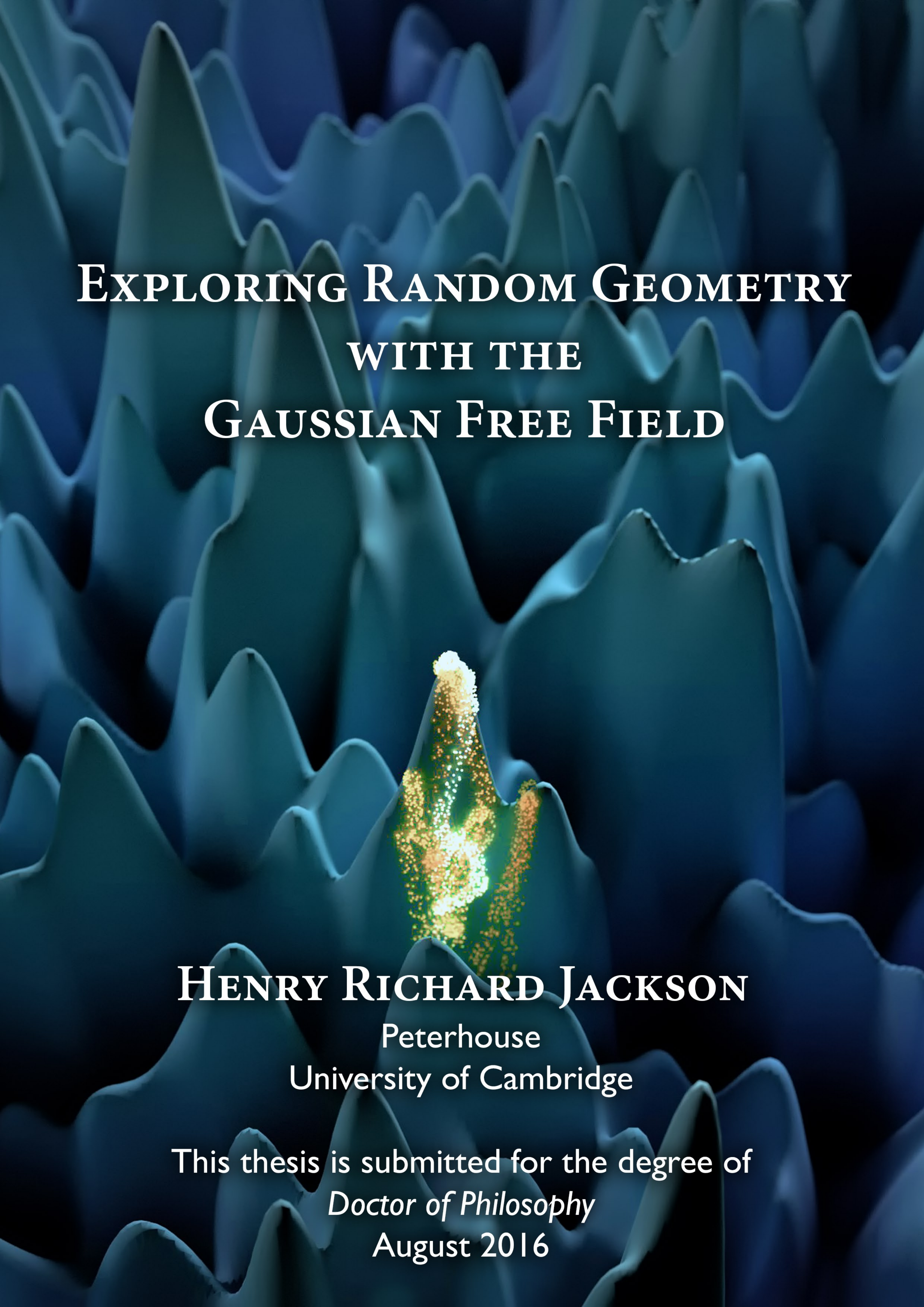

Exploring Random Geometry with the Gaussian Free Field

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Overview

This thesis studies the geometry of objects from 2-dimensional statistical physics in the continuum.

Chapter 1 is an introduction to Schramm-Loewner evolutions (SLE). SLEs are the canonical family of non-self-intersecting, conformally invariant random curves with a domain-Markov property. The family is indexed by a parameter, usually denoted by κ, which controls the regularity of the curve. We give the definition of the SLE_κ process, and summarise the proofs of some of its properties. We give particular attention to the Rohde-Schramm theorem which, in broad terms, tells us that an SLE_κ is a curve.

In Chapter 2 we introduce the Gaussian free field (GFF), a conformally invariant random surface with a domain-Markov property. We explain how to couple the GFF and an SLE_κ process, in particular how a GFF can be unzipped along a reverse SLE_κ to produce another GFF. We also look at how the GFF is used to define Liouville quantum gravity (LQG) surfaces, and how thick points of the GFF relate to the quantum gravity measure.

Chapter 3 introduces a diffusion on LQG surfaces, the Liouville Brownian motion (LBM). The main goal of the chapter is to complete an estimate given by N. Berestycki, which gives an upper bound for the Hausdorff dimension of times that a γ-LBM spends in α-thick points for $\gamma, \alpha \in [0, 2)$. We prove the corresponding, tight, lower bound.

In Chapter 4 we give a new proof of the Rohde-Schramm theorem (which tells us that an SLE_κ is a curve), which is valid for all values of κ except $\kappa = 8$. Our proof uses the coupling of the reverse SLE_κ with the free boundary GFF to bound the derivative of the inverse of the Loewner flow close to the origin. Our knowledge of the structure of the GFF lets us find bounds which are tight enough to ensure continuity of the SLE_κ trace.

Statement of Originality

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted or is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution. I further state that no part of my dissertation has already been submitted or is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution.

Chapters 1 and 2 predominantly consist of literature review. The results in Chapters 3 and 4 are original research. Chapter 3 appears in [Jac14], and Chapter 4 is joint work with Nathanaël Berestycki.

Acknowledgements

I would first of all like to thank my supervisor, Nathanaël Berestycki, for all of the help and support that he has given me throughout my time in Cambridge. I have learnt so much by working with him, and as an external source of confidence he is second to none!

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I would not have been able to finish writing this thesis without the support of my family. My parents have been hugely encouraging, and willing to provide food and a place to stay when I needed a break from Cambridge. Thank you especially to Mum for her careful reading of my *entire* thesis, and Dad for joining me for our fun writing days. My sisters have always been a cheerful voice on the end of the phone, and have been full of useful advice. And through the last 18 months, Caroline has been an amazing source of support.

Finally, I must thank Julie Saville. She was my first proper maths teacher, and the first person to show me how much fun maths could be. I am very lucky to have been one of her students.

Contents

1 Schramm-Loewner Evolutions

We will now present a brief introduction to the Schramm-Loewner evolution (SLE) process. The chapter begins with some motivation from discrete lattice models in statistical physics. In Section 1.2 we give the formal definition of the SLE process and look at some of its properties. Finally, in Section 1.3 we take a closer look at the Rohde-Schramm theorem which states, roughly, that an SLE process as defined in Section 1.2 is a curve. We will return to the Rohde-Schramm theorem in more detail in Chapter 4.

1.1 Motivation

In statistical physics there is often a physical reason for defining a model on a lattice. In some materials, molecules organise themselves into regular patterns to create crystals. In others, the lattice structure is not so rigid, but using a regular lattice is a reasonable simplifying assumption.

Even if there is no physical motivation for introducing a lattice, it can hugely simplify the mathematics. If we then take the limit as the lattice size tends to zero, the resulting model will often have all of the characteristics we want. In some way it "forgets" the lattice used in the definition.

1.1.1 Percolation

We will now introduce a simple lattice model to use as a concrete example. Specifically, we will look at the critical percolation model in two dimensions. This will be an informal introduction; the interested reader might like to look at [Gri99, GK12] for more details.

Let us suppose that we want to answer the question "how does water percolate through porous rock?" One way of modelling this phenomenon is to assume that the porous rock is a random conglomeration of small, solid rock particles, with empty space in between. We also assume that these rock particles are so small that their individual size and shape will not make a significant difference. So, we can simplify our model by saying that all of the rock particles (or empty spaces) are the same size and shape.

More precisely, we will assume that all of these shapes are hexagons organised into a (finite) honeycomb lattice. We will set all of the hexagons at the bottom of the lattice to be rock, all of the hexagons at the top to be empty space, and for each hexagon in the middle of the lattice we independently choose rock or empty space with probably 1/2. See Figure 1.1 for an example.

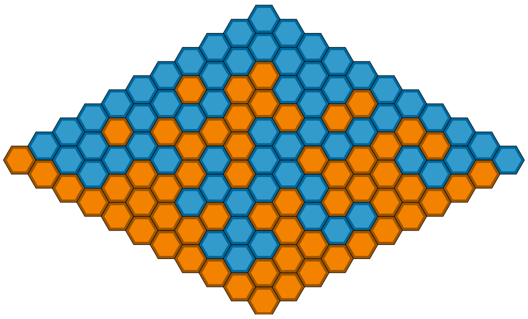


Figure 1.1: Example percolation configuration. Orange represents rock, blue represents empty space.

1.1.2 *Interface*

A question we can now ask is: what will the percolation interface look like? In other words, if we pour water in at the top of the lattice, what will the line that separates "wet" from "dry" look like? This "wet" and "dry" intuition helps us to see that the interface must exist (due to the boundary conditions we set) and must not cross itself.

Now let us explore the interface. Imagine that we are standing at the left hand side of Figure 1.1 on the edge which separates the leftmost orange hexagon and the blue one just "north-east" of it. Now, keeping orange on our right and blue on our left, we walk through the percolation configuration, tracing out the line seen in Figure 1.2.

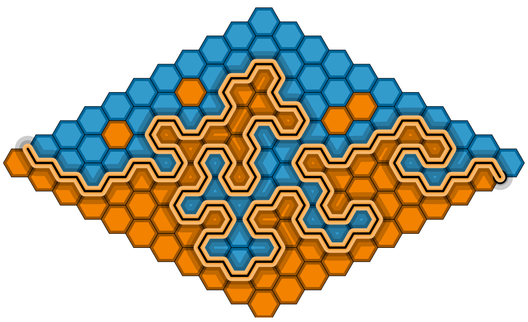


Figure 1.2: The same percolation configuration as Figure 1.1, with percolation interface drawn in black.

The idea behind exploring the interface in this way is to informally introduce the domain

Markov property. At each step, we know both where we are and where we will end up (the far right hand side of the lattice in this case). We also know that the remaining exploration of the lattice has the same type of randomness as the beginning part of it. So, let us imagine that we are half way through the exploration. What we have really done is changed the domain we are exploring by cutting a slit through it where we explored. We still have an orange boundary on our right and a blue boundary on our left, which meet at our target point. The remaining exploration has the same random law as the first half, but in a changed domain.

1.1.3 Scaling limit

The final thing do with this example is look at its scaling limit. What will happen to the interface when we take the hexagon size to zero? In the scaling limit, the exact shape of the original hexagons should not make a difference, so we could distort them under a conformal transformation without changing the limiting law.

So, the scaling limit of the interface should

- be non-self-crossing,
- have a domain Markov property, and
- be conformally invariant.

These are the properties which we take to characterise the Schramm-Loewner evolution (SLE). As we will see in Section 1.2, we can use these properties to define a process in the continuum (i.e. not on a lattice). SLEs are the canonical family of curves which have these properties. They are indexed by a parameter, usually denoted κ, which controls the regularity of the curve.

However, it is not obvious that an SLE defined using these three properties will actually be the scaling limit of the percolation interface! Luckily we know that, on the hexagonal lattice at least, the scaling limit is an SLE_6 curve [Smi01]. Figure 1.3 shows us a closer approximation of the scaling limit, and gives us an idea of what an SLE_6 might look like.

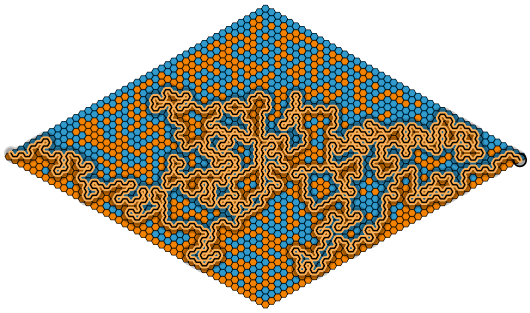


Figure 1.3: Percolation interface on a small lattice.

1.2 Formal definition and properties

Now that we understand where part of the motivation for studying SLE curves comes from, we look at the formal definition. The definition was originally given by O. Schramm in [Schoo] as a conjectural scaling limit for loop erased random walks (LERW), amongst other things. His insight was to use the Loewner differential equation to define the Loewner chain (or flow) of the SLE trace, rather than to try to define the trace itself. So, in order to understand the definition of an SLE, we must first introduce the Loewner differential equation.

1.2.1 Loewner's equation

In 1923, Charles Loewner introduced an ordinary differential equation for studying how "slit-maps" — conformal maps from a domain with a slit removed back to the domain itself — evolve as the slit they remove grows [Löw23]. We will restrict our attention to the case where the domain is the upper half plane, $\mathbb{H}$, and the slit we remove, (γ_t), starts at the origin and converges to infinity. For the rest of this section we also assume, for simplicity, that (γ_t) is a simple curve.

For $t > 0$ we define $H_t = \mathbb{H} \setminus \gamma(0,t]$ to be the upper half plane, cut along the curve γ from 0 to t. The slit map $g_t : H_t \to \mathbb{H}$ is a conformal map which removes the slit, and leaves us with the upper half plane. We ensure uniqueness of g_t by requiring that it converges to the identity as $z \to \infty$, so

$$g_t(z) - z \to 0 \quad \text{as} \quad z \to \infty. \tag{1.1}$$

We can also look at where the tip of the curve is taken at each time. Since $\gamma_t \notin H_t$, we can't look directly at $g_t(\gamma_t)$. However, we can define $\xi_t = \lim_{s \to 0} g_t(\gamma_{t+s})$ for all t. The function (ξ_t) is called the Loewner transform, or driving function, of the curve (γ_t). Its existence is guaranteed for any increasing family (K_t) of compact $\mathbb{H}$-hulls with a local growth property. The simple curve (γ_t) is a special case.

With this setup, we can introduce Loewner's equation.

Proposition 1.2.1 (Loewner's equation). *Let (K_t) be a family of compact $\mathbb{H}$-hulls with a local growth property, parameterised by their half-plane capacity. For any fixed $z \in \mathbb{H}$, the map $g_t : H_t \to \mathbb{H}$ satisfies*

$$\frac{d}{dt}(g_t(z)) = \frac{2}{g_t(z) - \xi_t},$$

where (ξ_t) is the Loewner transform of (K_t).

We will always take $(g_t(z))_{t \geq 0}$ to be the maximal solution of Loewner's equation and, for each $z \in \mathbb{H}$, we will call the lifetime of the solution $\tau(z)$. For a fixed time t, the subset of $\mathbb{H}$ for which Loewner's equation still has a valid solution is $\{z \in \mathbb{H} \,:\, t < \tau(z)\}$, and in fact we have equality between this set and H_t:

$$H_t = \{z \in \mathbb{H} \,:\, t < \tau(z)\}.$$

1.2.2 *Simple example*

We will now look at the simplest example we can. Rather than thinking of a straightforward curve as a candidate for (γ_t), we will instead use the simplest possible driving function that we can, $\xi_t \equiv 0$. We will see in Section 1.2.3 that starting with the driving function rather than the curve is a useful way of proceeding.

Taking $\xi_t \equiv 0$, the differential equation that (g_t) satisfies is

$$\frac{d}{dt}(g_t(z)) = \frac{2}{g_t(z)},$$

with $g_0(z) = z$. The solution is

$$g_t(z) = \sqrt{4t + z^2},$$

which is the map that "maps out" the vertical slit connecting the origin with $2i\sqrt{t}$. We also see that

$$g_t\left(2i\sqrt{t}\right) = 0,$$

which shows that the tip of the curve is mapped to the origin. See Figure 1.4.

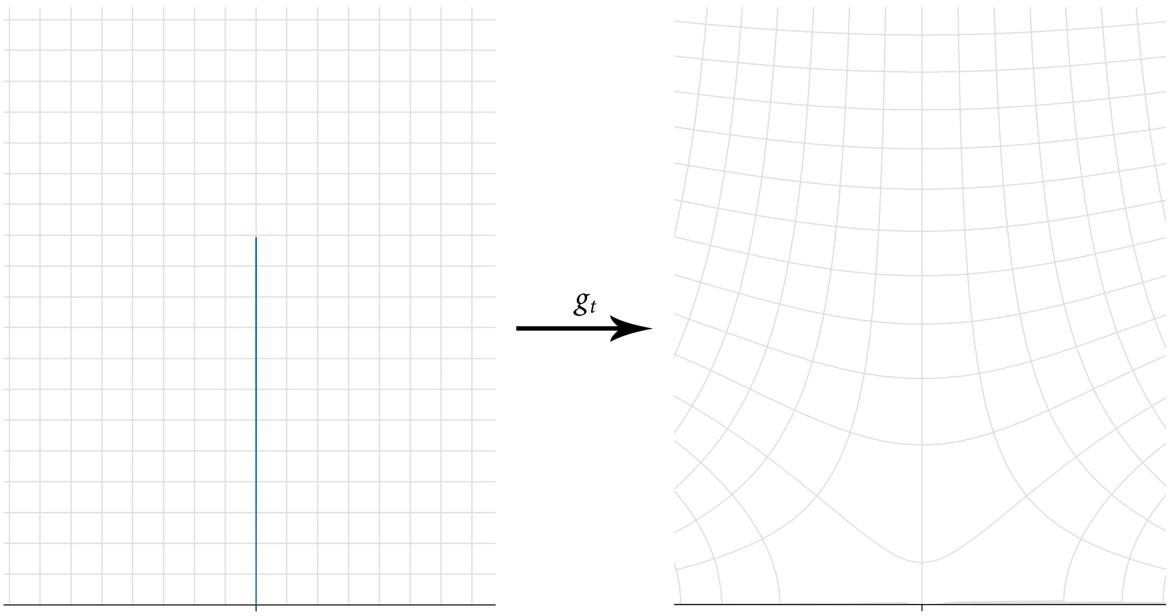


Figure 1.4: A visualisation of the slit map g_t.

1.2.3 Derivation of the driving function

We now define a random curve which, as we discussed in Section 1.1.2, is non-self-crossing, has a conformally invariant law, and has a domain Markov property. That the curve is non-self-crossing is taken care of by the fact that we are defining it through Loewner's equation. Conformal invariance comes from the way that we define an SLE on other complex domains: as the image under a conformal map of the process we define on $\mathbb{H}$. A careful choice of the driving function will give us the domain Markov property. The following derivation of the driving function is based on [Bef15].

We informally introduced the domain Markov property (DMP) in Section 1.1.2. We need to be more precise now. Assume that we have a random curve (γ_t) and the corresponding Loewner flow (g_t). We know that, for a fixed time $s > 0$, the function g_s maps out the curve $\gamma[0, s]$, sending the tip of the curve, γ_s, to the driving function, ξ_s. We say that (γ_t) has the domain Markov property if, conditional on $\gamma[0, s]$, the image of the remaining curve under g_s has the same law as the original curve (modulo the starting point). More formally, we say that (γ_t) has a domain Markov property if, for any fixed time $s > 0$,

$$\gamma[0, \infty) \stackrel{d}{=} g_s(\gamma[s, \infty)) - \xi_s. \tag{1.2}$$

See Figure 1.5 for an illustration.

Thanks to the right hand side of (1.2), it makes sense to define a centred map

$$h_t(\cdot) := g_t(\cdot) - \xi_t,$$

which always maps the tip of the curve to the origin. Then the DMP for (γ_t) reads

$$\gamma[0, \infty) \stackrel{d}{=} h_s(\gamma[s, \infty)).$$

This gives us a nice composition rule for the maps (h_t): let $t, s > 0$ and consider the map h_{t+s}. We can recover h_{t+s} by first mapping out $\gamma[0, s)$, and then mapping out $h_s(\gamma[s, t))$. But, the curve $h_s(\gamma[s, t + s))$ has the same law as an independent curve $\tilde{\gamma}[0, t)$. So, we can map out

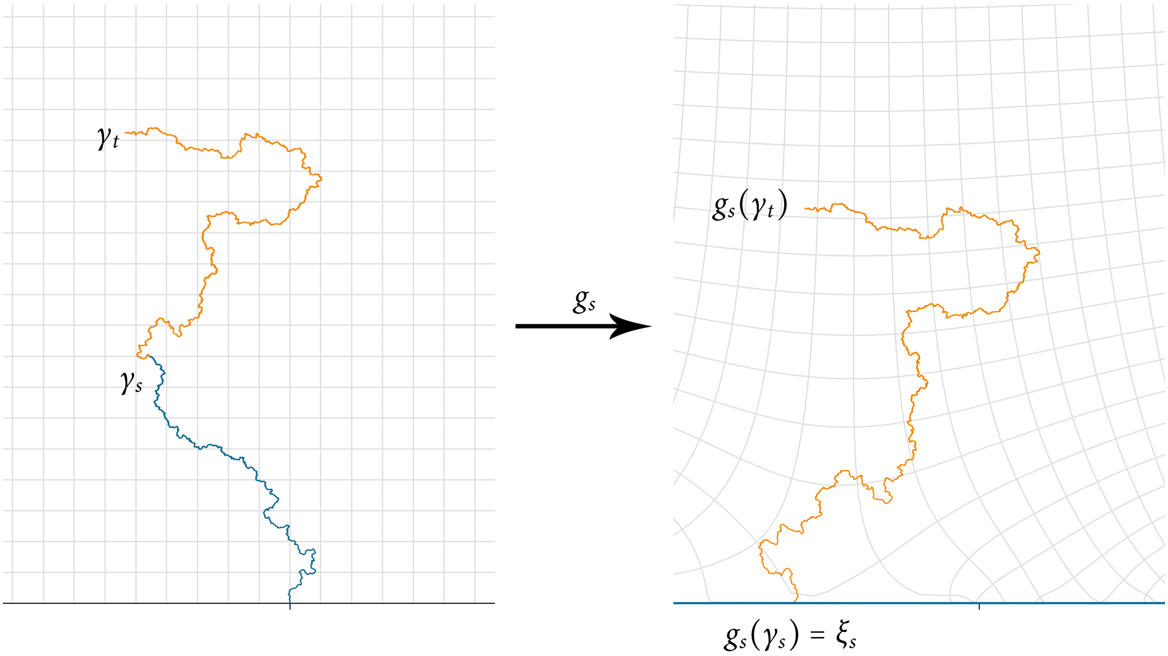


Figure 1.5: A visualisation of the DMP for an SLE_2. After the mapping out the blue curve $\gamma[0,s]$ by g_s, the orange curve has the law of an SLE_2 started at ξ_s.

the remaining curve with an independent map $\tilde{h}_t$, which has the same law as h_t. Therefore, we get the identity

$$h_{t+s} \stackrel{d}{=} \tilde{h}_t \circ h_s, \tag{1.3}$$

where $(\tilde{h}_t)$ and (h_t) are independent and identically distributed.

