L-SPACE INTERVALS FOR GRAPH MANIFOLDS AND CABLES

SARAH DEAN RASMUSSEN

Abstract. We present a graph manifold analog of the Jankins-Neumann classification of Seifert fibered spaces over S^2 admitting taut foliations, providing a finite recursive formula to compute the L-space Dehn-filling interval for any graph manifold with torus boundary. As an application of a generalization of this result to Floer simple manifolds, we compute the L-space interval for any cable of a Floer simple knot complement in a closed three-manifold in terms of the original L-space interval, recovering a result of Hedden and Hom as a special case.

1. Introduction

In the late 1990's, Thurston showed [36, 9] that any taut foliation on an atoroidal three-manifold M makes $\pi_1(M)$ act faithfully on the circle. This result came almost two decades after Eisenbud, Hirsch, and Neumann [12] encountered a complementary phenomenon: they proved that an oriented three-manifold M Seifert fibered over S^2 admits a co-oriented foliation transverse to the fiber if and only if $\pi_1(M)$ admits a representation in $\widetilde{\mathrm{Homeo}}_+S^1$ sending $\phi \mapsto \mathrm{sh}(1)$, where $\widetilde{\mathrm{Homeo}}_+S^1$ is the universal cover of the group of orientation-preserving homeomorphisms of the circle, ϕ is the regular fiber class, and $\mathrm{sh}(s) : t \mapsto t+s$ for any $s \in \mathbb{R}$, making $\widetilde{\mathrm{Homeo}}_+S^1$ the centralizer of $\mathrm{sh}(1)$ in $\mathrm{Homeo}_+\mathbb{R}$.

1.1. Jankins-Neumann Classification.

Inspired by this observation, Jankins and Neumann used Poincaré's "rotation number" invariant to generalize the criterion of [12] to a more local representation-theoretic condition in terms of meridians of exceptional fibers. This new formulation of the problem, in addition to a correct conjecture that the necessary representation-theoretic conditions could be met in $\widetilde{\mathrm{Homeo}}_+S^1$ if and only if they could also be met in a smooth Lie subgroup thereof, allowed them to work out a complete, explicit classification [23], which they proved in all but one special case, later proven by Naimi [28].

Theorem 1.1 ([23, 28]). *For $n > 1$, the manifold $M_{S^2}(y_0; y_1, \ldots, y_n)$ Seifert fibered over S^2 admits a co-oriented taut foliation if and only if $0 = y_- = y_+$ or $0 \in \langle y_+, y_- \rangle$, where*

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=0}^{n} \lfloor y_i k \rfloor\right), \qquad y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=0}^{n} \lceil y_i k \rceil\right).$$

Key words and phrases. graph manifold, taut foliation, L-space, Heegaard Floer.
The author was supported by EPSRC grant EP/M000648/1.

(In the above, and henceforth in this paper, we always regard k as an integer, writing $k > 0$ as shorthand for the restriction $k \in \mathbb{Z}_{>0}$.)

Since then, the Jankins-Neumann-Naimi classification has served as a Rosetta stone for certain *a priori* unrelated properties.

Theorem 1.2 ([12, 23, 28, 10, 30, 26, 27, 5, 33]). *If M is a closed oriented Seifert fibered space, then the following are equivalent:*

- *(1a) $\pi_1(M)$ admits a non-trivial representation in* $\mathrm{Homeo}_+\mathbb{R}$.
- *(1b) $\pi_1(M)$ is left-orderable.*
- *(2) M admits a co-oriented C^0 taut foliation.*
- *(3) M has non-trivial Heegaard Floer homology,* i.e.*, M fails to be an L-space.*

One often uses $(1a)$ as a proxy for $(1b)$, since a result of Boyer, Wiest, and Rolfsen [6, Theorem 1.1.1], combined with the well-known fact [25] that the set of countable left-orderable groups coincides with the set of countable nontrivial subgroups of $\mathrm{Homeo}_+\mathbb{R}$, shows that $(1a) = (1b)$ for every prime compact oriented three-manifold. Boyer, Gordon, and Watson have conjectured that $(1) = (3)$ for any prime compact oriented three-manifold [5], and quite recently, Kazez and Roberts [24], and independently Bowden [3], have extended a C^2 foliations result of Ozsváth and Szabó [30] to show that $(2) \Rightarrow (3)$ for any compact oriented three-manifold. (For this reason, all foliations in this paper are assumed to be C^0 unless otherwise stated.)

The implication $(3) \Rightarrow (2)$, however, is entirely more mysterious. In particular, all known proofs [26, 27, 33] that non-L-space oriented Seifert fibered spaces admit co-oriented taut foliations rely on an explicit comparison of sets of manifolds: one works out the classification of Seifert fibered manifolds over S^2 with non-trivial Heegaard Floer homology, and observes that this classification coincides with the Jankins-Neumann-Naimi classification of oriented Seifert fibered spaces over S^2 admitting co-oriented taut foliations. (The implication $(3) \Rightarrow (2)$ holds vacuously for closed oriented Seifert fibered spaces with $b_1 > 0$, all of which admit co-oriented taut foliations [14], and for oriented Seifert fibered spaces over $\mathbb{RP}^2$, all of which are L-spaces [5].)

1.2. **Graph Manifolds.** Boyer and Clay recently brought insight to this question by introducing a relative version of the problem, studying the gluing behavior of properties $(1a)$, $(1b)$, and (2) along the incompressible tori separating Seifert fibered components of graph manifolds. By showing that these three properties glue in an identical manner along boundaries of JSJ components of rational homology sphere graph manifolds, they were able to prove the equivalence of these three properties for any closed graph manifold [4]. Boyer and Clay also conjectured that property (3) should obey the same gluing behavior.

In answer, Hanselman and Watson [19], and independently J. Rasmussen and the author [33], were able to confirm this gluing conjecture for a larger class of three-manifolds with torus boundary, but subject to certain hypotheses, which one can show are safe to remove in the case of graph manifolds. The four of us [18] were therefore able to prove the following.

Theorem 1.3 ([18])**.** *A graph manifold is an L-space if and only if it fails to admit a co-oriented taut foliation.*

The current paper follows an independent trajectory from the work of [18], launched before the author joined the other collaboration. Although the two papers overlap in one or two results, including slightly variant proofs of Theorem 1.3 and the below gluing criterion, the main result of the current paper is the generalization of the Jankins-Neumann classification formula to graph manifolds, for which we now introduce some notation.

Definition 1.4. *If Y is a compact oriented three-manifold with torus boundary, then the* L-space interval *of Y is the space $\mathcal{L}(Y) \subset \mathbb{P}(H_1(\partial Y))$ of L-space Dehn filling slopes of Y.*

We call $\mathcal{L}(Y)$ an interval because if it contains more than one point, then it is the intersection of $\mathbb{P}(H_1(\partial Y))$ with either a closed interval in $\mathbb{P}(H_1(\partial Y;\mathbb{R}))$ or the complement of a single point in $\mathbb{P}(H_1(\partial Y;\mathbb{R}))$. It therefore makes sense to speak of the *interior* $\mathcal{L}^\circ(Y)$ of $\mathcal{L}(Y)$. If $\mathcal{L}^\circ(Y)$ is nonempty, we call Y *Floer simple.*

Proposition 1.5. *If Y_1 and Y_2 are non-solid-torus graph manifolds with torus boundary, then the union $Y_1 \cup_\varphi Y_2$, with gluing map $\varphi : \partial Y_1 \to -\partial Y_2$, is an L-space if and only if*

$$\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2)).$$

In particular, Floer simplicity is not required.

A graph manifold Y with torus boundary and $b_1(Y) > 1$ has $\mathcal{L}(Y) = \emptyset$. If $b_1(Y) = 1$, then the graph for Y is a tree, and we choose to root this tree at the JSJ component $\hat{Y}$ containing ∂Y. Writing $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ for the n_{G} components of $Y \setminus (\hat{Y} \setminus (\partial \hat{Y} \setminus \partial Y))$, we then regard Y as the union

$$Y = \hat{Y} \cup_{\boldsymbol{\varphi}} \coprod_{i=1}^{n_{\mathrm{G}}} Y_i, \quad \varphi_i : \partial Y_i \to -\partial_i \hat{Y},$$

with $\hat{Y}$ Seifert fibered over an n_{G}+1-punctured S^2 or $\mathbb{RP}^2$. Note that each Y_i is again a graph manifold with torus boundary and $b_1 = 1$, hence is endowed with its own tree graph rooted at the JSJ component containing ∂Y_i, but with the height of this tree strictly less than the height of the tree for Y, so that a recursive computation of $\mathcal{L}(Y)$ in terms of the $\mathcal{L}(Y_i)$ is a finite process.

For any (necessarily toroidal) boundary component of an oriented Seifert fibered space, we fix the reverse-oriented homology basis $(\tilde{f}, -\tilde{h})$, where $\tilde{h}$ is the meridian of the excised regular fiber, and $\tilde{f}$ is the lift dual to $\tilde{h}$ of the regular fiber class, so that we can express any slope $r\tilde{f} - s\tilde{h} \in \mathbb{P}(H_1(\partial Y))$ as $\frac{r}{s} \in \mathbb{Q} \cup \{\infty\}$. For any Y_i with nonempty $\mathcal{L}(Y_i)$, we then write

$$\varphi_{i*}^{\mathbb{P}}(\mathcal{L}(Y_i)) =: \begin{cases} [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] & \mathcal{L}^\circ(Y_i) \neq \emptyset \\ \{y_{i-}^{\mathrm{G}}\} = \{y_{i+}^{\mathrm{G}}\} & \mathcal{L}^\circ(Y_i) = \emptyset, \end{cases}$$

where we use the notation $[[y_-, y_+]] \subset \mathbb{Q} \cup \{\infty\}$ to denote the L-space interval with left-hand endpoint y_- and right-hand endpoint y_+, since any L-space interval with nonempty interior is uniquely specified by its endpoints.

We also write $y_0^{\rm D}, \ldots, y_{n_{\rm D}}^{\rm D}$ for the Seifert data of $\hat{Y}$, so that $\hat{Y}$ is the complement of $n_{\rm G}+1$ regular fibers in either $M_{S^2}(y_0^{\rm D}; y_1^{\rm D}, \ldots, y_{n_{\rm D}}^{\rm D})$ or $M_{\mathbb{RP}^2}(y_0^{\rm D}; y_1^{\rm D}, \ldots, y_{n_{\rm D}}^{\rm D})$, depending on whether $\hat{Y}$ has orientable or non-orientable base. (These $y_i^{\rm D}$ can also be regarded as Dehn-filling slopes in terms of the basis $(\tilde{f}_i^{\rm D}, -\tilde{h}_i^{\rm D})$ described above. See Section 2.2 for notation and homology conventions for Seifert fibered spaces.) We can now state our main result.

Theorem 1.6. *Suppose that Y is not a solid torus and that $\mathcal{L}(Y)$ is nonempty. Then*

$$\mathcal{L}(Y) = \begin{cases} \langle -\infty, +\infty \rangle & \hat{Y} \text{ has non-orientable base} \\ [[y_-, y_+]] & \hat{Y} \text{ has orientable base, } \mathcal{L}^\circ(Y) \neq \emptyset \\ \{y_-\} = \{y_+\} & \hat{Y} \text{ has orientable base, } \mathcal{L}^\circ(Y) = \emptyset, \end{cases}$$

where

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=0}^{n_{\rm D}} \lfloor y_i^{\rm D} k \rfloor + \sum_{i=1}^{n_{\rm G}} \left(\lceil y_{i+}^{\rm G} k \rceil - 1\right)\right),$$

$$y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=0}^{n_{\rm D}} \lceil y_i^{\rm D} k \rceil + \sum_{i=1}^{n_{\rm G}} \left(\lfloor y_{i-}^{\rm G} k \rfloor + 1\right)\right).$$

(In the above, we define $y_- := \infty$ or $y_+ := \infty$, respectively, if any infinite terms appear as summands of y_- or y_+, respectively.)

Whereas every oriented Seifert fibered space over the disk or Möbius strip is Floer simple, *i.e.* has $\mathcal{L}^\circ \neq \emptyset$, the story for graph manifolds is more complicated. Consider the following examples, for all of which we take $\hat{Y}$ to have orientable base.

$$\begin{aligned}
\mathcal{L}(Y) = [-\infty, 96]: \quad & n_{\rm D} = 3, \ \ (y_1^{\rm D}, y_2^{\rm D}, y_3^{\rm D}) = \left(\tfrac{1}{3}, -\tfrac{2}{5}, \tfrac{3}{2}\right); \\
& n_{\rm G} = 2, \ \ \varphi_{1*}^{\mathbb{P}}(\mathcal{L}(Y_1)) = [-100, +\infty], \ \varphi_{2*}^{\mathbb{P}}(\mathcal{L}(Y_2)) = \left[\left[\tfrac{2}{5}, -20\right]\right]. \\
\mathcal{L}(Y) = \{0\}: \quad & n_{\rm D} = 1, \ \ y_1^{\rm D} = \tfrac{1}{3}; \\
& n_{\rm G} = 1, \ \ \varphi_{1*}^{\mathbb{P}}(\mathcal{L}(Y_1)) = \left[-\tfrac{1}{3}, 0\right]. \\
\mathcal{L}(Y) = \emptyset: \quad & n_{\rm D} = 3, \ \ (y_1^{\rm D}, y_2^{\rm D}, y_3^{\rm D}) = \left(\tfrac{1}{3}, -\tfrac{2}{5}, \tfrac{3}{2}\right); \\
& n_{\rm G} = 2, \ \ \varphi_{1*}^{\mathbb{P}}(\mathcal{L}(Y_1)) = [-100, +\infty], \ \varphi_{2*}^{\mathbb{P}}(\mathcal{L}(Y_2)) = \left[-\tfrac{1}{3}, 0\right].
\end{aligned}$$

Above, we see examples in which Y is Floer simple, has an isolated L-space filling, or has empty L-space interval. One cannot use Theorem 1.6 without first knowing which of these three cases occurs for Y. We therefore provide Proposition 4.7, which lists explicit criteria for the multiple mutually exclusive cases in which Y is Floer simple or in which Y has an isolated L-space filling. In the complement of these criteria, $\mathcal{L}(Y)$ is empty.

In fact, the validity of Theorem 4.6 extends beyond the realm of graph manifolds.

Proposition 1.7. *Theorem 1.6 holds for any boundary incompressible Floer simple three-manifolds $Y_1, \ldots, Y_{n_{\rm G}}$, provided that Y satisfies the criteria in Proposition 4.7 for $\mathcal{L}(Y)$ to be nonempty.*

One immediate application of this generalization is the computation of L-space intervals for cables of Floer simple knot complements.

1.3. **Cables.** The (p,q)-cable $Y^{(p,q)} \subset X$ of a knot complement $Y := X \setminus \nu(K) \subset X$ is given by the knot complement $Y^{(p,q)} := X \setminus \nu(K^{(p,q)})$, where $K^{(p,q)} \subset X$ is the image of the (p,q)-torus knot embedded in the boundary of Y. Since one can realize any cable of $Y \subset X$ by gluing an appropriate Seifert fibered space onto Y, we can use the above generalization of Theorem 1.6 to prove the following result.

Theorem 1.8. *Suppose that $p,q \in \mathbb{Z}$ with $p > 1$ and $\gcd(p,q) = 1$, and that $Y = X \setminus \nu(K)$ is a boundary incompressible Floer simple knot complement in an L-space X, with L-space interval $\mathcal{L}(Y) = [[\frac{a_-}{b_-}, \frac{a_+}{b_+}]]$, written in terms of the surgery basis $\mu, \lambda \in H_1(\partial Y)$ for K, with μ the meridian of K and λ a choice of longitude. Then in terms of the surgery basis produced by cabling, the (p,q)-cable $Y^{(p,q)} \subset X$ of $Y \subset X$ has L-space interval*

$$\mathcal{L}(Y^{(p,q)}) = \begin{cases} \{\infty\} & \frac{a_-}{b_-} \in \left[\frac{p^*}{q^*}, \infty\right],\ \frac{a_+}{b_+} \in \left[\frac{q-p^*}{p-q^*}, \frac{q}{p}\right\rangle \cup \{\infty\} \\ [[1/y_+, 1/y_-]] & \textit{otherwise}, \end{cases}$$

where $p^*, q^* \in \mathbb{Z}$ are defined to satisfy $pp^* - qq^* = 1$ with $0 < q^* < p$, and where we define $y_- := \max_{k>0} y_-(k)$ and $y_+ := \min_{k>0} y_+(k)$, with

$$y_-(k) := \frac{1}{k}\left(\left\lceil \frac{q^*}{p}k \right\rceil - \lceil y_{1+}k \rceil\right), \quad y_+(k) := \frac{1}{k}\left(\left\lfloor \frac{q^*}{p}k \right\rfloor - \lfloor y_{1-}k \rfloor\right),$$
$$\textit{and} \quad y_{1\pm} := \frac{a_\pm q^* - b_\pm p^*}{a_\pm p - b_\pm q} = \frac{q^*}{p}\left(1 - \frac{b_\pm}{q^*(a_\pm p - b_\pm q)}\right).$$

We also prove a slightly more general version of Theorem 1.8 which does not require that X be an L-space, and which holds for any $\frac{p}{q} \in \mathbb{Q} \cup \{\infty\}$.

A brief application of the theorem, followed by an appropriate change of basis, recovers the following result of Hedden [21] and Hom [22].

Corollary 1.9. *Suppose $Y := S^3 \setminus \nu(K)$ is a boundary incompressible Floer simple positive knot complement in S^3. If $p > 0$ and $\gcd(p,q) = 1$, then in terms of the conventional basis for knot complements in S^3, $Y^{(p,q)}$ has L-space interval*

$$\mathcal{L}(Y^{(p,q)}) = \begin{cases} \{\infty\} & 2g(K) - 1 > \frac{q}{p} \\ [pq - p - q + 2g(K)p,\ \infty] & 2g(K) - 1 \leq \frac{q}{p}. \end{cases} \tag{1}$$

By *positive*, we simply mean that Y has an L-space Dehn filling $Y(N)$ for some $N > 0$.

Note that equating $pq - p - q + 2g(K)p$ with $2g(K^{(p,q)}) - 1$ recovers the formula for the genus of the (p,q)-cable of $K \subset S^3$. Note also that since $\frac{p^*}{q^*} - \frac{q}{p} = \frac{1}{pq^*}$, with $\frac{p^*}{q^*}, \frac{q}{p} \notin \mathbb{Z}$, the domain specified for Floer simple cables in the above corollary is equivalent to the condition $2g(K) - 1 \notin [\frac{p^*}{q^*}, \infty]$, matching Theorem 1.8.

1.4. **Generalized Solid Tori.** A recent result of Gillespie [17] states that a compact oriented three manifold Y with torus boundary satisfies $\mathcal{L}(Y) = \mathbb{P}(H_1(\partial Y)) \setminus \{l\}$ if and only if Y has genus 0 and an L-space filling, where l denotes the rational longitude of Y. Such manifolds are called *generalized solid tori* in [33] and are of independent interest [2, 15, 11, 1, 19].

Using the version of Theorem 1.8 that does not require X to be an L-space, along with some incremental results from the proof of Theorem 1.6, we are able to show the following.

Theorem 1.10. *If Y is a generalized solid torus, then any cable of $Y \subset Y(l)$ is a generalized solid torus. If Y is a graph manifold with torus boundary, $b_1(Y)=1$, and rational longitude other than the regular fiber, then Y is a generalized solid torus if and only if it is homeomorphic to an iterated cable of the regular fiber complement in $S^1 \times S^2$.*

Similarly, for any class of manifolds for which the gluing result in Proposition 1.5 holds without the requirement of Floer simplicity—such as graph manifolds—one has the result that if Y has an isolated L-space filling, *i.e.*, if $\mathcal{L}(Y) = \{\mu\}$ for some $\mu \in \mathbb{P}(H_1(\partial Y))$, then any cable of $Y \subset Y(\mu)$ has $Y(\mu)$ as an isolated L-space filling.

1.5. **Floer simple knot complements.** Whereas the regular fiber complement in a rational homology sphere Seifert fibered space could arguably be called the prototypical Floer simple manifold, not every regular fiber complement in an L-space graph manifold is Floer simple, due to the existence of isolated L-space fillings. However, the next best thing is true.

Given a closed graph manifold X, call an exceptional fiber $f_{\mathrm{E}} \subset X$ *invariantly exceptional* if the JSJ component $\hat{Y} \subset X$ containing f_{E} has more than one exceptional fiber. Note that if $\hat{Y}$ has only one exceptional fiber, then $\hat{Y}$ is either a lens space (if X is Seifert fibered) or a punctured solid torus. Since the solid torus has nonunique Seifert structure, one can show that if X is not a lens space, it is homeomorphic to a graph manifold X' in which the image f'_{E} of f_{E} is a regular fiber. excluding this scenario allows us to show the following.

Theorem 1.11. *Every invariantly exceptional fiber complement in an L-space graph manifold is Floer simple.*

There are also Floer simple knot complements traversing the graph structure of X.

Proposition 1.12. *If X is an L-space graph manifold, then for every incompressible torus $T \subset X$, there is a knot $K \subset X$ transversely intersecting T for which the complement $X \setminus \nu(K)$ is Floer simple.*

The same occurs for an arbitrary L-space X, provided that X decomposes as a union of Floer simple manifolds along T (see Proposition 6.6).

The above results, together with the evidence of various other classes of L-spaces, and a certain degree of optimism, motivate the following:

Conjecture 1.13. *Every L-space admits a Floer simple knot complement.*

1.6. **Organization.** In Section 2, we introduce our conventions for Seifert fibered spaces and provide a lengthy discussion of the Jankins-Neumann problem, since we cannot hope for Theorem 1.6 to provide insight if the original theorem of Jankins, Neumann and Naimi is opaque to the reader.

Section 3 reviews some basic facts about L-space intervals, including the independent results of Hanselman and Watson [19] and J. Rasmussen and the author [33] about L-space criteria for unions of Floer simple manifolds.

Section 4 is where we prove our main graph manifold results, including Theorems 1.3 and 1.6 in the forms of Theorems 4.5 and 4.6. This section also derives Proposition 4.7's classification of single-boundary-component graph manifolds with nonempty L-space intervals.

Our main cabling results reside in Section 5, although the proof of Theorem 1.10, for generalized solid tori, is relegated to Section 6.

Lastly, Section 6 justifies Proposition 1.7's generalization of our Jankins-Neumann graph manifold result to the union of a Seifert fibered space with Floer simple manifolds. This final section also lists an array of applications of the paper's main results, including the aforementioned generalized solid torus cabling result and proofs of the Floer simple knot complement results from Theorem 1.11 and Proposition 1.12.

Acknowledgements. I would like to thank Jonathan Bowden, Thomas Gillespie, Jake Rasmussen, and Rachel Roberts for helpful conversations, Liam Watson and Jonathan Hanselman for comments on an earlier draft, and the referee for helpful suggestions.

2. Foliations on Seifert fibered spaces

A *graph manifold* is a prime compact oriented three-manifold which admits a JSJ decomposition—which in this case, we take to be a *minimal* cutting apart along incompressible tori into disjoint pieces—such that each JSJ component is an oriented Seifert fibered space. The data for reassembling these components into the original manifold are encoded in a labeled graph, where each vertex corresponds to a Seifert fibered JSJ component, and each edge corresponds to a gluing of two Seifert fibered pieces along an incompressible torus.

2.1. Restricting taut foliations to JSJ components.

Questions about taut foliations on graph manifolds can often be reduced to questions about taut foliations on Seifert fibered spaces, due in part to the following result.

Proposition 2.1 ([34, 35, 8]). *If Y is a compact oriented three-manifold admitting a taut foliation F transverse to ∂Y, then every incompressible separating torus in Y can be isotoped to be everywhere transverse to F.*

Proof. Roussarie [34] showed that if F is C^2, then each incompressible torus $T \subset Y$ can be isotoped to be either everywhere transverse to F or a leaf of F. A later theorem of Brittenham and Roberts [8] extends the validity of this proposition to C^0 foliations. Thus, since a taut foliation has no compact separating leaves, an incompressible separating torus cannot be isotoped to be a leaf of F, and so it must be possible to isotop any incompressible separating torus to be everywhere transverse to F. This is also believed to have been known by Thurston [35]. □

As noted by Brittenham, Naimi, and Roberts [7], this result has major consequences for graph manifolds:

Corollary 2.2. *If Y is a graph manifold with tree graph and F is a taut foliation on Y transverse to ∂Y, then F can be isotoped so that it restricts to boundary-transverse taut foliations on the Seifert fibered JSJ components of Y.*

When a closed graph manifold has positive first Betti number, the question of existence of taut foliations becomes trivial, since a result of Gabai states that any such manifold admits a co-oriented taut foliation [14]. Correspondingly, any closed oriented three manifold with $b_1 > 0$ has non-trivial Heegaard Floer homology, hence is not an L-space. We therefore restrict attention to rational homology sphere graph manifolds, hence to oriented Seifert fibered spaces over S^2 or $\mathbb{RP}^2$, and regular fiber complements thereof.

