Tautological rings for high dimensional manifolds

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Abstract

We study tautological rings for high dimensional manifolds, that is, for each smooth manifold M the ring $R^*(M)$ of those of characteristic classes of smooth fibre bundles with fibre M which is generated by generalised Miller–Morita–Mumford classes. We completely describe these rings modulo nilpotent elements, when M is a connected sum of copies of $S^n \times S^n$ for n odd.

1. Introduction

Let E^{k+d} and B^k be connected compact smooth oriented manifolds, and $\pi : E \to B$ be a smooth fibre bundle (i.e. a proper submersion) with fibre M^d. Then there is a d-dimensional oriented vector bundle $T_\pi = \mathrm{Ker}(D\pi)$ over E. If $c \in H^*(BSO(d);\mathbb{Q})$ is a rational characteristic class of such vector bundles then we define the associated *generalised Miller–Morita–Mumford class* to be the fibre integral

$$\kappa_c(\pi) = \int_\pi c(T_\pi) \in H^{*-d}(B;\mathbb{Q}).$$

For a basis $\mathcal{B} \subset H^*(BSO(d);\mathbb{Q})$ this defines a ring homomorphism

$$\begin{aligned}\mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}] &\longrightarrow H^*(B;\mathbb{Q})\\ \kappa_c &\longmapsto \kappa_c(\pi).\end{aligned}$$

We let $I_M \subset \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$ be the ideal consisting of those polynomials in the classes κ_c which vanish on every such smooth fibre bundle with fibre M, and define the *tautological ring* of M to be the associated quotient ring

$$R^*(M) = \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]/I_M.$$

The name "tautological ring" is borrowed from the case $d = 2$, where it usually refers to the subring of the cohomology ring (or Chow ring) of moduli spaces of Riemann surfaces which is generated by certain tautological classes κ_i (which in our notation correspond to $\kappa_{e^{i+1}}$). Our definition coincides with the usual one in this case. There is a large literature on these rings; see [Mum83, Loo95, Fab99, Mor03].

We shall be interested in manifolds of even dimension $d = 2n$, in which case $H^*(BSO(2n);\mathbb{Q}) = \mathbb{Q}[p_1, p_2, \ldots, p_{n-1}, e]$ and we take $\mathcal{B}$ to be the basis of monomials in these classes. Writing W_g for the connected sum of g copies of $S^n \times S^n$, our goal in this paper is to study the structure of $R^*(W_g)$ modulo the nilradical $\sqrt{0}$, i.e. the ideal of nilpotent elements, and our main result can be stated as follows.

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THEOREM 1.1. *Let n be odd. Then*

- *(i)* $R^*(W_0)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}]$,
- *(ii)* $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,
- *(iii)* $R^*(W_g)/\sqrt{0} = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_{n-1}}]$ *for* $g > 1$.

In fact, (i) already holds before taking the quotient by the nilradical, and holds for n both even and odd (see Section 5.3).

As the Krull dimension of a ring is unchanged by taking the quotient by the nilradical, we may also conclude that $R^*(W_0)$ has Krull dimension n, $R^*(W_1)$ has Krull dimension 0, and $R^*(W_g)$ has Krull dimension $n-1$ as long as $g > 1$.

We shall also study two closely related tautological rings. Firstly, we may consider fibre bundles $\pi : E^{k+2n} \to B^k$ with fibre M^{2n} equipped with a section $s : B \to E$. To such a fibre bundle and a $c \in \mathcal{B}$ we may also associate the characteristic class $c(\pi, s) = s^*(c(T_\pi))$, so in this case there is defined a ring homomorphism

$$\begin{aligned} \mathbb{Q}[c, \kappa_c \,|\, c \in \mathcal{B}] &\longrightarrow H^*(B;\mathbb{Q}) \\ c &\longmapsto c(\pi, s) \\ \kappa_c &\longmapsto \kappa_c(\pi) \end{aligned}$$

and we let $I_{(M,\star)} \subset \mathbb{Q}[c, \kappa_c \,|\, c \in \mathcal{B}]$ denote the ideal of those polynomials in the c and κ_c which vanish on all such bundles; we write $R^*(M, \star)$ for the associated quotient ring.

THEOREM 1.2. *Let n be odd.*

- *(i) For each* $c \in \mathcal{B}$, $(2-2g) \cdot c = \kappa_{ec} \in R^*(W_g, \star)/\sqrt{0}$.
- *(ii) For* $g \neq 1$, *the map*

  $$R^*(W_g)/\sqrt{0} \longrightarrow R^*(W_g, \star)/\sqrt{0}$$

  is an isomorphism.
- *(iii) The class* $e \in R^*(W_1, \star)$ *is nilpotent, and the map*

  $$\mathbb{Q}[p_1, p_2, \ldots, p_{n-1}] \longrightarrow R^*(W_1, \star)/\sqrt{0}$$

  is an isomorphism.

Secondly, we may consider fibre bundles $\pi : E^{k+2n} \to B^k$ with fibre M^{2n} equipped with an embedding $S : D^{2n} \times B \to E$ over B (or equivalently, with a normally framed section). The restriction $s = S|_{\{0\}\times B}$ defines a section, but the fact that it is framed means that $c(\pi, s) = 0$. Thus in this case we denote by $I_{(M,D^{2n})} \subset \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$ the ideal of polynomials in the κ_c which vanish on all such bundles, and write $R^*(M, D^{2n})$ for the associated quotient ring. The following is immediate from Theorem 1.1 and Theorem 1.2 (i).

COROLLARY 1.3. *If n is odd then* $R^*(W_g, D^{2n})/\sqrt{0} = \mathbb{Q}$ *for all* g.

The second-named author has shown [Gri13, Theorem 1.1] that for n odd the ring $R^*(W_g)$ is finitely-generated as a $\mathbb{Q}$-algebra, and hence $R^*(W_g, D^{2n})$ is too. (In fact, Theorem 1.1 of [Gri13] only shows this for $g > 1$, but in Proposition 5.7 we will show that $R^*(W_g, \star)$, and hence $R^*(W_g, D^{2n})$, is finitely-generated for all g.) Corollary 1.3 therefore implies the following.

COROLLARY 1.4. *If n is odd then* $R^*(W_g, D^{2n})$ *is a finite-dimensional rational vector space for all* g.

It would be interesting to have a computation of the ring $R^*(W_g, D^{2n})$ for some $g > 1$. In proving these theorems, we obtain results about the vanishing of certain elements in $R^*(M)/\sqrt{0}$ for any manifold M (in Theorem 2.1), and about the algebraic independence of certain elements in $R^*(W_g)/\sqrt{0}$ for n even (in Theorem 4.1). We cannot obtain results as conclusive as Theorem 1.1 for n even, as our argument relies on [Gri13] which does not apply in this case.

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2. Nilpotence of the Hirzebruch $\mathcal{L}$-classes

In this section, we prove that certain generalised Miller–Morita–Mumford classes are nilpotent in $R^*(M)$ for *any* smooth even-dimensional manifold M^{2n}. We will later apply this together with results from [Gri13] to the manifolds W_g in order to obtain an upper bound on $R^*(W_g)/\sqrt{0}$.

