

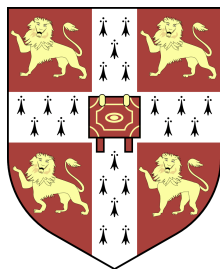
On the initial value problem in general relativity and wave propagation in black-hole spacetimes

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Declaration

This dissertation is based on research done while a graduate student at the Department of Applied Mathematics and Theoretical Physics, University of Cambridge, in the period between October 2011 and August 2014.

None of the material presented in this thesis is the outcome of work done in collaboration.

Chapter 2 of this thesis is based on the paper

On the Existence of a Maximal Cauchy Development for the Einstein Equations - a Dezornification (2013), arXiv:1309.7591v2.

and Chapter 3 (without the appendices 3.D and 3.E) is based on the paper

Characterisation of the Energy of Gaussian Beams on Lorentzian Manifolds - with Applications to Black Hole Spacetimes (2013), arXiv:1311.2477v1.

The content of the appendices 3.D and 3.E, and of Chapter 4 is unpublished in any form at the time of submission.

This dissertation has not been submitted for any other degree or qualification.

J. J. Sbierski
Cambridge
1st September 2014

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Summary

The first part of this thesis is concerned with the question of global uniqueness of solutions to the initial value problem in general relativity. In 1969, Choquet-Bruhat and Geroch proved, that in the class of globally hyperbolic Cauchy developments, there is a unique maximal Cauchy development. The original proof, however, has the peculiar feature that it appeals to Zorn's lemma in order to guarantee the existence of this maximal development; in particular, the proof is not constructive. In the first part of this thesis we give a proof of the above mentioned theorem that avoids the use of Zorn's lemma.

The second part of this thesis investigates the behaviour of so-called Gaussian beam solutions of the wave equation - highly oscillatory and localised solutions which travel, for some time, along null geodesics. The main result of this part of the thesis is a characterisation of the temporal behaviour of the energy of such Gaussian beams in terms of the underlying null geodesic. We conclude by giving applications of this result to black hole spacetimes. Recalling that the wave equation can be considered a "poor man's" linearisation of the Einstein equations, these applications are of interest for a better understanding of the black hole stability conjecture, which states that the exterior of our explicit black hole solutions is stable to small perturbations, while the interior is expected to be unstable.

The last part of the thesis is concerned with the wave equation in the interior of a black hole. In particular, we show that under certain conditions on the black hole parameters, waves that are compactly supported on the event horizon, have finite energy near the Cauchy horizon. This result is again motivated by the investigation of the conjectured instability of the interior of our explicit black hole solutions.

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Chapter 1

Introduction

In 1915, Albert Einstein put forward his general theory of relativity - at this time a novel theory of gravity which incorporates his special theory of relativity, dissolves the mystery of the apparent equality of inertial mass and gravitational mass, and contains Newton's classical theory of gravity in a weak-field and slow-motion limit. According to Einstein's new theory, space and time no longer form a fixed background against which the drama of physics is enacted, but they become dynamic themselves; the gravitational 'force' is modelled by a curved background, a 3 + 1 dimensional Lorentzian manifold (M, g) , where the metric g obeys the Einstein equations

$$Ric_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 2T_{\mu\nu} . \quad (1.0.1)$$

Here, rationalised units (i.e. $c \equiv 1$ and $4\pi G \equiv 1$) are used, $Ric_{\mu\nu}$ denotes the Ricci curvature of the metric g , R the scalar curvature, and $T_{\mu\nu}$ is the stress-energy tensor of a suitable matter model which acts as a source for the gravitation. The left hand side of equation (1.0.1) is a second order nonlinear partial differential operator applied to the metric g . By virtue of the differential Bianchi equations, the divergence of the left hand side vanishes identically - and thus the divergence of the stress-energy tensor has to vanish as well. The equation

$$\nabla^\mu T_{\mu\nu} = 0 \quad (1.0.2)$$

thus yields a partial differential equation for the matter fields, which however, is coupled to the spacetime metric g ! Hence, the equations (1.0.1) and (1.0.2) have to be considered together; as a coupled system of partial differential equations for the spacetime metric g and the matter fields. John Wheeler expressed this interplay between the 'gravitational potential' g and the matter fields in the following words:

Spacetime tells matter how to move; matter tells spacetime how to curve.¹

¹See page 235 in 'Geons, Black Holes, and Quantum Foam', by J. A. Wheeler and K. Ford, W. W. Norton & Company (2010)

Interestingly, and in contrast to Machian beliefs prevalent in the early years of general relativity, gravitation can also source itself, i.e., in Wheeler's words, gravitation/curvature *itself* also tells spacetime how to curve - and this in a non-trivial way! In other words, even in the absence of matter (i.e. $T_{\mu\nu} = 0$), the Einstein equations allow for a wide variety of solutions - and *not* only the flat Minkowski spacetime which can be considered free of gravitational forces. For convenience, let us restrict our following discussion to the vacuum Einstein equations $Ric_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0$, which, by taking the trace, can be shown to be equivalent to

$$Ric_{\mu\nu} = 0 . \tag{1.0.3}$$

Einstein himself argued in [29], based on perturbative arguments, that these equations allow for wave-like solutions. This was one hint among several to come that the Einstein equations (1.0.3) are of hyperbolic nature. However, since the mathematical analysis of this second order, nonlinear partial differential equation for the metric g turns out to be extremely intricate, it took 37 years to understand in what sense exactly the Einstein equations are hyperbolic and how the initial value problem they allow for has to be posed. This task was achieved in 1952 by Choquet-Bruhat in her seminal work [12]. We postpone a more detailed discussion of the initial value problem to Section 2.2 in Chapter 2. Let us just mention here that her proof is based on the fact that the Einstein equations (1.0.1) in harmonic coordinates² take the form

$$g^{\mu\nu} \partial_\mu \partial_\nu g_{\kappa\rho} + \mathcal{N}_{\kappa\rho}(g, \partial g) = 0 , \tag{1.0.5}$$

where $\mathcal{N}_{\kappa\rho}(g, \partial g)$ is a term nonlinear in the metric g and its first derivative, but does not contain any second derivatives. We see that the Einstein equations (1.0.3) in harmonic gauge (1.0.4) have taken the form of a coupled system of quasilinear wave equations! The idea is now to appeal to (or to prove...) a local well-posedness statement for a system of quasilinear wave equations. The gist of her argument however is the insight, that if the initial data for the Einstein equations satisfies certain constraint equations (cf. Section 2.2, equations (2.2.1)), then the gauge propagates and hence, solutions to the system of quasilinear wave equations (1.0.5) are also solutions to the Einstein equations (1.0.3)! In this way, the local aspects of the initial value problem for the Einstein equations were understood for the first time.

Subsequently, Choquet-Bruhat initiated the investigation of some global aspects of the initial value problem in general relativity, which culminated in the paper [13] from

²*Harmonic coordinates* are coordinate functions x^μ on the manifold M in which the metric g satisfies the condition

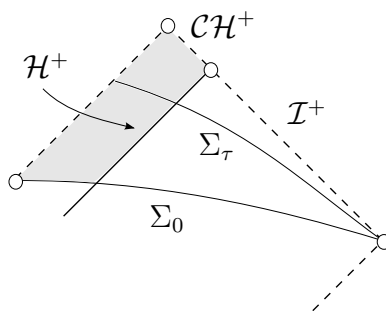
$$\square_g x^\mu := \frac{1}{\sqrt{-\det g}} \partial_\nu (g^{\nu\mu} \sqrt{-\det g}) = 0 \tag{1.0.4}$$

for $\mu = 0, \dots, 3$.

1969 in collaboration with Geroch, in which they showed that for given initial data for the Einstein equations, there exists a unique maximal globally hyperbolic Cauchy development³. In particular, this implies that as long as the development of given initial data remains globally hyperbolic, the development is unique. Chapter 2 of this thesis is concerned with this theorem. The original proof given by Choquet-Bruhat and Geroch has the peculiar and unsatisfactory feature that it relies crucially on the axiom of choice in the form of Zorn’s lemma. In Chapter 2 of this thesis we present a proof that avoids the use of Zorn’s lemma. In particular, we provide an explicit *construction* of this maximal globally hyperbolic development.

Chapters 3 and 4 of the thesis are motivated by the famous black hole stability conjecture, which we introduce in the following.

A particular class of important solutions to the vacuum Einstein equations (1.0.3) are the so called *black hole* solutions. Loosely speaking, a black hole is a spacetime, which (in space) asymptotically approaches the flat Minkowski spacetime and moreover, contains a region which is causally disconnected from the asymptotically flat region in the sense that no causal curves (in particular no light rays) starting in this region can ever reach the asymptotically Minkowskian region. It thus models an isolated gravitating system whose gravitational force becomes so strong in a certain region that not even light can escape it! A celebrated family of explicit black-hole solutions to the vacuum Einstein equations (1.0.3) is given by the so-called *Kerr family*, discovered in 1963 by Kerr, [38].⁴ This family is a two parameter family, the parameters being the mass and the angular momentum of the black hole. For a more detailed discussion of these solutions we refer the reader to Section 3.3.2 in Chapter 3 and to [34]. Here, we content ourselves with giving a Penrose diagrammatic representation of a Kerr black hole:⁵



Here, \mathcal{I}^+ represents the asymptotically flat region; the black hole region, i.e., the region from which not even light can escape, is represented by the shaded region; its boundary, called the event horizon, is denoted by \mathcal{H}^+ . The black hole spacetime at an ‘instant of time’ is given by the Cauchy hypersurface Σ_0 ; at a later instant of time

³The reader unfamiliar with this terminology is referred to Section 2.2 of Chapter 2.

⁴It is actually conjectured, that these are the only (stationary) vacuum black hole solutions to the Einstein equations.

⁵The reader unfamiliar with such Penrose diagrams is again referred to [34].

it is given by Σ_τ . For the purpose of the following discussion, our interest lies only with the black hole spacetime to the future of Σ_0 , and only with the structure of the interior ‘close’ to the event horizon \mathcal{H}^+ . We begin with the discussion of the exterior of the black hole.

Observational evidence suggests that our universe is populated by many black holes; in particular, black holes which are thought of not having changed much over a long period of time. Since small perturbations are omnipresent in nature, this implies in particular that these objects have to be stable to such small perturbations.

If these observed black and massive objects are indeed modelled correctly by the Kerr family, then their stability property to small perturbations should be reflected in the mathematical model, i.e., one should be able to prove that small deviations of the Kerr-black-hole data on Σ_0 lead, under Cauchy evolution, to a black hole spacetime, which, for late instances of time Σ_τ , asymptotes towards an exact Kerr black hole. However, let us stress here, that since one can only observe the exterior of a black hole (unless one undertakes the perilous journey into the interior of a black hole), the experimental evidence only suggests that the exterior is stable. Precisely this is the content of the famous black hole stability conjecture. If resolved in the positive, and given the observational evidence, it can be thought of as yet another verification of conclusions drawn from Einstein’s theory of gravity.

Before we relate this conjecture to the research presented in this thesis, let us briefly discuss the interior of a black hole. As already mentioned, our interest lies here with the structure of the interior ‘close’ to the event horizon. In particular, we will not discuss any expectations one could have on the interior towards the left boundary in the above diagram - we will focus instead on the so-called Cauchy horizon, denoted by \mathcal{CH}^+ in the Penrose diagram. It is important to stress, that in the exact Kerr model, this boundary is completely regular - indeed, one can even extend the spacetime across \mathcal{CH}^+ . However, a peculiar feature of the Cauchy horizon is, that it takes an observer a finite time to travel from Σ_0 to \mathcal{CH}^+ , while an observer in the exterior needs an infinite time to travel from Σ_0 to \mathcal{I}^+ . Based on this striking difference, and more elaborated upon in Section 3.3.1 of Chapter 3, Penrose conjectured that already a small (generic) perturbation of the black hole data on Σ_0 should suffice to turn, under Cauchy evolution, the completely regular Cauchy horizon into a singular Cauchy horizon. In contrast to the exterior of a Kerr black hole, Penrose thus argued on purely theoretical grounds, that the interior of a Kerr black hole, as given, is not a realistic model of the black holes in our universe. However, there is hope that by understanding the perturbations of the interior of a Kerr black hole, one can understand the structure of the interior of a ‘real’ black hole.

We now proceed to explain the connection between the black hole stability conjecture, ‘Penrose’s instability conjecture’, and the work presented in the Chapters 3 and 4 of this thesis.

In order to address either of these conjectures, one would need to understand the Cauchy evolution of perturbations under the Einstein equations (1.0.3). This, however, is an extremely difficult problem. Taking guidance from the monumental work of Christodoulou and Klainerman, [14], in which they proved the stability of Minkowski space, one should first try to understand the linearisation of the nonlinear problem. Linearising the Einstein equations, for instance in harmonic gauge, (1.0.5), around a Kerr metric, yields a coupled system of wave equations on the Kerr spacetime with lower order terms - still a very challenging equation to understand! A “poor man’s” linearisation is to forget about the tensorial structure, the coupling, and the lower order terms - hence ending up just with the wave equation

$$\square_{g_{Kerr}} u = 0$$

on the Kerr spacetime. Solutions of this equation already exhibit several novel features which are not shared by waves propagating in Minkowski space. Moreover, one can expect that many of these new features persist when one finally studies solutions to the ‘proper’ linearisation of the Einstein equations. Thus, it seems worthwhile, tractable, and instructive first to study the wave equation on a Kerr background.

In Chapter 3, using the Gaussian beam approximation, we study highly oscillatory and localised solutions to the wave equation on general globally hyperbolic Lorentzian manifolds. It is already known that using the Gaussian beam approximation one can show that there exist waves whose energy is localised along a given null geodesic for a finite, but arbitrarily long time. In Section 3.2.3 of Chapter 3, we show that the energy of such a localised solution to the wave equation is determined by the energy of the underlying null geodesic. This result opens the door to various applications of Gaussian beams on Lorentzian manifolds that *do not admit a globally timelike Killing vector field*. In particular, we can understand some of the above-mentioned novel features of waves on a Kerr background quantitatively: for example, the Kerr geometry can ‘trap’ waves on an orbit - we show that this phenomenon translates into the quantitative statement that a local energy decay statement necessarily comes with a ‘loss of derivative’. We also give a simple mathematical realisation of the heuristics given by Penrose for the behaviour of waves near the Cauchy horizon in the interior of a black hole: we construct a sequence of solutions to the wave equation whose initial energies are uniformly bounded, whereas the energy near the Cauchy horizon goes to infinity.

Chapter 4 of this thesis is concerned with the wave equation in the interior of a black hole. Under certain assumptions on the parameters of the black hole, we show that waves that are compactly supported on the event horizon of the black hole, have finite energy on a null hypersurface intersecting the Cauchy horizon. Extrapolating this result to the nonlinear theory, it suggests that perturbations which are compactly supported on the event horizon lead to a Cauchy horizon which is less singular than

the one forming under generic perturbations.

Chapter 2

On the existence of a maximal Cauchy development for the Einstein equations - a dezornification

2.1 Introduction

This chapter is concerned with the initial value problem for the vacuum Einstein equations, $Ric(g) = 0$. In her seminal paper [12] from 1952, Choquet-Bruhat showed that the initial value problem is *locally* well-posed, i.e., in particular she proved a local existence and a local uniqueness statement. Global aspects of the Cauchy problem in general relativity were explored in the paper [13] by Choquet-Bruhat and Geroch from 1969, where they showed that for given initial data there exists a (unique) *maximal* globally hyperbolic development (MGHD), i.e., a globally hyperbolic development (GHD) which is an extension of any other GHD of the same initial data. The existence of the MGHD not only implies ‘global uniqueness’ for the Cauchy problem in general relativity within the class of globally hyperbolic developments, but it also defines the object whose properties one needs to understand for answering further questions about the initial value problem⁶ - thus turning the MGHD into a central object in mathematical general relativity.

The proof of the existence of the MGHD, as given by Choquet-Bruhat and Geroch in [13], has the unsatisfactory feature that it relies heavily on the axiom of choice in the form of Zorn’s lemma, which they invoke in order to ensure the existence of such a maximal element *without actually finding it*. In this chapter we present another proof of the existence of the MGHD which does not appeal to Zorn’s lemma at all and, in

⁶Prominent and important examples are here the weak and the strong cosmic censorship conjectures, which are both concerned with the properties of the MGHD (for more details see Section 2.1.1).

fact, *constructs* the MGHD.

Outline of this chapter

In the next subsection we elaborate more on the importance of the MGHD by discussing the role it plays in the global theory of the Cauchy problem for the Einstein equations. Our motivation for giving another proof of the existence of the MGHD is discussed in Section 2.1.2. Thereafter, we briefly recall the original proof by Choquet-Bruhat and Geroch. The impatient or knowledgeable reader is invited to skip directly to Section 2.1.4, where we sketch the idea of the proof given in this chapter and exhibit the analogy of this new proof with the elementary proof of the existence of a unique MGHD for, say, a quasilinear wave equation on a fixed background. Finally, Section 2.1.5 gives a brief schematic comparison of the original and the new proof.

In Section 2.2, we introduce the necessary definitions and state the main theorems, which are then proved in Section 2.3.

2.1.1 The maximal globally hyperbolic development in the global theory of the Cauchy problem in general relativity

In the following we give a brief overview of the global aspects of the Cauchy problem in general relativity, focussing on the role played by the MGHD. Let us first discuss the aspect of ‘global uniqueness’. In the paper [13], Choquet-Bruhat and Geroch raised the following question:

A priori, it might appear possible that, once the solution has been integrated beyond a certain point in some region, the option, previously available, of further evolution in some quite different region has been destroyed⁷.

First of all it is clear, by looking at the Kerr solution for example, that one can only hope to obtain a global uniqueness result if one restricts consideration to *globally hyperbolic* developments of initial data⁸. That under this restriction, however, a global uniqueness statement indeed holds, was first proven in 1969 by Choquet-Bruhat and Geroch in the above cited paper. They actually proved a stronger statement than

⁷The possible scenario they describe here is well illustrated by the example of the simple ordinary differential equation $\dot{x} = 3x^{2/3}$. If we prescribe, for instance, at time $t = -1$ the initial data $x(t = -1) = -1$, then there is a *unique solution up to time* $t = 0$, given by $x(t) = t^3$. At time $t = 0$, however, one can continue x as a C^2 solution of the ODE in infinitely many ways, for example just by setting it to zero for all positive times.

⁸A globally hyperbolic development is not just a ‘development’ which is globally hyperbolic, but one also requires that the initial data embeds as a Cauchy hypersurface. See Section 2.2 for the precise definition of *GHD*.

global uniqueness, namely they showed the existence of the MGHD, from which it follows trivially that global uniqueness holds. But the MGHD also furnishes the central object for the study of further global aspects of the Cauchy problem in general relativity. First and foremost one should mention here the *weak* and the *strong cosmic censorship conjectures*. The latter states that for *generic* initial data, the MGHD cannot be isometrically embedded into a strictly larger spacetime (of a certain regularity). A positive resolution of the strong cosmic censorship conjecture would thus imply, that global uniqueness holds generically *even if we lift the restriction to globally hyperbolic developments*.

We now come to the more subtle aspect of ‘global existence’. In fact, the sheer notion of a spacetime existing for ‘all time’ is already non-trivial due to the absence of a fixed background. However, the *completeness of all causal geodesics* is a geometric invariant, which, moreover, accurately captures the physical concept of the spacetime existing for all time. And indeed, there are a few results which establish that global existence in this sense holds for small neighbourhoods of special initial data (see for example the monumental work of Christodoulou and Klainerman on the stability of Minkowski space, [14]). On the other hand, there are explicit solutions to the Einstein equations which do not enjoy this causal geodesic completeness, showing that one cannot possibly hope to establish ‘global existence’ in this sense for all initial data. Moreover, Penrose’s famous singularity theorem, see [54], shows that global existence in this sense cannot even hold generically⁹.

If we restrict, however, our attention to asymptotically flat initial data, that is data which models isolated gravitational systems, one could make the physically reasonable conjecture that at least the observers far out (at infinity) live for all time. Under the assumption that strong cosmic censorship holds for asymptotically flat initial data, the mathematical equivalent of this physical conjecture is that null infinity of the corresponding MGHD is complete - which, for *generic* asymptotically flat initial data, is the content of the weak cosmic censorship conjecture. Thus, the weak cosmic censorship conjecture should be thought of as conjecturing ‘global existence’.

2.1.2 Why another proof?

Our motivation for giving another proof of the existence of the MGHD is mainly based on the following three arguments:

- i) A constructive proof is more natural and, from an epistemological point of view, more satisfying than a non-constructive one, since one can actually *find* or *construct* the object one seeks instead of inferring a contradiction by assuming its

⁹Penrose’s singularity assumes that the development is globally hyperbolic, but recall from our discussion of global uniqueness, that this is the class of spacetimes we are interested in.

non-existence. Moreover, a direct construction usually provides not only more insight, but also more information.

- ii) In his lecture notes [35], David Hilbert distinguishes between two aspects of the mathematical method¹⁰: He first mentions the *progressive task* of mathematics, which is to establish a suitable set of postulates as the foundations of a theory, and then to investigate the theory itself by finding the logical consequences of its axioms. Hilbert then goes on to elaborate on the *regressive task* of mathematics, which he says is to find and exhibit the logical dependency of the theorems on the postulates, which, in particular, leads to a clarification of the strength and the necessity of each axiom of the theory.

The work in this chapter is motivated by the regressive task, we show that the existence of the MGHD for the Einstein equations does *not* rely on the axiom of choice. Besides a purely *mathematical* motivation for investigating the strength and the necessity of each axiom of a theory, there is also an important *physical* reason for doing so: The question whether an axiom or a theory is ‘true’ is beyond the realm of mathematics. However, a *physical* theory can be judged in accordance with its agreement with our perception of reality. For example, one would have a reason to dismiss the axiom of choice from the foundations of the *physical* theory¹¹, if its inclusion in the remaining postulates of our physical theory allowed the deduction of a statement which is in serious disagreement with our perception of reality. On the other hand, it would be reasonable to include the axiom of choice in our axiomatic framework of the physical theory, if one could not prove a theorem, that is crucial for the physical theory, without it.

To the best of our knowledge, there are neither very strong arguments for embracing nor for rejecting the axiom of choice in general relativity. But if it had been

¹⁰For Hilbert’s original words on this matter see [35], page 17:

Der Mathematik kommt hierbei eine zweifache Aufgabe zu: Einerseits gilt es, die Systeme von Relationen zu entwickeln und auf ihre logischen Konsequenzen zu untersuchen, wie dies ja in den rein mathematischen Disziplinen geschieht. Dies ist die *progressive Aufgabe* der Mathematik. Andererseits kommt es darauf an, den an Hand der Erfahrung gebildeten Theorien ein festes Gefüge und eine möglichst einfache Grundlage zu geben. Hierzu ist es nötig, die Voraussetzungen deutlich herauszuarbeiten, und überall genau zu unterscheiden, was Annahme und was logische Folgerung ist. Dadurch gewinnt man insbesondere auch Klarheit über die unbewußt gemachten Voraussetzungen, und man erkennt die Tragweite der verschiedenen Annahmen, so daß man übersehen kann, was für Modifikationen sich ergeben, falls eine oder die andere von diesen Annahmen aufgehoben werden muß. Dies ist die *regressive Aufgabe* der Mathematik.

¹¹Here, the ‘foundations of the physical theory’ should be thought of as ‘mathematics with all its axioms together with those postulates within mathematics that actually model the physical theory’.

the case that the axiom of choice had been needed for ensuring the existence of the MGHD, this would have been a strong reason for including it into the postulates of general relativity.

- iii) The structure of the original proof of the existence of the MGHD is in stark contrast to the straightforward and elementary construction of the MGHD for, say, a quasilinear wave equation on a fixed background; in the latter case one constructs the MGHD by taking the union of all GHDs (see also Section 2.1.4). The proof given in this chapter embeds the construction of the MGHD for the Einstein equations in the general scheme for constructing MGHDs by showing that an analogous construction to ‘taking the union of all GHDs’ works.

We conclude with some formal set theoretic remarks: The results from PDE theory and causality theory we resort to in our proof do not require more choice than the *axiom of dependent choice* (DC). Disregarding such ‘black box results’ we refer to, our proof only needs the *axiom of countable choice* (CC).¹² We can thus conclude that the existence of the MGHD is a theorem of¹³ $\mathbf{ZF}+\mathbf{DC}$; and checking how much choice is actually required for proving the ‘black box results’ we resort to might even reveal that the existence of the MGHD is provable in $\mathbf{ZF}+\mathbf{CC}$.

We have made no effort to avoid the axiom of *countable* choice in our proof - mainly for two reasons: Firstly, the axiom of countable choice is needed for many of the standard results and techniques in mathematical analysis. Thus, investigating whether the ‘black box results’ we resort to can be proven even without the axiom of countable choice promises to be a rather tedious undertaking, while the gained insight might not be that enlightening. Secondly, while the axiom of choice has rather wondrous consequences, the implications of the axiom of countable (or dependent) choice seem, so far, to be less foreign to human intuition.

2.1.3 Sketch of the proof given by Choquet-Bruhat and Geroch

The original proof by Choquet-Bruhat and Geroch can be divided into two steps. In the first step, they invoke Zorn’s lemma to ensure the existence of a *maximal* element in the class of all developments; and in the second step, which is more difficult, they show that actually any other development embeds into this maximal element. Let us recall their proof in some more detail¹⁴:

First step: Consider the set \mathcal{M} of all globally hyperbolic developments of certain

¹²For one application of it see for example the proof of Lemma 2.3.11.

¹³ \mathbf{ZF} stands here for the Zermelo-Fraenkel set theory.

¹⁴The reader, who is not familiar with the terminology used below, is referred to the definitions made in Section 2.2.

fixed initial data. Define a partial ordering on this set by $M \leq M'$ iff M' is an extension of M . Since a chain is by definition *totally ordered*, it is not difficult to glue all the elements of a chain together¹⁵ to construct a bound for the chain in question. Zorn's lemma then implies that there is at least one maximal element in \mathcal{M} . Pick one and call it M .¹⁶

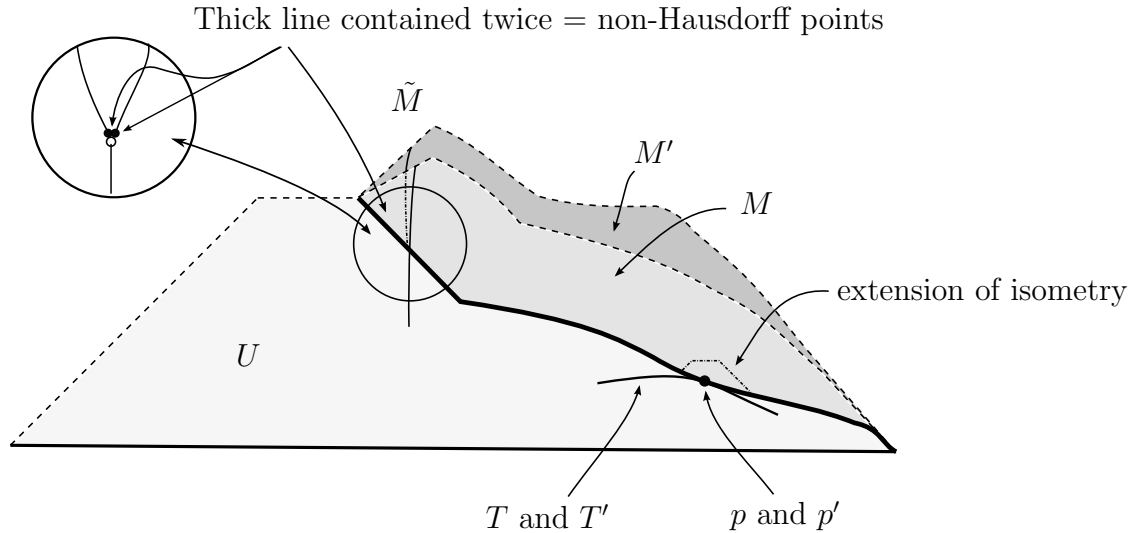
Second step: Let M' be another element of \mathcal{M} . Choquet-Bruhat and Geroch set up another partially ordered set, namely the set of all common globally hyperbolic developments of M and M' , where the partial order is given by inclusion. Using the same argument as in Corollary 2.3.2, they again argue that every chain is bounded, since one can just take the union of its elements. By appealing to Zorn's lemma once more, they establish the existence of a *maximal* common globally hyperbolic development U , and argue that it must be unique.

Now, one glues M and M' together along U . The resulting space \tilde{M} can be endowed in a natural way with the structure needed for turning it into a globally hyperbolic development, which, however, might a priori be non-Hausdorff. Establishing that \tilde{M} is indeed Hausdorff is at the heart of their argument. Once this is shown, the resulting development is trivially an extension of M - and since M is maximal, we must have had $U = M'$, i.e., M' embeds into M .

The proof of \tilde{M} being Hausdorff goes by contradiction. If it were not Hausdorff, then one shows that this would be due to pairs of points on the boundary of U in M and M' , respectively (cf. the picture below). One then has to ensure the existence of a 'spacelike' part of this non-Hausdorff boundary. Given a 'non-Hausdorff pair' $[p], [p'] \in \tilde{M}$, one then constructs a spacelike slice T in M that goes through p and such that $T \setminus \{p\}$ is contained in U . If ψ denotes the isometric embedding of U into M' , this also gives rise to a spacelike slice $T' := \psi(T \setminus \{p\}) \cup \{p'\}$ in M' .

¹⁵Given two sets A and B , and an identification of points in A with points in B , we can *glue* these two sets together by first taking the disjoint union $A \sqcup B$ of A and B and then forming the quotient space $(A \sqcup B)/\sim$, where the equivalence relation \sim is generated by the given identification of points in A with points in B . For a detailed presentation of the glueing construction we refer the reader to the beginning of the proof of Theorem 2.2.8 on page 31 (and also to the proof of Theorem 2.2.9 on page 33). In the case of a chain, the identification of points is given by the isometric embedding of one space in the other. In particular it is trivial to show that the so obtained space is Hausdorff!

¹⁶The collection of all globally hyperbolic developments of given initial data is actually a proper class and not a set (see also footnote 23 and the discussion above the proof of Theorem 2.2.9 on page 33). In order to justify the above steps within **ZFC** (the Zermelo-Fraenkel set theory with the axiom of choice) one can perform a reduction to a *set* of globally hyperbolic developments analogous to the reduction used in the proof of Theorem 2.2.9.



Clearly, the induced initial data on T and T' are isometric. Appealing to the local uniqueness statement for the initial value problem for the Einstein equations, one thus finds that one can actually extend the isometric identification of M with M' to a small neighbourhood of p - in contradiction with U being the *maximal* common globally hyperbolic development.

Let us remark that the proof of \tilde{M} being Hausdorff is rather briefly presented in the original paper by Choquet-Bruhat and Geroch. A very detailed proof is found in Ringström's [62], which is an errata to his book [63].

2.1.4 Outline of the proof presented in this chapter

We first discuss a proof of global uniqueness and of the existence of a MGH for the case of a quasilinear wave equation on a fixed background. Our proof for the case of the Einstein equations will then naturally arise by analogy.

The case of a quasilinear wave equation

Let us consider a quasilinear wave equation for $u : \mathbb{R}^{3+1} \rightarrow \mathbb{R}$,

$$g^{\mu\nu}(u, \partial u) \partial_\mu \partial_\nu u = F(u, \partial u), \quad (2.1.1)$$

where g is a Lorentz metric valued function. Under suitable conditions on g and F one can prove local existence and uniqueness of solutions to the Cauchy problem¹⁷. Such

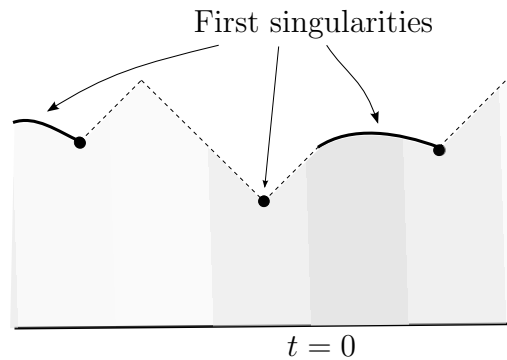
¹⁷We are not concerned with regularity questions here, all initial data can be assumed to be smooth.

a statement takes the following form (see for example [66]):

Given initial data $f, h \in C_0^\infty(\mathbb{R}^3)$ there exists a $T > 0$ and a unique solution $u \in C^\infty([0, T] \times \mathbb{R}^3)$ of (2.1.1) with $u(0, \cdot) = f(\cdot)$ and $\partial_t u(0, \cdot) = h(\cdot)$.

Moreover, if T^* denotes the supremum of all such $T > 0$ then we have (2.1.2) either $T^* = \infty$ or the $L^\infty(\mathbb{R}^3)$ norm of $u(t, \cdot)$ and/or of some derivatives of u blows up for $t \rightarrow T^*$.

However, in the case of $T^* < \infty$, in general $u(x, t)$ will not become singular for all $x \in \mathbb{R}^3$ for $t \rightarrow T^*$. The points $x \in \mathbb{R}^3$ where it becomes singular are called *first singularities* - at regular spacetime points (T^*, x) we can extend the solution.



A natural question is then: does there exist a unique maximal globally hyperbolic¹⁸ solution of (2.1.1) with initial values f and h ? In the following we sketch a construction of such an object.

First step: We show that *global uniqueness* holds, i.e., given two solutions $u_1 : U_1 \rightarrow \mathbb{R}$ and $u_2 : U_2 \rightarrow \mathbb{R}$ to the above Cauchy problem, where U_i is globally hyperbolic with respect to u_i and with Cauchy surface $\{t = 0\}$, the two solutions then agree on $U_1 \cap U_2$.

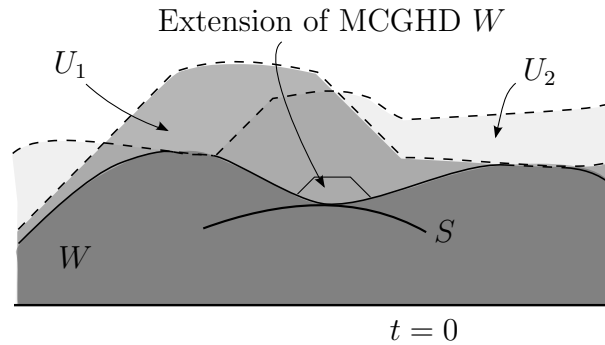
There are different ways to establish global uniqueness. One could for example prove this using energy estimates. Note, however, that such a proof is necessarily local by character, since $U_1 \cap U_2$ is not a priori globally hyperbolic with respect to either of the solutions.

The proof we sketch in the following is based on a continuity argument and only appeals to the local uniqueness statement. By this statement, we know that there is some open and globally hyperbolic neighbourhood $V \subset U_1 \cap U_2$ of $\{t = 0\}$ on which the two solutions agree (note that ‘global hyperbolicity’ is here well-defined since the two solutions agree on the domain in question). Let us take the union W of all such *common globally hyperbolic developments* (CGHD) of (U_1, u_1) and (U_2, u_2) . By definition this set is clearly maximal, i.e., it is the biggest globally hyperbolic set

¹⁸Note that it depends on the solution u whether a subset of \mathbb{R}^{3+1} is globally hyperbolic or not.

on which u_1 and u_2 agree. We also call it the *maximal common globally hyperbolic development* (MCGHD).

Assume the so obtained set is not equal to $U_1 \cap U_2$. Then, as in the picture below, we can take a small spacelike slice S that touches $\partial W \cap U_1 \cap U_2$.¹⁹



By assumption u_1 and u_2 agree in W , thus by continuity they also agree on the slice S . We now consider the initial value problem with the induced data on S .²⁰ Clearly, u_1 and u_2 are solutions, and thus, by the local uniqueness theorem, they agree in a small neighbourhood of S . This, however, contradicts the maximality of W . Hence, u_1 and u_2 agree on $U_1 \cap U_2$.²¹

Second step: Having proved global uniqueness, the construction of the MGHD is now a trivial task: We consider the set of all globally hyperbolic developments $\{U_\alpha, u_\alpha\}_{\alpha \in A}$ of the initial data f, h and note that this set is non-empty by the local existence theorem. We then take the union $U := \bigcup_{\alpha \in A} U_\alpha$ of all the domains U_α and define

$$u(x) := u_\alpha(x) \text{ for } x \in U_\alpha .$$

By global uniqueness, this is well-defined. Moreover, it is easy to see that the set U is globally hyperbolic with respect to u and that this development is maximal by construction.

¹⁹This step actually requires a bit of care...

²⁰Note that a local uniqueness and existence statement for the initial value problem on S can be derived from (2.1.2) by introducing slice coordinates for S and by appealing to the domain of dependence property.

²¹The proof we just sketched yielded $W = U_1 \cap U_2$ by *contradiction*. One should be aware, however, that one can also prove $W = U_1 \cap U_2$ *directly* by the following continuity argument: To begin with, the local uniqueness theorem shows that the set on which two solutions agree is not empty. By continuity of the solutions, we know then that the two solutions must also agree on the closure of this set, which furnishes the closedness part of the argument. Openness is achieved by restarting the local uniqueness argument from (spacelike slices that touch) the boundary, as in the above picture. Note however, that in order to obtain openness across null boundaries, one has to “work one’s way upwards” along the null boundary, which makes this direct argument a bit more complicated. Also note that this continuity argument is qualitatively the same as the one already encountered in proving uniqueness of solutions to the initial value problem for regular ODEs.

The case of the Einstein equations

Our proof of the existence of the MGHD for the Einstein equations can be viewed as an ‘imitation’ of the scheme just presented. To understand better the problems that have to be overcome, however, let us first qualitatively compare the Einstein equations with a quasilinear wave equation on a fixed background: A solution to the Einstein equations is given by a pair (M, g) , where M is a manifold and g a Lorentzian metric on M . The background manifold M is *not* fixed here. The diffeomorphism invariance of the Einstein equations states that if ϕ is a diffeomorphism from M to a manifold N , then (N, ϕ_*g) is also a solution to the Einstein equations. Physically, these two solutions are indistinguishable - which suggests that one should consider the Einstein equations as ‘equations for isometry classes of Lorentzian manifolds’ (cf. also Remark 2.2.6). It is also only then that the Einstein equations become hyperbolic. Moreover, it is well-known that breaking the diffeomorphism invariance by imposing a harmonic gauge (this should be thought of as picking a representative of the isometry class) turns the Einstein equations into a system of quasilinear wave equations. It is thus reasonable to expect that the only problems caused in transferring the construction of the MGHD from Section 2.1.4 to the Einstein equations are due to the fact that, while in the case of the quasilinear wave equation the objects one works with are functions defined on subsets of a fixed background, for the Einstein equations one actually would have to consider isometry classes of Lorentzian manifolds. In particular we face the following two problems:

- i) Already the *definition* of ‘global uniqueness’ does not transfer directly to the Einstein equations, since $U_1 \cap U_2$ is not a priori defined for two GHDs U_1 and U_2 for the Einstein equations.
- ii) Since there is no fixed ambient space in the context of the Einstein equations, one cannot just take the union of all GHDs of given initial data in order to construct the MGHD.

We discuss the first problem first. For the case of the quasilinear wave equation on a fixed background, a trivially equivalent formulation of ‘global uniqueness’ is that there is a globally hyperbolic development (U, u) of the initial data such that $U_1 \cup U_2$ is contained in U and such that $u = u_1$ on U_1 and $u = u_2$ on U_2 .

This formulation of ‘global uniqueness’ *does* transfer to the Einstein equations: *Given two globally hyperbolic developments of the same initial data, there exists a globally hyperbolic development in which both isometrically embed.* This statement is the content of Theorem 2.2.8. Moreover, it is exactly this notion of global uniqueness that is crucial for the existence of the MGHD.

Let us first motivate the method used in this chapter for constructing this common extension of two GHDs for the Einstein equations: In the case of a quasilinear wave equation on a fixed background, we would construct a common extension of (U_1, u_1)

and (U_2, u_2) by first showing that the solutions agree on $U_1 \cap U_2$ - as we did in Section 2.1.4 - and thereafter extending both solutions to $U_1 \cup U_2$. Let us observe here that instead of constructing the bigger space $U_1 \cup U_2$ by taking the union of U_1 and U_2 , we can also glue them together along $U_1 \cap U_2$ - which yields the same result. However, for the construction of the common extension, both operations only make sense, if we already know that the solutions agree on $U_1 \cap U_2$. We can, however, still *glue* along an a priori smaller set on which we know that the two solutions agree, i.e., along a common globally hyperbolic development V of U_1 and U_2 . In general, the so obtained space will not be Hausdorff due to the presence of ‘corresponding boundary points’, i.e., a point in ∂V that lies in U_1 as well as in U_2 . The same argument which established global uniqueness above (cf. the last picture) shows, however, that if this is the case, then we can actually find a bigger CGHD along which we can glue.

Let us now directly glue U_1 and U_2 together along the *maximal* CGHD (recall, that this was defined as the union of all CGHDs). Again, the same argument that corresponds to the last picture shows that the MCGHD of (U_1, u_1) and (U_2, u_2) *cannot have corresponding boundary points*²² since this would violate the maximality of the MCGHD. In particular, we see that glueing along the MCGHD yields a Hausdorff space.

This reinterpretation of the construction of the common extension $U_1 \cup U_2$ of U_1 and U_2 for the case of a quasilinear wave equation can be transferred to the Einstein equations: In Section 2.3.1 we establish the existence of the MCGHD for two given GHDs for the Einstein equations. Note that this is also proved in the original paper by Choquet-Bruhat and Geroch - however, they appeal to Zorn’s lemma. Here, we construct the MCGHD of two GHDs U_1 and U_2 by taking the union of all CGHDs (that are subsets of U_1) in U_1 . In Section 2.3.2 we then give the rigorous proof that the MCGHD does not possess corresponding boundary points, i.e., that the space obtained by glueing along the MCGHD, lets call it \tilde{M} , is then indeed Hausdorff. Moreover, it is more or less straightforward to show that \tilde{M} satisfies all other properties of a GHD, see Section 2.3.3, which then finishes the construction of the common extension and thus proves global uniqueness for the Einstein equations.

