$C_{0\alpha1\beta} = \delta_{\alpha\beta}\left(C_1\frac{\partial_z E_0}{2E_0} - \frac{\Lambda}{d-1}\right), \tag{3.78}$$

$$C_{\alpha\beta\gamma\delta} = 2\delta_{\alpha[\delta}\delta_{\gamma]\beta}\left(\frac{\Lambda}{d-1} + \frac{(\partial_z E_0)^2}{E_0^2} - \frac{1}{E_0^2}\right), \tag{3.79}$$

$$C_{1\alpha\beta2} = \frac{\delta_{\alpha\beta}}{2}\left(C_1\frac{\partial_v E_0}{E_0} + 2\frac{\partial_v\partial_z E_0}{E_0}\right) = -\frac{\delta_{\alpha\beta}}{2(d-3)}(\partial_z U_1 + \partial_v C_1), \tag{3.80}$$

$$C_{1\alpha1\beta} = \delta_{\alpha\beta}\left\{-\frac{1}{2(d-3)}\partial_z(\partial_z U_0 + C_1 U_0) + \left[U_2\frac{\partial_v E_0}{E_0} + \left(\frac{1}{2}\partial_z U_1 + \partial_v C_1\right)\frac{\partial_z E_0}{E_0}\right] r\right\}. \tag{3.81}$$

Note that while C_{0101}, C_{0112}, C_{0212}, C_{1212} and $C_{\alpha2\beta2}$ are nonzero, they are related to the above components by the tracefree property of the Weyl tensor. The first two Weyl components written above are of boost weight 0, while the remaining two are of boost weight -1 and -2 respectively. Hence the solutions we are considering here are at least type II, confirming the general result of Ref. [88].

Type III and N

Consider the case in which the solution is type III, or more special. In this case, the Weyl components of boost weight 0 vanish. This gives, for $d > 4$, the following equations

$$C_1\frac{\partial_z E_0}{E_0} = \frac{2\Lambda}{d-1}, \tag{3.82}$$

$$\frac{(\partial_z E_0)^2}{E_0^2} = \frac{1}{E_0^2} - \frac{\Lambda}{(d-1)}. \tag{3.83}$$

Note that equation (3.83) is not present for $d = 4$. Solving this equation gives

$$E_0(v,z) = \begin{cases} L\sin(z/L) & \text{if } \Lambda > 0, \\ z & \text{if } \Lambda = 0, \\ L\sinh(z/L) & \text{if } \Lambda < 0, \end{cases} \tag{3.84}$$

where $L > 0$ is defined by (3.46), and we have used the freedom (3.61) to eliminate an arbitrary function of v (we've also fixed signs using $z \to \pm z$). Equation (3.82) now determines C_1:

$$C_1(v,z) = \begin{cases} (2/L)\tan(z/L) & \text{if } \Lambda > 0, \\ 0 & \text{if } \Lambda = 0, \\ -(2/L)\tanh(z/L) & \text{if } \Lambda < 0. \end{cases} \tag{3.85}$$

Since E_0 is independent of v, equation (3.80) gives that $C_{1\alpha\beta2} = 0$, hence these solutions are type N or O. There are no axisymmetric type III Kundt solutions for $d > 4$. However, such solutions do exist for $d = 4$ [35].

Continuing the analysis, note that the coefficient of r in $C_{1\alpha1\beta}$ (given by (3.81)) vanishes, and so $C_{1\alpha1\beta}$ reduces to

$$C_{1\alpha1\beta} = -\frac{\delta_{\alpha\beta}}{2(d-3)}\partial_z(\partial_z U_0 + C_1 U_0). \tag{3.86}$$

$U_2(v,z)$ can be calculated using (3.66):

$$U_2(v,z) = \begin{cases} -(1/L^2)\sec^2(z/L) & \text{if } \Lambda > 0, \\ 0 & \text{if } \Lambda = 0, \\ (1/L^2)\,\mathrm{sech}^2(z/L) & \text{if } \Lambda < 0. \end{cases} \tag{3.87}$$

Equation (3.67) implies that $U_1(v,z)$ is independent of z: $U_1(v,z) = U_1(v)$. We can then use the transformation $v \to V(v)$, $r \to r/\partial_v V$ to arrange that

$$U_1 \equiv 0. \tag{3.88}$$

Finally, equation (3.68) can be solved to determine U_0. For $\Lambda = 0$, the solution is, for $d > 4$

$$U_0 = \frac{F(v)}{z^{d-4}} + G(v), \tag{3.89}$$

(for $d = 4$, the first term is replaced by $F(v)\log z$). A shift $r \to r - f(v)$ can be used to set $G(v) \equiv 0$. The only independent non-zero component of the Weyl tensor is (3.86). This reveals that the solution is type O if, and only if, $F(v) \equiv 0$. Therefore, the general axisymmetric type N Kundt solution with $\Lambda = 0$, is given by the following metric (for $d > 4$)

$$ds^2 = -\frac{F(v)}{z^{d-4}}dv^2 + 2dvdr + dz^2 + z^2 d\Omega^2. \tag{3.90}$$

The null vector field ℓ is covariantly constant, and so the solution above belongs to the family of pp-waves [83].

For $\Lambda < 0$, the solution for U_0 is

$$U_0 = \cosh^2(z/L)\left[F(v)I_-(z) + G(v)\right], \tag{3.91}$$

where

$$I_-(z) = \int_z^\infty \frac{dz}{\cosh^2(z/L)\sinh^{(d-3)}(z/L)}, \tag{3.92}$$

Define a new coordinate R by

$$r = R\cosh^2(z/L). \tag{3.93}$$

The metric becomes

$$ds^2 = \cosh^2(z/L)\left[-(F(v)I_-(z) + G(v) + R^2/L^2)dv^2 + 2dvdR\right] + dz^2 + L^2\sinh^2(z/L)d\Omega^2. \tag{3.94}$$

Now the transformations $R \to R - f(v)$ followed by $v \to V(v)$, $R \to R/\partial_v V$ can be used to eliminate $G(v)$, giving the final form of the solution:

$$ds^2 = \cosh^2(z/L)\left[-(F(v)I_-(z) + R^2/L^2)dv^2 + 2dvdR\right] + dz^2 + L^2\sinh^2(z/L)d\Omega^2. \quad (3.95)$$

A similar analysis for $\Lambda > 0$ (or $L \to iL$) gives

$$ds^2 = \cos^2(z/L)\left[-(F(v)I_+(z) - R^2/L^2)dv^2 + 2dvdR\right] + dz^2 + L^2\sin^2(z/L)d\Omega^2, \quad (3.96)$$

where

$$I_+(z) = \int \frac{dz}{\cos^2(z/L)\sin^{d-3}(z/L)}. \quad (3.97)$$

These solutions are type O, i.e., isometric to (anti-)de Sitter space, if, and only if, $F(v) \equiv 0$. If $F(v)$ is not identically zero then these metrics are the general axisymmetric type N Kundt solutions for $\Lambda \neq 0$, $d > 4$.[10] It seems natural to interpret the type N solutions as describing gravitational waves propagating in a type O background.

Type D

Now consider type D solutions, for which there exists a second multiple WAND n'. Note that n' need not coincide with the null basis vector n defined above. If n' were not axisymmetric then the solution would be encompassed by the analysis of section 3.4. However, the results of that section reveal that, in this case, *both* multiple WANDs would fail to be axisymmetric, which is not the case here. Hence we can assume that n' is axisymmetric. The most general form it can take is

$$n' = n - \frac{1}{2}a(v,r,z)^2\,\ell + a(v,r,z)\,m_2 \quad (3.98)$$

where $a(v,r,z)$ is arbitrary. Let us change to a new null frame (ℓ', n', m_i'), with

$$\ell' = \ell, \qquad m_2' = m_2 - a\ell, \qquad m_\alpha' = m_\alpha. \quad (3.99)$$

Note that $a \equiv 0$ corresponds to the frame used above. The fact that ℓ is a multiple WAND guarantees that Weyl components of boost weight 0 are the same in the two frames. The negative boost weight components in the new frame are related to the components in the

[10] For $d = 4$, these solutions are a special case of more general type N Kundt solutions discussed in Ref. [105]. For $d > 4$, they are a special case of more general (non-axisymmetric) solutions given in eqs (66) and (68) of Ref. [106].

old frame by

$$\begin{aligned} C'_{1\alpha\beta 2} &= C_{1\alpha\beta 2} - aC_{2\alpha 2\beta} + aC_{0\alpha 1\beta}, \\ C'_{1\alpha 1\beta} &= C_{1\alpha 1\beta} - 2aC_{1\alpha\beta 2} - a^2 C_{0\alpha 1\beta} + a^2 C_{2\alpha 2\beta}. \end{aligned} \tag{3.100}$$

Type D solutions are those for which Weyl tensor components of boost weight -2 and -1 (in the new frame) vanish, giving

$$C_1 \partial_v E_0 + 2\partial_v \partial_z E_0 + a\left(C_1 \partial_z E_0 + 2\partial_z^2 E_0\right) = 0, \tag{3.101}$$

$$C_{1\alpha 1\beta} - 2aC_{1\alpha\beta 2} - a^2 C_{0\alpha 1\beta} + a^2 C_{2\alpha 2\beta} = 0, \tag{3.102}$$

where the second equation has not been written explicitly for brevity. Note that equation (3.101) implies either $a = a(v,z)$; or $C_{2\alpha 2\beta} = C_{0\alpha 1\beta}$ and $C_{1\alpha\beta 2} = 0$. In the latter case, equation (3.102) implies $C_{1\alpha 1\beta} = 0$, and then one finds that n is a multiple WAND, i.e., one can set $a \equiv 0$. Hence, in either case, we have $a = a(v,z)$. Therefore, in equation (3.102), the only term with r dependence is that contained in $C_{1\alpha 1\beta}$, which must vanish, giving (from equation (3.81))

$$2U_2 \frac{\partial_v E_0}{E_0} + (\partial_z U_1 + 2\partial_v C_1) \frac{\partial_z E_0}{E_0} = 0. \tag{3.103}$$

To simplify the analysis, assume that the spacetime is not $dS_3 \times S^{d-3}$ (which we already know is type D). The results of Ref. [65] imply that the second multiple WAND n' must be geodesic. Axisymmetry implies that the geodesic equation reduces to

$$m_2'^{\,a} n'^{b} \nabla_b n'_a = 0. \tag{3.104}$$

The LHS is linear in r so this gives two equations:

$$2a\partial_z a + 2\partial_v a + a^2 C_1 - aU_1 + \partial_z U_0 + C_1 U_0 = 0, \tag{3.105}$$

$$\partial_z U_1 + 2\partial_v C_1 - 2aU_2 = 0. \tag{3.106}$$

Proceed by simplifying equation (3.103) using equation (3.106):

$$2U_2 \left(a + \frac{\partial_v E_0}{\partial_z E_0} \right) = 0. \tag{3.107}$$

Note that $\partial_z E_0$ is not identically zero, since otherwise equation (3.63) implies that E_0 is constant, which gives $dS_3 \times S^{d-3}$. There are two cases.

Case 1. $\partial_v E_0 = -a\partial_z E_0$. Using this to eliminate $\partial_v E_0$ from equation (3.101) gives $\partial_z a = 0$, so $a = a(v)$. Now we can use a transformation of the form (3.61) to reach a gauge in which $\partial_v E_0 \equiv 0$, i.e., $a \equiv 0$. Substituting (3.64) into (3.106) then gives $\partial_v C_1 = 0$. Now equation (3.67) gives $\partial_z U_1 = 0$. Define a positive function $W(z)$ by equation (3.70). Equation (3.66) reveals that U_2 is independent of v so equation (3.74) implies (3.75) as before (using the freedom to rescale W by a constant). Equation (3.105) gives $U_0 = F(v)W(z)$. Defining the coordinate R by (3.76), the metric can be brought to the form (3.77):

$$ds^2 = W(z)\left\{-\left[F(v) + RU_1(v) - kR^2\right]dv^2 + 2dvdR\right\} + dz^2 + E_0(z)^2 d\Omega^2. \tag{3.108}$$

The transformation $R \to R - G(v)$ can be used to set $F(v) \equiv 0$, then a transformation $v \to V(v)$, $R \to R/\partial_v V$ can be used to set $U_1 \equiv 0$. The metric is then

$$ds^2 = W(z)d\Sigma^2 + dz^2 + E_0(z)^2 d\Omega^2, \tag{3.109}$$

where $d\Sigma^2$ is the metric on a 2d Lorentzian space with Ricci scalar $2k$, i.e., Minkowski or (anti-) de Sitter spacetime.

Case 2. $U_2 \equiv 0$. Equations (3.66) and (3.64) imply

$$\partial_z\left(\frac{C_1}{E_0^{d-3}}\right) = 2(d-3)\frac{\partial_z^2 E_0}{E_0^{d-2}}, \tag{3.110}$$

while equations (3.106) and (3.67) imply

$$\partial_v\left(\frac{C_1}{E_0^{d-3}}\right) = 2(d-3)\frac{\partial_v\partial_z E_0}{E_0^{d-2}} \tag{3.111}$$

The integrability condition for these equations is

$$\partial_z\left(\frac{\partial_v E_0}{\partial_z E_0}\right) = 0, \tag{3.112}$$

hence $\partial_v E_0 = h(v)\partial_z E_0$ for some function $h(v)$. A gauge transformation of the form (3.61) can be used to reach a gauge in which $\partial_v E_0 \equiv 0$, i.e., $h(v) \equiv 0$. Equation (3.111) now gives $\partial_v C_1 \equiv 0$ and (3.67) gives $\partial_z U_1 \equiv 0$. Now if $a \equiv 0$ then we are back to case 1, so assume $a \neq 0$. Then the coefficient of a in equation (3.101) must vanish. But this is the case discussed below equation (3.102), where n is a multiple WAND, so one can set $a \equiv 0$ after all, leading back to case 1.

In summary, we have shown that, for a general type D axisymmetric Kundt metric, one can find a gauge in which E_0 and C_1 are independent of v and $U_0 = U_1 = 0$. The metric can be transformed to the form (3.109).[11] Conversely, the warped product structure of (3.109) implies that any such solution is type D or O [85]. Type O corresponds to the solutions for $E_0(z)$ and $C_1(z)$ found in our discussion of type N, i.e., equations (3.84), (3.85).

3.4 Axisymmetric solutions with non-axisymmetric WAND

Consider the Kaluza-Klein bubble spacetime [99] (generalized to include Λ) obtained by analytic continuation of the Schwarzschild solution:

$$ds^2 = r^2 ds^2(dS_{d-2}) + \frac{dr^2}{U(r)} + U(r)dz^2, \qquad U(r) = 1 - \frac{m}{r^{d-3}} - \frac{\Lambda r^2}{d-1}, \tag{3.113}$$

where $m \neq 0$ is a constant, and dS_{d-2} is $(d-2)$ dimensional de Sitter space:

$$ds^2(dS_{d-2}) = -dt^2 + \cosh^2 t \, d\Omega^2. \tag{3.114}$$

This spacetime is obviously axisymmetric. It is a warped product of dS_{d-2} and R^2 and is hence type D [85]. We did not discover this spacetime above. This is because the multiple WANDs live in the dS_{d-2} directions and hence must have non-vanishing components along S^{d-3}, i.e., they are not axisymmetric. Here this is possible because the axisymmetry $SO(d-2)$ is part of the bigger $SO(1, d-2)$ de Sitter symmetry. We shall show that this symmetry enhancement is necessary for a *multiple* WAND to be non-axisymmetric.

Consider first the case in which we have a non-axisymmetric (multiple) WAND ℓ that is everywhere orthogonal to the S^{d-3} orbits of $SO(d-2)$, i.e., the only non-zero components of the WAND (in the coordinates of (3.4)) are $\ell^a = \ell^a(x, \Omega)$, where Ω refers to the coordinates on S^{d-3}. Now, since the Weyl tensor is axisymmetric, it is clear that $\ell^a(x, \Omega_0)$ is also a (multiple) WAND where Ω_0 is an arbitrary point on S^{d-3}. But this new WAND does not vary on S^{d-3}, i.e., it is axisymmetric. Hence we conclude that, if the WAND is everywhere orthogonal to S^{d-3} then there is no loss of generality in assuming that it is axisymmetric.

Assume instead that we have an axisymmetric spacetime with metric (3.4) and that a WAND ℓ is *not* orthogonal to S^{d-3} at some point. Then the same must hold in

[11] Note that the special case $dS_3 \times S^{d-3}$ can be written in the form (3.109) (with constant E_0 and $k = 1$).

a neighbourhood of that point. Consider the "unphysical" spacetime $M_3 \times S^{d-3}$ with the product metric obtained by multiplying (3.4) by $E(x)^{-2}$. We shall work with this spacetime for most of this section. Obviously ℓ is a WAND of this spacetime. By rescaling ℓ we can ensure that the projection of ℓ onto S^{d-3} is a unit vector (in our neighbourhood). Hence we can write $\ell = (e_0 + e_3)/\sqrt{2}$ where e_0 is a timelike unit vector in M_3 and e_3 a unit vector on S^{d-3}. Choose $n = (-e_0 + e_3)/\sqrt{2}$. Choose e_1 and e_2 so that $\{e_0, e_1, e_2\}$ is an orthonormal basis for M_3, and choose $e_4 \ldots e_{d-1}$ so that $\{e_3 \ldots, e_{d-1}\}$ is an orthonormal basis for S^{d-3}. Now take $\{m_i\} = \{e_1, e_2, e_4, \ldots, e_{d-1}\}$. Let $\hat{a}, \hat{b}$ take values $1, 2$ and let $\hat{\alpha}, \hat{\beta}$ take values $4 \ldots, (d-1)$ and α, β take values $3, \ldots, d-1$. By axisymmetry we have that C_{abcd} vanishes if there are an odd number of indices of the form α, β. We also have

$$C_{0\alpha 0\beta} = a\delta_{\alpha\beta}, \qquad C_{\hat{a}\alpha\hat{b}\beta} = C_{\hat{a}\hat{b}}\delta_{\alpha\beta}, \qquad C_{\alpha\beta\gamma\delta} = b\left(\delta_{\alpha\gamma}\delta_{\beta\delta} - \delta_{\alpha\delta}\delta_{\beta\gamma}\right), \tag{3.115}$$

for some quantities a, $C_{\hat{a}\hat{b}}$, b. Now the WAND condition $C_{\mu\nu\rho\sigma}\ell^\mu m_i^\nu \ell^\rho m_j^\sigma = 0$ reduces to

$$C_{\hat{a}\hat{b}} = -C_{0\hat{a}0\hat{b}}, \qquad b = -a. \tag{3.116}$$

The first equation follows from choosing $i, j = \hat{a}, \hat{b}$ in the WAND condition and the second by choosing $i, j, = \hat{\alpha}, \hat{\beta}$.

Now, since we have a product metric, C_{abcd} is fully determined by the Ricci tensor of M_3. Hence these conditions give conditions on this Ricci tensor. Using the formulae in [85] (and $R_{\alpha\beta} = (d-4)\delta_{\alpha\beta}$), we find that the Ricci tensor of M_3 must obey

$$R_{00} = 2, \qquad R_{\hat{a}\hat{b}} = \frac{1}{2}R_{\hat{c}\hat{c}}\,\delta_{\hat{a}\hat{b}}. \tag{3.117}$$

Similarly, the additional condition for ℓ to be a multiple WAND reduces to

$$R_{0\hat{a}} = 0. \tag{3.118}$$

Note that these conditions are invariant under $e_0 \to -e_0$, which implies that if ℓ is a multiple WAND then so is n. Hence the spacetime is type D or more special. From now on, we assume that ℓ is indeed a multiple WAND. Note that we can argue as we did in the second paragraph of this section to deduce that there is no loss of generality in assuming that e_0 is axisymmetric, which we shall assume henceforth.

Using capital letters $M, N, \ldots$ to denote coordinate indices in M_3, we can summarize the form of the 3d Ricci tensor as

$$R_{MN} = (\mu + 1)u_M u_N + (\mu - 1)g_{MN}, \tag{3.119}$$

where $u^M \equiv e_0^M$. This is the Ricci tensor that would arise from a solution of the 3d Einstein equations sourced by a perfect fluid with energy density μ and pressure $p \equiv 1$ (with $8\pi G_3 = 1$). The contracted Bianchi identity (or stress tensor conservation) gives

$$(\mu + 1)u \cdot \nabla u^M = 0, \tag{3.120}$$

and

$$(\mu + 1)\theta + u \cdot \nabla \mu = 0, \tag{3.121}$$

where the expansion θ is defined by $\theta = \nabla \cdot u$. The first of these equations implies that either $\mu \equiv -1$ or u^M is tangent to affinely parametrized geodesics in M_3. In the former case, we have $R_{MN} = -2g_{MN}$, which implies that M_3 is locally isometric to AdS_3 with unit radius, which implies that $M_3 \times S^{d-3}$ is conformally flat, so the physical spacetime is type O. Assume henceforth that this is not the case, so $\mu \neq -1$ and u^M is geodesic in M_3.

The Einstein equation for the physical metric $E^2 g$ is

$$\Lambda \delta^\mu_\nu = E^{-2} R^\mu_\nu + (d-2)E^{-1}\nabla^\mu \nabla_\nu E^{-1} - \frac{1}{d-2} E^{-d} \nabla^2 E^{d-2} \delta^\mu_\nu, \tag{3.122}$$

where ∇ is the covariant derivative with respect to the unphysical metric g, and indices are raised and lowered with this metric. The components tangent to the sphere give

$$\frac{1}{d-2} E^{-d} \nabla^2 E^{d-2} = (d-4)E^{-2} - \Lambda. \tag{3.123}$$

Using this, and (3.119), the components tangent to M_3 give

$$E^{-1}\left[(\mu+1)u_M u_N + (\mu - (d-3))g_{MN}\right] + (d-2)\nabla_M \nabla_N E^{-1} = 0. \tag{3.124}$$

We now act on this with ∇_P, antisymmetrize on MP, and use the fact that the Riemann tensor in 3d is determined by the Ricci tensor, which is given by (3.119). This results in the equation

$$\begin{aligned} 0 \;\; &= \;\; (\mu+1)\nabla_{[P} u_{M]} u_N - (d-3)(\mu+1)\nabla_{[P} E^{-1} g_{M]N} + E^{-1}(\nabla_{[P}\mu) u_{M]} u_N \\ &+ \;\; E^{-1}(\mu+1)B_{[MP]} u_N + E^{-1}(\mu+1)B_{N[P} u_{M]} + E^{-1}(\nabla_{[P}\mu) g_{M]N} \\ &- \;\; (d-2)(\mu+1) g_{N[M} u_{P]} u \cdot \nabla E^{-1} - (d-2)(\mu+1)\nabla_{[P} E^{-1} u_{M]} u_N \end{aligned} \tag{3.125}$$

where

$$B_{MN} = \nabla_N u_M. \tag{3.126}$$

Contracting with u^N, this equation reduces to $(\mu+1)B_{[MP]} = 0$. Since we are assuming that $\mu + 1$ is not identically zero, we must have

$$B_{[MN]} = 0, \tag{3.127}$$

i.e. $du = 0$, so u^M is hypersurface-orthogonal. Define the projector

$$h_{MN} = g_{MN} + u_M u_N \tag{3.128}$$

and now contract (3.125) with h^N_Q to get

$$\begin{aligned} 0 &= -(d-3)(\mu+1)\nabla_{[P}E^{-1}h_{M]Q} + E^{-1}(\nabla_{[P}\mu)h_{M]Q} + E^{-1}(\mu+1)\hat{B}_{Q[P}u_{M]} \\ &- (d-2)(\mu+1)h_{Q[M}u_{P]}u\cdot\nabla E^{-1}, \end{aligned} \tag{3.129}$$

where

$$\hat{B}_{MN} = h^P_M h^Q_N B_{PQ}. \tag{3.130}$$

We can define the expansion and shear of the geodesic congruence tangent to u^M in terms of the trace and traceless parts of $\hat{B}_{MN}$:

$$\hat{B}_{MN} = \frac{1}{2}\theta h_{MN} + \sigma_{MN}. \tag{3.131}$$

Contracting (3.129) with u^M gives $(\mu+1)\hat{B}_{PQ} \propto h_{PQ}$, hence the congruence is shear-free:

$$\sigma_{MN} = 0. \tag{3.132}$$

Equation (3.129) now reduces to

$$X_{[P}h_{M]Q} = 0, \tag{3.133}$$

where

$$X_P = -(d-3)(\mu+1)\nabla_P E^{-1} + E^{-1}\nabla_P\mu - \frac{1}{2}\theta E^{-1}(\mu+1)u_P - (d-2)(\mu+1)u_P u\cdot\nabla E^{-1}. \tag{3.134}$$

However, contracting (3.133) with h^P_N reveals that $X_P = 0$. Decomposing this into a part orthogonal to u^M and a part parallel to u^M gives

$$h_M^N \nabla_N \left(E^{d-3}(\mu + 1) \right) = 0, \tag{3.135}$$

$$\theta = -2E^{-1} u \cdot \nabla E \tag{3.136}$$

where (3.121) was used to simplify the second equation.

Let Σ_0 be a surface orthogonal to u^M (recall that u^M is hypersurface orthogonal), let x^i be coordinates on Σ_0. Assign coordinates (T, x^i) to the point proper time T along the geodesic tangent to u^M starting at the point on Σ_0 with coordinates x^i. In this chart, the metric is

$$ds^2 = -dT^2 + h_{ij}(T, x) dx^i dx^j, \tag{3.137}$$

and $u = \partial/\partial T$. From the definition of $\hat{B}_{MN}$ and using the fact that the rotation and shear of the geodesics vanish, and equation (3.136) we deduce that

$$h_{ij}(T, x) = E^{-2} H_{ij}(x), \tag{3.138}$$

for some 2-metric H_{ij} independent of T. Eliminating θ between equations (3.121) and (3.136) gives $(\mu + 1) = f(x)E^2$ for some (non-zero) function f. Substituting this into equation (3.135) and integrating gives

$$E^{d-1} = \frac{g(T)}{f(x)}, \tag{3.139}$$

for some function $g(T)$. Now contracting (3.124) with $u^M u^N$ gives

$$\partial_T^2 E^{-1} + E^{-1} = 0. \tag{3.140}$$

Using (3.139) and the freedom to shift T by a constant we can solve to obtain

$$E^{-1} = r(x)^{-1} \sin T, \tag{3.141}$$

for some non-zero function $r(x)$.

Putting everything together, the physical metric is

$$ds^2 = r(x)^2 \left(\frac{-dT^2 + d\Omega^2}{\sin^2 T} \right) + H_{ij}(x) dx^i dx^j. \tag{3.142}$$

The metric in brackets is the metric of $(d-2)$-dimensional de Sitter space. The full metric is invariant under the de Sitter isometry group. Hence if we Wick rotate to Euclidean signature then obtain a spherically symmetric spacetime so we can apply Birkhoff's theorem to deduce that the above metric must be either the Kaluza-Klein bubble spacetime (3.113), or (if $r(x)$ is constant and $\Lambda > 0$) $dS_{d-2} \times S^2$.

3.A Curvature tensors

In this Appendix, we record the non-zero components of the Riemann tensor of the metric (3.28) describing an axisymmetric spacetime with an axisymmetric geodesic WAND, using the null basis (3.26) (with $B \equiv 0$)

$$R_{0101} = \frac{1}{4}\left(\frac{(\partial_r C)^2}{D^2} + 2\partial_r^2 U\right), \tag{3.143}$$

$$R_{0102} = -\frac{1}{2D}\left(\partial_r^2 C - \frac{\partial_r D}{D}\partial_r C\right), \tag{3.144}$$

$$R_{0202} = -\frac{\partial_r^2 D}{D}, \tag{3.145}$$

$$R_{0\alpha0\beta} = -\delta_{\alpha\beta}\frac{\partial_r^2 E}{E}, \tag{3.146}$$

$$\begin{aligned} R_{1012} = -\frac{1}{4D^2}&\left[2CD\partial_r^2 U - DU\partial_r^2 C - 2D\partial_r\partial_z U - 2D\partial_v\partial_r C - 2(C\partial_r U - \partial_z U)\partial_r D\right. \\ &\left.+4\partial_v C\partial_r D - 2\partial_r C\partial_v D + U\partial_r C\partial_r D\right], \end{aligned} \tag{3.147}$$

$$\begin{aligned} R_{1212} = \frac{1}{4D^3}&\left[2D(C^2\partial_r^2 U + \partial_z^2 U) - 2CDU\partial_r^2 C - D^2(4\partial_v^2 D + U^2\partial_r^2 D) - 4CD\partial_r\partial_z U\right. \\ &- 4D^2 U\partial_v\partial_r D - 2D(2C\partial_v\partial_r C - 2\partial_v\partial_z C - U\partial_r\partial_z C) - 2C^2\partial_r U\partial_r D \\ &+ 2CU\partial_r C\partial_r D - 2\partial_z U\partial_z D - 2D^2(\partial_v U\partial_r D - \partial_r U\partial_v D) \\ &-2D(\partial_r U\partial_z C - \partial_z U\partial_r C) + 2C(\partial_r U\partial_z D + \partial_z U\partial_r D) \\ &\left.+4\partial_v C(C\partial_r D - \partial_z D) - 2U\partial_r C\partial_z D\right], \end{aligned} \tag{3.148}$$

$$\begin{aligned} R_{1\alpha1\beta} = \frac{\delta_{\alpha\beta}}{4D^2 E}&\left[-D^2(4\partial_v^2 E + U^2\partial_r^2 E + 4U\partial_v\partial_r E) + 2C^2\partial_r U\partial_r E - 2CU\partial_r C\partial_r E\right. \\ &+ 2\partial_z U\partial_z E - 2D^2(\partial_v U\partial_r E - \partial_r U\partial_v E) + 4\partial_v C(\partial_z E - C\partial_r E) \\ &\left.+2(U\partial_r C - C\partial_r U)\partial_z E - 2C\partial_z U\partial_r E\right], \end{aligned} \tag{3.149}$$

$$R_{2021} = \frac{1}{4D^3}\left[-2D(C\partial_r^2 C + DU\partial_r^2 D) - 4D^2\partial_v\partial_r D + 2D\partial_r\partial_z C \right.$$
$$\left. -D(\partial_r C)^2 + 2\partial_r C(C\partial_r D - \partial_z D) - 2D^2\partial_r U\partial_r D\right], \tag{3.150}$$

$$R_{2\alpha 2\beta} = \frac{\delta_{\alpha\beta}}{D^3 E}\left[-D(C^2\partial_r^2 E + \partial_z^2 E) + 2CD\partial_r\partial_z E - CD\partial_r C\partial_r E + (C^2 - D^2 U)\partial_r D\partial_r E \right.$$
$$+\partial_z D\partial_z E + (D\partial_z C - C\partial_z D)\partial_r E$$
$$\left. -D^2(\partial_v D\partial_r E + \partial_r D\partial_v E) - C\partial_r D\partial_z E\right], \tag{3.151}$$

$$R_{\alpha 0\beta 1} = -\frac{\delta_{\alpha\beta}}{2D^2 E}\left[D^2(U\partial_r^2 E + 2\partial_v\partial_r E + \partial_r U\partial_r E) + C\partial_r C\partial_r E - \partial_r C\partial_z E\right], \tag{3.152}$$

$$R_{\alpha 0\beta 2} = \frac{\delta_{\alpha\beta}}{2D^2 E}\left[2D(\partial_r^2 EC - \partial_r\partial_z E) + (D\partial_r C - 2C\partial_r D)\partial_r E + 2\partial_r D\partial_z E\right], \tag{3.153}$$

$$R_{\alpha 1\beta 2} = \frac{\delta_{\alpha\beta}}{4D^2 E}\left[2CDU\partial_r^2 E + 4CD\partial_v\partial_r E - 4D\partial_v\partial_z E - 2DU\partial_r\partial_z E - 2D\partial_r C\partial_v E \right.$$
$$+(2CD\partial_r U - 2D\partial_z U - DU\partial_r C - 4C\partial_v D - 2CU\partial_r D)\partial_r E$$
$$\left. +2(2\partial_v D + U\partial_r D)\partial_z E\right], \tag{3.154}$$

$$R_{\alpha\beta\gamma\delta} = \frac{2\delta_{[\alpha|\gamma|}\delta_{\beta]\delta}}{D^2 E^2}\left[D^2 - (C^2 + D^2 U)(\partial_r E)^2 - (\partial_z E)^2 - 2\partial_r E(D^2\partial_v E - C\partial_z E)\right]. \tag{3.155}$$

The non-zero components of the Ricci tensor are

$$R_{00} = -\frac{\partial_r^2 D}{D} - (d-3)\frac{\partial_r^2 E}{E}, \tag{3.156}$$

$$\begin{aligned} R_{01} = &-\frac{1}{2D^3}\left[D^3\partial_r^2 U + CD\partial_r^2 C + D^2 U\partial_r^2 D + 2D^2\partial_v\partial_r D - D\partial_r\partial_z C\right.\\ &\left.+D(\partial_r C)^2 - \partial_r C(C\partial_r D - \partial_z D) + D^2\partial_r U\partial_r D\right]\\ &-\frac{(d-3)}{2D^2 E}\left[D^2 U\partial_r^2 E + 2D^2\partial_v\partial_r E + \partial_r C(C\partial_r E - \partial_z E) + D^2\partial_r U\partial_r E\right], \end{aligned} \tag{3.157}$$

$$\begin{aligned} R_{02} = \frac{1}{2D}&\left[\partial_r^2 C - \left(\frac{\partial_r D}{D} - (d-3)\frac{\partial_r E}{E}\right)\partial_r C + 2(d-3)\left(\frac{\partial_r^2 E}{E} - \frac{\partial_r D\partial_r E}{DE}\right)C\right.\\ &\left.+2(d-3)\left(\frac{\partial_r D\partial_z E}{DE} - \frac{\partial_r\partial_z E}{E}\right)\right], \end{aligned} \tag{3.158}$$

$$\begin{aligned} R_{11} = &\frac{1}{4D^3}\left[2D(C^2\partial_r^2 U + \partial_z^2 U) - 2CDU\partial_r^2 C - D^2(4\partial_v^2 D + U^2\partial_r^2 D) - 4CD\partial_r\partial_z U\right.\\ &- 4D^2 U\partial_v\partial_r D - 2D(2C\partial_v\partial_r C - 2\partial_v\partial_z C - U\partial_r\partial_z C) - 2C^2\partial_r U\partial_r D\\ &+ 2CU\partial_r C\partial_r D - 2\partial_z U\partial_z D - 2D^2(\partial_v U\partial_r D - \partial_r U\partial_v D)\\ &- 2D(\partial_r U\partial_z C - \partial_z U\partial_r C) + 2C(\partial_r U\partial_z D + \partial_z U\partial_r D)\\ &\left.+4\partial_v C(C\partial_r D - \partial_z D) - 2U\partial_r C\partial_z D\right]\\ &+\frac{(d-3)}{4D^2 E}\left[-D^2(4\partial_v^2 E + U^2\partial_r^2 E + 4U\partial_v\partial_r E) + 2C^2\partial_r U\partial_r E - 2CU\partial_r C\partial_r E\right.\\ &+ 2\partial_z U\partial_z E - 2D^2(\partial_v U\partial_r E - \partial_r U\partial_v E) + 4\partial_v C(\partial_z E - C\partial_r E)\\ &\left.+2(U\partial_r C - C\partial_r U)\partial_z E - 2C\partial_z U\partial_r E\right], \end{aligned} \tag{3.159}$$

$$\begin{aligned} R_{12} = &\frac{1}{4D^2}\left[2CD\partial_r^2 U - DU\partial_r^2 C - 2D\partial_r\partial_z U - 2D\partial_v\partial_r C - 2(C\partial_r U - \partial_z U)\partial_r D\right.\\ &\left.+4\partial_v C\partial_r D - 2\partial_r C\partial_v D + U\partial_r C\partial_r D\right]\\ &+\frac{(d-3)}{4D^2 E}\left[2CDU\partial_r^2 E + 4CD\partial_v\partial_r E - 4D\partial_v\partial_z E - 2DU\partial_r\partial_z E - 2D\partial_r C\partial_v E\right.\\ &+(2CD\partial_r U - 2D\partial_z U - DU\partial_r C - 4C\partial_v D - 2CU\partial_r D)\partial_r E\\ &\left.+2(2\partial_v D + U\partial_r D)\partial_z E\right], \end{aligned} \tag{3.160}$$

$$
\begin{aligned}
R_{22} = \frac{1}{2D^3} & \left[-2D(C\partial_r^2 C + DU\partial_r^2 D) - 4D^2\partial_v\partial_r D + 2D\partial_r\partial_z C \right. \\
& \left. -D(\partial_r C)^2 + 2\partial_r C(C\partial_r D - \partial_z D) - 2D^2\partial_r U\partial_r D\right] \\
+ \frac{(d-3)}{D^3 E} & \left[-D(C^2\partial_r^2 E + \partial_z^2 E) + 2CD\partial_r\partial_z E - CD\partial_r C\partial_r E + (C^2 - D^2 U)\partial_r D\partial_r E \right. \\
& \left. +\partial_z D\partial_z E + (D\partial_z C - C\partial_z D)\partial_r E - D^2(\partial_v D\partial_r E + \partial_r D\partial_v E) - C\partial_r D\partial_z E\right], \qquad (3.161)
\end{aligned}
$$

$$
\begin{aligned}
R_{\alpha\beta} = \delta_{\alpha\beta}\Bigg(& \frac{1}{D^3 E}\ \left[-D(C^2 + D^2 U)\partial_r^2 E - D\partial_z^2 E - 2D^3\partial_v\partial_r E + 2CD\partial_r\partial_z E \right. \\
& - D^3\partial_r U\partial_r E - 2CD\partial_r C\partial_r E + (C^2 - D^2 U)\partial_r D\partial_r E + \partial_z D\partial_z E \\
& - D^2(\partial_v D\partial_r E + \partial_r D\partial_v E) + D(\partial_r C\partial_z E + \partial_z C\partial_r E) \\
& \left. -C(\partial_r D\partial_z E + \partial_z D\partial_r E)\right] \\
& + \frac{(d-4)}{D^2 E^2}\left[D^2 - (C^2 + D^2 U)(\partial_r E)^2 - (\partial_z E)^2 - 2\partial_r E(D^2\partial_v E - C\partial_z E)\right]\Bigg). \qquad (3.162)
\end{aligned}
$$

Chapter 4

Perturbation of solutions in higher dimensions

4.1 Introduction

Soon after the discovery, by Kerr, of a solution to the vacuum Einstein equation representing an isolated rotating black hole [22], the status of its classical stability spawned an area of active research in general relativity. Significant progress in this direction was made by Teukolsky, who realised that in the case of algebraically special solutions [34, 35], of which the Kerr solution is an example, one is able to derive a decoupled equation, satisfied by Weyl scalar Ψ_0, from the original perturbation equation [31, 32]. The validity of a separability ansatz, which is related to the existence of hidden symmetries [41, 40], allowed Press and Teukolsky to use the decoupled equation to provide strong evidence for the linear (mode) stability of the Kerr solution under non-algebraically special perturbations [29]. The mode stability of the Kerr solution has been demonstrated by Whiting [30].

The existence of a decoupled equation was later shown to be related to the existence of gauge invariant quantities [107]. Given that the Weyl scalar Ψ_0, which solves the decoupled equation, is gauge invariant and has the same number of degrees of freedom as the perturbed metric, *viz.* 2, one may suspect that Ψ_0 encodes all information regarding the perturbation, i.e. a solution of the perturbation equation can be constructed given the existence of a decoupled equation. This was shown to be true by Kegeles and Cohen [108, 109] and Chrzanowski [110]. They outlined a constructive procedure (the Hertz map) for finding solutions of perturbation equations for an algebraically special background given some technical assumptions. A very short and elegant proof of these statements was provided by Wald some time later [111].

The study of higher dimensional gravity, particularly its black hole solutions, has become an active area of research in recent years. One of the most natural questions that one can ask is whether known black holes solutions, such as the Myers-Perry [48], or the black ring [50] solutions are classically stable. The stability of the Schwarzschild-Tangherlini solution, which can be thought of as belonging to the Myers-Perry family has been demonstrated [47].

However, it has been known for some time, by analogy with Gregory-Laflamme (G-L) type instabilities [55], that one would expect instabilities to occur in certain other regimes of the Myers-Perry and black ring families. For example, one would expect thin black rings to suffer from the same type of G-L instability that occurs for a black string [50]. Or, for ultraspinning ($d \geq 6$) Myers-Perry black holes to suffer from the same kind of G-L type instability that one finds for p-branes [56]. There has been a great deal of recent progress in using numerical methods to tackle and confirm these conjectures, at least, as far as the Myers-Perry solution is concerned [57].

Nevertheless, experience from 4d GR suggests that there may be a more simple framework (*à la* Teukolsky) in which the stability of higher dimensional black hole solutions can be addressed. The motivation for such a consideration would not only be that such a framework would facilitate a much simpler study of perturbations of regimes currently under investigation using numerical methods but that it may allow a study of regimes that are currently inaccessible to numerical investigations.

The higher dimensional generalisation of the Teukolsky decoupled equation was first considered in Ref. [68]. In higher dimensions, one can construct the analogue of Ψ_0 by choosing a null frame with null vectors ℓ and n such that $\ell \cdot n = 1$ and complete this frame with $d-2$ orthonormal spacelike vectors $m_{(i)}$ $(i = 2, \ldots, d-1)$ that are orthogonal to ℓ and n. Now, we define the higher dimensional generalisation of Ψ_0 to be $\Omega_{ij} = \ell^a {m_{(i)}}^b \ell^c {m_{(j)}}^d C_{abcd}$.[1] Furthermore, for algebraically special solutions the perturbed value of Ω_{ij} is gauge invariant (under infinitesimal diffeomorphisms and frame transformations) [68]. Thus, as the higher dimensional generalisation of Ψ_0, Ω_{ij} is the most natural candidate to consider decoupling in higher dimensions.

By studying the conditions required for Ω_{ij} to decouple, it was found that in higher dimensions, it is not sufficient for the background solution to be algebraically special. In

[1] Ω_{ij} is the higher dimensional generalisation of Ψ_0 in the sense that they are both the boost weight +2 components of the Weyl tensor.

addition, there must exist a geodesic null congruence with vanishing expansion, shear and twist. Equivalently, the solution must be Kundt.

With the above motivation in mind, the strong restrictions found in Ref. [68], which a background must satisfy for Ω_{ij} to decouple, can be thought of as being disappointing. However, this result can still be used to study perturbations of a particular class of higher dimensional black holes. This is because the near-horizon geometry (NHG) of extreme black hole solutions are known to be Kundt. This fact was used by Durkee and Reall to find instabilities of the NHG of cohomogeneity-1 Myers-Perry solutions [69]. A conjecture relating instabilities of the NHG and the full solution for perturbations preserving certain symmetries was then used to predict an instability of extremal and near-extremal cohomogeneity-1 Myers-Perry solutions in seven or more dimensions [69]. These predictions were later confirmed in Ref. [112].

Even so, one would like a decoupled equation that allows the study of a greater class of solutions within the family of the Myers-Perry solution. While, it is true that Ω_{ij} does not decouple it may be possible to construct some other gauge invariant quantity from the Weyl tensor that does. The first aim of this chapter is to consider this possibility for the Schwarzschild-Tangherlini background as a simple class within the Myers-Perry family. The existence of gauge invariant quantities constructed from the Weyl tensor that decouple on the Schwarzschild-Tangherlini background would give us hope of finding analogous quantities for more general solutions within the Myers-Perry family. Equivalently, the absence of such quantities for the Schwarzschild-Tangherlini solution would indicate their absence for more general solutions.

In section 4.2, we begin by considering whether other gauge invariant quantities can be added to Ω_{ij}, so that the new quantity decouples. The strategy we take is to find the obstruction to the decoupling of Ω_{ij}. Then, we consider the decoupling of this new quantity and find the obstruction to its decoupling. Once this iterative process terminates we are left with a set of gauge invariant quantities that form an obstruction to each others' decoupling. For the case of Ω_{ij}, there are three basic gauge invariant quantities including Ω_{ij} that must be considered. However, we find that a linear combination of these quantities can never decouple. Hence, there is no hope of constructing a gauge invariant quantity that decouples using Ω_{ij}.

We then consider other gauge invariant quantities constructed from components of the Weyl tensor and find that a particular set of components $\Phi^{\mathrm{A}}_{ij} \equiv \ell^a n^b m_{([i)}{}^c m_{(j])}{}^d C_{abcd}$ do decouple. Metric perturbations of the Schwarzschild-Tangherlini solution have been studied by Ishibashi and Kodama [47] using a gauge invariant analysis that is analogous to the 4d gauge invariant approach developed by Moncrief [113]. This approach uses the spherical symmetry of the solution to construct gauge invariant quantities from the metric perturbations. These quantities can be classified into tensor, vector and scalar modes depending on how they behave on the sphere. We argue that Φ^{A}_{ij} is related to the vector modes of Ishibashi and Kodama.

This is completely analogous to what is known to happen in four dimensions. On the Schwarzschild background, the perturbed value of the imaginary part of the Weyl scalar Ψ_2, which is gauge invariant and has Φ^{A}_{ij} as its higher dimensional generalisation, satisfies a decoupled equation. Furthermore, this decoupled equation is equivalent to the Regge-Wheeler equation [27] describing vector mode (or axial) perturbations of the Schwarzschild background [114].

Our results decrease the likelihood of finding a gauge invariant quantity constructed from Weyl tensor components that decouples on the more general Myers-Perry background. This is because Φ^{A}_{ij} is generally non-zero for the Myers-Perry solution, so its perturbation is not gauge invariant. It is not inconceivable that there exists a gauge invariant quantity constructed from Φ^{A}_{ij} that decouples. However, we have not been able to come up with a suitable candidate.

The second aim of the chapter is to construct the Hertz potential map for constructing solutions of the perturbation equations of higher dimensional Kundt backgrounds in the manner proposed by Wald [111].