2.1 (Modified) Hirzebruch $\mathcal{L}$-classes

We first define certain cohomology classes $\tilde{\mathcal{L}}_i \in H^{4i}(BSO(2n);\mathbb{Q})$. Consider the graded ring $\mathbb{Q}[x_1, \ldots, x_n]$ in which each variable $x_1, \ldots, x_n$ has degree 2. We define homogenous symmetric polynomials $\tilde{\mathcal{L}}_i$ by the expression

$$\tilde{\mathcal{L}} = 2^n + \tilde{\mathcal{L}}_1 + \tilde{\mathcal{L}}_2 + \cdots = \prod_{i=1}^{n} \frac{x_i}{\tanh x_i/2}.$$

(The function $\frac{x}{\tanh x/2}$ is even, so the above expresses $\tilde{\mathcal{L}}_i$ as a symmetric polynomial in $x_1^2, x_2^2, \ldots, x_n^2$.) The subring of symmetric polynomials is generated by the elementary symmetric polynomials $\sigma_{i,n}(x_1^2, \ldots, x_n^2)$, so we may express $\tilde{\mathcal{L}}_i$ as a polynomial $\tilde{\mathcal{L}}_i(\sigma_{1,n}, \ldots, \sigma_{n,n})$, and we write

$$\tilde{\mathcal{L}}_i = \tilde{\mathcal{L}}_i(p_1, p_2, \ldots, p_n) \in H^{4i}(BSO(2n);\mathbb{Q})$$

for this polynomial evaluated at the Pontrjagin classes. Note that these differ from the usual Hirzebruch L-classes, $\mathcal{L}_i$, by $\tilde{\mathcal{L}}_i = 2^{n-2i}\mathcal{L}_i$. These classes may be written as

$$\tilde{\mathcal{L}}_i = 2^n(2^{2i-1} - 1)\frac{B_i}{(2i)!} \cdot p_i + \text{(polynomial in lower Pontrjagin classes)}$$

where B_i is the ith Bernoulli number, cf. [MS74, Problem 19-C] (this uses the convention of [MS74, Appendix B], in which $B_i = (-1)^i \cdot 2i \cdot \zeta(1-2i)$). In particular, these leading coefficients are never zero, so $\tilde{\mathcal{L}}_1, \tilde{\mathcal{L}}_2, \ldots, \tilde{\mathcal{L}}_n$ generate the same subring of $H^*(BSO(2n);\mathbb{Q})$ as $p_1, p_2, \ldots, p_n$.

2.2 Nilpotence due to the Hirzebruch signature thorem

For any manifold M^{2n} the characteristic classes $\tilde{\mathcal{L}}_i \in H^{4i}(BSO(2n);\mathbb{Q})$ give rise to the corresponding generalised Miller–Morita–Mumford classes $\kappa_{\tilde{\mathcal{L}}_i} \in R^*(M)$. Our present goal is to prove the following.

THEOREM 2.1. *The classes $\kappa_{\tilde{\mathcal{L}}_i} \in R^*(M)$ are nilpotent for all natural numbers $i \geqslant 1$ such that $4i - 2n \neq 0$ (all natural numbers if n odd).*

The main ingredient in the proof is a parametrised version of the Hirzebruch signature theorem, due to Atiyah [Ati69], which relates the classes $\kappa_{\tilde{\mathcal{L}}_i}$ with classes arising from the cohomology of an arithmetic group.

Let M^{2n} be an oriented manifold, and let $H = H^n(M;\mathbb{Z})/\text{torsion}$. It is a consequence of Poincaré duality that the $(-1)^n$-symmetric pairing

$$\lambda : H \otimes_{\mathbb{Z}} H \xrightarrow{\cup} H^{2n}(M;\mathbb{Z})/\text{torsion} = H^{2n}(M;\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}$$

is non-degenerate. Any homotopy equivalence of M preserving its orientation therefore induces an isometry of the $(-1)^n$-symmetric form (H, λ). In particular if $\pi : E \to B$ is a smooh oriented fibre bundle with fibre M, there is a homomorphism

$$\phi : \pi_1(B) \longrightarrow \mathrm{Aut}(H, \lambda)$$

from the fundamental group of B to the group of isometries of (H, λ), inducing a map $\phi : B \to B\mathrm{Aut}(H, \lambda)$.

We may further consider the induced $(-1)^n$-symmetric form $\lambda_{\mathbb{R}}$ on $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$. There are now two cases, depending on the parity of n. If n is odd then $(H_{\mathbb{R}}, \lambda_{\mathbb{R}})$ is a non-degenerate skew-symmetric form over $\mathbb{R}$, and so is determined by its rank. This identifies $\mathrm{Aut}(H_{\mathbb{R}}, \lambda_{\mathbb{R}}) = \mathrm{Sp}_{2g}(\mathbb{R})$, where $H_{\mathbb{R}}$ has dimension $2g$. If n is even then $(H_{\mathbb{R}}, \lambda_{\mathbb{R}})$ is a non-degenerate symmetric form over $\mathbb{R}$, and so is determined by its rank and signature. This identifies $\mathrm{Aut}(H_{\mathbb{R}}, \lambda_{\mathbb{R}}) = \mathrm{O}_{p,q}(\mathbb{R})$, where $H_{\mathbb{R}}$ has dimension $p + q$ and signature $p - q$. We obtain a composition

$$B \xrightarrow{\phi} B\mathrm{Aut}(H, \lambda) \longrightarrow B\mathrm{Aut}(H_{\mathbb{R}}, \lambda_{\mathbb{R}}) = \begin{cases} B\mathrm{Sp}_{2g}(\mathbb{R}) \\ B\mathrm{O}_{p,q}(\mathbb{R}). \end{cases}$$

Now Atiyah's work [Ati69, Section 4] implies that the classes $\kappa_{\tilde{\mathcal{L}}_i}$ are pulled back via this composition, and so in particular they are in the image of $\phi^* : H^*(B\mathrm{Aut}(H, \lambda);\mathbb{Q}) \to H^*(B;\mathbb{Q})$. Theorem 2.1 is then an immediate consequence of the fact that all positive degree elements of $H^*(B\mathrm{Aut}(H, \lambda);\mathbb{Q})$ are nilpotent: this is because $\mathrm{Aut}(H, \lambda)$ is an arithmetic group, so has finite virtual cohomological dimension and so in particular finite $\mathbb{Q}$-cohomological dimension.

3. Relations modulo nilpotence

The main result (Theorem 2.7) of [Gri13] implies that, when n is odd, for any oriented manifold bundle $E \to B$ with fibre W_g and even-dimensional classes $a, b \in H^*(E;\mathbb{Q})$ such that $\pi_!(a) = 0$ and $\pi_!(b) = 0$, the class $\pi_!(ab)$ is nilpotent. We shall use the following stronger statement, which will be deduced quite formally from [Gri13, Theorem 2.7].

Theorem 3.1. *Let $\pi : E \to B$ be a fibration with homotopy fibre homotopy equivalent to $W_g = g(S^n \times S^n)$, for n odd, such that the action of $\pi_1(B, b)$ on $H^{2n}(\pi^{-1}(b);\mathbb{Q})$ is trivial. For classes $a, b \in H^*(E;\mathbb{Q})$ such that $A = |\pi_!(a)|$ and $B = |\pi_!(b)|$ are even, if $\pi_!(a)^k = 0$ and $\pi_!(b)^l = 0$, then $\pi_!(ab)^{(2g+1)(Ak+Bl)} = 0$.*

If $g \neq 1$, this is a consequence of [Gri13, Lemma 5.7]. The proof we present here is independent of genus.