Let us summarise the main idea that guided the way for the construction of the common extension of two GHDs for the Einstein equations:

In the case of the Einstein equations, the appropriate analogue of ‘taking the union’ of two GHDs is to glue them together along their MCGHD.

(2.1.3)

This statement, in spite of its simplicity, should be considered as the main new idea of this chapter. It also leads straightforwardly to the construction of the MGHD in the

²²In particular we inferred that thus the MCGHD must be equal to $U_1 \cap U_2$.

case of the Einstein equations by proceeding in analogy to the case of a quasilinear wave equation on a fixed background: for given initial data, we glue ‘all’ GHDs together along their MCGHDs, see Section 2.3.3.²³

2.1.5 Schematic comparison of the two proofs

Original proof	New proof
Ensure existence of a maximal element M in the set of all GHDs (using Zorn’s lemma).	
Ensure existence of a MCGHD of two GHDs (using Zorn’s lemma).	Construct MCGHD of two GHDs by taking the union (literally!) of all CGHDs.
Show that M is indeed the MGHD by ‘taking the union’ (in the sense of (2.1.3)).	Prove global uniqueness by ‘taking the union’ (in the sense of (2.1.3)) of two GHDs.
(Infer global uniqueness from the existence of the MGHD.)	Construct MGHD by ‘taking the union’ (in the sense of (2.1.3)) of ‘all’ GHDs.

2.2 The basic definitions and the main theorems

Let us start with some words about the stipulations we make:

- This chapter is only concerned with the smooth case, i.e., we only consider smooth initial data for the Einstein equations. In particular, the MGHD we construct is, a priori, only maximal among *smooth* GHDs. This raises the question whether one could extend the MGHD to a bigger GHD that is, however, less regular.

An answer to this question is provided by the *low regularity* local well-posedness theory for quasilinear wave equations, which in particular entails that *as long*

²³Let us already remark here the following subtlety: The collection of *all* GHDs of given initial data forms a *proper* class, i.e., it is too ‘large’ for being a set and, hence, also for performing the glueing construction using the axioms of **ZF**. In Section 2.3.3 we show that it suffices to work with an appropriate subclass of all GHDs, which actually is a *set*.

as the solution remains in the low regularity class local well-posedness is proven in, any additional regularity is preserved. The classical approach using energy estimates yields such a local well-posedness statement for very general quasilinear wave equations in $H^{5/2+\varepsilon}$. For the special case of the Einstein equations, the recent resolution of the bounded L^2 curvature conjecture by Klainerman, Rodnianski and Szeftel, [40], pushed this low-regularity well-posedness even further.

Regarding the technique of the proof given in this chapter, it heavily depends on the causality theory developed for at least C^2 -regular Lorentzian metrics. But as long as the initial data is such that it gives rise to a GHD of regularity at least C^2 , basically the same proof as given in this chapter goes through. For work on the existence of the MGHD for rougher initial data along the lines of the original Choquet-Bruhat Geroch style argument using Zorn's lemma, see [15]. Here one should mention that up to a few years ago the proof of local uniqueness (which plays, not surprisingly, a central role for proving global uniqueness) required one degree of differentiability more than the proof of local existence. This issue was overcome by Planchon and Rodnianski ([56]).

Having made these comments, we stipulate that all manifolds and tensor fields considered in this chapter are smooth, even if this is not mentioned explicitly.

- We moreover assume that all Lorentzian manifolds we consider are connected and time oriented. The dimension of the Lorentzian manifolds is denoted by $d + 1$, where $d \geq 1$.
- For simplicity of exposition we restrict our consideration to the *vacuum* Einstein equations $Ric(g) = 0$. However, the inclusion of matter and/or of a cosmological constant does not pose any additional difficulty as long as a local existence and uniqueness statement holds. In fact, exactly the same proof applies.

The Einstein equations are of hyperbolic character, they allow for a well-posed initial value problem. *Initial data* $(\bar{M}, \bar{g}, \bar{k})$ for the vacuum Einstein equations consists of a d -dimensional Riemannian manifold (\bar{M}, \bar{g}) together with a symmetric 2-covariant tensor field \bar{k} on \bar{M} that satisfy the *constraint equations*:

$$\begin{aligned} \bar{R} - |\bar{k}|^2 + (\text{tr}\bar{k})^2 &= 0 \\ \bar{\nabla}^i \bar{k}_{ij} - \bar{\nabla}_j \text{tr}\bar{k} &= 0, \end{aligned} \tag{2.2.1}$$

where \bar{R} denotes the scalar curvature and $\bar{\nabla}$ denotes the Levi-Civita connection on \bar{M} .

Definition 2.2.2. A globally hyperbolic development (*GHD*) (M, g, ι) of initial data $(\bar{M}, \bar{g}, \bar{k})$ is a time oriented, globally hyperbolic Lorentzian manifold (M, g) that satisfies the vacuum Einstein equations, together with an embedding $\iota : \bar{M} \rightarrow M$ such that

1. $\iota^*(g) = \bar{g}$
2. $\iota^*(k) = \bar{k}$, where k denotes the second fundamental form of $\iota(\bar{M})$ in M
3. $\iota(\bar{M})$ is a Cauchy surface in (M, g) .

Definition 2.2.3. *Given two GHDs (M, g, ι) and (M', g', ι') of the same initial data, we say that (M', g', ι') is an extension of (M, g, ι) iff there exists a time orientation preserving isometric embedding²⁴ $\psi : M \rightarrow M'$ that preserves the initial data, i.e. $\psi \circ \iota = \iota'$.*

Definition (First version) 2.2.4. *Given two GHDs (M, g, ι) and (M', g', ι') of initial data $(\bar{M}, \bar{g}, \bar{k})$, we say that a GHD (U, g_U, ι_U) of the same initial data is a common globally hyperbolic development (CGHD) of (M, g, ι) and (M', g', ι') iff both (M, g, ι) and (M', g', ι') are extensions of (U, g_U, ι_U) .*

Paraphrasing Definition 2.2.4, a GHD U is a CGHD of GHDs M and M' if, and only if, U is ‘contained’ in M as well as in M' . Here we have just written M instead of (M, g, ι) , etc. We will from now on often use this shorthand notation.

We now give a slightly different definition of a common globally hyperbolic development and discuss the relation with the previous definition thereafter in Remark 2.2.6.

Definition (Second version) 2.2.5. *Given two GHDs (M, g, ι) and (M', g', ι') of initial data $(\bar{M}, \bar{g}, \bar{k})$, we say that a GHD $(U \subseteq M, g|_U, \iota)$ is a common globally hyperbolic development (CGHD) of (M, g, ι) and (M', g', ι') iff (M', g', ι') is an extension of (U, g_U, ι_U) .*

Remark 2.2.6. *1. The diffeomorphism invariance of the Einstein equations implies that if M is a GHD of certain initial data, then so is any spacetime that is isometric to M . From a physical point of view, isometric spacetimes should be considered to be the same, i.e., one should actually consider the isometry class of a GHD to be the solution to the Einstein equations. It is easy to check that the Definitions 2.2.3 and 2.2.4 also descend to the isometry classes of GHDs, i.e., they do not depend on the chosen representative of the isometry class. It is also only when one considers isometry classes that one can prove uniqueness for the initial value problem to the Einstein equations in the strict meaning of this word. However, working with isometry classes has a decisive disadvantage for the purposes of this chapter: the isometry class of a given GHD is a proper class, i.e., not a set. Thus, if we considered an infinite number of isometry classes, not even the full axiom of choice would be strong enough to pick a representative of each*

²⁴We lay down some terminology here: An *isometry* is a diffeomorphism that preserves the metric. An *isometric immersion* is an immersion that preserves the metric. Finally, an *isometric embedding* is an isometric immersion that is a diffeomorphism onto its image.

- and we need a representative to work with. We thus refrain from considering isometry classes of GHDs.

2. As just mentioned, Definition 2.2.4 is diffeomorphism invariant. In Definition 2.2.5 we break the diffeomorphism invariance by requiring that a CGHD U of M and M' is realised as a subset of M . However, this is not a serious restriction, since given any CGHD U of M and M' in the sense of Definition 2.2.4, we can isometrically embed U into M by using the isometric embedding that is provided by M being an extension of U .

Although Definition 2.2.5 is a bit less natural, we will choose it over Definition 2.2.4 in this chapter since, for our purposes, it is more convenient to work with. Also note that while Definition 2.2.4 is symmetric in M and M' , i.e., U being a CGHD of M and M' is the same as U being a CGHD of M' and M , the symmetry is broken in Definition 2.2.5.

In 1952 Choquet-Bruhat proved local existence and uniqueness of solutions to the initial value problem for the vacuum Einstein equations, see [12]:

Theorem 2.2.7. *Given initial data for the vacuum Einstein equations, there exists a GHD, and for any two GHDs of the same initial data, there exists a CGHD.*

The next two theorems are the main theorems of this chapter.

Theorem 2.2.8 (Global uniqueness). *Given two GHDs M and M' of the same initial data, there exists a GHD \tilde{M} that is an extension of M and M' .*

Theorem 2.2.9 (Existence of MGHD). *Given initial data there exists a GHD \tilde{M} that is an extension of any other GHD of the same initial data. The GHD \tilde{M} is unique up to isometry and is called the maximal globally hyperbolic development (MGHD) of the given initial data.*

Note that Theorem 2.2.9 clearly implies Theorem 2.2.8. In the original proof by Choquet-Bruhat and Geroch, Theorem 2.2.9 was proven without first proving Theorem 2.2.8. In our approach, however, we first establish Theorem 2.2.8 - Theorem 2.2.9 then follows easily.

2.3 Proving the main theorems

2.3.1 The existence of the maximal common globally hyperbolic development

In this section we construct the unique maximal common globally hyperbolic development of two GHDs. We start with a couple of lemmata that are needed for this construction.

Lemma 2.3.1. *Let (M, g) and (M', g') be Lorentzian manifolds, where M is connected. Furthermore, let $\psi_1, \psi_2 : M \rightarrow M'$ be two isometric immersions with $\psi_1(p) = \psi_2(p)$ and $d\psi_1(p) = d\psi_2(p)$ for some $p \in M$. It then follows that $\psi_1 = \psi_2$.*

Proof. One shows that the set

$$A = \{x \in M \mid \psi_1(x) = \psi_2(x) \text{ and } d\psi_1(x) = d\psi_2(x)\}$$

is open, closed and non-empty, from which it then follows that $A = M$. In order to show openness, let $x_0 \in A$ be given and choose a normal neighbourhood U of x_0 . For $x \in U$, there is then a geodesic $\gamma : [0, \varepsilon] \rightarrow U$ with $\gamma(0) = x_0$ and $\gamma(\varepsilon) = x$. Since ψ_1 and ψ_2 are both isometric immersions, we have that both $\psi_1 \circ \gamma$ and $\psi_2 \circ \gamma$ are geodesics. Moreover, since by assumption we have $(\psi_1 \circ \gamma)(0) = (\psi_2 \circ \gamma)(0)$ and $d(\psi_1 \circ \gamma)(0) = d(\psi_2 \circ \gamma)(0)$, the two geodesics agree. In particular, we obtain $\psi_1(x) = (\psi_1 \circ \gamma)(\varepsilon) = (\psi_2 \circ \gamma)(\varepsilon) = \psi_2(x)$.

Closedness of A is by smoothness of ψ_1 and ψ_2 , and non-emptiness holds by assumption. \square

Corollary 2.3.2. *Let (M, g) be a globally hyperbolic, time oriented Lorentzian manifold with Cauchy surface Σ and (M', g') another time oriented Lorentzian manifold. Moreover, say $U_1, U_2 \subseteq M$ are open and globally hyperbolic with Cauchy surface Σ , and $\psi_i : U_i \rightarrow M'$, $i = 1, 2$, are time orientation preserving isometric immersions that agree on Σ .*

Then ψ_1 and ψ_2 agree on $U_1 \cap U_2$.

Proof. Since ψ_1 and ψ_2 agree on Σ , their differentials agree on Σ if evaluated on vectors tangent to Σ . Moreover, since the isometric immersion preserve the time orientation, they both map the future normal of Σ onto the future normal of $\psi_1(\Sigma) = \psi_2(\Sigma)$. Thus, the differentials of ψ_1 and ψ_2 agree on Σ . The corollary now follows from Lemma 2.3.1. \square

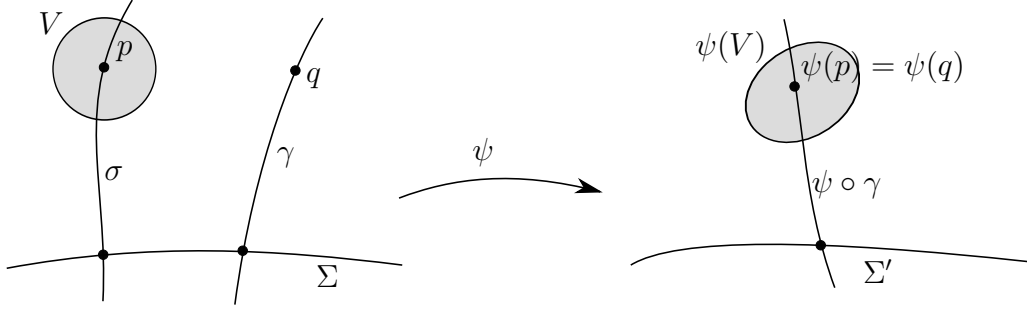
Lemma 2.3.3. *Say (M, g) and (M', g') are two globally hyperbolic spacetimes with Cauchy surfaces Σ and Σ' , respectively. Let $\psi : M \rightarrow M'$ be an isometric immersion such that $\psi|_{\Sigma} : \Sigma \rightarrow \Sigma'$ is a diffeomorphism.*

Then ψ is an isometric embedding.

Note that this shows in particular that in Definition 2.2.3 one does not need to require ψ to be an isometric embedding - ψ being an isometric immersion suffices.

Proof. It suffices to show that ψ is injective. So let p, q be points in M with $\psi(p) = \psi(q)$. Consider an inextendible timelike geodesic $\gamma : (a, b) \rightarrow M$ with $\gamma(0) = q$, where $-\infty \leq a < 0 < b \leq \infty$. Since (M, g) is globally hyperbolic, γ intersects Σ exactly once; say $\gamma(\tau_0) \in \Sigma$, where $\tau_0 \in (a, b)$. Note that since ψ is an isometric immersion, $\psi \circ \gamma : (a, b) \rightarrow M'$ is also a timelike geodesic. We now choose a neighbourhood V of p such that $\psi|_V : V \rightarrow \psi(V)$ is a diffeomorphism and we pull back the velocity vector

of $\psi \circ \gamma$ at $\psi(p)$ to p . Let $\sigma : (c, d) \rightarrow M$ denote the inextendible timelike geodesic with $\sigma(0) = p$ and $\dot{\sigma}(0) = d\psi|_V^{-1}(\psi \circ \dot{\gamma})|_{\psi(q)}$, where $-\infty \leq c < 0 < d \leq \infty$. Again, by M being globally hyperbolic, σ intersects Σ exactly once; say at $\sigma(\tau_1) \in \Sigma$, with $c < \tau_1 < d$. Clearly, the geodesics $\psi \circ \gamma$ and $\psi \circ \sigma$ agree on their common domain, since they share the same initial data.



By the global hyperbolicity of (M', g') , the geodesics $\psi \circ \gamma$ and $\psi \circ \sigma$ cannot intersect Σ' more than once, which implies that $\tau_0 = \tau_1$. Moreover, since $\psi|_{\Sigma} : \Sigma \rightarrow \Sigma'$ is a diffeomorphism, we have $\sigma(\tau_0) = \gamma(\tau_0)$. Now making use again of ψ being a local diffeomorphism at $\sigma(\tau_0)$, one infers that $\dot{\sigma}(\tau_0) = \dot{\gamma}(\tau_0)$ also holds. It follows that $\sigma = \gamma$ and in particular that $p = \sigma(0) = \gamma(0) = q$. \square

We can finally prove the main result of this section:

Theorem 2.3.4 (Existence of MCGHD). *Given two GHDs M and M' of the same initial data, there exists a unique CGHD U of M and M' with the property that if V is another CGHD of M and M' , then U is an extension of V .*

We call U the maximal common globally hyperbolic development (MCGHD) of M and M' .

The original proof of this theorem, i.e., as it is found in [13] or [62] for example, appeals to Zorn's lemma. The much simpler method of taking the union of all CGHDs of M and M' however works:

Proof. We consider the set $\{U_\alpha \subseteq M \mid \alpha \in A\}$ of all CGHDs of M and M' . By Theorem 2.2.7 this set is non-empty. We show that

$$U := \bigcup_{\alpha \in A} U_\alpha$$

is the MCGHD of M and M' .

1. It is clear that U is open and thus a time-oriented Ricci-flat Lorentzian manifold.
2. U is globally hyperbolic with Cauchy surface $\iota(\overline{M})$: Let γ be an inextendible timelike curve in U . Take a point on γ ; it lies in some U_α and the corresponding curve segment in U_α can be considered to be an inextendible timelike curve in U_α

and thus has to meet $\iota(\overline{M})$. Note that γ cannot meet $\iota(\overline{M})$ more than once, since γ is also a segment of an inextendible timelike curve in M - and M is globally hyperbolic.

3. It follows that U is a GHD of the given initial data.
4. U is a CGHD of M and M' : It suffices to give an isometric immersion $\psi : U \rightarrow M'$ that respects the embedding of \overline{M} and the time orientation. Note that by Lemma 2.3.3 ψ is then automatically an isometric embedding.

For each $\alpha \in A$ there is such an isometric immersion $\psi_\alpha : U_\alpha \rightarrow M'$. We define

$$\psi(p) := \psi_\alpha(p) \quad \text{for } p \in U_\alpha.$$

By Corollary 2.3.2 this is well-defined and clearly ψ is an isometric immersion that respects the embedding of \overline{M} and the time orientation.

5. That U is maximal follows directly from its definition. It then also follows that U is the unique CGHD with this maximality property.

□

2.3.2 The maximal common globally hyperbolic development does not have corresponding boundary points

In this section we prove that the MCGHD of two GHDs M and M' does not have ‘corresponding boundary points’. Most of the proofs found in this section are based on proofs from Ringström’s exposition [62].

Definition 2.3.5. *Let U be a CGHD of M and M' , and let us denote the isometric embedding of U into M' with ψ . Two points $p \in \partial U \subseteq M$ and $p' \in \partial\psi(U) \subseteq M'$ are called corresponding boundary points iff for all neighbourhoods V of p and for all neighbourhoods V' of p' one has*

$$\psi^{-1}(V' \cap \psi(U)) \cap V \neq \emptyset.$$

The main theorem of this section is

Theorem 2.3.6. *Let M and M' be GHDs of the same initial data, and say U is a CGHD of M and M' . If U possesses corresponding boundary points in M and M' , then there exists a strictly larger extension of U that is also a CGHD of M and M' . In particular, U is not the MCGHD of M and M' .*

Before we give the proof of Theorem 2.3.6, we need to establish some results concerning the structure and properties of corresponding boundary points. Let us begin

by giving a different characterisation of corresponding boundary points using timelike curves, which will often prove more convenient.

Proposition 2.3.7. *Let U be a CGHD of M and M' with isometric embedding $\psi : U \subseteq M \rightarrow M'$. The following statements are equivalent:*

- i) The points $p \in \partial U$ and $p' \in \partial\psi(U)$ are corresponding boundary points.*
- ii) If $\gamma : (-\varepsilon, 0) \rightarrow U$ is a timelike curve with $\lim_{s \nearrow 0} \gamma(s) = p$, then $\lim_{s \nearrow 0} (\psi \circ \gamma)(s) = p'$.*
- iii) There is a timelike curve $\gamma : (-\varepsilon, 0) \rightarrow U$ with $\lim_{s \nearrow 0} \gamma(s) = p$ such that $\lim_{s \nearrow 0} \psi \circ \gamma(s) = p'$.*

In particular it follows from ii) and iii) that $p \in \partial U$ has at most one corresponding boundary point.

Before we give the proof, let us recall some notation from causality theory on time oriented Lorentzian manifolds²⁵: we write

1. $p \ll q$ iff there is a future directed timelike curve from p to q
2. $p < q$ iff there is a future directed causal curve from p to q
3. $p \leq q$ iff $p < q$ or $p = q$.

Proof of Proposition 2.3.7: The implications *ii) \implies iii)* and *iii) \implies i)* are trivial. We prove *i) \implies ii)*: Without loss of generality let us assume that p and p' lie to the future of the Cauchy surfaces $\iota(\overline{M})$ and $\iota'(\overline{M}')$, respectively²⁶. Let $\gamma : (-\varepsilon, 0) \rightarrow U$ be now a (necessarily) future directed timelike curve with $\lim_{s \nearrow 0} \gamma(s) = p$.

We first show that²⁷ $\psi(I^-(p, M) \cap U) = I^-(p', M') \cap \psi(U)$.

So let $q \in I^-(p, M) \cap U$. Then $I^+(q, M)$ is an open neighbourhood of p . Moreover, let $t'_1 \in M'$ with $t'_1 \gg p'$. Then $I^-(t'_1, M')$ is an open neighbourhood of p' . Since p and p' are corresponding boundary points, it follows that $\psi^{-1}(I^-(t'_1, M') \cap \psi(U)) \cap I^+(q, M) \neq \emptyset$. Thus we can find an $r'_1 \in \psi(U)$ with $\psi(q) \ll r'_1 \ll t'_1$; hence, in particular, $\psi(q) \leq t'_1$. Taking a sequence $t'_i \gg p'$, $i \in \mathbb{N}$, with $t'_i \rightarrow p'$ for $i \rightarrow \infty$, we get $\psi(q) \leq p'$ since the relation \leq is closed on globally hyperbolic manifolds²⁸.

In order to get $\psi(q) \ll p'$, take an $s \in U$ with $q \ll s \ll p$ and repeat the argument above with s instead of q . This then gives $\psi(q) \ll \psi(s) \leq p'$, and thus²⁹ $\psi(q) \ll p'$.

²⁵For a detailed discussion of causality theory on Lorentzian manifolds the reader is referred to Chapter 14 of [52].

²⁶It follows directly from Definition 2.3.5 that one cannot have one lying to the future and the other to the past.

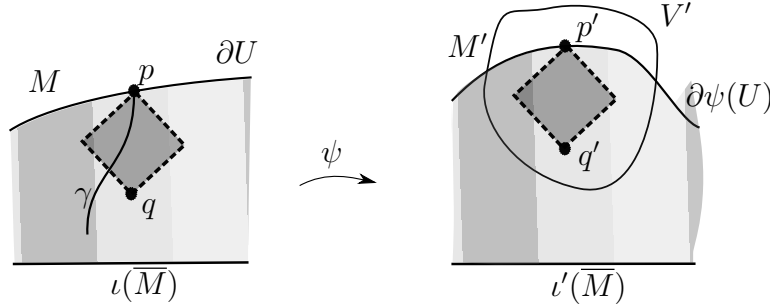
²⁷Although actually no confusion can arise, we write $I^-(p, M)$ to emphasise that this denotes the past of p in M .

²⁸Cf. Lemma 22 in Chapter 14 of [52].

²⁹Cf. Proposition 46 in Chapter 10 of [52].

Hence, we have shown $\psi(I^-(p, M) \cap U) \subseteq I^-(p', M') \cap \psi(U)$. The other inclusion follows by symmetry.

Let now $\gamma : (-\varepsilon, 0) \rightarrow M$ be a future directed timelike curve with $\lim_{s \nearrow 0} \gamma(s) = p$. Then $\psi \circ \gamma|_{(-\varepsilon, 0)}$ is a timelike curve in $I^-(p', M')$ and we claim that $\lim_{t \nearrow 0} (\psi \circ \gamma)(t) = p'$. To see this, let V' be an open neighbourhood of p' . Since M' satisfies the strong causality condition, we can find a $q' \in V' \cap I^-(p', M')$ such that $I^+(q', M') \cap I^-(p', M') \subseteq V'$.³⁰



From what we first showed, we know that $q := \psi^{-1}(q') \in I^-(p, M)$. Since $I^+(q, M)$ is an open neighbourhood of p , there exists a $\delta > 0$ such that $\gamma(s) \in I^+(q, M) \cap I^-(p, M)$ for all $-\delta < s < 0$. Moreover, we have $\psi(I^+(q, M) \cap I^-(p, M)) = I^+(q', M') \cap I^-(p', M')$, from which it follows that $(\psi \circ \gamma)(s) \in V'$ for all $-\delta < s < 0$. \square

If U is a CGHD of M and M' with isometric embedding $\psi : U \subseteq M \rightarrow M'$, we denote the set of points in ∂U that possess a corresponding boundary point in $\partial\psi(U)$ with C .

Lemma 2.3.8. *Let U be a CGHD of M and M' with isometric embedding $\psi : U \subseteq M \rightarrow M'$. Then the set C is open in ∂U and the isometric embedding $\psi : U \rightarrow M'$ extends smoothly to $\psi : U \cup C \rightarrow M'$.*

Proof. Assume that there exists a pair $p \in \partial U$ and $p' \in \partial\psi(U)$ of corresponding boundary points, otherwise there is nothing to show.

Let $V \subseteq M$ be a convex³¹ neighbourhood of p and $V' \subseteq M'$ be a convex neighbourhood of p' . Consider a future directed timelike geodesic $\gamma : [-\varepsilon, 0) \rightarrow U$ with $\lim_{s \nearrow 0} \gamma(s) = p$. Then, by Proposition 2.3.7, $\gamma' := \psi \circ \gamma$ is a future directed timelike geodesic in M' with $\lim_{s \nearrow 0} \gamma'(s) = p'$. Without loss of generality we may assume that $\varepsilon > 0$ is so small that $\gamma([-\varepsilon, 0)) \subseteq V$ and $\gamma'([-\varepsilon, 0)) \subseteq V'$.

³⁰Recall that the strong causality condition is satisfied at the point p' iff for all neighbourhoods V' of p' there is a neighbourhood W' of p' such that all causal curves with endpoints in W' are entirely contained in V' . In order to prove the just made claim, it remains to pick a point $q' \in W' \cap I^-(p', M')$.

³¹Recall that an open set is called convex iff it is a normal neighbourhood of each of its points. For the existence of convex neighbourhoods we refer the reader to Proposition 7 of Chapter 5 of [52].

Let $p \in W \subseteq V$ be a small open neighbourhood of p such that $W \subseteq I^+(\gamma(-\varepsilon))$ and

$$\psi_*[\exp_{\gamma(-\varepsilon)}^{-1}(W)] \subseteq \exp_{\gamma'(-\varepsilon)}^{-1}(V').$$

We can now define the smooth extension $\hat{\psi} : W \rightarrow M'$ by

$$\hat{\psi}(q) := \exp_{\gamma'(-\varepsilon)}(\psi_*(\exp_{\gamma(-\varepsilon)}^{-1}(q))).$$

This is clearly a smooth diffeomorphism onto its image and it also agrees with ψ on $W \cap U$, since the exponential map commutes with isometries: Let $q \in W \cap U$ and say $X \in T_{\gamma(-\varepsilon)}M$ is such that $q = \exp_{\gamma(-\varepsilon)}(X)$. We then have

$$\psi(q) = \psi(\exp_{\gamma(-\varepsilon)}(X)) = \exp_{(\psi \circ \gamma)(-\varepsilon)}(\psi_*(X)) = \hat{\psi}(q).$$

Moreover, using the same argument, we have $W \cap \partial U \subseteq C$, since for $q \in W \cap \partial U$ and $X := \exp_{\gamma(-\varepsilon)}^{-1}(q)$, we have that $s \mapsto \gamma(s) = \exp_{\gamma(-\varepsilon)}(s \cdot X)$ is a timelike curve that converges to q for $s \nearrow 1$, while $(\psi \circ \gamma)(s)$ converges to a point in $\partial\psi(U)$ for $s \nearrow 1$. By Proposition 2.3.7, point *iii*), q thus has a corresponding boundary point. Hence, C is open in ∂U . \square

Note that in the case that C is non-empty, this lemma allows to extend the identification of M with M' . It thus furnishes the closure part of the analogy to the method of continuity referred to in the introduction. Pursuing this analogy, the next two lemmata lay the foundation for restarting the local uniqueness argument again, i.e., they lay the foundation for the openness part.

Lemma 2.3.9. *Let U be a CGHD of M and M' with isometric embedding $\psi : U \subseteq M \rightarrow M'$. Assume that $C \cap J^+(\iota(\overline{M}))$ is non-empty. Then there exists a point $p \in C$ with the property*

$$J^-(p) \cap \partial U \cap J^+(\iota(\overline{M})) = \{p\}. \quad (2.3.10)$$

Whenever C is non-empty, we can assume without loss of generality (otherwise we reverse the time orientation) that we have in fact $C \cap J^+(\iota(\overline{M})) \neq \emptyset$. In this case, the above lemma ensures the existence of a ‘spacelike’ part of the boundary - only those parts are suitable for restarting the local uniqueness argument.

Proof. So assume that $C \cap J^+(\iota(\overline{M}))$ is non-empty. Let $p \in C \cap J^+(\iota(\overline{M}))$ and we have to deal with the case that $(J^-(p) \cap \partial U \cap J^+(\iota(\overline{M}))) \setminus \{p\} \neq \emptyset$. So let $q \in (J^-(p) \cap \partial U \cap J^+(\iota(\overline{M}))) \setminus \{p\}$. Thus, there exists a past directed causal curve γ from p to q . Since $\partial U \cap J^+(\iota(\overline{M}))$ is achronal, γ must be a null geodesic³². Let

³²That $\partial U \cap J^+(\iota(\overline{M}))$ is achronal follows from \ll being an open relation, see Lemma 3 in Chapter 14 of [52]: If there were two points $x, y \in \partial U \cap J^+(\iota(\overline{M}))$ with $x \ll y$, then we could also find $x' \in U^c \cap J^+(\iota(\overline{M}))$ close to x and $y' \in U \cap J^+(\iota(\overline{M}))$ close to y such that $x' \ll y'$. This, however, gives rise to an inextendible timelike curve in U which does not

$\gamma : [0, a) \rightarrow M$, where $a > 1$, be a parameterization of the past inextendible null geodesic γ with $\gamma(0) = p$ and $\gamma(1) = q$. Moreover, note that $\gamma([0, 1]) \subseteq \partial U$. Since if there were a $0 < t < 1$ with $\gamma(t) \in U$ then global hyperbolicity of U would imply that $\gamma(1) = q \in U$ as well. On the other hand, if $\gamma(t) \in U^c \setminus \partial U$ then we could find a closeby point $r \in U^c \setminus \partial U$ that could be connected by a timelike curve to p . But then, we could also find a point $s \in U$ close by to p such that r and s could be connected by a timelike curve - again a contradiction to the global hyperbolicity of U .

Let $[0, b] := \gamma^{-1}(\partial U)$. Since ∂U is closed in M , this is indeed a closed interval - and exactly the same argument as above shows that it is connected. In the following we show that $\gamma(b)$ has the wanted property, namely

$$\gamma(b) \in C \text{ and } J^-(\gamma(b)) \cap \partial U \cap J^+(\iota(\overline{M})) = \{\gamma(b)\} .$$

We first show that $J := \{t \in [0, b] \mid \gamma(t) \in C\}$ is equal to $[0, b]$. Since $\gamma(0) \in C$, J is non-empty. By Lemma 2.3.8 we know that C is open in ∂U , so J is open in $[0, b]$. It remains to show that J is closed in $[0, b]$ in order to deduce that $J = [0, b]$.

Since by Lemma 2.3.8 ψ extends to an isometric embedding on $U \cup C$, $\gamma'|_J := \psi \circ \gamma|_J$ is a null geodesic in M' . Denote with γ' the corresponding past inextendible null geodesic in M' . So let $t_j \in J$, $j \in \mathbb{N}$, be a sequence with $t_j \rightarrow t_\infty$ in $[0, b]$ for $j \rightarrow \infty$. We then claim that $\gamma'(t_\infty)$ and $\gamma(t_\infty)$ are corresponding boundary points. This is seen as follows: let $V \subseteq M$ be a neighbourhood of $\gamma(t_\infty)$ and $V' \subseteq M'$ a neighbourhood of $\gamma'(t_\infty)$. Consider now a sequence of future directed timelike curves $\alpha_j : (-\varepsilon, 0) \rightarrow U$, $j \in \mathbb{N}$, with $\lim_{s \nearrow 0} \alpha_j(s) = \gamma(t_j)$. Then for j large enough and $\sigma < 0$ close enough to zero, we have $\alpha_j(\sigma) \in V \cap \psi^{-1}(V' \cap \psi(U))$. This finally shows that $\gamma(b) \in C$.

That $\gamma(b)$ lies to the future of $\iota(\overline{M})$ is immediate, since γ cannot cross $\iota(\overline{M})$ as long as it lies in ∂U .

In order to show that $J^-(\gamma(b)) \cap \partial U \cap J^+(\iota(\overline{M})) = \{\gamma(b)\}$, assume that there were a $q \in (J^-(\gamma(b)) \cap \partial U \cap J^+(\iota(\overline{M}))) \setminus \{\gamma(b)\}$. Then there is a past directed null geodesic from $\gamma(b)$ to q . Concatenate $\gamma|_{[0, b]}$ and this null geodesic. Note that by definition of $[0, b]$ this null geodesic must be broken. But then we can connect p and q by a timelike curve³³, which, as before, leads to a contradiction to U being globally hyperbolic. \square

Lemma 2.3.11. *Let U be a GHD of some initial data and $M \supseteq U$ an extension of U . Suppose that there exists a $p \in \partial U$ that satisfies (2.3.10). Then for every open neighbourhood W of p in M there exists a point $q \in I^+(p) \subseteq M$ such that*

$$J^-(q) \cap U^c \cap J^+(\iota(\overline{M})) \subseteq W .$$

intersect the Cauchy hypersurface $\iota(\overline{M})$ - a contradiction to the global hyperbolicity of U . That γ must be a null geodesic is an easy consequence of the fundamental Proposition 46 in Chapter 10 of [52].

³³See again Proposition 46 in Chapter 10 of [52].

Proof. So let p satisfy $J^-(p) \cap \partial U \cap J^+(\iota(\overline{M})) = \{p\}$. Let $\gamma : [0, \varepsilon] \rightarrow M$ be a future directed timelike curve with $\gamma(0) = p$. Then we have $\gamma((0, \varepsilon]) \subseteq U^c$. Let $W \subseteq M$ be an open neighbourhood of p . If the lemma were not true, then there is a sequence $t_j \in (0, \varepsilon]$, $j \in \mathbb{N}$, with $t_j \rightarrow 0$ in $[0, \varepsilon]$ for $j \rightarrow \infty$, and a sequence of points $\{q_j\}_{j \in \mathbb{N}}$ with

$$q_j \in J^-(\gamma(t_j)) \cap U^c \cap J^+(\iota(\overline{M})) \cap W^c.$$

Since M is globally hyperbolic, $J^-(\gamma(\varepsilon)) \cap J^+(\iota(\overline{M}))$ is compact, thus $J^-(\gamma(\varepsilon)) \cap U^c \cap J^+(\iota(\overline{M})) \cap W^c$ is compact, and we can assume without loss of generality that $q_j \rightarrow q \in J^-(\gamma(\varepsilon)) \cap U^c \cap J^+(\iota(\overline{M})) \cap W^c$. Since the relation \leq is closed, we obtain $q \leq p$, and thus clearly $q < p$. But this leads again to a contradiction: We cannot have $q \in \partial U$ by assumption, thus $q \in U^c \setminus \partial U$. This, however, contradicts the global hyperbolicity of U in the same way as we argued in the proof of Lemma 2.3.9. \square

We are finally well-prepared for the proof of Theorem 2.3.6.

Proof of Theorem 2.3.6: Recall that M and M' are GHDs, and $U \subseteq M$ is a CGHD of M and M' that possesses corresponding boundary points. Without loss of generality we can assume that $C \cap J^+(\iota(\overline{M}))$ is non empty, and thus, by Lemma 2.3.9, we can find a $p \in C$ which satisfies $J^-(p) \cap \partial U \cap J^+(\iota(\overline{M})) = \{p\}$. Since by Lemma 2.3.8 C is open in ∂U , we can find a convex neighbourhood $V \subseteq M$ of p such that $V \cap \partial U \subseteq C$. Since the strong causality condition holds at p , we can find a causally convex neighbourhood W of p whose closure is compact and completely contained in V .³⁴ Let $q \in I^+(p)$ be a point with the property that $J^-(q) \cap U^c \cap J^+(\iota(\overline{M})) \subseteq W$, whose existence is guaranteed by Lemma 2.3.11.

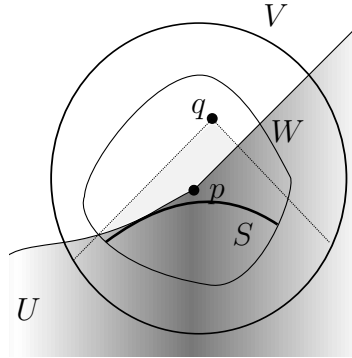
Let us denote with $\tau_q : M \rightarrow [0, \infty)$ the *time separation* from q , i.e.

$$\tau_q(r) := \sup\{L(\gamma) : \gamma \text{ is a future directed causal curve segment from } r \text{ to } q\},$$

where $L(\gamma)$ denotes the length of γ . If $r \notin J^-(q)$ we set $\tau_q(r)$ equal to zero. Note that τ_q restricted to W can be explicitly given by the exponential map based at q : Given $r \in W$, there exists, by the global hyperbolicity of M , a geodesic from r to q whose length equals the time separation from r to q . Since W is causally convex, this geodesic must be completely contained in W - and since $V \supseteq W$ is convex, this geodesic is a radial one in the exponential chart centred at q .

³⁴Recall that an open set $W \subseteq M$ is called *causally convex* iff every causal curve in M with endpoints in W is entirely contained in W . That we can find such a causally convex neighbourhood follows from the strong causality condition: Let V_1 be a neighbourhood of p whose closure is compact and completely contained in V . By the strong causality condition we can find a neighbourhood $V_2 \subseteq V_1$ of p with the property that every causal curve with endpoints in V_2 is completely contained in V_1 . Pick now two points $p_1, p_2 \in V_2$ such that $p_1 \ll p \ll p_2$. It follows that $W := I^+(p_1) \cap I^-(p_2)$ is an open neighbourhood of p which is completely contained in V_1 and thus has compact closure. Moreover, W is causally convex: Let γ be a causal curve with endpoints $x \leq y \in W$ and let z be a point on γ . We then have $p_1 \ll x \leq z \leq y \ll p_2$, and by Proposition 46 of Chapter 10 in [52] it follows that $z \in W$.

In particular τ_q is smooth in $I^-(q) \cap W$ and, by the global hyperbolicity of M , continuous in V .³⁵



Since \overline{W} is compact, τ_q takes on its maximum on $\overline{W} \cap U^c \cap J^+(\iota(\overline{M}))$. Let us denote this maximum by τ_0 . Clearly, we have $\tau_0 > 0$. Moreover, one has $\tau_q(r) = \tau_0$ only for $r \in \partial U \cap W \cap J^+(\iota(\overline{M}))$, since if this were not the case, using normal coordinates around q , one could continue the length maximising geodesic from r_0 to q a bit to the past, staying in $W \cap U^c$, which would lead to a longer timelike curve.

We now define

$$S := \tau_q^{-1}(\tau_0) \cap W \cap I^+(\iota(\overline{M})).$$

It is easy to see that S is smooth and spacelike and contains at least one point of ∂U . Moreover, S is contained in $\overline{U} \cap J^+(\iota(\overline{M}))$, since $\tau_q(r)$ is only greater than zero for $r \in J^-(q)$, and on $J^-(q) \cap U^c \cap J^+(\iota(\overline{M})) \subseteq W$ we only have $\tau_q(r) = \tau_0$ for $r \in \partial U$ as argued above.

Using Lemma 2.3.8 (and therefore the fact that $V \cap \partial U \subseteq C$) we can thus map³⁶ S isometrically to $\psi(S) \subseteq M'$ - and suitable neighbourhoods of S in M and of $\psi(S)$ in M' are GHDs of (S, \bar{g}_S, k_S) (where \bar{g}_S is the induced metric from the ambient spacetime M and k_S is the second fundamental form of S in M). By Theorem 2.2.7 there exists a globally hyperbolic development $N \subseteq M$ of (S, \bar{g}_S, k_S) together with an isometric embedding $\phi : N \rightarrow M'$ such that $\phi|_S = \psi|_S$. By Corollary 2.3.2 we have $\psi = \phi$ in $N \cap \overline{U}$, and so we can extend ψ to an isometric embedding $\Psi : U \cup N \rightarrow M'$. Moreover, note that $U \cup N$ is globally hyperbolic with Cauchy surface $\iota(\overline{M})$ and that $U \cup N$ is strictly bigger than U since S contains at least one point in ∂U . Hence, $U \cup N \subseteq M$ is a strictly larger CGHD of M and M' than the CGHD U we started with. \square

Invoking the tertium non datur, Theorem 2.3.6 implies

Theorem 2.3.12. *Let M and M' be GHDs of the same initial data, and let U be the MCGHD of M and M' . Then U does not possess corresponding boundary points in M and M' .*

³⁵Cf. Lemma 21 in Chapter 14 of [52].

³⁶Recall that we denote the isometric embedding of U into M' by ψ .

2.3.3 Finishing off the proof of the main theorems

From here on, the proof of Theorem 2.2.8 is straightforward:

Proof of Theorem 2.2.8: As already outlined in the introduction, we will construct the common extension of M and M' by glueing them together along their MCGHD. Theorem 2.3.12 will yield that this space is Hausdorff. It then remains to show that this quotient space comes with enough natural structure that turns it into a GHD.

