On the statistical properties of random causal frequency response functions

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Abstract

In nuclear physics it is known that under broad restrictions a random scattering matrix element H satisfies a condition known as the analyticity-ergodicity (AE) requirement, which states that $E[f(H)] = f(E[H])$, where f is some function of H , and $E[\cdot]$ represents the ensemble average. A scattering matrix element is directly analogous to a vibrational frequency response function, and it is of significant interest to consider whether the AE requirement is also applicable to random engineering systems. The proof of the AE condition in nuclear physics rests on the assumptions that H is causal and ergodic: causality implies that a Lorentzian frequency average satisfies the AE equation, and ergodicity implies that Lorentzian frequency averages are equal to ensemble averages. In vibrational systems it is readily shown that a typical frequency response function is non-stationary and non-ergodic, so that the Lorentzian and ensemble averages can differ significantly, and this means that the standard proof of the AE requirement breaks down. The question then arises as to whether the AE requirement might nonetheless apply to vibrational systems. It is shown in the present paper that the requirement does apply providing that the random point process representing the system natural frequencies is at least locally stationary (which is a much weaker condition than local stationarity of the frequency response function), and a number of the implications of this result are explored.

Keywords: frequency response functions; random systems; response statistics

1. Introduction

The vibration response of an engineering system can be extremely sensitive to manufacturing variability, to the extent that the frequency response functions (FRFs) of the system can be considered to be random over an ensemble of manufactured items. Ideally the statistical properties of the FRFs should be calculated at the design stage to ensure that the reliability and performance of the system will meet the required targets. In general, the statistics of the FRFs will depend on the statistical distribution of the manufacturing uncertainties, and it can be an extremely difficult task to (i) statistically quantify the manufacturing uncertainties, and (ii) propagate these uncertainties through a computational model of the system to yield the response statistics. However, if the system is sufficiently random then a number of “universal” statistical laws may be applicable, making it possible to avoid much of the difficulty involved in (i) and (ii). For example, it is known that the higher

natural frequencies and mode shapes of a random system often conform to the statistics of the Gaussian Orthogonal Ensemble (GOE) [1,2], and this fact can be used in conjunction with energy flow models (in particular, Statistical Energy Analysis [3]) to yield the mean and variance of the vibrational energies of the various system components [4]. The concern of the present paper is not with the system energy distribution, but rather with the statistical properties of the complex FRFs, and in particular, the aim is to explore possible universal properties of these functions. Previous work in this area includes that of Lyon regarding the variance of the modulus of a FRF [5] and the statistics of the phase of a FRF [6], and Skudrzyk [7] and Cremer and Heckl [8] regarding the mean value of a complex FRF. As explained below, these investigations are extended here by considering the potential applicability of a powerful statistical property of a FRF known as the analyticity-ergodic requirement.

A fundamental statistical property of a random FRF is the average value, and the average considered might be either a frequency average or an ensemble average. A frequency average can be taken over a rectangular frequency window, or for mathematical convenience a more complex window such as the Lorentzian weighting function can be used [9]. The Lorentzian weighting function, yielding the Lorentzian average, has the same shape as the Cauchy distribution [10] and is a bell-shaped curve with specified centre frequency and half-power bandwidth; the function has a single pole in the lower complex half-plane (and also a single pole in the upper half-plane), and because of this the Lorentzian average has remarkable properties when applied to a causal FRF (i.e. a function that is analytic in the lower half-plane). If the symbol $\langle x \rangle$ is used to represent the Lorentzian average of x , then under loose restrictions it is found that $\langle f(H) \rangle = f(\langle H \rangle)$ for a function f of a causal FRF H [9]. In considering a random FRF in an engineering context, the ensemble statistics of the function are of more concern than Lorentzian frequency averages taken on a single realization, and it is therefore of significant interest to consider whether the ensemble average will have the same properties as the frequency average. This issue has been addressed in nuclear physics literature in the context of random scattering matrices: it is generally assumed that H is an ergodic random function [11], so that the frequency and ensemble averages are equal (for a sufficiently wide frequency averaging window) meaning that the ensemble average has exactly the same properties as the Lorentzian average [9]. The condition $\langle f(H) \rangle = f(\langle H \rangle)$, when applied to ensemble averages, is known as the analyticity-ergodicity (AE) requirement, and Mello et. al. [9] have used this requirement in conjunction

with the principle of maximum entropy to yield the probability density function of a scattering matrix. More generally, the literature on random scattering matrices, with potential application to vibrational FRFs, is very extensive: references [12-14] are examples of review papers, and a very recent contribution [15] has derived exact results for the probability density function of the real or imaginary part of an off-diagonal element of a scattering matrix (analogous to a cross-admittance in vibration theory). The results reported in reference [15] are based on the assumption that the scattering object has either Gaussian Orthogonal Ensemble (GOE) or Gaussian Unitary Ensemble (GUE) statistical properties, and the AE requirement is not considered explicitly. The analysis of reference [15] is not immediately applicable to engineering systems, and despite the extent of the literature on random scattering matrices it is not clear whether the AE requirement will apply to vibrational FRFs, as discussed in what follows.

As noted above, the AE requirement is justified in nuclear physics by arguing that the properties of the Lorentzian average also hold for the ensemble average. For this to be a valid argument H must be ergodic, and this implies that H must also be a stationary random function [11], or at least approximately stationary over the window of the frequency averaging function. While this condition is likely to be met in nuclear physics, the condition is much less likely to apply in vibration and acoustics; in fact, in Section 2.2 of the present paper it is shown by example that the Lorentzian and ensemble averages can yield very different results for a simple vibrating system. It is therefore not possible to draw conclusions regarding the properties of the ensemble averaging process by appealing to ergodicity arguments. Nonetheless, simulations of random vibrational systems indicate a strong tendency for the ensemble average $E[\cdot]$ to have the same properties as the Lorentzian average, so that $E[f(H)] = f(E[H])$, and this raises the question of whether this is in some way a universal result, and if so, what conditions must apply for the result to hold. This issue is addressed in the present paper.

The properties of the Lorentzian average of an FRF, and its relation to the ensemble average, is considered in Section 2. It is shown that under certain conditions the two averages will be equal, but the conditions required for equality do not readily arise for vibrational systems. The properties of the ensemble average are then explored in Section 3 by using random point process theory [16] to model the system randomness. It is found that the ensemble average will have the same properties as the Lorentzian average providing the kernel functions that appear in the random point process theory are stationary. The

stationarity of the kernel functions does not imply that the FRF is stationary, and the condition is actually quite weak: the kernels need to be either locally stationary, or stationary under an unfolding transformation [1,17], and there is evidence to suggest that these conditions are met for highly random vibratory systems. It is further shown that stationarity of the kernel functions constrains, but does not fully determine, the statistical distribution of an FRF. A number of example applications are considered in Section 4, and the degree of randomness required for the predicted properties to hold is explored numerically. A summary of the findings of the work is then given in Section 5.