Proof. First suppose that both A and B are at least 2. Let F be defined by the homotopy fibre sequence

$$F \longrightarrow K(\mathbb{Q}, A) \times K(\mathbb{Q}, B) \xrightarrow{\iota_A^k \times \iota_B^l} K(\mathbb{Q}, kA) \times K(\mathbb{Q}, lB)$$

where the map between the Eilenberg–MacLane spaces is given by the product of the the appropriate powers of the fundamental cohomology classes. If A and B are even it is an elementary calculation to see that

$$H^*(F;\mathbb{Q}) = \mathbb{Q}[\iota_A, \iota_B]/(\iota_A^k, \iota_B^l),$$

and that F is also simply-connected. Thus F has $\mathbb{Q}$-cohomological dimension $Ak + Bl - 1$.

As $\pi_!(a)^k = 0$ and $\pi_!(b)^l = 0$, the map

$$(\pi_!(a), \pi_!(b)) : B \longrightarrow K(\mathbb{Q}, A) \times K(\mathbb{Q}, B)$$

representing $(\pi_!(a), \pi_!(b))$ lifts to a map $f : B \to F$, and we let B' denote the homotopy fibre of f, and $i : B' \to B$ the evaluation map. The class $i^*(\pi_!(ab))$ is the same as $\pi'_!(a'b')$, where $\pi' : E' \to B'$ is the pullback of π along i, and a' and b' are the pullbacks of a and b along the map $E' \to E$ covering i. The classes $\pi'_!(a')$ and $\pi'_!(b')$ are trivial, so by [Gri13, Theorem 2.7] we have $\pi'_!(a'b')^{2g+1} = 0$. Thus $\pi_!(ab)^{2g+1}$ lies in the kernel of i^*, so has positive Serre filtration with respect to the map f. Thus $\pi_!(ab)^{(2g+1)(Ak+Bl)}$ has Serre filtration at least $Ak + Bl$, which is beyond the $\mathbb{Q}$-cohomological dimension of F, so it is zero as required.

Finally, if $A = 0$ and $B = 0$ then this becomes [Gri13, Theorem 2.7], which leaves the case $A = 0$ and $B \geqslant 2$. In this case we must have $\pi_!(a) = 0$, and the above argument can be followed using the homotopy fibre of $\iota_B^l : K(\mathbb{Q}, B) \to K(\mathbb{Q}, Bl)$ in the place of F. ☐

Let us also recall some further results from [Gri13], and a convenient relation between $R^*(W_g)$ and $R^*(W_g, \star)$. We shall often write $\chi = \chi(W_g) = 2 - 2g$ for the Euler characteristic of W_g.

LEMMA 3.2. *Let n be odd.*

(i) For any $c \in \mathcal{B}$, we have

$$\chi^2 c - \chi\kappa_{ec} - \chi e\kappa_c + \kappa_{e^2}\kappa_c = 0 \in R^*(M_g, \star)/\sqrt{0}. \tag{1}$$

(ii) We have

$$(\chi - 2)\chi e + \kappa_{e^2} = 0 \in R^*(M_g, \star)/\sqrt{0}. \tag{2}$$

(iii) If $g \neq 1$ then the map

$$R^*(W_g) \longrightarrow R^*(W_g, \star)$$

is injective.

Proof. Item (i) is Example 5.19 of [Gri13], but taking $a = \chi\nu_{(1*)} - e_{(*)}$ and $b = \chi p_{(*)} - e_{(*)}\kappa_p$ to avoid dividing by χ. Item (ii) is Example 5.17 of [Gri13].

For item (iii), suppose that $x \in R^*(W_g)$ is a tautological class which vanishes in $R^*(W_g, \star)$, and let $\pi : E \to B$ be a fibre bundle with fibre W_g. Then the pullback fibre bundle $\pi' : \pi^*E \to E$ is a fibre bundle with fibre W_g and canonical section, so $\pi^*(x(\pi)) = 0$. But then

$$(2 - 2g) \cdot x(\pi) = \pi_!(e(T_\pi) \cdot \pi^*(x(\pi))) = 0$$

and $2 - 2g \neq 0$, so $x(\pi) = 0$. This holds for any fibre bundle π, so $x = 0 \in R^*(W_g)$. ☐

PROPOSITION 3.3. *Let n be odd, $I = (i_1, i_2, \ldots, i_n)$ be a sequence, $p_I = p_1^{i_1} p_2^{i_2} \cdots p_n^{i_n}$ be the associated monomial in the Pontrjagin classes, and write $|I| = \sum_{j=1}^n i_j$.*

(i) The class κ_{p_I} is nilpotent in $R^*(W_g)$.

(ii) We have

$$\chi^{|I|} \cdot \kappa_{ep_I} = \prod_{j=1}^{n} \kappa_{ep_j}^{i_j} \in R^*(W_g)/\sqrt{0}.$$

(iii) If $g > 1$ then the class e is nilpotent in $R^*(W_g, \star)$.

(iv) If $g \geqslant 1$ then for all $k > 1$ the class κ_{e^k} is nilpotent in $R^*(W_g)$.

In Proposition 5.5 we will show that $e \in R^*(W_1, \star)$ is also nilpotent, though the proof is very different to that of this proposition.

Proof.

(i) By Theorem 2.1, the class $\kappa_{\tilde{\mathcal{L}}_i}$ is nilpotent for all i. Therefore, for any monomial $\tilde{\mathcal{L}}_J = \tilde{\mathcal{L}}_1^{j_1}\tilde{\mathcal{L}}_2^{j_2}\cdots\tilde{\mathcal{L}}_n^{j_n}$, the class $\kappa_{\tilde{\mathcal{L}}_J}$ is nilpotent by Theorem 3.1. As the $\tilde{\mathcal{L}}_i$ and the p_i generate the same subring of $H^*(BSO(2n);\mathbb{Q})$, it follows that any monomial p_I may be written as a polynomial in the $\tilde{\mathcal{L}}_i$, and hence that each class κ_{p_I} is nilpotent.

(ii) Let $\pi : E \to B$ be a fibre bundle with fibre W_g, and for a monomial p_J in Pontrjagin classes let us write

$$(p_J)' = \chi e p_J - e\pi^*(\kappa_{ep_J}).$$

We compute that

$$\kappa_{(p_J)'p_i} = \pi_!(\chi e p_i p_J - e p_i \pi^*(\kappa_{ep_J})) = \chi\kappa_{ep_ip_J} - \kappa_{ep_i}\kappa_{ep_J}$$

and that $\kappa_{(p_J)'} = 0$ because $\pi_!(e) = \chi$. By part (i) the class κ_{p_i} is nilpotent, so applying Theorem 3.1 it follows that $\kappa_{(p_J)'p_i}$ is nilpotent. So $\chi\kappa_{ep_ip_J} = \kappa_{ep_i}\kappa_{ep_J}$ modulo nilpotents. These identities hold for any such fibre bundle, so hold in $R^*(W_g)$, which by induction proves (ii).

(iii) By (i), the class $\kappa_{e^2} = \kappa_{p_n}$ is nilpotent, so by equation (2) it follows that $e \in R^*(W_g, \star)$ is also nilpotent as long as $g > 1$.

(iv) Suppose first that $g = 1$, so that $\kappa_e = 0$. Then as

$$\kappa_{e^{2l}} = \kappa_{p_n^l} \quad \text{and} \quad \kappa_{e^{2l+1}} = \kappa_{ep_n^l}$$

it follows from (i) that the former are nilpotent, and then from Theorem 3.1 that the latter are too.