We ensured that the maps (g_t) were unique by the condition in (1.1), which states that g_t converges to the identity as $z \to \infty$. This gives us a condition on $h_t = g_t - \xi_t$, namely that

$$h_t(z) = z - \xi_t + \mathcal{O}\left(z^{-1}\right). \tag{1.4}$$

Now, the condition in (1.3) combined with the expansion in (1.4) shows us that

$$\tilde{h}_t \circ h_s(z) = z - \xi_s - \tilde{\xi}_t + \mathcal{O}\left(z^{-1}\right),$$

where $(\tilde{\xi}_t)$ is an independent driving process with the same law as (ξ_t). Looking just at the driving function, we see that

$$\xi_{s+t} = \xi_s + \tilde{\xi}_t,$$

which implies that $\xi_{s+t} - \xi_s = \tilde{\xi}_t$. In other words, (ξ_t) is a continuous random process with independent, stationary increments. Therefore, (ξ_t) must be a constant multiple of a Brownian motion with drift.

If we want to ensure that the random process we define is invariant under reflections around the imaginary axis (which usually holds for discrete processes), then we must have that $(\xi_t) \overset{d}{=} (-\xi_t)$. Therefore, the drift term must be zero. We will make this assumption throughout.

We can at last give the definition of the SLE flow:

Definition 1.2.2 (SLE flow)**.** Let (g_t) be a family of conformal maps which, for any $z \in \mathbb{H}$, are the maximal solution to the SDE

$$dg_t(z) = \frac{2dt}{g_t(z) - \xi_t}, \tag{1.5}$$

with $g_0(z) = 0$, where $\xi_t = \sqrt{\kappa} B_t$ for $\kappa \geq 0$ and (B_t) a standard Brownian motion. Then (g_t) is an SLE flow.

1.2.4 Basic properties

As we now have the definition of the SLE process, we can begin to study its properties. Unless otherwise stated, the properties of SLE we discuss in this section were first shown in [RS11]. First of all, we will refresh our ideas of the domain of definition of the solution. Heuristically, the solution to (1.5) is defined, for each $z \in \mathbb{H}$, up to the first time that "$g_t(z) = \xi_t$". We define the time $\tau(z)$, for each $z \in \mathbb{H}$, as the lifetime of the solution. Using the lifetime, we can define the hulls

$$K_t = \{z \in \mathbb{H} \; : \; t \geq \tau(z)\},$$

which are the points "eaten" by the SLE flow. We also define $H_t = \mathbb{H} \setminus K_t$ as the region on which the solution is still defined. We know further that $g_t : H_t \to \mathbb{H}$.

For the SLE process defined as in Definition 1.2.2 to be a good model of random curves and interfaces, we need to check that it satisfies some basic conditions. One of the properties that we need is that an SLE hull is generated by a curve, i.e. there exists a curve (γ_t) such that, for each t, the hull K_t is equal to the union of $\gamma(0, t]$ and the bounded components of $\mathbb{H} \setminus \gamma[0, t]$. This might seem like an obvious fact, given the way that we introduced Loewner's equation in Section 1.2.1. However, Loewner's equation as we used in in Section 1.2.1 started with the curve (γ_t), defined the maps (g_t) from the curve, and then found the driving function (ξ_t). We are now starting with the driving function, calculating the maps, and hoping that we get a curve at the end. (Even the fact that we can define K_t as we have is non-trivial, and comes from the Loewner-Kufareff theorem [Kuf43].)

That the SLE hulls are generated by a curve is such an important fact that we will come back to it in much more detail in Section 1.3. For now, we will simply assume that such a curve (γ_t) does exist.

Because we have assumed the existence of the curve, we can talk about how the curve depends on the parameter κ, the parameter that scales the Brownian motion in the driving function. The behaviour splits into three phases:

- For $\kappa \in [0, 4]$, the curve (γ_t) is simple and $K_t = \gamma[0, t]$.
- For $\kappa \in (4, 8)$, the curve is no longer simple.
- For $\kappa \in [8, \infty)$, the curve is space filling.

The second case needs a little more detail. If $\kappa \in (4, 8)$, then for any $z \in \mathbb{H}$ we know that $z \notin \gamma[0, \infty)$ almost surely. However $z \in \bigcup_{t>0} K_t$ almost surely. So, the curve will avoid any point you pick in advance, but the curve will end up "swallowing" the entire half plane. See Figure 1.6 for examples of the first two phases.

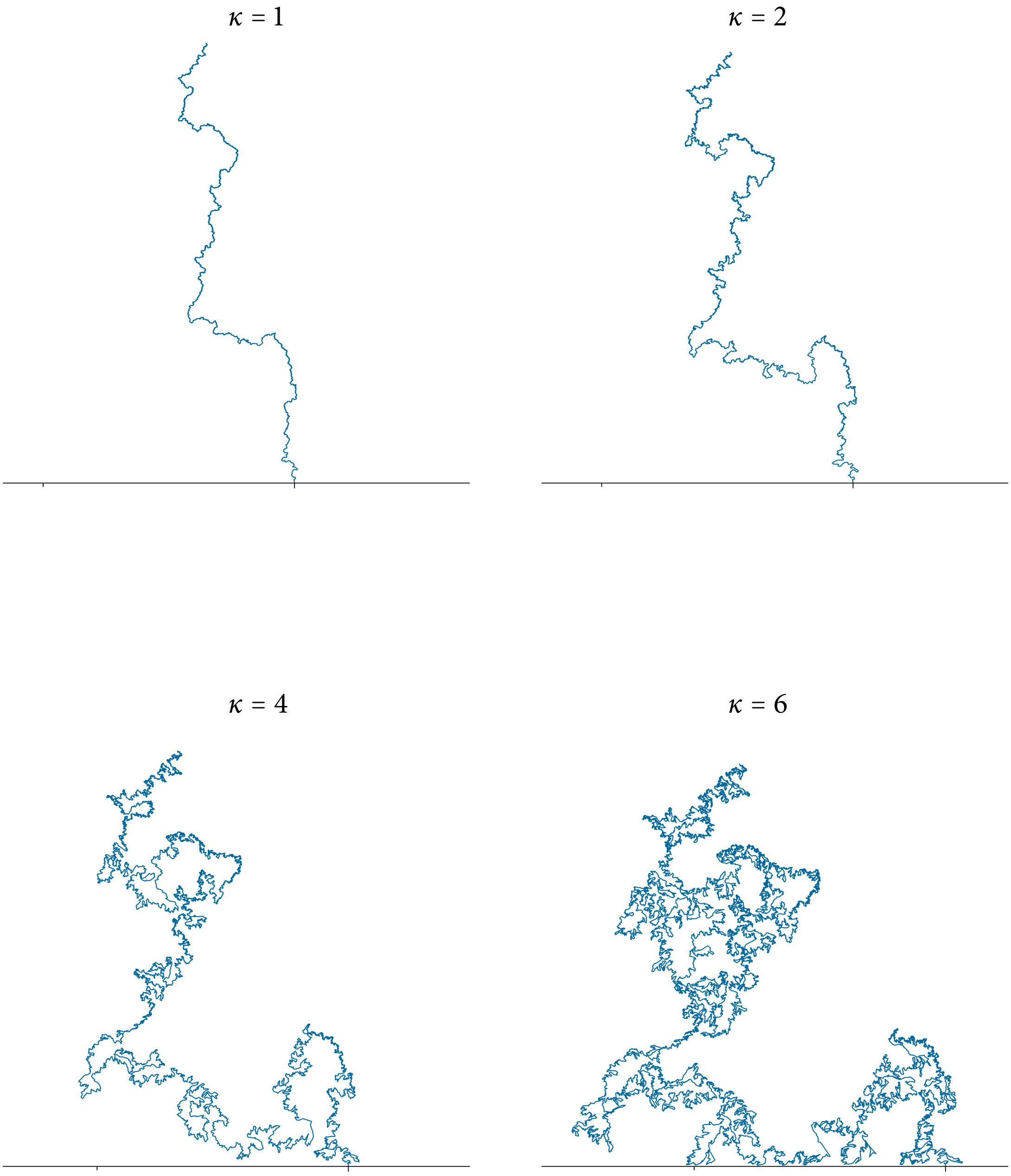


Figure 1.6: Examples of SLE processes for $\kappa = 1, 2, 4, 6$, all driven by the same Brownian motion. All SLE simulations based on algorithms in [Ken07].

1.2.5 More complex properties and generalisations

We saw in Section 1.1.2 that SLE_6 is the scaling limit of the interface in critical percolation on a triangular lattice [Smi01]. There are a few other relationships between scaling limits of discrete models and SLE curves. Schramm's original motivation for introducing SLE processes was to study loop erased random walks.

The loop erased random walk (LERW) is constructed using a simple random walk on a lattice. As the name implies, whenever the simple random walk creates a loop, that loop is deleted. The remainder of the curve forms the LERW. Schramm conjectured that the scaling limit was equal to an SLE_2 curve under the assumption that the scaling limit was conformally invariant and, in [LSW11], Lawler, Schramm and Werner completed the proof. More precisely, they show the following.

Theorem 1.2.3. *Let $D \subsetneq \mathbb{C}$ be a simply connected domain containing the origin. Let μ_δ be the law of a LERW on the grid $\delta\mathbb{Z}^2$ started at the origin and stopped when it hits ∂D. Let ν be the law of a radial SLE_2 in D starting at the origin. Then μ_δ converges weakly to ν as $\delta \to 0$ under the metric on unparameterised curves given by*

$$\rho(\gamma, \beta) = \inf \sup_{t \in [0,1]} |\hat{\gamma}_t - \hat{\beta}_t|,$$

where the infimum is over all parameterisations $\hat{\gamma}$ and $\hat{\beta}$ in $[0,1]$ of γ and β respectively.

Also conjectured by Schramm in [Sch00] and proved by Lawler, Schramm and Werner in [LSW11] is the relationship between SLE_8 and the Peano curve of a uniform spanning tree (UST). A spanning tree is a subset of a finite, connected graph $G = (V, E)$ such that for every pair of vertices $u, v \in V$, there is exactly one path connecting u and v. The UST on a graph is a spanning tree chosen uniformly at random from the set of all spanning trees of that graph. If G is a planar graph then we can define the Peano curve as the boundary between the UST and its dual.

With these definitions, Lawler, Schramm and Werner prove the following:

Theorem 1.2.4. *Let $D \subset \mathbb{C}$ be a domain with a C^1 boundary consisting of a simple curve, and let $a, b \in \partial D$ be distinct. Let α, β denote the two arcs of ∂D whose endpoints are a and b. Finally, let G_δ be the graph approximation of D in the grid $\delta\mathbb{Z}^2$, and let γ_δ denote the Peano curve of a UST on G_δ with wired (i.e. conditionally closed) edges close to α and free (conditionally open) edges close to β. Then γ_δ converges to the law of a chordal SLE_8 connecting a and b.*

The mode of convergence in Theorem 1.2.4 is the same as that in Theorem 1.2.3. The fact that scaling limits of the LERW and UST appear in the same paper is due to the relationship between them: if we pick two vertices $u, v \in V$ and look at the path on the UST connecting u and v, that path has the same law as a LERW between u and v. Indeed, Wilson's algorithm uses this fact to construct the UST from successive LERWs [Wil96].

The original proof of the Rohde-Schramm theorem (in [RS11]) works only for $\kappa \neq 8$. The extension to the $\kappa = 8$ case comes from estimates about the UST; in the $\kappa = 8$ case, the scaling limit helps us to deduce the existence of the curve, rather than the other way around [LSW11].

The Rohde-Schramm theorem for $\kappa \geq 0$ as proved in [RS11, LSW11] tells us that, for a forward chordal SLE defined in the upper half plane $\mathbb{H}$, the SLE hull (K_t) is generated by a continuous curve (γ_t). For any suitably smooth domain $D \subsetneq \mathbb{C}$, the existence of the chordal SLE curve can be deduced from its existence in $\mathbb{H}$ and conformal invariance of the law of SLE. However, if the boundary of the domain D is not regular, for example if it contains the "Topologists' sine curve" $\{z \in \mathbb{C} \,:\, z = x + i\sin(1/x),\ x > 0\}$, then the existence of the curve γ in this domain is no longer a simple consequence of its existence in $\mathbb{H}$. The proof for existence of the curve in general domains of this type was given in [GRS12].

There is also evidence to support the conjecture that the scaling limit of self-avoiding walks (SAW) is an $SLE_{8/3}$ curve. In [LSW02a], Lawler, Schramm and Werner show that "if the scaling limit of SAWs exists and is conformally covariant, then the scaling limit of SAWs is $SLE_{8/3}$". In [Ken02], Kennedy gives further numerical evidence to support the claim. An $SLE_{8/3}$ also has a restriction property. The restriction property roughly says that the law of an $SLE_{8/3}$ conditioned to be in some subset of $\mathbb{H}$ is identical to the law of an $SLE_{8/3}$ defined in that subset. More precisely, in [LSW03] the authors show the following:

Theorem 1.2.5 (Restriction). *Let K be the hull of an $SLE_{8/3}$ in $\mathbb{H}$. Then for any simply connected subset $H \subset \mathbb{H}$ such that $\mathbb{H} \setminus H$ is bounded and bounded away from the origin, the law of K conditioned on $K \subset H$ is equal to the law of $\Phi(K)$, where $\Phi : \mathbb{H} \to H$ is a conformal map that preserves 0 and ∞. Furthermore,*

$$\mathbb{P}\left[K \subset H\right] = \Phi'(0)^{5/8},$$

when Φ is chosen so that $\Phi(z)/z \to 1$ as $z \to \infty$.

The theory in [LSW03] is more general than our statement of Theorem 1.2.5. The restriction measures that Lawler, Schramm and Werner studied are a one parameter family of laws $\mathcal{P}_\alpha$ on "closed, random subsets K of $\mathbb{H}$ such that $\overline{K} \cap \mathbb{R} = 0$, K is unbounded and $\mathbb{H} \setminus K$ has two connected components." Then with H and Φ chosen as in Theorem 1.2.5, $\mathcal{P}_\alpha$ is defined as

$$\mathcal{P}_\alpha(K \subset H) = \Phi'(0)^\alpha.$$

They show that the only measure $\mathcal{P}_\alpha$ that is supported on simple curves is $\mathcal{P}_{5/8}$, and that is the law of chordal $SLE_{8/3}$.

An SLE_6 has something known as the locality property. Informally, that means that "an SLE_6 process does not feel where the boundary of the domain lies as long as it does not hit it" [LSW01b]. The exact formulation is

Theorem 1.2.6 (Locality). *Let $f : D \to \mathbb{H}$ be a conformal homeomorphism from a domain $D \subset \mathbb{C}$ onto $\mathbb{H}$. Suppose that N is a (suitably nice) neighbourhood of 0 in $\mathbb{H}$. Define $D^* = f^{-1}(N)$ and let f^* be the conformal homeomorphism $\psi_N \circ f$ from D^* onto $\mathbb{H}$, where $\psi_N : N \to \mathbb{H}$ is such that $\psi_N(0) = 0$ and $\psi_N'(0) = 1$. Let $K_t \subset D$ be the hull of SLE_6 starting at $f^{-1}(0)$, and let $\tau := \sup\left\{t \;:\; \overline{K}_t \cap \partial D^* \cap \partial D = \varnothing\right\}$. Let K_t^* denote SLE_6 in D^* started at $f^{*(-1)}(0)$, and let $\tau^* := \sup\left\{t \;:\; \overline{K}_t^* \cap \partial D^* \cap \partial D = \varnothing\right\}$.*

Then the law of $(K_t, t < \tau)$ is that of a time change of $(K_t^*, t < \tau^*)$.

The locality property is one of the reasons that SLE_6 was seen as the natural candidate for the scaling limit of the percolation interface. Recall from Section 1.1.2 that the percolation interface was explored from its starting point by looking only at the hexagons one step ahead of it. For a given realisation of the percolation configuration, we could change the configuration however we liked away from the interface without affecting the interface itself.

Along with the restriction property of $SLE_{8/3}$, the Hausdorff dimension of the curve γ was found in [LSW03] to be $4/3$. The Hausdorff dimension of SLE_6 was then found to be $7/4$ in [Bef04]. Full generality for $\kappa > 0$ was achieved in [Bef08] with the following theorem:

Theorem 1.2.7 (Hausdorff dimension)**.** *Let (K_t) be an SLE_κ in the upper half plane with $\kappa > 0$, let γ be its trace and let $\mathcal{H} := \gamma([0,\infty))$. Then, almost surely,*

$$\dim_H(\mathcal{H}) = 2 \wedge \left(1 + \frac{\kappa}{8}\right).$$

The definition of SLE has been extended from the one that we gave in Definition 1.2.2. The introduction of extra marked points on the boundary of the domain that the SLE is growing in gave rise to the $SLE_\kappa(\rho)$ processes. The value of ρ at the marked points gives a force, either towards or away from, the marked point, allowing finer control of the shape of the SLE curve [LSW03].

Furthermore, multiple SLEs can now be defined in the same domain at once [Dub07a]. So long as the values of κ and ρ are chosen carefully, the order in which growth of the $SLE_\kappa(\rho)$ processes is viewed does not matter. Dubedat showed this by considering the partition function of SLEs and finding relationships between κ and ρ to form a set of "commutation relations".

He also showed how, once we can define multiple SLEs at once, we can couple them together in useful ways. As we saw in Section 1.2.3, SLEs form the one parameter family of conformally invariant, non-self-intersecting random curves with a domain Markov property. Consider the outer boundary of an SLE_κ hull for $\kappa > 8$. That outer boundary will be conformally invariant and non-self-intersecting. Therefore, we would expect them to be related to an SLE. Indeed, the outer boundary is an $SLE_{\hat{\kappa}}(\rho)$ process, with $\hat{\kappa} = 16/\kappa$ [Dub07b].

From the same theory we can also deduce reversibility for SLE: a chordal SLE in a domain D connecting the point $a \in \partial D$ to $b \in \partial D$ has the same law as a chordal SLE connecting the point b to a. This was proved earlier, for $\kappa \le 4$, in [Zha08].

SLEs have been a useful tool in studying other objects. The series of papers that we have cited several times already, [LSW01b, LSW01c, LSW02b], use SLE theory to derive Brownian intersection exponents in various (planar) settings. These are summarised in [LSW01a]. Amongst many other results, they complete a long standing conjecture by Benoit Mandelbrot from 1983 that the Hausdorff dimension of the frontier of a planar Brownian motion is 4/3 [MW83].

There have also been many advances more recently using couplings between SLE processes and the Gaussian free field, for example [SS09] and [SS13]. We will discuss them more in Chapters 2 and 4.

1.3 The Rohde-Schramm theorem

We will now summarize the proof of the Rohde-Schramm theorem found in [RS11]. First of all, we state the condition that we will check to prove that an SLE hull is generated by a curve. It is found as Theorem 4.1 in [RS11]. It is worth noting its generality: there are few assumptions made on the driving function. In particular, the driving function is not necessarily random.

Theorem 1.3.1. *Let $\xi : [0,\infty) \to \mathbb{R}$ be continuous, and let g_t be the corresponding solution of Loewner's equation,* (1.5). *Assume that*

$$\gamma(t) := \lim_{y\to 0} g_t^{-1}(iy + \xi_t)$$

exists for all $t \in [0,\infty)$ and is continuous. Then g_t^{-1} extends continuously to $\overline{\mathbb{H}}$ and H_t is the unbounded connected component of $\mathbb{H} \setminus \gamma([0,t])$, for every $t \in [0,\infty)$.

Throughout the rest of this section, we will define

$$f_t := g_t^{-1} \qquad \text{and} \qquad \hat{f}_t(z) := f_t(z + \xi_t) = g_t^{-1}(z + \xi_t), \tag{1.6}$$

and we consider the driving function $\xi_t = \sqrt{\kappa} B_t$, where (B_t) is a standard Brownian motion and $\kappa \geq 0$.

1.3.1 *Theorem statement*

The following theorem is essentially the proof of assumption in Theorem 1.3.1 in the specific case of the SLE driving function. It appears as Theorem 3.6 in [RS11], but we have simplified the proof a little.

Theorem 1.3.2. *Define*

$$H(y,t) := \hat{f}_t(iy) \quad \text{for} \quad (y,t) \in (0,\infty) \times [0,\infty).$$

If $\kappa \neq 8$, then almost surely $H(y,t)$ extends continuously to $[0,\infty) \times [0,\infty)$.

The simplification of our proof comes from the fact that we have made the rectangle $\mathcal{S}$ that appears in the proof a tiny bit bigger. The sides of our rectangle have lengths which decay like $2^{-j(1-\varepsilon)}$, while in the original the sides decay like 2^{-j}. This lets us use a simpler result about the regularity of Brownian motion (Lemma 1.3.3, which gives an easier estimate on the probability that the increment of the driving function stays within the rectangle $\mathcal{S}$. This is used just after (1.14).

We now give the lemma about the regularity of Brownian motion. The modulus of continuity and local Hölder properties of Brownian motion are well known. The following is a quantitative bound for the smallest scale on a specific partition at which these might fail.

Lemma 1.3.3. *Let B be a standard Brownian motion, fix $\varepsilon > 0$ and let*

$$J = \sup\left\{ j \in \mathbb{N} \ : \ \exists k \in [1, 2^{2j}] \cap \mathbb{N},\ s \in [0, 2^{-2j}] \ : \ |B_{k2^{-2j}} - B_{k2^{-2j}+s}| \geq 2^{-j(1-\varepsilon)} \right\}.$$

Then for $n \in \mathbb{N}$,

$$\mathbb{P}[J \geq n] \lesssim 2^{-cn}$$

for any constant $c > 0$.

Proof. Fix a constant $c > 0$. Let

$$p(k,j) = \mathbb{P}\left[\exists s \in [0, 2^{-2j}] \ : \ |B_{k2^{-2j}+s} - B_{k2^{-2j}}| \geq 2^{-j(1-\varepsilon)}\right].$$

The Markov property and scale invariance of Brownian motion tells us that

$$\begin{aligned} p(k,j) &= \mathbb{P}\left[\exists s \in [0,1] : |B_{2^{2j}s}| \geq 2^{-j(1-\varepsilon)}\right] \\ &= \mathbb{P}\left[\max_{s \in [0,1]} |B_s| \geq 2^{\varepsilon j}\right] \\ &\lesssim \exp\left(-2^{2\varepsilon j-1}\right). \end{aligned}$$

Therefore, a simple union bound gives us, for a fixed $j \in \mathbb{N}$,

$$\begin{aligned} \mathbb{P}\left[\exists k \in [1, 2^{2j}] \cap \mathbb{N},\ s \in [0,1] \ : \ |B_{k2^{-2j}+s} - B_{k2^{-2j}}| \geq 2^{-j(1-\varepsilon)}\right] &\leq \sum_{k=1}^{2^{2j}} p(k,j) \\ &\lesssim 2^{2j} \exp\left(-2^{2\varepsilon j-1}\right) \\ &= \exp\left(-2^{2\varepsilon j-1} + 2j \log 2\right). \end{aligned}$$

For large enough j, we know $2^{2\varepsilon j-1} \geq (c+2) j \log 2$. And so we see that

$$\mathbb{P}\left[\exists k \in [1, 2^{2j}] \cap \mathbb{N},\ s \in [0,1] \ : \ |B_{k2^{-2j}+s} - B_{k2^{-2j}}| \geq 2^{-j(1-\varepsilon)}\right] \lesssim 2^{-cj}.$$

Again, a union bound shows us that, for $n \in \mathbb{N}$,

$$\mathbb{P}\left[\exists j \geq n,\ k \in [1, 2^{2j}] \cap \mathbb{N},\ s \in [0,1] \ : \ |B_{k2^{-2j}+s} - B_{k2^{-2j}}| \geq 2^{-j(1-\varepsilon)}\right] \lesssim \sum_{j=n}^{\infty} 2^{-cj}. \tag{1.7}$$

The event we are considering on the left hand side of (1.7) is exactly the event $\{J \geq n\}$. So we see that

$$\mathbb{P}[J \geq n] \lesssim \sum_{j=n}^{\infty} 2^{-cj} \lesssim 2^{-cn}.$$

Since the constant $c > 0$ was chosen arbitrarily, we are done. □

The proof of Theorem 1.3.2 also needs a result about the tail behaviour of $|\hat{f}_t'(iy)|$ for small y. We will state the result now, but postpone the proof to Section 1.3.2. The following is a simplified version of Corollary 3.5 in [RS11]. (The simplification in this case comes only from the fact that we are not stating it in as much generality as the original.)

