K-Theory of Fermat Curves

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration. It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or a diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University or similar institution.

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Summary

I investigate the K_2 groups of the quotients of Fermat curves given in projective coordinates by the equation $F_n : X^n + Y^n = Z^n$. On any quotient where the number of known elements is equal to the rank predicted by Beilinson's Conjecture I verify numerically that the determinant of the matrix of regulator values agrees with the leading coefficient of the L-function up to a simple rational number.

The main source of K_2 elements are the so-called "symbols with divisorial support at infinity" that were found by Ross in the 1990's. These consist of symbols of the form $\{f, g\}$ where f and g have divisors whose points P all satisfy $XYZ(P) = 0$. The image of this subgroup under the regulator is computed and is found to be of rank predicted by Beilinson's Conjecture on eleven non-isomorphic quotients of dimension greater than one. The L-functions of these quotients are computed using Dokchitser's ComputeL package and Beilinson's Conjecture is verified numerically to a precision of 200 decimal digits.

In chapter five, with careful analysis of a certain 2×2 determinant it is shown that a particular hyperelliptic quotient of all the Fermat curves has K_2 group of rank at least two.

In the last chapter of the dissertation, a computational method is used in order to discover new elements of K_2. These elements are rigorously proven to be tame and allow for the full verification of Beilinson's Conjecture on the Fermat curves F_7 and F_9. Also the method allows us to verify Beilinson's Conjecture on certain hyperelliptic quotients of F_8 and F_{10}. Quotients where a similar method might be successful are also suggested.

Contents

Chapter 1

Introduction

Let k be a number field with r_1 real embeddings and r_2 complex embeddings so that $[k:\mathbb{Q}] = r_1 + 2r_2$. It is well known that if $\mathcal{O}_k$ is the ring of integers of k that $\mathcal{O}_k^\times$ is a finitely generated abelian group of rank $r = r_1 + r_2 - 1$. If $u_1, \ldots, u_r$ are a basis of $\mathcal{O}_k^\times$ modulo torsion and $\sigma_1, \ldots, \sigma_{r+1}$ are the embeddings of k into $\mathbb{C}$ up to complex conjugation then define integers N_j to equal 1 if σ_j is a real embedding and 2 otherwise. Now consider the $r \times (r+1)$ matrix

$$\begin{pmatrix} N_1 \log|\sigma_1(u_1)| & \cdots & N_{r+1} \log|\sigma_{r+1}(u_1)| \\ \vdots & & \vdots \\ N_1 \log|\sigma_1(u_r)| & \cdots & N_{r+1} \log|\sigma_{r+1}(u_r)| \end{pmatrix}$$

Since the (u_i) are units, each row of the matrix sums to zero and from this we can see that the absolute value R of the determinant of the submatrix formed by removing any single column is independent of the column chosen.

The value R, called the *regulator*, is independent of the choice of the (u_i) and we have the *class number formula* which states that

$$\mathrm{Res}_{s=1}\, \zeta_k(s) = \frac{2^{r_1}\,(2\pi)^{r_2}\, hR}{w\sqrt{\Delta}}$$

Here ζ_k is the Dedekind zeta function of the number field k, h is the class number of $\mathcal{O}_k$, w is the number of roots of unity in k and Δ is the absolute value of the discriminant of k.

Much later, Borel [5] found a generalisation of the above formula that relates the value of a certain regulator on the higher K-group $K_{2n-1}(\mathcal{O}_k)$ to the values of $\zeta_k(n)$ at integers with $n \geq 2$. Inspired by this, Bloch [4] (also see Rohrlich [20]) considered K_2 of elliptic curves over $\mathbb{Q}$ with complex multiplication and showed that there is a regulator defined on K_2E which relates to the value of $L(E,2)$. Beilinson proposed a very general extension of these ideas. We give a sketch of the full conjecture before describing the case that concerns this thesis in more detail.

1.1 Beilinson's Conjecture

Beilinson's Conjecture relates the special values of the L-functions of a proper smooth variety X to the covolume of the image of a certain regulator map [2], [3]. We denote $\Lambda(n) = (2\pi i)^n\Lambda$ whenever Λ is a subring of $\mathbb{C}$.

The *motivic cohomology group* of X is defined to be $H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) = K^{(j)}_{2j-i}(X)$. This is a certain eigenspace of $K_{2j-i}(X) \otimes \mathbb{Q}$ under the action of the Adams operators.

We define the real *Deligne cohomology* $H^i_{\mathcal{D}}(X_{\mathbb{C}}, \mathbb{R}(j))$ of X to be the cohomology of the complex

$$\mathbb{R}(j) \to \mathcal{O}_{X_{\mathbb{C}}} \to \Omega^1_{X_{\mathbb{C}}} \to \cdots \to \Omega^{j-1}_{X_{\mathbb{C}}} \to 0$$

and define $H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j))$ to be the subspace which is invariant under the action of complex conjugation on the pair $(X_{\mathbb{C}}, \Omega^{\cdot})$. See [24] for more details.

Using the Chern class map of K-theory Beilinson constructed a regulator map

$$\text{reg} : H^i_{\mathcal{M}}(X, \mathbb{Q}(j)) \to H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j))$$

and conjectured that for $i+1 < 2j$ the regulator map is injective when restricted to the "integral" part of $H^i_{\mathcal{M}}(X, \mathbb{Q}(j))$ and its image gives a $\mathbb{Q}$-structure of $H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j))$.

Further, assuming that the Hasse-Weil conjectures are true for $L(X, s)$ if necessary, he conjectured that up to a rational multiple the covolume of the image of the integral part in $H^i_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(j))$ is equal to the first non-vanishing term of $L(H^{i-1}(X), i - j)$.

1.1.1 The Conjecture for K_2 of curves over $\mathbb{Q}$

When X is a smooth projective curve defined over $\mathbb{Q}$ there is a simpler definition of the regulator map. In the case we are interested in $i = j = 2$ and we have that $H^2_{\mathcal{D}}(X_{\mathbb{R}}, \mathbb{R}(2))$ is isomorphic to $H^1(X(\mathbb{C}), \mathbb{R}(1))^+$ where the $+$ denotes the subspace fixed under the action of complex conjugation on both $X(\mathbb{C})$ and $\mathbb{R}(1)$. We will define the regulator to be a map

$$\mathrm{reg} : K_2X \to H^1(X(\mathbb{C}), \mathbb{R}(1))^+$$

and interpret the Beilinson conjecture in this case. Note that the dimension over $\mathbb{R}$ of $H^1(X(\mathbb{C}), \mathbb{R}(1))^+$ is equal to the genus of X.

When F is a field, Matsumoto's Theorem [15] gives $K_2F \cong (F^\times \otimes F^\times)/R$, where R is the subgroup of $F^\times \otimes F^\times$ generated by tensors of the form $f \otimes (1-f)$ with $f \neq 0, 1$. We denote the image of $f \otimes g$ in K_2F by $\{f, g\}$.

The localisation sequence of K-theory gives rise to the exact sequence

$$\coprod_{P \in X(\bar{\mathbb{Q}})} K_2\mathbb{Q}(P) \to K_2X \to K_2\mathbb{Q}(X) \xrightarrow{\tau} \coprod_{P \in X(\bar{\mathbb{Q}})} \mathbb{Q}(P)^\times$$

where τ is a map from $K_2\bar{\mathbb{Q}}(X)$ defined by $\tau = \coprod_{P \in X(\bar{\mathbb{Q}})} \tau_P$, with τ_P being the *tame symbol* at P:

$$\tau_P\{f, g\} = (-1)^{(\mathrm{ord}_P f)(\mathrm{ord}_P g)} \frac{f^{\mathrm{ord}_P g}}{g^{\mathrm{ord}_P f}}(P)$$

By Garland's theorem [9], K_2 of a number field is torsion and hence we have that $\ker\tau \cap K_2\mathbb{Q}(X)$ and K_2X agree up to torsion. We use this to identify $(\ker\tau \cap K_2\mathbb{Q}(X))\otimes\mathbb{Q}$ and $K_2X\otimes\mathbb{Q}$ and we will speak of an element of $K_2\mathbb{Q}(X)$ "being tame" or "being an element of K_2X" almost interchangeably.

The exposition of the regulator given here is largely based on that in [8]. Define a map from $\bar{\mathbb{Q}}(X)^\times \times \bar{\mathbb{Q}}(X)^\times$ to the group of almost everywhere defined 1-forms on the Riemann surface $X(\mathbb{C})$ by

$$\omega(a,b) = \frac{1}{2\pi i}\left(\log|a|d\arg b - \log|b|d\arg a\right) \tag{1.1}$$

where $\arg a$ is the argument of a. This is a well-defined and smooth 1-form on the complement of the set of zeros and poles of a and b. It is clear that ω induces a map on $\bar{\mathbb{Q}}(X)^\times \otimes \bar{\mathbb{Q}}(X)^\times$ and that $\omega(a,b)$ is closed since $d\omega(a,b) = \frac{1}{2\pi i}\,\mathrm{Im}\,(d\log a \wedge d\log b)$.

For any smooth 1-form η defined on the complement of a finite set $S \subset X(\mathbb{C})$ and any smooth oriented loop γ in $X(\mathbb{C}) \setminus S$, we have a pairing

$$(\gamma,\eta) = \int_\gamma \eta$$

which depends only on the homology class of γ in $X \setminus S$. As γ moves across a point x in S, the value of (γ,η) jumps by (C_x,η) where C_x denotes a small circle around x. A simple calculation shows that $(C_x, \omega(a,b)) = -i\log|T_x(\{a,b\})|$ where T_x is the tame symbol at x. It follows that if $\alpha = \sum_i a_i \otimes b_i$ is an element of $\ker\tau$ then $(\cdot,\omega(\alpha))$ is a well-defined map from $H_1(X(\mathbb{C}),\mathbb{Z})$ to $\mathbb{R}(1) = 2\pi i\mathbb{R}$.

Furthermore, from the fact that $\omega(a, 1-a) = dD(a)$ where $D(z)$ is the Bloch-Wigner dilogarithm we see that the pairing vanishes on elements of the form $a\otimes(1-a)$ and so we end up with a pairing

$$\begin{aligned} H_1(X(\mathbb{C}),\mathbb{Z}) \times \ker\tau &\longrightarrow \mathbb{R}(1) \\ (\gamma,\alpha) &\longmapsto \int_\gamma \omega(\alpha) \end{aligned}$$

Equivalently, by viewing the space $H^1(X(\mathbb{C}), \mathbb{C})$ as the space of $\mathbb{C}$-valued functionals on $H_1(X(\mathbb{C}), \mathbb{Z})$, we have the regulator map

$$\mathrm{reg} : \ker \tau \longrightarrow H^1(X(\mathbb{C}), \mathbb{R}(1))$$

When restricting to K_2X we can say slightly more. The composition of complex conjugation on $\mathbb{R}(1)$ and complex conjugation on $X(\mathbb{C})$ acts on $H^1(X(\mathbb{C}), \mathbb{R}(1))$ and we see that the image of the regulator is invariant under this action: conjugation on $\mathbb{R}(1)$ sends i to $-i$ while conjugation on $X(\mathbb{C})$ sends $d\arg$ to $-d\arg$. We denote this subspace by $H^1(X(\mathbb{C}), \mathbb{R}(1))^+$ and so we can say that the regulator is a map

$$\mathrm{reg} : K_2X \longrightarrow H^1(X(\mathbb{C}), \mathbb{R}(1))^+$$

Let $\mathcal{X}$ be a regular proper flat model of X over $\operatorname{Spec} \mathbb{Z}$. The canonical morphism $X \to \mathcal{X}$ induces a map $K_2\mathcal{X} \to K_2X$, the image of which is, up to torsion, independent of $\mathcal{X}$. Composing this with the regulator we obtain a map $K_2\mathcal{X} \to H^1(X(\mathbb{C}), \mathbb{R}(1))^+$, which we shall also denote by reg. Further, if $\operatorname{Jac} X$ has everywhere potentially good reduction, then K_2X and $K_2\mathcal{X}$ agree up to torsion.

When X is over $\mathbb{Q}$ the L-function can be defined as Dirichlet series that converges in the half-plane $\operatorname{Re} s > \frac{3}{2}$ as follows. For primes l such that the reduction of X modulo l is non-singular over $\mathbb{F}_l$ we define the local zeta function $Z(X, \mathbb{F}_l, T)$ by

$$Z(X, \mathbb{F}_l, T) = \exp\left(\sum_{n=1}^{\infty} \frac{\#X\left(\mathbb{F}_{l^n}\right)}{n} \cdot T^n\right)$$

From part of the Weil conjectures we know that if X is of genus g that $Z(X, \mathbb{F}_l, T)$ is a rational function of the form $P_l(T)/\left((1-T)(1-lT)\right)$ where P_l is a polynomial with integer coefficients having degree $2g$ and constant term equal to 1. Except for a finite number of primes the L-function of X is defined by

$$L(X, s) = \prod_l P_l(p^{-s})^{-1}$$

and we have the Hasse-Weil conjecture that predicts a functional equation for $L(X,s)$.

Conjecture 1.1.1 (Hasse-Weil)**.** *With the definitions above there exists an integer N called the conductor and an integer $w \in \{+1,-1\}$ called the root number such that the function defined by*

$$L^*(X,s) = \frac{N^{s/2}}{(2\pi)^{gs}}\Gamma(s)^g L(X,s) \tag{1.2}$$

extends to an entire function of s and satisfies $L^*(X,s) = w \cdot L^*(X,2-s)$.

In this case the Beilinson conjecture states that

1. $\mathrm{reg}(K_2\mathcal{X})$ is a lattice in $H^1(X(\mathbb{C}),\mathbb{R}(1))^+$
2. Denote the volume of a lattice L by $\Lambda^g L$ and define $c \in \mathbb{R}^\times/\mathbb{Q}^\times$ by

$$\Lambda^g\left[(\mathrm{reg}\, K_2\mathcal{X}) \otimes \mathbb{Q}\right] = c \cdot \Lambda^g H^1(X(\mathbb{C}),\mathbb{Q}(1))^+$$

   Then $c \equiv L^{(g)}(X,0) \mod \mathbb{Q}^\times$.

Note that because of the appearance of the Gamma function in equation (1.2) we would have $L(X,0) = L'(X,0) = \ldots = L^{(g-1)}(X,0) = 0$ and

$$\frac{L^{(g)}(X,0)}{g!} = \lim_{s\to 0}\frac{L(X,s)}{s^g} = L^*(X,0) = wL^*(X,2) = \frac{wN}{(2\pi)^{2g}}L(X,2) \neq 0.$$

and so we could state Beilinson's conjecture without any dependence on the conjecture of Hasse-Weil. It turns out that the L-functions of Fermat curves can be defined in terms of Hecke L-functions that have a known functional equation (see Chapter 4) and so the difference is moot for this thesis.

Chapter 2

Introducing Fermat Curves

2.1 Quotients of Fermat Curves

The nth Fermat curve is defined in projective coordinates by the equation

$$X^n + Y^n = Z^n$$

It is a non-singular curve of genus $(n-1)(n-2)/2$. Let $\zeta = e^{\frac{2\pi i}{n}}$ and $\xi = e^{\frac{\pi i n'}{n}}$ where n' is the largest odd factor of n. Let Γ_n denote a subgroup of automorphisms of F_n over $\mathbb{C}$ generated by the elements A, B, σ and η defined as follows:

$$A : (x, y) \mapsto (\zeta x, y) \tag{2.1}$$

$$B : (x, y) \mapsto (x, \zeta y) \tag{2.2}$$

$$\sigma : (x, y) \mapsto (y, x) \tag{2.3}$$

$$\eta : (x, y) \mapsto \left(\frac{1}{y}, \frac{\xi x}{y}\right) \tag{2.4}$$

Then Γ_n is isomorphic to the semi-direct product of $(\mathbb{Z}/n\mathbb{Z})^2$ and the symmetric group S_3. Each element has a unique representation of the form $A^a B^b \eta^j \sigma^\epsilon$ with $0 \le a, b < n$, $0 \le j < 3$ and $\epsilon = 0, 1$.

The subgroup Γ_n would be the same if we had chosen any ξ such that $\xi^n = -1$. The specific definition chosen above will be beneficial later since it is desirable to have the degree of ξ over $\mathbb{Q}$ as small as possible. For example, when n is odd we have $\xi = -1$ so that the map η is in fact defined over $\mathbb{Q}$. Generally, ξ is defined over the field $\mathbb{Q}(\mu_{2n/n'})$ of degree n/n'.

Define the path $\gamma : [0,1] \longrightarrow F_n(\mathbb{C})$ by

$$\gamma(t) = \left(t^{1/n}, (1-t)^{1/n}\right)$$

and the loop κ in $H_1(F_n(\mathbb{C}), \mathbb{Z})$ by $\kappa = \gamma - A\gamma + AB\gamma - B\gamma$. Then in [10] it is proven that $H_1(F_n(\mathbb{C}), \mathbb{Z})$ is a cyclic $\mathbb{Z}[A,B]$ module generated by κ. Throughout this thesis will denote $\kappa_{a,b} = A^a B^b \kappa$.

Define an *admissible triple* (r,s,t) to be a triple of positive integers such that $r+s+t = n$. Then as shown in [10] a basis of the regular differential forms on F_n is given by the elements

$$\omega_{r,s,t} = x^{r-1} y^{n-s} dx$$

where (r,s,t) is any admissible triple. The $\omega_{r,s,t}$ are eigenforms for the action of A and B on the regular differentials since it is easy to check that

$$A^a B^b \omega_{r,s,t} = \zeta^{ar+bs} \omega_{r,s,t}$$

The periods of the $\omega_{r,s,t}$ with respect to the loops $\kappa_{a,b}$ are determined in [10] as given here.

Lemma 2.1.1. *Let $B(x,y)$ be the classical beta function defined by*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt$$

then we have

$$\int_{\kappa_{a,b}} \omega_{r,s,t} = \frac{1}{n}\zeta^{ar+bs}\left(1-\zeta^r\right)\left(1-\zeta^s\right)B\left(\frac{r}{n},\frac{s}{n}\right)$$

Proof. Integrating $\omega_{r,s,t}$ on the loop κ gives

$$\begin{aligned}\int_\kappa \omega_{r,s,t} &= \left(1-\zeta^r+\zeta^{r+s}-\zeta^s\right)\int_0^1 t^{(r-1)/n}(1-t)^{(s-n)/n}\,d\left(t^{1/n}\right)\\ &= \frac{1}{n}\left(1-\zeta^r\right)\left(1-\zeta^s\right)\int_0^1 t^{r/n-1}(1-t)^{s/n-1}\,dt\\ &= \frac{1}{n}\left(1-\zeta^r\right)\left(1-\zeta^s\right)B\left(\frac{r}{n},\frac{s}{n}\right)\end{aligned}$$

and we establish the result for non-zero a and b since

$$\int_{\kappa_{a,b}} \omega_{r,s,t} = \int_\kappa \left(A^aB^b\right)^* \omega_{r,s,t} = \zeta^{ar+bs}\int_\kappa \omega_{r,s,t}$$

■

Thus the period lattice $\mathcal{L}$ of F_n relative to our chosen basis of regular differentials is spanned by the vectors

$$\mathbf{v}_{a,b} = \left(\ldots,\frac{1}{n}\zeta^{ar+bs}\left(1-\zeta^r\right)\left(1-\zeta^s\right)B\left(\frac{r}{n},\frac{s}{n}\right),\ldots\right)_{\substack{0<r,s,t<n\\ r+s+t=n}}$$

for all $0\le a,b<n$.

Now write $\langle x\rangle$ for the unique integer congruent to x modulo n satisfying $0\le x<n$. Define an equivalence relation on admissible triples by saying that

$$(r,s,t)\sim(r',s',t')$$

if and only if $(r',s',t')=(\langle hr\rangle,\langle hs\rangle,\langle ht\rangle)$ for some $h\in(\mathbb{Z}/n\mathbb{Z})^\times$. Put $n_0 = n/\gcd(r,s,t)$ and define

$$H_{r,s,t} = \left\{h\in(\mathbb{Z}/n_0\mathbb{Z})^\times : \langle hr\rangle+\langle hs\rangle+\langle ht\rangle = n\right\}$$

Note that for all $h \in (\mathbb{Z}/n_0\mathbb{Z})^\times$ we have $0 < \langle hr \rangle + \langle hs \rangle + \langle ht \rangle < 3n$ and $\langle hr \rangle + \langle hs \rangle + \langle ht \rangle \equiv 0 \pmod{n}$. Therefore $\langle hr \rangle + \langle hs \rangle + \langle ht \rangle \in \{n, 2n\}$. From this we see that if $h \in (\mathbb{Z}/n_0\mathbb{Z})^\times$ then $H_{r,s,t}$ either contains h or $n_0 - h$ and so has size $\phi(n_0)/2$.

In [10] Rohrlich states that $\mathcal{L}$ is contained with finite index in the product of certain lattices $L_{r,s,t}$. We state the result here and give a sketch of its proof.

Proposition 2.1.2. *For any admissible triple (r,s,t) define the lattice $\mathcal{L}_{r,s,t} = \{\mathbf{w}_{r,s,t}(z) : z \in \mathbb{Z}[\zeta_{n_0}]\} \subset \mathbb{C}^{\phi(n_0)/2}$ where*

$$\mathbf{w}_{r,s,t}(z) = \left(\ldots, \frac{1}{n} z^{\sigma_h} \left(1 - \zeta^{rh}\right) \left(1 - \zeta^{sh}\right) B\left(\frac{\langle rh \rangle}{n}, \frac{\langle sh \rangle}{n}\right), \ldots\right)_{h \in H_{r,s,t}}$$

Then, after making appropriate identifications, the period lattice $\mathcal{L}$ is contained with finite index in the product of lattices $\mathcal{L}_{r,s,t}$ where the product contains one admissible triple (r,s,t) per equivalence class.

Proof. It is clear that $\mathcal{L}$ is contained in the product of the $\mathcal{L}_{r,s,t}$ since we can obtain the vector $\mathbf{v}_{a,b}$ by choosing $z = \zeta^{ar+bs}$ in each lattice $\mathcal{L}_{r,s,t}$.

To prove that $\mathcal{L}$ is contained in the product of the $L_{r,s,t}$ with finite index we must show that there is an integer N such that for any vector $\mathbf{w}$ in $\mathcal{L}_{r,s,t}$ there exists a vector $\mathbf{v}$ in $\mathcal{L}$ such that $\mathbf{v}$ is equal to $N\mathbf{w}$ on those components that are equivalent to (r,s,t) and equal to zero elsewhere.

When n is prime the method of achieving this is relatively simple to describe. When n is composite however, we must use an inclusion-exclusion type argument that is quite unwieldy. Instead of giving a full proof in this case we simply give a sketch of how the inclusion of lattices works when $n = 12$. F_{12} is the most complicated Fermat curve that will be explicitly mentioned in this thesis.

So assume that n is prime. In that case any admissible triple is equivalent to a triple of the form $(1, s', t')$ for some $1 \leq s' \leq n - 2$.

Omitting the repeated term $\frac{1}{n}\left(1 - \zeta^r\right)\left(1 - \zeta^s\right) B\left(\frac{r}{n}, \frac{s}{n}\right)$, the component at index (r,s,t) of the vector

$$\sum_{l=1}^{n} \mathbf{v}_{a+ls',-l}$$

is equal to

$$\sum_{l=1}^{n} \zeta^{(a+ls')r-ls} = \zeta^{ar} \sum_{l=1}^{n} \left(\zeta^{s'r-s}\right)^{l} = \begin{cases} n\zeta^{ar} & \text{if } s'r \equiv s \pmod{n} \\ 0 & \text{otherwise} \end{cases}$$

But $s'r \equiv s \pmod{n}$ if and only if $(r,s) = (\langle r \rangle, \langle s'r \rangle)$ so

$$\sum_{l=1}^{n} \mathbf{v}_{a+ls',-l} = n \cdot \mathbf{w}_{1,s',t'}\left(\zeta^{a}\right)$$

and we have the inclusions $\frac{1}{n} \prod \mathcal{L}_{r,s,t} \subset \mathcal{L} \subset \prod \mathcal{L}_{r,s,t}$.

For the case $n = 12$ we shall show that the lattice $72 \cdot \mathcal{L}_{1,1,10}$ is contained in $\mathcal{L}$ and hope that this gives the reader confidence that the statement of the proposition is true in general.

First of all, for integers a and b define $\mathbf{u}_{a,b}$ by

$$\mathbf{u}_{a,b} = \sum_{l=1}^{12} \mathbf{v}_{al,-bl}$$

Then the value of $\mathbf{u}_{a,b}$ at the component with index (r,s,t) is equal to

$$\left(1-\zeta^{r}\right)\left(1-\zeta^{s}\right) B\left(\frac{r}{n}, \frac{s}{n}\right)$$

if $ar \equiv bs \pmod{12}$ and is equal to 0 otherwise. For example, $\mathbf{u}_{1,1}$ is non-zero at components $(r,s) \in \{(1,1),(2,2),(3,3),(4,4),(5,5)\}$. Since $H_{1,1,10} = \{1,5\}$ the vector $\mathbf{u}_{1,1}$ can be used to make $\mathbf{w}_{1,1,10}(1)$ provided we can find a method to "turn off" the components at $(2,2)$, $(3,3)$ and $(4,4)$.

The component at $(3,3)$ is the easiest to isolate: the only non-zero component of the vector $\mathbf{u}_{1,1} + \mathbf{u}_{1,5} + \mathbf{u}_{1,9} + \mathbf{u}_{9,1} - \mathbf{u}_{3,3}$ is at the index corresponding to $(r,s,t) = (3,3,6)$. In fact the vector is equal to $36 \cdot \mathbf{w}_{3,3,6}(1)$.