For $g > 1$ the class κ_e is not itself nilpotent. However, writing

$$\kappa_{e^{2l}} = \kappa_{p_n^l} \quad \text{and} \quad \kappa_{e^{2l+1}} = \kappa_{e^3p_n^{l-1}}$$

we see that the same argument will go through as soon as we show that κ_{e^3} is nilpotent.

Applying equation (1) with $c = p_n = e^2$ gives that

$$\chi^2 p_n - \chi\kappa_{ep_n} - \chi e\kappa_{p_n} + \kappa_{e^2}\kappa_{p_n} = 0 \in R^*(M_g, \star)/\sqrt{0}.$$

The class κ_{p_n} is nilpotent by (i), and so we find that $\kappa_{ep_n} = \chi p_n \in R^*(W_g, \star)/\sqrt{0}$. By (iii) the class $p_n = e^2$ is nilpotent, and therefore so is $\kappa_{ep_n} = \kappa_{e^3}$ in the ring $R^*(W_g, \star)$. By Lemma 3.2 (iii) the natural ring map $R^*(W_g) \to R^*(W_g, \star)$ is injective, so κ_{e^3} is nilpotent in $R^*(W_g)$ too, as required. ☐

Corollary 3.4. *Let n be odd.*

(i) $R^*(W_0)/\sqrt{0}$ is generated by $\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}$,

(ii) $R^*(W_1)/\sqrt{0} = \mathbb{Q}$,

(iii) $R^*(W_g)/\sqrt{0}$ *is generated by* $\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_{n-1}}$ *for* $g > 1$.

Proof. It follows from Proposition 3.3 (i) and (ii) that for any g the ring $R^*(W_g)/\sqrt{0}$ is generated by the elements $\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}$.

For $g = 1$ we have $\kappa_e = 0$. It follows from Theorem 3.1 and Proposition 3.3 (i) that the κ_{ep_i} are nilpotent, and so $R^*(W_1)/\sqrt{0} = \mathbb{Q}$.

For $g > 1$ it follows from Proposition 3.3 (iv) that $\kappa_{ep_n} = \kappa_{e^3}$ is nilpotent. So $R^*(W_g)/\sqrt{0}$ is generated by the elements $\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_{n-1}}$. ☐

Corollary 3.5. *Let n be odd and $c \in \mathcal{B}$, then $\chi c = \kappa_{ec} \in R^*(W_g, \star)/\sqrt{0}$.*

Proof. If $g = 1$ then $\chi c = 0$, but also κ_{ec} is nilpotent, by Corollary 3.4 (ii). If $g \neq 1$ then $\chi \neq 0$, and by equation (1) we have

$$\chi^2 c - \chi\kappa_{ec} - \chi e \kappa_c + \kappa_{e^2}\kappa_c = 0 \in R^*(W_g, \star)/\sqrt{0}.$$

The class $\kappa_{e^2} = \kappa_{p_n}$ is nilpotent by Proposition 3.3 (i), and the class e is nilpotent by Proposition 3.3 (iii), this gives the required equation. ☐

Proof of Theorem 1.2 (i) and (ii). Part (i) is Corollary 3.5. For part (ii), note that if $g \neq 1$ it follows that the map

$$R^*(W_g)/\sqrt{0} \longrightarrow R^*(W_g, \star)/\sqrt{0}$$

is surjective, and under the same condition this map is injective by Lemma 3.2 (iii). ☐

In order to prove Theorem 1.1, we must show that the generators of the rings $R^*(W_g)/\sqrt{0}$ given in Corollary 3.4 are algebraically independent. That is the subject of the next section.

4. Algebraic independence

In this section, we finish the proof of Theorem 1.1 by showing the following.

Theorem 4.1. *Let n be either odd or even, and set $\epsilon = 1$ if n is odd and $\epsilon = 0$ if n is even. Then*

(i) the map $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}] \to R^*(W_0)/\sqrt{0}$ *is injective,*

(ii) the map $\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_{n-\epsilon}}] \to R^*(W_g)/\sqrt{0}$ *is injective for* $g > 1$.

When n is odd, we have shown in Corollary 3.4 that these maps are both surjective, and so Theorem 1.1 follows immediately. We first explain the proof of this theorem in case (i), as it is a simpler instance of the strategy we shall use to prove case (ii). Consider the fibre bundle

$$S^{2n} \longrightarrow BSO(2n) \xrightarrow{\pi} BSO(2n+1),$$

which defines a ring homomorphism $R^*(W_0) \to H^*(BSO(2n+1); \mathbb{Q})$. The target is the ring $\mathbb{Q}[p_1, p_2, \ldots, p_n]$ which contains no nilpotent elements, so the nilradical is in the kernel of this homomorphism. We may therefore consider the composition

$$\mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}] \longrightarrow R^*(W_0)/\sqrt{0} \xrightarrow{\phi} H^*(BSO(2n+1); \mathbb{Q}).$$

The fibre bundle π arises as the unit sphere bundle of the tautological bundle $\gamma_{2n+1} \to BSO(2n+1)$, so there is a bundle isomorphism $\epsilon^1 \oplus T_\pi \cong \pi^*\gamma_{2n+1}$, and so $p_i(T_\pi) = \pi^*(p_i)$. Thus we compute

$$\phi(\kappa_{ep_i}) = \pi_!(e(T_\pi) \cdot \pi^*(p_i)) = \pi_!(e(T_\pi)) \cdot p_i = 2 \cdot p_i$$

(as $\chi(S^{2n}) = 2$) so the composition is injective, which proves (i).

Remark 4.2. This fibre bundle also shows that Proposition 3.3 (iii) and (iv) cannot be improved in the case $g = 0$. Indeed, the vertical tangent bundle for the fibre bundle above is easily seen to be the tautological $2n$-dimensional bundle over $BSO(2n)$, so that $e(T_\pi) = e \in H^{2n}(BSO(2n);\mathbb{Q})$, which is not nilpotent. Furthermore, $\kappa_{e^{2k+1}} = \kappa_{ep_n^k} = 2 \cdot p_n^k \in H^*(BSO(2n+1);\mathbb{Q})$ is not nilpotent either.

Our strategy for proving (ii) will be similar, and we begin by constructing a fibre bundle with fibre W_g and base the classifying space of a Lie group.

4.1 Constructing the Lie group action

Let G be the Lie group $SO(n) \times SO(n)$. An action of G on the manifold W_g gives rise to a smooth fibre bundle $E \to BG$ with fiber W_g. We will construct an example of such an action such that the characteristic classes κ_{ep_i} with $1 \leqslant i \leqslant d - \epsilon$ are algebraically independent in $H^*(BG;\mathbb{Q})$. In our construction, W_g will appear as a boundary of another G-invariant manifold.

The standard n-dimensional representation of $SO(n)$ gives rise to two n-dimensional representations of $G = SO(n) \times SO(n)$. For $i = 1, 2$, we let V_i be the representation of G where the ith copy of $SO(n)$ acts in the standard way and the other copy acts trivially. Let $\mathbb{R}$ denote the trivial 1-dimensional representation.

Proposition 4.3. *The G-representation $W = V_1 \oplus V_2 \oplus \mathbb{R}$ contains an embedded compact smooth manifold with boundary $H_g \subset W$ enjoying the following properties:*

- *(i)* $\dim H_g = \dim W = 2n + 1$,
- *(ii)* *H_g is G-invariant, so ∂H_g is also G-invariant,*
- *(iii)* *∂H_g is a smooth manifold diffeomorphic to W_g.*

Proof. We first illustrate the idea behind the construction of H_g by presenting a construction that compromises on the smoothness conditions, but is correct up to homeomorphism.