Theorem 1.3.4. *Let $\kappa \neq 8$, and let $\hat{f}_t$ be the centred inverse of the Loewner flow as defined in* (1.6)*. Then there exist constants $\varepsilon > 0$, $\delta > 0$ and $C > 0$ such that*

$$\mathbb{P}\left[|\hat{f}_t'(iy)| > y^{-(1-\varepsilon)}\right] \leq Cy^{2+\delta}$$

for all $t \in [0,1]$ and $y \in (0,1)$.

Assuming Theorem 1.3.4 for now, we can give the proof of Theorem 1.3.2.

Proof of Theorem 1.3.2. First we note that, by scale invariance, it is enough to prove that H is continuous on $[0,1) \times [0,1)$. Since we already know continuity of H for $y > 0$, our area of focus will be very close to the origin.

Now, fix $\kappa \neq 8$. Then, for $j, k \in \mathbb{N}$ with $k < 2^{2j}$, let $R(j,k)$ be the rectangle

$$R(j,k) := [2^{-j-1}, 2^{-j}] \times [k2^{-2j}, (k+1)2^{-2j}],$$

and let

$$d(j,k) := \operatorname{diam} H(R(j,k)).$$

The set $H(R(j,k))$ represents how the vertical line segment $[i2^{-j-1}, i2^{-j}]$ is "smeared out" under the map $\hat{f}_t$ for $t \in [k2^{-2j}, (k+1)2^{-2j}]$. Our aim is to show that these sets decrease in diameter fast enough as $j \to \infty$ for us to deduce the continuity of H close to $y = 0$. We will do this by showing

$$\sum_{j=0}^{\infty} \sum_{k=0}^{2^{2j}-1} \mathbb{P}\left[d(j,k) \geq 2^{-j\sigma}\right] < \infty \tag{1.8}$$

for some $\sigma > 0$, allowing us to use a Borel-Cantelli argument.

We start by fixing a pair (j,k). We want to prove that

$$\mathbb{P}\left[d(j,k) \geq 2^{-j\sigma}\right] \leq C2^{-(2+\delta)j} \tag{1.9}$$

for some $\delta > 0$ and some constant $C > 0$, which will be enough to show that (1.8) holds.

We would like to look at a finite set of times rather than the entire interval $[k2^{-2j}, (k+1)2^{-2j}]$ or, ideally, a single time. We will do this by giving ourselves more freedom in space to compensate for less freedom in time. We want to choose a rectangle $S \subset \mathbb{C}$ and a time $\hat{t} \in [k2^{-2j}, (k+1)2^{-2j}]$ in such a way that

$$H(R(j,k)) \subset \hat{f}_{\hat{t}}(S). \tag{1.10}$$

We will find that choosing the rectangle

$$S := \left\{x + iy \in \mathbb{C} \ : \ |x| \leq 2^{3-j(1-\varepsilon)},\ y \in \left[2^{-1-j(1-\varepsilon)}, 2^{3-j(1-\varepsilon)}\right]\right\} \tag{1.11}$$

will work, for some small $\varepsilon > 0$ to be specified later. We will now show that (1.10) holds. Let $\hat{t} = k2^{-2j}$ and let $t \in [k2^{-2j}, (k+1)2^{-2j}]$, and choose $y \in [2^{-j-1}, 2^{-j}]$. Then, using the definitions from (1.6), we can write

$$\hat{f}_t(iy) = \hat{f}_{\hat{t}}\left(g_{\hat{t}}\left(\hat{f}_t(iy)\right) - \xi_{\hat{t}}\right).$$

If we can show that $g_{\hat{t}}\left(\hat{f}_t(iy)\right) - \xi_{\hat{t}} \in S$, then that is enough to prove (1.10). We know that if $\hat{t}$ is very close to t, then $g_{\hat{t}}\left(\hat{f}_t(iy)\right) - \xi_{\hat{t}}$ is very close to iy, a point in the interior of S. So, we write

$$g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}} - iy = g_{\hat{t}}(\hat{f}_t(iy)) - (iy + \xi_t) + \xi_t - \xi_{\hat{t}}. \tag{1.12}$$

Now let $\varphi(s) = g_s\left(\hat{f}_t(iy)\right)$ for $s \leq t$. Then $\varphi(t) = iy + \xi_t$, and

$$\varphi(\hat{t}) - \varphi(t) = g_{\hat{t}}(\hat{f}_t(iy)) - (iy + \xi_t). \tag{1.13}$$

Combining (1.12) and (1.13) and using the triangle inequality gives us the bound

$$|g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}} - iy| \leq |\varphi(\hat{t}) - \varphi(t)| + |\xi_t - \xi_{\hat{t}}|. \tag{1.14}$$

By Lemma 1.3.3 we know that, for large enough j, the second term on the right hand side of (1.14) is bounded by $|\xi_t - \xi_{\hat{t}}| \leq 2^{-j(1-\varepsilon)}$. We will now bound the first term.

Since g_s satisfies Loewner's equation, we know that φ_s does also:

$$\frac{\partial}{\partial s}\left(\varphi(s)\right) = \frac{2}{\varphi(s) - \xi_s}. \tag{1.15}$$

As with the standard Loewner flow, we can see that $\Im\left(\partial_s \varphi(s)\right) < 0$, and so the imaginary part of $\varphi(s)$ is decreasing. Therefore $\Im(\varphi(s)) \geq \Im(\varphi(t)) \geq 2^{-j-1}$. Substituting this inequality into (1.15) gives us the bound

$$\left|\frac{\partial}{\partial s}\left(\varphi(s)\right)\right| \leq 2^{2+j}. \tag{1.16}$$

Since we have also ensured that $t - \hat{t} \leq 2^{-2j}$, (1.16) lets us deduce that

$$|\varphi(\hat{t}) - \varphi(t)| \leq 2^{2-j}. \tag{1.17}$$

This shows that

$$|\Im(g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}} - iy)| = |\varphi(\hat{t}) - \varphi(t)| \leq 2^{2-j},$$

which, since the imaginary part of g_t decreases and $\hat{t} \leq t$, implies that

$$\Im(g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}}) \in [2^{-1-j(1-\varepsilon)}, 2^{3-j(1-\varepsilon)}]. \tag{1.18}$$

Substituting the bound (1.17) into (1.14) lets us see that

$$|g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}} - iy| \leq 2^{2-j} + 2^{-j(1-\varepsilon)} \leq 2^{3-j(1-\varepsilon)}. \tag{1.19}$$

Therefore, we certainly have

$$|\Re(g_{\hat{t}}(\hat{f}_t(iy)) - \xi_{\hat{t}})| \leq 2^{3-j(1-\varepsilon)}.$$

Combining (1.18) and (1.3.1) shows that $\hat{f}_t(iy) \in \hat{f}_{\hat{t}}(S)$, and so (1.10) holds.

So our focus shifts from $d(j,k)$ to $\operatorname{diam}(\hat{f}_{\hat{t}}(S))$. Koebe's distortion theorem lets us say that the ratio $|\hat{f}_t'(z)|/|\hat{f}_t'(i2^{-j})|$ is bounded by some constant if $z \in S$. Therefore, there is a constant C such that

$$\begin{aligned}\operatorname{diam}(\hat{f}_t(S)) &\leq C2^{-3} \operatorname{diam}(S)\, |\hat{f}_t'(i2^{-j})| \\ &= C2^{-j(1-\varepsilon)} |\hat{f}_t'(i2^{-j})|.\end{aligned}$$

In order to show (1.9) therefore, it is sufficient to show

$$\mathbb{P}\left[C2^{-j(1-\varepsilon)} |\hat{f}_{k2^{-2j}}'(i2^{-j})| > 2^{-\sigma j}\right] \leq \tilde{C} 2^{-(2+\delta)j} \tag{1.20}$$

for constants $C, \tilde{C}$ and some $\delta > 0, \sigma > 0$ and $\varepsilon > 0$ which we are free to choose. Rearranging (1.20) we see

$$\mathbb{P}\left[C2^{-j(1-\varepsilon)} |\hat{f}_{k2^{-2j}}'(i2^{-j})| > 2^{-\sigma j}\right] = \mathbb{P}\left[|\hat{f}_{k2^{-2j}}'(i2^{-j})| > 2^{(1-\sigma-\varepsilon)j}/C\right].$$

Theorem 1.3.4 tells us that we can choose suitable $\varepsilon, \sigma > 0$ to ensure the existence of $\delta > 0$, proving (1.20). Combined with the regularity result from Lemma 1.3.3, we can see that

$$\begin{aligned}\mathbb{P}\left[d(j,k) \geq 2^{-j\sigma}\right] &\leq \mathbb{P}\left[|\hat{f}'_{k2^{2j}}(i2^{-j})| > 2^{(1-\sigma-\varepsilon)j}/C\right] + \mathbb{P}\left[J \geq j\right] \\ &\lesssim 2^{-(2+\delta)j},\end{aligned}$$

showing that 1.9 holds.

Now that we know that (1.8) holds, the Borel-Cantelli lemma tells us that there are finitely many pairs (j,k) such that $d(j,k) \geq 2^{-j\sigma}$. So, $d(j,k) \leq C2^{-j\sigma}$ for some (random) constant $C > 0$. This lets us deduce the continuity of H as $y \to 0$. Let (y', t') and (y'', t'') be in the square $(0,1)^2$, and define

$$j_0 = \min\left\{j \in \mathbb{N} \;:\; j > \max\left\{-\log_2 y',\, -\log_2 y'',\, -\frac{1}{2}\log_2|t'-t''|\right\}\right\}.$$

The definition of j_0 means that any rectangle $R(j_0, k)$ has its largest y value greater than both y' and y'', since both $y' < 2^{-j_0}$ and $y'' < 2^{-j_0}$. It also ensures that the rectangles $R(j_0, k')$ and $R(j_0, k'')$ which contain the times t' and t'' respectively are adjacent (or the same rectangle), since $|t' - t''| < 2^{-2j_0}$. Then we know that

$$|H(y',t') - H(y'',t'')| \leq \sum_{j \geq j_0} \left(d(j,k'_j) + d(j,k''_j)\right) \lesssim C2^{-\sigma j_0}, \tag{1.21}$$

which shows that for every $t_0 \in [0,1)$ the limit of $H(y,t)$ exists as $(y,t) \to (0,t_0)$. See Figure 1.7 for an illustration of the sum in (1.21). This is enough to extend the definition of H to $[0,\infty) \times [0,1)$, completing the proof. □

1.3.2 *Tail bounds of $|\hat{f}'_t(iy)|$*

We now restate and prove Theorem 1.3.4, the theorem needed to show (1.20) in the proof of Theorem 1.3.2. We give a simplified version of the original proof from [RS11] here (omitting some of the more technical results), and a new proof using the Gaussian free field in Chapter 4.

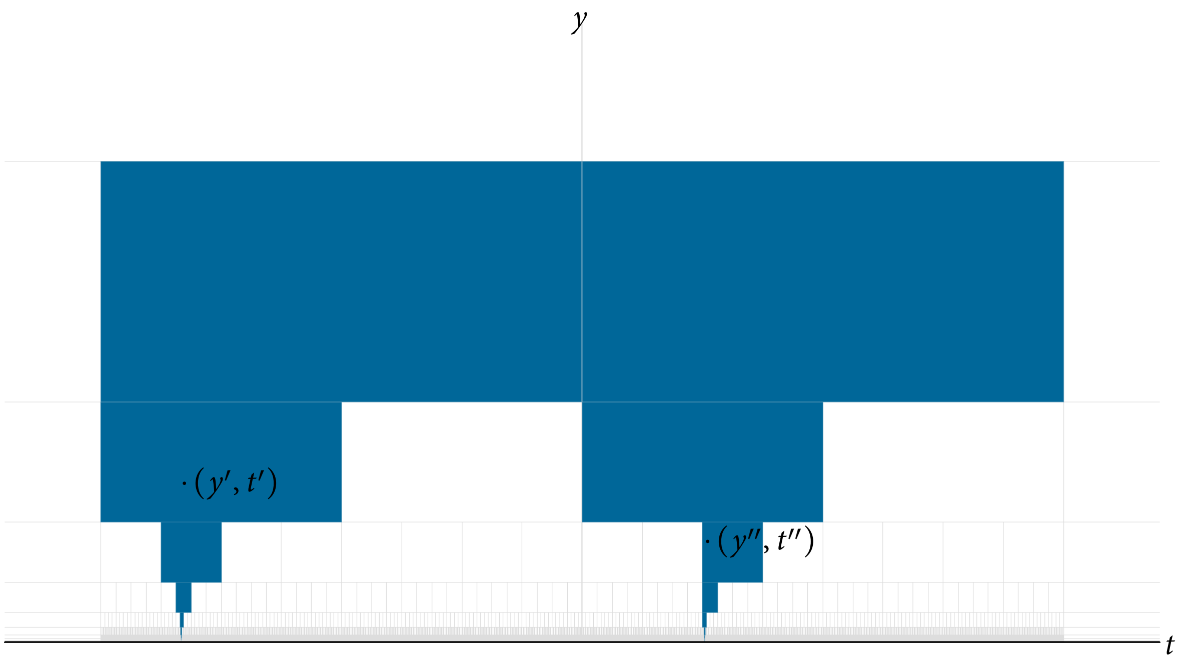


Figure 1.7: The sum in (1.21).

Theorem 1.3.5. *Let $\kappa \neq 8$, and let $\hat{f}_t$ be the centred inverse of the Loewner flow as defined in (1.6). Then there exist constants $\varepsilon > 0$ and $\delta > 0$ and $C > 0$ such that*

$$\mathbb{P}\left[|\hat{f}_t'(iy)| > y^{-(1-\varepsilon)}\right] \leq C y^{2+\delta}$$

for all $t \in [0,1]$ and $y \in (0,1)$.

We need some supporting results before we prove Theorem 1.3.5. First we introduce a time change. For $z \in \mathbb{H}$ and $u \in \mathbb{R}$, define the time $T_u(z)$ to be

$$T_u(z) = \sup\left\{t \in R \;:\; \Im(g_t(z)) \geq e^u\right\}. \tag{1.22}$$

Recall that $\Im(g_t(z))$ is decreasing. So, if u is chosen so that $e^u > \Im(z)$, there is no positive time at which we will have $\Im(g_t(z)) \geq e^u$. We get around this by using the fact that Brownian motion is reversible and we can easily define B_{-t}. This lets us run the Loewner flow (1.5) in reverse, and so we can make sense of negative values of $T_u(z)$.

We can do more than just run the Loewner flow in reverse. A result that we will use in this section, and go through more thoroughly in Chapter 4, is that, for all fixed $t > 0$, the maps

$$z \mapsto g_{-t}(z)$$

and

$$z \mapsto \hat{f}_t(z) - \xi_t$$

have the same distribution.

Now, fix $\hat{z} = \hat{x} + i\hat{y} \in \mathbb{H}$. With the time change from t to u as defined in (1.22), then for every $u \in \mathbb{R}$ set

$$Z_u = g_{T_u(\hat{z})}(\hat{z}) - \xi_{T_u(\hat{z})},$$

the position of the original point $\hat{z}$ at time u under the centred Loewner flow. It will be useful to decompose Z into its real and imaginary parts:

$$X_u = \Re(Z_u) \qquad Y_u = \Im(Z_u) = e^u.$$

We will also need the function

$$\phi(u) = \frac{\hat{y}}{Y_u}|g'_{T_u(\hat{z})}(\hat{z})|.$$

We claim that $T_u(z) \neq \pm\infty$, almost surely. (For proof of the claim, see [RS11].) That means that the map $u \mapsto T_u(\hat{z})$ is a bijection, and so all of the processes defined in the previous paragraph can be run at the original speed t as well. We will sometimes need to take derivatives with respect to the original clock t, and will try to make it clear from context when we do this. In that case, we simply write

$$Z_t = X_t + iY_t = g_t(\hat{z}) - \xi_t.$$

We can now state and prove the following, a slight simplification of Theorem 3.2 in [RS11].

Proposition 1.3.6. *Let $\hat{z} = \hat{x} + \hat{y} \in \mathbb{H}$, and assume that $y \in (0,1)$. Let $b \in \mathbb{R}$ and define a and λ by*

$$a = 2b + \kappa b(1-b)/2 \qquad \lambda = 4b + \kappa b(1-2b)/2.$$

Set

$$F(\hat{z}) = \hat{y}^a \mathbb{E}\left[(1+X_0^2)^b |g'_{T_0(\hat{z})}(\hat{z})|^a\right].$$

Then

$$F(\hat{z}) = \left(1 + (\hat{x}/\hat{y})^2\right)^b \hat{y}^\lambda.$$

Proof. First, define

$$\hat{F}(x+iy) = \left(1 + (x/y)^2\right)^b y^\lambda, \tag{1.23}$$

and then set

$$M_u = \phi(u)^a \hat{F}(Z_u).$$

Our aim is to show that M_u is a martingale. We will show that it is a local martingale in the original t-clock, and omit the technical details used to show that it is a true martingale in the u-clock. Itô's formula tells us that

$$dM = \hat{F}(Z) d\phi^a + \phi^a d\hat{F}(Z) + d\langle F(Z), \phi^a \rangle. \tag{1.24}$$

Now, we break (1.24) down further. To do that, we need to calculate dX and dY in the t-clock. We know that Z_t satisfies Loewner's equation, (1.5), and so taking real and imaginary parts lets us see that

$$dX_t = \frac{2X_t}{X_t^2 + Y_t^2} dt - d\xi_t, \qquad dY_t = -\frac{2Y_t}{X_t^2 + Y_t^2} dt.$$

Therefore, we can write

$$\begin{aligned} d\hat{F}(Z_t) &= \partial_x \hat{F} dX_t + \partial_y \hat{F} dY_t + \frac{1}{2}\partial_x^2 \hat{F} d\langle X\rangle_t \\ &= \partial_x \hat{F}\left(\frac{2X_t}{X_t^2+Y_t^2} dt - d\xi_t\right) + \partial_y \hat{F}\left(-\frac{2Y_t}{X_t^2+Y_t^2} dt\right) - \frac{\kappa}{2}\partial_x^2 \hat{F} dt \\ &= -\partial_x \hat{F} d\xi_t + \left(2(X_t^2+Y_t^2)^{-1}(X_t \partial_x \hat{F} - Y_t \partial_y \hat{F}) - \frac{\kappa}{2}\partial_x^2 \hat{F}\right) dt. \end{aligned} \tag{1.25}$$

Calculating $d\phi^a$ takes a couple of steps. Firstly, it will be easiest to calculate

$$d\log\phi_t = d\log|g_t'(\hat{z})| - d\log Y_t.$$

We see that

$$\begin{aligned}\partial_t \log|g_t'(\hat{z})| &= \Re\left(\frac{\partial_z\partial_t g_t(\hat{z})}{g_t'(\hat{z})}\right)\\ &= \Re\left(g_t'(\hat{z})^{-1}\partial_z\frac{2}{g_t(\hat{z})-\xi_t}\right)\\ &= -2\Re\left((g_t(\hat{z})-\xi_t)^{-2}\right)\\ &= -2\Re\left((X_t+iY_t)^{-2}\right)\\ &= -2\frac{X_t^2-Y_t^2}{(X_t^2+Y_t^2)^2}.\end{aligned} \tag{1.26}$$

Therefore,

$$\begin{aligned}d\log\phi_t &= 2\frac{Y_t^2-X_t^2}{(X_t^2+Y_t^2)^2}dt - d\log Y_t\\ &= \left(2\frac{Y_t^2-X_t^2}{(X_t^2+Y_t^2)^2}+\frac{2}{X_t+Y_t}\right)dt\\ &= \frac{4Y_t^2}{(X_t^2+Y_t^2)^2}dt.\end{aligned}$$

Now, we can calculate

$$\begin{aligned}d\phi_t^a &= a\phi_t^a d\log\phi_t\\ &= 4a\phi_t^a\frac{Y_t^2}{(X_t^2+Y_t^2)^2}dt.\end{aligned} \tag{1.27}$$

Substituting (1.25) and (1.27) into (1.24), we see that

$$\begin{aligned}dM_t &= \hat{F}(Z_t)\left(4a\phi_t^a\frac{Y_t^2}{(X_t^2+Y_t^2)^2}dt\right)+\\ &\quad+\phi_t^a\left(-\partial_x\hat{F}d\xi_t+\left(2(X_t^2+Y_t^2)^{-1}(X_t\partial_x\hat{F}(Z_t)-Y_t\partial_y\hat{F}(Z_t))-\frac{\kappa}{2}\partial_x^2\hat{F}(Z_t)\right)dt\right)\\ &= -\phi_t^a\partial_x\hat{F}(Z_t)d\xi_t+\\ &\quad+\phi_t^a\left(\frac{4aY_t^2}{(X_t^2+Y_t^2)^2}\hat{F}(Z_t)+\frac{2X_t}{X_t^2+Y_t^2}\partial_x\hat{F}(Z_t)-\frac{2Y_t}{X_t^2+Y_t^2}\partial_y\hat{F}(Z_t)-\frac{\kappa}{2}\partial_x^2\hat{F}(Z_t)\right)dt.\end{aligned} \tag{1.28}$$

To show that M_t is a local martingale, we need to check that the drift term in (1.28) is equal to zero. Recalling the definition of $\hat{F}(x+iy)$ from (1.23), we can calculate

$$\partial_x\hat{F}(x+iy) = \frac{2bx}{x^2+y^2}\hat{F}(x+iy),$$

$$\partial_y \hat{F}(x+iy) = \frac{1}{y(x^2+y^2)}\left((\lambda - 2b)(x^2+y^2) + 2by^2\right)\hat{F}(x+iy),$$

and

$$\partial_x^2 \hat{F}(x+iy) = \frac{1}{(x^2+y^2)^2}\left(2by^2 - 2b(1-2b)x^2\right)\hat{F}(x+iy).$$

We now look at the differential operator from the drift term of (1.28):

$$\begin{aligned}
&\frac{4ay^2}{(x^2+y^2)^2}\hat{F} + \frac{2x}{x^2+y^2}\partial_x \hat{F} - \frac{2y}{x^2+y^2}\partial_y \hat{F} - \frac{\kappa}{2}\partial_x^2 \hat{F} = \\
&= (x^2+y^2)^{-2}\left(4ay^2 + 4bx^2 - 2\left((\lambda-2b)(x^2+y^2) + 2by^2\right) - \kappa\left(by^2 - b(1-2b)x^2\right)\right)\hat{F} \\
&= (x^2+y^2)^{-2}\left((4b - 2(\lambda-2b) + \kappa b(1-2b))x^2 + (4a - 2(\lambda-2b) - 4b - b\kappa)y^2\right)\hat{F} \\
&= (x^2+y^2)^{-2}\left((8b + \kappa b(1-2b) - 2\lambda)x^2 + (4a - \kappa b - 2\lambda)y^2\right)\hat{F}. \qquad (1.29)
\end{aligned}$$

We can see from the definition of $\lambda = 4b + \kappa b(1-2b)/2$ that the x^2 coefficient in (1.29) is zero. Recalling that $a = 2b + \kappa b(1-b)/2$, we see that

$$\begin{aligned}
4a - \kappa b &= 8b + 2\kappa b(1-b) - \kappa b \\
&= 8b + \kappa b(1-2b),
\end{aligned}$$

and so the y^2 coefficient in (1.29) is also equal to zero. Therefore, we see that M_t is indeed a local martingale.