2. The Lorentzian frequency average

2.1 The properties of the Lorentzian average

The Lorentzian average of a function of frequency ω , $H(\omega)$ say, is defined as a weighted integral in the form

$$\langle H(\omega) | \omega_0, \Gamma \rangle = \int_{-\infty}^{\infty} H(\omega) W(\omega, \omega_0, \Gamma) d\omega / \int_{-\infty}^{\infty} W(\omega, \omega_0, \Gamma) d\omega, \quad (1)$$

where the Lorentzian weighting function W is given by

$$W(\omega, \omega_0, \Gamma) = \left[(\omega - \omega_0)^2 + \Gamma^2 \right]^{-1}. \quad (2)$$

The parameter ω_0 locates the peak of the weighting function, while the parameter Γ determines the width of the function: the half-power bandwidth (i.e. the width of the function at half the peak amplitude) is given by 2Γ . If $H(\omega)$ represents a frequency response function, then the associated impulse response function $h(t)$ is given by the Fourier transform relation

$$h(t) = \int_{-\infty}^{\infty} H(\omega) e^{i\omega t} d\omega. \quad (3)$$

If the impulse response is causal then by definition $h(t)$ is zero for $t < 0$ and this implies that $H(\omega)$ is analytic in the lower half-plane. The integrand in the numerator of Eq. (1) therefore has a single pole in the lower half-plane, located at $\omega = \omega_0 - i\Gamma$, and contour integration around a contour enclosing this half-plane yields

$$\langle H(\omega) | \omega_0, \Gamma \rangle = H(\omega_0 - i\Gamma). \quad (4)$$

It can similarly be demonstrated that the Lorentzian average of any causal function of $H(\omega)$ can be expressed as

$$\langle f(H) | \omega_0, \Gamma \rangle = f[H(\omega_0 - i\Gamma)], \quad (5)$$

from which it follows that

$$\langle f(H) | \omega_0, \Gamma \rangle = f(\langle H | \omega_0, \Gamma \rangle). \quad (6)$$

Thus the Lorentzian average of a function of H is equal to the value of the function when evaluated at the Lorentzian average of H . At first sight Eq. (6) is a very unexpected result, since it implies, for example

$$\langle H^2(\omega) | \omega_0, \Gamma \rangle = \langle H(\omega) | \omega_0, \Gamma \rangle^2. \quad (7)$$

Were $H(\omega)$ real then Eq. (7) would imply that the variance of the function along the frequency axis is zero, and therefore the function has a constant value. However, it should be recalled that in general $H(\omega)$ is a causal complex frequency response function; the real and imaginary parts of a causal function are related by the Kramers-Kronig relation [18], and it is this dependence between the real and imaginary parts that allows Eq. (7) to apply even when the function is non-constant.

It is interesting to consider whether Eq. (6) might apply to the *ensemble* average of a random frequency response function at a single specified frequency ω . This question provides the motivation for the following analysis: the conditions needed for the Lorentzian and ensemble averages of $H(\omega)$ to coincide are considered in the following sub-section, while the more general ensemble statistical properties of causal transfer functions are considered in Section 3.

2.2 Relation to the ensemble average

If the function $H(\omega)$ is a stationary random process with a correlation function that decays at large arguments, then it can be shown from the analysis presented in [11] and [19] that the function is ergodic, and the Lorentzian and ensemble averages are equal for $\Gamma \rightarrow \infty$. The random frequency response functions encountered in structural dynamics are generally not stationary over an extended frequency range, and the aim of the present section is to explore the relation between the two averages for this case. If $H(\omega)$ has singularities in the complex plane that are limited to simple poles, then the function can be expressed in the form of a Mittag-Leffler expansion [20], so that

$$H(\omega) = H(0) + \sum_j a_j \left(\frac{1}{\omega - \omega_j} + \frac{1}{\omega_j} \right), \quad (8)$$

where ω_j and a_j are respectively the j th pole and the associated residue. The fact that the impulse response function given by Eq. (3) is a real function implies that $H(\omega)$ must have the property $H(\omega) = H^*(-\omega)$, from which it follows that any pole ω_j must be accompanied by a second pole $-\omega_j^*$. This implies that Eq. (8) can be written in the form

$$H(\omega) = \sum_j \left\{ \frac{a_j}{\omega - \omega_j} - \frac{a_j^*}{\omega + \omega_j^*} \right\}, \quad (9)$$

where the summation now covers half the range of that in Eq. (8). In deriving Eq. (9) it has been assumed that $H(\infty) = 0$, so that the terms involving $H(0)$ and a_j / ω_j cancel in Eq. (8); this is equivalent to assuming that the impulse response function is finite at $t = 0$, which is the case for physical systems of interest. An example of Eq. (9) is afforded by a multi-degree-of-freedom vibrational system with proportional damping, for which the response to applied harmonic loading of frequency ω can be written in the form

$$H(\omega) = \sum_j \frac{g_j}{\omega_{nj}^2 - \omega^2 + 2i\beta_j \omega_{nj} \omega}, \quad (10)$$

where g_j depends on the force acting on the system, and ω_{nj} and β_j are respectively the undamped natural frequency and the damping ratio associated with the j th mode of vibration. Equation (10) can be rewritten as

$$H(\omega) = -(1/2) \sum_j \left\{ \frac{g_j \omega_{nj}^{-1} (1 - \beta_j^2)^{-1/2}}{\omega - [i\beta_j \omega_{nj} + \omega_{nj} (1 - \beta_j^2)^{1/2}]} - \frac{g_j \omega_{nj}^{-1} (1 - \beta_j^2)^{-1/2}}{\omega - [i\beta_j \omega_{nj} - \omega_{nj} (1 - \beta_j^2)^{1/2}]} \right\}, \quad (11)$$

which has exactly the form of Eq. (9), with

$$\omega_j = \omega_{nj} e^{i\phi_j}, \quad \phi_j = \tan^{-1}[\beta_j / (1 - \beta_j^2)^{1/2}]. \quad (12,13)$$

If the system damping is not proportional then Eq. (10) no longer applies; in this case the response of the system can be represented in terms of complex modes [21] and the resulting expression again has exactly the form of Eq. (9). For generality Eq. (9) rather than Eq. (10) will be adopted as the starting point for the following analysis.

The poles ω_j that appear in Eq. (9) can be written in the form

$$\omega_j = z_j e^{i\phi_j} \in \text{RUHP}, \quad (14)$$

where z_j is real and positive; with this convention, ω_j lies in the right upper half-plane (RUHP), while $-\omega_j^*$ lies in the left upper half-plane (LUHP). If the response function $H(\omega)$ is associated with a system that has random properties then the residues a_j and the amplitude and phase of the poles, z_j and ϕ_j , will vary randomly over the ensemble of systems. In this case the amplitudes z_j can be represented as a random point process along the positive real axis, i.e. the points z_j , ordered with $z_j < z_{j+1}$, fall randomly on the positive real axis with a statistical distribution which is determined by the ensemble statistics of the system properties. A key descriptor for a random point process is the number count $N(z)$, which gives the average number of points that fall below a specified value z ; the derivative of $N(z)$ gives the average density of the points, so that $n(z) = dN(z)/dz$ is the average number of points that lie in a unit interval centred on z . In structural dynamics $n(z)$ is referred to as the modal density of the system. Another key descriptor is the half-power bandwidth of a pole, B_j . An expression for this parameter can be derived by considering the modulus of a typical term in Eq. (9):

$$q(\omega) = \left| \frac{a_j}{\omega - \omega_j} \right|^2 = \frac{|a_j|^2}{\omega^2 + z_j^2 - 2\omega z_j \cos \phi_j} . \quad (15)$$

The function $q(\omega)$ is maximum when $\omega = z_j \cos \phi_j$, and the function has half the maximum value when $\omega = z_j \cos \phi_j \pm z_j \sin \phi_j$. The “width” of $q(\omega)$ between the two half-maximum points is by definition the half-power bandwidth B_j , and this is given by $2z_j \sin \phi_j$; for a proportionally damped structural dynamic system that obeys Eqs. (12) and (13) the result is $B_j = 2\beta_j \omega_{nj}$. The half-power bandwidth and the modal density can be combined to yield a third parameter known as the modal overlap factor $m(z_j) = B_j n(z_j)$, which represents the average number of poles that lie within the half-power bandwidth. Physically the modal overlap factor gives an indication of the number of poles that contribute significantly to Eq. (9): the excitation frequency ω lies inside the half-power bandwidth of approximately $m(\omega)$ poles. If the modal overlap factor is very low, then the function $H(\omega)$ displays very sharp peaks when plotted against frequency; if the modal overlap is high, then the function is a much smoother function of frequency. Given this background to the statistical features of the frequency response function $H(\omega)$, the extent to which the Lorentzian average of a single

realization of the function approximates the ensemble average of the function will be considered in what follows.