In section 4.3, we begin by presenting Wald's argument for constructing solutions of a general perturbation equation given a decoupled equation. It was shown in [68] that a decoupled equation for electromagnetic and gravitational perturbations exist for Kundt backgrounds. Thus, we apply this method to construct the Hertz map for electromagnetic (section 4.3.1) and gravitational (section 4.3.2) perturbations of Kundt backgrounds in higher dimensions.

As stated above, the NHG of extremal black hole solutions is Kundt. In particular, the decoupled equation for NHG of cohomogeneity-1 Myers-Perry solutions was studied in [69]. As an application, we use the results of [69] to determine the asymptotic behaviour of metric perturbations of the NHG of the 5d cohomogeneity-1 Myers-Perry black hole. This is, in part, motivated by a recent paper concerning the entropy counting of such black hole solutions [78]. We find that there exist modes that violate the boundary conditions required in Ref. [78]. Thus, at higher orders, it will not be possible to deform the NHG.

4.2 Results for Schwarzschild-Tangherlini solution

In this chapter, we define the gauge invariance of a perturbed quantity following Stewart and Walker [107]. Given a spacetime scalar X, we can decompose this into its background and perturbed value, i.e. to first order $X = X^{(0)} + X^{(1)}$. We are interested in how the perturbed value of X, $X^{(1)}$ changes under infinitesimal coordinate and frame transformations. We say that $X^{(1)}$ is gauge invariant if and only if it remains invariant under coordinate and frame transformations. This condition places constraints on the background value of X. For example, under a coordinate transformation parametrised by vector field ξ, $X^{(1)} \to X^{(1)} + \xi \cdot \partial X^{(0)}$. Hence, for $X^{(1)}$ to remain invariant under infinitesimal coordinate transformations $X^{(0)}$ must be constant. The invariance of the perturbed quantity $X^{(1)}$ under infinitesimal frame transformations is verified by performing infinitesimal frame transformations given in equations (B.2)—(B.5) of appendix B. In Ref. [68], it was shown that the perturbed value of Ω_{ij} is gauge invariant if and only if the background solution is algebraically special. Similar arguments to those presented in Ref. [68] can be used to demonstrate the gauge invariance of other Weyl components given a background. For brevity, we will suppress superscript labels on quantities indicating whether they are background or perturbed values since this will be apparent from the context.

In Ref. [68], Durkee and Reall consider the status of Teukolsky's decoupling result in higher dimensions. They do this by taking an algebraically special background, in which the perturbed value of Ω_{ij} is gauge invariant. Using the Bianchi identity equations of the higher dimensional GHP formalism[2] [67], they derive a second order coupled differential equation, where all second order derivatives act on the gauge invariant quantity Ω_{ij}. For Ω_{ij} to decouple, all other terms involving other components of the Weyl tensor must either be re-expressed in terms of terms involving Ω_{ij} or vanish. Once all the Newman-Penrose and Bianchi equations have been exhausted, they conclude that for Ω_{ij} to decouple, the background must be Kundt. This means that the background admits a null geodesic congruence with vanishing expansion, shear and twist.

Here, we shall take a different approach. Instead of forcing a particular gauge invariant quantity (in their case Ω_{ij}) to decouple and finding restrictions on the background, we fix the background (in this case the Schwarzschild-Tangherlini solution) and try to find gauge invariant quantities constructed from Weyl tensor components that decouple on this background.

Thus, we will use the higher dimensional GHP formalism developed in Ref. [67] to

[2] See appendix B for a review of the higher dimensional GHP formalism.

address the possibility that there exists a gauge invariant quantity constructed from Weyl tensor components that decouples on the Schwarzschild-Tangherlini background.[3]

We begin by considering the derivation of the decoupling result in Ref. [68] for the Schwarzschild-Tangherlini background. Up to equation (5.31) in [68], the only assumption made is that the background admits a null geodesic congruence with vanishing shear and rotation. This is satisfied by the Schwarzschild-Tangherlini solution. Removing other second order terms from equation (5.31) gives

$$\left(2\text{þ}'\text{þ} + \eth_k\eth_k + \rho'\text{þ} + \tfrac{d+6}{d-2}\rho\text{þ}' + \tfrac{2}{d-2}\rho\rho' + \tfrac{4(d-2)}{d-3}\Phi\right)\Omega_{ij} + \tfrac{2\rho^2}{d-2}\left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) = 0. \quad (4.1)$$

Thus, the trace-free part of Φ^{S}_{ij} is the obstruction for the decoupling of Ω_{ij}.[4] We know that this vanishes on the background (see appendix 4.A). Hence, it is invariant under coordinate transformations. By applying infinitesimal frame transformations (B.2)–(B.5) and using the fact that the background solution is type D, we can simply verify that $\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij}$ is a gauge invariant quantity.

We proceed by finding the obstruction to the decoupling of the gauge invariant quantity

$$\rho^2\left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right).$$

It can be shown that this quantity satisfies[5]

$$\left(2\text{þ}'\text{þ} + \eth_k\eth_k + \tfrac{d+6}{d-2}(\rho'\text{þ} + \rho\text{þ}') + \tfrac{4\Phi}{\rho}\text{þ} + \tfrac{2(3d+5)}{(d-2)^2}\rho\rho' - \tfrac{4(3d-8)}{(d-2)(d-3)}\Phi\right)\left[\rho^2\left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right)\right] + \tfrac{(d-4)}{(d-2)^2}\rho^2\left(\rho'^2\Omega_{ij} + \rho^2\Omega'_{ij}\right) = 0. \quad (4.2)$$

Hence, the obstructions to the decoupling of $\rho^2\left(\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij}\right)$ are $\rho^2\rho'^2\Omega_{ij}$ and $\rho^4\Omega'_{ij}$, which can simply be shown to be gauge invariant. An equation of the form above can be derived for these quantities using equation (4.1). However, this procedure will lead to yet another new quantity to consider: $\rho^4\rho'^2\left(\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij}\right)$.

Thus, we find that this iterative process generates boost weight +2 terms of the form Ω_{ij}, $\rho^2\left(\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij}\right)$ and $\rho^4\Omega'_{ij}$ multiplied by various factors of $\rho\rho'$. Therefore, we take

$$X_{ij} = \Omega_{ij} + f\ \rho^2\left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) + g\ \rho^4\Omega'_{ij} \quad (4.3)$$

[3]See appendix 4.A for a review of the Schwarzschild-Tangherlini solution.
[4]Note that in 4d this term vanishes identically leading to the decoupling of Ψ_0.
[5]The derivation of this equation is given in appendix 4.B.

as our ansatz for the decoupled gauge invariant quantity, where f and g are GHP scalars of boost and spin weight 0.

For X_{ij} to decouple, there must exist f and g such that

$$\left(2þ'þ + ð_k ð_k\right) X_{ij} = \mathcal{O} X_{ij}, \tag{4.4}$$

where $\mathcal{O}$ is some first order differential operator.

The only boost and spin weight 0 GHP scalars that do not vanish on the background are $\varsigma = \rho\rho'$ and Φ. Hence,

$$f = f(\varsigma, \Phi), \quad g = g(\varsigma, \Phi). \tag{4.5}$$

In particular, equations (NP3), (NP3)$'$ and (B5) evaluated on the background imply that on the background

$$ð_i f = 0, \quad ð_i g = 0. \tag{4.6}$$

Using equation (4.1), its primed version for $\rho^4 \Omega'_{ij}$ and eq. (4.2), we find

$$\begin{aligned} \left(2þ'þ + ð_k ð_k\right) X_{ij} = &\left\{\rho' þ + \tfrac{d+6}{d-2} þ' + \dots\right\} \Omega_{ij} \\ &+ \left\{\left(-2\tfrac{þ' f}{f} + \tfrac{d+6}{d-2}\rho' + \tfrac{4\Phi}{\rho}\right) þ + \left(-2\tfrac{þ f}{f} + \tfrac{d+6}{d-2}\rho\right) þ' + \dots\right\} \left[f\rho^2 \left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right)\right] \\ &+ \left\{\left(-2\tfrac{þ' g}{g} + \tfrac{d+14}{d-2}\rho' + \tfrac{8\Phi}{\rho}\right) þ + \left(-2\tfrac{þ g}{g} + \tfrac{d+6}{d-2}\rho\right) þ' + \dots\right\} \left(g\rho^4 \Omega'_{ij}\right), \end{aligned} \tag{4.7}$$

where the ellipses indicate other scalar terms that have been omitted for brevity. Necessarily, for X_{ij} to decouple, the coefficients of the derivative operators in the 3 terms on the right hand side must be equal, i.e.

$$\begin{aligned} \rho' = -2\tfrac{þ' f}{f} + \tfrac{d+6}{d-2}\rho' + \tfrac{4\Phi}{\rho} &= -2\tfrac{þ' g}{g} + \tfrac{d+14}{d-2}\rho' + \tfrac{8\Phi}{\rho}, \\ \tfrac{d+6}{d-2}\rho = -2\tfrac{þ f}{f} + \tfrac{d+6}{d-2}\rho &= -2\tfrac{þ g}{g} + \tfrac{d+6}{d-2}\rho. \end{aligned} \tag{4.8}$$

These imply that on the background

$$þ f = 0, \quad þ g = 0, \quad þ' f = 2\left(\tfrac{2}{d-2}\rho' + \tfrac{\Phi}{\rho}\right) f, \quad þ' g = 4\left(\tfrac{2}{d-2}\rho' + \tfrac{\Phi}{\rho}\right) g. \tag{4.9}$$

Equation (B2) evaluated on the background implies that

$$þ'\Phi = -\tfrac{d-1}{d-2}\rho\Phi. \tag{4.10}$$

This, along with the equations above, can be used to show that

$$[\text{þ}', \text{þ}]f = \left(\tfrac{4}{(d-2)^2}\rho\rho' + \tfrac{2(d-1)}{d-2}\Phi - \tfrac{1}{d-2}\right) f, \tag{4.11}$$

which is non-vanishing on the background. However, considering commutator equation (C1) on f, on the background, gives that this should vanish. Thus, we encounter a contradiction. We would arrive also at a contradiction had we considered a commutator on g.

Therefore, there do not exist scalars f and g such that the gauge invariant quantity X_{ij}, given by eq. (4.3), decouples.

In summary, we took Ω_{ij} as the initial ansatz for a decoupled equation. This does not work as there is another gauge invariant quantity that obstructs the decoupling of Ω_{ij}. Then, we considered the decoupling of this new quantity and found other quantities that obstruct its decoupling. Repeating this iterative process we found that all obstructions are of a form given by one of three basic gauge invariant quantities constructed from Ω_{ij}, Φ^{S}_{ij} and Ω'_{ij}. Thus, we take a gauge invariant quantity constructed from a linear combination of these three basic quantities as our new ansatz and find that such a quantity cannot decouple. Hence, a decoupled equation for the Schwarzschild-Tangherlini solution cannot be found by completing Ω_{ij} with other gauge invariant quantities.

It is clear from the above calculation that gauge invariant quantities formed from Ω_{ij} or the trace-free part of Φ^{S}_{ij} will not decouple. However, one can consider other gauge invariant quantities. For example, one can consider boost weight $+1$ components of the Weyl tensor: Ψ_{ijk}. Although Ψ_{ijk} is gauge invariant under infinitesimal coordinate transformations, since it vanishes on the background, its gauge invariance under infinitesimal tetrad transformations is not so clear. Under a null rotation about n parametrised by z_i, Ψ_i and Ψ_{ijk} transform as [67]

$$\Psi_i \mapsto \Psi_i - z_i\Phi + \ldots, \quad \Psi_{ijk} \mapsto \Psi_{ijk} + z_{[k}\Phi_{j]i} + z_l\Phi_{lijk} + \ldots, \tag{4.12}$$

where the ellipses refer to terms that vanish on the background and need not be considered. Thus, in order to be gauge invariant, the boost weight $+1$ ansatz must be of the form

$$\Psi_{ijk} + \tfrac{2}{d-3}\delta_{i[j}\Psi_{k]}, \tag{4.13}$$

which vanishes in 4d. Looking at the Bianchi equations (B1)–(B7), it is not clear how they can be manipulated to give a decoupling result for the quantity above.

Alternatively, we can use the Penrose wave equation [20]

$$g^{ef}\nabla_e\nabla_f R_{abcd} + R_{abef}R^{ef}_{cd} + 2(R_{aecf}R^{ef}_{bd} - R_{aedf}R^{ef}_{bc}) = 0 \tag{4.14}$$

to confirm that decoupling is indeed not possible. Shortly after the decoupling result of Teukolsky, it was shown by Ryan, that the Teukolsky decoupled equation can be derived by projecting the Penrose wave equation on the appropriate null tetrad component and linearising [115]. Note, that since we are going to consider this equation to first order in perturbed quantities and, furthermore, that the perturbed solution continues to satisfy the vacuum Einstein equation, we can replace the Riemann tensors in the equation above with Weyl tensors.

Contracting the Penrose wave equation (4.14) with

$$\ell^a {m_{(i)}}^b {m_{(j)}}^c {m_{(k)}}^d + \tfrac{2}{d-3}\ell^a n^b \ell^c \delta_{i[j} {m_{(k])}}^d$$

gives

$$\mathcal{O}\left(\Psi_{ijk} + \tfrac{2}{d-3}\delta_{i[j}\Psi_{k]}\right) = \tfrac{2\rho}{d-2}\left(ð_i \Phi^{\mathrm{A}}_{jk} - \tfrac{1}{d-3}\delta_{i[j}ð_{k]}\Phi\right) + \ldots, \tag{4.15}$$

where we have used the Bianchi identities to simplify the expression on the right hand side. The ellipses indicate terms that are of zero order in derivatives. The Bianchi identities cannot be used to transform the first order in derivatives obstruction on the right hand side into a terms involving $\Psi_{ijk} + \frac{2}{d-3}\delta_{i[j}\Psi_{k]}$. Thus, we conclude that this boost weight $+1$ gauge invariant quantity cannot decouple.

We will not consider gauge invariant quantities constructed from Φ_{ijkl} since it is not clear whether such a quantity would be simpler to work with than the original metric perturbation.

The only components left to consider are Φ^{A}_{ij}. It is simple to verify that Φ^{A}_{ij} is gauge invariant. We proceed by using the Bianchi equations to derive an equation in which second order derivatives act only on Φ^{A}_{ij}. As before, we throw away terms that are clearly of quadratic order or above in the perturbation expansion.

Antisymmetrising over ij in (B2) gives

$$-þ\Phi^{\mathrm{A}}_{ij} + ð_{[i}\Psi_{j]} = \tfrac{3}{d-2}\rho\Phi^{\mathrm{A}}_{ij} + \tfrac{d-1}{d-2}\Phi\rho_{[ij]}, \tag{4.16}$$

while antisymmetrising over ij in (B5) gives

$$ð_k \Phi^A_{ij} + ð_{[i}\Phi_{j]k} + þ'\Psi_{[ij]k} = -\tfrac{2}{d-2}\rho'\Psi_{[ij]k} - \tfrac{d-1}{(d-2)(d-3)}\Phi\delta_{k[i}\tau_{j]} - \tfrac{\rho}{d-2}\left(\Psi'_{[ij]k} - \delta_{k[i}\Psi'_{j]}\right). \quad (4.17)$$

Contracting over indices jl in (B3) and antisymmetrising over the remaining indices gives

$$ð_{[i}\Psi_{j]} - ð_k\Psi_{[ij]k} = \tfrac{(d-1)(d-4)}{(d-2)(d-3)}\Phi\rho_{[ij]} - \tfrac{d-4}{d-2}\rho\Phi^A_{ij}. \quad (4.18)$$

Now, consider $ð_k(4.17)\ -2þ'(4.16)$

$$(2þ'þ + ð_kð_k)\Phi^A_{ij} + þ'ð_k\Psi_{[ij]k} - 2þ'ð_{[i}\Psi_{j]} + ð_{[i}ð_{|k|}\Phi_{j]k} - [þ', ð_k]\Psi_{[ij]k} + [ð_k, ð_{[i}]\Phi_{j]k} = \dots, \quad (4.19)$$

where the ellipses refer to terms on the right hand side. All such terms have only single derivatives. In addition, commutator equations (C2) and (C3) can be used to rewrite the final two terms on the right hand side as an expression involving only single derivatives. Furthermore, equations (4.89) and (4.18) imply that

$$þ'ð_k\Psi_{[ij]k} - 2þ'ð_{[i}\Psi_{j]} + ð_{[i}ð_{|k|}\Phi_{j]k} = -[þ', ð_{[i}]\Psi_{j]} + \dots,$$

which can be converted into an expression with single derivatives using (C2)$'$. Thus, we have an equation in which all double derivatives act on Φ^A_{ij}

$$(2þ'þ + ð_kð_k)\Phi^A_{ij} = \dots.$$

On the right hand side, terms of the form $ð_k\Psi_{[ij]k}$ can be removed by using equation (4.18). Other terms can be simplified by rearranging terms such that one of the factors in a term vanishes on the background. Then, the other factor can be assumed to take its background value. Bianchi and Newman-Penrose equations on the background can be used to further simply such terms.

Having simplified terms on the right hand side of the equation above, we find that the expression simplifies greatly and Φ^A_{ij} decouples

$$\left(2þ'þ + ð_kð_k + \tfrac{d+2}{d-2}(\rho þ' + \rho' þ) + \tfrac{4(d-1)}{(d-2)^2}\rho\rho' - \tfrac{2(d-1)}{d-3}\Phi\right)\Phi^A_{ij} = 0. \quad (4.20)$$

Metric perturbations of the Schwarzschild-Tangherlini solution have been studied by Ishibashi and Kodama [47] using a gauge invariant analysis that is analogous to the 4d approach developed by Moncrief [113]. Φ^A_{ij} is a 2-form on the $(d-2)$−sphere. However, their analysis does not find any 2-form type modes on the sphere. They classify the perturbations into three types of tensorial, vector and scalar modes depending on their behaviour on the $(d-2)-$sphere, which is parametrised, here, by i type indices. The tensor modes decompose into a trace-free transverse symmetric tensor, a transverse vector and two other scalars, while the vector modes decompose into a transverse vector and a scalar.

Performing a similar decomposition of Φ^{A}_{ij} gives

$$\Phi^{\mathrm{A}}_{ij} = \Phi^{\mathrm{A}}_{T\,ij} + \eth_{[i} V_{j]}, \tag{4.21}$$

where $\Phi^{\mathrm{A}}_{T\,ij}$ is transverse, i.e.

$$\eth_i \Phi^{\mathrm{A}}_{T\,ij} = 0 \tag{4.22}$$

and V_i is some vector mode. Since transverse 2-form modes on the sphere are not found in the Ishibashi and Kodama analysis, we can only conclude that $\Phi^{\mathrm{A}}_{T\,ij}$ parametrise trivial metric perturbations. An intuitive reason for why this should be the case is that to construct (symmetric) metric perturbations from $\Phi^{\mathrm{A}}_{T\,ij}$, we will need to contract it with a $\eth_i$ derivative, which gives zero. This leaves V_i, which correspond to the vector modes of Ishibashi and Kodama.

As mentioned in the introduction, this is analogous to what is found in 4d. On the Schwarzschild background, the perturbed value of the imaginary part of the Weyl scalar Ψ_2, which is gauge invariant and has Φ^{A}_{ij} as its higher dimensional generalisation, satisfies a decoupled equation [114]. Furthermore, this decoupled equation is equivalent to the Regge-Wheeler equation [27] describing vector mode (or axial) perturbations of the Schwarzschild background [114]. The real part of the Weyl scalar Ψ_2, which has $\Phi \equiv \ell^a n^b \ell^c n^d C_{abcd}$ as its higher dimensional generalisation, is not gauge invariant. From this, one can construct a gauge invariant quantity using metric perturbations and show that this also decouples and the decoupled equation is equivalent to the Zerilli equation [28] describing scalar mode (or polar) perturbations of the Schwarzschild background [116]. Analogously, Φ is not gauge invariant on a Schwarzschild-Tangherlini background. However, we do not attempt to construct a gauge invariant quantity from this since we are only considering gauge invariant quantities constructed from the Weyl tensor. Had we done so, we would presumably find that this satisfies the scalar mode master equation of Ishibashi and Kodama [47].

In conclusion, we find that the only gauge invariant quantities constructed from Weyl tensor components that decouple on the Schwarzschild background are Φ^{A}_{ij}. These components are shown to be related to the vector modes of Ishibashi and Kodama [47].

4.3 Constructing solutions of perturbation equations

The existence of a map in 4d for constructing exact linear perturbations given the existence of a decoupled equation was demonstrated, given some technical assumptions, by Kegeles and Cohen [108, 109] and Chrzanowski [110]. However, Wald put this map on a firmer basis with his elegant proof of it [111]. In this section, we review Wald's result [111] regarding the construction of solutions of perturbation equations, given the existence of decoupled equations, emphasising the generality of his argument.

We consider as our background, a solution of the vacuum Einstein equation

$$\bar{R}_{ab} = \Lambda \bar{g}_{ab}. \tag{4.23}$$

We would like to find a solution of some linear (perturbation) equation

$$\mathcal{E}(f) = 0 \tag{4.24}$$

on this background, where f is the perturbation of some field. For example, for an electromagnetic perturbation, f is the 1-form potential A and the equation above reduces to the Maxwell equation

$$\nabla^a(\nabla_a A_b - \nabla_b A_a) = 0. \tag{4.25}$$

In the case of gravitational perturbations, f is the associated metric perturbation, usually denoted by h, while $\mathcal{E}$ is the linearised Einstein operator

$$[\mathcal{G}^{(1)}(h)]_{ab} = \nabla_c \nabla_{(a} h_{b)}{}^c - \frac{1}{2}(\nabla^2 h_{ab} + \nabla_a \nabla_b h) - \frac{1}{2}\bar{g}_{ab}\left(\nabla_c \nabla_d h^{cd} - \nabla^2 h\right) + \frac{1}{2}\Lambda\left(h\bar{g}_{ab} - d h_{ab}\right), \tag{4.26}$$

where all indices above have been raised using the background metric $\bar{g}$, ∇ is the Levi-Civita connection of $\bar{g}$ and h is the trace of h_{ab}.

Now, assume that we can construct another equation, which is satisfied by a quantity $\mathcal{T}(f)$,

$$\mathcal{O}\left(\mathcal{T}(f)\right) = 0, \tag{4.27}$$

where $\mathcal{O}$ and $\mathcal{T}$ are some linear operators. Such an equation, where $\mathcal{T}(f)$ is a more simple quantity than f, is usually referred to as a decoupled equation, in the sense that the quantity $\mathcal{T}(f)$ "decouples" from the perturbation equation. In 4d, a decoupled equation for electromagnetic and gravitational perturbations can only be derived for vacuum solutions admitting a null geodesic, shear-free congruence [31, 32]. By the Goldberg-Sachs theorem

[36], this is equivalent to the condition that the solution be algebraically special (i.e. Petrov type II or more special).

In higher dimensions, the existence of a decoupled equation for electromagnetic and gravitational perturbations places more restrictive conditions on the background solution [68]. For $d > 4$, the background solution must admit a null geodesic congruence with vanishing optics, i.e. the background solution must be Kundt. Kundt solutions are necessarily of CMPP [58] type II [88].

The existence of a decoupled equation implies the existence of a linear operator $\mathcal{S}$ such that

$$\mathcal{S}\mathcal{E} = \mathcal{O}\mathcal{T}. \tag{4.28}$$

Simply put, operator $\mathcal{S}$ describes how the decoupled equation is derived from the perturbation equation.

The key idea in Wald's derivation is the use of operator adjoints. Given a linear differential operator $\mathcal{O}$ acting on a tensor field $t \in T$ and taking it to a tensor field $u \in U$, the unique adjoint operator $\mathcal{O}^\dagger$ is defined via

$$(u, \mathcal{O}t) = (\mathcal{O}^\dagger u, t) \tag{4.29}$$

up to a total divergence, where the inner product between any tensor fields v_1 and v_2 of rank m is defined by

$$(v_1, v_2) \equiv \int {v_1}^{a_1 \dots a_m} v_{2 a_1 \dots a_m} d(Vol). \tag{4.30}$$

We proceed by taking the adjoint of the operator equivalence (4.28) and using the fact that $(\mathcal{O}_1\mathcal{O}_2)^\dagger = \mathcal{O}_2^\dagger \mathcal{O}_1^\dagger$ to get

$$\mathcal{E}^\dagger \mathcal{S}^\dagger = \mathcal{T}^\dagger \mathcal{O}^\dagger. \tag{4.31}$$

The operator equivalence above implies that given a solution ψ of

$$\mathcal{O}^\dagger \psi = 0, \tag{4.32}$$

$f = \mathcal{S}^\dagger \psi$ solves the equation

$$\mathcal{E}^\dagger (f) = 0. \tag{4.33}$$

For all cases of interest here, the operator $\mathcal{E}$ is self-adjoint, i.e. $\mathcal{E}^\dagger = \mathcal{E}$. Thus,

$$\mathcal{O}^\dagger \psi = 0 \implies \mathcal{E}(\mathcal{S}^\dagger \psi) = 0. \tag{4.34}$$

The solution ψ is called the Hertz potential. Furthermore,

$$\mathcal{O}(\mathcal{T}\mathcal{S}^\dagger \psi) = (\mathcal{O}\mathcal{T})\mathcal{S}^\dagger \psi = (\mathcal{S}\mathcal{E})\mathcal{S}^\dagger \psi = \mathcal{S}\mathcal{E}(\mathcal{S}^\dagger \psi) = 0, \tag{4.35}$$

where we have used (4.28) in the third equality and (4.34) in the final equality. Hence, $\varphi = \mathcal{T}\mathcal{S}^\dagger \psi$ solves the decoupled equation

$$\mathcal{O}\varphi = 0. \tag{4.36}$$

4.3.1 Electromagnetic perturbations of Kundt solutions

In [68], it was shown that for electromagnetic perturbations, a higher dimensional analogue of the decoupled equation found in 4d by Teukolsky [32] can only be derived for Kundt solutions. The higher dimensional decoupled equation is analogous to the Teukolsky equation in the sense that in both cases $\mathcal{T}(A)$ is the highest boost weight component of the Maxwell field $F = dA$. Hence,

$$\mathcal{T}_E(A)_i \equiv \varphi_i = \ell^a \, m_i{}^b \, (\nabla_a A_b - \nabla_b A_a). \tag{4.37}$$

The decoupled equation is

$$\mathcal{O}_E(\varphi)_i = 0, \tag{4.38}$$

where

$$\mathcal{O}_E(\varphi)_i = \left(2\text{þ}'\text{þ} + \eth_j\eth_j + \rho'\text{þ} - 4\tau_j\eth_j + \Phi - \tfrac{2d-3}{d-1}\,\Lambda\right)\varphi_i + 2(-2\tau_{[i}\eth_{j]} + \Phi^{\rm S}_{ij} + 2\Phi^{\rm A}_{ij})\varphi_j. \tag{4.39}$$

As noted before,

$$\mathcal{E}_E(A)_b = \nabla^a(\nabla_a A_b - \nabla_b A_a) = 0. \tag{4.40}$$

Operator $\mathcal{S}_E$ is found by considering the derivation of the decoupled equation in [68]. To simplify the derivation of $\mathcal{S}_E$, we assume from the beginning that the background under consideration is Kundt, i.e. $\kappa^{(0)} = \rho^{(0)}{}_{ij} = 0$.

Also, in the derivation in [68], a vector potential A is not introduced. Hence, the Bianchi identity $dF = 0$ is non-trivial. Here we let $F = dA$. Thus, of the Maxwell equations (4.4)–(4.7) in [68], only (4.4), (4.6) and their associated primed equations are non-trivial:

$$
\begin{aligned}
(4.4) &\iff -\ell^a\,\mathcal{E}_E(A)_a = 0, && (4.41)\\
(4.4)' &\iff -n^a\,\mathcal{E}_E(A)_a = 0, && (4.42)\\
(4.6) \iff (4.6)' &\iff m_i{}^a\,\mathcal{E}_E(A)_a = 0. && (4.43)
\end{aligned}
$$

The derivation begins by considering the combination $\text{þ}(4.6) + \eth_j\,[\delta_{ij}(4.4) - (4.5)]$, which results in equation (4.9) of [68][6]

$$
\begin{aligned}
0 \;=\;& (2\text{þ}'\text{þ} + \eth_j\eth_j)\varphi_i + 2[\text{þ},\text{þ}']\varphi_i - [\text{þ},\eth_j](F_{ij} + F\delta_{ij}) + [\eth_i,\eth_j]\varphi_j\\
&+\text{þ}\big(-(2\rho'_{[ij]} - \rho'\delta_{ij})\varphi_j + 2(F_{ij} + F\delta_{ij})\tau_j\big)\\
&-\eth_i\big(\tau'_j\varphi_j\big) + \eth_j\big(2\tau'_{[i}\varphi_{j]}\big),
\end{aligned}
\tag{4.44}
$$

where, as noted before, we assume $\kappa = \rho_{ij} = 0$. This is equivalent to considering

$$\text{þ}\,(m_i{}^a\,\mathcal{E}_E(A)_a) - \eth_i\,(\ell^a\,\mathcal{E}_E(A)_a)\,. \tag{4.45}$$

The next manipulation that involves equations (4.4) and (4.6) is eliminating the combination $\text{þ}(F_{ij} + F\delta_{ij})$. This is done by adding $-(2\tau_j + \tau'_j)(\delta_{ij}(4.4) - (4.5))$ to (4.44) or equivalently (4.45). Hence, we have

$$\text{þ}\,(m_i{}^a\,\mathcal{E}_E(A)_a) - \eth_i\,(\ell^a\,\mathcal{E}_E(A)_a) + (2\tau_i + \tau'_i)\,(\ell^a\,\mathcal{E}_E(A)_a)\,. \tag{4.46}$$

The rest of the derivation does not make use of the Maxwell equation, that is it involves either the Bianchi identity or the Newman-Penrose equations. Hence, operator $\mathcal{S}_E$ is given by equation (4.46)

$$\mathcal{S}_E(J)_i = \text{þ}(m_i{}^a J_a) - (\eth_i - 2\tau_i - \tau'_i)(\ell^a J_a). \tag{4.47}$$

The Hertz potential ψ_H is given by solving

$$\mathcal{O}^\dagger_E(\psi_H)_i = 0. \tag{4.48}$$

[6] We use the notation of [68] to denote components of the Maxwell field strength; that is $\varphi_i = \ell^a m_{(i)}{}^b F_{ab}$, $F = \ell^a n^b F_{ab}$, $F_{ij} = m_{(i)}{}^a m_{(j)}{}^b F_{ab}$ and $\varphi'_i = n^a m_{(i)}{}^b F_{ab}$.

We derive $\mathcal{O}_E^\dagger$ by considering the inner product

$$(\psi_i, \mathcal{O}_E(\varphi)_i)\,. \tag{4.49}$$

For the inner product to be well-defined ψ_i must be boost weight -1 since $\mathcal{O}_E(\varphi)_i$ is boost weight 1. Using equation (4.39)

$$\begin{aligned}(\psi_i, \mathcal{O}_E(\varphi)_i) = \Big(\Big[2þ^\dagger þ'^\dagger + ð_j{}^\dagger ð_j{}^\dagger + þ^\dagger \rho' + -4ð_j{}^\dagger \tau_j + \Phi - \tfrac{2d-3}{d-1}\Lambda\Big]\,\psi_i \\ + 2\left[-2ð_{[i}{}^\dagger \tau_{j]} + \Phi^{\mathrm{S}}_{ij} - 2\Phi^{\mathrm{A}}_{ij}\right]\psi_j, \varphi_i\Big)\end{aligned} \tag{4.50}$$

Using equations (4.97)–(4.101) for the adjoint of the GHP covariant derivatives, commutator (C1) and the NP equation (NP4) gives

$$\mathcal{O}_E^\dagger(\psi)_i = \left(2þ'þ + ð_j ð_j + \rho' þ + \Phi + \tfrac{\Lambda}{d-1}\right)\psi_i + 2(-2\tau_{[i}ð_{j]} + \Phi^{\mathrm{S}}_{ij} - 2\Phi^{\mathrm{A}}_{ij})\psi_j. \tag{4.51}$$

To find a solution of the electromagnetic perturbation equation (4.40), we also need to calculate the adjoint of $\mathcal{S}$. We do this by considering the inner product

$$(\psi_i, \mathcal{S}(A)_i) = \left(\left[m_i{}^a þ^\dagger + \ell^a(-ð_i{}^\dagger + 2\tau_i + \tau_i')\right]\psi_i, A_a\right) \tag{4.52}$$

where we have used (4.47). Using equations (4.97) and (4.101) gives

$$\mathcal{S}^\dagger(\psi)_a = \left[-m_{ia}\,þ + \ell_a(ð_i + \tau_i)\right]\psi_i. \tag{4.53}$$

Thus, using the results of section 4.3, we have that if $\psi_{H\,i}$ is a solution to (4.48), where $\mathcal{O}_E^\dagger$ is given in equation (4.51), then

$$\mathcal{S}_E^\dagger(\psi_H)_a = \left[-m_{ia}\,þ + \ell_a(ð_i + \tau_i)\right]\psi_{H\,i} \tag{4.54}$$

is a solution of the electromagnetic perturbation equation (4.40).

Also,

$$\mathcal{T}_E \mathcal{S}_E^\dagger(\psi_H)_i = þ^2(\psi_H)_i \tag{4.55}$$

is a solution of the decoupled equation $\mathcal{O}_E(\varphi)_i = 0$.

Consider a doubly Kundt solution. This is a solution with two null geodesic congruences with vanishing optics, i.e. we also have $\kappa_i'^{(0)} = \rho_{ij}'^{(0)} = 0$. In this case, the boost weight -1 component of the Maxwell field φ' also satisfies a decoupled equation—the prime of the decoupled equation (4.39)

$$\mathcal{O}'_E(\varphi')_i = 0, \tag{4.56}$$

where (using equation (4.39) and commutator (C1))

$$\begin{aligned}\mathcal{O}'_E(\varphi')_i = &\left(2\text{þ}'\text{þ} + \eth_j\eth_j - 2(\tau_j + \tau'_j)\eth_j + 2\tau_j\tau'_j - \Phi - \tfrac{2d-3}{d-1}\Lambda\right)\varphi'_i \\ &+2\left(-2\tau'_{[i}\eth_{j]} + \Phi^{\text{S}}_{ij} + 4\Phi^{\text{A}}_{ij} + 2\tau'_{[i}\tau_{j]}\right)\varphi'_j.\end{aligned} \tag{4.57}$$

In this case, if ψ'_{Hi} is a solution to the prime of equation (4.48), then $\mathcal{S}'^{\dagger}_{\mathcal{E}}(\psi'_H)_a$ is also a solution of the perturbation equation.

4.3.2 Gravitational perturbations of Kundt solutions

For gravitational perturbations we assume that the perturbed solution also satisfies the vacuum Einstein equation. The metric perturbation h satisfies

$$\mathcal{E}_G(h)_{ab} = 0, \tag{4.58}$$

where $\mathcal{E}_G$ is given in equation (4.26).

In [68], it was shown that as with electromagnetic perturbations an analogue of the Teukolsky decoupled equation for gravitational perturbations [32] only exists for Kundt solutions. The decoupled equation is solved by the boost weight +2 components of the perturbed Weyl tensor Ω_{ij}, which generalises the complex Weyl scalar Ψ_0 of the 4d NP formalism to higher dimensions.

$$\begin{aligned}\mathcal{T}_G(h)_{ij} \equiv \Omega_{ij} = &-\frac{1}{2}\left\{\left(\eth_{(i}\eth_{j)} - \frac{\delta_{ij}}{d-2}\eth_k\eth_k\right) - 2\left(\tau'_{(i}\eth_{j)} - \frac{\delta_{ij}}{d-2}\tau'_k\eth_k\right)\right. \\ &\left.-\left(\rho'_{ij} - \frac{\delta_{ij}}{d-2}\rho'\right)\text{þ} - 2\left(\text{þ}\rho'_{ij} - \frac{\delta_{ij}}{d-2}\text{þ}\rho'\right)\right\}(\ell^a\ell^b h_{ab}) \\ &-\frac{1}{2}\text{þ}^2\left\{\left({m_i}^a{m_j}^b - \frac{\delta_{ij}}{d-2}{m_k}^a{m_k}^b\right)h_{ab}\right\} \\ &+\left\{\text{þ}\eth_{(i} - \tau'_{(i}\text{þ} - (\text{þ}\tau'_{(i)})\right\}(\ell^a {m_{j)}}^b h_{ab}) - \frac{\delta_{ij}}{d-2}\left\{\text{þ}\eth_k - \tau'_k\text{þ} - (\text{þ}\tau'_k)\right\}(\ell^a {m_k}^b h_{ab}).\end{aligned} \tag{4.59}$$

The decoupled equation is

$$\mathcal{O}_G(\Omega)_{ij} = 0, \tag{4.60}$$

where

$$\mathcal{O}_G(\Omega)_{ij} = \left(2þ'þ + ð_kð_k + \rho'þ - 6\tau_kð_k + 4\Phi - \tfrac{2d}{d-1}\Lambda\right)\Omega_{ij} \\ +4\left(\tau_kð_{(i|} - \tau_{(i|}ð_k + \Phi^{\mathrm{S}}_{(i|k} + 4\Phi^{\mathrm{A}}_{(i|k}\right)\Omega_{k|j)} + 2\Phi_{ikjl}\,\Omega_{kl}. \tag{4.61}$$

To derive operator $\mathcal{S}_G$, we assume that the gravitational perturbation also generates a first order energy-momentum tensor. Then, $\mathcal{S}_G$ is the operator acting on the first order energy-momentum tensor in the inhomogeneous decoupled equation

$$\mathcal{O}_G(\Omega)_{ij} = 8\pi\,\mathcal{S}_G\left(T^{(1)}_{ab}\right)_{ij}. \tag{4.62}$$

Therefore, we need to rederive the decoupling result of Ref. [68] assuming a non-zero first order energy-momentum tensor. Thus, we must use the more general (*viz.* including matter) NP equations, Bianchi identities and commutator relations found in Ref. [67]. In order, to simplify the calculation, we may assume from the onset that the background is Kundt.

Going through the derivation of the decoupled equation for gravitational perturbations given in Ref. [68], except using the more general equations that include matter terms gives

$$\mathcal{S}_G\,(T)_{ij} = \tfrac{1}{d-2}\,\delta_{ij}\left(2þ'þ + ð_kð_k + \rho'þ - 6\tau_kð_k + 4\Phi\right)(\ell^a\ell^bT_{ab}) - \left(\Phi^{\mathrm{S}}_{ij} - þ\rho'_{(ij)}\right)(\ell^a\ell^bT_{ab}) \\ +2\left(þð_{(i} - (2\tau_{(i} + \tau'_{(i})þ - (þ\tau'_{(i})\right)(\ell^a m_{j)}{}^bT_{ab}) - þþ(m_i{}^am_j{}^bT_{ab}) + \tfrac{1}{d-2}\,\delta_{ij}\,þþ(g^{ab}T_{ab}). \tag{4.63}$$

The adjoints of $\mathcal{O}_G$ and $\mathcal{S}_G$ can be derived in a fashion analogous to the electromagnetic case

$$\mathcal{O}^\dagger_G(\Pi)_{ij} = \left(2þ'þ + ð_kð_k + \rho'þ + 2\tau_kð_k + 4\Phi + \tfrac{2(d-4)}{d-1}\Lambda\right)\Pi_{ij} \\ +4\left(\tau_kð_{(i|} - \tau_{(i|}ð_k + \Phi^{\mathrm{S}}_{(i|k} - 4\Phi^{\mathrm{A}}_{(i|k}\right)\Pi_{k|j)} + 2\Phi_{ikjl}\,\Pi_{kl}, \tag{4.64}$$

$$\mathcal{S}^\dagger_G\,(\Pi)_{ab} = -\ell_a\ell_b\left(\Phi^{\mathrm{S}}_{ij} + \rho'_{(ij)}þ\right)\Pi_{ij} + 2\ell_{(a}m_{|j|b)}\left(þð_i + (\tau_i + \tau'_i)þ\right)\Pi_{ij} - m_{ia}m_{jb}\,þ^2\,\Pi_{ij}. \tag{4.65}$$

Thus, given a Hertz potential Ω_H that is a solution of

$$\mathcal{O}^{\dagger}_{G}(\Omega_H)_{ij} = 0, \tag{4.66}$$

where operator $\mathcal{O}^{\dagger}_{G}$ is given in eq. (4.64), then

$$\mathcal{S}^{\dagger}_{G}\left(\Omega_H\right)_{ab} \tag{4.67}$$

where $\mathcal{S}^{\dagger}_{G}$ is given in eq. (4.65), is a solution of the gravitational perturbation equation (4.58).

Furthermore,

$$\mathcal{T}_G\mathcal{S}^{\dagger}_{G}\left(\Omega_H\right)_{ij} = \frac{1}{2}\text{þ}^4\Omega_{H\,ij} \tag{4.68}$$

is a solution of the decoupled equation (4.60).

In the case of a background that is doubly Kundt, the prime of all the above equations hold also.

For the case of doubly Kundt solutions, the operators $\mathcal{O}$, $\mathcal{O}'$, $\mathcal{O}^{\dagger}$ and $\mathcal{O}'^{\dagger}$ encode the same physical information. Therefore, there should be some connection between them. Indeed, in 4d, for type D solutions, it is known that the solution of the equations corresponding to the above operators are related by various factors of the background value of the Weyl scalar Ψ_2 (see Refs. [117, 118, 111] and references therein). It is very simple to prove such relations using the Bianchi identities. However, in higher dimensions the situation is more complicated. We have not been able to derive similar expressions relating the solutions to the various equations defined by the four operators above.

4.3.3 Asymptotic behaviour of metric perturbations of near horizon geometry of 5d cohomogeneity-1 extreme Myers-Perry solutions

In this section, we consider as an application of the Hertz potential map developed above, the asymptotic behaviour of the metric perturbation of the near horizon geometry of 5d cohomogeneity-1 extreme Myers-Perry black hole. This consideration is motivated by a recent proposal that quantum gravity on the near horizon of a class of 5d solutions of which the above solution is an example, with appropriate asymptotic fall-off conditions on the metric perturbation is equivalent to a CFT, which can be used to calculate the Bekenstein-Hawking entropy of the original solution [78]. Thus, giving a statistical counting of the black hole's degrees of freedom.

Turning off the graviphoton charge in the solution discussed in [78] gives the 5d cohomogeneity-1 extreme Myers-Perry black hole solution with near-horizon geometry

$$ds^2 = \frac{r_+^2}{4}\left\{-R^2dT^2 + \frac{dR^2}{R^2} + 2(d\psi + \cos\theta d\phi + RdT)^2 + d\Omega^2_{(2)}\right\}, \tag{4.69}$$

where r_+ is the event horizon radius and $d\Omega^2_{(2)}$ is the round metric on S^2. This (doubly Kundt) solution is studied in Ref. [69], where the decoupling result of [68] is used to predict an instability of the corresponding extreme Myers-Perry solution. Choose the following null frame

$$\begin{aligned} \ell = \frac{r_+}{2\sqrt{2}}\left(-R\,dT + \frac{dR}{R}\right), \quad n = \frac{r_+}{2\sqrt{2}}\left(R\,dT + \frac{dR}{R}\right), \\ m_2 = \frac{r_+}{\sqrt{2}}(d\psi + \cos\theta d\phi + RdT), \quad m_\alpha = \frac{r_+}{2}\hat{e}_\alpha, \end{aligned} \tag{4.70}$$

where $\alpha = 3, 4$ and $\hat{e}_\alpha$ form an orthonormal basis on S^2.

In Ref. [69], it was shown that for the geometry with metric (4.69) with basis chosen as above

$$\begin{aligned} \kappa_i = \kappa'_i = 0, \quad \rho_{ij} = \rho'_{ij} = 0, \quad \tau_i + \tau'_i = 0, \\ \Omega_{ij} = \Omega'_{ij} = 0, \quad \Psi_{ijk} = \Psi'_{ijk} = 0, \quad \Phi_{ijkl} = \hat{R}_{ijkl}, \quad \Phi^{\mathrm{A}}_{ij} = -\frac{4}{r_+^2}(m_3 \wedge m_4)_{ij}, \end{aligned} \tag{4.71}$$

where $\hat{R}_{ijkl}$ is the Riemann tensor of the three dimensional space $\mathcal{H}$ with metric

$$ds^2_{\mathcal{H}} = \frac{r_+^2}{4}\left\{2(d\psi + \cos\theta d\phi)^2 + d\Omega^2_{(2)}\right\}. \tag{4.72}$$

In order to determine the asymptotic behaviour of the metric perturbation of the near horizon geometry with metric (4.69), we must first solve the Hertz potential equation

$$\mathcal{O}^\dagger_G(\Pi)_{ij} = 0, \tag{4.73}$$

where operator $\mathcal{O}^\dagger_G$ is given in (4.64). Then,

$$h_{ab} = \frac{1}{2}\,\ell_a\ell_b(\hat{R}_{ij}\Pi_{ij}) + 2\ell_{(a}m_{|j|b)}\text{þ}\eth_i\Pi_{ij} - m_{ia}m_{jb}\,\text{þ}^2\,\Pi_{ij} \tag{4.74}$$

is a solution of the gravitational perturbation equation in the ingoing radiation gauge, where we have used eq. (4.65) and eqs. (4.71) above. Space $\mathcal{H}$ is not an Einstein solution.

Thus, the $\ell_a \ell_b$ component of h_{ab} is non-zero.