As we said at the start, we will admit the fact that M_u is a true martingale. The details can be found in Lemma 2.2 of [RS11]. Given that it is a martingale, we of course know that $M_u = \mathbb{E}[M_0]$. Expanding this in terms of $\hat{F}$, we see that

$$\phi(\hat{u})^a \hat{F}(\hat{z}) = \mathbb{E}\left[\phi(0)^a \hat{F}(z(0))\right], \qquad (1.30)$$

where $\hat{u} = \log \Im(\hat{z})$. Substituting in the definition of ϕ and $\hat{F}$, (1.30) reads

$$\hat{F}(\hat{z}) = \hat{y}^a \mathbb{E}\left[(1+X_0^2)|g'_{T_u(\hat{z})}(\hat{z})\right],$$

which shows that $\hat{F}$ satisfies the definition of F from the statement of this proposition, and the proof is finished. □

We can now give the proof of our required tail bound.

Proof of Theorem 1.3.5. We use the fact that, for a fixed time t, the maps $z \mapsto \hat{f}_t'(z)$ and $z \mapsto g_{-t}'(z)$ have the same distribution to prove the result that we want for the map g_{-t}', i.e. there exist constants $\varepsilon > 0$, $C > 0$ and $\delta > 0$ such that

$$\mathbb{P}\left[|g_{-t}'(iy)| > y^{-(1-\varepsilon)}\right] \le Cy^{2+\delta}$$

for all $t \in [0,1]$ and $y \in (0,1)$.

Recall that in (1.26) we calculated that

$$\partial_t \log|g_t'(z)| = -2\Re\left((g_t(z) - \xi_t)^{-2}\right).$$

We also defined the time $u = u(z,t)$ as $u = \log \Im(g_t(z))$, and can calculate

$$\begin{aligned}
\partial_t u &= \frac{\partial_t \Im(g_t(z))}{\Im(g_t(z))} \\
&= \frac{1}{\Im(g_t(z))}\Im\left(\frac{2}{g_t(z) - \xi_t}\right) \\
&= \frac{1}{\Im(g_t(z))}\frac{-2\left(\Im(g_t(z)) - \xi_t\right)}{|g_t(z) - \xi_t|^2} \\
&= -2|g_t(z) - \xi_t|^{-2}.
\end{aligned}$$

Therefore, a simple application of the chain rule gives

$$\partial_u \log|g_t'(z)| = \frac{\Re\left((g_t(z) - \xi_t)^{-2}\right)}{|g_t(z) - \xi_t|^{-2}}, \tag{1.31}$$

which lets us see immediately that $\partial_u \log|g_t'(z)| \le 1$.

Setting $\tilde{u} = \log \Im(g_{-t}(iy))$, the bound inferred from (1.31) lets us see that

$$\log|g_{-t}'(iy)| - \log|g_{T_u(iy)}'(iy)| \le |\tilde{u} - u|$$

which, in turn, shows

$$\left|\frac{g_{-t}'(iy)}{g_{T_u(iy)}'(iy)}\right| \le \exp\left(|\tilde{u} - u|\right).$$

In particular, we see that

$$\mathbb{P}\left[|g_{-t}'(iy)| > y^{-(1-\varepsilon)}\right] \le \mathbb{P}\left[e^{|\tilde{u}|}|g_{T_0}'(iy)| > y^{-(1-\varepsilon)}\right].$$

So, we need to control $\tilde{u}$ to complete the proof. Since we have assumed that $y \le 1$ and $t < 1$, we know that there is some constant c such that $\tilde{u} \le c$ almost surely, uniformly in t and y. Also, on the event $\{|g'_{-t}(iy)| > y^{-(1-\varepsilon)}\}$ Koebe's 1/4 theorem tells us that $\Im(g_t(iy)) \ge y^{\varepsilon}/4$. Therefore, on the event $\{|g'_{-t}(iy)| > y^{-(1-\varepsilon)}\}$ we know that $\tilde{u} \ge \varepsilon \log y - \log 4$. Combining the two bounds shows that there is some constant C such that, on the event $\{|g'_{-t}(iy)| > y^{-(1-\varepsilon)}\}$, we have $e^{|\tilde{u}|} \le Cy^{-\varepsilon}$. So we can write

$$\begin{aligned}\mathbb{P}\left[|g'_{-t}(iy)| > y^{-(1-\varepsilon)}\right] &\le \mathbb{P}\left[Cy^{1-\varepsilon}|g'_{T_0}(iy)| > 1\right]\\ &\le C^a y^{a(1-\varepsilon)}\mathbb{E}\left[|g'_{T_0}(iy)|^a\right], \end{aligned} \tag{1.32}$$

where the last line comes from Markov's inequality and holds for any $a \in \mathbb{R}$.

If we choose a, b and λ as in Proposition 1.3.6, we can re-write the right hand side of (1.32) as

$$C^a y^{a(1-\varepsilon)}\mathbb{E}\left[|g'_{T_0}(iy)|^a\right] = C^a y^{-a\varepsilon}F(iy) = C^a y^{\lambda - a\varepsilon},$$

and so

$$\mathbb{P}\left[|g'_{-t}(iy)| > y^{-(1-\varepsilon)}\right] \le C^a y^{\lambda - a\varepsilon}.$$

Optimising over our choice of b, we find that the value which gives the largest λ is

$$b = (8+\kappa)/(4\kappa),$$

at which point

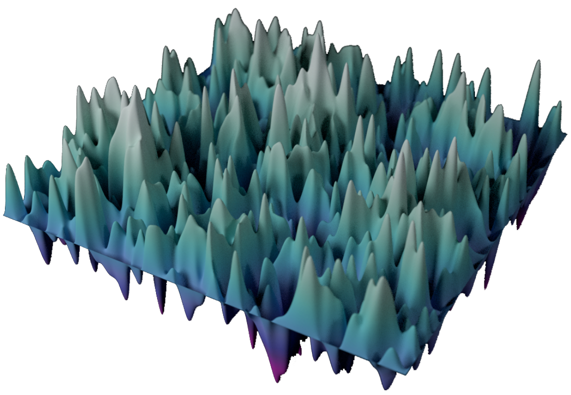
$$\lambda = 1 + \frac{4}{\kappa} + \frac{\kappa}{16}.$$

We can see that $\lambda > 2$ for every value of $\kappa > 0$ (except $\kappa = 8$, at which point $\lambda = 2$). Therefore, for $\kappa \ne 8$, we can choose $\varepsilon > 0$ small enough to ensure that $\lambda - a\varepsilon > 2$, completing the proof. □

It is worth noting at this point that the bound for $\mathbb{P}\left[|\hat{f}_t(iy)| > y^{-(1-\varepsilon)}\right]$, namely $Cy^{1+\frac{4}{\kappa}+\frac{\kappa}{16}-a\varepsilon}$ is, up to logarithmic terms, exactly the same as the bound that we derive in Chapter 4, using totally different methods.

2 Liouville Quantum Gravity and the Gaussian Free Field

We will now give a brief introduction to the Gaussian free field and related objects, including quantum gravity surfaces and Liouville Brownian motion.



2.1 Motivation

Before we give the formal definition of the Gaussian free field (GFF), we will put it in context and explain to the reader how it fits into the subject as a whole.

2.1.1 *Conformally invariant random surface*

We met the idea of conformal invariance for systems at criticality with our percolation example in Section 1.1.2, and from that we motivated the construction of random curves with a conformally invariant law. The link between conformal invariance and statistical physics systems at criticality is much more general than that, however. It was introduced in [BPZ84], and there has been much work since then.

One example is the KPZ equation[1] from [KPZ88]. It is a non-linear relationship between scaling exponents, which gives us a one-to-one correspondence between the scaling exponent of an object when viewed in Euclidean geometry and the scaling exponent of the same object when viewed in random (Liouville quantum gravity) geometry. If x is the scaling exponent of an object in Euclidean geometry and Δ is the scaling exponent of the same object in Liouville quantum gravity (LQG) geometry, then the KPZ equation tells us that

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta, \tag{2.1}$$

where γ is a parameter related to the "central charge" of the random geometry. Although known for a long time in the Physics literature, the KPZ equation in (2.1) has only recently been rigorously proven (see [DS11, RV11, BGRV14]).

[1]This is the KPZ equation associated with Knizhnik, Polyakov and Zamolodchikov, not to be confused with the identically named equation due to Kardar, Parisi and Zhang.

2.1.2 *Discretising the surface*

Early work on random surfaces in the mathematics literature dealt with discrete random surfaces, or random lattices. In [AS03], Angel and Schramm study random triangulations of the sphere. They show the existence of a weak limit of random triangulations of the sphere with n vertices as $n \to \infty$, which can be viewed as "a probability measure on random triangulations of the plane." A similar random graph, the uniform infinite planar quadrangulation (UIPQ) was introduced in [Kri05].

A purely mathematical reason to introduce random lattices such as these is that there is no good reason to prefer a regular lattice, say $\mathbb{Z}^2$, over any other. Why choose $\mathbb{Z}^2$ and not a triangular lattice, for instance? Picking a lattice uniformly at random gets around this difficulty.

It can also make the mathematics easier. Benjamini and Curien studied the set of "pioneer points" of a random walk on the UIPQ [BC13]. The set of pioneer points of a simple random walk is essentially the outer boundary of the trace of the random walk. For a simple random walk (SRW) on a regular lattice, the evolution of the set of pioneer points will depend heavily on its shape. However, they found that the evolution of the set of pioneer points of a SRW on the UIPQ depends only on its boundary length and not on its shape, thanks to a spatial Markov property of the UIPQ.

2.1.3 *Taking the scaling limit*

From the discrete lattices discussed in Section 2.1.2, we would like to obtain continuous random surfaces. We will give a brief description of how we do this now. Much more detail can be found in the set of lecture notes [Mie15], for example.

The first step is to find the limiting radius of the random quadrangulations. For a given quadrangulation $Q = (V(Q), E(Q))$ and a vertex $v \in V(Q)$, the radius of Q as seen from v

is defined as

$$R(Q,v) = \max_{u \in V(Q)} d_Q(u,v),$$

where d_Q is the graph distance on the quadrangulation Q. Using that definition, we can state the following theorem from [CS04].

Theorem 2.1.1. *Let $\mathcal{Q}_n$ be the set of all planar quadrangulations with n faces. Let $Q_n \in \mathcal{Q}_n$ be chosen uniformly at random and, conditionally on Q_n, let v^* be chosen uniformly at random from the set of its vertices, $V(Q_n)$. Then*

$$\left(\frac{9}{8n}\right)^{1/4} R(Q_n, v^*) \to \sup Z - \inf Z,$$

where Z is the head of a Brownian snake with lifetime 1.

The proof uses the Cori-Vauquelin-Schaeffer (CVS) bijection between rooted planar quadrangulations with n faces and one distinguished vertex, and rooted plane trees with n edges with suitably labelled vertices and one extra vertex [CV81, Sch98]. The labels on the vertices are "suitable" if the difference in labels on two neighbouring vertices is in $\{-1, 0, +1\}$. To obtain the quadrangulation from the tree, explore the vertices in contour order, clockwise, starting from the root. Connect each vertex visited to the next vertex encountered in the exploration with a label exactly one less than the vertex to be attached. See Figure 2.1 for an example. The edges we form as we complete these attachments form the edges of our quadrangulation.

By keeping track of the distance from the root of our tree as we do the contour exploration we get a contour process of length $2n$. If the tree we explore, T_n, is chosen uniformly at random from the set of all rooted plane trees with n edges, then the contour process is uniformly chosen from the set of excursions of length $2n$. We will call this contour process C_n.

If, instead of keeping track of the distance from the root as we explore, we keep track of the label, then we get a label process L_n. If we have chosen our labels uniformly at random amongst the set of admissible labels then, conditionally on the tree, we can see that the process L_n behaves like a random walk with jumps of $\{-1, 0, +1\}$ along each branch of the tree. Note that L_n gives us the graph distance from the distinguished vertex in the corresponding quadrangulation.

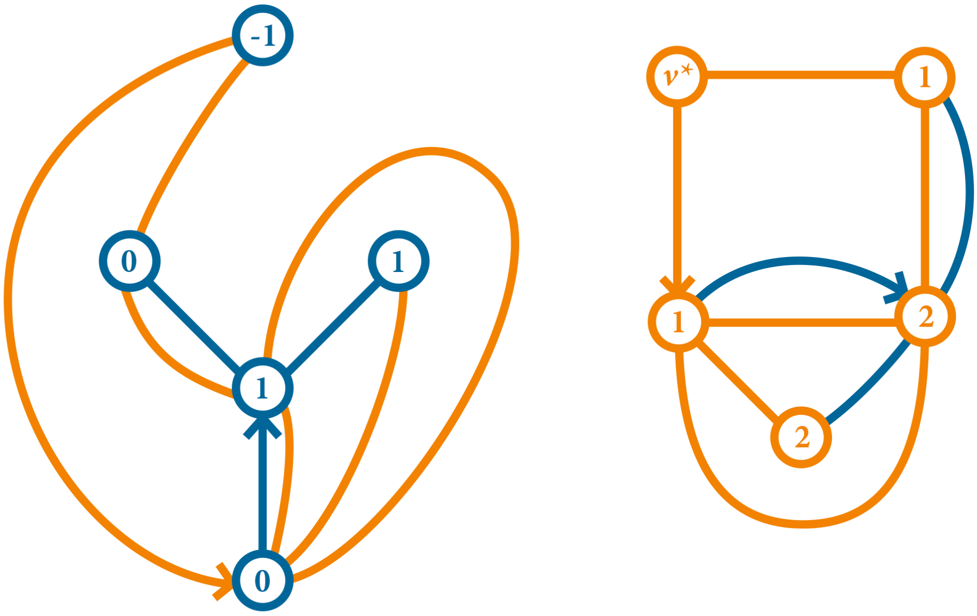


Figure 2.1: The CVS bijection. Left: a rooted tree with an added vertex is drawn in blue, and the corresponding rooted quadrangulation is drawn around it in orange. Right: the same quadrangulation is drawn in orange, with the tree drawn inside it in blue.

We get the following convergence. The proof can be found in, for example [LGM12]

Theorem 2.1.2. *Let C_n be the contour process of a tree T_n chosen uniformly at random from the set of all rooted plane trees with n vertices. Then*

$$\left(\frac{C_n(2nt)}{\sqrt{2n}}\right)_{t\in[0,1]} \xrightarrow{d} \mathbf{e} \qquad as \qquad n \to \infty,$$

where $\mathbf{e}$ *is a Brownian excursion.*

We get a similar convergence result for the label process L_n. But, L_n behaves like a random walk along branches of the tree, conditional on the tree. The tree already has a "central limit theorem" style rescaling, so rather than scaling L_n by $n^{-1/2}$ we need to rescale by $(n^{-1/2})^{-1/2} = n^{-1/4}$. Then we get the following.

Theorem 2.1.3. *Let C_n and L_n be the contour process and label process of a tree chosen uniformly at random from the set of all rooted plane trees with n vertices. Then*

$$\left(\frac{C_n(2nt)}{\sqrt{2n}}, \left(\frac{9}{8n}\right)^{1/4} L_n(2nt)\right)_{t\in[0,1]} \xrightarrow{d} (\mathbf{e}, Z) \qquad as \qquad n \to \infty,$$

where $\mathbf{e}$ *is a Brownian excursion and, conditional on* $\mathbf{e}$*, Z is a continuous, centred Gaussian process with covariance function*

$$\mathrm{Cov}(Z_s, Z_t) = \inf\{\mathbf{e}_u \ : \ u \in [s,t]\},$$

for $s,t \in [0,1]$ with $s \leq t$.

These distributional properties were used in [LG13, Mie13] to show convergence of UIPQ, suitably renormalised, to the Brownian map, in the Gromov-Hausdorff topology. Some unexpected things appear in this scaling limit: the Brownian map is homeomorphic to the sphere not the plane [LGP08, Mie08], and it has a Hausdorff dimension of 4 and not 2 [LG07].

2.1.4 Starting in the continuum

Another approach to introducing random surfaces is to work directly in the continuum. This was done by mathematicians, in some way, before they explicitly talked about random surfaces. In the early work of Lawler and Werner, [LW99], Duplantier "recognised the emergence of an underlying quantum gravity structure" [Dup06].

It was constructed more explicitly in the work of Duplantier and Sheffield [DS11]. They constructed the volume form of a random manifold out of the exponential of the Gaussian free field and, using that volume form, were able to show a version of the KPZ equation that we saw in Section 2.1.1. A slightly different version of the KPZ equation was proven concurrently in [RV11], using Kahane's theory of multiplicative chaos [Kah85].

For now though, we concentrate on the Duplantier–Sheffield construction. (A good introduction can be found in the survey [Gar12].) A difficulty that we will come across in Section 2.2 is that the GFF is not defined pointwise, so its exponential is not well defined. However, with a sequence of regularisations, the measure converges.

We will discuss the construction of the measure more thoroughly in Section 2.4.1. For now, we will simply assume the existence of a measure μ which can be thought of, formally, as

$$\mu(dz) = e^{\gamma h(z)} dz,$$

where h is a Gaussian free field.

Since Duplantier and Sheffield constructed the volume form for the random manifold rather than a metric, in order to prove the KPZ equation they had to use a slightly modified version of the definition of quantum scaling exponent. Let $B_\varepsilon(z)$ be the ball of radius ε centred around the point z. Then, if $z \in D$, the "isothermal quantum ball" of area δ is $B^\delta(z) = B_{\tilde{\varepsilon}}(z)$, where $\tilde{\varepsilon} = \sup\{\varepsilon \ : \ \mu(B_\varepsilon(z)) \le \delta\}$. For $X \subset D$ the Euclidean and quantum neighbourhoods are defined as

$$B_\varepsilon(X) = \{z \ : \ B_\varepsilon(z) \cap X \neq \varnothing\}$$

and

$$B^\delta(X) = \left\{z \ : \ B^\delta(z) \cap X \neq \varnothing\right\},$$

respectively. From these we define the Euclidean scaling exponent x and quantum scaling exponent Δ as

$$\lim_{\varepsilon \to 0} \frac{\log \mathbb{E}\left[\mathcal{L}(B_\varepsilon(X))\right]}{\log \varepsilon^2} =: x$$

and

$$\lim_{\delta \to 0} \frac{\log \mathbb{E}\left[\mu(B^\delta(X))\right]}{\log \delta} =: \Delta,$$

respectively. With these definitions, Duplantier and Sheffield obtained the KPZ equation as follows.

Theorem 2.1.4. *Fix $\gamma \in [0, 2)$ and a compact subset $\tilde{D} \subset D$. Let $X \subset D$ be a (possibly random) set, independent of the measure μ. If $X \cap \tilde{D}$ has a Euclidean scaling exponent x then it has a* *quantum scaling exponent* Δ*, where* Δ *is the non-negative solution to the KPZ equation*

$$x = \frac{\gamma^2}{4}\Delta^2 + \left(1 - \frac{\gamma^2}{4}\right)\Delta.$$

Note that the quantum neighbourhood of X is defined in terms of Euclidean balls. It is thought that, as $\delta \to 0$, these Euclidean balls will approximate true quantum balls.

2.1.5 *Linking the two approaches*

We have now met two different ways of defining a random surface, both motivated from different but related ideas from statistical physics. The scaling limit approach of Section 2.1.3 has the advantage, amongst other things, of being equipped with a metric space structure. An advantage of the approach of starting in the continuum is, again amongst others, that the space comes with a conformal structure inherited from the GFF. However, we cannot view the random surface as a metric space yet.

There has recently been significant progress in linking the two spaces. For $\gamma = \sqrt{8/3}$, Miller and Sheffield have been able to show that the Brownian map and Liouville quantum gravity can be endowed with the other's structure in a consistent way [MS15, MS16a, MS16b].

2.1.6 *The Gaussian free field*

We briefly mentioned the Gaussian free field in the construction of the LQG measure from Section 2.1.4. That is far from its only use! It appears in areas as diverse as the limit of the height function in random domino tilings (dimers) [Ken01], fluctuations of the Hastings–Levitov growth model [Sil15], random matrix theory [RV07], fluctuations for the Ginzburg-Landau interface model [Mil11] and Internal DLA [JLS14].

2.2 Definition and Properties

We will now give the definition of the Gaussian free field. It does not actually exist as a function, but instead as a distribution[2], and we will make it clear in exactly what space we are viewing the GFF. A more thorough introduction to the GFF can be found in [Ber15c], for example. Our presentation follows those notes, along with elements from [Mil13].

We concentrate mostly on the zero (or Dirichlet) boundary Gaussian free field in this subsection. The Neumann (or free) boundary GFF is constructed in a similar way, but with a different Green function. The Green function we consider in Section 2.2.1 is that of a stopped Brownian motion. To define the Neumann boundary GFF, we would use the Green function for a reflected Brownian motion.

We will meet the Neumann boundary GFF again in Section 2.3, and examine it more thoroughly in Chapter 4.

2.2.1 *Green function*

Before we can define the GFF, we need to spend some time looking at the Green function defined from a stopped Brownian motion. Let $D \subset \mathbb{R}^d$ be some domain and, for $x, y \in D$, let $p_t^D(x, y)$ be the transition probability for a Brownian motion killed when it leaves D. Then the Green function G_D is given by

$$G_D(x, y) = \pi \int_0^\infty p_t^D(x, y) dt. \tag{2.2}$$

The factor of π in (2.2) is to make some calculations later on easier. No matter what the domain D is, G_D will always be infinite on the diagonal. We can see this by noting that

$$p_t^D(x, y) = p_t(x, y)\pi_t^D(x, y),$$

where p_t is the standard Brownian transition density and $\pi_t^D(x, y)$ is the probability that a Brownian bridge from x to y with duration t stays in D. Clearly, $\pi_t^D(x, y) \to 1$ as $t \to 0$.

[2] This is distribution in the sense of generalised function, not in the sense of the law of a random variable.

Combining that with the fact that $p_t(x,x) = (2\pi t)^{-d/2}$ shows that $G_D(x,x)$ is not finite.

However, as soon as we are off the diagonal, $G_D(x,y)$ might be finite. We will call any domain D for which G_D is finite off the diagonal a Greenian domain. All bounded domains will be Greenian, but there are unbounded Greenian domains: the upper half plane $\mathbb{H}$ in $\mathbb{C}$ is an example. A reflection argument lets us see that $p_t^{\mathbb{H}}(x,y) = p_t(x,y) - p_t(x,\overline{y})$, which lets us calculate explicitly that

$$G_{\mathbb{H}}(x,y) = \log \frac{|x-\overline{y}|}{|x-y|}. \tag{2.3}$$

In two dimensions, when we view $\mathbb{R}^2$ as the complex plane, the Green function inherits a conformal invariance property from Brownian motion (see Theorem 7.20 of [MP10] for example).

Proposition 2.2.1. *Let $D \subset \mathbb{C}$ be a domain and $\phi : D \to \phi(D)$ a conformal map. Then for all $x, y \in D$,*

$$G_D(x,y) = G_{\phi(D)}(\phi(x),\phi(y)).$$

From the explicit form of $G_{\mathbb{H}}$ in (2.3) and Proposition 2.2.1 we can calculate the Green function G_D for any simply connected domain $D \subsetneq \mathbb{C}$. However, there is an even more explicit form, which arises from the following. More detail can be found in section 1.2 of [Ber15c].