When the excitation frequency ω is positive, the poles in the LUHP will contribute little to the response yielded by Eq. (9) if the excitation frequency lies well outside the half-power bandwidth of each these poles, i.e. if $\omega > z_j(\sin \phi_j - \cos \phi_j)$ for each pole j . The right hand side of this inequality is negative unless $\phi_j > \pi/4$, which corresponds to $\beta_j > 1/\sqrt{2}$ in the case of a structural dynamic system. Typical systems have damping levels that are very much less than this value, and thus as an algebraic simplification it will be assumed that Eq. (9) can be approximated as

$$H(\omega) = \sum_j \left\{ \frac{a_j}{\omega - \omega_j} - \frac{a_j^*}{\omega + \omega_j^*} \right\} \approx \sum_j \frac{a_j}{\omega - \omega_j}. \quad (16)$$

The Lorentzian average of the frequency response function can then be expressed as

$$\langle H(\omega) | \omega_0, \Gamma \rangle = H(\omega_0 - i\Gamma) = \sum_j \frac{a_j}{\omega_0 - i\Gamma - \omega_j}. \quad (17)$$

The effect of Γ in the denominator of Eq. (17) is to increase the half-power bandwidth of a pole from $B_j = 2z_j \sin \phi_j$ to approximately $B_j = 2(\Gamma + z_j \sin \phi_j)$, which causes an increase in the modal overlap factor of $m_{\text{add}} = 2\Gamma n(z)$. This implies that the increase in the modal overlap factor (the average number of modes in the half-power bandwidth of a pole) is given by the average number of modes in the half-power bandwidth (2Γ) of the Lorentzian weighting function. Under certain conditions Eq. (17) can be viewed as a trapezium approximation to the following integral:

$$H(\omega_0 - i\Gamma) \approx \int_0^\infty \left\{ \frac{\bar{a}(z)}{\omega_0 - i\Gamma - z \exp[i\phi(z)]} \right\} n(z) dz, \quad (18)$$

where

$$\bar{a}(z) \approx (1/m_{\text{eff}}) \sum_{j=j(z)-m_{\text{eff}}/2}^{j(z)+m_{\text{eff}}/2} a_j. \quad (19)$$

The term $\bar{a}(z)$ represents an average of the residues taken over those poles that make the most contribution to Eq. (17), i.e. those poles whose half-power bandwidth encloses ω ; following the earlier discussion there are approximately m_{eff} of these poles, where $m_{\text{eff}} = m_{\text{add}} + m$ is the total effective modal overlap of the Lorentzian average of the response. The conversion of a sum in the form of Eq. (17) to an integral in the form of Eq. (18) has

been performed previously by Cremer and Heckl [8], and the main requirement for accuracy is that the integrand in Eq. (18) is a smooth function of z , to the point where the function can be linearly interpolated between sample points $z = z_j$ with a high degree of accuracy. As discussed in reference [8], this requires $m_{\text{eff}} \gg 1$, and this condition will be met in the present case if $m_{\text{add}} = 2\Gamma n(z) \gg 1$, which means that the half-power bandwidth of the Lorentzian weighting function contains a large number of poles (modes). If in addition to this condition the filter centre frequency is such that $\omega_0 \gg \Gamma$, so that

$$1/(2n) \gg \Gamma, \quad (20)$$

then Eq. (18) can be approximated to

$$H(\omega_0 - i\Gamma) \approx \int_0^\infty \left\{ \frac{\bar{a}(z)}{\omega_0 - z \exp[i\phi(z)]} \right\} n(z) dz. \quad (21)$$

Now the *ensemble* average of the response given by Eq. (16) can be expressed immediately by using random point process theory [16]. The standard result (as given for example by Eq. (10) of reference [22]) is

$$E[H(\omega)] = \int_0^\infty \left\{ \frac{E[a(z)]}{\omega - z \exp[i\phi(z)]} \right\} n(z) dz. \quad (22)$$

If $m_{\text{eff}} \gg 1$ and the residues are ergodic, then Eq. (19) will approximate the ensemble average of the residues, so that $\bar{a}(z) \approx E[a(z)]$. Equations (21) and (22) then imply that

$$\langle H(\omega) | \omega_0, \Gamma \rangle \approx E[H(\omega_0)]. \quad (23)$$

Thus if the conditions indicated by Eq. (20) are met, together with the condition on $\bar{a}(z)$, then the Lorentzian average of $H(\omega)$ will be approximately equal to the ensemble average of the function. The full set of conditions can be stated as: (i) the half-power bandwidth of the Lorentzian weighting function contains a significant number of poles (modes), (ii) the excitation frequency of interest is much larger than the half-power bandwidth of the Lorentzian weighting function, (iii) the residues are approximately ergodic over the half-power bandwidth of the Lorentzian weighting function.

The validity of Eq. (23) can be explored by considering a numerical example concerning the bending vibration of a simply supported plate that is randomized by the addition of small masses. Full details of the plate are given in Section 4.1, but it is sufficient to note here that: the mean spacing of the plate resonances is 61.13 rad/s; the frequency range of interest is 0 to 6000 rad/s, and this range contains approximately 100 resonances; the

modal overlap factor varies linearly over the frequency range from 0 to 3. A single realization of the drive point response of the plate is shown in Fig. 1, together with the Lorentzian average of this realization and an ensemble average taken over 2000 realizations. The Lorentzian bandwidth factor Γ was set to 10 times the mean spacing, so that 20 modes are included in the half-power bandwidth. It can be seen that the Lorentzian average is in general close to, but not exactly equal to, the ensemble average – increasing or decreasing the value of Γ does not lead to improved agreement. The cross-response between a force and a response point that are separated by approximately 30% of the width of the plate is shown in Fig. 2. It can be noted that the two averaged curves tend to lie below the single realization; this can be explained by noting that neither the real nor the imaginary part of the frequency response function is positive definite for this example, and hence the modulus of the average value tends to be lower than the modulus of any one realization. For the drive point response shown in Fig.1, the imaginary part of the frequency response function is always negative (because the force must always input positive power), and the reduction in the overall level of the response due to averaging is less pronounced. For the case shown in Fig. 2 the Lorentzian average is in fairly poor agreement with the ensemble average, and the agreement could not be improved by increasing or decreasing the value of Γ . The disagreement in this case can be traced to the properties of the residues a_j : it can be shown [23] that the ensemble average of the residues in the vicinity of a frequency ω is proportional to the diffuse field correlation function $J_0(kr)$, where k is the wavenumber of the plate and r is the distance between the force and response points. When the distance is zero, as for the drive point response, then the result is independent of frequency, and the residues are ergodic. For the cross-response the ensemble average of the residues may vary over the Lorentzian averaging bandwidth, and the approximation $\bar{a}(z) \approx E[a(z)]$ is less likely to be accurate. Moreover, the residues are always positive for the drive point response, whereas they can oscillate in sign for the cross-response, meaning that in the latter case Eq. (17) may require a significant number of terms m_{eff} to produce a converged estimate of $\bar{a}(z)$; this requires a large averaging bandwidth, and $H(\omega)$ can be significantly non-stationary over this bandwidth. This example demonstrates that for structural dynamics it *cannot* be assumed that the Lorentzian and ensemble averages will be equal, and therefore it certainly cannot be assumed that the properties of the Lorentzian average represented by Eq. (6) will be shared by the ensemble average. However, it is shown in the next section that Eq. (6) *can* be applied to

ensemble averages, but the proof of this result does not rely on any equality between Lorentzian and ensemble averages.