Thus, let us take the disjoint union $M \sqcup M'$ of M and M' and endow it with the natural topology. Let us denote the MCGHD of M and M' by U (the existence of such a CGHD is guaranteed by Theorem 2.3.4) and the isometric embedding of U into M' by ψ . We now consider the following equivalence relation on $M \sqcup M'$: For $p, q \in M \sqcup M'$ we define $p \sim q$ if and only if

$$\begin{array}{ll} p \in U \subseteq M \text{ and } q = \psi(p) & \text{or} \quad q \in U \subseteq M \text{ and } p = \psi(q) \\ & \text{or} \quad p = q. \end{array}$$

We then take the quotient $(M \sqcup M')/\sim =: \tilde{M}$, endowed with the quotient topology. The following elementary remark will come useful at various points in the proof:

$$\text{The maps } \pi \circ j \text{ and } \pi \circ j' \text{ are homeomorphisms onto their image.} \quad (2.3.13)$$

$$\begin{array}{ccc} M & \xrightarrow{j} & M \sqcup M' \xrightarrow{\pi} (M \sqcup M')/\sim \\ & \nearrow^{j'} & \\ M' & & \end{array}$$

Here the maps j and j' denote the canonical inclusion maps. Verifying (2.3.13) is an easy exercise in set topology: Clearly the maps are continuous and injective. We show that they are also open: for $A \subseteq M$ open we have, with slight abuse of notation, that $M \cap [\pi^{-1}((\pi \circ j)(A))] = A$ is open and so is $M' \cap [\pi^{-1}((\pi \circ j)(A))] = \psi(U \cap A)$.

We now show that the quotient topology on \tilde{M} is indeed Hausdorff. Using (2.3.13), we can easily separate two points $[p] \neq [q] \in \tilde{M}$, if

1. $p \neq q \in M$: In this case we separate p and q in M and then use the fact that $\pi \circ j$ is a homeomorphism in order to push forward the separating neighbourhoods to \tilde{M} .
2. $p \in M \setminus \bar{U}$ and $q \in M' \setminus \psi(U)$: we choose a neighbourhood of p in M that lies entirely in $M \setminus \bar{U}$ and an arbitrary neighbourhood of q in M' . Pushing forward these neighbourhoods via the homeomorphisms, we obtain separating neighbourhoods in \tilde{M} .

Trivial permutations or modifications of these two possibilities leave only open the task to separate $[p]$ and $[q]$ if $p \in \partial U$ and $q \in \partial\psi(U)$, or $q \in \partial U$ and $p \in \partial\psi(U)$. So

suppose we could not separate these two points in, without loss of generality, the case $p \in \partial U$ and $q \in \partial\psi(U)$. For all neighbourhoods V of p and V' of p' , we then have $(\pi \circ j)(V) \cap (\pi \circ j')(V') \neq \emptyset$. This, however, implies that $\psi^{-1}(V' \cap \psi(U)) \cap V \neq \emptyset$, i.e., p and q are corresponding boundary points of U - in contradiction to Theorem 2.3.12. Thus, \tilde{M} is indeed Hausdorff.

In the remaining part of the proof we show that \tilde{M} possesses a natural structure that turns it into a common extension of M and M' .

1. \tilde{M} is locally euclidean and has a natural smooth structure: We have to give an atlas for \tilde{M} . Let $\{V_i, \varphi_i\}_{i \in \mathbb{N}}$ be an atlas for M and $\{V'_k, \varphi'_k\}_{k \in \mathbb{N}}$ an atlas for M' , where the φ 's are here homeomorphisms from some open subset of \mathbb{R}^{d+1} to the V 's. We then define an atlas for \tilde{M} by

$$\left\{ (\pi \circ j)(V_i), \pi \circ j \circ \varphi_i \right\}_{i \in \mathbb{N}} \cup \left\{ (\pi \circ j')(V_k), \pi \circ j' \circ \varphi_k \right\}_{k \in \mathbb{N}}.$$

By (2.3.13) this furnishes an open covering of \tilde{M} and it is easy to check that the transition functions are either of the form $\varphi_{i_0}^{-1} \circ \varphi_{i_1}$ with $i_0, i_1 \in \mathbb{N}$, the primed analogue, or $(\varphi'_{k_0})^{-1} \circ \psi \circ \varphi_{i_0}$ with $i_0, k_0 \in \mathbb{N}$, which are all smooth diffeomorphisms.

2. \tilde{M} is second countable: This follows directly from the previous construction.
3. \tilde{M} has a natural smooth Lorentzian metric that is Ricci-flat: Since $\pi \circ j$ and $\pi \circ j'$ are smooth diffeomorphism onto their image, we can endow \tilde{M} with a smooth Lorentzian metric by pushing forward g and g' . On $(\pi \circ j)(U)$ the two metrics obtained in this way agree since ψ is an isometry, thus this yields a smooth Lorentzian Ricci-flat metric \tilde{g} on \tilde{M} . Moreover, note that this turns $\pi \circ j$ and $\pi \circ j'$ into isometries.
4. (\tilde{M}, \tilde{g}) is globally hyperbolic with Cauchy surface $\tilde{\iota}(\overline{M})$: Here we have defined $\tilde{\iota} := \pi \circ j \circ \iota : \overline{M} \rightarrow \tilde{M}$. So let $\gamma : I \rightarrow \tilde{M}$ be an inextendible timelike curve, where $I \subseteq \mathbb{R}$. Take $t_0 \in I$ and, without loss of generality, assume $\gamma(t_0) \in (\pi \circ j)(M)$. If we denote with $J \ni t_0$ the maximal connected subinterval of I such that $\gamma(J) \subseteq (\pi \circ j)(M)$, then $\gamma|_J$ can be considered as an inextendible timelike curve in M and thus has to intersect $\iota(\overline{M})$. Hence, γ intersects $\tilde{\iota}(\overline{M})$ at least once.

Let us now assume that γ intersected $\tilde{\iota}(\overline{M})$ more than once. We can find $t_1 < t_3 \in I$ with $\gamma(t_1), \gamma(t_3) \in \tilde{\iota}(\overline{M})$ and $\gamma(t) \notin \tilde{\iota}(\overline{M})$ for $t_1 < t < t_3$. Since M and M' are globally hyperbolic, $\gamma|_{[t_1, t_3]}$ cannot be contained entirely in $\pi \circ j(M)$ or $\pi \circ j'(M')$. Thus, there must be t_2, t_{12}, t_{23} with $t_1 < t_{12} < t_2 < t_{23} < t_3$ such that $\gamma(t_2) \in (\pi \circ j)(U)$ and, without loss of generality, $\gamma(t_{12}) \notin (\pi \circ j')(M')$ and

$\gamma(t_{23}) \notin (\pi \circ j)(M)$.³⁷ But this leads to an inextendible timelike curve in U that does not intersect $\iota(\bar{M})$, a contradiction, since U is globally hyperbolic.

5. (\tilde{M}, \tilde{g}) has a natural time orientation: Since M and M' are time oriented, there exist continuous timelike vector fields T on M and T' on M' . Since $\psi : U \rightarrow M'$ preserves the time orientation, at each point $\psi_*(T|_U)$ and $T'|_{\psi(U)}$ lie in the same component of the set of all timelike tangent vectors at this point. Thus, pushing forward T and T' via $\pi \circ j$ and $\pi \circ j'$ we can consistently single out a future direction at each point of \tilde{M} . It remains to show that this choice is continuous. But since this is a local property, this follows immediately from $(\pi \circ j)_*(T)$ and $(\pi \circ j')_*(T')$ being continuous.

We have thus shown that $(\tilde{M}, \tilde{g}, \tilde{\iota})$ is a GHD of $(\bar{M}, \bar{g}, \bar{k})$ and, moreover, it is an extension of M and M' , where the isometric embeddings are given by the maps $\pi \circ j$ and $\pi \circ j'$. This finishes the proof of Theorem 2.2.8. \square

As outlined in the introduction, we would like to construct now the MGHD by glueing all GHDs together along their MCGHDs. However, the following subtlety arises: the collection of all GHDs of given initial data is not a set, but a *proper* class - and thus we cannot use the axioms of the Zermelo-Fraenkel set theory for justifying the glueing construction we have in mind. Fortunately, there is an easy way to circumvent this obstacle: Instead of considering *all* GHDs of given initial data $(\bar{M}, \bar{g}, \bar{k})$, we only consider those whose underlying manifold is a subset of $\bar{M} \times \mathbb{R}$.³⁸ This collection X of GHDs is indeed a set (as we will show below), and thus we can glue all such GHDs together along their MCGHDs. In order to justify that the so obtained GHD \tilde{M} is indeed the MGHD, we just note that any GHD of the same initial data is isometric to one in X , and hence isometrically embeds into \tilde{M} .

Proof of Theorem 2.2.9: We consider fixed initial data $(\bar{M}, \bar{g}, \bar{k})$. In the following we argue that *the collection X of all GHDs M whose underlying manifold is an open neighbourhood of $\bar{M} \times \{0\}$ in $\bar{M} \times \mathbb{R}$ and whose embeddings $\iota : \bar{M} \rightarrow M$ of the initial data into M are given by $\iota(x) = (x, 0)$, where $x \in \bar{M}$, is a set.*

To see this, consider the set $Y := T^*(\bar{M} \times \mathbb{R}) \otimes T^*(\bar{M} \times \mathbb{R})$, i.e., the tensor product of the cotangent bundle of $\bar{M} \times \mathbb{R}$ with itself. Each of the members of X is given by a subset of Y . The *axiom of power set* ensures that there is a set $\mathcal{P}(Y)$ containing all subsets of Y . The *axiom schema of specification* now ensures that

$$X := \{M \in \mathcal{P}(Y) \mid \bar{M} \times \{0\} \subseteq M \subseteq \bar{M} \times \mathbb{R} \text{ is a GHD of the given initial data} \\ \text{and the initial data embeds canonically into } \bar{M} \times \{0\} \subseteq M\}$$

³⁷The other possibility is $\gamma(t_{12}) \notin (\pi \circ j)(M)$ and $\gamma(t_{23}) \notin (\pi \circ j')(M')$ and leads in the same way to a contradiction.

³⁸We will in fact impose some further restrictions on the GHDs, which are, however, not strictly necessary.

is a set.

To simplify notation, let us now write $X = \{M_\alpha \mid \alpha \in A\}$. We denote the MCGHD of M_{α_i} and M_{α_k} with $U_{\alpha_i\alpha_k} \subseteq M_{\alpha_i}$ and the corresponding isometry with $\psi_{\alpha_i\alpha_k} : U_{\alpha_i\alpha_k} \rightarrow M_{\alpha_k}$. We define an equivalence relation \sim on $\bigsqcup_{\alpha \in A} M_\alpha$ by

$$M_{\alpha_i} \ni p_{\alpha_i} \sim q_{\alpha_k} \in M_{\alpha_k} \text{ iff } p_{\alpha_i} \in U_{\alpha_i\alpha_k} \text{ and } \psi_{\alpha_i\alpha_k}(p_{\alpha_i}) = q_{\alpha_k} \quad (2.3.14)$$

and take the quotient $(\bigsqcup_{\alpha \in A} M_\alpha)/\sim =: \tilde{M}$ with the quotient topology. Note that (2.3.14) is indeed an equivalence relation. For the transitivity observe that if $p_{\alpha_i} \in M_{\alpha_i}$, $p_{\alpha_k} \in M_{\alpha_k}$ and $p_{\alpha_l} \in M_{\alpha_l}$ with $p_{\alpha_i} \sim p_{\alpha_k}$ and $p_{\alpha_k} \sim p_{\alpha_l}$, then we have that $U_{\alpha_i\alpha_k} \cap \psi_{\alpha_i\alpha_k}^{-1}(U_{\alpha_k\alpha_l})$ together with the composition $\psi_{\alpha_k\alpha_l} \circ \psi_{\alpha_i\alpha_k}$ is a CGHD of M_{α_i} and M_{α_l} that contains p_{α_i} and identifies it with p_{α_l} - so certainly the MCGHD of M_{α_i} and M_{α_l} leads to the same identification.

1. \tilde{M} is Hausdorff: Let $[p_{\alpha_i}] \neq [q_{\alpha_k}] \in \tilde{M}$ with $p_{\alpha_i} \in M_{\alpha_i}$ and $q_{\alpha_k} \in M_{\alpha_k}$. We show that we can find open neighbourhoods in \tilde{M} that separate these points.

$$\begin{array}{ccccc}
 M_{\alpha_i} & \xrightarrow{j_i} & \bigsqcup_{\alpha \in A} M_\alpha & \xrightarrow{\pi} & (\bigsqcup_{\alpha \in A} M_\alpha)/\sim \\
 & \searrow j_i & \uparrow j_{ik} & \nearrow \pi \circ j_{ik} & \uparrow \tilde{j}_{ik} \\
 M_{\alpha_k} & \xrightarrow{j_k} & M_{\alpha_i} \sqcup M_{\alpha_k} & \xrightarrow{\pi} & (M_{\alpha_i} \sqcup M_{\alpha_k})/\sim
 \end{array}$$

Here, all j 's denote canonical inclusion maps, the π 's denote projection maps, the lower equivalence relation is defined as in the proof of Theorem 2.2.8 and it is easy to check that the map $\pi \circ j_{ik}$ descends to the quotient, i.e. to \tilde{j}_{ik} .

As for (2.3.13) one checks that $\pi \circ j_{ik}$ is an open map. Thus, \tilde{j}_{ik} is open as well. Since \tilde{j}_{ik} is also continuous and injective, it is a homeomorphism onto its image.

In Theorem 2.2.8 we proved that the quotient topology on $(M_{\alpha_i} \sqcup M_{\alpha_k})/\sim$ is Hausdorff - thus we can find open neighbourhoods that separate $[p_{\alpha_i}]$ and $[q_{\alpha_k}]$ in $(M_{\alpha_i} \sqcup M_{\alpha_k})/\sim$. Pushing forward these neighbourhoods to $(\bigsqcup_{\alpha \in A} M_\alpha)/\sim$ via \tilde{j}_{ik} we obtain separating open neighbourhoods of $[p_{\alpha_i}]$ and $[q_{\alpha_k}]$ in \tilde{M} .

2. \tilde{M} is locally euclidean and has a natural smooth structure: This is seen exactly as in the proof of Theorem 2.2.8.
3. \tilde{M} has a natural smooth Lorentzian metric that is Ricci-flat and comes with a natural time orientation: Again, this is seen exactly as before.
4. (\tilde{M}, \tilde{g}) is globally hyperbolic with Cauchy surface $\tilde{i}(\overline{M})$: Here, $\tilde{i} := \pi \circ j_i \circ \iota_i$ for some $\alpha_i \in A$. This definition does obviously not depend on $\alpha_i \in A$.

The proof is also nearly the same as before. Let $\gamma : I \rightarrow \tilde{M}$ be an inextendible timelike curve. For $t_0 \in I$ we have, say, $\gamma(t_0) \in M_{\alpha_i}$. Let $J \ni t_0$ denote the maximal connected subinterval of I such that $\gamma(J) \subseteq (\pi \circ j_i)(M_{\alpha_i})$. We can then pull back $\gamma|_J$ via $\pi \circ j_i$ to M_{α_i} , which gives rise to an inextendible timelike curve in M_{α_i} that has to intersect $\iota_i(\overline{M})$. Thus γ intersects $\tilde{\iota}(\overline{M})$.

Assume γ intersected $\tilde{\iota}(\overline{M})$ more than once. Again, we can find $t_1 < t_4 \in I$ with $\gamma(t_1), \gamma(t_4) \in \tilde{\iota}(\overline{M})$ and $\gamma(t) \notin \tilde{\iota}(\overline{M})$ for $t_1 < t < t_4$. Since γ is continuous and $[t_1, t_4]$ is compact, $\gamma([t_1, t_4])$ is contained in finitely many $\pi \circ j_\alpha(M_\alpha)$. But since each of these M_α 's is globally hyperbolic one can actually reduce this cover to just two elements, since otherwise one would get an inextendible timelike curve of the form $\gamma|_{[t_2, t_3]}$ in some M_α , where $t_1 < t_2 < t_3 < t_4$, that does not intersect $\iota_\alpha(\overline{M})$.

From here on, one follows the remaining argument from point 4 of the proof of Theorem 2.2.8.

5. \tilde{M} is second countable: This follows directly from a Theorem of Geroch, see the appendix of [32], where he shows that any manifold that is connected³⁹, Hausdorff and locally euclidean and which, moreover, admits a smooth Lorentzian metric, is also second countable.
6. \tilde{M} is an extension of any GHD of the same initial data: Let (M, g, ι) be a GHD of the same initial data. Since M is second countable and time oriented, we can find a globally timelike vector field T on M . Let us denote with $I_x \subseteq \mathbb{R}$ the maximal time interval of existence of the integral curve of T starting at $x \in M$.⁴⁰ In the following we recall some results from standard ODE theory: The set $\mathcal{D} := \{(x, t) \in M \times \mathbb{R} \mid t \in I_x\}$ is open and the flow $\Phi : \mathcal{D} \rightarrow M$ of T is smooth. Moreover, if we fix $t \in \mathbb{R}$ and regard $\Phi_t(\cdot) := \Phi(\cdot, t)$ as a function from some open subset of M to M , then Φ_t is a local diffeomorphism.

We now define $\mathcal{D}_{\iota(\overline{M})} := \{(x, t) \in \iota(\overline{M}) \times \mathbb{R} \mid t \in I_x\}$, which is an open neighbourhood of $\iota(\overline{M}) \times \{0\}$ in $\iota(\overline{M}) \times \mathbb{R}$ (again by standard ODE theory), and claim that $\chi := \Phi|_{\mathcal{D}_{\iota(\overline{M})}} : \mathcal{D}_{\iota(\overline{M})} \rightarrow M$ is a diffeomorphism.

The smoothness of χ follows directly from the smoothness of Φ , and the bijectivity follows from the global hyperbolicity of M . More precisely, since every maximal integral curve of T (which is, in particular, an inextendible timelike curve) has to intersect $\iota(\overline{M})$, χ is surjective; and since every such curve intersects $\iota(\overline{M})$ exactly once, we obtain the injectivity. In order to see that χ is a local

³⁹That \tilde{M} is connected here follows trivially from it being globally hyperbolic, hence path connected (recall that we assumed that \overline{M} is connected).

⁴⁰Note that the existence of such a maximal time interval follows from an elementary 'taking the union of all time intervals of existence argument' - without appealing to Zorn's lemma.

diffeomorphism, let $(x, t) \in \mathcal{D}_{\iota(\overline{M})}$ and choose a basis (Z_1, \dots, Z_d) of $T_x \iota(\overline{M})$. We have

$$\chi_*|_{(x,t)}(Z_i) = (\Phi_t)_*|_x(Z_i) \quad \text{and} \quad \chi_*|_{(x,t)}(\partial_t) = T|_{\Phi_t(x)} = (\Phi_t)_*|_x(T|_x). \quad (2.3.15)$$

Since $\iota(\overline{M})$ is spacelike, (Z_1, \dots, Z_d, T_x) forms a basis for $T_x M$; and since Φ_t is a local diffeomorphism, it follows from (2.3.15) that χ_* is surjective. Thus, we have shown that χ is a diffeomorphism.

It now follows that $\chi \circ (\iota \times \text{id})$ is a diffeomorphism from some open neighbourhood of $\overline{M} \times \{0\}$ in $\overline{M} \times \mathbb{R}$ to M which maps $\overline{M} \times \{0\}$ on $\iota(\overline{M})$. Pulling back the Lorentzian metric, we obtain that there is an $M_{\alpha_i} \in X$ that is isometric to M via $\chi \circ (\iota \times \text{id})$. The isometric embedding of M into \tilde{M} is now given by $\pi \circ j_i \circ (\chi \circ (\iota \times \text{id}))^{-1}$.

Finally, it is straightforward to deduce from this maximality property that \tilde{M} is, up to isometry, the only GHD with this property.

This finally finishes the proof of the existence of the MGHD. □

Chapter 3

Characterisation of the energy of Gaussian beams on Lorentzian manifolds - with applications to black hole spacetimes

3.1 Introduction

Part I of this chapter is concerned with the study of the temporal behaviour of Gaussian beams on *general* globally hyperbolic Lorentzian manifolds. Here, a Gaussian beam is a highly oscillatory wave packet of the form

$$\tilde{u}_\lambda = \frac{1}{\sqrt{E(\lambda, a, \phi)}} \cdot a \cdot e^{i\lambda\phi} ,$$

where $E(\lambda, a, \phi)$ is a renormalisation factor keeping the initial energy of \tilde{u}_λ independent of $\lambda \in \mathbb{R}^+$, and the complex valued functions a and ϕ are chosen in such a way that for $\lambda \gg 0$ the Gaussian beam \tilde{u}_λ is an approximate solution to the wave equation on the underlying Lorentzian manifold (M, g) . The failure of \tilde{u}_λ being an actual solution to the wave equation

$$\square_g u = 0 \tag{3.1.1}$$

is measured in terms of an energy norm - and this error can be made arbitrarily small up to a finite, but arbitrarily long time by choosing λ large enough. The construction of the functions a and ϕ allows for restricting the support of a to a small neighbourhood of a given null geodesic. Thus, one can infer from \tilde{u}_λ being an approximate solution with respect to some energy norm, that⁴¹

⁴¹Cf. Theorem 3.2.1.

there exist *actual* solutions of the wave equation (3.1.1) whose ‘energy’ is localised along a given null geodesic up to some finite, (3.1.2) but arbitrarily long time.

This is, roughly, the state of the art knowledge of Gaussian beams (see for instance [61]).

The main new result of Part I of this chapter is to provide a geometric characterisation of the temporal behaviour of the localised energy of a Gaussian beam. More precisely, given a timelike vector field N (with respect to which we measure the energy) and a Gaussian beam \tilde{u}_λ supported in a small neighbourhood of an affinely parametrised null geodesic γ , we show in Theorem 3.2.36 that

$$\int_{\Sigma_\tau} J^N(\tilde{u}_\lambda) \cdot n_{\Sigma_\tau} \approx -g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_\tau} \quad (3.1.3)$$

holds up to some finite time T . Here, we consider a foliation of the Lorentzian manifold (M, g) by spacelike slices Σ_τ , $J^N(\tilde{u}_\lambda)$ denotes the contraction of the stress-energy tensor⁴² of \tilde{u}_λ with N , and n_{Σ_τ} is the normal of Σ_τ . The left hand side of (3.1.3) is called the N -energy of the Gaussian beam \tilde{u}_λ . The approximation in (3.1.3) can be made arbitrarily good and the time T arbitrarily large if we only take $\lambda > 0$ to be big enough and a to be supported in a small enough neighbourhood of γ . This characterisation of the energy allows then for a refinement of (3.1.2):⁴³

There exist (actual) solutions of the wave equation (3.1.1) whose N -energy is localised along a given null geodesic γ and behaves approximately like $-g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_\tau}$ up to some finite, (3.1.4) but arbitrarily large time T . Here, $\dot{\gamma}$ is with respect to some affine parametrisation of γ .

It is worth emphasising that the need for an understanding of the temporal behaviour of the energy only arises for Gaussian beams on Lorentzian manifolds that *do not admit a globally timelike Killing vector field* - otherwise there is a canonical energy which is conserved for solutions to the wave equation (3.1.1). Thus, for the majority of problems which so far found applications of Gaussian beams, for example the obstacle problem or the wave equation in time-independent inhomogeneous media, the question of the temporal behaviour of the energy did not arise (since it is trivial). However, understanding this behaviour on general Lorentzian manifolds is crucial for widening the application of Gaussian beams to problems arising, in particular, from general relativity.

⁴²We refer the reader to (3.1.7) in Section 3.1.5 for the definition of the stress-energy tensor.

⁴³Cf. Theorem 3.2.43.

In Part II of this chapter, by applying (3.1.4), we derive some new results on the study of the wave equation on the familiar Schwarzschild, Reissner-Nordström, and Kerr black hole backgrounds (see [34] for an introduction to these spacetimes):

1. It is well-known folklore that the trapping⁴⁴ at the *photon sphere* in Reissner-Nordström and in Kerr necessarily leads to a ‘loss of derivative’ in a local energy decay (LED) statement. We give a rigorous proof of this fact.
2. We also show that the trapping at the *horizon* of an extremal Reissner-Nordström (and Kerr) black hole necessarily leads to a loss of derivative in an LED statement.
3. When solving the wave equation (3.1.1) on the exterior of a Schwarzschild black hole backwards in time, the red-shift effect at the event horizon turns into a blue-shift: we construct solutions to the backwards problem whose energies grow exponentially for a finite, but arbitrarily long time. This demonstrates the obstruction formed by the red-shift effect at the event horizon to scattering constructions from the future.
4. Finally, we give a simple mathematical realisation of the heuristics for the blue-shift effect near the Cauchy horizon of (sub)-extremal Reissner-Nordström and Kerr black holes: we construct a sequence of solutions to the wave equation whose initial energy is uniformly bounded whereas the energy near the Cauchy horizon goes to infinity.

Outline of this chapter

We start by giving a short historical review of Gaussian beams in Section 3.1.1. Thereafter we briefly explain how the notion of ‘energy’ arises in the study of the wave equation and why it is important. We also discuss how the results obtained in this chapter allow to disprove certain uniform statements about the temporal behaviour of the energy of waves. Section 3.1.3 elaborates on the wide applicability of the Gaussian beam approximation and explains its advantage over the geometric optics approximation. In the physics literature a similar ‘characterisation of the energy of high frequency waves’ is folklore - we discuss its origin in Section 3.1.4 and put it into context with the work presented in this chapter. Section 3.1.5 lays down the notation we use.

Part I of this chapter discusses the theory of Gaussian beams on Lorentzian manifolds. Sections 3.2.1 and 3.2.2 establish Theorem 3.2.1 which basically says (3.1.2) and is more or less well-known. Although the proof of Theorem 3.2.1 can be reconstructed from the literature (cf. especially [61]), we could not find a complete and self-contained

⁴⁴We do not intend to give a precise definition here of what we mean by ‘trapping’. However, loosely speaking ‘trapping’ refers here to the presence of null geodesics that stay for all time in a compact region of ‘space’.

proof of this statement. Moreover, there are some important subtleties (cf. footnote 53) which are not discussed elsewhere. For these reasons, and moreover for making the chapter self-contained, we have included a full proof of Theorem 3.2.1. In Section 3.2.3 we characterise the energy of a Gaussian beam, which is the main result of Part I of this chapter. This result is then incorporated into Theorem 3.2.1, which yields Theorem 3.2.43 (or (3.1.4)). Moreover, Section 3.2.4 contains some general theorems which are tailored to the needs of many applications.

In Part II of this chapter, we prove the above mentioned new results on the behaviour of waves on various black hole backgrounds. The important ideas are first introduced in Section 3.3.1 by the example of the Schwarzschild and Reissner-Nordström family, whose simple form of the metric allows for an uncomplicated presentation. Thereafter, in Section 3.3.2, we proceed to the Kerr family.

The main purpose of the first part of the appendix is to contrast the Gaussian beam approximation with the much older geometric optics approximation. In Appendix 3.A, we recall the basics of the geometric optics approximation. Appendix 3.B discusses Ralston's work [59] from 1969, where he showed that trapping in the obstacle problem necessarily leads to a loss of differentiability in an LED statement. This proof made use of the geometric optics approximation and we explain why it does not transfer directly to general Lorentzian manifolds. We conclude in Appendix 3.C with giving a sufficient criterion for the formation of caustics, i.e., a breakdown criterion for solutions of the eikonal equation.

In Appendix 3.D we extend the results obtained in Part I of this chapter to Gaussian beams for a wave equation with lower order terms, and in Appendix 3.E we give an application of Gaussian beams to the Teukolsky equation.

3.1.1 A brief historical review of Gaussian beams

The ansatz

$$u_\lambda = e^{i\lambda\phi} \left(a_0 + \frac{1}{\lambda} a_1 + \dots + \frac{1}{\lambda^N} a_N \right) \quad (3.1.5)$$

for either an highly oscillatory approximate solution to some PDE or for an highly oscillatory approximate eigenfunction to some partial differential operator, is known as the *geometric optics ansatz*. Here, $N \in \mathbb{N}$, ϕ is a real function (called the eikonal), the a_k 's are complex valued functions, and λ is a positive parameter determining how quickly the function u_λ oscillates. In the widest sense, we understand under a Gaussian beam a function of the form (3.1.5) with a *complex* valued eikonal ϕ that is real valued along a bicharacteristic and has growing imaginary part off this bicharacteristic. This then leads to an exponential fall off in λ away from the bicharacteristic.

The use of a complex eikonal, although in a slightly different context, appears already in work of Keller from 1956, see [37]. It was, however, only in the 1960's that the method of Gaussian beams was systematically applied and explored - mainly from

a physics perspective. For more on these early developments we refer the reader to [4], Chapter 4, and references therein. A general, mathematical theory of Gaussian beams, or what he called the complex WKB method, was developed by Maslov, see the book [44] for an overview and also for references therein. Several of the later papers on Gaussian beams have their roots in this work.

The earliest application of the Gaussian beam method was to the construction of *quasimodes*, see for example the paper [60] by Ralston from 1976. Quasimodes approximately satisfy some type of Helmholtz equation, and thus they give rise to time harmonic, approximate solutions to a wave equation. In this way quasimodes can be interpreted as standing waves. Later, various people used the Gaussian beam method for the construction of Gaussian *wave packets* (but also called ‘Gaussian beams’) which form approximate solutions to a hyperbolic PDE⁴⁵. Those wave packets, in contrast to quasimodes, are not stationary waves, but they move through space, the trajectory in spacetime being a bicharacteristic of the partial differential operator. A detailed reference for this construction is the work [61] by Ralston, which goes back to 1977. Another presentation of this construction scheme was given in 1981 by Babich and Ulin, see [6].

Since then, there have been a lot of papers applying Gaussian beams to various problems⁴⁶. For instance, in quantum mechanics Gaussian beams correspond to semi-classical approximate solutions to the Schrödinger equation and thus help understand the classical limit; or in geophysics, one models seismic waves using the Gaussian beam approximation for solutions to a wave equation in an inhomogeneous (time-independent) medium.

3.1.2 Gaussian beams and the energy method

The energy method as a versatile method for studying the wave equation

The study of the wave equation on various geometries has a long history in mathematics and physics. A very successful and widely applicable method for obtaining quantitative results on the long-time behaviour of waves is the energy method. It was pioneered by Morawetz in the papers [48] and [49], where she proved pointwise decay results in the context of the obstacle problem. In [50] she established what is now known as *integrated local energy decay* (ILED) for solutions of the Klein-Gordon equation (and thus inferring decay). In the past ten years her methods were adapted and extended by many people in order to prove boundedness and decay of waves on various (black hole) spacetimes - a study which is mainly motivated by the black hole stability conjecture

⁴⁵It is this sort of ‘Gaussian beam’ that is the subject of this chapter for the case of the wave equation on Lorentzian manifolds. More appropriately, one could name them ‘Gaussian wave packets’ or ‘Gaussian pulses’ to distinguish them from the standing waves - which are actually *beams*. However, we stick to the standard terminology.

⁴⁶We refer the reader to [44] for a list of references.

(cf. the introduction of [26]). A small selection of examples is [39], [22], [24], [23], [25], [1], [67], [43], [64], [2], [36], [16] and [28].

The philosophy of the energy method is first to derive estimates on a suitable energy (and higher order energies⁴⁷) and then to establish pointwise estimates using Sobolev embeddings. Thus, given a spacetime on which one intends to study the wave equation using the energy method, one first has to set up such a suitable energy (and higher order energies - but in this work we focus on the first order energy). A general procedure is to construct an energy from a foliation of the spacetime by spacelike slices Σ_τ together with a timelike vector field N , see (3.1.8) in Section 3.1.5. We refrain from discussing here what choices of foliation and timelike vector field lead to a ‘suitable’ notion of energy⁴⁸. Let us just mention here that in the presence of a globally timelike Killing vector field T one obtains a particularly well behaved energy by choosing $N = T$ and a foliation that is invariant under the flow of T .⁴⁹ We invite the reader to convince him- or herself that the familiar notions of energy for the wave equation on the Minkowski spacetime or in time independent inhomogeneous media arise as special cases of this more general scheme.

Gaussian beams as obstructions to certain uniform behaviour of the energy of waves

The approximation with Gaussian beams allows to construct solutions to the wave equation whose energy is localised for an arbitrarily long, but finite time along a null geodesic. Such solutions form naturally an obstruction to certain uniform statements about the temporal behaviour of the energy of waves. A classical example is the case in which one has a null geodesic that does not leave a compact region in ‘space’ and which has constant energy⁵⁰. Such null geodesics form obstructions to certain formulations of local energy decay being true⁵¹. However, it is very important to be aware of the fact, that in general none of the solutions from (3.1.4) has localised energy for *all* time. Thus, in order to contradict, for instance, an LED statement, it is in general inevitable to resort to a *sequence* of solutions of the form (3.1.4) which exhibit the contradictory behaviour in the limit. For this scheme to work, however, it is clearly crucial that the LED statement in question is *uniform* with respect to some energy which is left constant by the sequence of Gaussian beam solutions. Note here that

⁴⁷A *first order* energy controls the *first* derivatives of the wave and is referred to in the following just as ‘energy’. *Higher order* energies control *higher* derivatives of the wave. A special case of the energy method is the so-called *vector field method*. Higher order energies arise there naturally by commutation with suitable vector fields, see [39].

⁴⁸However, see Section 3.3 for some examples and footnote 72 for some further comments.

⁴⁹Such a choice corresponds to what we denoted in the introduction as a ‘canonical energy’.

⁵⁰We refer to the right hand side of (3.1.3) as the N -energy of the null geodesic.

⁵¹A classic regarding such a result is the work [59] by Ralston. However, he does not use the Gaussian beam approximation in this work, but the geometric optics approximation. We discuss his work in some detail in Appendix 3.B.

(3.1.4) in particular states that the time T , up to which one has good control over the wave, can be made arbitrarily large without changing the initial energy! Higher order initial energies, however, will blow up when T is taken bigger and bigger. In this work we restrict our consideration to disproving statements that are uniform with respect to the first order energy. In Sections 3.3.1, 3.3.1 and 3.3.2 we demonstrate this important application of Gaussian beams: we show that certain (I)LED statements derived by various people in the presence of ‘trapping’ are sharp in the sense that *some* loss of derivative is necessary (however, one does not necessarily need to lose a whole derivative, cf. the discussion at the end of Section 3.3.1).

We conclude this section with the remark that in the presence of a globally timelike Killing vector field one can already infer such obstructions from (3.1.2), since the (canonical) energy of solutions to the wave equation is then constant. In this way one can easily infer from (3.1.2) alone that an LED statement in Schwarzschild has to lose differentiability due to the trapping at the photon sphere. But already for trapping in Kerr one needs to know how the ‘trapped’ energy of the solutions referred to in (3.1.2) behaves in order to infer the analogous result. This knowledge is provided by (3.1.3) and/or (3.1.4).

3.1.3 Gaussian beams are parsimonious

The approximation by Gaussian beams can be carried out on a Lorentzian manifold (M, g) under minimal assumptions:

1. One needs a well-posed initial value problem. This is ensured by requiring that (M, g) is globally hyperbolic⁵². However, one can also replace the well-posed initial value problem by a well-posed initial-boundary value problem - and one can obtain, with small changes and some additional work in the proof, qualitatively the same results.
2. Having fixed a choice of N -energy one intends to work with, one needs that this choice allows for a global energy estimate (cf. (3.2.8)). This can be ensured by conditions on the vector field N and the foliation by spacelike slices (cf. (3.2.3)). The energy estimate (3.2.8) allows us to infer that the approximation by the Gaussian beam is *global* in space. In particular, it is only under this condition that it is justified to say in (3.1.2) and (3.1.4) that the energy of the actual solution is *localised* along a null geodesic⁵³. However, as we show

⁵²The assumption of global hyperbolicity has another simplifying, but not essential, feature; cf. the discussion after Definition 3.2.35.

⁵³That one needs condition (3.2.3) for ensuring that the energy is indeed localised is in fact another minor novelty in the study of Gaussian beams on *general* Lorentzian manifolds (note that in the case of N being a Killing vector field, condition (3.2.3) is trivially satisfied). For an example for a violation of condition (3.2.3) we refer to the discussion after (3.3.8) on page 75.

in Remark 3.2.9 one *always* has a *local* approximation, which is, together with the geometric characterisation of the energy, sufficient for obtaining control of the wave in a small neighbourhood of the underlying null geodesic *regardless of condition* (3.2.3). This then allows to establish, for example, the very general Theorem 3.2.47 which only requires global hyperbolicity (or some other form of well-posedness for the wave equation, cf. point 1.).

In particular the method of Gaussian beams is not in need of any special structure on the Lorentzian manifold like Killing vector fields (as for example needed for the mode analysis or for the construction of quasimodes).

We would also like to emphasise here that in order to apply (3.1.4) *one only needs to understand the behaviour of the null geodesics of the underlying Lorentzian manifold!* This knowledge is often in reach and thus Gaussian beams provide in many cases an easy and feasible way for obtaining control of highly oscillatory solutions to the wave equation. In this sense the theory presented in Part I of this chapter forms a good ‘black box result’ which can be applied to various different problems.

We conclude this section with a brief comparison of the Gaussian beam approximation with the geometric optics approximation: Although the geometric optics approximation applies under the same general conditions as the Gaussian beam approximation, in general the time T , up to which one has good control over the solution, cannot be chosen arbitrarily large since the approximate solution breaks down at caustics. In Appendix 3.C we show that caustics necessarily form along null geodesics that possess conjugate points. A prominent example of such null geodesics are the trapped null geodesics at the photon sphere in the Schwarzschild spacetime (cf. Section 3.3.1 for the proof that these null geodesics have conjugate points). It follows that the geometric optics approximation is in particular not suitable for proving that a local energy decay statement in Schwarzschild has to lose differentiability.

The breakdown at caustics forms a serious restriction of the range of applicability of the geometric optics approximation - a restriction which is *not* shared by the Gaussian beam approximation.

3.1.4 ‘High frequency’ waves in the physics literature

In physics, the notion of a *local observer’s energy* arose with the emergence of Einstein’s theory of relativity: Suppose an observer travels along a timelike curve $\sigma : I \rightarrow M$ with unit velocity $\dot{\sigma}$. Then, with respect to a Lorentz frame of his, he measures the local energy density of a wave u to be $\mathbb{T}(u)(\dot{\sigma}, \dot{\sigma})$, where $\mathbb{T}(u)$ is the stress-energy tensor of the wave u , cf. (3.1.7) in Section 3.1.5. By considering the three parameter family of observers whose velocity vector field is given by the normal n_{Σ_τ} to a foliation of M by spacelike slices Σ_τ , the physical definition of energy is contained in the mathematical

one (which is given by (3.1.8)).

The prevalent description of highly oscillatory (or ‘high frequency’) waves in the physics literature is that the waves (or ‘photons’) propagate along null geodesics γ and each of these rays (or photons) carries an energy-momentum 4-vector $\dot{\gamma}$, where the dot is with respect to some affine parametrisation. In the high frequency limit, the number of photons is preserved. Thus, the energy of the wave, as measured by a local observer with world line σ , is determined by the energy component $-g(\dot{\gamma}, \dot{\sigma})$ of the momentum 4-vector $\dot{\gamma}$. By considering a highly oscillatory wave that ‘gives rise to just one photon’, one recovers the characterisation of the energy of a Gaussian beam, (3.1.3), given in this chapter.

In the physics literature (see for example the classic [47], Chapter 22.5), this description is justified using the geometric optics approximation. For the following brief discussion we refer the reader unfamiliar with the geometric optics approximation to Appendix 3.A.

The conservation law

$$\operatorname{div}(a^2 \operatorname{grad} \phi) = 0, \quad (3.1.6)$$

which can be easily inferred from equation (3.A.2), is interpreted as the conservation of the number-flux vector $S = a^2 \operatorname{grad} \phi$ of the photons. The leading component in λ of the renormalised⁵⁴ stress-energy tensor $T(u_\lambda)$ of the wave $u_\lambda = a \cdot e^{i\lambda\phi}$ in the geometric optics limit is then given by

$$\mathbb{T}(u_\lambda) = \operatorname{grad} \phi \otimes S,$$

from which it then follows that each photon carries a 4-momentum $\operatorname{grad} \phi = \dot{\gamma}$.

In particular, making use of the conservation law (3.1.6), it is not difficult⁵⁵ to prove a geometric characterisation of the energy of waves in the geometric optics limit analogous to the one we prove in this chapter for Gaussian beams. However, as we have mentioned in the previous section, the geometric optics approximation has the undesirable feature that it breaks down at caustics.

The characterisation of the energy of Gaussian beams is more difficult since (3.1.6) is replaced only by an *approximate* conservation law⁵⁶. Moreover, it provides a rigorous justification of the temporal behaviour of the local observer’s energy of photons, *which also applies to photons along whose trajectory caustics would form*.

3.1.5 Notation

Whenever we are given a time oriented Lorentzian manifold (M, g) that is (partly) foliated by spacelike slices $\{\Sigma_\tau\}_{\tau \in [0, \tau^*)}$, $0 < \tau^* \leq \infty$, we denote the future directed

⁵⁴i.e. divided by λ^2

⁵⁵Although, to the best of our knowledge, it is nowhere done explicitly.

⁵⁶Cf. the discussion below (3.2.40) in Section 3.2.3.

unit normal to the slice Σ_τ with n_{Σ_τ} . Moreover, the induced Riemannian metric on Σ_τ is then denoted by \bar{g}_τ and we set $R_{[0,T]} := \bigcup_{0 \leq \tau \leq T} \Sigma_\tau$.

For $u \in C^\infty(M, \mathbb{C})$ we define the *stress-energy tensor* $\mathbb{T}(u)$ by

$$\mathbb{T}(u) := \frac{1}{2} \overline{du} \otimes du + \frac{1}{2} du \otimes \overline{du} - \frac{1}{2} g(\cdot, \cdot) g^{-1}(du, \overline{du}) . \quad (3.1.7)$$

Given a vector field N the current $J^N(u)$ is defined by

$$J^N(u) := [\mathbb{T}(u)(N, \cdot)]^\sharp ,$$

where \sharp and \flat denote the canonical isomorphisms induced by the metric g between the vector fields and the covector fields on M . For N future directed timelike, we define the *N-energy* of u at time τ to be

$$E_\tau^N(u) := \int_{\Sigma_\tau} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_{\bar{g}_\tau} , \quad (3.1.8)$$

where \cdot stands for the inner product on (M, g) between two vectors (but \cdot is also used for the inner product between two covectors) and $\text{vol}_{\bar{g}_\tau}$ denotes the volume element corresponding to the metric \bar{g}_τ . If $A \subseteq \Sigma_\tau$, then $E_{\tau,A}^N(u)$ denotes the *N-energy* of u at time τ in the volume A , i.e., the integration in (3.1.8) is only over A .