2.2. **Conventions for Seifert fibered spaces.** If $\hat{M}$ denotes the trivial circle fibration over the $n+1$-punctured two-sphere,

$$\hat{M} := S^1 \times \left(S^2 \setminus \coprod_{i=0}^{n} D_i^2 \right), \qquad \partial_i \hat{M} := -\partial(S^1 \times D_i^2), \;\; i \in \{0, \ldots, n\}, \tag{2}$$

then writing $-\tilde{h}_i :\in H_1(\partial_i \hat{M})$ for the meridian of each excised solid torus $S^1 \times D_i^2$, we have

$$-\sum_{i=1}^{n} \tilde{h}_i = p_{S^1} \times \partial(S^2 \setminus D_0^2) = p_{S^1} \times -\partial D_0^2 = \tilde{h}_0 \tag{3}$$

for any point class $p_{S^1} \in H_0(S^1)$ of the circle fiber. For each $i \in \{0, \ldots, n\}$, if we write $\iota_i : H_1(\partial_i \hat{M}) \to H_1(\hat{M})$ for the map induced by inclusion, then there is a lift $\tilde{f}_i \in \iota_i^{-1}(f)$ of the regular fiber class $f \in H_1(\hat{M})$ satisfying $(-\tilde{h}_i \cdot \tilde{f}_i)|_{\partial_i \hat{M}} = 1$. The reverse-oriented basis $(\tilde{f}_i, -\tilde{h}_i)$ for $H_1(\partial_i \hat{M})$ induces a projectivization map

$$\pi_i : H_1(\partial_i \hat{M}) \setminus \{0\} \to \mathbb{P}(H_1(\partial_i \hat{M})) \xrightarrow{\sim} \mathbb{Q} \cup \{\infty\}, \quad r_i \tilde{f}_i - s_i \tilde{h}_i \mapsto \tfrac{r_i}{s_i}, \tag{4}$$

by which we identify Seifert invariants with slopes, and slopes with $\mathbb{Q} \cup \{\infty\}$.

The Seifert fibered space $M_{S^2}(\frac{r_*}{s_*}) := M_{S^2}(e_0 = \frac{r_0}{s_0}; \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$ over S^2 is the Dehn filling of $\hat{M}$ along the slopes $\mu_i := r_i \tilde{f}_i - s_i \tilde{h}_i$, with $\pi_i(\mu_i) = \frac{r_i}{s_i}$, subject to the convention that $e_0 = \frac{r_0}{s_0} \in \mathbb{Z}$ and $\frac{r_i}{s_i} \notin \mathbb{Z} \cup \{\infty\}$. Setting each $h_i := \iota_i(\tilde{h}_i)$ then gives the presentation

$$H_1(M_{S^2}(\tfrac{r_*}{s_*})) = \left\langle f, h_0, \ldots, h_n \,\middle|\, \sum_{i=0}^{n} h_i = e_0 f - h_0 = r_1 f - s_1 h_1 = \ldots = r_n f - s_n h_n = 0 \right\rangle. \tag{5}$$

Likewise, if we respectively lift f and each h_i to generators ϕ and η_i for $\pi_1(M_{S^2}(\frac{r_*}{s_*}))$ and substitute ϕ^{e_0} for η_0, then we obtain the fundamental group presentation

$$\pi_1(M_{S^2}(\tfrac{r_*}{s_*})) = \left\langle \phi, \eta_1, \ldots, \eta_n \,\middle|\, \phi \text{ central}, \prod_{i=1}^{n} \eta_i = \phi^{-e_0}, \, \eta_1^{s_1} = \phi^{r_1}, \, \ldots, \, \eta_n^{s_n} = \phi^{r_n} \right\rangle. \tag{6}$$

For a manifold $M_{\mathbb{RP}^2}(\frac{r_*}{s_*})$ Seifert fibered over $\mathbb{RP}^2$, we adopt the same homology and slope conventions for the boundary of a regular fiber complement, but the global homology is slightly different. Since, this time, $\hat{M}$ is the *twisted* circle bundle over a punctured $\mathbb{RP}^2$, the fiber class f is now 2-torsion. Also, since puncturing $\mathbb{RP}^2$ once gives a Möbius strip instead of a disk, the sum $\sum_{i=1}^{n} \tilde{h}_i$ differs from $p_{\mathbb{RP}^2} \times \partial(\mathbb{RP}^2 \setminus$

D^2_0) by twice the one-cell c glued to the disk to make $\mathbb{RP}^2$, yielding a homology presentation of the form

(7)
$$H_1(M_{\mathbb{RP}^2}(\tfrac{r_*}{s_*})) = \langle f, c, h_0, \ldots, h_n | 2f = 0, 2c + \textstyle\sum_{i=0}^n h_i = 0,\ \iota_0(\mu_0) = \ldots = \iota_n(\mu_n) = 0\rangle\,.$$

For either type of base, the Seifert fibration is invariant under any reparameterization $M(\frac{r_0}{s_0}, \ldots, \frac{r_n}{s_n}) \to M(\frac{r_0}{s_0} + z_0, \ldots, z_n + \frac{r_n}{s_n})$ with $\sum_{i=0}^n z_i = 0$ and each $z_i \in \mathbb{Z}$. The manifold also admits orientation-reversing homeomorphism, $M(\frac{r_0}{s_0}, \ldots, \frac{r_n}{s_n}) \to M(-\frac{r_0}{s_0}, \ldots, -\frac{r_n}{s_n})$.

A *regular fiber complement* $Y := M \setminus \nu(f)$ in a rational homology sphere Seifert fibered space has $b_1(Y) = 1$, hence has a well-defined rational longitude.

Definition 2.3. *Any compact oriented three manifold Y with torus boundary and $b_1(Y) = 1$ has a* rational longitude*, a unique class $l \in \mathbb{P}(H_1(\partial Y))$ such that representatives in $H_1(\partial Y)$ have torsion image in $H_1(Y)$ under the homomorphism induced by inclusion of the boundary.*

It is straightforward to show (see, *e.g.*, [33]) that

$$M_{S^2}(\tfrac{r_*}{s_*}) \setminus \nu(f) \quad \text{has rational longitude} \quad l = -\sum_{i=0}^n \tfrac{r_i}{s_i}. \tag{8}$$

A mild generalization of the calculation in [33] shows that the above result also holds if each solid torus $S^1 \times D^2_i$ is replaced with an arbitrary compact oriented three-manifold Y_i with torus boundary and $b_1(Y_i) = 1$, with $\frac{r_i}{s_i}$ the image of the rational longitude of Y_i. By contrast, if the JSJ component containing ∂Y has non-orientable base, then $l = \pi(\tilde{f}) = \infty$.

Remark. The requirement that $\frac{r_i}{s_i} \neq \infty$ for $i \in \{0, \ldots, n\}$ is a necessary (assuming $n > 1$) and sufficient condition for the resulting Seifert fibered space to be prime—an important property for manifolds serving as building blocks in combinatorial constructions. To understand necessity, let $\hat{M}^\infty$ denote the result of Dehn filling $\hat{M}$ with slope ∞ along $\partial_0\hat{M}$. If $\hat{M}$ is fibered over a punctured S^2, then $\hat{M}^\infty$ is a connected sum of n solid tori, each with longitude of slope ∞. Similarly, if $\hat{M}$ is fibered over a punctured $\mathbb{RP}^2$, then $\hat{M}^\infty$ is the connected sum of an $S^1 \times S^2$ with n solid tori, each with longitude of slope ∞.

Primality is especially important in the context of foliations, since, by Novikov [29], no reducible manifold except $S^1 \times S^2$ admits a co-oriented taut foliation. On the other hand, not all connected sums are L-spaces, so any correspondence between being an L-space and failing to admit a co-oriented taut foliation breaks down beyond the realm of prime manifolds.

2.3. **Rotation Number, Shift, and Foliation Slope.**

One of the key insights of Jankins and Neumann into the work of Eisenbud, Hirsch, and Neumann on taut foliations on Seifert fibered spaces was the need for a better invariant on $\widetilde{\mathrm{Homeo}}_+ S^1$. Whereas the latter group relied on the invariants $\underline{m}, \overline{m} : \widetilde{\mathrm{Homeo}}_+ S^1 \to \mathbb{R}$, with $\underline{m}(\gamma) := \min_{t\in\mathbb{R}} \gamma(t) - t$ and $\overline{m}(\gamma) := \max_{t\in\mathbb{R}} \gamma(t) - t$, Jankins and Neumann introduced the problem to a more precise invariant of circle actions: a conjugacy invariant called the (Poincaré) *rotation number*,

$$\text{rot} : \widetilde{\text{Homeo}}_+(S^1) \to \mathbb{R}, \quad \text{rot}(\gamma) = \lim_{k\to\infty} \frac{1}{k}(\gamma^k(t) - t), \tag{9}$$

which is independent of $t \in \mathbb{R}$, and rational if and only if γ has some closed orbit [16]. The rotation number is not, in general, a homomorphism. However, it restricts to a homomorphism on any amenable, hence any abelian, subgroup [16]. In particular, it restricts to a homomorphism on any representation of the fundamental group of a torus.

The simplest element of $\widetilde{\text{Homeo}}_+(S^1)$ is a *rotation*, or *shift*,

$$\text{sh}(s) : t \mapsto t + s, \quad t \in \mathbb{R}. \tag{10}$$

Whereas rot $\circ$ sh = id, not every element of $\widetilde{\text{Homeo}}_+(S^1)$ is conjugate to a rotation. It is a classic result, however, that every element of $\widetilde{\text{Homeo}}_+(S^1)$ with irrational rotation number is left and right semiconjugate to a shift of the same rotation number [16].

Rotation numbers can also be used to associate slopes to taut foliations on tori.

Definition 2.4. *For the two-torus T, there is a canonical map*

$$\alpha : \left\{ \, C^0 \textit{ codimension-one foliations on } T \, \right\} \; \to \; \mathbb{P}(H_1(T; \mathbb{R})), \tag{11}$$

constructed below, which respects isotopy. We call $\alpha(F)$ the slope *of F.*

If F has Reeb compenents, then $\alpha(F)$ is given by the class of any closed leaf of F. All Reebless foliations on tori are taut. Thus, if F is Reebless, then there is a curve, say C_λ of primitive class $\lambda \in H_1(T)$, which intersects every leaf transversely, and F can be realized as the suspension of a circle homeomorphism $\gamma_{F,\lambda} \in \text{Homeo}_+ S^1$ from C_λ to itself [20]. A choice of $\mu \in H_1(T)$ with $\mu \cdot \lambda = 1$ induces a lift of this suspension to a suspension from a universal cover $\tilde{C}_\lambda$ of C_λ to its translate by μ in the universal cover of T. That is, if we regard $\tilde{C}_\lambda$ as the real vector space $\{t\lambda\}_{t\in\mathbb{R}}$ spanned by λ, with $C_\lambda \cong \tilde{C}_\lambda / \lambda\mathbb{Z}$, then one can lift the foliation F to the universal cover of T by iteratively suspending the map $t\lambda \mapsto \tilde{\gamma}_{F,\lambda,\mu}(t)\lambda + \mu$, for an appropriate lift $\tilde{\gamma}_{F,\lambda,\mu} \in \widetilde{\text{Homeo}}_+ S^1 \subset \text{Homeo}_+\mathbb{R}$ of $\gamma_{F,\lambda} \in \text{Homeo}_+ S^1$.

This lifted suspsension, in turn, induces a representation

$$\rho^F_\lambda : \pi_1(T) \to \widetilde{\text{Homeo}}_+ S^1, \quad [\lambda] \mapsto \text{sh}(1), \quad [-\mu] \mapsto \tilde{\gamma}_{F,\lambda,\mu}, \tag{12}$$

where $[\lambda]$ and $[\mu]$ denote the lifts of λ and μ to $\pi_1(T)$. One can regard ρ^F_λ as describing how to traverse the line $\{t\lambda\}_{t\in\mathbb{R}} \subset \langle \mu, \lambda \rangle_{\mathbb{R}}$ by traveling only along foliation leaves or integer multiples of λ or μ. That is, if one starts at some $t_0\lambda$, hops by $a\lambda + b\mu$ for some $a, b \in \mathbb{Z}$, takes the foliation leaf intersecting this new point, and follows this leaf back to the line $\{t\lambda\}_{t\in\mathbb{R}}$, then one will arrive at $\rho^F_\lambda(a\lambda + b\mu)(t_0)\lambda$. Note that while ρ^F_λ is independent of the choice of μ, and is determined up to conjugacy by a choice of λ, it still depends on λ.

On the other hand, when we define the slope $\alpha(F)$ of F to be

$$\alpha(F) := \ker\, (\text{rot} \circ \rho^F_\lambda) = \big\langle (\text{rot}(\rho^F_\lambda(-\mu))\lambda + \mu \big\rangle \in \mathbb{P}(H_1(T; \mathbb{R})), \tag{13}$$

then the rotation number washes out all dependence on λ and choice of suspension. That is, one can use the definition of rotation number to compute $\mathrm{rot}(\rho^F_\lambda(-\mu))$ in terms of the rotation number associated to a different choice of basis and suspension for F, and obtain the same answer for $\alpha(F)$ in both cases. Alternatively, any suspension homeomorphism with rational rotation number has a periodic orbit, hence realizes a foliation with a compact leaf of slope $\alpha(F)$. If the suspension homeomorphism has irrational rotation number, then it is semiconjugate to a shift of matching rotation number [16], giving rise to a linear foliation of slope $\alpha(F)$.

2.4. **Restricting Seifert fibered space foliations to torus foliations.** If a compact oriented three-manifold Y admits a co-oriented taut foliation transverse to ∂Y, then Gabai tells us that ∂Y can only have toroidal components [14]. Thus, we often encounter foliations on tori as boundary restrictions of foliations on three-manifolds. Moreover, on a Seifert fibered space, any taut foliation transverse to the boundary restricts to taut foliations on boundary components.

Suppose F is a co-oriented taut foliation transverse to the fibration of the Seifert fibered Dehn filling $M_{S^2}(\frac{r_*}{s_*})$ along the slopes $\frac{r_*}{s_*} = (\frac{r_0}{s_0}, \dots, \frac{r_n}{s_n})$ of the trivial circle fibration $\hat{M}$ over an $n+1$-punctured S^2, according to the conventions of Section 2.2. For each boundary component $\partial_i \hat{M}$, we regard the foliation $F \cap \partial_i \hat{M}$ as a suspension of a homemorphism of the curve of class $\tilde{f}_i$ to itself, and since the class $-\tilde{h}_i$ satisfies $-\tilde{h}_i \cdot \tilde{f}_i = 1$, it specifies a lift of this suspension to a suspension of an element $\gamma_{F,\tilde{f}_i,-\tilde{h}_i} \in \widetilde{\mathrm{Homeo}}_+ S^1$. To this suspension we associate the representation

$$\rho_i := \rho^{F\cap\partial_i\hat{M}}_{\tilde{f}_i} : \pi_1(\partial_i \hat{M}) \to \widetilde{\mathrm{Homeo}}_+ S^1, \quad [\tilde{f}_i] \to \mathrm{sh}(1), \quad [\tilde{h}_i] \to \gamma_{F,\tilde{f}_i,-\tilde{h}_i}, \tag{14}$$

allowing us to express the slope $\alpha(F \cap \partial_i \hat{M})$ of $F \cap \partial_i \hat{M}$ as

$$\alpha(F \cap \partial_i \hat{M}) = \pi_i(\mathrm{rot}(\rho_i([\tilde{h}_i]))\tilde{f}_i - \tilde{h}_i) = \mathrm{rot}(\rho_i([\tilde{h}_i])). \tag{15}$$

The construction of Eisenbud, Hirsch, and Neumann [12] associating a representation $\rho : \pi_1(M(\frac{r_i}{s_i})) \to \widetilde{\mathrm{Homeo}}_+ S^1$ to F, sending the fiber class $\phi = [f]$ to $\mathrm{sh}(1)$, is sufficiently compatible with the construction of each ρ_i above that, possibly after conjugation of each ρ_i, ρ can be chosen to satisfy $\rho_i = \rho \circ \iota_i^{\pi_1}$, with $\iota_i^{\pi_1} : \pi_1(\partial_i \hat{M}) \to \pi_1(M_{S^2}(\frac{r_*}{s_*}))$ the homomorphism induced by inclusion. The presentation (6) for $\pi_1(M_{S^2}(\frac{r_*}{s_*}))$ then places the following restrictions on ρ, as observed by Jankins and Neumann [23]:

$$\textstyle\prod_{i=1}^n \eta_i = \phi^{-e_0} \implies \bullet\ \mathrm{rot}(\prod_{i=1}^n \rho(\eta_i)) = -e_0 = -\frac{r_0}{s_0}, \tag{16}$$

$$i \in \{0, \dots, n\}: \ \eta_i^{s_i} = \phi^{r_i} \implies \begin{cases} \bullet\ \mathrm{rot}(\rho(\eta_i)) = \frac{r_i}{s_i}, \\ \bullet\ \rho(\eta_i) \text{ is conjugate to } \mathrm{sh}(\frac{r_i}{s_i}). \end{cases} \tag{17}$$

Jankins and Neumann mostly focused on the case of $n = 3$, $e_0 = -1$, and $0 < \frac{r_1}{s_1}, \frac{r_2}{s_2}, \frac{r_3}{s_3} < 1$, but their above observation holds in general.

Whereas the first condition enforces a global restriction on F, the latter two conditions provide local restrictions at each $F \cap \partial_i \hat{M}$, which we could recover simply by considering Dehn fillings. The solid torus admits only one taut foliation, namely, the product foliation with slope given by the rational longitude. As a consequence, the co-oriented taut foliation $F \cap \partial_i \hat{M}$ extends to a co-oriented taut foliation on the Dehn filling $\partial_i \hat{M}(\frac{r_i}{s_i})$ if and only if $F \cap \partial_i \hat{M}$ is the product foliation of slope $\frac{r_i}{s_i}$, which occurs if and only if $\rho_i([\tilde{h}_i]) = \rho(\eta_i)$ is conjugate to $\mathrm{sh}(\frac{r_i}{s_i})$, a condition which requires $\alpha(F \cap \partial_i \hat{M}) = \mathrm{rot}(\rho(\eta_i)) = \frac{r_i}{s_i}$. In particular, the j^{th} shift conjugacy condition, that $\rho(\eta_j)$ be conjugate to $\mathrm{sh}(\frac{r_j}{s_j})$, is due solely to the fact that $\partial_j \hat{M}$ is glued to a solid torus. As emphasized by Boyer and Clay [4], when one relaxes the j^{th} shift conjugacy condition, one can still find manifolds Y with torus boundary for which F extends to a taut foliation on $\hat{M} \cup_{\partial_j \hat{M}} Y$, for a suitable choice of gluing map.

It is presumably for this reason that Jankins and Neumann focused on the more general condition of J-*realizability* for an $n{+}1$-tuple $\frac{r_*}{s_*} := (e_0{=}\frac{r_0}{s_0}, \frac{r_1}{s_1}, \ldots, \frac{r_n}{s_n})$, given a subset $J \subset \{1, \ldots, n\}$. They deem $\frac{r_*}{s_*}$ J-realizable if there is a representation $\rho^J : \langle \eta_1, \ldots \eta_n \rangle \to \widetilde{\mathrm{Homeo}}_+ S^1$ such that ρ^J meets the $\frac{r_*}{s_*}$ rotation number condition, that $\mathrm{rot}(\prod_{i=1}^n \rho(\eta_i)) = -e_0$ with $\mathrm{rot}(\rho(\eta_i)) = \frac{r_i}{s_i}$ for each $i \in \{1, \ldots, n\}$, and such that ρ^J meets the j^{th} shift conjugacy condition for each $j \in J$. We have already shown that J-realizability is a necessary condition for $\hat{M}$ to admit a taut foliation F of slopes $\alpha(F \cap \partial_i \hat{M}) = \frac{r_i}{s_i}$ which extends to a taut foliation on the partial Dehn filling of $\hat{M}$ along the slopes $\frac{r_j}{s_j} \in \mathbb{P}(H_1(\partial_j \hat{M}))$ for all $j \in \{0\} \cup J$. With the help of Naimi, Jankins and Neumann showed that the condition is also sufficient [23, 28].

2.5. **Solutions for J-realizability.**

Jankins and Neumann conjectured that a slope $\frac{r_*}{s_*}$ is J-realizable in $\widetilde{\mathrm{Homeo}}_+ S^1$ if and only if it is J-realizable in a smooth Lie subgroup of $\widetilde{\mathrm{Homeo}}_+ S^1$. Observing that any smooth Lie subgroup of $\widetilde{\mathrm{Homeo}}_+ S^1$ is conjugate to $\widetilde{PSL}_k(2, \mathbb{R})$ for some $k \in \mathbb{Z}_{>0}$, where $\widetilde{PSL}_k(2, \mathbb{R}) = \psi_k^{-1} \widetilde{PSL}(2, \mathbb{R}) \psi_k$ for $(\psi_k : t \mapsto kt) \in \mathrm{Homeo}_+ \mathbb{R}$, they computed

$$\max \ \mathrm{rot} \left(\prod_{i=1}^{n} \gamma_i \right) = \frac{1}{k} \left(-1 + \sum_{i=1}^{n} \left(\left\lfloor \frac{r_i k}{s_i} \right\rfloor + 1 \right) \right) \tag{18}$$

for the maximum rotation number of a product of elements $\gamma_i \in \widetilde{PSL}_k(2, \mathbb{R})$ with each $\mathrm{rot}(\gamma_i) = \frac{r_i}{s_i}$. They then proved the above conjecture in all but one case, later proven by Naimi [28]. More recently, in Theorem 3.9 of [10] (appropriately generalized from 2 to n), Calegari and Walker rederived (18) (with the maximum taken over $k \in \mathbb{Z}_{>0}$) for $\widetilde{\mathrm{Homeo}}_+ S^1$, without appealing to $\widetilde{PSL}_k(2, \mathbb{R})$, by using dynamical techniques similar to those of Naimi.

One obtains the analogous minimum rotation number of a product by sending $\frac{r_i}{s_i} \mapsto -\frac{r_i}{s_i}$ in (18). Demanding that $-e_0$ lie between the minimum and maximum rotation numbers for $\prod_{i=1}^n \rho(\eta_i)$, and multiplying the resulting inequality by -1, implies that a representation in $\widetilde{\mathrm{Homeo}}_+ S^1$ can only satisfy the rotation number condition for $\frac{r_*}{s_*} = (e_0, \frac{r_1}{s_1}, \dots, \frac{r_n}{s_n})$ if

$$\min_{k>0} -\frac{1}{k}\left(-1+\sum_{i=1}^{n}\left(\left\lfloor\frac{r_ik}{s_i}\right\rfloor+1\right)\right) \le e_0 \le \max_{k>0} -\frac{1}{k}\left(1+\sum_{i=1}^{n}\left(\left\lceil\frac{r_ik}{s_i}\right\rceil-1\right)\right), \tag{19}$$

a criterion which Jankins and Neumann prove is also sufficient [23]. Moreover, $\frac{r_*}{s_*}$ is J-realizable in $\widetilde{PSL}_k(2,\mathbb{R})$, for some $k \in \mathbb{Z}_{>0}$, if and only if

$$-\frac{1}{k}\left(-1+\sum_{i=0}^{n}\left(\left\lfloor\frac{r_ik}{s_i}\right\rfloor+1\right)\right) \le 0 \le -\sum_{i=0}^{n}\frac{r_i}{s_i} \;\text{ or }\; -\sum_{i=0}^{n}\frac{r_i}{s_i} \le 0 \le -\frac{1}{k}\left(1+\sum_{i=0}^{n}\left(\left\lceil\frac{r_ik}{s_i}\right\rceil-1\right)\right). \tag{20}$$

The shift conjugacy condition is easier to apply: one can approximate an element of $\widetilde{\mathrm{Homeo}}_+S^1$ with a shift-conjugate element of arbitrarily close rotation number. Jankins and Neumann used this fact to show that if one fixes all $\frac{r_i}{s_i}$ with $i \ne j$, for some fixed $j \in \{1,\dots,n\}$, then imposing the j^{th} shift conjugacy condition is equivalent to restricting to the interior of the interval of $\frac{r_j}{s_j} \in \mathbb{R}$ for which the $\frac{r_*}{s_*}$ rotation number condition is satisfied. More generally, Calegari and Walker have shown that the same principle holds for the the rotation number condition associated to any fixed positive word [10, Lemma 3.31].

Since for any $r \in \mathbb{R}$ and $z \in \mathbb{Z}$, we have

$$z \le \lfloor r\rfloor \iff z \le r, \qquad \lceil r\rceil \le z \iff r \le z, \tag{21}$$

it follows, for any $j \in \{1,\dots,n\}$, that if we fix all $\frac{r_i}{s_i}$ with $i \ne j$, then the interval of $\frac{r_j}{s_j} \in \mathbb{R}$ satisfying (19) is closed. On the other hand, for any $r \in \mathbb{R}$ and $z \in \mathbb{Z}$, we know that

$$z \le \lceil r\rceil - 1 \iff z < r, \qquad \lfloor r\rfloor + 1 \le z \iff r < z. \tag{22}$$

These identities led Jankins and Neumann to produce the formulas in the following result.