Proposition 2.2.2. *Let $D \subset \mathbb{C}$ be a bounded domain. For any fixed $x \in D$, the Green function $G_D(x,\cdot)$*

- *is harmonic in $D \setminus \{x\}$,*
- *is equal to zero on the boundary, and*
- *satisfies $\Delta G_D(x,\cdot) = -2\pi\delta_x(\cdot)$ in the distributional sense.*

Proposition 2.2.2 tells us that for any bounded domain $D \subset \mathbb{C}$,

$$G_D(x,y) = -\log|x-y| - \tilde{G}(x,y),$$

where $\tilde{G}(x,y)$ is the harmonic extension of $-\log|x-\cdot|$ from ∂D to D.

2.2.2 *Definition*

We would like to define the Gaussian free field as a Gaussian process on a Greenian domain $D \subset \mathbb{C}$ whose covariance function is equal to the Green function G_D. This leads us into some difficulties, because the Green function G_D is not finite on the diagonal, so our Gaussian process will not have a well-defined variance. As a consequence, it will not be a function. We can, however, define it as a stochastic process indexed by signed measures, or as a distribution in an appropriate generalised function space.

First, we need to be clear about the variance of the process. Let $\mathcal{M}_+$ be the set of positive measures with finite energy

$$\mathcal{E}_D(\rho) = \iint_{D^2} G_D(x,y)\rho(dx)\rho(dy), \tag{2.4}$$

and let $\mathcal{M}$ be the set of signed measures ρ which can be decomposed as $\rho = \rho_+ - \rho_-$, where $\rho_+, \rho_- \in \mathcal{M}_+$. We can also talk about the energy of a function f, so long as the measure given by $\rho(dx) = f(x)dx$ is in $\mathcal{M}$. In that case, we will write $\mathcal{E}_D(f) = \mathcal{E}_D(\rho)$.

Also, for $\rho_1, \rho_2 \in \mathcal{M}$, we define the cross energy of ρ_1 and ρ_2 as

$$\mathcal{E}_D(\rho_1,\rho_2) = \iint_{D^2} G_D(x,y)\rho_1(dx)\rho_2(dy).$$

Note that $\mathcal{E}_D(\cdot,\cdot)$ is symmetric and bilinear.

We can now define the zero boundary Gaussian free field.

Definition 2.2.3. The zero boundary Gaussian free field is a stochastic process $(h_\rho)_{\rho\in\mathcal{M}}$ indexed by the set of measures $\mathcal{M}$, where each random variable h_ρ is a centred Gaussian with variance equal to the energy $\mathcal{E}_D(\rho)$. Furthermore, for $\rho_1, \rho_2 \in \mathcal{M}$, the random variables h_{ρ_1} and h_{ρ_2} have covariance equal to the cross energy $\mathcal{E}_D(\rho_1,\rho_2)$.

We need to check whether Definition 2.2.3 actually makes sense. The first point we need to check is that $\mathcal{E}_D(\rho) \geq 0$ for all $\rho \in \mathcal{M}$. For that, we need the Gauss–Green formula which states that

$$\int_D \nabla f \cdot \nabla g = -\int_D f \Delta g,$$

for all suitably smooth function f, g which are compactly supported in D. We will also use the Dirichlet inner product, defined as

$$\langle f, g \rangle_\nabla = \frac{1}{2\pi} \int_D \nabla f(x) \cdot \nabla g(x) dx, \tag{2.5}$$

where the factor $(2\pi)^{-1}$ is to make calculations later on simpler.

The following proposition is well known, but we include the proof for completeness.

Proposition 2.2.4. *The energy $\mathcal{E}_D(\rho)$ is non-negative for all $\rho \in \mathcal{M}$.*

In order to prove Proposition 2.2.4, we will use a mollification argument. We will need the following result.

Lemma 2.2.5. *Let $\theta : \mathbb{R}^2 \to \mathbb{R}$ be a smooth, radially symmetric, positive function with compact support and*

$$\int_{\mathbb{R}^2} \theta(x) dx = 1$$

For $\varepsilon > 0$, let $\theta_\varepsilon(x) = \varepsilon^{-2}\theta(x/\varepsilon)$. Define the function $G_D^\varepsilon(x, y)$ by

$$G_D^\varepsilon(x, y) = \iint G_D(u, v)\theta_\varepsilon(u - x)\theta_\varepsilon(v - y) du dv.$$

Then

$$G_D^\varepsilon(x, y) \leq G_D(x, y)$$

for all $x, y \in D$.

Proof. We will prove that

$$\int G_D(x, v)\theta_\varepsilon(v - y) dv \leq G_D(x, y). \tag{2.6}$$

Iterating (2.6) is enough to prove the lemma.

Let $\delta = \text{dist}(x, y)$. The integral changes behaviour depending on whether or not $v \in B_\delta(y)$. We will split the integral up and treat the cases separately. Start by writing

$$\int G_D(x,v)\theta_\varepsilon(v-y)dv = \int_{v\in B_\delta(y)} G_D(x,v)\theta_\varepsilon(v-y)dv + \int_{v\notin B_\delta(y)} G_D(x,v)\theta_\varepsilon(v-y)dv. \quad (2.7)$$

We know that $G_D(x,\cdot)$ is harmonic in $D \setminus \{x\}$. We have ensured that $B_\delta(y) \subset D \setminus \{x\}$, and so $G_D(x,\cdot)$ is harmonic in $B_\delta(y)$. Because θ is radially symmetric, we can use the circle average property of harmonic functions to see that

$$\int_{v\in B_\delta(y)} G_D(x,v)\theta_\varepsilon(v-y)dv = G_D(x,y)\int_{v\in B_\delta(y)} \theta_\varepsilon(v-y)dv. \quad (2.8)$$

We cannot use the circle average argument to get a similar equality for the second term in (2.7), because any circle that we integrate over will encircle the point x. However, the circle average argument gives us an inequality. Consider a circle of radius $r > \text{dist}(x, y)$, centred at y. Let F_r be the harmonic extension of $G_D(x,\cdot)$ from $\partial B_r(y)$ to $B_r(y)$. Then we certainly have

$$F_r(y) = \frac{1}{2\pi}\int_0^{2\pi} G_D(x, y + re^{i\psi})d\psi.$$

The function F_r is harmonic in $B_r(y)$ and agrees with $G_D(x,\cdot)$ on $\partial B_r(y)$. Because $G_D(x,\cdot)$ has an infinite singularity in $B_r(y)$, we see that

$$F_r(y) \le G_D(x,y).$$

The choice of $r > \text{dist}(x, y)$ was arbitrary, and so we can bound the second term of (2.7) by

$$\int_{v\notin B_\delta(y)} G_D(x,v)\theta_\varepsilon(v-y)dv \le G_D(x,y)\int_{v\notin B_\delta(y)} \theta_\varepsilon(v-y)dv. \quad (2.9)$$

Substituting (2.8) and (2.9) into (2.7) gives

$$\begin{aligned}\int G_D(x,v)\theta_\varepsilon(v-y)dv &\le G_D(x,y)\int_{v\in B_\delta(y)} \theta_\varepsilon(v-y)dv + G_D(x,y)\int_{v\notin B_\delta(y)} \theta_\varepsilon(v-y)dv \\ &= G_D(x,y),\end{aligned}$$

finishing the proof. □

Proof of Proposition 2.2.4. First, assume that ρ is a smooth, compactly supported function in D and let f be a function which satisfies $\Delta f = 2\pi\rho$ in D, and $f = 0$ on ∂D. Then

$$\begin{aligned}
\mathcal{E}_D(\rho) &= \iint_{D^2} G_D(x,y)\rho(x)\rho(y)dxdy \\
&= \int_D \rho(x)\left(\int_D G_D(x,y)\rho(y)dy\right)dx \\
&= -\int_D \rho(x)f(x)dx && \text{by Proposition 2.2.2} \\
&= -\frac{1}{2\pi}\int_D f(x)\Delta f(x)dx \\
&= \frac{1}{2\pi}\int_D \nabla f(x)\cdot\nabla f(x)dx && \text{using integration by parts} \\
&= \frac{1}{2\pi}\int_D |\nabla f(x)|^2 dx
\end{aligned}$$

The final line shows us that $\mathcal{E}_D(\rho) \geq 0$.

To extend from smooth functions to all measures $\rho \in \mathcal{M}$, let θ be a smooth mollifier satisfying the conditions in Lemma 2.2.5, define $\theta_\varepsilon(x) = \varepsilon^{-2}\theta(x/\varepsilon)$, and

$$\rho_\varepsilon(x) = \int_D \theta_\varepsilon(y-x)\rho(dy).$$

We know, from the calculations above, that $\mathcal{E}_D(\rho_\varepsilon) \geq 0$. Now, note that

$$\begin{aligned}
\mathcal{E}_D(\rho_\varepsilon) &= \iint_{D^2} G_D(x,y)\rho_\varepsilon(x)\rho_\varepsilon(y)dxdy \\
&= \iint_{D^2} G_D(x,y)\left(\int_D \theta_\varepsilon(u-x)\rho(du)\right)\left(\int_D \theta_\varepsilon(v-y)\rho(dv)\right)dxdy \\
&= \iint_{D^2}\left(\iint_{D^2} G_D(x,y)\theta_\varepsilon(u-x)\theta_\varepsilon(v-y)dxdy\right)\rho(du)\rho(dv).
\end{aligned}$$

Using the notation from Lemma 2.2.5, we see

$$\mathcal{E}_D(\rho_\varepsilon) = \iint_{D^2} G_D^\varepsilon(u,v)\rho(du)\rho(dv).$$

Therefore, using the result from Lemma 2.2.5, we can use the dominated convergence theorem to see that

$$\lim_{\varepsilon\to 0}\mathcal{E}_D(\rho_\varepsilon) = \mathcal{E}_D(\rho),$$

and so $\mathcal{E}_D(\rho) \geq 0$. □

Bilinearity of $\mathcal{E}_D(\cdot,\cdot)$ lets us see that, for any $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{R}$ and $\rho_1, \rho_2, \ldots, \rho_n \in \mathcal{M}$, we have

$$\sum_{i=1}^{n} \lambda_i h_{\rho_i} \overset{d}{=} h_{\sum_{i=1}^{n} \lambda_i \rho_i}. \tag{2.10}$$

Positivity of $\mathcal{E}_D$ from Proposition 2.2.4 shows us that $\mathcal{E}_D\left(\sum_{i=1}^{n} \lambda_i \rho_i\right) \geq 0$, so the right hand side of (2.10) is well defined. Therefore, we see that

$$\begin{aligned}\mathbb{E}\left[\exp\left(i\sum_{i=1}^{n} \lambda_i h_{\rho_i}\right)\right] &= \mathbb{E}\left[\exp\left(i h_{\sum_{i=1}^{n} \lambda_i \rho_i}\right)\right] \\ &= \exp\left(-\frac{1}{2}\mathcal{E}_D\left(\sum_{i=1}^{n} \lambda_i \rho_i\right)\right),\end{aligned}$$

and so the finite dimensional distributions of $(h_\rho)_{\rho\in\mathcal{M}}$ are well defined.

2.2.3 Viewing the Gaussian free field as a distribution

Suppose that ρ is a smooth, compactly supported function on D, and f is a function that satisfies $\Delta f = 2\pi\rho$. Then, in the proof of Proposition 2.2.4, we derived the identity

$$\mathcal{E}_D(\rho) = \langle f, f\rangle_\nabla .$$

This motivates an alternative definition of the Gaussian free field.

Let $C_0^\infty(D)$ be the set of smooth, compactly supported functions on D, and let $H_0^1(D)$ be the closure of $C_0^\infty(D)$ under the Dirichlet inner product (2.5). Let $\{f_n\}$ be an orthonormal basis of $H_0^1(D)$.

Definition 2.2.6. Let $\{f_n\}$ be an orthonormal basis of $H_0^1(D)$, and $\{\alpha_n\}$ a sequence of i.i.d. standard Gaussian random variables. We define the GFF as the formal sum

$$h = \sum_n \alpha_n f_n.$$

The first thing to note here is that the GFF h as defined in Definition 2.2.6 is not a member of $H_0^1(D)$. Its squared $H_0^1(D)$ norm is

$$\|h\|_\nabla^2 = \sum_n \alpha_n^2 \|f_n\|_\nabla^2 = \sum_n \alpha_n^2,$$

which diverges almost surely.

Instead, let us define the partial sum $h^N = \sum_n^N \alpha_n f_n$, which is a member of $H_0^1(D)$. Then for a function $f \in H_0^1(D)$, we can consider the series

$$\left\langle h^N, f\right\rangle_\nabla = \sum_{n=1}^N \alpha_n \langle f_n, f\rangle_\nabla .$$

By independence and symmetry of the $\{\alpha_n\}$, we know that $(\langle h^N, f\rangle_\nabla)_{N\in\mathbb{N}}$ is a martingale. Furthermore, we see that

$$\begin{aligned}
\mathbb{E}\left[\left\langle h^N, f\right\rangle_\nabla^2\right] &= \sum_{n=1}^N \langle f_n, f\rangle_\nabla \mathbb{E}\left[\alpha_n^2\right] \\
&= \sum_{n=1}^N \langle f_n, f\rangle_\nabla \\
&\leq \|f\|_\nabla^2 < \infty,
\end{aligned}$$

i.e. $(\langle h^N, f\rangle_\nabla)_{n\in\mathbb{N}}$ is an $L^2(\mathbb{P})$ bounded martingale. Therefore, it converges almost surely and in $L^2(\mathbb{P})$ by the martingale convergence theorem. Its limit, which we write as $\langle h, f\rangle_\nabla$, is a Gaussian random variable whose variance is $\sum_n \langle f_n, f\rangle_\nabla = \|f\|_\nabla^2$.

For functions $f, g \in H_0^1(D)$, we can use the same definition using partial sums to see that

$$\langle h, f+g\rangle_\nabla = \langle h, f\rangle_\nabla + \langle h, g\rangle_\nabla .$$

Therefore, we can use the polarisation identity to see that

$$\begin{aligned}
\mathrm{Cov}\left(\langle h, f\rangle_\nabla, \langle h, g\rangle_\nabla\right) &= \frac{1}{4}\left(\mathrm{Var}(\langle h, f+g\rangle_\nabla) - \mathrm{Var}(\langle h, f-g\rangle_\nabla)\right) \\
&= \frac{1}{4}\left(\|f+g\|_\nabla^2 - \|f-g\|_\nabla^2\right) \\
&= \langle f, g\rangle_\nabla .
\end{aligned}$$

If we use suitably normalised eigenfunctions of the negative Laplacian as our basis functions $\{f_n\}$, then we can use Weyl's law on asymptotic behaviour of eigenvalues to see that the GFF is in $H_0^{-\varepsilon}(D)$ for any $\varepsilon > 0$. In particular, we can construct a version which is a continuous linear functional on $H_0^1(D)$. For more detail, see Theorem 15.5 of [BN14].

Importantly, Definitions 2.2.3 and the distribution definition of the GFF are consistent. Suppose we take a GFF h, viewed as a stochastic process on $\mathcal{M}$ as we have in Definition 2.2.3. Let $\mathcal{M}^*$ be the set of measures $\rho \in \mathcal{M}$ which are regular enough to be written as $\rho = \frac{1}{2\pi}(\Delta)^{-1} f$ for some function $f \in H_0^1(D)$. Then there is a version of the process $(h_\rho)_{\rho\in\mathcal{M}}$ which, when restricted to $\mathcal{M}^*$, is a member of $H_0^{-1}(D)$ and can be defined as in Definition 2.2.6.

2.2.4 Basic properties

We saw in Section 2.1.1 that a vital property for the GFF to have is conformal invariance. Luckily, conformal invariance is inherited from the Dirichlet inner product used in the covariance structure. All of the results in this subsection can be found in more detail in [She07], for example.

Theorem 2.2.7 (Conformal Invariance)**.** *Let D and D' be two domains in $\mathbb{C}$, and let ψ be a bijective, conformal map from D to D'. Further, let h be a GFF on D, and h' a GFF on D'. Then $h \overset{d}{=} h' \circ \psi$ in the sense that, for any function $f \in \mathrm{H}_0^1(D)$, we have*

$$\langle h, f\rangle_\nabla \overset{d}{=} \left\langle h', f \circ \psi^{-1}\right\rangle_\nabla .$$

Proof. The claim above follows directly from conformal invariance of the Dirichlet inner product, i.e. for $f, g \in \mathrm{H}_0^1(D)$ and ψ as in the statement of the proposition, we have

$$\int_{D'} \nabla\left(f \circ \psi^{-1}\right) \cdot \nabla\left(g \circ \psi^{-1}\right) dx = \int_D \nabla f \cdot \nabla g \, dx. \tag{2.11}$$

□

Another very important property of the GFF is the domain Markov property. Suppose that h is a GFF defined on a domain D and $U \subset D$ is some subdomain. The domain Markov property essentially says that, if we condition on the values of h on ∂U, then its distribution inside U is that of a zero boundary GFF plus the harmonic extension of the boundary values. As the GFF is not a function, conditioning on the values that it takes on ∂U is not really possible. However, it is a good way of understanding the following theorem intuitively.

Theorem 2.2.8 (Domain Markov Property)**.** *Let D be a domain, and $U \subset D$ a subdomain. Let h be a GFF on D. Then we can write $h = h_1 + h_2$, where*

- *h_1 and h_2 are independent,*
- *h_2 is a GFF on U and zero on $D \setminus U$,*
- *$h_1, h_2 \in \mathrm{H}_0^{-\varepsilon}(D)$, and*
- *h_1 is harmonic on U and agrees with h on $D \setminus U$.*

The proof (which we will show soon) involves decomposing the space $\mathrm{H}_0^1(D)$ into orthogonal subspaces, and projecting the basis functions used to define the GFF onto those spaces. So, for a domain D and a subdomain $U \subset D$, let $\mathrm{Harm}(U)$ be the functions in $\mathrm{H}_0^1(D)$ which are harmonic in U, and let $\mathrm{Supp}(U)$ be the functions in $\mathrm{H}_0^1(D)$ which are supported in U. These are orthogonal and spanning subspaces of $\mathrm{H}_0^1(D)$, as we will now see.

Lemma 2.2.9. *Using the same sets D and U as in the previous proposition. Then*

$$\mathrm{H}_0^1(D) = \mathrm{Harm}(U) \oplus \mathrm{Supp}(U),$$

i.e. $\mathrm{Harm}(U)$ *and* $\mathrm{Supp}(U)$ *are orthogonal and span* $\mathrm{H}_0^1(D)$.

Proof. The proof splits into two parts.

Orthogonal: Let $f \in \mathrm{Harm}(U)$ and $g \in \mathrm{Supp}(U)$. Then we calculate the Dirichlet inner product:

$$\begin{aligned} \int_D \nabla f \cdot \nabla g \, dx &= \int_U \nabla f \cdot \nabla g \, dx && \text{because } g = 0 \text{ in } D \setminus U, \\ &= -\int_U (\Delta f)\, g \, dx && \text{by integration by parts ,} \\ &= 0 && \text{because } f \text{ is harmonic on } U. \end{aligned}$$

Therefore $\mathrm{Harm}(U)$ and $\mathrm{Supp}(U)$ are orthogonal. For the "integration by parts" section in the argument, we are using a consequence of the Gauss-Green theorem. See Theorem 3 in Appendix C of [Eva10].

Span: Now let $f \in \mathrm{H}_0^1(D)$, and let f_1 be the orthogonal projection of f onto $\mathrm{Supp}(U)$. We want to show that $f_2 := f - f_1$ is harmonic on U. Let $\phi \in \mathrm{Harm}(U)$ be a function such that $\phi|_{D \setminus U} = f|_{D \setminus U}$. Then we can write $f_2 = \phi + \eta$ for some function $\eta \in \mathrm{Supp}(U)$. We calculate the norm of f_2 in terms of ϕ and η to see the following inequality.

$$\begin{aligned} \|f_2\|_\nabla^2 &= \langle \phi + \eta, \phi + \eta \rangle_\nabla \\ &= \|\phi\|_\nabla^2 + 2\langle \phi, \eta \rangle_\nabla + \|\eta\|_\nabla^2 \\ &= \|\phi\|_\nabla^2 + \|\eta\|_\nabla^2 && \text{by orthogonality} \\ &> \|\phi\|_\nabla^2 && \text{if } \eta \not\equiv 0. \end{aligned}$$

However, since f_1 is defined as the function which satisfies

$$\|f - f_1\|_\nabla = \inf_{g \in \mathrm{Supp}(U)} \|f - g\|_\nabla = \inf_{\substack{\tilde{\phi} \in \mathrm{H}_0^1(D), \\ \tilde{\phi}|_{D \setminus U} = f|_{D \setminus U}}} \left\|\tilde{\phi}\right\|_\nabla ,$$

we know that $\|f_2\|_\nabla \leq \|\phi\|_\nabla$. Therefore, η must be identically zero, and so f_2 is harmonic in U. □

We can now prove the Markov property.

Proof of Theorem 4.2.15. Let $\{\phi_n\}$ be any orthonormal basis of $\mathrm{H}_0^1(D)$ and let $h = \sum_n \alpha_n \phi_n$ be a GFF (so the α_n values are i.i.d. standard normal random variables). Further, let $\{\phi_n^1\}$ be an orthonormal basis of $\mathrm{Harm}(U)$, and $\{\phi_n^2\}$ an orthonormal basis of $\mathrm{Supp}(U)$. Then, by Lemma 2.2.9, $\{\phi_n^1\} \cup \{\phi_n^2\}$ is an orthonormal basis of $\mathrm{H}_0^1(D)$. We can change basis to see that

$$\begin{aligned} h &= \sum_n \alpha_n \phi_n \\ &= \sum_n \alpha_n^1 \phi_n^1 + \sum_n \alpha_n^2 \phi_n^2, \end{aligned}$$

where $\alpha_n^i = \langle h, \phi_n^i \rangle_\nabla$ for $i = 1, 2$. A simple calculation shows that the new coefficients (α_n^i) are also i.i.d. standard normal random variables.

Define the random variables h_1 and h_2 by

$$h_1 := \sum_n \alpha_n^1 \phi_n^1 \quad \text{and} \quad h_2 := \sum_n \alpha_n^2 \phi_n^2.$$

First, note that h_1 and h_2 are independent, by the independence of the sets of random variables $\{\alpha_n^1\}$ and $\{\alpha_n^2\}$. Further, it follows from the definition of the basis $\{\phi_n^2\}$ that h_2 is a GFF on U and zero on $D \setminus U$. Therefore, we know that $h_2 \in \mathrm{H}_0^{-\varepsilon}(U)$ and so we have $h_2 \in \mathrm{H}_0^{-\varepsilon}(D)$, from which it follows that $h_1 = h - h_2 \in \mathrm{H}_0^{-\varepsilon}(D)$ also.