The components $\mathbf{w}_{2,2}$ and $\mathbf{w}_{4,4}$ are slightly trickier. Define vectors $\mathbf{x}_0, \ldots, \mathbf{x}_3$ by

$$\begin{aligned}
\mathbf{x}_0 &= \mathbf{u}_{2,2} + \mathbf{u}_{2,8} + \mathbf{u}_{8,2} - \mathbf{u}_{4,4} \\
\mathbf{x}_1 &= \mathbf{u}_{1,1} + \mathbf{u}_{1,7} - \mathbf{u}_{2,2} \\
\mathbf{x}_2 &= \mathbf{u}_{1,4} + \mathbf{u}_{1,10} - \mathbf{u}_{2,8} \\
\mathbf{x}_3 &= \mathbf{u}_{4,1} + \mathbf{u}_{10,1} - \mathbf{u}_{8,2}
\end{aligned}$$

then it can be checked that

$$\begin{aligned}
\mathbf{x}_0 &= 24\left(\mathbf{w}_{4,4}(1) + \mathbf{w}_{2,2}(1) + \mathbf{w}_{2,8}(1) + \mathbf{w}_{8,2}(1)\right) \\
\mathbf{x}_1 &= 24\left(\mathbf{w}_{4,4}(1) - \mathbf{w}_{2,2}(1) + \mathbf{w}_{2,8}(1) + \mathbf{w}_{8,2}(1)\right) \\
\mathbf{x}_2 &= 24\left(\mathbf{w}_{4,4}(1) + \mathbf{w}_{2,2}(1) + \mathbf{w}_{2,8}(1) - \mathbf{w}_{8,2}(1)\right) \\
\mathbf{x}_3 &= 24\left(\mathbf{w}_{4,4}(1) + \mathbf{w}_{2,2}(1) - \mathbf{w}_{2,8}(1) + \mathbf{w}_{8,2}(1)\right)
\end{aligned}$$

and from these vectors each of $\mathbf{w}_{4,4}(1), \mathbf{w}_{2,2}(1), \mathbf{w}_{2,8}(1)$ and $\mathbf{w}_{8,2}(1)$ may be obtained. We can get the other vectors $\mathbf{w}_{1,1,10}(\zeta^k)$ by making analogous definitions

$$\mathbf{u}_{a,b,k} = \sum_{l=1}^{12} \mathbf{v}_{k+al,-bl}$$

and proceeding in the same manner. ∎

Proposition 2.1.3. *There are varieties $A_{r,s,t}$ defined over $\mathbb{Q}$ and isomorphic to $\mathbb{C}^{\phi(n_0)/2}/\mathcal{L}_{r,s,t}$ such that the Jacobian of the Fermat curve F_n is isogenous over $\mathbb{Q}$ to the product of varieties $A_{r,s,t}$, taking one admissible triple per equivalence class.*

Proof. The preceding proposition shows that the Jacobian of F_n is isogenous over $\mathbb{C}$ to the product of varieties $\mathbb{C}^{\phi(n_0)/2}/\mathcal{L}_{r,s,t}$ and we must show that this isogeny can be defined over $\mathbb{Q}$.

First of all note that if $\gcd(r,s,t)=d$ then $d\mathcal{L}_{r,s,t}=\mathcal{L}_{r/d,s/d,t/d}$ so we can assume that $\gcd(r,s,t)=1$.

Let $C_{r,s,t}$ (note that we will often omit the third subscript t if the value of $r+s+t=n$ is understood) be the curve defined by the affine equation

$$v^n = u^r(1-u)^s$$

then we have the rational map

$$\begin{aligned}\varphi_{r,s,t} : F_n &\longrightarrow C_{r,s,t}\\ (x,y) &\longmapsto (x^n, x^r y^s)\end{aligned}$$

For any m dividing n there is an obvious map

$$\begin{aligned}F_n &\longrightarrow F_{n/m}\\ (x,y) &\longmapsto (x^m, y^m)\end{aligned}$$

and so we obtain a map on Jacobians:

$$\operatorname{Jac} F_{n/m} \longrightarrow \operatorname{Jac} F_n \longrightarrow \operatorname{Jac} C_{r,s,t}.$$

Define the variety $J_{r,s,t}$ to be the subvariety of $\operatorname{Jac} C_{r,s,t}$ generated by the images of the above maps for all proper divisors m of n and define $A_{r,s,t}$ to be the quotient of $\operatorname{Jac} C_{r,s,t}$ by $J_{r,s,t}$. We will show that $A_{r,s,t}$ is isogenous to $\mathbb{C}^{\phi(n)/2}/\mathcal{L}_{r,s,t}$.

It is easy to check that the regular differential $\omega_{r',s',t'}$ on $F_{n/m}$ pulls back to $m\cdot\omega_{mr',ms',mt'}$ on F_n and so a differential $\omega_{r',s',t'}$ on F_n is the pull back of a differential from some $F_{n/m}$ if and only if $\gcd(r',s',t')>1$.

Next we examine the behaviour of differentials $\omega_{r',s',t'}$ on F_n under the pushforward $\varphi_{r,s,t*}$. If $\varphi_{r,s,t}(x,y)=(u,v)$ with $x,y\neq 0$ then the other preimages of (u,v) are given by $\left(\zeta^j x, \zeta^{-k} y\right)$ where j and k are integers such that

$jr \equiv ks \pmod{n}$.

Since $\gcd(r,s,t) = 1$ we can find integers a and b such that $ar + bs \equiv 1 \pmod{n}$. If $jr \equiv ks \pmod{n}$ then we can compute directly that

$$\begin{aligned}(ak + bj)s &= aks + bjs \\ &\equiv ajr + bjs \\ &= j(ar + bs) \\ &\equiv j \pmod{n}\end{aligned}$$

and similarly

$$\begin{aligned}(ak + bj)r &= akr + bjr \\ &\equiv akr + bks \\ &= k(ar + bs) \\ &\equiv k \pmod{n}.\end{aligned}$$

Therefore the pairs (j,k) satisfying $jr \equiv ks \pmod{n}$ are precisely the pairs $(\langle hs \rangle, \langle hr \rangle)$ for $0 \leq h < n$. The push-forward ${\varphi_{r,s,t}}_* \omega_{r',s',t'}$ can now be computed as

$$\begin{aligned}{\varphi_{r,s,t}}_* \omega_{r',s',t'} &= \sum_{h=0}^{n-1} \zeta^{h\left(sr'-rs'\right)} x^{r'-1} y^{s'-n} dx \\ &= \begin{cases} n \cdot x^{r'-1} y^{s'-n} dx & \text{if } sr' \equiv rs' \pmod{n} \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

We see that the $\omega_{r',s',t'}$ mapping to a non-zero differential under ${\varphi_{r,s,t}}_*$ are precisely those differentials of the form $\omega_{\langle hr \rangle, \langle hs \rangle, \langle ht \rangle}$. If we write $\langle hr \rangle = hr + l_1 n$ and $\langle hs \rangle = hs + l_2 n$ then remembering that $u = x^n$, $1 - u = y^n$ and $v = x^r y^s$ it is in fact easy to verify that

$${\varphi_{r,s,t}}_* \omega_{\langle hr \rangle, \langle hs \rangle, \langle ht \rangle} = u^{l_1 - 1} (1-u)^{l_2 - 1} v^h \, du$$

Altogether we see that those differentials on $C_{r,s,t}$ that survive after taking the quotient by $J_{r,s,t}$ are those differentials of the form ${\varphi_{r,s,t}}_* \omega_{\langle hr \rangle, \langle hs \rangle, \langle ht \rangle}$ with h coprime to n. But these differentials give period vectors of the form

$$\left(\ldots, \frac{1}{n} z^{\sigma_h} \left(1 - \zeta^{rh}\right) \left(1 - \zeta^{sh}\right) B \left(\frac{\langle rh \rangle}{n}, \frac{\langle sh \rangle}{n} \right), \ldots \right)_{h \in H_{r,s,t}}$$

and so $A_{r,s,t}$ is isogenous to $\mathbb{C}^{\phi(n)/2} / \mathcal{L}_{r,s,t}$. ■

Remark 2.1.4. If n is prime then every admissible triple (r, s, t) is equivalent to a triple of the form $C_{1,s,n-s-1}$ for some s with $1 \leq s \leq n-2$. When n is prime the subvariety $J_{r,s,t}$ is obviously trivial so $A_{r,s,t} = \operatorname{Jac} C_{r,s,t}$ and the genus of $C_{r,s,t}$ is equal to $(n-1)/2$. We see that the Jacobian of F_n is isogenous to the product of $n-2$ Jacobians of dimension $(n-1)/2$.

The following lemma tells us about the genus of the curves $C_{r,s,t}$ in the general case.

Lemma 2.1.5. *If* $\gcd(r, s, t) = 1$ *then the genus of the curve* $C_{r,s,t}$ *is equal to*

$$\frac{n - \gcd(r, n) - \gcd(s, n) - \gcd(t, n)}{2} + 1$$

Proof. We prove the formula by counting the number of regular differentials on $C_{r,s,t}$. In the proof of the previous proposition we saw that the regular differentials on $C_{r,s,t}$ were the push-forwards of differentials of the form $\omega_{\langle hr \rangle, \langle hs \rangle, \langle ht \rangle}$ that satisfy $\langle hr \rangle, \langle hs \rangle, \langle ht \rangle \neq 0$ and $\langle hr \rangle + \langle hs \rangle + \langle ht \rangle = n$.

Let $a = \gcd(r, n)$, then it is clear that $\langle hr \rangle = 0$ if and only if h is a multiple of n/a. Similarly if $b = \gcd(s, n)$ then $\langle hs \rangle = 0$ if and only if h is a multiple of n/b. Because $\gcd(r, s, t) = 1$ we know that a and b are coprime and we can prove that multiples of n/a and multiples of n/b are distinct modulo n except at 0:

$$\left\{ \begin{array}{c} (n/a) \mid m \\ (n/b) \mid m \end{array} \right\} \iff \left\{ \begin{array}{c} n \mid am \\ n \mid bm \end{array} \right\} \iff n \mid m \iff m \equiv 0 \pmod{n}$$

If we similarly define $c = \gcd(t, n)$ then we see that there will be $(n-1) - (a-1) - (b-1) - (c-1)$ values of h that guarantee $\langle hr \rangle, \langle hs \rangle, \langle ht \rangle \neq 0$. Half of them will have $\langle hr \rangle + \langle hs \rangle + \langle ht \rangle = n$, the other half $\langle hr \rangle + \langle hs \rangle + \langle ht \rangle = 2n$ and so the result is as claimed. ■

Remark 2.1.6. The previous result could also be obtained by considering the rational map

$$C_{r,s,t} \longrightarrow \mathbb{P}^1$$
$$(u, v) \longmapsto u$$

which is ramified at the points 0, 1 and infinity. Above 0 there are $\gcd(r, n)$ branches and the ramification index is $n/\gcd(r, n)$. Similar things happen above the points 1 and infinity and the result in the lemma can be obtained by using the Riemann-Hurwitz formula.

2.2 Isomorphisms between the $A_{r,s,t}$

Proposition 2.2.1. *For all admissible triples (r, s, t) we have that $C_{r,s,t}$ is birationally equivalent to $C_{s,r,t}$ and birationally equivalent to $C_{t,s,r}$ if s is even or if $r + s + t$ is odd.*

Proof. To prove the first claim we see that $C_{r,s,t}$ and $C_{s,r,t}$ are isomorphic via the map

$$C_{r,s} \longrightarrow C_{s,r}$$
$$(u, v) \longmapsto (1 - u, v)$$

If s is even then $C_{r,s,t}$ and $C_{t,s,r}$ are isomorphic via the map

$$\begin{aligned} C_{r,s,t} &\longrightarrow C_{t,s,r} \\ (u,v) &\longmapsto \left(\frac{1}{u}, \frac{v}{u}\right) \end{aligned}$$

It is easy to check that this really is a map between the specified curves: for a point (u,v) on $C_{r,s,t}$ we have

$$\left(\frac{v}{u}\right)^n = \frac{v^n}{u^{r+s+t}} = \frac{u^r\,(1-u)^s}{u^{r+s+t}} = \left(\frac{1}{u}\right)^t \left(\frac{1}{u}-1\right)^s = \left(\frac{1}{u}\right)^t \left(1-\frac{1}{u}\right)^s$$

Finally, if $n = r+s+t$ is odd then $C_{r,s,t}$ is isomorphic to $C_{t,s,r}$ via the map

$$\begin{aligned} C_{r,s,t} &\longrightarrow C_{t,s,r} \\ (u,v) &\longmapsto \left(\frac{1}{u}, (-1)^s \cdot \frac{v}{u}\right) \end{aligned}$$

The calculation to check this is almost exactly the same as above but relies on the fact that $(-1)^{sn} = (-1)^s$ when n is odd. ∎

Corollary 2.2.2. *If n has fewer than three prime factors then any variety $A_{r,s,t}$ with $\gcd(r,s,t) = 1$ is isomorphic to a variety of the form $A_{1,s,n-s-1}$.*

Proof. If r is coprime to n then set h equal to $r^{-1} \mod n$. We see that $A_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}$ is isomorphic to $A_{r,s,t}$ and of the desired form.

If s is coprime to n then since $A_{r,s,t}$ is always isomorphic to $A_{s,r,t}$ we can set h equal to $s^{-1} \mod n$ and operate as above.

If t is coprime to n then we must consider two separate cases. If n is odd then $A_{r,s,t}$ is isomorphic to $A_{t,s,r}$ and we may proceed as previously. If n is even then t is necessarily odd and so one of r and s must be even. Again we see that $A_{r,s,t}$ is isomorphic to $A_{t,s,r}$ and we get a variety of the desired form by inverting t modulo n.

Therefore, if the variety $A_{r,s,t}$ is not isomorphic to any variety of the desired form, we must have that r, s and t all share a common factor with n. Since

$\gcd(r, s, t) = 1$ these factors must be coprime and so n must have at least three prime factors. ■

Remark 2.2.3. Perhaps the simplest case when $A_{r,s,t}$ is not isomorphic to any variety of the form $A_{1,s,n-s-1}$ is when $n = 30$ and $(r, s, t) = (2, 3, 25)$. Generally, if s is invertible modulo n then $A_{1,s,n-s-1}$ is isomorphic to $A_{1,s',n-s'-1}$ where s' satisfies $ss' \equiv 1 \mod n$. If n is odd then $A_{1,s,n-s-1}$ is isomorphic to $A_{1,n-s-1,s}$.

With these isomorphisms in mind we give a table detailing the primitive quotients of the Fermat curves for n up to 10 and for $n = 12$. The second column gives a list of all s such that $A_{1,s,n-s-1}$ is not isomorphic to any $A_{1,s',n-s'-1}$ with s' smaller than s and the third column gives a list of all inequivalent admissible triples that give varieties isomorphic to $A_{1,s,n-s-1}$. The fourth column gives the number of such triples.

n	s	r,s,t	$\#N$
3	1	(1,1,1)	1
4	1	(1,1,2)	1
	2	(1,2,1), (2,1,1)	2
5	1	(1,1,3), (1,2,2), (1,3,1)	3
6	1	(1,1,4)	1
	2	(1,2,3), (2,1,3), (2,3,1)	3
	3	(1,3,2), (3,1,2), (3,2,1)	3
	4	(1,4,1), (4,1,1)	2
7	1	(1,1,5), (1,3,3), (1,5,1)	3
	2	(1,2,4), (1,4,2)	2
8	1	(1,1,6)	1
	2	(1,2,5), (2,1,5)	2
	3	(1,3,4)	1
	4	(1,4,3), (4,1,3)	2
	5	(1,5,2)	1
	6	(1,6,1), (6,1,1)	2
9	1	(1,1,7), (1,4,4), (1,7,1)	3
	2	(1,2,6), (1,3,5), (1,5,3) (1,6,2), (3,1,5), (6,1,2)	6

n	s	r,s,t	$\#N$
10	1	(1,1,8)	1
	2	(1,2,7), (2,1,7), (1,6,3), (6,1,3)	4
	3	(1,3,6), (1,7,2)	2
	4	(1,4,5), (4,1,5), (4,5,1), (5,4,1)	4
	5	(1,5,4), (5,1,4)	2
	8	(1,8,1), (8,1,1)	2
12	1	(1,1,10)	1
	2	(1,2,9), (2,1,9), (2,9,1), (9,2,1)	4
	3	(1,3,8), (3,1,8)	2
	4	(1,4,7), (4,1,7)	2
	5	(1,5,6)	1
	6	(1,6,5), (6,1,5)	2
	7	(1,7,4)	1
	8	(1,8,3), (8,1,3), (8,3,1), (3,8,1)	4
	9	(1,9,2), (9,1,2)	2
	10	(1,10,1), (10,1,1)	2

Chapter 3

On a subgroup of K_2F_n

3.1 Symbols with divisorial support at infinity

In this chapter we introduce the main elements of K-theory that we are going to be working with. Initial work in this direction was done by Ross [22]. There he considered the so-called *symbols with support at infinity* and showed that they were a good source of elements of K_2F_n. In the case where n was an odd prime p he proved that the rank of the subgroup generated by these symbols and defined over $\mathbb{Q}$ was less than or equal to $3(p-1)$. Eventually we will show that this bound can be reduced to $3(p-3)$.

By a *point at infinity* we mean points on F_n such that $XYZ(P) = 0$. Keeping the definitions $\zeta = e^{\frac{2\pi i}{n}}$ and $\xi = e^{\frac{\pi i n'}{n}}$ from the previous chapter, these are the $3n$ points given in projective coordinates by $(0, \zeta^j, 1)$, $(\zeta^j, 0, 1)$ and $(\xi\zeta^j, 1, 0)$ for $0 \leq j < n$.

Let $\mathcal{D}_\infty$ be the group of divisors of degree zero consisting of points at infinity. This group has rank $3n-1$. It can be shown that the divisors of the functions x, y and

$$x - \zeta^j, \qquad y - \zeta^j \qquad x - \xi\zeta^j y \qquad 0 \leq j < n \tag{3.1}$$

span a subgroup of $\mathcal{D}_\infty$ which also has rank $3n-1$. Thus the quotient is finite and modulo constants any function supported on $\mathcal{D}_\infty$ is a root of some product of the above functions. In his paper [19] Rohrlich computes exactly which functions occur in this way but we do not need the exact details here.

Let $k = \mathbb{Q}(\mu_{2n})$ and choose an embedding $\iota : F_n \hookrightarrow \operatorname{Jac} F_n$ defined over $\mathbb{Q}$ that sends some point at infinity to the origin. Also from [19] we see that $\iota(P)$ is torsion when P is any point at infinity. Using this fact Ross uses Bloch's trick [4] to define a certain subgroup $\mathcal{N}$ of $K_2k(F_n) \cap \ker \tau$ using symbols consisting of the functions x, y and those in (3.1). The exact definition of $\mathcal{N}$ depends on the choices made when invoking Bloch's trick but for the Fermat curves Ross conjectures that the differences are trivial. See his paper [22] for more details.

Let $G = \operatorname{Gal}(k/\mathbb{Q})$ and $\Sigma = \sum_{\sigma \in G} \sigma$ then the image of $\mathcal{N}$ under the trace map $\operatorname{Tr} : K_2k(F_n) \to K_2\mathbb{Q}(F_n)$ is equal to $\mathcal{N}^\Sigma$. Since Σ commutes with τ we can identify this with a subgroup of $K_2F_n \otimes \mathbb{Q}$ which Ross calls the subgroup of $K_2F_n \otimes \mathbb{Q}$ *with divisorial support at infinity.*

Ross also gives a more concrete description of $\mathcal{N}$ which is what we will be dealing with in the rest of this chapter. First of all he discovered that (a suitable power of) the symbol $\alpha = \{1-x, 1-y\}$ lies in the kernel of the tame symbol and and hence defined an element of K_2F_n. Recall the group Γ_n of automorphisms of F_n introduced in the previous chapter that was generated by the elements

$$A : (x,y) \mapsto (\zeta x, y) \tag{3.2}$$

$$B : (x,y) \mapsto (x, \zeta y) \tag{3.3}$$

$$\sigma : (x,y) \mapsto (y, x) \tag{3.4}$$

$$\eta : (x,y) \mapsto \left(\frac{1}{y}, \frac{\xi x}{y}\right) \tag{3.5}$$

and define $\mathcal{S} = \mathbb{Q}[\Gamma_n] \cdot \alpha$. Then Ross shows that $\mathcal{N} \otimes \mathbb{Q} = \mathcal{S}$.

In the next section we prove upper bounds for the rank of $\mathcal{S}^\Sigma$ for odd values of n. Later in the chapter we consider even values of n but we only give upper bounds for the rank of $\mathcal{S}^\Sigma$ after projection under the regulator.

3.2 Bounding the rank of $\mathcal{S}^\Sigma$ for odd n

Firstly, note that because $\{x,y\} = -\{y,x\}$ we have $\sigma\alpha = -\alpha$ and hence $\mathcal{S}$ is generated by the $3n^2$ elements

$$A^a B^b \eta^j \alpha \qquad 0 \le a, b < n \qquad 0 \le j < 3$$

Remember that when n is odd we have $\xi = -1$. To ease notation let us define

$$\beta = \eta\alpha = \{1 - 1/y, 1 + x/y\}$$
$$\delta = \eta^2\alpha = \{1 + y/x, 1 - 1/x\}$$

For odd n we have $\mathrm{Gal}(k/\mathbb{Q}) \cong (\mathbb{Z}/n\mathbb{Z})^\times$. Let $\sigma_d \in G$ be such that $\sigma_d(\zeta) = \zeta^d$ then notice that if $c \in \mathbb{Z}/n\mathbb{Z}^\times$ that

$$\begin{aligned}
(A^{ac}B^{bc}\alpha)^\Sigma &= \sum_{d \in \mathbb{Z}/n\mathbb{Z}^\times} \{1 - \zeta^{ac}x, 1 - \zeta^{bc}y\}^{\sigma_d} \\
&= \sum_{d \in \mathbb{Z}/n\mathbb{Z}^\times} \{1 - \zeta^{adc}x, 1 - \zeta^{bdc}y\} \\
&= \sum_{d \in \mathbb{Z}/n\mathbb{Z}^\times} \{1 - \zeta^{ad}x, 1 - \zeta^{bd}y\} \\
&= (A^a B^b \alpha)^\Sigma
\end{aligned}$$

An equivalent result holds for β and δ so, for example, if n is an odd prime p then the subgroup is generated by the $3(p+2)$ elements

$$\alpha \qquad (B\alpha)^\Sigma \qquad (AB^j\alpha)^\Sigma \qquad 0 \le j < p \tag{3.6}$$
$$\beta \qquad (B\beta)^\Sigma \qquad (AB^j\beta)^\Sigma \qquad 0 \le j < p \tag{3.7}$$
$$\delta \qquad (B\delta)^\Sigma \qquad (AB^j\delta)^\Sigma \qquad 0 \le j < p \tag{3.8}$$

Note that because of the calculation

$$\begin{aligned}\sum_{j=0}^{p-1}\{1-x,1-\zeta^j y\} &= \left\{1-x,\prod_{j=0}^{p-1}\left(1-\zeta^j y\right)\right\}\\ &= \{1-x,1-y^p\}\\ &= \{1-x,x^p\}\\ &= 0\end{aligned}$$

we have $(B\alpha)^{\Sigma} = -\alpha$. The relations $(A\alpha)^{\Sigma} = -\alpha$ and $\sum(AB^j\alpha)^{\Sigma} = 0$ can be proved in the same fashion. Also there are analogous results for β and δ so that the maximum possible rank of the subgroup is $3(p-1)$. These were the results obtained by Ross and allowed him to conclude that the symbols supported at infinity could not generate the whole of K_2F_p for all odd primes greater than 7 if Beilinson's conjectures are to be true.

Note that because the Jacobian of the Fermat curves (and all of the quotients $A_{r,s,t}$) have complex multiplication they have potentially good reduction everywhere and so by the remark in the introduction K_2F_n agrees up to torsion with $K_2\mathcal{F}_n$ where $\mathcal{F}_n$ is any regular proper flat model of F_n.

The rest of this section will be devoted to proving the following proposition which shows that the rank of $\mathcal{S}^{\Sigma}$ for odd n is at most three times the number of equivalence classes of admissible triples.

Proposition 3.2.1. *For odd n the subgroup $\mathcal{S}^{\Sigma}$ is generated by the elements*

$$\varphi^*_{r,s,t}\varphi_{r,s,t_*}\alpha, \qquad \varphi^*_{r,s,t}\varphi_{r,s,t_*}\beta, \qquad \varphi^*_{r,s,t}\varphi_{r,s,t_*}\delta$$

where

$$\alpha = \{1-x,1-y\}, \qquad \beta = \{1-1/y,1+x/y\}, \qquad \delta = \{1+y/x,1-1/x\}$$

and the (r,s,t) take on one element per equivalence class of admissible triples.

In particular if n is a prime p we know that there are only $p-2$ inequivalent admissible triples. Therefore the maximum rank of $\mathcal{S}^{\Sigma}$ is at most $3(p-2)$. This is already an improvement on Ross' bound.

We have shown that $\mathcal{S}^{\Sigma}$ is generated by the symbols $(A^rB^s\alpha)^{\Sigma}$, $(A^rB^s\beta)^{\Sigma}$ and $(A^rB^s\delta)^{\Sigma}$ for $0 \leq r, s < n$ and that if d is coprime to n then $(A^{dr}B^{ds}\alpha)^{\Sigma} = (A^rB^s\alpha)^{\Sigma}$.

Let us denote

$$S_n^{r,s} = (A^rB^s\alpha)^{\Sigma} = \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^{\times}} \{1-\zeta^{dr}x, 1-\zeta^{ds}y\}$$

Now since $C_{r,s}$ is the quotient of F_n by the group of automorphisms generated by A^sB^{-r} we see that $\varphi^*_{s,-r}\varphi_{s,-r*}\alpha$ is equal to

$$\sum_{d\in\mathbb{Z}/n\mathbb{Z}} \{1-\zeta^{dr}x, 1-\zeta^{ds}y\}$$

which we shall denote $R_n^{r,s}$.

Given the similarity of the definitions of $R_n^{r,s}$ and $S_n^{r,s}$ it is not surprising that they both span the same subgroup as the following "Moebius inversion" type argument shows. Equivalent results hold for$(A^rB^s\beta)^{\Sigma}$ and $(A^rB^s\delta)^{\Sigma}$ so this will complete the proof of Proposition 3.2.1.

Proposition 3.2.2. *The subgroup generated by the elements $S_n^{r,s}$ as r and s range through the integers modulo n is the same as that spanned by the equivalent elements $R_n^{r,s}$.*

Proof. The idea is to find a relation expressing the R's in terms of the S's and then to use induction on the value of $\gcd(r, s, n)$ to show that these relations allow us to express the S's in terms of the R's.