Theorem 2.5 (Jankins, Neumann, Naimi [23, 28]; *c.f.* Calegari, Walker [10])**.** *For any $n \ge 2$, partition $J \amalg \bar{J} = \{0,\dots,n\}$ with $0 \in J$, and $n{+}1$-tuple $\frac{r_*}{s_*} := (\frac{r_0}{s_0},\dots,\frac{r_n}{s_n}) \in \mathbb{Q}^{n+1}$ with $\frac{r_0}{s_0} \in \mathbb{Z}$ and $\frac{r_i}{s_i} \notin \mathbb{Z}$ for $i > 0$, the trivial circle fibration $\hat{M}$ over an $n+1$-punctured S^2 admits a co-oriented taut foliation F transverse to the boundary, with slopes $\alpha(F \cap \partial_i\hat{M}) = \frac{r_i}{s_i}$ for $i \in \{0,\dots,n\}$, and with F extending to a co-oriented taut foliation on the Dehn filling of $\partial_j\hat{M}$ of slope $\frac{r_j}{s_j}$ for each $j \in J$, if and only if $0 = y_- = y_+$ or $0 \in \langle y_+, y_-\rangle$, with*

$$y_- := \max_{1\le k\le s} -\frac{1}{k}\left(1+\sum_{j\in J}\left\lfloor\frac{r_jk}{s_j}\right\rfloor+\sum_{\bar{j}\in\bar{J}}\left(\left\lceil\frac{r_{\bar{j}}k}{s_{\bar{j}}}\right\rceil-1\right)\right), \tag{23}$$

$$y_+ := \min_{1\le k\le s} -\frac{1}{k}\left(-1+\sum_{j\in J}\left\lceil\frac{r_jk}{s_j}\right\rceil+\sum_{\bar{j}\in\bar{J}}\left(\left\lfloor\frac{r_{\bar{j}}k}{s_{\bar{j}}}\right\rfloor+1\right)\right),$$

where s is the least common positive multiple of the s_i.

2.6. **Dehn fillings and $\bar{N}$-fillings.** For any particular $i \in \{1, \dots, n\}$ in the above theorem, if one fixes the remaining slopes, one finds that the space of slopes in $\mathbb{P}(H_1(\partial_i \hat{M}))$ for which the desired taut foliation exists is often an interval. We now introduce some notation to describe such spaces of slopes in general.

Definition 2.6. *If Y is a compact oriented three-manifold with torus boundary, then we define the sets $\mathcal{F}^L(Y) \subset \mathcal{F}^D(Y) \subset \mathcal{F}(Y) \subset \mathbb{P}(H_1(\partial Y; \mathbb{Z}))$ of rational foliation slopes as follows:*

$$\mathcal{F}^L(Y) := \left\{ \alpha(F \cap \partial Y) \;\middle|\; \begin{array}{l} F \textit{ is a co-oriented taut foliation on } Y \textit{ transverse to } \partial Y, \\ \textit{restricting to a rational co-oriented } \text{linear } \textit{foliation on } \partial Y \end{array} \right\},$$
$$\mathcal{F}^D(Y) := \{ \quad \mu \quad | \textit{ The Dehn filling } Y(\mu) \textit{ admits a co-oriented taut foliation } \},$$
$$\mathcal{F}(Y) := \{ \alpha(F \cap \partial Y) \mid F \textit{ is a co-oriented taut foliation on } Y \textit{ transverse to } \partial Y \}.$$

All linear foliations, even irrational ones, are taut, but rational linear foliations are product foliations, hence extend to co-oriented taut foliations on Dehn fillings of matching slope, implying $\mathcal{F}^L(Y) \subset \mathcal{F}^D(Y) \subset \mathcal{F}(Y)$. In fact, the work of Jankins and Neumann tells us that $\mathcal{F}^L = \mathcal{F}^D$ for manifolds Seifert fibered over the disk, and that the analogous result holds for manifolds Seifert fibered over a punctured S^2. Since the same also holds for manifolds Seifert fibered over a punctured $\mathbb{RP}^2$ [4], and since Corollary 2.2 tells us that taut foliations on homology sphere graph manifolds isotop to restrict to taut foliations transverse to boundaries on Seifert fibered JSJ components, we additionally have $\mathcal{F}^L = \mathcal{F}^D$ for any graph manifold with torus boundary and $b_1 = 1$.

In this latter case, it is natural to ask whether $\mathcal{F}(Y)$ admits a description analogous to the Dehn filling characterization for $\mathcal{F}^L(Y)$. That is, can $\mathcal{F}(Y)$ be characterized in terms of taut foliations on some closed union of Y with some other manifold? Boyer and Clay answer this question affirmatively [4], as we shall see.

Let $\bar{N}$ denote the regular fiber complement

$$\bar{N} := M_{S^2}(0, -\tfrac{1}{2}, \tfrac{1}{2}) \setminus (S^1 \times D_0^2), \tag{24}$$

which Boyer and Clay call the "twisted I-bundle over the Klein bottle," or N_2. The manifold $\bar{N}$ can play a role analogous to that of the solid torus for Dehn fillings.

Definition 2.7. *Suppose Y is an oriented three-manifold with toroidal boundary component $\partial_i Y$. We call any union $Y \cup_\varphi \bar{N}$ with gluing map $\varphi : \partial \bar{N} \to -\partial_i Y$ an* $\bar{N}$-filling *of Y along $\mu \in \mathbb{P}(H_1(\partial_i Y))$, where $\mu := \varphi_*^{\mathbb{P}}(l)$ is the image of the rational longitude l of $\bar{N}$. If Y has (single) torus boundary, we denote an $\bar{N}$-filling of Y along $\mu \in \mathbb{P}(H_1(\partial Y))$ by $Y^{\bar{N}}(\mu)$.*

More generally, if $\partial Y = \coprod_{i=1}^n \partial_i Y$ is a disjoint union of tori, then given a slope $\mu_* := (\mu_1, \dots, \mu_n) \in \prod_{i=1}^n \mathbb{P}(H_1(\partial_i Y))$ and subset $\bar{J} \subset \{1, \dots, n\}$, we denote by $Y^{\bar{N}}(\bar{J}; \mu_*)$ any manifold resulting from $\bar{N}$-filling Y along $\mu_{\bar{\jmath}}$ in $\partial_{\bar{\jmath}} Y$ for each $\bar{\jmath} \in \bar{J}$.

We then have the following result for $\bar{N}$-fillings.

Proposition 2.8. *Suppose Y is a prime compact oriented manifold with boundary a disjoint union $\coprod_{i=1}^n \partial_i Y$ of tori, with some given slope $\mu_* := (\mu_1, \dots, \mu_n) \in$*

$\prod_{i=1}^{n} \mathbb{P}(H_1(\partial_i Y))$. *Moreover, suppose either that* $b_1(Y(\mu_*)) > 0$ *for the Dehn filling* $Y(\mu_*)$, *or that* Y *is a graph manifold, and there is some (possibly empty)* $\bar{J} \subset \{1, \ldots, n\}$, *and* $\bar{N}$*-filling* $Y^{\bar{N}}(\bar{J}; \mu_*)$ *of* Y *along* $\mu_{\bar{\jmath}}$ *in* $\partial_{\bar{\jmath}} Y$ *for each* $\bar{\jmath} \in \bar{J}$, *such that* $Y^{\bar{N}}(\bar{J}; \mu_*)$ *admits a co-oriented taut foliation* F *transverse to the boundary, with* $\alpha(F \cap \partial_j Y) = \mu_j$ *for each* $j \in J := \{1, \ldots, n\} \setminus \bar{J}$.

Then, for every $\bar{J} \subset \{1, \ldots, n\}$, every $\bar{N}$*-filling* $Y^{\bar{N}}(\bar{J}; \mu_*)$ *(including* $Y := Y^{\bar{N}}(\emptyset; \mu_*)$*) admits a co-oriented taut foliation* F *transverse to the boundary, with* $\alpha(F \cap \partial_j Y) = \mu_j$ *for each* $j \in J := \{1, \ldots, n\} \setminus \bar{J}$.

Proof. Part (2) of Gabai's main theorem in [14] tells us that any prime oriented three-manifold with $b_1 > 0$ and boundary a (possibly empty) union of tori admits a co-oriented taut foliation transverse to the boundary. Thus, if $b_1(Y(\mu_*)) > 0$, then any $\bar{N}$-filling $Y^{\bar{N}}(\mu_*) := Y^{\bar{N}}(\{1, \ldots, n\}; \mu_*)$ has $b_1 > 0$, hence admits a co-oriented taut foliation F. Since each $\partial_i Y$ is an incompressible separating torus in this $\bar{N}$-filling, Proposition 2.1 allows us to isotop these separating tori so that they are everywhere transverse to F. Restricting F to any sub-$\bar{N}$-filling $Y^{\bar{N}}(\bar{J}; \mu_*) \subset Y^{\bar{N}}(\mu_*)$ then gives the desired taut foliation on $Y^{\bar{N}}(\bar{J}; \mu_*)$.

If instead, Y is a graph manifold with $b_1(Y(\mu_*)) = 0$, and we are given $\bar{J} \subset \{1, \ldots, n\}$ and a co-oriented taut foliation F on some $\bar{N}$-filling $Y^{\bar{N}}(\bar{J}; \mu_*)$, with F transverse to the boundary and with $\alpha(F \cap \partial_j Y) = \mu_j$ for each $j \in J := \{1, \ldots, n\} \setminus \bar{J}$, then Proposition 2.1 again allows us to isotop each separating torus $\partial_{\bar{\jmath}} Y$ so that F restricts to a co-oriented taut foliation on Y, transverse to ∂Y, with $\alpha(F \cap \partial_i Y) = \mu_i$ for each $i \in \{1, \ldots, n\}$.

We then apply the the foliation gluing theorem of Boyer and Clay [4, Theorem 9.5.2]. That is, for each $i \in \{1, \ldots, n\}$, Theorem 2.5 computes that $\mathcal{F}(\bar{N}_i) = \{l_i\}$, with l_i (of slope 0) the rational longitude of of the i^{th} copy $\bar{N}_i$ of $\bar{N}$. Thus, for any gluing maps $\varphi_i : \bar{N}_i \to -\partial_i Y$ sending $l_i \mapsto \mu_i$ in homology, Boyer's and Clay's gluing theorem tells us that there exist co-oriented taut foliations F' on Y and F_i on $\bar{N}_i$, transverse to respective boundaries, with $\alpha(F' \cap \partial_i Y) = \mu_i = \varphi_{i*}^{\mathbb{P}}(\alpha(F_j \cap \partial \bar{N}_j))$ for each $i \in \{1, \ldots, n\}$, such that the F_i and F' glue together to form a co-oriented taut foliation on the $\bar{N}$-filling $Y^{\bar{N}}(\mu_*)$ specified by the φ_i. After isotoping the $\partial_i Y$ to be transverse to this foliation, we can restrict this foliation to any any sub-$\bar{N}$-filling $Y^{\bar{N}}(\bar{J}'; \mu_*) \subset Y^{\bar{N}}(\mu_*)$. □

In particular, for a graph manifold Y with torus boundary, we have $\mu \in \mathcal{F}(Y)$ if and only if an $\bar{N}$-filling $Y^{\bar{N}}(\mu)$ admits a co-oriented taut foliation.

3. L-space intervals

An L-space is a closed oriented three manifold whose Heegaard Floer homology is trivial, in the sense that for each Spin^c structure, the hat Heegaard Floer homology looks like the singular homology of a point. The reader unfamiliar with L-spaces could consult [31, 32] for an introduction to Heegaard Floer homology, or [33] for a treatment of L-space Dehn fillings. For present purposes, we shall only need the classification of Seifert fibered L-spaces, some formal properties of sets of L-space Dehn-filling slopes, and some basic gluing results, all of which we catalog below.

3.1. L-space Dehn fillings and $\bar{N}$-fillings.

Definition 3.1. *If Y is a compact oriented three-manifold with torus boundary, then we define the* L-space interval *of Y to be*

$$\mathcal{L}(Y) := \{\mu \in \mathbb{P}(H_1(\partial Y)) | \, Y(\mu) \text{ is an L-space}\}. \tag{25}$$

We shall write $\mathcal{L}^\circ(Y)$ for the interior of $\mathcal{L}(Y)$ in $\mathbb{P}(H_1(\partial Y))$.

Thus, $\mathcal{L}(Y)$ is analogous to, and often complementary to, $\mathcal{F}^D(Y)$, especially when Y has no reducible non-L-space Dehn-fillings. Moreover, since the set of slopes of co-oriented taut foliations meeting a generalized rotation number condition is often the closure of the set of product foliation slopes meeting that condition [10, Lemma 3.31], it is natural to ask if $\mathcal{F}(Y)$ bears any relation to the complement of $\mathcal{L}^\circ(Y)$. In fact, we have the following result.

Proposition 3.2 ([33])**.** *If Y is a compact oriented three manifold with torus boundary, then an $\bar{N}$-filling $Y^{\bar{N}}(\mu)$ is an L-space if and only if $\mu \in \mathcal{L}^\circ(Y)$.*

Proof. In [33, Proposition 7.9], J. Rasmussen and the author prove the above result with $\bar{N}$ replaced by any member of a more general class of manifolds dubbed *generalized solid tori*. Since $\bar{N}$ is a generalized solid torus as defined in [33], the result follows. □

Thus $\mathcal{L}^\circ(Y)$ can be regarded as the L-space $\bar{N}$-filling interval of Y.

3.2. **L-space gluing.** Our primary tool for characterizing when a union of three-manifolds along a torus boundary gives an L-space is the following joint result of J. Rasmussen and the author [33]. Hanselman and Watson have proven a similar result in [19].

Proposition 3.3 ([33])**.** *If each of Y_1 and Y_2 is a compact oriented three-manifold with torus boundary, then for any gluing map $\varphi : \partial Y_1 \to -\partial Y_2$ with $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2) \neq \emptyset$, the union $Y_1 \cup_\varphi Y_2$ is an L-space if and only if $\mathbb{P}(H_1(\partial Y_2)) = \varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2)$ if both Y_i are boundary incompressible, and if and only if $\mathbb{P}(H_1(\partial Y_2)) = \varphi_*^{\mathbb{P}}(\mathcal{L}(Y_1)) \cup \mathcal{L}(Y_2)$ otherwise.*

Proof. Recall that a compact three-manifold with torus boundary is boundary incompressible if and only if it is not a connected sum of a solid torus with a (possibly empty) closed three-manifold. The above proposition replicates Theorem 6.2 from [33], except with the hypothesis of boundary incompressibility of each Y_i replacing an *a priori* more technical condition that certain subsets $\mathcal{D}^\tau(Y_i) \subset H_1(Y_i)$ be nonempty. Thomas Gillespie has recently shown [17] these two conditions to be equivalent. None of our gluing arguments involving graph manifolds actually make use of his result, but our later cabling results, which in principle require unions with non graph manifolds, do require Gillespie's result.

□

We shall later show that in the case of non-solid-torus graph manifolds Y_i with torus boundary, various foliation results allow us to drop some of the above hypotheses, so that one obtains an L-space if and only if $\mathbb{P}(H_1(\partial Y_2)) = \varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2)$.

In rather the opposite direction, if Y is Seifert fibered over a punctured S^2 or $\mathbb{RP}^2$, then a Dehn filling Y' of Y fails to be a graph manifold if and only if Y' fails to be prime, if and only if $Y' \neq S^1 \times S^2$ and the Dehn filling, in some $\partial_i Y$, was along the fiber lift $\tilde{f}_i \in H_1(\partial_i Y)$ of slope $\pi_i(\tilde{f}_i) = \infty$ (see the Remark in Section 2.2). In this case, Y' is neither a graph manifold nor a habitat for taut foliations, but since it has *compressible boundary*, its L-space gluing properties simplify, due to the following result.

Proposition 3.4. *Suppose that each of Y_1 and Y_2 is a compact oriented three-manifold with torus boundary, and that Y_1 has compressible boundary. Then the union $Y_1 \cup_\varphi Y_2$ is an L-space if and only if $X_1, \ldots, X_N$ are all L-spaces and $\varphi_*^{\mathbb{P}}(l_1) \in \mathcal{L}(Y_2)$, where l_1 is the rational longitude of Y_1, and where Y_1 decomposes as $Y_1 = (S^1 \times D^2) \# (X_1 \# \cdots \# X_N)$.*

Proof. A union along toroidal boundaries with a solid torus is just a Dehn filling, so we have $Y_1 \cup_\varphi Y_2 = Y_2(\varphi_*^{\mathbb{P}}(l_1)) \# X_1 \# \cdots \# X_N$, and a connected sum of closed manifolds is an L-space if and only if each summand is an L-space. □

The above result explains why not every graph manifold Y with torus boundary satisfies $\mathcal{F}^D(Y) \amalg \mathcal{L}(Y) = \mathbb{P}(H_1(\partial Y))$. That is, no reducible manifold (besides $S^1 \times D^2$) admits a co-oriented taut foliation, but there exist graph manifolds with reducible Dehn fillings which are not $S^1 \times D^2$ or an L-space.

3.3. Floer simple manifolds and L-space intervals.

It is not known, in general, what forms the sets $\mathcal{F}(Y)$ or $\mathcal{F}^D(Y)$ can take for an arbitrary compact oriented three-manifold Y with torus boundary, but the situation for L-spaces is better understood. As shown in [33] by J. Rasmussen and the author, $\mathcal{L}(Y)$ can only be empty, the set of a single point, a closed interval, or the complement of the rational longitude in $\mathbb{P}(H_1(\partial Y))$. For historical reasons, we call Y *Floer simple* in the latter two cases. Equivalently, we could define Floer simple manifolds as follows.

Definition 3.5. *A compact oriented three-manifold Y with torus boundary is* Floer simple *if $\mathcal{L}^\circ(Y) \neq \emptyset$.*

In particular, if Y is Floer simple, then its space $\mathcal{L}(Y)$ of L-space Dehn filling slopes can be specified entirely in terms of the left-hand and right-hand endpoints of $\mathcal{L}(Y)$, in a sense we can make precise, prefaced with the introduction of an abbreviative notation for the closed interval with infinite endpoint.

Definition 3.6. *For $y \in \mathbb{Q}$, we shall write $[-\infty, y]$, $[y, +\infty]$, $[-\infty, y\rangle$, and $\langle y, +\infty]$ for the following intervals in $(\mathbb{Q} \cup \{\infty\}) \subset (\mathbb{R} \cup \{\infty\})$:*

$$[-\infty, y] := \{\infty\} \cup \langle -\infty, y] , \qquad [-\infty, y\rangle := \{\infty\} \cup \langle -\infty, y\rangle ,$$
$$[y, +\infty] := [y, +\infty\rangle \cup \{\infty\}, \qquad \langle y, +\infty] := \langle y, +\infty\rangle \cup \{\infty\}.$$

Definition 3.7. *If $y_-, y_+ \in \mathbb{Q} \cup \{\infty\}$, then we define the* L-space interval from y_- to y_+, *denoted $[[y_-, y_+]] \subset \mathbb{Q} \cup \{\infty\}$, as follows:*

$$[[y_-, y_+]] := \begin{cases} \langle -\infty, +\infty \rangle & \infty = y_- \ , \ y_+ = \infty \\ \langle y_-, +\infty] \cup [-\infty, y_+\rangle & \mathbb{Q} \ni y_- = y_+ \in \mathbb{Q} \\ [y_-, +\infty] \cup [-\infty, y_+] & \mathbb{Q} \ni y_- > y_+ \in \mathbb{Q} \\ [y_-, +\infty] \cap [-\infty, y_+] & \mathbb{Q} \ni y_- < y_+ \in \mathbb{Q} \\ [-\infty, y_+] & \infty = y_- \ , \ y_+ \in \mathbb{Q} \\ [y_-, +\infty] & \mathbb{Q} \ni y_- \ , \ y_+ = \infty \end{cases} . \tag{26}$$

In other words, $[[y_-, y_+]]$ is the unique interval with left-hand endpoint y_- and right-hand endpoint y_+ which is closed if $y_- \neq y_+$ and open otherwise.

Remark: In practice, we extend the above definition to allow $y_- = -\infty$ or $y_+ = +\infty$, which we treat as identical to the respective cases of $y_- = \infty$ or $y_+ = \infty$.

Proposition 3.8. *Suppose, for some compact oriented three-manifold Y with torus boundary, that we are given an identification $\mathbb{P}(H_1(\partial Y)) \cong \mathbb{Q} \cup \{\infty\}$. If Y is Floer simple, then there are unique $y_-, y_+ \in \mathbb{Q} \cup \{\infty\}$ such that $\mathcal{L}(Y) = [[y_-, y_+]] \subset \mathbb{Q} \cup \{\infty\}$. Conversely, if there are $y_-, y_+ \in \mathbb{Q} \cup \{\infty\}$ for which $\mathcal{L}(Y) = [[y_-, y_+]]$, then Y is Floer simple.*

The above follows from the aforementioned result, proven in [33], that if $\mathcal{L}(Y)$ contains more than one point, then $\mathcal{L}(Y)$ is either a closed interval or the complement of a point in $\mathbb{P}(H_1(\partial Y))$. The following computation of L-space intervals for Seifert fibered spaces demonstrates one use of this "$[[\cdot, \cdot]]$" notation.

Proposition 3.9. *If Y is a regular fiber complement in a Seifert fibered rational homology sphere, then $\mathcal{L}^{\circ}(Y) \amalg \mathcal{F}(Y) = \mathbb{P}(H_1(\partial Y))$. If Y has non-orientable base, we have $\mathcal{L}(Y) = \langle -\infty, +\infty \rangle$ and $\mathcal{F}^D(Y) = \emptyset$ (unless Y is the twisted S^2-bundle over the Möbius strip, in which case $\mathcal{F}^D(Y) = \{\infty\}$) If Y is a regular fiber complement in $M_{S^2}(y_*)$, for some $y_* = (y_0, \ldots, y_n) \in \mathbb{Q}^{n+1}$, then $\mathbb{P}(H_1(\partial Y)) \setminus \mathcal{F}^D(Y) = \mathcal{L}(Y) = [[y_-, y_+]]$, where*

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=0}^{n} \lfloor y_i k \rfloor\right), \qquad y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=0}^{n} \lceil y_i k \rceil\right), \tag{27}$$

unless Y is a solid torus, in which case $y_- := y_+ := -\sum_{i=0}^{n} y_i$ is the rational longitude of Y.

Proof. For the case of Y with non-orientable base, see the work of Boyer, Gordon, and Watson [5] and Boyer and Clay [4]. For Y with orientable base, the foliations result is due to Jankins, Neumann, and Naimi [23, 28], and the L-space result is originally due to the combined work of Jankins, Neumann, and Naimi [23, 28], Eliashberg and Thurston [13], Ozsváth and Szabó [30], Lisca and Matić [26], and Lisca and Stipsicz [27]. Alternatively, J. Rasmussen and the author offer a recent stand-alone proof of the L-space result [33]. □

4. L-space Intervals and Foliation Slopes for Graph Manifolds

This is the section in which we prove most of our main results. We begin, however, by introducing the notion of *L/NTF-equivalence*, the presence of which makes gluing easier. We further pause in Section 4.2, to establish some conventions for graph manifolds with torus boundary and $b_1 = 1$.

4.1. L/NTF-equivalence and Gluing.

For a pair of manifolds spliced together along torus boundaries, we can often prove stronger gluing results about the existence of co-oriented taut foliations or non-trivial Heegaard Floer homology if we are able to use gluing theorems from both areas of mathematics. In general, however, this strategy only works if we know that each manifold behaves in a suitably complementary manner with respect to co-oriented taut foliations and L-space Dehn fillings, a notion which we now make precise.

Definition 4.1. *If Y is a prime compact oriented three-manifold with torus boundary, then we call Y* L/NTF-equivalent *if $\mathcal{F}(Y) \amalg \mathcal{L}^{\circ}(Y) = \mathbb{P}(H_1(\partial Y))$.*

In certain circumstances, one can characterize L/NTF-equivalence in terms of $\bar{N}$-fillings.

Proposition 4.2. *Suppose Y is a prime compact oriented three-manifold with torus boundary. If $b_1(Y) > 1$ or Y is a graph manifold, then Y is L/NTF-equivalent if and only if each $\bar{N}$-filling of Y is an L-space precisely when it fails to admit a co-oriented taut foliation.*

Proof. This follows immediately from Propositions 2.8 and 3.2. □

There are some classes of manifold which we already know to be L/NTF-equivalent.

Proposition 4.3. *Suppose Y is a prime compact oriented three-manifold with torus boundary. If $b_1(Y) > 1$, or if Y is the union of a Seifert fibered space with zero or more copies of $\bar{N}$, then Y is L/NTF-equivalent.*

Proof. If $b_1(Y) > 1$, then since no Dehn filling of Y is a rational homology sphere, we have $\mathcal{L}(Y) = \emptyset$, implying $\mathcal{L}^{\circ}(Y) = \emptyset$. Correspondingly, Proposition 2.8 implies $\mathcal{F}(Y) = \mathbb{P}(H_1(\partial Y))$.

Suppose Y has $b_1(Y) = 1$ and is the union of a Seifert fibered space with zero or more copies of $\bar{N}$. Then, using [33, Proposition 7.9] of J. Rasmussen and the author to replace Boyer's and Clay's "N_t" manifolds with $\bar{N}$, we invoke the "slope detection" theorem of Boyer and Clay [4, Theorem 8.1] to deduce L/NTF-equivalence for Y.