We define the *wave operator* \square_g on the Lorentzian manifold (M, g) by

$$\square_g u := \nabla^\mu \nabla_\mu u ,$$

where ∇ denotes the Levi-Civita connection on (M, g) . However, we will omit from here on the index g on \square_g , since it is clear from the context which Lorentzian metric is referred to.

The notion (3.1.8) of the *N-energy* of a function u is especially helpful whenever we have an adequate knowledge of $\square u$, since one can then infer detailed information about the behaviour of the *N-energy* (cf. the energy estimate (3.2.8) in the next section), and thus also about the behaviour of u itself. Hence, the stress-energy tensor (3.1.7) together with the notion of the *N-energy* is particularly useful for solutions u of the *wave equation*

$$\square u = 0. \quad (3.1.9)$$

For more on the stress-energy tensor and the notion of energy, we refer the reader to [68], chapters 2.7 and 2.8.

Given a Lorentzian manifold (M, g) and $A \subseteq M$, we denote with $J^+(A)$ the causal future of A , i.e., all the points $x \in M$ such that there exists a future directed causal curve starting at some point of A and ending at x . The causal past of A , $J^-(A)$, is

defined analogously⁵⁷. Finally, C and c will always denote positive constants.

We remark that for simplicity of notation we restrict our considerations to $3 + 1$ -dimensional Lorentzian manifolds (M, g) . However, all results extend in an obvious way to dimensions $n + 1$, $n \geq 1$. Moreover, all given manifolds, functions and tensor fields are assumed to be smooth, although this is only for convenience and clearly not necessary.

3.2 Part I: The theory of Gaussian beams on Lorentzian manifolds

3.2.1 Solutions of the wave equation with localised energy

This section and the next are devoted to the proof of Theorem 3.2.1, which summarises the state of the art knowledge concerning the construction of solutions with localised energy using the approximation by Gaussian beams.

Theorem 3.2.1. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface⁵⁸. Furthermore, let γ be a null geodesic that intersects Σ_0 and N a timelike, future directed vector field.*

For any neighbourhood \mathcal{N} of γ , for any $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$, and for any $\mu > 0$ there exists a solution $v \in C^\infty(M, \mathbb{C})$ of the wave equation (3.1.9) with $E_0^N(v) = 1$ and a $\tilde{u} \in C^\infty(M, \mathbb{C})$ with $\text{supp}(\tilde{u}) \subseteq \mathcal{N}$ such that

$$E_\tau^N(v - \tilde{u}) < \mu \quad \forall 0 \leq \tau \leq T, \quad (3.2.2)$$

provided that we have on $R_{[0, T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$

$$\frac{1}{|n_{\Sigma_\tau}(t)|} \leq C, \quad g(N, N) \leq -c < 0, \quad -g(N, n_{\Sigma_\tau}) \leq C$$

and

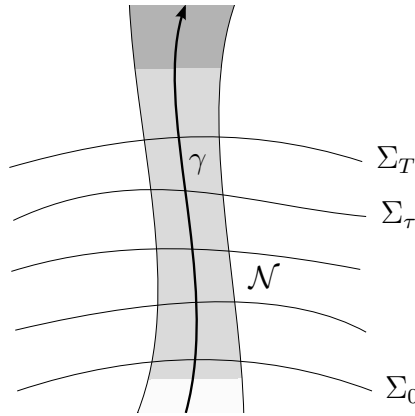
$$(3.2.3)$$

$$|\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})|, |\nabla N(n_{\Sigma_\tau}, e_i)|, |\nabla N(e_i, e_j)| \leq C \quad \text{for } 1 \leq i, j \leq 3,$$

where c and C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ is an orthonormal frame.

⁵⁷See also Chapter 14 in [52].

⁵⁸Note that [7] shows that every globally hyperbolic Lorentzian manifold admits a smooth time function.



Note that (3.2.2) together with $\text{supp}(\tilde{u}) \subseteq \mathcal{N}$ make the statement, that the solution v hardly disperses up to time T , rigorous. The energy of the solution v stays localised for finite time.

Proof. The function \tilde{u} in the theorem is the Gaussian beam, the *approximate* solution to the wave equation (3.1.9) which we need to construct. Recall that a Gaussian beam $u_\lambda \in C^\infty(M, \mathbb{C})$ is of the form

$$u_\lambda(x) = a_{\mathcal{N}}(x)e^{i\lambda\phi(x)}, \quad (3.2.4)$$

where $\lambda > 0$ is a parameter that determines how quickly the Gaussian beam oscillates, and $a_{\mathcal{N}}$ and ϕ are smooth, complex valued functions on M , that do *not* depend on λ . However, $a_{\mathcal{N}}$ depends on the neighbourhood \mathcal{N} of the null geodesic γ . In Section 3.2.2 we construct the functions $a_{\mathcal{N}}$ and ϕ in such a way that u_λ satisfies the following three conditions: The *first condition* is

$$\|\square u_\lambda\|_{L^2(R_{[0,T]})} \leq C(T), \quad (3.2.5)$$

where the constant $C(T)$ depends on $a_{\mathcal{N}}$, ϕ and T , but not on λ . The *second condition* is

$$E_0^N(u_\lambda) \rightarrow \infty \quad \text{for } \lambda \rightarrow \infty, \quad (3.2.6)$$

where N is the timelike vector field from Theorem 3.2.1. Finally, the *third condition* is

$$u_\lambda \text{ is supported in } \mathcal{N}. \quad (3.2.7)$$

Assuming for now that we have already found functions $a_{\mathcal{N}}$ and ϕ such that the conditions (3.2.5), (3.2.6) and (3.2.7) are satisfied, we finish the proof of Theorem 3.2.1. In order to normalise the initial energy of the approximate solutions u_λ , we define

$$\tilde{u}_\lambda := \frac{u_\lambda}{\sqrt{E_0^N(u_\lambda)}},$$

which, moreover, yields

$$\|\square\tilde{u}_\lambda\|_{L^2(R_{[0,T]})} \rightarrow 0 \quad \text{for } \lambda \rightarrow \infty.$$

This says that when the Gaussian beam becomes more and more oscillatory (i.e. for bigger and bigger λ), the closer it comes to being a proper solution to the wave equation.

We now define the actual solution v_λ of the wave equation - the one that is being approximated by the \tilde{u}_λ - to be the solution of the following initial value problem:

$$\begin{aligned} \square v &= 0 \\ v|_{\Sigma_0} &= \tilde{u}_\lambda|_{\Sigma_0} \\ n_{\Sigma_0} v|_{\Sigma_0} &= n_{\Sigma_0} \tilde{u}_\lambda|_{\Sigma_0}. \end{aligned}$$

Here, we make use of the fact that the Lorentzian manifold (M, g) is globally hyperbolic and thus allows for a well-posed initial value problem for the wave equation. Moreover, the condition (3.2.3) ensures that we have an energy estimate of the form

$$\begin{aligned} \int_{\Sigma_\tau} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_{\tilde{g}_\tau} \leq C(T, N, \{\Sigma_\tau\}) \left(\int_{\Sigma_0} J^N(u) \cdot n_{\Sigma_0} \text{vol}_{\tilde{g}_0} + \|\square u\|_{L^2(R_{[0,T]})}^2 \right) \\ \forall 0 \leq \tau \leq T \end{aligned} \tag{3.2.8}$$

at our disposal (see for example [68], chapter 2.8). Thus, we obtain

$$E_\tau^N(v_\lambda - \tilde{u}_\lambda) \leq C(T, N, \Sigma_\tau) \cdot \|\square\tilde{u}_\lambda\|_{L^2(R_{[0,T]})}^2 \quad \forall 0 \leq \tau \leq T,$$

which goes to zero for $\lambda \rightarrow \infty$. Given now $\mu > 0$, it suffices to choose $\lambda_0 > 0$ big enough and to set $\tilde{u} := \tilde{u}_{\lambda_0}$ and $v := v_{\lambda_0}$, which then finishes the proof under the assumption of the conditions (3.2.5), (3.2.6) and (3.2.7). \square

We end this section with a couple of remarks about Theorem 3.2.1:

Remark 3.2.9. *As already mentioned, the condition (3.2.3) ensures that we have the energy estimate (3.2.8). Note that it is automatically satisfied if the region under consideration, $R_{[0,T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$, is relatively compact, which will be the case in many concrete applications.*

Moreover, by choosing, if necessary, \mathcal{N} a bit smaller, we can always arrange that $\Sigma_T \cap \mathcal{N}$ is relatively compact and that $\mathcal{N} \cap R_{[0,T]} \subseteq J^-(\Sigma_T \cap \mathcal{N})$. Doing then the energy estimate in the relatively compact region $J^-(\Sigma_T \cap \mathcal{N}) \cap J^+(\Sigma_0)$, we obtain

$$E_{\tau, \mathcal{N} \cap \Sigma_\tau}^N(v - \tilde{u}) < \mu \quad \forall 0 \leq \tau \leq T \tag{3.2.10}$$

independently of (3.2.3). Of course, the information given by (3.2.10) is not interesting here, since Theorem 3.2.1 does not provide more information about \tilde{u} than its region of support. However, in Section 3.2.3 we will derive more information about the approximate solution \tilde{u} and then (3.2.10) will tell us about the temporal behaviour of the localised energy of v , cf. Theorem 3.2.43.

Remark 3.2.11. By taking the real or the imaginary part of \tilde{u}_λ and v_λ it is clear that we can choose \tilde{u} and v in Theorem 3.2.1 to be real valued.

3.2.2 The construction of Gaussian beams

Before we start with the construction of Gaussian beams, let us mention that other presentations of this subject can be found for example in [5] or [61]. The latter reference also includes the construction of Gaussian beams for more general hyperbolic PDEs.

Given now a neighbourhood \mathcal{N} of a null geodesic γ , we will construct functions $a_{\mathcal{N}}$, $\phi \in C^\infty(M, \mathbb{C})$ such that the approximate solution $u_\lambda = a_{\mathcal{N}} \cdot e^{i\lambda\phi}$ satisfies the conditions (3.2.5), (3.2.6) and (3.2.7). This will then finish the proof of Theorem 3.2.1. We compute

$$\square u_\lambda = -\lambda^2(d\phi \cdot d\phi)a_{\mathcal{N}}e^{i\lambda\phi} + i\lambda\square\phi \cdot a_{\mathcal{N}}e^{i\lambda\phi} + 2i\lambda \operatorname{grad} \phi(a_{\mathcal{N}}) \cdot e^{i\lambda\phi} + \square a_{\mathcal{N}} \cdot e^{i\lambda\phi}. \quad (3.2.12)$$

If we required $d\phi \cdot d\phi = 0$ (eikonal equation) and $2\operatorname{grad} \phi(a_{\mathcal{N}}) + \square\phi \cdot a_{\mathcal{N}} = 0$, we would be able to satisfy (3.2.5).⁵⁹ This, however, would lead us to the geometric optics approximation (see Appendix 3.A), whose major drawback is that in general the solution ϕ of the eikonal equation breaks down at some point along γ due to the formation of caustics. The method of Gaussian beams takes a slightly different approach. We only require an *approximate* solution $\phi \in C^\infty(M, \mathbb{C})$ of the eikonal equation in the sense that

$$d\phi \cdot d\phi \text{ vanishes on } \gamma \text{ to high order.}^{60}$$

Moreover, we demand that

$$\phi|_\gamma \text{ and } d\phi|_\gamma \text{ are real valued} \quad (3.2.13)$$

$$\operatorname{Im}(\nabla\nabla\phi|_\gamma) \text{ is positive definite on a 3-dimensional subspace transversal to } \dot{\gamma}, \quad (3.2.14)$$

where $\operatorname{Im}(\nabla\nabla\phi|_x)$, $x \in M$, denotes the imaginary part of the bilinear map $\nabla\nabla\phi|_x : T_xM \times T_xM \rightarrow \mathbb{C}$. Let us assume for a moment that (3.2.13) and (3.2.14) hold. Taking

⁵⁹We would also be able to satisfy (3.2.6) and, at least up to some finite time T , (3.2.7), see Appendix 3.A.

⁶⁰The exact order to which we require $d\phi \cdot d\phi$ to vanish on γ will be determined later.

slice coordinates for γ , i.e., a coordinate chart (U, φ) , $\varphi : U \subseteq M \rightarrow \mathbb{R}^4$, such that $\varphi(\text{Im}(\gamma) \cap U) = \{x_1 = x_2 = x_3 = 0\}$, we obtain

$$\text{Im}(\phi)(x) \geq c \cdot (x_1^2 + x_2^2 + x_3^2), \quad (3.2.15)$$

at least if we restrict ϕ to a small enough neighbourhood of γ . Note that such slice coordinates exist, since the global hyperbolicity of (M, g) implies that γ is an embedded submanifold of M . This is easily seen by appealing to the strong causality condition⁶¹. Let us now denote the real part of ϕ by ϕ_1 and the imaginary part by ϕ_2 . We then have

$$u_\lambda = a_N \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2}.$$

We see that the last factor imposes the shape of a Gaussian on u_λ , centred around γ – this explains the name. Moreover, for λ large this Gaussian will become more and more narrow, i.e., less and less weight is given to the values of a_N away from γ .

We rewrite (3.2.12) as

$$\begin{aligned} \square u_\lambda = & -\lambda^2 \underbrace{(d\phi \cdot d\phi)} \cdot a_N e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2} + i\lambda \underbrace{(2\text{grad } \phi(a_N) + \square\phi \cdot a_N)} \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2} \\ & + \square a_N \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2}. \end{aligned} \quad (3.2.16)$$

Intuitively, if we can arrange for the underbraced terms to vanish on γ to some order and if we choose large λ , then we will pick up only very small contributions. The next lemma makes this rigorous:

Lemma 3.2.17. *Let $f \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{C})$ vanish along $\{x_1 = x_2 = x_3 = 0\}$ to order S , i.e., all partial derivatives up to and including the order S of f vanish along $\{x_1 = x_2 = x_3 = 0\}$. We then have*

$$\int_{[0, T] \times \mathbb{R}^3} \left(|f(x)| e^{-\lambda(x_1^2 + x_2^2 + x_3^2)} \right)^2 dx \leq C \lambda^{-(S+1) - \frac{3}{2}},$$

where C depends on f (and on T).

Proof. Introduce stretched coordinates $y_0 := x_0$, $y_i := \sqrt{\lambda}x_i$ for $i = 1, 2, 3$. Since f vanishes along the x_0 axis to order S and has compact support, we get $|f(x)| \leq C \cdot |\underline{x}|^{S+1}$ for all $x = (x_0, \underline{x}) \in [0, T] \times \mathbb{R}^3$; thus

$$\left| f\left(y_0, \frac{\underline{y}}{\sqrt{\lambda}}\right) \right| \leq C \cdot \frac{|y|^{S+1}}{\lambda^{\frac{S+1}{2}}}.$$

⁶¹Cf. for example [52], Chapter 14, for more on the strong causality condition.

This yields

$$\int_{[0,T] \times \mathbb{R}^3} |f(x)|^2 e^{-2\lambda|\underline{x}|^2} dx \leq \int_{[0,T] \times \mathbb{R}^3} C \cdot |\underline{y}|^{2(S+1)} e^{-2|\underline{y}|^2} dy \cdot \lambda^{-(S+1)-\frac{3}{2}}. \quad (3.2.18)$$

□

We summarise the approach taken by the Gaussian beam approximation in the following

Lemma 3.2.19. *Within the setting of Theorem 3.2.1, assume we are given $a, \phi \in C^\infty(M, \mathbb{C})$ which satisfy (3.2.13) and (3.2.14). Moreover, assume*

$$d\phi \cdot d\phi \quad \text{vanishes to second order along } \gamma \quad (3.2.20)$$

$$2\text{grad } \phi(a) + \square\phi \cdot a \quad \text{vanishes to zeroth order along } \gamma \quad (3.2.21)$$

$$a(\text{Im}(\gamma) \cap \Sigma_0) \neq 0 \quad \text{and} \quad d\phi(\text{Im}(\gamma) \cap \Sigma_0) \neq 0 \quad (3.2.22)$$

Given a neighbourhood \mathcal{N} of γ , we can then multiply a by a suitable bump function $\chi_{\mathcal{N}}$ which is equal to one in a neighbourhood of γ and satisfies $\text{supp}(\chi_{\mathcal{N}}) \subseteq \mathcal{N}$, such that

$$u_\lambda = u_{\lambda, \mathcal{N}} = a_{\mathcal{N}} e^{i\lambda\phi}$$

satisfies (3.2.5), (3.2.6) and (3.2.7), where $a_{\mathcal{N}} := a \cdot \chi_{\mathcal{N}}$.

Proof. Cover γ by slice coordinate patches and let $\tilde{\chi}$ be a bump function which meets the following three requirements:

i) $\tilde{\chi}$ is equal to one in a neighbourhood of γ

ii) (3.2.15) is satisfied for all $x \in \text{supp}(\tilde{\chi})$

iii) $R_{[0,T]} \cap \text{supp}(\tilde{\chi})$ is relatively compact in M for all $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$.

Pick now a second bump function $\tilde{\chi}_{\mathcal{N}}$ which is again equal to one in a neighbourhood of γ and is supported in \mathcal{N} . We then define $\chi_{\mathcal{N}} := \tilde{\chi} \cdot \tilde{\chi}_{\mathcal{N}}$. Clearly, (3.2.7) is satisfied.

In order to see that (3.2.5) holds, note that the conditions (3.2.13), (3.2.14), (3.2.20) and (3.2.21) are still satisfied by the pair $(a_{\mathcal{N}}, \phi)$. Moreover note that due to condition iii) the integrand is supported in a compact region for each $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$. Thus, the spacetime volume of this region is finite. We thus obtain (3.2.5) from (3.2.16) and Lemma 3.2.17.

Finally, we have

$$E_0^N(u_\lambda) \geq C \cdot (\lambda^{\frac{1}{2}} - 1).$$

This follows since the highest order term in λ in $E_0^N(u_\lambda)$ is

$$\lambda^2 \cdot \int_{\Sigma_0} |a|^2 N(\phi) \cdot n_{\Sigma_0}(\phi) e^{-2\lambda \text{Im}(\phi)} \text{vol}_{\tilde{g}_0},$$

and the same scaling argument used in the proof of Lemma 3.2.17 shows that the term $e^{-2\lambda\text{Im}(\phi)}$ leads to a $\lambda^{-\frac{3}{2}}$ damping - and only to a $\lambda^{-\frac{3}{2}}$ damping due to condition (3.2.22) (together with (3.2.20) and (3.2.13)). Thus, (3.2.6) is satisfied as well and the lemma is proved. \square

Given a null geodesic γ on (M, g) , we now construct a Gaussian beam along γ , i.e., we construct functions $a, \phi \in C^\infty(M, \mathbb{C})$ which satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21) and (3.2.22). By Lemma 3.2.19, this then finishes the proof of Theorem 3.2.1.

Note that the conditions (3.2.13), (3.2.14), (3.2.20), (3.2.21) and (3.2.22) only depend on the derivatives of ϕ and a on γ . This allows for, instead of constructing ϕ and a directly, constructing compatible first and second derivatives of ϕ along γ and the function a along γ such that the above conditions are satisfied. With the first and second derivatives of ϕ being *compatible* we mean the following consistency statement

$$\partial_\mu \partial_\nu \phi(\gamma(s)) \dot{\gamma}^\nu(s) = \frac{d}{ds} \partial_\mu \phi(\gamma(s)) . \quad (3.2.23)$$

From this data we can then build functions $\phi, a \in C^\infty(M, \mathbb{C})$ whose derivatives along γ agree with the constructed ones⁶² - and thus, ϕ and a will satisfy the above requirements. We start with the construction of ϕ .

Let s be some affine parameter for the future directed null geodesic γ such that $\gamma(0) \in \Sigma_0$. We set⁶³

$$d\phi(s) := \dot{\gamma}^\flat(s) . \quad (3.2.24)$$

Moreover, we require that $\phi(0) \in \mathbb{R}$. The definition (3.2.24) then determines $\phi(s) \in \mathbb{R}$ for all s ; hence (3.2.13) is satisfied. Since $\dot{\gamma}$ is a null vector, we clearly have $d\phi \cdot d\phi = 0$ along γ . We now pick a slice coordinate chart that covers part of γ and set $f(x) := \frac{1}{2} g^{\mu\nu}(x) \partial_\mu \phi(x) \partial_\nu \phi(x)$. Note that the notion of ‘vanishing to second order’ is independent of the choice of coordinates. In order to find the conditions that the second derivative of ϕ has to satisfy, we compute

$$0 \stackrel{!}{=} \partial_\kappa f|_\gamma = \frac{1}{2} (\partial_\kappa g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi|_\gamma + g^{\mu\nu} \partial_\mu \phi \partial_\kappa \partial_\nu \phi|_\gamma = -(\partial_\kappa \dot{\phi}) + \dot{\gamma}^\nu \partial_\nu \partial_\kappa \phi , \quad (3.2.25)$$

where we have used that we have already fixed (3.2.24) and that γ is a null geodesic, thus it satisfies the equations (3.A.3) of the geodesic flow on T^*M . The condition (3.2.25) is exactly the compatibility condition (3.2.23), thus f vanishes to first order along γ if we choose the second derivatives of ϕ to be compatible with the first ones.

⁶²This construction is known as Borel’s Lemma.

⁶³By slight abuse of notation we will denote the covector field along γ which will later be the differential of ϕ already by $d\phi$. Similarly for the second derivatives.

Moreover, we compute

$$\begin{aligned}
0 \stackrel{!}{=} \partial_\kappa \partial_\rho f|_\gamma &= \frac{1}{2} (\partial_\kappa \partial_\rho g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi|_\gamma + (\partial_\kappa g^{\mu\nu}) \underbrace{\partial_\rho \partial_\mu \phi \partial_\nu \phi|_\gamma}_{=} + (\partial_\rho g^{\mu\nu}) \partial_\mu \phi \underbrace{\partial_\kappa \partial_\nu \phi|_\gamma}_{=} \\
&\quad + g^{\mu\nu} \underbrace{\partial_\rho \partial_\mu \phi \partial_\kappa \partial_\nu \phi|_\gamma}_{=} + \underbrace{g^{\mu\nu} \partial_\mu \phi}_{=:\dot{\gamma}^\nu} \partial_\rho \partial_\kappa \partial_\nu \phi|_\gamma .
\end{aligned} \tag{3.2.26}$$

The condition (3.2.26) has actually a lot of structure. In order to see this more clearly, let $H : T^*M \rightarrow \mathbb{R}$ be given by $H(\zeta) := \frac{1}{2} g^{-1}(\zeta, \zeta)$, and having chosen a coordinate system $\{x^\mu\}$ for part of M we denote the corresponding canonical coordinate system on part of T^*M by $\{x^\mu, p^\nu\} = \{\xi^\alpha\}$, where $\mu, \nu \in \{0, \dots, 3\}$ and $\alpha \in \{0, \dots, 7\}$. We define the following matrices

$$\begin{aligned}
A_{\kappa\rho}(s) &:= \frac{1}{2} (\partial_\kappa \partial_\rho g^{\mu\nu}) \partial_\mu \phi \partial_\nu \phi (\gamma(s)) &= \frac{\partial^2 H}{\partial x^\kappa \partial x^\rho} (\gamma(s)) \\
B_{\kappa\rho}(s) &:= \partial_\kappa g^{\rho\nu} \partial_\nu \phi (\gamma(s)) &= \frac{\partial^2 H}{\partial x^\kappa \partial p^\rho} (\gamma(s)) \\
C_{\kappa\rho}(s) &:= g^{\kappa\rho} (\gamma(s)) &= \frac{\partial^2 H}{\partial p^\kappa \partial p^\rho} (\gamma(s)) \\
M_{\kappa\rho}(s) &:= \partial_\kappa \partial_\rho \phi (\gamma(s)) ,
\end{aligned}$$

and rewrite (3.2.26) as

$$0 = A + BM + MB^T + MCM + \frac{d}{ds} M . \tag{3.2.27}$$

This quadratic ODE for the matrix M is called a *Riccati equation*. We would like to ensure that we can find a global solution that satisfies (3.2.14) and is compatible with the first derivatives.

There is a well-known way to solve (3.2.27), which boils down here to finding a suitable set of Jacobi fields - or using the language of Appendix 3.C, a suitable Jacobi tensor. We consider the system of matrix ODEs

$$\begin{aligned}
\dot{J} &= B^T J + CV \\
\dot{V} &= -AJ - BV ,
\end{aligned} \tag{3.2.28}$$

where J and V are 4×4 matrices. If J is invertible then it is an easy exercise to verify that $M := VJ^{-1}$ solves (3.2.27). We will show that we can choose initial data such that J is invertible for all time. But first let us make some remarks about (3.2.28).

Although (3.2.27) depends on the choice of coordinates and thus has no geometric interpretation, a vector solution of the system of ODEs (3.2.28) is a geometric quantity: Let us denote the Hamiltonian flow of H by $\Psi_t : T^*M \rightarrow T^*M$, which is exactly the geodesic flow on T^*M . The vector solutions of (3.2.28) are exactly those flow lines of the lifted flow $(\Psi_t)_* : T(T^*M) \rightarrow T(T^*M)$ that project down on the lifted geodesic

$s \mapsto \dot{\gamma}^b(s) \in T^*M$. In order to see this, let⁶⁴

$$\tilde{X} = \tilde{X}^\alpha \frac{\partial}{\partial \xi^\alpha} \Big|_{\dot{\gamma}^b(0)} = \tilde{J}^\mu \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(0)} + \tilde{V}^\nu \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(0)} \in T_{\dot{\gamma}^b(0)}(T^*M).$$

The pushforward via Ψ_t is then a vector field along $\dot{\gamma}^b(s)$,

$$(\Psi_s)_* \tilde{X} = \frac{\partial \Psi_s^\alpha}{\partial \xi^\beta} \Big|_{\dot{\gamma}^b(0)} \tilde{X}^\beta \frac{\partial}{\partial \xi^\alpha} \Big|_{\dot{\gamma}^b(s)} =: \tilde{J}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + \tilde{V}^\nu(s) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)},$$

whose x^ρ component satisfies

$$\begin{aligned} \frac{d}{ds} \Big|_{s=s_0} (\Psi_s)_* \tilde{X}(x^\rho) &= \frac{\partial}{\partial s} \Big|_{s=s_0} \left[\frac{\partial(x^\rho \circ \Psi_s)}{\partial \xi^\alpha} \Big|_{\dot{\gamma}^b(0)} \tilde{X}^\alpha \right] \\ &= \frac{\partial}{\partial \xi^\alpha} \Big|_{\dot{\gamma}^b(0)} \frac{\partial}{\partial s} \Big|_{s=s_0} (x^\rho \circ \Psi_s) \tilde{X}^\alpha \\ &= \frac{\partial}{\partial \xi^\alpha} \Big|_{\dot{\gamma}^b(0)} \left(\frac{\partial H}{\partial p^\rho} \circ \Psi_{s_0} \right) \tilde{X}^\alpha \\ &= \frac{\partial^2 H}{\partial \xi^\alpha \partial p^\rho} \Big|_{\Psi_{s_0}(\dot{\gamma}^b(0))} \cdot \frac{\partial \Psi_{s_0}^\alpha}{\partial \xi^\beta} \Big|_{\dot{\gamma}^b(0)} \tilde{X}^\beta \\ &= \frac{\partial^2 H}{\partial \xi^\alpha \partial p^\rho} \Big|_{\dot{\gamma}^b(s_0)} \cdot [(\Psi_{s_0})_* \tilde{X}]^\alpha \\ &= \frac{\partial^2 H}{\partial x^\kappa \partial p^\rho} \Big|_{\dot{\gamma}^b(s_0)} \tilde{J}^\kappa(s_0) + \frac{\partial^2 H}{\partial p^\kappa \partial p^\rho} \Big|_{\dot{\gamma}^b(s_0)} \tilde{V}^\kappa(s_0). \end{aligned}$$

Here, we have used

$$\frac{d}{ds} \Big|_{s=s_0} (x^\rho \circ \Psi_s)(\xi_0) = \frac{\partial H}{\partial p^\rho} \Big|_{\Psi_{s_0}(\xi_0)},$$

see equation (3.A.3). The computation for the $\frac{\partial}{\partial p^\rho}$ components is analogous. Thus, if

$$\tilde{X}(s) = \tilde{J}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + \tilde{V}^\nu(s) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)}$$

is a vector solution of (3.2.28), we see that

$$\pi_* \tilde{X}(s) = \tilde{J}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\gamma(s)}$$

is a Jacobi field along γ , where $\pi : T^*M \rightarrow M$ is the canonical projection map. Hence, we can construct a matrix solution (J, V) with invertible J if, and only if, we can find four everywhere linearly independent Jacobi fields along γ . This shows that if we demanded J to be real valued, we would encounter the same obstruction as in the geometric optics approach, i.e., the solution M would break down at caustics.

⁶⁴In the following vectors are denoted by tilded capital letters in order to distinguish them from the untilded matrices.

Moreover, note that the Hamiltonian flow Ψ_t leaves the symplectic form ω on T^*M invariant, which is given in $\{x^\mu, p^\nu\}$ coordinates by

$$\begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

So in particular, given two vector valued solutions $\tilde{X}(s) = \tilde{J}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + \tilde{V}^\nu(s) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)}$ and $\hat{X}(s) = \hat{J}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + \hat{V}^\nu(s) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)}$ of (3.2.28), we have that

$$\omega(\tilde{X}(s), \hat{X}(s)) = \begin{pmatrix} \tilde{J}(s) & \tilde{V}(s) \end{pmatrix} \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} \begin{pmatrix} \hat{J}(s) \\ \hat{V}(s) \end{pmatrix} \quad \text{is constant.}^{65} \quad (3.2.29)$$

We now prescribe suitable initial data for (3.2.28) such that $J(s)$ is invertible for all s and $\sum_{\mu=0}^3 V_{\kappa\mu} J_{\mu\rho}^{-1}(s) = M_{\kappa\rho}(s) =: \partial_\kappa \partial_\rho \phi(\gamma(s))$ is symmetric, satisfies (3.2.27), (3.2.14), and (3.2.23). Therefore choose $M(0)$ such that⁶⁶

- i) $M(0)$ is symmetric
- ii) $M(0)_{\mu\nu} \dot{\gamma}^\nu = (\partial_\mu \phi)(0)$
- iii) $\text{Im}(M(0)_{\mu\nu}) dx^\mu \Big|_{\gamma(0)} \otimes dx^\nu \Big|_{\gamma(0)}$ is positive definite on a three dimensional subspace of $T_{\gamma(0)}M$ that is transversal to $\dot{\gamma}$

⁶⁵The reader might find the following remark instructive: Although solutions of (3.2.28) are geometric quantities, the splitting in J and V is not a geometric one. To be more precise, while J gives rise to a vector field on M , V depends on the choice of the coordinates. One can, however, turn this splitting into a geometric one, namely by making use of the splitting of $T_{\dot{\gamma}^b(s)}(T^*M)$ in a vertical and a horizontal subspace, which is induced by the Levi-Civita connection. In this approach, one considers the second covariant derivative of f instead of the partial derivatives in (3.2.26) and thus obtains an ODE for $\nabla\nabla\phi$. Again, one can reduce the so obtained equation to a system of linear ODEs for a 1-contravariant and 1-covariant tensor J and a 2-covariant tensor V along γ such that $\text{tr} V \otimes J^{-1}$ solves again the original equation for $\nabla\nabla\phi$. The system of linear ODEs is now equivalent to the Jacobi equation (for a Jacobi tensor J)

$$D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$$

with $V = D_t J^b$. The background for the reduction of the nonlinear ODE for $\nabla\nabla\phi$ to a linear second order ODE is provided by equation (3.C.3) of Appendix 3.C. Given two solutions $J(s)$ and $J'(s)$ of the Jacobi equation, we obtain that

$$g(D_t J(s), J'(s)) - g(J(s), D_t J'(s)) \quad \text{is constant.}$$

This follows either from (3.2.29) or by a direct computation, making use of the Jacobi equation and the symmetry properties of the Riemannian curvature tensor. In this slightly more geometric approach, the following discussion is then analogous.

⁶⁶Note that the right hand side of ii) is determined by (3.2.24).

and solve (3.2.28) with initial data

$$\begin{pmatrix} J(0) \\ V(0) \end{pmatrix} = \begin{pmatrix} \mathbb{1} \\ M(0) \end{pmatrix}. \quad (3.2.30)$$

Since (3.2.28) is a linear ODE, we get, in the chart we are working with, a global solution

$$[0, s_{\max}) \ni s \mapsto \begin{pmatrix} J(s) \\ V(s) \end{pmatrix}.$$

We show that $J(s)$ is invertible for all s by contradiction. Thus, assume there is an $s_0 > 0$ such that $J(s_0)$ is degenerate, i.e., there is a column vector $0 \neq f \in \mathbb{C}^4$ such that $J(s_0)f = 0$. We define

$$\tilde{X}_f(s) = (J(s)f)^\mu \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + (V(s)f)^\nu \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)}, \quad (3.2.31)$$

which is a vector solution to (3.2.28). Using (3.2.29), we compute

$$\begin{aligned} 0 &= \omega(\tilde{X}_f(s_0), \overline{\tilde{X}_f(s_0)}) = \omega(\tilde{X}_f(0), \overline{\tilde{X}_f(0)}) = [J(0)f] \cdot \overline{[V(0)f]} - [V(0)f] \cdot \overline{[J(0)f]} \\ &= -2if \cdot [\operatorname{Im}(M(0))] \bar{f}, \end{aligned}$$

where we used that $M(0)$ is symmetric. Since $\operatorname{Im}(M(0))$ is positive definite on a three dimensional subspace transversal to $\dot{\gamma}$, this yields $f^\mu \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}(0)} = z \cdot \dot{\gamma}(0)$, for some $0 \neq z \in \mathbb{C}$. Without loss of generality we can assume that $z = 1$, since if necessary we consider $z^{-1} \cdot f$ instead of f . Using (3.2.30) and ii) of the properties of $M(0)$, we infer that

$$\tilde{X}_f(0) = \dot{\gamma}^\mu(0) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(0)} + \sum_{\nu=0}^3 d\phi_\nu(0) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(0)}.$$

On the other hand, since $s \mapsto \dot{\gamma}^b(s) =: \sigma(s) \in T^*M$ is a flow line of Ψ_t , we have that

$$\dot{\sigma}(s) = (\Psi_s)_*(\dot{\sigma}(0)),$$

and thus, $s \mapsto \dot{\sigma}(s) \in T(T^*M)$ is a solution of (3.2.28). Written out in components, we have

$$\dot{\sigma}(s) = \dot{\gamma}^\mu(s) \frac{\partial}{\partial x^\mu} \Big|_{\dot{\gamma}^b(s)} + \sum_{\nu=0}^3 d\phi_\nu(s) \frac{\partial}{\partial p^\nu} \Big|_{\dot{\gamma}^b(s)}, \quad (3.2.32)$$

and thus in particular $\dot{\sigma}(0) = \tilde{X}_f(0)$. Since two solutions of (3.2.28) that agree initially are actually equal, we infer that

$$\tilde{X}_f(s) = \dot{\sigma}(s) \text{ for all } s. \quad (3.2.33)$$

Projecting (3.2.33) down on TM using π_* , we obtain the contradiction

$$0 = (J(s_0)f)^\mu \frac{\partial}{\partial x^\mu} \Big|_{\gamma(s_0)} = \pi_* \tilde{X}_f(s_0) = \pi_* \dot{\sigma}(s_0) = \dot{\gamma}(s_0) \neq 0 .$$

This shows that $J(s)$ is invertible for all $s \in [0, s_{\max})$ and hence, we obtain a global solution $M(s)$ to the Riccati equation.

Since $M(0)$ is chosen to be symmetric and the Riccati equation (3.2.27) is invariant under transposition, it follows that $M(s)$ is symmetric for all s .

In order to see that this choice of second derivatives of ϕ is compatible with our prescription of the first derivatives of ϕ , (3.2.24), i.e., in order to show that (3.2.23) holds, we choose $f \in \mathbb{C}^4$ such that $f^\mu = \dot{\gamma}^\mu(0)$. Recall that we were also led to this choice in the proof of J being invertible, and so we can deduce from (3.2.31), (3.2.32) and (3.2.33) that

$$\begin{pmatrix} \dot{\gamma}^\mu(s) \\ \dot{d}\phi_\nu(s) \end{pmatrix} = \begin{pmatrix} J(s)_{\mu\rho} \dot{\gamma}^\rho(0) \\ V(s)_{\nu\rho} \dot{\gamma}^\rho(0) \end{pmatrix} . \quad (3.2.34)$$

Using this, the compatibility (3.2.23) follows:

$$M_{\mu\nu}(s) \dot{\gamma}^\nu(s) = \sum_{\rho, \kappa=0}^3 V_{\mu\rho}(s) J_{\rho\kappa}^{-1}(s) J_{\kappa\eta}(s) \dot{\gamma}^\eta(0) = (\partial_\mu \dot{\phi})(s) .$$

Finally, for showing that (3.2.14) holds, we compute for $f \in \mathbb{C}^4$ and using the notation from (3.2.31)

$$\begin{aligned} \omega(\tilde{X}_f(s), \overline{\tilde{X}_f(s)}) &= [J(s)f] \cdot \overline{[V(s)f]} - [V(s)f] \cdot \overline{[J(s)f]} \\ &= [J(s)f] \cdot \overline{[M(s)J(s)f]} - [M(s)J(s)f] \cdot \overline{[J(s)f]} \\ &= -2i [\operatorname{Im}(M(s))J(s)f] \cdot \overline{[J(s)f]} , \end{aligned}$$

where we made use of the symmetry of $M(s)$. Together with (3.2.29), we obtain

$$-2i [\operatorname{Im}(M(0))f] \cdot \overline{[f]} = -2i [\operatorname{Im}(M(s))J(s)f] \cdot \overline{[J(s)f]} .$$

Since $J(s)$ is an isomorphism for all s , this shows that $\operatorname{Im}(M(s))$ stays positive definite on a three dimensional subspace transversal to $\dot{\gamma}(s)$, where we also use (3.2.34). This finishes the construction of the second derivatives of ϕ in a coordinate chart.

Staying in this chart, the condition (3.2.21) is a linear first order ODE for a function $a(s)$ along γ , and thus prescribing initial data $a(0) \neq 0$, the existence of a global solution $a(s)$ with respect to this chart is guaranteed. Writing down the formal Taylor series up to order two for ϕ and up to order zero for a in the slice coordinates (special case of Borel's Lemma), we construct two functions $a, \phi \in C^\infty(U, \mathbb{C})$ that satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21) and (3.2.22), where U is the domain of the slice coordinate chart.

Let $\gamma : [0, S) \rightarrow M$ be the affine parametrisation of γ , where $0 < S \leq \infty$. Let us for the following presentation assume that $S = \infty$ – the case $S < \infty$ is even simpler. We cover $\text{Im}(\gamma)$ by slice coordinate charts (U_k, φ_k) , $k \in \mathbb{N}$, such that there is a partition of $[0, \infty)$ by intervals $[s_{k-1}, s_k]$ with $s_0 = 0$ and $s_{k-1} < s_k$ that satisfies $\gamma([s_{k-1}, s_k]) \subseteq U_k$. We then construct functions $a_k, \phi_k \in C^\infty(U_k, \mathbb{C})$ that satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21) (and (3.2.22) for $k = 1$) as follows: The case $k = 1$ was presented above. For $k > 1$ we repeat the construction from above with some slight modifications: If $M_{k-1}(s_k)$ denotes the solution of (3.2.27) in the chart U_{k-1} at time s_{k-1} , we now express $M_{k-1}(s_k)$ in the φ_k coordinates⁶⁷ and solve (3.2.27) in both time directions. We proceed analogously for a .

Extending $\{U_k\}_{k \in \mathbb{N}}$ to an open cover of M by $U_0 \subseteq M$ in such a way that $U_0 \cap \text{Im}(\gamma) = \emptyset$ and taking a partition of unity $\{\eta_k\}_{k \in \mathbb{N}_0}$ subordinate to this open cover, we glue all the local functions ϕ_k and a_k together to obtain $\phi := \sum_{k=1}^{\infty} \phi_k \eta_k$ and $a := \sum_{k=1}^{\infty} a_k \eta_k$, which are in $C^\infty(M, \mathbb{C})$ and satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21) and (3.2.22). This finally completes the proof of Theorem 3.2.1.

For future reference, we make the following

Definition 3.2.35. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$. Furthermore, let $\gamma : [0, S) \rightarrow M$ be an affinely parametrised future directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$, and let N be a timelike, future directed vector field.*

Given functions $a, \phi \in C^\infty(M, \mathbb{C})$ that satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21), $a(\text{Im}(\gamma) \cap \Sigma_0) \neq 0$ and (3.2.24), we call the function

$$u_{\lambda, \mathcal{N}} = a_{\mathcal{N}} e^{i\lambda\phi}$$

a Gaussian beam along γ with structure functions a and ϕ and with parameters λ and \mathcal{N} . Here, $a_{\mathcal{N}} = a \cdot \chi_{\mathcal{N}} = a \cdot \tilde{\chi} \cdot \tilde{\chi}_{\mathcal{N}}$ with $\tilde{\chi}$ and $\tilde{\chi}_{\mathcal{N}}$ as in the proof of Lemma 3.2.19. Moreover, we call the function

$$\tilde{u}_{\lambda, \mathcal{N}} = \frac{u_{\lambda, \mathcal{N}}}{\sqrt{E_0^{\mathcal{N}}(u_{\lambda, \mathcal{N}})}} \cdot \sqrt{E}$$

a Gaussian beam along γ with structure functions a and ϕ , with parameters λ and \mathcal{N} , and with initial N -energy E , where E is a strictly positive real number. Let us emphasise, that when we say ‘a Gaussian beam along γ ’, γ encodes here not only the image of γ , but also the affine parametrisation.