Let $Z_{n,f}$ denote the set $\{d \in \mathbb{Z} : 0 \leq d < n, \gcd(d, n) = f\}$. This set is in bijection with $(\mathbb{Z}/(n/f)\mathbb{Z})^\times$ via the map

$$\begin{aligned} Z_{n,f} &\longrightarrow (\mathbb{Z}/(n/f)\mathbb{Z})^\times \\ d &\longmapsto d/f \end{aligned}$$

On the other hand, the map

$$\begin{aligned} (\mathbb{Z}/n\mathbb{Z})^\times &\longrightarrow Z_{n,f} \\ d &\longmapsto df \end{aligned}$$

is a surjective map for which every element of $Z_{n,f}$ has $\phi(n)/\phi(n/f)$ preimages in $(\mathbb{Z}/n\mathbb{Z})^\times$.

Note that $\mathbb{Z}/n\mathbb{Z} = \bigcup_{f|n} Z_{n,f}$ so that

$$\begin{aligned} R_n^{r,s} &= \sum_{d\in\mathbb{Z}/n\mathbb{Z}} \{1-\zeta^{dr}x, 1-\zeta^{ds}y\} \\ &= \sum_{f|n}\sum_{d\in Z_{n,f}} \{1-\zeta^{dr}x, 1-\zeta^{ds}y\} \end{aligned}$$

But, by our previous observation, we see that as d runs through $(\mathbb{Z}/n\mathbb{Z})^\times$, df runs through $Z_{n,f}$ $\phi(n)/\phi(n/f)$ times so we get

$$\begin{aligned} R_n^{r,s} &= \sum_{f|n} \frac{\phi(n/f)}{\phi(n)} \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^\times} \{1-\zeta^{dfr}x, 1-\zeta^{dfs}y\} \\ &= \sum_{f|n} \frac{\phi(n/f)}{\phi(n)} S_n^{fr,fs} \end{aligned}$$

From this we can see that the subgroup generated by the R's is a subgroup of that generated by the S's. An inductive argument on the value of $\gcd(r,s,n)$ allows us to prove the reverse inclusion.

First of all we need to investigate the behaviour of $S_n^{fr,fs}$. If $\gcd(r,s,n) = n/k$ and $\gcd(f,k) = 1$ then define integers r' and s' by $r = r'n/k$ and $s = s'n/k$.

We see that

$$\begin{aligned} S_n^{fr,fs} &= \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^\times} \{1-\zeta^{dfr}x, 1-\zeta^{dfs}y\} \\ &= \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^\times} \{1-\zeta^{dfr'n/k}x, 1-\zeta^{dfs'n/k}y\} \end{aligned}$$

But $\zeta^{n/k}$ is a kth root of unity so that the sum only depends on d modulo k. We get:

$$S_n^{fr,fs} = \frac{\phi(n)}{\phi(k)} \sum_{d\in(\mathbb{Z}/k\mathbb{Z})^\times} \{1-\zeta^{dfr'n/k}x, 1-\zeta^{dfs'n/k}y\}$$

But f is coprime to k so that as d runs through $(\mathbb{Z}/k\mathbb{Z})^\times$, so does df. Hence we get:

$$\begin{aligned} S_n^{fr,fs} &= \frac{\phi(n)}{\phi(k)} \sum_{d\in(\mathbb{Z}/k\mathbb{Z})^\times} \{1-\zeta^{dr'n/k}x, 1-\zeta^{ds'n/k}\} \\ &= \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^\times} \{1-\zeta^{dr'n/k}x, 1-\zeta^{ds'n/k}y\} \\ &= \sum_{d\in(\mathbb{Z}/n\mathbb{Z})^\times} \{1-\zeta^{dr}x, 1-\zeta^{ds}y\} \\ &= S_n^{r,s} \end{aligned}$$

Now we are ready for the induction. Assume that all $S_n^{r,s}$ such that $\frac{n}{\gcd(r,s,n)}$ strictly divides k can be expressed in terms of the R's. This is clearly satisfied when k is prime since $R_n^{0,0}$ and $S_n^{0,0}$ are both multiples of α. Now if we are given a pair (r, s) satisfying $\gcd(r, s, n) = \frac{n}{k}$ we already know that

$$R_n^{r,s} = \sum_{f|n} \frac{\phi(n/f)}{\phi(n)} S_n^{fr,fs}$$

Those f which satisfy $\gcd(f, k) = 1$ have that $S_n^{fr,fs} = S_n^{r,s}$ while those f which share a common factor with k result in $\gcd(fr, fs, n)$ being a strict multiple of $\gcd(r, s, n)$. Therefore $S_n^{r,s}$ can be expressed in terms of $R_n^{r,s}$ and those S such that their values of r and s have a greater value of $\gcd(r, s, n)$. By induction, these S could already be expressed in terms of the R and so the induction is complete. ■

3.3 Expressions for the regulators

We start this section with a lemma that allows us to express integrals of the regulator defined by equation (1.1) in a more convenient form.

Lemma 3.3.1. *Suppose we have a curve X and functions f, g in $\mathbb{Q}(X)^\times$ with zeroes contained in a finite set S. Suppose that γ is a loop in $X(\mathbb{C})\backslash S$ based at a point $P_0 \in X$ and choose fixed branches of $\log f$ and $\log g$ on some neighbourhood of γ. Then with $\omega\{f, g\}$ defined as in equation* (1.1) *we have*

$$\int_\gamma \omega\{f, g\} = \frac{1}{2\pi i} \operatorname{Im}\left[\int_\gamma \log f \, d\log g - \log|g(P_0)| \int_\gamma d\log f\right] \tag{3.9}$$

Proof. From the definition of ω and expanding out the logarithms in terms of their real and imaginary parts we can compute

$$\begin{aligned}
&\int_\gamma \omega\{f, g\} - \frac{1}{2\pi i} \operatorname{Im}\int_\gamma \log f d\log g \\
=&\frac{1}{2\pi i}\int_\gamma \big[\log|f|\, d\arg g - \log|g|\, d\arg f\big] - \big[\log|f|\, d\arg g + \arg f d\log|g|\big] \\
=&\frac{-1}{2\pi i}\int_\gamma \big[\log|g|\, d\arg f + \arg f d\log|g|\big] \\
=&\frac{-1}{2\pi i}\int_\gamma d\big(\log|g| \cdot \arg f\big)
\end{aligned}$$

which gives our result. ■

Recall our definitions $\gamma : t \mapsto \big(t^{1/n}, (1-t)^{1/n}\big)$ and $\kappa = \gamma - A\gamma + AB\gamma - B\gamma$ from chapter 2. The following calculation up to equation (3.10) is due to Ross in [22]. He showed that when calculating $\int_\kappa \operatorname{reg}\alpha$ that the second integral in equation (3.9) is equal to zero. Using the Beta function identity $B(u + 1, v) =$

$\frac{u}{u+v}B(u,v)$ he calculates the first integral as follows

$$\begin{aligned}
\operatorname{Im}\int_\gamma \log(1-\zeta^a x)d\log(1-\zeta^b y) &= \operatorname{Im}\int_0^1 \log(1-\zeta^a(1-t)^{1/n})d\log(1-\zeta^b t^{1/n}) \\
&= \operatorname{Im}\int_0^1 \sum_{j=1}^\infty \frac{-1}{j}\zeta^{aj}(1-t)^{j/n} d\left(\sum_{k=1}^\infty \frac{-1}{k}\zeta^{bk}t^{k/n}\right) \\
&= \frac{1}{n}\operatorname{Im}\int_0^1 \sum_{j=1}^\infty\sum_{k=1}^\infty \frac{1}{j}\zeta^{aj+bk}(1-t)^{j/n}t^{k/n-1}dt \\
&= \frac{1}{n}\operatorname{Im}\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{1}{j}\zeta^{aj+bk}\int_0^1 (1-t)^{j/n}t^{k/n-1}dt \\
&= \frac{1}{n}\operatorname{Im}\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{1}{j}\zeta^{aj+bk}B\left(\frac{j}{n}+1,\frac{k}{n}\right) \\
&= \frac{1}{n}\operatorname{Im}\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{1}{j+k}\zeta^{aj+bk}B\left(\frac{j}{n},\frac{k}{n}\right)
\end{aligned}$$

So that

$$\begin{aligned}
\int_\kappa \operatorname{reg} A^a B^b \alpha &= \int_\gamma \operatorname{reg}\left(A^a B^b - A^{a+1}B^b + A^{a+1}B^{b+1} - A^a B^{b+1}\right)\alpha \\
&= \frac{1}{2\pi i n}\operatorname{Im}\sum_{j=1}^\infty\sum_{k=1}^\infty \frac{1}{j+k}\zeta^{aj+bk}(1-\zeta^j)(1-\zeta^k)B\left(\frac{j}{n},\frac{k}{n}\right)
\end{aligned} \tag{3.10}$$

If for $x, y > 0$ we define

$$F(x,y) = \sum_{j,k=0}^\infty \frac{B(x+j, y+k)}{x+j+y+k}$$

then we can write

$$\int_{\kappa_{a,b}} \operatorname{reg}\{1-x, 1-y\} = \frac{1}{2\pi i n^2}\operatorname{Im}\sum_{j,k=1}^n \zeta^{aj+bk}(1-\zeta^j)(1-\zeta^k)F\left(\frac{j}{n},\frac{k}{n}\right) \tag{3.11}$$

First of all note that all terms with j or k equal to n in (3.11) are zero and because of complex conjugation and the fact that $F(x,y) = F(y,x)$ we see that the sum of all terms with $j+k=n$ is also equal to zero:

$$\begin{aligned}
&\operatorname{Im} \sum_{j+k=n} \zeta^{aj+bk}(1-\zeta^j)(1-\zeta^k)F\left(\frac{j}{n},\frac{k}{n}\right) \\
=&\operatorname{Im} \sum_{j+k=n} \zeta^{-ak-bj}(1-\zeta^{-j})(1-\zeta^{-k})F\left(\frac{n-j}{n},\frac{n-k}{n}\right) \\
=&\operatorname{Im} \sum_{j+k=n} \zeta^{-ak-bj}(1-\zeta^{-j})(1-\zeta^{-k})F\left(\frac{k}{n},\frac{j}{n}\right) \\
=&-\operatorname{Im} \sum_{j+k=n} \zeta^{aj+bk}(1-\zeta^j)(1-\zeta^k)F\left(\frac{j}{n},\frac{k}{n}\right)
\end{aligned}$$

In similar fashion, conjugating all terms with $j+k>n$ allows us to write the regulator purely using terms with $j+k<n$:

$$\int_{\kappa_{a,b}} \operatorname{reg}\alpha = \frac{1}{2\pi i n^2}\operatorname{Im}\sum_{j+k<n}\zeta^{aj+bk}(1-\zeta^j)(1-\zeta^k)\left(F\left(\frac{j}{n},\frac{k}{n}\right)-F\left(1-\frac{j}{n},1-\frac{k}{n}\right)\right) \tag{3.12}$$

In what follows, for $\omega \in H^1(X(\mathbb{C}),\mathbb{C})$ we will understand $\operatorname{Im}\omega$ to mean $(\omega-\bar{\omega})/2$. From Lemma 2.1.1 we will have

$$\int_{\kappa_{a,b}} \operatorname{Im}\omega_{r,s,t} = \frac{i}{n}\operatorname{Im}\zeta^{ar+bs}(1-\zeta^r)(1-\zeta^s)B\left(\frac{r}{n},\frac{s}{n}\right) \tag{3.13}$$

and so after making the convenient definition

$$G(x,y)=\frac{F(x,y)-F(1-x,1-y)}{B(x,y)}$$

we can compare equations (3.12) and (3.13) to obtain

$$\operatorname{reg}\alpha = \frac{-1}{2\pi n}\sum_{j+k<n} G\left(\frac{j}{n},\frac{k}{n}\right)\cdot \operatorname{Im}\omega_{j,k,n-j-k} \tag{3.14}$$

3.3.1 Action of η on the $\omega_{r,s,t}$

The action of η on $\omega_{r,s,t}$ is as follows:

$$\begin{aligned}\eta\,\omega_{r,s,t} &= \eta(x^{r-1}y^{s-n}dx) = \left(\frac{1}{y}\right)^{r-1}\left(\frac{\xi x}{y}\right)^{s-n}\left(-\frac{1}{y^2}\right)dy \\ &= \xi^{s-n}y^{n-r-s+1}x^{s-n}\frac{-1}{y^2}dy \\ &= \xi^{s}y^{t-1}x^{s-n}dy\end{aligned}$$

Now from the fact that $x^n + y^n = 1$ we see that

$$dy = -\left(\frac{x}{y}\right)^{n-1} dx$$

Hence

$$\eta\,\omega_{r,s,t} = -\xi^s x^{s-1}y^{t-n}dx = -\xi^s\omega_{s,t,r}$$

After checking carefully that

$$\eta^2\omega_{r,s,t} = -\xi^s\eta\,\omega_{s,t,r} = \xi^{s+t}\omega_{t,r,s} = \xi^{n-r}\omega_{t,r,s} = -\xi^{-r}\omega_{t,r,s}$$

then since the $G\left(\frac{j}{n},\frac{k}{n}\right)$ are real-valued in equation (3.14) we get the following expressions for $\operatorname{reg}\eta\alpha$ and $\operatorname{reg}\eta^2\alpha$

$$\operatorname{reg}\eta\alpha = \frac{1}{2\pi n}\sum_{j+k<n} G\left(\frac{j}{n},\frac{k}{n}\right)\cdot \operatorname{Im}\xi^k\omega_{k,n-j-k,j} \tag{3.15}$$

$$\operatorname{reg}\eta^2\alpha = \frac{1}{2\pi n}\sum_{j+k<n} G\left(\frac{j}{n},\frac{k}{n}\right)\cdot \operatorname{Im}\xi^{-j}\omega_{n-j-k,j,k} \tag{3.16}$$

Note when n is even η is not defined over $\mathbb{Q}$ and so we do not expect that the above two expressions should be members of $H^1(X(\mathbb{C}),\mathbb{R}(1))^+$.

3.3.2 Projection on to the quotients

Let $\varphi_{r,s,t}$ be the canonical projection

$$\varphi_{r,s,t} : \mathrm{Jac}(F_n) \longrightarrow A_{r,s,t}$$

and remember that the $\omega_{r,s,t}$ on F_n that do not vanish under this projection are of the form $\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}$ for $h \in H_{r,s,t}$. To compute expressions for $\operatorname{reg}\varphi_{r,s,t}\eta^l\alpha$ for $l \in 0,1,2$ we must use equations (3.14), (3.15) and (3.16) and remove all differentials that vanish under the projection.

The differentials that remain from equation (3.14) will come from indices (j,k) of the form $(\langle hr\rangle, \langle hs\rangle)$ for h in $H_{r,s,t}$. Similarly, for equations (3.15) and (3.16) the relevant indices will be of the form $(\langle ht\rangle, \langle hr\rangle)$ and $(\langle hs\rangle, \langle ht\rangle)$ respectively. Namely we will have

$$\begin{aligned}
\operatorname{reg}\varphi_{r,s,t}\alpha &= \frac{-1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle hr\rangle}{n}, \frac{\langle hs\rangle}{n}\right)\cdot \operatorname{Im}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}\\
\operatorname{reg}\varphi_{r,s,t}\eta\alpha &= \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n}, \frac{\langle hr\rangle}{n}\right)\cdot \operatorname{Im}\xi^{\langle hr\rangle}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}\\
\operatorname{reg}\varphi_{r,s,t}\eta^2\alpha &= \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle hs\rangle}{n}, \frac{\langle ht\rangle}{n}\right)\cdot \operatorname{Im}\xi^{-\langle hs\rangle}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}
\end{aligned}$$

More generally, for any integers a and b we have

$$\begin{aligned}
\operatorname{reg}\varphi_{r,s,t}A^aB^b\alpha &= \frac{-1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle hr\rangle}{n}, \frac{\langle hs\rangle}{n}\right)\cdot \operatorname{Im}\zeta^{ahr+bhs}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}\\
\operatorname{reg}\varphi_{r,s,t}A^aB^b\eta\alpha &= \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n}, \frac{\langle hr\rangle}{n}\right)\cdot \operatorname{Im}\xi^{\langle hr\rangle}\zeta^{ahr+bhs}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}\\
\operatorname{reg}\varphi_{r,s,t}A^aB^b\eta^2\alpha &= \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle hs\rangle}{n}, \frac{\langle ht\rangle}{n}\right)\cdot \operatorname{Im}\xi^{-\langle hs\rangle}\zeta^{ahr+bhs}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}
\end{aligned}$$

We would like to investigate how these regulators behave under the trace $\mathrm{Tr} : A_{r,s,t}\otimes\mathbb{Q}(\mu_{2n}) \longrightarrow A_{r,s,t}$. Note that when $c \in \mathbb{Q}(\mu_{2n})$ and ω is defined over $\mathbb{Q}$ we have $\mathrm{Tr}\,(\operatorname{Im} c\,\omega) = \mathrm{Tr}\, c\cdot\operatorname{Im}\omega$.

The case of $\operatorname{reg}\varphi_{r,s,t}A^aB^b\alpha$ is easiest to deal with. Since h is coprime to n we see that the map $\zeta \longmapsto \zeta^h$ is in $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ and so $\mathrm{Tr}(\zeta^{h(ar+bs)}) = \mathrm{Tr}(\zeta^{ar+bs})$. The projection is therefore

$$\frac{-1}{2\pi n}\,\mathrm{Tr}(\zeta^{ar+bs}) \sum_{h\in H_{r,s,t}} G\left(\frac{\langle hr\rangle}{n}, \frac{\langle hs\rangle}{n}\right) \cdot \mathrm{Im}\,\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}$$

and no matter what values are taken by a and b we will always end up with an element of $H^1(A_{r,s,t}(\mathbb{C}), \mathbb{R}(1))$ that is linearly equivalent to the case $a = b = 0$.

The character of the expressions for $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta\alpha$ and $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta^2\alpha$ are very similar and we will only explicitly deal with the former. The behaviour of this expression when projected onto $\mathbb{Q}$ is quite different when n is odd compared to when n is even. Let us assume that n is odd first of all.

In that case $n' = n$ and so $\xi = -1$. Therefore, very similarly to before, we have

$$\mathrm{Tr}\left(\xi^{\langle hr\rangle}\zeta^{ahr+bhs}\right) = (-1)^{\langle hr\rangle}\,\mathrm{Tr}\left(\zeta^{h(ar+bs)}\right) = (-1)^{\langle hr\rangle}\,\mathrm{Tr}\left(\zeta^{ar+bs}\right)$$

and the projection onto $\mathbb{Q}$ is

$$\frac{1}{2\pi n}\,\mathrm{Tr}\left(\zeta^{ar+bs}\right) \sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n}, \frac{\langle hr\rangle}{n}\right) \cdot (-1)^{\langle hr\rangle}\,\mathrm{Im}\,\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}.$$

Again we see that nothing is gained by letting a and b vary and that everything is covered by the case $a = b = 0$.

When n is even taking the transfer to $\mathbb{Q}$ is more problematic. The following lemma will show that if n is even and r is odd then taking the transfer to $\mathbb{Q}$ of $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta\alpha$ yields zero.

Lemma 3.3.2. *If j is any integer, k is odd and n is even then*

$$\mathrm{Tr}\left(\xi^k\zeta^j\right) = 0$$

Proof. Let $\theta = \xi^k \zeta^j$, then for any odd integer l we see that

$$\theta^{2l} = \left(\xi^k \zeta^j\right)^{2l} = \zeta^{(n'k+2j)l} \neq 1$$

since $(n'k+2j)l$ is odd and n is even. Therefore 4 divides the order of θ. Suppose the order of θ is $4l$ then the map $\theta \longmapsto \theta^{2l+1} = -\theta$ is a member of $\mathrm{Gal}(\mathbb{Q}(\theta)/\mathbb{Q})$ and so $\mathrm{Tr}\,\theta = 0$. ∎

If n is even then all members of $H_{r,s,t}$ are odd. Therefore if r is odd then the above lemma can be applied to all terms of the form $\xi^{\langle hr\rangle}\zeta^{ar+bs}$ in the expression for $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta\alpha$ which therefore maps to zero when transferred to $\mathbb{Q}$. Similarly if s is odd then all elements of the form $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta^2\alpha$ vanish after transferring to $\mathbb{Q}$.

If n is even then primitive triples (r,s,t) have two odd elements and one even element amongst the three. Thus in one-third of the cases we will have r and s both odd and both $\eta\alpha$ and $\eta^2\alpha$ will yield trivial results. In the other cases we can hope to find non-trivial elements from $\eta\alpha$ or $\eta^2\alpha$ but never both.

From now on let us assume that r and n are both even. Since $\xi^{\langle hr\rangle - hr}$ is rational (in fact it is equal to $(-1)^{\lfloor hr/n\rfloor}$ where $\lfloor\cdot\rfloor$ denotes the integer part) and the map $\zeta \longmapsto \zeta^h$ is a member of $\mathrm{Gal}(\mathbb{Q}(\zeta)/\mathbb{Q})$ we see that

$$\begin{aligned}\mathrm{Tr}\left(\xi^{\langle hr\rangle}\zeta^{ahr+bhs}\right) &= \mathrm{Tr}\left(\xi^{\langle hr\rangle - hr}\cdot\zeta^{h(ar+bs+n'r/2)}\right)\\ &= (-1)^{\lfloor hr/n\rfloor}\cdot\mathrm{Tr}\left(\zeta^{ar+bs+n'r/2}\right)\end{aligned}$$

and so the projection of $\mathrm{reg}\,\varphi_{r,s,t}A^aB^b\eta\alpha$ is

$$\frac{1}{2\pi n}\,\mathrm{Tr}\left(\zeta^{ar+bs+n'r/2}\right)\sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n},\frac{\langle hr\rangle}{n}\right)(-1)^{\lfloor hr/n\rfloor}\,\mathrm{Im}\,\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle} \tag{3.17}$$

These expressions are linearly related no matter what values are taken by a and b thus we can only hope to get a single independent element of K_2 from elements of this form.

If we are dealing with a primitive triple and r is even then we must have s odd. Since $n/(2n')$ is an integer with no odd factors we can define $a = 0$ and b by

$$b = n' \cdot [-r/(2s) \bmod (n/(2n'))]$$

so that $ar + bs + n'r/2$ is some multiple of $n/2$ and we are taking the trace of ± 1 in equation (3.17).

The preceding work is summarised by the following proposition.

Proposition 3.3.3. *Suppose we have a primitive triple (r, s, t) with $r + s + t = n$. Let n' be the largest odd factor of n and define the rational element $\alpha = \{1 - x, 1 - y\}$ in $K_2 F_n$.*

If n is odd then the image of $\mathcal{S}^\Sigma$ under the regulator is generated by $\operatorname{reg} \alpha$, $\operatorname{reg} \eta\alpha$ *and* $\operatorname{reg} \eta^2 \alpha$.

If n is even and r and s are odd then the image of $\mathcal{S}^\Sigma$ under the regulator in $H^1(A_{r,s,t}(\mathbb{C}), \mathbb{R}(1))$ is generated by only $\operatorname{reg} \alpha$.

If one of r and s is even then without loss of generality we may assume that it is r and that s is odd. Define $b = n' \cdot [-r/(2s) \bmod (n/2n')]$ then the automorphism B^b is defined over the field $\mathbb{Q}(\mu_{n/n'})$. Define β in $K_2 F_n$ by

$$\beta = \operatorname{Tr}_{\mathbb{Q}(\mu_{2n/n'})/\mathbb{Q}} B^b \eta \alpha$$

then the image of $\mathcal{S}^\Sigma$ under the regulator in $H^1(A_{r,s,t}(\mathbb{C}), \mathbb{R}(1))$ is generated by $\operatorname{reg} \alpha$ *and* $\operatorname{reg} \beta$.

Now that the fields of definition of all our elements are known we can write concrete and final expressions for the regulators. For any n we have

$$\operatorname{reg} \varphi_{r,s,t} \alpha = \frac{-1}{2\pi n} \sum_{h \in H_{r,s,t}} G\left(\frac{\langle hr \rangle}{n}, \frac{\langle hs \rangle}{n} \right) \cdot \operatorname{Im} \omega_{\langle hr \rangle, \langle hs \rangle, \langle ht \rangle}. \tag{3.18}$$

For odd n we have

$$\operatorname{reg}\varphi_{r,s,t}\eta\alpha = \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n},\frac{\langle hr\rangle}{n}\right)\cdot(-1)^{\langle hr\rangle}\operatorname{Im}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle} \tag{3.19}$$

$$\operatorname{reg}\varphi_{r,s,t}\eta^2\alpha = \frac{1}{2\pi n}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle hs\rangle}{n},\frac{\langle ht\rangle}{n}\right)\cdot(-1)^{\langle hs\rangle}\operatorname{Im}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle} \tag{3.20}$$

and when r and n are both even we have (up to sign at least)

$$\operatorname{reg}\varphi_{r,s,t}\operatorname{Tr}B^b\eta\alpha = \frac{1}{2\pi n'}\sum_{h\in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n},\frac{\langle hr\rangle}{n}\right)(-1)^{\lfloor hr/n\rfloor}\operatorname{Im}\omega_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}. \tag{3.21}$$

3.4 Some further relations

The following proposition details some relations between the stated elements when projecting onto the curves $A_{1,1,n-2}$, $A_{1,n-2,1}$ and $A_{n-2,1,1}$ (which is isomorphic to $A_{1,(n-1)/2,(n-1)/2}$ when n is odd). This means that the rank of the images will be one less than we could hope for based on the previous proposition.

Proposition 3.4.1. *For n odd we have*

$$\operatorname{reg}\varphi_{1,1,n-2}\left(\eta\alpha-\eta^2\alpha\right)=0$$
$$\operatorname{reg}\varphi_{1,n-2,1}\left(\alpha+\eta^2\alpha\right)=0$$
$$\operatorname{reg}\varphi_{n-2,1,1}\left(\alpha+\eta\alpha\right)=0$$

And for n even we have

$$n\operatorname{reg}\varphi_{n-2,1,1}\alpha = n'\operatorname{reg}\varphi_{n-2,1,1}\beta$$

up to sign.