Alternatively, one could prove the same result by inductively performing $\bar{N}$-fillings in regular fiber complements, starting with Proposition 3.9 for the Seifert fibered base case, and using the gluing result in Proposition 4.4 below, together with (21) and (22), to evolve (27) to match (23). Similar inductive arguments appear in the proof of Theorem 4.6. □

Remark. We later prove L/NTF-equivalence for all graph manifolds with torus boundary.

We are now ready to state our main gluing result.

Proposition 4.4. *Suppose Y_1 and Y_2 are non-solid-torus L/NTF-equivalent graph manifolds with torus boundary. Then for any union $Y_1 \cup_\varphi Y_2$, $\varphi : \partial Y_1 \to -\partial Y_2$, the following are equivalent:*

(*i*) $Y_1 \cup_\varphi Y_2$ *is an L-space.*
(*ii*) $Y_1 \cup_\varphi Y_2$ *does not admit a co-oriented taut foliation.*
(*iii*) $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2))$.

Proof. Suppose (*iii*) holds, so that Proposition 3.3 implies $Y_1 \cup_\varphi Y_2$ is an L-space. Ozsváth and Szabó have shown [30] that an L-space does not admit C^2 co-oriented taut foliations, and this result has been improved to C^0 co-oriented taut foliations by Bowden [3] and independently by Kazez and Roberts [24].

Suppose (*iii*) fails to hold. If $b_1(Y_1 \cup_\varphi Y_2) > 0$, then Gabai [14] tells us there is a co-oriented taut foliation on $Y_1 \cup_\varphi Y_2$, and we also know that $Y_1 \cup_\varphi Y_2$ is not an L-space. Suppose instead that $b_1(Y_1 \cup_\varphi Y_2) = 0$. Then L/NTF-equivalence implies $\varphi_*^{\mathbb{P}}(\mathcal{F}(Y_1)) \cap \mathcal{F}(Y_2) \neq \emptyset$, and so there are co-oriented taut foliations F_i on Y_i transverse to ∂Y_i such that $\varphi_*^{\mathbb{P}}(\alpha(F_1 \cap \partial Y_1)) = \alpha(F_2 \cap \partial Y_2)$. By Corollary 2.2, we can isotop the incompressible tori in Y_1 and Y_2 so that each F_i restricts to boundary-transverse co-oriented taut foliations on each of the JSJ components of each Y_i. This means that the JSJ components of $Y_1 \cup_\varphi Y_2$ each admit boundary-transverse co-oriented taut foliations which restrict to boundary foliations of matching slopes with respect to boundary-gluing maps. We can therefore invoke the foliation gluing theorem of Boyer and Clay [4, Theorem 9.5.2] to assert the existence of a co-oriented taut foliation on all of $Y_1 \cup_\varphi Y_2$. Again, this co-oriented taut foliation implies that $Y_1 \cup_\varphi Y_2$ is not an L-space. □

4.2. **Graph manifold conventions.** Any closed graph manifold can be regarded as a Dehn filling of a graph manifold Y with torus boundary. If $b_1(Y) > 1$, then $\mathcal{L}(Y) = \mathcal{L}^\circ(Y) = \emptyset$ and $\mathcal{F}(Y) = \mathbb{P}(H_1(\partial Y))$ (see Proposition 4.3), which is not very interesting.

If $b_1(Y) = 1$, then we call Y a *tree manifold*, since Y admits a rational homology sphere Dehn filling, corresponding to a tree graph. Rooting the tree graph for Y at the Seifert fibered piece containing ∂Y provides a recursive construction for Y,

$$Y = \hat{M} \cup \left(\coprod_{i=1}^{n_{\mathrm{D}}} (S^1 \times D_i^2) \ \amalg \ \coprod_{i=1}^{n_{\mathrm{G}}} Y_i \right), \tag{28}$$

where each of $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ is a non-solid-torus tree manifold with torus boundary. Since $b_1(Y) = 1$, $\hat{M}$ is the trivial circle fibration over an $n+1$-punctured S^2 or the twisted circle fibration over an $n+1$-punctured $\mathbb{RP}^2$, with boundary components $\partial_i^{\mathrm{D}} \hat{M} := \partial_i \hat{M}$ for $i \in \{1, \ldots, n_{\mathrm{D}}\}$, $\partial_i^{\mathrm{G}} \hat{M} := \partial_{n_{\mathrm{D}}+i} \hat{M}$ for $i \in \{1, \ldots, n_{\mathrm{G}}\}$, and $\partial Y := \partial_{n+1} \hat{M} =: \partial_{n_{\mathrm{G}}+1}^{\mathrm{G}} \hat{M} =: \partial_{n_{\mathrm{D}}+1}^{\mathrm{D}} \hat{M}$, with $n := n_{\mathrm{D}} + n_{\mathrm{G}}$. We shall sometimes call $\hat{M}$ the "foundation" for Y.

Since edges in the graph for Y correspond to gluings along incompresible tori, each gluing map $\varphi_i : \partial Y_i \to -\partial_i^{\mathrm{G}} \hat{M}$, for $i \in \{1, \ldots, n_{\mathrm{G}}\}$, labels one of the n_{G} edges descending from the root of the graph for Y. We call $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ the *daughter subtrees* of Y. Each Y_i is a tree manifold with torus boundary and $b_1(Y_i) = 1$, with tree rooted at the Seifert fibered piece containing ∂Y_i, giving rise to a recursive description for Y_i analogous to that for Y in (28). For any $i \in \{1, \ldots, n_{\text{G}}\}$ for which Y_i is Floer simple, *i.e.*, for which $\mathcal{L}^\circ(Y_i) \neq \emptyset$, then we invoke Proposition 3.8 to write

$$[[y^{\text{G}}_{i-}, y^{\text{G}}_{i+}]] := \varphi^{\mathbb{P}}_{i*}(\mathcal{L}(Y_i)). \tag{29}$$

If instead, $\mathcal{L}(Y_i) \neq \emptyset$ for some non-Floer-simple Y_i, then we write

$$\{y^{\text{G}}_{i-}\} := \{y^{\text{G}}_{i+}\} := \varphi^{\mathbb{P}}_{i*}(\mathcal{L}(Y_i)). \tag{30}$$

Since the n_{D} solid tori glued to $\hat{M}$ create the exceptional fibers of the Seifert fibered "root" $\hat{M} \cup \coprod_{i=1}^{n_{\text{D}}} (S^1 \times D^2_i)$ of our tree, we record the Seifert data of this Seifert fibered space by labeling the root vertex with the Dehn filling slopes $y^{\text{D}}_1, \ldots, y^{\text{D}}_{n_{\text{D}}} \in \mathbb{Q}$. More explicitly, to each solid torus $S^1 \times D^2_i$ in (28), we associate the gluing map $\varphi^{\text{D}}_i : \partial(S^1 \times D^2_i) \to -\partial^{\text{D}}_i \hat{M}$, and set $y^{\text{D}}_i := \varphi^{\text{D}\mathbb{P}}_{i*}(l_i)$, for l_i the rational longitude of $S^1 \times D^2_i$. As usual, we demand that each $y^{\text{D}}_i \neq \infty$. We stray slightly from our earlier convention by allowing $y^{\text{D}}_i \in \mathbb{Z}$, but this allows us to fix $e_0 := y^{\text{D}}_0 := 0$ and then forget the 0^{th} fiber complement altogether, without loss of generality.

4.3. **Statement of Main Results.** We first show that all graph manifolds with torus boundary are L/NTF-equivalent, making our main gluing tool, Proposition 4.4, applicable for all such non-solid-torus graph manifolds. We then can make the inductive gluing arguments necessary to calculate L-space intervals for graph manifolds with torus boundary.

Theorem 4.5. *Every graph manifold Y with torus boundary is L/NTF equivalent,* i.e.*, satisfies $\mathcal{F}(Y) \amalg \mathcal{L}^\circ(Y) = \mathbb{P}(H_1(\partial Y))$. Moreover, if we let $\mathcal{R}(Y)$ denote the set of slopes of reducible (and not $S^1 \times D^2$) Dehn fillings of Y, then $\mathcal{F}^D(Y) \amalg (\mathcal{L}(Y) \cup \mathcal{R}(Y)) = \mathbb{P}(H_1(\partial Y))$.*

The above also implies that the following calculation of $\mathcal{L}(Y)$ for graph manifolds Y with torus boundary completely determines both $\mathcal{F}(Y)$ and $\mathcal{F}^D(Y)$.

Theorem 4.6. *Suppose Y is a graph manifold with torus boundary and nonempty $\mathcal{L}(Y)$. If the Seifert fibered component of Y containing ∂Y has non-orientable base, then $\mathcal{L}(Y) = \langle -\infty, +\infty \rangle$. Otherwise, we have*

$$\mathcal{L}(Y) = \begin{cases} [[y_-, y_+]] & Y \textit{ Floer simple}, \\ \{y_-\} = \{y_+\} & Y \textit{ not Floer simple}, \end{cases}$$

where, for $n_{\text{D}}, n_{\text{G}}, y^{\text{D}}_i, y^{\text{G}}_{i-}$, and y^{G}_{i+} as defined in Section 4.2, we define $y_-, y_+ \in \mathbb{Q} \cup \{\infty\}$ as

$$y_- := \max_{k>0} -\frac{1}{k}\left(1 + \sum_{i=1}^{n_{\text{D}}} \lfloor y^{\text{D}}_i k \rfloor + \sum_{i=1}^{n_{\text{G}}} \left(\lceil y^{\text{G}}_{i+} k \rceil - 1\right)\right),$$
$$y_+ := \min_{k>0} -\frac{1}{k}\left(-1 + \sum_{i=1}^{n_{\text{D}}} \lceil y^{\text{D}}_i k \rceil + \sum_{i=1}^{n_{\text{G}}} \left(\lfloor y^{\text{G}}_{i-} k \rfloor + 1\right)\right),$$

unless Y is a solid torus, in which case $y_- := y_+ := -\sum_{i=1}^{n_{\text{D}}} y^{\text{D}}_i$.

Remark. The above formulae for $y_{\mp}$ are finitely computable. In particular, the maximum (respectively, minimum) is realized for $k \leq s_{\pm}$, where $s_{\pm}$ is the least common positive multiple of the denominators of $y_1^{\mathrm{D}}, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}}$ and $y_{1\pm}^{\mathrm{G}}, \ldots, y_{n_{\mathrm{G}}\pm}^{\mathrm{G}}$, with $\mp$ and $\pm$ cases taken respectively from top to bottom. If computation is not the goal, then one can avoid treating the solid torus case separately by replacing "max" with "sup" and "min" with "inf."

Unlike the case of oriented Seifert fibered spaces over the Möbius strip or disk, not all graph manifolds with torus boundary are Floer simple. We therefore need a companion result to characterize precisely when $\mathcal{L}^{\circ}$ or $\mathcal{L}$ is nonempty.

Proposition 4.7. *Suppose Y is a graph manfold with torus boundary. If $b_1(Y) > 1$, then $\mathcal{L}(Y) = \emptyset$. Suppose $b_1(Y) = 1$, so that Y admits the recursive description in Section 4.2.*

If the JSJ component containing ∂Y has non-orientable base, then $\mathcal{L}(Y) \neq \emptyset$ if and only if Y is Floer simple, if and only if the following holds:

(FS0) *All daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ are Floer simple;*
$\infty \in ([[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]])^{\circ}$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\}$.

If the JSJ component containing ∂Y has orientable base, then Y is Floer simple if and only if the daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ are each Floer simple and one of the following holds:

(FS1) $[[y_{j-}^{\mathrm{G}}, y_{j+}^{\mathrm{G}}]] = \langle -\infty, +\infty \rangle$ *for some* $j \in \{1, \ldots, n_{\mathrm{G}}\}$;
$\infty \in ([[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]])^{\circ}$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\} \setminus \{j\}$.
(FS2) $[[y_{j-}^{\mathrm{G}}, y_{j+}^{\mathrm{G}}]] = [y_{j-}^{\mathrm{G}}, y_{j+}^{\mathrm{G}}]$ *for some* $j \in \{1, \ldots, n_{\mathrm{G}}\}$;
$\infty \in ([[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]])^{\circ}$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\} \setminus \{j\}$;
$y_- < y_+$.
(FS3) *At least one of* $\{i : [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] = [-\infty, y_{i+}^{\mathrm{G}}]\}$ *and* $\{i : [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] = [y_{i-}^{\mathrm{G}}, +\infty]\}$ *is the empty set;*
$\infty \in [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]]$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\}$.

If the JSJ component containing ∂Y has orientable base, then $\mathcal{L}(Y) \neq \emptyset$ with Y not *Floer simple if and only if one of the following holds:*

(NFS1) $n_{\mathrm{G}} = 1$, $|\{i : y_i^{\mathrm{D}} \in \mathbb{Q} \setminus \mathbb{Z}\}| \leq 1$;
$\varphi_{1*}^{\mathbb{P}}(\mathcal{L}(Y_1)) = \{y_1^{\mathrm{G}}\}$ *for some* $y_1^{\mathrm{G}} \in \mathbb{Q}$;
$\sum_{i=1}^{n_{\mathrm{D}}} y_i^{\mathrm{D}} + y_1^{\mathrm{G}} \in \mathbb{Z}$ *or all* $y_1^{\mathrm{D}}, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}} \in \mathbb{Z}$.
(NFS2) *All daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ are Floer simple;*
$[[y_{j-}^{\mathrm{G}}, y_{j+}^{\mathrm{G}}]] = [y_{j-}^{\mathrm{G}}, y_{j+}^{\mathrm{G}}]$ *for some* $j \in \{1, \ldots, n_{\mathrm{G}}\}$;
$\infty \in ([[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]])^{\circ}$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\} \setminus \{j\}$;
$y_- = y_+$.
(NFS3) *All daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ are Floer simple;*
$\{i : [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] = [-\infty, y_{i+}^{\mathrm{G}}]\} \neq \emptyset$ *and* $\{i : [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] = [y_{i-}^{\mathrm{G}}, +\infty]\} \neq \emptyset$;
$\infty \in [[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]]$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\}$.
(NFS4) $\varphi_{j*}^{\mathbb{P}}(\mathcal{L}(Y_j)) = \{\infty\}$ *for some* $j \in \{1, \ldots, n_{\mathrm{G}}\}$;
$\infty \in \varphi_{i*}^{\mathbb{P}}(\mathcal{L}(Y_i))$ *for all* $i \in \{1, \ldots, n_{\mathrm{G}}\}$.

Note that all eight (FS) and (NFS) conditions are mutually exclusive. Note also that the isolated L-space fillings described in (NFS3) and (NFS4) are not graph manifolds.

We now proceed to prove our main results, starting with that of L/NTF-equivalence.

4.4. **Proof of Theorem 4.5.** Proposition 4.3 gives the desired result for $b_1(Y) > 1$.

We therefore restrict attention to graph manifolds Y with $b_1(Y) = 1$ and torus boundary ∂Y, so that Y admits the recursive description in (28), with tree graph rooted at the Seifert fibered piece containing ∂Y. Inductively assume that any such Y with tree height $\leq k-1$ is L/NTF-equivalent and satisfies $\mathcal{F}(Y) \amalg (\mathcal{L}(Y) \cup \mathcal{R}(Y)) = \mathbb{P}(H_1(\partial Y))$, noting that Proposition 3.9 covers the case of trees of height zero. Fix an arbitrary tree manifold Y with $b_1(Y) = 1$, torus boundary, and tree height $k > 0$, and parameterize its data as in Section 4.2.

To each $j \in \{1, \ldots, n_{\text{G}}\}$, $y^j_* := (y^j_{j+1}, \ldots, y^j_{n_{\text{G}}+1}) \in (\mathbb{Q} \cup \{\infty\})^{n_{\text{G}}+1-j}$, and $\theta \in \{0, 1\}$, we associate a manifold $Y^j_\theta[y^j_*]$, constructed as follows. Starting with $\hat{M}$, first perform Dehn fillings of slopes $y^{\text{D}}_1, \ldots, y^{\text{D}}_{n_{\text{D}}}$ along the respective boundary components $\partial^{\text{D}}_1 \hat{M}, \ldots, \partial^{\text{D}}_{n_{\text{D}}} \hat{M}$ in $\hat{M}$. Next, attach the graph manifolds $Y_1, \ldots, Y_{j-1}$, via the respective gluing maps $\varphi_1, \ldots, \varphi_{j-1}$, to the resulting manifold. Leaving the boundary component $\partial^{\text{G}}_j \hat{M} =: \partial Y^j[y^j_*]$ unfilled, lastly perform $\bar{N}$-fillings of slopes $y^j_{j+1}, \ldots, y^j_{n_{\text{G}}+1}$ along the respective boundary components $\partial^{\text{G}}_{j+1} \hat{M}, \ldots, \partial^{\text{G}}_{n_{\text{G}}+1} \hat{M}$ of the resulting manifold, and call the result $Y^j_0[y^j_*]$. To form $Y^j_1[y^j_*]$, replace the $\bar{N}$-filling of slope $y^j_{n_{\text{G}}+1}$ in $\partial^{\text{G}}_{n_{\text{G}}+1} \hat{M} \subset Y^j_0[y^j_*]$ with the Dehn filling of slope $y^j_{n_{\text{G}}+1}$. In addition, set $y^{n_{\text{G}}+1}_* := \emptyset$ and $Y^{n_{\text{G}}+1}_0[\emptyset] := Y$.

For positive $j \leq n_{\text{G}}$, inductively assume that for any $y^j_* \in (\mathbb{Q} \cup \infty)^{n_{\text{G}}+1-j}$ and $\theta \in \{0, 1\}$, any manifold of the form $Y^j_\theta[y^j_*]$ is L/NTF-equivalent if it is prime, noting that Proposition 4.3 covers the base case of $j = 1$. For any $y \in \mathbb{Q} \cup \{\infty\}$, $y^{j+1}_* := (y^j_{j+2}, \ldots, y^j_{n_{\text{G}}+1}) \in (\mathbb{Q} \cup \{\infty\})^{n_{\text{G}}+1-j}$, $\theta \in \{0, 1\}$, and manifold of the form $Y^{j+1}_\theta[y^{j+1}_*]$, we can make matching choices of $\bar{N}$-filling gluing maps to obtain

$$Y^{j+1}_\theta[y^{j+1}_*]^{\bar{N}}(y) = Y_j \cup Y^j_\theta[(y, y^{j+1}_*)]. \tag{31}$$

Suppose $Y^{j+1}_\theta[y^{j+1}_*]^{\bar{N}}(y)$ is prime, implying $Y^j_\theta[(y, y^{j+1}_*)]$ is prime and hence L/NTF-equivalent by inductive assumption. Since $Y^j_\theta[(y, y^{j+1}_*)]$ and Y_j are L/NTF-equivalent non-solid-torus graph manifolds with torus boundary, Proposition 4.4 makes $Y^{j+1}_\theta[y^{j+1}_*]^{\bar{N}}(y)$ an L-space if and only if it fails to admit a co-oriented taut foliation. Since this holds for arbitrary $y \in \mathbb{Q} \cup \{\infty\}$, Proposition 4.2 tells us $Y^{j+1}_\theta[y^{j+1}_*]$ is L/NTF-equivalent.

Completing our induction on j, we conclude that $Y := Y^{n_{\text{G}}+1}_0[\emptyset]$ is L/NTF-equivalent, and that $Y^{n_{\text{G}}}_1[y]$ is L/NTF-equivalent for any $y \in \mathbb{Q} \cup \{\infty\}$ for which $Y^{n_{\text{G}}}_1[y]$ is prime. Thus, for any prime Dehn filling $Y(y)$, Proposition 4.4 tells us that the union

$$Y(y) = Y_{n_{\text{G}}} \cup_{\varphi_{n_{\text{G}}}} Y^{n_{\text{G}}}_1[y] \tag{32}$$

is an L-space if and only if it fails to admit a co-oriented taut foliation, and so

$$\mathbb{P}(H_1(\partial Y)) \setminus (\mathcal{L}(Y) \cup \mathcal{R}(Y)) = \mathcal{F}(Y) \setminus \mathcal{R}(Y) = \mathcal{F}(Y). \tag{33}$$

Inducting on tree height k then completes the proof. □

We next prove Theorem 4.6 and Proposition 4.7 in tandem over the course of Sections 4.5 – 4.8. The inductive program laid out in Section 4.5 spans all three of the subsequent subsections.

4.5. **Inductive Set-up for Proof of Theorem 4.6 and Proposition 4.7.** Both results hold automatically when $b_1(Y)>1$. This leaves the case of $b_1(Y)=1$, so that Y admits the recursive description in (28), with tree rooted at the Seifert fibered piece containing ∂Y.

Inductively assume that both Theorem 4.6 and Proposition 4.7 hold for all tree manifolds with torus boundary, $b_1 = 1$, and tree height $\leq k-1$, noting that Proposition 3.9 covers the height zero case. In addition, inductively assume that Theorem 4.6 and Proposition 4.7 hold for any tree manifold with torus boundary, $b_1=1$, tree height $\leq k$, and $\leq n_{\mathrm{G}}-1$ daughter subtrees, noting that Proposition 3.9 also covers the case of zero daughter subtrees.

For the remainder of the proof, we fix an arbitrary height k tree manifold Y with torus boundary and $b_1(Y) = 1$, as described in Section 4.2. Thus, Y has n_{G} daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$, attached via respective gluing maps $\varphi_1, \ldots, \varphi_{n_{\mathrm{G}}}$, and the Seifert fibered piece containing ∂Y, at which we root the tree for Y, is the Dehn filling of slope $y_*^{\mathrm{D}} = (y_1^{\mathrm{D}}, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}})$ of the "foundation" $\hat{M}$ of Y, where $\hat{M}$ is either the trivial S^1-fibration over an $n := n_{\mathrm{D}} + n_{\mathrm{G}} + 1$-punctured S^2 or the twisted S^1-fibration over an n-punctured $\mathbb{RP}^2$. For any $y \in \mathbb{Q} \cup \{\infty\}$, let $\hat{Y}[y]$ denote the complement of $Y_{n_{\mathrm{G}}} \setminus \partial Y_{n_{\mathrm{G}}}$ in the Dehn filling $Y(y)$, so that we regard $Y(y)$ as the union

$$Y(y) = \hat{Y}[y] \cup_{\varphi_{n_{\mathrm{G}}}} Y_{n_{\mathrm{G}}}. \tag{34}$$

For any $y \in \mathbb{Q}$, our inductive assumptions make Theorem 4.6 and Proposition 4.7 hold for $\hat{Y}[y]$ and $Y_{n_{\mathrm{G}}}$, since $Y_{n_{\mathrm{G}}}$ has tree height $\leq k-1$, and since for $y \neq \infty$, $\hat{Y}[y]$ is a $b_1 = 1$ tree manifold with torus boundary, $n_{\mathrm{G}} - 1$ daughter subtrees, and tree height $\leq k$.

4.6. **Non-orientable Base.** Consider the case in which $\hat{M}$ is S^1-fibered over a punctured $\mathbb{RP}^2$. First note that since the regular fiber class is torsion, its primitive lift $\tilde{f}_{n_{\mathrm{G}}}^{\mathrm{G}} \in H_1(\partial Y)$, of slope ∞, is the rational longitude, which means that $\infty \notin \mathcal{L}(Y)$.

Suppose there is some $y \neq \infty$ for which $\mathcal{L}(\hat{Y}[y]) \neq \emptyset$. Then, by inductive assumption, Proposition 4.7 tells us that the daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}-1}$ are Floer simple, with $\infty \neq y_{i-}^{\mathrm{G}} \geq y_{i+}^{\mathrm{G}} \neq \infty$ for all $i \in \{1, \ldots, n_{\mathrm{G}}-1\}$, where $[[y_{i-}^{\mathrm{G}}, y_{i+}^{\mathrm{G}}]] := \varphi_i(\mathcal{L}(Y_i))$. This conversely implies that $\hat{Y}[y]$ is Floer simple for all $y \in \langle -\infty, +\infty \rangle$. Now, for each $y \in \langle -\infty, +\infty \rangle$, $\hat{Y}[y]$ is a non-solid-torus graph manifold, and so Proposition 4.4 tells us that the union $Y(y) = \hat{Y}[y] \cup Y_{n_{\mathrm{G}}}$ in (34) is an L-space if and only if

$$\mathcal{L}^{\circ}(\hat{Y}[y]) \cup \varphi_{n_{\mathrm{G}}*}^{\mathbb{P}}(\mathcal{L}^{\circ}(Y_{n_{\mathrm{G}}})) = \mathbb{P}(H_1(\partial_{n_{\mathrm{G}}}^{\mathrm{G}} \hat{M})). \tag{35}$$

Since, by inductive assumption, Theorem 4.6 implies $\mathcal{L}^\circ(\hat{Y}[y]) = \langle -\infty, +\infty\rangle$ for all $y \in \langle -\infty, +\infty\rangle$, (35) holds if and only if $Y_{n_{\mathrm{G}}}$ is Floer simple and has $\infty \neq y^{\mathrm{G}}_{n_{\mathrm{G}}-} \geq y^{\mathrm{G}}_{n_{\mathrm{G}}+} \neq \infty$, in which case we consequently have $\mathcal{L}(Y) = \langle -\infty, +\infty\rangle$.