3. The ensemble statistics of causal functions

3.1 Properties of the ensemble average

This section will consider the statistical properties of a set of causal frequency response functions $\{H_1(\omega), H_2(\omega), \dots, H_M(\omega)\}$ which all have the same set of simple poles. Each function can be written in the form of Eq. (16), which can be generalised slightly to read

$$H_r(\omega) = \sum_j a_j^{(r)} h_r(\omega, \omega - \omega_j), \quad r = 1, 2, \dots, M, \quad (24)$$

where the function $h_r(\omega, \omega - \omega_j)$ has a simple pole at $\omega = \omega_j$. The poles lie in the right upper half-plane, and they can be written in the form of Eq. (14) so that

$$z_j = |\omega_j|, \quad \phi_j = \arg(\omega_j). \quad (25a,b)$$

If the function $u_r(\omega, \phi_j, z_j)$ is defined so that

$$u_r(\omega, \phi_j, z_j) \equiv h_r(\omega, \omega - z_j e^{i\phi_j}), \quad (26)$$

then Eq. (24) can be written as

$$H_r(\omega) = \sum_j a_j^{(r)} u_r(\omega, \phi_j, z_j). \quad (27)$$

For future reference it can be noted that when viewed as a function of z_j , at fixed ω and ϕ_j , the function $u_r(\omega, \phi_j, z_j)$ will have a simple pole in the right *lower* half-plane.

The joint statistics of the frequency response functions can be explored by considering the characteristic functional [16,24], which is defined as

$$\varphi[\theta_1(\omega), \theta_2(\omega), \dots, \theta_M(\omega)] = E \left[\exp \left\{ i \sum_{r=1}^M \int_{-\infty}^{\infty} H_r(\omega) \theta_r(\omega) d\omega \right\} \right]. \quad (28)$$

The characteristic functional can be expressed in terms of the joint cumulants of the frequency response functions in the form [16,24]

$$\varphi = \exp \left\{ i \sum_{r=1}^M \int_{-\infty}^{\infty} \kappa_1[H_r(\omega)] \theta_r(\omega) d\omega + \frac{i^2}{2} \sum_{r=1}^M \sum_{s=1}^M \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \kappa_2[H_r(\omega'), H_s(\omega'_2)] \theta_r(\omega'_1) \theta_s(\omega'_2) d\omega'_1 d\omega'_2 + \dots \right\} \quad (29)$$

where $\kappa_n[\cdot]$ represents the n th joint cumulant: for example, $\kappa_1[H_r(\omega)]$ is the mean value of $H_r(\omega)$ and $\kappa_2[H_r(\omega'_1), H_s(\omega'_2)]$ is the covariance of the two functions H_r and H_s when evaluated at the frequencies ω'_1 and ω'_2 respectively. If the quantities z_j are taken to constitute a random point process and the phase quantities ϕ_j are (initially) taken to be deterministic, then the joint cumulants that appear in Eq. (29) can be written in the form [16]

$$\kappa_\alpha[H_{n_1}(\omega'_1), H_{n_2}(\omega'_2), \dots, H_{n_\alpha}(\omega'_\alpha)] = \int_0^\infty \int_0^\infty \dots \int_0^\infty k_{n_1 n_2 \dots n_\alpha}^{(\alpha)}(z_1, z_2, \dots, z_\alpha, \mathbf{a}) \times \\ u_{n_1}(\omega'_1, \phi_1, z_1) u_{n_2}(\omega'_2, \phi_2, z_2) \dots u_{n_\alpha}(\omega'_\alpha, \phi_\alpha, z_\alpha) dz_1 dz_2 \dots dz_\alpha, \quad (30)$$

where the kernel functions $k_{n_1 n_2 \dots n_\alpha}^{(\alpha)}(z_1, z_2, \dots, z_\alpha, \mathbf{a})$ are determined by the statistical properties of the random point process and the statistics of the residues $a_j^{(r)}$. The residues are assumed to have statistical properties that are independent of the index j , and in what follows, for ease of notation, a quantity such as $E[a_j^{(r)}]$ is abbreviated to $E[a^{(r)}]$. This assumption amounts to considering the residues $a_j^{(r)}$ to be at least locally stationary over the range of modes that contribute significantly to Eq. (27). This is generally a much weaker condition than assuming that the residues are stationary over the half-power bandwidth of the Lorentzian weighting function, which is a required condition for the validity of the relation $\bar{a}(z) \approx E[a(z)]$ of the previous section. Following Stratonovich [16], the first three kernel functions that appear in Eq. (30) can be written as

$$k_{n_1}^{(1)}(z_1, \mathbf{a}) = E[a^{(n_1)}] g_1(z_1), \quad (31)$$

$$k_{n_1 n_2}^{(2)}(z_1, z_2, \mathbf{a}) = E[a^{(n_1)} a^{(n_2)}] g_1(z_1) \delta(z_1 - z_2) + E[a^{(n_1)}] E[a^{(n_2)}] g_2(z_1, z_2), \quad (32)$$

$$k_{n_1 n_2 n_3}^{(3)}(z_1, z_2, z_3, \mathbf{a}) = E[a^{(n_1)} a^{(n_2)} a^{(n_3)}] g_1(z_1) \delta(z_1 - z_2) \delta(z_2 - z_3) \\ + 3 \{ E[a^{(n_1)} a^{(n_2)}] E[a^{(n_3)}] g_2(z_1, z_3) \delta(z_1 - z_2) \}_s \\ + E[a^{(n_1)}] E[a^{(n_2)}] E[a^{(n_3)}] g_3(z_1, z_2, z_3), \quad (33)$$

and the higher order kernels follow a similar pattern. In Eq. (33) the notation $\{\}_s$ denotes an average taken over the distinct permutations of the arguments, and the functions g_n are referred to by Stratonovich [16] as the correlation functions of the point process. The same functions are referred to by Lin [24] as the cumulant functions, and in random matrix theory [17] (with a very minor change in definition) the functions are referred to as the n -level cluster functions. Whatever terminology is adopted, the functions are dependent on the

statistical distribution of the poles of the system, which is in turn determined by the ensemble statistics of the system properties. It can be noted that: (i) in all cases the function g_1 corresponds to the modal density of the points, (ii) for a Poisson point process, $g_n = 0$ for $n > 1$, (iii) for a point process conforming to the Gaussian Orthogonal Ensemble, the functions g_n are available from random matrix theory [17]. Now by analogy with the approximation that underlies Eq. (16), it can be assumed that

$$u_r(-\omega, \phi_j, z_j) \approx u_r(\omega, \phi_j, z_j), \quad (34)$$

so that the integrals in Eq. (30) can be extended to an infinite range to yield

$$\begin{aligned} \kappa_\alpha[H_{n_1}(\omega'_1), H_{n_2}(\omega'_2), \dots, H_{n_\alpha}(\omega'_\alpha)] &\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} k_{n_1 n_2 \dots n_\alpha}^{(\alpha)}(z_1, z_2, \dots, z_\alpha, \mathbf{a}) \times \\ &u_{n_1}(\omega'_1, \phi_1, z_1) u_{n_2}(\omega'_2, \phi_2, z_2) \dots u_{n_\alpha}(\omega'_\alpha, \phi_\alpha, z_\alpha) dz_1 dz_2 \dots dz_\alpha. \end{aligned} \quad (35)$$