Let $B(V_1, 1)$ and $B(V_2 \oplus \mathbb{R}, 1)$ denote the unit balls in the respective representations. For any $g \in \mathbb{N}$ and a sufficiently small $\varepsilon > 0$, it is possible to embed g balls of radius ε inside $B(V_2 \oplus \mathbb{R}, 1)$ as disjoint and $SO(n)$-invariant subspaces. We can thus define the following subspaces, both invariant under the G-action (X_g is pictured on Figure 1).

$$\begin{aligned} X_g &= B(V_2 \oplus \mathbb{R}, 1) \backslash \left(\bigsqcup_g B(V_2 \oplus \mathbb{R}, \varepsilon) \right) \subset V_2 \oplus \mathbb{R} \\ H'_g &= B(V_1, 1) \times X_g \subset V_1 \oplus V_2 \oplus \mathbb{R}. \end{aligned}$$

The manifold $H'_g \subset W$ has codimension 0 and is G-invariant. Moreover, $\partial H'_g$ is by definition the boundary of a $(2n+1)$-manifold obtained by attaching g trivial unlinked n-handles to D^{2n+1}, so is homeomorphic to W_g.

The construction so far does not prove the proposition, as $\partial H'_g$ is not a smooth submanifold of W. We would like to smooth out the corners of the manifold H'_g to obtain a manifold H_g that is again G-invariant, but has smooth boundary. The boundary of H'_g is not smooth at the set $\partial B(V_1, 1) \times \partial X_g \subset \partial H_g$.

The Riemannian metrics on $B(V_1, 1)$ and X_g induced by those of V_1 and $V_2 \oplus \mathbb{R}$ are $SO(n)$-invariant. Choosing inwards-pointing, nowhere-vanishing, $SO(n)$-invariant vector fields on $\partial B(V_1, 1)$ and ∂X_g (which may be achieved by choosing an inwards-pointing, nowhere-vanishing

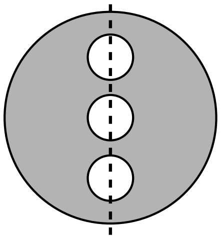

Figure 1: A cross-section of $X_g \subset \mathbb{R}^3$ for $(n, g) = (2, 3)$. $SO(2)$ acts by rotating about the axis.

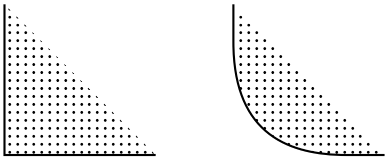

Figure 2: The corner C and the rounded corner S.

vector field, and then averaging over $SO(n)$) and integrating them, we may find a G-invariant neighbourhood

$$U' = \partial B(V_1, 1) \times \partial X_g \times C \hookrightarrow V_1 \oplus V_2 \oplus \mathbb{R}$$

where $C = \{(x, y) \in \mathbb{R}^2 \mid x, y \geqslant 0\}$ and has the trivial G-action.

Let $S \subset C$ be a strictly convex subset of C that agrees with C in a complement of a compact set around the origin, but has a smooth boundary diffeomorphic to $\mathbb{R}$ (see Figure 2). We replace U' with its (G-invariant) subset $U = \partial B(V_1, 1) \times \partial X_g \times S$, to obtain H_g. It clearly satisfies conditions (i) and (ii) of Proposition 4.3. Finally, ∂H_g is by definition the boundary of a smooth $(2n+1)$-manifold obtained by attaching g trivial unlinked n-handles to D^{2n+1} and smoothing corners, so is diffeomorphic to W_g. ☐

4.2 Proof of Theorem 4.1 (ii)

Let us write

$$W_g \longrightarrow E \xrightarrow{\pi} BSO(n) \times BSO(n)$$

for the fibre bundle of Proposition 4.3, which, as $H^*(BSO(n) \times BSO(n); \mathbb{Q})$ has no nilpotent elements, gives a ring homomorphism

$$\phi : R^*(W_g)/\sqrt{0} \longrightarrow H^*(BSO(n) \times BSO(n); \mathbb{Q}).$$

Theorem 4.1 (ii) will follow if we show that $\phi(\kappa_{ep_1}), \phi(\kappa_{ep_2}), \ldots, \phi(\kappa_{ep_{n-\epsilon}})$ are algebraically independent.

By Proposition 4.3, the fibre bundle π arises as a codimension 1 subbundle of the vector bundle $V_1 \oplus V_2 \oplus \mathbb{R}$. We thus obtain a bundle isomorphism $\mathbb{R} \oplus T_\pi \cong \pi^*(V_1 \oplus V_2 \oplus \mathbb{R})$, and hence $p_i(T_\pi) = \pi^*(p_i(V_1 \oplus V_2))$. Thus

$$\phi(\kappa_{ep_i}) = \pi_!(e(T_\pi)) \cdot p_i(V_1 \oplus V_2) = \chi \cdot p_i(V_1 \oplus V_2).$$

As $\chi \neq 0$, because we have assumed $g > 1$, the following lemma finishes the proof.

Lemma 4.4. *The classes $p_i(V_1 \oplus V_2)$ for $1 \leqslant i \leqslant n-\epsilon$ are algebraically independent in $H^*(BSO(n) \times BSO(n); \mathbb{Q})$.*

Proof. Recall from [MS74, Theorem 15.21] that in terms of the Pontrjagin and Euler classes of the tautological bundle over $BSO(d)$, we have

$$H^*(BSO(d); \mathbb{Q}) = \begin{cases} \mathbb{Q}[p_1, \ldots, p_{\frac{d-1}{2}}] & d \text{ odd} \\ \mathbb{Q}[p_1, \ldots, p_{\frac{d}{2}-1}, e] & d \text{ even.} \end{cases}$$

Moreover, if d is even then $p_{\frac{d}{2}} = e^2$. In all cases, if $i > \frac{d-\epsilon}{2}$ then $p_i = 0$.

The block sum map $s : SO(n) \times SO(n) \to SO(2n)$ gives a map

$$Bs^* : R = H^*(BSO(2n); \mathbb{Q}) \longrightarrow S = H^*(BSO(n) \times BSO(n); \mathbb{Q})$$

on cohomology, and the claim is that it is injective when restricted to the subalgbra $\mathbb{Q}[p_1, p_2 \ldots, p_{n-\epsilon}]$. As the block sum map is faithful, it follows from a theorem of Venkov [Ven59] that S is finitely-generated as a module over R. By the above: S is a polynomial ring on $(n-1)$ generators if n is odd, or n generators if n is even; R is a polynomial algebra on n generators.

We shall use the following lemma from commutative algebra. If $f : U \to V$ is a finite morphism between polynomial rings on the same (finite) number of generators, then it is injective. If it were not, then $\mathrm{Ker}(f)$ would be a proper prime ideal (because V, and hence $\mathrm{Im}(f)$, is an integral domain) so $U/\mathrm{Ker}(f)$ would have strictly smaller Krull dimension than U. But as V is also finite over $U/\mathrm{Ker}(f)$ it too would have strictly smaller Krull dimension than U, a contradiction.