Conversely, suppose that $\mathcal{L}(\hat{Y}[y]) = \emptyset$ for all $y \in \mathbb{Q}$. Then for all $y \in \mathbb{Q}$, Proposition 4.4 implies that $Y(y) = \hat{Y}[y] \cup Y_{n_{\mathrm{G}}}$ is not an L-space, and so $\mathcal{L}(Y) = \emptyset$. Moreover, since there is $y \in \mathbb{Q}$ with $\mathcal{L}(\hat{Y}[y]) = \emptyset$, our inductive assumption for Proposition 4.7 tells us that either there is some $i \in \{1, \ldots, n_{\mathrm{G}}-1\}$ for which Y_i is not Floer simple, or there is some Floer simple Y_i failing to satisfy $\infty \neq y^{\mathrm{G}}_{i-} \geq y^{\mathrm{G}}_{i+} \neq \infty$.

Thus, in either case, both Theorem 4.6 and Proposition 4.7 hold for Y.

4.7. **Orientable Base: Cases in which $\hat{Y}[y]$ is never a solid torus.** From now on, we assume that the JSJ component of Y containing ∂Y has orientable base.

In this subsection of the proof, we consider the case in which $\hat{Y}[y]$ is not a solid torus for any $y \in \mathbb{Q} \cup \{\infty\}$. More precisely, we consider the case of a fixed tree manifold Y with torus boundary and $b_1(Y) = 1$, parameterized as in Section 4.2, with tree height $k > 0$ and $n_{\mathrm{G}} > 0$ daughter subtrees, where we demand that if $n_{\mathrm{G}} - 1 = 0$, then $y^{\mathrm{D}}_i \notin \mathbb{Z}$ for at least two distinct values of $i \in \{1, \ldots, n_{\mathrm{D}}\}$.

We begin by fixing some notation. For all $k \in \mathbb{Z}_{>0}$, define $\hat{y}^0_-(k), \hat{y}^0_+(k) \in \mathbb{Q} \cup \{\infty\}$ by

$$\hat{y}^0_-(k) := -\frac{1}{k}\left(\ \ 1 + \sum_{i=1}^{n_{\mathrm{D}}} \lfloor y^{\mathrm{D}}_i k \rfloor + \sum_{i=1}^{n_{\mathrm{G}}-1} \left(\lceil y^{\mathrm{G}}_{i+} k\rceil - 1\right)\right), \tag{36}$$
$$\hat{y}^0_+(k) := -\frac{1}{k}\left(-1 + \sum_{i=1}^{n_{\mathrm{D}}} \lceil y^{\mathrm{D}}_i k \rceil + \sum_{i=1}^{n_{\mathrm{G}}-1} \left(\lfloor y^{\mathrm{G}}_{i-} k\rfloor + 1\right)\right).$$

The endpoints $y_-, y_+ \in \mathbb{Q} \cup \{\infty\}$ defined in Theorem 4.6 are then given by

$$y_- := \max_{k>0} \left(\hat{y}^0_-(k) - \tfrac{1}{k}(\lceil y^{\mathrm{G}}_{n_{\mathrm{G}}+} k\rceil - 1)\right), \quad y_+ := \min_{k>0} \left(\hat{y}^0_+(k) - \tfrac{1}{k}(\lfloor y^{\mathrm{G}}_{n_{\mathrm{G}}-} k\rfloor + 1)\right). \tag{37}$$

Moreover, if we define the functions $\hat{y}_-, \hat{y}_+ \in \mathbb{Q} \cup \{\infty\}$ of $y \in \mathbb{Q} \cup \{\infty\}$ by

$$\hat{y}_- := \max_{k>0} \left(-\tfrac{1}{k}\lfloor yk\rfloor + \hat{y}^0_-(k)\right), \quad \hat{y}_+ := \min_{k>0} \left(-\tfrac{1}{k}\lceil yk\rceil + \hat{y}^0_+(k)\right), \tag{38}$$

then by inductive assumption, we have

$$\mathcal{L}(\hat{Y}[y]) = \begin{cases} \{\hat{y}_-\} = \{\hat{y}_+\} & \hat{Y}[y] \text{ not Floer simple} \\ [[\hat{y}_-, \hat{y}_+]] & \hat{Y}[y] \text{ Floer simple} \end{cases} \tag{39}$$

for all $y \in \mathbb{Q}$ for which $\mathcal{L}(\hat{Y}[y]) \neq \emptyset$. Since $\hat{Y}[y]$ and $Y_{n_{\mathrm{G}}}$ are each non-solid-torus graph manifolds for all $y \in \mathbb{Q}$, Proposition 4.4 then implies, for each $y \in \mathbb{Q}$, that

$$y \in \mathcal{L}(Y) \iff \mathcal{L}^\circ(\hat{Y}[y]) \cup \varphi^{\mathbb{P}}_{n_{\mathrm{G}}*}(\mathcal{L}^\circ(Y_{n_{\mathrm{G}}})) = \mathbb{Q} \cup \{\infty\} \tag{40}$$

for all $y \in \mathbb{Q}$. Note that $\hat{Y}[\infty]$ is not a graph manifold, being a non-solid-torus with compressible boundary, hence not prime.

We next prove some basic rules about the behavior of y_- and y_+.

Claim 1. *If $y^{\mathrm{G}}_{n_{\mathrm{G}}+}, \hat{y}_-, y_-, y \in \mathbb{Q}$, then*

$$y^{\mathrm{G}}_{n_{\mathrm{G}}+} > \hat{y}_- \iff y \in [y_-, +\infty]. \tag{41}$$

If $y^{\mathrm{G}}_{n_{\mathrm{G}}-}, \hat{y}_+, y_+, y \in \mathbb{Q}$*, then*

$$y^{\mathrm{G}}_{n_{\mathrm{G}}-} < \hat{y}_+ \iff y \in [-\infty, y_+]. \tag{42}$$

Proof of Claim. Suppose that $y^{\mathrm{G}}_{n_{\mathrm{G}}+}, \hat{y}_-, y_-, y \in \mathbb{Q}$. Then

$$\begin{aligned}
& & y^{\mathrm{G}}_{n_{\mathrm{G}}+} &> \hat{y}_- \\
(43)\quad & \iff & y^{\mathrm{G}}_{n_{\mathrm{G}}+} &> -\tfrac{1}{k}\lfloor yk \rfloor + \hat{y}^0_-(k) & \forall\, k \in \mathbb{Z}_{>0} \\
(44)\quad & \iff & \lceil y^{\mathrm{G}}_{n_{\mathrm{G}}+} k \rceil - 1 &\geq -\lfloor yk \rfloor + \hat{y}^0_-(k)k & \forall\, k \in \mathbb{Z}_{>0} \\
(45)\quad & \iff & yk &\geq \hat{y}^0_-(k)k - (\lceil y^{\mathrm{G}}_{n_{\mathrm{G}}+} k \rceil - 1) & \forall\, k \in \mathbb{Z}_{>0} \\
(46)\quad & \iff & y &\geq y_-,
\end{aligned}$$

where (37) implies (43), (22) implies (44), (21) implies (45), and (38) implies (46). The proof of (42) is nearly identical, but with signs reversed. ☐

Claim 2. *If* $y^{\mathrm{D}}_1, \ldots, y^{\mathrm{D}}_{n_{\mathrm{D}}}, y^{\mathrm{G}}_{1+}, \ldots, y^{\mathrm{G}}_{n_{\mathrm{G}}+} \in \mathbb{Q}$*, then*

$$y_- > -\sum_{i=1}^{n_{\mathrm{D}}} y^{\mathrm{D}}_i - \sum_{i=1}^{n_{\mathrm{G}}} y^{\mathrm{G}}_{i+}. \tag{47}$$

If $y^{\mathrm{D}}_1, \ldots, y^{\mathrm{D}}_{n_{\mathrm{D}}}, y^{\mathrm{G}}_{1-}, \ldots, y^{\mathrm{G}}_{n_{\mathrm{G}}-} \in \mathbb{Q}$*, then*

$$y_+ < -\sum_{i=1}^{n_{\mathrm{D}}} y^{\mathrm{D}}_i - \sum_{i=1}^{n_{\mathrm{G}}} y^{\mathrm{G}}_{i-}. \tag{48}$$

Proof of Claim. Writing $[\cdot] : \mathbb{Q} \to [0,1\rangle$ for the map sending $q \mapsto [q] := q - \lfloor q \rfloor$, define

$$y'_-(k) := \frac{1}{k}\left(-1 + \sum_{i=1}^{n_{\mathrm{D}}} [y^{\mathrm{D}}_i k] + \sum_{i=1}^{n_{\mathrm{G}}} (1 - [-y^{\mathrm{G}}_{i+} k])\right) \tag{49}$$

for each $k \in \mathbb{Z}_{>0}$, so that

$$y_- = -\sum_{i=1}^{n_{\mathrm{D}}} y^{\mathrm{D}}_i - \sum_{i=1}^{n_{\mathrm{G}}} y^{\mathrm{G}}_{i+} + \max_{k>0} y'_-(k). \tag{50}$$

In addition, let

$$s_+ := \min\left\{k \in \mathbb{Z}_{>0} \,|\, y^{\mathrm{D}}_1 k, \ldots, y^{\mathrm{D}}_{n_{\mathrm{D}}} k, y^{\mathrm{G}}_{1+} k, \ldots, y^{\mathrm{G}}_{n_{\mathrm{G}}+} k \in \mathbb{Z}\right\} \tag{51}$$

denote the least common positive multiple of the denominators of the $\{y^{\mathrm{D}}_i\}$ and $\{y^{\mathrm{G}}_{i+}\}$.

Suppose (47) fails to hold. Then since $y'_-(k) \leq 0$ for all $k \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned}
0 &\geq y'_-(1) + (s_+ - 1)y'_-(s_+ - 1) \\
&= -2 + \sum_{i=1}^{n_{\mathrm{D}}} ([y^{\mathrm{D}}_i] + [-y^{\mathrm{D}}_i]) + \sum_{i=1}^{n_{\mathrm{G}}} (1 + (1 - [-y^{\mathrm{G}}_{i+}] - [y^{\mathrm{G}}_{i+}])) \\
&\geq -2 + |\{i : y^{\mathrm{D}}_i \in \mathbb{Q} \setminus \mathbb{Z}\}| + n_{\mathrm{G}}, \qquad (52)
\end{aligned}$$

and likewise, we have

$$0 \geq s_+ y'_-(s_+) = -1 + \sum_{i=1}^{n_{\mathrm{G}}} 1 = n_{\mathrm{G}} - 1. \tag{53}$$

The hypotheses of Section 4.7, however, demand that either $|\{i: y_i^{\text{D}} \in \mathbb{Q}\setminus\mathbb{Z}\}| \geq 2$ and $n_{\text{G}} = 1$, contradicting (52), or $n_{\text{G}} > 1$, contradicting (53). Thus (47) holds, and a similar argument proves (48). □

For the proof that Theorem 4.6 and Proposition 4.7 hold for Y, we divide our argument into two main cases, depending on whether or not $\infty \in \mathcal{L}(Y)$.

Proposition 4.8. *$\infty \in \mathcal{L}(Y)$ if and only if either condition* (FS3) *from Proposition 4.7 holds, in which case $\infty \in \mathcal{L}(Y) = [[y_-, y_+]]$, or condition* (NFS3) *or* (NFS4) *from Proposition 4.7 holds, in which case $\mathcal{L}(Y) = \{y_-\} = \{y_+\} = \{\infty\}$.*

Proof. Since $\hat{Y}[\infty]$ is not a graph manifold, our inductive assumptions fail to hold for $\hat{Y}[\infty]$, but fortunately, $Y(\infty)$ has a simple structure. The remark in Section 2.2 implies

$$Y(\infty) = \left(\mathop{\#}_{i=1}^{n_{\text{D}}} L(s_i^{\text{D}}, r_i^{\text{D}})\right) \# \left(\mathop{\#}_{i=1}^{n_{\text{G}}} Y_i\left((\varphi_{i*}^{\mathbb{P}})^{-1}(\infty)\right)\right), \tag{54}$$

where we have written $y_i^{\text{D}} = r_i^{\text{D}}/s_i^{\text{D}}$, $L(s_i^{\text{D}}, r_i^{\text{D}})$ denotes the lens space of slope $s_i^{\text{D}}/r_i^{\text{D}}$, and $Y_i\left((\varphi_{i*}^{\mathbb{P}})^{-1}(\infty)\right)$ is the Dehn filling of the inverse image of the slope ∞. Thus,

$$\infty \in \mathcal{L}(Y) \quad \iff \quad \infty \in \varphi_{i*}^{\mathbb{P}}(\mathcal{L}(Y_i)) \;\text{ for all }\; i \in \{1, \ldots, n_{\text{G}}\}. \tag{55}$$

Conditions (NFS3) and (FS3) jointly exhaust the cases in which the right-hand condition of (55) holds and all daughter subtrees $Y_1, \ldots, Y_{n_{\text{G}}}$ are Floer simple. Condition (NFS4), on the other hand, describes all cases in which the the right-hand condition of (55) holds and at least one Y_j is not Floer simple.

If (NFS4) holds, then, permuting the daughter subtrees without loss of generality so that $\varphi_{n_{\text{G}}*}^{\mathbb{P}}(\mathcal{L}(Y_{n_{\text{G}}})) = \{\infty\}$, we have $\mathcal{L}^{\circ}(Y_{n_{\text{G}}}) = \emptyset$, which, by (40), implies $\mathcal{L}(Y) \cap \mathbb{Q} = \emptyset$, so that $\mathcal{L}(Y) = \{\infty\}$. Moreover, the fact that $\varphi_{n_{\text{G}}*}^{\mathbb{P}}(\mathcal{L}(Y_{n_{\text{G}}})) = \{\infty\}$ implies, by inductive assumption, that $y_{n_{\text{G}}-}^{\text{G}} = y_{n_{\text{G}}+}^{\text{G}} = \infty$. Thus y_- and y_+ each have infinite summands, and so $y_- = y_+ = \infty$.

Next suppose that (NFS3) holds, and set

$$I_{-\infty} := \{i : [[y_{i-}^{\text{G}}, y_{i+}^{\text{G}}]] = [-\infty, y_{i+}^{\text{G}}]\}, \quad I_{+\infty} := \{i : [[y_{i-}^{\text{G}}, y_{i+}^{\text{G}}]] = [y_{i-}^{\text{G}}, +\infty]\}. \tag{56}$$

We can also define the analogous sets for $\hat{Y}[y]$:

$$I_{-\infty}^{\hat{Y}[y]} := I_{-\infty} \cap \{1, \ldots, n_{\text{G}} - 1\}, \quad I_{+\infty}^{\hat{Y}[y]} := I_{+\infty} \cap \{1, \ldots, n_{\text{G}} - 1\}. \tag{57}$$

Since $I_{-\infty}$ and $I_{+\infty}$ are nonempty for (NFS3), we know that y_- and y_+ each have infinite summands, implying $y_- = y_+ = \infty$. Assume without loss of generality that $n_{\text{G}} \in I_{+\infty}$. Thus $I_{-\infty}^{\hat{Y}[y]} \neq \emptyset$, and $I_{+\infty}^{\hat{Y}[y]}$ is either empty or not. By inductive assumption, Proposition 4.7 holds for $\hat{Y}[y]$ for all $y \in \mathbb{Q}$. Thus, for each $y \in \mathbb{Q}$, either $I_{-\infty}^{\hat{Y}[y]}$ and $I_{+\infty}^{\hat{Y}[y]}$ are both nonempty, making (NFS3) hold for $\hat{Y}[y]$, so that $\mathcal{L}(\hat{Y}[y]) = \{\infty\}$; or, $I_{-\infty}^{\hat{Y}[y]} \neq \emptyset$ and $I_{+\infty}^{\hat{Y}[y]} = \emptyset$, making (FS3) hold for $\hat{Y}[y]$, so that $\mathcal{L}(\hat{Y}[y]) = [\hat{y}_-, +\infty]$. In both cases, the right-hand side of (40) fails to hold for all $y \in \mathbb{Q}$, and we are left with $\mathcal{L}(Y) = \{\infty\} = \{y_-\} = \{y_+\}$.

This leaves us with the case in which (FS3) holds, for which we first consider the subcase $I_{+\infty} = \emptyset$ and $I_{-\infty} \neq \emptyset$, implying $[[y_-, y_+]] = [y_-, +\infty]$. Without loss of generality, assume $n_{\text{G}} \in I_{-\infty}$, so that $[[y^{\text{G}}_{n_{\text{G}}-}, y^{\text{G}}_{n_{\text{G}}+}]] = [-\infty, y^{\text{G}}_{n_{\text{G}}+}]$. Since $\mathcal{L}(\hat{Y}[y]) = [[\hat{y}_-, \hat{y}_+]]$ for all $y \in \mathbb{Q}$ by inductive assumption, we know that either

(a) $I^{\hat{Y}[y]}_{-\infty} = I^{\hat{Y}[y]}_{+\infty} = \emptyset$, or
(b) $I^{\hat{Y}[y]}_{-\infty} \neq \emptyset$, with $\mathcal{L}(\hat{Y}[y]) = [\hat{y}_-, +\infty]$.

In case (a), the condition $I^{\hat{Y}[y]}_{-\infty} = I^{\hat{Y}[y]}_{+\infty} = \emptyset$, together with (55), implies

$$\infty \neq y^{\text{G}}_{i-} \geq y^{\text{G}}_{i+} \neq \infty \text{ for all } i \in \{1, \ldots, n_{\text{G}} - 1\}. \tag{58}$$

Thus, if we apply Claim 2 to (38), respectively substituting $\hat{y}_-$, $\hat{y}_+$, $n_{\text{D}} + 1$, $n_{\text{G}} - 1$, and $(y, y^{\text{D}}_1, \ldots, y^{\text{D}}_{n_{\text{D}}})$ for y_-, y_+, n_{D}, n_{G} and $(y^{\text{D}}_1, \ldots, y^{\text{D}}_{n_{\text{D}}})$ in the statement of the claim, then we obtain

$$\infty \neq \hat{y}_- \;>\; -y - \sum_{i=1}^{n_{\text{D}}} y^{\text{D}}_i - \sum_{i=1}^{n_{\text{G}}-1} y^{\text{G}}_{i+} \;\geq\; -y - \sum_{i=1}^{n_{\text{D}}} y^{\text{D}}_i - \sum_{i=1}^{n_{\text{G}}-1} y^{\text{G}}_{i-} \;>\; \hat{y}_+ \neq \infty, \tag{59}$$

with the outer, strict, inequalities resulting from Claim 2, and the middle, non-strict, inequality resulting from (58). Thus, for any $y \in \mathbb{Q}$, we have

$$\mathcal{L}^\circ(\hat{Y}[y]) \cup \varphi^{\mathbb{P}}_{n_{\text{G}}*}(\mathcal{L}^\circ(Y_{n_{\text{G}}})) = (\langle \hat{y}_-, +\infty] \cup [-\infty, \hat{y}_+\rangle) \cup \left\langle -\infty, y^{\text{G}}_{n_{\text{G}}+} \right\rangle \tag{60}$$

by inductive assumption, and so (40) implies that $y \in \mathcal{L}(Y)$ if and only if $y^{\text{G}}_{n_{\text{G}}+} > \hat{y}_-$, which, by Claim 1, occurs if and only if $y \in [y_-, +\infty]$. On the other hand, in case (b), with $\mathcal{L}^\circ(\hat{Y}[y]) = \langle \hat{y}_-, +\infty\rangle$, (40) again implies, for any $y \in \mathbb{Q}$, that $y \in \mathcal{L}(Y)$ if and only if $y^{\text{G}}_{n_{\text{G}}+} > \hat{y}_-$, which again occurs if and only if $y \in [y_-, +\infty]$. Thus, whether case (a) or (b) holds, we always have $\infty \in \mathcal{L}(Y) = [y_-, +\infty] = [[y_-, y_+]]$.

If (FS3) holds with $I_{+\infty} \neq \emptyset$ and $I_{-\infty} = \emptyset$, then an argument precisely analogous to that in the preceding paragraph shows that $\infty \in \mathcal{L}(Y) = [-\infty, y_+] = [[y_-, y_+]]$.

Lastly, suppose that (FS3) holds with $I_{+\infty} = I_{-\infty} = \emptyset$. This, as well as the fact that $\infty \in [[y^{\text{G}}_{i-}, y^{\text{G}}_{i+}]]$ for each $i \in \{1, \ldots, n_{\text{G}}\}$, implies that $\infty \neq y^{\text{G}}_{i-} \geq y^{\text{G}}_{i+} \neq \infty$ for all $i \in \{1, \ldots, n_{\text{G}}\}$, which, by Claim 2, implies that $\infty \neq y_- > y_+ \neq \infty$. Moreover, applying Claim 2 to (38) as in case (a) above yields (59), so that we also have $\infty \neq \hat{y}_- > \hat{y}_+ \neq \infty$. By inductive assumption, we then have

$$\mathcal{L}^\circ(\hat{Y}[y]) \cup \varphi^{\mathbb{P}}_{n_{\text{G}}*}(\mathcal{L}^\circ(Y_{n_{\text{G}}})) = (\langle \hat{y}_-, +\infty] \cup [-\infty, \hat{y}_+\rangle) \cup \left(\left\langle y^{\text{G}}_{n_{\text{G}}-}, +\infty\right] \cup \left[-\infty, y^{\text{G}}_{n_{\text{G}}+}\right\rangle\right) \tag{61}$$

for any $y \in \mathbb{Q}$, and so (40) implies that $y \in \mathcal{L}(Y)$ if and only if $y^{\text{G}}_{n_{\text{G}}+} > \hat{y}_-$ or $y^{\text{G}}_{n_{\text{G}}-} < \hat{y}_+$ which, by Claim 1, occurs if and only if $y \in [y_-, +\infty] \cup [-\infty, y_+]$. We therefore have $\infty \in \mathcal{L}(Y) = [[y_-, y_+]] = [y_-, +\infty] \cup [-\infty, y_+]$, completing our proof. □

Proposition 4.9. *Suppose $\infty \notin \mathcal{L}(Y)$. Then $\mathcal{L}(Y) \neq \emptyset$ if and only if either* (NFS2) *holds for Y, in which case $\mathcal{L}(Y) = \{y_-\} = \{y_+\}$, or* (FS1) *or* (FS2) *holds for Y, in which case $\mathcal{L}(Y) = [[y_-, y_+]]$.*

Proof. We first observe that if any daughter subtree of Y fails to be Floer simple, then $\mathcal{L}(Y) = \emptyset$. That, if we choose $Y_{n_{\text{G}}}$ to be non Floer simple, then by (40),

$\mathcal{L}^\circ(Y_{n_{\mathrm{G}}}) = \emptyset$ implies $\mathcal{L}(Y) \cap \mathbb{Q} = \emptyset$, which, since $\infty \notin \mathcal{L}(Y)$, implies $\mathcal{L}(Y) = \emptyset$. Thus, we henceforth assume the daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ are all Floer simple.

Let $I_{\pm\infty}, I_{\cap} \subset \{1, \ldots, n_{\mathrm{G}}\}$ denote the sets

(62)
$$I_{\pm\infty} := \left\{i : [[y^{\mathrm{G}}_{i-}, y^{\mathrm{G}}_{i+}]] = \langle -\infty, +\infty\rangle \right\},$$
$$I_{\cap} := \left\{i : [[y^{\mathrm{G}}_{i-}, y^{\mathrm{G}}_{i+}]] = [y^{\mathrm{G}}_{i-}, +\infty] \cap [-\infty, y^{\mathrm{G}}_{i+}] \right\}.$$

Then by (55) from the proof of Proposition 4.8, we know that

(63)
$$I_{\pm\infty} \cup I_{\cap} \neq \emptyset.$$

Suppose $I_{\pm\infty} \neq \emptyset$. We claim that in this case, $\mathcal{L}(Y) \neq \emptyset$ if and only if (FS1) holds for Y, in which case $\mathcal{L}(Y) = [[y_-, y_+]]$. First note that Y fails to satisfy any of the conditions for nonempty $\mathcal{L}(Y)$ in Proposition 4.7 except possibly (FS1). Suppose that Y satisfies (FS1). Then for each $i \in \{1, \ldots, n_{\mathrm{G}} - 1\}$, we have $\infty \in [[y^{\mathrm{G}}_{i-}, y^{\mathrm{G}}_{i+}]]^\circ$, implying $\infty \neq y^{\mathrm{G}}_{i-} \geq y^{\mathrm{G}}_{i+} \neq \infty$, which by Claim 2 implies $\infty \neq \hat{y}_- > \hat{y}_+ \neq \infty$ for all $y \in \mathbb{Q}$. By inductive assumption, we then have $\infty \in \mathcal{L}^\circ(\hat{Y}[y])$ for all $y \in \mathbb{Q}$. Thus, since (40) tells us that

(64)
$$\mathcal{L}^\circ(\hat{Y}[y]) \cup \langle -\infty, +\infty\rangle = \mathbb{Q} \cup \{\infty\} \quad \iff \quad y \in \mathcal{L}(Y),$$

we have $\mathcal{L}(Y) = \mathbb{Q} = \langle -\infty, +\infty\rangle = [[y_-, y_+]]$. Suppose instead we are given that $\mathcal{L}(Y) \neq \emptyset$. Then (64) tells us that $\infty \in \mathcal{L}^\circ(\hat{Y}[y])$ for all $y \in \mathcal{L}(Y)$. By inductive assumption this implies, for each $y \in \mathcal{L}(Y)$, that (FS3) holds for $\hat{Y}[y]$, with $\infty \in [[\hat{y}_-, \hat{y}_+]]^\circ$. Since $\infty \neq \{\hat{y}_-, \hat{y}_+\}$, we know that $\hat{y}_-$ and $\hat{y}_+$ cannot have infinite summands, and so $I^{\hat{Y}[y]}_{-\infty} = I^{\hat{Y}[y]}_{+\infty} = \emptyset$. This, in addition to the fact that that (FS3) holds for $\hat{Y}[y]$, implies that $\infty \in [[y^{\mathrm{G}}_{i-}, y^{\mathrm{G}}_{i+}]]^\circ$ for all $i \in \{1, \ldots, n_{\mathrm{G}} - 1\}$, and thus (FS1) holds for Y.