If the random point process is now taken to be stationary, then the kernel functions have the property

$$k_{n_1 n_2 \dots n_\alpha}^{(\alpha)}(z_1, z_2, \dots, z_\alpha, \mathbf{a}) = k_{n_1 n_2 \dots n_\alpha}'^{(\alpha)}(z_2 - z_1, \dots, z_\alpha - z_1, \mathbf{a}), \quad (36)$$

where $k_{n_1 n_2 \dots n_\alpha}'^{(\alpha)}$ is a modified kernel involving only the separation of the various poles. The joint cumulants of the frequency response functions can then be written in the form

$$\begin{aligned} \kappa_\alpha[H_{n_1}(\omega'_1), H_{n_2}(\omega'_2), \dots, H_{n_\alpha}(\omega'_\alpha)] &\approx \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} k_{n_1 n_2 \dots n_\alpha}'^{(\alpha)}(y_1, y_2, \dots, y_{\alpha-1}, \mathbf{a}) \times \\ &\left\{ \int_{-\infty}^{\infty} u_{n_1}(\omega'_1, \phi_1, z_1) u_{n_2}(\omega'_2, \phi_2, y_1 + z_1) \dots u_{n_\alpha}(\omega'_\alpha, \phi_\alpha, y_{\alpha-1} + z_1) dz_1 \right\} dy_1 dy_2 \dots dy_{\alpha-1}. \end{aligned} \quad (37)$$

It is important to note that Eq. (36) is a statement regarding the stationarity of the random point process, rather than the stationarity of $H_r(\omega)$: depending on the nature of $u_r(\omega, \phi_j, z_j)$, a stationary point process may lead to a non-stationary frequency response function $H_r(\omega)$. Regarding the stationarity of the point process, it is known that the modal density $n(z) \equiv g_1(z)$ of a number of common structural or acoustic components is not constant with frequency, and strictly Eq. (36) does not apply to such cases. The modal density of a bending beam, for example, is proportional to $z^{-1/2}$, while the modal density of an acoustic volume is proportional to z^2 [3,8]. However, there are two ways in which the use of Eq. (36) can be justified even for components of this type: (1) if the frequency of interest is sufficiently high, then the modal density can be considered to be locally constant over the modes that significantly contribute to the system response, and the same approximation can be applied to

the higher order kernel functions; (2) the process of *unfolding* can be applied, in which the variable z is transformed to a new variable (the number count N) which has a constant modal density – the transformed kernels are then found to be stationary to a remarkable degree for many systems [17]. If Eqs. (36) and (37) are accepted, then progress can be made by noting that: (i) each function $u_{n_j}(\omega'_j, \phi_j, y_{j-1} + z_1)$ has a single pole in the complex z_1 -plane and, following the discussion below Eq. (27), this pole lies in the lower half-plane, (ii) each function $u_{n_j}(\omega'_j, \phi_j, y_{j-1} + z_1)$ is analytic in the upper half-plane, (iii) the integral over z_1 around a half-circle in the upper half-plane, centred at the origin and of infinite radius, is zero providing $r > 1$, (iv) for $r=1$, the integral around the half-circle has a finite, non-zero, value. By performing the integral over z_1 around a contour enclosing the upper-half plane it then follows from Eq. (37) that

$$\kappa_\alpha[H_{n_1}(\omega'_1), H_{n_2}(\omega'_2), \dots, H_{n_\alpha}(\omega'_\alpha)] = 0 \quad \alpha > 1. \quad (38)$$

$$\kappa_1[H_r(\omega)] = E[H_r(\omega)] \neq 0. \quad (39)$$

Given that only the first cumulants are non-zero, it follows from Eqs. (28) and (29) that the characteristic functional of the set of frequency response functions has the form

$$\begin{aligned} \phi[\theta_1(\omega), \theta_2(\omega), \dots, \theta_M(\omega)] &\equiv E \left[\exp \left\{ i \sum_{r=1}^M \int_{-\infty}^{\infty} H_r(\omega) \theta_r(\omega) d\omega \right\} \right] \\ &= \exp \left\{ i \sum_{r=1}^M \int_{-\infty}^{\infty} \kappa_1[H_r(\omega)] \theta_r(\omega) d\omega \right\}. \end{aligned} \quad (40)$$

It follows immediately from this result (for example by a Taylor series expansion of the exponential function) that

$$E[H_{n_1}(\omega'_1) H_{n_2}(\omega'_2) \dots H_{n_\alpha}(\omega'_\alpha)] = E[H_{n_1}(\omega'_1)] E[H_{n_2}(\omega'_2)] \dots E[H_{n_\alpha}(\omega'_\alpha)], \quad (41)$$

$$E[H_r^s(\omega)] = E[H_r(\omega)]^s. \quad (42)$$

This implies that the property noted previously for the Lorentzian average in Eq. (6) also applies to the ensemble average if the function f is a polynomial function. As noted above Eq. (30), it has been assumed in the derivation of Eqs. (41) and (42) that the phase quantities ϕ_j (which determine the degree of damping in the system) are deterministic. This implies that the expectations in Eqs. (41) and (42) should be interpreted as conditional expectations, conditional upon specified values of the random phase quantities. However, it is well known that ensemble average quantities such as $E[H_{n_1}(\omega'_1)]$ are analogous to a non-resonant “direct field” response, or equivalently, to the response of an infinitely extended system [8,22]. This

response is independent of the system damping providing the damping is within reasonable limits, and hence the expectations on the right hand sides of Eqs. (41) and (42) can be interpreted as unconditional expectations. The left hand sides of these equations must also be independent of damping, and thus again the expectations can be interpreted as unconditional values. It then follows that Eqs. (41) and (42) are also valid in the case of random damping, expressed through random phase quantities ϕ_j .