Proof. The first identity follows from equations (3.19) and (3.20) and the symmetry of G. For the remaining identities we note that $H_{1,n-2,1}$ and $H_{n-2,1,1}$ consist of those numbers h that are prime to n and satisfy $0<h<\frac{n}{2}$. This observation leads to the fact that

$$(-1)^{\langle h(n-2)\rangle} = (-1)^{n-2h} = -1$$

when n is odd and

$$(-1)^{\lfloor h(n-2)/n\rfloor} = (-1)^{\lfloor h-2h/n\rfloor} = (-1)^{h-1} = 1$$

when n is even since all h are odd in that case.

From these calculations the claimed identities easily follow from equations (3.18)-(3.21). ■

3.4.1 Calculating traces in K_2

Once we have explained how to compute traces in K_2, the relations given in Proposition 3.4.1 for odd values of n can actually be proven in terms of K-theory instead of just on the regulator level. Suppose $E \subset F$ is a finite field extension and let

$$\mathrm{Tr} : K_2(F) \to K_2(E)$$

be the trace map. By Matsumoto's theorem it is possible to express the trace of any symbol in $K_2(F)$ as a sum of symbols in $K_2(E)$. In [23] Rosset and Tate describe a reciprocity law which gives rise to an algorithm for computing such an expression.

Before describing the algorithm let us first introduce some notation. If a polynomial p can be written as

$$p(T) = a_n T^n + a_{n-1} T^{n-1} + \cdots + a_m T^m$$

with $n \geq m$ and $a_m a_n \neq 0$ then we let

$$p^*(T) = (a_m T^m)^{-1} p(T)$$
$$c(p) = (-1)^n a_n$$

Given these definitions we have the following proposition

Proposition 3.4.2 (Rosset and Tate)**.** *Let $E \subset F$ be a finite extension of fields and let $x, y \in F^\times$.*

Let $g(T) \in E[T]$ be the monic irreducible polynomial with root x and let $f(T) \in E[T]$ be the polynomial of smallest degree such that $N_{F/E(x)} y = f(x)$.

Finally, let g_0, g_1, …, $g_m \neq 0$, $g_{m+1} = 0$ be the sequence of polynomials of strictly decreasing degree defined by

$$g_0 = g \qquad\qquad g_1 = f$$

and for $i \geq 1$ by

$$g_{i+1} \equiv g_{i-1}^* \mod g_i \tag{3.22}$$

provided $g_i \neq 0$. Then we have that

$$1 \leq m \leq \deg g = [E(x) : E] \leq [F : E]$$

and

$$\mathrm{Tr}_{F/E}\{x, y\} = -\sum_{i=1}^{m} \{c(g_{i-1}^*), c(g_i)\}$$

3.4.2 Proof that $\varphi_{1,1}\eta\,\alpha = \varphi_{1,1}\eta^2\alpha$

We are now ready to prove the relation that we are seeking.

Proposition 3.4.3. *For odd n we have $\varphi_{1,1}\eta\,\alpha = \varphi_{1,1}\eta^2\alpha$ up to torsion in $C_{1,1,n-2}$.*

Proof. Let us recall that $\eta\alpha = \{1 - 1/y, x + y\}$ and $\eta^2\alpha = \{x + y, 1 - 1/x\}$, thus we are given the task of proving that ${\varphi_{1,1}}_*\{x+y, \frac{x-1}{x}\frac{y-1}{y}\}$ is torsion in $K_2C_{1,1}$.

The function field of the Fermat curve F_n is given by $\mathbb{Q}(x,y)$, while the function field of the quotient curve $C_{1,1}$ under the natural inclusion induced by $\varphi_{1,1}$ is given by $\mathbb{Q}(x^n, xy)$. The proof that the aforementioned trace is torsion hinges on the fact that the minimal polynomial of $x+y$ over the subfield $\mathbb{Q}(x^n, xy)$ is actually contained in $\mathbb{Q}(xy)[T]$. This can be seen as follows.

First, observe that

$$x^{k+1} + y^{k+1} = (x+y)(x^k + y^k) - xy(x^{k-1} + y^{k-1})$$

Thus, if we define polynomials $h_k \in \mathbb{Q}(xy)[T]$ by $h_0(T) = 2$, $h_1(T) = T$ and for $k \geq 1$ by

$$h_{k+1}(T) = Th_k(T) - xyh_{k-1}(T)$$

then by induction we have that $x^k + y^k = h_k(x+y)$. Since $x^n + y^n = 1$ and $h_n(T)$ is monic of degree n we see that $x+y$ satisfies the polynomial $h_n(T) - 1$.

We can show that this is the minimal polynomial of $x+y$ as follows. The minimal polynomial will not change when we extend scalars to $\mathbb{C}$. Since $\mathbb{C}(x,y)$ contains the n-th roots of unity the extension $\mathbb{C}(x,y)$ over $\mathbb{C}(xy, x^n)$ is a Kummer extension having Galois group $\mathbb{Z}/n\mathbb{Z}$, with $j \in \mathbb{Z}/n\mathbb{Z}$ mapping $x+y$ to $\zeta^j x + \zeta^{-j} y$, which is never equal to $x+y$ for $j \neq 0$ since x/y is non-constant. Therefore $x+y$ has degree n over $\mathbb{C}(xy, x^n)$ and thus $h_n(T) - 1$ is its minimal polynomial.

We want to compute the trace of the element $\{x+y, \frac{1-(x+y)+xy}{xy}\}$. The polynomial $f \in \mathbb{Q}(x^n, xy)[T]$ of minimal degree satisfying $f(x+y) = \frac{1-(x+y)+xy}{xy}$ is clearly $\frac{1-T+xy}{xy}$. Thus f also belongs to $\mathbb{Q}(xy)[T]$ and so all polynomials encountered during the Rosset-Tate algorithm will be in $\mathbb{Q}(xy)[T]$ too. Consequently all symbols in the representation of the result will lie in $\mathbb{Q}(xy)$. Therefore the result can be expressed as a pull-back from an element of $K_2\mathbb{P}^1$ which is known to be torsion. ■

Applying the transformation η to the relation in previous proposition will yield the other relations from the first part of Proposition 3.4.1. In the case of an odd prime p this will reduce our upper bound for the rank of $\mathcal{S}^{\Sigma}$ to $3(p-3)$ so for $n = 7$ the maximum possible rank of $\mathcal{S}^{\Sigma}$ is equal to 12 – three less than the rank of 15 that is predicted by Beilinson's conjecture.

3.4.3 An aside on $\varphi_{1,1}\alpha$

While we are computing traces let us try to find an expression for $\varphi_{1,1}\alpha$. It will turn out that this has a useful application in the next section.

Proposition 3.4.4. *$\varphi_{1,1}\alpha = 3n\{1 - xy, x\}$ up to torsion in $K_2C_{1,1}$.*

Proof. Under the natural inclusion induced by $\varphi_{1,1}$, the function field of $C_{1,1}$ sits as $E = \mathbb{Q}(x^n, xy)$ inside the field $F = \mathbb{Q}(x, y)$ that is the function field of F_n.

Using the fact that $\{1-x, x\} = 0$ we see that $\{1-x, 1-y\} = \{1-x, x-xy\}$ and it will be easier for us to compute the trace of this symbol.

The minimal polynomial of $1 - x$ over E is given by $g(T) = (T - 1)^n - (-1)^n x^n$. Thus $E(1 - x) = F$ and so we require $f \in E[T]$ to be the polynomial of smallest degree satisfying $f(1 - x) = x - xy$. Clearly we must have $f(T) = 1 - T - xy$.

We are now ready to begin the algorithm. We have

$$g_0(T) = (T-1)^n - (-1)^n x^n$$
$$g_0^*(T) = \frac{(-1)^n}{1-x^n} g_0(T) = \frac{(-1)^n}{y^n} g_0(T)$$

$$g_1(T) = 1 - T - xy$$
$$g_1^*(T) = \frac{-T}{1-xy} + 1$$

$$g_2(T) = g_0^* \mod g_1 = \frac{(-1)^n}{y^n}((-xy)^n - (-1)^n x^n) = \frac{(xy)^n - x^n}{y^n} = \frac{-x^{2n}}{y^n}$$
$$g_2^*(T) = 1$$

$$g_3(T) = g_1^* \mod g_2 = 0$$

Thus $m = 2$ and

$$\begin{aligned}
\mathrm{Tr}\{1-x, 1-y\} &= -\{c(g_0^*), c(g_1)\} - \{c(g_1^*), c(g_2)\} \\
&= -\{\frac{1}{y^n}, 1\} - \{\frac{1}{1-xy}, \frac{-x^{2n}}{y^n}\} \\
&= \{1-xy, \frac{-x^{3n}}{(xy)^n}\} \\
&= 3n\{1-xy, -x\}
\end{aligned}$$

which clearly gives the desired result up to torsion. ■

Remark 3.4.5. The element $\{1-xy, x\}$ was actually already known in the literature. Even before Ross introduced the element $\{1-x, 1-y\}$ he gave the element $\{1-xy, x\}$ in [21] and proved that its regulator was non-zero. Later on, Kimura [13] used this element together with another to numerically confirm Beilinson's conjecture on the genus 2 quotient $C_{1,1,3}$ of F_5. After taking the trace from F_5 down to $C_{1,1,3}$ he proved that those two elements spanned a rank

2 subgroup of $C_{1,1,3}$ and that the determinant of the change of basis matrix involved was equal, up to a simple rational number, to the value of the leading coefficient of the L-function of $C_{1,1,3}$ at $s = 2$ to 12 or so decimal places.

Now that we know that $\{1 - xy, x\}$ is just an expression for the projection of $\{1 - x, 1 - y\}$ under the map $\varphi_{1,1}$ it is clear that we cannot use it to increase the known rank of $K_2 F_n$. On the positive side the above calculation gives us some insight for how to evaluate the regulators on a computer as we shall now see.

3.5 Evaluating the regulators on computer

Following Ross' calculation of the regulator of $\{1 - xy, x\}$ in [21] we have

$$\begin{aligned}
\operatorname{Im} \int_\gamma \log(1 - \zeta^a xy) \operatorname{dlog} x &= \operatorname{Im} \int_0^1 \log(1 - \zeta^a t^{1/n} (1-t)^{1/n}) \operatorname{dlog} (1-t)^{1/n} \\
&= \frac{1}{n} \operatorname{Im} \int_0^1 \sum_{j=1}^{\infty} \frac{-1}{j} \zeta^{aj} t^{j/n} (1-t)^{j/n} \frac{-1}{1-t} dt \\
&= \frac{1}{n} \operatorname{Im} \sum_{j=1}^{\infty} \frac{1}{j} \zeta^{aj} B\left(\frac{j}{n} + 1, \frac{j}{n}\right) \\
&= \frac{1}{2n} \operatorname{Im} \sum_{j=1}^{\infty} \frac{1}{j} \zeta^{aj} B\left(\frac{j}{n}, \frac{j}{n}\right)
\end{aligned}$$

which results in

$$\int_{\kappa_{a,b}} \operatorname{reg}\{1 - xy, x\} = \frac{1}{4\pi i n} \operatorname{Im} \sum_{j=1}^{\infty} \frac{1}{j} \zeta^{(a+b)j} (1 - \zeta^j)^2 B\left(\frac{j}{n}, \frac{j}{n}\right) \tag{3.23}$$

But from the previous proposition we know that $3n\{1 - xy, x\}$ is equal to $\varphi_{1,1}^* \varphi_{1,1*} \alpha$. Since $C_{1,1}$ is the quotient of F_n by the group of automorphisms generated by AB^{-1} we have

$$\varphi_{1,1}^* \varphi_{1,1*} \alpha = \sum_{l=0}^{n-1} A^l B^{-1} \alpha$$

and using equation (3.10) we can compute

$$\begin{aligned}
\int_{\kappa_{a,b}} \operatorname{reg}\{1-xy,x\} &= \frac{1}{3n}\int_{\kappa} \operatorname{reg}\sum_{l=0}^{n-1} A^{a+l}B^{b-l}\alpha \\
&= \frac{1}{6\pi i n^2}\sum_{l=0}^{n-1}\sum_{j,k=1}^{\infty}\frac{1}{k+j}B\left(\frac{k}{n},\frac{j}{n}\right)(1-\zeta^k)(1-\zeta^j)\zeta^{(a+l)k+(b-l)j} \\
&= \frac{1}{6\pi i n^2}\sum_{j,k=1}^{\infty}\frac{1}{k+j}B\left(\frac{k}{n},\frac{j}{n}\right)(1-\zeta^k)(1-\zeta^j)\zeta^{ak+bj}\sum_{l=0}^{n-1}\zeta^{l(k-j)} \\
&= \frac{1}{6\pi i n}\sum_{k\equiv j \pmod n}\frac{1}{k+j}B\left(\frac{k}{n},\frac{j}{n}\right)(1-\zeta^k)(1-\zeta^j)\zeta^{ak+bj}
\end{aligned}$$

The fact that this equation and equation (3.23) are equivalent but feature a two dimensional sum and a one dimensional sum respectively was intriguing. Eventually a series identity was discovered that could be used to prove their equivalence in a direct manner.

Proposition 3.5.1. *For $x, y > 0$*

$$\sum_{k=0}^{\infty}\frac{B(x,y+k)}{x+y+k} = \frac{B(x,y)}{x}$$

Proof. Let $c_k = \frac{B(x,y+k)}{x+y+k} = \frac{\Gamma(x)\Gamma(y+k)}{\Gamma(x+y+k+1)}$. Then

$$\begin{aligned}
\frac{c_{k+1}}{c_k} &= \frac{\Gamma(x)\Gamma(y+k+1)}{\Gamma(x+y+k+2)}\left(\frac{\Gamma(x)\Gamma(y+k)}{\Gamma(x+y+k+1)}\right)^{-1} \\
&= \frac{y+k}{x+y+k+1}
\end{aligned}$$

Therefore we have a hypergeometric series and

$$\sum_{k=0}^{\infty}\frac{B(x,y+k)}{x+y+k} = c_0 \cdot {}_2F_1(y,1;x+y+1;1)$$

However, Gauss' Hypergeometric Theorem [1] gives

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

for $\mathrm{Re}\,(c-a-b)>0$ so we see that

$$\sum_{k=0}^{\infty}\frac{B(x,y+k)}{x+y+k}=\frac{B(x,y)}{x+y}\cdot\frac{\Gamma(x+y+1)\Gamma(x)}{\Gamma(x+1)\Gamma(x+y)}=\frac{B(x,y)}{x}$$

■

The benefit of this identity is that we can use it to write the function F in such a way that it can be computed as an exponentially decreasing sum.

If we define

$$K(x,y)=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{B\left(x+j,y+k\right)}{x+j+y+k}$$

then we have

$$\begin{aligned}F(x,y)&=\sum_{j,k=0}^{\infty}\frac{B\left(x+j,y+k\right)}{x+j+y+k}\\&=\sum_{k\geq j}\frac{B\left(x+j,y+k\right)}{x+j+y+k}+\sum_{k<j}\frac{B\left(x+j,y+k\right)}{x+j+y+k}\\&=K(x,y)+\sum_{j-1\geq k}\frac{B\left(x+j,y+k\right)}{x+j+y+k}\\&=K(x,y)+K(y+1,x)\end{aligned}$$

but using our series identity we see that

$$\begin{aligned}K(x,y)&=\sum_{j=0}^{\infty}\sum_{k=j}^{\infty}\frac{B\left(x+j,y+k\right)}{x+j+y+k}\\&=\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{B\left(x+j,y+j+k\right)}{x+j+y+j+k}\\&=\sum_{j=0}^{\infty}\frac{B\left(x+j,y+j\right)}{x+j}\end{aligned}$$

The first term in the sum for $K(x,y)$ is $\frac{1}{x}B(x,y)$ and the ratio of the subsequent

Beta terms is given by

$$\begin{aligned}
&\frac{B(x+j+1, y+j+1)}{B(x+j, y+j)} \\
=&\frac{\Gamma(x+j+1)\Gamma(y+j+1)}{\Gamma(x+y+2j+2)} \frac{\Gamma(x+j+2j)}{\Gamma(x+j)\Gamma(y+j)} \\
=&\frac{(x+j)(y+j)}{(x+y+2j+1)(x+y+2j)}
\end{aligned}$$

which tends to $\frac{1}{4}$ as j tends to infinity.

In terms of computer code we can write

```
K(x, y, tolerance) {
  s = beta(x,y);
  t = 0;
  while(abs(s) > tolerance,
      t += s / x;
      s *= x * y / ((x+y+1) * (x+y));
      x += 1;
      y += 1;
  );
  return t;
}
```

which is both simple and quickly convergent.

3.5.1 Determining the rank of K_2 numerically

Now that we can easily compute the regulator values we can determine the rank of $\mathcal{S}^{\Sigma}$ for small values of n. Remember that Proposition 3.3.3 states that for n odd the rank of $\mathcal{S}^{\Sigma}$ on each quotient is at most 3. When n is even and one of r and s is even it is at most 2 and when both r and s are odd it is at most 1.

Proposition 3.4.1 shows that on quotients of the form $A_{1,1,n-2}$, $A_{1,n-2,1}$ and $A_{n-2,1,1,}$ the rank of the projection of $\mathcal{S}^{\Sigma}$ is at most 2 when n is odd. Also it shows that when n is even, the rank for quotients of the form $A_{n-2,1,1}$ and $A_{1,n-2,1}$ is at most 1.

For all primitive triples (r, s, t) with $n = r + s + t \leq 100$ we computed regulator values for the elements stated in Proposition 3.3.3. Except for the relations outlined in Proposition 3.4.1 no other relations were found between any of the K_2 elements except for when $n = 3$ and $n = 6$.

When $n = 3$ the only quotient is the curve $C_{1,1,1}$ of genus 1. The relations stated in Proposition 3.4.1 are actually all taking place under the projection by $\varphi_{1,1,1}$ and we evidently have the relations $\operatorname{reg} \eta\alpha = \operatorname{reg} \eta^2\alpha = -\operatorname{reg} \alpha$.

When $n = 6$ we might expect that there can be two independent elements on the quotient $A_{2,1,3}$. If we actually compute expressions for the projections using equations (3.18) and (3.21) we get $\frac{1}{6}G\left(\frac{2}{6}, \frac{1}{6}\right) \cdot \operatorname{Im} \omega_{2,1,3}$ and $\frac{1}{3}G\left(\frac{3}{6}, \frac{2}{6}\right) \cdot \operatorname{Im} \omega_{2,1,3}$ respectively. It is clear that these will be related over $\mathbb{R}$ but in fact they appear to be related over $\mathbb{Q}$ due to the apparent identity $G\left(\frac{2}{6}, \frac{1}{6}\right) / G\left(\frac{3}{6}, \frac{2}{6}\right) = 3$. This appears to be a non-trivial identity of hypergeometric functions.

In table 3.1 we detail the rank of the subgroups $\mathcal{S}^\Sigma$ for all values of n up to 12. The first column gives the value of n; the second column gives the dimension of the primitive quotients of F_n; the third column gives the number of primitive quotients; the fourth column gives the rank of $\mathcal{S}^\Sigma$ that comes from the primitive quotients; the fifth column gives the rank of $\mathcal{S}^\Sigma$ that comes from any other quotients; the sixth column gives the total rank of $\mathcal{S}^\Sigma$; and the seventh column gives the rank of K_2F_n predicted by Beilinson's conjecture.

If n is odd and greater than 5 there should be three independent elements for each primitive quotient except for the three special cases given by Proposition 3.4.1: indeed we see that $Rk_{prim} = 3 \cdot (N_{prim} - 1)$ in all cases.

If n is even then primitive quotients (r, s, t) must have one even element and two odd elements amongst the three. In other words, exactly one third of the primitive quotients are of the form (r, s, t) with r and s both odd. On these quotients we expect to find just one independent element of K_2 and on the others (except for $A_{1,n-2,1}$ and $A_{n-2,1,1}$) we expect to find two. Indeed, checking the table we see that the formula $Rk_{prim} = \frac{5}{3}N_{prim} - 2$ is satisfied by all even n greater than 6.

On any quotient $A_{r,s,t}$ such that the known rank of K_2 is equal to the dimension $\varphi(r+s+t)/2$ predicted by Beilinson we can attempt a numerical verification of Beilinson's conjecture by comparing the determinant of the regulator values

n	g	N_{prim}	Rk_{prim}	Rk_{other}	Rk_{total}	Rk_{pred}
3	1	1	1	0	1	1
4	1	3	3	0	3	3
5	2	3	6	0	6	6
6	1	9	9	1	10	10
7	3	5	12	0	12	15
8	2	9	13	3	16	21
9	3	9	24	1	25	28
10	2	15	23	6	29	36
11	5	9	24	0	24	45
12	2	21	33	$3+10$	46	55

Table 3.1: Details of the ranks of the subgroups $\mathcal{S}^{\Sigma}$ for n up to 12

to the value of the L-function. This will be attempted in the next chapter. Ignoring elliptic curves, we are able to find enough independent elements of K_2 in the following non-isomorphic cases:

- $A_{1,1,3}$ a genus 2 quotient of F_5.
- $A_{1,2,4}$ a genus 3 quotient of F_7.
- $A_{1,2,5}$ and $A_{1,4,3}$ genus 2 quotients of F_8.
- $A_{1,2,6}$ a genus 3 quotient of F_9.
- $A_{1,2,7}$ and $A_{1,4,5}$ genus 2 quotients of F_{10}.
- $A_{1,2,9}$, $A_{1,4,7}$, $A_{1,6,5}$ and $A_{1,8,3}$ genus 2 quotients of F_{12}.

(It should be noted that $A_{1,6,3}$, the quotient of F_{10}, is isomorphic to $A_{1,2,7}$ via $A_{1,6,3} \longrightarrow A_{3,6,1} \longrightarrow A_{21,42,7} \longrightarrow A_{1,2,7}$)

3.5.2 Comparison to previous work

As already mentioned, Beilinson's conjecture was verified numerically on the quotient $A_{1,1,3}$ by Kimura in [13]. In this thesis we repeat the numerical verification but to a much higher precision.

More recently the same subgroup $\mathcal{S}^{\Sigma}$ of K_2F_n was studied by Otsubo in [17] and [18]. When n is odd our results are identical but when n is even the results given in this thesis are more complete.

Instead of the automorphism η that we defined by $\eta(x,y) = (1/y, \xi x/y)$, Otsubo uses an automorphism η' of order 2 defined by

$$\eta'(x,y) = \left(\frac{\zeta}{x}, \frac{\xi' y}{x}\right)$$

where $\xi' = e^{\frac{i\pi}{n}}$. Otsubo considers the elements $\operatorname{Tr}\eta'\alpha$ but does not examine elements of the form $\operatorname{Tr}B^b\eta'\alpha$. Due to these differences Otsubo only finds potentially non-trivial elements of K_2 on the quotient $A_{r,s,t}$ when r is congruent to -2 modulo n/n'.

If n is divisible by 2 but not 4 (for example on F_{10}) then $n/n' = 2$ and Otsubo finds just as many elements as in this work. When n is divisible by 4 he misses some elements that are non-zero.

For example, when $n = 12$ we have $n/n' = 4$ and Otsubo is able to find two independent elements of K_2 on the quotients $A_{1,2,9}$ and $A_{1,6,5}$ (on the quotient $A_{1,10,1}$ there is only one known element due to Proposition 3.4.1). He numerically verifies Beilinson's conjecture in these cases but is unable to do the same for the quotients $A_{1,4,7}$ and $A_{1,8,3}$.

When $n = 8$ we have $n/n' = 8$ and unfortunately for Otsubo the only quotient with $r \equiv -2 \pmod 8$ is $A_{6,1,1}$. By Proposition 3.4.1 we know that there can only be a single element there. As a result Otsubo is unable to verify the conjecture on quotients $A_{1,2,5}$ and $A_{1,4,3}$.

Remark 3.5.2. Some historical details are perhaps worth mentioning. I first verified Beilinson's conjecture on $A_{1,2,4}$ in 2005 and on $A_{1,2,6}$ in 2012. Both of these were before Otsubo's publication. My work with n even took place later than Otsubo's publication but was almost completely independent. I found the elements in Proposition 3.3.3 that have $b = 0$ in late 2015 and very soon afterwards became aware of Otsubo's second paper. The differences between our approaches urged me to investigate elements of the form $B^b\eta\alpha$ and the full version of Proposition 3.3.3 was then proved.

Chapter 4

L-functions of Fermat Curves

The L-function of the variety $A_{r,s,t}$ has an Euler product of the form

$$L_{r,s,t}(s) = \prod_{l} P_l(l^{-s})^{-1}$$

where P_l is a polynomial with integral coefficients of degree $\phi(n) = 2g$. The complete description of $P_l(T)$ is given by Davenport-Hasse [6] and Weil [28]. To begin with we restrict ourselves to primes l that do not divide n and explain what happens at the other primes later. If $\mathfrak{l}$ is a prime in $\mathbb{Q}(\mu_n)$ dividing l, let $\chi_{\mathfrak{l}}$ be the n-th power residue symbol given by

$$\chi_{\mathfrak{l}}(a) = \zeta^k \iff a^{\frac{N\mathfrak{l}-1}{n}} \equiv \zeta^k \pmod{\mathfrak{l}}.$$

The Jacobi sum

$$\tau_{r,s,t}(\mathfrak{l}) = -\sum_{a \in \mathcal{O}/\mathfrak{l}} \chi_{\mathfrak{l}}^r(a)\chi_{\mathfrak{l}}^s(1-a)$$

is an integer in $\mathbb{Q}(\mu_n)$ with absolute value $(N\mathfrak{l})^{1/2}$ in a any complex embedding and we have the basic formula

$$P_l(T) = \prod_{\mathfrak{l}|l} \left(1 - \tau_{r,s,t}(\mathfrak{l})T^f\right)$$

where f is the order of $l \bmod n$.

If we let

$$\Lambda_{r,s,t}(s) = L_{r,s,t}(s) N_{r,s,t}^{s/2} \left((2\pi)^{-s}\Gamma(s)\right)^g$$

where $N_{r,s,t}$ is the conductor of $J_{r,s,t}$ then we will show that Λ satisfies the functional equation

$$\Lambda_{r,s,t}(s) = W_{r,s,t}\Lambda_{r,s,t}(2-s)$$

where $W_{r,s,t} \in \{\pm 1\}$ is the root number.