Next, consider the case in which $I_{\pm\infty} = \emptyset$, so that by (63), we have $I_{\cap} \neq \emptyset$. Assume, without loss of generality, that $n_{\mathrm{G}} \in I_{\cap}$. Then since $\infty \notin [[y^{\mathrm{G}}_{n_{\mathrm{G}}-}, y^{\mathrm{G}}_{n_{\mathrm{G}}+}]]^\circ$, the L-space gluing condition in (40) again tells us that $\infty \in \mathcal{L}^\circ(\hat{Y}[y])$ for all $y \in \mathcal{L}(Y)$. Just as in the preceding paragraph, we deduce from this that $\infty \in [[y^{\mathrm{G}}_{i-}, y^{\mathrm{G}}_{i+}]]^\circ$ for all $i \in \{1, \ldots, n_{\mathrm{G}} - 1\}$, and that this implies that $\mathcal{L}^\circ(\hat{Y}[y]) = [[\hat{y}_-, \hat{y}_+]]$, with $\infty \neq \hat{y}_- > \hat{y}_+ \neq \infty$, for all $y \in \mathbb{Q}$. Since y_- and y_+ have only finite summands, we also know that $y_-, y_+ \in \mathbb{Q}$. We therefore have

(65)
$$\begin{aligned} y \in \mathcal{L}(Y) \quad &\iff \quad (\langle \hat{y}_-, +\infty] \cup [-\infty, \hat{y}_+\rangle) \cup \left(\langle y^{\mathrm{G}}_{n_{\mathrm{G}}-}, +\infty\right] \cap \left[-\infty, y^{\mathrm{G}}_{n_{\mathrm{G}}+}\rangle\right) = \mathbb{Q} \cup \{\infty\} \\ &\iff \quad y^{\mathrm{G}}_{n_{\mathrm{G}}+} > \hat{y}_- \text{ and } y^{\mathrm{G}}_{n_{\mathrm{G}}-} < \hat{y}_+ \\ &\iff \quad y \in [y_-, +\infty] \cap [-\infty, y_+], \end{aligned}$$

where the first line is due to (40), and the third line is due to Claim 1. Thus, if $y_- > y_+$, then $\mathcal{L}(Y) = \emptyset$; if $y_- = y_+$, then (NFS2) holds for Y, with $\mathcal{L}(Y) = \{y_-\} = \{y_+\}$; and if $y_- < y_+$, then (FS2) holds for Y, with $\mathcal{L}(Y) = [[y_-, y_+]] = [y_-, y_+]$. □

4.8. **Orientable Base: Cases involving solid tori $\hat{Y}[y]$.** In this final subsection of the proof of Theorem 4.6 and Proposition 4.7, we consider all cases of Y for which $\hat{Y}[y]$, defined in (34), is a solid torus for some $y \in \mathbb{Q} \cup \{\infty\}$. Recall that we have fixed a tree manifold Y with torus boundary, $b_1(Y) = 1$, tree height $k > 0$, and $n_{\mathrm{G}} > 0$ daughter subtrees, as parameterized in Section 4.2. Since $\hat{Y}[y]$ contains no incompressible tori, we must have $n_{\mathrm{G}} - 1 = 0$. Thus, for any $y \in \mathbb{Q}$, $\hat{Y}[y]$ is Seifert fibered over the disk, and is a solid torus if and only if it has one or fewer exceptional fibers. This occurs for $y \in \mathbb{Z}$ if and only if the set $\{y_1^{\mathrm{D}}, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}}\}$ contains ≤ 1 nonintegers. Since $\hat{Y}[y]$ and Y are invariant under reparameterizations $(y_1^{\mathrm{D}}, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}}) \mapsto (y_1^{\mathrm{D}} + z_1, \ldots, y_{n_{\mathrm{D}}}^{\mathrm{D}} + z_{n_{\mathrm{D}}})$ with $\sum_{i=1}^{n_{\mathrm{D}}} z_i = 0$ and each $z_i \in \mathbb{Z}$, or under the operation of forgetting a fiber complement with Dehn filling of slope $\pi_i(-\tilde{h}_i) = 0$, we may assume, without loss of generality, that $n_{\mathrm{D}} = 1$. Thus $n_{\mathrm{G}} = n_{\mathrm{D}} = 1$.

If the unique daughter subtree Y_1 satisfies $\mathcal{L}(Y_1) = \emptyset$, then the gluing Propositions 4.4 and 3.4 imply that $\mathcal{L}(Y) = \emptyset$, as predicted for Y by Proposition 4.7.

We therefore henceforth assume $\mathcal{L}(Y_1) \neq \emptyset$. Since Y has tree height k, Y_1 has tree height $k - 1$. Thus, by inductive assumption as laid out in Section 4.5, Y_1 satisfies Theorem 4.6 and Proposition 4.7, with

$$\mathcal{L}(Y_1) = \begin{cases} \{y_{1-}^{\mathrm{G}}\} = \{y_{1+}^{\mathrm{G}}\} & Y_1 \text{ not Floer simple,} \\ [[y_{1-}^{\mathrm{G}}, y_{1+}^{\mathrm{G}}]] & Y_1 \text{ Floer simple.} \end{cases} \tag{66}$$

We proceed, once again, by fixing some notation. Recall the definitions of y_- and y_+:

$$\begin{aligned} y_- &:= \max_{k>0} -\tfrac{1}{k}(1 + \lfloor y_1^{\mathrm{D}} k \rfloor + (\lceil y_{1+}^{\mathrm{G}} k \rceil - 1)), \\ y_+ &:= \min_{k>0} -\tfrac{1}{k}(-1 + \lceil y_1^{\mathrm{D}} k \rceil + (\lfloor y_{1-}^{\mathrm{G}} k \rfloor + 1)), \end{aligned} \tag{67}$$

where, as always, we define y_- or y_+ to be infinite if any of its summands is infinite.

For any $y \neq \infty$, $\hat{Y}[y]$ is Seifert fibred over the disk, hence is Floer simple, allowing us to write $[[\hat{y}_-, \hat{y}_+]] := \mathcal{L}(\hat{Y}[y])$. Moreover, when $\hat{Y}[y]$ is *not* a solid torus, we have

$$\{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} = \emptyset; \qquad \begin{aligned} \hat{y}_- &= \max_{k>0} -\tfrac{1}{k}(1 + \lfloor yk \rfloor + \lfloor y_1^{\mathrm{D}} k \rfloor), \\ \hat{y}_+ &= \min_{k>0} -\tfrac{1}{k}(-1 + \lceil yk \rceil + \lceil y_1^{\mathrm{D}} k \rceil), \end{aligned} \tag{68}$$

but when $\hat{Y}[y]$ *is* a solid torus, we have

$$\{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} \neq \emptyset; \qquad \hat{y}_- = \hat{y}_+ = -y - y_1^{\mathrm{D}}. \tag{69}$$

We next prove the analogs of Claims 1 and 2 from Section 4.7

Claim 3. *For any $y_1^{\mathrm{D}}, y \in \mathbb{Q}$, we have*

$$y \in [y_-, +\infty] \iff \begin{cases} y_{1+}^{\mathrm{G}} > \hat{y}_- & \text{if } \{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} = \emptyset \\ y_{1+}^{\mathrm{G}} \geq \hat{y}_- & \text{if } \{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} \neq \emptyset \end{cases} \qquad \text{if } y_{1+}^{\mathrm{G}} \in \mathbb{Q}; \tag{70}$$

$$y \in [-\infty, y_+] \iff \begin{cases} y_{1-}^{\mathrm{G}} < \hat{y}_+ & \text{if } \{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} = \emptyset \\ y_{1-}^{\mathrm{G}} \leq \hat{y}_+ & \text{if } \{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} \neq \emptyset \end{cases} \qquad \text{if } y_{1-}^{\mathrm{G}} \in \mathbb{Q}. \tag{71}$$

Proof of Claim. To understand (70), note that due to (21), we have $y \geq y_-$ if and only if

$$\lfloor yk \rfloor \geq -(\lfloor y_1^{\mathrm{D}} k \rfloor + \lceil y_{1+}^{\mathrm{G}} k \rceil) \quad \text{for all } k > 0. \tag{72}$$

When $\{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} = \emptyset$, (22) implies that (72) holds if and only if $y_{1+}^{\mathrm{G}} > \hat{y}_-$. On the other hand, if $\{y, y_1^{\mathrm{D}}\} \cap \mathbb{Z} \neq \emptyset$, then $\lfloor yk \rfloor + \lfloor y_1^{\mathrm{D}} k \rfloor = \lfloor (y + y_1^{\mathrm{D}})k \rfloor$, and so applying (21) to (72) yields

$$y + y_1^{\mathrm{D}} \;\geq\; \max_{k>0} -\tfrac{1}{k}\lceil y_{1+}^{\mathrm{G}} k \rceil \;=\; -y_{1+}^{\mathrm{G}}, \tag{73}$$

which is equivalent to the inequality $y_{1+}^{\mathrm{G}} \geq \hat{y}_-$, completing the proof of (70). One can then obtain (71) by replacing y, y_{1+}^{G}, and y_1^{D} in (70) with $-y$, $-y_{1-}^{\mathrm{G}}$, and $-y_1^{\mathrm{D}}$, respectively. □

Claim 4. *If* $y_1^{\mathrm{D}}, y_{1+}^{\mathrm{G}} \in \mathbb{Q}$, *then*

$$\begin{aligned} -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}} \;\leq\; y_- \;\leq\; -\lfloor y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \rfloor, &\quad \textit{with} \\ -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}} \;=\; y_- \quad \iff \quad y_1^{\mathrm{D}} \in \mathbb{Z} &\quad \textit{or} \quad y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \in \mathbb{Z}. \end{aligned}$$

If $y_1^{\mathrm{D}}, y_{1-}^{\mathrm{G}} \in \mathbb{Q}$, *then*

$$\begin{aligned} -\lceil y_1^{\mathrm{D}} + y_{1-}^{\mathrm{G}} \rceil \;\leq\; y_+ \;\leq\; -y_1^{\mathrm{D}} - y_{1-}^{\mathrm{G}}, &\quad \textit{with} \\ y_+ \;=\; -y_1^{\mathrm{D}} - y_{1-}^{\mathrm{G}} \quad \iff \quad y_1^{\mathrm{D}} \in \mathbb{Z} &\quad \textit{or} \quad y_1^{\mathrm{D}} + y_{1-}^{\mathrm{G}} \in \mathbb{Z}. \end{aligned}$$

Proof of Claim. Just as in the proof of Claim 2, we define

$$y'_-(k) := \frac{1}{k}\left([y_1^{\mathrm{D}} k] - [-y_{1+}^{\mathrm{G}} k]\right) \tag{74}$$

for each $k \in \mathbb{Z}_{>0}$, so that

$$y_- \;=\; -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}} + \max_{k>0} y'_-(k). \tag{75}$$

Demanding $y_1^{\mathrm{D}}, y_{1+}^{\mathrm{G}} \in \mathbb{Q}$, we again set $s_+ := \min\left\{k \in \mathbb{Z}_{>0} \,|\, y_1^{\mathrm{D}} k, y_{1+}^{\mathrm{G}} k \in \mathbb{Z}\right\}$.

Since $y'_-(s_+) = 0$, we already know that $y_- \geq -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}}$ for all $y_1^{\mathrm{D}}, y_{1+}^{\mathrm{G}} \in \mathbb{Q}$. If $y_i^{\mathrm{D}} \in \mathbb{Z}$, then $y'_-(k) \leq 0$ for all $k \in \mathbb{Z}_{>0}$, implying $y_- \geq -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}}$. Suppose that $y_- = -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}}$ for some $y_i^{\mathrm{D}} \in \mathbb{Q} \setminus \mathbb{Z}$. Then since $y'_-(k) \leq 0$ for all $k \in \mathbb{Z}_{>0}$, we have

$$\begin{aligned} 0 &\geq y'_-(1) + (s_+ - 1) y'_-(s_+ - 1) \\ &= ([y_1^{\mathrm{D}}] + [-y_1^{\mathrm{D}}]) - ([-y_{1+}^{\mathrm{G}}] + [y_{1+}^{\mathrm{G}}]) \\ &\geq 0. \end{aligned} \tag{76}$$

Thus, the top line of (76) must be an equality, which, since $y'_-(k) \leq 0$ for all $k \in \mathbb{Z}_{>0}$, implies $y'_-(1) = y'_-(s_+ - 1) = 0$. In particular, the fact that $y'_-(1) = 0$ implies that $y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \in \mathbb{Z}$. Conversely, if $y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \in \mathbb{Z}$, then $y'_-(k) \equiv 0$ for all $k \in \mathbb{Z}_{>0}$, implying $y_- = -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}}$.

We have shown that $y_- \geq -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}}$, with equality if and only if $y_1^{\mathrm{D}} \in \mathbb{Z}$ or $y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \in \mathbb{Z}$, and so it remains to show that $y_- \leq -\lfloor y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}} \rfloor$. Note that for each

$k \in \mathbb{Z}_{>0}$, we have

$$y'_{-}(k) = \begin{cases} \frac{1}{k}\lceil y_1^{\mathrm{D}}k + y_{1+}^{\mathrm{G}}k\rceil & y_{1+}^{\mathrm{G}}k \in \mathbb{Z} \\ \frac{1}{k}(\lceil y_1^{\mathrm{D}}k\rceil + \lceil y_{1+}^{\mathrm{G}}k\rceil - 1) & y_{1+}^{\mathrm{G}}k \notin \mathbb{Z} \end{cases} \tag{77}$$
$$\leq \frac{1}{k}\lceil (y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}})k\rceil$$
$$\leq \lceil y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}}\rceil.$$

Thus, by (75), we have

$$y_{-} \leq -y_1^{\mathrm{D}} - y_{1+}^{\mathrm{G}} + \lceil y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}}\rceil = -\lfloor y_1^{\mathrm{D}} + y_{1+}^{\mathrm{G}}\rfloor. \tag{78}$$

A similar argument proves all of the analogous results for y_{+}. □

For the proof that Theorem 4.6 and Proposition 4.7 hold for Y, we divide our argument into three main cases, first according to whether or not Y_1 is Floer simple, and then according to whether $\infty \in \mathcal{L}(Y)$.

Proposition 4.10. *Suppose $\mathcal{L}(Y_1) \neq \emptyset$ with Y_1 not Floer simple. Then $\mathcal{L}(Y) \neq \emptyset$ if and only if condition* (NFS1) *or* (NFS4) *from Proposition 4.7 holds, in which case $\mathcal{L}(Y) = \{y_{-}\} = \{y_{+}\}$.*

Proof. For brevity, set $y_1^{\mathrm{G}} := y_{1-}^{\mathrm{G}} = y_{1+}^{\mathrm{G}}$, so that $\varphi_{1*}^{\mathbb{P}}(\mathcal{L}(Y_1)) = \{y_1^{\mathrm{G}}\}$. We first note that if $\hat{Y}[y]$ has incompressible boundary, in which case $\hat{Y}[y]$ is a non-solid-torus graph manifold, then Proposition 4.4 implies that the union $Y(y) = \hat{Y}[y] \cup_{\varphi_1} Y_1$ is not an L-space. $\hat{Y}[y]$ has compressible boundary if and only if $y_1^{\mathrm{D}} \in \mathbb{Z}$ or $y \in \mathbb{Z} \cup \{\infty\}$. Thus, for any $y \in \mathbb{Q} \cup \{\infty\}$, Proposition 3.4 implies

$$y \in \mathcal{L}(Y) \qquad \Longleftrightarrow \qquad \hat{y}_{-} = \hat{y}_{+} = y_1^{\mathrm{G}} \quad \text{and} \quad y_1^{\mathrm{D}} \in \mathbb{Z} \text{ or } y \in \mathbb{Z} \cup \{\infty\}. \tag{79}$$

Condition (NFS4) holds if and only if $y_1^{\mathrm{G}} = \infty$. Since $\hat{y}_{-} = \hat{y}_{+} = \infty$ if and only if $y = \infty$, we then have $\mathcal{L}(Y) = \{\infty\} = \{y_{-}\} = \{y_{-}\}$ when $y_1^{\mathrm{G}} = \infty$.

We henceforth demand $y_1^{\mathrm{G}} \in \mathbb{Q}$. If $y_1^{\mathrm{D}} \in \mathbb{Z}$, then (NFS1) holds, and we have

$$\hat{y}_{-} = \hat{y}_{+} = -y_1^{\mathrm{D}} - y, \qquad y_{-} = y_{+} = -y_1^{\mathrm{D}} - y_1^{\mathrm{G}}. \tag{80}$$

Statement (79) then tells us that $y \in \mathcal{L}(Y)$ if and only if $-y_1^{\mathrm{D}} - y = y_1^{\mathrm{G}}$, which occurs if and only if $y = -y_1^{\mathrm{D}} - y_1^{\mathrm{G}} = y_{-} = y_{+}$, which means that $\mathcal{L}(Y) = \{y_{-}\} = \{y_{+}\}$.

Lastly, suppose that $y_1^{\mathrm{D}} \in \mathbb{Q} \setminus \mathbb{Z}$, with $y_1^{\mathrm{G}} \in \mathbb{Q}$. Since $\hat{y}_{-} = \hat{y}_{+} = \infty \neq y_1^{\mathrm{G}}$ when $y = \infty$, (79) tells us that $\infty \notin \mathcal{L}(Y)$. Thus $y \in \mathcal{L}(Y)$ if and only if $y \in \mathbb{Z}$ and $\hat{y}_{-} = \hat{y}_{+} = y_1^{\mathrm{G}}$. Since $\hat{y}_{-} = \hat{y}_{+} = -y_1^{\mathrm{D}} - y$ when $y \in \mathbb{Z}$, this means $y \in \mathcal{L}(Y)$ if and only if $-y_1^{\mathrm{D}} - y_1^{\mathrm{G}} = y \in \mathbb{Z}$. That is,

$$\mathcal{L}(Y) = \begin{cases} -y_1^{\mathrm{D}} - y_1^{\mathrm{G}} & -y_1^{\mathrm{D}} - y_1^{\mathrm{G}} \in \mathbb{Z} \\ \emptyset & -y_1^{\mathrm{D}} - y_1^{\mathrm{G}} \notin \mathbb{Z}. \end{cases} \tag{81}$$

Thus, if $y_1^{\mathrm{D}} \in \mathbb{Q} \setminus \mathbb{Z}$ and $y_1^{\mathrm{G}} \in \mathbb{Q}$, then $\mathcal{L}(Y)$ is nonempty if and only if (NFS1) holds, in which case, since $y_1^{\mathrm{D}} + y_1^{\mathrm{G}} \in \mathbb{Z}$, Claim 4 implies $y_{-} = y_{+} = -y_1^{\mathrm{D}} - y_1^{\mathrm{G}}$, so that $\mathcal{L}(Y) = \{y_{-}\} = \{y_{+}\}$.

□

We henceforth demand that Y_1 be Floer simple, in which case Propositions 4.4 and 3.4 tell us that for any $y \in \mathbb{Q} \cup \{\infty\}$, the union $Y(y) = \hat{Y}[y] \cup_{\varphi_1} Y_1$ is an L-space if and only if

(82)
$$\begin{cases} [[\hat{y}_-, \hat{y}_+]]^\circ \cup [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]^\circ = \mathbb{Q} \cup \{\infty\} & \text{if } \hat{Y}[y] \text{ has incompressible boundary} \\ \hat{y}_- = \hat{y}_+ \in [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] & \text{if } \hat{Y}[y] \text{ has compressible boundary.} \end{cases}$$

Note that $\hat{Y}[y]$ has compressible boundary if and only if $\{y, y^{\mathrm{D}}_1\} \cap \mathbb{Z} \neq \emptyset$ or $y = \infty$, with the former condition accounting for the case of solid torus $\hat{Y}[y]$, and the latter condition accounting for the case in which $\hat{Y}[\infty]$ is either a solid torus (when $y^{\mathrm{D}}_1 \in \mathbb{Z}$) or the connected sum of a solid torus with a lens space (when $y^{\mathrm{D}}_1 \notin \mathbb{Z}$).

Proposition 4.11. *Suppose Y_1 is Floer simple. Then $\infty \in \mathcal{L}(Y)$ if and only if condition* (FS3) *from Proposition 4.7 holds, in which case $\mathcal{L}(Y) = [[y_-, y_+]]$.*

Proof. The first part of the statement is immediate. That is, since $\hat{Y}[\infty]$ has compressible boundary, with $\hat{y}_- = \hat{y}_+ = \infty$, we have $\infty \in \mathcal{L}(Y)$ if and only if $\infty \in [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$, which occurs if and only if (FS3) holds. For the remainder of the proof, we assume (FS3) holds.

Consider the case in which $y^{\mathrm{G}}_{1-} \neq y^{\mathrm{G}}_{1+}$. Since $\infty \in [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$, $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$ takes one of the forms (a) $[y^{\mathrm{G}}_{1-}, +\infty]$, (b) $[-\infty, y^{\mathrm{G}}_{1+}]$, or (c) $[y^{\mathrm{G}}_{1-}, +\infty] \cup [-\infty, y^{\mathrm{G}}_{1+}]$, in which cases $[[y_-, y_+]]$ takes the respective forms (a) $[-\infty, y_+]$, (b) $[y_-, +\infty]$, (c) $[y_-, +\infty] \cup [-\infty, y_+]$, with Case (c) due to the fact that $\infty \neq y^{\mathrm{G}}_{1-} > y^{\mathrm{G}}_{1+} \neq \infty$ implies $\infty \neq y_- > y_+ \neq \infty$ by Claim 4. Thus, for $y \in \mathbb{Q}$, the condition $y \in [[y_-, y_+]]$ is respectively equivalent to the right-hand conditions of (a) (70), (b) (71), or (c) (70) or (71) from Claim 3, each of which conditions, given the respective form of $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$, is equivalent to (82), which holds if and only if $Y(y)$ is and L-space, and we conclude that $\mathcal{L}(Y) = [[y_-, y_+]]$.

Next, suppose that (FS3) holds with $y^{\mathrm{G}}_{1-} = y^{\mathrm{G}}_{1+} =: y^{\mathrm{G}}_1$. Since $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]^\circ = \mathbb{Q} \cup \{\infty\} \setminus y^{\mathrm{G}}_1$, the L-space condition in (82) takes the following form. For any $y \in \mathbb{Q}$,

$$y \in \mathcal{L}(Y) \iff \begin{cases} y^{\mathrm{G}}_1 \in [[\hat{y}_-, \hat{y}_+]]^\circ & \{y^{\mathrm{D}}_1, y\} \cap \mathbb{Z} = \emptyset \\ y \neq -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1 & \{y^{\mathrm{D}}_1, y\} \cap \mathbb{Z} \neq \emptyset. \end{cases} \tag{83}$$

The derivation of the $\{y^{\mathrm{D}}_1, y\} \cap \mathbb{Z} = \emptyset$ case from (82) is immediate. In the $\{y^{\mathrm{D}}_1, y\} \cap \mathbb{Z} \neq \emptyset$ case, (82) tells us that $y \in \mathcal{L}(Y)$ if and only if $\hat{y}_- \neq y^{\mathrm{D}}_1$. Since $\hat{y}_- = \hat{y}_+ = -y^{\mathrm{D}}_1 - y$, this occurs if and only if $y \neq -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1$, completing the proof of (83).

Now, since (FS3) demands that $\infty \in [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$, implying $y^{\mathrm{G}}_1 \in \mathbb{Q}$, Claim 4 tells us that $y_+ \leq -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1 \leq y_-$, with $y_- = y_+$ if and only if $y^{\mathrm{D}}_1 \in \mathbb{Z}$ or $y^{\mathrm{D}}_1 + y^{\mathrm{G}}_1 \in \mathbb{Z}$. If $y^{\mathrm{D}}_1 \in \mathbb{Z}$, then (83) implies $\mathcal{L}(Y) = \mathbb{Q} \cup \{\infty\} \setminus \{-y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1\} = [[y_-, y_+]]$. Suppose instead that $y^{\mathrm{D}}_1 \notin \mathbb{Z}$. If $y^{\mathrm{D}}_1 + y^{\mathrm{G}}_1 \in \mathbb{Z}$, then for each $y \in [[y_-, y_+]] \cap \mathbb{Q}$, either $y \notin \mathbb{Z}$, in which case, the fact that $y \in [y_-, +\infty] \cup [-\infty, y_+]$ makes Claim 3 tell us that $y^{\mathrm{G}}_1 \in [[\hat{y}_-, \hat{y}_+]]^\circ$, so that (83) implies $y \in \mathcal{L}(Y)$; or $y \in \mathbb{Z}$, in which case (83) tells us that $y \in \mathcal{L}(Y)$ if and only if $y \neq -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1 = y_- = y_+$. Combining these two results makes $\mathcal{L}(Y) = [[y_-, y_+]]$.