The fact that h_2 is zero on $D \setminus U$ gives us that h_1 agrees with h on $D \setminus U$. The last thing left to show is that h_1 is harmonic on U. Since $h_1 \in \mathrm{H}_0^{-\varepsilon}(D)$, we know that

$$\sum_{n=1}^{N} \alpha_n^1 \phi_n^1 \to h_1$$

in $\mathrm{H}_0^{-\varepsilon}(D)$ as $N \to \infty$. It therefore also converges in the sense of distributions, and so for any function $f \in \mathrm{C}_0^{\infty}(U)$ we can calculate

$$\begin{aligned} \langle h_1, \Delta f \rangle &= \lim_{N\to\infty} \left\langle \sum_{n=1}^{N} \alpha_n^1 \phi_n^1, \Delta f \right\rangle \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n^1 \langle \phi_n^1, \Delta f \rangle \\ &= \lim_{N\to\infty} \sum_{n=1}^{N} \alpha_n^1 \langle \Delta \phi_n^1, f \rangle \qquad \text{by integration by parts,} \\ &= 0, \end{aligned}$$

where the last equality comes from the fact that each ϕ_n^1 is harmonic on U. However

$$\langle h_1, \Delta f \rangle = 0 \quad \text{for all} \quad f \in \mathrm{C}_0^{\infty}(U)$$

is exactly what it means for h_1 to be harmonic on U in the sense of distributions, and so we are done. □

2.2.5 *Regularisation*

We met one technique used for regularising the GFF in Section 2.2.3, which was to define the partial sum h^N. Another way of regularising the GFF into a surface that we need to understand is the circle average process. We want to look at the GFF when applied to uniform measure on a circle. For $\varepsilon > 0$ and $z \in D$, let ρ_ε^z be uniform measure on a circle of radius ε centred at z.

We will now look at some of the properties of the circle average process given by

$$h_\varepsilon(z) := \langle h, \rho_\varepsilon^z \rangle$$

for $z \in D$, $0 < \varepsilon < \mathrm{dist}(z, \partial D)$, and h a GFF on D.

The first property of the circle average process we study is its continuity. The following appears as Proposition 3.1 in [DS11].

Proposition 2.2.10. *The process $(h_\rho)_{\rho \in \mathcal{M}}$ has a modification such that*

$$(h_\varepsilon(z) \ : \ z \in D, \ \varepsilon \in (0, \mathrm{dist}(z, \partial D))$$

is Hölder continuous on all compact subsets of

$$\{(z, \varepsilon) \ : \ z \in D, \ 0 < \varepsilon < \mathrm{dist}(z, \partial D)\}$$

for every $\eta < 1/2$.

The proof of Proposition 2.2.10 involves showing a Lipschitz condition the variance of the difference:

$$\mathrm{Var}(h_{\varepsilon_1}(z_1) - h_{\varepsilon_2}(z_2)) \leq K(|z_1 - z_2| + |\varepsilon_1 - \varepsilon_2|), \tag{2.12}$$

for all (z_1, ε_1) and (z_2, ε_2) in a compact subset of

$$\{(z, \varepsilon) \ : \ z \in D, \ 0 < \varepsilon < \mathrm{dist}(z, \partial D)\}$$

Condition (2.12) comes from estimates of the covariance which we know is given by

$$\mathrm{Cov}(h_{\varepsilon_1}(z_1), h_{\varepsilon_2}(z_2)) = \iint_{D^2} G_D(x, y) \rho_{\varepsilon_1}^{z_1}(dx) \rho_{\varepsilon_2}^{z_2}(dy). \tag{2.13}$$

Regularity of the Green function G_D then ensures that condition (2.12) can be met on compact sets, with a large enough constant K.

Once we know that (2.12) is true, we can combine it with the fact that $h_{\varepsilon_1}(z_1)$ and $h_{\varepsilon_2}(z_2)$ are Gaussian to use the multi-parameter Kolmogorov-Čentsov theorem. From that, the result follows. (See Theorem 2.8 of [KS91] for more detail on the Kolmogorov-Čentsov theorem.)

Related to the idea of the continuity of the circle average process that we saw in Proposition 2.2.10 and the Markov property of the GFF that we saw in Section 2.2.4 is the idea that we can recover a Brownian motion from the circle averages around a point $z \in D$. This is a well known result (see [She07, DS11] amongst others) but we include the proof for completeness.

Lemma 2.2.11. *Let $z \in D$ and fix $\varepsilon_0 > 0$ small enough so that the circle of radius ε_0 centred at z is contained in D. Then the process defined for $t > 0$ by*

$$B_t := h_{\varepsilon_0 e^{-t}}(z) - h_{\varepsilon_0}(z)$$

is a standard Brownian motion.

Proof. From Proposition 2.2.10 we know that (B_t) is continuous. Now, let $\varepsilon_1, \varepsilon_2 \in (0, \varepsilon_0)$ with $\varepsilon_1 > \varepsilon_2$, and consider the increment

$$h_{\varepsilon_2}(z) - h_{\varepsilon_1}(z). \tag{2.14}$$

The Markov property of the GFF lets us write $h = h^i + h^o$, where h^i is a zero boundary GFF inside the ball of radius ε_1 centred at z, $B_{\varepsilon_1}(z)$, h^o is harmonic in that ball and agrees with h outside of it, and h^i and h^o are independent. As h^o is harmonic in $B_{\varepsilon_1}(z)$, the circle average property of harmonic functions tells us that $h^o_{\varepsilon_1}(z) = h^o_{\varepsilon_2}(z) = h^o(z)$. Therefore, we can write the increment (2.14) as

$$h_{\varepsilon_2}(z) - h_{\varepsilon_1}(z) = h^i_{\varepsilon_2}(z) - h^i_{\varepsilon_1}(z).$$

This depends only on the GFF h^i, and is therefore independent of h^o, the GFF outside the ball $B_{\varepsilon_1}(z)$. Translating back to the process (B_t), what we have shown is that for $s > t > 0$, the increment $B_s - B_t$ is independent of $(B_u : u \in (0,t))$. And therefore (B_t) has independent increments.

We know at this point that (B_t) is a Brownian motion, up to a time change. To check that we are looking at the correct time scale for (B_t) to be a Brownian motion, we need to calculate the variance of the increment $h_{\varepsilon_2}(z) - h_{\varepsilon_1}(z) = h^i_{\varepsilon_2}(z) - h^i_{\varepsilon_1}(z) = h^i_{\varepsilon_2}(z)$. We can calculate variance of the right hand side explicitly:

$$\begin{aligned}\operatorname{Var}(h^i_{\varepsilon_2}(z)) &= \iint_{D^2} G_{B_{\varepsilon_1}(z)}(x,y)\rho^z_{\varepsilon_2}(dx)\rho^z_{\varepsilon_2}(dy)\\ &= \int_D G_{B_{\varepsilon_1}(z)}(x,0)\rho^z_{\varepsilon_2}(dx),\end{aligned}$$

where we have used the circle average property of harmonic functions for the second inequality. Now, for x on the circle of radius ε_2 centred at z, we know that

$$G_{B_{\varepsilon_1}(z)}(x,0) = -\log\frac{\varepsilon_2}{\varepsilon_1},$$

and so we see that

$$\operatorname{Var}(h_{\varepsilon_2}(z) - h_{\varepsilon_1}(z)) = -\log\varepsilon_2 + \log\varepsilon_1.$$

Translating this back to the (B_t) process, we find that for $s > t > 0$,

$$\operatorname{Var}(B_s - B_t) = s - t,$$

and so (B_t) is a Brownian motion. □

With more careful calculations involving the Green function, we can find the following result on the variance of the circle average. See Proposition 3.2 of [DS11] for all of the details.

Lemma 2.2.12. *Let $z \in D$ and let $\varepsilon \in (0, \operatorname{dist}(z, \partial D))$. Then the circle average $h_\varepsilon(z)$ has variance*

$$\operatorname{Var}(h_\varepsilon(z)) = -\log\varepsilon + \log R(z; D),$$

where $R(z;D)$ is the conformal radius of the domain D viewed from z.

2.3 Coupling with SLE

We will now show how the GFF can be related to the SLE curves from Chapter 1. We will see two similar results. Roughly stated, we can either “zip up” a zero boundary GFF using a forward SLE to obtain another zero boundary GFF, or we can “cut” a Neumann boundary GFF using a reverse SLE to obtain another Neumann boundary GFF. The Theorem statements are originally from [She10], and the proofs are a combination of those in [She10], [BN14] and [Mil13].

2.3.1 Theorem statements

We now state the forward and reverse coupling theorems. The forward coupling allows us to "zip up" a zero boundary GFF using a forward SLE. See Figure 2.2. In this section, we break from the usual notation of g_t for the forward SLE flow and f_t for the reverse. We will, instead, abuse notation and call them both f_t. Most of the calculations that we will perform are almost identical for both cases, and so using the same notation for both cases will ease notation later on.

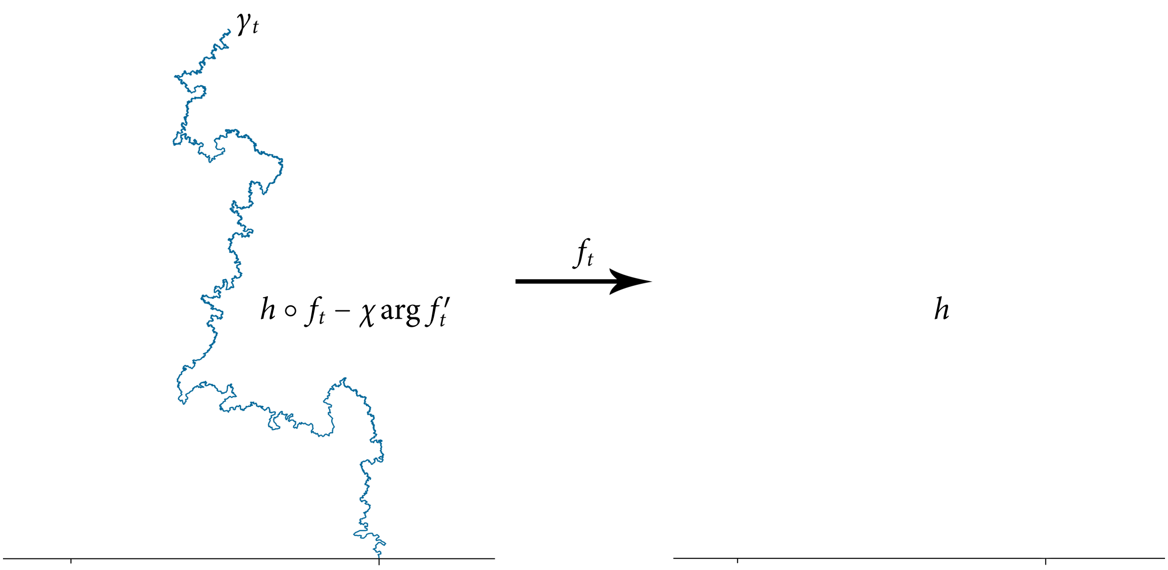


Figure 2.2: A visualisation of the forward coupling.

Theorem 2.3.1. *Fix $\kappa \in (0,4]$ and let γ_T be the segment of SLE_κ generated by the Loewner flow*

$$df_t(z) = \frac{2}{f_t(z)} - \sqrt{\kappa} dB_t, \tag{2.15}$$

up to a fixed time $T > 0$. Write

$$u_0(z) := \frac{-2}{\sqrt{\kappa}} \arg z, \qquad \chi := \frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2},$$

$$u_t(z) := u_0(f_t(z)) - \chi \arg f_t'(z),$$

and let $\tilde{h}$ be an instance of the zero boundary GFF on $\mathbb{H}$, independent of B. Then the following *two random distributions on $\mathbb{H}$ agree in law:*

$$h := u_0 + \tilde{h}.$$
$$h \circ f_T - \chi \arg f_T' = u_T + \tilde{h} \circ f_T.$$

The statement of the reverse coupling is similar. It lets us cut through a Neumann boundary GFF with a reverse SLE to obtain another Neumann boundary GFF. See Figure 2.3.

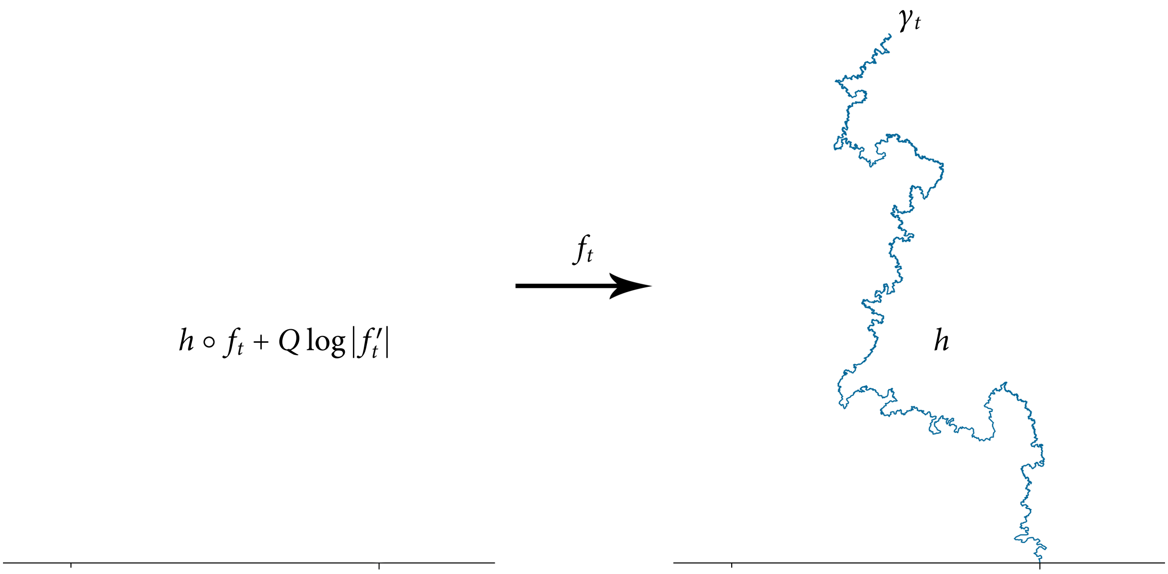


Figure 2.3: A visualisation of the reverse coupling.

Theorem 2.3.2. *Fix $\kappa > 0$ and let γ_T be the segment of SLE_κ generated by the reverse Loewner flow*

$$df_t(z) = \frac{-2}{f_t(z)} - \sqrt{\kappa} dB_t, \tag{2.16}$$

up to a fixed time $T > 0$. Write

$$u_0(z) := \frac{2}{\sqrt{\kappa}} \log |z|, \qquad Q := \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2},$$
$$u_t(z) := u_0(f_t(z)) + Q \log |f_t'(z)|,$$

and let $\tilde{h}$ be an instance of the free boundary GFF on $\mathbb{H}$, independent of B. Then the following two random distributions (modulo additive constants) on $\mathbb{H}$ agree in law:

$$h := u_0 + \tilde{h}.$$

$$h \circ f_T + Q \log |f_T'| = u_T + \tilde{h} \circ f_T.$$

There is some difficulty with the statement of Theorem 2.3.1 because the distribution $h \circ f_T - \chi \arg f_T'$ is only a priori defined as a distribution on $\mathbb{H} \setminus \gamma_T$, and we want to view it as a distribution on $\mathbb{H}$. It is easiest if we view it as $u_T + \tilde{h} \circ f_T$ and show that both terms are well defined distributions on $\mathbb{H}$.

We will deal with the second term first. First note that, by conformal invariance of the GFF, $\tilde{h} \circ f_T$ is a zero boundary GFF on $\mathbb{H} \setminus \gamma_T$. To extend it to a distribution on $\mathbb{H}$, we use the following. This was originally Proposition 2.2 in [SS13], but our statement and proof is closer to Proposition 15.8 in [BN14].

Proposition 2.3.3. *Let D be a subdomain of $\mathbb{H}$. Let h be a zero boundary GFF on D. Then h extends uniquely almost surely to a random variable $\overline{h}$ in $\mathcal{D}'(\mathbb{H})$, the space of distributions on $\mathbb{H}$, such that $\langle \overline{h}, \rho \rangle$ is a centred Gaussian r.v. with variance*

$$\mathcal{E}_D(\rho) = \iint_{D \times D} \rho(x) G_D(x,y) \rho(y) dx dy,$$

where G_D is the Green function on D.

2.3.2 *Informal justification*

The proof that u_T is a well defined distribution on $\mathbb{H}$ will come later. For now, we assume that it is a well defined distribution on $\mathbb{H}$ to derive the form of the two Theorems.

Most of the following calculations are common to both the forward and reverse couplings. When we talk about the Green function, we will abuse notation again by having a common letter for both the forward and reverse cases. Let G_0 be the Green function for the upper half plane, and G_t the Green function for $H_t = \mathbb{H} \setminus \gamma_t$. By conformal invariance, $G_t(x, y) = G_0(f_t(x), f_t(y))$.

We will show that both distributions in the Theorem statements have the same finite dimensional distributions by considering their characteristic functions. Let $\rho \in C_c^\infty(\mathbb{H})$ be a compactly supported, smooth test function. We know that $\left\langle \tilde{h} + u_0, \rho \right\rangle$ is Gaussian with mean

$$m_0 = \langle u_0, \rho \rangle$$

and variance

$$E_0 = \iint_{\mathbb{H}} \rho(x) G_0(x, y) \rho(y) dx dy.$$

So, $\mathbb{E}\left[e^{i\theta \langle \tilde{h} + u_0, \rho \rangle} \right] = e^{i\theta m_0 - \frac{1}{2}\theta^2 E_0}$.

Conditional on $\mathcal{F}_T$, Proposition 2.3.3 tells us that $\left\langle \tilde{h} \circ f_T, \rho \right\rangle$ is a centred Gaussian with variance

$$E_T = \iint_{H_T} \rho(x) G_T(x, y) \rho(y) dx dy.$$

Therefore, we can calculate

$$\begin{aligned} \mathbb{E}\left[\exp\left(i\theta \left\langle \tilde{h} \circ f_T + u_T, \rho \right\rangle \right) \right] &= \mathbb{E}\left[e^{i\theta \langle u_T, \rho \rangle} \mathbb{E}\left[\exp\left(i\theta \left\langle \tilde{h} \circ f_T, \rho \right\rangle \right) | F_T \right] \right] \\ &= \mathbb{E}\left[\exp\left(i\theta \langle u_T, \rho \rangle - \frac{1}{2}\theta^2 E_T \right) \right]. \end{aligned}$$

To show equality in distribution, we would like to show that

$$\mathbb{E}\left[\exp\left(i\theta \langle u_T, \rho \rangle - \frac{1}{2}\theta^2 E_T \right) \right] = \exp\left(i\theta \langle u_0, \rho \rangle - \frac{1}{2}\theta^2 E_0 \right). \tag{2.17}$$

It will therefore be sufficient to show that $\langle u_T, \rho \rangle$ is a martingale with quadratic variation given by

$$\langle \langle u_t, \rho \rangle \rangle = -E_t = -\iint \rho(x) G_t(x, y) \rho(y) dx dy.$$

If it is, then Ito's isometry tells us that

$$\mathbb{E}\left[\langle u_t, \rho \rangle^2 \right] = \mathbb{E}\left[\iint \rho(x) u_t(x) u_t(y) \rho(y) dx dy \right] = -\iint \rho(x) G_t(x, y) \rho(y) dx dy,$$

which is enough to see (2.17).

2.3.3 *Derivation of u_t*

What we will in fact show is that for any fixed $x, y \in \mathbb{H}$, $u_t(x)$ and $u_t(y)$ are martingales with cross variation

$$d\left\langle u_t(x), u_t(y)\right\rangle = -dG_t(x, y)$$

and then argue that we can interchange the order of integration.

First, we need to calculate dG_t. As most of the calculations are common to all cases, we will consider them all at once. When we need to distinguish between forward and reverse couplings, we will use blue and red respectively, and when we distinguish between zero and free boundary conditions we will use orange and green, respectively.

Note that the Green function takes the form

$$G_t(x, y) = G_0(f_t(x), f_t(y)) = \pm \log\left|f_t(x) - \overline{f_t(y)}\right| - \log\left|f_t(x) - f_t(y)\right|.$$

Now, what is $d\log\left(f_t(x) - f_t(y)\right)$? It has finite variation (and is in fact smooth), so we can use standard calculus to see that

$$\begin{aligned} d\log\left(f_t(x) - f_t(y)\right) &= \frac{1}{f_t(x) - f_t(y)} d\left(f_t(x) - f_t(y)\right) \\ &= \pm \frac{1}{f_t(x) - f_t(y)} \left(\frac{2}{f_t(x)} - \frac{2}{f_t(y)}\right) dt \\ &= \mp \frac{2}{f_t(x) f_t(y)} dt. \end{aligned}$$

Therefore,

$$\begin{aligned} dG_t(x, y) &= \mathrm{Re}\left(\mp \frac{2}{f_t(x)}\left(\pm \frac{1}{\overline{f_t(y)}} - \frac{1}{f_t(y)}\right)\right) dt \\ &= \mp \mathrm{Re}\left(\frac{2}{f_t(x)} \begin{matrix} -2i\,\mathrm{Im}\left(\frac{1}{f_t(y)}\right) \\ -2\mathrm{Re}\left(\frac{1}{f_t(y)}\right) \end{matrix}\right) dt \\ &= \pm \mathrm{Re}\left(\frac{2}{f_t(x)} i\right) \begin{matrix} \mathrm{Im}\left(\frac{2}{f_t(y)}\right) \\ \mathrm{Re}\left(\frac{2}{f_t(y)}\right) \end{matrix} \; dt \\ &= \mp \begin{matrix} -\mathrm{Im}\left(\frac{2}{f_t(x)}\right)\mathrm{Im}\left(\frac{2}{f_t(y)}\right) \\ \mathrm{Re}\left(\frac{2}{f_t(x)}\right)\mathrm{Re}\left(\frac{2}{f_t(y)}\right) \end{matrix} \; dt. \end{aligned}$$

We summarise these calculations in the following table.

$-dG_t(x,y)$	Zero	Free
Forward	$\mathrm{Im}\left(\frac{2}{f_t(x)}\right)\mathrm{Im}\left(\frac{2}{f_t(y)}\right)dt$	$-\mathrm{Re}\left(\frac{2}{f_t(x)}\right)\mathrm{Re}\left(\frac{2}{f_t(y)}\right)dt$
Reverse	$-\mathrm{Im}\left(\frac{2}{f_t(x)}\right)\mathrm{Im}\left(\frac{2}{f_t(y)}\right)dt$	$\mathrm{Re}\left(\frac{2}{f_t(x)}\right)\mathrm{Re}\left(\frac{2}{f_t(y)}\right)dt$

It is clear from this that, as we want $-dG_t(x,y)$ to be the cross variation of two martingales, and therefore the product of two real expressions, we cannot work with the elements in the table above which have a negative coefficient. Therefore, we use the zero boundary GFF in the forward case and the free boundary GFF in the reverse.

It is also clear that we want to construct our martingale $u_t(x)$ to have diffusivity equal to the real or imaginary part of $\frac{2}{f_t(x)}$. So, let's check $\log f_t(x)$:

$$\begin{aligned} d\log(f_t(x)) &= \frac{1}{f_t(x)}df_t(x) - \frac{1}{2f_t(x)^2}d\langle f_t(x)\rangle \\ &= \frac{1}{f_t(x)}\left(\pm\frac{2}{f_t(x)}dt - \sqrt{\kappa}dB_t\right) - \frac{\kappa}{2f_t(x)^2}dt \\ &= \frac{\pm 4-\kappa}{2f_t(x)^2}dt - \frac{\sqrt{\kappa}}{f_t(z)}dB_t. \end{aligned}$$

That is almost what we need, but in the $\kappa \neq 4$ case, we need a correction term. By differentiating the Loewner equation with respect to space, we find that

$$df_t'(x) = \mp\frac{2f_t'(x)}{f_t(x)^2}dt,$$

and so

$$d\log f_t'(x) = \frac{1}{f_t'(x)}df_t'(x) = \mp\frac{2}{f_t(x)^2}dt.$$

Therefore, we find that

$$d\left(\log f_t(x) + \left(1\mp\frac{\kappa}{4}\right)\log f_t'(x)\right) = -\frac{\sqrt{\kappa}}{f_t(x)}dB_t,$$

and hence

$$d\left(\frac{2}{\sqrt{\kappa}}\log f_t(x) + \left(\frac{2}{\sqrt{\kappa}}\mp\frac{\sqrt{\kappa}}{2}\right)\log f_t'(x)\right) = -\frac{2}{f_t(x)}dB_t.$$

Taking the real or imaginary parts as necessary gives the form of u_t given in the statements of the theorems.