Assume the following separability ansatz for Π_{ij}

$$\Pi_{ij} = \chi(T,R) Y_{ij}(\theta,\phi,\psi). \tag{4.75}$$

Substituting this ansatz into the equation for Π_{ij}, i.e. eq. (4.66) gives[7]

$$(D^2 - q^2 - \lambda)\chi = 0, \qquad \mathcal{O}^{(2)} Y_{ij} = \lambda Y_{ij}, \tag{4.76}$$

where D is some charge covariant derivative for some AdS_2 scalar with charge q defined in Ref. [69] and $\mathcal{O}^{(2)}$ is some operator given in Ref. [69], which we do not need to know about in detail here. Hence, χ solves the equation for a massive, charged scalar in an AdS_2 background with homogeneous electric field. Such an equation has been studied by a number of authors [119, 120, 121, 69]. At large R,

$$\chi \sim R^{-\Delta_\pm}, \quad \Delta_\pm = \frac{1}{2} \pm \sqrt{\lambda + \frac{1}{4}}. \tag{4.77}$$

Hence, from the form of h_{ab} given in (4.74) we can conclude that for large R

$$h_{ab} \sim R^{\frac{1}{2} \pm \frac{1}{2}\eta} \begin{pmatrix} \mathcal{O}(1) & \underline{\mathcal{O}(\frac{1}{R^2})} & \mathcal{O}(\frac{1}{R}) & \underline{\mathcal{O}(\frac{1}{R})} \\ & \underline{\mathcal{O}(\frac{1}{R^3})} & \mathcal{O}(\frac{1}{R^3}) & \mathcal{O}(\frac{1}{R^3}) \\ & & \mathcal{O}(\frac{1}{R^2}) & \underline{\mathcal{O}(\frac{1}{R})} \\ & & & \underline{\mathcal{O}(\frac{1}{R})} \end{pmatrix}, \qquad \eta = \sqrt{1 + 4\lambda}. \tag{4.78}$$

where the columns and rows indicate the T, R, ψ and θ and ϕ (collectively labelled α) components, respectively. The same result would be found if we considered n as the WAND of choice, that is if h_{ab} were in outgoing gauge.

Comparing this with the fall-off conditions in Ref. [78], we find that the TR, $T\psi$, RR, $\psi\alpha$ and $\alpha\beta$ components are the most restrictive. These components have been underlined in the matrix above. Thus, in order to satisfy the fall-off conditions, η must be real. We must also choose the lower sign (corresponding to normalisable modes) and require that $\eta \geq 1$ or $\lambda \geq 0$. Recall that λ is the eigenvalue of operator $\mathcal{O}^{(2)}$. The spectrum of operator $\mathcal{O}^{(2)}$ was studied in appendix B of Ref. [69]. It is clear from the study of gravitational

[7] The steps involved in this computation are almost identical to that given in appendix A of Ref. [69].

scalar modes that there exists modes for which

$$\lambda = 2 + \kappa(\kappa + 1) + |m|(\kappa + \tfrac{1}{2}) - m^2/8, \tag{4.79}$$

where m is an integer and κ is a positive integer. For $\kappa = 1$, $m \geq 15$, $\lambda < 0$. Hence, there exist modes that violate the fall-off conditions. It was shown in Ref. [69] that for all axisymmetric modes ($m = 0$) $\lambda \geq 0$. Hence, all axisymmetric modes satisfy the boundary conditions.

There is a similar proposal for the entropy counting of the 4d extremal Kerr solution [122]. In Ref. [122], as in the 5d case, the metric perturbation of the NHG is assumed to satisfy a certain asymptotic form. The asymptotic behaviour of the NHG of the extremal Kerr solution has been studied in [120] and [121]. The results we find for the 5d case are the same as they found in the 4d case.

4.A Schwarzschild-Tangherlini solution

The Schwarzschild-Tangherlini black hole is an example of a higher dimensional type D solution. The Schwarzschild-Tangherlini metric in Schwarzschild coordinates is

$$ds^2 = -f(r)dt^2 + dr^2/f(r) + r^2 d\Omega^2_{(d-2)}, \qquad f(r) = 1 - \frac{\mu}{r^{d-3}}, \tag{4.80}$$

where $d\Omega^2_{(d-2)}$ is the round metric on a unit radius $(d-2)-$sphere. In ingoing Eddington-Finkelstein coordinates, the WANDs of the solution are [123]

$$\ell = \frac{\partial}{\partial r}, \quad n = \frac{\partial}{\partial v} + \tfrac{1}{2} f \frac{\partial}{\partial r}. \tag{4.81}$$

Defining $(d-2)$ orthonormal spacelike vectors

$$m_i = r\hat{e}_i \qquad (i = 2, \cdots, d), \tag{4.82}$$

completes a null frame (ℓ, n, m_i) for the solution.

Cartan's first equation of structure $de^\mu + \omega^\mu{}_\nu e^\nu = 0$ can be used to find the optical scalars associated with WANDs ℓ and n

$$\begin{gathered} L_{11} = -N_{01} = -\tfrac{1}{2}\partial_r f, \\ \kappa_i = \tau_i = 0, \quad \rho_{ij} = \frac{\rho}{d-2}\delta_{ij}, \quad \rho = (d-2)/r, \\ \kappa'_i = \tau'_i = 0, \quad \rho'_{ij} = \frac{\rho'}{d-2}\delta_{ij}, \quad \rho' = (d-2)f/(2r). \end{gathered} \tag{4.83}$$

Thus, the solution is an example of a Robinson-Trautman solution, that is there exists a null geodesic congruence with vanishing shear and rotation, but non-vanishing expansion.

The curvature tensors can be derived from Cartan's second equation of structure $R_{\mu\nu} = d\omega_{\mu\nu} + \omega_\mu{}^\rho \wedge \omega_{\rho\nu}$. Or, alternatively, one can read off the curvature tensors from appendix A of [1]

$$\begin{gathered} \Omega_{ij} = \Psi_{ijk} = 0, \quad \Omega'_{ij} = \Psi'_{ijk} = 0, \\ \Phi_{ijkl} = \frac{-4\Phi\delta_{i[k}\delta_{l]j}}{(d-2)(d-3)}, \quad \Phi^{\mathrm{S}}_{ij} = -\tfrac{1}{2}\Phi_{ikjk}, \quad \Phi^{\mathrm{A}}_{ij} = 0, \quad \Phi = -\frac{(d-2)(d-3)\mu}{2r^{d-1}}. \end{gathered} \tag{4.84}$$

In particular,

$$\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij} = 0. \tag{4.85}$$

4.B Derivation of equation (4.2)

This appendix is dedicated to the derivation of equation (4.2). We proceed by deriving an equation in which second order derivatives act only on $\Phi^{S}_{ij} - \frac{\Phi}{d-2}\delta_{ij}$. To simplify the derivation we shall neglect from the beginning any terms that are clearly of quadratic order or above when quantities are decomposed into background plus perturbation parts.

Contracting (B3), taking its symmetric part and removing its trace gives

$$\begin{aligned} 2þ\left(\Phi^{S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) + ð_k\left(\delta_{k(i}\Psi_{j)} - \Psi_{(ij)k} - \tfrac{2\Psi_k}{d-2}\delta_{ij}\right) \\ = \rho_{(kl)}\left(\Phi_{ikjl} + \tfrac{2\Phi^{S}_{kl}}{d-2}\delta_{ij}\right) - \rho\left(\Phi_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) \\ - \Phi\left(\rho_{ij} - \tfrac{\rho}{d-2}\delta_{ij}\right) - \tfrac{d-4}{d-2}\rho'\Omega_{ij}. \end{aligned} \tag{4.86}$$

Symmetrising over the ij indices in (B5) and removing its trace gives

$$\begin{aligned} ð_k\left(\Phi^{S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) - ð_l\left(\delta_{l(i}\Phi_{j)k} - \tfrac{\Phi_{lk}}{d-2}\delta_{ij}\right) + þ'\left(\Psi_{(ij)k} + \tfrac{\Psi_k}{d-2}\delta_{ij}\right) \\ = \tfrac{\rho}{d-2}\left(\Psi'_{(i}\delta_{j)k} - \Psi'_{(ij)k} - \tfrac{2\Psi'_k}{d-2}\delta_{ij}\right) - \tfrac{2\rho'}{d-2}\left(\Psi_{(ij)k} + \tfrac{\Psi'_k}{d-2}\delta_{ij}\right) \\ - \tfrac{(d-1)\Phi}{(d-2)(d-3)}\left(\tau_{(i}\delta_{j)k} - \tfrac{\tau_k}{d-2}\delta_{ij}\right). \end{aligned} \tag{4.87}$$

Now consider $þ'(4.86) + ð_k(4.87)$, the left hand side of which is equal to

$$\begin{aligned} (2þ'þ + ð_kð_k)\left(\Phi^{S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) - [þ', ð_k]\left(\Psi_{(ij)k} + \tfrac{\Psi'_k}{d-2}\delta_{ij}\right) \\ - ð_k\left(ð_{(i}\Phi_{j)k} - \tfrac{ð_l\Phi_{lk}}{d-2}\delta_{ij}\right) + þ'\left(ð_{(i}\Psi_{j)} - \tfrac{ð_k\Psi_k}{d-2}\delta_{ij}\right). \end{aligned} \tag{4.88}$$

The top line is precisely what we want since commutator equation (C2)′ can be used to convert the second term to an expression with only first order derivatives. The second line can be simplified using the trace of (B5) and full contraction of (B7), which when added together give

$$ð_k\Phi_{ik} - þ'\Psi_i = \rho'\Psi_i + \tfrac{d-1}{d-2}\Phi\tau_i. \tag{4.89}$$

Applying $ð_j$ to this, symmetrising over ij and removing the trace gives

$$\begin{aligned} ð_k\left(ð_{(i}\Phi_{j)k} - \tfrac{ð_l\Phi_{lk}}{d-2}\delta_{ij}\right) - þ'\left(ð_{(i}\Psi_{j)} - \tfrac{ð_k\Psi_k}{d-2}\delta_{ij}\right) \\ + [ð_k, ð_l]\left(\delta_{k(i}\Phi_{j)l} - \tfrac{\Phi_{kl}}{d-2}\delta_{ij}\right) + [þ', ð_k]\left(\delta_{k(i}\Psi_{j)} - \tfrac{\Psi_k}{d-2}\delta_{ij}\right) \\ = ð_k\left[\rho'\left(\delta_{k(i}\Psi_{j)} - \tfrac{\Psi_k}{d-2}\delta_{ij}\right)\right] + \tfrac{d-1}{d-2}ð_k\left[\Phi\left(\delta_{k(i}\tau_{j)} - \tfrac{\tau_k}{d-2}\delta_{ij}\right)\right]. \end{aligned} \tag{4.90}$$

Commutator equations (C3) and (C2)′ can be used to rewrite the second line in terms of first order derivative terms. Thus, the equation above can be used to rewrite the second line of (4.88) in terms of first order derivative terms. To summarise, we have

$$(2\text{þ}'\text{þ} + \eth_k\eth_k)\left(\Phi^{\rm S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right) = \ldots, \tag{4.91}$$

where the right hand side of the equation above involves only first order derivatives.

Terms of the form $\eth_k\Psi_{(ij)k}$ and $\eth_k\Psi'_{(ij)k}$ can be removed by using equation (4.86), while a term of the form $\text{þ}'\Omega_{ij}$ can be removed by using the symmetrisation of (B2).

The remaining terms can be simplified by rewriting them such that one has a factor that vanishes on the background. Since, we are neglecting terms of order quadratic or above, this means the other factor takes its background value. For example, a term of the form

$$\begin{aligned}\rho_{(kl)}\text{þ}'\left(\Phi_{ikjl} + \tfrac{2\Phi^{\rm S}_{kl}}{d-2}\delta_{ij}\right) &= \left(\rho_{(kl)} - \tfrac{\rho\delta_{kl}}{d-2}\right)\text{þ}'\left(\Phi_{ikjl} + \tfrac{2\Phi^{\rm S}_{kl}}{d-2}\delta_{ij}\right) + \tfrac{\rho\delta_{kl}}{d-2}\text{þ}'\left(\Phi_{ikjl} + \tfrac{2\Phi^{\rm S}_{kl}}{d-2}\delta_{ij}\right)\\ &= \tfrac{2}{(d-2)(d-3)}\left(\rho_{(ij)} - \tfrac{\rho}{d-2}\delta_{ij}\right)\text{þ}'\Phi - \tfrac{2\rho}{d-2}\text{þ}'\left(\Phi^{\rm S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right)\\ &= -\tfrac{2(d-1)}{(d-2)^2(d-3)}\rho'\Phi\left(\rho_{(ij)} - \tfrac{\rho}{d-2}\delta_{ij}\right) - \tfrac{2\rho}{d-2}\text{þ}'\left(\Phi^{\rm S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right),\end{aligned} \tag{4.92}$$

where in the second equality, we have used the fact that $\rho_{(kl)} - \frac{\rho}{d-2}\delta_{kl}$ vanishes on the background and since we are only considering terms that are of linear order in the perturbation expansion, we can take the background value of the factor multiplying this term. In the final line we use Bianchi equation (B2)′ evaluated on the background to simplify $\text{þ}'\Phi$. Using this trick to simplify all such terms and, also, using (NP4) to eliminate terms of the form $\eth_i\tau_j$, we find that the equation simplifies significantly

$$\begin{aligned}\left(2\text{þ}'\text{þ} + \eth_k\eth_k + \tfrac{d+2}{d-2}(\rho'\text{þ} + \rho\text{þ}') + \tfrac{2(d-1)}{(d-2)^2}\rho\rho' - \tfrac{2(d-1)(d-4)}{(d-2)(d-3)}\Phi\right)\left(\Phi^{\rm S}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right)&\\ + \tfrac{(d-4)}{(d-2)^2}\left(\rho'^2\Omega_{ij} + \rho^2\Omega'_{ij}\right) = 0.&\end{aligned} \tag{4.93}$$

Of course, we would like to derive an equation satisfied by $\rho^2\left(\Phi^{\rm S}_{ij} - \frac{\Phi}{d-2}\delta_{ij}\right)$. Letting

$\Phi^{\mathrm{S}t}{}_{ij} = \left(\Phi^{\mathrm{S}}_{ij} - \frac{\Phi}{d-2}\delta_{ij}\right)$,

$$\begin{aligned}
\rho^2 þ' þ \Phi^{\mathrm{S}t}{}_{ij} &= þ' \left(\rho^2 þ \Phi^{\mathrm{S}t}{}_{ij}\right) - (þ'\rho^2) þ \Phi^{\mathrm{S}t}{}_{ij} \\
&= þ' þ \left(\rho^2 \Phi^{\mathrm{S}t}{}_{ij}\right) - (þ'\rho^2) þ \Phi^{\mathrm{S}t}{}_{ij} - (þ\rho^2) þ' \Phi^{\mathrm{S}t}{}_{ij} - (þ' þ \rho^2) \Phi^{\mathrm{S}t}{}_{ij}, \\
\rho^2 þ \Phi^{\mathrm{S}t}{}_{ij} &= þ \left(\rho^2 \Phi^{\mathrm{S}t}{}_{ij}\right) - (þ\rho^2) \Phi^{\mathrm{S}t}{}_{ij}, \\
\rho^2 þ' \Phi^{\mathrm{S}t}{}_{ij} &= þ' \left(\rho^2 \Phi^{\mathrm{S}t}{}_{ij}\right) - (þ'\rho^2) \Phi^{\mathrm{S}t}{}_{ij}.
\end{aligned}$$

Equations (NP1) and (NP2) evaluated on the background give

$$þ\rho = -\tfrac{\rho^2}{d-2}, \quad þ'\rho = -\tfrac{\rho\rho'}{d-2} - \Phi. \tag{4.94}$$

Thus, equation (4.93) is equivalent to equation (4.2)

$$\begin{aligned}
\Big(2þ'þ + ð_k ð_k + \tfrac{d+6}{d-2}(\rho' þ + \rho þ') + \tfrac{4\Phi}{\rho} þ + \tfrac{2(3d+5)}{(d-2)^2}\rho\rho' - \tfrac{4(3d-8)}{(d-2)(d-3)}\Phi\Big) \left[\rho^2 \left(\Phi^{\mathrm{S}}_{ij} - \tfrac{\Phi}{d-2}\delta_{ij}\right)\right] \\
+ \tfrac{(d-4)}{(d-2)^2}\rho^2 \left(\rho'^2 \Omega_{ij} + \rho^2 \Omega'_{ij}\right) = 0.
\end{aligned}$$

4.C Adjoints of GHP covariant derivatives

In this appendix, we derive the adjoints of the GHP covariant derivatives. First, consider the adjoint of operator þ. Let $\eta_{i_1 \ldots i_s}$ and $\zeta_{i_1 \ldots i_s}$ be GHP scalars of spin s and boost weights b and $-(b+1)$, respectively and consider the inner product

$$\begin{aligned}
&(\zeta_{i_1\ldots i_s}, þ\, \eta_{i_1\ldots i_s}) = \Big(\zeta_{i_1\ldots i_s}, \ell\cdot\partial\, \eta_{i_1 i_2\ldots i_s} - bL_{10}\eta_{i_1 i_2\ldots i_s} + \textstyle\sum_{r=1}^s \overset{k}{M}_{i_r 0}\, \eta_{i_1\ldots i_{r-1} k i_{r+1}\ldots i_s}\Big) \\
&= \Big(-bL_{10}\zeta_{i_1 i_2\ldots i_s} - \textstyle\sum_{r=1}^s \overset{k}{M}_{i_r 0}\, \zeta_{i_1\ldots i_{r-1} k i_{r+1}\ldots i_s}, \ell\cdot\partial\, \eta_{i_1 i_2\ldots i_s}\Big) \\
&= \Big(-[(\ell\cdot\partial + \nabla\cdot\ell)\, \zeta_{i_1 i_2\ldots i_s} + bL_{10}\zeta_{i_1 i_2\ldots i_s} + \textstyle\sum_{r=1}^s \overset{k}{M}_{i_r 0}\, \zeta_{i_1\ldots i_{r-1} k i_{r+1}\ldots i_s}], \eta_{i_1 i_2\ldots i_s}\Big) \\
&= \Big(-[(\ell\cdot\partial + \rho)\, \zeta_{i_1 i_2\ldots i_s} + (b+1)L_{10}\zeta_{i_1 i_2\ldots i_s} + \textstyle\sum_{r=1}^s \overset{k}{M}_{i_r 0}\, \zeta_{i_1\ldots i_{r-1} k i_{r+1}\ldots i_s}], \eta_{i_1 i_2\ldots i_s}\Big) \\
&= \left(-[þ + \rho]\zeta_{i_1 i_2\ldots i_s}, \eta_{i_1 i_2\ldots i_s}\right),
\end{aligned}$$

where the first equality uses the definition of operator þ given in eq. (B.10), the second equality uses the property that

$$\overset{i}{M}_{j\mu} + \overset{j}{M}_{i\mu} = 0 \tag{4.95}$$

and the third inequality is obtained using integration by parts and ignoring divergence terms, since operator adjoints are defined up to such terms. The penultimate equality uses the geodesity of ℓ to deduce that

$$\nabla \cdot \ell = L_{10} + \rho \tag{4.96}$$

and the final equality uses the definition of operator þ given in eq. (B.10). Hence,

$$þ^\dagger = -(þ + \rho). \tag{4.97}$$

Taking the prime of this equation gives the adjoint of $þ'$

$$þ'^\dagger = -(þ' + \rho'). \tag{4.98}$$

Now, consider the inner product of $ð_{i_1}\eta_{i_2\ldots i_{s+1}}$ with $\xi_{i_1\ldots i_{s+1}}$, a GHP scalar with boost weight $-b$ and spin $s+1$

$$\begin{aligned}
&(\xi_{i_1\ldots i_{s+1}}, ð_{i_1}\eta_{i_2\ldots i_{s+1}}) \\
&= \left(\xi_{i_1\ldots i_{s+1}}, [m_{(i_1)}\cdot\partial - bL_{1i_1}]\eta_{i_2\ldots i_{s+1}} + \textstyle\sum_{r=2}^{s+1} \overset{k}{M}_{i_r i_1}\eta_{i_2\ldots i_{r-1}k i_{r+1}\ldots i_{s+1}}\right) \\
&= \left(-[bL_{1i_1} - \overset{i_1}{M}_{kk}]\xi_{i_1\ldots i_{s+1}} - \textstyle\sum_{r=1}^{s+1} \overset{k}{M}_{i_r i_1}\xi_{i_1..i_{r-1}k i_{r+1}\ldots i_{s+1}}, m_{(i_1)}\cdot\partial\eta_{i_2\ldots i_{s+1}}\right) \\
&= \Big(-[m_{(i_1)}\cdot\partial + \nabla\cdot m_{(i_1)} + bL_{1i_1} - \overset{i_1}{M}_{kk}]\,\xi_{i_1\ldots i_{s+1}} \\
&\qquad\qquad - \textstyle\sum_{r=1}^{s+1} \overset{k}{M}_{i_r i_1}\xi_{i_1..i_{r-1}k i_{r+1}\ldots i_{s+1}}, \eta_{i_2\ldots i_{s+1}}\Big) \\
&= \left(-[ð_{i_1} + \nabla\cdot m_{(i_1)} - \overset{i_1}{M}_{kk}]\xi_{i_1\ldots i_{s+1}}, \eta_{i_2\ldots i_{s+1}}\right) \\
&= \left(-[ð_{i_1} - \tau_{i_1} - \tau'_{i_1}]\xi_{i_1\ldots i_{s+1}}, \eta_{i_2\ldots i_{s+1}}\right),
\end{aligned}$$

where the first equality uses the definition of operator ð given in eq. (B.12), the second equality uses the property that

$$\overset{i}{M}_{j\mu} + \overset{j}{M}_{i\mu} = 0 \tag{4.99}$$

and the third inequality is obtained using integration by parts. The penultimate equality uses the definition of operator ð and the final equality uses the fact that

$$\nabla\cdot m_{(i)} = \overset{i}{M}_{kk} - \tau_i - \tau'_i. \tag{4.100}$$

Thus,

$$\eth_i{}^{\dagger} = -\eth_i + \tau_i + \tau'_i. \tag{4.101}$$

Chapter 5

Peeling of the Weyl tensor in higher dimensions

5.1 Introduction

In 4 spacetime dimensions, the Weyl tensor of an asymptotically flat spacetime exhibits the "peeling" property:

$$C_{\mu\nu\rho\sigma} = \lambda^{-1} C^{(N)}_{\mu\nu\rho\sigma} + \lambda^{-2} C^{(III)}_{\mu\nu\rho\sigma} + \lambda^{-3} C^{(II)}_{\mu\nu\rho\sigma} + \lambda^{-4} C^{(I)}_{\mu\nu\rho\sigma} + \mathcal{O}(\lambda^{-5}) \tag{5.1}$$

where indices $\mu, \nu, \ldots$ refer to a basis parallely transported along an outgoing null geodesic with affine parameter λ. In the first term, $C^{(N)}_{\mu\nu\rho\sigma}$ is a Weyl tensor of algebraic type N and the subsequent terms involve Weyl tensors of algebraic types III, II and I. The tangent to the geodesics is the repeated principal null direction for the type N, III and II terms, and a principal null direction for the type I term.

This result was originally derived using Bondi coordinates [42, 43]. In this approach, one assumes that the metric components can be expanded in inverse powers of a coordinate r. A more geometrical proof can be given using the definition of asymptotic flatness in terms of a conformal compactification [44, 45]. In this case, the result follows from the assumed smoothness of the unphysical spacetime. This smoothness assumption (or the assumption of an expansion in inverse powers of r) excludes some spacetimes in which radiation is present near spatial infinity. In this case, the peeling property is modified by an $\mathcal{O}(\lambda^{-4} \log \lambda)$ term [124, 26] (see also [125]).[1]

[1] We thank M. Dafermos for pointing out these references.

In $d > 4$ dimensions, a definition of asymptotic flatness at null infinity using conformal compactification is possible only for even d [126, 127]. It has been shown that this definition is preserved by linearized metric perturbations arising from compactly supported initial data [126]. It has also been argued that a vacuum spacetime satisfying this definition arises from initial data describing a small (but finite) perturbation of Minkowski spacetime that coincides with Schwarzschild initial data outside some compact set [128]. Just as in 4d, there are more general initial data that do not give a smooth null infinity [129].

For odd d, conformal compactification is unsatisfactory because the unphysical spacetime cannot be smooth in any radiating spacetime [130]. Instead, one can follow the older approach of defining asymptotic flatness at null infinity using Bondi coordinates [131, 132]. In section 5.2 we will weaken this definition slightly and demonstrate equivalence of the conformal and Bondi definitions for even d. For odd d, it remains to be shown that there exists an interesting class of initial data that gives rise to a spacetime which satisfies this definition.

The goal of this chapter is to determine how the Weyl tensor peels near null infinity in a spacetime satisfying one of the above definitions of asymptotic flatness. As just mentioned, at least for even d, this includes a large class of physically interesting spacetimes, but probably also excludes some physically interesting spacetimes. However, we can hope that in the latter case, just as in 4d, the peeling behaviour is modified only at a sufficiently high order that our result is still useful.

Two previous papers have investigated peeling using the conformal approach to asymptotic flatness [133, 134]. Both papers concluded that peeling is similar to the $d = 4$ case. They started from the assumption that all components of the unphysical Weyl tensor decay at the same (unspecified) rate near null infinity. However, Ref. [130] showed that this assumption is not true even for linearized perturbations of Minkowski spacetime, and argued that peeling should be qualitatively different for $d > 4$. This is what we find.

In section 5.3, we determine the behaviour of the Weyl tensor near null infinity in a spacetime satisfying the "Bondi definition" of asymptotic flatness (since this is valid for odd or even d and equivalent to the conformal definition for even d). For $d \geq 6$, we find the following result:

$$C_{\mu\nu\rho\sigma} = \lambda^{-(d/2-1)} C^{(N)}_{\mu\nu\rho\sigma} + \lambda^{-d/2} C^{(II)}_{\mu\nu\rho\sigma} + \lambda^{-(d/2+1)} C^{(G)}_{\mu\nu\rho\sigma} + \dots \tag{5.2}$$

Again λ is an affine parameter along a null geodesic and $\mu, \nu, \dots$ refer to a parallely transported basis. The superscripts N,II,G refer to the higher-dimensional classification of the Weyl tensor developed in Ref. [58], based on the concept of Weyl Aligned Null Directions (WANDs). A type N or type II Weyl tensor admits a “multiple WAND”, in this case it is the tangent to the geodesic. The type II term in (5.2) is not the most general type II Weyl tensor: it obeys additional restrictions explained below. Type G denotes an algebraically general Weyl tensor. The ellipsis in (5.2) denotes terms of order $\lambda^{-(d/2+2)}$ (even d) or $\lambda^{-(d/2+3/2)}$ (odd d).

For even d, the derivation of this result requires no more than the definition of asymptotic flatness. We do not use the Einstein equation, so this result is valid for any energy-momentum tensor consistent with asymptotic flatness. For odd d, we need to use some additional information from the Einstein equation: a mild condition on the decay of the Ricci tensor near null infinity is required to eliminate a term of order $\lambda^{-(d/2+1/2)}$ from (5.2).

The case $d = 5$ is exceptional. In this case, the Einstein equation no longer eliminates the term of order $\lambda^{-(d/2+1/2)} = \lambda^{-3}$. Instead, it fixes this term to be quadratic in the leading order metric perturbation and hence non-zero in any radiating spacetime. The result is that an additional type N term appears between the type II and type G terms:

$$C_{\mu\nu\rho\sigma} = \lambda^{-3/2} C^{(N)}_{\mu\nu\rho\sigma} + \lambda^{-5/2} C^{(II)}_{\mu\nu\rho\sigma} + \lambda^{-3} C^{(N)'}_{\mu\nu\rho\sigma} + \lambda^{-7/2} C^{(G)}_{\mu\nu\rho\sigma} + \mathcal{O}(\lambda^{-4}). \tag{5.3}$$

The subleading type N term is distinct from the leading order type N term. The presence of this term can be attributed to the nonlinearity of the Einstein equation. For $d > 5$, nonlinear effects decay faster and this term does not arise.

Refs. [126, 127, 131, 132] gave expressions for the rate of change of the Bondi energy at null infinity. For applications (e.g. higher-dimensional numerical relativity) it is convenient to have results that can be calculated easily and do not refer to a particular coordinate chart. This can be achieved by writing the result in terms of the asymptotic Weyl tensor components. We do this in section 5.4.

5.2 Definitions of asymptotic flatness

5.2.1 Conformal definition for even d

For even $d > 4$, Ref. [126] defined a spacetime (M, g) to be asymptotically flat at null infinity as follows. Given the (physical) metric g and the Minkowski metric η, we would like to specify the precise rate at which g approaches η asymptotically. We do this by conformally compactifying both M and Minkowski spacetime so that "infinity" is now at a finite metric distance. Thus, we obtain the "unphysical" spacetime $(\tilde{M}, \tilde{g})$ and the "background" spacetime $(\bar{M}, \bar{g})$, where the metrics $\tilde{g}$ and $\bar{g}$ are related to the respective physical and flat metrics via

$$\tilde{g}_{ab} = \Omega^2 g_{ab}, \qquad \bar{g}_{ab} = \Omega^2 \eta_{ab} \tag{5.4}$$

with the conformal factor Ω^2 satisfying the usual suitable properties.

Now, the spacetime is defined to be asymptotically flat at null infinity if

$$\begin{aligned} \tilde{g}_{ab} - \bar{g}_{ab} = \mathcal{O}(\Omega^{d/2-1}), \qquad \tilde{\epsilon}_{a_1 \dots a_d} - \bar{\epsilon}_{a_1 \dots a_d} = \mathcal{O}(\Omega^{d/2}), \\ (\tilde{g}^{ab} - \bar{g}^{ab})(\mathrm{d}\Omega)_a = \mathcal{O}(\Omega^{d/2}), \qquad (\tilde{g}^{ab} - \bar{g}^{ab})(\mathrm{d}\Omega)_a(\mathrm{d}\Omega)_b = \mathcal{O}(\Omega^{d/2+1}), \end{aligned} \tag{5.5}$$

where $\tilde{g}^{ab}$ and $\bar{g}^{ab}$ are the inverse metrics of $\tilde{g}$ and $\bar{g}$, respectively and $\tilde{\epsilon}$ and $\bar{\epsilon}$ are the volume forms on $(\tilde{M}, \tilde{g})$ and $(\bar{M}, \bar{g})$, respectively. Following Ref. [45], if $L_{ab\dots c}$ is a tensor field on $\tilde{M}$ then the notation $L_{ab\dots c} = \mathcal{O}(\Omega^s)$ means that $\Omega^{-s} L_{ab\dots c}$ is smooth at future null infinity.

5.2.2 Definition using Bondi coordinates

For general $d > 4$, Ref. [132] defined a spacetime to be asymptotically flat at future null infinity if, outside some cylindrical world tube, coordinates (u, r, x^I) can be introduced following [43] such that the metric takes the form

$$ds^2 = -Ae^B du^2 - 2e^B du dr + r^2 h_{IJ}(dx^I + C^I du)(dx^J + C^J du) \tag{5.6}$$

with

$$\det h_{IJ} = \det \omega_{IJ} \tag{5.7}$$

where $\omega_{IJ}(x)$ is the unit round metric on S^{d-2}. Surfaces of constant u are null with topology $\mathbb{R} \times S^{d-2}$ where x^I are coordinates on S^{d-2} and $\mathbb{R}$ corresponds to the null geodesic generators of the surface. These generators are given by u, x^I = constant and r is a (non-affine) parameter along the generators. A, B, C^I and h_{IJ} are functions of all of the coordinates. It is assumed that, at large r, they be expanded in inverse powers of r (even d) or $\sqrt{r}$ (odd d) with[2]

$$\begin{aligned} A &= 1 + \mathcal{O}(r^{-(d/2-1)}), \qquad B = \mathcal{O}(r^{-d/2}), \\ C^I &= \mathcal{O}(r^{-d/2}), \qquad h_{IJ} = \omega_{IJ} + \mathcal{O}(r^{-(d/2-1)}). \end{aligned} \tag{5.8}$$

For odd d it appears that an extra condition is required (discussed for $d = 5$ in Ref. [131]). One way of seeing this is to note that the results of Refs. [126, 130] suggest that, for linearized perturbations of Minkowski spacetime (arising from compactly supported initial data), the components of the metric perturbation each will be some half-integer power of $1/r$ times a *smooth* function of $1/r$. Hence each component will involve either integer powers of $1/r$ or half-odd-integer powers, but not both. Therefore, the presence of both integer and half-odd-integer powers in the expansions of individual metric components can be attributed to nonlinear effects. One would expect these only to affect terms beyond a certain order in the above expansions. If so, at low enough order, these expansions should contain only integer powers, or only half-odd-integer powers. This is indeed the case if one imposes the additional boundary condition that the expansion of h_{IJ} in inverse powers of $\sqrt{r}$ contains no term of order $r^{-(d/2-1/2)}$ (see below).

5.2.3 Equivalence of definitions for even d

Starting from the Bondi definition, define $\Omega = 1/r$. It is straightforward to show that this satisfies the conformal definition with (conformally flat) background metric

$$\bar{g} = -\Omega^2 du^2 + 2du d\Omega + \omega_{IJ} dx^I dx^J. \tag{5.9}$$

Now consider a spacetime that is asymptotically flat according to the conformal definition. Write the flat metric in the form

$$\eta = -dU^2 - 2dUdR + R^2 \omega_{IJ}(X) dX^I dX^J. \tag{5.10}$$

[2]Ref. [132] took $B = \mathcal{O}(r^{-d})$, which was obtained by solving the vacuum Einstein equation. We have weakened this condition since we don't want to assume vacuum.

Now define $\Omega = 1/R$. The background spacetime is

$$\bar{g} = \Omega^2 \eta = -\Omega^2 dU^2 + 2dU d\Omega + \omega_{IJ}(X) dX^I dX^J. \tag{5.11}$$

$\mathcal{I}^+$ is at $\Omega = 0$ and $\Omega > 0$ corresponds to the spacetime interior.

In these coordinates, the definition of asymptotic flatness reduces to the following conditions on the unphysical spacetime

$$\begin{aligned} \tilde{g}_{UU} &= -\Omega^2 + \mathcal{O}(\Omega^{d/2+1}), \qquad \tilde{g}_{U\Omega} = 1 + \mathcal{O}(\Omega^{d/2}), \qquad \tilde{g}_{UI} = \mathcal{O}(\Omega^{d/2}) \\ \tilde{g}_{\Omega\Omega} &= \mathcal{O}(\Omega^{d/2-1}), \qquad \tilde{g}_{\Omega I} = \mathcal{O}(\Omega^{d/2-1}), \qquad \tilde{g}_{IJ} = \omega_{IJ} + \mathcal{O}(\Omega^{d/2-1}) \end{aligned} \tag{5.12}$$

and

$$\det \tilde{g}_{IJ} = \det \omega_{IJ} + \mathcal{O}(\Omega^{d/2}). \tag{5.13}$$

Now convert to Gaussian null coordinates based on the null surface $\mathcal{I}^+$ in the unphysical spacetime as follows. Consider the (past-directed) null geodesic (of $\tilde{g}$) that passes through the point on $\mathcal{I}^+$ with coordinates $(u, 0, x^I)$ and has tangent vector $\partial/\partial\Omega$ there. Let λ denote the affine parameter along the geodesic. Since $\tilde{g}$ is required to be smooth near $\mathcal{I}^+$ it follows that the coordinates along the geodesic are smooth functions of λ in a neighbourhood of $\lambda = 0$. Expanding them in a Taylor series in λ and substituting into the geodesic equations gives

$$U = u + \mathcal{O}(\lambda^{d/2}), \qquad \Omega = \lambda + \mathcal{O}(\lambda^{d/2+1}), \qquad X^I = x^I + \mathcal{O}(\lambda^{d/2}). \tag{5.14}$$

We take (u, λ, x^I) as new coordinates. In these coordinates, the unphysical metric is

$$\begin{aligned} \tilde{g} &= \left[-\lambda^2 + \mathcal{O}(\lambda^{d/2+1})\right] du^2 + 2du d\lambda + \mathcal{O}(\lambda^{d/2}) du dx^I \\ &+ \left[\omega_{IJ}(x) + \mathcal{O}(\lambda^{d/2-1})\right] dx^I dx^J \end{aligned} \tag{5.15}$$

where all components are smooth at $\lambda = 0$ and

$$\det \tilde{g}_{IJ} = \det \omega_{IJ} + \mathcal{O}(\lambda^{d/2}). \tag{5.16}$$

We now replace λ with a non-affine parameter r defined by

$$r = \Omega^{-1} \left(\frac{\det \tilde{g}_{IJ}}{\det \omega_{IJ}} \right)^{1/(2(d-2))} = \lambda^{-1} \left(1 + \mathcal{O}(\lambda^{d/2}) \right) \tag{5.17}$$

so

$$\lambda = r^{-1}\left(1 + \mathcal{O}(r^{-d/2})\right) \tag{5.18}$$

and

$$\Omega^{-1} = r\left(1 + \mathcal{O}(r^{-d/2})\right). \tag{5.19}$$

In coordinates (u, r, x^I) the physical metric takes the Bondi form (5.6,5.7) with metric coefficients that are smooth functions of $1/r$ respecting the fall-off conditions (5.8).

5.3 The Weyl tensor

In this section, we determine the asymptotic fall off of Weyl tensor components for asymptotically flat spacetimes, as defined above. We will use Bondi coordinates since this allows us to treat even and odd d simultaneously. We perform our calculations using the higher dimensional Geroch-Held-Penrose (GHP) formalism of Ref. [67] (see appendix B).

5.3.1 Expansion of metric

We begin with the metric written in Bondi coordinates (5.6,5.7). From the definition of asymptotic flatness we have [132][3]

$$\begin{aligned} h_{IJ} = \omega_{IJ}(x) + \sum_{k\geq 0} \frac{h_{IJ}^{(k+1)}(u,x)}{r^{d/2+k-1}}, \qquad A = 1 + \sum_{k\geq 0} \frac{A^{(k+1)}(u,x)}{r^{d/2+k-1}}, \\ B = \sum_{k\geq 0} \frac{B^{(k+1)}(u,x)}{r^{d/2+k}}, \qquad C^I = \sum_{k\geq 0} \frac{C^{(k+1)I}(u,x)}{r^{d/2+k}}, \end{aligned} \tag{5.20}$$

where in all of the summations $k \in \mathbb{Z}$ for even d and $2k \in \mathbb{Z}$ for odd d. Equation (5.7) implies that

$$\omega^{IJ} h_{IJ}^{(k+1)} = 0 \quad \text{for} \quad k < d/2 - 1 \tag{5.21}$$

where ω^{IJ} is the inverse of ω_{IJ}. In particular we have, for $d \geq 5$

$$\omega^{IJ} h_{IJ}^{(1)} = \omega^{IJ} h_{IJ}^{(3/2)} = \omega^{IJ} h_{IJ}^{(2)} = 0 \tag{5.22}$$

and

$$\omega^{IJ} h_{IJ}^{(5/2)} = 0 \quad (d > 5), \qquad \omega^{IJ} h_{IJ}^{(5/2)} = \frac{1}{2} h^{(1)IJ} h_{IJ}^{(1)} \quad (d = 5), \tag{5.23}$$

[3] Our notation differs slightly from that of Ref. [132], notably in the expansion coefficients of B.

where all indices on $h^{(k+1)}_{IJ}$ are raised using ω^{IJ}. Some of the coefficients here have special significance. The Bondi mass is defined as [132]

$$M(u) = -\frac{d-2}{16\pi}\int_{S^{d-2}} A^{(d/2-1)} d\omega \tag{5.24}$$

where the integral is taken over a sphere at null infinity. In vacuum it obeys the mass decrease law [132]

$$\dot{M}(u) = -\frac{1}{32\pi}\int_{S^{d-2}} \dot{h}^{(1)}_{IJ}\dot{h}^{(1)IJ} d\omega. \tag{5.25}$$

This demonstrates that the quantity $h^{(1)}_{IJ}$ will be non-zero when gravitational radiation is present. Ref. [132] showed that $h^{(1)}_{IJ}$ is not constrained by the asymptotic vacuum Einstein equation: it is a free function in the Bondi approach, just as in 4d [42, 43]. $\dot{h}^{(1)}_{IJ}$ corresponds to Bondi's "news function".

5.3.2 Null frame and connection components

We choose a null frame $(\ell, n, m_{(i)})$ for the metric given by

$$\ell = m_{(0)} = -\frac{\partial}{\partial r}, \quad n = m_{(1)} = e^{-B}\left(\frac{\partial}{\partial u} - \tfrac{1}{2}A\tfrac{\partial}{\partial r} - C^I \tfrac{\partial}{\partial x^I}\right), \quad m_{(i)} = e_i^I \frac{\partial}{\partial x^I},$$
$$\ell^\flat = e^B du, \quad n^\flat = -(dr + \tfrac{1}{2}A du), \quad {m_{(i)}}^\flat = e_{iI}(dx^I + C^I du), \tag{5.26}$$

where e_i form a vielbein for the metric h_{IJ} on S^{d-2}: $h_{IJ} = e_{iI}e_{jJ}\delta_{ij}$. We choose this vielbein by using the Gram-Schmidt algorithm starting from the basis $r^{-1}\hat{e}_i^I$ where the vectors $\hat{e}_i^I(x)$ form an orthonormal basis for the metric ω_{IJ} on S^{d-2}. This gives an expansion in inverse powers of r (even d) or $\sqrt{r}$ (odd d):

$$e_{iI} = r\left(\hat{e}_{iI} + \frac{e^{(1)}_{iI}}{r^{d/2-1}}\right) + \mathcal{O}(r^{-(d-5)/2}), \quad e_i^I = r^{-1}\left(\hat{e}_i^I - \frac{e_i^{(1)I}}{r^{d/2-1}}\right) + \mathcal{O}(r^{-(d-1)/2}), \tag{5.27}$$

where $2\hat{e}_{i(I}e^{(1)}_{|j|J)}\delta_{ij} = h^{(1)}_{IJ}$ and $e_i^{(1)I} = \omega^{IJ}e^{(1)}_{iJ}$.

Using the definition of the connection components given in appendix B and the null frame given in (5.26), we find that the GHP covariant connection components are

$$\kappa_i = 0, \quad \rho_{ij} = -\tfrac{1}{2} e_i^I e_j^J \partial_r \left(r^2 h_{IJ}\right), \quad \rho = -(d-2)/r, \quad \tau_i = -\tfrac{1}{2}\left(e_i \cdot \partial B + e^{-B} e_{iI} \partial_r C^I\right)$$
$$\kappa'_i = \tfrac{1}{2} e^{-B} e_i \cdot \partial A, \quad \rho'_{ij} = -e^{-B} e_{(i|K|} e_{j)} \cdot \partial C^K + \tfrac{1}{2} e_i^I e_j^J \, n \cdot \partial \left(r^2 h_{IJ}\right),$$
$$\tau'_i = -\tfrac{1}{2}\left(e_i \cdot \partial B - e^{-B} e_{iI} \partial_r C^I\right). \tag{5.28}$$

The non-covariant coefficients are

$$L_{10} = -\partial_r B, \quad L_{11} = \tfrac{1}{2} e^{-B} \partial_r A, \quad L_{1i} = -\tau'_i$$
$$\overset{i}{M}_{j0} = e^I_{[i} \partial_r e_{j]I}, \quad \overset{i}{M}_{j1} = e^{-B} e_{[i} \cdot \partial C^K e_{j]K} - e^I_{[i} n \cdot \partial e_{j]I}$$
$$\overset{i}{M}_{jk} = r^2 e_i^I e_j^J e_k^K \partial_{[I} h_{J]K} - e^I_{[i} e_{|k|} \cdot \partial e_{j]I}. \tag{5.29}$$

Using the asymptotic behaviour of the metric components given in (5.20) gives

$$\rho_{ij} = -\frac{\delta_{ij}}{r} + \frac{\hat{e}_i^I \hat{e}_j^J}{4}\left((d-2)\frac{h^{(1)}_{IJ}}{r^{d/2}} + (d-1)\frac{h^{(3/2)}_{IJ}}{r^{(d+1)/2}}\right) + \mathcal{O}(r^{-(d+2)/2}), \quad \rho = -(d-2)/r,$$
$$\rho'_{ij} = -\frac{1}{2}\frac{\delta_{ij}}{r} + \frac{\hat{e}_i^I \hat{e}_j^J}{2}\left(\frac{\dot{h}^{(1)}_{IJ}}{r^{d/2-1}} + \frac{\dot{h}^{(3/2)}_{IJ}}{r^{(d-1)/2}}\right) - \hat{e}^I_{(i} e^{(1)J}_{j)} \frac{\dot{h}^{(1)}_{IJ}}{r^{d-2}} + \mathcal{O}(r^{-d/2}),$$
$$\rho' = -\frac{(d-2)}{2r}\left(\frac{1}{r} + \frac{A^{(1)}}{r^{d/2-1}} + \frac{A^{(3/2)}}{r^{(d-1)/2}}\right) + \mathcal{O}(r^{-d/2}), \quad \kappa'_i = \mathcal{O}(r^{-d/2}),$$
$$\tau_i = \frac{d}{4}\frac{\hat{e} \cdot C^{(1)}}{r^{d/2}} + \frac{d+1}{4}\frac{\hat{e} \cdot C^{(3/2)}}{r^{(d+1)/2}} + \mathcal{O}(r^{-(d/2+1)}),$$
$$\tau'_i = -\frac{d}{4}\frac{\hat{e} \cdot C^{(1)}}{r^{d/2}} - \frac{d+1}{4}\frac{\hat{e} \cdot C^{(3/2)}}{r^{(d+1)/2}} + \mathcal{O}(r^{-(d/2+1)}). \tag{5.30}$$

where a dot denotes a partial derivative with respect to u. Also,

$$L_{10} = \frac{d}{2}\frac{B^{(1)}}{r^{d/2+1}} + \frac{(d+1)}{2}\frac{B^{(3/2)}}{r^{(d+3)/2}} + \mathcal{O}(r^{-(d/2+2)}),$$
$$L_{11} = -\frac{d-2}{4}\frac{A^{(1)}}{r^{d/2}} - \frac{d-1}{4}\frac{A^{(3/2)}}{r^{(d+1)/2}} + \mathcal{O}(r^{-(d/2+1)}), \quad L_{1i} = \mathcal{O}(r^{-d/2}),$$
$$\overset{i}{M}_{j0} = \mathcal{O}(r^{-d/2}), \quad \overset{i}{M}_{j1} = -\frac{\hat{e}_i^I \dot{e}^{(1)}_{jI}}{r^{d/2-1}} + \mathcal{O}(r^{-(d-1)/2}),$$
$$\eth_i = r^{-1} \hat{e}_i \cdot \nabla + \mathcal{O}(r^{-d/2}), \tag{5.31}$$

where ∇_I denotes the covariant derivative induced by ω_{IJ}. Of course, terms with half-odd-integer powers appear only for odd d.