If we define $m = \sum_{h \in H_{r,s,t}} \left\langle h^{-1} \right\rangle$ then it is shown in [11] that when n is a prime p that

$$N_{r,s,t} = \begin{cases} p^{p-1} & \text{if } p \mid m \\ p^p & \text{otherwise} \end{cases}$$

$$W_{r,s,t} = \begin{cases} \left(\frac{2}{p}\right) & \text{if } p \mid m \\ (-1)^m \left(\frac{-m}{p}\right) & \text{otherwise} \end{cases}$$

We are interested in computing the conductor and root number in cases where n is not prime and to achieve this goal we need to investigate the L-function's interpretation as a Hecke L-function.

Another remark is that if f is even and n is prime then Gross and Rohrlich show that

$$P_l(T) = \left(1 + l^{f/2}T^f\right)^{(n-1)/f}$$

If we replace $n-1$ with $\phi(n)$ then this equation appears to be true for composite values of n provided $l^{f/2} \equiv -1 \pmod{n}$ although I do not have a proof.

4.1 Introducing Hecke L-functions

In this section we follow the exposition of Hecke L-functions in [25]. Let $\mathbb{K}$ be a number field of over $\mathbb{Q}$ with r_1 real embeddings and r_2 pairs of complex conjugate embeddings of $\mathbb{K}$ into $\mathbb{C}$. For an element $\alpha \in \mathbb{K}$ denote the conjugates of α by

$$\alpha_1, \ldots, \alpha_{r_1}, \alpha_{r_1+1}, \overline{\alpha_{r_1+1}}, \ldots, \alpha_{r_1+r_2}, \overline{\alpha_{r_1+r_2}}$$

where $\alpha_j \in \mathbb{R}$ for $1 \leq j \leq r_1$.

Let $\mathfrak{f}$ be an integral ideal in $\mathbb{K}$ and denote by $\mathbb{I}(\mathfrak{f})$ the multiplicative group generated by all ideals coprime to $\mathfrak{f}$. Define the *principal ray class* $\mathbb{P}(\mathfrak{f})$ to be the subgroup of $\mathbb{I}(\mathfrak{f})$ consisting of all principal ideals of the form (α/β) satisfying

- $0 \neq \alpha, \beta \in \mathcal{O}_{\mathbb{K}}$;
- $\alpha \equiv \beta \pmod{\mathfrak{f}}$;
- α/β is totally positive, i.e. all of its real conjugates are positive.

Suppose we have numbers a_j and ν_k satisfying

- $a_j \in 0, 1$ for $1 \leq j \leq r_1$ and $a_j \in \mathbb{Z}$ for $r_1 < j \leq r_1 + r_2$;
- $\nu_k \in \mathbb{R}$ for $1 \leq k \leq r_1 + r_2$ and $\nu_1 + \cdots + \nu_{r_1+r_2} = 0$.

Then we define a function $\chi_\infty : \mathbb{K}^\times \to \mathbb{C}^\times$ by

$$\chi_\infty(\alpha) = \prod_{k=1}^{r_1+r_2} \left|\alpha^{(k)}\right|^{i\nu_k} \prod_{j=1}^{r_1+r_2} \left(\frac{\alpha^{(j)}}{\left|\alpha^{(j)}\right|}\right)^{a_j}$$

Suppose that $\chi_\infty(\epsilon) = 1$ for all units $\epsilon \equiv 1 \pmod{\mathfrak{f}}$ that are totally positive, then χ_∞ induces a character on $\mathbb{P}(\mathfrak{f})$.

If a non-trivial homomorphism $\chi : \mathbb{I}(\mathfrak{f}) \to \mathbb{C}^\times$ agrees with χ_∞ on $\mathbb{P}(\mathfrak{f})$ i.e.

$$\chi(\mathfrak{a}) = \chi_\infty(\alpha)$$

for all $\mathfrak{a} = (\alpha) \in \mathbb{P}(\mathfrak{f})$ then χ is said to be a *grossencharacter* modulo $\mathfrak{f}$. If $\mathfrak{f} \subset \mathfrak{f}^*$ then $\mathbb{I}(\mathfrak{f}) \subset \mathbb{I}(\mathfrak{f}^*)$ and if χ^* is a grossencharacter modulo $\mathfrak{f}^*$ and $\chi^* = \chi$ on $\mathbb{I}(\mathfrak{f})$

then χ is said to be induced by χ^*, otherwise we say that χ is *primitive* and $\mathfrak{f}$ is its *conductor*.

Next we define the Hecke L-function associated with a grossencharacter χ by

$$L(s,\chi) = \prod_{\mathfrak{p}} \left(1 - \chi(\mathfrak{p})\,(N\mathfrak{p})^{-s}\right)^{-1} \tag{4.1}$$

where the product is taken over all prime ideals $\mathfrak{p}$ coprime to $\mathfrak{f}$.

Hecke in [12] proved that $L(s,\chi)$ extends to an entire function and satisfies a functional equation of the usual type if χ is primitive. Define

$$\gamma(\chi) = \prod_{k=r_1+1}^{r_1+r_2} 2^{i\nu_k/2}, \quad A(\mathfrak{f}) = \left(\frac{|\mathrm{disc}\,\mathbb{K}|\,N(\mathfrak{f})}{\pi^{r_1+2r_2}}\right)^{\frac{1}{2}} \cdot 2^{-r_2},$$

$$\Gamma(s,\chi) = \prod_{j=1}^{r_1} \Gamma\left(\frac{s+a_j-i\nu_j}{2}\right) \prod_{j=r_1+1}^{r_1+r_2} \Gamma\left(s+\frac{|a_j|-i\nu_j}{2}\right)$$

and

$$\Lambda(s,\chi) = \gamma(\chi)\,A(\mathfrak{f})^s\,\Gamma(s,\chi)\,L(s,\chi)$$

Then

$$\Lambda(1-s,\chi) = \omega(\chi)\Lambda(s,\overline{\chi})$$

where $\omega(\chi)$ is a complex number of magnitude 1 depending only on χ.

4.2 Jacobi sums as Grossencharacter

Let us fix $\mathbb{K}$ as $\mathbb{Q}(\mu_n)$ of degree $\phi(n) = 2g$ over $\mathbb{Q}$. This field has no real embeddings and g pairs of complex conjugate embeddings into $\mathbb{C}$. Define the element $\sigma_t \in \mathrm{Gal}(\mathbb{Q}(\mu_n)/\mathbb{Q})$ by $\sigma_t : \zeta \mapsto \zeta^t$ and the group-ring element

$$\omega_{r,s,t} = \sum_{t\in(\mathbb{Z}/n\mathbb{Z})^\times} \left\lfloor \frac{\langle tr\rangle + \langle ts\rangle}{n} \right\rfloor \sigma_{-t}^{-1}$$

where the $\lfloor . \rfloor$ denotes the integer part. If we extend the definition of $\tau_{r,s,t}$ multiplicatively to non-prime ideals then we have the so-called Stickelberger relation [26] which is a prime ideal decomposition of the Jacobi sum $\tau_{r,s,t}(\mathfrak{a})$ in terms of the above group-element:

$$(\tau_{r,s,t}(\mathfrak{a})) = \mathfrak{a}^{\omega_{r,s,t}}$$

In Weil's 1952 paper [29] he goes further by proving that

$$\tau_{r,s,t}((\alpha)) = \alpha^{\omega_{r,s,t}} \tag{4.2}$$

when $\alpha \equiv 1 \pmod{n^2}$.

Looking at the definition of $\omega_{r,s,t}$ more closely we notice that each integer part is either 0 or 1 and that

$$\frac{\langle tr \rangle + \langle ts \rangle}{n} + \frac{\langle -tr \rangle + \langle -ts \rangle}{n} = 2.$$

Therefore if σ_{-t}^{-1} appears in the sum then σ_t^{-1} will not (and vice versa).

Numbering the complex embeddings of $\mathbb{Q}(\mu_n)$ carefully we will be able to write

$$\tau_{r,s,t}((\alpha)) = \prod_{j=1}^{g} \alpha_j$$

when $\alpha \equiv 1 \pmod{n^2}$ so that

$$(N\alpha)^{-\frac{1}{2}}\, \tau_{r,s,t}((\alpha)) = \prod_{j=1}^{g} \frac{\alpha_j}{|\alpha_j|}$$

and thus $(N\mathfrak{a})^{-\frac{1}{2}}\, \tau_{r,s,t}(\mathfrak{a})$ is a grossencharacter modulo (n^2) in the notation of the previous section. Suppose that the conductor of the grossencharacter is $\mathfrak{f}$ and denote $N_{r,s,t} = |\mathrm{disc}\, \mathbb{Q}(\mu_n)|\, N\mathfrak{f}$. The functional form of the Hecke L-function will be

$$\Lambda(s,\chi) = \left(\frac{N_{r,s,t}}{(2\pi)^g}\right)^{\frac{s}{2}} \Gamma\left(s + \frac{1}{2}\right)^g L(s,\chi).$$

If we define

$$\Lambda_{r,s,t}(s) = \left(\frac{N_{r,s,t}}{\pi^{2g}}\right)^{\frac{s}{2}} \left[\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)\right]^g L_{r,s,t}(s)$$

then using the identities

$$L_{r,s,t}(s) = L(s-\tfrac{1}{2},\chi) \quad \text{and} \quad \Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right) = \sqrt{\pi}\cdot 2^{1-s}\cdot\Gamma(s)$$

we can easily verify that the functions $\Lambda_{r,s,t}(s)$ and $\Lambda(s-\frac{1}{2},\chi)$ are different by a constant:

$$\Lambda_{r,s,t}(s) = 2^{\frac{g}{2}} N_{r,s,t}^{\frac{1}{4}}\,\Lambda(s-\tfrac{1}{2},\chi)$$

and as a consequence the functional equation

$$\Lambda_{r,s,t}(2-s) = \omega(\chi)\Lambda_{r,s,t}(s)$$

is satisfied.

4.3 Finding the conductors

In order to find the conductor $N_{r,s,t}$ in the functional equation we need to be able to find the largest defining ideal $\mathfrak{f}$ for our grossencharacter. That is we must find the largest ideal $\mathfrak{f}$ such that equation

$$\tau_{r,s,t}((\alpha)) = \alpha^{\omega_{r,s,t}} \tag{4.3}$$

holds for all $\alpha \equiv 1 \pmod{\mathfrak{f}}$. As previously mentioned we know that $n^2\mathcal{O}_{\mathbb{K}} \subset \mathfrak{f}$.

For simplicity we stick to the case when $\mathbb{Q}(\mu_n)$ is a principal ideal domain which is true for $n < 23$ at least. Suppose we can factor n^2 into a product of primes

$$n^2 = \pi_1^{e_1}\cdot\pi_2^{e_2}\cdot\ldots\cdot\pi_k^{e_k}$$

Then our task is, when given a prime π_l and a number f such that $\pi_l f\mathcal{O}_{\mathbb{K}}$

satisfies the equation (4.3), to determine if the same also holds for $f\mathcal{O}_{\mathbb{K}}$.

For an element $\alpha \in \mathcal{O}_{\mathbb{K}}$ define $\mathcal{O}_\alpha$ to be the multiplicative subgroup of the field of fractions of $\mathcal{O}_{\mathbb{K}}$ with elements whose support is disjoint from that of α. Define $\mathcal{D}_\alpha$ to be the fractions of $\mathcal{O}_{\mathbb{K}}$ having denominator coprime to α and define the map $\mu : \mathcal{O}_{\pi_l f} \to \mathbb{K}^\times$ by

$$\mu(\alpha) = \alpha^{-\omega_{r,s,t}} \tau_{r,s,t}((\alpha)).$$

We know that the subgroup $1+\pi_l f\mathcal{D}_{\pi_l f}$ lies inside $\ker \mu$. Let $\mathcal{A}$ be the subgroup $\mathcal{O}_{\pi_l} \cap (1+f\mathcal{D}_f)$. If we can find a set of elements in $\mathcal{O}_{\pi_l f}$ that span the quotient $\mathcal{A}\,/\,(1+\pi_l f\mathcal{D}_{\pi_l f})$ then we can determine whether equation (4.3) is valid on $\mathcal{A}$ by calculating μ on this set of elements. The following lemma shows that there are two distinct cases to consider when analysing the quotient group.

Lemma 4.3.1. *If π_l divides f then*

$$\mathcal{A}\,/\,(1+\pi_l f\mathcal{D}_{\pi_l f}) \xrightarrow{\sim} \mathcal{O}_{\mathbb{K}}/\pi_l\mathcal{O}_{\mathbb{K}}$$

otherwise

$$\mathcal{A}\,/\,(1+\pi_l f\mathcal{D}_{\pi_l f}) \xrightarrow{\sim} (\mathcal{O}_{\mathbb{K}}/\pi_l\mathcal{O}_{\mathbb{K}})^\times$$

Proof. In the first case $(1+f\mathcal{D}_f) \subset \mathcal{O}_{\pi_l}$ and we have the map

$$\begin{aligned} 1+f\mathcal{D}_f &\longrightarrow \mathcal{O}_{\mathbb{K}}/\pi_l\mathcal{O}_{\mathbb{K}} \\ x &\longmapsto (x-1)/f + \pi_l\mathcal{O}_{\mathbb{K}} \end{aligned}$$

This is a well-defined homomorphism since if $x, y \in 1+f\mathcal{D}_f$ then because

$$xy-1 = (x-1)(y-1) + (x-1) + (y-1)$$

and $\pi_l f$ divides f^2 which divides $(x-1)(y-1)$ we see that $(x-1)(y-1)/f$ is zero in $\mathcal{O}_{\mathbb{K}}/\pi_l\mathcal{O}_{\mathbb{K}}$. The homomorphism is surjective with kernel $1+\pi_l f\mathcal{D}_{\pi_l f}$

and so we have proved the first claim.

In the second case we have the map

$$\mathcal{A} \longrightarrow (\mathcal{O}_{\mathbb{K}}/\pi_l \mathcal{O}_{\mathbb{K}})^\times$$
$$x \longmapsto x + \pi_l \mathcal{O}_{\mathbb{K}}.$$

The fact that the kernel of this map is $(1 + \pi_l f \mathcal{D}_{\pi_l f})$ comes directly from the Chinese Remainder Theorem. ■

The second case in the lemma can be especially interesting. Suppose π_l and f are coprime and that (4.3) is satisfied by all elements congruent to 1 (mod f). Choose an element θ that is congruent to π_l modulo f and that is not a multiple of π_l. Then we can extend the definition of μ (and hence of $\tau_{r,s,t}$) to π_l by defining $\mu(\pi_l) = \mu(\theta)$. It is easy to check that this definition is independent of the choice of θ, that it extends μ to $\mathcal{O}_f$ and that the kernel of μ contains $1 + f\mathcal{D}_f$.

In fact this does occur in a couple of the cases that we are interested in, both times when $n = 12$. Let ζ be a primitive 12-th root of unity then $1 + \zeta^3$ is the unique prime above 2. When $(r, s, t) = (1, 4, 7)$ it turns out that the conductor is the ideal generated by 3. One can check that $1 + \zeta^3 \equiv (1 + \zeta)^3 \pmod 3$ and that $\omega_{1,4,7} = \sigma_1 + \sigma_7$. Since $1 + \zeta$ is a unit we have $\tau_{1,4,7}((1 + \zeta)) = 1$ and we get $\tau_{1,4,7}((1 + \zeta^3)) = \left[(1 + \zeta^3)/(1 + \zeta)^3\right]^{\sigma_1+\sigma_7} = -2$.

When $(r, s, t) = (1, 8, 3)$ the conductor is the ideal generated by $1+\zeta^2$ of norm 9. One can check that $1+\zeta^3 \equiv 1-\zeta \pmod{(1+\zeta^2)}$, that $\omega_{1,8,3} = \sigma_1+\sigma_5$ and that $1-\zeta$ is a unit. We end up with $\tau_{1,8,3}((1+\zeta^3)) = \left[(1 + \zeta^3)/(1 - \zeta)\right]^{\sigma_1+\sigma_5} = -2$.

Looking at the definition of the Hecke L-function from (4.1) we have an Euler factor of

$$1 + \frac{2}{4^s}$$

in both cases since the norm of $1 + \zeta^3$ is equal to 4. For the curves we looked at, these proved to be the only occasions that non-trivial Euler factors were discovered at primes with bad reduction.

4.4 Implementation on Computer

As a practical consideration it takes time proportional to $N\mathfrak{l}$ to evaluate the Jacobi sum

$$\tau_{r,s,t}(\mathfrak{l}) = -\sum_{a\in\mathcal{O}/\mathfrak{l}} \chi_{\mathfrak{l}}^r(a)\chi_{\mathfrak{l}}^s(1-a).$$

when $\mathfrak{l}$ is a prime ideal and therefore it is preferable if our elements generating $\mathcal{A}$ factor into elements of small norm.

We implemented these calculations in PARI/GP which automatically calculates an LLL reduced basis for $\mathcal{O}_{\mathbb{K}}$. Denote such a basis by $z_1, z_2, \ldots, z_{2g}$ and consider the set of elements

$$S = \left\{ 1 + f \cdot \left(\sum_{k=1}^{2g} a_k z_k \right) : a_k \in \mathbb{Z}, |a_k| \leq 2 \right\}$$

Then it was always possible in the cases we considered to find a set of generators for $\mathcal{A}$ consisting of elements of S that factor into primes of norm less than or equal to 10^4. In this way the computational time was very reasonable and the conductors for $n \leq 12$ and $\varphi(n) \in \{4, 6\}$ are shown in table 4.1.

Note that the numbers in the column $W_{r,s,t}$ are the root numbers in the functional equation and were computed with the ComputeL package as we shall now describe.

4.5 The ComputeL package

Dokchitser [7] has implemented a PARI/GP [27] package which computes values of L-functions numerically to high precision. Required as input to the package are the sign of the functional equation, the conductor, the weight of the L-function, a description of the Gamma factors in the functional equation, together of course with as many terms of the L-series required to calculate the value of

n	g	r,s,t	$\lvert\operatorname{disc}\mathbb{Q}(\mu_n)\rvert$	$N\mathfrak{f}$	$N_{r,s,t}$	$W_{r,s,t}$
5	2	1,1,3	5^3	5^2	5^5	1
7	3	1,1,5	7^5	7^2	7^7	1
7	3	1,2,4	7^5	7^1	7^6	1
8	2	1,1,6	2^8	2^8	2^{16}	1
8	2	1,2,5	2^8	2^6	2^{14}	1
8	2	1,3,4	2^8	2^8	2^{16}	-1
8	2	1,4,3	2^8	2^4	2^{12}	1
8	2	1,5,2	2^8	2^8	2^{16}	-1
8	2	1,6,1	2^8	2^6	2^{14}	1
9	3	1,1,7	3^9	3^6	3^{15}	-1
9	3	1,2,6	3^9	3^4	3^{13}	1
10	2	1,1,8	5^3	$2^8 \cdot 5^2$	$2^8 \cdot 5^5$	1
10	2	1,2,7	5^3	$2^4 \cdot 5^2$	$2^4 \cdot 5^5$	1
10	2	1,3,6	5^3	$2^8 \cdot 5^2$	$2^8 \cdot 5^5$	-1
10	2	1,4,5	5^3	$2^4 \cdot 5^1$	$2^4 \cdot 5^4$	1
10	2	1,5,4	5^3	$2^8 \cdot 5^1$	$2^8 \cdot 5^4$	-1
10	2	1,8,1	5^3	$2^4 \cdot 5^2$	$2^4 \cdot 5^5$	-1
12	2	1,1,10	$2^4 \cdot 3^2$	$2^8 \cdot 3^4$	$2^{12} \cdot 3^6$	-1
12	2	1,2,9	$2^4 \cdot 3^2$	$2^6 \cdot 3^2$	$2^{10} \cdot 3^4$	1
12	2	1,3,8	$2^4 \cdot 3^2$	$2^8 \cdot 3^2$	$2^{12} \cdot 3^4$	1
12	2	1,4,7	$2^4 \cdot 3^2$	$2^0 \cdot 3^4$	$2^4 \cdot 3^6$	1
12	2	1,5,6	$2^4 \cdot 3^2$	$2^8 \cdot 3^2$	$2^{12} \cdot 3^4$	1
12	2	1,6,5	$2^4 \cdot 3^2$	$2^6 \cdot 3^2$	$2^{10} \cdot 3^4$	-1
12	2	1,7,4	$2^4 \cdot 3^2$	$2^8 \cdot 3^4$	$2^{12} \cdot 3^6$	1
12	2	1,8,3	$2^4 \cdot 3^2$	$2^0 \cdot 3^2$	$2^4 \cdot 3^4$	1
12	2	1,9,2	$2^4 \cdot 3^2$	$2^8 \cdot 3^2$	$2^{12} \cdot 3^4$	-1
12	2	1,10,1	$2^4 \cdot 3^2$	$2^6 \cdot 3^4$	$2^{10} \cdot 3^6$	1

Table 4.1: Conductors of all quotients with $n \leq 12$ and $\varphi(n) \in \{4, 6\}$. The root numbers $W_{r,s,t}$ in the final column were computed with the ComputeL package.

the L-function to the current precision.

Actually, if the L-function is known to obey a functional equation with a certain value for the conductor, then ComputeL can be used to find the root number from the binary choice $\{\pm 1\}$ by checking a certain equality based on theta series. We take advantage of this feature as the easiest way for finding the root number.

The remaining task is therefore to compute enough values of the Dirichlet series in order for ComputeL to be able to compute our L-values.

To compute the Jacobi sum

$$\tau_{r,s,t}(\mathfrak{l}) = -\sum_{a \in \mathcal{O}/\mathfrak{l}} \chi_{\mathfrak{l}}^r(a)\chi_{\mathfrak{l}}^s(1-a)$$

in the general case, we first note that $\mathcal{O}/\mathfrak{l}$ is isomorphic to the finite field of l^f elements where l is the rational prime above $\mathfrak{l}$ and f is the order of l modulo n.

We begin by searching for a primitive element a modulo $\mathfrak{l}$ and compute the n-th power residue symbol $\chi_{\mathfrak{l}}(a)$ with a search for equality modulo $\mathfrak{l}$ between the value $a^{(l^f-1)/n}$ and the n values $1, \zeta, \ldots, \zeta^{n-1}$.

Next we determine a bijection between $\mathcal{O}/\mathfrak{l}$ and the integers $0, 1, \ldots, l^f - 1$. To help with this PARI/GP provides a convenient function that gives generators of the ideal $\mathfrak{l}$ in terms of the integral basis of $\mathcal{O}$ and in Hermite Normal Form.

A table of size l^f is allocated and we proceed to fill this table with values of $\chi_{\mathfrak{l}}$ using the obvious equality $\chi_{\mathfrak{l}}\left(a^k\right) = \chi_{\mathfrak{l}}(a)^k$.

Once all the values for $\chi_{\mathfrak{l}}$ have been determined the Jacobi sum can be evaluated in straightforward fashion.

It should be noted that if we require all of the terms of the Dirichlet series up to index N then primes l where $f > 1$ will make no contribution if $l^2 > N$. Therefore the bulk of the work is taken up by determining the coefficient of T in the expression

$$P_l(T) = \prod_{\mathfrak{l}|l} \left(1 - \tau_{r,s,t}(\mathfrak{l})\, T^f\right)$$

when $f = 1$ and $l > \sqrt{N}$. In that case we must have $l \equiv 1 \pmod{n}$ and the coefficient of T is going to be given by

$$\sum_{x \in \mathbb{F}_l \setminus \{0,1\}} \operatorname{Tr}\left(\chi_l^r(x)\chi_l^s(1-x)\right)$$

This expression is so simple that it was possible to program directly in the low-level computer language “C” giving a great speed advantage.

4.6 Fast modular arithmetic on a computer

For reasonably small primes l we describe a method to perform multiplication of an arbitrary integer modulo l by a fixed integer modulo l in just three fast instructions on modern computer hardware. It is inspired by Montgomery multiplication [16] but has been further optimised according to our specific needs.

Let l be an odd positive integer and suppose that $a = -y/2^{32} \mod l$ where $ly < 2^{32}$. Define $z := y/l \mod 2^{32}$. If $0 \leq x < l$ then compute $w := xz \mod 2^{32}$ and define q and r with $0 \leq r < 2^{32}$ by

$$2^{32}q + r = lw.$$

Then $q < l$ (since $w < 2^{32}$) and r satisfies

$$r \equiv lw \equiv lzx \equiv xy \pmod{2^{32}}.$$

But $xy < ly < 2^{32}$ so r is precisely equal to xy and we see that

$$q = \frac{lw - r}{2^{32}} \equiv \frac{-r}{2^{32}} = \frac{-xy}{2^{32}} \equiv ax \pmod{l}.$$

Thus if the 32-bit value z has been pre-computed we can compute $q := ax \mod l$ very quickly on a computer with the operations

```
w := x * z mod 2^32;
```

```
q := (l * w) // 2^32;
```

where `//` represents the operation "divide and round down to nearest integer".

For all primes l less than 36 million it was possible to find y satisfying $yl < 2^{32}$ and $-y/2^{32}$ being a primitive root modulo l, therefore we could use the above multiplication trick when computing our table of values for each χ_l. In fact, the multiplication operation is so fast that the limiting factor of the program's speed was the time required to write the χ_l values into the table.

4.7 The results

With the computational methods outlined in previous sections we were able to compute L-values for all quotients mentioned at the end of section 3.5.1 to a precision of 200 decimal digits. The most difficult value to compute was on the quotient $A_{1,2,6}$ of genus 3 and conductor 3^{13}. Roughly 23 million terms of the Dirichlet series were required which took around 24 hours of CPU time.