We end this section with the remark that for the sole *construction* of the Gaussian

⁶⁷The transformation is of course given by the rule by which second coordinate derivatives of scalar functions transform.

beams the assumption of the global hyperbolicity of (M, g) can be replaced by the assumption that the null geodesic $\gamma : \mathbb{R} \supseteq I \rightarrow M$ is a smooth embedding, i.e., in particular $\gamma(I)$ being an embedded submanifold. Moreover, note that if $\gamma : \mathbb{R} \supseteq I \rightarrow M$ is a smooth injective immersion and if $[a, b] \subseteq I$ with $a, b \in \mathbb{R}$, then $\gamma|_{(a,b)} : (a, b) \rightarrow M$ is a smooth embedding. It thus follows that the above construction is always possible for null geodesics with no self-intersections on general Lorentzian manifolds - at least up to some finite affine time in the domain of γ .

3.2.3 Geometric characterisation of the energy of Gaussian beams

In this section we characterise the energy of a Gaussian beam in terms of the energy of the underlying null geodesic. The following theorem is the main result of Part I of this chapter:

Theorem 3.2.36. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$. Moreover, let N be a timelike future directed vector field and $\gamma : [0, S) \rightarrow M$ an affinely parametrised future directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$.*

For any $T > 0$ with $\text{Im}(\gamma) \cap \Sigma_T \neq \emptyset$ and for any $\mu > 0$ there exists a neighbourhood \mathcal{N}_0 of γ and a $\lambda_0 > 0$ such that any Gaussian beam $\tilde{u}_{\lambda, \mathcal{N}}$ along γ with structure functions a and ϕ , with parameters $\lambda \geq \lambda_0$ and \mathcal{N}_0 , and with initial N -energy equal to $-g(N, \dot{\gamma})|_{\gamma(0)}$ satisfies

$$\left| E_\tau^N(\tilde{u}_{\lambda, \mathcal{N}_0}) - [-g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_\tau}] \right| < \mu \quad \forall 0 \leq \tau \leq T. \quad (3.2.37)$$

Before we give the proof, we make a couple of remarks:

- i) The only information about a Gaussian beam we made use of in Theorem 3.2.1, apart from it being an approximate solution, was that it is supported in a given neighbourhood \mathcal{N} of the null geodesic γ . This then yielded, together with (3.2.2), an estimate on the energy outside of the neighbourhood \mathcal{N} of the actual solution to the wave equation, i.e., we could construct solutions to the wave equation with *localised* energy. However, Theorem 3.2.1 does not make any statement about the *temporal behaviour* of this localised energy. The above theorem fills this gap by investigating the temporal behaviour of the energy of the approximate solution, i.e., of the Gaussian beam. Together with (3.2.2) (or even with (3.2.10)!) this then gives an estimate on the temporal behaviour of the localised energy of the actual solution to the wave equation.
- ii) Note that if N is a timelike Killing vector field, the N -energy $-g(N, \dot{\gamma})$ of the null geodesic γ is constant, and thus, so is approximately the N -energy of the Gaussian beam.

- iii) By our Definition 3.2.35 a Gaussian beam is a complex valued function. However, by taking the real or the imaginary part, one can also define a real valued Gaussian beam. The result of Theorem 3.2.36 also holds true in this case, and can be proved using exactly the same technique - only the computations become a bit longer, since we have to deal with more terms.
- iv) Although we have stated the above theorem again using the general assumptions needed for Theorem 3.2.1, we actually do not need more assumptions than we need for the construction of a Gaussian beam, cf. the final remark of the previous section.

Proof. Recall from Definition 3.2.35 that a Gaussian beam $\tilde{u}_{\lambda, \mathcal{N}}$ along γ with structure functions a and ϕ , with parameters \mathcal{N} and λ , and with initial N -energy equal to $-g(N, \dot{\gamma})|_{\gamma(0)}$ is a function

$$\tilde{u}_{\lambda, \mathcal{N}} = \frac{u_{\lambda, \mathcal{N}}}{\sqrt{E_0^N(u_{\lambda, \mathcal{N}})}} \cdot \sqrt{-g(N, \dot{\gamma})|_{\gamma(0)}} = \frac{a_{\mathcal{N}} e^{i\lambda\phi}}{\sqrt{E_0^N(u_{\lambda, \mathcal{N}})}} \cdot \sqrt{-g(N, \dot{\gamma})|_{\gamma(0)}},$$

where the functions $a_{\mathcal{N}}$ and ϕ satisfy (3.2.13), (3.2.14), (3.2.20), (3.2.21), (3.2.22), $\text{supp}(a_{\mathcal{N}}) \subseteq \mathcal{N}$, $\mathcal{N} \cap R_{[0, T]}$ is relatively compact for all $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$, and for a cover of γ with slice coordinate patches (3.2.15) holds for all $x \in \text{supp}(a_{\mathcal{N}})$.

In order to estimate the energy of $\tilde{u}_{\lambda, \mathcal{N}}$ it suffices to consider the leading order term in λ of the energy of $u_{\lambda, \mathcal{N}}$, since all lower order terms are negligible for large λ . We compute

$$\begin{aligned} J^N(u_{\lambda, \mathcal{N}}) \cdot n_{\Sigma_\tau} &= \text{Re}(Nu_{\lambda, \mathcal{N}} \cdot \overline{n_{\Sigma_\tau} u_{\lambda, \mathcal{N}}}) - \frac{1}{2}g(N, n_{\Sigma_\tau}) du_{\lambda, \mathcal{N}} \cdot \overline{du_{\lambda, \mathcal{N}}} \\ &= \lambda^2 |a_{\mathcal{N}}|^2 N \phi_1 \cdot n_{\Sigma_\tau} \phi_1 \cdot e^{-2\lambda\phi_2} + \lambda^2 |a_{\mathcal{N}}|^2 N \phi_2 \cdot n_{\Sigma_\tau} \phi_2 \cdot e^{-2\lambda\phi_2} \\ &\quad + \mathcal{O}(\lambda) \cdot e^{-2\lambda\phi_2} - \frac{1}{2}g(N, n_{\Sigma_\tau}) \left[\lambda^2 |a_{\mathcal{N}}|^2 (d\phi_1 \cdot d\phi_1) e^{-2\lambda\phi_2} \right. \\ &\quad \left. + \lambda^2 |a_{\mathcal{N}}|^2 (d\phi_2 \cdot d\phi_2) e^{-2\lambda\phi_2} + \mathcal{O}(\lambda) \cdot e^{-2\lambda\phi_2} \right]. \end{aligned}$$

Note that $d\phi_2|_{\gamma(\tau)} = 0$, so these terms are of lower order after integration over Σ_τ . The same holds for the $d\phi_1 \cdot d\phi_1$ term. Thus, we get

$$E_\tau^N(u_{\lambda, \mathcal{N}}) = \underbrace{\lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 N \phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau}}_{=\mathcal{O}(\lambda^{\frac{1}{2}})} + \underbrace{\text{lower order terms}}_{=\mathcal{O}(1)}. \quad (3.2.38)$$

The main part of the proof is an approximate conservation law. Recall that $a_{\mathcal{N}}$

and ϕ satisfy (3.2.20) and (3.2.21). These equations yield

$$\begin{aligned} \text{grad } \phi (|a_{\mathcal{N}}|^2) &= \text{grad } \phi (a_{\mathcal{N}}) \cdot \overline{a_{\mathcal{N}}} + a_{\mathcal{N}} \cdot \text{grad } \phi (\overline{a_{\mathcal{N}}}) \\ &= -\frac{1}{2} (\square \phi \cdot a_{\mathcal{N}} \overline{a_{\mathcal{N}}} + a_{\mathcal{N}} \overline{\square \phi} \cdot \overline{a_{\mathcal{N}}}) = -\text{Re}(\square \phi) |a_{\mathcal{N}}|^2 \quad \text{along } \gamma \end{aligned} \quad (3.2.39)$$

and

$$d\phi \cdot d\phi = (d\phi_1 + id\phi_2) \cdot (d\phi_1 + id\phi_2) = d\phi_1 \cdot d\phi_1 - d\phi_2 \cdot d\phi_2 + 2i d\phi_1 \cdot d\phi_2$$

vanishes to second order along γ , thus in particular

$$d\phi_1 \cdot d\phi_2 = \text{grad} \phi_1 (\phi_2) \quad \text{vanishes along } \gamma \text{ to second order.} \quad (3.2.40)$$

Together, (3.2.39) and (3.2.40) show that the current

$$X_{\lambda, \mathcal{N}} = \lambda^2 \cdot |a_{\mathcal{N}}|^2 e^{-2\lambda\phi_2} \text{grad } \phi_1$$

is approximately conserved in the sense that⁶⁸

$$\begin{aligned} \int_{R_{[0, \tau]}} \text{div } X_{\lambda, \mathcal{N}} \text{ vol}_g &= \lambda^2 \cdot \int_{R_{[0, \tau]}} \left(\underbrace{[\text{grad} \phi_1 (|a_{\mathcal{N}}|^2) + \square \phi_1 \cdot |a_{\mathcal{N}}|^2] e^{-2\lambda\phi_2}}_{=\lambda^{-\frac{1}{2}} \cdot \lambda^{-\frac{3}{2}} = \lambda^{-2} \text{ after integration}} \right. \\ &\quad \left. - \underbrace{2\lambda \text{grad} \phi_1 (\phi_2) \cdot |a_{\mathcal{N}}|^2 e^{-2\lambda\phi_2}}_{=\lambda \cdot \lambda^{-\frac{3}{2}} \cdot \lambda^{-\frac{3}{2}} = \lambda^{-2} \text{ after int.}} \right) \text{ vol}_g \\ &= \mathcal{O}(1), \end{aligned}$$

but

$$\int_{\Sigma_\tau} X_{\lambda, \mathcal{N}} \cdot n_{\Sigma_\tau} \text{ vol}_{\tilde{g}_\tau} = \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{ vol}_{\tilde{g}_\tau} = \mathcal{O}(\lambda^{\frac{1}{2}}).$$

In particular, we obtain⁶⁹

$$\begin{aligned} &\left| \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{ vol}_{\tilde{g}_\tau} - \lambda^2 \cdot \int_{\Sigma_0} |a_{\mathcal{N}}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{ vol}_{\tilde{g}_0} \right| \\ &= \left| \int_{R_{[0, \tau]}} \text{div } X_{\lambda, \mathcal{N}} \text{ vol}_g \right| = \mathcal{O}(1). \end{aligned} \quad (3.2.41)$$

⁶⁸Here, we use a slight reformulation of Lemma 3.2.17, namely if $f \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{C})$ vanishes to order S along $\{x_1 = x_2 = x_3 = 0\}$, then

$$\int_{[0, T] \times \mathbb{R}^3} |f(x)| e^{-2\lambda(x_1^2 + x_2^2 + x_3^2)} dx \leq C \lambda^{-\frac{S+1}{2} - \frac{3}{2}}.$$

This is proved in exactly the same way.

⁶⁹In the geometric optics approximation we have indeed a proper conservation law, which is interpreted in the physics literature as conservation of photon number, cf. for example [47], Chapter 22.5.

Note that these boundary integrals appear in the energy. They are multiplied (under the integral) by $N\phi_1$. It basically remains now to choose a neighbourhood \mathcal{N}_0 so small that $N\phi_1|_{\Sigma_\tau \cap \mathcal{N}_0}$ is roughly constant.

Since $\text{supp}(a_{\mathcal{N}}) \cap R_{[0,T]}$ is compact, $N\phi_1$ is uniformly continuous in $\text{supp}(a_{\mathcal{N}}) \cap R_{[0,T]}$. Thus, for given $\delta > 0$ we can find a neighbourhood $\mathcal{N}_1 = \mathcal{N}_1(\delta) \subseteq \mathcal{N}$ of γ such that the following holds true for all $0 \leq \tau \leq T$:

$$\left| N\phi_1(x) - N\phi_1|_{\gamma_\tau} \right| \leq \delta \quad \forall x \in \Sigma_\tau \cap \mathcal{N}_0,$$

where we have introduced the notation $\gamma_\tau := \text{Im}(\gamma) \cap \Sigma_\tau$. Using (3.2.41) we compute

$$\begin{aligned} & \left| \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} - \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 N\phi_1|_{\gamma_\tau} \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \right| \\ & \leq -\delta \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \\ & \leq -\delta \lambda^2 \cdot \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} + \mathcal{O}(1). \end{aligned} \tag{3.2.42}$$

Finally,

$$\begin{aligned} & \left| -g(N, \dot{\gamma})|_{\gamma_0} \lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \right. \\ & \quad \left. - (-g(N, \dot{\gamma})|_{\gamma_\tau}) \lambda^2 \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} \right| \\ & \leq -g(N, \dot{\gamma})|_{\gamma_0} \left| \lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \right. \\ & \quad \left. - N\phi_1|_{\gamma_\tau} \lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \right| \\ & \quad + g(N, \dot{\gamma})|_{\gamma_0} N\phi_1|_{\gamma_\tau} \left| \lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}_1}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} \right. \\ & \quad \left. - \lambda^2 \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} \right| \\ & \quad - N\phi_1|_{\gamma_\tau} \left| \lambda^2 \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 N\phi_1|_{\gamma_0} \cdot n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} \right. \\ & \quad \left. - \lambda^2 \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} \right|. \end{aligned}$$

Let us denote the first two rows after the inequality sign by I_1 , the next two rows by I_2 , and the last two rows by I_3 . In order to obtain (3.2.37) (modulo lower order terms in λ) we divide the inequality above by $E_0^N(u_{\lambda, \mathcal{N}_1})$, and again, we only have to consider

the leading order term

$$J := \lambda^2 \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 N\phi_1 \cdot n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0}$$

of $E_0^N(u_{\lambda, \mathcal{N}_1})$. We estimate the single terms, using (3.2.41) and (3.2.42):

- $\frac{I_1}{J} \leq -g(N, \dot{\gamma})|_{\gamma_0} \frac{\left[-\delta\lambda^2 \cdot \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0} + \mathcal{O}(1) \right]}{\lambda^2 \min_{x \in \Sigma_0 \cap \mathcal{N}_1} \{-N\phi_1(x)\} (-1) \int_{\Sigma_0} |a_{\mathcal{N}_1}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_0}}$
- $\frac{I_2}{J} = \mathcal{O}(\lambda^{-\frac{1}{2}})$
- $\frac{I_3}{J} \leq \frac{-N\phi_1|_{\gamma_\tau}}{\min_{x \in \Sigma_0 \cap \mathcal{N}_1} \{-N\phi_1(x)\}} \cdot \delta$

Recall that $-g(N, \dot{\gamma})|_{\gamma_s}$ is strictly positive for all $s \in [0, S)$. For $0 < \delta < -\frac{N\phi_1|_{\gamma_0}}{2}$ we thus have

$$\min_{x \in \Sigma_0 \cap \mathcal{N}_1(\delta)} \{-N\phi_1(x)\} \geq -\frac{N\phi_1|_{\gamma_0}}{2} > c > 0.$$

Moreover, note that given the $T > 0$ from the theorem, we obtain

$$-g(N, \dot{\gamma}(\tau)) \leq C \quad \text{for } 0 \leq \tau \leq T$$

for some constant $C > 0$. Given now the $\mu > 0$, pick $\delta > 0$ so small such that $\frac{C}{c}\delta < \frac{\mu}{4}$ and $0 < \delta < -\frac{N\phi_1|_{\gamma_0}}{2}$ hold. We can now set $\mathcal{N}_0 := \mathcal{N}_1(\delta)$. Finally choose λ_0 sufficiently large. This finishes the proof of Theorem 3.2.36. \square

3.2.4 Some general theorems about the Gaussian beam limit of the wave equation

We can now make a much more detailed statement about the behaviour of solutions v of the wave equation in the Gaussian beam limit than Theorem 3.2.1 does:

Theorem 3.2.43. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let $\gamma : [0, S) \rightarrow M$ be an affinely parametrised future directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$. Finally, let N be a timelike, future directed vector field.*

For any neighbourhood \mathcal{N} of γ , for any $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$, and for any $\mu > 0$, there exists a solution $v \in C^\infty(M, \mathbb{C})$ of the wave equation (3.1.9) with

$E_0^N(v) = -g(N, \dot{\gamma})|_{\gamma(0)}$ such that

$$\left| E_{\tau, \mathcal{N} \cap \Sigma_\tau}^N(v) - [-g(N, \dot{\gamma})|_{\text{Im} \gamma \cap \Sigma_\tau}] \right| < \mu \quad \forall 0 \leq \tau \leq T \quad (3.2.44)$$

and⁷⁰

$$E_{\tau, \mathcal{N}^c \cap \Sigma_\tau}^N(v) < \mu \quad \forall 0 \leq \tau \leq T, \quad (3.2.45)$$

provided that we have on $R_{[0, T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$

$$\frac{1}{|n_{\Sigma_\tau}(t)|} \leq C, \quad g(N, N) \leq -c < 0, \quad -g(N, n_{\Sigma_\tau}) \leq C$$

and (3.2.46)

$$|\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})|, |\nabla N(n_{\Sigma_\tau}, e_i)|, |\nabla N(e_i, e_j)| \leq C \quad \text{for } 1 \leq i, j \leq 3,$$

where c and C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ is an orthonormal frame.

Moreover, by choosing \mathcal{N} , if necessary, a bit smaller, (3.2.44) holds independently of (3.2.46).

Proof. This follows easily from Theorem 3.2.1, Theorem 3.2.36, the second part of Remark 3.2.9 and the triangle inequality for the square root of the N -energy. \square

Let us again remark that the solution v of the wave equation in Theorem 3.2.43 can also be chosen to be real valued.

The next theorem is a direct consequence of Theorem 3.2.43 and can be used in particular, but not only for, proving upper bounds on the rate of the energy decay of waves on globally hyperbolic Lorentzian manifolds if we only allow the initial energy on the right hand side of the decay statement.

Theorem 3.2.47. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let \mathcal{T} be an open subset of M . Assume there is an affinely parametrised future directed null geodesic $\gamma : [0, S) \rightarrow M$ with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$, that is completely contained in \mathcal{T} . Let*

$$\tau^* := \sup \{ \hat{\tau} \in [0, \infty) \mid \text{Im}(\gamma) \cap \Sigma_\tau \neq \emptyset \text{ for all } 0 \leq \tau < \hat{\tau} \}.$$

Moreover, let N be a timelike, future directed vector field and $P : [0, \tau^*) \rightarrow (0, \infty)$.

If there is no constant $C > 0$ such that

$$-g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_\tau} \leq P(\tau)C$$

⁷⁰We denote the complement of \mathcal{N} in M with \mathcal{N}^c .

holds for all $0 \leq \tau < \tau^*$, then there exists no constant $C > 0$ such that

$$E_{\tau, \mathcal{T} \cap \Sigma_\tau}^N(u) \leq P(\tau) C E_0^N(u) \quad (3.2.48)$$

holds for all solutions u of the wave equation (3.1.9) for $0 \leq \tau < \tau^*$.

Proof. Assume the contrary, i.e., that there exists a constant $C_0 > 0$ such that (3.2.48) holds. There is then a $0 \leq \tau_0 < \tau^*$ with $-g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_{\tau_0}} > -g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_0} C_0 P(\tau_0)$. Choosing now $\mu > 0$ small enough and a neighbourhood $\mathcal{N} \subseteq \mathcal{T}$ of γ small enough such that (3.2.44) of Theorem 3.2.43 applies without reference to (3.2.46), we obtain a contradiction. \square

A very robust method for proving decay of solutions of the wave equation was given in [21] by Dafermos and Rodnianski (but also see [46]). This method requires in particular an integrated local energy decay (ILED) statement (possibly with loss of derivative), i.e., a statement of the form (3.2.50). The next theorem gives a sufficient criterion for an ILED statement having to lose regularity.

Theorem 3.2.49. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let \mathcal{T} be an open subset of M . Assume there is an affinely parametrised future directed null geodesic $\gamma : [0, S) \rightarrow M$ with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$, that is completely contained in \mathcal{T} . Let N be a timelike, future directed vector field and set*

$$\tau^* := \sup \{ \hat{\tau} \in [0, \infty) \mid \text{Im}(\gamma) \cap \Sigma_\tau \neq \emptyset \text{ for all } 0 \leq \tau < \hat{\tau} \} .$$

If

$$\int_0^{\tau^*} -g(N, \dot{\gamma})|_{\text{Im}(\gamma) \cap \Sigma_\tau} d\tau = \infty ,$$

where $\dot{\gamma}$ is with respect to some affine parametrization, then there exists no constant $C > 0$ such that

$$\int_0^{\tau^*} \int_{\Sigma_\tau \cap \mathcal{T}} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_{\bar{g}_\tau} d\tau \leq C E_0^N(u) \quad (3.2.50)$$

holds for all solutions u of the wave equation (3.1.9).

The proof of this theorem goes along the same lines as the one of Theorem 3.2.47. The reader might have noticed that whether an ILED statement of the form (3.2.50) exists or not depends heavily on the choice of the time function. On the other hand, it also depends heavily on the choice of the time function whether an ILED statement is helpful or not. So, for instance, we only have an estimate of the form

$$\int_{\mathcal{T} \cap R_{[0, \tau^*]}} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_g \leq C \cdot \int_0^{\tau^*} \int_{\Sigma_\tau \cap \mathcal{T}} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_{\bar{g}_\tau} d\tau ,$$

where $C > 0$, if the time function t is chosen such that $\frac{1}{|n_{\Sigma_\tau}(t)|} \leq C$ is satisfied for all $0 \leq \tau \leq \tau^*$. Such an estimate, together with an ILED statement, is very convenient whenever one needs to control spacetime integrals that are quadratic in the first derivatives of the field.

3.3 Part II: Applications to black hole spacetimes

In the following we give a selection of applications of Theorems 3.2.43, 3.2.47 and 3.2.49. A rich variety of behaviours of the energy is provided by black hole spacetimes arising in general relativity⁷¹. Although we will briefly introduce the Lorentzian manifolds that represent these black hole spacetimes, the reader completely unfamiliar with those is referred to [34] for a more detailed discussion, including the concept of a so called Penrose diagram and an introduction to general relativity.

We first restrict our considerations to the 2-parameter family of Reissner-Nordström black holes, which are exact solutions to the Einstein-Maxwell equations. The spherical symmetry of these spacetimes (and the accompanying simplicity of the metric) allows for an easy presentation without hiding any crucial details. In Section 3.3.2 we then discuss the Kerr family and show that analogous results hold.

3.3.1 Applications to Schwarzschild and Reissner-Nordström black holes

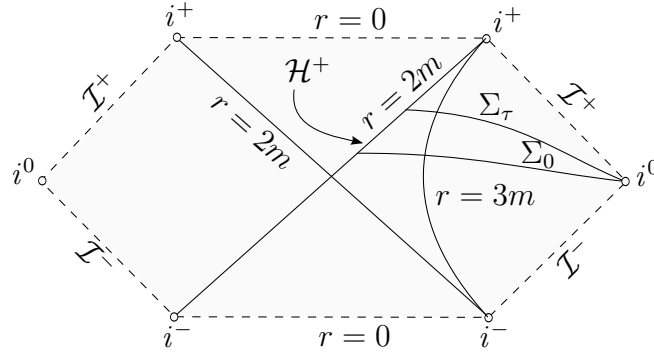
The 2-parameter family of Reissner-Nordström spacetimes is given by

$$g = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2, \quad (3.3.1)$$

initially defined on the manifold $M := \mathbb{R} \times (m + \sqrt{m^2 - e^2}, \infty) \times \mathbb{S}^2$, for which (t, r, θ, φ) are the standard coordinates. We restrict the real parameters m and e , which model the mass and the charge of the black hole, respectively, to the range $0 \leq e \leq m$, $m \neq 0$.

For $e = 0$ we obtain the 1-parameter Schwarzschild subfamily which solves the vacuum Einstein equations. The manifold M and the metric (3.3.1) can be analytically extended (such that they still solve the Einstein equations). The so called Penrose diagram of the maximal analytic extension of the Schwarzschild family is given below:

⁷¹Another physically interesting application would be for example to the study of waves in time dependent inhomogeneous media



The diamond shaped region to the right corresponds to the Lorentzian manifold (M, g) we started with; it represents the exterior of the black hole. The triangle to the top corresponds to the interior of the black hole, which is separated from the exterior by the so called event horizon, the line from the centre to the top-right i^+ . The remaining parts of the Penrose diagram play no role in the following discussion.

The black hole stability problem (see the introduction of [26]) motivates the study of the wave equation in the exterior of the black hole (the event horizon included). In accordance with our discussion in Section 3.1.2, we consider the framework of the energy method for the study of the wave equation. A suitable notion of energy for the black hole exterior is obtained via (3.1.8) through the foliation given by $\Sigma_\tau = \{t^* = \tau\}$ for $t^* \geq c > -\infty$, where $t^* = t + 2m \log(r - 2m)$, together with the timelike vector field $N := -(dt^*)^\sharp$.⁷²

Trapping at the photon sphere

There are null geodesics in the Schwarzschild spacetime that stay forever on the photon sphere at $\{r = 3m\}$. Indeed, one can check that the curve γ , given by

$$\gamma(s) = (s, 3m, \frac{\pi}{2}, (27m^2)^{-\frac{1}{2}}s)$$

⁷² We are intentionally quite vague about what we mean by ‘suitable notion of energy’. Instead of considering a foliation that ends at spacelike infinity ι^0 , it is sometimes desirable to work with a foliation that ends at future null infinity \mathcal{I}^+ . In a stationary spacetime, however, it is always convenient (and indeed ‘suitable’...) to work with a foliation and an energy measuring vector field N both of which are invariant under the flow of the Killing vector field. The obvious advantage is that the constants in Sobolev embeddings do not depend on the leaf - of course provided that higher energy norms are also defined accordingly. The precise choice of the timelike vector field N in a *compact* region of one leaf is completely irrelevant, since all the energy norms are equivalent in a compact region. In particular one can deduce that the following result about trapping at the photonsphere in Schwarzschild remains unchanged if we choose a different timelike vector field N which commutes with ∂_t and a different foliation by spacelike slices. In fact note that the behaviour of the energy of the null geodesic, $-g(N, \dot{\gamma})$, does not depend at all on the choice of the foliation!

in (t, r, θ, φ) coordinates is an affinely parametrised null geodesic, whose N -energy is given by $-g(N, \dot{\gamma}) = 1$. We now apply Theorem 3.2.47: The time oriented⁷³ globally hyperbolic Lorentzian manifold can be taken to be the domain of dependence $\mathcal{D}(\Sigma_0)$ of Σ_0 in (M, g) . Moreover, we choose the time function to be given by the restriction of t^* to $\mathcal{D}(\Sigma_0)$, and the vector field N and null geodesic $\gamma(s)$ in Theorem 3.2.47 are given by N and $\gamma(s - 2m \log(m))$ from above. Since $-g(N, \dot{\gamma}) = 1$ holds, Theorem 3.2.47 now states that given any open neighbourhood \mathcal{T} of $\text{Im}(\gamma)$ in $\mathcal{D}(\Sigma_0)$, there is no function $P : [0, \infty) \rightarrow (0, \infty)$ with $P(\tau) \rightarrow 0$ for $\tau \rightarrow \infty$ such that

$$E_{\tau, \mathcal{T} \cap \Sigma_\tau}^N(u) \leq P(\tau) E_0^N(u)$$

holds for all solutions u of the wave equation for all $\tau \geq 0$. It follows, that an LED statement for such a region can only hold if it loses differentiability. One can infer the analogous result about ILED statements from Theorem 3.2.49.

Let us mention here that γ has conjugate points. Indeed, the Jacobi field J with initial data $J(0) = 0$ and $D_s J(0) = \partial_\theta|_{\gamma(0)}$ vanishes in finite affine time $s > 0$: First note that the vector field

$$s \mapsto \partial_\theta|_{\gamma(s)}$$

along γ is parallel, i.e., $D_s \partial_\theta|_{\gamma(s)} = 0$. Moreover, a direct computation yields

$$R(\partial_\theta, \dot{\gamma}) \dot{\gamma}|_{\gamma(s)} = \frac{1}{27m^2} \partial_\theta|_{\gamma(s)},$$

where $R(\cdot, \cdot) \cdot$ is the Riemann curvature endomorphism. Thus, it follows that the vector field

$$J(s) = (27m^2)^{\frac{1}{2}} \sin((27m^2)^{-\frac{1}{2}} s) \cdot \partial_\theta|_{\gamma(s)}$$

satisfies the Jacobi equation $D_t^2 J + R(J, \dot{\gamma}) \dot{\gamma} = 0$. Moreover, it clearly satisfies the above initial conditions and vanishes in finite affine time.

It now follows from Theorem 3.C.1 that one cannot construct localised solutions to the wave equation along the trapped null geodesic γ by *just* using the geometric optics approximation, since caustics will form. In order to make the geometric optics approach work here, one would need to find a way to bridge these caustics.

That one can indeed prove an (I)LED statement with a loss of derivative was shown in [22] (see also [9]). In fact, it is sufficient to lose only an ε of a derivative, see [8] and also [26]. For a numerical study of the behaviour of a wave trapped at the photon sphere we refer the interested reader to [70].

Other, similar, examples are trapping at the photon sphere in higher dimensional Schwarzschild [64] or in Reissner-Nordström [2] and [8].

⁷³The time orientation is given by the timelike vector field N .

The red-shift effect at the event horizon - and its relevance for scattering constructions from the future

Another kind of behaviour of the energy is exhibited by the trapping occurring at the event horizon of the Schwarzschild spacetime. Recall that the event horizon \mathcal{H}^+ at $\{r = 2m\}$ is a null hypersurface, spanned by null geodesics. In $(t^*, r, \theta, \varphi)$ coordinates the affinely parametrised generators are given by

$$\gamma(s) = \left(\frac{1}{\kappa} \log(s), 2m, \theta_0, \varphi_0 \right),$$

where $\kappa = \frac{1}{4m}$ is the surface gravity, $s \in (0, \infty)$ and θ_0, φ_0 are constants. Thus, we have

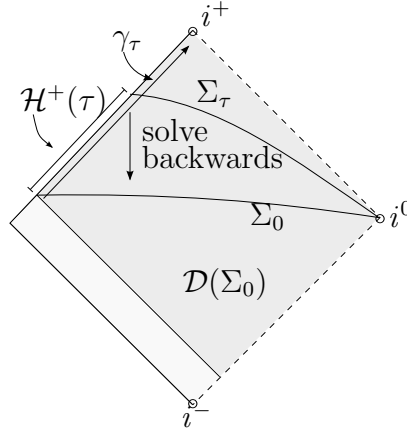
$$-(\dot{\gamma}(s), N) = \frac{1}{\kappa s} = \frac{1}{\kappa} e^{-\kappa t^*}, \quad (3.3.2)$$

i.e., the energy of the corresponding Gaussian beam decays exponentially - a direct manifestation of the celebrated red-shift effect. For more on the impact of the red-shift effect on the study of the wave equation on Schwarzschild we refer the reader to the original paper [22] by Dafermos and Rodnianski, but also see their [26].

Let us emphasise again that the null geodesics at the photon sphere as well as those at the horizon are trapped in the sense that they never escape to null infinity - but only those at the photon sphere form an obstruction for an LED statement without loss of differentiability to hold; the ‘trapped’ energy at the horizon decays exponentially. This is in stark contrast to the obstacle problem where every trapped light ray automatically leads to an obstruction for an LED statement without loss of derivatives to hold (see [59]). This new variety of how the ‘trapped’ energy behaves is due to the lack of a global timelike Killing vector field.

Let us now investigate the role played by the red-shift effect in scattering constructions from the future. While the red-shift effect is conducive to proving bounds on solutions to the wave equation in the ‘forward problem’, it turns into a blue-shift in the ‘backwards problem’⁷⁴; it amplifies energy near the horizon.

⁷⁴We call the initial value problem on Σ_0 to the future the ‘forward problem’, while solving a mixed characteristic initial value problem on $\mathcal{H}^+(\tau) \cup \Sigma_\tau$ to the past (or indeed a scattering construction from the future with data on \mathcal{H}^+ and \mathcal{I}^+) is called the ‘backwards problem’. Here, we have denoted the (closed) portion of the event horizon \mathcal{H}^+ that is cut out by Σ_0 and Σ_τ by $\mathcal{H}^+(\tau)$.



Proposition 3.3.3. *For every $\mu > 0$ and for every $\tau > 0$ there exists a smooth solution⁷⁵ $v \in C^\infty(\overline{\mathcal{D}(\Sigma_0)}, \mathbb{R})$ to the wave equation (3.1.9) with $E_\tau^N(v) = 1$ and $\int_{\mathcal{H}^+(\tau)} J^N(v) \lrcorner \text{vol}_g < \mu$, which satisfies $E_0^N(v) \geq e^{\kappa\tau} - \mu$, where $\kappa = \frac{1}{4m}$ is the surface gravity of the Schwarzschild black hole.*

Here, $J^N(v) \lrcorner \text{vol}_g$ denotes the 3-form obtained by inserting the vector field $J^N(v)$ into the first slot of vol_g . Let us also remark that μ should be thought of as a small positive number, while τ rather as a big one.

Proof. As in Section 3.3.1 we consider the Lorentzian manifold $\mathcal{D}(\Sigma_0)$ with time function t^* and timelike vector field N . Since geodesics depend smoothly on their initial data, it follows from (3.3.2) that we can find for every $\tau > 0$ an affinely parametrised radially outgoing null geodesic⁷⁶ γ_τ in $\mathcal{D}(\Sigma_0)$ with $|\langle \dot{\gamma}_\tau, N \rangle|_{\text{Im}(\gamma_\tau) \cap \Sigma_0} - e^{\kappa\tau}| < \frac{\mu}{2}$ and $-\langle \dot{\gamma}_\tau, N \rangle|_{\text{Im}(\gamma_\tau) \cap \Sigma_\tau} = 1$. We note that for our choice of time function and vector field N the condition (3.2.3) is satisfied, which does not only give us the energy estimate (3.2.8), but here also the refined version

$$\int_{\mathcal{H}^+(\tau)} J^N(u) \lrcorner \text{vol}_g + E_\tau^N(u) \leq C(\tau) (E_0^N(u) + \|\square u\|_{L^2(R_{[0, T]})}^2), \quad (3.3.4)$$

which holds in $\overline{\mathcal{D}(\Sigma_0)}$ for all $\tau > 0$ and for all $u \in C^\infty(\overline{\mathcal{D}(\Sigma_0)}, \mathbb{R})$. The estimate (3.3.4) is derived in the same way as (3.2.8), namely by an application of Stokes' theorem to $J^N(u) \lrcorner \text{vol}_g$, followed by Gronwall's inequality. The estimate (3.3.4) gives in addition to (3.2.2) in Theorem 3.2.1 the estimate

$$\int_{\mathcal{H}^+(\tau)} J^N(v - \tilde{u}) \lrcorner \text{vol}_g < \mu, \quad (3.3.5)$$

⁷⁵We denote with $\overline{\mathcal{D}(\Sigma_0)}$ the closure of $\mathcal{D}(\Sigma_0)$ in the maximal analytic extension of Schwarzschild, see the Penrose diagram on page 68.

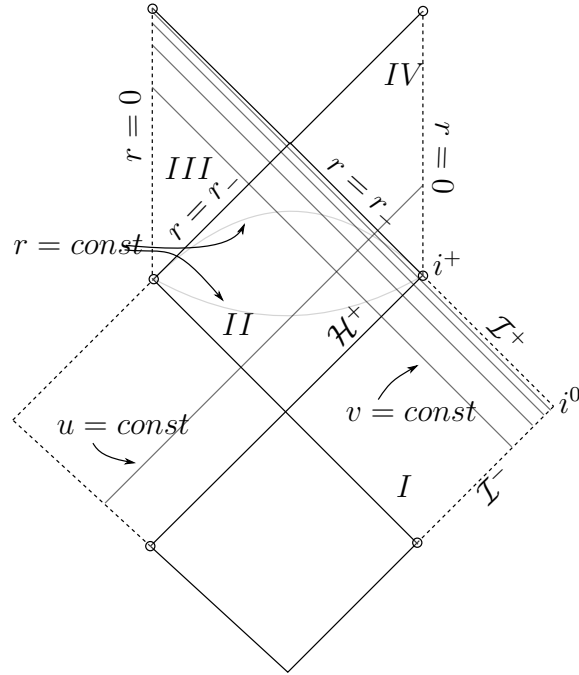
⁷⁶Radially outgoing null geodesics are the lines parallel to, and to the right of, \mathcal{H}^+ in the Penrose diagram. In (u, r, θ, φ) coordinates, where $u(t, r, \theta, \varphi) := t - 2m \log(r - 2m) - r$, these null geodesics are tangent to $\frac{\partial}{\partial r}$.

where \tilde{u} is the Gaussian beam and v is the actual solution, as in Theorem 3.2.1. We now apply Theorem 3.2.43, where the Lorentzian manifold is given by $\mathcal{D}(\Sigma_0)$, the time function by t^* , the timelike vector field by N and for given $\tau > 0$, the affinely parametrised null geodesic is taken to be γ_τ from above. For our purposes we can choose any neighbourhood \mathcal{N} of $\text{Im}(\gamma_\tau)$ in $\mathcal{D}(\Sigma_0)$. Theorem 3.2.43 then ensures the existence of a solution $v \in C^\infty(\mathcal{D}(\Sigma_0), \mathbb{R})$ to the wave equation with $E_0^N(v) \geq e^{\kappa\tau} - \mu$ and $E_\tau^N(v) = 1$ – possibly after renormalising the energy at time τ of v to be exactly 1. It is not difficult to show, for example by considering the Cauchy problem for a slightly larger globally hyperbolic Lorentzian manifold which contains the event horizon, that v can be chosen to extend smoothly to the event horizon. We then obtain $\int_{\mathcal{H}^+(\tau)} J^N(v) \lrcorner \text{vol}_g < \mu$ from (3.3.5), since recall that the Gaussian beam \tilde{u} in Theorem 3.2.1 is supported in \mathcal{N} , which is disjoint from \mathcal{H}^+ . This finishes the proof. \square

The above proposition shows that for every $\tau > 0$ one can prescribe initial data for the mixed characteristic initial value problem on $\mathcal{H}^+ \cup \Sigma_\tau$ such that the total initial energy is equal to one, while the energy of the solution obtained by solving backwards grows exponentially to $\approx e^{\kappa\tau}$ on Σ_0 . In [19], Dafermos, Holzegel and Rodnianski approach the scattering problem from the future for the Einstein equations (with initial data prescribed on \mathcal{H}^+ and \mathcal{I}^+) by considering it as the limit of finite backwards problems, which – for the wave equation – are qualitatively the same as the backwards problem with initial data on $H^+(\tau)$ and Σ_τ . In order to take the limit of the finite problems, uniform control over the solutions is required: Dafermos et al. use a backwards energy estimate which bounds the energy on Σ_0 by the initial energy on \mathcal{H}^+ and Σ_τ , multiplied by $C \cdot \exp(c\tau)$, where c and C are constants that are independent of τ . Proposition 3.3.3 shows now that this estimate is sharp in the sense that one cannot avoid exponential growth (at least not as long as one does not sacrifice regularity in the estimate). In particular, working with this estimate enforces the assumption of exponential decay on the scattering data in [19].

The blue-shift near the Cauchy horizon of a sub-extremal Reissner-Nordström black hole

We now move on to the sub-extremal Reissner-Nordström black hole, i.e., to the parameter range $0 < e < m$ in (3.3.1). More precisely, we consider again its maximal analytic extension. Part of the Penrose diagram is given below:



Again, the diamond-shaped region I represents the black hole exterior and corresponds to the Lorentzian manifold on which the metric g from (3.3.1) was initially defined. The regions II , III and IV represent the black hole interior. Recall that Reissner-Nordström is a spherically symmetric spacetime. The ‘radius’ of the spheres of symmetry is given by a globally defined function r . We write $D(r) := 1 - \frac{2m}{r} + \frac{e^2}{r^2}$ and denote the two roots of D with $r_{\pm} = m \pm \sqrt{m^2 - e^2}$. The future Cauchy horizon⁷⁷ is given by $r = r_-$. The coordinate functions (θ, φ) parametrise the spheres of symmetry in the usual way and are globally defined up to one meridian. Regions $I - III$ are covered by a (v, r, θ, φ) coordinate chart, where in the region I , the function v is given by $v = t + r_I^*$, where r_I^* is a function of r , satisfying $\frac{dr_I^*}{dr} = \frac{1}{D}$. With respect to these coordinates, the Lorentzian metric takes the form

$$g = -D dv^2 + dv \otimes dr + dr \otimes dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2 .$$

Introducing a function r_{II}^* in region II , which satisfies $\frac{dr_{II}^*}{dr} = \frac{1}{D}$ in this region, and defining a function⁷⁸ $t := v - r_{II}^*$, we obtain a (t, r, θ, φ) coordinate system for region II in which the metric g is again given by the algebraic expression (3.3.1). The regions II and IV are covered by a coordinate system (u, r, θ, φ) , where the function u is given in region II by $u = t - r_{II}^*$.

⁷⁷We consider a Cauchy surface Σ_0 of the big diamond shaped region as shown in the next diagram, i.e., a Cauchy surface of the region pictured in the above diagram without the regions III and IV ,.

⁷⁸One could also assign the functions t an index, specifying in which region they are defined. Note that these different functions t do *not* patch together to give a globally defined smooth function!