5.3.3 Parallely transported frame

The null basis introduced above is convenient for calculations but it is not parallely transported along the geodesics. A parallely transported basis is one for which, in addition to the geodesic equation $\kappa_i = 0$, we have $\tau'_i = L_{10} = \overset{i}{M}_{j0} = 0$. Any such basis will be related to ours by a boost, spin and null rotation (see Appendix). Let $(\hat{\ell}, \hat{n}, \hat{m}_i)$ be such a basis. $\hat{\ell}$ must be parallel to ℓ, with the coefficient fixed by requiring that $\hat{\ell}$ correspond to *affine* parameterization of the geodesics, ensuring $\hat{\kappa}_i = \hat{L}_{10} = 0$. This gives $\hat{\ell} = e^{-B}\ell$, corresponding to a boost with parameter e^{-B}. τ'_i is invariant under a boost and transforms covariantly under a spin. But under a null rotation with parameters z_i it transforms inhomogeneously [67] so z_i is determined by $\hat{\tau}'_i = 0$. This gives

$$z_i = c_i + \mathcal{O}(r^{-(d/2-1)}) \tag{5.32}$$

where the parameters c_i are independent of r. Finally, $\overset{i}{M}_{j0}$ transforms homogeneously under a boost and trivially under a null rotation but inhomogeneously under a spin. Requiring $\overset{i}{\hat{M}}_{j0}= 0$ determines the spin matrix to be

$$X_{ij} = O_{ij} + \mathcal{O}(r^{-(d/2-1)}) \tag{5.33}$$

where O_{ij} is a r-independent orthogonal matrix.

Our strategy will be to determine curvature components in the basis defined previously and then transform our results to a parallely transported frame by first performing a boost with parameter $e^{-B} = 1 + \mathcal{O}(r^{-d/2})$, then a null rotation with parameters z_i and finally a spin with parameters X_{ij} as given above.

5.3.4 Calculation of curvature components

In the GHP formalism the Weyl tensor components are denoted

$$\begin{gathered}
\Omega_{ij} = C_{0i0j}, \qquad \Psi_{ijk} = C_{0ijk}, \qquad \Psi_i = C_{010i} = \Psi_{jij}, \\
\Phi_{ijkl} = C_{ijkl}, \qquad \Phi_{ij} = C_{0i1j}, \qquad \Phi = \Phi_{ii} = C_{0101}, \\
(2\Phi^{\mathrm{S}}_{ij} = 2\Phi_{(ij)} = -\Phi_{ikjk}, \quad 2\Phi^{\mathrm{A}}_{ij} = 2\Phi_{[ij]} = C_{01ij}), \\
\Omega'_{ij} = C_{1i1j}, \qquad \Psi'_{ijk} = C_{1ijk}, \qquad \Psi'_i = C_{101i} = \Psi'_{jij}
\end{gathered} \tag{5.34}$$

and the Ricci tensor components are

$$\omega = R_{00}, \quad \psi_i = R_{0i}, \quad \phi_{ij} = R_{ij}, \quad \phi = R_{01}, \quad \psi'_i = R_{1i}, \quad \omega' = R_{11}. \tag{5.35}$$

The Newman-Penrose equations (see Appendix) are used to determine all of these quantities except for those of boost weight zero (i.e. those written with the letters Φ or ϕ). To determine the latter we used the Bianchi equation (B3) from the Appendix.[4]

5.3.5 Results: even d

In our basis (5.26), we find that the Ricci tensor components are smooth functions of $1/r$ with

$$\begin{aligned} \omega = \mathcal{O}(r^{-(d/2+2)}), \qquad \psi_i = \mathcal{O}(r^{-(d/2+1)}), \qquad \phi_{ij} = \mathcal{O}(r^{-(d/2+1)}), \\ \phi = \mathcal{O}(r^{-(d/2+1)}), \qquad \psi'_i = \mathcal{O}(r^{-d/2}), \qquad \omega' = \mathcal{O}(r^{-d/2}). \end{aligned} \tag{5.36}$$

The Weyl tensor components are smooth functions of $1/r$ with

$$\begin{aligned} \Omega_{ij} &= -\frac{(d-2)(d-4)}{8}\frac{\hat{e}_i^I \hat{e}_j^J h_{IJ}^{(1)}}{r^{d/2+1}} + \mathcal{O}(r^{-(d/2+2)}), \\ \Psi_{ijk} &= \mathcal{O}(r^{-(d/2+1)}), \qquad \Psi_i = \mathcal{O}(r^{-(d/2+1)}), \\ \Phi^A_{ij} &= \mathcal{O}(r^{-(d/2+1)}), \qquad \Phi = \mathcal{O}(r^{-(d/2+1)}), \\ \Phi^S_{ij} &= -\frac{(d-4)}{4}\frac{\hat{e}_i^I \hat{e}_j^J \dot{h}_{IJ}^{(1)}}{r^{d/2}} + \mathcal{O}(r^{-(d/2+1)}), \\ \Phi_{ijkl} &= (\hat{e}_i^I \hat{e}^J_{[k}\delta_{l]j} - \hat{e}_j^I \hat{e}^J_{[k}\delta_{l]i})\frac{\dot{h}_{IJ}^{(1)}}{r^{d/2}} + \mathcal{O}(r^{-(d/2+1)}), \\ \Psi'_{ijk} &= \mathcal{O}(r^{-d/2}), \qquad \Psi'_i = \mathcal{O}(r^{-d/2}), \\ \Omega'_{ij} &= -\frac{1}{2}\frac{\hat{e}_i^I \hat{e}_j^J \ddot{h}_{IJ}^{(1)}}{r^{d/2-1}} + \mathcal{O}(r^{-d/2}). \end{aligned} \tag{5.37}$$

Recall that $h_{IJ}^{(1)}$ is non-zero in any spacetime containing outgoing gravitational radiation, and it is not determined by the asymptotic Einstein equation.

Now we transform to a parallely transported frame as determined above. The boost and spin are easy to deal with since the curvature components transform covariantly with respect to these. Formulae for the transformation under a null rotation are given in the Appendix. Using these results, we see that the transformation to a parallely transported frame does not change any of these results (aside from acting with the rotation matrix O_{ij} on the indices i, j, k etc).

[4]This involves an integration with respect to r, introducing a homogeneous term decaying as $1/r$ into the boost weight zero quantities. This is not compatible with asymptotic flatness so the coefficient of this term must vanish. This could be shown e.g. by using the commutator (C3) of Ref. [67].

Finally we have to convert from our parameter r to an affine parameter along the geodesics. Denote the latter by λ. Then (up to the freedom to multiply by a quantity independent of r)

$$\lambda = \int e^B dr = r + c + \mathcal{O}(r^{-(d/2-1)}) \tag{5.38}$$

where c is independent of r. Inverting gives

$$r = \lambda - c + \mathcal{O}(\lambda^{-(d/2-1)}). \tag{5.39}$$

If we substitute this into the above expressions for the Weyl components then they become smooth functions of $1/\lambda$ with leading order behaviour given by replacing r with λ in these expressions. Hence the leading order term in the Weyl tensor is of order $\lambda^{-(d/2-1)}$ and the only non-vanishing components at this order are Ω'_{ij}, which (from (5.37)) is generically non-zero. But this is precisely the definition of a type N Weyl tensor with ℓ (the tangent to the geodesics) a multiple Weyl Aligned Null Direction (WAND) [58].

The next non-vanishing terms in the Weyl arise at order $\lambda^{-d/2}$. Such terms can arise from Ω'_{ij}, Ψ'_{ijk}, Ψ'_i, Φ_{ijkl} and Φ^S_{ij}. So, at this order, we have $\Omega_{ij} = \Psi_{ijk} = \Psi_i = 0$ and hence the Weyl tensor is type II with multiple WAND ℓ. It cannot be type III because (5.37) shows that Φ^S_{ij} is generically non-vanishing. However, it is not the most general possible type II Weyl tensor because (5.37) shows that it has vanishing Φ and Φ^A_{ij}.

After this, we have terms of order $\lambda^{-(d/2+1)}$. At this order, any of the Weyl components can be non-zero. In particular, the above expression shows that Ω_{ij} is generically non-zero, which implies that the Weyl tensor at this order is type G (i.e. ℓ is not a WAND).

In summary, for even $d > 4$, we have demonstrated that, in a spacetime satisfying the definition of asymptotic flatness at null infinity of Refs. [126, 132], the Weyl tensor exhibits the peeling behaviour described around equation (5.2), with the type II obeying the additional conditions $\Phi = \Phi^A_{ij} = 0$.

When $d = 4$, our results for Ω'_{ij}, Ψ'_{ijk} and Ψ'_i are consistent with the 4d peeling property. The boost-weight zero terms also are consistent: in 4d, all such terms are determined by Φ and Φ^A_{ij}, which vanish at order $\lambda^{-d/2} = \lambda^{-2}$. Hence at order λ^{-2} we have a Weyl tensor of type III instead of type II. More explicitly, in 4d, Φ_{ijkl} is determined by its trace Φ^S_{ij}. But the first term in the expansion of Φ^S_{ij} in (5.37) comes with a coefficient of $d-4$. Similar results hold for the other Weyl components (e.g. the above expression for Ω_{ij} has a factor $d-4$). This is why peeling is qualitatively different when $d = 4$.

5.3.6 Results: odd d

As discussed above, for odd d there is an additional condition in the definition of asymptotic flatness at future null infinity, that the term of order $r^{-(d/2-1/2)}$ in the expansion of h_{IJ} should be absent, i.e.,

$$h_{IJ}^{(3/2)} = 0. \tag{5.40}$$

With this condition, we find that the results (5.36,5.37) for the Ricci and Weyl components are valid also for odd d, with the understanding that these formulae now refer to expansions in inverse powers of $\sqrt{r}$. (Without (5.40), there would be e.g. a term of order $r^{-(d/2-1/2)}$ in the expansion of Ω'_{ij}.) Following the same steps as for even d, converting to a parallely transported frame and affine parameterization, we find, just as before, that the leading components of the Weyl tensor arise at order $\lambda^{-(d/2-1)}$ and this term is type N as before. There are no terms at order $\lambda^{-(d/2-1/2)}$ so the next term is at order $\lambda^{-d/2}$ which is type II with $\Phi^A_{ij} = \Phi = 0$, again as for even d.

A difference between even and odd d arises at next order: for odd d there is the possibility of terms of order $r^{-(d/2+1/2)}$. For example, we find that

$$\begin{aligned}\Omega'_{ij} + \frac{\omega'}{d-2}\delta_{ij} =& -\frac{1}{2}r^{-(d/2-1)}\hat{e}_i^I\hat{e}_j^J\ddot{h}_{IJ}^{(1)} + r^{-d/2}Y_{ij} \\ & -\frac{1}{2}r^{-(d/2+1/2)}\left(\hat{e}_i^I\hat{e}_j^J\ddot{h}_{IJ}^{(5/2)} - \dot{A}^{(3/2)}\delta_{ij} - 2\hat{e}_{(i|K|}\hat{e}_{j)}\cdot\nabla\dot{C}^{(3/2)K}\right) \\ & +\frac{1}{4}r^{-(d-2)}\left(4\hat{e}^I_{(i}e^{(1)J}_{j)}\ddot{h}^{(1)}_{IJ} + \hat{e}_i^I\hat{e}_j^J\omega^{KL}\dot{h}^{(1)}_{IK}\dot{h}^{(1)}_{JL}\right) + \mathcal{O}(r^{-(d/2+1)}),\end{aligned} \tag{5.41}$$

where Y_{ij} is a quantity whose explicit form we will not need. The Weyl components Ω'_{ij} are obtained by taking the traceless part of this equation and the Ricci component ω' by taking the trace. We have retained a term of order $r^{-(d-2)}$ because $r^{-(d-2)} = r^{-(d/2+1/2)}$ if $d = 5$.

The only other Weyl components containing terms of order $r^{-(d/2+1/2)}$ are Ψ'_{ijk} and Ψ'_i, which can be obtained from

$$\begin{aligned}\Psi'_{ijk} + \frac{2}{d-2}\psi'_{[j}\delta_{k]i} =& r^{-d/2}\hat{e}_i^I\hat{e}_{[j}^J\hat{e}_{k]}\cdot\nabla\dot{h}_{IJ}^{(1)} + \mathcal{O}(r^{-(d/2+1)}), \\ \Psi'_i - \frac{1}{d-2}\psi'_i =& \frac{d}{4}r^{-d/2}\hat{e}_i\cdot\dot{C}^{(1)} + \frac{(d+1)}{4}r^{-(d/2+1/2)}\hat{e}_i\cdot\dot{C}^{(3/2)} + \mathcal{O}(r^{-(d/2+1)})\end{aligned} \tag{5.42}$$

where the Weyl and Ricci components can be disentangled by taking a trace of the first equation and combining with the second equation.

We will now argue that terms of order $r^{-(d/2+1/2)}$ can be eliminated for $d > 5$ by exploiting the Einstein equation (which we did not use for even d). We will assume that the Ricci tensor (and hence the energy-momentum tensor) decays faster near infinity than the rate which is given by asymptotic flatness alone (equation (5.36)). The rate that we require is faster by a factor $1/r$:[5]

$$\begin{aligned} \omega &= \mathcal{O}(r^{-(d/2+3)}), & \psi_i &= \mathcal{O}(r^{-(d/2+2)}), & \phi_{ij} &= \mathcal{O}(r^{-(d/2+2)}), \\ \phi &= \mathcal{O}(r^{-(d/2+2)}), & \psi'_i &= \mathcal{O}(r^{-(d/2+1)}), & \omega' &= \mathcal{O}(r^{-(d/2+1)}). \end{aligned} \tag{5.43}$$

Imposing these conditions implies that the first few coefficients in the expansions of the metric components must satisfy the same equations as in a *vacuum* spacetime, as determined in Ref. [132]:

$$B^{(1)} = 0, \qquad A^{(1)} = -\frac{2}{d-2}\nabla\cdot C^{(1)} = -\frac{4}{d(d-2)}\nabla^I\nabla^J h^{(1)}_{IJ}, \qquad C^{(1)I} = \frac{2}{d}\nabla_J h^{(1)IJ},$$
$$\begin{cases} \dot{A}^{(3/2)} = \frac{1}{6}\dot{h}^{(1)IJ}\dot{h}^{(1)}_{IJ} & d=5 \\ A^{(3/2)} = 0 & d>5 \end{cases}, \qquad B^{(3/2)} = \begin{cases} -\frac{1}{16}h^{(1)IJ}h^{(1)}_{IJ} & d=5 \\ 0 & d>5 \end{cases},$$
$$C^{(3/2)I} = 0, \qquad \dot{h}^{(5/2)}_{IJ} = \begin{cases} \omega^{KL}h^{(1)}_{K(I}\dot{h}^{(1)}_{J)L} & d=5 \\ 0 & d>5 \end{cases} \tag{5.44}$$

and an equation relating $\dot{h}^{(2)}_{IJ}$ to $A^{(1)}$, $C^{(1)I}$ and $h^{(1)}_{IJ}$. Note that the asymptotic Einstein equation implies no restriction on $h^{(1)}_{IJ}$. Recall that for $d = 5$, $A^{(3/2)}$ determines the Bondi mass via (5.24).[6]

Using these results, we see that the term of order $r^{-(d/2+1/2)}$ in (5.42) is absent and hence such terms do not appear in Ψ'_{ijk} and Ψ'_i. However, terms of this order are absent from (5.41) if, and only if, $d > 5$. Hence, for $d > 5$, such terms are absent from Ω'_{ij}. Transforming to a parallely transported frame and affine parameterization, similar arguments to those used for the even d case establish the peeling result given in equation (5.2) for odd $d > 5$. As for even d, the type II term obeys the additional restrictions $\Phi^{\mathrm{A}}_{ij} = \Phi = 0$.

[5]For $d > 5$, the constraints on ω, ψ_i and ϕ, necessarily imply the constraints on ψ'_i and ω'. For $d = 5$, the constraint on ψ_i implies the constraint on ψ'_i.

[6] Note also that $\dot{h}^{(5/2)}_{IJ} = 0$ for $d > 5$ implies that one can impose the additional boundary condition $h^{(5/2)}_{IJ} = 0$ for $d > 5$. Ref. [132] examined the vacuum Einstein equations to higher order and the results suggest that the definition of asymptotic flatness for odd d should be augmented with the condition $h^{(k+1)}_{IJ} = 0$ for $k = 1/2, 3/2, \ldots, d/2-2$ although we will not assume any more than (5.40).

Finally we must discuss the $d = 5$ case. For $d = 5$, terms of order $r^{-(d/2+1/2)} = r^{-3}$ do *not* drop out of Ω'_{ij}:

$$\begin{aligned}\Omega'_{ij} = & -\frac{1}{2}\frac{\hat{e}_i^I \hat{e}_j^J \ddot{h}_{IJ}^{(1)}}{r^{3/2}} + \frac{Y_{ij}}{r^{5/2}} \\ & - \frac{1}{2r^3}\left\{\hat{e}_i^I \hat{e}_j^J \left(\omega^{KL} h_{IK}^{(1)} \ddot{h}_{JL}^{(1)} + \tfrac{1}{2}\omega^{KL}\dot{h}_{IK}^{(1)}\dot{h}_{JL}^{(1)} - \tfrac{1}{6}\omega_{IJ}\dot{h}^{(1)KL}\dot{h}_{KL}^{(1)}\right) - 2\hat{e}_{(i}^I e_{j)}^{(1)J}\ddot{h}_{IJ}^{(1)}\right\} \\ & + \mathcal{O}(r^{-7/2}). \end{aligned} \tag{5.45}$$

Note that the coefficient of r^{-3} is quadratic in $h_{IJ}^{(1)}$ and its time derivatives and hence generically it is non-zero if gravitational radiation is present.

Now we must transform to a parallely transported frame. As for $d > 5$, the boost and null rotation do not change our results. But note that the spin matrix (5.33) involves a term of order $r^{-3/2}$. Hence when the spin acts on Ω'_{ij} this term will combine with the leading term in Ω'_{ij} to produce a new term of order r^{-3} in $\hat{\Omega}'_{ij}$. Could this new term cancel the terms already present? Generically no: the new term will involve $\ddot{h}_{IJ}^{(1)}$ whereas some of the terms already present involve only first derivatives of $h_{IJ}^{(1)}$. Since $\dot{h}_{IJ}^{(1)}$ is a free function in the Bondi approach, these terms will not cancel in general. For example, one could choose $\ddot{h}_{IJ}^{(1)}$ to be zero somewhere, with $\dot{h}_{IJ}^{(1)}$ non-zero.

The last step is to convert to affine parameterization using (5.39), which does not change anything. We conclude that for $d = 5$, the Weyl tensor satisfies the peeling property (5.3) described in the introduction. Again the type II term obeys the additional restrictions $\Phi_{ij}^{A} = \Phi = 0$.

5.4 Bondi flux

In 4d, the rate of decrease of the Bondi energy at future null infinity is given in terms of the Newman-Penrose Weyl scalar Ψ_4 as

$$\dot{M}(u) = -\lim_{r\to\infty}\frac{r^2}{4\pi}\int_{S^2}\left|\int_{-\infty}^{u}\Psi_4(\hat{u}, r, x)d\hat{u}\right|^2 d\omega \tag{5.46}$$

where $d\omega$ is the volume element on a unit S^2. In $d > 4$ dimensions, the rate of decrease of the Bondi energy at future null infinity is given by (5.25) [132]. We can rewrite this in terms of Ω'_{ij} (the analogue of Ψ_4) as follows. Assume that the Bondi flux vanishes in the far past, i.e. $\dot{h}_{IJ}^{(1)} \to 0$ as $u \to -\infty$. Then from (5.37) (which holds for even or odd d) we have

$$\hat{e}_i^I \hat{e}_j^J \dot{h}_{IJ}^{(1)}(u,x) = -2 \lim_{r\to\infty} r^{d/2-1} \int_{-\infty}^{u} \Omega'_{ij}(\hat{u},r,x)\, d\hat{u} \tag{5.47}$$

and hence

$$\dot{M}(u) = -\lim_{r\to\infty} \frac{r^{d-2}}{8\pi} \int_{S^{d-2}} \left(\int_{-\infty}^{u} \Omega'_{ij}(\hat{u},r,x)\, d\hat{u} \right)^2 d\omega \tag{5.48}$$

where $d\omega$ is the volume element on a unit S^{d-2} and $(Y_{ij})^2 \equiv Y_{ij}Y_{ij}$. In practice, the RHS is computed by choosing coordinates so that the asymptotic metric takes the form

$$ds^2 \sim -du^2 - 2du dr + r^2 d\omega^2. \tag{5.49}$$

One then chooses a null vector field n that approaches $\pm(\partial/\partial u - \frac{1}{2}\partial/\partial r)$ asymptotically (the sign does not matter here) and a set of orthonormal spacelike vectors $m_{(i)}$ ($i = 2, \ldots d-1$) such that $n \cdot m_{(i)} = 0$. Then $\Omega'_{ij} = C_{abcd} n^a m^b_{(i)} n^c m^d_{(j)}$.

5.5 Discussion

We derived our result using Bondi coordinates since this allows us to treat even and odd d together for much of the analysis. However, for even d it would be more elegant to derive the peeling property using the conformal approach. It would be nice to see this worked out.

For odd $d > 5$, our result (5.2) involves only inverse half-odd-integer powers of λ. Inverse integer powers will appear if one continues to higher orders in the expansion. It would be interesting to know at what order inverse integer powers first appear. If one strengthens the definition of asymptotic flatness as suggested in footnote 6 then it seems likely that the first such terms will appear at order $\lambda^{-(d-2)}$, in agreement with our result for $d = 5$.

Ref. [66] studied asymptotically flat solutions in $d > 4$ dimensions that are *algebraically special*. It was found that the latter condition is incompatible with gravitational radiation (in contrast with the $d = 4$ case). We can see a similar result here: if ℓ is a WAND then Ω_{ij} must vanish. For $d > 4$, (5.37) then requires $h^{(1)}_{IJ} = 0$, which implies vanishing Bondi energy flux, i.e., no gravitational radiation.

Chapter 6

Spinor classification of the Weyl tensor

6.1 Introduction

The De Smet classification [70] generalises the concept of a Petrov Weyl spinor to five dimensions. The classification uses the 5d Clifford algebra to define a totally symmetric 4-spinor, called the *Weyl spinor*, that is equivalent to the Weyl tensor. A given solution is classified by studying how its Weyl spinor factorises. Or more precisely, how the fourth order quartic homogeneous polynomial formed from its Weyl spinor factorises. The fact that the Weyl spinor is generally complex means that it must satisfy a reality condition and this reduces the number of possible types.

In five dimensions, it is known that the De Smet and CMPP classification schemes are not equivalent; that is they do not agree on the definition of an "algebraically special" solution. An example is known that is algebraically special in the CMPP classification, but algebraically general in the De Smet classification [135] and vice versa [1]. The presence of two inequivalent classification schemes in five dimensions presents us with the opportunity of studying solutions that are algebraically general in one scheme and special in the other.

Apart from a classification of static axisymmetric solutions belonging to two particular algebraic types [70, 71], the De Smet classification has not been studied much. The aim of this chapter is to better understand the De Smet classification and its relation to the 4d Petrov and 5d CMPP classifications. We shall find that the previously overlooked reality condition will play an important part in this study.

As a way of highlighting the most important characteristics of the spinor classification of the Weyl tensor, we shall also consider the spinor classification of two-form fields, where it is much easier to appreciate subtle issues such as reality conditions. This is because, we shall be dealing only with a bispinor, rather than a 4-spinor as is the case in the Weyl classification.

Therefore, we begin, in section 6.2.1, with a derivation of a spinor classification of 2-form fields. We construct a bispinor equivalent of the 2-form and use properties of the Clifford algebra to show that it is symmetric and satisfies a reality condition. This leads to a classification of 2-forms based on whether the equivalent bispinor factorises or not.

Then, in section 6.2.2, we move on to derive the spinor classification of the Weyl tensor due to De Smet [70] in similar vein to the derivation of the spinor classification of 2-forms in section 6.2.1. We define the Weyl spinor, and show that it is totally symmetric and satisfies a reality condition. The reality condition reduces the number of algebraically special types.

The De Smet classification is intended to be a generalisation of the spinor formulation of the Petrov classification to five dimensions. It is not clear, though, how the two schemes are related and in what sense the De Smet classification is a generalisation of Petrov's beyond the superficial link that they both deal with the factorisability properties of a totally symmetric spinor quantity. This issue is addressed in section 6.2.3, where it is shown that one can define an analogue of the De Smet Weyl spinor in 4d and that the classification of the Weyl tensor based on this can be thought of as a classification using Majorana spinors. Recall that in the Petrov classification, one uses chiral spinors.

In section 6.2.4, we use the results obtained in section 6.2.3 to study direct product solutions. We find that the De Smet type of solutions with a 4d factor is equal to the De Smet type of the 4d submanifold. Thus, the analysis reduces to that done in section 6.2.3. For the case with 2d and 3d Lorentzian factors with non-zero cosmological constant, the Weyl spinor factorises into two proportional bispinors that cannot be further factorised. These results are similar to those found in the study of warped product manifolds in the context of the CMPP classification in [85].

In section 6.3.1, we consider the connection between the tensor and spinor classifications of a 2-form, where the tensor classification is based on the CMPP classification. We find that a solution of any spinor algebraic type may be algebraically general in the tensorial sense. For solutions that are algebraically special in the spinorial sense, what determines whether they are algebraically special or general in the tensorial sense is whether the vector that can be formed from the spinor that we have from the factorisation of the bispinor is null or timelike.

We find that similar statements can be made regarding the relation between the De Smet and 5d CMPP classifications of the Weyl tensor in section 6.3.2. However, in this case we cannot study all types fully. Thus, we begin by assuming that the solution is of type N, III or D and derive the general De Smet polynomials for the respective cases. Considering the factorisability properties of these general polynomials gives the possible spinor types that they can have. We show that type III and D solutions may be algebraically general in the spinor classification, while for type N solutions, the De Smet polynomial is guaranteed to factorise into linear factors, so type N solutions are also algebraically special in the spinor classification. We do not consider more general types due to the complexities of factorising a general polynomial. Then, we go on to consider the reverse case, i.e. assuming a particular De Smet type and examining what this implies about the CMPP type. Since the general form of Weyl tensor is important for this analysis, we can only do this for the case where the Weyl spinor factorises into two bispinors, or a more special case of this, using the general form of the Weyl tensor of such solutions derived in section 6.2.2. Thus, we do not consider the case where the solution is algebraically general, i.e. the Weyl spinor does not factorise nor the case where it factorises into a rank-3 spinor and a univalent spinor. We find that any spinor type may be algebraically general in the CMPP sense.

An important motivation, given above, for understanding the De Smet classification and its relation to the 5d CMPP classification was that this may allow us to study 5d solutions that are algebraically general in one classification scheme and special in the other. Furthermore, the result found in section 6.3.2 that any spinor type may be algebraically general in the CMPP sense strengthens this motivation. The black ring [50] is a well-known example of a CMPP algebraically general five dimensional solution [136][1]. Therefore, it would be desirable to know the De Smet type of the black ring solution. It is shown in section 6.3.3 that the black ring is also, unfortunately, algebraically general in the De Smet classification.

In section 6.4.1, we consider the constraints imposed on a spacetime by the existence of an algebraically special 2-form solving Maxwell-type equations. An algebraically special

[1]In [136], it is shown that WANDs can only be found in certain regions for the black ring and it is claimed that the black ring is type I. However, if we take the strict definition of the classification, which states that the algebraic type of the spacetime corresponds to the type of its most algebraically general point, then the black ring is type G. The black ring is an example of a solution that is type G in one open region and type I in another. This kind of behaviour is discussed in [1].

2-form is defined by a single spinor. In 4d, the existence of an algebraically special Maxwell field is equivalent to the spacetime being algebraically special. This follows from the Mariot-Robinson [137, 35] and Goldberg-Sachs [36] thereoms. Thus, studying the existence of algebraically special fields can shed light on the status of the Goldberg-Sachs theorem in higher dimensions. The 2-form field is assumed to satisfy the Bianchi identity and a general equation of motion that includes Maxwell theory as well as minimal supergravity [138]. The analysis splits into two cases of whether the vector derived from the spinor that defines the 2-form field is null or timelike. From section 6.3.1, we know that if it is null then this is equivalent to the field being algebraically special in the CMPP sense. This analysis has already been done in [67]. The null vector defines a geodesic congruence with constraints on its optical properties, which are explained in section 6.4.1. If the vector is timelike, then the solution admits a timelike geodesic congruence and an almost-Kähler structure.

Finally, in section 6.4.2, we undertake a classification of solutions belonging to the most special type, that is type $\underline{11}\ \underline{11}$ solutions. These are defined as those for which the Weyl spinor factorises into two proportional bispinors, which factorise further into spinors. The reality condition gives that the Weyl tensor is fully determined from a single spinor. We use the Bianchi identity to find constraints on this spinor for a vacuum Einstein solution. As in section 6.4.1, the analysis divides into two cases of whether the vector defined from the spinor is null or timelike. If the vector is null, then we have a type N Kundt solution satisfying further conditions that are explained in section 6.4.2. The timelike case reveals more structure. The spacetime is found to be a cosmological solution with spatial geometry a type (D,O) Einstein solution. The solutions in section 6.4.2 are more constrained that those found in section 6.4.1.

The index conventions in this chapter are as follows: indices $a, b, c \dots$ refer to orthonormal or null frame basis vectors and generally take values from 0 to 4, although this is not always the case. Indices $i, j, k \dots$ refer to spacelike basis vectors and generally take values from 2 to 4. In section 6.3.2, where we move between orthonormal and null frame bases, indices $a, b, c \dots$ refer to null frame basis vectors, while $\mu, \nu, \rho \dots$ refer to orthonormal basis vectors. $\alpha, \beta, \gamma \dots$ and $\dot{\alpha}, \dot{\beta}, \dot{\gamma} \dots$ label left and right-handed chiral spinor indices in four dimensions and run from 1 to 2, while $A, B, C \dots$ and $\dot{A}, \dot{B}, \dot{C} \dots$ label Dirac and Dirac complex conjugate indices in four and five dimensions and run from 1 to 4. There are additional index conventions in section 6.2.4, which are explained separately in that section.

6.2 Spinor classification of two-form and Weyl tensor

6.2.1 Spinor classification of two-form

Let F_{ab} be a real two-form. We can construct a bispinor ϵ_{AB} that is equivalent to the 2-form [2]

$$\epsilon_{AB} = \frac{i}{8} F_{ab} \Gamma^{ab}{}_{AB}, \tag{6.1}$$

where $\Gamma^{ab} = \Gamma^{[a}\Gamma^{b]}$. As explained in appendix 6.A, for brevity, we omit factors of C and C^{-1} where it is clear from the index structure that charge conjugation matrices have been used. Thus, $\Gamma^{ab}{}_{AB} = (C\Gamma^{ab})_{AB} = C_{AC}\Gamma^{ab\,C}{}_{B}$.

It can be shown, using the antisymmetry of the charge conjugation matrix C that $\Gamma^{ab}{}_{AB}$ is symmetric in its spinor indices: using the definition of C, we find that

$$\Gamma^t_{ab} = -C\Gamma_{ab}C^{-1}, \tag{6.2}$$

which implies

$$(C^t\Gamma_{ab})^t = -(C\Gamma_{ab}),$$

i.e.

$$(C\Gamma_{ab})^t = (C\Gamma_{ab}). \tag{6.3}$$

Thus, the bispinor ϵ_{AB} is symmetric.

Using properties of *gamma*-matrices, we can invert equation (6.1)

$$F_{ab} = i\, tr(\Gamma_{ab}\epsilon), \tag{6.4}$$

where $tr(\Gamma_{ab}\epsilon) = tr(\Gamma_{ab}C^{-1}\epsilon) = \Gamma_{ab}{}^{A}{}_{B}C^{BC}\epsilon_{CA}$.

Note that while the 2-form is real, the bispinor is generally complex since there is no Majorana representation of the Clifford algebra in five dimensions. A complex bispinor has 20 real independent components, whereas a real 2-form has 10 real components. Therefore, the bispinor must satisfy a reality condition, which halves its number of independent components.

Using the definitions of the Dirac and charge conjugation matrices (see appendix

[2] See appendix 6.A for conventions used for the 5d Clifford algebra.

6.A) one can derive the following relation between Γ^*_{ab} and Γ_{ab}

$$\Gamma^*_{ab} = A\,\Gamma_{ab}\,A^{-1}, \tag{6.5}$$

where $A = (CB^{-1})^t$. Now, taking the complex conjugate of equation (6.1) and using the equation above gives

$$\epsilon_{\dot{A}\dot{B}} = -\frac{i}{8} F_{ab}\, A_{\dot{A}}{}^{A} \Gamma^{ab}{}_{AB} (A^{-1})^{B}{}_{\dot{B}},$$

where $A_{\dot{A}}{}^{A} = (CAC^{-1})_{\dot{A}}{}^{A} = -(A^{-1})^{A}{}_{\dot{A}}$ [139]. Then re-arranging the above equation gives

$$\epsilon_{AB} = \bar{\epsilon}_{AB}, \tag{6.6}$$

where $\bar{\epsilon}_{AB} \equiv \epsilon_{\dot{A}\dot{B}} A^{\dot{A}}{}_{A} A^{\dot{B}}{}_{B}$.

A 2-form field is said to be *algebraically special* if the bispinor factorises. If this is the case, then the reality condition, equation (6.6), implies that [3]

$$\epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)}. \tag{6.7}$$

Then, the 2-form F is of the form

$$F_{ab} = i\bar{\epsilon}\Gamma_{ab}\epsilon \tag{6.8}$$

We can also form a real scalar and vector

$$f = \bar{\epsilon}\epsilon, \qquad V^a = i\bar{\epsilon}\Gamma^a\epsilon. \tag{6.9}$$

The Fierz identity can be used to relate the above three quantities [138]

$$V^2 = -f^2 \tag{6.10}$$
$$F^2 = F^{ab}F_{ab} = 4f^2 \tag{6.11}$$
$$\iota_V F = 0 \tag{6.12}$$
$$F_a{}^c F_c{}^b = -f^2 \delta_a{}^b - V_a V^b \tag{6.13}$$
$$F \wedge F = 2f \star V \tag{6.14}$$
$$fF = \star(V \wedge F). \tag{6.15}$$

Note that the above equations are not independent. In fact, equations (6.10) and (6.15)

[3] A sketch of the proof of this result is given in appendix 6.B.

can be used to derive equations (6.11)–(6.14).

6.2.2 De Smet classification

In four dimensions, the Petrov classification is most simply derived by defining a totally symmetric *Weyl spinor* $\Psi_{\alpha\beta\gamma\delta}$ and considering the *Weyl polynomial*

$$\Psi(\chi) = \Psi_{\alpha\beta\gamma\delta}\chi^\alpha\chi^\beta\chi^\gamma\chi^\delta$$

formed from the Weyl spinor, where χ^α is a general chiral spinor. The fundamental theorem of algebra ensures the factorisability of the polynomial and the Petrov classification reduces to an analysis of the multiplicity of the factors.

In similar vein, the spinor classification of the Weyl tensor in five dimensions [70] uses a spinorial approach to the classification of the Weyl tensor. Define the Weyl spinor, associated with the Weyl tensor, to be

$$C_{ABCD} = C_{abcd}\Gamma^{ab}{}_{AB}\Gamma^{cd}{}_{CD}. \tag{6.16}$$

The Weyl spinor is symmetric in its first and last pair of indices since $C\Gamma_{ab}$ is symmetric (see section 6.2.1). Also, using the symmetries of the Weyl tensor, it is symmetric under interchange of AB and CD. In five dimensions, the Fierz identity can be used to show that it is totally symmetric. The five dimensional Fierz identity is

$$M_{AB}N_{CD} = \frac{1}{4}C_{AD}(NM)_{CB} + \frac{1}{4}\Gamma_{eAD}(N\Gamma^e M)_{CB} - \frac{1}{8}\Gamma_{ef\,AD}(N\Gamma^{ef}M)_{CB}. \tag{6.17}$$

Letting $M = \Gamma^{ab}$ and $N = \Gamma^{cd}$, and multiplying by C_{abcd} gives

$$C_{abcd}\Gamma^{ab}{}_{AB}\Gamma^{cd}{}_{CD} = \frac{1}{4}C_{abcd}C_{AD}(\Gamma^{cd}\Gamma^{ab})_{CB} + \frac{1}{4}C_{abcd}\Gamma_{eAD}(\Gamma^{cd}\Gamma^e\Gamma^{ab})_{CB} - \frac{1}{8}C_{abcd}\Gamma_{ef\,AD}(\Gamma^{cd}\Gamma^{ef}\Gamma^{ab})_{CB}. \tag{6.18}$$

The trace free property of the Weyl tensor implies that $C_{abcd}\Gamma^a\Gamma^b\Gamma^c = C_{abcd}\Gamma^{abc} = C_{a[bcd]}\Gamma^{abc}$. Thus, the Bianchi identity gives

$$C_{abcd}\Gamma^a\Gamma^b\Gamma^c = C_{abcd}\Gamma^b\Gamma^c\Gamma^d = 0.$$

Therefore, equation (6.18) reduces to

$$C_{ABCD} = \frac{1}{4} C_{abcd} \Gamma_{e\,AD} ([\Gamma^{cd}, \Gamma^{e}] \Gamma^{ab})_{CB} - \frac{1}{8} C_{abcd} \Gamma_{ef\,AD} ([\Gamma^{cd}, \Gamma^{ef}] \Gamma^{ab})_{CB}. \tag{6.19}$$

Using the following identities

$$[\Gamma^{ab}, \Gamma^{c}] = 2(g^{bc}\Gamma^{a} - g^{ac}\Gamma^{b}), \tag{6.20}$$

$$[\Gamma^{ab}, \Gamma^{cd}] = 2(g^{bc}\Gamma^{ad} + g^{ad}\Gamma^{bc} - g^{ac}\Gamma^{bd} - g^{bd}\Gamma^{ac}), \tag{6.21}$$

equation (6.19) reduces to

$$C_{ABCD} = C_{ADCB}.$$

Therefore, the Weyl spinor is totally symmetric [4]

$$C_{ABCD} = C_{(ABCD)}. \tag{6.22}$$

As with the case of the 2-form in section 6.2.1, the Weyl tensor is real, while the Weyl spinor will in general be complex. The complex Weyl spinor has 70 real independent components, while the 5d Weyl tensor has 35 independent components. Thus, the Weyl spinor satisfies a reality condition, which halves its number of independent components.

Taking the complex conjugate of equation (6.16) and using equation (6.5) gives

$$C_{ABCD} = C_{\dot{A}\dot{B}\dot{C}\dot{D}} A^{\dot{A}}{}_{A} A^{\dot{B}}{}_{B} A^{\dot{C}}{}_{C} A^{\dot{D}}{}_{D}. \tag{6.23}$$

The De Smet classification involves the factorisability properties of the invariant Weyl polynomial

$$C(\psi) = C_{ABCD}\psi^{A}\psi^{B}\psi^{C}\psi^{D}, \tag{6.24}$$

where ψ is a general Dirac spinor. In contrast to the Petrov classification, in general, the polynomial above will not factorise. If it does factorise, the solution is said to be *algebraically special*. Each polynomial factor in the product is distinguished by its degree and multiplicity. There are 12 possibilities, as depicted in figure 6.1 [70].

The notation is such made that a number represents the degree of the polynomial

[4] The fact that C_{ABCD} as defined by equation (6.16) is totally symmetric depends very much on properties of the 5d Clifford algebra and the 5d Fierz identity. At least, with regard to the antisymmetry property of C, which is crucial in ensuring that $C\Gamma_{ab}$ is symmetric, this does not hold in $d = 7, 8, 9 \bmod 8$ [139]. That is, in these dimensions, a representation of the Clifford algebra for which C is antisymmetric does not exist.

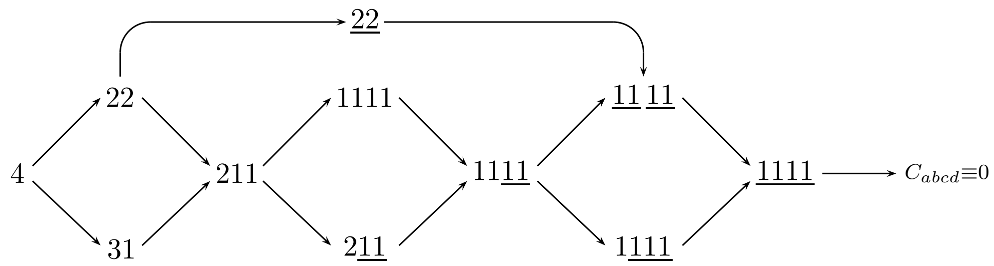


Fig. 6.1: The 12 different algebraic types in the spinor classification.

factor and an underline represents its multiplicity. For example, type $\underline{22}$ corresponds to the case where the Weyl polynomial factorises into two quadratic factors that are proportional to one another and cannot be further factorised. Type 4 solutions (for which the polynomial does not factorise) are said to be *algebraically general.*

For type 22 or more special solutions, we can learn more. For all such solutions, the Weyl spinor is of the form

$$C_{ABCD} = \epsilon_{(AB}\eta_{CD)}. \tag{6.25}$$

The reality condition, equation (6.23), reduces to

$$\epsilon_{(AB}\eta_{CD)} = \bar{\epsilon}_{(AB}\bar{\eta}_{CD)}, \tag{6.26}$$

where

$$\bar{\epsilon}_{AB} \equiv \epsilon_{\dot{A}\dot{B}} A^{\dot{A}}{}_{A} A^{\dot{B}}{}_{B}.$$

It can be shown that this implies that either [5]

$$\epsilon_{AB} = \bar{\epsilon}_{AB}, \qquad \eta_{AB} = \bar{\eta}_{AB}, \tag{6.27}$$

or

$$\epsilon_{AB} = \bar{\eta}_{AB}. \tag{6.28}$$

We can invert equation (6.16), so that the Weyl tensor is given in terms of the Weyl spinor, i.e.

$$C_{abcd} = \frac{1}{64}(\Gamma_{ab})^{AB}(\Gamma_{cd})^{CD}C_{ABCD}. \tag{6.29}$$

Using equation (6.25) and the 5d Fierz identity, one can derive the general form of the

[5] A sketch of the proof of this result is given in appendix 6.C.

Weyl tensor of type 22 or more special solutions [6]

$$\begin{aligned} C_{abcd} =& A_{a[c}B_{d]b} + B_{a[c}A_{d]b} - A_{ab}B_{cd} - B_{ab}A_{cd} - \frac{1}{2}A^{ef}B_{ef}g_{a[c}g_{d]b} \\ &- A_{ae}B^{e}{}_{[c}g_{d]b} - B_{ae}A^{e}{}_{[c}g_{d]b} + A_{be}B^{e}{}_{[c}g_{d]a} + B_{be}A^{e}{}_{[c}g_{d]a}, \end{aligned} \tag{6.30}$$

where

$$A_{ab} = i\,tr(\Gamma_{ab}\epsilon) \quad \text{and} \quad B_{ab} = i\,tr(\Gamma_{ab}\eta).$$

From the derivation of the reality condition in section 6.2.1, we find that reality conditions (6.27) and (6.28) translate to

$$A^*_{ab} = A_{ab}, \quad B^*_{ab} = B_{ab} \tag{6.31}$$

and

$$A^*_{ab} = B_{ab}, \tag{6.32}$$

respectively.

The reality condition constrains the Weyl spinor and we can use the results above to show that some types are not possible.

Type 1111

The Weyl spinor of type 1111 solutions is of the form

$$C_{ABCD} = \epsilon_A\epsilon_B\epsilon_C\epsilon_D. \tag{6.33}$$

Letting

$$\epsilon_{AB} = \epsilon_A\epsilon_B, \quad \eta_{AB} = \epsilon_A\epsilon_B,$$

reality conditions (6.27) and (6.28) both reduce to

$$\epsilon_A\epsilon_B = \bar{\epsilon}_A\bar{\epsilon}_B.$$

In appendix 6.B, we show that this implies a Majorana condition on ϵ

$$\epsilon_A \propto \bar{\epsilon}_A,$$

which gives that $\epsilon = 0$ since the Majorana condition has no non-trivial solutions in 5d.

[6] See appendix 6.D for the derivation of the form of the Weyl tensor of type 22 or more special solutions.

$\epsilon = 0$ contradicts the assumption that the solution is not conformally flat.

Type $1\underline{111}$

The Weyl spinor of type $1\underline{111}$ solutions is of the form

$$C_{ABCD} = \eta_{(A}\epsilon_B\epsilon_C\epsilon_{D)}, \tag{6.34}$$

where $\eta \not\propto \epsilon$. We can choose

$$\epsilon_{AB} = \epsilon_A\epsilon_B, \quad \eta_{AB} = \eta_{(A}\epsilon_{B)}.$$

Reality condition (6.27) gives two constraints, one of which is

$$\epsilon_A\epsilon_B = \bar{\epsilon}_A\bar{\epsilon}_B,$$

which contradicts the original assumption, as we showed for type $\underline{1111}$ solutions.