If n is even then $Bs^* : R \to S$ is a finite morphism between polynomial rings on the same number of generators, so is injective. If n is odd, note that $e \in H^{2n}(BSO(2n); \mathbb{Q}) = R$ lies in the kernel of Bs^* (as $V_1 \oplus V_2$ has trivial Euler class), so instead $R/(e) \to S$ is a finite morphism between polynomial rings on the same number of generators, so is injective. In either case, the subalgebra $\mathbb{Q}[p_1, p_2 \ldots, p_{n-\epsilon}]$ of $H^*(BSO(2n); \mathbb{Q})$ injects under the block sum map. □

5. Addenda

5.1 Classifying spaces

We have defined the tautological ring $R^*(M)$ as a quotient of the abstract polynomial ring $\mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]$, but it may also be described as a subring of the cohomology of the classifying space $B\,\mathrm{Diff}^+(M)$ of the group of orientation-preserving diffeomorphisms of M.

Precisely, if $\pi : E \to B$ is equipped with the structure of a smooth oriented numerable fibre bundle with fibre M^d, though without assuming that E or B are themselves oriented compact smooth manifolds, then we still have a vertical tangent bundle T_π and a cohomological pushforward $\pi_!(-)$, so we may still define

$$\kappa_c(\pi) = \pi_!(c(T_\pi)) \in H^*(B; \mathbb{Q}).$$

The projection map

$$E\,\mathrm{Diff}^+(M) \times_{\mathrm{Diff}^+(M)} M \longrightarrow B\,\mathrm{Diff}^+(M)$$

is the universal example of such a fibre bundle, so there are universal classes $\kappa_c \in H^*(B\,\mathrm{Diff}^+(M); \mathbb{Q})$.

Lemma 5.1. *The ring homomorphism $\mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}] \to H^*(B\,\mathrm{Diff}^+(M); \mathbb{Q})$ has kernel I_M, and hence image isomorphic to $R^*(M)$.*

Proof. The only point which needs discussion is that we defined I_M in terms of the collection of all smooth fibre bundles over compact smooth oriented manifolds (in other words, proper submersions $\pi : E^{k+d} \to B^k$), and $B\,\mathrm{Diff}^+(M)$ is not one.

Let us write I'_M for the kernel of this ring homomorphism. As $B\,\mathrm{Diff}^+(M)$ carries the universal smooth oriented fibre bundle with fibre M, any class in I'_M must be trivial when evaluated on any proper submersion $\pi : E^{k+d} \to B^k$, so $I'_M \subset I_M$. The reverse inclusion holds because a rational cohomology class is zero if and only if it evaluates to zero on every rational homology class, and every rational homology class is represented by a map from a smooth (stably framed) manifold. □

The analogous argument identifies $R^*(M, \star)$ with a subring of the cohomology of $B\operatorname{Diff}^+(M, \star)$, the classifying space of the group of orientation preserving diffeomorphisms of M which fix a point $\star \in M$, and identifies $R^*(M, D^d)$ with a subring of the cohomology of $B\operatorname{Diff}^+(M, D^d)$, the classifying space of the group of orientation preserving diffeomorphisms of M which fix a disc $D^d \subset M$

5.2 Other classes of fibre bundles

We have defined the tautological ring $R^*(M) = \mathbb{Q}[\kappa_c \,|\, c \in \mathcal{B}]/I_M$ in terms of the ideal I_M of polynomials in the κ_c which vanish when evaluated on every smooth oriented fibre bundle with fibre M. By varying this condition, we may form the following related tautological rings:

(i) $R^*_{\text{torelli}}(M)$, defined in terms of the ideal of polynomials in the κ_c which vanish on every smooth oriented fibre bundle with fibre M and homologically trivial action of the fundamental group of the base,

(ii) $R^*_0(M)$, defined in terms of the ideal of polynomials in the κ_c which vanish on every smooth oriented fibre bundle with fibre M and isotopically trivial action of the fundamental group of the base.

These are related by surjective ring homomorphisms

$$R^*(M) \longrightarrow R^*_{\text{torelli}}(M) \longrightarrow R^*_0(M).$$

It follows from our results that for $M = W_g$ and n odd, these maps all become isomorphisms after dividing out each ring by its nilradical: Sections 2 and 3 give relations in $R^*(W_g)/\sqrt{0}$ whereas Section 4 shows algebraic independence of classes in $R^*_0(W_g)/\sqrt{0}$. In fact, this is true quite generally.

Proposition 5.2. *Let M be a simply-connected manifold of dimension $d > 5$. Then*

$$R^*(M)/\sqrt{0} \longrightarrow R^*_{\mathit{torelli}}(M)/\sqrt{0} \longrightarrow R^*_0(M)/\sqrt{0}$$

are isomorphisms. The same holds for $(M, \star)$ and (M, D^d).

Proof. Both maps are epimorphisms by definition, so it is enough to show that the composition is an isomorphism. Sullivan has shown [Sul77, Theorem 13.3] that under the stated conditions the mapping class group $\Gamma_M = \pi_0(\operatorname{Diff}^+(M))$ is commensurable to an arithmetic group. In particular, it has finite virtual cohomological dimension, and so finite $\mathbb{Q}$-cohomological dimension.

Letting $\operatorname{Diff}^+_0(M)$ denote the path component of the identity, there is a fibration sequence

$$B\operatorname{Diff}^+_0(M) \xrightarrow{i} B\operatorname{Diff}^+(M) \xrightarrow{p} B\Gamma_M$$

and an associated Serre spectral sequence. We then proceed much as in the proof of Theorem 3.1: any class in

$$\operatorname{Ker}(i^* : H^*(B\operatorname{Diff}^+(M);\mathbb{Q}) \to H^*(B\operatorname{Diff}^+_0(M);\mathbb{Q}))$$

has positive Serre filtration, so some power of it has Serre filtration beyond the $\mathbb{Q}$-cohomological dimension of $B\Gamma_M$. Such a power is therefore zero, showing that $\operatorname{Ker}(i^*)$ consists of nilpotent elements, and hence that the map

$$i^* : H^*(B\operatorname{Diff}^+(M);\mathbb{Q})/\sqrt{0} \longrightarrow H^*(B\operatorname{Diff}^+_0(M);\mathbb{Q})/\sqrt{0}$$

is injective. The result now follows by the discussion of classifying spaces in the previous section.

For $(M, \star)$ and (M, D^d) the argument is identical, but using the relative mapping class groups of these pairs. For M simply-connected the natural map $\Gamma_{(M,\star)} \to \Gamma_M$ is an isomorphism, as may be seen by the fibration sequence $M \to B\,\mathrm{Diff}^+(M, \star) \to B\,\mathrm{Diff}^+(M)$. The fibration $SO(d) \to B\,\mathrm{Diff}^+(M, D^d) \to B\,\mathrm{Diff}^+(M, \star)$ shows that $\Gamma_{(M,D^d)} \to \Gamma_{(M,\star)}$ is onto with kernel of order at most 2. Thus $\Gamma_{(M,D^d)}$ still has finite $\mathbb{Q}$-cohomological dimension. ☐

5.3 Genus zero

For the manifold $W_0 = S^{2n}$, we can improve Theorem 1.1 to say that

$$R^*(W_0) = \mathbb{Q}[\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}],$$

that is, we do not need to divide by the nilradical, nor need a condition on the parity of n. Let $\pi : E \to B$ be a fibre bundle with fibre S^{2n}. We use the following two strengthenings of our results.

Lemma 5.3. $\kappa_{\tilde{\mathcal{L}}_i}(\pi) = 0 \in H^*(B; \mathbb{Q})$.