At any specified frequency, a general function of a single frequency response function can be expanded as a Taylor series so that

$$f(H) = \sum_n c_n (H - H_0)^n. \quad (43)$$

where H_0 is the expansion point. The Taylor series will always converge providing

$$|H - H_0| < |H_p - H_0|, \quad (44)$$

for any realisation of the frequency response function H , where H_p is the pole of $f(H)$ in the complex H -plane that falls closest to H_0 . The fact that $H(\omega)$ is a random function of frequency implies that $f(H)$ can also be viewed as a random function of frequency, $f(H) \equiv f_1(\omega)$ say. The principle concern is whether Eq. (43) is convergent at a single specified frequency ω' say, where the argument of the function ranges over the ensemble of random values of H that can be realized at this frequency. If a random value should coincide with a pole of the function, so that $H(\omega') = H_p$, then this implies that the associated realization of the random function $f_1(\omega)$ has a singularity on the real axis at $\omega = \omega'$. This cannot occur if $f_1(\omega)$ is a causal function having all poles located in the upper half-plane. Thus, if $f(H)$ is a causal frequency response function then, at any specified frequency, the argument H can never fall on a pole H_p of the function; it is therefore likely that a value of H_0 can be found to ensure that the condition specified in Eq. (44) is met, and that Eq. (43) will converge. In this case, Eq. (42) implies that

$$E[f(H)] = f(E[H]). \quad (45)$$

Similarly, Eq. (41) implies that

$$E[f(H_{n_1}, H_{n_2}, \dots, H_{n_\alpha})] = f(E[H_{n_1}], E[H_{n_2}], \dots, E[H_{n_\alpha}]), \quad (46)$$

providing f can be expressed as a convergent Taylor series for any realisation of the set of frequency response functions. These results imply that the Lorentzian and ensemble

averages share the same properties, even though the results yielded by the two averaging processes may differ. The validity of Eqs. (45) and (46) is explored numerically in section 4 for a range of example systems.

3.2 The statistical distribution of causal functions

It is of interest to consider the degree to which Eq. (40) imposes constraints on the statistical distribution of a causal frequency response function. If the function is written in terms of the real and imaginary parts so that

$$H(\omega) = H_R(\omega) + iH_I(\omega), \quad (47)$$

and a single fixed frequency ω is considered, then the characteristic function, rather than the characteristic functional, can be considered. The characteristic function $M(\theta_R, \theta_I)$ is the Fourier transform of the joint probability density function of H_R and H_I , so that

$$M(\theta_R, \theta_I) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} p(H_R, H_I) \exp\{i[\theta_R H_R + \theta_I H_I]\} dH_R dH_I = E[\exp\{i[\theta_R H_R + \theta_I H_I]\}]. \quad (48)$$

It is implied by Eq. (40) that the characteristic function must have the property

$$M(\theta, i\theta) = \exp\{i\theta E[H_R + iH_I]\}, \quad (49)$$

which can also be expressed in the form

$$\ln M(\theta, i\theta) = i\theta \kappa_1[H_R] - \theta \kappa_1[H_I]. \quad (50)$$

Now in general the log of the characteristic function can be expressed in terms of the joint cumulants in the form

$$\ln M(\theta_1, \theta_2) = \sum_{\alpha=1}^{\infty} \left\{ \frac{1}{\alpha!} \sum_{n_1=1}^2 \sum_{n_2=1}^2 \dots \sum_{n_\alpha=1}^2 (i\theta_{n_1}) (i\theta_{n_2}) \dots (i\theta_{n_\alpha}) \kappa_\alpha[H_{n_1}, H_{n_2}, \dots, H_{n_\alpha}] \right\}, \quad (51)$$

where the notation $H_1 \equiv H_R$ and $H_2 \equiv H_I$ has been adopted for convenience. It is clear that Eq. (50) can only be satisfied if there are relations between the various joint cumulants of the frequency response function. For example, by comparing second order terms ($\alpha = 2$) in Eqs. (50) and (51) it can be deduced that

$$\kappa_2[H_R, H_R] = \kappa_2[H_I, H_I], \quad \kappa_2[H_I, H_R] = 0. \quad (52), (53)$$

Thus the real and imaginary parts of the frequency response function must be of equal variance, and they must be uncorrelated. Comparing terms of third order ($\alpha = 3$) in Eqs. (50) and (51) yields

$$\kappa_3[H_R, H_R, H_R] = 3\kappa_3[H_R, H_I, H_I], \quad \kappa_3[H_I, H_I, H_I] = 3\kappa_3[H_I, H_R, H_R]. \quad (54), (55)$$

Thus, if (for example) the skewness of the real part of the frequency response function is non-zero, then the joint skewness of the real and imaginary parts must also be non-zero. Within the constraints imposed by Eq. (50), the frequency response function can have a wide range of statistical distributions. This result is consistent with the fact that the frequency response function satisfies the Kramers-Kronig relations [18]: given that $H(\omega)$ is analytic in the lower half-plane, it follows that the real and imaginary parts of the function form a Hilbert transform pair, so that

$$H_I(\omega) = \frac{1}{\pi} \text{PV} \int_{-\infty}^{\infty} \frac{H_R(\omega')}{\omega - \omega'} d\omega' . \quad (56)$$

Equation (56) implies that the random process $H_R(\omega)$ completely determines the random process $H_I(\omega)$, and equations such as Eqs. (52) and (53) can be derived directly from this constraint between the two functions.

4. Numerical examples

4.1 Random plate with two ports

To illustrate the foregoing theory, an example system is considered which consists of a simply supported flat plate which is randomised by the addition of a number of small masses that are attached in random locations to generate an ensemble of systems. The out-of-plane motion of the plate at two locations $\mathbf{x}_1$ and $\mathbf{x}_2$ is investigated: the response at these locations (q_1 and q_2 say) to applied forces F_1 and F_2 of frequency ω can be written in the form

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \mathbf{H}(\omega) \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \quad (57)$$

where the admittance matrix $\mathbf{H}$ is given by

$$\mathbf{H}(\omega) = \sum_j \left\{ \left(\frac{1}{\omega_{nj}^2 - \omega^2 + 2i\beta_j\omega_{nj}\omega} \right) \begin{pmatrix} \varphi_j^2(\mathbf{x}_1) & \varphi_j(\mathbf{x}_1)\varphi_j(\mathbf{x}_2) \\ \varphi_j(\mathbf{x}_1)\varphi_j(\mathbf{x}_2) & \varphi_j^2(\mathbf{x}_2) \end{pmatrix} \right\}. \quad (58)$$

Here ω_{nj} is the j th natural frequency of the plate, φ_j is the j th (mass normalized) mode shape, β_j is the associated damping ratio, and proportional damping has been assumed. The modal parameters are random due to the presence of the small masses, and they can be computed by formulating the equations of motion of the plate using the Lagrange-Rayleigh-Ritz method,

with the modes of the bare plate employed as trial functions. The dynamic stiffness matrix of the plate is given by

$$\mathbf{D}(\omega) = [\mathbf{H}(\omega)]^{-1}. \quad (59)$$

Numerical simulations of the random matrices $\mathbf{H}$ and $\mathbf{D}$ can be used to test various findings of the foregoing analysis. To this end a steel rectangular plate of length $l_1 = 0.9\text{m}$, width $l_2 = 0.7\text{m}$ and thickness 2mm is considered. The Young's modulus is $2 \times 10^{11} \text{N/m}^2$, the density is 7800kg/m^3 , the Poisson ratio is 0.3, and the damping factor is set at $\beta_j = 0.015$. The two ports are located at $\mathbf{x}_1 = (0.3l_1 \ 0.43l_2)$ and $\mathbf{x}_2 = (0.23l_1 \ 0.71l_2)$. The randomization is realized by the addition of 10 point masses, randomly located on the plate with a uniform distribution, with each mass having 2% of the mass of the bare plate. In the results that follow, unless otherwise stated, averages have been taken over an ensemble of 2000 realizations. The modal density of the bare plate is $n(\omega) = 0.0164 \text{ modes/rad/s}$, which means that the average modal spacing is 61.13 rad/s. The frequency range considered (0 to 6000 rad/s) covers approximately 100 resonant frequencies, and over this frequency range the modal overlap factor $m = 2\beta_j\omega_j n$ varies linearly from 0 to 3.