Lastly, consider the case in which $y^{\mathrm{G}}_{1-} = y^{\mathrm{G}}_{1+} =: y^{\mathrm{G}}_1 \in \mathbb{Q}$, $y^{\mathrm{D}}_1 \notin \mathbb{Z}$, and $y^{\mathrm{D}}_1 + y^{\mathrm{G}}_1 \notin \mathbb{Z}$. Since in this case, $y \in \mathbb{Z}$ automatically implies $y \neq -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1$, we deduce that the second line of (83) is vacuous. That is, $y \in \mathcal{L}(Y)$ for all $y \in \mathbb{Z}$. Accordingly, since Claim 4 tells us that

$$-\lceil y^{\mathrm{D}}_1 + y^{\mathrm{G}}_1 \rceil \leq y_+ < -y^{\mathrm{D}}_1 - y^{\mathrm{G}}_1 < y_- \leq -\lfloor y^{\mathrm{D}}_1 + y^{\mathrm{G}}_1 \rfloor, \tag{84}$$

we observe that the complement of $[[y_-, y_+]]$ contains no integers, and so $\mathbb{Z} \subset [[y_-, y_+]]$. On the other hand, for any $y \in \mathbb{Q} \setminus \mathbb{Z}$, (83) tells us that $y \in \mathcal{L}(Y)$ if and only if $y^{\mathrm{G}}_1 \in [[\hat{y}_-, \hat{y}_+]]^\circ$, which, by Claim 3, occurs if and only if $y \in [y_-, +\infty] \cup [-\infty, y_+] = [[y_-, y_+]]$. Thus, once again, $\mathcal{L}(Y) = [[y_-, y_+]]$, completing the proof of the proposition. □

Proposition 4.12. *Suppose Y_1 is Floer simple and $\infty \notin \mathcal{L}(Y)$. Then $\mathcal{L}(Y) \neq \emptyset$ if and only if either condition* (NFS2) *from Proposition 4.7 holds, in which case $\mathcal{L}(Y) = \{y_-\} = \{y_+\}$, or condition* (FS1) *or* (FS2) *holds, in which case $\mathcal{L}(Y) = [[y_-, y_+]]$.*

Proof. Since Y_1 is Floer simple, and since $\infty \notin \mathcal{L}(Y)$ implies that (FS3) fails to hold, we know that $\infty \notin [[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]]$. Thus, $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = \langle -\infty, +\infty \rangle$ or $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = [y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]$.

Suppose that $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = \langle -\infty, +\infty \rangle$, which occurs if and only if condition (FS1) holds. For any $y \in \mathbb{Q}$, we have $\infty \neq \hat{y}_- \geq \hat{y}_+ \neq \infty$, implying condition (82) holds, making $Y(y)$ an L-space. Thus, since $y^{\mathrm{G}}_{1-} = y^{\mathrm{G}}_{1+} = \infty$ implies $y_- = y_+ = \infty$, we have $\mathcal{L}(Y) = \mathbb{Q} = \langle -\infty, +\infty \rangle = [[y_-, y_+]]$.

Lastly, suppose $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = [y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]$. For any $y \in \mathbb{Q}$, we have $y \in [y_-, +\infty] \cap [-\infty, y_+]$ if and only if the right-hand conditions of (70) and (71) from Claim 3 hold, which, since $[[y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}]] = [y^{\mathrm{G}}_{1-}, +\infty] \cap [y^{\mathrm{G}}_{1-}, y^{\mathrm{G}}_{1+}] \neq \emptyset$, occurs if and only if (82) holds, which happens precisely when $Y(y)$ is an L-space. Thus, $\mathcal{L}(Y) = [y_-, +\infty] \cap [-\infty, y_+]$, or in other words,

$$\mathcal{L}(Y) = \begin{cases} [[y_-, y_+]] & y_- < y_+ \\ \{y_-\} = \{y_+\} & y_- = y_+ \\ \emptyset & y_- > y_+ \end{cases}. \tag{85}$$

That is, $\mathcal{L}(Y)$ is nonempty if and only if either (FS2) holds, in which case $\mathcal{L}(Y) = [[y_-, y_+]]$, or (NFS2) holds, in which case $\mathcal{L}(Y) = \{y_-\} = \{y_+\}$. □

The combined results of Sections 4.6, 4.7, and 4.8 prove that Theorem 4.6 and Proposition 4.7 hold for any graph manifold Y with torus boundary, $b_1 = 1$, tree height $k > 0$, and $n_{\mathrm{G}} > 0$ daughter subtrees, given the inductive assumptions, laid out in Section 4.5, that Theorem 4.6 and Proposition 4.7 hold for any graph manifold with torus boundary, $b_1 = 1$, and either tree height k and $\leq n_{\mathrm{G}} - 1$ daughter subtrees, or tree height $\leq k - 1$.

For graph manifolds Y with torus boundary, $b_1 = 1$, and tree height k, inducting on the number of daughter subtrees n_{G} yields the result that any graph manifold with torus boundary, $b_1 = 1$, and tree height k satisfies Theorem 4.6 and Proposition 4.7. Inducting on tree height k then completes the proof of Theorem 4.6 and Proposition 4.7.

□

4.9. **Some technical results for $\boldsymbol{y_-}$ and $\boldsymbol{y_+}$.** We conclude this section with the proof of some basic facts about y_- and y_+ for later use.

Recall that y_- and y_+ are defined by $y_- := \max_{k>0} y_-(k)$ and $y_+ := \min_{k>0} y_+(k)$, where

$$
\begin{aligned}
(86) \qquad y_-(k) &:= -\frac{1}{k}\left(1 + \sum_{i=1}^{n_{\mathrm{D}}} \lfloor y_i^{\mathrm{D}} k \rfloor + \sum_{i=1}^{n_{\mathrm{G}}} \left(\lceil y_{i+}^{\mathrm{G}} k \rceil - 1\right)\right), \\
y_+(k) &:= -\frac{1}{k}\left(-1 + \sum_{i=1}^{n_{\mathrm{D}}} \lceil y_i^{\mathrm{D}} k \rceil + \sum_{i=1}^{n_{\mathrm{G}}} \left(\lfloor y_{i-}^{\mathrm{G}} k \rfloor + 1\right)\right).
\end{aligned}
$$

Let $k_-, k_+ \in \mathbb{Z}_{>0}$ denote the lowest values of k for which these extrema occur. That is, set

$$
(87) \qquad k_- := \min\{k \in \mathbb{Z}_{>0} | \, y_-(k) = y_-\}, \qquad k_+ := \min\{k \in \mathbb{Z}_{>0} | \, y_+(k) = y_+\}.
$$

We then have the following result.

Proposition 4.13. *If Y is not a solid torus, and $y_-, y_+ \in \mathbb{Q}$, then k_- and k_+ are the respective denominators of y_- and y_+. That is,*

$$
(88) \qquad k_- := \min\{k \in \mathbb{Z}_{>0} | \, y_- k \in \mathbb{Z}\}, \qquad k_+ := \min\{k \in \mathbb{Z}_{>0} | \, y_+ k \in \mathbb{Z}\}.
$$

Proof. Since this question is unaffected by an overall translation of y_- by an integer, we assume without loss of generality that $y_i^{\mathrm{D}} \in \langle 0, 1\rangle$ for all $i \in \{1, \ldots, n_{\mathrm{D}}\}$ and that $y_{i+}^{\mathrm{G}} \in [0, 1\rangle$ for all $i \in \{1, \ldots, n_{\mathrm{G}}\}$. In addition, we permute the daughter subtrees $Y_1, \ldots, Y_{n_{\mathrm{G}}}$ so that $y_{i+}^{\mathrm{G}} \in \langle 0, 1\rangle$ for all $i \in \{1, \ldots, \bar{n}_{\mathrm{G}}\}$, and $y_{i+}^{\mathrm{G}} = 0$ for all $i \in \{\bar{n}_{\mathrm{G}} + 1, \ldots, n_{\mathrm{G}}\}$, for some $\bar{n}_{\mathrm{G}} \leq n_{\mathrm{G}}$.

Setting $N_{\mathrm{G}} := n_{\mathrm{G}} - \bar{n}_{\mathrm{G}}$, we note that if $N_{\mathrm{G}} > 0$, then we obtain

$$
\begin{aligned}
(89) \qquad y_-(k) &:= \frac{1}{k}\left(N_{\mathrm{G}} - 1 - \sum_{i=1}^{n_{\mathrm{D}}} \lfloor y_i^{\mathrm{D}} k \rfloor - \sum_{i=1}^{\bar{n}_{\mathrm{G}}} \left(\lceil y_{1+}^{\mathrm{G}} k \rceil - 1\right)\right), \\
&\leq \frac{1}{k}(N_{\mathrm{G}} - 1) \ \leq N_{\mathrm{G}} - 1 \ = \ y_-(1)
\end{aligned}
$$

for all $k > 0$, making $y_- = y_-(1)$. On the other hand, if $N_{\mathrm{G}} = 0$, then the top line of (89) implies $y_-(k) \leq -\frac{1}{k} < 0$ for all $k > 0$, which, since $y_- \in \mathbb{Z}$, implies $y_- \leq -1 = y_-(1)$. Thus, in either case, we have $k_- = -1$, and a similar argument shows $k_+ = 1$ when $y_+ \in \mathbb{Z}$.

If $n_{\mathrm{G}} = 0$ and $n_{\mathrm{D}} \leq 1$, then Y is a solid torus, a case excluded by hypothesis. Suppose $n_{\mathrm{D}} = 0$ and $n_{\mathrm{G}} = 1$. If $y_{1+}^{\mathrm{G}} \in \mathbb{Z}$, then the above argument shows $y_- \in \mathbb{Z}$ and $k_- = 1$. If $y_{1+}^{\mathrm{G}} =: r_{1+}^{\mathrm{G}}/s_{1+}^{\mathrm{G}} \in \langle 0, 1\rangle$ with $r_{1+}^{\mathrm{G}}, s_{1+}^{\mathrm{G}} \in \mathbb{Z}_{>0}$ relatively prime, then for all $k > 0$, we have

$$
(90) \qquad y_-(k) = -y_{1+}^{\mathrm{G}} - \frac{1}{k}[-y_{1+}^{\mathrm{G}}] \leq -y_{1+}^{\mathrm{G}} = y_-(s_{1+}^{\mathrm{G}}).
$$

Thus $y_- = -y_{1+}^{\mathrm{G}} = y_-(s_{1+}^{\mathrm{G}})$, and since $y_-(k) < y_-$ for all $k \in \{1, \ldots, s_{1+}^{\mathrm{G}} - 1\}$, we also have $k_- = s_{1+}^{\mathrm{G}}$, which is the denominator of y_-. A similar argument shows that (88) also holds for k_+ when $n_{\mathrm{D}} = 0$ and $n_{\mathrm{G}} = 1$.

Finally, suppose we exclude all cases considered in the preceding paragraph, and all cases in which $y_- \in \mathbb{Z}$. Since $y_- \in \mathbb{Q}$ by hypothesis, the second paragraph implies we also have $y^{\rm G}_{i+} \in \mathbb{Q}\setminus\mathbb{Z}$ for all $i \in \{1,\dots,n_{\rm G}\}$. Since the problem is still unaffected by an overall integer translation of y_-, we demand without loss of generality that $y^{\rm D}_i, y^{\rm G}_{j+} \in \langle 0,1\rangle$ for all $i \in \{1,\dots,n_{\rm D}\}$ and $j \in \{1,\dots,n_{\rm G}\}$. Thus, after removing one more regular fiber neighborhood $\nu(f_0)$ from the JSJ component containing ∂Y, and Dehn filling this complement with slope $y^{\rm D}_0 := -1$, we may appeal to Theorem 3 from Jankins and Neumann [23], which is equivalent to the following statement.

If $y_- := \max_{k>0} y_-(k)$, where

$$y_-(k) := 1 - \frac{1}{k}\left(1 + \sum_{i=1}^{n_{\rm D}} \lfloor y^{\rm D}_i k\rfloor + \sum_{i=1}^{n_{\rm G}} \left(\lceil y^{\rm G}_{i+} k\rceil - 1\right)\right) \tag{91}$$

(with the initial 1 coming from $-y^{\rm D}_0$), and if k_- is defined as in (87), then there is a positive integer $c < k_-$ with $\gcd(c, k_-) = 1$, a permutation π on $n_{\rm D}+n_{\rm G}+1$ elements, and an $n_{\rm D}+n_{\rm G}+1$-tuple $a_* = (c, k_- - c, 1, \dots, 1)$, such that

$$\lfloor y^{\rm D}_i k_-\rfloor + 1 = a_{\pi(i)}, \qquad \lceil y^{\rm G}_{j+} k_-\rceil = a_{\pi(n_{\rm D}+j)}, \qquad y_- = \frac{a_{\pi(n_{\rm D}+n_{\rm G}+1)}}{k_-} \tag{92}$$

for all $i \in \{1,\dots,n_{\rm D}\}$ and $j \in \{1,\dots,n_{\rm G}\}$. In particular, $\gcd(a_{\pi(n_{\rm D}+n_{\rm G}+1)}, k_-) = 1$, making k_- satisfy (88). A similar argument shows that (88) holds for k_+, completing the proof. $\square$

The above result is useful for proving the following proposition, but first, we define

$$N_{\rm G} := \left|\{i : y^{\rm G}_{i+} \in \mathbb{Z}\}\right| + \left|\{i : y^{\rm G}_{i-} \in \mathbb{Z}\}\right|, \qquad \bar{n}_{\rm D} := \left|\{i : y^{\rm D}_i \in \mathbb{Q}\setminus\mathbb{Z}\}\right|. \tag{93}$$

Proposition 4.14. *Suppose that $\bar{n}_{\rm D} + N_{\rm G} > 0$, with Y not a solid torus. If $y_- = y_+ \in \mathbb{Q}$, then $y_- = y_-(1) = y_+(1) = y_+ \in \mathbb{Z}$.*

Proof. Since $y_- = y_+$, we have

$$0 \;=\; y_- - y_+ \;=\; \max_{k_1>0} y_-(k_1) - \min_{k_2>0} y_+(k_2) \;=\; \max_{k_1,k_2>0} \left(y_-(k_1) - y_+(k_2)\right), \tag{94}$$

with $y_-(k)$ and $y_+(k)$ as defined in (86). Defining $k_-, k_+ \in \mathbb{Z}_{>0}$ as in (87), we observe that since $y_- = y_+$, Proposition 4.13 implies $k_- = k_+$. In particular, the set of $(k_1,k_2) \in \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ for which $y_-(k_1) - y_+(k_2)$ is maximized has nontrivial intersection with the set of $(k_1,k_2) \in \mathbb{Z}_{>0}\times\mathbb{Z}_{>0}$ for which $k_1 = k_2$. We therefore have

$$\begin{aligned} 0 \;&=\; \max_{k>0}\left(y_-(k) + y_+(k)\right) \\ &= \max_{k>0} \frac{1}{k}\left(-2 + \sum_{i=1}^{n_{\rm D}} \left(\lceil y^{\rm D}_i k\rceil - \lfloor y^{\rm D}_i k\rfloor\right) - \sum_{i=1}^{n_{\rm G}} \left(\lceil y^{\rm G}_{i+} k\rceil - 1 + \lceil -y^{\rm G}_{i-} k\rceil - 1\right)\right) \\ &= -\sum_{i=1}^{\tilde{n}_{\rm G}} \lfloor \tilde{y}^{\rm G}_i\rfloor + \max_{k>0} \frac{1}{k}\left(N_{\rm G} - 2 + \sum_{i=1}^{n_{\rm D}} \left(\lceil y^{\rm D}_i k\rceil - \lfloor y^{\rm D}_i k\rfloor\right) - \sum_{i=1}^{\tilde{n}_{\rm G}} \left(\lceil [\tilde{y}^{\rm G}_i] k\rceil - 1\right)\right), \end{aligned} \tag{95}$$

where the second line uses the fact that $-\lfloor q\rfloor = \lceil -q\rceil$ for all $q \in \mathbb{Q}$, and where in the third line, if we set $\tilde{n}_{\rm G} := 2n_{\rm G} - N_{\rm G}$, then $\tilde{y}^{\rm G}_* \in \mathbb{Q}^{\tilde{n}_{\rm G}}$ is the $\tilde{n}_{\rm G}$-tuple obtained from deleting the $N_{\rm G}$ integer-valued entries from the $2n_{\rm G}$-tuple $(y^{\rm G}_{1+}, \dots, y^{\rm G}_{n_{\rm G}+}, -y^{\rm G}_{1-}, \dots, -y^{\rm G}_{n_{\rm G}-})$. The third line also makes use of the notation $[\cdot] : \mathbb{Q} \to [0,1\rangle$, $q \mapsto [q] := q - \lfloor q\rfloor$.

Since $\bar{n}_{\rm D} + N_{\rm G} > 0$, we know that $\bar{n}_{\rm D} + N_{\rm G} - 1 \geq 0$. Thus for all $k > 0$, we have

$$\begin{aligned} y_-(k) + y_+(k) &< -\sum_{i=1}^{\tilde{n}_{\rm G}} \lfloor \tilde{y}_i^{\rm G} \rfloor + \frac{1}{k}(\bar{n}_{\rm D} + N_{\rm G} - 1) \\ &\leq -\sum_{i=1}^{\tilde{n}_{\rm G}} \lfloor \tilde{y}_i^{\rm G} \rfloor \ + \ \bar{n}_{\rm D} + N_{\rm G} - 1 \\ &= \ y_-(1) + y_+(1) + 1 \in \mathbb{Z}, \end{aligned} \tag{96}$$

which, since $\max_{k>0}(y_-(k) + y_+(k)) \in \mathbb{Z}$, implies that $y_-(k) + y_+(k)$ is maximized at $k = 1$. Thus, $y_-(k_1)$ and $y_+(k_2)$ are respectively maximized and minimized at $k_1 = 1$ and $k_2 = 1$, completing the proof of the proposition.

□

5. Cabling

The (p, q)-cable $Y^{(p,q)} \subset k$ of a knot complement $Y := X \setminus \nu(K) \subset X$ is given by the knot complement $Y^{(p,q)} := X \setminus \nu(K^{(p,q)})$, where $K^{(p,q)} \subset X$ is the image of the (p, q)-torus knot embedded in the boundary of Y. I recently made the mundane, and almost certainly not novel, observation that one can realize any cable of $Y \subset X$ by gluing on an appropriate Seifert fibered space.

5.1. Cabling via gluing.

Suppose $Y := X \setminus \nu(K)$ is the knot complement of an arbitrary knot $K \subset X$ in an arbitrary closed oriented three-manifold X. We construct the (p, q)-cable $Y^{(p,q)} \subset X$ of $Y \subset X$ as follows.

Let $\mu \in H_1(\partial Y)$ denote the meridian of K, and let $\lambda \in H_1(\partial Y)$ denote a choice of longitude, so that $X = Y(\mu)$ and $\mu \cdot \lambda = 1$. Choosing $p^*, q^* \in Z$ such that $pp^* - qq^* = 1$, let $Y_{(-q^*,p)}$ denote the regular fiber complement

$$Y_{(-q^*,p)} := M_{S^2}(0; -q^*/p) \setminus \nu(f), \tag{97}$$

so that $Y_{(-q^*,p)}$ is a solid torus whose compressing disk has boundary of slope $\frac{q^*}{p}$.

To construct the (p, q)-cable $Y^{(p,q)} \subset X$, we form the union

$$Y^{(p,q)} := \hat{Y}_{(-q^*,p)} \cup_\varphi Y, \quad \varphi : \partial Y \to -\partial_1 \hat{Y}_{(-q^*,p)}, \tag{98}$$

where $\hat{Y}_{(-q^*,p)}$ is a regular fiber complement in $Y_{(-q^*,p)}$, with $\partial_1 \hat{Y}_{(-q^*,p)} := \partial Y_{(-q^*,p)}$, and where φ is chosen to induce the map φ_* on homology defined by

$$\varphi_*(\mu) := -q^* \tilde{f}_1 + p\tilde{h}_1, \ \ \varphi_*(\lambda) := p^* \tilde{f}_1 - q\tilde{h}_1. \tag{99}$$

Proposition 5.1. $Y^{(p,q)} \subset Y^{(p,q)}(0) = X$ *is the* (p, q)*-cable of the knot complement* $Y \subset X$.

Proof. To verify that $Y^{(p,q)}(0) = X$, note that $Y^{(p,q)}(0)$ is a union of Y with the solid torus $Y_{(-q^*,p)}$, such that μ is sent to the slope $\frac{q^*}{p}$ bounding the compressing disk of $Y_{(-q^*,p)}$. In other words, $Y^{(p,q)}(0)$ is the Dehn filling $Y(\mu) =: X$.

Since $Y^{(p,q)} = X \setminus \nu(f)$ is the complement of the regular fiber f, we next must verify that, in the boundary of the solid torus (ST) to which Y is glued, the regular fiber is of class $pm_{\text{ST}} + ql_{\text{ST}} \in H_1(\partial Y_{(-q^*,p)})$ in terms of the basis $l_{\text{ST}} := \varphi_*(\mu)$, $m_{\text{ST}} := \varphi_*(\lambda)$ specified by the meridian μ and longitude λ of K. Indeed, we have

(100)
$$pm_{\text{ST}} + ql_{\text{ST}} = p\varphi_*(\lambda) + q\varphi_*(\mu) = (pp^* - qq^*)\tilde{f}_1 = \tilde{f}_1. \qquad \square$$

5.2. **L-space intervals for cables.** Supposing $Y \subset X$ is Floer simple and boundary incompressible, write $a_-\mu + b_-\lambda$ and $a_+\mu + b_+\lambda$ for respective representatives in $H_1(\partial Y)$ of the left-hand and right-hand endpoints of the L-space interval $\mathcal{L}(Y) \subset \mathbb{P}(H_1(\partial Y))$. Now, φ is an orientation-reversing map, but since we change from a positively-oriented basis to a negatively-oriented basis, the induced map $\varphi_*^{\mathbb{P}}$ is orientation-preserving. We therefore have

(101)
$$\varphi_*^{\mathbb{P}}(\mathcal{L}(Y)) = [[y_{1-}, y_{1+}]], \qquad y_{1\pm} := \frac{a_\pm q^* - b_\pm p^*}{a_\pm p - b_\pm q} = \frac{q^*}{p}\left(1 - \frac{b_\pm}{q^*(a_\pm p - b_\pm q)}\right).$$

For $k \in \mathbb{Z}_{>0}$, define

(102)
$$y_-(k) := -\frac{1}{k}\left(\left\lfloor -\frac{q^*}{p}k \right\rfloor + \lceil y_{1+}k \rceil\right), \quad y_+(k) := -\frac{1}{k}\left(\left\lceil -\frac{q^*}{p}k \right\rceil + \lfloor y_{1-}k \rfloor\right).$$

which simplifies to

(103)
$$y_-(k) := \frac{1}{k}\left(\left\lceil \frac{q^*}{p}k \right\rceil - \lceil y_{1+}k \rceil\right), \quad y_+(k) := \frac{1}{k}\left(\left\lfloor \frac{q^*}{p}k \right\rfloor - \lfloor y_{1-}k \rfloor\right).$$

As usual, we also define

(104)
$$y_- := \max_{k>0} y_-(k), \qquad y_+ := \min_{k>0} y_+(k).$$

Since Y is Floer simple and boundary incompressible, and since every Dehn filling of $\hat{Y}_{(-q^*,p)}$ along $\partial Y^{(p,q)}$ is Floer simple, we can still invoke Theorem 4.6 and Proposition 4.7 to compute the L-space interval for $Y^{(p,q)}$ (see Corollary 6.2 for justification of this generalization). Thus, in terms of the Seifert basis $(\tilde{f}, -\tilde{h})$, $\hat{Y}_{(-q^*,p)}$ has L-space interval $[[y_-, y_+]]$ if it is Floer simple, and $\{y_-\} = \{y_+\}$ if it has an isolated L-space filling.

It is often more natural, however, to express this L-space interval in terms of the surgery basis for the cabled knot. Recall that $\hat{Y}_{(-q^*,p)} = X \setminus \nu(f)$. The natural surgery basis associated to the complement of the regular fiber is given by the meridian $\mu^{(p,q)} := -\tilde{h}$ and longitude $\lambda^{(p,q)} = \tilde{f}$, yielding the following result.

Theorem 5.2. *Suppose $Y = X \setminus \nu(K)$ is a boundary incompressible Floer simple knot complement with L-space interval $\mathcal{L}(Y) = [[\frac{a_-}{b_-}, \frac{a_+}{b_+}]]$ in terms of the surgery basis $\mu, \lambda \in H_1(\partial Y)$ for K, with μ the meridian of K and λ a choice of longitude. Then in terms of the surgery basis produced by cabling, the (p,q)-cable $Y^{(p,q)} \subset X$ of $Y \subset X$ has L-space interval*

(105)
$$\mathcal{L}(Y^{(p,q)}) = \begin{cases} \emptyset & \infty \neq y_{1-} < y_{1+} \neq \infty \ \textit{and} \ y_- > y_+ \\ \{1/y_-\} = \{1/y_+\} & \infty \neq y_{1-} < y_{1+} \neq \infty \ \textit{and} \ y_- = y_+ \\ [[1/y_+, 1/y_-]] & \textit{otherwise.} \end{cases}$$

5.3. **Knots in S^3.** As an illustration, we apply the above result to an arbitrary boundary incompressible Floer simple knot complement $Y := S^3 \setminus \nu(K)$ in S^3. The surgery basis for a knot complement in S^3 conventionally takes λ to be the rational longitude, which in S^3 is Seifert framed. The meridian μ of K is automatically dual to this λ.