Reality condition (6.28) reduces to

$$\epsilon_A\epsilon_B = \bar{\epsilon}_{(A}\bar{\eta}_{B)}. \tag{6.35}$$

Since $\epsilon \neq 0$, this gives

$$\epsilon_A = \alpha\,\bar{\epsilon}_A + \beta\,\bar{\eta}_A.$$

Substituting this into equation (6.35) gives

$$\alpha^2\bar{\epsilon}_A\bar{\epsilon}_B + (2\alpha\beta - 1)\bar{\epsilon}_{(A}\bar{\eta}_{B)} + \beta^2\bar{\eta}_A\bar{\eta}_B = 0.$$

Using similar techniques to those used in appendix 6.B, it is not too difficult to show that this implies

$$\alpha = 0 \quad \text{or} \quad \beta = 0.$$

$\beta = 0$ gives a Majorana condition on ϵ, so $\alpha = 0$. Then equation (6.35) becomes

$$\beta^2\,\bar{\eta}_A\bar{\eta}_B = \bar{\epsilon}_{(A}\bar{\eta}_{B)}.$$

Since $\bar{\eta} \neq 0$, this implies that $\epsilon \propto \eta$. However, this contradicts the assumption that the solution is type $1\underline{111}$.

Type $11\underline{11}$

The Weyl spinor of type $11\underline{11}$ solutions is of the form

$$C_{ABCD} = \eta_{(A}\kappa_B\epsilon_C\epsilon_{D)}, \tag{6.36}$$

where none of the spinors are proportional to one another. Choose

$$\epsilon_{AB} = \epsilon_A\epsilon_B, \quad \eta_{AB} = \eta_{(A}\kappa_{B)}.$$

As before, reality condition (6.27) gives two constraints, one of which is

$$\epsilon_A\epsilon_B = \bar{\epsilon}_A\bar{\epsilon}_B,$$

which gives a contradiction.

Reality condition (6.28) reduces to

$$\epsilon_A\epsilon_B = \bar{\eta}_{(A}\bar{\kappa}_{B)}. \tag{6.37}$$

Since $\epsilon \neq 0$, this gives

$$\epsilon_A = \alpha\,\bar{\eta}_A + \beta\,\bar{\kappa}_A.$$

The arguments used for the analysis of type $1\underline{111}$ solutions apply to give

$$\epsilon \propto \bar{\eta}, \quad \text{or} \quad \epsilon \propto \bar{\kappa}.$$

As before, both these conditions give that $\eta \propto \kappa$, which contradicts the assumption that the solution is type $11\underline{11}$.

Type $2\underline{11}$

The Weyl spinor of type $2\underline{11}$ solutions is of the form

$$C_{ABCD} = \epsilon_{(AB}\eta_C\eta_{D)}, \tag{6.38}$$

where ϵ_{AB} does not factorise. Reality condition (6.28) gives that ϵ_{AB} factorises, contradicting the assumption that the solution is type $2\underline{11}$. Reality condition (6.27) gives a Majorana condition on η, since it implies that

$$\eta_A\eta_B = \bar{\eta}_A\bar{\eta}_B.$$

Thus, type $2\underline{11}$ solutions are also not possible.

In summary, we have shown that the reality condition on the Weyl spinor means that a solution cannot be of types $\underline{1111}$, $1\underline{111}$, $11\underline{11}$ and $2\underline{11}$. Therefore, the number of possible types reduces to eight. A revised version of figure 6.1 is drawn in figure 6.2.

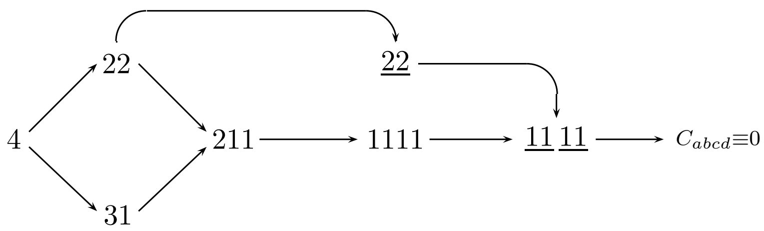


Fig. 6.2: Revised figure showing the 8 different algebraic types in the spinor classification.

Although, we considered the spinor classification of Lorentzian solutions in the analysis above, the spinor classification of Euclidean solutions is identical except that one uses the Euclidean Clifford algebra. The reason for this is that as with the 5d Lorentzian Clifford algebra, the 5d Euclidean Clifford algebra does not admit a Majorana representation. This means that the Weyl spinor of the Euclidean solution must satisfy the same reality condition as the Lorentzian case, i.e. equation (6.23), where A now defines a Majorana condition for the Euclidean Clifford algebra. However, since our arguments do not depend on specific properties of A, the same conclusions as those found above will follow.

6.2.3 Relation to Petrov classification

The five dimensional De Smet classification is a generalisation of the four dimensional Petrov classification insofar as it is concerned with the factorisability of a totally symmetric 4-spinor that is equivalent to the Weyl tensor. Here, we discuss the relation between the two classification schemes.

The 4d analogue of the De Smet Weyl spinor is

$$C_{ABCD} = C_{abcd}\gamma^{ab}{}_{AB}\gamma^{cd}{}_{CD}, \tag{6.39}$$

where γ^a form a representation of the 4d Clifford algebra. We need to show that the 4d De Smet Weyl spinor defined above is totally symmetric. We can do this by using the results found in 5d.

In the definition of the 5d Weyl spinor (equation (6.16)) restrict the indices to take values $0, \ldots, 3$ so that

$$C_{ABCD} = C_{abcd}(\gamma_0\gamma_5\gamma^{ab})_{AB}(\gamma_0\gamma_5\gamma^{cd})_{CD}, \tag{6.40}$$

where we have used the Clifford algebra representation defined in (6.169), for which $C = \gamma_0\gamma_5$. Lower case Latin indices range now from 0 to 3. Using the definition of γ_5 we find that

$$\gamma_5\gamma^{ab} = \frac{1}{2}\varepsilon^{ab}{}_{ef}\gamma^{ef},$$

where ε_{abcd} is the Levi-Civita or permutation tensor. Given the relation [140]

$$C_{efgh} = \frac{1}{4}\varepsilon^{ab}{}_{ef}\varepsilon^{cd}{}_{gh}C_{abcd},$$

equation (6.40) reduces to

$$C_{ABCD} = C_{abcd}(\gamma_0\gamma^{ab})_{AB}(\gamma_0\gamma^{cd})_{CD}.$$

Choosing the four-dimensional charge conjugation matrix, $C = \gamma_0$ gives the 4d De Smet Weyl spinor

$$C_{ABCD} = C_{abcd}\gamma^{ab}{}_{AB}\gamma^{cd}{}_{CD}, \tag{6.41}$$

which must be totally symmetric since the spinor we began with is totally symmetric.

In 4d, the Weyl tensor has 10 independent components, while a general totally symmetric 4-spinor has 35 complex independent components. However, the definition of the 4d De Smet Weyl spinor using the 4d Weyl tensor in equation (6.39) ensures that it has 10 complex independent components. Put another way, the symmetries of the Weyl tensor in 5d give that the Weyl spinor is totally symmetric, whereas in 4d, the symmetries give more constraints on the spinor, including the condition that it be totally symmetric. The fact that the Weyl tensor is real further constrains the De Smet Weyl spinor via a reality condition that halves its number of real independent components to 10.

In the Petrov classification, the homomorphism between $SL(2,\mathbb{C})$ and the Lorentz group is used to relate chiral spinor and Lorentz indices

$$X_{\alpha\dot{\alpha}} = iX^a\sigma_{a\,\alpha\dot{\alpha}}, \qquad X^a = \frac{i}{2}X_{\alpha\dot{\alpha}}\bar{\sigma}^{a\,\dot{\alpha}\alpha}, \tag{6.42}$$

where α and $\dot{\alpha}$ are left-handed and right-handed chiral spinor indices, respectively, $\sigma^a = (1, \sigma^i)$ and $\bar{\sigma} = (1, -\vec{\sigma})$.

Using the symmetries of the Weyl tensor and spinor calculus, it can be shown that the spinor equivalent of the Weyl tensor

$$C_{\alpha\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \sigma^a{}_{\alpha\dot{\alpha}}\sigma^b{}_{\beta\dot{\beta}}\sigma^c{}_{\gamma\dot{\gamma}}\sigma^d{}_{\delta\dot{\delta}}C_{abcd}, \tag{6.43}$$

is equivalent to a totally symmetric spinor $\Psi_{\alpha\beta\gamma\delta}$, known as the *Weyl spinor* in the Petrov classification [20, 141]

$$C_{\alpha\beta\gamma\delta\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}} = \Psi_{\alpha\beta\gamma\delta}\varepsilon_{\dot{\alpha}\dot{\beta}}\varepsilon_{\dot{\gamma}\dot{\delta}} + \varepsilon_{\alpha\beta}\varepsilon_{\gamma\delta}\bar{\Psi}_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}, \tag{6.44}$$

where $\varepsilon_{\alpha\beta} = \varepsilon_{\dot{\alpha}\dot{\beta}}$ are the alternating tensors, which can be used to lower undotted and dotted indices, i.e. they act as charge conjugation matrices for chiral spinors. Now, multiplying equation (6.43) with $\varepsilon^{\dot{\alpha}\dot{\beta}} = -\varepsilon_{\dot{\alpha}\dot{\beta}}$, and contracting over dotted indices gives

$$\Psi_{\alpha\beta\gamma\delta} = C_{abcd}\zeta^{ab}{}_{\alpha\beta}\zeta^{cd}{}_{\gamma\delta}, \tag{6.45}$$

where equation (6.44) has been used and

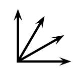
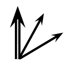
$$\zeta^{ab}{}_{\alpha\beta} = \frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^a{}_{\alpha\dot{\alpha}}\sigma^b{}_{\beta\dot{\beta}} \tag{6.46}$$

are the Lorentz algebra generators. The fundamental theorem of algebra guarantees that $\Psi_{\alpha\beta\gamma\delta}$ factorises

$$\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\beta_{\beta}\gamma_{\gamma}\delta_{\delta)}. \tag{6.47}$$

The Petrov classification concerns the multiplicity of the factors in (6.47), with the Petrov types defined in table 6.1 [35] (see figure 6.3).

Table 6.1: The Petrov classification of the Weyl tensor

Petrov type	Multiplicities	Diagram
I	(1,1,1,1)	
II	(2,1,1)	
D	(2,2)	
III	(3,1)	
N	(4)	
O	($C_{abcd} \equiv 0$)	–

Going back to the De Smet Weyl spinor in four dimensions,

$$C_{ABCD} = C_{abcd}\gamma^{ab}{}_{AB}\gamma^{cd}{}_{CD},$$

we work in a chiral representation given by

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ -\bar{\sigma}^a & 0 \end{pmatrix} \tag{6.48}$$

in the hope of relating the De Smet Weyl spinor to the Petrov Weyl spinor $\Psi_{\alpha\beta\gamma\delta}$ as defined in (6.45). The reality condition is

$$C_{ABCD} = C_{\dot{A}\dot{B}\dot{C}\dot{D}}A^{\dot{A}}{}_A A^{\dot{B}}{}_B A^{\dot{C}}{}_C A^{\dot{D}}{}_D, \tag{6.49}$$

where

$$A = \begin{pmatrix} 0 & -i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix}. \tag{6.50}$$

In the chiral representation, $\gamma^{ab}{}_{AB}$ are block diagonal. Thus, using the fact that C_{ABCD} is totally symmetric, we deduce that

$$C_{ABCD} = (C^{\alpha\beta\gamma\delta}, C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}), \tag{6.51}$$

where

$$C_{\alpha\beta\gamma\delta} = C_{abcd}\gamma^{ab}{}_{\alpha\beta}\gamma^{cd}{}_{\gamma\delta},$$

i.e. mixed components vanish. $C_{\alpha\beta\gamma\delta}$ and $C_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ are related via reality condition (6.49). But, $\gamma^{ab}{}_{\alpha\beta}$=$\zeta^{ab}{}_{\alpha\beta}$, and so

$$C_{\alpha\beta\gamma\delta} = \Psi_{\alpha\beta\gamma\delta}, \tag{6.52}$$

i.e. the undotted part of the De Smet Weyl spinor in four dimensions is the Petrov Weyl spinor. Equivalently, the Petrov polynomial

$$\Psi_{\alpha\beta\gamma\delta}\chi^\alpha\chi^\beta\chi^\gamma\chi^\delta = C_{ABCD}\psi^A\psi^B\psi^C\psi^D, \tag{6.53}$$

where $\psi = \begin{pmatrix} \chi \\ 0 \end{pmatrix}$.

We can move to a Majorana representation by performing a similarity transformation

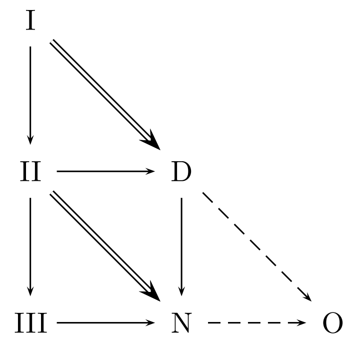


Fig. 6.3: Penrose diagram of the Petrov classification

such that a Majorana spinor in the chiral representation

$$\begin{pmatrix} \chi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix} \longrightarrow \sqrt{2} \begin{pmatrix} \mathrm{Re}\chi \\ \mathrm{Im}\chi \end{pmatrix} \tag{6.54}$$

in the Majorana representation. In a Majorana representation, the De Smet Weyl spinor is real. Thus, the De Smet polynomial

$$C(\psi) = C_{ABCD}\psi^A\psi^B\psi^C\psi^D, \tag{6.55}$$

where ψ is an arbitrary real (Majorana) spinor given by equation (6.54), is real. Thus, the De Smet classification in 4d can be viewed as a classification of the Weyl tensor using Majorana spinors, in contrast to the Petrov classification, which uses chiral spinors. Rewriting the polynomial as

$$C(\psi) = C_{abcd}\gamma^{ab}(\chi)\gamma^{cd}(\chi),$$

where $\gamma^{ab}(\chi) = \gamma^{ab}{}_{AB}\psi^A\psi^B$, it can be shown by direct calculation that

$$\gamma^{ab}(\chi) = \zeta^{ab}(\chi) + \zeta^{ab}(\chi)^*,$$

where $\zeta^{ab}(\chi) = \zeta^{ab}{}_{\alpha\beta}\chi^\alpha\chi^\beta$, so that

$$C(\psi) = \Psi(\chi) + \Psi(\chi)^* + 2C_{abcd}\zeta^{ab}(\chi)^*\zeta^{cd}(\chi),$$

where $\Psi(\chi) = C_{abcd}\zeta^{ab}(\chi)\zeta^{cd}(\chi)$ is the Petrov polynomial. The tracefree property of the Weyl tensor implies

$$C_{abcd}\zeta^{ab}(\chi)^*\zeta^{cd}(\chi) = 0,$$

which implies

$$C(\psi) = \Psi(\chi) + \Psi(\chi)^*. \tag{6.56}$$

This can be used to relate De Smet types in four dimensions to Petrov types. For example, assume that the solution is type N. Then,

$$\Psi(\chi) = \omega^4. \tag{6.57}$$

Equation (6.56) gives

$$C(\psi) = \omega^4 + \omega^{*\,4} = (\omega + \sqrt{-i}\,\omega^*)(\omega - \sqrt{-i}\,\omega^*)(\omega + \sqrt{i}\,\omega^*)(\omega - \sqrt{i}\,\omega^*). \tag{6.58}$$

Hence, type N solutions are type 1111 in the 4d De Smet classification.

We can consider the other Petrov types in a similar manner. The results are summarised in table 6.2. What we find is that type I, II and III solutions are all algebraically general in the 4d De Smet classification, while type D and N solutions are type 22 and 1111, respectively. Thus, the De Smet or Majorana spinor classification of the Weyl tensor in 4d is a coarse version of the Petrov or chiral spinor classification.

Table 6.2: Relation of De Smet classification in 4d to Petrov classification

De Smet type	Petrov types
4	I, II, III
22	D
1111	N

The reason why other De Smet types are not possible goes back to the definition of the 4d De Smet Weyl spinor via the 4d Weyl tensor, in equation (6.39). As discussed before, the symmetries of the 4d Weyl tensor imply not only that the spinor is totally symmetric, but give further conditions. It is these further conditions in addition to the reality condition that constrains the spinor in such a way that it can only admit three types.

For a Euclidean solution, we need to consider the Petrov classification of Euclidean solutions. Starting from the chiral representation of the Lorentzian Clifford algebra (6.48), we can define a chiral representation for the Euclidean Clifford algebra by setting $\gamma^4 = i\gamma^0$, i.e. γ^a is given by

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \quad \gamma^4 = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{6.59}$$

where $i = 1$, 2 or 3. As with the Lorentzian case, since $\gamma^{ab}{}_{AB}$ are block-diagonal and C_{ABCD} is totally symmetric, we conclude that

$$C_{ABCD} = (\Psi^{\alpha\beta\gamma\delta}, \Psi_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}), \tag{6.60}$$

where $\Psi_{\alpha\beta\gamma\delta}$ and $\Psi_{\dot{\alpha}\dot{\beta}\dot{\gamma}\dot{\delta}}$ are the spinor equivalents of the self-dual and anti-self-dual parts of the Weyl tensor. Unlike the Lorentzian case, these two parts are independent of each other and this is manifested in the reality condition. The reality condition on the Weyl spinor is

$$C_{ABCD} = C_{\dot{A}\dot{B}\dot{C}\dot{D}} A^{\dot{A}}{}_{A} A^{\dot{B}}{}_{B} A^{\dot{C}}{}_{C} A^{\dot{D}}{}_{D}, \tag{6.61}$$

where

$$A = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \tag{6.62}$$

i.e. for the Euclidean case, A, which defines the reality condition, is block-diagonal (compare this to (6.50)). Thus, rather than relating the two different parts of the Weyl spinor to one another, the reality condition places conditions on each part separately. Analysing these conditions leads to the Petrov classification of Euclidean geometries [142].

Table 6.3: De Smet classification and Petrov classification of 4d Euclidean metrics

De Smet type	Petrov types
4	(I,I), (I,D), (D,I)
22	(D,D)
1111	(I,O), (O,I)
<u>11</u> <u>11</u>	(D,O), (O,D)

In the Petrov classification of Euclidean metrics, the two independent parts are classified separately and the Petrov type of the geometry is given as a pair consisting of the Petrov type of each part. The reality condition implies that the Petrov types of the self-dual and anti-self-dual parts can only be I, D or O, leading to nine different cases. The relation of these types to the De Smet types of the 4d geometry is given in table 6.3.

6.2.4 Direct product solutions

In [85], direct and warped product solutions to vacuum Einstein equations are studied in the context of the CMPP classification and it is found that solutions with a one-dimensional Lorentzian factor are necessarily of types G, I, D or O. It is also found that solutions with a two-dimensional Lorentzian factor can only be of types D or O.

Here, we study direct product solutions in the context of De Smet classification. The results below directly generalise to warped product manifolds for which the conformally related product manifold is again an Einstein manifold, since the Weyl tensors are conformally related. Let

$$g_{ab}(x^a) = g_{\Gamma\Delta}(x^\Gamma) \oplus g_{\mu\nu}(x^\mu), \tag{6.63}$$

where $g_{\Gamma\Delta}$ is the metric of a n-dimensional Lorentzian manifold, and $g_{\mu\nu}$ is the metric of a $(5-n)$-dimensional Euclidean manifold. In such a setting, a tensor that splits like the metric—it has no mixed components and components belonging to one submanifold depend only on the coordinates covering that manifold—is known as a *product-object* or as being *decomposable*. The Riemann tensor and its contractions are product-objects [143]. However, the Weyl tensor is in general not decomposable.

Assuming that the spacetime solves the vacuum Einstein equations

$$R_{ab} = \Lambda g_{ab} \qquad \Longleftrightarrow \qquad R_{\Gamma\Delta} = \Lambda g_{\Gamma\Delta},\ R_{\mu\nu} = \Lambda g_{\mu\nu}, \tag{6.64}$$

the mixed Weyl tensor components are

$$\begin{aligned} C_{\Gamma\Delta\Theta\mu} =& C_{\Gamma\Delta\mu\nu} = C_{\Gamma\mu\nu\rho} = 0, \\ C_{\Gamma\mu\Delta\nu} =& -\frac{\Lambda}{4} g_{\Gamma\Delta} g_{\mu\nu}. \end{aligned} \tag{6.65}$$

The non-mixed Weyl tensor components are

$$\begin{aligned} C_{\Gamma\Delta\Theta\Lambda} &= \begin{cases} 0, & n=1, \\ {}^{(1)}C_{\Gamma\Delta\Theta\Lambda} + \dfrac{(5-n)\Lambda}{2(n-1)} g_{\Gamma[\Theta} g_{\Lambda]\Delta}, & n \geq 2, \end{cases} \\ C_{\mu\nu\rho\sigma} &= \begin{cases} {}^{(2)}C_{\mu\nu\rho\sigma} + \dfrac{n\Lambda}{2(4-n)} g_{\mu[\rho} g_{\sigma]\nu}, & n \leq 4, \\ 0, & n=4, \end{cases} \end{aligned} \tag{6.66}$$

where ${}^{(1)}C_{\Gamma\Delta\Theta\Lambda}$ and ${}^{(2)}C_{\mu\nu\rho\sigma}$ are the Weyl tensors derived from metrics $g_{\Gamma\Delta}$ and $g_{\mu\nu}$, respectively [85].

For $n = 1$ and $n = 4$, the vacuum Einstein equations (6.64) give that $\Lambda = 0$. For $n = 4$, the Weyl spinor is

$$C_{ABCD} = {}^{(1)}C_{ABCD}, \tag{6.67}$$

i.e. the De Smet type of the solution is equal to the De Smet type of the four dimensional submanifold. Thus, from section 6.2.3, we know that the De Smet type can only be one of 4, 22 or 1111. The relation between the De Smet type of the 5d solution and the Petrov type of the 4d factor is given in table 6.2. The black string is an example of a direct product solution (with $n = 4$). It is formed from the direct product of 4d type D Schwarzschild solution with a line, which means it is type 22 in De Smet classification [71].

For $n = 1$, the story is similar:

$$C_{ABCD} = {}^{(2)}C_{ABCD}, \tag{6.68}$$

and the De Smet type can only be one of 4, 22, 1111 or $\underline{11}\ \underline{11}$, as is shown in section 6.2.3. The relation between the De Smet type of the 5d solution and the Petrov type of the 4d factor is given in table 6.3.

For $n = 2$, ${}^{(1)}C_{\Gamma\Delta\Theta\Lambda} = {}^{(2)}C_{\mu\nu\rho\sigma} = 0$. Using a vielbein, the Weyl spinor is

$$C_{ABCD} = \frac{\Lambda}{4}\left[3\Gamma^{\Gamma\Delta}{}_{AB}\Gamma_{\Gamma\Delta CD} - 4\Gamma^{\Gamma\mu}{}_{AB}\Gamma_{\Gamma\mu CD} + 2\Gamma^{\mu\nu}{}_{AB}\Gamma_{\mu\nu CD}\right]. \tag{6.69}$$

If $\Lambda = 0$, the solution is conformally flat. If $\Lambda \neq 0$, the Weyl polynomial is

$$C(\psi) = -24\Lambda(vw - uz)^2, \tag{6.70}$$

where $\psi = (u, v, w, z)$. Hence, such solutions are type $\underline{22}$. This result mirrors that found in [85], where it is shown that $n = 2$ product solutions can only be types D and O.

For $n = 3$, as with the $n = 2$ case, we have ${}^{(1)}C_{\Gamma\Delta\Theta\Lambda} = {}^{(2)}C_{\mu\nu\rho\sigma} = 0$ and the Weyl spinor is

$$C_{ABCD} = \frac{\Lambda}{4}\left[2\Gamma^{\Gamma\Delta}{}_{AB}\Gamma_{\Gamma\Delta CD} - 4\Gamma^{\Gamma\mu}{}_{AB}\Gamma_{\Gamma\mu CD} + 3\Gamma^{\mu\nu}{}_{AB}\Gamma_{\mu\nu CD}\right]. \tag{6.71}$$

If the solution is not conformally flat, then $\Lambda \neq 0$ and

$$C(\psi) = -24\Lambda(uw - vz)^2. \tag{6.72}$$

Hence, such solutions are also type $\underline{22}$.

6.3 Connection between tensor and spinor classifications

6.3.1 Connection between tensor and spinor classifications of two-form

A non-vanishing real two-form F can be classified in two different ways. There is the tensorial approach analogous to the CMPP classification of the Weyl tensor whereby one looks for a null vector ℓ such that in the null frame (ℓ, n, m^i) with $\ell \cdot n = -1$, $F_{0i} = 0$. If this is the case, the 2-form is said to be aligned with respect to ℓ and of type I. If no such ℓ exists, then the 2-form is of type G. The 2-form is algebraically special (or type II) if $F_{0i} = F_{01} = F_{ij} = 0$.

We also have the spinorial approach outlined in section 6.2.1, whereby the 2-form is algebraically special if and only if its bispinor equivalent ϵ_{AB} factorises, i.e. $\epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)}$. The fact that the factors are conjugates of one another is implied by the reality condition, as shown in section 6.2.1. We call algebraically special 2-form fields type 11 fields in analogy with the De Smet classification of the Weyl tensor. Type 11 fields can be further classified by considering whether the scalar $f = \bar{\epsilon}\epsilon$ vanishes or not. If the bispinor does not factorise, we say that the field is type 2.

The question we address in this section is how are the different types in the two classification schemes related to one another?

From Fierz identities (6.10)–(6.12), we know that given a 2-form F with bispinor equivalent ϵ_{AB}

$$\epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)} \qquad \Longrightarrow \qquad V^a F_{ab} = 0 \;\text{ and }\; F^{ab}F_{ab} = -4V^2$$

for $V^a = i\bar{\epsilon}\Gamma^a\epsilon$. For the case where V is null, the converse can also be shown by direct computation, i.e. given a null vector V and 2-form F with bispinor equivalent ϵ_{AB}

$$V^a F_{ab} = 0 \;\text{ and }\; F^{ab}F_{ab} = 0 \qquad \Longrightarrow \qquad \epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)}.$$

We prove this by taking a general bispinor, finding its equivalent 2-form F and showing that constraining it as on the left hand side above gives that the bispinor factorises. The reality condition gives that the factorisation must be of the form given on the right hand side and it turns out that $V^a \propto i\bar{\epsilon}\Gamma^a\epsilon$.

From the above two results, we conclude that the 2-form field F is

$$\text{Type 11 } (f = 0) \quad \Longleftrightarrow \quad \text{Type II}, \tag{6.73}$$

$$\text{Type 11 } (f \neq 0) \quad \Longrightarrow \quad \text{Type G}. \tag{6.74}$$

Assume the field F is type 2. From (6.73), we know that the field cannot be type II. However, there could still exist a null vector ℓ such that in the null frame (ℓ, n, m^i) $F_{0i} = 0$, i.e. the field can be type I. If no such ℓ exists, then the 2-form is type G. These results are summarised in table 6.4.

Table 6.4: Relation between spinor and tensor types for 2-form

Spinor type	Possible tensor types
2	G, I
11	G, II

6.3.2 Connection between CMPP and De Smet classifications

It is known that the definition of algebraic specialness in the CMPP and De Smet classification schemes do not agree [135, 1]. Indeed, we showed in section 6.2.4 that the direct product of any type II or III 4d Ricci-flat solution with a line is algebraically general in the De Smet sense, but special in the CMPP sense. Moreover, it is not even the case that the CMPP classification is a refinement, because there are examples that are algebraically special in the De Smet classification and general in the CMPP classification. In this section, we investigate the connection between the two classification schemes.

Relation of CMPP types to De Smet types

We shall proceed by assuming a five dimensional solution to be of particular CMPP type and consider what this means in the De Smet classification. The vielbeins are chosen to be

$$e_{\hat{0}} = \frac{1}{\sqrt{2}}(\ell + n), \quad e_{\hat{1}} = \frac{1}{\sqrt{2}}(\ell - n), \quad e_{\hat{i}} = m_i,$$

where (ℓ, n, m^i) form a null frame, such that ℓ and n are null, $\ell \cdot n = -1$ and m^i are a set of $d-2$ orthonormal spacelike vectors orthogonal to ℓ and n. The implicit Latin letters in the equation above label null frame indices.

Now, using equations (6.16) and (6.24), where $\psi = (u, v, w, z)$ we derive the Weyl polynomials associated with type D, III and N solutions and consider how they may factorise.

Type N

For type N solutions the only non-zero components of the Weyl tensor are C_{1i1j}.

Using the trace-free property of the Weyl tensor ($C_{1i1}{}^{i} = 0$), we have five independent Weyl tensor components: C_{1313}, C_{1414}, C_{1213}, C_{1214}, and C_{1314}. Rotating $m_i \to m_i'$ such that C_{1i1j} is diagonal in the new frame and computing the Weyl polynomial gives

$$C(\psi) = 8\left[(2C_{1313} + C_{1414})(u^4 + v^4) - 6C_{1414}u^2v^2\right], \tag{6.75}$$

which factorises to give

$$C(\psi) = A(u + av)(u - av)(u + v/a)(u - v/a), \tag{6.76}$$

where $A = 8(2C_{1313} + C_{1414})$ and a is given by

$$a^2 + 1/a^2 = \frac{6C_{1414}}{2C_{1313} + C_{1414}},$$

assuming that $2C_{1313} + C_{1414} \neq 0$. If $2C_{1313} + C_{1414} = 0$, then $C(\psi) \propto u^2v^2$, which means the solution is type $\underline{11}\,\underline{11}$. Equation (6.76) implies that type N solutions are type 1111 or more special. If $a = \pm 1$, then the solution is of type $\underline{11}\,\underline{11}$. Hence, any type N solution must be of De Smet type 1111 or $\underline{11}\,\underline{11}$.

Type III

For type III solutions, the non-zero components of the Weyl tensor are C_{1i1j}, C_{1ijk} and C_{011i}. The thirteen independent components are chosen to be: C_{1212}, C_{1313}, C_{1213}, C_{1214}, C_{1314}, C_{1234}, C_{1342}, C_{1232}, C_{1242}, C_{1323}, C_{1343}, C_{1424} and C_{1434}. Again, rotating the frame as before so that C_{1i1j} is diagonal and computing the Weyl polynomial gives

$$\begin{aligned} C(\psi) =& 8\left\{(C_{1313} - C_{1212})(u^4 + v^4) + 6(C_{1212} + C_{1313})u^2v^2\right. \\ &- 2\sqrt{2}w[(C_{1342} - C_{1234} + iC_{1343} - iC_{1242})u^3 + 3(C_{1434} - iC_{1424})u^2v \\ &+ 3(C_{1234} + C_{1342} + iC_{1242} + iC_{1343})uv^2 + (2C_{1232} + C_{1434} + 2iC_{1323} + iC_{1424})v^3] \\ &+ 2\sqrt{2}z[(2C_{1232} + C_{1434} - 2iC_{1323} - iC_{1424})u^3 + 3(C_{1434} + iC_{1424})uv^2 \\ &\left.- 3(C_{1234} + C_{1342} - iC_{1242} - iC_{1343})u^2v - (C_{1342} - C_{1234} + iC_{1242} - iC_{1343})v^3]\right\}. \end{aligned} \tag{6.77}$$

Note that if all the coefficients of factors with w or z vanish, then this implies that the solution is type N giving a contradiction. It can be shown that the polynomial may factorise into cubic and linear factors. However, the conditions needed for the polynomial to factorise into two quadratic factors directly imply that one of the quadratic factors must factorise further into linear factors. The polynomial cannot be factorised any further without contradicting the assumption that the solution is type III. Hence, any type III solution must be of De Smet type 4, 31 or 211.

Type D

Finally, we consider type D solutions, for which the non-zero Weyl tensor components are C_{ijkl}, C_{0i1j}, C_{01ij} and C_{0101}. The nine independent components are chosen to be: C_{2323}, C_{2424}, C_{3434}, C_{0123}, C_{0124}, C_{0134}, C_{2324}, C_{3234} and C_{2434}. We can rotate the spacelike basis vectors m_i such that the symmetric part of $\Phi_{ij} \equiv C_{0i1j}$ is diagonal. Then, computing the Weyl polynomial gives

$$\begin{aligned} C(\psi) = 16 \{ & u^2[3(\Phi_{22} - \Phi_{33})w^2 + 2(\Phi_{24} + (2+i)\Phi_{34})wz \\ & - (3(\Phi_{22} + \Phi_{33}) + 6i\Phi_{23} + 4i\Phi_{24})z^2] \\ & + 2uv[-(\Phi_{24} + (2+i)\Phi_{34})w^2 + 6\Phi_{44}wz + (\Phi_{24} + (2-i)\Phi_{34})z^2] \\ & - v^2[(3(\Phi_{22} + \Phi_{33}) - 6i\Phi_{23} - 4i\Phi_{24})w^2 + 2(\Phi_{24} + (2-i)\Phi_{34})wz \\ & - 3(\Phi_{22} - \Phi_{33})z^2]\} . \end{aligned} \tag{6.78}$$

If the polynomial factorises, then, to avoid a contradiction, it does so into two non-factorisable quadratic factors that may or may not be proportional to one another, or into four independent linear factors. Hence, any type D solution must of De Smet type 4, 22, $\underline{22}$ or 1111.

The results found above are summarized in table 6.5.

Relation of De Smet types to CMPP types

Now, we shall consider the different De Smet types and study what possible CMPP types they imply. As with the previous section, we shall not be able to examine all De Smet types. However, we shall study all algebraically special types, except type 31. All such solutions can be regarded as special cases of type 22 solutions, for which the form of the Weyl tensor is given in equation (6.30)

$$\begin{aligned} C_{abcd} = & A_{a[c}B_{d]b} + B_{a[c}A_{d]b} - A_{ab}B_{cd} - B_{ab}A_{cd} - \frac{1}{2}A^{ef}B_{ef}g_{a[c}g_{d]b} \\ & - A_{ae}B^e{}_{[c}g_{d]b} - B_{ae}A^e{}_{[c}g_{d]b} + A_{be}B^e{}_{[c}g_{d]a} + B_{be}A^e{}_{[c}g_{d]a}, \end{aligned} \tag{6.79}$$

where 2-forms $A_{ab} = i\,tr(\Gamma_{ab}\epsilon)$ and $B_{ab} = i\,tr(\Gamma_{cd}\eta)$ satisfy one of the following reality conditions

$$A^*_{ab} = A_{ab}, \quad B^*_{ab} = B_{ab} \tag{6.80}$$

or

$$A^*_{ab} = B_{ab}. \tag{6.81}$$

Also, we shall require results (6.73) and (6.74) and the generalisation of result (6.73) for 2-forms that do not satisfy any reality condition. If a 2-form F is type I, then it can be shown by direct computation that

$$\epsilon_{AB} = \epsilon_{(A}\kappa_{B)} \qquad \Longleftrightarrow \qquad F \text{ is type II}, \tag{6.82}$$

where ϵ_{AB} is the spinor equivalent of F.

Type 22

Working in null frame (ℓ, n, m^i) such that $\ell\cdot n = -1$, the $+2$ boost weight components of the Weyl tensor $\Omega_{ij} \equiv C_{0i0j}$ are of the form

$$\Omega_{ij} = A_{0k}B_{0k}\delta_{ij} - 3A_{0(i}B_{|0|j)}.$$

$A_{0i} = 0$ or $B_{0i} = 0$ is sufficient for $\Omega_{ij} = 0$. However, one can show that it is also necessary. $\Omega_{ij} = 0$ gives

$$A_{0k}B_{0k}\delta_{ij} = 3A_{0(i}B_{|0|j)}. \tag{6.83}$$

Assume that neither A_{0i} nor B_{0i} vanish. Contracting the above equation with $A_{0i}A_{0j}$ gives

$$A_{0k}B_{0k} = 0. \tag{6.84}$$

Hence, from equation (6.83)

$$A_{0(i}B_{|0|j)} = 0.$$

Contracting this with B_{0i} and using equation (6.84) gives

$$A_{0i} = 0.$$

But, this contradicts the original assumption that $A_{0i} \neq 0$. Therefore, either $A_{0i} = 0$ or $B_{0i} = 0$.

This means that the solution is type I or more special if and only if $A_{0i} = 0$ or

$B_{0i} = 0$. If there does not exist a ℓ such that A_{0i} or B_{0i} vanish then the solution is type G. Now, assume there exists a ℓ such that without loss of generality $A_{0i} = 0$.

Given that $A_{0i} = 0$, the solution is type II if and only if $\Psi_{ijk} \equiv C_{0ijk} = 0$.

$$\Psi_{ijk} = B_{0[j}A_{k]i} - A_{jk}B_{0i} + A_{01}B_{0[j}\delta_{k]i} - B_{0l}A_{l[j}\delta_{k]i}.$$

It can be shown that

$$\Psi_{ijk} = 0 \quad \Longleftrightarrow \quad A_{01} = A_{ij} = 0, \quad \text{or} \quad B_{0i} = 0. \tag{6.85}$$

From the result in equation (6.82), $A_{01} = A_{ij} = 0$ would imply that bispinor ϵ factorises, contradicting the assumption that the solution is type 22. Thus, B_{0i} must also vanish for the solution to be type II. This would be a further condition if reality conditions (6.80) are satisfied. However, $A_{0i} = 0 \implies B_{0i} = 0$ if reality condition (6.81) is satisfied.

Given that the solution is type II, then it can be shown that boost weight 0 components vanish if and only if

$$A_{01} = A_{ij} = 0, \quad \text{or} \quad B_{01} = B_{ij} = 0, \tag{6.86}$$

which would imply (using (6.82)) that one of the bispinors associated with A or B factorises, contradicting the original assumption.

However, if there exists an n such that $A_{1i} = B_{1i} = 0$, then the results above apply directly to give that the solution is type D. As before, $A_{1i} = 0 \iff B_{1i} = 0$ for reality condition (6.81).

To summarise, type 22 solutions are of CMPP types G, I, II or D, depending on whether the 2-forms A and B are aligned. An example of a type 22 solution that is algebraically general in the CMPP sense is the 'homogeneous wrapped object' of [70] [1].

Type $\underline{22}$

A special case of type 22 is when $A \propto B$. In this case the solution is type $\underline{22}$ and the Weyl tensor is completely determined by 2-form F; using equation (6.79)

$$C_{abcd} = 2(F_{a[c}F_{d]b} - F_{ab}F_{cd} - F_{ae}F^{e}{}_{[c}g_{d]b} + F_{be}F^{e}{}_{[c}g_{d]a}) - \frac{1}{2}F^2 g_{a[c}g_{d]b}, \tag{6.87}$$

where $F^2 = F^{ef}F_{ef}$. The reality condition is simply that F is real. Using the results derived when analysing type 22 solutions, the solution is type II if and only if there exists a ℓ such that $F_{0i} = 0$. If no such ℓ exists then the solution is type G. If, in addition, there exists a n such that $F_{1i} = 0$, then and only then is the solution type D. Any further constraint on F contradicts the original assumptions.

Thus, in summary, type $\underline{22}$ solutions can only be of CMPP type G, II and D. An example of a type $\underline{22}$ solution is the 5d Myers-Perry solution [144] (CMPP type D [123]). For this solution, the 2-form F that squares to give the Riemann tensor is conformal to a test Maxwell field on it.

Type 211

The solution is type 211 if one of the bispinors, for example η, factorises. Reality condition (6.81) contradicts the assumption that only one of the spinors factorises. One of the reality conditions in (6.80) implies that $\eta_{AB} = \eta_{(A}\bar{\eta}_{B)}$ (using the results of section 6.2.1), so that $B_{ab} = i\bar{\eta}\Gamma_{ab}\eta$. We can also form a vector from η, $V^a = i\bar{\eta}\Gamma^a\eta$. The result in (6.73) gives that $B_{0i} = B_{01} = B_{ij} = 0$ if and only if V is null. Thus, the analysis splits to two cases of V timelike or null.

If V is timelike, then the solution is type I if and only there exists a ℓ such that $A_{0i} = 0$. Otherwise, result (6.74) implies that the solution is type G. Any other constraints on A or B contradict the original assumptions.

If V is null, then choosing $\ell = V$ gives that the solution is type II. If, in addition, $A_{0i} = 0$ then solution is type III. Any further constraints on the 2-forms give contradictions.

To summarise, type 211 solutions can only be of CMPP types G, I, II and III.

Type 1111

For type 1111 solutions, both bispinors ϵ and η factorise, i.e. $\epsilon_{AB} = \zeta_{(A}\kappa_{B)}$ and $\eta_{AB} = \lambda_{(A}\mu_{B)}$. Using arguments very similar to those used in section 6.2.2 to show that type $11\underline{11}$ solutions are not possible, one can show that reality condition (6.81) implies that at least two of the spinors coincide, contradicting the assumption that they are distinct.

Reality conditions (6.80) give that $\epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)}$ and $\eta_{AB} = \eta_{(A}\bar{\eta}_{B)}$. As above, we can form two vectors, $V^a = i\bar{\epsilon}\Gamma^a\epsilon$ and $W^a = i\bar{\eta}\Gamma^a\eta$.

The solution is type G if and only if V and W are timelike. If only one of the vectors is null, then and only then is the solution type II. The solution is type D if and only if both vectors are null but not proportional to one another. Finally, the solution is type N if and only if both vectors are null and proportional to one another.

Thus, type 1111 solutions are of CMPP types G, II, D or N.

Type $\underline{11}\,\underline{11}$

Type $\underline{11}\,\underline{11}$ solutions are special cases of type 1111 solutions for which two pairs of spinors coincide. However, it is more useful to think of them as special cases of type $\underline{22}$ solutions for which the bispinor ϵ factorises. The reality condition gives that $\epsilon_{AB} = \epsilon_{(A}\bar{\epsilon}_{B)}$. Thus, the Weyl tensor is determined only from one spinor ϵ. Forming a vector from this, $V^a = i\bar{\epsilon}\Gamma^a\epsilon$, we find that the solution is type G if and only if V is timelike and type N if and only if it is null.

The results found above are summarised in table 6.6. These results are consistent with those found in section 6.3.2 (see table 6.5).

Table 6.5: Possible De Smet types given CMPP type

CMPP type	Possible De Smet types
D	4, 22, $\underline{22}$, 1111
III	4, 31, 211
N	1111, $\underline{11}\,\underline{11}$

Table 6.6: Possible CMPP types given De Smet type

De Smet type	Possible CMPP types
22	G, I, II, D
$\underline{22}$	G, II, D
211	G, I, II, III
1111	G, II, D, N
$\underline{11}\,\underline{11}$	G, N

6.3.3 De Smet classification of black ring

The results above show that a type G solution could be of any type in the De Smet classification. This can be used to study algebraically general solutions as defined by the CMPP classification. The singly rotating black ring solution [50] is a well-known example of a CMPP algebraically general five dimensional solution [136]. Therefore, it would be desirable to know the De Smet type of the black ring solution. The metric of the black ring can be written as [145, 8]

$$ds^2 = -\frac{F(y)}{F(x)}\left(dt - C\,R\frac{1+y}{F(y)}d\psi\right)^2 + \frac{R^2}{(x-y)^2}F(x)\left(\frac{dx^2}{G(x)} - \frac{dy^2}{G(y)} + \frac{G(x)}{F(x)}d\phi^2 - \frac{G(y)}{F(y)}d\psi^2\right), \tag{6.88}$$

where

$$F(\zeta) = 1 + \lambda\zeta, \qquad G(\zeta) = (1-\zeta^2)(1+\nu\zeta), \qquad C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}},$$

The parameters λ and ν are not independent and are related via an equation that will not be given here. Furthermore, they satisfy $0 < \nu \leq \lambda < 1$. The coordinates x and y lie in the ranges $-1 \leq x \leq 1$ and $-\infty \leq y \leq -1$. Asymptotic infinity is at $x = y = -1$, the ergosurface is at $y = -1/\lambda$ and inside this is the horizon at $y = -1/\nu$.

Choosing the following vielbein

$$e_{\hat{0}} = \sqrt{\frac{F(y)}{F(x)}}\left(dt - C\,R\frac{1+y}{F(y)}d\psi\right), \quad e_{\hat{1}} = \frac{R}{x-y}\sqrt{\frac{F(x)}{G(x)}}dx, \quad e_{\hat{2}} = \frac{R}{x-y}\sqrt{\frac{-F(x)}{G(y)}}dy,$$
$$e_{\hat{3}} = \frac{R}{x-y}\sqrt{G(x)}d\phi, \quad e_{\hat{4}} = \frac{R}{x-y}\sqrt{\frac{-F(x)G(y)}{F(y)}}d\psi, \tag{6.89}$$

which is well-defined everywhere, except at the horizon and ergosurface, the Weyl polynomial is

$$C(\psi) = \frac{6(x-y)}{R^2F(x)^3F(y)}\left\{A_1(vw+uz)(uw-vz) + i(u^2-v^2+w^2-z^2)[A_2(vw+uz) + A_3(uw-vz)] + A_4(u^4+v^4+w^4+z^4) + A_5uvwz + A_6\,(u^2z^2+v^2w^2) + A_7(u^2w^2+v^2z^2) + A_8(u^2v^2+w^2z^2)\right\}, \tag{6.90}$$

where $\psi = (u, v, w, z)$ and A_i are expressions involving $x,\ y,\ \lambda$ and ν given in appendix 6.E.

The polynomial above does not factorise, in general. Thus, the solution is algebraically general in the De Smet sense. If we take the static limit $\nu \to \lambda$, for which $C = 0$, the polynomial is again not factorisable, so the static black ring is also algebraically general.

6.4 2-forms and Weyl tensors defined by a spinor

6.4.1 Algebraically special 2-form fields

In four dimensions, the algebraic specialness of the Weyl tensor of a solution is related to the admittance by the solution of algebraically special electromagnetic fields. The Mariot-Robinson theorem [137, 35] states that in a null frame defined by null vector field V, a test null electromagnetic field F has only negative boost weight components (F is *algebraically special*), if and only if V defines a shear-free geodesic null congruence. But the Goldberg-Sachs theorem [36] states that a vacuum solution admits a shear-free geodesic null congruence if and only if the solution is algebraically special. Thus in 4d the algebraic specialness of a 2-form test field satisfying Maxwell equations and the algebraic specialness of the Weyl tensor coincide. Considering the property of algebraically special p-form fields in higher dimensions could help clarify the status of a higher dimensional generalisation of the Goldberg-Sachs theorem. Some progress has been made in this regard [146, 67] using the CMPP classification.