$L^{(2)}_{1,1,3}(0) =$

```
3.9044666224382301406864995150031685591095140398523
222713454587777300823329670234928205093659253019576
212520371734060626609899047059517799317906004136057
531764120452506961029939912046439066735224068634
```

$L^{(3)}_{1,2,4}(0) =$

```
10.590031668633318742244645498163991186904708544985
401984039171033430170202561524939650304520192251498
070926360472831923604952776070681037913665805692357
350157697877977480567432482416937545494345289060
```

$L^{(2)}_{1,2,5}(0) =$

```
22.172122239977660314764280156454627467594809129906
```

494994918804153823719729902774472084256236009674597
581704153585743598113427921784464774855267331039165
307054455694409589146919959996750830406295056721

$L^{(2)}_{1,4,3}(0) =$

5.2172818564081939178064977263799176954883549978614
713574137277682984662111611274834399476192074954021
227803304147572566911180114474383433188877888541419
569989928896150810215065540432782139727745260828

$L^{(3)}_{1,2,6}(0) =$

158.08487420273125120081107516291898934764854087401
974765964325145996785542977682236515199498604103724
679168249281646834059704074014619968898789928291385
953692392493474430660535999464776467315360789548

$L^{(2)}_{1,2,7}(0) =$

65.222342356362598081358890537868167550811563016704
820563497513135668465956984156267451712773496359724
893203653153773248079890809508006666010012312630445
132023282319039879079909630941229361699650036122

$L^{(2)}_{1,4,5}(0) =$

13.143417042986172127876620474551928103156692503849
235530275498889649612689775490460124956432568867765
451991377471532513328899905762053891163674529958552
283404421555672258513604180255389494342730299433

$L^{(2)}_{1,2,9}(0) =$

108.40886795116844195660279639760193315805969644739
464535181442894276932699661874899946651241395904217
401854552424494401476572055226078059157740847945490
996441316925791893353480930687356337511113761853

$L^{(2)}_{1,4,7}(0) =$

13.946845434271244783299827284784853858026054773931

316221681237864761313284308863474711771696913608900

743732532561181586136784637166149560564164072088070

198411526552919213368411779818327736712185089043

$L^{(2)}_{1,6,5}(0) =$

-98.543759796628833194988868372296798911457293301 88

188954459068265832110950172076996809677431850456933

821600713573894345263576026088527975816024432292317

0605728459475911005020577930315219045864737621912

$L^{(2)}_{1,8,3}(0) =$

1.429084973559021176415886589031963242269313926829 7

145420204941078860672491886976414500889706540248102

930816184589527342161153835040656617716565598026264

20163959443077584549148291251517888581861 5696944

4.8 Comparisons to the regulator values

With regard to the elements of K_2 outlined in Proposition 3.3.3 and formulas (3.18)-(3.20) let $\alpha_1, \alpha_2, \ldots, \alpha_k$ be a list of elements in $K_2(F_n)$, let $\kappa_{a_1,b_1}, \ldots, \kappa_{a_k,b_k}$ be a list of loops in $H_1(F_n, \mathbb{Z})$. Define

$$\mathrm{Det}(A_{r,s,t};\ \alpha_1, \ldots, \alpha_k;\ \kappa_{a_1,b_1}, \ldots \kappa_{a_k,b_k})$$

to be the absolute value of the determinant of the matrix whose (i,j)-th element is the integral of $\operatorname{reg} \alpha_i$ along the loop κ_{a_j,b_j} when pushed down to the variety $A_{r,s,t}$.

A small note on which loops to choose is necessary. Let c denote complex conjugation on a curve X. Then for $\alpha \in K_2X$ and $\gamma \in H_1(X(\mathbb{C}), \mathbb{Z})$ and invariant under c we have that $\int_\gamma \operatorname{reg}\alpha = 0$ on account of the following calculation:

$$\int_\gamma \operatorname{reg}\alpha = \int_{c\circ\gamma} \operatorname{reg}\alpha = \int_\gamma c^*\left(\operatorname{reg}\alpha\right) = -\int_\gamma \operatorname{reg}\alpha$$

Therefore the loops chosen should be generators of $H_1(X(\mathbb{C}),\mathbb{Z})/H_1(X(\mathbb{C}),\mathbb{Z})^+$. For example, the loops could be chosen as generators of the eigenspace $H_1(X(\mathbb{C}),\mathbb{Z})^-$ if one was motivated to do so.

Then the following equalities, which give good evidence that Beilinson's conjecture is true for these curves, hold up to a precision of around 200 decimal digits.

$$\begin{array}{lcll}
L^{(2)}_{1,1,3}(0) & / & \operatorname{Det}(A_{1,1,3};\ \alpha,\eta\alpha;\ \kappa_{0,0},\kappa_{0,1}) & = \dfrac{8}{25} \\
L^{(3)}_{1,2,4}(0) & / & \operatorname{Det}(A_{1,2,4};\ \alpha,\eta\alpha,\eta^2\alpha;\ \kappa_{0,0},\kappa_{0,3},\kappa_{0,4}) & = \dfrac{48}{7^3} \\
L^{(2)}_{1,2,5}(0) & / & \operatorname{Det}(A_{2,1,5};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,2}) & = 4 \\
L^{(2)}_{1,4,3}(0) & / & \operatorname{Det}(A_{4,1,3};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = 1 \\
L^{(3)}_{1,2,6}(0) & / & \operatorname{Det}(A_{1,2,6};\ \alpha,\eta\alpha,\eta^2\alpha;\ \kappa_{0,0},\kappa_{0,1},\kappa_{0,4}) & = \dfrac{16}{3} \\
L^{(2)}_{1,2,7}(0) & / & \operatorname{Det}(A_{2,1,7};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = 16 \\
L^{(2)}_{1,4,5}(0) & / & \operatorname{Det}(A_{4,1,5};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = 4 \\
L^{(2)}_{1,2,9}(0) & / & \operatorname{Det}(A_{2,1,9};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = 48 \\
L^{(2)}_{1,4,7}(0) & / & \operatorname{Det}(A_{4,1,7};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,2}) & = \dfrac{32}{9} \\
L^{(2)}_{1,6,5}(0) & / & \operatorname{Det}(A_{6,1,5};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = -24 \\
L^{(2)}_{1,8,3}(0) & / & \operatorname{Det}(A_{8,1,3};\ \alpha,\beta;\ \kappa_{0,0},\kappa_{0,1}) & = \dfrac{8}{9}
\end{array}$$

Chapter 5

On the curves $C_{1,1,n-2}$

In this chapter we specifically discuss the hyperelliptic curves $C_{1,1,n-2}$. The main result will be to prove that for all odd primes $n \geq 5$ the two elements α and $\eta\alpha$ are independent elements of K_2 when projected on to the curve $C_{1,1,n-2}$. We do this by analysing a particular 2×2 matrix of regulator values so let us begin by gathering the formulas required.

5.1 Defining a certain determinant

Lemma 5.1.1. *For integers a and b we have*

$$\int_{\kappa_{a,b}} \hat{\omega}_{r,s,t} = \frac{-4i}{n} \cdot \sin \tfrac{\pi}{n} \left[(2a+1)r + (2b+1)s\right] \cdot \sin \tfrac{\pi r}{n} \cdot \sin \tfrac{\pi s}{n} \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right)$$

Proof. Going back to formula (3.13) we have the straightforward computation

$$\begin{aligned}
\int_{\kappa_{a,b}} \hat{\omega}_{r,s,t} &= \frac{i}{n} \operatorname{Im} \zeta^{ar+bs} \left(1-\zeta^r\right)\left(1-\zeta^s\right) B\left(\tfrac{r}{n}, \tfrac{s}{n}\right) \\
&= \frac{i}{n} \operatorname{Im} \zeta^{\left(a+\frac{1}{2}\right)r + \left(b+\frac{1}{2}\right)s} \left(\zeta^{-\frac{r}{2}} - \zeta^{\frac{r}{2}}\right)\left(\zeta^{-\frac{s}{2}} - \zeta^{\frac{s}{2}}\right) \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right) \\
&= \frac{-4i}{n} \operatorname{Im} \zeta^{\left(a+\frac{1}{2}\right)r + \left(b+\frac{1}{2}\right)s} \sin \tfrac{\pi r}{n} \cdot \sin \tfrac{\pi s}{n} \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right) \\
&= \frac{-4i}{n} \cdot \sin \tfrac{\pi}{n} \left[(2a+1)r + (2b+1)s\right] \cdot \sin \tfrac{\pi r}{n} \cdot \sin \tfrac{\pi s}{n} \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right)
\end{aligned}$$

$\blacksquare$

In the course of the proof we only need the case $a = \frac{n-1}{2}$ where we have

$$\begin{aligned}\int_{\kappa_{(n-1)/2,b}} \hat{\omega}_{r,s,t} &= \frac{-4i}{n} \cdot \sin \tfrac{\pi}{n}\left(nr + (2b+1)s\right) \cdot \sin \tfrac{\pi r}{n} \cdot \sin \tfrac{\pi s}{n} \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right) \\ &= \frac{-4i}{n} \cdot \sin \tfrac{\pi}{n}(2b+1)s \cdot \sin \tfrac{\pi r}{n} \cdot \sin \tfrac{\pi s}{n} \cdot B\left(\tfrac{r}{n}, \tfrac{s}{n}\right) \cdot (-1)^r\end{aligned}$$

To ease the notation slightly, define $P(b,n)$ and $Q(b,n)$ by

$$\begin{aligned}P(b,n) &= 2\pi i n \int_{\kappa_{(n-1)/2,b}} \operatorname{reg} \varphi_{1,1,n-2}\alpha \\ Q(b,n) &= -2\pi i n \int_{\kappa_{(n-1)/2,b}} \operatorname{reg} \varphi_{1,1,n-2}\eta\, \alpha\end{aligned}$$

The relevant formulas for computing the projection were (3.18) and (3.19) which we repeat here.

$$\begin{aligned}\operatorname{reg} \varphi_{r,s,t}\alpha &= \frac{-1}{2\pi n} \sum_{h \in H_{r,s,t}} G\left(\frac{\langle hr\rangle}{n}, \frac{\langle hs\rangle}{n}\right) \cdot \hat{\omega}_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle} \\ \operatorname{reg} \varphi_{r,s,t}\eta\alpha &= \frac{1}{2\pi n} \sum_{h \in H_{r,s,t}} G\left(\frac{\langle ht\rangle}{n}, \frac{\langle hr\rangle}{n}\right) \cdot (-1)^{\langle hr\rangle} \hat{\omega}_{\langle hr\rangle,\langle hs\rangle,\langle ht\rangle}\end{aligned}$$

Remember that $H_{1,1,n-2}$ is the set of all integers that are coprime to n and less than $\frac{n}{2}$. When n is prime this just means all integers in the range $1, \ldots, (n-1)/2$. With $(r,s,t) = (1,1,n-2)$ and integers h in this range we will have $\langle hr\rangle = \langle hs\rangle = h$ and $\langle ht\rangle = n - 2h$ giving

$$P(b,n) = -\frac{4}{n} \sum_{h=1}^{\frac{n-1}{2}} G\left(\tfrac{h}{n}, \tfrac{h}{n}\right) \cdot \sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot B\left(\tfrac{h}{n}, \tfrac{h}{n}\right) \cdot (-1)^h \tag{5.1}$$

and

$$Q(b,n) = -\frac{4}{n} \sum_{h=1}^{\frac{n-1}{2}} G\left(1 - \tfrac{2h}{n}, \tfrac{h}{n}\right) \cdot \sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot B\left(\tfrac{h}{n}, \tfrac{h}{n}\right) \tag{5.2}$$

Note that two occurrences of $(-1)^h$ cancelled each other out in equation (5.2).

The rest of this chapter will be devoted to proving the following proposition.

Proposition 5.1.2. *For odd primes n greater than 5 we have that $P(0,n)$, $P(1,n)$ and $Q(0,n)$ are positive while $Q(1,n)$ is negative so that the determinant of the matrix*

$$\begin{pmatrix} P(0,n) & P(1,n) \\ Q(0,n) & Q(1,n) \end{pmatrix}$$

is negative and so α and $\eta\alpha$ are independent elements of K_2 after the projection onto $C_{1,1,n-2}$.

5.2 Analysing $P(b,n)$

The summand in the expression for $P(b,n)$ is actually a Riemann-integrable function evaluated the points $\frac{h}{n}$ but multiplied by the alternating factor $(-1)^h$ and so we expect $P(b,n)$ to tend towards 0 as n tends to infinity. Nonetheless we can prove that $P(b,n)$ will be positive for $b \in \{0,1\}$ with some careful analysis.

With n being odd it can be checked that the expression

$$\sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot (-1)^h \tag{5.3}$$

is negated under the substitution $h \longrightarrow n-h$. Therefore, after recalling the definition of the function G

$$G(x,y) = \frac{F(x,y) - F(1-x,1-y)}{B(x,y)}$$

we can "unfold" the sum in equation (5.1) and obtain the following more convenient form

$$P(b,n) = -\frac{4}{n} \sum_{h=1}^{n-1} F\left(\tfrac{h}{n}, \tfrac{h}{n}\right) \cdot \sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot (-1)^h$$

In fact we can use Proposition 3.5.1 to find a simpler expression for F when both of its arguments are equal:

$$F(x,x) = \frac{3}{2}\sum_{k=0}^{\infty} \frac{B(x+k,x+k)}{x+k}$$

so we get the even simpler form

$$P(b,n) = -\frac{6}{n}\sum_{h=1}^{\infty} \tfrac{n}{h} \cdot B\left(\tfrac{h}{n},\tfrac{h}{n}\right) \cdot \sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot (-1)^h \tag{5.4}$$

since expression (5.3) is invariant under the substitution $h \longrightarrow n+h$. From here we are ready to analyse this expression using the Euler-Maclaurin formula.

5.2.1 The Euler-Maclaurin Formula

Suppose $f : [0,1] \to \mathbb{R}$ is differentiable and $x \in [0,1]$. Define $g : [0,1] \to \mathbb{R}$ by

$$g(t) = \begin{cases} t - x + \frac{1}{2}, & 0 \le t < x \\ t - x - \frac{1}{2}, & x < t \le 1. \end{cases}$$

When $g'(t)$ is defined it is equal to 1 so integration by parts gives

$$\begin{aligned}
\int_0^1 f(t)\,dt &= \int_0^x f(t)\,dt + \int_x^1 f(t)\,dt \\
&= \left[f(t)\left(t - x + \frac{1}{2}\right)\right]_0^x - \int_0^x f'(t)\left(t - x + \frac{1}{2}\right)\,dt \\
&+ \left[f(t)\left(t - x - \frac{1}{2}\right)\right]_x^1 - \int_x^1 f'(t)\left(t - x - \frac{1}{2}\right)\,dt \\
&= f(x) - \left(x - \frac{1}{2}\right)(f(1) - f(0)) - \int_0^1 f'(t)g(t)\,dt
\end{aligned}$$

So that

$$f(x) = \int_0^1 f(t)\,dt + \left(x - \frac{1}{2}\right)(f(1) - f(0)) + \int_0^1 f'(t)g(t)\,dt \tag{5.5}$$

Let us introduce the Bernoulli polynomials $B_j(x)$ defined by

$$\frac{te^{tx}}{e^t - 1} = \sum_{j=0}^{\infty} \frac{B_j(x)}{j!} t^j$$

This gives $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, $B_2(x) = x^2 - x + 1/6$ and so on. It follows easily from the definition of $B_j(x)$ that the Bernoulli numbers B_j satisfy $B_j = B_j(0)$ and that the following identities hold:

$$B_j(1 - x) = (-1)^j B_j(x) \tag{5.6}$$

$$B_j'(x) = jB_{j-1}(x) \tag{5.7}$$

Also let the Bernoulli periodic functions $\overline{B}_j(x)$ be functions of period 1 defined by

$$\overline{B}_j(x) = B_j(x), \qquad\qquad 0 \leq x < 1$$

Note that $g(t) = -\overline{B}_1(x - t)$ so we can rewrite equation (5.5) as

$$f(x) = \int_0^1 f(t)\,dt + B_1(x)\,(f(1) - f(0)) - \int_0^1 f'(t)\overline{B}_1(x - t)\,dt \tag{5.8}$$

Using equation 5.6 we see that $B_j(0) = B_j(1)$ for all even j. Moreover, since $B_j = 0$ for all odd $j > 1$ we in fact see that $B_j(0) = B_j(1)$ for all $j \geq 1$ and hence that $\overline{B}_j(x)$ is continuous for $j \geq 1$. In view of equation 5.7 we have that the derivative of $\overline{B}_j(x - t)$ with respect to t is $-j\overline{B}_{j-1}(x - t)$, thus if $f \in C^J[0, 1]$ then integration by parts applied to equation 5.8 leads to the following formula

$$\begin{aligned}\int_0^1 f'(t)\overline{B}_1(x - t)\,dt &= \left[f'(t).\left(-\overline{B}_2(x - t)\right)\right]_0^1 - \int_0^1 f''(t)\left(-\overline{B}_2(x - t)\right)\,dt \\ &= -\overline{B}_2(x)\,(f'(1) - f'(0)) + \int_0^1 f''(t)\overline{B}_2(x - t)\,dt\end{aligned}$$

and applying the same process inductively gives

$$f(x) = \int_0^1 f(t)\,dt + \sum_{j=1}^{J} \frac{B_j(x)}{j!}\left(f^{(j-1)}(1) - f^{(j-1)}(0)\right) - \int_0^1 \frac{\overline{B}_J(x-t)}{J!} f^{(J)}(t)\,dt$$

For any integer k we clearly also have the identity

$$\begin{aligned} f(k+x) = \int_k^{k+1} f(t)\,dt &+ \sum_{j=1}^{J} \frac{B_j(x)}{j!}\left(f^{(j-1)}(k+1) - f^{(j-1)}(k)\right) \\ &- \int_k^{k+1} \frac{\overline{B}_J(x-t)}{J!} f^{(J)}(t)\,dt \end{aligned}$$

Summing this expression over all $a \le k < b$ gives the Euler-Maclaurin formula

Theorem 5.2.1. *For $a, b \in \mathbb{Z}$, $x \in [0,1]$ and $f \in C^J[a,b]$ we have*

$$\begin{aligned} \sum_{a \le k < b} f(k+x) = \int_a^b f(t)\,dt &+ \sum_{j=1}^{J} \frac{B_j(x)}{j!}\left(f^{(j-1)}(b) - f^{(j-1)}(a)\right) \\ &- \int_a^b \frac{\overline{B}_J(x-t)}{J!} f^{(J)}(t)\,dt \end{aligned}$$

And we also have the following scaled version as a simple corollary

Corollary 5.2.2. *For n a positive integer, $x \in [0,1]$ and $f \in C^J[a,b]$ we have*

$$\begin{aligned} \frac{1}{n}\sum_{k=0}^{n-1} f\left(\frac{k+x}{n}\right) = \int_0^1 f(t)\,dt &+ \sum_{j=1}^{J} \frac{B_j(x)}{j!n^j}\left(f^{(j-1)}(1) - f^{(j-1)}(0)\right) \\ &- \int_0^1 \frac{\overline{B}_J(x-Jt)}{J!n^J} f^{(J)}(t)\,dt \end{aligned}$$

Applying Euler-Maclaurin to $P(b,n)$

If we define $f : \mathbb{C} \to \mathbb{C}$ by

$$f(z) = \frac{1}{z} \cdot B(z,z) \cdot \sin(2b+1)\pi z \cdot \sin^2 \pi z$$

then we can write equation (5.4) in the form

$$P(b,n) = -\frac{6}{n}\sum_{h=0}^{\infty}\left[f\left(\frac{2h}{n}\right) - f\left(\frac{2h+1}{n}\right)\right]$$

Define the partial sum $P_K(b,n)$ by

$$P_K(b,n) = \frac{1}{n/2}\sum_{h=0}^{nK-1}\left[f\left(\frac{h}{n/2}\right) - f\left(\frac{h+\frac{1}{2}}{n/2}\right)\right]$$

and note that $P(b,n) = -3\lim_{K\to\infty} P_K(b,n)$. Then applying Euler-Maclaurin gives

$$\begin{aligned} P_K(b,n) = &\sum_{j=1}^{J}\frac{B_j(0) - B_j\left(\frac{1}{2}\right)}{j!\,(n/2)^j}\left(f^{(j-1)}(K) - f^{(j-1)}(0)\right) \\ &- \int_0^K \frac{\overline{B}_J(-Jt) - \overline{B}_J\left(\frac{1}{2} - Jt\right)}{J!\,(n/2)^J}\cdot f^{(J)}(t)\,dt \end{aligned} \tag{5.9}$$

We would like to take the limit as J tends to infinity in the above expression. To do that we examine the Fourier expansion of the Bernoulli polynomials and then prove some bounds on the entire function f.

Proposition 5.2.3. *The Fourier expansion of $\overline{B}_j(x)$ is given by*

$$F_j(x) = -j!\sum_{l=-\infty}^{\infty}{}' \frac{1}{(2\pi i l)^j}e^{2\pi i l x}$$

where the prime indicates the omission of the term corresponding to $l = 0$. The equality $\overline{B}_j(x) = F_j(x)$ holds for all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$ except for when $j = 1$ and $x \in \mathbb{Z}$.

Proof. After checking by hand the validity of this expansion for $j = 1$ the expansion can be seen to be valid for $j > 1$ by observing the following

- $F_j(0) = \overline{B}_j(0)$ since for j odd $F_j(0) = 0 = \overline{B}_j(0)$ and for j even $F_j(0) = -\frac{2j!}{(2\pi i)^j}\zeta(j) = B_j = \overline{B}_j(0)$ by the famous formula of Euler.
- $F_j'(x) = jF_{j-1}(x)$

Therefore the equality $F_j(x) = \overline{B}_j(x)$ holds whenever $\overline{B}_j$ is continuous at x. This applies to all $x \in \mathbb{R}$ and $j \in \mathbb{Z}$ except for when $j = 1$ and $x \in \mathbb{Z}$. ∎

By bounding this series in the obvious way and using the fact that $\zeta(j) < 2$ for $j \geq 2$ we obtain the following corollary.

Corollary 5.2.4. *For all $x \in \mathbb{R}$ and $j \geq 2$*

$$\left| \frac{\overline{B}_j(x)}{j!} \right| < \frac{4}{(2\pi)^j} \tag{5.10}$$

Proposition 5.2.5. *Let $\Delta = 2\log 2$ then for all z we have*

$$|f(z)| \leq A|z|^{-\frac{1}{2}} e^{-\Delta \operatorname{Re} z + (2b+3)\pi|\operatorname{Im} z|}$$

for some constant A.

Proof. We know that

$$f(z) = \frac{1}{z} \cdot \frac{\Gamma(z)^2}{\Gamma(2z)} \cdot \sin(2b+1)\pi z \cdot \sin^2 \pi z$$

By Stirling's formula we know that

$$\log \Gamma(z) = \left(z - \frac{1}{2}\right) \log z - z + O(1)$$

for $\operatorname{Re} z > 0$. Some easy manipulation shows that

$$\begin{aligned} \log\left(\frac{\Gamma(z)^2}{\Gamma(2z)}\right) &= 2\log\Gamma(z) - \log\Gamma(2z) \\ &= -\tfrac{1}{2}\log z - 2\log 2 \cdot z + O(1) \end{aligned}$$

Therefore $|B(z,z)|$ is bounded above (and as will soon become relevant, bounded below) by some multiple of $|z|^{-\frac{1}{2}} e^{-\Delta \operatorname{Re} z}$. Together with the obvious bound $|\sin z| \leq e^{|\operatorname{Im} z|}$ we get the claimed result in the region $\operatorname{Re} z > 0$.

To get the result in the left half-plane we can use some Gamma identities to write

$$\frac{\Gamma(z)^2}{\Gamma(2z)} \cdot \sin^2 \pi z = \frac{\pi}{1-2z} \frac{\Gamma(2-2z)}{\Gamma(1-z)^2} \cdot \sin 2\pi z$$
$$= \frac{\pi}{1-2z} \cdot B(1-z, 1-z)^{-1} \cdot \sin 2\pi z$$

and the result follows in the region $\operatorname{Re} z \leq 0$ by making use of the remark in the previous discussion. ∎

Proposition 5.2.6. *Let* $\sigma = \left(\Delta^2 + (2b+3)^2\pi^2\right)^{\frac{1}{2}}$ *then*

$$\left| f^{(j)}(z) \right| \leq C\sigma^j e^{\sigma|z|}$$

for some constant C.

Proof. First of all, from the Cauchy-Schwarz inequality and the previous proposition we see that

$$|f(z)| \leq A\,|z|^{-\frac{1}{2}}\, e^{-\Delta \operatorname{Re} z + (2b+3)\pi|\operatorname{Im} z|} \leq A\,|z|^{-\frac{1}{2}}\, e^{\sigma|z|}$$

It can be seen that f is an entire function. Let us assume that it has Taylor series given by

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

For $r > 0$ we know from the Residue Theorem that

$$\left|\frac{a_n}{n!}\right| = \left|\frac{1}{2\pi i} \int_{|z|=r} \frac{f(z)}{z^{n+1}}\, dz\right| \leq \frac{1}{2\pi} \int_{|z|=r} \left|\frac{f(z)}{z^{n+1}}\right| dz \leq \frac{2\pi r}{2\pi} \cdot \frac{Ar^{-\frac{1}{2}} e^{\sigma r}}{r^{n+1}} = A\frac{e^{\sigma r}}{r^{n+\frac{1}{2}}}$$

Setting $r = \frac{n}{\sigma}$ gives

$$|a_n| \leq A \frac{n! e^n}{n^{n+\frac{1}{2}}} \sigma^{n+\frac{1}{2}}$$

Hence by Stirling's formula we see that $|a_n| \leq C\sigma^n$ for some constant C.

An expression for $f^{(j)}(z)$ is given by

$$f^{(j)}(z) = \sum_{n=0}^{\infty} \frac{a_{n+j}}{n!} z^n$$

and the above bound for $|a_n|$ allows us to give the following bound for $|f^{(j)}(z)|$ when $|z| \leq r$

$$|f^{(j)}(z)| \leq \sum_{n=0}^{\infty} \frac{|a_{n+j}|}{n!} |z|^n \leq C\sigma^j \sum_{n=0}^{\infty} \frac{(\sigma r)^n}{n!} = C\sigma^j e^{\sigma r}$$

■

Now we can go all the way back to equation (5.9) where we had the remainder term

$$-\int_0^K \frac{\overline{B}_J(-Jt) - \overline{B}_J\left(\frac{1}{2} - Jt\right)}{J!\,(n/2)^J} \cdot f^{(J)}(t)\,dt.$$

Using corollary 5.2.4 and the previous proposition we see that this remainder tends to zero in the limit $J \to \infty$ provided $\sigma < \pi n$. It is not difficult to see that this is satisfied when $n > 2b + 3$.