To recap, we have just shown the following:

Lemma 2.3.4. *For fixed $x, y \in \mathbb{H}$, the processes $u_t(x)$ and $u_t(y)$ as defined in Theorem 2.3.1 or 2.3.2 are continuous local martingales, with cross variation equal to*

$$d\langle u_t(x), u_t(y)\rangle = -dG_t(x,y).$$

Furthermore, the process $u_t(x)$ has quadratic variation given by

$$d\langle u_t(x)\rangle = -dC_t(x),$$

where

$$C_t(x) := \pm \log \operatorname{Im} f_t(x) - \operatorname{Re}\log f_t'(x).$$

The last result in Lemma 2.3.4 can be seen by similar Itô calculus calculations on $u_t(x)$ and $C_t(x)$.

2.3.4 *Extending to $\langle u_t, \rho\rangle$*

We now need to argue that the processes $u_t(x)$ are martingales (rather than just local martingales). We also need to show that they are martingales in some uniform sense, to make sure that $\langle u_t, \rho\rangle$ is a continuous martingale. We will concentrate on the forward case. The arguments in the reverse case are, in general, much simpler because the support of the test functions stay bounded away from the curve γ.

To do so, we will need the following:

Lemma 2.3.5. *For a fixed $t > 0$, the function given by*

$$C_t(x) = \log \operatorname{Im} f_t(x) - \operatorname{Re}\log f_t'(x)$$

has a law which decays exponentially, uniformly on compact sets $D \subset \mathbb{H}$.

Therefore, we can find $\alpha > 0$ so that

$$\mathbb{P}\left[-C_t(x) > s\right] \le e^{-\alpha s}$$

for all $x \in \text{supp}(\rho)$.

Corollary 2.3.6. *The law of $|u_t(x)|$ decays exponentially fast, uniformly for x in the support of the test function ρ.*

Proof. First use the Dubins-Schwartz theorem to see that we can write

$$\mathbb{P}\left[|u_t(x)| > c\right] = \mathbb{P}\left[|B_{-C_t(x)}| > c\right]$$

for some Brownian motion B. For some $s > 0$, we can then split the event up to see

$$\begin{aligned}\mathbb{P}\left[|B_{-C_t(x)}| > c\right] &= \mathbb{P}\left[|B_{-C_t(x)}| > c, -C_t(x) > s\right] + \mathbb{P}\left[|B_{-C_t(x)}| > c, -C_t(x) < s\right] \\ &\le \mathbb{P}\left[-C_t(x) > s\right] + \mathbb{P}\left[|B_s| > c\right] \\ &\le e^{-\alpha s} + 2e^{-c^2/(2s)}.\end{aligned}$$

Taking $s = c$ shows the exponential decay we were looking for. □

That exponential decay is enough for us to see that for a fixed t, the functions $u_t(x)$ are uniformly bounded in $L^1(\Omega)$ for every $x \in \text{supp}(\rho)$. We can now use Fubini's theorem to interchange the order of integration to see, for $s < t$

$$\begin{aligned}\mathbb{E}\left[\langle u_t, \rho\rangle \,|\mathcal{F}_s\right] &= \mathbb{E}\left[\int u_t(x)\rho(x)dx|\mathcal{F}_s\right] \\ &= \int \rho(x)\mathbb{E}\left[u_t(x)|\mathcal{F}_s\right] dx \\ &= \int \rho(x)u_s(x)dx = \langle u_s, \rho\rangle\,.\end{aligned}$$

In other words, the process $\langle u_t, \rho\rangle$ is a martingale.

We now need to show that it is a continuous martingale. We know (from the SDE definition) that $u_t(x)$ is almost surely a continuous martingale for any x, but only up to the time $\tau(x) = \inf\{t > 0 \,:\, x \in \gamma_t\}$.

In fact, because u_t is the real or imaginary part of a conformal function, it is harmonic on its domain of definition. Therefore, so long as the SLE path stays away from the support of the test function ρ, $u_t(x)$ is uniformly in t for $x \in \text{supp}(\rho)$, which implies that $\langle u_t, \rho \rangle$ is continuous.

Fix a realisation of γ, and let ρ_ε be equal to 0 in an ε-neighbourhood of γ and equal to ρ everywhere else. By the previous argument, we know that $\langle u_t, \rho_\varepsilon \rangle$ is continuous in t. We now want to show that it converges to $\langle u_t, \rho \rangle$ as $\varepsilon \to 0$ uniformly in t. That will show continuity of $\langle u_t, \rho \rangle$, as the uniform limit of continuous functions is continuous.

The uniform exponential decay in law that we saw in Corollary 6 implies that $\mathbb{E}\left[|u_t(x)|^p\right]$ is uniformly bounded for all x in a compact set, which implies that $u_t \in L^p_{loc}(\mathbb{H})$ almost surely, for any $p \geq 1$.

Doob's inequality tells us that

$$\mathbb{P}\left[\sup_{s\in[0,t]} |u_s(x)| > c\right] = \mathbb{P}\left[\sup_{s\in[0,t]} e^{\lambda|u_s(x)|} > e^{\lambda c}\right] \leq e^{-\lambda c}\mathbb{E}\left[e^{\lambda|u_t(x)|}\right].$$

Since $|u_t(x)|$ has exponentially decaying tails, $\mathbb{E}\left[e^{\lambda|u_t(x)|}\right]$ is finite for $\lambda > 0$ small enough. So we see that $\sup_{s\in[0,t]} |u_s(x)|$ also has exponentially decaying tails. And again, this decay is uniform for x in compact sets. So, by the same argument as above, we see that $\sup_{s\in[0,t]} |u_s(x)| \in L^p_{loc}(\mathbb{H})$ almost surely.

Now, fix $T > 0$. Then for $t \in [0, T]$, we see that

$$\langle u_t, \rho \rangle - \langle u_t, \rho_\varepsilon \rangle = \int_{\gamma^\varepsilon} u_t(x)\rho(x)dx$$

where γ^ε is the ε-neighbourhood of γ. We can bound the right hand side by

$$\int_{\gamma^\varepsilon} u_t(x)\rho(x)dx \leq \sup_x(\rho(x)) \int_{\gamma^\varepsilon} \sup_{s\in[0,T]} |u_s(x)|dx. \tag{2.18}$$

Because the path γ has Lebesgue measure 0 almost surely, we know that the area of γ^ε tends to 0 as $\varepsilon \to 0$. Therefore, since $\sup |u_s(x)| \in L^p_{loc}(\mathbb{H})$ for $p > 1$, a simple Hölder argument shows us that the right hand side of (2.18) tends to 0 as $\varepsilon \to 0$ as well. That convergence in uniform for $t \in [0, T]$, and so we know that $\langle u_t, \rho \rangle$ is continuous.

The final thing to do is to check that

$$\langle u_t, \rho\rangle^2 + \iint_{H_t} \rho(x) G_t(x,y)\rho(y)\,dx\,dy \tag{2.19}$$

is, in fact, a martingale. We already know that $u_t(x)u_t(y) + G_t(x,y)$ is. We also know that the functions u_t have exponentially decaying laws. Combining that with the fact that $G_t(x,y)$ is non-increasing in t lets us use Fubini's theorem again to deduce that (2.19) is a martingale.

2.4 Liouville quantum gravity

We want to create something that we can use as a random Riemannian surface, whose law is conformally invariant. The Riemann uniformisation theorem tells us that any simply connected Riemann surface is conformally equivalent to one of (a) the open unit disc, (b) the complex plane, or (c) the Riemann sphere. Two Riemannian metrics are conformally equivalent if the metric of one can be written as a positive function multiplied by the metric of the other. So any metric of our random Riemann surface, once it has been parameterised by (a), (b) or (c), should take the form $\rho(z)dz^2$ for some positive function ρ.

Since the GFF is a conformally invariant object, a natural proposal for the metric of the manifold is $e^{\gamma h(z)}dz^2$. This is, of course, not well defined. However, we can use the regularisation from Section 2.2.5 to successfully define something we can view as a volume form.

2.4.1 Liouville quantum gravity measure

To construct our volume form, we first give a definition of the regularised measure. This construction is due to Duplantier and Sheffield [DS11]. It gives the random measure on a domain D that we used while discussing the KPZ equation in Section 2.1.4.

Definition 2.4.1. Let D be a (Greenian) domain, and let h be a zero boundary GFF on D. Let $\gamma \in [0,2)$ and $\varepsilon > 0$. The ε-regularised Liouville measure, μ_ε, is given by

$$\mu_\varepsilon(dz) = \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(z)}dz,$$

where dz is standard Lebesgue measure on D.

The first thing to note here is the normalisation. We know that

$$\mathrm{Var}(h_\varepsilon(z)) = -\log\varepsilon + \log R(z;D),$$

and so we see that

$$\mathbb{E}\left[e^{\gamma h_\varepsilon(z)}\right] = \exp\left(\frac{\gamma^2}{2}(-\log\varepsilon + \log R(z;D)\right) = R(z;D)^{\gamma^2/2}\varepsilon^{-\gamma^2/2}.$$

Therefore the expected value of the Radon-Nikodym derivative of the measure μ_ε is constant in ε (ignoring slight boundary effects). We can go further: thanks to Lemma 2.2.11 we can view $\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}$ as the exponential martingale of a Brownian motion (suitably time-changed).

An application of Fubini's theorem then lets us deduce that the expected value for the measure is

$$\mathbb{E}[\mu_\varepsilon(A)] = \int_A R(z;D)^{\gamma^2/2}dz,$$

so long as the set A has a distance at least ε from ∂D. What we cannot do, however, is view $\mu_\varepsilon(A)$ as a martingale, despite the fact that it is the integral over a function which is a martingale when viewed pointwise. The difficulty is in constructing the filtration; the overlapping circles of the circle averages mean that we have to know, in some sense, the entire GFF in and around the set A in order to calculate $\mu_\varepsilon(A)$.

Because we can not view $\mu_\varepsilon(A)$ as a martingale, we are unable to use standard martingale convergence techniques to deduce the existence of $\lim_{\varepsilon\to 0}\mu_\varepsilon(A)$. However, the following Proposition from [DS11] gives us that result.

Proposition 2.4.2. *Fix $\gamma \in [0,2)$. Then the measures $\mu_\varepsilon := \varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}dz$ almost surely converge weakly inside D, as $\varepsilon \to 0$ along powers of 2, to a limiting measure $\mu = \mu_h$.*

We will explore other ways of constructing similar measures in Section 2.4.3. One of the features that sets this construction apart is the almost sure convergence of the measure. That means that the measure μ is a measurable function of the field h. Conversely, the fact that the field h is a measurable function of the measure μ was shown in [BSS14].

2.4.2 *Thick points*

We would like to study the set of points on which the Liouville quantum gravity measure is supported. For that, we need the idea of a thick point of the Gaussian free field. We follow the definition given in [HMP10].

Definition 2.4.3. Let $D \subset \mathbb{C}$ be a Greenian domain and let h be a zero boundary GFF in D. A point $z \in D$ is said to be a α-thick point if

$$\lim_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{-\log \varepsilon} = \alpha.$$

For $\alpha > 0$, the set of α-thick points, usually denoted by $\mathcal{T}_\alpha$, is a set where the field is unusually large. We know that the circle average process, $(h_\varepsilon(z))_{\varepsilon>0}$, behaves (up to a random starting point) like a standard Brownian motion run at speed $t = -\log \varepsilon$. So we can view the limit as

$$\lim_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{-\log \varepsilon} = \lim_{t \to \infty} \frac{B_t}{t}, \tag{2.20}$$

and the time inversion property of Brownian motion tells us that the right hand side of (2.20) is zero, almost surely (see Theorem 1.9 of [MP10]).

The set of α-thick points is not always empty, however, because we consider every $z \in D$, which is an uncountable number of points. Indeed, we know that the Hausdorff dimension of $\mathcal{T}_\alpha$ is non-zero for suitable α. The following theorem is from [HMP10]:

Theorem 2.4.4. *Let $\alpha \geq 0$, and let $\mathcal{T}_\alpha$ be the set of α-thick points as defined in Definition 2.4.3. Then the Hausdorff dimension of $\mathcal{T}_\alpha$ is*

$$\dim_H(\mathcal{T}_\alpha) = \max\left(0, 2 - \frac{\alpha^2}{2}\right).$$

(Similar results for general Gaussian multiplicative chaos measures can be found as Theorems 4.1 and 4.2 of [RV13].)

The set of thick points is important for the study of the Liouville quantum gravity measure because, as we mentioned at the start of this subsection, it is the carrier of the measure. The following can be deduced from Proposition 3.4 in [DS11]:

Proposition 2.4.5. *Let $D \subset \mathbb{C}$ be a Greenian domain and let h be a zero boundary GFF on D. Let $\gamma \in [0,2)$ and let μ be the γ-LQG measure from Proposition 2.4.2. Then μ is supported on the set of γ-thick points, $\mathcal{T}_\gamma$, almost surely.*

Sketch proof. The proof involves constructing a joint law on the set of points $z \in D$ and the GFF h simultaneously. One way to construct this measure is to choose the GFF h first, construct the regularised Liouville measure μ_ε, and then choose a point z according to the measure

$$\frac{\mu_\varepsilon(dz)}{\mu_\varepsilon(D)}.$$

This gives a measure Q_ε on the point z and the field h simultaneously. The key idea to the proof is then to swap the order we pick the points, so choose z first and then h conditional on z. We will find that, conditional on the point z, the field h has a log-singularity at z, meaning that z must be a thick point.

We can write the joint measure, informally, as

$$Q_\varepsilon(dz, dh) = \frac{1}{Z}\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}dz\mathbb{P}[dh],$$

where $\mathbb{P}$ is the law of the GFF. The marginal law of h under Q is

$$Q_\varepsilon(dh) = \frac{1}{Z}\mu_\varepsilon(D)\mathbb{P}[dh],$$

i.e. it has the law of a GFF weighted by the total mass it gives to D. So, the marginal law of h under Q_ε is absolutely continuous with respect to $\mathbb{P}$. The marginal law of z is equal to

$$Q_\varepsilon(dz) = \frac{dz}{Z}\mathbb{E}\left[\varepsilon^{\gamma^2/2}e^{\gamma h_\varepsilon(z)}\right],$$

which, thanks to the fact that $\mathrm{Var}(h_\varepsilon(z)) = -\log\varepsilon + R(z;D)$, is equal to

$$Q_\varepsilon(dz) = \frac{dz}{Z}R(z;D)^{\gamma^2/2}.$$

(In fact, it is only equal to this so long as z is further than ε from the boundary. We will ignore boundary effects to simplify this sketch.) So, as we said at the beginning of the sketch, we now reverse the procedure. Pick $z \in D$ randomly, with probability density proportional to the conformal radius. Next, pick h according to the measure

$$Q_\varepsilon(dh \mid z) \propto e^{\gamma h_\varepsilon(z)}\mathbb{P}[dh]. \tag{2.21}$$

Now, Girsanov's lemma tells us that, if we tilt a centred Gaussian random variable by the exponential of a Gaussian, the result will still be Gaussian and have the same variance, but a shifted mean. In our case, by tilting as we have in (2.21), h will have the same variance as a standard GFF, but a mean function given by

$$\mathbb{E}_Q\left[h(w)\right] = \mathrm{Cov}(h(w), \gamma h_\varepsilon(z)) \approx -\gamma \log|w-z|.$$

This shows that h has a γ-log singularity, almost surely, and so

$$\lim_{\varepsilon\to 0}\frac{h_\varepsilon(z)}{-\log\varepsilon} = \gamma.$$

So, a point z chosen according to the measure Q is almost surely a γ-thick point of the field, also chosen with respect to Q. Since the law of the field under Q is absolutely continuous with respect to the standard law of the GFF, we see that the Liouville measure is supported on γ-thick points, almost surely. □

2.4.3 Multiplicative chaos

Another way of constructing a measure out of Gaussian fields was introduced by J.P.Kahane in [Kah85]. He gave a rigorous way of defining the measure

$$\mu(dz) = e^{\gamma X(z) - \frac{\gamma^2}{2}\mathbb{E}[X(z)^2]}\sigma(dz), \tag{2.22}$$

on a metric space (D,ρ), where X is a Gaussian random field, σ is a reference measure in the class R_d^+ (a similar condition to requiring that σ has finite d-energy), and $\gamma > 0$.

One of the constraints the theory in [Kah85] is that, to ensure uniqueness of the measure, we need to assume that the covariance kernel

$$K(x,y) := \mathbb{E}\left[X(x)X(y)\right]$$

is of σ-positive type. The kernel $K : D \times D \to \mathbb{R} \cup \{\infty\}$ is σ-positive if K can be written as

$$K(x,y) = \sum_{n=0}^{\infty} K_n(x,y), \tag{2.23}$$

where the functions K_n are non-negative definite, continuous and pointwise non-negative. Kahane focussed on the case that K can be written as

$$K(x,y) = \log_+ \frac{T}{\rho(x,y)} + g(x,y), \tag{2.24}$$

where g is a bounded continuous function.

With K expanded as in (2.23), let Y_n be a Gaussian field with covariance kernel K_n. Then we can define an approximation to the field X as $X^N = \sum_{n=0}^{N} Y_n$. Then the sequence of measures given by

$$\mu^N(dz) = e^{\gamma X^N(z) - \mathbb{E}[X^N(z)^2]}\sigma(dz)$$

converge almost surely in the space of Radon measures to a random measure μ. The σ-positivity assumption (coupled with Kahane's convexity inequality) ensures that the law of the limiting measure does not depend on the σ-positive decomposition of K. However, it does not guarantee almost sure uniqueness of the limit.

In the same paper, he also derived conditions on γ which ensure that the limiting measure μ is non-degenerate: we must have $\gamma^2 < 2d$. Furthermore, he gave some results on the structure of the carrier of the measure μ which are analogous to those we saw in Section 2.4.2.

Robert and Vargas generalised Kahane's theory in [RV10], relaxing the σ-positivity assumption (which can be difficult to check). They showed that, using a positive definite function θ (with polynomial decay at infinity) as a mollifier, the sequence of random measures μ_ε, constructed from the field $X * \theta_\varepsilon$ as in (2.22), converge in law to a measure μ. Further, the law of the limiting measure μ is independent of the mollifier θ.

Shamov has since extended the theory of Robert and Vargas in [Sha16] to show that, for very general regularised fields X_ε, we can construct measures μ_ε as in (2.22) which converge in probability to a limiting measure μ, and that limit is independent of the mollifier. This implies that the limiting measure μ is a measurable function of the field X. Junnila and Saksman have shown similar results in [JS15], and their paper covers the critical case, $\gamma^2 = 2d$, as well.

A slightly weaker version of Shamov's results, with a significantly simpler proof, was shown in [Ber15b]. We will summarise some of these in a little more detail now, and use them to give a new presentation of the Liouville Brownian motion in Section 2.5.

We will make the following assumptions. Let $D \subset \mathbb{R}^k$, and suppose the metric ρ is the standard Euclidean metric. Without loss of generality, assume that D contains the ball of radius 10 centred at the origin. Suppose that the measure σ has dimension $d \le k$, so

$$\iint_{D^2} \frac{1}{|x-y|^{d-\varepsilon}} \sigma(dx)\sigma(dy) < \infty$$

for all $\varepsilon > 0$.

Let X be a generalised Gaussian function on D with covariance kernel given by (2.24), and assume that the function g in (2.24) is smooth on $\overline{D} \times \overline{D}$. Fix a function $\theta : \mathbb{R}^k \to [0,1]$ which satisfies

$$\int \theta(x)dx = 1,$$

and

$$\iint \log \frac{1}{|x-y|} \theta(x)\theta(y)dxdy < \infty.$$

Set $\theta_\varepsilon(z) = \varepsilon^{-d}\theta(z/\varepsilon)$, and define $X_\varepsilon(z) = X * \theta_\varepsilon(z)$. Finally, define the measure

$$\mu_\varepsilon(dz) = e^{\gamma X_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[X_\varepsilon(z)^2]}\sigma(dz).$$

Under these assumptions, the following theorem is given in [Ber15b].

Theorem 2.4.6. *Let $\gamma < \sqrt{2d}$. Let S be the unit cube in $\mathbb{R}^k$. Then $\mu_\varepsilon(S)$ converges in probability and L^1 to a limit μ. The limit μ is independent of the regularisation θ_ε.*

As with Shamov's theory, Theorem 2.4.6 is enough to show that the measure μ is a measurable function of the field X. (It is worth noting that a partial converse to this, that the field X is a measurable function of the measure μ when X is a Gaussian free field, is given in [BSS14].)

The proof of convergence involves showing that $\mu_\varepsilon(S)$ is uniformly integrable. In the case that $\gamma < \sqrt{d}$ the measures μ_ε are uniformly bounded in L^2 and the proof is more straightforward. In the general case, however, space is partitioned into "good points," called $G \subset S$, and the rest. The set of good points are those which are not too thick up to a certain scale, i.e. where regularisations of the field are not too big. The measure of the good points, $\mu_\varepsilon(G)$, is bounded in L^2 and the rest, $\mu_\varepsilon(S \setminus G)$, does not contribute to the L^1 limit.

In the course of the proof of existence and uniqueness of the measure, the following two lemmas were shown.

Lemma 2.4.7. *We can write $X = \sum_i X_i$ where the (X_i) are independent, continuous Gaussian fields, in the sense that for arbitrary, fixed $f \in L^2(D, dx)$, the sum $\sum_i \langle X_i, f\rangle$ converges almost surely and agrees with $\langle X, f\rangle$ almost surely.*

Using the decomposition from Lemma 2.4.7 we can define a measure in a similar way that Kahane did using his σ-positive decomposition. Importantly, the limiting measures from both constructions are the same.

Lemma 2.4.8. *Let X^N be the partial sum $X^N(z) = \sum_{i=0}^N X_i(z)$. Define $\mu^N(S)$ by*

$$\mu^N(S) = \int_S e^{\gamma X^N(z) - \frac{\gamma^2}{2}\mathbb{E}[X^N(z)]}\sigma(dz).$$

Then $\mu^N(S)$ is a uniformly integrable martingale, and converges to the same limit $\mu(S)$ that the sequence $\mu_\varepsilon(S)$ from Theorem 2.4.6 converges to. Furthermore, if we define the sequence of filtrations $(\mathcal{F}_n)$ by $\mathcal{F}_n := \sigma(X_1, X_2, \ldots, X_n)$, then $\mu(S)$ is $\mathcal{F}_\infty$ measurable.

2.5 Liouville Brownian motion

Chapter 3 studies properties of Brownian motion on the Liouville quantum gravity surface mentioned in Section 2.4 — the surface for which the Liouville measure of Section 2.4.1 is the volume form. In this section we will introduce this Brownian motion, the Liouville Brownian motion, defined simultaneously in [Ber15a] and [GRV13a], both following the definition in the physics literature by [Wat93]. The two papers give slightly different derivations of LBM, both of which end with essentially the same object. We will now give a new construction, made possible by the theory in [Ber15b].