Here, we consider the algebraic specialness of a 2-form field in the context of the De Smet classification. We define a real 2-form field F to be algebraically special if and only if its bispinor equivalent

$$\epsilon_{AB} = \frac{i}{8} F_{ab} \Gamma^{ab}{}_{AB}, \tag{6.91}$$

factorises, i.e.

$$\epsilon_{AB} = \eta_{(A}\epsilon_{B)}. \tag{6.92}$$

From section 6.2.1, we find that requiring F to be real implies that $\eta_A \propto \bar{\epsilon}_A$ and hence

$$F_{ab} = i\bar{\epsilon}\Gamma_{ab}\epsilon. \tag{6.93}$$

Furthermore, we can construct a scalar f and vector V (defined in (6.9)) that are related to eachother and F via the Fierz identities (equations (6.10)–(6.15)).

The equations of motion for F are given by

$$dF = 0, \tag{6.94}$$

$$d \star F = \lambda F \wedge F, \tag{6.95}$$

where $\lambda = 0$ corresponds to Maxwell theory and $\lambda = -2/\sqrt{3}$ corresponds to minimal supergravity [138]. The analysis divides naturally into two cases of f zero and non-zero, corresponding to V null and timelike, respectively.

V null

For the null case, we showed in section 6.3.1 that this is equivalent to the algebraic specialness of F in the CMPP sense. Also, equation (6.14) reduces to

$$F \wedge F = 0,$$

so that

$$d \star F = 0, \tag{6.96}$$

i.e. F solves Maxwell equations. From equation (6.15), we have

$$F = V \wedge W, \tag{6.97}$$

for some 1-form W. Equations (6.10) and (6.13) imply that

$$V \cdot W = 0 \quad \text{and} \quad W^2 = 1,$$

respectively.

Such Maxwell fields have been studied in [67], where it is shown that V is geodesic; W is an eigenvector of the shear matrix and the wedge product of the rotation matrix with W vanishes, i.e.

$$V \cdot \nabla V \propto V, \tag{6.98}$$

and in null frame (V, n, m^i), $\rho_{ij} = \nabla_j V_i$ satisfies

$$\rho_{(ij)} W_j = \frac{\rho}{2} W_i, \tag{6.99}$$

$$\rho_{[ij]} = \frac{1}{2}(Y \wedge W)_{ij}, \tag{6.100}$$

for some 1-form Y.

V timelike

Taking Hodge dual and then exterior derivative of equation (6.15) gives

$$\left[\lambda F + d\left(\frac{V}{f}\right)\right] \wedge F = 0. \tag{6.101}$$

The interior product of the above equation with F implies

$$\left(4\lambda f^3 + \iota_F dV\right) F = 2f \left[\iota_V d\left(\frac{V}{f}\right)\right] \wedge V, \tag{6.102}$$

and the interior product of this equation with V gives

$$V \cdot \nabla V = \frac{V \cdot \nabla f}{f}\, V, \tag{6.103}$$

i.e. V defines a timelike geodesic congruence.

Taking exterior derivative and then Hodge dual of equation (6.15) and using equations of motion gives

$$df^a V_{[a}F_{bc]} = f\nabla^a (V_{[a}F_{bc]}),$$

which is equivalent to

$$f\left(F_{c[a}\nabla^c V_{b]} + F_{c[a}\nabla_{b]}V^c\right) - df^c F_{c[a}V_{b]} - (V \cdot \nabla f)F_{ab} = 0.$$

Interior product of this with F gives

$$f\nabla \cdot V + V \cdot \nabla f = 0. \tag{6.104}$$

Introduce coordinates (t, x^m) such that $V = \partial/\partial t$. The metric can then be written as

$$ds^2 = -f^2 \left(dt + \omega(x^p)\right)^2 + f^{-1} h_{mn}(t, x^p) dx^m dx^n, \tag{6.105}$$

where the manifold with metric h_{mn} will be referred to as the base space. ω is a 1-form with components only on the base space.

Equation (6.12) implies that one can regard F as a 2-form on the base space. Then, equation (6.13) gives

$$F_m{}^p F_p{}^n = -\delta_m{}^n, \tag{6.106}$$

where indices have been raised with respect to h_{mn}. Thus, $F_m{}^n$ defines an almost complex structure with respect to h_{mn}. Moreover, the Bianchi identity on F implies that F_{mn} is closed, i.e.

$${}^{(h)}\nabla_{[p}F_{mn]} = 0. \tag{6.107}$$

Thus, h_{mn} is an almost-Kähler metric.

Defining

$$\xi = f^{-1}V,$$

so that $\xi^2 = -1$ and $\xi\cdot\nabla\xi = 0$, the expansion, shear and rotation of the timelike congruence are

$$\theta = \nabla_a \xi_b (g^{ab} + \xi^a \xi^b) = -2f^{-2} V \cdot \nabla f, \tag{6.108}$$

$$\begin{aligned}\sigma_{ab} &= \nabla_{(a}\xi_{b)} - \frac{1}{4}\theta(g_{ab} + \xi_a\xi_b) = f^{-1}\nabla_{(a}V_{b)} - f^{-2}V_{(a}df_{b)} - \frac{1}{4}f^{-1}\theta h_{ab} \\ &= \frac{1}{2}f^{-2}(\mathcal{L}_V h_{ab}),\end{aligned} \tag{6.109}$$

$$\omega_{ab} = -\nabla_{[a}\xi_{b]} = f^{-2}df_{[a}V_{b]} - f^{-1}\nabla_{[a}V_{b]}, \tag{6.110}$$

respectively, where we have used equation (6.104) in the first line above..

The Bianchi identity and equation (6.12) give that

$$\mathcal{L}_V F = 0. \tag{6.111}$$

Rewriting equation (6.106) as

$$F_{mn}F_{pq}h^{np} = h_{mq} \tag{6.112}$$

and taking Lie derivative with respect to V gives

$$F_m{}^n F_q{}^p (\mathcal{L}_V h_{np}) = -(\mathcal{L}_V h_{mq})$$

or

$$F_m{}^n F_q{}^p \sigma_{np} + \sigma_{mq} = 0, \tag{6.113}$$

i.e. the $(1,1)-$part of the shear vanishes.

Therefore, for a timelike 2-form field, we have that the solution admits a timelike geodesic congruence and that the base space is almost-Kähler. Furthermore, the $(1,1)-$part of the shear vanishes.

6.4.2 Classification of type $\underline{11}\ \underline{11}$ solutions

Type $\underline{11}\,\underline{11}$ solutions are significant in that their Weyl tensors are fully determined from a single Dirac spinor. In this section, we consider type $\underline{11}\,\underline{11}$ solutions and use the Bianchi identity to classify them. The spinor ϵ that defines the type $\underline{11}\,\underline{11}$ solution can be used to form a real scalar, vector and 2-form as follows

$$f = \bar{\epsilon}\epsilon, \qquad V^a = i\bar{\epsilon}\Gamma^a\epsilon, \qquad F_{ab} = i\bar{\epsilon}\Gamma_{ab}\epsilon. \tag{6.114}$$

As noted in section 6.2.1, these quantities are related via Fierz identities (6.10)–(6.15).

Simplifying the general form of the Weyl tensor of type 22 or more special solutions (equation (6.30)) using the Fierz identities, we find that the Weyl tensor of type $\underline{11}\,\underline{11}$ solutions is of the form

$$C_{abcd} = 2\left(F_{a[c}F_{d]b} - F_{ab}F_{cd} + V_aV_{[c}g_{d]b} - V_bV_{[c}g_{d]a} + f^2 g_{a[c}g_{d]b}\right). \tag{6.115}$$

Assuming a vacuum Einstein solution

$$R_{ab} = \Lambda g_{ab}, \tag{6.116}$$

the Bianchi identity reduces to

$$C_{ab[cd;e]} = 0. \tag{6.117}$$

We shall use the above equation to find restrictions on the spinor. As in section 6.4.1, the analysis divides into two cases of V null and timelike.

V null

As shown in section 6.3.2, this class of type $\underline{11}\,\underline{11}$ solutions are type N in the CMPP classification (V is a multiple WAND). Type N solutions to vacuum Einstein equations have been studied before [64, 67]. In [67], it is shown that in the null frame (V, n, m_i), the optical matrix $\rho_{ij} = \nabla_j V_i$ is of the form

$$\rho_{ij} = \frac{1}{2}\begin{pmatrix} \rho & a & 0 \\ -a & \rho & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and that if the expansion $\rho = 0$ then $a = 0$, giving a Kundt solution.

The Fierz identities give

$$F = V \wedge W, \tag{6.118}$$

for some W satisfying $V \cdot W = 0$ and $W^2 = 1$. The Weyl tensor is of the form

$$C_{abcd} = -3F_{ab}F_{cd} + 2(V_aV_{[c}g_{d]b} - V_bV_{[c}g_{d]a}). \tag{6.119}$$

Then, $V^b C^a{}_{bcd;a} = 0$ and $W^b C^a{}_{bcd;a} = 0$ give

$$V \cdot \nabla V = (\nabla \cdot V) V, \tag{6.120}$$

i.e. V is geodesic. This has been shown before [64, 1]. Now, $F^{cd} C_{ab[cd;e]} = 0$ gives

$$\nabla \cdot V = 0, \tag{6.121}$$

i.e. $\rho = 0$, which means that we have a Kundt solution.

Choosing null frame $(V, n, W, m^{\hat{i}})$ where $\hat{i} = 3, 4$ and $V \cdot n = -1$, $C^a{}_{bcd;a} = 0$ gives

$$2L_{1\hat{j}} - \tau_{\hat{j}} = 0, \tag{6.122}$$

where

$$\nabla_a V_b = L_{11} V_a V_b + L_{1i}\, m_a^{(i)} V_b + \tau_i\, V_a m_b^{(i)}. \tag{6.123}$$

Then, $W^a C_{ab[cd;e]} = 0$ gives

$$V \wedge dW = 0, \tag{6.124}$$

which implies

$$V \cdot \nabla W \propto V, \qquad \text{and} \qquad W \cdot \nabla W \propto V,$$

where we have used $\rho_{ij} = 0$.

The above equations give that

$$[V, W] \propto V, \tag{6.125}$$

which, by Frobenius' theorem, gives that the distribution spanned by $\{V, W\}$ is integrable. Moreover, using hyper-surface orthogonality of V and equation (6.124), the dual formulation of Frobenius' theorem implies that the distribution spanned by $\{V, m^{\hat{i}}\}$ is also integrable. We also have such a structure for the null case in section 6.4.1.

The Bianchi identity then reduces to

$$\nabla_a W_b = \frac{1}{3}(\tau_2 - 2L_{12})\, m^{(\hat{i})}{}_a m_{(\hat{i})\, b} - V_a\, n \cdot \nabla W_b - V_a\, V_b\, (n^c n^d \nabla_c W_d) - n^c \nabla_a W_c\, V_b. \tag{6.126}$$

Furthermore, the only non-zero components of the Weyl tensor, $\Omega'_{ij} \equiv C_{1i1j}$ are of the form

$$\Omega'_{ij} = -3 W_i W_j + \delta_{ij},$$

i.e. Ω'_{ij} has two equal eigenvalues.

The information above does not seem to be strong enough to allow a complete classification of all such solutions. In summary, type $\underline{11}\,\underline{11}$ solutions to vacuum Einstein equations with $f = 0$ are type N Kundt solutions. In addition, the WAND is in two orthogonal integrable null distributions of dimensions two and three and Ω'_{ij} has two equal eigenvalues.

V timelike

Contracting the Bianchi identity with the inverse metric and using the trace-free property of the Weyl tensor gives

$$C^a{}_{bcd;a} = 0.$$

Then $F^{bc}C^a{}_{bcd;a} = 0$ gives

$$3f^2F^a{}_{d;a} + (f df^a - V \cdot \nabla V^a)F_{ad} = 0, \tag{6.127}$$

and $V^cC^a{}_{bcd;a} = 0$ gives

$$\begin{aligned} f^2\nabla_a V_b &+ (F^c{}_aF^d{}_b + 2F^c{}_bF^d{}_a)\nabla_c V_d \\ &- \nabla \cdot V(f^2 g_{ab} + V_aV_b) + f\, V \cdot \nabla f\, g_{ab} + V_aV \cdot \nabla V_b - f df_a V_b = 0. \end{aligned} \tag{6.128}$$

Using the Fierz identities and equation (6.127) to simplify $F^{ab}F_{ea;b}$, $F^{ab}F^{cd}C_{ab[cd;e]} = 0$ reduces to

$$3\, V \cdot \nabla f + f\, \nabla \cdot V = 0, \tag{6.129}$$

and a geodesity condition on V

$$f\, V \cdot \nabla V = (V \cdot \nabla f)V\,. \tag{6.130}$$

Using equation (6.128) to simplify $V^aF^{cd}C_{ab[cd;e]} = 0$ gives

$$\nabla_{(a}(f^{-1}V)_{b)} = \frac{1}{4}\nabla \cdot (f^{-1}V)(g_{ab} + f^{-2}V_aV_b) \tag{6.131}$$

and $F^{ab}C_{ab[cd;e]} = 0$ reduces to

$$d(f^{1/3}F) = \frac{2}{3}f^{-3}(V \cdot \nabla f)\, V \wedge (f^{1/3}F). \tag{6.132}$$

To simplify the above equations, define new vector and 2-form

$$W = f^{-1}V \quad \text{and} \quad J = f^{1/3}F. \tag{6.133}$$

Then, the equations above give that W forms a shear-free affinely parametrised geodesic congruence

$$W^2 = -1, \qquad W \cdot \nabla W = 0, \qquad \nabla_{(a}W_{b)} = \frac{1}{4}\nabla \cdot W(g_{ab} + W_a W_b), \tag{6.134}$$

while J satisfies

$$d \star J = 0, \qquad dJ = \frac{2}{3}f^{-1}(W \cdot \nabla f)\, W \wedge J. \tag{6.135}$$

Also, we have

$$dW_{ab} - f^{-8/3}J^c{}_a J^d{}_b dW_{cd} = 0. \tag{6.136}$$

Introduce coordinates (t, x^m) such that $W = \partial/\partial t$. The metric can then be written as

$$ds^2 = -\left(dt + \omega(x^p)\right)^2 + f^{-4/3}k_{mn}(t, x^p)dx^m dx^n, \tag{6.137}$$

where the manifold with metric k_{mn} is called the base space, and ω is a 1-form with components only on the base space. J lives on the base space and equation (6.13) gives that $J_m{}^n$ is an almost complex structure on it, i.e.

$$J^m{}_p J^p{}_n = -\delta^m{}_n, \tag{6.138}$$

where base space indices have been raised using k^{mn}, and will be done so hereafter. Equation (6.136) implies that dW is a $(1,1)-$form. Written in terms of base space components, $V^e C_{ab[cd;e]} = 0$ reduces to

$$k_{m[p}dW_{q]n} - k_{n[p}dW_{q]m} - dW_{mr}J^r{}_{[p}J_{q]n} + dW_{nr}J^r{}_{[p}J_{q]m} = 0. \tag{6.139}$$

Contracting the above equation with k^{mp} gives

$$dW_{nq} + J^m{}_n J^p{}_q dW_{mp} + (dW_{mp}J^{mp})J_{nq} = 0. \tag{6.140}$$

Now, contracting this with J^{nq} gives

$$dW_{mp}J^{mp} = 0.$$

Hence, from equation (6.140), we have

$$dW_{mn} + J^p{}_m J^q{}_n dW_{pq} = 0, \tag{6.141}$$

i.e. the $(1,1)-$part of dW vanishes. But, we have from before that dW is a $(1,1)-$form. This means that

$$dW = 0. \tag{6.142}$$

Remaining coordinate freedom can then be used to set

$$\omega(x^p) \equiv 0. \tag{6.143}$$

Finally, $F^{bc}C_{ab[cd;e]} = 0$ gives

$$^{(h)}\nabla_p J_{mn} = 0, \tag{6.144}$$

i.e. 2-form J is a Kähler form on the base space.

Now, taking the exterior derivative of the second equation in (6.135) and using equations (6.135) and (6.15) gives

$$d(f^{-1}W \cdot \nabla f)^a J_{ab} = 0$$

or

$$d(f^{-1}W \cdot \nabla f) \propto W.$$

Note that equation (6.129) gives that the expansion of the congruence defined by W

$$\theta = -4f^{-1}W \cdot \nabla f. \tag{6.145}$$

Thus,

$$d\theta = -(W \cdot \nabla\theta)\, W,$$

i.e. the expansion depends only on t:

$$\theta = \theta(t). \tag{6.146}$$

Since $W^a C_{abcd} = 0$, we can think of the Weyl tensor as living on the base space

$$C_{mnpq} = -2f^{-2/3}\left(J_{mn}J_{pq} - J_{m[p}J_{q]n} - k_{m[p}k_{q]n}\right). \tag{6.147}$$

Equation (6.14) gives that

$$\varepsilon_{mnpq} = \frac{1}{2}(J \wedge J)_{mnpq}, \tag{6.148}$$

where $\varepsilon_{mnpq} = f^{8/3} W^a \varepsilon_{amnpq}$ is the Levi-Civita tensor on the base space. Equation (6.15) gives that J is self-dual

$$\star_4 J = J. \tag{6.149}$$

Using the above two equations gives

$$\star_4 C_{mnpq} = \frac{1}{2}\varepsilon_{mn}{}^{rs} C_{rspq} = C_{mnpq}, \tag{6.150}$$

i.e. the 5d Weyl tensor with components restricted to the base space with metric k is self-dual.

Rewriting the metric as

$$ds^2 = -dt^2 + \tilde{h}_{mn}(t, x^p) dx^m dx^n,$$

the final equation that gives the shear-free property of the congruence defined by W in (6.134) reduces to

$$\partial_t \tilde{h}_{mn} = \frac{1}{2}\theta(t)\, \tilde{h}_{mn},$$

where we have used the fact that W is hyper-surface orthogonal and the result in (6.146) that the expansion is only time-dependent. The equation above can be integrated for each component to give

$$\tilde{h}_{mn}(t, x^p) = A(t)^2 h_{mn}(x^p), \tag{6.151}$$

for some function $A(t)$.

Thus, the metric can be written as

$$ds^2 = -dt^2 + A(t)^2 h_{mn}(x^p) dx^m dx^n, \tag{6.152}$$

where the expansion

$$\theta(t) = 4\, A'(t)/A(t), \tag{6.153}$$

and h is a conformally Kähler metric.

Defining L by

$$\Lambda = \frac{4\epsilon}{L^2}, \qquad \epsilon \in \{-1, 0, 1\}, \tag{6.154}$$

Raychaudhuri's equation

$$W \cdot \nabla\theta = -\frac{1}{4}\theta^2 + \Lambda,$$

reduces to

$$A''(t) - \frac{\epsilon}{L^2}A(t) = 0.$$

The first integral of this equation gives

$$A'(t)^2 - \frac{\epsilon}{L^2}A(t)^2 = -\eta, \tag{6.155}$$

where we can use the freedom in soaking up constants into metric h to normalise $\eta \in \{-1, 0, 1\}$.

Using equation (6.155), Einstein equations reduce to a constraint on the Ricci tensor of metric h

$$^{(h)}R_{mn} = 3\eta\ h_{mn}, \tag{6.156}$$

i.e. h is an Einstein metric.

Solving equation (6.155) gives

$A(t)$	$\eta = -1$	$\eta = 0$	$\eta = 1$
$\epsilon = -1$	$L\sin(t/L)$		
$\epsilon = 0$	t	1	
$\epsilon = 1$	$L\sinh(t/L)$	$e^{\pm t/L}$	$L\cosh(t/L)$

Furthermore, because of the warped product nature of the metric and the duality property of the 5d Weyl tensor derived above, we can learn something about the duality property of the Weyl tensor of metric h. The Weyl tensor of the 5d metric (6.152) is proportional to the Weyl tensor of the conformally related direct product metric

$$ds^2 = -d\tilde{t}^2 + h_{mn}(x^p)dx^m dx^n,$$

where $d\tilde{t} = dt/A(t)$. Using equations derived in [85] and equation (6.156) one can show that

$$C_{mnpq} = {}^{(h)}C_{mnpq} - \eta\, h_{m[p}h_{q]n}. \tag{6.157}$$

Above, we showed that the 5d Weyl tensor with indices restricted to the 4d base space with metric k is self-dual. The Weyl tensor in the equation above is the 5d Weyl tensor with indices restricted to 4d base space with metric h. By finding the relation between these two Weyl tensors, one can show that the self-duality result above translates to

$$\star(C_{mnpq} + \eta\, h_{m[p}h_{q]n}) = C_{mnpq} + \eta\, h_{m[p}h_{q]n},$$

or

$$\star{}^{(h)}C_{mnpq} = {}^{(h)}C_{mnpq}, \tag{6.158}$$

i.e. the manifold with metric h has self-dual Weyl tensor. Since h is conformal to a Kähler metric k with Kähler form J self-dual (equation (6.149)), this means that the self-dual part of ${}^{(h)}C_{mnpq}$ is type D [147].

A simpler way of deriving this result is to absorb the Euclidean indices in equation (6.157) by multiplying with *gamma*-matrices so that

$$C_{ABCD} = {}^{(h)}C_{ABCD} - \eta\,\Gamma_{mn(AB}\Gamma^{mn}{}_{CD)}, \tag{6.159}$$

where we have used the definition of the Weyl spinor given in equation (6.16). By direct computation, or using the Fierz identity, one can show that

$$\Gamma_{mn(AB}\Gamma^{mn}{}_{CD)} = 0,$$

so that

$$C_{ABCD} = {}^{(h)}C_{ABCD}. \tag{6.160}$$

Since the solution is type $\underline{11}\,\underline{11}$, we find from the correspondence between the 4d De Smet classification and the Euclidean Petrov classification (table 6.3) that the solution with metric h is self-dual and of Petrov type (D,O).

In summary, all type $\underline{11}\,\underline{11}$ ($f \neq 0$) solutions to vacuum Einstein equations are warped product solutions of the form

$$ds^2 = -dt^2 + A(t)^2 h_{mn}(x^p)dx^m dx^n, \tag{6.161}$$

where $A(t)$ is one of the functions in the table above depending on the curvature of the 5d metric and the curvature of the 4d Euclidean metric h. The 4d manifold with metric h is a self-dual Einstein solution [7]. In addition, h is conformal to a Kähler metric k, thus the solution is of Petrov type (D,O). If $\eta = 0$, h is a self-dual Ricci-flat metric. Thus, it is hyper-Kähler (see e.g. [150]).

[7] In 4d, self-dual Einstein solutions are quaternion-Kähler [148]. The holonomy group of a quaternion-Kähler solution is a subgroup of $Sp(1)^2 \cong SU(2)^2$ [149].

Note that the $\epsilon = \eta = 0$ case for which $A(t) = 1$ corresponds to a direct product solution and agrees with the result found in section 6.2.4, where it was shown that for type $\underline{11}\,\underline{11}$ direct product solutions, the Euclidean base space has Petrov type (D,O) or (O,D) depending on the choice of orientation.

It is simple to show the converse, i.e. that the uplift of all type (D,O) Einstein solutions with metric (6.161) are type $\underline{11}\,\underline{11}$. Equation (6.160) gives that the De Smet type of the 5d solution coincides with the De Smet type of the 4d base space. Table 6.3 gives that the De Smet type of the 5d solution is $\underline{11}\,\underline{11}$. Therefore, we have that a solution is

$$\text{type } \underline{11}\,\underline{11}\ (f \neq 0) \quad \Longleftrightarrow \quad \begin{array}{l}\text{a cosmological solution with spatial geometry a} \\ \text{type (D,O) Einstein solution (metric (6.161)).}\end{array}$$

The Fubini-Study metric on $\mathbb{CP}^2$ is an example of a $\eta > 0$ type (D,O) Einstein solution. A $\eta < 0$ example is the Bergman metric on complex hyperbolic space $\mathbb{CH}^2$. An important example of a type D hyper-Kähler metric is the Euclidean Taub-NUT solution. The $\Lambda = 0$ 5d solution with metric (6.161), where h is the metric of Euclidean Taub-NUT is a magnetic monopole solution of Kaluza-Klein theory [104].

6.A 5d Clifford algebra

In five dimensions, the Clifford algebra is

$$\{\Gamma_a, \Gamma_b\} = 2g_{ab}. \tag{6.162}$$

The *gamma*-matrices have spinor index structure $(\Gamma_a)^A{}_B$. Given a Dirac spinor ψ, we can define its Majorana and Dirac conjugates as

$$\psi^{\mathrm{C}} = \psi^t C \quad (\psi_B = \psi^A C_{AB}) \tag{6.163}$$

and

$$\bar{\psi} = \psi^\dagger B \quad (\bar{\psi}_A = \psi^{\dot{A}} B_{\dot{A}A}) \tag{6.164}$$

respectively, where C is the charge conjugation matrix, defined by

$$\Gamma_a^t = C\Gamma_a C^{-1}, \tag{6.165}$$

and B is the Dirac conjugation matrix, defined by

$$\Gamma_a^\dagger = -B\Gamma_a B^{-1}. \tag{6.166}$$

Note that $\psi^{\dot{A}} \equiv \psi^{*A}$.

It follows from Schur's lemma that B and C are unique up to a phase factor. Moreover, B is Hermitian or anti-Hermitian, where we are free to choose which and C is antisymmetric [139]. Also, from the definition of Clifford algebra, we find that Γ_0 is anti-Hermitian, while Γ_i are Hermitian. Thus, from equation (6.166), we find that B is Γ_0 up to a phase. We choose B Hermitian so that assignment of indices is consistent, i.e. $\overline{B_{\dot{A}A}} = (B^t)_{\dot{A}A} = B_{A\dot{A}}$.

The Majorana condition is

$$\bar{\psi}C^{-1} = \psi \quad \text{or} \quad \psi^* = A\psi \quad (\psi^{\dot{A}} = A^{\dot{A}}{}_B \psi^B), \tag{6.167}$$

up to a phase, where $A = (CB^{-1})^t$. In 5d, the Majorana condition has no non-trivial solutions, i.e. it implies that $\psi = 0$. This is because $A^*A = -1$.

A convenient representation to use for the five dimensional Clifford algebra is to start with the Majorana representation for the four dimensional Clifford algebra

$$\gamma_0 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}, \ \gamma_1 = \begin{pmatrix} \sigma^1 & 0 \\ 0 & \sigma^1 \end{pmatrix}, \ \gamma_2 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix}, \ \gamma_3 = \begin{pmatrix} 0 & i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, \tag{6.168}$$

where $\sigma^{\hat{i}}$ for $\hat{i} = 1, 2, 3$ are the usual Pauli matrices and add $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$. Thus,

$$\Gamma_a = (\gamma_0, \gamma_{\hat{i}}, i\gamma_5). \tag{6.169}$$

Then

$$B = i\gamma_0, \qquad C = \gamma_0\gamma_5 \qquad \text{and} \qquad A = -i\gamma_5. \tag{6.170}$$

The five dimensional Fierz identity is

$$M_{AB}N_{CD} = \frac{1}{4}C_{AD}(NM)_{CB} + \frac{1}{4}\Gamma_{eAD}(N\Gamma^e M)_{CB} - \frac{1}{8}\Gamma_{ef\,AD}(N\Gamma^{ef}M)_{CB}. \tag{6.171}$$

For brevity, we omit factors of C and C^{-1} where it is clear that indices have been lowered or raised.

6.B Solving reality condition for 2-form

In this appendix, we outline how one can prove that the reality condition

$$\epsilon_{(A}\eta_{B)} = \bar{\epsilon}_{(A}\bar{\eta}_{B)} \tag{6.172}$$

implies

$$\eta = \bar{\epsilon}, \tag{6.173}$$

where ϵ and η are non-zero spinors. Letting $B = 1, 2, 3, 4$ in (6.172) gives four equations of the form

$$\epsilon_1\eta_A + \eta_1\epsilon_A = \bar{\epsilon}_1\bar{\eta}_A + \bar{\eta}_1\bar{\epsilon}_A.$$

If all four of the equations are dependent, that is they are proportional to one another, then by considering all possible cases one can show that [8]

$$\epsilon \propto \eta.$$

[8] In fact it is enough for only a pair of the equations above to be proportional to one another for this result to hold. We shall not use this, since we would like the sketch of the proof in this section to mirror that given in appendix 6.C.

Then, (6.172) becomes

$$\epsilon_{(A}\epsilon_{B)} \propto \bar{\epsilon}_{(A}\bar{\epsilon}_{B)}.$$

Since ϵ is non-zero, assume without loss of generality that $\epsilon_1 \neq 0$. Then letting $B = 1$ in the equation above gives

$$\bar{\epsilon} \propto \epsilon,$$

which is a Majorana condition on ϵ. The Majorana condition has no non-zero solutions in five dimensions. Therefore, $\epsilon \propto \eta$ implies a contradiction.

Now, assuming that two of the equation are independent gives

$$\bar{\eta} = \alpha\, \epsilon + \beta\, \eta, \tag{6.174}$$

where α and β are constants and $\alpha \neq 0$, otherwise we have a Majorana condition on η. Taking complex conjugate of the above equation and multiplying appropriately by A gives

$$\bar{\epsilon} = -\frac{1}{\alpha^*}\left(\alpha\beta^*\, \epsilon + (1 + |\beta|^2)\, \eta\right). \tag{6.175}$$

Substituting equations (6.174) and (6.175) into equation (6.172) gives

$$\alpha^2\beta^*\, \epsilon_{(A}\epsilon_{B)} + (\alpha + \alpha^* + 2\alpha|\beta|^2)\, \epsilon_{(A}\eta_{B)} + \beta(1 + |\beta|^2)\eta_{(A}\eta_{B)} = 0.$$

Letting $B = 1, 2, 3, 4$ gives four equations relating ϵ and η. Now, the analysis splits into three cases. The first case is that some coefficients in the equations are non-zero. This implies $\epsilon \propto \eta$, which gives a contradiction as shown above. The second case is that all coefficients in each of the equations vanish. But, it can be shown that this too implies that $\epsilon \propto \eta$. Thus, we are left with the final case that the coefficients in the equation above vanish. Since $\alpha \neq 0$, we have $\beta = 0$.

Then, equation (6.174) gives $\bar{\eta} \propto \epsilon$ or

$$\eta \propto \bar{\epsilon}. \tag{6.176}$$

6.C Solving reality condition for type 22

In this appendix, we outline how one can prove that the reality condition for type 22 solutions

$$\epsilon_{(AB}\eta_{CD)} = \bar{\epsilon}_{(AB}\bar{\eta}_{CD)}, \tag{6.177}$$

implies either

$$\epsilon_{AB} = \bar{\epsilon}_{AB}, \qquad \eta_{AB} = \bar{\eta}_{AB}, \tag{6.178}$$

or

$$\epsilon_{AB} = \bar{\eta}_{AB}, \tag{6.179}$$

where ϵ and η are non-zero bispinors, and

$$\bar{\epsilon}_{AB} \equiv \epsilon_{\dot{A}\dot{B}} A^{\dot{A}}{}_A A^{\dot{B}}{}_B.$$

The strategy used for the proof here is similar in nature to that used to prove the result in appendix 6.B, except that there are more cases to consider.

First consider the case $\eta \propto \epsilon$. Then, equation (6.177) becomes

$$\epsilon_{(AB}\epsilon_{CD)} = \bar{\epsilon}_{(AB}\bar{\epsilon}_{CD)}. \tag{6.180}$$

Assume $\epsilon_{11} \neq 0$. This is true unless $\epsilon_{AA} = 0$ for all A (no sum on A), in which case, it can be shown that $\epsilon_{AB} \propto \bar{\epsilon}_{AB}$, as required. Letting $A = B = C = D = 1$ in equation (6.180) gives

$${\epsilon_{11}}^2 = {\bar{\epsilon}_{11}}^2.$$

Now, letting $B = C = D = 1$ in equation (6.180) and using the equation above gives

$$\epsilon_{A1} = \pm\bar{\epsilon}_{A1}.$$

Finally, letting $C = D = 1$ in equation (6.180) and using the two equations above gives

$$\epsilon_{AB} = \bar{\epsilon}_{AB}. \tag{6.181}$$

Thus, we have proved the result above for $\eta \propto \epsilon$. Note that this is equivalent to solving the reality condition for type $\underline{22}$ solutions.

Now, assume $\eta \not\propto \epsilon$. Letting $B,\ C,\ D = 1,2,3,4$ in equation (6.177) with at least two of them coinciding gives 16 equations of the form

$$\epsilon_{A1}\eta_{11} + \epsilon_{11}\eta_{A1} = \bar{\epsilon}_{A1}\bar{\eta}_{11} + \bar{\epsilon}_{11}\bar{\eta}_{A1},$$

or

$$\epsilon_{A2}\eta_{11} + \epsilon_{A1}\eta_{21} + \epsilon_{21}\eta_{A1} + \epsilon_{11}\eta_{A2} = \bar{\epsilon}_{A2}\bar{\eta}_{11} + \bar{\epsilon}_{A1}\bar{\eta}_{21} + \bar{\epsilon}_{21}\bar{\eta}_{A1} + \bar{\epsilon}_{11}\bar{\eta}_{A2}.$$

If three of the 16 equations are not independent, then by considering all possible cases it can be shown that $\eta \propto \epsilon$, which contradicts the original assumption that $\eta \not\propto \epsilon$. Therefore, at least 15 of the equations are independent. They can be used to express $\epsilon_{A2}, \ldots, \eta_{A1}, \ldots, \bar{\epsilon}_{A1}, \ldots, \bar{\eta}_{A1}, \ldots, \bar{\eta}_{A4}$ in terms of ϵ_{A1}.

Now, let $C = D = 1, 2, 3, 4$ in equation (6.177) to give 4 equations of the form

$$\epsilon_{AB}\eta_{11} + \epsilon_{A1}\eta_{B1} + \epsilon_{B1}\eta_{A1} + \epsilon_{11}\eta_{AB} = \bar{\epsilon}_{AB}\bar{\eta}_{11} + \bar{\epsilon}_{A1}\bar{\eta}_{B1} + \bar{\epsilon}_{B1}\bar{\eta}_{A1} + \bar{\epsilon}_{11}\bar{\eta}_{AB}.$$

Similar to before, if two of these four equations are not independent then it can be shown by considering all the different possibilities that $\eta \propto \epsilon$, which contradicts the original assumption. Thus, three of the equations are independent, which means we can eliminate ϵ_{A1}, $\bar{\eta}_{AB}$ and use the last equation to show that

$$\bar{\epsilon} = \alpha\, \epsilon + \beta\, \eta, \tag{6.182}$$

where α and β are constants and if $\beta = 0$, it can be shown that

$$\bar{\eta} \propto \eta. \tag{6.183}$$

Thus $\beta = 0$ gives one of the possibilities allowed above: $\bar{\epsilon} \propto \epsilon$ and $\bar{\eta} \propto \eta$. Furthermore, $\alpha = 0$ gives the second possibility: $\bar{\epsilon} \propto \eta$. Using equations (6.182) and (6.177), one can show that

$$\alpha = 0 \iff |\beta|^2 = 1 \iff \beta = \pm 1, \tag{6.184}$$

and

$$\beta = 0 \iff |\alpha|^2 = 1 \iff \alpha = \pm 1. \tag{6.185}$$

Assume $\alpha \neq 0$ and $\beta \neq 0$. Taking complex conjugate of equation (6.182) and multiplying appropriately by a pair of A's gives

$$\bar{\eta} = \frac{1}{\beta^*}\left((1 - |\alpha|^2)\,\epsilon - \alpha^*\beta\,\eta\right). \tag{6.186}$$

Substituting equations (6.182) and (6.186) into equation (6.177) gives

$$\lambda\, \epsilon_{(AB}\epsilon_{CD)} + \mu\, \epsilon_{(AB}\eta_{CD)} + \nu\, \eta_{(AB}\eta_{CD)} = 0, \tag{6.187}$$

where

$$\lambda = \alpha(1 - |\alpha|^2), \qquad \mu = \beta - \beta^* - 2\beta|\alpha|^2, \qquad \nu = -\alpha^*\beta^2.$$

Using the first equivalence in (6.185) gives that $\lambda = 0$ implies that $\alpha = 0$ or $\beta = 0$, which contradicts the original assumption. This is also trivially true for $\nu = 0$. $\mu = 0$ implies

$$\beta(1 - 2|\alpha|^2) = \beta^*.$$

Multiplying the equation above with its complex conjugate gives

$$|\alpha|^2|\beta|^2 = 0.$$

Thus, $\mu = 0$ also contradicts the original assumption that $\alpha\beta \neq 0$.

Letting $B = C = D = 1, 2, 3, 4$ in equation (6.187) gives four equations of the form

$$(2\lambda\epsilon_{11} + \mu\eta_{11})\ \epsilon_{A1} + (2nu\eta_{11} + \mu\epsilon_{11})\ \eta_{A1} = 0.$$

The equations are independent unless the coefficients vanish. If this is the case, then we have

$$\mu^2 - 4\lambda\nu = 0.$$

It can be shown that this implies that $\alpha = 0$, which contradicts the original assumption. Thus, by considering different components of the four equations, one can show that, in general, they imply that $\eta \propto \epsilon$, which contradicts the original assumption. Although, one must also consider special cases, where, for example, $\epsilon_{A1} \neq 0$ only for $A = 1$. However, in these cases too, one can show that $\eta \propto \epsilon$.

Therefore, $\alpha\beta \neq 0$ contradicts the original assumption, which implies that $\alpha\beta = 0$.

6.D Weyl tensor of type 22 or more special solutions

In this appendix, we use the 5d Fierz identity to derive the form of the Weyl tensor of type 22 or more special solutions. For all such solutions, the Weyl spinor is of the form

$$C_{ABCD} = \epsilon_{(AB}\eta_{CD)}. \tag{6.188}$$

We can invert the definition of the Weyl spinor (equation (6.16)), so that given a Weyl spinor, the associated Weyl tensor is given by

$$C_{abcd} = \frac{1}{64}(\Gamma_{ab})^{AB}(\Gamma_{cd})^{CD}C_{ABCD}. \tag{6.189}$$

Using the form of the Weyl spinor (6.188) and equation (6.189), the Weyl tensor is

$$C_{abcd} = -\frac{1}{2}(A_{ab}B_{cd} + B_{ab}A_{cd}) + 2tr(\Gamma_{ab}\epsilon\Gamma_{cd}\eta), \tag{6.190}$$

where

$$A_{ab} = i\,tr(\Gamma_{ab}\epsilon) \quad \text{and} \quad B_{ab} = i\,tr(\Gamma_{ab}\eta),$$

and ϵ and η have been rescaled.

Using the 5d Fierz identity (equation (6.171)) with $M = \Gamma_{ab}$, $N = \Gamma_{cd}$ and using the fact that C and $C\Gamma^a$ are antisymmetric in their spinor indices, while $C\Gamma^{ab}$ and η are symmetric gives

$$tr(\Gamma_{ab}\epsilon\Gamma_{cd}\eta) = \frac{i}{8}B_{ef}tr(\Gamma_{cd}\Gamma^{ef}\Gamma_{ab}\epsilon). \tag{6.191}$$

Using the Fierz identity with $M = \Gamma_{cd}$, $N = \Gamma^{ef}$, contracting two spinor indices between Γ_{cd} and Γ^{ef}, and multiplying by Γ_{ab} gives an expression for $(\Gamma_{cd}\Gamma^{ef}\Gamma_{ab})_{CB}$, which when inserted into the equation above gives

$$\begin{aligned} tr(\Gamma_{ab}\epsilon\Gamma_{cd}\eta) = \frac{i}{32}B_{ef}\Big(&-i\,A_{ab}tr(\Gamma^{ef}\Gamma_{cd}) + tr(\Gamma_g\Gamma_{ab}\epsilon)\,tr(\Gamma^{ef}\Gamma^g\Gamma_{cd}) \\ &-\frac{1}{2}tr(\Gamma_{gh}\Gamma_{ab}\epsilon)tr(\Gamma^{ef}\Gamma^{gh}\Gamma_{cd})\Big). \end{aligned}$$

Again, using the Fierz identities in a similar way to that used to derive equation (6.191) and properties of *gamma*-matrices in 5d, in particular that

$$tr(\Gamma^a\Gamma^b\Gamma^c\Gamma^d\Gamma^e) = -4i\varepsilon^{abcde}, \tag{6.192}$$

gives

$$\begin{aligned} tr(\Gamma_{ab}\epsilon\Gamma_{cd}\eta) = &-\frac{1}{4}\left(A_{ab}B_{cd} + B_{ab}A_{cd} + A^{ef}B_{ef}g_{a[c}g_{d]b}\right) + \frac{1}{2}(A_{a[c}B_{d]b} + B_{a[c}A_{d]b}) \\ &-\frac{1}{2}\left(A_{ae}B^e{}_{[c}g_{d]b} + B_{ae}A^e{}_{[c}g_{d]b} - A_{be}B^e{}_{[c}g_{d]a} - B_{be}A^e{}_{[c}g_{d]a}\right). \end{aligned}$$

Equation (6.190) then gives

$$\begin{aligned} C_{abcd} =& A_{a[c}B_{d]b} + B_{a[c}A_{d]b} - A_{ab}B_{cd} - B_{ab}A_{cd} - \frac{1}{2}A^{ef}B_{ef}g_{a[c}g_{d]b} \\ &- A_{ae}B^e{}_{[c}g_{d]b} - B_{ae}A^e{}_{[c}g_{d]b} + A_{be}B^e{}_{[c}g_{d]a} + B_{be}A^e{}_{[c}g_{d]a}, \end{aligned} \tag{6.193}$$

i.e. the Weyl tensor of type 22 solutions is determined by two 2-forms A and B.

6.E Weyl polynomial of black ring

The Weyl polynomial of the singly rotating black ring (6.88), using the tetrad given in (6.89), is

$$\begin{aligned} C(\psi) = \frac{6(x-y)}{R^2F(x)^3F(y)} \big\{ & A_1(vw+uz)(uw-vz) + i(u^2-v^2+w^2-z^2)[A_2(vw+uz) \\ & + A_3(uw-vz)] + A_4(u^4+v^4+w^4+z^4) + A_5 uvwz \\ & + A_6\ (u^2z^2+v^2w^2) + A_7(u^2w^2+v^2z^2) + A_8(u^2v^2+w^2z^2)\big\}\,, \end{aligned} \tag{6.194}$$

where $\psi = (u, v, w, z)$ and

$$\begin{aligned}
A_1 &= 8(x-y)(1-\lambda)CF(y)\sqrt{G(x)}, \\
A_2 &= -4(x-y)(1-\lambda)CF(x)\sqrt{-G(y)}, \\
A_3 &= 4F(x)F(y)\sqrt{-G(x)G(y)}, \\
A_4 &= (1+x\lambda)^2[(x-y)^2\nu + (y^2-1)\lambda - (1-x^2)y\lambda\nu - 2(x-y)y^2\lambda\nu], \\
A_5 &= 8\big\{(1-\lambda^2)[2xy(\lambda-\nu) + (x^2+y^2)\nu] - \lambda(y^2-1)(1+x^2\lambda^2) \\
&\quad + 2\lambda(1+y\lambda)(y\lambda-x^2) + \lambda\nu x(1-x^2)(1+y\lambda)^2\big\}\,, \\
A_6 &= 2\big\{3(x-y)^2\nu + 2x\lambda^2(1-y^2) + x^3\lambda\nu(1+x\nu) + 2xy\lambda^2\nu - 2(x-y)^2\lambda^2\nu \\
&\quad + \lambda(1+y\nu) - 3x^2y\lambda\nu(1+y\lambda) - 2(x-y)^2\lambda(1-\lambda^2) + x^3\lambda\nu - y^2\lambda \\
&\quad + +x^3y\lambda^3\nu(x-y) + x^2\lambda^3(1-y^2)x^2y\lambda^3\nu(1-xy) + v^2(1+x\lambda)^2[(x-y)^2\nu \\
&\quad + \lambda(1-y^2) + y\lambda\nu(1-xy) + xy\lambda\nu(x-y)]\big\}\,, \\
A_7 &= 2\big\{(x-y)^2\nu + -4y\lambda^2(1-x^2) + 2x\lambda^2(y^2-1) - 4xy\lambda^2\nu(1-x^2) \\
&\quad + x^2\lambda^2\nu(x^2-y^2) - 2xy\lambda^2\nu(1+xy) + 2xy^3\lambda^2\nu - 3\lambda + y^2\lambda - 2x\lambda\nu \\
&\quad + 4x^3\lambda\nu - y\lambda\nu + 2y^3\lambda\nu(1+x\lambda) + 2x^2\lambda - 3x^2y\lambda\nu - 2y^2\lambda^3 + x^2\lambda^3(y^2-1) \\
&\quad + 2x^2\lambda^3 - x^2y\lambda^3\nu(1-x^2) - 2xy^2\lambda^3\nu(1-xy)\big\}\,, \\
A_8 &= 2(1+x\lambda)^2\,\big\{(x-y)^2\nu - \lambda(y^2-1) + y\lambda\nu(1-xy) + xy\lambda\nu(x-y)\big\}].
\end{aligned}$$

Appendix A

Geroch-Held-Penrose formalism

In this appendix, we review the GHP formalism [37]. Given a background solution, we choose a Newman-Penrose (NP) frame $(\ell, n, m, \bar{m})$ such that in this frame, the metric takes the form

$$g_{ab} = 2\ell_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}, \tag{A.1}$$

i.e. the only non-zero inner products between the basis vectors are $\ell \cdot n = 1$ and $m \cdot \bar{m} = -1$.

In the GHP formalism, one breaks complete covariance by singling out two null directions (ℓ and n) at each point, but preserves covariance in the remaining directions. This is in contrast to the NP formalism where none of the covariance is preserved.