Three realisations of the frequency response function $H_{11}(\omega)$ are shown in Fig. 3, together with the analytical result for the ensemble average value, which is given by $E[H_{11}(\omega)] = -i\pi n / (2\omega M_p)$, where M_p is the mass of the bare plate [8,22]. It can be seen that the randomization approach has produced a great deal of variability in the frequency response function. The modal density of the system has been estimated by calculating the ensemble average of the number count $N(\omega)$ (the number of natural frequencies below ω) and then differentiating numerically to yield $n(\omega) = dN / d\omega$. The result is shown in Fig. 4, where it can be seen that the calculated result tends to oscillate around the asymptotic estimate. The fluctuations are related to the geometry of the plate, and they persist despite the addition of the random masses.

In Fig. 5 the validity of Eq. (45) is explored for the particular case

$$f(H_{11}) = H_{11}^{-1} \quad \Rightarrow \quad E[H_{11}^{-1}] = E[H_{11}]^{-1}. \quad (60)$$

The two ensemble average results shown in the figure are converged (the results did not change appreciably with an increase in the size of the ensemble), and the oscillations in the curves mirror to some degree the modal density shown in Fig. 4. The accuracy of Eq. (60) is further explored in Fig. 6, where the percentage difference between the two average results is

shown, both for the default value of damping $\beta_j = 0.015$ and for an increased level $\beta_j = 0.03$. It can be seen that the level of departure from Eq. (60) is of the order of a few percent across most of the frequency range. The derivation of Eq. (60) presented in section 3.1 does not depend on the random natural frequency point processes having any particular distribution (such as the Gaussian Orthogonal Ensemble), but rather rests on the validity of the stationary condition, Eq. (36), either locally or in terms of unfolded variables. The results shown in Fig. 6 indicate that the stationary condition is met to an acceptable degree across most of the frequency range, despite the non-stationary nature of the modal density shown in Fig. 4. With increased damping the modal overlap factor increases, and there is a smoothing effect that leads to lower errors. The effect of the degree of randomization is explored in Fig. 7, which concerns the response of the system at $\omega = 610$ rad/s with $\beta_j = 0.03$. As usual, 10 masses are added to the system in random positions, but the total added mass is varied from 0% to 150% of the mass of the bare plate. The erratic nature of the curve shown in the figure is due to the very large number of samples needed for convergence – for this figure 80000 samples were employed (as opposed to the usual 2000 samples), and yet fully converged results have not been obtained. Nonetheless the curve is sufficiently accurate to demonstrate the physical effects of the randomisation. When no mass is added, then Eq. (60) holds perfectly, since the system is fully deterministic. The accuracy of Eq. (60) worsens with an increase in the added mass, since the system becomes random, but not random enough for the stationary condition, Eq. (36), to apply. When the added mass is increased further then the system becomes more random, Eq. (36) becomes valid, and hence the error in Eq. (60) becomes small. This reasoning also explains the error trend shown in Fig. 6: the short-wavelength higher modes are more easily randomized than the long-wavelength lower modes, and hence the error in Eq. (60) tends to decrease with increasing frequency.

The results shown in Fig. 8 relate to the off-diagonal component D_{12} of the dynamic stiffness matrix, where the concern is with the validity of the following expression:

$$E[f(\mathbf{H})] = E\left[\frac{-H_{12}}{H_{11}H_{22} - H_{12}H_{21}}\right] = \frac{-E[H_{21}]}{E[H_{11}]E[H_{22}] - E[H_{12}]E[H_{21}]} . \quad (61)$$

The numerical results confirm the validity of Eq. (61), despite the highly non-stationary character of the expectation. It can be noted that D_{12} is a causal function of the $\mathbf{H}$ matrix, and thus the existence of a convergent Taylor series expansion of the function can be expected, as noted below Eq. (44).

It is instructive to consider a case where Eq. (45) would *not* be expected to apply, namely

$$f(H_{12}) = H_{12}^{-1} . \quad (62)$$

The fact that the function H_{12} is a cross-admittance implies that the function can contain zeros in the lower half-plane. In this case $f(H_{12})$ contains poles in the lower half-plane and the function is not causal. It can therefore be anticipated that it is not possible to construct a convergent Taylor series around any point H_0 , and this will invalidate Eq. (45). This line of reasoning is confirmed by the results shown in Fig. 9, where there is little resemblance between the expectation of the function and the function evaluated at the expectation of the argument. The issues of a convergent Taylor series can be explored in more detail by considering the case

$$E[f(D_{11})] = E\left[\frac{1}{D_{11} - a}\right], \quad f(E[D_{11}]) = \frac{1}{E[D_{11}] - a} . \quad (63,64)$$

Numerical results for Eqs. (63) and (64) are compared in Fig. 10 for the two cases $a = 10^5 i$ and $a = 10^6 i$. For the larger value of a , Eqs. (63) and (64) are in agreement only at the highest frequencies; for the smaller value of a the results are in agreement only for frequencies higher than 1500 rad/s. These findings can be explained by reference to Fig. 11, which plots realizations of D_{11} in the complex plane for $\omega = 3000$ rad/s. Also indicated in the figure are the two values of a , in each case representing the location of the pole of the function $f(D_{11})$. It is clear that the larger pole lies within the cloud of points, while the smaller pole lies below the cloud of points; only in the second case can the function $f(D_{11})$ be expanded as a Taylor series which converges for all realisations of D_{11} , and thus Eqs. (63) and (64) can be expected to agree for this case only. The results shown in Fig. 10 are therefore fully consistent with the notion that Eq. (45) can only be applied if the function has a convergent Taylor series expansion.

4.2 Two-port deterministic/random system

The second example concerns a deterministic two-port system which is attached to the random plate of the previous example at a single point, as shown schematically in Fig. 12. If the two ports have displacement degrees of freedom q_1 and q_2 then the governing equations can be written in the form

$$\begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} + D_{\text{ran}} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ 0 \end{pmatrix}, \quad (65)$$

where K_{ij} are the entries of the dynamic stiffness matrix $\mathbf{K}$ of the deterministic system, and D_{ran} is the dynamic stiffness of the plate at the attachment point (D_{11} of the previous example). A force $F_1 = 1$ is applied to the top of the deterministic system: the force is harmonic with frequency ω , and both $\mathbf{K}$ and D_{ran} are functions of this frequency. The deterministic system is taken to be a mass-spring-damper system so that

$$\mathbf{K} = \begin{pmatrix} -\omega^2 M_{\text{det}} + i\omega C_{\text{det}} + K_{\text{det}} & -i\omega C_{\text{det}} - K_{\text{det}} \\ -i\omega C_{\text{det}} - K_{\text{det}} & i\omega C_{\text{det}} + K_{\text{det}} \end{pmatrix}, \quad (66)$$

where M_{det} , C_{det} and K_{det} are the mass, damping and stiffness parameters. As discussed by Woodhouse [25], this type of model can be used to represent a light bridge structure (the deterministic system) attached to the body of a violin (the plate), although the parameter values selected below are not representative of this application. The expected value of the response at the forcing point can be written as

$$E[q_1] = E \left[\frac{K_{22} + D_{\text{ran}}}{K_{11}(K_{22} + D_{\text{ran}}) - K_{12}^2} \right], \quad (67)$$

and if Eq. (45) is correct then this result should be equal to

$$q_1(E[D_{\text{ran}}]) = \frac{K_{22} + E[D_{\text{ran}}]}{K_{11}(K_{22} + E[D_{\text{ran}}]) - K_{12}^2}. \quad (68)$$

The equality of Eqs. (67) and (68) has been explored numerically for the case in which M_{det} is 0.5% of the mass of the plate, and the resonant frequency and damping ratio of the mass-spring-damper system are (when q_2 is restrained) respectively 2150 rad/s and 0.005. The plate has the properties considered in the previous section.