Without loss of generality (up to replacing K with its mirror image), we demand that K be *positive*, by which we mean that there exist positive $u, v \in \mathbb{Z}$ such that the Dehn filling $Y(u\mu + v\lambda)$ is an L-space. In terms of the projectivization map $x\mu + y\lambda \mapsto x/y \in \mathbb{Q} \cup \{\infty\}$, it is easy to show (see "example" in [33, Section 4]) that Y has L-space interval

$$\mathcal{L}(Y) = [N, +\infty], \quad N := 2g(K) - 1 = \deg(\Delta(K)) - 1, \tag{106}$$

where $g(K)$ and $\Delta(K)$ are the genus and Alexander polynomial of K.

Choosing $q^*, p^* \in \mathbb{Z}$ such that $pp^* - qq^* = 1$, and demanding $0 \leq q^* < p$, we then have

$$\varphi^{\mathbb{P}}_*(\mathcal{L}(Y)) = [[y_{1-}, y_{1+}]], \quad y_{1-} := \frac{Nq^* - p^*}{Np - q} = \frac{q^*}{p} + \frac{1}{p(q - Np)}, \quad y_{1+} := \frac{q^*}{p}. \tag{107}$$

From (103) and (104), we immediately compute that $y_- = 0$. For y_+, we have

$$y_+ = \min_{k>0} y_+(k), \quad y_+(k) := \frac{1}{k}\left(\left\lfloor \frac{q^*}{p}k \right\rfloor - \left\lfloor \left(\frac{q^*}{p} + \frac{1}{p(q - Np)}\right) k \right\rfloor\right). \tag{108}$$

If $q - Np < 0$, then $y_+(k) \geq 0$ for all $k > 0$, and so $y_+(k)$ is minimized at $y_+ = y_+(1) = 0$. Since $\infty \neq y_{1-} < y_{1+} \neq \infty$ in this case, we then have $\mathcal{L}(Y^{(p,q)}) = \{0\}$. If $q - Np = 0$, then $y_+ = \infty$, yielding $\mathcal{L}(Y^{(p,q)}) = [0, \infty]$. If $q - Np < 0$, then $y_+(k) \leq 0$ for all $k > 0$, and it is straightforward to show that $y_+(k)$ is minimized at the lowest value of $k_+ > 0$ for which $\lfloor \frac{q^*}{p} k_+ \rfloor \neq \lfloor y_{1-} k_+ \rfloor$. Since $y_{1-} - \frac{q^*}{p} < \frac{1}{p}$, a necessary condition for this to occur is to have

$$\left(y_{1-} - \frac{q^*}{p}\right) k_+ \geq \frac{1}{p}, \quad \text{which implies} \quad k_+ \geq q - Np. \tag{109}$$

Since $y_{1+}(q - Np) = p^* - Nq^* \in \mathbb{Z}$, setting $k_+ = q - Np$ is also sufficient:

$$y_+ = y_+(k_+) = \frac{1}{q - Np}\left((p^* - Nq^* - 1) - (p^* - Nq^*)\right) = -\frac{1}{q - Np}, \tag{110}$$

and we have $\mathcal{L}(Y^{(p,q)}) = [[0, -1/(q - Np)]]$.

As a final step, we re-express $\mathcal{L}(Y^{(p,q)})$ in terms of the conventional basis for knot complements in S^3. We again use the meridian $\mu^{(p,q)} := -\tilde{h}$, but the rational longitude is

$$l^{(p,q)} = -\left(-\frac{q^*}{p} + \varphi^{\mathbb{P}}_*([\lambda])\right) = -\left(-\frac{q^*}{p} + \frac{p^*}{q}\right) = -\frac{1}{pq}, \tag{111}$$

for which we choose the representative $\lambda_{\mathbb{Q}} := \tilde{f} + pq\tilde{h}$ to achieve $\mu^{(p,q)} \cdot \lambda_{\mathbb{Q}} = 1$. Performing the requisite change of basis on $\mathcal{L}(Y^{(p,q)})$ for the three cases described in the preceding paragraph then recovers the following result of Hedden [21] and Hom [22].

Corollary 5.3. *$Y^{(p,q)} \subset S^3$ has L-space interval*

$$\mathcal{L}(Y^{(p,q)}) = \begin{cases} \{\infty\} & 2g(K)-1 > \frac{q}{p} \\ [pq-p-q+2g(K)p,\ \infty] & 2g(K)-1 \leq \frac{q}{p}. \end{cases} \tag{112}$$

5.4. **Knots in L-spaces.** It is possible to prove an analogous result for boundary incompressible Floer simple knot complements in arbitrary L-spaces.

To simplify the statement of such a result, we discard cables with $p=0$, $p=1$, or $q=0$, since the zero-cable of a knot complement $Y \subset X$ is just the connected sum of X with the unknot complement in S^3; the $1/q$-cable is just a change of framing; and the $1/0$-cable, which changes the framing by zero, is the identity cable. We then have the following.

Theorem 5.4. *Suppose that $p, q \in \mathbb{Z}$ with $p > 1$ and $\gcd(p,q) = 1$, and that $Y = X \setminus \nu(K)$ is a boundary incompressible Floer simple knot complement in an L-space X, with L-space interval $\mathcal{L}(Y) = [[\frac{a_-}{b_-}, \frac{a_+}{b_+}]]$, written in terms of the surgery basis $\mu, \lambda \in H_1(\partial Y)$ for K, with μ the meridian of K and λ a choice of longitude. Then in terms of the surgery basis produced by cabling, the (p,q)-cable $Y^{(p,q)} \subset X$ of $Y \subset X$ has L-space interval*

$$\mathcal{L}(Y^{(p,q)}) = \begin{cases} \{\infty\} & \frac{a_-}{b_-} \in \left[\frac{p^*}{q^*}, \infty\right],\ \frac{a_+}{b_+} \in \left[\frac{q-p^*}{p-q^*}, \frac{q}{p}\right\rangle \cup \{\infty\} \\ [[1/y_+, 1/y_-]] & \textit{otherwise}, \end{cases} \tag{113}$$

where $pp^* - qq^* = 1$ with $0 < q^* < p$.

Proof. If $Y^{(p,q)}$ is Floer simple, then Theorem 5.2 implies $\mathcal{L}(Y^{(p,q)}) = [[1/y_+, 1/y_-]]$ in terms of the surgery basis produced by cabling.

Observe that $Y^{(p,q)}$ is not Floer simple if and only if $y_- = y_+ = 0$. That is, if $Y^{(p,q)}$ is not Floer simple, then the meridional filling X is the only L-space filling, and conversely if $y_- = y_+ = 0$, then since $0 \notin [[0,0]]$, we know that $Y^{(p,q)}$ is not Floer simple.

Choose $p^*, q^* \in \mathbb{Z}$ so that $pp^* - qq^* = 1$ with $0 < q^* < p$. Then for $y_{1\pm} \neq \infty$, one has

$$\begin{cases} y_- = -\lceil y_{1+} \rceil + 1 & 1 - [-y_{1+}] \geq \frac{q^*}{p} \\ y_- \notin \mathbb{Z} & 1 - [-y_{1+}] < \frac{q^*}{p}, \end{cases} \qquad \begin{cases} y_+ = -\lfloor y_{1-} \rfloor & [y_{1-}] \leq \frac{q^*}{p} \\ y_+ \notin \mathbb{Z} & [y_{1-}] > \frac{q^*}{p}. \end{cases} \tag{114}$$

That is, since

$$y_+ = -\lfloor y_{1-} \rfloor + \min_{k>0} \frac{1}{k} \left(\left\lfloor \frac{q^*}{p} k \right\rfloor - \lfloor [y_{1-}] k \rfloor \right), \tag{115}$$

the right-hand summand vanishes when $[y_{1-}] \leq \frac{q^*}{p}$, but when $[y_{1-}] > \frac{q^*}{p}$, the right-hand summand is not minimized at $k=1$. Thus $y_+ \neq y_+(1)$, and Proposition 4.13 tells us that $y_+ \notin \mathbb{Z}$. A similar argument holds for y_-.

We therefore have

$$y_- = 0 \iff \frac{q^*}{p} \leq y_{1+} \leq 1 \iff \frac{q-p^*}{p-q^*} \leq \frac{a_+}{b_+} < \frac{q}{p} \text{ or } \frac{a_+}{b_+} = \infty, \tag{116}$$

and similarly,

$$(117)\qquad y_+ = 0 \iff 0 \le y_{1-} \le \frac{q^*}{p} \iff \frac{p^*}{q^*} \le \frac{a_-}{b_-} \ne \infty \text{ or } \frac{a_-}{b_-} = \infty.$$

Thus $y_- = y_+ = 0$ if and only if

$$(118)\qquad \frac{a_-}{b_-} \in \left[\frac{p^*}{q^*}, \infty\right] \quad \text{and} \quad \frac{a_+}{b_+} \in \left[\frac{q-p^*}{p-q^*}, \frac{q}{p}\right\rangle \cup \{\infty\}.$$

□

6. Observations

Our demonstration of extended L/NTF-equivalence for graph manifolds in Theorem 4.5 gives a (mildly) alternate proof of the Theorem 1.3 statement that a graph manifold is an L-space if and only if it fails to admit a co-oriented taut foliation.

From a practical standpoint, however, the main utility of Theorem 4.5 for us was its implication that the gluing result in Proposition 4.4 holds for all graph manifolds:

Corollary 6.1. *If Y_1 and Y_2 are non-solid-torus graph manifolds with torus boundary, then the union $Y_1 \cup_\varphi Y_2$, with gluing map $\varphi : \partial Y_1 \to -\partial Y_2$, is an L-space if and only if*

$$\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2)).$$

Corollary 6.1 has two advantages over the more general L-space gluing criterion of Proposition 3.3: it removes the condition that $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2)$ be nonempty, and it allows one to prove that $Y_1 \cup Y_2$ is not an L-space in cases in which boundary incompressible Y_1 and Y_2 are not Floer simple.

6.1. Generalization of Theorem 4.6.

Nevertheless, while the L-space gluing result analogous to Proposition 3.3 proved by Hanselman and Watson in [19] replaces the hypothesis of Floer simplicity by a more technical condition, their gluing result does not impose the hypothesis of nonempty $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2)$ required by the gluing result of J. Rasmussen and the author in [33]. In [18], the four authors discuss how these two gluing results can be combined to prove a gluing result analogous to Proposition 3.3 which requires Floer simplicity but not nonempty $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2)$. Thus, the only real hypothesis we have circumvented is that of Floer simplicity. If we replace the condition that the Y_i glued to $\hat{M}$ be graph manifolds with the condition that they be Floer simple, then we can extend the domain of validity of Proposition 4.7 and Theorem 4.6 as follows.

Corollary 6.2. *Theorem 4.6 holds for any boundary incompressible Floer simple three-manifolds $Y_1, \ldots, Y_{n_G}$, provided that Y satisfies the criteria in Proposition 4.7 for $\mathcal{L}(Y)$ to be nonempty.*

6.2. Generalized Solid Tori.

A recent result of Gillespie [17] states that a compact oriented three-manifold Y with torus boundary satisfies $\mathcal{L}(Y) = \mathbb{P}(H_1(\partial Y)) \setminus \{l\}$ if and only if Y has genus 0 and an L-space filling. Such manifolds, called *generalized solid tori* in [33], are of independent interest.

In the proof of Theorem 4.6 and Proposition 4.7, we find many generalized solid tori with the regular fiber class as rational longitude, but there are limited circumstances in which other generalized solid tori appear. In fact, we can prove the following.

Theorem 6.3. *If Y is a graph manifold with torus boundary, $b_1(Y)=1$, and rational longitude other than the regular fiber, then Y is a generalized solid torus if and only if it is homeomorphic to an iterated cable of the regular fiber complement in $S^1\times S^2$.*

Proof. Suppose Y is a generalized solid torus graph manifold, with rational longitude l not coinciding with the regular fiber. Since Y is Floer simple with $\infty \in \mathcal{L}^{\circ}(Y)$, Y must satisfy (FS3) from Proposition 4.7, and since $y_-, y_+ \neq \infty$, we must have $\infty \neq y^{\text{G}}_{i-} \geq y^{\text{G}}_{i+} \neq \infty$. Claim 2 from Section 4.7 then implies $y_- > y_+$ unless $n_{\text{G}}=1$ with $n_{\text{D}} \leq 1$ or $n_{\text{G}}=0$ with $n_{\text{D}} \leq 2$.

If $n_{\text{G}} = 1$, then according to Claim 4 from Section 4.8, either $y^{\text{D}}_1 \in \mathbb{Z}$, in which case Y is homeomorphic to Y_1 and we should replace Y with Y_1 and begin again; or $y^{\text{G}}_{-1} = y^{\text{G}}_{+1} =: y^{\text{G}}_1$ with $y^{\text{D}}_1 + y^{\text{G}}_1 \in \mathbb{Z}$, in which case Y_1 is a generalized solid torus, and Proposition 5.1 implies $Y \subset Y(l)$ is a cable of $Y_1 \subset Y_1(l)$.

If $n_{\text{G}} = 0$, then Y is Seifert fibered with $y_- = y_+$, and so either from [33] or from a mildly modified version of Claim 4, we deduce that either $y^{\text{D}}_1 + y^{\text{D}}_2 \in \mathbb{Z}$, in which case Proposition 5.1 implies Y is a cable of the regular fiber complement in $S^1\times S^2$; or $\{y^{\text{D}}_1, y^{\text{D}}_2\} \cap \mathbb{Z} \neq \emptyset$, in which case Y is a solid torus, hence homeomorphic to the regular fiber complement in $S^1\times S^2$.

The converse is an immediate corollary of Proposition 5.1 and Theorem 4.6. □

We also have the following result for arbitrary generalized solid tori.

Proposition 6.4. *If Y is a generalized solid torus, then any cable of $Y \subset Y(l)$ is a generalized solid torus.*

Proof. If Y is boundary compressible, then it is the connected sum of a solid torus with lens spaces, and Theorem 6.3 implies that any cable of a solid torus within its longitudinal filling is a generalized solid torus. If Y is boundary incompressible, then the result is an immediate corollary of Theorem 5.2. □

Similarly, for any class of manifolds for which the gluing result in Proposition 1.5 holds without the requirement of Floer simplicity—such as graph manifolds—one has the result that if Y has an isolated L-space filling, *i.e.*, if $\mathcal{L}(Y) = \{\mu\}$ for some $\mu \in \mathbb{P}(H_1(\partial Y))$, then any cable of $Y \subset Y(\mu)$ has $Y(\mu)$ as an isolated L-space filling.

6.3. **Isolated L-space fillings.** A Seifert fiber complement in an L-space Seifert fibered manifold could justifiably be called the prototypical Floer simple manifold, just as a lens space is the prototypical L-space. It is therefore striking that we encounter isolated L-space fillings as regular fiber complements in graph manifolds. Fortunately, this still does not prevent L-space graph manifolds from admitting Floer simple Seifert fiber complements.

Given a closed graph manifold X, we shall call an exceptional fiber $f_{\text{E}} \subset X$ *invariantly exceptional* if the JSJ componenent $\hat{Y} \subset X$ containing f_{E} has more than one exceptional fiber. To motivate this name, note that if X has more than one JSJ component and $\hat{Y}$ has only one exceptional fiber, say, of slope $y_1^{\mathrm{D}} = y_0$, then since the punctured solid torus has nonunique Seifert structure, X is homemorphic to a graph manifold in which y_1^{D} is replaced with 0 and $\varphi_{1*}^{\mathbb{P}}$ is replaced with $\varphi_{1*}^{\mathbb{P}} + y_0$.

Theorem 6.5. *Every invariantly exceptional fiber complement in an L-space graph manifold is Floer simple.*

Proof. Suppose X is an L-space graph manifold. If X is Seifert fibered, then every Seifert fiber complement, regular or otherwise, is Floer simple.

Suppose X has more than one JSJ component, and let Y denote a non-Floer-simple complement of an invariantly exceptional fiber. Since $\mathcal{L}(Y) \neq \emptyset$, Y non-Floer-simple implies $\mathcal{L}(Y) = \{y_-\} = \{y_+\}$, with $y_\pm \in \mathbb{Q}$. However, since Y has at least one exceptional fiber, Proposition 4.14 tells us that $y_- = y_+ \in \mathbb{Z}$, contradicting the hypothesis that Y is an exceptional fiber complement of X. Thus the theorem holds. □

On the other hand, for a graph manifold with more than one JSJ component, Seifert fibers are not the only knots yielding Floer simple knot complements, due to the following result for arbitrary L-spaces.

Proposition 6.6. *If an L-space X decomposes as a union $Y_1 \cup_\varphi Y_2$ of Floer simple manifolds Y_i along an incompressible torus $T \subset X$, then there is a knot $K \subset X$ transversely intersecting T for which knot the complement $X \setminus \nu(K)$ is Floer simple.*

Proof. In fact, an analogous result holds for any toroidal L-space.

Suppose the above hypotheses hold. Then since Y_1 and Y_2 are Floer simple with incompressible boundary, Proposition 3.3 implies

$$\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cup \mathcal{L}^\circ(Y_2) = \mathbb{P}(H_1(\partial Y_2)). \tag{119}$$

Since $\varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1))$ and $\mathcal{L}^\circ(Y_2)$ are open, (119) implies they intersect. Choosing any $\mu_2 \in \varphi_*^{\mathbb{P}}(\mathcal{L}^\circ(Y_1)) \cap \mathcal{L}^\circ(Y_2)$, set $\mu_1 := \varphi_*^{\mathbb{P}-1}(\mu_2)$, and let K_i denote the knot core of $Y_i(\mu_i) \setminus Y_i$. As explained in more detail in the proof of [33, Theorem 6.2], X can be regarded as zero-surgery on the knot $K_1 \# K_2 \subset Y_1(\mu_1) \# Y_2(\mu_2)$. Thus, if we set $Y := Y_1(\mu_1) \# Y_2(\mu_2) \setminus \nu(K_1 \# K_2)$ and let K denote the knot core of $X \setminus Y$, then the knot complement $Y = X \setminus \nu(K)$ has at least two distinct L-space fillings, hence is Floer simple. Moreover, since K is dual to $K_1 \# K_2$ under surgery, K intersects the separating torus transversely. □

Corollary 6.7. *If X is an L-space graph manifold, then for every incompressible torus $T \subset X$, there is a knot $K \subset X$ transversely intersecting T for which knot the complement $X \setminus \nu(K)$ is Floer simple.*

Proof. Choose an arbitrary incompressible torus $T \subset X$, not necessarily one used in the minimal JSJ decomposition for X, and write $X = Y_1 \cup_T Y_2$. Since X is an L-space, Corollary 6.1 implies each Y_i has nonempty $\mathcal{L}^\circ(Y_i)$, hence is Floer simple. Thus, we can apply Proposition 6.6. □

This section has only cataloged the most obvious corollaries of the paper's main results. We invite the reader to find more.

References

[1] Kenneth Baker, Dorothy Buck, and Ana Lecuona. Some knots in $S^1 \times S^2$ with lens space surgeries. arXiv:1302.7011, 2013.

[2] John Berge. The knots in $D^2 \times S^1$ which have nontrivial Dehn surgeries that yield $D^2 \times S^1$. *Topology Appl.*, 38(1):1–19, 1991.

[3] Jonathan Bowden. Approximating C^0-foliations by contact structures. arXiv:1509.07709, 2015.

[4] Steven Boyer and Adam Clay. Foliations, orders, representations, L-spaces and graph manifolds. arXiv:1401.7726, 2014.

[5] Steven Boyer, Cameron McA. Gordon, and Liam Watson. On L-spaces and left-orderable fundamental groups. *Math. Ann.*, 356(4):1213–1245, 2013.

[6] Steven Boyer, Dale Rolfsen, and Bert Wiest. Orderable 3-manifold groups. *Ann. Inst. Fourier (Grenoble)*, 55(1):243–288, 2005.

[7] Mark Brittenham, Ramin Naimi, and Rachel Roberts. Graph manifolds and taut foliations. *J. Differential Geom.*, 45(3):446–470, 1997.

[8] Mark Brittenham and Rachel Roberts. When incompressible tori meet essential laminations. *Pacific J. Math.*, 190(1):21–40, 1999.

[9] Danny Calegari and Nathan M. Dunfield. Laminations and groups of homeomorphisms of the circle. *Invent. Math.*, 152(1):149–204, 2003. arXiv:math/0203192.

[10] Danny Calegari and Alden Walker. Ziggurats and rotation numbers. *J. Mod. Dyn.*, 5(4):711–746, 2011.

[11] Radu Cebanu. *A Generalization of Property R.* 2013. Thesis (Ph.D.)–UQAM.

[12] David Eisenbud, Ulrich Hirsch, and Walter Neumann. Transverse foliations of Seifert bundles and self-homeomorphism of the circle. *Comment. Math. Helv.*, 56(4):638–660, 1981.

[13] Yakov M. Eliashberg and William P. Thurston. *Confoliations*, volume 13 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1998.

[14] David Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18(3):445–503, 1983.

[15] David Gabai. Surgery on knots in solid tori. *Topology*, 28(1):1–6, 1989.

[16] Étienne Ghys. Groups acting on the circle. *Enseign. Math. (2)*, 47(3-4):329–407, 2001.

[17] Thomas Gillespie. L-space fillings and Generalized Solid Tori. In preparation, 2015.

[18] Jonathan Hanselman, Jacob Rasmussen, Sarah Dean Rasmussen, and Liam Watson. Taut foliations on graph manifolds. arXiv:1508.05911, 2015.

[19] Jonathan Hanselman and Liam Watson. A calculus for bordered Floer homology. Preprint, 2015.

[20] Gilbert Hector and Ulrich Hirsch. *Introduction to the geometry of foliations. Part A*, volume 1 of *Aspects of Mathematics*. Friedr. Vieweg & Sohn, Braunschweig, 1981. Foliations on compact surfaces, fundamentals for arbitrary codimension, and holonomy.

[21] Matthew Hedden. On knot Floer homology and cabling. II. *Int. Math. Res. Not. IMRN*, (12):2248–2274, 2009.

[22] Jennifer Hom. A note on cabling and L-space surgeries. *Algebr. Geom. Topol.*, 11(1):219–223, 2011.

[23] Mark Jankins and Walter D. Neumann. Rotation numbers of products of circle homeomorphisms. *Math. Ann.*, 271(3):381–400, 1985.

[24] William H. Kazez and Rachel Roberts. Approximating $C^{1,0}$ foliations. 2015. arXiv:1404.5919.

[25] Peter A. Linnell. Left ordered amenable and locally indicable groups. *J. London Math. Soc. (2)*, 60(1):133–142, 1999.

[26] Paolo Lisca and Gordana Matić. Transverse contact structures on Seifert 3-manifolds. *Algebr. Geom. Topol.*, 4:1125–1144 (electronic), 2004.

[27] Paolo Lisca and András I. Stipsicz. Ozsváth-Szabó invariants and tight contact 3-manifolds. III. *J. Symplectic Geom.*, 5(4):357–384, 2007.

[28] Ramin Naimi. Foliations transverse to fibers of Seifert manifolds. *Comment. Math. Helv.*, 69(1):155–162, 1994.

[29] S. P. Novikov. The topology of foliations. *Trudy Moskov. Mat. Obšč.*, 14:248–278, 1965.
[30] Peter. Ozsváth and Zoltán Szabó. Holomorphic disks and genus bounds. *Geom. Topol.*, 8:311–334, 2004. math.GT/0311496.
[31] Peter Ozsváth and Zoltán Szabó. An introduction to Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 3–27. Amer. Math. Soc., Providence, RI, 2006. http://math.mit.edu/ petero/Introduction.pdf.
[32] Peter Ozsváth and Zoltán Szabó. Lectures on Heegaard Floer homology. In *Floer homology, gauge theory, and low-dimensional topology*, volume 5 of *Clay Math. Proc.*, pages 29–70. Amer. Math. Soc., Providence, RI, 2006. http://math.mit.edu/ petero/Lectures.pdf.
[33] Jacob Rasmussen and Sarah Dean Rasmussen. Floer Simple Manifolds and L-space Intervals. arXiv:1508.05900, 2015.
[34] R. Roussarie. Plongements dans les variétés feuilletées et classification de feuilletages sans holonomie. *Inst. Hautes Études Sci. Publ. Math.*, (43):101–141, 1974.
[35] William P. Thurston. The geometry and topology of 3-manifolds. Lecture Notes, 1980.
[36] William P. Thurston. Three-manifolds, Foliations and Circles, II. Unfinished manuscript, 1998.

E-mail address: S.Rasmussen@dpmms.cam.ac.uk

DPMMS, University of Cambridge, UK