Proof. Observe that in the proof of Theorem 2.1, the group $\mathrm{Aut}(H, \lambda)$ is trivial when $g = 0$. Therefore, the classes $\kappa_{\tilde{\mathcal{L}}_i}(\pi)$ must be pulled back through the cohomology of the contractible space $B\mathrm{Aut}(H, \lambda)$. ☐

Lemma 5.4. *If $p, q \in H^*(E; \mathbb{Q})$ satisfy $\pi_!(p) = 0$ and $\pi_!(q) = 0$, then $\pi_!(pq) = 0$.*

Proof. It follows from the Gysin sequence for π that there is a $p' \in H^*(B; \mathbb{Q})$ such that $p = \pi^*(p')$, but then $\pi_!(pq) = \pi_!\,(\pi^*(p')q) = p' \cdot \pi_!(q) = 0$. ☐

It follows from Lemma 5.3 that $\kappa_{\tilde{\mathcal{L}}_i} = 0 \in R^*(W_0)$, and so, by Lemma 5.4, $\kappa_{\tilde{\mathcal{L}}_J} = 0 \in R^*(W_0)$ for any monomial $\tilde{\mathcal{L}}_J$. As these give a basis for the subring of $H^*(BSO(2n); \mathbb{Q})$ generated by Pontrjagin classes, we have that $\kappa_{p_I} = 0$ too. For a fibre bundle $\pi : E \to B$ with fibre S^{2n}, by the Gysin sequence we therefore have $p_I = \pi^*((p_I)')$ for some $(p_I)' \in H^*(B; \mathbb{Q})$, and so

$$\kappa_{ep_I}(\pi) = \pi_!(e(T_\pi)\pi^*((p_I)')) = \pi_!(e(T_\pi)) \cdot (p_I)' = 2(p_I)'.$$

In particular, we may express $\kappa_{ep_I}(\pi)$ by a universal formula in the $p'_i = \frac{1}{2}\kappa_{ep_i}(\pi)$, so $\kappa_{ep_1}, \kappa_{ep_2}, \ldots, \kappa_{ep_n}$ generate $R^*(W_0)$. They are algebraically independent in this ring as we have already show that they are algebraically independent in $R^*(W_0)/\sqrt{0}$.

5.4 Genus one

We have two outstanding claims to prove in the case $g = 1$. Firstly, Theorem 1.2 (iii), and secondly that $R^*(W_1, \star)$ is finitely-generated, so that $R^*(W_1, D^{2n})$ is too and we can deduce Corollary 1.4 from Corollary 1.3.

5.4.1 *The Euler class* We first prove the analogue of Proposition 3.3 (iii) for $g = 1$.

Proposition 5.5. *The class $e \in R^*(W_1, \star)$ is nilpotent.*

Our proof of this proposition will use the following, which would seem to be quite generally useful in the study of tautological rings.

Theorem 5.6. *Let B be a finite CW-complex, let $\pi : E \to B$ and $\pi' : E' \to B$ be smooth fibre bundles with closed d-manifold fibres, and let $f : E \to E'$ be a map over B which is a fibre homotopy equivalence. Then $f^*(e(T_{\pi'})) = e(T_\pi) \in H^d(E; \mathbb{Z})$.*

Proof. We refer to [BG76] for background on the following, and to [MS06, Chapter 15] for technical details of fibrewise Spanier–Whitehead duality. For a space X over B, let $X_+ = X \sqcup B$ be the associated ex-space, and $\Sigma^\infty_B X_+$ denote the fibrewise suspension spectrum. By [BG76, Theorem 4.7] or [MS06, Theorem 15.1.1] the parameterised spectrum $\Sigma^\infty_B E_+$ is dualisable, and in fact its dual can be made rather explicit. Choose a smooth embedding $E \subset B \times \mathbb{R}^N$ with tubular neighbourhood U and projection $p : U \to E$, and let U^+_B denote the 1-point compactification of U formed fibrewise over B, so there is a canonical section $s_\infty : B \to U^+_B$ given by the points at ∞. There is then a map

$$\begin{aligned} B \times S^N &\longrightarrow E_+ \wedge_B U^+_B \\ (b,x) &\longmapsto \begin{cases} s(b) \wedge_B s_\infty(b) & (b,x) \notin U \\ p(b,x) \wedge_B (b,x) & (b,x) \in U \end{cases} \end{aligned}$$

giving a map

$$\mu_E : \Sigma^\infty_B B_+ \longrightarrow \Sigma^\infty_B E_+ \wedge_B \Sigma^{\infty-N}_B U^+_B$$

of parameterised spectra which exhibits $\Sigma^\infty_B E_+$ and $\Sigma^{\infty-N}_B U^+_B$ as Spanier–Whitehead dual. There is therefore a complementary duality map

$$\hat{\mu}_E : \Sigma^{\infty-N}_B U^+_B \wedge_B \Sigma^\infty_B E_+ \longrightarrow \Sigma^\infty_B B_+$$

and the composition

$$\begin{aligned} \Sigma^\infty_B B_+ \xrightarrow{\mu_E} \Sigma^\infty_B E_+ \wedge_B \Sigma^{\infty-N}_B U^+_B &\xrightarrow{\cong} \Sigma^{\infty-N}_B U^+_B \wedge_B \Sigma^\infty_B E_+ \\ &\xrightarrow{\mathrm{Id}\wedge_B \Delta} \Sigma^{\infty-N}_B U^+_B \wedge_B \Sigma^\infty_B E_+ \wedge_B \Sigma^\infty_B E_+ \\ &\xrightarrow{\hat{\mu}_E \wedge_B \mathrm{Id}} \Sigma^\infty_B B_+ \wedge_B \Sigma^\infty_B E_+ \xrightarrow{\simeq} \Sigma^\infty_B E_+ \end{aligned}$$

is a lift of the Becker–Gottlieb transfer of π to parameterised spectra. Dualising this map gives a map of parameterised spectra

$$\epsilon_E : \Sigma^{\infty-N}_B U^+_B \longrightarrow \Sigma^\infty_B B_+$$

and so, base changing along $B \to \{*\}$ and then collapsing B to a point, we obtain a map of spectra

$$c : \Sigma^{\infty-N} U^+ \longrightarrow \Sigma^\infty B_+ \longrightarrow \Sigma^\infty S^0.$$

On the other hand, U is homeomorphic to the normal bundle ν_π of E in $B \times \mathbb{R}^N$, so U^+ is homeomorphic to the Thom space $\mathrm{Th}(\nu_\pi)$. The pullback along c of the canonical cohomology class in $H^0(\Sigma^\infty S^0;\mathbb{Z})$, followed by the Thom isomorphism (as T_π, and hence ν_π, is oriented), therefore gives a class in $H^d(E;\mathbb{Z})$. It follows from the Poincaré–Hopf theorem that this is $e(T_\pi)$.

The analogous construction, using a tubular neighbourhood $E' \subset V \subset B \times \mathbb{R}^N$, describes $e(T_{\pi'})$. The point is that the above construction used only fibrewise Spanier–Whitehead duality, and so if $D(f_+) : \Sigma^{\infty-N}_B V^+_B \to \Sigma^{\infty-N}_B U^+_B$ is the dual of f_+ then we have

$$\epsilon_E \circ D(f_+) \simeq \epsilon_{E'} : \Sigma^{\infty-N}_B V^+_B \longrightarrow \Sigma^\infty_B B_+$$

and so, as under the Thom isomorphism $D(f_+)^*$ induces the map $(f^*)^{-1} : H^*(E;\mathbb{Z}) \to H^*(E';\mathbb{Z})$, we have $f^*(e(T_{\pi'})) = e(T_\pi)$. ☐

The arguments of this proof can be refined to *define* an Euler class of the vertical tangent bundle, $e(T_\pi) \in H^d(E;\mathbb{Z})$, when $\pi : E \to B$ is a fibration whose fibre is a Poincaré duality space of dimension d. In particular, for such a Poincaré duality space M^d one can define the universal such class $e \in H^d(B\mathrm{hAut}^+(M,\star);\mathbb{Z})$.