Having laid down the coordinate functions we work with, we now investigate the family of affinely parametrised ingoing null geodesics, given in (v, r, θ, φ) coordinates by

$$\gamma_{v_0}(s) = (v_0, -s, \theta_0, \varphi_0) ,$$

where $s \in (-\infty, 0)$ and we keep θ_0, φ_0 fixed. Clearly, we have⁷⁹ $\dot{\gamma}_{v_0} = -\frac{\partial}{\partial r}\Big|_v$. We are interested in the energy of these null geodesics in region *II* close to i^+ (in the topology of the Penrose diagram), i.e. close to the Cauchy horizon separating region *II* from region *IV*. A suitable notion of energy is given by a regular vector field that is future directed timelike in a neighbourhood of i^+ . In order to construct such a vector field, we consider (u, v, θ, φ) coordinates in region *II*. A straightforward computation shows that

$$\begin{aligned} N &:= -\frac{1}{r_+ - r} \frac{\partial}{\partial u}\Big|_v + \frac{1}{r - r_-} \frac{\partial}{\partial v}\Big|_u \\ &= -\frac{1}{r_+ - r} \frac{\partial}{\partial u}\Big|_r - \frac{1}{2r^2}(r_+ - r_-) \frac{\partial}{\partial r}\Big|_u \\ &= \frac{r_- - r_+}{2r^2} \frac{\partial}{\partial r}\Big|_v + \frac{1}{r - r_-} \frac{\partial}{\partial v}\Big|_r \end{aligned}$$

is future directed timelike in a neighbourhood of i^+ intersected with region *II* and can be extended to a smooth timelike vector field defined on a neighbourhood of i^+ . We obtain

$$-(N, \dot{\gamma}_{v_0}) = \frac{1}{r - r_-} , \quad (3.3.6)$$

the N -energy of the null geodesics γ_{v_0} gets infinitely blue-shifted near the Cauchy horizon.

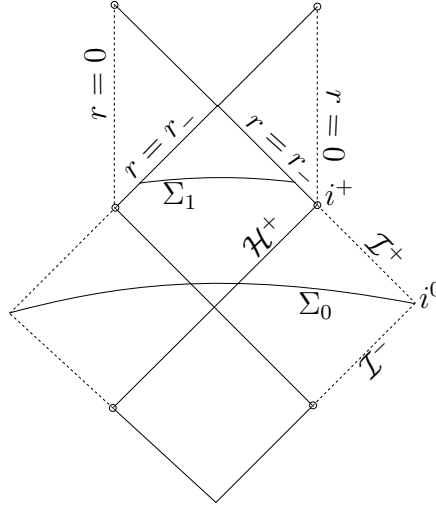
For later reference let us note that the rate, in advanced time v , with which the N -energy (3.3.6) of γ_{v_0} blows up along a hypersurface of constant u , is exponential. This is seen as follows: One has

$$r_{II}^*(r) = r + \frac{1}{2\kappa_+} \log(r_+ - r) + \frac{1}{2\kappa_-} \log(r - r_-) + const ,$$

where $\kappa_{\pm} = \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2}$ are the surface gravities of the event and the Cauchy horizon, respectively. Thus, for large r_{II}^* one has $\frac{1}{r - r_-}(r_{II}^*) \sim e^{-2\kappa_- r_{II}^*}$. Finally, along $\{u = u_0 = const\}$, we have $r_{II}^*(v) = \frac{1}{2}(v - u_0)$. It thus follows that the N -energy (3.3.6) of γ_{v_0} blows up like $e^{-\kappa_- v}$ along a hypersurface of constant u .

Let us now consider spacelike slices Σ_0 and Σ_1 as in the diagram below, where Σ_0 asymptotes to a hypersurface of constant t and Σ_1 is extendible as a smooth spacelike slice into the neighbouring regions.

⁷⁹Let us denote with a subscript on the partial derivative which other coordinate (apart from θ and φ) remains fixed.



Since the normal n_{Σ_1} of Σ_1 is also regular at the Cauchy horizon, it follows from (3.3.6) that the n_{Σ_1} -energy of the null geodesics γ_{v_0} blows up along Σ_1 when approaching the Cauchy horizon. Moreover, note that the n_{Σ_0} -energy of the geodesics γ_{v_0} along Σ_0 is uniformly bounded as $v_0 \rightarrow \infty$. We now apply Theorem 3.2.43 to the family of the null geodesics γ_{v_0} with the following further input: the Lorentzian manifold is given by the domain of dependence $\mathcal{D}(\Sigma_0)$ of Σ_0 , the time function is such that Σ_0 and Σ_1 are level sets, N is a timelike vector field that extends n_{Σ_0} and n_{Σ_1} , and finally \mathcal{N} is a small enough neighbourhoods of γ_{v_0} . This yields

Theorem 3.3.7. *Let Σ_0 and Σ_1 be spacelike slices in the sub-extremal Reissner-Nordström spacetime as indicated in the diagram below. Then there exists a sequence $\{u_i\}_{i \in \mathbb{N}}$ of solutions to the wave equation with initial energy $E_0^{n_{\Sigma_0}}(u_i) = 1$ on Σ_0 such that the n_{Σ_1} -energy on Σ_1 goes to infinity, i.e., $E_1^{n_{\Sigma_1}}(u_i) \rightarrow \infty$ for $i \rightarrow \infty$.*

In particular we can infer from Theorem 3.3.7 that there is no uniform energy boundedness statement – i.e., there is no constant $C > 0$ such that

$$\int_{\Sigma_1} J^{n_{\Sigma_1}}(u) \cdot n_{\Sigma_1} \leq C \int_{\Sigma_0} J^{n_{\Sigma_0}}(u) \cdot n_{\Sigma_0}, \quad (3.3.8)$$

holds for all solutions u of the wave equation.

Let us remark here that the non-existence of a uniform energy boundedness statement has in particular the following consequence: one cannot choose a time function for the region bounded by Σ_0 and Σ_1 for which these hypersurfaces are level sets and, moreover, extend the normals of Σ_0 and Σ_1 to a smooth timelike vector field N in such a way that an energy estimate of the form (3.2.8) holds. In particular this emphasises the importance of the condition (3.2.3) for the *global* approximation scheme on general Lorentzian manifolds and points out the necessity of a *local* understanding of the approximate solution provided by Theorem 3.2.36 and 3.2.43.

We would also like to bring to the reader's attention that one actually expects that there is no energy boundedness statement at all, no matter how many derivatives one

loses or whether one restricts the support of the initial data:

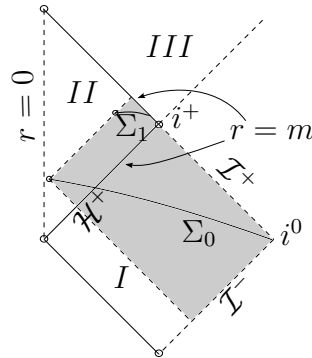
Conjecture 3.3.9. *For generic compactly supported smooth initial data on Σ_0 , the n_{Σ_1} -energy along Σ_1 of the corresponding solution to the wave equation is infinite.*

Let us remark here that the analysis carried out in [18] by Dafermos shows in particular that proving the above conjecture can be reduced to proving a lower bound on the decay rate of the spherical mean of the generic solution (as in Conjecture 3.3.9) on the horizon.

Before we elaborate in Section 3.3.1 on the mechanism that leads to the blow-up of the energy near the Cauchy horizon in Theorem 3.3.7, let us investigate the situation for *extremal* Reissner-Nordström black holes.

The blue-shift near the Cauchy horizon of an extremal Reissner-Nordström black hole

The extremal Reissner-Nordström black hole is given by the choice $m = e$ of the parameters in (3.3.1). We again consider the maximal analytic extension of the initially defined spacetime. Part of the Penrose diagram is given below:



The region I represents again the black hole exterior and corresponds to the Lorentzian manifold on which the metric g from (3.3.1) was initially defined. The black hole interior extends over the regions II and III . The discussion of the functions r, θ and φ carries over from the sub-extremal case. However, in the extremal case, $D(r)$ has a double zero at $r = m$, the value of the radius of the spheres of symmetry on the event, as well as on the Cauchy horizon. The regions I and II can be covered by ‘ingoing’ null coordinates (v, r, θ, φ) , where the function v is given in region I by $v = t + r_I^*$, where again $r_I^*(r)$ satisfies $\frac{dr_I^*}{dr} = \frac{1}{D}$. In the same way as in the sub-extremal case one introduces r_{II}^* and defines a (t, r, θ, φ) coordinate system for the region II . Finally, the regions II and III are covered by ‘outgoing’ null coordinates (u, r, θ, φ) , where we have $u = t - r_{II}^*$ in region II .

In ingoing null coordinates, the affinely parametrised radially ingoing null geodesics are given by $\gamma_{v_0}(s) = (v_0, -s, \theta_0, \varphi_0)$, where $s \in (-\infty, 0)$. Expressing the tangent

vector of γ_{v_0} in region II in outgoing coordinates, we obtain

$$\dot{\gamma}_{v_0} = -\frac{\partial}{\partial r}\Big|_v = \frac{2}{D} \frac{\partial}{\partial u}\Big|_r - \frac{\partial}{\partial r}\Big|_u, \quad (3.3.10)$$

which blows up at $r = m$. Thus, we have for any future directed timelike vector field N in region II which extends to a regular timelike vector field to region III , that the N -energy $-g(\dot{\gamma}_{v_0}, N)$ of γ_{v_0} blows up along the hypersurface Σ_1 for $v_0 \rightarrow \infty$. Choosing now a spacelike slice Σ_0 as in the above diagram, again asymptoting to a $\{t = \text{const}\}$ hypersurface at i^0 , and restricting consideration to its domain of dependence, we obtain a globally hyperbolic spacetime (the shaded region) with respect to which we can apply Theorem 3.2.43, inferring the analogon of Theorem 3.3.7 for extremal Reissner-Nordström black holes.

For the discussion in the next section, we again investigate the rate, in advanced time v , with which the N -energy $-g(\dot{\gamma}_{v_0}, N)$ blows up along a hypersurface of constant u : Here, we have

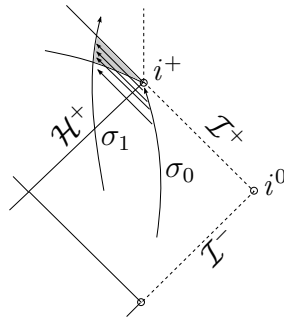
$$r_{II}^*(r) = r + m \log((r - m)^2) - \frac{m^2}{(r - m)} + \text{const}.$$

It follows that for large r_{II}^* one has $\frac{1}{D}(r_{II}^*) \sim (r_{II}^*)^2$. Moreover, along $\{u = u_0 = \text{const}\}$, we have $r_{II}^*(v) = \frac{1}{2}(v - u_0)$, from which it follows that the N -energy $-g(\dot{\gamma}_{v_0}, N)$ of the family of null geodesics γ_{v_0} blows up like v^2 .

The strong and the weak blue-shift – and their relevance for strong cosmic censorship

In the example of sub-extremal Reissner-Nordström as well as in the example of extremal Reissner-Nordström, the energy of the Gaussian beams is blue-shifted near the Cauchy horizon. Although not important for the proof of the *qualitative* result of Proposition 3.3.7 (and the analogous statement for the extremal case), the difference in the *quantitative* blow-up rate of the energy in the two cases is conspicuous.

Let us first recall the familiar heuristic picture that explains the basic mechanism responsible for the blue-shift effect in both cases⁸⁰:



⁸⁰Below, we give the picture for the sub-extremal case. However, the picture and the heuristics for the extremal case are exactly the same!

The observer σ_0 travels along a timelike curve of infinite proper time to i^+ and, in regular time intervals, sends signals of the same energy into the black hole. These signals are received by the observer σ_1 , who travels into the black hole and crosses the Cauchy horizon, within *finite* proper time - which leads to an infinite blue shift. This mechanism was first pointed out by Roger Penrose, see [55], page 222.⁸¹ Although the picture, along with its heuristics, allow for inferring the *presence* of a blue-shift near the Cauchy horizon, they do not reveal the *strength* of the blue-shift. For investigating the latter, it is important to note that the region in spacetime, which actually causes the blue shift, is a neighbourhood of the Cauchy horizon. This neighbourhood is not well-defined, however, one could think of it as being given by a neighbourhood of constant r - the shaded region in the diagram of sub-extremal Reissner-Nordström above. The crucial difference between the sub-extremal and the extremal case is that in the extremal case, the blue-shift degenerates *at the Cauchy horizon itself*, while in the sub-extremal case, it does not: the sub-extremal Cauchy horizon continues to blue-shift radiation. In particular, one can prove an analogous result to Proposition 3.3.3 there - but for the forward problem.

This degeneration of the blue-shift towards the Cauchy horizon in the extremal case leads to the (total) blue-shift being weaker than the blue-shift in the sub-extremal case. Thus, the geometry of spacetime near the Cauchy horizon is crucial for understanding the strength of the blue-shift effect, and hence the blow-up rate of the energy.

We now continue with a heuristic discussion of the importance of the different blow-up rates. The reader might have noticed that we only made Conjecture 3.3.9 for the sub-extremal case; and indeed, the analogous conjecture for the extremal case is expected to be false: While in our construction we consider a family of ingoing wave packets whose energy along a fixed outgoing null ray to \mathcal{I}^+ does *not* decay in advanced time v , the scattered ‘ingoing energy’ of a wave with initial data as in Conjecture 3.3.9 will decay in advanced time v along such an outgoing null ray. Thus, the blow-up of the energy near the Cauchy horizon can be counteracted by the decay of the energy of the wave towards null infinity. In the extremal case, the blow-up rate is v^2 , which does not dominate the decay rate of the energy towards null infinity; the exponential

⁸¹There, he describes the above scenario in the following, more dramatic language (he considers the scenario of gravitational collapse, where the Einstein equations are coupled to some matter model and denotes the Cauchy horizon with $H_+(\mathcal{H})$):

There is a further difficulty confronting our observer who tries to cross $H_+(\mathcal{H})$. As he looks out at the universe that he is “leaving behind,” he sees, in one final flash, as he crosses $H_+(\mathcal{H})$, the entire later history of the rest of his “old universe”. [...] If, for example, an unlimited amount of matter eventually falls into the star then presumably he will be confronted with an infinite density of matter along “ $H_+(\mathcal{H})$.” Even if only a finite amount of matter falls in, it may not be possible, in generic situations to avoid a curvature singularity in place of $H_+(\mathcal{H})$.

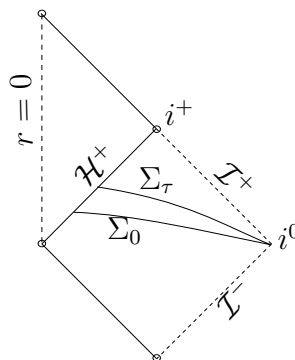
blow-up rate $e^{-\kappa-v}$, however, does. These are the heuristic reasons for only formulating Conjecture 3.3.9 for the sub-extremal case. We conclude with a couple of remarks: Firstly, one should actually compare the decay rate of the ingoing energy not along an outgoing null ray to \mathcal{I}^+ , but along the event horizon - or even better, along a spacelike slice in the interior of the black hole approaching i^+ in the topology of the Penrose diagram. Secondly, we would like to repeat and stress the point made, namely that the heuristics given in the very beginning of this section, and which solely ensure the *presence* of a blue-shift, are not sufficient to cause a C^1 instability of the wave at the Cauchy horizon. For this to happen, the local geometry of the Cauchy horizon is crucial. Finally, let us conjecture, based on the fact that in the extremal case one gains powers of v in the blow-up rate at the Cauchy horizon when considering higher order energies, that there is some natural number $k > 1$ such that waves with initial data as in Conjecture 3.3.9 exhibit a C^k instability at the Cauchy horizon.

We conclude this section with recalling that the study of the wave equation on black hole backgrounds serves as a source of intuition for the behaviour of gravitational perturbations of these spacetimes. Thus, the following expected picture emerges: Consider a *generic* dynamical spacetime which at late times approaches a sub-extremal Reissner-Nordström black hole. Then the Cauchy horizon is replaced by a weak null curvature singularity (for this notion see [18]).

If we restrict consideration to the class of dynamical spacetimes which at late times approach an extremal Reissner-Nordström black hole, then the *generic* spacetime *within this class* has a more regular Cauchy horizon, which in particular is not seen as a singularity from the point of view of the low regularity well-posedness theory for the Einstein equations, see the resolution [40] of the L^2 -curvature conjecture. This picture is also supported by the recent numerical work [51].

Trapping at the horizon of an extremal Reissner-Nordström black hole

We again consider the extremal Reissner-Nordström black hole, this time together with a foliation of the exterior region by spacelike slices given by $\Sigma_\tau = \{t^* = \tau\}$, where $t^* = v - r$ and v as in the previous section.



In [2] and [3] Aretakis investigated the behaviour of waves on this spacetime and obtained stability (i.e., boundedness and decay results) as well as instability results (blow-up of certain higher order derivatives along the horizon); for further developments see also [42]. The instability results originate from a conservation law on the extremal horizon once decay results for the wave are established. In order to obtain these stability results Aretakis followed the new method introduced by Dafermos and Rodnianski in [21].⁸² The first important step is to prove an ILED statement. As in the Schwarzschild spacetime we have trapping at the photon sphere (here at $\{r = 2m\}$), and as shown before, an ILED statement has to degenerate there in order to hold. The fundamentally new difficulty in the extremal setting arises from the degeneration of the red-shift effect at the horizon \mathcal{H}^+ , which was needed for proving an ILED statement that holds up to the horizon (see for example [26]). And indeed, the energy of the generators of the horizon is no longer decaying: In $(t^*, r, \theta, \varphi)$ coordinates, the affinely parametrised generators are given by

$$\gamma(s) = (s, m, \theta_0, \varphi_0) ,$$

where $s \in (-\infty, \infty)$ and again θ_0, φ_0 are fixed. A good choice of energy along the foliation $\bigcup_{\tau \geq 0} \Sigma_\tau$ is given by $N = -(dt^*)^\sharp$ and thus we see that the energy is constant: $-(N, \dot{\gamma}) = 1$.

If we consider a globally hyperbolic subset of the depicted part of extremal Reissner-Nordström that contains the horizon \mathcal{H}^+ , for example by extending Σ_0 a bit through the event horizon and then considering its domain of dependence, we can directly infer from Theorem 3.2.47 and 3.2.49, by applying it to the null geodesic γ from above, that every (I)LED statement which concerns a neighbourhood of the horizon, necessarily has to lose differentiability. However, we can also infer the same result for the wave equation on the Lorentzian manifold $\mathcal{D}(\Sigma_0)$, where ‘a neighbourhood of the horizon’ is ‘a neighbourhood of the horizon in the previous, bigger spacetime, intersected with $\mathcal{D}(\Sigma_0)$ ’: Analogous to the proof of Proposition 3.3.3, we consider a sequence of radially outgoing null geodesics in $\mathcal{D}(\Sigma_0)$ whose initial data on Σ_0 converges to the data of γ from above. For every ‘neighbourhood of the horizon’, for every $\tau_0 > 0$ and for every (small) $\mu > 0$ there is then an element γ_0 of the sequence such that $-(N, \dot{\gamma}_0)|_{\text{Im}(\gamma_0) \cap \Sigma_\tau} \in (1 - \mu, 1 + \mu)$ for all $0 \leq \tau \leq \tau_0$. This follows again from the smooth dependence of geodesics on their initial data. We now apply Theorem 3.2.43 to this sequence of null geodesics to infer that for every ‘neighbourhood of the horizon’ and for every $\tau_0 > 0$ we can construct a solution to the wave equation whose energy in this neighbourhood is, say, bigger than $\frac{1}{2}$ for all times τ with $0 \leq \tau \leq \tau_0$. This proves again that there is no non-degenerate (I)LED statement concerning ‘a neighbourhood of the horizon’ in $\mathcal{D}(\Sigma_0)$; the trapping at the event horizon obstructs local energy

⁸²Though in addition he had to work with a degenerate energy, which makes things more complicated.

decay - which is in stark contrast to sub-extremal black holes.

One should ask now whether an ILED statement with loss of derivative can actually hold. To answer this question, at least partially, it is helpful to decompose the angular part of the wave into spherical harmonics. In [2] Aretakis proved indeed an (I)LED statement with loss of one derivative for waves that are supported on the angular frequencies $l \geq 1$. By constructing a localised solution with vanishing spherical mean we can show that this result is optimal in the sense that *some* loss of derivative is again necessary. This can be done for instance by considering the superposition of two Gaussian beams that follow the generators $\gamma_1(s) = (s, m, \frac{\pi}{2}, \frac{\pi}{2})$ and $\gamma_2(s) = (s, m, \frac{\pi}{2}, \frac{3\pi}{2})$, where the initial value of beam one is exactly the negative of the initial value of beam two if translated in the φ variable by π .⁸³ The question whether one can prove an ILED statement with loss of derivative in the case $l = 0$ is still open, though it is expected that the answer is negative. In order to obtain stability results for waves supported on all angular frequencies Aretakis had to use the degenerate energy (of course these results are weaker than results one would obtain if an ILED statement for the case $l = 0$ actually held).

3.3.2 Applications to Kerr black holes

The Kerr family is a 2-parameter family of solutions to the vacuum Einstein equations. Let us fix the manifold $M := \mathbb{R} \times (m + \sqrt{m^2 - a^2}, \infty) \times \mathbb{S}^2$, where m and a are real parameters that will model the mass and the angular momentum per unit mass of the black hole, respectively, and which are restricted to the range $0 \leq a \leq m$, $0 \neq m$. Let (t, r, θ, φ) denote the standard coordinates on the manifold M and define functions

$$\begin{aligned} \rho^2 &:= r^2 + a^2 \cos^2 \theta \\ \Delta &:= r^2 - 2mr + a^2 \\ g_{tt} &:= -1 + \frac{2mr}{\rho^2} \\ g_{t\varphi} &:= -\frac{2mra \sin^2 \theta}{\rho^2} \\ g_{\varphi\varphi} &:= \left(r^2 + a^2 + \frac{2mra^2 \sin^2 \theta}{\rho^2} \right) \sin^2 \theta . \end{aligned}$$

⁸³Let us mention here that in this particular situation the approximation using geometric optics is easier. Indeed, one can easily write down a solution of the eikonal equation such that the characteristics are the outgoing null geodesics. First one has to prove then the analogue of Theorem 3.2.36, which is easier since the approximate conservation law we used in the case of Gaussian beams is replaced by an exact conservation law for the geometric optics approximation, cf. footnote 69. But then we can easily contradict the validity of (I)LED statements for any angular frequency: working in $(t^*, r, \theta, \varphi)$ coordinates, we choose the initial value of the function a (see Appendix 3.A) to have the angular dependence of a certain spherical harmonic and the radial dependence corresponds to a smooth cut-off, i.e., a initially is only non-vanishing for $r \in [m, m + \varepsilon)$.

The metric on M is then defined by

$$g = g_{tt} dt^2 - g_{t\varphi} (d\varphi \otimes dt + dt \otimes d\varphi) + g_{\varphi\varphi} d\varphi^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 .$$

As for the Reissner-Nordström family, one can (and should) extend these spacetimes in order to understand their physical interpretation as a black hole. For details, we refer the reader again to [34]. Fixing the θ coordinate to be $\frac{\pi}{2}$ and moding out the \mathbb{S}^1 corresponding to the φ coordinate, we again obtain pictorial representations of these spacetimes. For the sub-extremal case $0 < a < m$, the diagram is the same as the one depicted in Section 3.3.1, while in the extremal case $a = m$, one obtains the same diagram as in Section 3.3.1.

Trapping in (sub)-extremal Kerr

As in the case of the Schwarzschild spacetime there are trapped null geodesics in the domain of outer communications of the Kerr spacetime whose energy stays bounded away from zero and infinity if the energy measuring vector field N is sensibly chosen. In the case of $a > 0$, however, the set that accomodates trapped null geodesics is the closure of an *open* set in spacetime, which is in contrast to the 3-dimensional photonsphere in Schwarzschild and Reissner-Nordström. Before we explain in some more detail how to find the trapped geodesics, we set up a suitable choice of foliation and energy measuring vector field:

For (sub)-extremal Kerr we foliate the domain of outer communication (which is covered by the above (t, r, θ, φ) coordinates) in the same way as we did before for the Schwarzschild and the extremal Reissner-Nordström spacetimes, namely by first introducing an ingoing ‘null’ coordinate v and then subtracting off r to get a good time coordinate t^* . Slightly more general than needed at this point, let us define

$$v_+ := t + r^* \quad \text{and} \quad \varphi_+ := \varphi + \bar{r} ,$$

where r^* is defined up to a constant by $\frac{dr^*}{dr} = \frac{r^2+a^2}{\Delta}$ and \bar{r} is defined up to a constant by $\frac{d\bar{r}}{dr} = \frac{a}{\Delta}$. The set of functions $(v_+, r, \theta, \varphi_+)$ form ingoing ‘null’ coordinates (v_+ is here the ‘null’ coordinate, however, it does *not* satisfy the eikonal equation $d\phi \cdot d\phi = 0$), they cover the regions *I*, *II* and *III* in the spacetime diagram for sub-extremal Kerr⁸⁴ and the metric takes the form

$$g = g_{tt} dv_+^2 + g_{t\varphi} (dv_+ \otimes d\varphi_+ + d\varphi_+ \otimes dv_+) + (dv_+ \otimes dr + dr \otimes dv_+) - a \sin^2 \theta (dr \otimes d\varphi_+ + d\varphi_+ \otimes dr) + g_{\varphi\varphi} d\varphi_+^2 + \rho^2 d\theta^2 .$$

Finally, we define $t^* := v_+ - r$. That this is indeed a good time coordinate is easily seen

⁸⁴In the extremal case they cover all of the in Section 3.3.1 depicted spacetime diagram.

from writing the metric in $(t^*, r, \theta, \varphi_+)$ coordinates and restricting it to $\{t^* = \text{const}\}$ slices: One obtains

$$\bar{g} = (g_{tt} + 2) dr^2 + (g_{t\varphi} - a \sin^2 \theta) (d\varphi_+ \otimes dr + dr \otimes d\varphi_+) + \rho^2 d\theta^2 + g_{\varphi\varphi} d\varphi_+^2,$$

and the (θ, θ) minor of this matrix is found to be $2mr \sin^2 \theta + (r^2 + a^2) \sin^2 \theta - a^2 \sin^4 \theta$, which is positive away from the well understood coordinate singularity $\theta = \{0, \frac{\pi}{2}\}$. Hence, the slices $\Sigma_\tau := \{t^* = \tau\}$ are spacelike and it is easily seen that they asymptote to $\{t = \text{const}\}$ slices near spacelike infinity and end on the *future* event horizon. A suitable timelike vector field N for measuring the energy is again given by $N := -(dt^*)^\sharp$.

Let us now give a brief sketch of how one finds the trapped null geodesics. A detailed discussion of the geodesic flow on Kerr is found for example in [53] or [10]. The key insight is that the geodesic flow on Kerr separates. A *null* geodesic $\gamma(s) = ((t(s), r(s), \theta(s), \varphi(s)))$ satisfies the following first order equations:

$$\rho^2 \dot{t} = a\mathbb{D} + (r^2 + a^2) \frac{\mathbb{P}}{\Delta} \quad (3.3.11)$$

$$\rho^4 (\dot{r})^2 = R(r) := -\mathcal{K}\Delta + \mathbb{P}^2 \quad (3.3.12)$$

$$\rho^4 (\dot{\theta})^2 = \Theta(\theta) := \mathcal{K} - \frac{\mathbb{D}^2}{\sin^2 \theta} \quad (3.3.13)$$

$$\rho^2 \dot{\varphi} = \frac{\mathbb{D}}{\sin^2 \theta} + \frac{a\mathbb{P}}{\Delta},$$

where \mathcal{K} is the Carter constant of the geodesic, $\mathbb{P}(r) = (r^2 + a^2)E - La$ and $\mathbb{D}(\theta) = L - Ea \sin^2 \theta$. Here, $E = -g(\partial_t, \dot{\gamma})$ is the energy of the geodesic⁸⁵ and $L = g(\partial_\varphi, \dot{\gamma})$ is the angular momentum.

Clearly, for finding the *trapped* null geodesics, investigating equation (3.3.12) is most important. The crucial observation is that a simple zero of $R(r)$ corresponds to a turning point (in the r -coordinate) of the geodesic, while a double zero corresponds to an orbit of constant r . Since we can infer from equation (3.3.13) that $\mathcal{K} \geq 0$, $R(r)$ is positive in (r_-, r_+) , where r_- and r_+ denote the roots of Δ . It follows that $R(r)$ must have an even number of roots (counted with multiplicity) in (r_+, ∞) . However, we can exclude $R(r)$ having four roots in (r_+, ∞) , since the sum of the roots has to yield zero. Thus $R(r)$ has either zero or two roots in (r_+, ∞) . If the constants of motions (these are E, L and \mathcal{K}) are such that $R(r)$ has zero roots in (r_+, ∞) , the null geodesic is clearly not trapped. If $R(r)$ has two distinct roots in this interval, it follows that $R(r)$ is negative in the region bounded by the two roots, thus, again the null geodesic is not trapped. It remains to investigate the case of $R(r)$ having a double zero in (r_+, ∞) , which potentially correspond to orbits of constant r .

⁸⁵Note that ∂_t is not timelike everywhere! However, one still calls this quantity the ‘energy’ of the null geodesic.

Without loss of generality we can assume that $E = 1$. One then solves $R(r, L, \mathcal{K}) = 0$ and $\frac{dR}{dr}(r, L, \mathcal{K}) = 0$ in terms of $L_{\pm}(r)$ and $\mathcal{K}(r)$. By plugging these relations into equation (3.3.13), one can rule out the solution $L_+(r)$ completely and most of the r -values of the solution $(L_-(r), \mathcal{K}(r))$; only values of r in the interval $[r_{\delta}, r_{\rho}]$ remain and it is then not difficult to show that there are indeed null geodesics with constant r for all r in this closed interval. Here, r_{δ} and r_{ρ} are the roots of the polynomial $p(r) = r(r - 3m)^2 - 4a^2m$ that are in the interval $[r_+, \infty)$.

We now show that the N -energy of a trapped null geodesic γ_{r_0} , trapped on the hypersurface $\{r = r_0\}$ with $r_0 \in [r_{\delta}, r_{\rho}]$, is bounded away from zero and infinity. One way to do this is to compute the N -energy directly:

$$-(N, \dot{\gamma}) = (dt + dr^* - dr)(\dot{\gamma}) = \dot{t} = \frac{1}{\rho^2} [a\mathbb{D}(\theta) + (r_0^2 + a^2) \frac{\mathbb{P}(r_0)}{\Delta(r_0)}]$$

where we have used equation (3.3.11). A further analysis of the behaviour of the θ component of γ_{r_0} yields that its image is a closed subset of $[0, \pi]$, thus $-(N, \dot{\gamma})(\theta)$ takes on its minimum and maximum. Since $-(N, \dot{\gamma})$ is always strictly positive, this immediately yields that it is bounded away from zero and infinity.

Invoking Theorem 3.2.47 we thus obtain

Theorem 3.3.14 (Trapping in (sub)-extremal Kerr). *Let (M, g) be the domain of outer communications of a (sub)-extremal Kerr spacetime, foliated by the level sets of a time function t^* as above. Moreover, let N be the timelike vector field from above and \mathcal{T} an open set with the property that for all $\tau \geq 0$ we have $\mathcal{T} \cap \Sigma_{\tau} \cap [r_{\delta}, r_{\rho}] \neq \emptyset$. Then there is no function $P : [0, \infty) \rightarrow (0, \infty)$ with $P(\tau) \rightarrow 0$ for $\tau \rightarrow \infty$ such that*

$$E_{\tau, \mathcal{T} \cap \Sigma_{\tau}}^N(u) \leq P(\tau) E_0^N(u)$$

holds for all solutions u of the wave equation.

Note that the same remark as made in footnote 72 on page 68 applies, i.e., the theorem remains true if we choose a different timelike vector field N which commutes with the Killing vector field ∂_t and also if we choose a different foliation by timelike slices, i.e., a different time function⁸⁶.

Another way to show that the energy of the trapped null geodesic γ_{r_0} is bounded away from zero and infinity is to choose a different suitable vector field N . Recall that the vector fields ∂_t and ∂_{φ} are Killing, and that at each point in the domain of outer communications they also span a timelike direction. We can thus find a timelike vector field \tilde{N} that commutes with ∂_t and such that in a small r -neighbourhood of r_0 the vector field \tilde{N} is given by $\partial_t + k\partial_{\varphi}$ with $k \in \mathbb{R}$ a constant. Thus, \tilde{N} is Killing in this small r -neighbourhood and hence the \tilde{N} -energy of γ_{r_0} is constant.

⁸⁶In the latter case one may have to alter the decay statement for the function P , i.e., replace it with $P(\tau) \rightarrow 0$ for $\tau \rightarrow \tau^*$.

Blue-shift near the Cauchy horizon of (sub)-extremal Kerr

In this section we show that the results of Section 3.3.1 and 3.3.1 also hold for (sub)-extremal Kerr. The proof is completely analogous: In the above defined $(v_+, r, \theta, \varphi_+)$ coordinates a family of ingoing null geodesics with uniformly bounded energy on Σ_0 near spacelike infinity ι^0 is given by

$$\gamma_{v_+^0}(s) = (v_+^0, -s, \theta_0, \varphi_0),$$

where $s \in (-\infty, 0)$. The same pictures as in Sections 3.3.1 and 3.3.1 apply, along with the same spacelike hypersurfaces Σ_0 and Σ_1 . In order to obtain regular coordinates in a neighbourhood of the Cauchy horizon, we define, starting with (t, r, θ, φ) coordinates in region *II*, outgoing ‘null’ coordinates $(v_-, r, \theta, \varphi_-)$ by $v_- = t - r^*$ and $\varphi_- = \varphi - \bar{r}$. These coordinates cover the regions *II* and *IV* in the sub-extremal case and regions *II* and *III* in the extremal case. In these coordinates, the tangent vector of the null geodesic $\gamma_{v_+^0}$ takes the form

$$\dot{\gamma}_{v_+^0} = -\frac{\partial}{\partial r}\Big|_+ = 2\frac{r^2 + a^2}{\Delta}\frac{\partial}{\partial v_-}\Big|_- - \frac{\partial}{\partial r}\Big|_- + 2\frac{a}{\Delta}\frac{\partial}{\partial \varphi_-}\Big|_-, \quad (3.3.15)$$

which blows up at the Cauchy horizon. It is again easy to see that the inner product with a timelike vector field, which extends smoothly to a timelike vector field over the Cauchy horizon, necessarily blows up along Σ_1 for $v_+^0 \rightarrow \infty$. Thus, we obtain, after invoking Theorem 3.2.43,

Theorem 3.3.16 (Blue-shift near the Cauchy horizon in sub-extremal Kerr). *Let Σ_0 and Σ_1 be spacelike slices in the sub-extremal Kerr spacetime as indicated in the second diagram in Section 3.3.1. Then there exists a sequence $\{u_i\}_{i \in \mathbb{N}}$ of solutions to the wave equation with initial energy $E_0^{n_{\Sigma_0}}(u_i) = 1$ on Σ_0 such that the n_{Σ_1} -energy on Σ_1 goes to infinity, i.e., $E_1^{n_{\Sigma_1}}(u_i) \rightarrow \infty$ for $i \rightarrow \infty$.*

In particular, there is no energy boundedness statement of the form (3.3.8).

As before, let us state the following

Conjecture 3.3.17. *For generic compactly supported smooth initial data on Σ_0 , the n_{Σ_1} -energy along Σ_1 of the corresponding solution to the wave equation is infinite.*

Let us conclude this section with a couple of remarks:

- i) Obviously, an analogous statement to Theorem 3.3.16 is true for extremal Kerr, however, one has to introduce again a suitable globally hyperbolic subset in order to be able to apply Theorem 3.2.43.
- ii) The discussion in Section 3.3.1 carries over to the Kerr case. In particular let us stress that Conjecture 3.3.17 only concerns *sub*-extremal Kerr black holes - the

same statement for extremal Kerr black holes is expected to be false. However, as for Reissner-Nordström black holes, we conjecture a C^k instability (for some finite k) at the Cauchy horizon of extremal Kerr black holes.

- iii) We leave it as an exercise for the reader to convince him- or herself that analogous versions of the Theorems 3.3.14 and 3.3.16 also hold true for the Kerr-Newman family.

Appendix

The first part of the appendix compares the construction of localised solutions to the wave equation using the Gaussian beam approximation with the older method which employs the geometric optics approximation. In particular, we discuss Ralston's paper [59] from 1969, where he proves that trapping in the obstacle problem necessarily leads to a loss of derivative in a uniform LED statement and we argue that this older method does not directly transfer to general Lorentzian manifolds.

The second part of the appendix extends the analysis of Gaussian beams for the wave equation to equations of the form $\square u + Xu + fu = 0$, where X is a smooth vector field and f a smooth function.

3.A A sketch of the construction of localised solutions to the wave equation using the geometric optics approximation

In this short section we outline how one can construct localised solutions to the wave equation with the help of the geometric optics approximation. Although this approach is simpler than the Gaussian beam approximation we have presented, it alone is not strong enough to prove Theorem 3.2.1, since the geometric optics approximation, in contrast to the Gaussian beam approximation, breaks down at caustics.

As already mentioned at the beginning of Section 3.2.2, the geometric optics approximation also considers approximate solutions of the wave equation that are of the form $u_\lambda = a \cdot e^{i\lambda\phi}$. But, here it suffices to consider real valued functions a and ϕ . Also recall that we can satisfy (3.2.5), i.e., $\|\square u_\lambda\|_{L^2(R_{[0,T]})} \leq C$, if we require

$$d\phi \cdot d\phi = 0 \quad (\text{eikonal equation}) \quad (3.A.1)$$

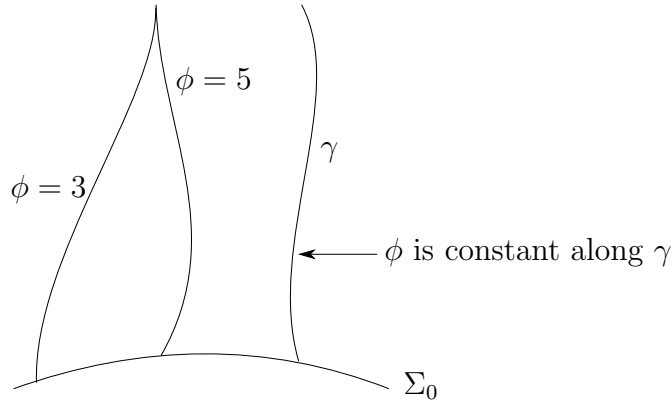
$$2\text{grad } \phi(a) + \square\phi \cdot a = 0 \quad (\text{transport equation}) . \quad (3.A.2)$$

Recall that one can solve the eikonal equation $H(x, p) = H(x, d\phi) = \frac{1}{2}g^{-1}(x)(d\phi, d\phi) =$

0 using the method of characteristics. The characteristic equations are

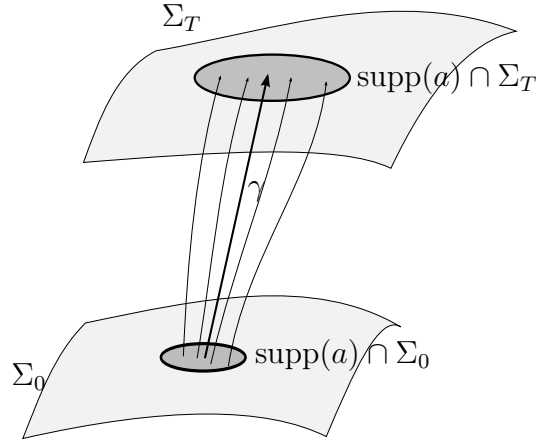
$$\begin{aligned} \dot{x}^\mu &= \frac{\partial H}{\partial p_\mu} = g^{\mu\nu} p_\nu \\ \dot{p}_\nu &= -\frac{\partial H}{\partial x^\nu} = -\frac{1}{2} \left(\frac{\partial}{\partial x^\nu} g^{\mu\kappa} \right) p_\mu p_\kappa. \end{aligned} \tag{3.A.3}$$

Given initial data $\phi|_{\Sigma_0}$ we choose $n_{\Sigma_0}\phi$ such that $d\phi \cdot d\phi = 0$ is satisfied on Σ_0 . Moreover, we assume that $\text{grad } \phi$ is transversal to Σ_0 . Then the integral curves of (3.A.3) sweep out a 4-dimensional submanifold of T^*M - and one can show that it is Lagrangian, i.e., it is locally the graph of a function ϕ which solves the eikonal equation. This ensures that a solution ϕ of (3.A.1) exists locally. In order to understand the obstruction for a global solution to exist, first note that (3.A.3) are just the equations for the geodesic flow in the cotangent bundle. In particular, the projections of the integral curves of (3.A.3) to M are geodesics γ with tangent vector $\dot{\gamma} = \text{grad } \phi$. Moreover, using the eikonal equation, it follows that ϕ is constant along those geodesics. Thus, if two of those geodesics cross (which is called a *caustic*) the solution of the eikonal equation breaks down.



Let us now consider the transport equation. Since $\dot{\gamma} = \text{grad } \phi$, a is transported along the geodesics determined by ϕ . Hence, the solution of (3.A.2) has the same domain of existence as the solution of (3.A.1), and thus we see that the geometric optics approximation only breaks down at caustics.

In the context of Theorem 3.2.1, i.e., for the purpose of the construction of localised solutions to the wave equation, recall that given a neighbourhood \mathcal{N} of a certain geodesic γ and a finite time $T > 0$, we aim for a solution a of (3.A.2) such that a is supported in \mathcal{N} up to time T . Therefore, we first prescribe initial data $\phi|_{\Sigma_0}$, $n_{\Sigma_0}\phi$ such that this particular geodesic is one of the integral curves of (3.A.3). Let us assume that there are no caustics up to time T , i.e., we obtain a solution ϕ of (3.A.1) in $R_{[0,T]}$. Secondly, notice that if a is initially zero at some point on Σ_0 , then it vanishes on the geodesic it is transported along. Thus, by continuity we can choose $\text{supp}(a|_{\Sigma_0})$ so small, centred around the base point of γ , such that $a|_{\Sigma_\tau}$ is supported in $\mathcal{N} \cap \Sigma_\tau$ up to time T , i.e., (3.2.7) is satisfied, at least up to time T .



Finally, we notice that the initial energy of u_λ grows like λ^2 , i.e., (3.2.6) is satisfied as well. This finishes then the construction of the approximate solution and one can now prove Theorem 3.2.1 *under the additional assumption that no caustics form up to time T* in the same way as before, using (3.2.5), (3.2.6) and (3.2.7). Note that, although we cannot prove Theorem 3.2.1 without any further assumptions by just using the geometric optics approximation, this construction is already sufficient for obtaining solutions to the wave equation with localised energy along light rays in the Minkowski spacetime for instance, since there we can avoid the formation of caustics by a suitable choice of initial data for ϕ . However, in general spacetimes one cannot exclude the possibility of the formation of caustics.