In conclusion we have the expression

$$P_K(b,n) = \sum_{j=1}^{\infty} \frac{B_j(0) - B_j\left(\frac{1}{2}\right)}{j!\,(n/2)^j} \left(f^{(j-1)}(K) - f^{(j-1)}(0)\right)$$

The next step in our analysis involves replacing the values of the Bernoulli polynomials with their Fourier expansions. Since $f(0) = f(K) = 0$ we can ignore the term corresponding to $j = 1$ and thus all sums are absolutely convergent in the following calculation:

$$\begin{aligned}
P_K(b,n) &= \sum_{j=2}^{\infty} \frac{f^{(j-1)}(K) - f^{(j-1)}(0)}{j!\,(n/2)^j} \sum_{l=-\infty}^{\infty} \frac{-j!}{(2\pi i l)^j} \left(1 - e^{\pi i l}\right) \\
&= 2 \sum_{l\,\text{odd}} \sum_{j=2}^{\infty} \frac{f^{(j-1)}(0) - f^{(j-1)}(K)}{(\pi i l n)^j} \\
&= 4 \sum_{\substack{l\,\text{odd} \\ l>0}} \mathrm{Re} \sum_{j=2}^{\infty} \frac{f^{(j-1)}(0) - f^{(j-1)}(K)}{(\pi i l n)^j}
\end{aligned}$$

It turns out that

$$\sum_{j=2}^{\infty} \frac{f^{(j-1)}(K)}{(\pi i l n)^j} \tag{5.11}$$

can be expressed in terms of the Borel transform of f which we now describe.

5.2.2 The Borel Transform

If $f(z)$ is an entire function with Taylor series given by

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$$

then the Borel Transform of f is defined by

$$\mathcal{B}_f(w) = \sum_{n=0}^{\infty} \frac{a_n}{w^{n+1}}$$

If $|f(z)|$ is bounded by some multiple of $e^{\sigma|z|}$ then this series is convergent for $|w| > \sigma$ but the function $\mathcal{B}_f$ can be potentially analytically continued inside this disc. This can be seen by observing an integral representation of $\mathcal{B}_f$ which we now explain.

Proposition 5.2.7. *If $w = re^{i\theta}$ then*

$$\mathcal{B}_f(w) = e^{-i\theta} \int_0^{\infty} f(te^{-i\theta}) e^{-rt} \, dt$$

Proof. We compute directly

$$\begin{aligned}
e^{-i\theta}\int_0^\infty f(te^{-i\theta})e^{-rt}\,dt &= e^{-i\theta}\int_0^\infty \sum_{n=0}^\infty \frac{a_n}{n!}t^n e^{-in\theta}e^{-rt}\,dt \\
&= \sum_{n=0}^\infty \frac{a_n}{n!}e^{-i(n+1)\theta}\int_0^\infty t^n e^{-rt}\,dt \\
&= \sum_{n=0}^\infty \frac{a_n}{n!}e^{-i(n+1)\theta}\int_0^\infty \left(\frac{t}{r}\right)^n e^{-t}\,d\left(\frac{t}{r}\right) \\
&= \sum_{n=0}^\infty \frac{a_n}{n!}\frac{1}{w^{n+1}}\int_0^\infty t^n e^{-t}\,dt \\
&= \sum_{n=0}^\infty \frac{a_n}{w^{n+1}} = \mathcal{B}_f(w)
\end{aligned}$$

■

In our situation it is a fact that $f(K+z)$ can be written

$$f(K+z) = \sum_{j=0}^\infty \frac{f^{(j)}(K)}{n!}z^j$$

hence we get that the sum in (5.11) is given by

$$\sum_{j=2}^\infty \frac{f^{(j-1)}(K)}{(\pi i l n)^j} = \int_0^\infty -if(K-it)e^{-\pi l n t}\,dt. \tag{5.12}$$

Summing this expression over odd positive values of l gives

$$\begin{aligned}
\sum_{\substack{l\,\text{odd}\\ l>0}}\sum_{j=2}^\infty \frac{f^{(j-1)}(K)}{(\pi i l n)^j} &= \int_0^\infty -if(K-it)\sum_{\substack{l\,\text{odd}\\ l>0}} e^{-\pi l n t}\,dt \\
&= \int_0^\infty -if(K-it)\frac{e^{-\pi n t}}{1-e^{-2\pi n t}}\,dt \\
&= -\frac{i}{2}\int_0^\infty \frac{f(K-it)}{\sinh \pi n t}\,dt
\end{aligned}$$

Now because of Proposition 5.2.5 we see that the integrand decays like 4^{-K} *independently* of t. We conclude that the limit of the expression as $K\to\infty$ is zero. In other words

$$\lim_{K\to\infty} P_K(b,n) = 4\,\mathrm{Re}\lim_{K\to\infty}\frac{-i}{2}\int_0^\infty \frac{f(-it)-f(K-it)}{\sinh \pi n t}\,dt$$
$$= 2\int_0^\infty \frac{\mathrm{Im}\, f(-it)}{\sinh \pi n t}\,dt$$

or after remembering the factor of -3 that disappears when moving from $P(b,n)$ to $P_K(b,n)$ we see that we have proved the following.

Proposition 5.2.8. *For $n > 2b+3$ we have*

$$P(b,n) = 6\int_0^\infty \frac{\mathrm{Im}\, f(it)}{\sinh \pi n t}\,dt$$

Now we need a couple of lemmas before finishing the proof.

Lemma 5.2.9. *In the case $b=0$ we have*

$$\left|\mathrm{Im}\, f(it) - 2\pi^3 t\right| \leq \tfrac{4}{3}\pi^5 t^3 \cosh 3\pi t$$

and when $b=1$ we have

$$\left|\mathrm{Im}\, f(it) - 6\pi^3 t\right| \leq 12\pi^5 t^3 \cosh 5\pi t$$

Proof. We will outline a proof of the first inequality that relies on being able to find explicit constants A and C from Propositions 5.2.5 and 5.2.6.

Suppose that the expression $\mathrm{Im}\, f(it) - 2\pi^3 t$ has a Taylor series of the form

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} t^n$$

As in the proof of Proposition 5.2.6 we can show that there exists C such that $|a_n| \leq C\sigma^n$ where $\sigma = \left(\Delta^2 + 9\pi^2\right)^{\frac{1}{2}}$. Assume that it safe to take $C = 100$ (computer calculations suggest that C could be made much lower).

Both sides of the inequality are odd functions so that only odd powers occur in the Taylor series of both sides. Furthermore, the right hand side has only positive Taylor coefficients and it is possible to check that these are at least $|a_n/n!|$ for $1 \leq n \leq 19$. We can bound the terms in the Taylor series of the left hand side when $n \geq 21$ and $t \leq 2$ as follows:

$$\left|\sum_{n=21}^{\infty} \frac{a_n}{n!} t^n\right| \leq \sum_{n=21}^{\infty} \frac{C\sigma^n}{n!} \cdot t^n \leq \frac{C \cdot \sigma^{21} t^{21}}{21!} \sum_{n=0}^{\infty} \left(\frac{2\sigma}{21}\right)^n \leq 10^4 \cdot t^{21}$$

But the Taylor coefficient of the t^{21} term in the right hand side is $\frac{4}{3}\pi^5 \left(3\pi\right)^{18}/18! \geq 2 \cdot 10^4$ so we have proven the inequality for $t \leq 2$ provided we accept the constant $C = 100$.

We expect that the inequality is true for large t since by Proposition 5.2.5 we see that the left hand side is $O\left(e^{3\pi t}\right)$. But already for $t = 2$ the ratio between the two sides of the inequality is greater than $2 \cdot 10^4$ so an explicit version of Proposition 5.2.5 would not need to be at all tight to give a complete proof. ■

Lemma 5.2.10. *We have the following definite integrals:*

$$\int_0^\infty \frac{t}{\sinh \pi t}\, dt = \frac{1}{4} \qquad\qquad \int_0^\infty \frac{t^3 \cosh \frac{\pi t}{3}}{\sinh \pi t}\, dt = \frac{1}{3}$$

Proof. We need the Hurwitz Zeta function which is defined by

$$\zeta(s, q) = \sum_{k=0}^{\infty} \frac{1}{(k+q)^s}$$

Other properties we need are that $\zeta(s, 1) = \zeta(s)$ and the following "multiplication formulas" that can be obtained via elementary manipulations.

$$\zeta\left(s, \tfrac{1}{2}\right) + \zeta(s, 1) = 2^s \zeta(s)$$
$$\zeta\left(s, \tfrac{1}{3}\right) + \zeta\left(s, \tfrac{2}{3}\right) + \zeta(s, 1) = 3^s \zeta(s)$$

For $q < 1$ it is easy to check that

$$\int_0^\infty \frac{t^n \cdot e^{q\pi t}}{\sinh \pi t}\, dt = \frac{2n!}{(2\pi)^{n+1}} \cdot \zeta\left(n+1, \frac{1-q}{2}\right)$$

Now remembering that $\zeta(2) = \pi^2/6$ and $\zeta(4) = \pi^4/90$ will give the required results after some further computation. ■

Finally we get

$$\left| P(0,n) - 6\int_0^\infty \frac{2\pi^3 t}{\sinh \pi n t}\, dt \right| \leq 6\int_0^\infty \frac{4\pi^5 t^3 \cosh 3\pi t}{3\sinh \pi n t}\, dt$$

Substituting $t \to t/n$ gives

$$\left| P(0,n) - \frac{12\pi^3}{n^2}\int_0^\infty \frac{t}{\sinh \pi t}\, dt \right| \leq \frac{8\pi^5}{n^4}\int_0^\infty \frac{t^3 \cosh 3\pi \frac{t}{n}}{\sinh \pi t}\, dt$$

and so when $n \geq 9$ we have $3\pi t/n \leq \pi t/3$ and we can use the previous lemma to write

$$\left| P(0,n) - \frac{3\pi^3}{n^2} \right| \leq \frac{8\pi^5}{n^4} \cdot \frac{1}{3} \leq \frac{8\pi^5}{243 n^2}$$

which implies that $P(0,n)$ is positive.

Very similarly for $P(1,n)$ we see that if $n \geq 15$ then

$$\left| P(1,n) - \frac{9\pi^3}{n^2} \right| \leq \frac{24\pi^5}{n^4} \leq \frac{8\pi^5}{75 n^2}$$

showing that $P(1,n)$ is positive in this region.

5.3 Analysing $Q(b,n)$

Remember all the way back to equation (5.2) for the following expression for $Q(b,n)$

$$Q(b,n) = -\frac{4}{n}\sum_{h=1}^{\frac{n-1}{2}} G\left(1 - \tfrac{2h}{n}, \tfrac{h}{n}\right) \cdot \sin(2b+1)\tfrac{\pi h}{n} \cdot \sin^2 \tfrac{\pi h}{n} \cdot B\left(\tfrac{h}{n}, \tfrac{h}{n}\right)$$

Define the function $g : (0, \frac{1}{2}) \to \mathbb{R}$ by

$$g(x) = 4 \cdot G(1-2x, x) \cdot \sin(2b+1)\pi x \cdot \sin^2 \pi x \cdot B(x,x)$$

Then $Q(b,n)$ is simply the Riemann-type sum

$$Q(b,n) = -\frac{1}{n}\sum_{h=1}^{\frac{n-1}{2}} g\left(\frac{h}{n}\right)$$

The function g is well-behaved near $x = 0$. Near $x = \frac{1}{2}$ the term

$$4 \cdot \sin(2b+1)\pi x \cdot \sin^2 \pi x \cdot B(x,x)$$

approaches $4\pi(-1)^b$. We now show that $G(1-2x,x)$ has a simple pole at $x = \frac{1}{2}$ by manipulating the problematic term $F(1-2x,x)$ inside G and using Proposition 3.5.1.

$$\begin{aligned}
G(1-2x,x) &= \frac{F(1-2x,x) - F(2x,1-x)}{B(1-2x,x)} \\
&= B(1-2x,x)^{-1}\left(\sum_{k=0}^{\infty}\frac{B(1-2x,x+k)}{1-2x+x+k} + F(2-2x,x) - F(2x,1-x)\right) \\
&= B(1-2x,x)^{-1}\left(\frac{B(1-2x,x)}{1-2x} + F(2-2x,x) - F(2x,1-x)\right) \\
&= \frac{1}{1-2x} + \frac{F(2-2x,x) - F(2x,1-x)}{B(1-2x,x)}
\end{aligned}$$

The second fraction in the above expression is bounded near $x = \frac{1}{2}$ therefore if we define $g^*(x)$ by

$$g^*(x) = g(x) - \frac{4\pi(-1)^b}{1-2x}$$

then g^* extends to a continuous function on all of $[0, \frac{1}{2}]$ and we have

$$Q(b,n) = \frac{1}{n}\sum_{h=1}^{\frac{n-1}{2}}\left(g^*\left(\frac{h}{n}\right) + \frac{4\pi(-1)^b}{1-2h/n}\right)$$

Lemma 5.3.1. *Let γ denote the Euler-Mascheroni constant then for $n \geq 3$ we have*

$$\sum_{h=1}^{\frac{n-1}{2}}\frac{1}{n-2h} \geq \frac{\log(2n-2)+\gamma}{2}$$

Proof. We will need the Digamma function that is defined by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$$

By taking the logarithmic derivative of the identity $\Gamma(z+1) = z\Gamma(z)$ we see that

$$\psi(x+1) - \psi(x) = \frac{1}{x}$$

Using this we can compute

$$\begin{aligned}
\sum_{h=1}^{\frac{n-1}{2}} \frac{1}{n-2h} &= \frac{1}{2} \sum_{h=1}^{\frac{n-1}{2}} \frac{1}{\frac{n}{2}-h} \\
&= \frac{1}{2} \sum_{h=1}^{\frac{n-1}{2}} \left(\psi\left(\tfrac{n}{2}-h+1\right) - \psi\left(\tfrac{n}{2}-h\right)\right) \\
&= \frac{1}{2}\left(\psi\left(\frac{n}{2}\right) - \psi\left(\frac{1}{2}\right)\right)
\end{aligned}$$

Finally it can be seen that $\psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2$ and that the inequality $\psi(x) \geq \log\left(x - \frac{1}{2}\right)$ holds for $x > \frac{1}{2}$. Together these give the claimed result. ■

Proposition 5.3.2. *For $n \geq 3$ we have $Q(0,n) > 0$ and for $n \geq 97$ we have $Q(1,n) < 0$.*

Proof. In the case $b = 0$ we have $g^*(x) \geq -4\pi$ for all $x \in [0, \frac{1}{2}]$ (in fact g^* appears to map $[0, \frac{1}{2}]$ bijectively onto the interval $[-4\pi, 8\pi \log 2]$). Therefore we have

$$\begin{aligned}
Q(0,n) &= \frac{1}{n} \sum_{h=1}^{\frac{n-1}{2}} \left(g^*\left(\frac{h}{n}\right) + \frac{4\pi}{1-2h/n}\right) \\
&\geq \frac{-4\pi}{2} + 4\pi \cdot \frac{\log(2n-2)+\gamma}{2}
\end{aligned}$$

which is positive for $n \geq 3$.

In the case $b = 1$ we can see from computer plot (!) that $g^*(x) \leq 73.25$ and we have

$$Q(1,n) = \frac{1}{n}\sum_{h=1}^{\frac{n-1}{2}}\left(g^*\left(\frac{h}{n}\right) - \frac{4\pi}{1-2h/n}\right)$$
$$\le \frac{73.25}{2} - 4\pi\cdot\frac{\log(2n-2)+\gamma}{2}$$

which is negative for $n \ge 97$. ■

So far we have proved Proposition 5.1.2 in all but a finite number of cases. The remaining cases can be checked with the help of a computer.

Chapter 6

New tame elements

Since the curves F_7 and F_9 are missing only a single independent element of K_2 on the three quotients isomorphic to $C_{1,1}$ it is natural to wonder where the other element could be hiding.

When n is odd, using the Rosset-Tate algorithm it is possible to find expressions for the two elements that are already known. The equation of the curve $C_{1,1}$ is given by $u(1-u) = v^n$ and we already have the elements $\{1-x, 1-y\}$ and $\{1-1/y, 1+x/y\}$ in $K_2(F_n)$.

The norm of the first element is basically $\{u, 1-v\}$ while the norm of the second is

$$\left\{u-(-v)^{(n-1)/2}, u(1+v)\right\} + \{u, 1+v\}$$

It is interesting to understand why these elements are tame. It is fairly easy to see that both are tame when $u = 0$. When $v = 1$ the function $1-v$ has a zero and u satisfies $u(1-u) = v^n = 1$. This actually implies that u is a sixth root of unity and so a multiple of the first element is tame.

Seeing that the second element is tame is not much more difficult. When $u = (-v)^{(n-1)/2}$ we have $u^2 = (-v)^{n-1} = v^{n-1}$ since n is odd. Therefore

$$u\,(1+v) = u\left(1+\frac{v^n}{v^{n-1}}\right) = u\left(1+\frac{u(1-u)}{u^2}\right) = 1$$

When $v = -1$ the tame symbol is $(u-1) \cdot u = -v^n = -(-1)^n = 1$.

These calculations led to thoughts about what other kinds of symbols might be useful in finding new tame elements. For example I found the identity

$$\begin{aligned}\left(u+v^k\right)\left(1-u-v^{n-k}\right) &= u(1-u) - v^n + v^k\left(1-u-uv^{n-2k}\right)\\ &= v^k\left(1-u\left(1+v^{n-2k}\right)\right)\end{aligned}$$

which implies that the symbol

$$\left\{v^{-k}\left(u+v^k\right)\left(1-u-v^{n-k}\right), u\left(1+v^{n-2k}\right)\right\}$$

is zero as an element of K_2. I wondered if a linear combination of symbols of a similar form could be used to make new tame elements.

6.1 A computer search for new tame elements

Suppose we have a curve C of genus g and that $g-1$ independent elements $\alpha_1, \alpha_2, \ldots, \alpha_{g-1}$ of K_2C are already known. Suppose we have list of candidate symbols $\beta_1, \beta_2, \ldots, \beta_k$ which can be used to make some unknown new tame element $\alpha_g = \Sigma\lambda_i\beta_i$. By Beilinson's conjecture we would expect that the determinant of the matrix formed by the integrals of the g elements (α_i) along each of g loops forming a basis of $H_1(C, \mathbb{Z})^-$ was equal to some rational multiple of $L^{(g)}(C, 0)$.

Let us treat the curve C, the basis of $H_1(C, \mathbb{Z})^-$ and the elements $(\alpha_i)_{1\le i<g}$ as fixed entities and view the resulting determinant solely as a function of α_g and denote this $V(\alpha_g)$. By linearity of the determinant we will have

$$b_1V(\beta_1) + b_2V(\beta_2) + \ldots + b_kV(\beta_k) = cL^{(g)}(C, 0)$$

where $b_1, b_2, \ldots, b_k$ and c are all integers.

The idea is to compute $L^{(g)}(C, 0)$ and all of the $V(\beta_i)$ numerically to high precision and use an integer relation algorithm to find approximate equalities between them having hopefully small integer coefficients. With any luck these relations would give insight on how new tame elements could be formed.

Suppose we have a set of real numbers $x_1, x_2, \ldots, x_k$. Choose a precision d and define integers X_i by $X_i = \lfloor 10^d x_i \rfloor$. One method to find small integer relations between the (x_i) is to form the $(k+1) \times k$ matrix

$$\begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & & 1 \\ X_1 & X_2 & \ldots & X_k \end{pmatrix}$$

and then apply the LLL lattice reduction algorithm [14] on its columns. If there really was a linear relationship between the (x_i) and the precision was high enough then it is very likely that such a relation would be obtained by reading the first column of the resulting matrix.

Luckily all of this has already been implemented by the function `lindep` in the computer algebra package PARI/GP. For example the following command recovers the well-known identity $\zeta(2) = \pi^2/6$:

```
? lindep([log(zeta(2)), log(Pi), log(2), log(3)])
%1 = [1, -2, 1, 1]~
```

All that remains is to decide upon the set of symbols (β_i) and find a basis for $H_1(C, \mathbb{Z})$.

6.2 A basis for $H_1(C_{1,1,n-2}, \mathbb{Z})^-$

Proposition 6.2.1. *Define the path γ_k on $C_{1,1,n-2}$ by*

$$\begin{aligned} \gamma_k : [0,1] &\longrightarrow C_{1,1,n-2} \\ t &\longmapsto \left(t, \zeta^k \left[t(1-t)\right]^{\frac{1}{n}}\right) \end{aligned}$$

where $\zeta = e^{\frac{2\pi i}{n}}$. Then the loops $\gamma_k - \gamma_{-k}$ for k satisfying $1 \leq k < \frac{n}{2}$ are a basis of $H_1(C_{1,1,n-2}, \mathbb{Z})^-$.

Proof. Recall that $H_1(F_n, \mathbb{Z})$ is spanned by paths of the form $t \mapsto \left(\zeta^j t^{\frac{1}{n}}, \zeta^k (1-t)^{\frac{1}{n}}\right)$. These paths map onto the paths γ_k under the projection $\varphi : F_n \to C_{1,1,n-2}$ given by $\varphi(x,y) = (x^n, xy)$.

The paths γ_k all start at the point $(u,v) = (0,0)$ and end at the point $(u,v) = (1,0)$ therefore the projection of any loop in F_n can be written as a combination of loops of the form $\gamma_{k_1} - \gamma_{k_2}$. These in turn can be written in terms of loops of the form $\gamma_k - \gamma_0$ for $1 \leq k < n$.

When n is even we have the equality $\sum_{k=1}^{n-1} (-1)^k (\gamma_k - \gamma_0) = \sum_{k=0}^{n-1} (-1)^k \gamma_k$. On account of the fact that regular differentials on $C_{1,1,n-2}$ are spanned by differentials of form $v^k du$ and because of the identity $\sum_{k=0}^{n-1} (-\zeta)^k = 0$ it is easy to show that this loop is trivial in $H_1(C_{1,1,n-2}, \mathbb{Z})$.

When n is odd the number of independent loops is $n-1$ and when n is even it is $n-2$. Therefore if we omit the loop $\gamma_{n/2} - \gamma_0$ when n is even, loops of the form $\gamma_k - \gamma_0$ form a basis of $H_1(C_{1,1,n-2}, \mathbb{Z})$ for all values of n.

The loop $\gamma_k - \gamma_0$ maps to $\gamma_{n-k} - \gamma_0$ under complex conjugation so any loop belonging to the -1 eigenspace of $H_1(C_{1,1,n-2}, \mathbb{Z})$ will require the coefficient of $\gamma_k - \gamma_0$ to be the negative of the coefficient of $\gamma_{n-k} - \gamma_0$ and so we see that loops of the form $\gamma_k - \gamma_{-k}$ with $1 \leq k < \frac{n}{2}$ are a basis for $H_1(C_{1,1,n-2}, \mathbb{Z})^-$. ■

6.3 Choosing which symbols to use

After much experimentation it was decided to use symbols of the following form

$$\begin{aligned} &\{u, 1+v^{2k}\}\,, && \{u, 1\pm v^{2k+1}\} \\ &\{u\pm v^{j}, u\}\,, && \{u\pm v^{j}, v\} \\ &\{u\pm v^{j}, 1+v^{2k}\}\,, && \{u\pm v^{j}, 1\pm v^{2k+1}\} \end{aligned}$$

It is easy to verify that all of the above symbols are tame for $(u, v) \in \{(0,0),(1,0)\}$ (More correctly, the tame symbols are equal to ± 1 at those points.) This means that when performing the integration, any zeros of the above functions at those points are relatively easy to deal with. Throughout the rest of this chapter it means that there is no need to check the value of the tame symbol when $v = 0$. Also, except for when u and v^j have the same order at infinity (this happens when n is even and $j = \frac{n}{2}$), we can find a local parameter, f, such that all functions used in the above symbols behave like ± 1 multiplied by some power of f at infinity. This means that tameness at infinity is automatic unless we use the functions $u \pm v^{\frac{n}{2}}$ or introduce any functions with leading coefficient different to ± 1.

Note that u is always real on the paths of integration and so the functions $u \pm v^j$ can only have a zero on path γ_k if $\zeta^{jk} \in \mathbb{R}$. For example, this can never happen if n is prime. Nevertheless, care was taken to properly handle zeros on the path of integration in the cases where they occurred.

If the integer relation algorithm produced a relation between the $V(\beta_i)$ that did not include the term $L^{(g)}(0)$ then the "most complicated" symbol that appeared in the relation would be dismissed and the process repeated with a smaller set of symbols.

6.4 Verifying Beilinson's conjecture on F_7

The first target was to find the missing element of K_2 on the hyperelliptic quotient $C_{1,1,5}$ of F_7. Because of the transformation η available on F_7 this would yield the three missing elements of K_2 on the isomorphic quotients $C_{1,3,3}$ and $C_{1,5,1}$ and therefore complete the verification of Beilinson's conjecture on F_7.

The element in the following proposition came almost immediately from the computer search even though it took some time to believe that it could actually be tame.

Proposition 6.4.1. *If $n \equiv \pm 1 \pmod 6$ then the following is a tame element on the curve $u(1-u) = v^n$.*

$$\left\{u - (-v)^{(n-3)/2}, u(1+v^3)\right\} + 4\left\{u, 1+v\right\} - 3\left\{u, 1+v^3\right\}$$

Proof. When $u = (-v)^{(n-3)/2}$ we have $u^2 = (-v)^{n-3} = v^{n-3}$ since n is odd. We see that the element is tame at these points with a calculation very similar to before:

$$u(1+v^3) = u(1+v^n/u^2) = u(1+(1-u)/u) = 1$$

When $v = -1$ all three functions appearing on the right hand side of the symbols have simple zeros. The tame symbol at these points is therefore

$$\left(u - (-v)^{(n-3)/2}\right) \cdot u^4 \cdot u^{-3} = (u-1) \cdot u = -v^n = -(-1)^n = 1$$

From now on we only need to deal with points where $v^3 = -1$ but $v \neq -1$. This implies that $v^2 - v + 1 = 0$. The tame symbol at these points is

$$\frac{u - (-v)^{(n-3)/2}}{u^3}$$

If we assume that $n \equiv 1 \pmod 6$ then it is clear that $v^n = v$ and not much more difficult to check that $(-v)^{(n-3)/2} = v^2$. Using the fact that $u^2 = u - v^n = u - v$

we can prove the identity $vu^3 = u - v^2$ as follows

$$vu^3 = vu(u-v) = v(u^2 - uv) = v(u - v - uv) = u(v - v^2) - v^2 = u - v^2$$

so that

$$\frac{u - (-v)^{(n-3)/2}}{u^3} = \frac{u - v^2}{u^3} = \frac{vu^3}{u^3} = v$$

which is a root of unity.