At any point, the Lorentz group divides into

- boosts (r a real function):

$$\ell \to r\,\ell, \quad n \to r^{-1}n, \quad m \to m, \tag{A.2}$$

- spatial rotations (θ a real function):

$$\ell \to \ell, \quad n \to n, \quad m \to e^{i\theta}m, \tag{A.3}$$

- null rotations about ℓ (z a complex function):

$$\ell \to \ell, \quad n \to n + \bar{z}m + z\bar{m} + |z|^2\ell, \quad m \to m + z\ell, \tag{A.4}$$

- null rotations about n (z a complex function):

$$\ell \to \ell + \bar{z}m + z\bar{m} + |z|^2 n, \quad n \to n, \quad m \to m + zn, \tag{A.5}$$

We would like to keep the subgroup that preserves the null directions, i.e. the 2-parameter subgroup given by boosts and spatial rotations. Thus, we would like to work with objects that transform covariantly under this 2-parameter subgroup.

Define any scalar η that transforms covariantly as

$$\eta \to \lambda^p \bar{\lambda}^q \eta, \qquad (\lambda = \sqrt{r}\, e^{i\theta/2}) \tag{A.6}$$

under boosts and spatial rotations, a GHP scalar of type $\{p,q\}$. Evidently, the product of two GHP scalars of types $\{p,q\}$ and $\{r,s\}$ is a GHP scalar of type $\{p+r, q+s\}$.

Given this definition, clearly, we may regard the basis vectors ℓ, n, m and $\bar{m}$ as GHP vectors of types $\{1,1\}$, $\{-1,-1\}$, $\{1,-1\}$ and $\{-1,1\}$, respectively.

Also, we define a prime operation

$$' \;:\; \ell \to n, \quad n \to \ell, \quad m \to \bar{m}, \quad \bar{m} \to m. \tag{A.7}$$

As summarised in table A.1, it can be shown that eight of the twelve spin coefficients defined in the NP formalism are GHP scalars. Note that complex conjugation of a type $\{p,q\}$ GHP scalar gives a type $\{q,p\}$ GHP scalar. Thus, for example, $\bar{\kappa}$ is a GHP scalar of type $\{1,3\}$.

The non-GHP covariant spin coefficients β, ϵ and their primes can be used to construct GHP covariant derivatives . For a GHP scalar η of type $\{p,q\}$, we define its GHP covariant derivatives to be[1]

$$\begin{aligned} \text{þ}\eta &= (\ell\cdot\nabla - p\epsilon - q\bar{\epsilon})\,\eta, \qquad & \text{þ}'\eta &= (n\cdot\nabla + p\epsilon' + q\bar{\epsilon}')\,\eta, \\ \eth\eta &= (m\cdot\nabla - p\beta + q\bar{\beta}')\,\eta, \qquad & \eth'\eta &= (\bar{m}\cdot\nabla + p\beta' - q\bar{\beta})\,\eta. \end{aligned} \tag{A.8}$$

Note that the spin weight of the derivatives is as follows

$$\text{þ} : \{1,1\}, \quad \text{þ}' : \{-1,-1\}, \quad \eth : \{1,-1\}, \quad \eth' : \{-1,1\}. \tag{A.9}$$

[1] Symbols þ and ð, pronounced "thorn" and "eth", respectively are old Germanic letters that have been retained in the Icelandic alphabet.

Table A.1: GHP type of the NP spin coefficients

Spin coefficient	NP notation	GHP type
κ	$\kappa = m^a \ell \cdot \nabla \ell_a$	$\{3, 1\}$
κ'	$-\nu = \bar{m}^a n \cdot \nabla n_a$	$\{-3, -1\}$
σ	$\sigma = m^a m \cdot \nabla \ell_a$	$\{3, -1\}$
σ'	$-\lambda = \bar{m}^a \bar{m} \cdot \nabla n_a$	$\{-3, 1\}$
ρ	$\rho = m^a \bar{m} \cdot \nabla \ell_a$	$\{1, 1\}$
ρ'	$-\mu = \bar{m}^a m \cdot \nabla n_a$	$\{-1, -1\}$
τ	$\tau = m^a n \cdot \nabla \ell_a$	$\{1, -1\}$
τ'	$-\pi = \bar{m}^a \ell \cdot \nabla n_a$	$\{-1, 1\}$
β	$\beta = \frac{1}{2}(n^a m \cdot \nabla \ell_a - \bar{m}^a m \cdot \nabla m_a)$	—
β'	$-\alpha = \frac{1}{2}(\ell^a \bar{m} \cdot \nabla n_a - m^a \bar{m} \cdot \nabla \bar{m}_a)$	—
ϵ	$\epsilon = \frac{1}{2}(n^a \ell \cdot \nabla \ell_a - \bar{m}^a \ell \cdot \nabla m_a)$	—
ϵ'	$-\gamma = \frac{1}{2}(\ell^a n \cdot \nabla n_a - m^a n \cdot \nabla \bar{m}_a)$	—

Now define 5 complex scalar that are equivalent to the Weyl tensor:

$$\Psi_0 = C_{abcd}\ell^a m^b \ell^c m^d, \quad \Psi_1 = C_{abcd}\ell^a n^b \ell^c m^d, \quad \Psi_2 = C_{abcd}\ell^a m^b \bar{m}^c n^d,$$
$$\Psi_1' = C_{abcd}\ell^a n^b \bar{m}^c n^d, \quad \Psi_0' = C_{abcd} n^a \bar{m}^b n^c \bar{m}^d. \tag{A.10}$$

These Weyl scalars, as they are called, are GHP scalars of type $\{4, 0\}$, $\{2, 0\}$, $\{0, 0\}$, $\{-2, 0\}$ and $\{-4, 0\}$, respectively. The Petrov classification reduces to the problem of choosing a frame (or more precisely null vectors ℓ and n) such that certain Weyl scalars vanish. It turns out that a ℓ can always be chosen such that $\Psi_0 = 0$. This is best understand from the spinor approach, where it is as a consequence of the fundamental theorem of algebra. The different Petrov types are summarised in table A.2.

Table A.2: Possible Petrov types

Petrov type	Vanishing scalar
I	Ψ_0
II	Ψ_0, Ψ_1
D	$\Psi_0, \Psi_1, \Psi_3, \Psi_4$
III	Ψ_0, Ψ_1, Ψ_2
N	$\Psi_0, \Psi_1, \Psi_2, \Psi_3$
O	$C_{abcd} \equiv 0$

The Newmann-Penrose equations, Bianchi identities and commutator equations can now be written compactly in GHP notation [37]. We write these for a vacuum Einstein solution.

A.1 Newmann-Penrose equations

$$\eth\rho - \eth'\sigma = (\rho - \bar{\rho})\tau + (\bar{\rho}' - \rho')\kappa - \Psi_1, \tag{A.11}$$
$$þ\rho - \eth'\kappa = \rho^2 + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa\tau', \tag{A.12}$$
$$þ\sigma - \eth\kappa = (\bar{\rho} + \rho)\sigma - (\tau + \bar{\tau}')\kappa + \Psi_0, \tag{A.13}$$
$$þ\tau - þ'\kappa = (\tau - \bar{\tau}')\rho + (\bar{\tau} - \tau')\sigma + \Psi_1, \tag{A.14}$$
$$\eth\tau - þ'\sigma = -\rho'\sigma - \rho\bar{\sigma}' + \tau^2 + \kappa\bar{\kappa}', \tag{A.15}$$
$$þ'\rho - \eth'\tau = \rho\bar{\rho}' + \sigma\sigma' - \tau\bar{\tau} - \kappa\kappa' - \Psi_2 - \Lambda/3. \tag{A.16}$$

A further six equations are obtained by applying the prime operation to the equations above.

A.2 Bianchi equation

$$þ\Psi_1 - \eth'\Psi_0 = -\tau'\Psi_0 + 4\rho\Psi_1 - 3\kappa\Psi_2, \tag{A.17}$$
$$þ\Psi_2 - \eth'\Psi_1 = \sigma'\Psi_0 - 2\tau'\Psi_1 + 3\rho\Psi_2 - 2\kappa\Psi_1', \tag{A.18}$$
$$þ\Psi_1' - \eth'\Psi_2 = 2\sigma'\Psi_1 - 3\tau'\Psi_2 + 2\rho\Psi_1' - \kappa\Psi_0', \tag{A.19}$$
$$þ\Psi_0' - \eth'\Psi_1' = 3\sigma'\Psi_2 - 4\tau'\Psi_1' + \rho\Psi_0'. \tag{A.20}$$

As before, a further four equations are obtained by applying the prime operation to the equations above.

A.3 Commutators of derivatives

For some arbitrary GHP scalar of type $\{p, q\}$,

$$\begin{aligned}(þþ' - þ'þ)\eta = \big[&(\bar{\tau} - \tau')\eth + (\tau - \bar{\tau}')\eth' \\ &-p(\kappa\kappa' - \tau\tau' + \Psi_2 - \Lambda) - q(\bar{\kappa}\bar{\kappa}' - \bar{\tau}\bar{\tau}' + \bar{\Psi}_2 - \Lambda)\big]\,\eta,\end{aligned} \tag{A.21}$$
$$\begin{aligned}(þ\eth - \eth þ)\eta = \big[&\bar{\rho}\eth + \sigma\eth' - \bar{\tau}'þ - \kappa þ' \\ &-p(\rho'\kappa - \tau'\sigma + \Psi_1) - q(\bar{\sigma}'\bar{\kappa} - \bar{\rho}\bar{\tau}')\big]\,\eta,\end{aligned} \tag{A.22}$$
$$\begin{aligned}(\eth\eth' - \eth'\eth)\eta = \big[&(\bar{\rho}' - \rho')þ + (\rho - \bar{\rho})þ' \\ &-p(\rho\rho' - \sigma\sigma' + \Psi_2 - \Lambda) - q(\bar{\rho}\bar{\rho}' - \bar{\sigma}\bar{\sigma}' + \bar{\Psi}_2 - \Lambda)\big]\,\eta.\end{aligned} \tag{A.23}$$

Appendix B

Higher dimensional GHP formalism

In this appendix, we review the higher dimensional GHP formalism of [67]. Given a background solution, we choose a null frame $(\ell, n, m_{(i)})$ such that in this frame, the metric takes the form

$$g_{ab} = 2\ell_{(a}n_{b)} + m_{(i)_a} m_{(i)_b}. \tag{B.1}$$

As in the 4d case, we would like to break complete covariance by singling out two null directions (ℓ and n) at each point, but preserve covariance in the remaining directions.

At any point, the Lorentz group divides into

- boosts (λ a real function):

$$\ell \to \lambda \ell, \quad n \to \lambda^{-1} n, \quad m_{(i)} \to m_{(i)}, \tag{B.2}$$

- spins ($X_{ij} \in SO(d-2)$):

$$\ell \to \ell, \quad n \to n, \quad m_{(i)} \to X_{ij} m_{(j)}, \tag{B.3}$$

- null rotations about ℓ (z_i $d-2$ real functions):

$$\ell \to \ell, \quad n \to n + z_i m_{(i)} - \tfrac{1}{2} z^2 \ell, \quad m_{(i)} \to m_{(i)} - z_i \ell, \tag{B.4}$$

- null rotations about n (z_i $d-2$ real functions):

$$\ell \to \ell + z_i m_{(i)} - \tfrac{1}{2}z^2 n, \quad n \to n, \quad m_{(i)} \to m_{(i)} - z_i n, \tag{B.5}$$

where $\lambda \neq 0$ and X_{ij} is some position-dependent orthogonal matrix.

We would like to keep the subgroup that preserves the null directions, i.e. the subgroup given by boosts and spins (spatial rotations). Thus, we would like to work with objects that transform covariantly under this subgroup.

Define any scalar $\eta_{i_1 \dots i_s}$ that transforms covariantly as

$$\eta_{i_1\dots i_s} \to \lambda^b \eta_{i_1\dots i_s} \tag{B.6}$$

under boosts and

$$\eta_{i_1\dots i_s} \to X_{i_1 j_1} \cdots X_{i_s j_s} \eta_{j_1 \dots j_s} \tag{B.7}$$

under spins, a GHP scalar of boost weight b and spin s. Evidently, the product of two GHP scalars of boost weights b_1 and b_2 and spins s_1 and s_2, respectively, gives a GHP scalar of boost weight $b_1 + b_2$ and spin $s_1 + s_2$.

Defining the covariant derivatives of the basis vectors as

$$L_{ab} = \nabla_b \ell_a, \quad N_{ab} = \nabla_b n_a, \quad \overset{i}{M}_{ab} = \nabla_b {m_{(i)}}_a, \tag{B.8}$$

one finds that not all the scalars formed from the projection of these objects into the basis are GHP scalars. Those that are GHP scalars are listed in table B.1 [67].

Table B.1: GHP scalars constructed from covariant derivatives of the basis vectors.

Spin coefficient	GHP notation	Boost weight b	Spin s	Interpretation
L_{ij}	ρ_{ij}	1	2	expansion, shear and twist of ℓ
L_{ii}	$\rho = \rho_{ii}$	1	0	expansion of ℓ
L_{i0}	κ_i	2	1	non-geodesity of ℓ
L_{i1}	τ_i	0	1	transport of ℓ along n
N_{ij}	ρ'_{ij}	-1	2	expansion, shear and twist of n
N_{ii}	$\rho' = \rho'_{ii}$	-1	0	expansion of n
N_{i1}	κ'_i	-2	1	non-geodesity of n
N_{i0}	τ'_i	0	1	transport of n along l

Notice that we have used a prime operation, which interchanges the null basis vectors

$$' \; : \; \ell \leftrightarrow n. \tag{B.9}$$

The prime operation is especially useful when considering type D backgrounds, since in this case ℓ and n are essentially equivalent.

The non-GHP covariant scalars formed from the covariant derivative of the basis vectors can be used to construct GHP covariant derivatives . For a GHP scalar $\eta_{i_1 \dots i_s}$ of boost weight b and spin s, we define its GHP covariant derivatives to be

$$\text{þ} T_{i_1 i_2 \dots i_s} \equiv \ell \cdot \partial T_{i_1 i_2 \dots i_s} - b L_{10} T_{i_1 i_2 \dots i_s} + \sum_{r=1}^{s} \overset{k}{M}_{i_r 0} T_{i_1 \dots i_{r-1} k i_{r+1} \dots i_s}, \tag{B.10}$$

$$\text{þ}' T_{i_1 i_2 \dots i_s} \equiv n \cdot \partial T_{i_1 i_2 \dots i_s} - b L_{11} T_{i_1 i_2 \dots i_s} + \sum_{r=1}^{s} \overset{k}{M}_{i_r 1} T_{i_1 \dots i_{r-1} k i_{r+1} \dots i_s}, \tag{B.11}$$

$$\eth_i T_{j_1 j_2 \dots j_s} \equiv m_{(i)} \cdot \partial T_{j_1 j_2 \dots j_s} - b L_{1i} T_{j_1 j_2 \dots j_s} + \sum_{r=1}^{s} \overset{k}{M}_{j_r i} T_{j_1 \dots j_{r-1} k j_{r+1} \dots j_s}. \tag{B.12}$$

In GHP notation, the Newman-Penrose, Bianchi and the commutator equations are much more compact that in the NP formalism [64]. For convenience, we write these equations, here, for an Einstein spacetime [68].

B.1 Newman-Penrose equations

$$\text{þ} \rho_{ij} - \eth_j \kappa_i = -\rho_{ik} \rho_{kj} - \kappa_i \tau'_j - \tau_i \kappa_j - \Omega_{ij}, \tag{NP1}$$

$$\text{þ} \tau_i - \text{þ}' \kappa_i = \rho_{ij}(-\tau_j + \tau'_j) - \Psi_i, \tag{NP2}$$

$$2\eth_{[j|} \rho_{i|k]} = 2\tau_i \rho_{[jk]} + 2\kappa_i \rho'_{[jk]} - \Psi_{ijk}, \tag{NP3}$$

$$\text{þ}' \rho_{ij} - \eth_j \tau_i = -\tau_i \tau_j - \kappa_i \kappa'_j - \rho_{ik} \rho'_{kj} - \Phi_{ij} - \tfrac{\Lambda}{d-1} \delta_{ij}. \tag{NP4}$$

Another four equations can be obtained by taking the prime $'$ of these four (i.e. by exchanging the vectors ℓ and n).

B.2 Bianchi equations

Boost weight +2:

$$\begin{aligned} \text{þ}\Psi_{ijk} - 2\eth_{[j}\Omega_{k]i} &= (2\Phi_{i[j|}\delta_{k]l} - 2\delta_{il}\Phi^{\mathrm{A}}_{jk} - \Phi_{iljk})\kappa_l \\ &\quad -2(\Psi_{[j|}\delta_{il} + \Psi_i\delta_{[j|l} + \Psi_{i[j|l} + \Psi_{[j|il})\rho_{l|k]} + 2\Omega_{i[j}\tau'_{k]}, \end{aligned} \tag{B1}$$

Boost weight +1:

$$\begin{aligned} -\text{þ}\Phi_{ij} - \eth_j\Psi_i + \text{þ}'\Omega_{ij} &= -(\Psi'_j\delta_{ik} - \Psi'_{jik})\kappa_k + (\Phi_{ik} + 2\Phi^{\mathrm{A}}_{ik} + \Phi\delta_{ik})\rho_{kj} \\ &\quad +(\Psi_{ijk} - \Psi_i\delta_{jk})\tau'_k - 2(\Psi_{(i}\delta_{j)k} + \Psi_{(ij)k})\tau_k - \Omega_{ik}\rho'_{kj}, \end{aligned} \tag{B2}$$

$$\begin{aligned} -\text{þ}\Phi_{ijkl} + 2\eth_{[k}\Psi_{l]ij} &= -2\Psi'_{[i|kl}\kappa_{|j]} - 2\Psi'_{[k|ij}\kappa_{|l]} \\ &\quad +4\Phi^{\mathrm{A}}_{ij}\rho_{[kl]} - 2\Phi_{[k|i}\rho_{j|l]} + 2\Phi_{[k|j}\rho_{i|l]} + 2\Phi_{ij[k|m}\rho_{m|l]} \\ &\quad -2\Psi_{[i|kl}\tau'_{|j]} - 2\Psi_{[k|ij}\tau'_{|l]} - 2\Omega_{i[k|}\rho'_{j|l]} + 2\Omega_{j[k}\rho'_{i|l]}, \end{aligned} \tag{B3}$$

$$-\eth_{[j|}\Psi_{i|kl]} = 2\Phi^{\mathrm{A}}_{[jk|}\rho_{i|l]} - 2\Phi_{i[j}\rho_{kl]} + \Phi_{im[jk|}\rho_{m|l]} - 2\Omega_{i[j}\rho'_{kl]}, \tag{B4}$$

Boost weight 0:

$$\begin{aligned} \text{þ}'\Psi_{ijk} - 2\eth_{[j|}\Phi_{i|k]} &= 2(\Psi'_{[j|}\delta_{il} - \Psi'_{[j|il})\rho_{l|k]} + (2\Phi_{i[j}\delta_{k]l} - 2\delta_{il}\Phi^{\mathrm{A}}_{jk} - \Phi_{iljk})\tau_l \\ &\quad +2(\Psi_i\delta_{[j|l} - \Psi_{i[j|l})\rho'_{l|k]} + 2\Omega_{i[j}\kappa'_{k]}, \end{aligned} \tag{B5}$$

$$-2\eth_{[i}\Phi^{\mathrm{A}}_{jk]} = 2\Psi'_{[i}\rho_{jk]} + \Psi'_{l[ij|}\rho_{l|k]} - 2\Psi_{[i}\rho'_{jk]} - \Psi_{l[ij|}\rho'_{l|k]}, \tag{B6}$$

$$\begin{aligned} -\eth_{[k|}\Phi_{ij|lm]} &= -\Psi'_{i[kl|}\rho_{j|m]} + \Psi'_{j[kl|}\rho_{i|m]} - 2\Psi'_{[k|ij}\rho_{|lm]} \\ &\quad -\Psi_{i[kl|}\rho'_{j|m]} + \Psi_{j[kl|}\rho'_{i|m]} - 2\Psi_{[k|ij}\rho'_{|lm]}. \end{aligned} \tag{B7}$$

Another five equations are obtained by applying the prime operator to equations (B1)-(B5) above.

B.3 Commutators of derivatives

Acting on a GHP scalar of boost weight b and spin s, commutators of GHP derivatives can be simplified by:

$$\begin{aligned}
[þ, þ']T_{i_1\dots i_s} &= (-\tau_j+\tau'_j)ð_j T_{i_1\dots i_s} + b\left(-\tau_j\tau'_j+\kappa_j\kappa'_j+\Phi-\tfrac{\Lambda}{d-1}\right)T_{i_1\dots i_s}\\
&\quad+\sum_{r=1}^{s}\left(\kappa_{i_r}\kappa'_j-\kappa'_{i_r}\kappa_j+\tau'_{i_r}\tau_j-\tau_{i_r}\tau'_j+2\Phi^{\mathrm{A}}_{i_r j}\right)T_{i_1\dots j\dots i_s}, &\text{(C1)}\\
[þ, ð_i]T_{k_1\dots k_s} &= -(\kappa_i þ'+\tau'_i þ+\rho_{ji}ð_j)T_{k_1\dots k_s}+b\left(-\tau'_j\rho_{ji}+\kappa_j\rho'_{ji}+\Psi_i\right)T_{k_1\dots k_s}\\
&\quad+\sum_{r=1}^{s}\left(\kappa_{k_r}\rho'_{li}-\rho_{k_r i}\tau'_l+\tau'_{k_r}\rho_{li}-\rho'_{k_r i}\kappa_l-\Psi_{ilk_r}\right)T_{k_1\dots l\dots k_s}, &\text{(C2)}\\
[ð_i, ð_j]T_{k_1\dots k_s} &= \left(2\rho_{[ij]}þ'+2\rho'_{[ij]}þ\right)T_{k_1\dots k_s}+b\left(2\rho_{l[i|}\rho'_{l|j]}+2\Phi^{\mathrm{A}}_{ij}\right)T_{k_1\dots k_s}\\
&\quad+\sum_{r=1}^{s}\left(2\rho_{k_r[i|}\rho'_{l|j]}+2\rho'_{k_r[i|}\rho_{l|j]}+\Phi_{ijk_rl}+\tfrac{2\Lambda}{d-1}\delta_{[i|k_r}\delta_{|j]l}\right)T_{k_1\dots l\dots k_s} &\text{(C3)}
\end{aligned}$$

The result for $[þ', ð_i]$ can be obtained from (C2)$'$.

Null rotations

Under a null rotation about ℓ of the form given by equation (B.4) the Weyl tensor components transform as:

$$\begin{aligned}
\Omega_{ij} &\mapsto \Omega_{ij}, &\text{(A.13)}\\
\Psi_i &\mapsto \Psi_i+\Omega_{ij}z_j, &\text{(A.14)}\\
\Psi_{ijk} &\mapsto \Psi_{ijk}+2\Omega_{i[j}z_{k]}, &\text{(A.15)}\\
\Phi &\mapsto \Phi+2z_i\Psi_i+z_i\Omega_{ij}z_j, &\text{(A.16)}\\
\Phi_{ij} &\mapsto \Phi_{ij}+z_j\Psi_i+z_k\Psi_{ikj}+Z_{jk}\Omega_{ik}, &\text{(A.17)}\\
\Phi_{ijkl} &\mapsto \Phi_{ijkl}-2z_{[k}\Psi_{l]ij}-2z_{[i}\Psi_{j]kl}-2z_jz_{[k}\Omega_{l]i}+2z_iz_{[k}\Omega_{l]j}, &\text{(A.18)}\\
\Psi'_i &\mapsto \Psi'_i-z_i\Phi+3\Phi^{\mathrm{A}}_{ij}z_j-\Phi^{\mathrm{S}}_{ij}z_j-2Z_{ij}\Psi_j-Z_{jk}\Psi_{jki}-z_jZ_{ik}\Omega_{jk}, &\text{(A.19)}\\
\Psi'_{ijk} &\mapsto \Psi'_{ijk}+2z_{[k}\Phi_{j]i}+2z_i\Phi^{\mathrm{A}}_{jk}+z_l\Phi_{lijk}+2z_iz_{[k}\Psi_{j]}+2z_lz_{[k}\Psi_{j]li}+Z_{il}\Psi_{ljk}\\
&\quad+2Z_{il}z_{[k}\Omega_{j]l}, &\text{(A.20)}\\
\Omega'_{ij} &\mapsto \Omega'_{ij}-2z_{(j}\Psi'_{i)}+2z_k\Psi'_{(i|k|j)}+2Z_{(i|k}\Phi_{k|j)}+z_iz_j\Phi-4z_kz_{(i}\Phi^{A}_{j)k}+z_kz_l\Phi_{kilj}\\
&\quad+2z_{(i}Z_{j)k}\Psi_k+2z_lZ_{(i|k}\Psi_{kl|j)}+Z_{ik}Z_{jl}\Omega_{kl}. &\text{(A.21)}
\end{aligned}$$

Part III

M-theory dualities and generalised geometry

Chapter 7

Introduction

The bosonic part of the action of 11d supergravity is [151]

$$S_{11dSUGRA} = \int d^{11}x \left\{ \sqrt{g} \left(R - \frac{1}{48} F_{abcd} F^{abcd} \right) + \lambda \eta^{a_1 \ldots a_{11}} C_{a_1 a_2 a_3} F_{a_4 \ldots a_7} F_{a_8 \ldots a_{11}} \right\}, \tag{7.1}$$

where the 4-form field F is the exterior derivative of the 3-form potential C, i.e.

$$F_{abcd} = 4\partial_{[a} C_{bcd]}, \tag{7.2}$$

η is the 11d alternating tensor density and $\lambda = 2^{-7}3^{-4}$.

Initial interest in 11d supergravity lay in the idea that it is the unique supergravity theory that can be used to construct a four dimensional theory with gravity and standard model forces via a dimensional reduction. This is because the requirement that one has particles with spin less than or equal to two places an upper bound of 11 on the dimension of the theory [152], while the phenomenological requirement that the theory be big enough to contain the standard model gauge groups SU(3) $\times$ SU(2) $\times$ U(1) means that the theory must be at least 11 dimensional [153]. These two results suggested the uniqueness of 11d supergravity. Furthermore, the compactification down to four dimensions seemed very natural given the result that any compactification from eleven dimensions that preserves all the supersymmetry generators must leave only four or seven macroscopic dimensions [154]. However, the appeal of this idea waned as problems soon became apparent and it was realised that the constraints described above can be avoided if one is willing to work in a more general setting.

Today, the main motivation for the study of eleven dimensional supergravity is that it is the low energy effective theory associated with M-theory [155, 156, 157], which is conjectured to embody all the different types of string theories. It has been known for some time that reducing 11-dimensional supergravity on tori of various dimensions leads to a host of symmetries, referred to as M-theory dualities [158, 159, 160].

For concreteness, consider the case where one compactifies 11d supergravity on a 7-torus [159]. The resulting theory in four dimensions is maximal ($\mathcal{N} = 8$) supergravity. The bosonic field content of this theory consists of a graviton, 21+7=28 vectors and 35+28+7=70 scalars, where the 35 of the scalars are pseudo-scalars.

The 11d theory has local SO(1,10) and global $\mathrm{GL}(11, \mathbb{R})$ symmetries. The local symmetry can always be used to work in a gauge where the 11-bein that is now split into a 4d part and a 7d part is upper triangular, i.e. $e_m^\alpha = 0$, where m denotes curved space indices on the 7-torus, while α denotes tangent spacetime indices on the 4d part. This gauge is preserved by O(7) transformations[1]. Also, one is left with a global $\mathrm{GL}(7, \mathbb{R})$ symmetry. Thus, reducing the 11d action given in equation (7.1) leads to a theory in four dimensions involving the fields described above with explicit O(7) local and $\mathrm{GL}(7, \mathbb{R})$ global invariances.

However, if we consider the full theory including the fermionic part, we find that the reduced theory has $\mathcal{N} = 8$ extended local supersymmetry, as eluded to above. This means that the action should be invariant under the local action of the R-symmetry group SO(8), which leads us to conclude that the theory has further hidden symmetries. Indeed, ignoring the 35 pseudo-scalars for now, the scalar part of the reduced Lagrangian can be re-written as a non-linear sigma model where the sigma function is an element of the coset space $\mathrm{SL}(8, \mathbb{R})/\mathrm{SO}(8)$ and is parameterised by the 35 non-pseudo-scalars. This is, of course, consistent with the dimension of the coset space. The vector part of the Lagrangian in the absence of pseudo-scalars can also be shown to be invariant under local SO(8) and global $\mathrm{SL}(8, \mathbb{R})$ transformations.

Now, the problem is how to incorporate the pseudo-scalars into this framework and this is where Cremmer and Julia [159] provide a series of remarkable insights that leads to the full symmetry group of the theory. The only possible solution for the incorporation of the pseudo-scalars is that the theory is invariant under a yet larger local symmetry group of which SO(8) is a subgroup. Given that all the fields in the 4d theory are massless and that massless theories usually have $\mathrm{SU}(n)$ or $\mathrm{U}(n)$ symmetries [161] leads to the conjecture

[1] The reason why the local symmetry group that one gets is O(7) rather than SO(7) is that one is always free to perform conformal or Weyl transformations.

that the enlarged local symmetry group is SU(8)[2]. If all the scalars are to parameterise a sigma function that is an element of some coset space G/SU(8), the dimension of group G must be 63+70=133. Furthermore, G must contain as subgroups $SL(8, \mathbb{R})$ and SU(8). This lead Cremmer and Julia to conclude that G must be the exceptional group E_7.

In conclusion, we find that 11d supergravity reduced on a 7-torus leads to a theory with local SU(8) and global E_7 symmetries[3].

One finds similar hidden symmetries for other toroidal compactification of the 11d theory down to the various dimensions [160]. These symmetry groups are summarised in table 7.1. Soon after these symmetries were found, de Wit and Nicolai speculated that these symmetries are fundamental to the 11-dimensional theory as a whole rather than a particularity of the toroidal compactification [162, 163], which has lead to much research in this direction (see e.g. [164, 165, 166, 167]). Recently, generalised geometry has been used to reformulate 11d supergravity in such a way as to make manifest most of these duality symmetries [168, 169, 170].

Table 7.1: Table of M-theory dualities

torus dimension (d)	global symmetry	local symmetry
1	$SO(1,1)$	1
2	$SL(2) \times \mathbb{R}^+$	SO(2)
3	SL(3)× SL(2)	SO(3)× SO(2)
4	SL(5)	SO(5)
5	$SO(5,5)$	SO(5)× SO(5)
6	E_6	USp(8)
7	E_7	SU(8)
8	E_8	$Spin(16)/\mathbb{Z}_2$

7.1 Generalised geometry

The basic idea in generalised geometry [171, 172, 173, 174] is to enlarge the tangent space of a manifold by adding $p-$forms as well vectors. Mathematically, this corresponds to replacing the tangent bundle $T(M)$ with a direct sum of the tangent bundle and the bundle of $p-$forms and their direct products. Thus, instead of considering extra structures

[2]The group must be SU(8) rather than U(8) due to the existence of a certain self-duality relation.

[3]While, the derivation above is rather sketchy, the main aim is to convey the general arguments and ideas that lead to the discovery of the hidden symmetries rather than to dwell on the many technical details of the actual computation.

on the tangent bundle of the manifold, what one is doing is to unify the relevant structures and treat fields as sections of this new more general bundle. Let's consider a few examples, which will hopefully clarify the abstract notions described above.

First consider enlarging the tangent bundle of a $d-$dimensional manifold by adding the cotangent bundle

$$T(M) \longrightarrow T(M) \oplus T^{*}(M). \tag{7.3}$$

A generalised vector in this geometry $V = (v, \eta)$, where v is a vector and η is a 1-form. This new generalised geometry was first studied by Siegel [171] who realised its importance in string theory. The full mathematics of the geometry has been developed by Hitchen and Gaultieri [172, 173]. Endowing the fibers with an inner product such that for two generalised vectors $V = (v, \eta)$ and $W = (w, \xi)$

$$\langle V, W\rangle = \langle v + \eta, w + \xi\rangle = \eta(w) + \xi(v) \tag{7.4}$$

induces a natural action of the group $\mathrm{O}(d, d)$ with invariant metric

$$\eta = \begin{pmatrix} 0 & \mathbb{1}_{d\times d} \\ \mathbb{1}_{d\times d} & 0 \end{pmatrix}. \tag{7.5}$$

In addition, one can define a bracket on the space of generalised vectors that replaces the usual Lie bracket on the space of tangent vectors. The Courant bracket [175] of two generalised vectors $V = (v, \eta)$ and $W = (w, \xi)$ is

$$[V, W]_C = [v + \eta, w + \xi]_C = [v, w] + \mathcal{L}_v\xi - \mathcal{L}_w\eta + \frac{1}{2}\mathrm{d}\left(i_w\eta - i_v\xi\right), \tag{7.6}$$

where $[v, w]$ denotes the Lie bracket between two vector fields v and w, d is the exterior derivative and i is the interior product. The properties of the Courant bracket will be discussed in more detail in the next chapter. However, for now it suffices to note that it includes both the ordinary diffeomorphisms and the gauge transformations of a 2-form field. Thus, this particular generalised geometry is well suited to describe theories with a metric and a 2-form field in which the group $\mathrm{O}(d, d)$ plays an important role. Of course, such geometries arise in string theory, where the 2-form field can be recognised as the NS-NS B field and the group $\mathrm{O}(d, d)$ is the global T-duality group.

The application of these ideas has lead to the doubled geometry approach [176, 177, 178, 179], which makes the T-duality group a manifest symmetry of closed bosonic string theory and also naturally combines the metric and NS-NS 2-form field into a single geometric object known as the generalised metric. The extra d coordinates that are the duals of the original coordinates are naturally interpreted as the winding coordinates of the string.

Of course one cannot just double the dimension of the space and remain with the same theory. One needs a constraint, which we refer to as the physical section condition, that imposes that physics lives in a section of the doubled space. A solution of the section condition gives a submanifold of the original dimension. Different solutions of this section condition correspond to different duality frames. One obvious solution of the section condition is that all fields are independent of the winding coordinates. In this sense, duality is a spontaneously broken symmetry with solutions to the section condition breaking the duality invariance of the theory.

One might suspect that this construction of string theory is somewhat artificial and alien to the usual formulation of string theory but in fact the structure of doubled geometry is present in a truncation of closed string field theory [180]. In particular, apart from the inclusion of winding modes to provide the coordinates of the doubled space, closed string field theory gives the local symmetries, the subsequent Courant algebra and the section condition of doubled geometry first found by Siegel [171]. From the string field theory perspective, the section condition is nothing more than an immediate consequence of level matching for the closed string.

Now consider replacing the tangent bundle by

$$T(M) \longrightarrow T(M) \oplus \Lambda^2(T^*(M)). \tag{7.7}$$

A generalised vector V in this geometry is composed of a vector field and a 2-form. In four dimensions, the dimension of the tangent space increases from 4 to $4 + 6 = 10$. The extra six coordinates can be thought of as representing the winding modes of an $M2$-brane—one of the fundamental constituents of M-theory. By considering duality transformations on the world-volume theory of an $M2$-brane that rotate field equations and Bianchi identities into one another, one can derive a so-called generalised metric that unifies the metric and 3-form potential C associated with the $M2$-brane into a single geometric structure [181]

$$M_{MN} = \begin{pmatrix} g_{ij} + \frac{1}{2}C_i{}^{kl}C_{jkl} & \frac{1}{\sqrt{2}}C_i{}^{kl} \\ \frac{1}{\sqrt{2}}C^{mn}{}_j & g^{mn,kl} \end{pmatrix}, \tag{7.8}$$

where g_{ij} is the metric, C_{ijk} is the 3-form potential and $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$. In four dimensions the generalised metric acts on a $\mathbf{10}$ of SL(5) and parameterises the coset space SL(5)/SO(5), which as one can see from table 7.1 is the duality symmetry found when reducing 11d supergravity on a 4-torus.

Using the generalised metric described above, Berman and Perry have been able to reformulate 11d supergravity in such a way as to make the SL(5)/SO(5) duality of 11d supergravity manifest, without requiring Killing directions [168]. The approach they take is the complement of the dimensional reduction approach that first lead to the appearance of these dualities in that they split the theory into 4+7 dimensions and consider the 4d part that is purely spatial.

One starts by constructing the Hamiltonian form of the theory. The reason for this is that duality transformations on timelike directions leads to complications, such as the complexification of fields [182]. To avoid such complications a canonical approach is preferred. The spacetime metric g_{ab} is decomposed into a purely spatial metric γ_{ij}, a lapse function α and a shift vector β^i [183] so that

$$g_{ab} = \begin{pmatrix} -\alpha^2 + \beta_k\beta^k & \beta_j \\ \beta_i & \gamma_{ij} \end{pmatrix}, \tag{7.9}$$

where here indices $a, b, c \dots$ run from $0, \dots, 11$ and indices $i, j, k \dots$ run from $1, \dots, 11$ and are lowered and raised using the spatial metric γ_{ij} and its inverse γ^{ij}, respectively. Similarly the 3-form potential C_{abc} is split up into its purely spatial components and the remainder. Hence,

$$C_{abc} \to \begin{cases} C_{ijk} \\ C_{0ij} \equiv B_{ij} \end{cases} . \tag{7.10}$$

The Hamiltonian of the theory is then constructed in an approach similar to that done for gravity [184, 183, 185]. The Hamiltonian consists of two parts: a kinetic part[4] and a potential part V

$$V = \gamma^{1/2}\left(-R(\gamma) + \frac{1}{48}F^{ijkl}F_{ijkl}\right), \tag{7.11}$$

where γ is the determinant of the metric γ_{ij}, $R(\gamma)$ is the associated Ricci scalar and $F_{ijkl} = 4\partial_{[i}C_{jkl]}$. Indices $i, j, k \dots$ now run from $1, \dots, 4$, because of the 4+7 split described above.

Now consider a potential constructed purely from the generalised metric M_{MN} given

[4]For brevity, we shall only cover the calculation for the potential part in this review. For details of the construction of the Hamiltonian and the kinetic part see [168].

in equation (7.8)

$$V = \gamma^{1/2}\left(\frac{1}{12}M^{MN}(\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2}M^{MN}(\partial_M M^{KL})(\partial_K M_{LN}) \right.$$
$$\left. + \frac{1}{12}M^{MN}(M^{KL}\partial_M M^{KL})(M^{RS}\partial_N M^{RS}) + \frac{1}{4}M^{MN}M^{PQ}(M^{KL}\partial_P M^{KL})(\partial_M M^{NQ})\right), \tag{7.12}$$

where $\partial_M = (\frac{\partial}{\partial x^i}, \frac{\partial}{\partial y_{ij}})$. Assuming that the fields do not depend on the winding modes of the $M2$-brane, i.e. that $\partial_y = 0$, the potential above reduces to the original potential, given in (7.11), derived from the canonical approach. This condition is similar to the physical section condition discussed in the case of the $\mathrm{O}(d,d)$ symmetry. It allows one to reduce to original theory from the enlarged one and it is always needed for the enlarged theory to make any sense. Also, a similar argument can be made for the kinetic part of the Hamiltonian. Thus, the Hamiltonian formulation of 11d supergravity has been reformulated using generalised geometry in such a way as to make manifest its SL(5) symmetry.

However, a few important questions regarding the precise nature of the generalised geometry at work remain unanswered in the construction above. For example, what is the generalised algebra of diffeomorphisms that gives rise to the expected local symmetries of the theory and how does it compare with the Courant algebra that one finds in the case of the $\mathrm{O}(d,d)$ generalised geometry? Or, how does the section condition arise? One would expect the condition used above to be a particular solution among many possibilities. What are those possibilities and how are they related to this particular condition? These questions will be addressed in chapter 8.

The generalised metric for the other duality groups up to E_7 has been constructed in [169, 170] and the manifest invariance of 11d supergravity under transformations belonging to these groups has been established. For example, the generalised geometry relevant for the E_7 symmetry group is where the tangent bundle is replaced by

$$T(M) \longrightarrow T(M) \oplus \Lambda^2(T^*(M)) \oplus \Lambda^5(T^*(M)) \oplus \big(T^*(M) \otimes \Lambda^7(T^*(M))\big). \tag{7.13}$$

The generalised coordinates

$$x^i, y_{ij}, w^{ij}, z_i, \tag{7.14}$$

where indices $i, j, \ldots$ run now from $1, \ldots, 7$, correspond to the usual space coordinates (x^i), the windings of the $M2$-brane (y_{ij}) as we had before, the winding of the $M5$-brane

(w^{ij}) and the windings of the dual graviton. The reason we did not have the windings of the $M5$-brane and the dual graviton in the SL(5) case above is that there, the spatial dimension is four and so these objects are irrelevant.

The generalised metric for the other groups is found from a non-linear realisation of the duality groups using a procedure pioneered by Isham, Salam and Strathdee [186]. The application of the non-linear realisation technique [187, 188, 189, 190] to external symmetries has a long and rich history, which, unfortunately we shall not be able to expound on here (see e.g. [186, 191, 192, 193, 194, 195, 196, 197]).

Chapter 8

SL(5) generalised geometry

8.1 Introduction

In this chapter, the local symmetry associated with the SL(5) generalised geometry is constructed. The group of local diffeomorphisms contains both ordinary diffeomorphisms and gauge transformations of the 3-form potential. The generalised Lie derivative is written in an SL(5) covariant manner and the generalised (Courant) bracket is determined by considering the antisymmetrisation of the generalised Lie derivative. The algebra is shown to close if a certain quadratic condition is satisfied. This condition is taken to be the physical section condition. It has as one of its solutions, the condition used in [168] to show the equivalence of the generalised geometry formulation and the canonical formulation of 11d supergravity, i.e. that no field has dependence on winding coordinates of the $M2$-brane. Consistency is checked by comparing these results with those already obtained for the generalised geometry of string theory through dimensional reduction (section 8.3). Finally, in section 8.4, general solutions of the section condition are discussed.

8.2 Generalised Lie derivative and Courant Bracket

The SL(5) duality group is found in the reduction of eleven-dimensional supergravity to seven dimensions. In [168], this duality group is made manifest in the directions of the 4-torus without assuming the existence of isometries. Since the duality group is made to act along four directions, the only supergravity fields that are non-vanishing are the metric and the 3-form potential C. The corresponding generalised geometry involves the extension of vectors, making up the tangent space, by 2-forms. The two-forms are the windings of the membrane, which source C. Furthermore, it was shown in [168] that the diffeomorphism and gauge symmetry of the 3-form potential are a result of reparametrisations of the ordinary space coordinates and winding coordinates, respectively. These form a Courant bracket algebra, which is exactly the same algebra found in generalised geometry [175, 172, 173].

The analogue of a Lie derivative in generalised geometry is a generalised Lie derivative [198], which encodes reparametrisations of the generalised coordinates as well as ordinary coordinate transformations. For generalised vector fields $V = (v, \mu)$ and $X = (x, \lambda)$, where v and x are vector fields and μ and λ are 2-forms, the generalised Lie derivative is defined to be

$$\hat{\mathcal{L}}_X V = \mathcal{L}_x v + \mathcal{L}_x \mu - i_v \mathrm{d}\lambda. \tag{8.1}$$

The first term is the Lie derivative of the vector field v along x. This reproduces the transformation of a vector field under coordinate transformations. The second and third terms in the generalised Lie derivative give the transformation of a two-form field, μ, under coordinate transformations generated by x and gauge transformations generated by λ.

The antisymmetrisation of the generalised Lie derivative gives rise to a Courant bracket, as opposed to a Lie bracket,

$$[X, Y]_C = \frac{1}{2}\left(\hat{\mathcal{L}}_X Y - \hat{\mathcal{L}}_Y X\right) \tag{8.2}$$

$$= [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y \lambda + \frac{1}{2}\mathrm{d}\left(i_y \lambda - i_x \eta\right), \tag{8.3}$$

where $X = (x, \lambda)$, $Y = (y, \eta)$ and $[x, y]$ denotes the Lie bracket of vector fields x and y. One may view the Courant bracket as describing the algebra of combined diffeomorphisms and gauge transformations. Thus the algebra of diffeomorphisms is exactly as one would expect as can be seen from the first term on the right-hand side of (8.3) and is given by the Lie bracket. The second and third terms on the right-hand side of (8.3) are what one would expect from a gauge transformation followed by a diffeomorphism. The last term is perhaps a little surprising in that it is exact. A gauge transformation that is exact will have no effect on the three-form potential C.

Given this, one might wonder if the Jacobi transformations hold. The Jacobiator of the Courant bracket is defined by

$$J(X, Y, Z) = [[X, Y]_C, Z]_C + [[Y, Z]_C, X]_C + [[Z, X]_C, Y]_C \tag{8.4}$$

and measures by how much the Jacobi identities fail. Using $X = (x, \lambda), Y = (y, \mu)$ and $Z = (z, \kappa)$ we find

$$J(X, Y, Z) = \frac{1}{4} d[(\iota_x L_y - \iota_y L_x)\kappa + (\iota_y L_z - \iota_z L_y)\lambda + (\iota_z L_x - \iota_x L_z)\mu]. \tag{8.5}$$

Since J is exact, the Jacobi identity holds when restricted to being evaluated on the field C which is the only field that changes under a gauge transformation. It trivially holds on all the other fields that do not transform under this gauge transformation.

The generalised vector fields can be twisted by a 3-form C in the following way:

$$\rho_C(X, \lambda) = (x, \lambda + \frac{1}{\sqrt{2}} i_X C).$$

The Courant bracket in terms of the twisted vector fields now reproduces the algebra of diffeomorphisms and gauge symmetries [168]. The generalised Lie derivative and Courant bracket above, equations (8.1) and (8.3) respectively, treat each "component" of the generalised vector field separately. The distinction made between the coordinates and windings is unnatural from the perspective of making duality a manifest symmetry. We would like a more democratic, or covariant, formulation of these objects. This allows us to find the section condition for the generalised geometry of the SL(5) duality group.