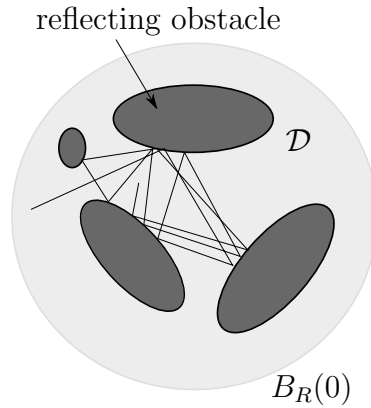
3.B Discussion of Ralston's proof that trapping forms an obstruction to LED in the obstacle problem

The *obstacle problem* is the study of the wave equation

$$-\frac{\partial^2}{\partial t^2}u + \left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2} \right)u = 0$$

on $\mathbb{R} \times \mathcal{D}$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial\mathcal{D}$, where $\mathcal{D} \subseteq \mathbb{R}^3$ is an open set with smooth boundary and bounded complement.

Let us define an *admissible light ray* to be a straight line in \mathcal{D} that is nowhere tangent to $\partial\mathcal{D}$ and that is reflected off the boundary \mathcal{D} by the classical laws of ray optics. Moreover, let ℓ_R denote the supremum of the lengths of all admissible light rays that are contained in $B_R(0)$, where $R > 0$.



In [59], Ralston proved the following

Theorem 3.B.1. *If $\ell_R = \infty$, then there is no uniform decay of the energy contained in $B_R(0)$ with respect to the initial energy, i.e., there is no constant $C > 0$ and no function $P : [0, \infty) \rightarrow (0, \infty)$ with $P(t) \rightarrow 0$ for $t \rightarrow \infty$ such that*

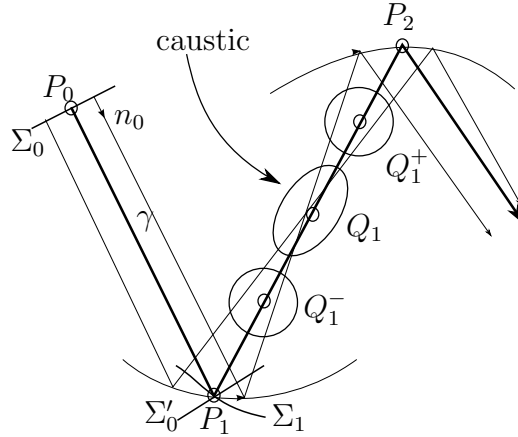
$$\int_{B_R(0)} |\nabla u|^2(t, x) + |\partial_t u|^2(t, x) dx \leq P(t) \cdot C \cdot \int_{\mathcal{D}} |\nabla u|^2(0, x) + |\partial_t u|^2(0, x) dx$$

holds for all solutions u of the wave equation that vanish on $\mathbb{R} \times \partial\mathcal{D}$ and whose initial data (prescribed on $\{t = 0\}$) is contained in $B_\rho(0)$ for some large, but fixed ρ .

His work was motivated by a conjecture of Lax and Phillips from 1967, see [41], Chapter 5.3. They conjectured that an even stronger theorem holds, namely that the theorem above is true even without the assumption that the light rays contributing to ℓ_R are nowhere tangent. However, the behaviour of such grazing rays is in general quite complicated and is still not completely understood.

In the following we will give a very brief sketch of his proof and discuss why it does not transfer directly to more general spacetimes. The idea, Ralston followed, to contradict the uniform rate of the local energy decay is to construct, using the geometric optics approximation, solutions with localised energy that follow one of those trapped light rays. One can implement reflections at $\partial\mathcal{D}$ into the construction of localised solutions via geometric optics without problems, see for example [68], Chapter 6.6. However, one has to expect the formation of caustics and thus the breakdown of the approximate solution - and this is exactly the difficulty that he had to overcome.

Ralston starts by constructing the optical path: Given a light ray γ that starts at P_0 , he considers a small 2-surface Σ_0 through P_0 such that γ points in the normal direction n_0 . This gives rise to a whole bunch of light rays that start off Σ_0 in normal direction.



If the principal curvatures of Σ_0 at P_0 are distinct from $\frac{1}{l_0} := \frac{1}{|P_1 - P_0|}$ (which can be achieved, of course), then the normal translate Σ'_0 of Σ_0 at P_1 exists if we choose Σ_0 small enough. We then ‘reflect’ Σ'_0 at the boundary $\partial\mathcal{D}$ and obtain in this way the surface Σ_1 . This procedure is repeated, and by slight perturbations of the already constructed surfaces we can ensure the condition on the principal curvatures. Caustics are forming in a neighbourhood of Q_1 , which is at distance $\frac{1}{\kappa_1}$, where κ_1 is one of the principal curvatures of Σ_1 at P_1 . Here, the normal translate of Σ_1 fails to exist, even if we choose Σ_1 very small. Ralston then considered two points, Q_1^- and Q_1^+ on γ , that are at distance δ of Q_1 . The construction with the 2-surfaces allows for an explicit construction of the phase function in the geometric optics approximation away from Q_1 . The phase is such that $\text{grad}_{\mathbb{R}^3}\phi$ points exactly along the light rays we have constructed. Thus, via geometric optics we can obtain a localised, approximate solution u_λ that propagates from a neighbourhood of P_0 to a neighbourhood of Q_1^- .

To bridge the caustics, Ralston uses the explicit representation formula for solutions of the wave equation in \mathbb{R}^{3+1} with initial data $u_\lambda(t = \tau_1)$ and $\partial_t u_\lambda(t = \tau_1)$.⁸⁷ Let us denote this solution with u_λ^f . Since the initial data of u_λ^f is highly oscillatory, one can use the method of stationary phase to approximate $u_\lambda^f(t = \tau_1 + 2\delta)$. Ralston does this in a uniform way and finds that $u_\lambda^f(t = \tau_1 + 2\delta)$ is approximately localised around Q_1^+ and also has approximately the correct phase dependence to continue propagating along the preassigned optical path. Moreover, it is clear by the domain of dependence property that, if we choose δ (and thus the 2-surfaces Σ_i) small enough, u_λ^f will stay for the time $t \in [\tau_1, \tau_1 + 2\delta]$ in a preassigned small neighbourhood of γ .

At Q_1^+ we go over to an approximation via geometric optics again, etc. This scheme yields a localised, approximate solution W_λ up to some finite time T which is patched together by the geometric optics approximations and the free space solutions. Hence, Ralston obtained $\|\square W_\lambda\|_{L^2(\mathcal{D} \times [0, T])} \leq C$, and as for the proof of Theorem 3.2.1 in Section 3.2.1 we get our proper solution v to the initial boundary value problem with initial energy equal to one. Note that in this setting it is trivial to ensure that the

⁸⁷Actually, Ralston only considers the leading order in λ for the time derivative. This simplifies the computations, and works equally well, since later one takes large λ anyway.

energy of v , that is localised in a neighbourhood of γ , does not decay: equation (3.2.2) states in particular that the energy of v that is outside the neighbourhood in question is smaller than μ . But the energy of v is constant and equal to one. Thus, the energy inside the neighbourhood of γ must be bigger than $1 - \mu$. In this way Ralston contradicts the uniform local energy decay statement and proves Theorem 3.B.1.

Since we are interested in the wave equation on general Lorentzian manifolds, we also have to expect the formation of caustics (see Appendix 3.C). However, we do not have an explicit representation formula for solutions of the wave equation that would help us mimic Ralston's proof for the obstacle problem. Thus, Ralston's proof does not directly transfer to more general spacetimes. Moreover, note that the absence of a globally timelike Killing vector field allows for the phenomenon that the 'trapped' energy decays or blows up. Hence, a theorem of the form 3.2.36 is not needed for the obstacle problem, but it is essential for the general Lorentzian case.

3.C A breakdown criterion for solutions of the eikonal equation

We give a breakdown criterion for solutions of the eikonal equation for which a given null geodesic is a characteristic.

Theorem 3.C.1. *Let (M, g) be a Lorentzian manifold and $\gamma : [0, a) \rightarrow M$ an affinely parametrised null geodesic, $a \in (0, \infty]$. If γ has conjugate points then there exists no solution $\phi : U \rightarrow \mathbb{R}$ of the eikonal equation $d\phi \cdot d\phi = 0$ with $\text{grad } \phi|_{\text{Im } \gamma} = \dot{\gamma}$, where U is a neighbourhood of $\text{Im } \gamma$.*

The theorem is motivated by the construction of localised solutions to the wave equation using geometric optics, where we need to find a solution of the eikonal equation for which a given null geodesic is a characteristic. It is well known that solutions of the eikonal equation break down whenever characteristics cross. However, by choosing the initial data (and thus the neighbouring characteristics) suitably one can try to avoid crossing characteristics. This is for example possible in the Minkowski spacetime. The theorem gives a sufficient condition for when no such choice is possible.

Our proof is a minor adaptation of Riemannian methods to the Lorentzian null case, see for example [30], in particular their Proposition 3.

First we need some groundwork. We pull back the tangent bundle TM via γ and denote the subbundle of vectors that are orthogonal to $\dot{\gamma}$ by $N(\gamma)$. The vectors that are proportional to $\dot{\gamma}$ give rise to a subbundle of $N(\gamma)$, which we quotient out to obtain the quotient bundle $\bar{N}(\gamma)$. It is easy to see that the metric g induces a positive definite metric \bar{g} on $\bar{N}(\gamma)$ and that the bundle map $R_\gamma : N(\gamma) \rightarrow \bar{N}(\gamma)$, where

$R_\gamma(X) = R(X, \dot{\gamma})\dot{\gamma}$ and R is the Riemann curvature tensor, induces a bundle map \bar{R}_γ on $\bar{N}(\gamma)$ and finally that the Levi-Civita connection ∇ induces a connection $\bar{\nabla}$ for $\bar{N}(\gamma)$.

Definition 3.C.2. $\bar{J} \in \text{End}(\bar{N}(\gamma))$ is a Jacobi tensor class iff⁸⁸ $\bar{D}_t^2 \bar{J} + \bar{R}_\gamma \bar{J} = 0$.

A Jacobi tensor class should be thought of as a variation field of γ that arises from a many-parameter variation by geodesics. It generalizes the notion of a Jacobi field (class), an infinitesimal 1-parameter variation. Indeed, a solution ϕ of the eikonal equation for which γ is a characteristic gives rise to a Jacobi tensor class \bar{J} :

We denote the flow of $\text{grad } \phi$ by Ψ_t and define $J \in \text{End}(N(\gamma))$ by

$$J_t(X_t) := (\Psi_t)_*(X_0),$$

where we extend $X_t \in N(\gamma)_t$ by parallel propagation to a vector field X along γ whose value at 0 is X_0 . Note that J is well-defined, i.e., we have $J_t(X_t) \in N(\gamma)$: Given $X_0 \in T_{\gamma(0)}M$, extend it to a vector field \tilde{X} on M with $[\tilde{X}, \text{grad } \phi] = 0$, i.e., along γ we have $\tilde{X}|_{\gamma(t)} = (\Psi_t)_*(X_0)$. Then

$$0 = \nabla_{\tilde{X}}(\text{grad } \phi, \text{grad } \phi) = 2(\nabla_{\tilde{X}} \text{grad } \phi, \text{grad } \phi) = 2\nabla_{\text{grad } \phi}(\tilde{X}, \text{grad } \phi),$$

from which it follows that $\tilde{X}|_{\gamma(t)}$ is orthogonal to $\text{grad } \phi|_{\gamma(t)}$. Moreover, J is a Jacobi tensor.⁸⁹ Let X be a parallel section along γ and \tilde{X} an extension of X_0 as above. Then

$$(D_t J)(X) = D_t(JX) = D_t(\Psi_t^* X_0) = \nabla_{\text{grad } \phi} \tilde{X} = \nabla_{\tilde{X}} \text{grad } \phi = \nabla_{JX} \text{grad } \phi.$$

Thus,

$$D_t J = (\nabla \text{grad } \phi) \circ J. \quad (3.C.3)$$

Differentiating once more gives

$$(D_t^2 J)(X) = \nabla_{\text{grad } \phi}(\nabla_{JX} \text{grad } \phi) = R(\text{grad } \phi, JX) \text{grad } \phi = -R_\gamma \circ J(X).$$

Using that $(\Psi_t)_*(\text{grad } \phi|_{\gamma(0)}) = \text{grad } \phi|_{\gamma(t)}$, it is now clear that J descends to a Jacobi tensor class \bar{J} . Moreover, \bar{J} is non-singular, i.e., \bar{J}^{-1} exists. Since the metric \bar{g} is non-degenerate, we can form adjoints of sections of $\text{End}(\bar{N}(\gamma))$, what we will denote by $*$. Note also that $(\bar{D}_t \bar{J})\bar{J}^{-1}$ is self-adjoint. This follows from (3.C.3) and the fact that $\nabla \text{grad } \phi$ is symmetric. We now prove the theorem.

Proof of Theorem 3.C.1: Assume there exists such a solution ϕ of the eikonal equation. Say the points $\gamma(t_0)$ and $\gamma(t_1)$ are conjugate, $0 \leq t_0 < t_1 < a$, and \bar{J} is the Jacobi

⁸⁸Here and in what follows we write \bar{D}_t for $\bar{\nabla}_{\partial_t}$.

⁸⁹This notion is analogous to Definition 3.C.2, without taking the quotient.

tensor class induced by ϕ as discussed above. Using the identification of $\text{End}(\bar{N}(\gamma)_t)$ with $\text{End}(\bar{N}(\gamma)_{t_0})$ via parallel translation, we write

$$\bar{K}(t) := \bar{J}(t)C \int_{t_0}^t (\bar{J}^* \bar{J})^{-1}(\tau) d\tau ,$$

where $C = \bar{J}^{-1}(t_0)\bar{J}^*(t_0)\bar{J}(t_0)$. A straightforward computation shows that \bar{K} is a Jacobi tensor class with $\bar{K}(t_0) = 0$ and $\bar{D}_t \bar{K}(t_0) = \text{id}$. Moreover, $\bar{K}(t)$ is non-singular for $t > t_0$.

On the other hand there exists a Jacobi field Y with $Y(t_0) = 0$ and $Y(t_1) = 0$. In particular, this implies that Y is a section of $N(\gamma)$. The Jacobi field Y induces a non-trivial Jacobi field class \bar{Y} that vanishes at t_0 and t_1 . However, a Jacobi field class is uniquely determined by its value and velocity at a point. Parallely propagating $\bar{D}_t \bar{Y}|_{t_0}$ gives rise to a vector field class \bar{Z} . $\bar{K}\bar{Z}$ is then a Jacobi field class that has the same value and velocity as \bar{Y} at $t = t_0$, thus $\bar{K}\bar{Z} = \bar{Y}$. This, however, contradicts \bar{K} being non-singular for $t > t_0$. \square

3.D Gaussian beams for the wave equation with lower order terms

In this appendix we discuss the construction and the characterisation of the energy of Gaussian beams for the wave equation on a globally hyperbolic Lorentzian manifold with lower order terms, i.e., for the equation

$$Pu := \square u + Xu + fu = 0 , \tag{3.D.1}$$

where X is a smooth, possibly complex valued vector field on M , and $f \in C^\infty(M, \mathbb{C})$. The following theorem, which corresponds to Theorem 3.2.43, but for the equation (3.D.1), can be proven:

Theorem 3.D.2. *Let (M, g) be a time oriented globally hyperbolic Lorentzian manifold with time function t , foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let $\gamma : [0, S) \rightarrow M$ be an affinely parametrised future directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$, and let N be a timelike, future directed vector field. Moreover, let X be a smooth complex valued vector field and $f \in C^\infty(M, \mathbb{C})$.*

For any neighbourhood \mathcal{N} of γ , for any $T > 0$ with $\Sigma_T \cap \text{Im}(\gamma) \neq \emptyset$, and for any $\mu > 0$, there exists a solution $v \in C^\infty(M, \mathbb{C})$ of the equation (3.D.1) with $E_0^N(v) = -g(N, \dot{\gamma})|_{\gamma(0)}$ such that

$$\left| E_{\tau, \mathcal{N} \cap \Sigma_\tau}^N(v) - [-g(N, \dot{\gamma})|_{\text{Im}\gamma \cap \Sigma_\tau}] \cdot |m_X(\tau)|^2 \right| < \mu \tag{3.D.3}$$

holds for all $0 \leq \tau \leq T$, where

$$|m_X(\tau)|^2 = \exp \left(- \int_0^\tau \operatorname{Re}[g(X, \dot{\gamma})] \Big|_{\operatorname{Im}(\gamma) \cap \Sigma_{\tau'}} \cdot \frac{1}{\dot{\gamma}(t)} \Big|_{\operatorname{Im}(\gamma) \cap \Sigma_{\tau'}} d\tau' \right),$$

and

$$E_{\tau, \mathcal{N}^c \cap \Sigma_\tau}^N(v) < \mu \quad \forall 0 \leq \tau \leq T, \quad (3.D.4)$$

provided that we have on $R_{[0, T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$

$$\begin{aligned} \frac{1}{|n_{\Sigma_\tau}(t)|} + |g(N, n_{\Sigma_\tau})| &\leq C < \infty \quad \text{and} \quad 0 < c \leq |g(N, N)| \\ |\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})| + \sum_{i=1}^3 |\nabla N(n_{\Sigma_\tau}, e_i)| + \sum_{i,j=1}^3 |\nabla N(e_i, e_j)| &\leq C < \infty \\ |g(X, n_{\Sigma_\tau})| + \bar{g}_\tau(X, \bar{X}) &\leq C < \infty \\ |f| &\leq C < \infty \end{aligned} \quad (3.D.5)$$

where c and C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ is an orthonormal frame.

Moreover, by choosing \mathcal{N} , if necessary, a bit smaller, (3.D.3) holds independently of (3.D.5).

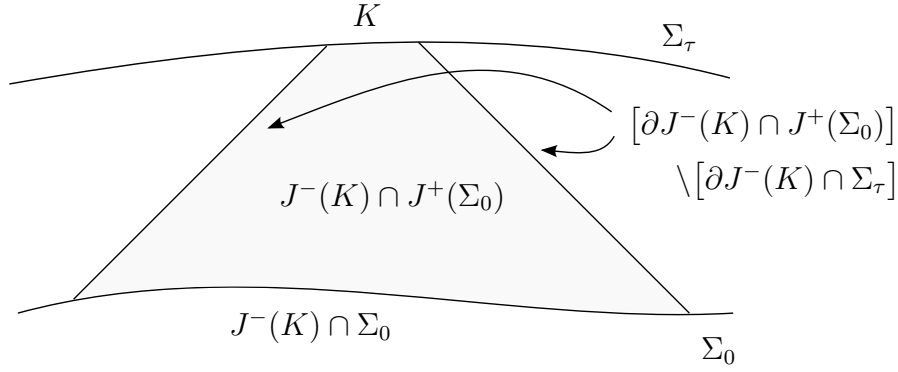
Let us remark here, that although we consider a slightly different partial differential equation, our Definitions 3.1.7 and 3.1.8 of the stress-energy tensor and the N -energy remain unchanged. In the following we will sketch the proof of Theorem 3.D.2. The next section gives in particular also a sketch of the proof of the energy estimate (3.2.8) for the wave equation under the condition (3.2.46), which was skipped in the main part of Chapter 3.

The energy estimate

Again, the condition (3.D.5) ensures that we have a ‘global’ energy estimate, which for the equation (3.D.1) takes the form

$$\int_{\Sigma_\tau} J^N(u) \cdot n_{\Sigma_\tau} \operatorname{vol}_{\bar{g}_\tau} \leq C(T, N, \{\Sigma_\tau\}) \left(\int_{\Sigma_0} [J^N(u) \cdot n_{\Sigma_0} + |u|^2] \operatorname{vol}_{\bar{g}_0} + \|Pu\|_{L^2(R_{[0, T]})}^2 \right) \quad \forall 0 \leq \tau \leq T. \quad (3.D.6)$$

In the following, we sketch how (3.D.6) is obtained. Let K be a compact subset of Σ_τ .



By integrating the divergence of $J^N(u)$ over the region $J^-(K) \cap J^+(\Sigma_0)$, we obtain

$$\begin{aligned}
& \int_K J^N(u) \cdot n_{\Sigma_\tau} + \int_{[\partial J^-(K) \cap J^+(\Sigma_0)] \setminus [\partial J^-(K) \cap \Sigma_\tau]} J^N(u) \cdot n_{\partial J^-(K)} + \int_{J^-(K) \cap J^+(\Sigma_0)} \nabla^\mu N^\nu \mathbb{T}_{\mu\nu}(u) \\
& \quad + \int_{J^-(K) \cap J^+(\Sigma_0)} Nu \cdot \square u \\
& = \int_{J^-(K) \cap \Sigma_0} J^N(u) \cdot n_{\Sigma_0} .
\end{aligned} \tag{3.D.7}$$

Note here that $[\partial J^-(K) \cap J^+(\Sigma_0)] \setminus [\partial J^-(K) \cap \Sigma_\tau]$ is a Lipschitz manifold⁹⁰, thus differentiable almost everywhere. Dropping the positive term $\int_{[\partial J^-(K) \cap J^+(\Sigma_0)] \setminus [\partial J^-(K) \cap \Sigma_\tau]} J^N(u) \cdot n_{\partial J^-(K)}$ in (3.D.7), letting K exhaust Σ_τ and substituting from the definition of P , we obtain

$$\begin{aligned}
\int_{\Sigma_\tau} J^N(u) \cdot n_{\Sigma_\tau} & \leq \int_{R_{[0,\tau]}} |\nabla^\mu N^\nu \mathbb{T}_{\mu\nu}(u)| + \int_{R_{[0,\tau]}} |Nu| \cdot (|Pu| + |Xu| + |fu|) + \int_{\Sigma_0} J^N(u) \cdot n_{\Sigma_0} \\
& \leq \underbrace{\int_{R_{[0,\tau]}} (|\nabla^\mu N^\nu \mathbb{T}_{\mu\nu}(u)| + 2|Nu|^2 + |Xu|^2 + |fu|^2)}_{:=I_0} \\
& \quad + \int_{R_{[0,\tau]}} |Pu|^2 + \int_{\Sigma_0} J^N(u) \cdot n_{\Sigma_0} .
\end{aligned}$$

In the following we show that the conditions (3.D.5) yield the estimate

$$I_0 \leq C \int_0^\tau \int_{\Sigma_{\tau'}} J^N(u) \cdot n_{\Sigma_{\tau'}} \text{vol}_{\bar{g}_{\tau'}} d\tau' + C \int_{\Sigma_0} |u|^2 \text{vol}_{\bar{g}_0} ,$$

such that (3.D.6) follows by Gronwall's inequality.

⁹⁰Cf. [52], Chapter 14, 25. Proposition

Let $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ be an orthonormal basis. In terms of this basis, the timelike vector field N takes the form $N = N^0 n_{\Sigma_\tau} + \sum_{i=1}^3 N^i e_i$. We have $g(N, N) = -(N^0)^2 + \sum_{i=1}^3 (N^i)^2$, and since N is future directed, we have $N^0 > 0$. We compute

$$\begin{aligned}
J^N(u) \cdot n_{\Sigma_\tau} &= \mathbb{T}(u)(N, n_{\Sigma_\tau}) \\
&= \operatorname{Re}(Nu \cdot \overline{n_{\Sigma_\tau} u}) - \frac{1}{2} g(N, n_{\Sigma_\tau}) g^{-1}(du, \overline{du}) \\
&= \frac{1}{2} N^0 (|n_{\Sigma_\tau} u|^2 + \sum_{i=1}^3 |e_i u|^2) + \operatorname{Re} \left(\sum_{i=1}^3 N^i e_i u \cdot \overline{n_{\Sigma_\tau} u} \right) \\
&\geq \frac{1}{2} N^0 (|n_{\Sigma_\tau} u|^2 + \sum_{i=1}^3 |e_i u|^2) - \sqrt{\sum_{i=1}^3 (N^i)^2} \cdot \sqrt{\sum_{i=1}^3 |e_i u|^2} \cdot |n_{\Sigma_\tau} u| \\
&\geq \frac{1}{2} (|n_{\Sigma_\tau} u|^2 + \sum_{i=1}^3 |e_i u|^2) \cdot [N^0 - \sqrt{\sum_{i=1}^3 (N^i)^2}] .
\end{aligned} \tag{3.D.8}$$

Under the assumptions

$$|g(N, N)| \geq c_1 > 0 \quad \text{and} \quad |g(N, n_{\Sigma_\tau})| \leq c_2 < \infty , \tag{3.D.9}$$

where c_1 and c_2 are constants, it follows that

$$N^0 - \sqrt{\sum_{i=1}^3 (N^i)^2} = N^0 - \sqrt{g(N, N) + (N^0)^2} \geq c_0 > 0 .$$

Here, c_0 is another constant. Thus, under the assumptions (3.D.9), $J^N(u) \cdot n_{\Sigma_\tau}$ controls $|n_{\Sigma_\tau} u|^2$ and $(\bar{g}_\tau)^{-1}(du, \overline{du})$ uniformly.

The first condition in (3.D.9) ensures that N does not go to zero, nor tends to a (regular) null vector in the $(n_{\Sigma_\tau}, e_1, e_2, e_3)$ frame. The second condition in (3.D.9) ensures that N is also not tending to a ‘singular’ null vector⁹¹.

Let us remark, that by virtue of the Cauchy-Schwarz inequality⁹² for timelike vectors N and n , i.e.,

$$|g(N, n)|^2 \geq |g(N, N)| |g(n, n)| ,$$

and the fact that $g(n_{\Sigma_\tau}, n_{\Sigma_\tau}) = -1$, the bounds (3.D.9) imply that

$$0 < c_1 \leq |g(N, N)| \leq |g(N, n_{\Sigma_\tau})| \leq c_2 < \infty . \tag{3.D.10}$$

⁹¹By this we mean, that also after a possible renormalisation (multiplication by a function), N does not tend to a (regular) null vector.

⁹²Cf. [52], Chapter 5, 30. Proposition

Furthermore, note that

$$\text{vol}_g = \frac{1}{dt(n_{\Sigma_\tau})} dt \wedge \text{vol}_{\bar{g}_\tau}, \quad (3.D.11)$$

and thus under the assumption

$$\frac{1}{dt(n_{\Sigma_\tau})} \leq c_3 < \infty \quad (3.D.12)$$

and

$$|\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})| + \sum_{i=1}^3 |\nabla N(n_{\Sigma_\tau}, e_i)| + \sum_{i,j=1}^3 |\nabla N(e_i, e_j)| \leq c_4, \quad (3.D.13)$$

we obtain from (3.D.8)

$$\int_{R_{[0,\tau]}} |\nabla^\mu N^\nu \mathbb{T}_{\mu\nu}(u)| \text{vol}_g \leq C \int_0^\tau \int_{\Sigma_{\tau'}} J^N(u) \cdot n_{\Sigma_{\tau'}} \text{vol}_{\bar{g}_{\tau'}} d\tau'.$$

Recalling the assumptions (3.D.9) and using that $\bar{g}_\tau(N, N) \leq |g(N, n_{\Sigma_\tau})|^2$, we obtain the analogous estimate for the term $\int_{R_{[0,\tau]}} |Nu|^2$; and making the assumptions

$$|g(X, n_{\Sigma_\tau})| + \bar{g}_\tau(X, \bar{X}) \leq c_5 < \infty,$$

the term $\int_{R_{[0,\tau]}} |Xu|^2$ is estimated the same way. It remains to estimate the term $\int_{R_{[0,\tau]}} |fu|^2$.

Integrating the divergence of $|u|^2 N$ over the region $R_{[0,\tau]}$, we obtain⁹³

$$\int_{\Sigma_{\bar{\tau}}} |u|^2 \cdot g(-N, n_{\Sigma_{\bar{\tau}}}) \text{vol}_{\bar{g}_{\bar{\tau}}} = \int_{\Sigma_0} |u|^2 \cdot g(-N, n_{\Sigma_0}) \text{vol}_{\bar{g}_0} + \int_{R_{[0,\bar{\tau}]}} \text{div}(|u|^2 N) \text{vol}_g.$$

By virtue of (3.D.10), we have a bound on $g(-N, n_{\Sigma_{\bar{\tau}}})$ away from zero, and by (3.D.13) and (3.D.9) we have upper bounds on $\text{div}(N)$ and $g(-N, n_{\Sigma_{\bar{\tau}}})$. Thus,

$$\int_{\Sigma_{\bar{\tau}}} |u|^2 \text{vol}_{\bar{g}_{\bar{\tau}}} \leq C \int_{\Sigma_0} |u|^2 \text{vol}_{\bar{g}_0} + C \int_{R_{[0,\bar{\tau}]}} |Nu|^2 \text{vol}_g + C \int_0^{\bar{\tau}} \int_{\Sigma_{\tau'}} |u|^2 \text{vol}_{\bar{g}_{\tau'}} d\tau'.$$

⁹³Actually, one should first fix a compact set K in Σ_0 , and integrate over the region between Σ_0 and Σ_τ flown out by the integral curves of N that start in K . The boundaries to which N is tangent do not appear in the divergence estimate. One then exhausts Σ_0 by bigger and bigger K .

Gronwall's inequality yields

$$\int_{\Sigma_{\tilde{\tau}}} |u|^2 \text{vol}_{\tilde{g}_{\tilde{\tau}}} \leq C(T) \cdot \left(\int_{\Sigma_0} |u|^2 \text{vol}_{\tilde{g}_0} + \int_{R_{[0, \tilde{\tau}]}} |Nu|^2 \text{vol}_g \right) \quad (3.D.14)$$

for all $0 \leq \tilde{\tau} \leq T$. Recalling (3.D.11) and (3.D.12), and after an integration in $\tilde{\tau}$ from 0 to τ , (3.D.14) yields

$$\int_{R_{[0, \tau]}} |u|^2 \text{vol}_g \leq C(T) \left(\int_{\Sigma_0} |u|^2 \text{vol}_{\tilde{g}_0} + \int_{R_{[0, \tau]}} |Nu|^2 \text{vol}_g \right)$$

for all $0 \leq \tau \leq T$. The last term has already been estimated. Hence, making the assumption

$$\|f\|_{L^\infty} \leq c_6$$

finishes the sketch of the proof of the 'global' energy estimate (3.D.6).

Given $K \subseteq \Sigma_T$ compact, the local energy estimate

$$\int_{\Sigma_\tau \cap J^-(K)} J^N(u) \cdot n_{\Sigma_\tau} \text{vol}_{\tilde{g}_\tau} \leq C(T) \left(\int_{\Sigma_0 \cap J^-(K)} [J^N(u) \cdot n_{\Sigma_0} + |u|^2] \text{vol}_{\tilde{g}_0} + \|Pu\|_{L^2(J^-(K) \cap J^+(\Sigma_0))}^2 \right) \quad (3.D.15)$$

holds true for all $0 \leq \tau \leq T$ independently of (3.D.5), since by the global hyperbolicity of (M, g) , the set $J^-(K) \cap J^+(\Sigma_0)$ is compact, and thus all the necessary bounds follow by continuity.

The construction of the Gaussian beam

The construction of Gaussian beams for the equation (3.D.1) is nearly exactly the same as for the wave equation. One considers complex valued functions of the form $u_\lambda(x) = a_\mathcal{N}(x)e^{i\lambda\phi(x)}$ and constructs functions $a_\mathcal{N}, \phi \in C^\infty(M, \mathbb{C})$ such that

$$\|Pu_\lambda\|_{L^2(R_{[0, T]})} \leq C(T) \quad (3.D.16)$$

holds together with (3.2.6) and (3.2.7). The argument from Section 3.2.1 together with the energy estimate (3.D.6) (or (3.D.15)) shows then that for λ large the function

$$\tilde{u}_\lambda := \frac{u_\lambda}{\sqrt{E_0^N(u_\lambda)}}$$

is a very good approximate solution of (3.D.1) with localised energy.

In order to find functions a and ϕ such that (3.D.16), (3.2.6) and (3.2.7) are satis-

fied, we compute

$$\begin{aligned} Pu_\lambda &= -\lambda^2(d\phi \cdot d\phi)a_N e^{i\lambda\phi} + i\lambda \left(2\text{grad } \phi(a_N) + \square\phi \cdot a_N + (X\phi) \cdot a_N \right) \cdot e^{i\lambda\phi} \\ &\quad + (\square a_N + Xa_N + fa_N) e^{i\lambda\phi}. \end{aligned}$$

The construction of the phase function ϕ is exactly the same as for the wave equation in Section 3.2.2. The only difference is that instead of requiring $2\text{grad } \phi(a) + \square\phi \cdot a$ to vanish along γ to zeroth order (cf. (3.2.21) in Lemma 3.2.19), one requires

$$2\text{grad } \phi(a) + \square\phi \cdot a + g(X, \text{grad } \phi) \cdot a \quad \text{to vanish to zeroth order along } \gamma. \quad (3.D.17)$$

Solving this linear ordinary differential equation for a (instead of (3.2.21)), one constructs a function a_N (cf. Lemma 3.2.19) and thus finishes the construction of the Gaussian beam as before in Section 3.2.2.

We make the following remark: if s denotes the affine parameter of γ such that $\dot{\gamma} = \text{grad } \phi$ holds, then we can write the ordinary differential equation (3.D.17) as

$$\frac{d}{ds}a = -\frac{1}{2} \left(\square\phi + g(X, \dot{\gamma}) \right) \cdot a.$$

Its solution with initial value $a(0) = a_0$ is given by

$$a(s) = a_0 \cdot \exp \left(-\frac{1}{2} \int_0^s [\square\phi(s') + g(X, \dot{\gamma})(s')] ds' \right).$$

Note that $ds = \frac{ds}{dt} \cdot dt = \frac{1}{\dot{\gamma}(t)} \cdot dt$. Hence, if we define a function $m_X : I \subseteq \mathbb{R} \rightarrow \mathbb{C}$ by

$$m_X(\tau) := \exp \left(-\frac{1}{2} \int_0^\tau g(X, \dot{\gamma}) \Big|_{\text{Im}(\gamma) \cap \Sigma_{\tau'}} \cdot \frac{1}{\dot{\gamma}(t)} \Big|_{\text{Im}(\gamma) \cap \Sigma_{\tau'}} d\tau' \right),$$

we obtain

Lemma 3.D.18. *The function $w_{\lambda, \mathcal{N}} = \hat{a}_N \cdot e^{i\lambda\phi}$ is a Gaussian beam along γ for the wave equation (3.1.9) if, and only if, $u_{\lambda, \mathcal{N}} = \hat{a}_N(m_X \circ t) \cdot e^{i\lambda\phi}$ is a Gaussian beam along γ for the equation (3.D.1).*

Here, $t \in C^\infty(M, \mathbb{C})$ is the time function from Theorem 3.D.2.

The characterisation of the energy

The characterisation of the energy of Gaussian beams for the equation (3.D.1) now follows easily from Lemma 3.D.18. Recall from the proof of Theorem 3.2.36, in particular (3.2.38), that if $u_{\lambda, \mathcal{N}} = \hat{a}_N(m_X \circ t) \cdot e^{i\lambda\phi}$ is a Gaussian beam for the equation

(3.D.1), then

$$\begin{aligned} E_\tau^N(u_{\lambda, \mathcal{N}}) &= \lambda^2 \underbrace{\int_{\Sigma_\tau} |\hat{a}_\mathcal{N} \cdot (m_X \circ t)|^2 N \phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau}}_{=\mathcal{O}(\lambda^{\frac{1}{2}})} + \underbrace{\text{lower order terms}}_{=\mathcal{O}(1)} \\ &= \lambda^2 \cdot |m_X(\tau)|^2 \cdot \int_{\Sigma_\tau} |\hat{a}_\mathcal{N}|^2 N \phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \text{vol}_{\bar{g}_\tau} + \text{lower order terms} \end{aligned}$$

The characterisation of the energy of Gaussian beams for the equation (3.D.1) now follows from Lemma 3.D.18 together with the characterisation of the energy of Gaussian beams for the wave equation, Theorem 3.2.36. This finishes the sketch of the proof of Theorem 3.D.2.

3.E An application of Gaussian beams to the Teukolsky equation

In this appendix we give an application of Theorem 3.D.2 to the Teukolsky equation. We consider the exterior of a (sub)-extremal Kerr spacetime, which we introduced in Section 3.3.2. Recall that the motivation for studying the wave equation (also referred to as the ‘spin-0’ equation) on a Kerr background was that it constitutes a “poor man’s” linearisation of the Einstein equations. This poor man’s linearisation in the full sub-extremal range $|a| < m$ was well understood in a series of papers [24], [23], [25], [65], and [27] by Dafermos, Rodnianski and Shlapentokh-Rothman. The next step towards a resolution of the black hole stability conjecture is to prove stability results for the *proper* linearisation of the Einstein equations. Recall, however, that the linearisation of a partial differential equation always depends on what one considers the dynamical variable to be. The viewpoint we have taken in the introduction in Chapter 1 was that the equations (1.0.3) form a system of partial differential equations for the spacetime metric g . A different viewpoint is to consider the Riemann curvature tensor (together with the connection coefficients) as the dynamical variables. This is achieved by complementing the equations (1.0.3) by the Bianchi equations⁹⁴

$$\nabla_{[\mu} R_{\nu\kappa]\rho\sigma} = 0, \quad (3.E.1)$$

which now form *differential* equations for the Riemann curvature tensor, while the Einstein equations (1.0.3) form *algebraic* equations (in addition one complements these equations by differential equations for the connection coefficients - for the details we refer the reader to the exposition of the so-called Newman-Penrose formalism in [10])

⁹⁴Here, $R_{\mu\nu\kappa\rho}$ denotes the Riemann curvature tensor, and the square brackets denote antisymmetrisation of the indices enclosed.

or [69].). This viewpoint proved to be advantageous for the sake of proving estimates, and in particular, was also taken in the monumental work of the stability of Minkowski space by Christodoulou and Klainerman, [14].

The Kerr spacetime is an algebraically special spacetime, it admits two repeated principal null directions, which are used to construct a (complex) frame field with distinguished properties. Scalarising the (tensorial) equations for the curvature and the connection coefficients by projecting out the components with respect to this frame field, one obtains a coupled system of nonlinear scalar partial differential equations. In [69], Teukolsky showed, that after linearising this system, there are two components of the perturbed curvature which decouple, i.e., he showed that they satisfy a scalar wave equation with lower order terms - now called the Teukolsky equation, which in Boyer-Lindquist (t, r, θ, φ) coordinates reads

$$\begin{aligned} \mathcal{T}_s u &:= \square_g u + \frac{2s}{\rho^2} \cdot (r - m) \partial_r u + \frac{2s}{\rho^2} \left[\frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right] \partial_\varphi u \\ &+ \frac{2s}{\rho^2} \left[\frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \partial_t u + \frac{1}{\rho^2} (s - s^2 \cot^2 \theta) u \\ &= 0 . \end{aligned} \quad (3.E.2)$$

One should mention, that these decoupled components of the curvature contain complete information of the linearised gravitational field - hence the usefulness of Teukolsky's insight. The parameter s in the equation (3.E.2) refers to the 'spin' of the field. The above discussed gravitational perturbation correspond to spin $s = 2$.⁹⁵

We see, that the Teukolsky equation is an equation of the form $\square u + Xu + fu = 0$ studied in Appendix 3.D with

$$X = \frac{2s}{\rho^2} \left[\frac{m(r^2 - a^2)}{\Delta} - r - ia \cos \theta \right] \partial_t + \frac{2s}{\rho^2} \cdot (r - m) \partial_r + \frac{2s}{\rho^2} \left[\frac{a(r - m)}{\Delta} + i \frac{\cos \theta}{\sin^2 \theta} \right] \partial_\varphi$$

and

$$f = \frac{1}{\rho^2} (s - s^2 \cot^2 \theta) .$$

In the following, we investigate the temporal behaviour of the energy of Gaussian beams for the Teukolsky equation along trapped null geodesics in the (sub)-extremal Kerr spacetime - in analogy to the discussion in Section 3.3.2 for the wave equation⁹⁶.

⁹⁵For a fixed metric g , the linear equation (3.E.1) for a field $S_{\nu\kappa\rho\sigma}$, that has the same symmetries as the Weyl curvature tensor, is known as the spin-2 equation.

Maxwell's equation for the field $F_{\mu\nu}$ is known as the spin-1 equation. In [69], Teukolsky also showed that after projecting $F_{\mu\nu}$ on the frame constructed from the principal null directions special to the Kerr spacetime, one can again find two components which satisfy a decoupled equation. This equation corresponds to the case $s = 1$ in (3.E.2).

Finally, $s = 0$ is just the ordinary wave equation.

⁹⁶Let us stress again that we consider the same notion of energy for solutions of the Teukolsky equation as we did for solutions of the wave equation, i.e. Definition (3.1.8)

We prove the analogue of Theorem 3.3.14:

Theorem 3.E.3. *Let (M, g) be the domain of outer communications of a (sub)-extremal Kerr spacetime, foliated by the level sets of a time function t^* as in Section 3.3.2. Moreover, let N be the timelike vector field from Section 3.3.2 and \mathcal{T} an open set with the property that for all $\tau \geq 0$ we have $\mathcal{T} \cap \Sigma_\tau \cap [r_\delta, r_\rho] \neq \emptyset$. Then there is no function $P : [0, \infty) \rightarrow (0, \infty)$ with $P(\tau) \rightarrow 0$ for $\tau \rightarrow \infty$ such that*

$$E_{\tau, \mathcal{T} \cap \Sigma_\tau}^N(u) \leq P(\tau) \left(E_0^N(u) + \int_{\Sigma_0} |u|^2 \text{vol}_{\bar{g}_0} \right)$$

holds for all solutions u of the Teukolsky equation (3.E.2).