Three realisations of the frequency response function $q_1(\omega)$ are shown in Fig. 13, together with the analytical result obtained when D_{ran} is replaced in Eq. (65) by the analytical ensemble average, $E[D_{\text{ran}}] = i2\omega M_p / (\pi n)$. Clearly the frequency response function is highly random up to around 3000 rad/s, beyond which the function D_{ran} makes a small contribution to Eq. (65). The fact that the analytical curve appears to follow the mean value of the realizations suggests that Eq. (45) applies to this example, and this is confirmed by the

results shown in Fig. 14, which demonstrate the equality of Eqs. (67) and (68). The validity of Eq. (45) is to be expected, given that $q_1(\omega)$ is a causal function.

5. Conclusions

The main conclusions of the present work are as follows:

- 1) The Lorentzian and ensemble averages of a random FRF can differ significantly in vibration theory, meaning that vibrational FRFs are not ergodic in general.
- 2) Under very broad conditions, the analyticity-ergodicity condition of scattering matrix theory, $E[f(H)] = f(E[H])$, is applicable to a random causal FRF $H(\omega)$ even if the FRF is non-stationary and non-ergodic. The same result applies to functions of multiple FRFs, so that $E[\mathbf{f}(\mathbf{H})] = \mathbf{f}(E[\mathbf{H}])$ for a vector or matrix $\mathbf{H}(\omega)$ of causal FRFs. The result can also be applied to FRFs evaluated at different excitation frequencies, as in Eq. (41).
- 3) The broad conditions alluded to above are: (i) the point process describing the statistics of the system natural frequencies (poles) must be stationary, or approximately stationary for natural frequencies in the vicinity of the excitation frequency; (ii) the function $f(\mathbf{H})$ must have a convergent Taylor series expansion for all realisable values of $\mathbf{H}$. It can be noted that condition (i) does not require the FRFs to be stationary, and furthermore the condition can be weakened by requiring the point process to be stationary after “unfolding” [1,17]. It can also be noted that the condition does not require the system natural frequencies to conform to the GOE or some other universal distribution. Condition (ii) is likely to be met if $f(\mathbf{H})$ is causal, meaning that it is analytic in the lower half-plane when viewed as a function of ω .
- 4) It can be noted that the modal overlap factor plays a minimal role in determining the validity of the AE condition. The present analysis does not explicitly employ this parameter, and the AE condition has been confirmed numerically over a wide range of values of the modal overlap factor. Having said this, it can be noted that if the modal overlap is extremely high then the AE condition will be applicable in a trivial way: for very high modal overlap the ensemble variance of an FRF is extremely small [4] and the function becomes deterministic, so that $E[f(H)] = f(E[H])$ is clearly a true statement.

5) Numerical simulations have shown that the analyticity-ergodicity condition is surprisingly robust. The least favourable conditions are those in which the system natural frequencies are affected by randomness but the random shift in the natural frequencies is less than the natural frequency spacing, so that a near-stationary natural frequency point process is not achieved. For less randomness the condition is met because the system is virtually deterministic, for more randomness the condition is met because the point process becomes stationary. The effective randomness of a system can be measured by the “statistical overlap factor”, which is proportional to the standard deviation of a particular natural frequency divided by the mean frequency spacing. When the statistical overlap factor is high, then there is a high degree of mixing of modes across the ensemble, and this promotes stationary statistics. The effectiveness of a given amount of randomisation generally increases with increasing excitation frequency, since the higher modes of the structure are generally more sensitive to random perturbations in the system properties.

6) The analyticity-ergodicity condition implies that the characteristic function $M(\theta_r, \theta_i)$ of a complex FRF must have the form of Eq. (49). This specifies the function only for argument combinations of the type $M(\theta, i\theta)$, and so the condition does not fully determine the statistical distribution of the FRF. A number of constraints on the cumulants of the FRF are listed in section 3.2, and it is notable that the real and imaginary parts must be uncorrelated and be of equal variance.

These results have application to the statistical analysis of complex engineering systems that have manufacturing imperfections, and they complement existing results on the statistical properties of energy metrics [3,4].

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Fig. 1 The drive point response of a random plate: light solid curve – modulus of one realization of the response; dashed curve – modulus of the Lorentzian average of the realization; heavy solid curve – modulus of the ensemble average taken over 2000 realizations.

Fig. 2 The cross-response of a random plate: light solid curve – modulus of one realization of the response; dashed curve – modulus of the Lorentzian average of the realization; heavy solid curve – modulus of the ensemble average taken over 2000 realizations.

Fig.3 Three samples of the random frequency response function $|H_{11}(\omega)|$. The dashed curve represents the analytical prediction of $|E[H_{11}]|$ based on the asymptotic modal density.

Fig. 4 Numerical estimate of the modal density of the random plate (solid line) compared with the asymptotic modal density (dashed line).

Fig. 5 Modulus of each of the two ensemble averages $E[H_{11}]$ and $E[H_{11}^{-1}]^{-1}$ (solid curves, the first average has deeper troughs at low frequencies). The dashed curve represents the analytical prediction of the ensemble average based on the asymptotic modal density.

Fig. 6 Percentage difference between $E[H_{11}]$ and $E[H_{11}^{-1}]^{-1}$, defined in terms of the complex frequency response functions as $100 \times |(E[H_{11}] - E[H_{11}^{-1}]^{-1}) / E[H_{11}]|$. The solid curve is for $\beta_j = 0.015$, the dashed curve is for $\beta_j = 0.03$.

Fig. 7 Percentage difference $100 \times |(E[H_{11}] - E[H_{11}^{-1}]^{-1}) / E[H_{11}]|$ as a function of the total amount of random mass added to the plate (in 10 point masses). The results relate to $\beta_j = 0.03$ and $\omega = 610$ rad/s.

Fig. 8 Modulus of each of the two ensemble averages $E[D_{12}]$ and $(E[\mathbf{H}]^{-1})_{12}$ (the first average has slightly lower troughs at low frequencies) .

Fig. 9 Modulus of each of the two ensemble averages $E[H_{12}^{-1}]$ and $E[H_{12}]^{-1}$ (the second average is the higher curve) .

Fig. 10 Modulus of each of the two functions $E[(D_{11} - a)^{-1}]$ and $(E[D_{11}] - a)^{-1}$. The solid curves relate to $a = 10^5 i$, the dashed curves relate to $a = 10^6 i$. In each case the upper curve corresponds to $(E[D_{11}] - a)^{-1}$.

Fig. 11 Random samples of the function $D_{11}(\omega)$ shown in the complex plane, for the frequency $\omega = 3000$ rad/s. The two circles $\circ$ correspond to the two points $D_{11} = 10^5 i$ and $D_{11} = 10^6 i$.

Fig. 12 Schematic of a two-port deterministic system coupled to a random plate.

Fig.13 Three samples of the random frequency response function $|q(\omega)|$. The dashed curve represents the analytical prediction of the ensemble average based on the asymptotic modal density.

Fig. 14 Modulus of each of the two ensemble averages $E[q(D_{\text{ran}})]$ and $q(E[D_{\text{ran}}])$ (the first average has slightly lower troughs).