In components, the generalised vector field X is (x^i, μ_{ij}), where $i, j = 1, \dots, 4$. The 2-form can be Hodge dualised with the alternating symbol η ($\eta^{1234} = 1$) so that the indices on X become SL(5) indices, viz.

$$X^{ab} = \begin{cases} X^{i5} = x^i \\ X^{5i} = -x^i \\ X^{ij} = \frac{1}{2}\eta^{ijkl}\mu_{kl} \end{cases}, \tag{8.6}$$

where $a = (i, 5)$. The generalised Lie derivative, defined in equation (8.1), can be written as two pieces with different tensor structures

$$\left(\hat{\mathcal{L}}_X V\right)^i = x^k \partial_k v^i - v^k \partial_k x^i, \tag{8.7}$$

$$\left(\hat{\mathcal{L}}_X V\right)_{ij} = x^k \partial_k \mu_{ij} + \mu_{ik} \partial_j x^k + \mu_{kj} \partial_i x^k - 3 v^k \partial_{[k} \lambda_{ij]}. \tag{8.8}$$

Now, the Hodge dual of the second equation above is

$$\begin{aligned}\left(\hat{\mathcal{L}}_X V\right)^{ij} &= \frac{1}{2}\eta^{ijkl}\left(x^m\partial_m\mu_{kl} + \mu_{km}\partial_l x^m + \mu_{ml}\partial_k x^m - 3v^m\partial_{[m}\lambda_{kl]}\right) \\ &= x^m\partial_m V^{ij} + 2V^{k[i}\partial_k x^{j]} + V^{ij}\partial_k x^k + 2v^{[i}\partial_k X^{j]k}, \end{aligned} \tag{8.9}$$

where $V^{ij} = \frac{1}{2}\eta^{ijkl}\mu_{kl}$ and similarly $X^{ij} = \frac{1}{2}\eta^{ijkl}\lambda_{kl}$. We would like to write the generalised Lie derivative in terms of indices a, b that run from 1 to 5, $(\hat{\mathcal{L}}_X V)^{ab}$, so that if b is 5 we get the expression on the right-hand side of equation (8.7) and if a, b take values from 1 to 4 then we get the expression on the right-hand side of equation (8.9). The expression that reduces to equations (8.7) and (8.9) is

$$(\hat{\mathcal{L}}_X V)^{ab} = \frac{1}{2}X^{cd}\partial_{cd}V^{ab} + \frac{1}{2}V^{ab}\partial_{cd}X^{cd} + V^{ac}\partial_{cd}X^{db} - V^{cb}\partial_{cd}X^{ad}, \tag{8.10}$$

assuming that $\partial_{ij} = 0$. Although to write the generalised Lie derivative in terms of the generalised fields we assumed that $\partial_{ij} = 0$, in what follows we drop this condition. The requirement that $\partial_{ij} = 0$ is one particular choice to make the restriction from the 10-dimensional extended space to the four-dimensional physical space. This is one solution of the section condition, but there are other choices that can be made.

The generalised Lie bracket that reduces to the Courant bracket, equation (8.3), when particular components are considered is, as in the case of the Lie or Courant bracket, the antisymmetrisation of the corresponding Lie derivative, (8.2). Therefore, antisymmetrising the generalised Lie derivative, defined in equation (8.10), gives the generalised Lie bracket

$$\begin{aligned}[X, Y]_G^{ab} = \frac{1}{4}X^{cd}\partial_{cd}Y^{ab} - \frac{1}{4}Y^{cd}\partial_{cd}X^{ab} + X^{[a|c}\partial_{cd}Y^{|b]d} \\ -Y^{[a|c}\partial_{cd}X^{|b]d} - \frac{1}{4}X^{ab}\partial_{cd}Y^{cd} + \frac{1}{4}Y^{ab}\partial_{cd}X^{cd}. \end{aligned} \tag{8.11}$$

For particular choices of indices a, b, the above reduces to the Courant bracket, equation (8.3). For example, letting $a = i$ and $b = 5$ in the above we get $[x, y]^i$, which agrees with the Courant bracket. A similar check can easily be done for the choice $ab = ij$. If we think of a, b as ordinary coordinate indices, then we should only get ordinary differential geometry. In the case of Riemannian geometry one should only find the first two terms, which are the Lie bracket of X and Y.

The Jacobi identity for the generalised Lie bracket is not satisfied. However, the Jacobi identity holds on fields up to terms that vanish by the section condition (see appendix 8.A). Physically, the Jacobiator can be shown to be a pure gauge transformation on fields.

The algebra of generalised diffeomorphisms, which includes diffeomorphisms and gauge symmetries, must be closed. However, we find that

$$\begin{aligned}\Big([\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y]V\Big)^{ab} = \Big(\hat{\mathcal{L}}_{[X,Y]_G}V\Big)^{ab} + \frac{3}{4}\bigg[\Big(& X^{ef}\partial_{[ef}Y^{cd}\cdot\partial_{cd]}V^{ab} \\ &+ V^{ab}X^{ef}\partial_{[ef}\partial_{cd]}Y^{cd} + V^{ab}\partial_{[ef}X^{ef}\cdot\partial_{cd]}Y^{cd} \\ &- 4V^{c[a}\partial_{[ef}Y^{b]d}\cdot\partial_{cd]}X^{ef} + 2X^{ef}V^{c[a}\partial_{[ef}\partial_{cd]}Y^{b]d} \\ &+ 2V^{c[a}X^{b]d}\partial_{[ef}\partial_{cd]}Y^{ef}\Big) - (X\leftrightarrow Y)\bigg]. \qquad (8.12)\end{aligned}$$

Since closure of the algebra requires that

$$[\hat{\mathcal{L}}_X,\hat{\mathcal{L}}_Y]V = \hat{\mathcal{L}}_{[X,Y]_G}V,$$

the rest of the terms on the right-hand side of equation (8.12) must vanish which leads us to the section condition for the SL(5) generalised geometry. All of the extra terms vanish if

$$\partial_{[ab}\partial_{cd]}X = 0 \qquad \text{and} \qquad \partial_{[ab}X\cdot\partial_{cd]}Y = 0, \qquad (8.13)$$

where X and Y are arbitrary fields. This is the section condition that has to be satisfied by the generalised fields in the SL(5) generalised geometry. Letting the fields depend only on the x^i coordinates and imposing

$$\frac{\partial}{\partial y_{ij}} = 0$$

on all fields is one solution to the section condition, which was the one considered in [168], but there are also other possibilities. Solutions to the section condition will be considered in section 8.4.

The generalised Lie derivative has been defined with respect to the antisymmetric representation of SL(5). We can in principle write the generalised Lie derivative of a field in any representation of SL(5), including the fundamental representation. This is analogous to the spinorial Lie derivative in differential geometry. For the generalised field in the fundamental representation the generalised Lie derivative takes the form

$$(\hat{\mathcal{L}}_X V)^a = \frac{1}{2}X^{cd}\partial_{cd}V^a + \frac{1}{4}V^a\partial_{cd}X^{cd} - V^c\partial_{cd}X^{ad}. \qquad (8.14)$$

Evaluating the generalised Lie derivative of $U^{[a}V^{b]}$ using the above equation one can easily see that this definition is consistent with the generalised Lie derivative of a field in the antisymmetric representation.

The generalised Lie derivative of other SL(5) objects can be computed by a simple application of the Leibniz rule and by assuming that the generalised Lie derivative on a scalar is equal to its partial derivative, i.e.

$$\hat{\mathcal{L}}_X S = \frac{1}{2}X^{cd}\partial_{cd}S. \tag{8.15}$$

For example, the generalised Lie derivative on an SL(5) covector V_a is

$$(\hat{\mathcal{L}}_X V)_a = \frac{1}{2}X^{cd}\partial_{cd}V_a - \frac{1}{4}V_a\partial_{cd}X^{cd} + V_c\partial_{ad}X^{cd}. \tag{8.16}$$

Or, the generalised Lie derivative on an antisymmetric object V_{ab} is

$$(\hat{\mathcal{L}}_X V)_{ab} = \frac{1}{2}X^{cd}\partial_{cd}V_{ab} - \frac{1}{2}V_{ab}\partial_{cd}X^{cd} + V_{ad}\partial_{cb}X^{cd} - V_{bd}\partial_{ca}X^{cd}. \tag{8.17}$$

The generalised Lie derivative is supposed to encode all gauge freedoms associated with the theory. In particular, one would expect the generalised Lie derivative on the generalised metric to induce diffeomorphism and gauge transformations of the metric and 3-form potential.

The generalised metric for the SL(5) generalised geometry given in [168] is

$$\begin{pmatrix} g_{ij} + \frac{1}{2}C_i{}^{kl}C_{jkl} & \frac{1}{\sqrt{2}}C_i{}^{kl} \\ \frac{1}{\sqrt{2}}C^{mn}{}_j & g^{mn,kl} \end{pmatrix}, \tag{8.18}$$

where g_{ij} is the metric, C_{ijk} is the 3-form potential and $g^{mn,kl} = \frac{1}{2}(g^{mk}g^{nl} - g^{ml}g^{nk})$. However, in order to make connection with the SL(5) group, we dualised the winding coordinates and combined them with the ordinary coordinates to construct a object, X^{ab}, with SL(5) indices, equation (8.6). The metric that acts on X^{ab}, or equivalently the **10** of SL(5), is the generalised metric[1]

$$M_{ab,\,cd} = \begin{pmatrix} M_{i5,\,j5} & M_{i5,\,mn} \\ M_{kl,\,j5} & M_{kl,\,mn} \end{pmatrix} = \begin{pmatrix} g_{ij} + \frac{1}{2}C_i{}^{pq}C_{jpq} & -\frac{1}{2\sqrt{2}}C_i{}^{pq}\eta_{pqmn} \\ -\frac{1}{2\sqrt{2}}C_j{}^{pq}\eta_{pqkl} & g^{-1}g_{kl,mn} \end{pmatrix}, \tag{8.19}$$

[1] In terms of the M2-brane picture from which the original generalised metric was derived in [168], this generalised metric can be derived by considering the Hodge dual of the Lagrange multiplier used in the membrane action.

where $g = \det(g_{ij})$ and $g_{kl,mn} = \frac{1}{2}(g_{mk}g_{nl} - g_{ml}g_{nk})$ so that $g^{kl,mn}g_{mn,pq} = \frac{1}{2}(\delta^k_p\delta^l_q - \delta^k_q\delta^l_p)$.

The generalised metric above is determined by g_{ij} and C_{ijk}. Hence, it has $10+4=14$ number of independent components. A general symmetric 10×10 matrix has 55 independent components. However, the metric M is constrained to parametrise the coset SL(5)/SO(5), which is a $24-10=14$ dimensional space.

Using equation (8.17), we find that under a generalised Lie derivative, the variation of the generalised metric is

$$\begin{aligned}(\hat{\mathcal{L}}_X M)_{ab,\,cd} =& \frac{1}{2}X^{ef}\partial_{ef}M_{ab,\,cd} - M_{ab,\,cd}\partial_{ef}X^{ef} \\ &+ M_{af,\,cd}\partial_{eb}X^{ef} - M_{bf,\,cd}\partial_{ea}X^{ef} \\ &+ M_{ab,\,cf}\partial_{ed}X^{ef} - M_{ab,\,df}\partial_{ec}X^{ef}. \end{aligned} \tag{8.20}$$

In particular, taking the $ab = i5$, $cd = j5$ components gives

$$\begin{aligned}(\hat{\mathcal{L}}_X M)_{i5,\,j5} =& x^k\partial_k(g_{ij} + \tfrac{1}{2}C_i{}^{pq}C_{jpq}) + (g_{kj} + \tfrac{1}{2}C_k{}^{pq}C_{jpq})\partial_i x^k \\ &+ (g_{ik} + \frac{1}{2}C_i{}^{pq}C_{kpq})\partial_j x^k \\ &- \frac{1}{2\sqrt{2}}\partial_k X^{kl}(C_i{}^{pq}\eta_{pqjl} + C_j{}^{pq}\eta_{pqil}) + \ldots, \end{aligned} \tag{8.21}$$

where we have used equation (8.6), notably that $X^{i5} = x^i$, and the form of the generalised metric given in (8.19). The ellipses denote terms involving derivatives with respect to winding coordinates, i.e. ∂_{ij}, which we are not interested in.

Letting $X^{ij} = -\frac{1}{\sqrt{2}}\eta^{ijkl}\Lambda_{kl}$, the expression above reduces to

$$(\hat{\mathcal{L}}_X M)_{i5,\,j5} = \mathcal{L}_x(g_{ij} + \tfrac{1}{2}C_i{}^{pq}C_{jpq}) + \tfrac{1}{2}(C_i{}^{pq}\partial_{[j}\Lambda_{pq]} + C_j{}^{pq}\partial_{[i}\Lambda_{pq]}) + \ldots, \tag{8.22}$$

allowing us to conclude that the generalised Lie derivative does indeed induce diffeomorphism and gauge transformations in a manner in which we would expect it to. Similar computations on the other components of the generalised metric result in the expected transformations.

The form of the generalised metric in equation (8.19) allows us to write it as a pair of objects acting on the **5** of SL(5). That is,

$$M_{ab,\,cd} = m_{ac}m_{bd} - m_{ad}m_{bc}, \tag{8.23}$$

where the symmetric metric m is

$$m_{ab} = \begin{pmatrix} \frac{1}{\sqrt{2}} g^{-1/2} g_{ij} & V_i \\ V_j & g^{1/2}\sqrt{2}(1 + g_{ij} V^i V^j) \end{pmatrix}. \tag{8.24}$$

The SL(4) vector

$$V^i = \frac{1}{6}\epsilon^{ijkl} C_{jkl}, \tag{8.25}$$

where $\epsilon^{ijkl} = g^{-1/2}\eta^{ijkl}$ and is the alternating tensor. The fact that M can be written as in equation (8.23) is related to the generalised metric parametrising the coset space SL(5)/SO(5) and can be thought of as the constraint that reduces the number of degrees of freedom of M from 55 to 14.

A natural question to consider is whether the structure in equation (8.23) is preserved under generalised diffeomorphisms. In other words, does the metric m transform as one would expect its components to transform. The generalised Lie derivative of m is

$$(\hat{\mathcal{L}}_X m)_{ab} = \frac{1}{2} X^{cd}\partial_{cd} m_{ab} - \frac{1}{2} m_{ab}\partial_{cd} X^{cd} + m_{cb}\partial_{ad} X^{cd} + m_{ac}\partial_{bd} X^{cd}. \tag{8.26}$$

Taking the $a = i$, $b = j$ component of the equation above gives

$$\begin{aligned}(\hat{\mathcal{L}}_X m)_{ij} =& \tfrac{1}{\sqrt{2}}[x^k\partial_k(g^{-1/2}g_{ij}) + g^{-1/2}g_{kj}\partial_i x^k + g^{-1/2}g_{ik}\partial_j x^k - g^{-1/2}g_{ij}\partial_k x^k] + \dots \\ =& \mathcal{L}_x(\tfrac{1}{\sqrt{2}} g^{-1/2} g_{ij}) + \dots . \end{aligned} \tag{8.27}$$

Hence, the generalised Lie derivative on m reproduces diffeomorphisms. Now, taking $a = i$, $b = 5$,

$$\begin{aligned}(\hat{\mathcal{L}}_X m)_{i5} =& x^k\partial_k(V_i) + V_k\partial_i x^k - \tfrac{1}{\sqrt{2}} g^{-1/2} g_{ik}\partial_l X^{kl} + \dots \\ =& (\mathcal{L}_x V)_i + \delta_\Lambda V_i + \dots , \end{aligned} \tag{8.28}$$

where

$$\delta_\Lambda V_i = \frac{1}{6} g_{ik}\epsilon^{klmn}\delta_\Lambda C_{lmn} = \frac{1}{2} g_{ik}\epsilon^{klmn}\partial_l\Lambda_{mn} = -\tfrac{1}{\sqrt{2}} g^{-1/2} g_{ik}\partial_l X^{kl}, \tag{8.29}$$

for $X^{kl} = -\frac{1}{\sqrt{2}}\eta^{klmn}\Lambda_{mn}$. A similar result is obtained if we choose the 5 5 component of equation (8.26). We conclude that equation (8.23) is preserved under generalised diffeomorphisms. In other words, generalised diffeomorphisms preserve the coset structure in which M lies.

The crucial test is whether the SL(5) invariant action is actually invariant under these generalised diffeomorphisms. For the metric $M_{IJ} = M_{ab,cd}$ given by equation (8.19), the potential is given by:

$$V = \int M^{-\frac{1}{4}} \left(\frac{1}{12} M^{MN} (\partial_M M^{KL})(\partial_N M_{KL}) - \frac{1}{2} M^{MN} (\partial_N M^{KL})(\partial_L M_{MK}) \right. \\ \left. - \frac{1}{4} (\partial_M M^{MN})(M^{KL} \partial_N M_{KL}) + \frac{1}{12} M^{MN} (M^{KL} \partial_M M_{KL})(M^{PQ} \partial_N M_{PQ}) \right) . \quad (8.30)$$

This is same as in [168], which was written with respect to the generalised metric in equation (8.18). Here we have also included the measure factor, $(\det M)^{-\frac{1}{4}}$. One might be worried that it is a strange power but substituting the metric M_{IJ}, reveals that this is equivalent to the usual measure $(\det g)^{\frac{1}{2}}$. A long and involved calculation then confirms that this action is invariant under the local transformations of the generalised metric, equation (8.20).

8.3 Relation to O(d, d) generalised geometry

In this section, the generalised Lie derivative and bracket of the SL(5) generalised geometry, (8.10) and (8.11), are related to the corresponding objects of the O(d, d) generalised geometry [171, 179, 199]. The dimensional reduction of the SL(5) duality manifest description of M-theory should be related to the O(3,3) structure in string theory. Therefore, by dimensional reduction of the objects describing the SL(5) generalised geometry we should recover the O(3,3) generalised geometry.

First, let us consider the generalised Lie derivative. We reduce along the $a = 4$ direction, so we let any derivative with 4 as an index vanish, and consider the fields along the first three directions which we label by $\alpha, \beta, \cdots = 1, 2, 3$. Let $a = \alpha$ and $b = 5$ in equation (8.10),

$$(\hat{\mathcal{L}}_X V)^{\alpha 5} = \frac{1}{2} X^{cd} \partial_{cd} V^{\alpha 5} + \frac{1}{2} V^{\alpha 5} \partial_{cd} X^{cd} - V^{\alpha c} \partial_{cd} X^{5d} - V^{c5} \partial_{cd} X^{\alpha d}.$$

Note that because of the reduction ansatz the indices on the derivatives ∂_{cd} can only take values along the first three directions or the fifth direction, viz. $c, d = \gamma$ or 5, but because of the antisymmetry both c and d cannot be 5. Hence, denoting

$$V^{\alpha 5} = V^{\alpha}, \quad X^{\alpha 5} = X^{\alpha} \quad \text{and} \quad \partial_{\alpha 5} = \partial_{\alpha},$$

write

$$
\begin{aligned}
(\hat{\mathcal{L}}_X V)^\alpha =& X^\gamma\partial_\gamma V^\alpha + \frac{1}{2}X^{\gamma\delta}\partial_{\gamma\delta}V^\alpha + V^\alpha\partial_\gamma X^\gamma + \frac{1}{2}V^\alpha\partial_{\gamma\delta}X^{\gamma\delta} \\
&- V^{\alpha 5}\partial_{5\gamma}X^{5\gamma} - V^{\alpha\gamma}\partial_{\gamma\delta}X^{5\delta} - V^{\gamma 5}\partial_{\gamma 5}X^{\alpha 5} - V^{\gamma 5}\partial_{\gamma\delta}X^{\alpha\delta} \\
=& X^\gamma\partial_\gamma V^\alpha + \frac{1}{2}X^{\gamma\delta}\partial_{\gamma\delta}V^\alpha + \frac{1}{2}V^\alpha\partial_{\gamma\delta}X^{\gamma\delta} \\
&+ V^{\alpha\gamma}\partial_{\gamma\delta}X^\delta - V^\gamma\partial_\gamma X^\alpha - V^\gamma\partial_{\gamma\delta}X^{\alpha\delta}.
\end{aligned}
$$

Identifying

$$
\tilde{V}_\alpha = \frac{1}{2}\eta_{\alpha\beta\gamma}V^{\beta\gamma}, \quad \tilde{X}_\alpha = \frac{1}{2}\eta_{\alpha\beta\gamma}X^{\beta\gamma} \quad \text{and} \quad \tilde{\partial}^\alpha = \frac{1}{2}\eta^{\alpha\beta\gamma}\partial_{\beta\gamma},
$$

where $\eta_{\alpha\beta\gamma}$ is the alternating symbol, the expression for the O(3,3) generalised Lie derivative acting on a vector V^α is

$$
\begin{aligned}
(\hat{\mathcal{L}}_X V)^\alpha =& X^\gamma\partial_\gamma V^\alpha + \tilde{X}_\gamma\tilde{\partial}^\gamma V^\alpha - \tilde{V}_\gamma\tilde{\partial}^\gamma X^\alpha + \tilde{V}_\gamma\tilde{\partial}^\alpha X^\gamma - V^\gamma\partial_\gamma X^\alpha + V^\gamma\tilde{\partial}^\gamma\tilde{X}_\gamma \\
=& X^\Pi\partial_\Pi V^\alpha - V^\Pi\partial_\Pi X^\alpha + V^\Pi\tilde{\partial}^\alpha X_\Pi, \qquad (8.31)
\end{aligned}
$$

where

$$
X^\Pi = (X^\alpha, \tilde{X}_\alpha), \qquad X_\Pi = (\tilde{X}_\alpha, X^\alpha)
$$

and

$$
\partial_\Pi = (\partial_\alpha, \tilde{\partial}^\alpha), \qquad \partial^\Pi = (\tilde{\partial}^\alpha, \partial_\alpha).
$$

Similarly, we find the O(3,3) generalised Lie derivative acting on a field with lowered indices by considering the Hodge dual of equation (8.10) with $a = \beta$ and $b = \gamma$,

$$
\begin{aligned}
(\hat{\mathcal{L}}_X \tilde{V})_\alpha =& \frac{1}{2}\eta_{\alpha\beta\gamma}(\hat{\mathcal{L}}_X V)^{\beta\gamma} \\
=& X^\Pi\partial_\Pi\tilde{V}_\alpha - V^\Pi\partial_\Pi\tilde{X}_\alpha + V^\Pi\partial_\alpha X_\Pi. \qquad (8.32)
\end{aligned}
$$

Therefore, putting together equations (8.31) and (8.32) we find the O(3,3) covariant generalised Lie derivative given in equation (3.22) of [179],

$$
(\hat{\mathcal{L}}_X V)^\Sigma = X^\Pi\partial_\Pi V^\Sigma - V^\Pi\partial_\Pi X^\Sigma + V^\Pi\partial^\Sigma X_\Pi.
$$

The dimensional reduction of the generalised Lie derivative on a generalised vector field in the antisymmetric representation of SL(5) corresponds to the generalised Lie derivative on a generalised vector field in the fundamental of O(3,3). Therefore, one would expect that dimensionally reducing the expression for the generalised Lie derivative on a generalised field in the fundamental representation of SL(5) is related to the generalised Lie derivative of a generalised field in some other representation of O(3,3). This is indeed the case as we will now show. A field in the antisymmetric representation, W^{ab}, can be constructed from fields in the fundamental representation, U^a and V^a, by simply taking the antisymmetrisation of the two fields,

$$W^{ab} = U^{[a}V^{b]}.$$

We showed that under dimensional reduction W^{ab} becomes a field in the vector of O(3,3), which we denote W^{Π}. When $a = \alpha$ and $b = 5$, we have the following picture

$$\begin{array}{ccc} W^{ab} & = & U^{[a}V^{b]} \\ \downarrow & & \downarrow \\ W^{\alpha} & & U^{[\alpha}V^{5]} \end{array} .$$

The above diagram shows that the O(3,3) vector field W^{α} must also be decomposable in terms of fields U and V that are in a lower dimensional representation of O(3,3). The only non-trivial representation of O(3,3) that is possible is the Majorana-Weyl representation of O(3,3)[2]. Therefore, the O(3,3) vector field W^{α} is given by a product of Majorana-Weyl spinors U and V

$$W^{\alpha} = U^{[\alpha}V^{5]} \propto U^A \gamma^{\alpha}{}_{AB} V^B,$$

where uppercase Latin letters label spinor indices and take values in $\{1, 2, 3, 5\}$. Hence, we deduce that

$$\gamma^{\alpha}{}_{AB} \propto \delta^{[\alpha}_{[A}\delta^{5]}_{B]}. \tag{8.33}$$

Similarly, if we consider $W^{ab} = U^{[a}V^{b]}$ for $a = \alpha$ and $b = \beta$, we deduce that

$$\tilde{\gamma}_{\alpha AB} \propto \begin{cases} \eta_{\alpha AB} & \text{for } A, B = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}, \tag{8.34}$$

where η is the alternating symbol.

We have defined gamma matrices, but we have not shown that they satisfy the

[2]More technically, it is the Majorana-Weyl representation of the double cover of O(3,3), Pin(3,3), that is being considered. However, following common parlance we neglect this distinction.

Clifford algebra. The 4×4 O(3,3) γ-matrices act on Majorana-Weyl spinors, so the corresponding 8×8 Γ-matrices, which act on Dirac spinors, are

$$\Gamma^{\Pi} = \begin{pmatrix} 0 & (\gamma^{\Pi})^{AB} \\ (\gamma^{\Pi})_{AB} & 0 \end{pmatrix}, \tag{8.35}$$

which satisfy the Clifford algebra

$$\{\Gamma^{\Pi}, \Gamma^{\Sigma}\} = 2\eta^{\Pi\Sigma}\mathbb{I}_8,$$

where

$$\eta = \begin{pmatrix} 0 & \mathbb{I}_3 \\ \mathbb{I}_3 & 0 \end{pmatrix}$$

is the O(3,3) invariant. In terms of γ-matrices the Clifford algebra reads

$$(\gamma^{\Pi})^{AC}(\gamma^{\Sigma})_{CB} + (\gamma^{\Sigma})^{AC}(\gamma^{\Pi})_{CB} = 2\eta^{\Pi\Sigma}\delta^A_B. \tag{8.36}$$

Using the matrices $\Gamma^{\Pi} = (\Gamma^{\alpha}, \tilde{\Gamma}_{\alpha})$, define matrices

$$\begin{aligned} \Gamma^{(-)\alpha} &= \frac{1}{\sqrt{2}}(\Gamma^{\alpha} - \tilde{\Gamma}_{\alpha}) \\ \Gamma^{(+)\alpha} &= \frac{1}{\sqrt{2}}(\Gamma^{\alpha} + \tilde{\Gamma}_{\alpha}), \end{aligned} \tag{8.37}$$

which satisfy the Clifford algebra with the diagonal metric, diag(-1,-1,-1,1,1,1),

$$\{\Gamma^{(-)\alpha}, \Gamma^{(-)\beta}\} = -2\delta^{\alpha\beta}\mathbb{I}_8, \quad \{\Gamma^{(+)\alpha}, \quad \Gamma^{(+)\beta}\} = 2\delta^{\alpha\beta}\mathbb{I}_8, \quad \{\Gamma^{(-)\alpha}, \Gamma^{(+)\beta}\} = 0.$$

Since the $\Gamma^{(-)}$ matrices square to -1 and $\Gamma^{(+)}$ matrices square to 1, we can assume that the former are antisymmetric while the latter are symmetric. Indeed, choosing the charge conjugation matrix

$$C = \Gamma^{(-)1}\Gamma^{(-)2}\Gamma^{(-)3},$$

then

$$(\Gamma^{(\pm)\alpha})^T = -C\Gamma^{(\pm)\alpha}C^{-1}.$$

Now, from equation (8.37) we can deduce that

$$(\Gamma^{\alpha})^T = \tilde{\Gamma}_{\alpha}, \qquad (\tilde{\Gamma}_{\alpha})^T = \Gamma^{\alpha}.$$

Hence, looking at equation (8.35),

$$\begin{pmatrix} 0 & (\gamma^\alpha)_{BA} \\ (\gamma^\alpha)^{BA} & 0 \end{pmatrix} = \begin{pmatrix} 0 & (\tilde{\gamma}_\alpha)^{AB} \\ (\tilde{\gamma}_\alpha)_{AB} & 0 \end{pmatrix}. \tag{8.38}$$

The Clifford algebra for the γ-matrices, equation (8.36), can now, using the above relation be written as

$$\begin{aligned} (\gamma^\alpha)_{CA}(\gamma^\beta)_{CB} + (\tilde{\gamma}_\beta)_{CA}(\tilde{\gamma}_\alpha)_{CB} &= 2\delta^\beta_\alpha \delta_{AB}, \\ (\tilde{\gamma}_\alpha)_{CA}(\gamma^\beta)_{CB} + (\tilde{\gamma}_\beta)_{CA}(\gamma^\alpha)_{CB} &= 0. \end{aligned} \tag{8.39}$$

The following gamma-matrices satisfy the Clifford algebra relations above:

$$\gamma^\alpha{}_{AB} = 2\sqrt{2}\delta^{[\alpha}_{[A}\delta^{5]}_{B]} \quad \text{and} \quad \tilde{\gamma}_{\alpha AB} = \begin{cases} \sqrt{2}\eta_{\alpha AB} & \text{for } A, B = 1,2,3 \\ 0 & \text{otherwise} \end{cases}, \tag{8.40}$$

which are consistent with equations (8.33) and (8.34).

We have shown that under dimensional reduction an SL(5) generalised field in the fundamental representation becomes an O(3,3) spinor field. Therefore, the generalised Lie derivative of a field in the fundamental representation of SL(5), equation (8.14), should reduce to the spinorial Lie derivative in O(3,3) [200, 199, 201]

$$\hat{\mathcal{L}}_X V = X^\Pi \partial_\Pi V + \frac{1}{4}(\partial_\Pi X_\Sigma - \partial_\Sigma X_\Pi)\Gamma^{\Pi\Sigma} V. \tag{8.41}$$

To show this we require the expression for $\Gamma^{\Pi\Sigma}$ in the representation in which the dimensionally reduced SL(5) field is related to the O(3,3) spinor, i.e. the representation given in equation (8.40). Using equation (8.35),

$$\left(\Gamma^{\Pi\Sigma}\right)^A{}_C = \frac{1}{2}\left(\gamma^{\Pi AB}\gamma^\Sigma{}_{BC} - \gamma^{\Sigma AB}\gamma^\Pi{}_{BC}\right).$$

The relations in equation (8.38) can now be used to find $\Gamma^{\Pi\Sigma}$ in the representation of the Clifford algebra given in equation (8.40)

$$\begin{aligned} (\Gamma^{\alpha\beta})^A{}_B = 2\eta^{\alpha\beta A}\delta^5_B, \qquad (\Gamma_{\alpha\beta})^A{}_B &= -2\eta_{\alpha\beta B}\delta^A_5, \\ (\Gamma^\alpha{}_\beta)^A{}_B = \delta^\alpha_\beta(\delta^A_\gamma\delta^\gamma_B - \delta^A_5\delta^5_B) &- 2\delta^\alpha_B\delta^A_\beta, \end{aligned} \tag{8.42}$$

where

$$\Gamma^{\alpha\beta} = \frac{1}{2}(\Gamma^\alpha\Gamma^\beta - \Gamma^\beta\Gamma^\alpha), \quad \Gamma^\alpha{}_\beta = \frac{1}{2}(\Gamma^\alpha\tilde{\Gamma}_\beta - \tilde{\Gamma}_\beta\Gamma^\alpha), \quad \Gamma_{\alpha\beta} = \frac{1}{2}(\tilde{\Gamma}_\alpha\tilde{\Gamma}_\beta - \tilde{\Gamma}_\beta\tilde{\Gamma}_\alpha).$$

Now, we are ready to check the consistency of the two generalised Lie derivatives (8.14) and (8.41). Inserting equations (8.42) into the O(3,3) spinorial Lie derivative, (8.41), we obtain

$$\begin{aligned}(\hat{\mathcal{L}}_X V)^A =& X^\Pi\partial_\Pi V^A + \frac{1}{2}\delta^A_\gamma(\partial_\beta X^\beta - \tilde{\partial}^\beta\tilde{X}_\beta)V^\gamma - \delta^A_\beta(\partial_\gamma X^\beta - \tilde{\partial}^\beta\tilde{X}_\gamma)V^\gamma \\ &+ \eta^{A\beta\gamma}(\partial_\beta\tilde{X}_\gamma)V^5 - \delta^A_5\eta_{\beta\gamma\sigma}(\tilde{\partial}^\beta X^\gamma)V^\sigma - \frac{1}{2}\delta^A_5(\partial_\beta X^\beta - \tilde{\partial}^\beta\tilde{X}_\beta)V^5. \end{aligned} \quad (8.43)$$

This should be compared to the dimensional reduction of the generalised Lie derivative on a field in the fundamental of SL(5). We use the same reduction ansatz as before, namely that derivatives along the fourth direction vanish, and evaluate the Lie derivative, equation (8.14), for the index $a = \alpha$ and $a = 5$, respectively,

$$\begin{aligned}(\hat{\mathcal{L}}_X V)^\alpha =& X^\Pi\partial_\Pi V^\alpha + \frac{1}{2}(\partial_\beta X^\beta - \tilde{\partial}^\beta\tilde{X}_\beta)V^\alpha \\ &\qquad - (\partial_\beta X^\alpha - \tilde{\partial}^\alpha\tilde{X}_\beta)V^\beta + \eta^{\alpha\beta\gamma}(\partial_\beta\tilde{X}_\gamma)V^5, \\ (\hat{\mathcal{L}}_X V)^5 =& X^\Pi\partial_\Pi V^5 - \eta_{\beta\gamma\sigma}(\tilde{\partial}^\beta X^\gamma)V^\sigma - \frac{1}{2}(\partial_\beta X^\beta - \tilde{\partial}^\beta\tilde{X}_\beta)V^5.\end{aligned}$$

Comparing the above equations with (8.43), we conclude that the dimensionally reduced SL(5) generalised Lie derivative acting on a field in the fundamental representation is equal to the O(3,3) generalised spinorial Lie derivative.

In this section, we have shown that under dimensional reduction the SL(5) generalised field in the antisymmetric representation becomes an O(3,3) vector, while an SL(5) vector field becomes an O(3,3) spinor. Furthermore, by dimensionally reducing the SL(5) generalised Lie derivatives, (8.10) and (8.14), we find the O(3,3) generalised Lie derivatives on vector fields and spinors. The equality between these objects was established without the use of any section condition. However, the section condition seems to play an integral role in generalised geometry. In particular, applications of generalised geometry to physics rely on specific solutions of the section condition. For example, the rewriting of supergravity actions in terms of a generalised metric uses a particular solution in which the fields only depend on the spacetime coordinates and not on the brane windings. In the next section, we will consider the section condition, giving examples of other solutions where the fields depend on brane windings.

8.4 Section condition

The section condition for the SL(5) generalised geometry,

$$\partial_{[ab}\partial_{cd]}X = 0 \qquad \text{on all fields } X, \tag{8.44}$$

was found in section 8.2 by requiring closure of the algebra of generalised diffeomorphisms. The second equation in (8.13) is necessarily satisfied given the first. The section condition can be dimensionally reduced to obtain the $\mathrm{O}(3,3)$ section condition

$$\tilde{\partial}^{\alpha}\partial_{\alpha} = 0.$$

Consider the operator constraint

$$\eta^{abcde}\partial_{bc}\partial_{de} = 0.$$

Reducing this along the fourth direction with the ansatz that derivatives along this direction vanishes, the section condition becomes

$$\eta^{4\alpha\beta\gamma5}\partial_{\alpha\beta}\partial_{\gamma} = 0,$$

which implies the O(3,3) section condition

$$\tilde{\partial}^{\alpha}\partial_{\alpha} = 0.$$

The $\mathrm{O}(d,d)$ section condition is a Laplace equation in a Kleinian space. The general solution to this type of differential equation was found in [202] in the context of $\mathcal{N} = 2$ strings.

One can use the Weyl group to investigate solutions to equation (8.44). The Weyl group permutes the coordinates into each other. For the SL(5) group the Weyl group is the permutation group of 5 elements S_5. Consider the Fourier transform of the section condition (8.44), which implies that

$$p_{[ab}p_{cd]} = 0. \tag{8.45}$$

Since the Weyl group takes into account the redundancy in our labelling of coordinates, pick the indices in the above equation to be $1, \ldots, 4$. Therefore, equation (8.45) becomes

$$\mathbf{p}^T \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \mathbf{p} = 0, \tag{8.46}$$

where $\mathbf{p} = (p_{12}, p_{13}, p_{14}, p_{23}, p_{24}, p_{34})$. The matrix above can be diagonalised so that the above equation reads

$$a^2 + b^2 + c^2 - j^2 - k^2 - l^2 = 0, \tag{8.47}$$

where

$$\begin{aligned} a &= p_{12} + p_{34}, \qquad & b &= p_{13} + p_{24}, \qquad & c &= p_{14} + p_{23} \\ j &= p_{12} - p_{34}, \qquad & k &= p_{13} - p_{24}, \qquad & l &= p_{14} - p_{23}. \end{aligned} \tag{8.48}$$

Hence the SL(5) section condition is also the Laplace equation on a Kleinian space. Consider now the other 4 equations in (8.45). These equations can be written as a matrix equation

$$N\mathbf{p}^{(5)} = 0,$$

where $\mathbf{p}^{(5)} = (p_{15}, p_{25}, p_{35}, p_{45})$, and

$$N = \begin{pmatrix} 0 & p_{34} & -p_{24} & p_{23} \\ -p_{34} & 0 & p_{14} & -p_{13} \\ p_{24} & -p_{14} & 0 & p_{12} \\ -p_{23} & p_{13} & -p_{12} & 0 \end{pmatrix}.$$

If we think of $\mathbf{p}$ as solving equation (8.46), then the determinant of the matrix N vanishes, so it is necessarily degenerate. We can use the Gauss elimination method to find the rank of the matrix. Assuming that it has non-zero rank, we can without loss of generality take $p_{34} \neq 0$, in which case the matrix N reduces to its row echelon form

$$\begin{pmatrix} 1 & 0 & -p_{14}/p_{34} & p_{13}/p_{34} \\ 0 & 1 & -p_{24}/p_{34} & p_{23}/p_{34} \\ 0 & 0 & p_{34}p_{12} - p_{24}p_{13} + p_{23}p_{24} & 0 \\ 0 & 0 & 0 & p_{34}p_{12} - p_{24}p_{13} + p_{23}p_{24} \end{pmatrix}. \tag{8.49}$$

The expression $p_{34}p_{12} - p_{24}p_{13} + p_{23}p_{24}$ vanishes by equation (8.46), hence the rank of N is less than or equal to two. But because the matrix N is antisymmetric it cannot be of rank one. Therefore, N has rank zero or two. The former case corresponds to letting the fields be independent of the winding coordinates. This is the section condition that was used in [168] to recover the supergravity action from the duality invariant formulation in terms of the generalised metric. The latter case, when the rank of N is two, gives alternative section conditions where the fields can depend on winding coordinates. For example, the choice

$$p_{12}, p_{23}, p_{13}, p_{15}, p_{25}, p_{35} = 0$$

solves the SL(5) condition, and so the fields depend on the coordinates

$$x^4, y_{14}, y_{24}, y_{34}.$$

It is possible to consider alternative solutions of the section condition and find the theory to which the duality-invariant formulation reduces. This leads to different duality frames of eleven-dimensional supergravity.

8.5 Discussion

In this chapter, we constructed the generalised geometry associated to the SL(5) duality group. This is the duality group that appears on the reduction of eleven-dimensional supergravity to seven dimensions. An SL(5) covariant generalised Lie derivative was constructed by dualising the winding coordinates. The generalised Lie derivative on a field in the antisymmetric and vector representations was shown to give the generalised Lie derivative on an O(3,3) vector and spinor. Therefore, an SL(5) vector field becomes an O(3,3) spinor under dimensional reduction. This is perhaps related to the fact that 3-form potential in eleven-dimensional supergravity gives rise to both NS-NS and R-R fields under dimensional reduction.

The generalised Lie derivative generates generalised diffeomorphisms which encode the diffeomorphisms and U(1) gauge transformations. The closure of the algebra of generalised diffeomorphisms is only satisfied up to a constraint, which we identify with the section condition. The section condition has solutions whereby the fields can also depend on the winding coordinates. However, unlike the O(d,d) section condition the SL(5) section condition does not admit solutions whereby the fields only depend on winding coordinates. An interesting prospect to explore is to reduce the duality-invariant dynamics with respect to these alternative section choices. These different section choices will lead to different duality frames for the theory and allow for the definition of non-geometries based on transition functions that are given by SL(5) transformations.

Furthermore, it will be interesting to extend this work to the other dualities of M-theory. It is likely that the section conditions for the larger duality groups, $SO(5,5), E_6, E_7$ and E_8, can also be found by imposing closure of the algebra of generalised diffeomorphisms.

The section condition is a quantum off-shell condition rather than an on-shell condition. Equation (8.45) must be read as a quantum mechanical condition as $p = i\hbar\frac{\partial}{\partial x}$.

The type of constraint given by the section condition has been observed before in the context of half-BPS states and U-duality [203],[204]. It would be of great interest to see if this connection remains true for the larger duality groups.

8.A Jacobiator of the generalised Lie bracket

In this appendix, we consider the status of the Jacobi identity for the generalised Lie bracket, equation (8.11).

Define the Jacobiator of the generalised Lie bracket to be

$$J(X,Y,Z)^{ab} = ([[X,Y]_G, Z]_G + [[Y,Z]_G, X]_G + [[Z,X]_G, Y]_G)^{ab}\,, \tag{8.50}$$

where the bracket $[\ ,\ ]_G$ is defined in equation (8.11) and X, Y, Z are SL(5) bivectors, i.e. they have index structure $X^{ab} = X^{[ab]}$. The Jacobi identity for the generalised Lie bracket would be

$$J(X,Y,Z)^{ab} = 0. \tag{8.51}$$

Using the fact that the generalised Lie bracket is given by antisymmetrising a generalised Lie derivative, equation (8.2), i.e.

$$[X,Y]_G = \frac{1}{2}(\hat{\mathcal{L}}_X Y - \hat{\mathcal{L}}_Y X), \tag{8.52}$$

we rewrite the Jacobiator

$$\begin{aligned} J(X,Y,Z)^{ab} =& \frac{1}{4}([\hat{\mathcal{L}}_X, \hat{\mathcal{L}}_Y]Z + [\hat{\mathcal{L}}_Y, \hat{\mathcal{L}}_Z]X + [\hat{\mathcal{L}}_Z, \hat{\mathcal{L}}_X]Y)^{ab} \\ &- \frac{1}{2}(\hat{\mathcal{L}}_{[X,Y]_G} Z + \hat{\mathcal{L}}_{[Y,Z]_G} X + \hat{\mathcal{L}}_{[Z,X]_G} Y)^{ab}. \end{aligned} \tag{8.53}$$

Using, equation (8.12), the above expression reduces to

$$J(X,Y,Z)^{ab} = -\frac{1}{4}(\hat{\mathcal{L}}_{[X,Y]_G} Z + \hat{\mathcal{L}}_{[Y,Z]_G} X + \hat{\mathcal{L}}_{[Z,X]_G} Y)^{ab} + \ldots, \tag{8.54}$$

where the ellipses here and below denote terms that vanish if the section condition is assumed to hold.

Expanding out the terms above using the definition of the generalised Lie derivative given in equation (8.10), we find that

$$\begin{aligned} J(X,Y,Z)^{ab} = -\frac{1}{8}\Big\{ & \Big(Z^{ab}\partial_{cd}[X,Y]_G^{cd} + 4Z^{c[a}\partial_{cd}[X,Y]_G^{b]d} \Big) \\ & + \Big(X^{ab}\partial_{cd}[Y,Z]_G^{cd} + 4X^{c[a}\partial_{cd}[Y,Z]_G^{b]d} \Big) \\ & + \Big(Y^{ab}\partial_{cd}[Z,X]_G^{cd} + 4Y^{c[a}\partial_{cd}[Z,X]_G^{b]d} \Big) \\ & + [X,Y]_G^{cd}\partial_{cd}Z^{ab} + [Y,Z]_G^{cd}\partial_{cd}X^{ab} + [Z,X]_G^{cd}\partial_{cd}Y^{ab} \Big\} + \ldots. \end{aligned} \tag{8.55}$$

$J(X,Y,Z)^{ab}$ does not vanish even if the section condition is assumed. However, one can show that the Jacobi identity is satisfied when acting on fields modulo terms that vanish by the section condition. Using the following identity

$$6Z^{ab}\partial_{[cd}[X,Y]^{cd}_G\partial_{ab]}F = \left\{\left(Z^{ab}\partial_{cd}[X,Y]^{cd}_G + 4Z^{ca}\partial_{cd}[X,Y]^{bd}_G\right) + Z^{cd}\partial_{cd}[X,Y]^{ab}_G\right\}\partial_{ab}F$$

repeatedly in the expression for $J(X,Y,Z)^{ab}$ above, (8.55), gives

$$\begin{aligned} J(X,Y,Z)^{ab}\partial_{ab}F =& \frac{1}{8}\Big\{Z^{cd}\partial_{cd}[X,Y]^{ab}_G + X^{cd}\partial_{cd}[Y,Z]^{ab}_G + Y^{cd}\partial_{cd}[Z,X]^{ab}_G \\ & -[X,Y]^{cd}_G\partial_{cd}Z^{ab} - [Y,Z]^{cd}_G\partial_{cd}X^{ab} - [Z,X]^{cd}_G\partial_{cd}Y^{ab}\Big\}\partial_{ab}F + \dots, \end{aligned} \tag{8.56}$$

where F is any SL(5) covariant object, where indices have been suppressed. Finally, using the operator identity

$$[X,Y]^{cd}_G\partial_{cd} = \frac{1}{2}(X^{ef}\partial_{ef}Y^{cd} - Y^{ef}\partial_{ef}X^{cd})\partial_{cd} + \dots,$$

to simplify all six terms in the expression for $J(X,Y,Z)^{ab}\partial_{ab}F$ above allows us to conclude that

$$J(X,Y,Z)^{ab}\partial_{ab}F = 0, \tag{8.57}$$

modulo terms that vanish by the section condition.

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