Proof. By the proof of Theorem 3.3.2 and by Theorem 3.D.2, it suffices to show that

$$\exp \left(- \int_0^\tau \text{Re} [g(X, \dot{\gamma})] \Big|_{\text{Im}(\gamma) \cap \Sigma_{\tau'}} \cdot \frac{1}{\dot{\gamma}(t)} \Big|_{\text{Im}(\gamma) \cap \Sigma_{\tau'}} d\tau' \right)$$

is bounded away from 0. Recall that the energy $E = -g(\partial_t, \dot{\gamma})$ and the angular momentum $L = g(\partial_\phi, \dot{\gamma})$ are constant along a geodesic, and that we can choose without loss of generality $E = 1$. Considering the trapped null geodesics on orbits of constant r , discussed in Section 3.3.2, we obtain

$$\begin{aligned} \text{Re} g(X, \dot{\gamma}) &= \frac{2s}{\rho^2} \left(\frac{a(r-m)}{\Delta} \cdot L - \frac{m(r^2 - a^2)}{\Delta} + r \right) \\ &= \frac{2s}{\rho^2 \cdot \Delta} \left(a(r-m) \cdot L - 3mr^2 + a^2(m+r) + r^3 \right). \end{aligned} \quad (3.E.4)$$

Hence, we need to know the value of L for the null geodesics with $E = 1$, trapped at a surface of constant r (in Section 3.3.2 we denoted this value of L by $L_-(r)$). This value is given by

$$L(r) = \frac{1}{a(r-m)} [m(r^2 - a^2) - r\Delta], \quad (3.E.5)$$

cf. [10], page 351, equation (224). Inserting (3.E.5) into (3.E.4) yields

$$\text{Re} g(X, \dot{\gamma}) = 0.$$

This concludes the proof of Theorem 3.E.3. \square

Chapter 4

Aspects of wave propagation in the interior of a sub-extremal Reissner-Nordström black hole

4.1 Introduction

This chapter is concerned with the wave equation in the interior of a sub-extremal Reissner-Nordström black hole. We study the characteristic initial value problem, where data is prescribed on the event horizon and on a null hypersurface transversal to it that penetrates the black hole interior. The results obtained in this chapter show in particular, that if the wave is compactly supported on the event horizon and the mass m and electric charge e of the black hole satisfy $\frac{e^2}{m^2} > \frac{4\sqrt{2}}{3+2\sqrt{2}}$, then the energy of the wave along a null hypersurface intersecting the Cauchy horizon is finite.

In the following section we discuss the motivation and the implications of this result. Section 4.3 fixes the notation. The main theorem is stated in Section 4.4 and is proved in Section 4.5.

4.2 The wave equation in the black hole interior and the mass inflation scenario

Recall from Section 3.3.1 the heuristic picture given by Penrose in 1968 for the instability of the Cauchy horizon, triggered by the blue-shift effect. Around 1980, the following results on the appearance of the blue-shift effect in the *linear theory* were obtained: In [45], McNamara showed that for the wave equation on a sub-extremal Reissner-Nordström background, there exists initial data on the event horizon (or also on past null infinity \mathcal{I}^-) such that at the bifurcation sphere in the black hole interior,

the derivative transversal to the Cauchy horizon of the wave blows up pointwise⁹⁷. He showed, that the initial data can be chosen to exhibit an arbitrarily fast polynomial fall-off. Moreover, he proved an analogous result for electromagnetic and gravitational perturbations on a sub-extremal Kerr background.

The later papers [33] and [11] argue, that on a sub-extremal Reissner-Nordström background, one has pointwise blow up of the transversal derivative at the bifurcation sphere *even* if the initial data is *compactly supported* on the event horizon.

Results on the manifestation of the blue-shift effect in the *non-linear theory* are known only in spherical symmetry. In the paper [57], Poisson and Israel investigated the spherically symmetric Einstein–Maxwell–null-dust system and put forward the so-called ‘mass inflation’ scenario according to which a combination of infalling and outgoing radiation leads to a blow up of the ‘mass parameter’ of the black hole at the Cauchy horizon. In their analysis they assumed a late-time polynomial fall-off of the ingoing radiation across the event horizon.

The first mathematically rigorous treatment of this mass inflation scenario was given by Dafermos in [17] and [18], where he considers the spherically symmetric Einstein-Maxwell-scalar field system. In [18] he shows that if the scalar field ψ satisfies

$$0 < cv^{-3p+\varepsilon} \leq |\partial_v \psi| \leq Cv^{-p} \quad (4.2.1)$$

on the event horizon for large enough Eddington Finkelstein coordinate v , where c , C , and ε are positive constants, then the Hawking mass $m_H = \frac{r}{2}(1 - dr \cdot dr)$ blows up at the Cauchy horizon⁹⁸. Note that the condition (4.2.1) implies that eventually the outgoing derivative $\partial_v \psi$ has a sign on the event horizon. This fact is crucial in Dafermos’ proof.

The result obtained shows in particular, that the metric does not extend as a C^1 metric across the Cauchy horizon. If one can establish, that generic perturbations in the exterior of the black hole indeed exhibit a power law decay compatible with (4.2.1), then this would yield a proof of a suitable formulation of strong cosmic censorship for Dafermos’ model⁹⁹.

Even if one can prove that generic perturbations in the spherically symmetric Einstein-Maxwell-scalar field model eventually acquire a sign on the event horizon, it is doubtful that this property will carry over even to linear scalar perturbations of the Kerr spacetime. Thus, the question arises, whether one can replace the condition (4.2.1) by a less restrictive condition that is expected to carry over to Kerr.

⁹⁷The initial data on the null hypersurface transversal to the event horizon or past null infinity is taken to be trivial.

⁹⁸Note that this statement already implies that a Cauchy horizon forms. For this proof, Dafermos only requires the upper bound in (4.2.1).

⁹⁹In [20], Dafermos and Rodnianski proved that the *upper* bound holds for $p < 3$. However, no mathematical proof of the lower bound has been given so far - cf. however the analysis of Price [58].

Recall from the discussion of the results obtained for the linear theory, that even perturbations which are compactly supported on the event horizon are expected to blow up *pointwise* at the Cauchy horizon. But would this suffice for triggering mass inflation?

In order to obtain a better understanding of this question, we sketch the heuristics underlying the mass inflation scenario in the Einstein-Maxwell-scalar field model. Consider double null coordinates (u, v) (think Eddington Finkelstein coordinates) and introduce the renormalised Hawking mass¹⁰⁰ $\varpi = m_H + \frac{e^2}{2r}$. As usual, r is the area radius of the spheres of spherical symmetry. The Einstein equations imply¹⁰¹

$$\partial_v \varpi = \frac{1}{2\partial_v r} \left(1 - \frac{2m_H}{r}\right) r^2 [\partial_v \psi]^2 .$$

Hence, *ignoring the back-reaction of the scalar field on the geometry*, we see that an infinite energy of the scalar field along an outgoing null hypersurface intersecting the Cauchy horizon,

$$\sim \int_{v_0}^{v(\mathcal{CH}^+)} (\partial_v \psi)^2 dv = \infty ,$$

would lead to mass inflation. However, in this first order perturbative analysis, the pointwise blow up of $\partial_v \psi$ does not allow us to make any predictions on whether mass inflation sets in or not!

Thus, the natural question arises, whether perturbations that are compactly supported on the event horizon have infinite energy on a null hypersurface intersecting the Cauchy horizon. Theorem 4.4.1 shows, that if the mass m and charge e of the sub-extremal Reissner-Nordström black hole satisfy $\frac{e^2}{m^2} > \frac{4\sqrt{2}}{3+2\sqrt{2}}$, this energy is finite. Extrapolating this result to the non-linear theory suggests, that for a wide range of black hole parameters m and e , compactly supported perturbations do *not* trigger mass inflation.

4.3 The interior of a Reissner-Nordström black hole and notation

Recall that the *interior of a Reissner-Nordström black hole* is the Lorentzian manifold (M, g) with underlying manifold $M := \mathbb{R} \times (r_-, r_+) \times \mathbb{S}^2$. Here r_- and r_+ are the roots of $D(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2}$, i.e., $r_{\pm} = m \pm \sqrt{m^2 - e^2}$, and the differential structure is given by the standard coordinates (t, r, θ, φ) . In these coordinates, the Lorentzian metric g is given by $g = -D(r) dt^2 + \frac{1}{D(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2$. Note that the vector field $\partial_t =: T$ is a Killing vector field. Also recall that the function $r^* : (r_-, r_+) \rightarrow \mathbb{R}$ is

¹⁰⁰The ‘mass parameter’ in the work of Poisson and Israel is exactly this renormalised Hawking mass.

¹⁰¹Cf. [17], page 886

defined by

$$r^*(r) := r + \frac{1}{2\kappa_+} \log(r_+ - r) + \frac{1}{2\kappa_-} \log(r - r_-) + c, \quad (4.3.1)$$

where c is a fixed but arbitrary constant and $\kappa_{\pm} := \frac{r_{\pm} - r_{\mp}}{2r_{\pm}^2}$. We now define the null coordinate functions $u := t - r^*$ and $v := t + r^*$. In (v, r, θ, φ) coordinates the metric reads

$$g = -D dv^2 + dv \otimes dr + dr \otimes dv + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$

and its inverse is given by

$$g^{-1} = \partial_v \otimes \partial_r + \partial_r \otimes \partial_v + D \partial_r^2 + r^{-2} \partial_\theta^2 + r^{-2} \sin^{-2} \theta \partial_\varphi^2. \quad (4.3.2)$$

Note that the metric in (v, r, θ, φ) coordinates is regular at $r = r_+$. We can thus attach the null hypersurface $\{r = r_+\} =: \mathcal{H}^+$ to our manifold M , and we call \mathcal{H}^+ the event horizon.

In (u, r, θ, φ) coordinates the metric reads

$$g = -D du^2 - du \otimes dr - dr \otimes du + r^2 d\theta^2 + r^2 \sin^2 \theta d\varphi^2,$$

and its inverse is given by

$$g^{-1} = -\partial_u \otimes \partial_r - \partial_r \otimes \partial_u + D \partial_r^2 + r^{-2} \partial_\theta^2 + r^{-2} \sin^{-2} \theta \partial_\varphi^2. \quad (4.3.3)$$

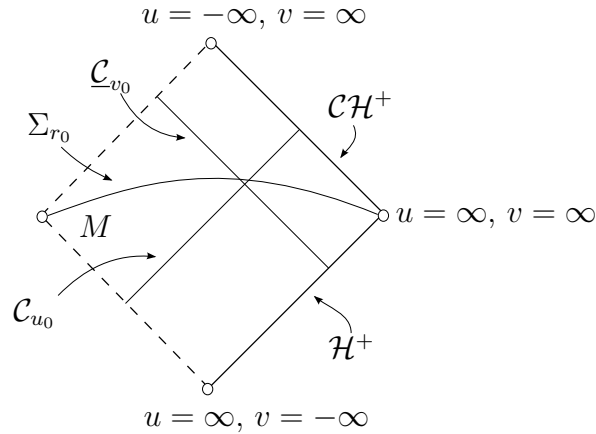
We see that the metric in (u, r, θ, φ) coordinates is regular at $r = r_-$. Hence, we can also attach the null hypersurface $\{r = r_-\} =: \mathcal{CH}^+$ to our manifold M , and we call \mathcal{CH}^+ the Cauchy horizon. In the Penrose diagrammatic representation of Section 3.3.1 in Chapter 3, the Lorentzian manifold with boundary $(M \cup \mathcal{H}^+ \cup \mathcal{CH}^+, g)$ corresponds to the region *II* with the two right boundaries of the diamond attached. A time orientation on $M \cup \mathcal{H}^+ \cup \mathcal{CH}^+$ is defined by stipulating that T is future directed on \mathcal{H}^+ (and hence past directed on \mathcal{CH}^+). We would like to bring to the reader's attention that the null coordinate function u increases to the past.

For later use we define the following hypersurfaces in $M \cup \mathcal{H}^+ \cup \mathcal{CH}^+$:

$$\mathcal{C}_{u_0} := \{u = u_0\}, \quad \underline{\mathcal{C}}_{v_0} := \{v = v_0\}, \quad \Sigma_{r_0} := \{r = r_0\},$$

$$\mathcal{C}_{u_0}(v_0) := \mathcal{C}_{u_0} \cap \{v \geq v_0\}, \quad \underline{\mathcal{C}}_{v_0}(u_0) := \underline{\mathcal{C}}_{v_0} \cap \{u \geq u_0\}, \quad \Sigma_{r_0}(u_0) := \Sigma_{r_0} \cap \{u \geq u_0\},$$

$$\mathcal{H}^+(v_0) := \mathcal{H}^+ \cap \{v \geq v_0\}, \quad \mathcal{CH}^+(u_0) := \mathcal{CH}^+ \cap \{u \geq u_0\}.$$



4.4 The main theorem

The following theorem is the main theorem of this chapter.

Theorem 4.4.1. *Let $\psi \in C^\infty(M \cup \mathcal{H}^+, \mathbb{C})$ satisfy $\square\psi = 0$.*

- i) If there is a $v_0 > 0$ such that ψ vanishes along $\mathcal{H}^+(v_0)$ and if the mass m and the charge e of the black hole are such that $2\kappa_+ > -\kappa_-$, then for any smooth and future directed timelike vector field N in $M \cup \mathcal{CH}^+$ with $[N, T] = 0$ and for any $u_1, v_1 \in \mathbb{R}$ there exists a constant $C > 0$ such that*

$$\int_{\mathcal{C}_{u_1}(v_1)} J^N(\psi) \cdot n_{\mathcal{C}_{u_1}} + \int_{\mathcal{CH}^+(u_1)} J^N(\psi) \cdot n_{\mathcal{CH}^+} \leq C. \quad (4.4.2)$$

- ii) If there is a $v_0 > 0$ such that ψ vanishes along $\mathcal{H}^+(v_0)$ to infinite order, then for any smooth timelike vector field N in $M \cup \mathcal{CH}^+$ with $[N, T] = 0$ and for any $u_1, v_1 \in \mathbb{R}$ there exists a constant $C > 0$ such that (4.4.2) holds.*

We make the following remarks:

1. The condition $2\kappa_+ > -\kappa_-$ in part *i)* of Theorem 4.4.1 is equivalent to $\frac{r_-}{r_+} > \frac{1}{\sqrt{2}}$, which in turn is equivalent to $\frac{e^2}{m^2} > \frac{4\sqrt{2}}{3+2\sqrt{2}}$.
2. In the above theorem we only distinguished the two cases ‘ ψ vanishing to zeroth order’ and ‘ ψ vanishing to infinite order along $\mathcal{H}^+(v_0)$ ’. Given that the wave ψ vanishes to some order k along $\mathcal{H}^+(v_0)$, we leave it to the interested reader to infer from the proof the relation needed between e and m in order to ensure that (4.4.2) holds.
3. The analogous theorem also holds for the interior of sub-extremal Kerr black holes. Here, one would replace the Killing vector field T in the above statement by the corresponding ‘Hawking vector field’ at the Cauchy horizon. The proof needs to be only minimally modified.

4. A related result was obtained by Franzen in [31], where she showed in particular that if the energy of the wave decays polynomially along the event horizon at a sufficiently fast rate, i.e., if $\int_{\mathcal{H}^+(v_0)} v^{p+1} J^N(\psi) \cdot n_{\mathcal{H}^+} < \infty$ holds, where $p > 1$ and N is a future directed timelike vector field invariant under the flow of T , then the *degenerate* energy $\int_{\mathcal{C}_{u_0}(v_0)} -D(r)v^p J^{-\frac{\partial}{\partial r}}|_u(\psi) \cdot n_{\mathcal{C}_{u_0}}$ is finite. Here, u_0 and v_0 are arbitrary constants.

4.5 The proof of the main theorem

The proof of Theorem 4.4.1 is given as a series of lemmata.

Lemma 4.5.1. *Let $\psi \in C^\infty(M \cup \mathcal{H}^+, \mathbb{C})$ and let N be a smooth and future directed timelike vector field in $M \cup \mathcal{H}^+$. Given a fixed but arbitrary u_0 , there exists then a $C_\psi > 0$ such that*

$$\int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \cdot n_{\underline{\mathcal{C}}} \leq C_\psi e^{-\kappa_+ u}$$

holds for all $u \geq u_0$.

If moreover ψ vanishes on $\underline{\mathcal{C}}_{v_0} \cap \mathcal{H}^+$ to order k , then for given u_0 there exists a $C_\psi > 0$ such that the following holds for all $u \geq u_0$:

$$\int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \cdot n_{\underline{\mathcal{C}}} \leq C_\psi e^{-(2k+1)\kappa_+ u}.$$

Proof. Recall that $\int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \cdot n_{\underline{\mathcal{C}}}$ is a shorthand notation for $\int_{\underline{\mathcal{C}}_{v_0}(u)} *J^N(\psi) = \int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \lrcorner \text{vol}$, where $*$ is the Hodge-star operator and \lrcorner inserts the vector field to its left into the first slot of the form to its right. Let $m_{\underline{\mathcal{C}}}$ denote a vector field that is transversal to $\underline{\mathcal{C}}_{v_0}$, past directed, and satisfies $g(n_{\underline{\mathcal{C}}}, m_{\underline{\mathcal{C}}}) = 1$. We then have

$$\int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \lrcorner \text{vol} = \int_{\underline{\mathcal{C}}_{v_0}(u)} (J^N(\psi) \cdot n_{\underline{\mathcal{C}}}) m_{\underline{\mathcal{C}}} \lrcorner \text{vol}.$$

The orientation of $\underline{\mathcal{C}}_{v_0}$ is assumed to be such that $m_{\underline{\mathcal{C}}} \lrcorner \text{vol}$ is a positive volume form¹⁰².

We now choose a normal $n_{\underline{\mathcal{C}}}$ of $\underline{\mathcal{C}}_{v_0}(u_0)$ and a transversal vector field $m_{\underline{\mathcal{C}}}$ which are regular at \mathcal{H}^+ . For the sake of explicitness, let us choose $n_{\underline{\mathcal{C}}} = -\frac{\partial}{\partial r}|_v$ and $m_{\underline{\mathcal{C}}} = -\frac{\partial}{\partial v}|_r$. We obtain

$$\int_{\underline{\mathcal{C}}_{v_0}(u)} (J^N(\psi) \cdot n_{\underline{\mathcal{C}}}) m_{\underline{\mathcal{C}}} \lrcorner \text{vol} = \int_{r(u)}^{r_+} \int_0^\pi \int_0^{2\pi} (J^N(\psi) \cdot n_{\underline{\mathcal{C}}})|_{v=v_0} r^2 \sin \theta \, d\varphi \, d\theta \, dr.$$

Moreover, there exists a constant $C(\psi) > 0$ such that $J^N(\psi) \cdot n_{\underline{\mathcal{C}}} \leq C(\psi)$ holds

¹⁰²This is the Stokes' orientation in the energy estimate (4.5.3) in the proof of the next lemma.

on $\underline{\mathcal{C}}_{v_0}(u_0)$. In case that ψ vanishes to order¹⁰³ k on $\underline{\mathcal{C}}_{v_0}(u_0) \cap \mathcal{H}^+$, we even have $J^N(\psi) \cdot n_{\underline{\mathcal{C}}} \leq C(\psi) \cdot (r_+ - r)^{2k}$ on $\underline{\mathcal{C}}_{v_0}(u_0)$ for some $C(\psi) > 0$.

We thus obtain

$$\int_{r(u)}^{r_+} \int_0^\pi \int_0^{2\pi} (J^N(\psi) \cdot n_{\underline{\mathcal{C}}})|_{v=v_0} r^2 \sin \theta \, d\varphi \, d\theta \, dr \leq \int_{r(u)}^{r_+} C(\psi) \, dr \leq C(\psi) [r_+ - r(u)]$$

in the general case, and

$$\begin{aligned} \int_{r(u)}^{r_+} \int_0^\pi \int_0^{2\pi} (J^N(\psi) \cdot n_{\underline{\mathcal{C}}})|_{v=v_0} r^2 \sin \theta \, d\varphi \, d\theta \, dr &\leq \int_{r(u)}^{r_+} C(\psi) [r_+ - r]^{2k} \, dr \\ &\leq C(\psi) [r_+ - r(u)]^{2k+1} \end{aligned}$$

in the case that ψ vanishes to order k along $\underline{\mathcal{C}}_{v_0}(u_0) \cap \mathcal{H}^+$.

To conclude the lemma, we find the dependence of r on u : from (4.3.1) it follows that there is a $C > 0$ such that

$$r^*(r) + C \geq \frac{1}{2\kappa_+} \log(r_+ - r) \quad \text{on } \underline{\mathcal{C}}_{v_0}(u_0).$$

Hence, on $\underline{\mathcal{C}}_{v_0}(u_0)$ we have

$$r_+ - r \leq e^{2\kappa_+ r^* + 2\kappa_+ C} \leq C \cdot e^{2\kappa_+ r^*}.$$

Finally, we note that on $\underline{\mathcal{C}}_{v_0}(u_0)$ one has the relation $r^* = \frac{v_0 - u}{2}$, and hence

$$r_+ - r \leq C \cdot e^{-\kappa_+ u}.$$

This proves the lemma. □

The next lemma captures the red-shift effect at the event horizon \mathcal{H}^+ . It shows that if the wave propagates along the event horizon, and we measure its energy along a surface of constant r close enough to the event horizon \mathcal{H}^+ , we pick up additional exponential decay in u .

Lemma 4.5.2. *For all $\delta > 0$ there exists a smooth and future directed timelike vector field N in $M \cup \mathcal{H}^+$ with $[N, T] = 0$, an $r_0 < r_+$ (close to r_+) and a constant $C > 0$ (depending on v_0) such that*

$$\int_{\Sigma_{r_0}(u)} J^N(\psi) \cdot n_{\Sigma_{r_0}} \leq C \cdot e^{-(1-\delta)\kappa_+ \cdot u} \int_{\underline{\mathcal{C}}_{v_0}(u)} J^N(\psi) \cdot n_{\underline{\mathcal{C}}}$$

¹⁰³Recall that we say that a function vanishes on some subset to order k , if all its partial derivatives up to and including order k vanish on this subset.

holds for all $\psi \in C^\infty(M \cup \mathcal{H}^+, \mathbb{C})$ satisfying the wave equation $\square\psi = 0$ and vanishing on $\mathcal{H}^+(v_0)$.

Proof. Let us recall the construction of the red-shift vector field due to Dafermos and Rodnianski, see [26] and [22], Chapter 3.3.2: for every $\sigma > 0$ one can find a spherically symmetric and time translation invariant vector field Y which satisfies on the event horizon \mathcal{H}^+

- $\langle Y, Y \rangle = 0$, $\langle Y, T \rangle = -1$ and Y is orthogonal to the spheres of spherical symmetry.
- $$K^Y(\psi) \geq \kappa_+ \mathbb{T}(\psi)(Y, Y) + \sigma \mathbb{T}(\psi)(T, T + Y) - c \mathbb{T}(\psi)(T, T + Y) - c \sqrt{\mathbb{T}(\psi)(T, T + Y) \mathbb{T}(\psi)(Y, Y)},$$
 where $c > 0$ is independent of σ .¹⁰⁴

Given $\delta > 0$ we now choose σ big enough such that

$$K^Y(\psi) \geq \left(1 - \frac{\delta}{2}\right) \kappa_+ \left[\mathbb{T}(\psi)(Y, Y) + \mathbb{T}(\psi)(T, T + Y) \right]$$

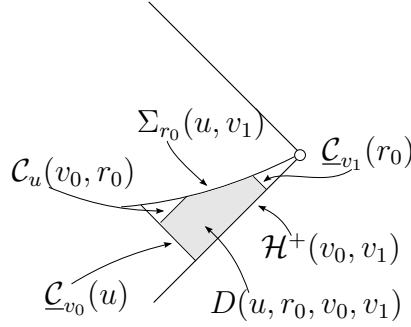
holds on \mathcal{H}^+ . We now set $N = Y + T$ and $N_w := e^{(1-\delta)\kappa_+ v} \cdot N$. Since by (4.3.2) we have $(dv)^\sharp = \frac{\partial}{\partial r}|_v = -Y$ on the event horizon, it follows that

$$\begin{aligned} K^{N_w}(\psi) &= e^{(1-\delta)\kappa_+ v} \left(K^N(\psi) + (1 - \delta) \kappa_+ \mathbb{T}(\psi)(N, (dv)^\sharp) \right) \\ &\geq e^{(1-\delta)\kappa_+ v} \left(\left(1 - \frac{\delta}{2}\right) \kappa_+ \left[\mathbb{T}(\psi)(Y, Y) + \mathbb{T}(\psi)(T, T + Y) \right] \right. \\ &\quad \left. - (1 - \delta) \kappa_+ \left[\mathbb{T}(\psi)(Y, Y) + \mathbb{T}(\psi)(T, Y) \right] \right) \\ &\geq e^{(1-\delta)\kappa_+ v} \frac{\delta}{2} \kappa_+ \left[\mathbb{T}(\psi)(Y, Y) + \mathbb{T}(\psi)(T, T + Y) \right] \end{aligned}$$

holds on the event horizon \mathcal{H}^+ . Since the right hand side is positive definite in $d\psi$, and T and Y do not depend on v , there is an $r_0 < r_+$ such that $K^{N_w}(\psi) \geq 0$ in $\{r_0 \leq r \leq r_+\}$. The energy estimate with multiplier N_w in the shaded region depicted below reads

$$\begin{aligned} \int_{\Sigma_{r_0}(u, v_1)} J^{N_w}(\psi) \cdot n_{\Sigma_{r_0}} + \int_{\mathcal{C}_u(v_0, r_0)} J^{N_w}(\psi) \cdot n_{\mathcal{C}_u} + \int_{\underline{\mathcal{C}}_{v_1}(r_0)} J^{N_w}(\psi) \cdot n_{\underline{\mathcal{C}}_{v_1}} \\ + \int_{D(u, r_0, v_0, v_1)} K^{N_w}(\psi) \\ = \int_{\underline{\mathcal{C}}_{v_0}(u)} J^{N_w}(\psi) \cdot n_{\underline{\mathcal{C}}_{v_0}} + \int_{\mathcal{H}^+(v_0, v_1)} J^{N_w}(\psi) \cdot n_{\mathcal{H}^+}. \end{aligned} \tag{4.5.3}$$

¹⁰⁴Here we have used the notation $K^Y(\psi) := \mathbb{T}(\psi)_{\mu\nu} \nabla^\mu Y^\nu$, where $\mathbb{T}(\psi)$ is the stress-energy tensor of ψ .



After taking the limit $v_1 \rightarrow \infty$ we thus obtain

$$\int_{\Sigma_{r_0}(u)} e^{(1-\delta)\kappa_+ \cdot v} J^N(\psi) \cdot n_{\Sigma_{r_0}} \leq \int_{\mathcal{C}_{v_0}(u)} e^{(1-\delta)\kappa_+ \cdot v_0} J^N(\psi) \cdot n_{\mathcal{C}_{v_0}} .$$

On $\{r = r_0\}$ we have $u = v - 2r^*(r_0)$ and hence

$$e^{(1-\delta)\kappa_+[u+2r^*(r_0)]} \int_{\Sigma_{r_0}(u)} J^N(\psi) \cdot n_{\Sigma_{r_0}} \leq e^{(1-\delta)\kappa_+ \cdot v_0} \int_{\mathcal{C}_{v_0}(u)} J^N(\psi) \cdot n_{\mathcal{C}_{v_0}} ,$$

from which the lemma follows. \square

Let us summarise the progress in the proof of Theorem 4.4.1 we have made so far. Under the assumptions *i*) of Theorem 4.4.1, the Lemmata 4.5.1 and 4.5.2 show that for all $\delta > 0$ there is a timelike vector field N with $[N, T] = 0$, an $r_0 < r_+$, and a constant $C > 0$ (depending on ψ) such that the following holds:

$$\int_{\Sigma_{r_0}(u)} J^N(\psi) \cdot n_{\Sigma_{r_0}} \leq C \cdot e^{-(2-\delta)\kappa_+ \cdot u} . \quad (4.5.4)$$

Moreover, given the assumptions *ii*) of Theorem 4.4.1, the Lemmata 4.5.1 and 4.5.2 show that there is a timelike vector field N with $[N, T] = 0$, an $r_0 < r_+$, and for every $k \in \mathbb{N}$ there is a constant $C > 0$ (depending also on ψ) such that

$$\int_{\Sigma_{r_0}(u)} J^N(\psi) \cdot n_{\Sigma_{r_0}} \leq C \cdot e^{-k \cdot \kappa_+ \cdot u} \quad (4.5.5)$$

holds.

The next lemma shows that the statements (4.5.4) and (4.5.5) also hold true with r_0 replaced by an arbitrary $r_1 > r_-$. Note here that the energies $\int_{\Sigma_{r_0}(u)} J^{N_1}(\psi) \cdot n_{\Sigma_{r_0}}$ and $\int_{\Sigma_{r_0}(u)} J^{N_2}(\psi) \cdot n_{\Sigma_{r_0}}$ are comparable for all smooth future directed timelike vector fields N_1 and N_2 that satisfy $[N_1, T] = 0 = [N_2, T]$.

Lemma 4.5.6. *Given $r_- < r_1 < r_0 < r_+$ and a smooth and future directed timelike*

vector field N with $[N, T] = 0$, there exists a constant $C > 0$ such that

$$\int_{\Sigma_{r_1}(u)} J^N(\psi) \cdot n_{\Sigma_{r_1}} \leq C \cdot \int_{\Sigma_{r_0}(u)} J^N(\psi) \cdot n_{\Sigma_{r_0}}$$

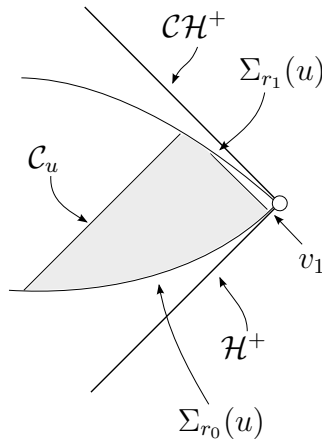
holds for all solutions $\psi \in C^\infty(M, \mathbb{C})$ of the wave equation $\square\psi = 0$.

Proof. Since N is invariant under the flow of the Killing vector field T , one has

$$K^N(\psi) \geq -C(r_0, r_1)\mathbb{T}(\psi)(N, (dr)^\sharp)$$

for some constant $C(r_0, r_1) > 0$. The lemma follows from the energy estimate with multiplier $N_w := e^{C(r_0, r_1) \cdot r} \cdot N$ in the shaded region depicted below after letting v_1 go to infinity. Note here that

$$K^{N_w}(\psi) = e^{C(r_0, r_1) \cdot r} \left[K^N(\psi) + C(r_0, r_1)\mathbb{T}(\psi)(N, (dr)^\sharp) \right] \geq 0.$$



□

Continuing the proof of Theorem 4.4.1, and remembering what we have proved so far, the next lemma implies that under the assumptions *i)* of Theorem 4.4.1, we have that for all $\delta > 0$, for all $r_1 > r_-$ and for all smooth and future directed timelike vector fields N that satisfy $[N, T] = 0$, there exists a constant $C > 0$ such that the following holds:

$$\int_{\Sigma_{r_1}(u)} e^{(2-\delta)\kappa_+ \cdot u} \cdot J^N(\psi) \cdot n_{\Sigma_{r_1}} \leq C.$$

Under the assumptions *ii)*, we obtain that for all $r_1 > r_-$, for all smooth and future directed timelike vector fields N that satisfy $[N, T] = 0$, and for all $k \in \mathbb{N}$, there exists a constant $C > 0$ such that

$$\int_{\Sigma_{r_1}(u)} e^{k \cdot \kappa_+ \cdot u} \cdot J^N(\psi) \cdot n_{\Sigma_{r_1}} \leq C$$

holds.

Lemma 4.5.7. *Let $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be a measurable function and $a > 0$ such that $\int_u^\infty f(\tilde{u}) d\tilde{u} \leq C \cdot e^{-a \cdot u}$ holds for all $u \in \mathbb{R}_+$, where $C > 0$ is some constant. It then follows that*

$$\int_{u_0}^\infty e^{b \cdot u} f(u) du < \infty$$

holds for all $b < a$ and $u_0 \in \mathbb{R}_+$.

Proof. Without loss of generality assume that $b > 0$. Then

$$\begin{aligned} \int_{u_0}^\infty e^{b \cdot u} f(u) du &\leq \sum_{n \in \mathbb{N}} \int_n^{n+1} e^{b \cdot u} f(u) du \\ &\leq \sum_{n \in \mathbb{N}} e^{b \cdot (n+1)} \int_n^{n+1} f(u) du \\ &\leq \sum_{n \in \mathbb{N}} e^{b \cdot (n+1)} \cdot C \cdot e^{-a \cdot n} \\ &\leq C \cdot e^b \cdot \sum_{n \in \mathbb{N}} e^{-(a-b) \cdot n} \\ &< \infty \end{aligned}$$

□

The next proposition finishes the proof of Theorem 4.4.1.

Proposition 4.5.8. *For every $\kappa < \kappa_- < 0$ there is an $r_1 > r_-$ (close to r_-), a smooth and future directed timelike vector field N in $M \cup \mathcal{CH}^+$ with $[N, T] = 0$, and a constant $C > 0$ such that*

$$\int_{\mathcal{C}_{u_0}(r_1)} J^N(\psi) \cdot n_{\mathcal{C}_{u_0}} + \int_{\mathcal{CH}^+(u_0)} J^N(\psi) \cdot n_{\mathcal{CH}^+(u_0)} \leq C \cdot \int_{\Sigma_{r_1}(u_0)} e^{-\kappa u} J^N(\psi) \cdot n_{\Sigma_{r_1}} \quad (4.5.9)$$

holds for all solutions $\psi \in C^\infty(M \cup \mathcal{H}^+, \mathbb{C})$ of the wave equation $\square \psi = 0$.

Here, we have used the notation $\mathcal{C}_{u_0}(r_1) := \mathcal{C}_{u_0} \cap \{r_- \leq r \leq r_1\}$.

Let us remark that we actually prove the stronger statement (4.5.11) which contains an exponentially weighted energy on $\mathcal{CH}^+(u_0)$. However, for us the most interesting aspect of the statement (4.5.9) is the boundedness of the energy flux through \mathcal{C}_{u_0} , i.e., the first term in (4.5.9).

Also note that a priori the wave ψ is not defined on the Cauchy horizon \mathcal{CH}^+ . The second term in (4.5.9) is to be interpreted as the limit

$$\limsup_{r_2 \searrow r_-} \int_{\Sigma_{r_2}(u_0)} J^N(\psi) \cdot n_{\Sigma_{r_2}},$$

which will become clear from the proof.

Proof. Let us define $\bar{T} := -T = -\frac{\partial}{\partial t}$, which is future directed at \mathcal{CH}^+ . In the following we construct a time-translation invariant vector field N in an r -neighbourhood of the horizon such that $K^N(\psi)|_{\mathcal{CH}^+}$ is negative only in the derivatives of ψ that are transversal to the horizon, and positive in all the derivatives of ψ that are tangential to the horizon. The construction is very similar to the construction of the red-shift vector field by Dafermos and Rodnianski.

First note that one can find a time-translation invariant and spherically symmetric vector field Y that satisfies on the Cauchy horizon \mathcal{CH}^+

- $\langle Y, Y \rangle = 0$, $\langle Y, \bar{T} \rangle = -1$, and Y is orthogonal to the spheres of spherical symmetry
- $\nabla_Y Y = -\sigma(Y + \bar{T})$.

Choosing a local frame field E_1, E_2 for the orbits of spherical symmetry which commutes with \bar{T} , and noting that on the Cauchy horizon \mathcal{CH}^+ one has $\nabla_{\bar{T}} \bar{T} = \kappa_- \bar{T}$, it is easy to show that there are real numbers $a^1, a^2, h_1^1, h_1^2, h_2^1, h_2^2$ such that the following holds on \mathcal{CH}^+ :

$$\nabla_{\bar{T}} Y = -\kappa_- Y + a^1 E_1 + a^2 E_2$$

$$\nabla_Y Y = -\sigma \bar{T} - \sigma Y$$

$$\nabla_{E_1} Y = h_1^1 E_1 + h_1^2 E_2 - a^1 Y$$

$$\nabla_{E_2} Y = h_2^1 E_1 + h_2^2 E_2 - a^2 Y .$$

It follows that

$$\begin{aligned} K^Y(\psi) &= \kappa_- \mathbb{T}(Y, Y) - a^1 \mathbb{T}(Y, E_1) - a^2 \mathbb{T}(Y, E_2) \\ &\quad + \sigma \mathbb{T}(\bar{T}, \bar{T}) + \sigma \mathbb{T}(\bar{T}, Y) \\ &\quad + h_1^1 \mathbb{T}(E_1, E_1) + h_1^2 \mathbb{T}(E_1, E_2) - a^1 \mathbb{T}(E_1, Y) \\ &\quad + h_2^1 \mathbb{T}(E_2, E_1) + h_2^2 \mathbb{T}(E_2, E_2) - a^2 \mathbb{T}(E_2, Y) \end{aligned}$$

holds on the Cauchy horizon \mathcal{CH}^+ . We now note that only the first term on the right hand side contains $(Y\psi)^2$, and that $\mathbb{T}(\bar{T}, \bar{T} + Y)$ controls $(\bar{T}\psi)^2$, $(E_1\psi)^2$, and $(E_2\psi)^2$. We can thus find a constant $c > 0$ (that is independent of σ !) such that the following holds on \mathcal{CH}^+ :

$$K^Y(\psi) \geq \kappa_- \mathbb{T}(Y, Y) + \sigma \mathbb{T}(\bar{T}, \bar{T} + Y) - c \mathbb{T}(\bar{T}, \bar{T} + Y) - c \sqrt{\mathbb{T}(\bar{T}, \bar{T} + Y) \mathbb{T}(Y, Y)} .$$

We now choose a $\delta > 0$ such that $\kappa < \kappa_-(1 + \delta)$ and choose $\sigma > 0$ big enough such that

$$K^Y(\psi) \geq \kappa_- \left(1 + \frac{\delta}{2}\right) \mathbb{T}(Y, Y) - \kappa_- \frac{\delta}{2} \mathbb{T}(\bar{T}, \bar{T} + Y)$$

holds on \mathcal{CH}^+ . The vector field $N := Y + \bar{T}$ is then timelike in a sufficiently small r -neighbourhood of the Cauchy horizon. Defining $N_w := e^{-\kappa_-(1+\delta)u} \cdot N$, we compute on \mathcal{CH}^+

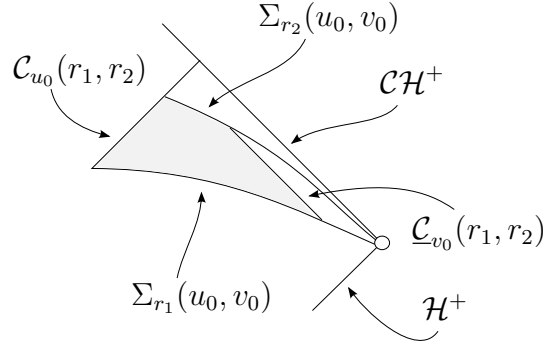
$$\begin{aligned} K^{N_w}(\psi) &= e^{-\kappa_-(1+\delta)u} \left[K^N(\psi) - \kappa_-(1+\delta) \mathbb{T}(N, (du)^\#) \right] \\ &\geq e^{-\kappa_-(1+\delta)u} \left[\kappa_-(1 + \frac{\delta}{2}) \mathbb{T}(Y, Y) - \kappa_- \frac{\delta}{2} \mathbb{T}(\bar{T}, \bar{T} + Y) - \kappa_-(1+\delta) \mathbb{T}(Y + \bar{T}, Y) \right] \\ &\geq e^{-\kappa_-(1+\delta)u} \left[-\kappa_- \frac{\delta}{2} (\mathbb{T}(Y, Y) + \mathbb{T}(\bar{T}, \bar{T} + Y)) \right], \end{aligned}$$

where we have used that $(du)^\# = -\frac{\partial}{\partial r}|_u = Y$ on the Cauchy horizon, cf. (4.3.3). The term in the square brackets is positive definite in $d\psi$, and since Y and \bar{T} are invariant under the flow of T , there is an $r_1 > r_-$ (close to r_-) such that $K^{N_w}(\psi) \geq 0$ in $\{r_- \leq r \leq r_1\}$.

The energy estimate with multiplier N_w in the shaded region depicted below yields

$$\begin{aligned} &\int_{\mathcal{C}_{u_0}(r_1, r_2)} e^{-\kappa_-(1+\delta)u_0} J^N(\psi) \cdot n_{\mathcal{C}_{u_0}} + \int_{\Sigma_{r_2}(u_0, v_0)} e^{-\kappa_-(1+\delta)u} J^N(\psi) \cdot n_{\Sigma_{r_2}} \\ &\leq \int_{\Sigma_{r_1}(u_0, v_0)} e^{-\kappa_-(1+\delta)u} J^N(\psi) \cdot n_{\Sigma_{r_1}}. \end{aligned} \quad (4.5.10)$$

Here we have used the notation $\Sigma_r(u_0, v_0) := \Sigma_r \cap \{u \geq u_0\} \cap \{v \leq v_0\}$, $\mathcal{C}_{u_0}(r_1, r_2) := \mathcal{C}_{u_0} \cap \{r_1 \leq r \leq r_2\}$, and $\mathcal{C}_{v_0}(r_1, r_2) := \mathcal{C}_{v_0} \cap \{r_1 \leq r \leq r_2\}$.



Letting first tend v_0 to infinity in (4.5.10), and thereafter $r_2 \rightarrow r_-$, we obtain

$$\begin{aligned} &e^{-\kappa_-(1+\delta)u_0} \int_{\mathcal{C}_{u_0}(r_1)} J^N(\psi) \cdot n_{\mathcal{C}_{u_0}} + \int_{\mathcal{CH}^+(u_0)} e^{-\kappa_-(1+\delta)u} J^N(\psi) \cdot n_{\mathcal{CH}^+} \\ &\leq \int_{\Sigma_{r_1}(u_0)} e^{-\kappa_-(1+\delta)u} J^N(\psi) \cdot n_{\Sigma_{r_1}}. \end{aligned} \quad (4.5.11)$$

This finishes the proof of Proposition 4.5.8. \square

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