Proof of Proposition 5.5. Let $\pi : E^{k+2n} \to B^k$ be a homologically trivial smooth oriented fibre bundle with fibre W_1, and let $s : B \to E$ be a section. We will show that $s^*(e(T_\pi)) = 0 \in H^{2n}(B;\mathbb{Q})$, so that $e = 0 \in R^*_{\mathrm{torelli}}(W_1, \star)$. Hence, by Proposition 5.2, e is nilpotent in $R^*(W_1, \star)$.

Under the stated assumptions the Serre spectral sequence for π has a product structure and collapses, giving classes $a, b \in H^n(E;\mathbb{Z})$ which restrict to a basis of $H^n(W_1;\mathbb{Z})$. The obstructions to finding a lift

$$\begin{tikzcd}
 & S^n \times S^n \arrow[d] \\
E \arrow[ur, dashed] \arrow[r, "(a,b)"'] & K(\mathbb{Z},n) \times K(\mathbb{Z},n)
\end{tikzcd}$$

lie in $H^{i+1}(E;\pi_i(S^n \times S^n))$ for $n < i < k+2n$, which are all torsion groups.

Let $\cdots \to B_2 \xrightarrow{f_1} B_1 \xrightarrow{f_0} B_0 = B$ be a tower of finite k-dimensional CW-complexes in which each $f_i^* : H^*(B_i;\mathbb{Z}) \to H^*(B_{i+1};\mathbb{Z})$ annihilates all torsion but is rationally injective. (Such a tower may be formed as follows: an n-torsion class $x \in H^j(B_i;\mathbb{Z})$ arises via Bockstein from a class $x' \in H^{j-1}(B_i;\mathbb{Z}/n)$ represented by a map $g_x : B_i \to K(\mathbb{Z}/n, j-1)$, and B_{i+1} may be taken to be a k-skeleton of the homotopy fibre of the product of the maps g_x over all torsion classes $x \in H^*(B_i;\mathbb{Z})$.) We may pull back π and s to obtain homologically trivial bundles $\pi_i : E_i \to B_i$ with sections s_i. Each of these also have Serre spectral sequences which have a product structure and collapse, and they are connected by maps $\hat{f}_i : E_{i+1} \to E_i$ covering the f_i.

It follows that $\hat{f}_i^*$ always sends a torsion class to one of higher Serre filtration, and so in particular (as the Serre spectral sequence for each π_i has three rows) that the composition of three such maps annihilates any torsion class. Therefore a finite composition of such maps annihilates all obstructions for finding the dashed lift above, so for some i we have a map $t : E_i \to S^n \times S^n$ splitting the inclusion of the fibre. It follows that π_i is fibre homotopy equivalent to the trivial W_1-bundle, and hence by Theorem 5.6 we have

$$e(T_{\pi_i}) = 0 \in H^n(E_i;\mathbb{Z})$$

and so $s_i^* e(T_{\pi_i}) = 0 \in H^n(B_i;\mathbb{Z})$. The maps f_i^* were rationally injective, and so $s^*e(T_\pi) = 0 \in H^{2n}(B;\mathbb{Q})$, as required. ☐

Finally, we can prove the third part of Theorem 1.2.

Proof of Theorem 1.2 (iii). We have shown that e is nilpotent in $R^*(W_1, \star)$. Consider the fibre bundle

$$S^n \times S^n \longrightarrow BSO(n) \times BSO(n) \xrightarrow{\pi} BSO(n+1) \times BSO(n+1)$$

and let $\pi' : E \to BSO(n) \times BSO(n)$ be the fibre bundle obtained by pulling π back along itself. Then π' has a section s', and the ring $H^*(BSO(n) \times BSO(n);\mathbb{Q})$ has no nilpotent elements, so there is a ring homomorphism

$$\phi : R^*(W_1, \star)/\sqrt{0} \longrightarrow H^*(BSO(n) \times BSO(n);\mathbb{Q}).$$

There is a bundle isomorphism $(s')^*(T_{\pi'}) \cong V_1 \oplus V_2$, so $\phi(p_i) = p_i(V_1 \oplus V_2)$. It follows from Lemma 4.4 that $p_1, p_2, \ldots, p_{n-1} \subset R^*(W_1, \star)/\sqrt{0}$ are algebraically independent. ☐

5.4.2 *Finite generation* The proof in [Gri13] showing that $R^*(W_g)$ is finitely generated does not apply for $g \leqslant 1$. We have computed $R^*(W_0)$ completely, and can observe that it is finitely generated, which leaves only the case $g = 1$. In fact, we do not know whether this ring is finitely generated. However, we have the following.

Proposition 5.7. *For all $g \geqslant 0$, the ring $R^*(W_g, \star)$ is finitely generated.*

Proof. Our argument follows [Gri13, Example 5.19 and Lemma 5.20] and we shall freely use the language and notation of that paper. We work in the space $\mathcal{M}_g(\{1, 2, \star\})$ with three marked points $1, 2$, and $\star$. Let $p \in H^*(BSO(2n); \mathbb{Q})$ be even-dimensional, and define

$$a = p_{(\star)} - \nu_{(1\star)}\kappa_p \in H^*(\mathcal{M}_g(\{1, 2, \star\}); \mathbb{Q})$$
$$b = \nu_{(1\star)} - \nu_{(2\star)} \in H^*(\mathcal{M}_g(\{1, 2, \star\}); \mathbb{Q}).$$

Both of these classes push forward to zero in $H^*(\mathcal{M}_g(\{1, 2\}); \mathbb{Q})$. According to [Gri13, Theorem 2.7] we therefore have

$$0 = \left(\left(\pi^{\{1,2,\star\}}_{\{1,2\}}\right)_! \left(\left(p_{(\star)} - \nu_{(1\star)}\kappa_p\right) \left(\nu_{(1\star)} - \nu_{(2\star)}\right)\right)\right)^{2g+1} =$$
$$= \left(p_{(1)} - p_{(2)} - e_{(1)}\kappa_p + \nu_{(12)}\kappa_p\right)^{2g+1} \in H^*(\mathcal{M}_g(\{1, 2\}); \mathbb{Q}).$$

For any $q \in H^*(BSO(2n); \mathbb{Q})$ the above relation can be multiplied by $q_{(2)}$ and pushed forward to $H^*(\mathcal{M}_g(1); \mathbb{Q}))$ to obtain a relation that expresses $\kappa_{p^{2g+1}q}$ in terms of decomposable elements in the tautological subring $R^*(W_g, \star) \subset H^*(\mathcal{M}_g(1); \mathbb{Q})$. (Recall that this ring is generated by the κ_c as well as c for $c \in H^*(BSO(2n); \mathbb{Q})$.) As explained in [Gri13, Proof of Theorem 1.1], this is sufficient to show that the ring is finitely generated. ☐

It follows that the ring $R^*(W_g, D^{2n})$ is also finitely generated.

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