When $n \equiv -1 \pmod 6$ we can check that $v^n = v^{-1}$ and $(-v)^{(n-3)/2} = v^{-2}$ so the calculations go exactly the same as before but with v replaced by its conjugate $1/v$. ■

Corollary 6.4.2. *The element*

$$\left\{u - v^2, u(1+v^3)\right\} + 4\left\{u, 1+v\right\} - 3\left\{u, 1+v^3\right\}$$

is a member of $K_2C_{1,1,5}$. It is independent of the two elements that were previously known and completes the verification of Beilinson's conjecture on the Fermat curve F_7.

6.5 Verifying Beilinson's conjecture on F_9

The Fermat curve F_9 has genus 28 and consists of the Fermat curve F_3 of genus 1 together with 9 quotients of genus 3. Six of the quotients are isomorphic to $C_{1,2,6}$ for which we have already verified Beilinson's conjecture, while the other three are isomorphic to the quotient of the hyperelliptic curve $C_{1,1,7}$ by $C_{1,1,1}$.

We will now verify Beilinson's conjecture on the genus 4 hyperelliptic curve $C_{1,1,7}$ defined by the equation $u(1-u) = v^9$. Since the curve is not "primitive" the L-value will be the product of the L-values from the sub-curves: $L'_{1,1,1}(0) \times L^{(3)}_{1,1,7}(0)$.

Again the element below came almost directly from computer search. The third symbol was adjusted by hand in order to make the element tame without altering the lattice after projection under the regulator.

Proposition 6.5.1. $K_2(C_{1,1,7})$ *contains the following four independent elements and their images under the regulator generate a lattice having volume consistent with Beilinson's conjecture.*

$$\{u, 1-v\}, \quad \{u, 1-v^3\}, \quad \{u-v^4, u(1+v)\} + \{u, 1+v\},$$
$$2\left\{u+v^3, \frac{(1+v)^2(1-v)^3\left(1+v^2\right)}{v^2\left(1-v^3\right)}\right\} + 4\left\{u, \frac{1+v}{1+v^2}\right\} - \left\{3u, \frac{(1-v)^3}{1-v^3}\right\}$$

Proof. The first and third elements in this list are the norms of $\{1-x, 1-y\}$ and $\eta\{1-x, 1-y\}$ respectively from $K_2(F_9)$ to $K_2(C_{1,1,7})$. The second element is the norm of $\{1-x, 1-y\}$ from $K_2(F_3)$ to $K_2(C_{1,1,7})$. The fourth element is new and we proceed to show that it is in the kernel of the tame symbol.

Note that we have introduced the function $3u$ in the third symbol. This does not present any problem at infinity because the function on the right hand side of the symbol is of order 0 there.

At points where $u = -v^3$ but $v \neq 0$ we can use the equation of the curve to show that v satisfies $v^6 + v^3 + 1 = 0$. Now it is possible to verify that

$$\begin{aligned}
&(1+v)^2(1-v)^3(1+v^2) + v^2(1-v^3) \\
&\qquad = (1-v)\left[(1+v)^2(1-v)^2(1+v^2) + v^2(1+v+v^2)\right] \\
&\qquad = (1-v)\left[(1-v^2)^2(1+v^2) + v^2(1+v+v^2)\right] \\
&\qquad = (1-v)\left[1 - 2v^2 + v^4 + v^2 - 2v^4 + v^6 + v^2 + v^3 + v^4\right] \\
&\qquad = (1-v)(v^6 + v^3 + 1) \\
&\qquad = 0
\end{aligned}$$

so that when $u = -v^3 \neq 0$ we have

$$\frac{(1+v)^2(1-v)^3(1+v^2)}{v^2(1-v^3)} = -1$$

(Note that the denominator cannot be zero since the polynomial $X^2(1-X^3)$ is coprime to $X^6 + X^3 + 1$).

When $v = -1$ it is easy to see that the element is tame:

$$\left(u + v^3\right)^4 \cdot u^4 = (u-1)^4 u^4 = v^{36} = 1$$

When $v^2 = -1$ the tame symbol can be seen to be equal to 1 as follows

$$\frac{\left(u + v^3\right)^2}{u^4} = \left(\frac{u + v^2 \cdot v}{u - v^9}\right)^2 = \left(\frac{u - v}{u - v}\right)^2 = 1$$

Last of all we look at the places where $v^3 = 1$. At these points u satisfies $u(1-u) = v^9 = 1$ and the function $\frac{(1-v)^3}{1-v^3}$ either has a zero of order two or a pole. Regardless, the tame symbol will be equal to 1 at these points because of the following calculation:

$$\frac{\left(u + v^3\right)^2}{3u} = \frac{(u+1)^2}{3u} = \frac{u^2 + 2u + 1}{3u} = \frac{3u - u(1-u) + 1}{3u} = \frac{3u}{3u} = 1$$

■

6.6 Verifying Beilinson's conjecture on $C_{1,1,6}$

The same method was used on the hyperelliptic curve $C_{1,1,6}$ of genus 3. Since the curve $C_{1,1,6}$ is not primitive (it has the genus 1 curve $C_{1,1,2}$ as a quotient) the correct L-value is $L'_{1,1,2}(0) \times L^{(2)}_{1,1,6}(0)$. A new tame element was found without any major difficulty.

Proposition 6.6.1. *The following three elements are independent elements of $K_2(C_{1,1,6})$ whose image under the regulator generates a lattice having volume consistent with Beilinson's conjecture.*

$$\begin{aligned} &\{u, 1-v\}, \quad \{u, 1+v\}, \\ &2\left\{u - v^3, \frac{(1-v)(1+v)^2}{u^2(1+v^3)}\right\} + 3\left\{u, 1+v^3\right\} + \left\{1+v, \frac{1+v^3}{1+v}\right\} \end{aligned}$$

Proof. The equation of the curve is $u(1-u) = v^8$. This implies that when $u = v^3 \neq 0$ we have $v^5 = 1 - v^3$. We use this identity a couple of times in the following calculation

$$\begin{aligned}
u^2(1+v^3) &= v^6(1+v^3) = v \cdot v^5(1+v^3) = v(1-v^3)(1+v^3) = v(1-v^6) \\
&= v(1-v^2)(1+v^2+v^4) = (1-v^2)(v+v^3+v^5) \\
&= (1-v^2)(1+v) = (1-v)(1+v)^2
\end{aligned}$$

So the element is tame when $u = v^3$.

When $v = 1$ the tame symbol is $(u-1)^2$. Since $u(1-u) = v^8 = 1$ this is a root of unity and so some multiple of the given element is tame. Alternatively we could add the symbol $2\{u, 1-v\}$ to make the element tame.

When $v = -1$ the first symbol gives $(u+1)^2$ to the tame symbol, the second gives u^3 while the third gives $\frac{1}{3}$ since the *left* side has a simple zero while the right side is equal to 3. Similar to what we have seen before this implies the tame symbol is

$$\frac{1}{3}(u+1)^2u^3 = \frac{1}{3}u^3(u^2+2u+1) = \frac{1}{3}u^3(3u+u^2-u+1) = u^4$$

Again this is a root of unity or it could be made tame by subtracting the symbol $4\{u, 1+v\}$.

Finally, when $v^3 = -1$ with $v \neq -1$ we can verify that the element is tame by using the identities $v^2 - v + 1 = 0$, the identity $u^2 = u - v^2$ and some fairly tedious algebra:

$$\begin{aligned}
u^3(1+v) &= u(u-v^2)(1+v) \\
&= (u^2-uv^2)(1+v) \\
&= (u-v^2-uv^2)(1+v) \\
&= u(1-v^2)(1+v) - v^2(1+v) \\
&= u(1+v-v^2-v^3) - v^2 - v^3 \\
&= 3u+1-v^2 \\
&= 3u+1-(u-u^2) \\
&= u^2+2u+1 \\
&= (u+1)^2 \\
&= (u-v^3)^2
\end{aligned}$$

■

Remark 6.6.2. Since the previous calculation is not very enlightening we can try to explain why this kind of thing should not be totally unexpected.

When $v^3 = -1$ and $u(1-u) = v^8$ the field $\mathbb{Q}(u)$ is of degree 4 with 2 complex embeddings and so has unit group of rank 1. We can see that u is a unit since it divides v^8. The roots of unity are generated by v and it turns out out that the group of all units is generated by u and v together.

The norm of $u - v^3$ from $\mathbb{Q}(u)$ down to $\mathbb{Q}(v)$ is $(u-v^3)(1-u-v^3)$ which is equal to $v^8 - v^3 + v^6 = v^2 + 2 = 1 + v$. In turn, one finds that $1 + v$ has norm 3. There is a unique prime above 3 since one can check that $(u-v)^4 = -3u^2$ therefore $\left(u-v^3\right)^2/(1+v)$ is a unit and so it has to be expressible as a product of powers of u and v. It is a little surprising that no power of v occurs in this instance but, even if it did, v is a root of unity in this situation and so we could take some multiple of the element instead.

6.7 Verifying Beilinson's conjecture on $C_{1,1,8}$

The curve $C_{1,1,8}$ defined by the equation $u(1-u)=v^{10}$ is a curve of genus 4 having the curve $C_{1,1,3}$ defined by $u(1-u)=v^5$ as a genus 2 quotient. We have two independent elements of K_2 on the variety $A_{1,1,3}$ but only one on $A_{1,1,8}$ so that we are missing a single element of K_2 on $C_{1,1,8}$. The correct L-value will be $L^{(2)}_{1,1,3}(0)\times L^{(2)}_{1,1,8}(0)$ and we are ready to apply our computational method again. We managed to find a new element but it is so unwieldy that the proof of its tameness relies on computer algebra in a couple of places.

Proposition 6.7.1. *The following three elements are independent elements of $K_2(C_{1,1,8})$ and together with a complicated element to be described in the proof their image under the regulator generates a lattice having volume consistent with Beilinson's conjecture.*

$$\begin{gathered}\{u,1-v\}\,,\quad \{u,1+v\}\,,\\ \left\{u-v^4,u\left(1+v^2\right)\right\}+\left\{u,1+v^2\right\}\end{gathered}$$

Proof. The first element in the list is just the norm of $\{1-x,1-y\}$ from F_{10} down to $C_{1,1,8}$. Similarly the norm of $\{1-x,1-y\}$ from F_5 via F_{10} is $\{u,1-v^2\}$. Clearly $\{u,1+v\}$ is just the difference of $\{u,1-v\}$ and $\{u,1-v^2\}$. The third element is the other element of K_2 that comes from $C_{1,1,3}$.

A computer search showed that there was a linear relationship between the L-value and the determinant of the regulators of the above elements together with the following element which we will denote by α.

$$\begin{aligned}\alpha=-7\left\{u,1-v^3\right\}+\left\{u-v,\frac{(1+v)(1-v^5)(1+v^5)}{1+v^3}\right\}\\ +\left\{u-v^3,\frac{(1-v)^5(1+v)^7(1+v^5)^3}{v^{38}(1+v^3)^5(1-v^5)^2}\right\}\end{aligned}$$

It is possible to see that the above is tame when $u=v\neq 0$. At these points we have

$$\left(1-v^5\right)\left(1+v^5\right) = 1-v^{10} = 1-u(1-u) = 1-v+v^2 = \frac{1+v^3}{1+v}$$

Incredibly, the element is tame when $u = v^3$ but I can only offer a brief explanation of the situation at these points before relying on some computer algebra to complete the proof.

We can say that when $u = v^3$ with $v \neq 0$ that v satisfies the polynomial $v^7 + v^3 - 1 = 0$. The number field $\mathbb{Q}(v)$ has degree 7 with 1 real embedding and 3 complex embeddings so the group of units has rank 3.

It turns out that v, $1-v$ and $1-v^5$ are units that generate the units of $\mathbb{Q}(v)$ modulo ± 1. Meanwhile the expression $\frac{(1+v)^7(1+v^5)^3}{(1+v^3)^5}$ turns out to be a unit in $\mathbb{Q}(v)$ and so it has a unique expression in terms of v, $1-v$ and $1-v^5$. The fact that v appears with exponent 38 is obviously a surprise. Here is computer input that shows that the symbol really is tame at these points:

```
? f = v^7 + v^3 - 1;
? g = (1-v)^5 * (1+v)^7 * (1+v^5)^3;
? h = v^38 * (1+v^3)^5 * (1-v^5)^2;
? print(g / h % f);
1
```

Continuing, it was found that α is not tame when the various factors on the right hand side of the symbols are zero. Instead we need to make use of the transformation $\sigma : v \mapsto -v$ that is available on the quotient curves when n is even.

Consider integrating the regulator of a symbol β along path γ_k. Then it is fairly simple to prove that

$$\int_{\gamma_k} \operatorname{reg}\beta = -\int_{\gamma_{n/2-k}} \operatorname{reg}\sigma\beta$$

The first two of our elements of K_2 map to each other under σ while the third is invariant under σ. This means that the first three columns of the matrices we form will look something like this:

$$\begin{pmatrix} a & -d & e \\ b & -c & f \\ c & -b & -f \\ d & -a & -e \end{pmatrix}$$

The matrices that we form when including the elements α and $\sigma\alpha$ will look like the following two matrices respectively:

$$\begin{pmatrix} a & -d & e & g \\ b & -c & f & h \\ c & -b & -f & i \\ d & -a & -e & j \end{pmatrix} \qquad \begin{pmatrix} a & -d & e & -j \\ b & -c & f & -i \\ c & -b & -f & -h \\ d & -a & -e & -g \end{pmatrix}$$

Flipping the second matrix from top to bottom (which is an even permutation of the rows) and then swapping the first two columns will give a matrix that is the negative of the first one and hence it will have the same determinant. The fact that we used an odd number of swaps means that the original two matrices have determinants that are negatives of each other.

Using this property it is far easier to study the element $\alpha - \sigma\alpha$ which will give a determinant twice as big as previous but will still have an apparent relationship with the L-value.

Concretely we now study the element

$$\begin{aligned} -7\left\{u, \frac{1-v^3}{1+v^3}\right\} &+ \left\{u-v, \frac{(1+v)(1-v^5)(1+v^5)}{1+v^3}\right\} \\ &- \left\{u+v, \frac{(1-v)(1-v^5)(1+v^5)}{1-v^3}\right\} \\ &+ \left\{u-v^3, \frac{(1-v)^5(1+v)^7(1+v^5)^3}{v^{38}(1+v^3)^5(1-v^5)^2}\right\} \\ &- \left\{u+v^3, \frac{(1+v)^5(1-v)^7(1-v^5)^3}{v^{38}(1-v^3)^5(1+v^5)^2}\right\} \end{aligned}$$

Because of the symmetry under σ we only need to show that this element is tame (or can be made tame) at points where $v^5 = 1$, where $v^3 = 1$ and then separately where $v = 1$.

First of all suppose that $v^5 = 1$ with $v \neq 1$. At these points $u(1-u) = v^{10} = 1$ and so u is a sixth root unity and the field $\mathbb{Q}(u, v)$ is the 15-th cyclotomic field $\mathbb{Q}(\mu)$ having degree 8. Indeed, let us suppose that $u = -\mu^5$ and $v = \mu^3$. The tame symbol at points with $v^5 = 1$ and $v \neq 1$ is equal to

$$\frac{u - v}{(u + v)\,(u - v^3)^2\,(u + v^3)^3}$$

and it is possible to check in $\mathbb{Q}(\mu)$ that this is equal to

$$\frac{-v}{u\,(1 + v)^2}.$$

This fact allows us to subtract the element

$$\left\{\frac{-v}{u\,(1 + v)^2}, \frac{1 - v^5}{1 - v}\right\}$$

which corrects the tame symbol at the desired points while being itself tame at all points where the left-hand side of the above symbol has zeros or poles.

Also this symbol does not affect the lattice after projection under the regulator. There are three reasons for this. First, applying the substitution $u \mapsto 1 - u$ to a symbol β means that $\operatorname{reg}\beta$ maps to $-\operatorname{reg}\beta$. This is because of the symmetry under the transformation $t \mapsto 1 - t$ in the definition of the γ_k. In particular any symbols that contain only functions of v map to 0 under the regulator. Second, the functions u and $1 - v^5$ are always real on the paths γ_k so that the integrands are always precisely zero. Finally, the symbol $\{u, 1 - v\}$ is already present in our list of elements and so it will be irrelevant once the determinant is taken.

Now let us consider points where $v^3 = 1$ and $v \neq 1$. At these points $u(1-u) =$

$v^{10} = v$ so $\mathbb{Q}(u, v)$ is a degree 2 extension of $\mathbb{Q}(\mu_3)$. The tame symbol here is

$$u^{-7}(u+v)\left(u+v^3\right)^5$$

and it is possible to check that this is equal to $(v-1)^3$; for example with the following PARI/GP input:

```
? f = u^-7 * (u+v) * (u+v^3)^5;
? g = (v-1)^3;
? print((f - g) * Mod(1, u*(1-u) - v) * Mod(1, v^2 + v + 1))
Mod(0, -u^2 + u - v)
```

Therefore we can correct the tame symbol at the desired points by subtracting the symbol

$$3\left\{v-1, \tfrac{1}{3}\left(v^2+v+1\right)\right\}.$$

Again this is trivial under the regulator since both sides are functions of v. The symbol is tame at points we have already examined but not at infinity. This will be fixed shortly.

Finally the tame symbol at $v = 1$ is

$$u^{-7}(u-v)(u+v)^{-1}\left(u-v^3\right)^3\left(u+v^3\right)^{-5} = u^{-7}(u-1)^4(u+1)^{-6} = \frac{-u}{27}$$

so that we can correct the tame symbol at these points by using the symbol $\{\frac{-u}{27}, 1-v\}$. This does not affect the tame symbol at any point already shown to be tame, it doesn't change the lattice after projection under the regulator and because of the fact that $3^3 = 27$ it is easy to check that the element has become tame at infinity. ■

6.8 On the curves $u(1-u) = -v^n$

When n is even we have only a single independent element of K_2 on the curves $C_{1,n-2,1}$. These curves are not isomorphic to $C_{1,1,n-2}$ but instead isomorphic to a curve defined by the equation

$$u(1-u) = -v^n$$

We have the projection $\varphi : F_n \longrightarrow C$ given by

$$\varphi(x,y) = \left(\frac{1}{y^n}, \frac{x}{y^2}\right)$$

Using the Rosset-Tate algorithm we can take the norm of our usual element $\{1-x, 1-y\}$ and we end up with

$$\left\{\frac{v}{1-v}, \frac{u-v^{n/2}}{u-1}\right\}$$

There are a couple of interesting things to note about this result. First, the only zero of the function $u - v^{n/2}$ is when $u = v = 0$. It is easiest to see this from the factorisation

$$u = u^2 - v^n = \left(u - v^{n/2}\right)\left(u + v^{n/2}\right)$$

Second, we might wonder why the fraction $\frac{v}{1-v}$ is required. Could we not just get away with using $1-v$ on its own? The answer appears to be "no" because when n is even there are two branches of the function u at infinity: one where u behaves like $v^{n/2}$ and one where it behaves like $-v^{n/2}$.

Indeed we can see from the identity

$$u - v^{n/2} = \frac{u}{u + v^{n/2}}$$

that $u - v^{n/2}$ tends to $\frac{1}{2}$ on the branch where $u \sim v^{n/2}$. This reveals why the ratio of v and $1-v$ is necessary: it is always equal to 1 at infinity.

To describe the loops on the curve $C : u(1-u) = -v^n$ we have the following proposition.

Proposition 6.8.1. *Let* $\epsilon = e^{\frac{\pi i}{n}}$ *and* $\zeta = e^{\frac{2\pi i}{n}}$. *Define the paths* γ_k *by*

$$\begin{aligned} \gamma_k : [0,1] &\longrightarrow C \\ t &\longmapsto \left(t, \epsilon\zeta^k \left(t(1-t)\right)^{\frac{1}{n}}\right) \end{aligned}$$

then the loops $\gamma_k - \gamma_{-1-k}$ *for* k *satisfying* $0 \leq k < n/2$ *span* $H_1(C,\mathbb{Z})^-$ *with the only relation between these loops being*

$$\sum_{k=0}^{n/2-1} (-1)^k \left(\gamma_k - \gamma_{-1-k}\right) = 0$$

Proof. It is clear that loops of the form $\gamma_k - \gamma_{-1-k}$ belong to $H_1(C,\mathbb{Z})^-$. Since n is even we have

$$\sum_{k=0}^{n/2-1} (-1)^k \left(\gamma_k - \gamma_{-1-k}\right) = \sum_{k=0}^{n-1} (-1)^k \gamma_k$$

and after applying the obvious isomorphism between C and $C_{1,1,n-2}$ given by $(u,v) \mapsto (u, \epsilon v)$ this is exactly the same relation mentioned in the proof of Proposition 6.2.1. ■

Now we are ready to use the same technique as in the previous sections. The following new element was found originally when $n = 8$. In fact it generalises to any even n that is not a multiple of three, thus giving a verification of Beilinson's conjecture when $n = 10$ as an added bonus.

Proposition 6.8.2. *When* $n \equiv 2, 4 \pmod 6$ *the following is an element of* K_2 *of the curve* $u(1-u) = -v^n$.

$$\left\{u - v^{n/2}, \frac{(1-v)^3}{1-v^3}\right\} + \left\{u, \frac{\left(1-v^3\right)^3}{1-v}\right\}$$

Proof. We have already seen that $u - v^{n/2}$ is only zero when $u = v = 0$ so tameness there is easy to check. At infinity $u - v^{n/2}$ has different behaviour on either of the two branches but this is irrelevant because the right hand side of the first symbol is equal to 1 at infinity.

When $u = 0$ the element is tame by design. When $v = 1$ the right hand functions of both symbols have zeroes of order 2 and so the tame symbol is

$$\left(u - v^{n/2}\right)^2 u^2 = (u-1)^2 u^2 = v^{2n} = 1$$

All that remains is to check points where $v^3 = 1$ but $v \neq 1$. At these points we have $v^2 + v + 1 = 0$. If $n \equiv 2 \pmod 6$ then $v^{n/2} = v$ so the tame symbol is

$$\frac{u^3}{u - v}$$

Using $u^2 = u + v^n = u + v^2$ we can compute

$$u^3 = u(u + v^2) = u^2 + uv^2 = u + v^2 + uv^2 = -uv + v^2 = -v(u - v)$$

so that

$$\frac{u^3}{u - v} = -v$$

is a root of unity. When $n \equiv 4 \pmod 6$ we have $v^{n/2} = v^2 = 1/v$ so the same calculation will be repeated but with v replaced by its conjugate $1/v$. ■

Corollary 6.8.3. *The following three elements are independent elements of K_2 of the curve $u(1-u) = -v^8$ and their images under the regulator generate a lattice having volume consistent with Beilinson's conjecture.*

$$\left\{\frac{u - v^4}{u - 1}, \frac{v}{v - 1}\right\}, \qquad \left\{\frac{u - v^4}{u - 1}, \frac{v}{v + 1}\right\},$$
$$\left\{u - v^4, \frac{1 - v^3}{(1 - v)^3}\right\} + \left\{u, \frac{\left(1 - v^3\right)^3}{1 - v}\right\}.$$

Corollary 6.8.4. *The following four elements are independent elements of K_2 of the curve $u(1-u) = -v^{10}$ and their images under the regulator generate a* *lattice having volume consistent with Beilinson's conjecture.*

$$\left\{u, 1+v^2\right\}, \qquad \left\{u-v^4, u(1-v^2)\right\} + \left\{u, 1-v^2\right\},$$

$$\left\{\frac{u-v^5}{u-1}, \frac{v}{v-1}\right\}, \qquad \left\{u-v^5, \frac{(1-v)^3}{1-v^3}\right\} + \left\{u, \frac{\left(1-v^3\right)^3}{1-v}\right\}.$$

6.9 Summary and further possibilities

We have seen that Proposition 3.3.3 gives a set of generators of $\operatorname{reg}\mathcal{S}^\Sigma$, where $\mathcal{S}^\Sigma$ is the subgroup of K_2F_n that is generated by symbols with divisorial support at infinity and defined over $\mathbb{Q}$. For $n \leq 100$ at least, computer calculations show that the only other relations between these generators are given by Proposition 3.4.1 except for when n equals 3 or 6. We used this source of elements to verify Beilinson's conjecture on 11 non-isomorphic quotients of the Fermat curves as detailed in section 4.8.

In this chapter we found new elements of K_2 that allowed us to give a full verification of Beilinson's conjecture on the Fermat curves F_7 and F_9 and also on certain hyperelliptic quotients of F_8 and F_{10}. Of course there are other quotients of Fermat curves where the known rank of K_2 is one less than the rank predicted by Beilinson's conjecture. It is possible that the method introduced in this chapter will be successful there as well.

Looking back to the tables in Remark 2.2.3 we see that we have the desired number of elements of K_2 on all quotients of F_8 except for $A_{1,3,4}$ and $A_{1,5,2}$. Therefore the known rank of K_2F_8 is currently 19 from an expected 21 with just a single element of K_2 being missing from each of those genus 2 quotients.

The situation is similar on F_{10}. There we are missing a single element of K_2 on two pairs of genus 2 quotients isomorphic to $A_{1,3,6}$ and $A_{1,5,4}$ so that the known rank of K_2F_{10} is 32 from a predicted rank of 36.

The K_2 groups on F_{12} were less well explored than I intended. A putative relation between regulator values and the L-function on the curve $C_{1,1,10}$ was discovered but a proof that the suggested element is tame was not finalised and included in this thesis.

Even beyond $n = 12$ there are some situations where the rank of known elements is one smaller than the rank predicted by Beilinson. The primitive quotients of the Fermat curves F_{14} and F_{18} have genus 3. On quotients of the form $A_{1,2s,n-2s-1}$ with $1 \leq s \leq \frac{n}{2} - 2$ there are two independent elements α and β as given in Proposition 3.3.3. It may be interesting to try the method of this chapter on those curves. Furthermore, on account of Proposition 6.8.2, I speculate that there are two independent elements of K_2 on the quotient $A_{1,12,1}$ of F_{14} and the method of this chapter could yield a third. Finally, all quotients of F_{15} except for those isomorphic to $A_{1,1,13}$ have 3 independent elements of K_2 whereas Beilinson conjectures a rank of 4.

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