2.5.1 *Intuition*

We now want to construct a Brownian motion on the Riemann surface parameterised by some domain $D \subset \mathbb{C}$ whose metric tensor is given (formally) by

$$e^{\gamma h(z)}(dx^2 + dy^2), \tag{2.25}$$

where h is a GFF on D. Let us, for the rest of this subsection, assume that h (and, indeed, anything else that we use) is as smooth as we need it to be. We will see how to deal with it rigorously in Section 2.5.2.

One way of constructing a Brownian motion on a Riemannian manifold is to construct a diffusion whose generator is the Laplace-Beltrami operator on the manifold. Since the metric (2.25) is conformally equivalent to the "flat" metric on D, the matrix representation of the metric tensor is simply a scaling of the identity matrix. That simplifies the Laplace-Beltrami operator that we need to deal with. For a suitably smooth function $f : M \to \mathbb{R}$ defined on the manifold, the Laplace–Beltrami operator in local coordinates on D is

$$\Delta_M f(z) = \frac{1}{\sqrt{e^{\gamma h(z)}}} \Delta f(z).$$

Therefore, the Liouville Brownian motion in the local coordinates of D should be the solution

Z to the SDE

$$dZ_t = e^{-\frac{\gamma}{2}h(Z_t)}d\overline{B}_t,$$

where $\overline{B}$ is a standard Brownian motion. Therefore, we see that Z is a continuous local martingale, and we can write

$$Z_t = B_{\langle Z \rangle_t}$$

where B is a standard Brownian motion.

The study of Liouville Brownian motion Z therefore comes down to the study of its quadratic variation, $\langle Z \rangle$. We know that we can write

$$\langle Z \rangle_t = \int_0^t e^{-\gamma h(Z_s)} ds \tag{2.26}$$

but we cannot use the expression (2.26) as the definition of the quadratic variation of Z since the right hand side still depends on Z. We claim that

$$\langle Z \rangle_t := \inf \left\{ s > 0 \ : \ \int_0^s e^{\gamma h(B_u)} du \right\} \tag{2.27}$$

satisfies (2.26).

To see that, first note that (2.27) implies

$$\int_0^{\langle Z \rangle_t} e^{\gamma h(B_u)} du = t$$

which, after differentiating with respect to t and rearranging slightly, gives us

$$\frac{d \langle Z \rangle_t}{dt} = e^{-\gamma h(B_{\langle Z \rangle_t})}.$$

Therefore, we can integrate again to see

$$\langle Z \rangle_t = \int_0^t e^{-\gamma h(B_{\langle Z \rangle_s})} ds,$$

which is (2.26), as we wanted.

2.5.2 *Rigorous construction*

We will now make rigorous the notion of the function

$$F(t) = \int_0^t e^{\gamma h(B_s)} ds$$

used in the definition of the quadratic variation of the LBM Z, from (2.27). Once we have shown the existence of such a function, we will be able to define the LBM as

$$Z_t = B_{F^{-1}(t)},$$

where the Brownian motion B is the same as that used in the definition of F.

Our presentation of this construction is new in that is shows convergence of regularised functions F_ε in probability and L^1 to a limit F. In particular, that implies that the limiting function F is a measurable function of the underlying field. The presentation uses the theory found in [Ber15b] that we summarised in Section 2.4.3.

It is worth noting that the existence of the limit F does need the underlying field h to be a GFF, only that it should be a log correlation Gaussian field, as discussed in Section 2.4.3. Further properties of Liouville Brownian motion such as conformal invariance, however, do depend on the underlying field being a GFF.

Let h be a GFF in D, let (B_t) be an independent Brownian motion started in D, and let $T = \inf\{t > 0\, B_t \notin D\}$ be the first exit time of the Brownian motion from D. Let ν_t^T be the occupation measure of the killed Brownian motion at time t so that, for an integrable function f,

$$\int_0^{t\wedge T} f(B_s)ds = \int_D f(z)\nu_t^T(dz).$$

In order to use the theory in Section 2.4.3, we need to check that the stopped occupation measure has dimension 2, i.e. for all $\varepsilon > 0$ we have

$$\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|^{2-\varepsilon}} \nu_t^T(dx)\nu_t^T(dy) < \infty. \tag{2.28}$$

We will check (2.28) by comparing the stopped measure ν_t^T with the standard occupation measure for Brownian motion, which we denote by ν_t. If we use the same Brownian motion B to construct both the measure ν_t and the stopped measure ν_t^T, we can easily see that for any positive function f we have

$$\begin{aligned}\int_{\mathbb{R}^2} f(z)\nu_t^T(dz) &= \int_0^{t\wedge T} f(B_s)ds \\ &\le \int_0^t f(B_s)ds \\ &= \int_{\mathbb{R}^2} f(z)\nu_t(dz).\end{aligned}$$

Therefore, if we can show that the measure ν_t has dimension 2, we will know that the measure ν_t^T does also. We can compute the expectation:

$$\begin{aligned}\mathbb{E}\left[\iint_{\mathbb{R}^2\times\mathbb{R}^2} \frac{1}{|x-y|^{2-\varepsilon}}\nu_t(dx)\nu_t(dy)\right] &= \mathbb{E}\left[\int_0^t\int_0^t \frac{1}{|B_u-B_v|^{2-\varepsilon}}dudv\right] \\ &= \int_0^t\int_0^t \mathbb{E}\left[\frac{1}{|B_u-B_v|^{2-\varepsilon}}\right]dudv \\ &= \int_0^t\int_0^t \mathbb{E}\left[\frac{1}{|B_1|^{2-\varepsilon}|u-v|^{(2-\varepsilon)/2}}\right]dudv \\ &= \mathbb{E}\left[\frac{1}{|B_1|^{2-\varepsilon}}\right]\int_0^t\int_0^t \frac{1}{|u-v|^{(2-\varepsilon)/2}}dudv \\ &< \infty.\end{aligned}$$

So we see that the occupation measure ν_t has dimension 2 almost surely, and therefore so does the stopped measure ν_t^T.

By a similar comparison argument we can show that, if ν_t^T has dimension 2 then, for all $s \in (0,t)$, the measure ν_s^T has dimension 2. Taking a countable sequence $(t_n) \subset \mathbb{R}$ converging to infinity, we can see that all the measures $(\nu_{t_n}^T)$ have dimension 2, almost surely. And therefore, by the comparison argument, ν_t^T has dimension 2 for all $t \ge 0$, almost surely.

Now, let θ be a mollifier satisfying the assumptions detailed in Section 2.4.3, and define

$h_\varepsilon(z) = h * \theta_\varepsilon(z)$. Then define the regularised function[3]

$$F_\varepsilon(t) := \int_0^{t\wedge T} e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]} ds$$
$$= \int_D e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(z)^2]} \nu_t^T(dz). \tag{2.29}$$

By Theorem 2.4.6, we know that $F_\varepsilon(t)$ converges in probability and in L^1 to a limit which is independent of the choice of θ, almost surely in B. We will define the function F as this limit:

$$F(t) := \lim_{\varepsilon\to 0} F_\varepsilon(t). \tag{2.30}$$

Now that we know how to define the Liouville Brownian motion Z, we would like to show that it is

- continuous, and
- it does not get stuck.

Continuity of $Z_t = B_{F^{-1}(t)}$ is implied by continuity of $F^{-1}(t)$ which we get so long as F is strictly increasing. (Discontinuities of F^{-1} correspond to flat areas of F.) Similarly, Z does not get stuck if F^{-1} is strictly increasing, i.e. if F is continuous. We will omit the proof of continuity. See Section 3.4 of [Ber15a], for example.

We now show that F is strictly increasing. The proof we give is similar to that of Theorem 2.7 in [GRV13a]. It is worth noting that the proof we present here works only for a LBM started from a single point. Generalising the proof to work for all starting points simultaneously (which is needed in order to say that LBM is a Feller process, and is done in [GRV13a]) involves a significant jump in complexity.

Proposition 2.5.1. *The function F defined in* (2.30) *is strictly increasing on the interval* $[0, T]$, *where T is first exit time of the Brownian motion B from the domain D, almost surely in h and B.*

[3] In order for our definition to agree completely with that of [Ber15a], we should weight the measure by $R(z;D)^{\gamma^2/2}$, the conformal radius raised to the power $\gamma^2/2$. We do not do this to keep the notation cleaner.

Proof. First, fix a realisation of the Brownian motion B, and pick times s, t so that $0 \le s < t < T$. We will show that, for this realisation of B, we have a positive difference, $F(t) - F(s) > 0$, almost surely in h. First, note that $(F_\varepsilon(s))$ and $(F_\varepsilon(t))$ are uniformly integrable, so we can write

$$\begin{aligned}
\mathbb{E}_h\left[F(t) - F(s)\right] &= \lim_{\varepsilon \to 0} \mathbb{E}_h\left[F_\varepsilon(t) - F_\varepsilon(s)\right] \\
&= \lim_{\varepsilon \to 0} \mathbb{E}_h\left[\int_D e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(z)^2]} \left(\nu_t^T(dz) - \nu_s^T(dz)\right)\right] \\
&= \lim_{\varepsilon \to 0} \int_D \mathbb{E}\left[e^{\gamma h_\varepsilon(z) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(z)^2]}\right] \left(\nu_t^T(dz) - \nu_s^T(dz)\right) \\
&= \int_D \nu_t^T(dz) - \nu_s^T(dz) \\
&= t - s > 0.
\end{aligned}$$

Now, we can apply Lemma 2.4.7 to regularise the GFF into partial sums $h^N = \sum_{i=0}^N h_i$. Applying Lemma 2.4.8 to the fields h^N and the reference measure $\sigma(dz) = \nu_t^T(dz) - \nu_s^T(dz)$ we can see that, if we define

$$M_N := \int_D e^{\gamma h^N(dz) - \frac{\gamma^2}{2}\mathbb{E}[h^N(z)^2]} \sigma(dz)$$

then (M_N) is a uniformly integrable martingale with limit $F(t) - F(s)$.

The event $\{F(t) - F(s) = 0\}$ is measurable with respect to the tail σ-algebra generated by the random fields (h_i), i.e.

$$\{F(t) - F(s) = 0\} \in \bigcap_{N \in \mathbb{N}} \sigma\left(h_i \,:\, i > N\right).$$

Therefore, Kolmogorov's 0-1 law tells us that

$$\mathbb{P}_h\left[F(t) - F(s) = 0\right] \in \{0, 1\}.$$

Since we have shown that $F(t) - F(s)$ has positive expectation, we know that $F(t) - F(s) > 0$, almost surely in h.

Taking countable, dense pairs in $(s, t) \in [0, T]^2$ shows that F is strictly increasing in the interval $[0, T]$ h-almost surely. The only property of the Brownian path B that we used here, other than its exit time T, was that its occupation measure has dimension 2. Since this is true almost surely for Brownian motion, we can see that F is strictly increasing up to the exit time of the Brownian motion B, almost surely in h and B. □

2.5.3 *Known properties of LBM*

Several properties of the Liouville Brownian motion were derived in the introductory papers [Ber15a, GRV13a]. Most importantly, the fact that the clock function exists when the regularisation process is taken to its limit. Both showed that the clock function is continuous and strictly increasing, so the LBM is a continuous process which does not get stuck.

In [Ber15a], an upper bound for the Hausdorff dimension of times that an LBM stays in thick points was derived. It is strong enough to show that γ-LBM spends Lebesgue-all of its time in γ-thick points, almost surely. We will discuss this upper bound in much more detail in Chapter 3. In [GRV13a], they showed that LBM is a Feller and that the Liouville measure μ is invariant. This invariance, combined with our knowledge of the carrier of μ from Section 2.4.2, is also enough to see that LBM spends Lebesgue-all of its time in γ-thick points, almost surely.

Since its introduction, other interesting properties of Liouville Brownian motion have been found. The existence of the heat kernel and Dirichlet form for LBM were shown in [GRV14]. The regularity of the heat kernel, and upper and lower bounds for it were found in [MRVZ14], and was further bounded in [AK15]. It was constructed on the sphere in [DKRV14], and at criticality, i.e. when $\gamma = 2$, in [RV15].

It has also been used to derive a more intrinsic version of the KPZ equation [BGRV14]. By using the heat kernel of LBM to define a quantum scaling exponent, the authors were able to derive the KPZ equation without using the Euclidean metric in the quantum definitions that we saw in Section 2.1.4.

3 Liouville Brownian Motion and Thick Points of the Gaussian Free Field

In this chapter, we find a lower bound for the Hausdorff dimension of times that a Liouville Brownian motion spends in α-thick points of the Gaussian free field, where α is not necessarily equal to the parameter used in the construction of the geometry. This completes a conjecture in [Ber15a], where the corresponding upper bound was shown.

In the course of the proof, we obtain estimates on the (Euclidean) diffusivity exponent, which depends strongly on the nature of the starting point. For a Liouville typical point, it is $1/(2-\frac{\gamma^2}{2})$. In particular, for $\gamma > \sqrt{2}$, the path is Lebesgue-almost everywhere differentiable, almost surely. This provides a detailed description of the multifractal nature of Liouville Brownian motion.

3.1 Introduction

The goal of this chapter is to study the multifractal nature of Liouville Brownian motion. This is a process which was introduced in [Ber15a] and [GRV13a] as the canonical diffusion in planar Liouville quantum gravity. For instance, it is the conjectured scaling limit of a simple random walk on a uniform random triangulation, conformally embedded into the plane (via circle packing, for example). Liouville quantum gravity and its geometry has itself been at the centre of remarkable developments. We point out, among many other works, [RV11, DS11, BGRV14].

Liouville Brownian motion is a useful tool for studying the geometry of Liouville quantum gravity. In fact, Watabiki has already considered the object (in a non-rigorous way) in an attempt to describe the metric and fractal structure of Liouville quantum gravity [Wat93]. This led him to propose a formula for the Hausdorff dimension of the random metric space. The paper [BGRV14] may be viewed as a first rigorous step in studying such multifractal aspects using Liouville Brownian motion. The current chapter addresses a similar point, but from a different perspective.

The general structure of the chapter is as follows. In the remainder of this section, we state the main results and try to give the intuitive idea behind the proof. In Section 3.2 we will briefly introduce the objects and definitions we use throughout the chapter, providing references for the reader should they need more detail. In Section 3.3.1 we show that the time change function has finite moments around times when the Brownian motion is conditioned to be in a thick point, and derive crude tail estimates for the time change process from those bounds. Section 3.3.2 is spent proving a simple large-deviation type result for the supremum of the harmonic projection of the GFF on to a disc, to use as an analogue of the scaling relation enjoyed by exactly stochastically scale invariant fields. In Section 3.3.3 we combine the results from the previous sections to show Hölder like properties of the time change function F_γ. Finally, in Sections 3.3.4 and 3.3.5, we prove the main theorems, using the regularity results obtained in Section 3.3.3.

3.1.1 Statement of results

Let h be a zero boundary GFF, defined in a simply connected proper domain $\mathcal{D} \subset \mathbb{C}$. One of the difficulties of working with a GFF is that it is not defined as a function, so we cannot say what value $h(z)$ takes, for $z \in \mathcal{D}$. However, it is regular enough that we can talk about its average value on a set. We will usually take that set to be the circle of radius $\varepsilon > 0$ centred at a point $z \in \mathcal{D}$, and call that average $h_\varepsilon(z)$. Let $\{h_\varepsilon(z)\ ;\ \varepsilon > 0\}_{z\in\mathcal{D}}$ be the circle averages of h. We will define both the GFF and its averages more precisely in Section 3.2.1. For $\alpha > 0$, the set $\mathcal{T}_\alpha$ of α-thick points is given by

$$\mathcal{T}_\alpha = \left\{ z \in \mathcal{D} \,:\, \lim_{\varepsilon\to 0} \frac{h_\varepsilon(z)}{\log \frac{1}{\varepsilon}} = \alpha \right\}.$$

By a theorem in [HMP10], it is known that the Hausdorff dimension of $\mathcal{T}_\alpha$ is

$$\dim_H(\mathcal{T}_\alpha) = \max\left(0, 2 - \frac{\alpha^2}{2}\right),$$

almost surely.

Let $0 < \gamma < 2$. We will denote by Z^γ a γ-Liouville Brownian motion, formally defined as follows. Let B be a Brownian motion killed upon leaving $\mathcal{D}$. We define its clock process, F_γ, to be

$$F_\gamma(t) = \int_0^t e^{\gamma h(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h(B_s)^2]} ds,$$

and the LBM is given by $Z_t^\gamma = B_{F_\gamma^{-1}(t)}$. It is not trivial to make sense of this definition. This was done in [Ber15a] and [GRV13a], where further properties were also proved. We recall the construction more precisely in Section 3.2.3.

The main goal of this chapter is to prove the following bound:

Theorem 3.1.1. *Let $\alpha, \gamma \in [0, 2)$, and let Z^γ denote a γ-Liouville Brownian motion. Then*

$$\dim_H(\{t \,:\, Z_t^\gamma \in \mathcal{T}_\alpha\}) \geq \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}},$$

almost surely, where $\dim_H$ refers to the Hausdorff dimension.

The proof of Theorem 3.1.1 follows similar lines as the proof of Theorem 4.1 in [RV13]. Long range correlations introduced by the Brownian motion created a few more technical difficulties to overcome. The authors proved their result for an exactly stochastically scale invariant field, and claimed that the result generalised from that to all log-correlated Gaussian fields. We were also unable to follow the generalisation of their proof, which used a non-trivial application of Kahane's convexity inequality – we instead had to rely on Lemma 3.3.5.

Theorem 3.1.1, combined with Theorem 1.4 in [Ber15a], gives us the following corollary:

Corollary 3.1.2. *Let $\alpha, \gamma \in [0,2)$, and let Z^γ denote a γ-Liouville Brownian motion. Then*

$$\dim_H(\{t : Z_t^\gamma \in \mathcal{T}_\alpha\}) = \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}},$$

almost surely.

Similar results for diffusions on deterministic fractals were given in [HK03].

The key to our proof is good estimates on the regularity of the time change F_γ around α-thick points. For a given α, we do not get the regularity results around all of the α-thick points, but we do get it around almost all of them, for the correct choice of measure. If we define the measure μ_α by setting, for $s \le t$,

$$\mu_\alpha([s,t]) = F_\alpha(t) - F_\alpha(s),$$

then we will show that F_γ behaves polynomially for μ_α-almost every t, in the following sense:

Theorem 3.1.3. *For μ_α-almost every $t > 0$, the change of time F_γ has the following growth rate:*

$$\lim_{r \to 0} \frac{\log|F_\gamma(t) - F_\gamma(t+r)|}{\log|r|} = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4},$$

almost surely.

When we combine the regularity of the time change function F_γ with known regularity properties of Brownian motion, we are able to find a bound on the small time behaviour of the LBM. Let us call M_α the Liouville measure constructed from a GFF with parameter α, which is formally defined as

$$M_\alpha(dz) = e^{\alpha h(z)} dz.$$

The measure M_α is almost surely supported on the set of α-thick points and so, if we choose a point in $\mathcal{D}$ according to M_α, it will almost surely be an α-thick point of the GFF.

Corollary 3.1.4. *Suppose that the starting point of a γ-Liouville Brownian motion is chosen according to M_α, i.e. $Z_0^\gamma \sim M_\alpha$. Then*

$$\limsup_{t \to 0} \frac{\log |Z_t^\gamma|}{\log t} = \frac{1}{2 - \alpha\gamma + \frac{\gamma^2}{2}},$$

almost surely.

Remark 3.1.5. When $\alpha = \gamma$ (which will be the typical case), the diffusivity exponent is $2 - \frac{\gamma^2}{2}$. Also observe that a single process can be both superdiffusive (e.g. when $\alpha = 0$) and subdiffusive (e.g. when $\alpha = \gamma$).

Finally, we will show the following result about the differentiability of a Liouville Brownian motion, for certain values of the parameter γ.

Corollary 3.1.6. *Let $\gamma \in (\sqrt{2}, 2)$. Then the γ-Liouville Brownian motion Z^γ is Lebesgue-almost everywhere differentiable with derivative zero, almost surely.*

3.1.2 Intuition behind the proof

Since we are looking at the dimension of times that a γ-LBM spent in α-thick points, it helps us to first note the following lemma. It is not used in the proofs of the main theorems, but it provides motivation for them.

Lemma 3.1.7. *Let $\alpha \in [0, 2)$. The Hausdorff dimension of time that a Brownian motion B spends in the α-thick points of a GFF is given by*

$$\dim_H(\{t : B_t \in \mathcal{T}_\alpha\}) = 1 - \frac{\alpha^2}{4},$$

almost surely.

Proof. Let $[B]$ denote the path of the Brownian motion B. Kauffman's dimension doubling formula for Brownian motion (see, for example, Theorem 9.28 of [MP10]), tells us that

$$2\dim_H(\{t : B_t \in \mathcal{T}_\alpha\}) = \dim_H(\mathcal{T}_\alpha \cap [B])$$

almost surely. But then, since B is independent of the GFF and hence $\mathcal{T}_\alpha$, by a theorem due to Hawkes [Haw71a, Haw71b] (clearly stated and proved as Corollary 5.2 in [Per96]), we know that

$$\dim_H(\mathcal{T}_\alpha \cap [B]) = \dim_H(\mathcal{T}_\alpha)$$

almost surely. The result in [HMP10] gives us that

$$\dim_H(\mathcal{T}_\alpha) = 2 - \frac{\alpha^2}{2}$$

almost surely, which completes our proof. □

Recall the result that, if a function f is β-Hölder continuous, then for any suitable set E we have the bound

$$\dim_H(f(E)) \leq \frac{1}{\beta}\dim_H(E). \tag{3.1}$$

Let us call the set $T_\alpha = \{t : B_t \in \mathcal{T}_\alpha\}$. Then notice that $F_\gamma(T_\alpha)$ is the set of time spent by γ-Liouville Brownian motion in α-thick points. We will show that the inverse of the change of time, F_γ^{-1} is $\frac{1}{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}}$-Hölder continuous around α thick points, allowing us to see that

$$1 - \frac{\alpha^2}{4} = \dim_H(T_\alpha) = \dim_H(F_\gamma^{-1}(F_\gamma(T_\alpha))) \leq \left(1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}\right)\dim_H(F_\gamma(T_\alpha)),$$

where the final inequality comes from a result very similar to that in (3.1). (We cannot use that result exactly, since the Hölder continuity property of F_γ^{-1} is restricted to a subset of its domain. We will discuss this further in Section 3.2.4.)

Rather than showing the regularity of F_γ^{-1} directly, we will find properties of F_γ and use them to deduce results about the inverse. However, showing the regularity properties of F_γ around a single thick point, while useful, is not enough. We want to look at the regularity of F_γ simultaneously around all α-thick points that the Brownian motion B visits. This is where we use the upper bound that was previously found in [Ber15a]. The upper bound is enough to show that a γ-LBM, Z^γ, spends Lebesgue-almost all of its time in γ-thick points, almost surely. Our trick, therefore, is to construct two Liouville Brownian motion processes simultaneously on the same underlying path B; one will use the parameter γ, the other will use the parameter α. Then, if we sample a time uniformly at random and look at where the process Z^α is, it will almost surely be an α-thick point. Since the process is constructed using the Brownian motion B, we know that B must pass through that particular α-thick point at some time t, say. But then we know that $F_\gamma(t)$ corresponds to a time that Z^γ is in an α-thick point.

Using this procedure, we can construct a measure on the (Euclidean) set of times that Z^γ spends in α-thick points. We can then sample a time at random from this measure, and look at the regularity properties of F_γ around that time. This idea is more thoroughly fleshed out in Section 3.3.3.

3.2 Setup

We will now collect a few of the definitions and results that we use throughout Section 3.3. Throughout, we let $\mathcal{D}$ be a simply connected, proper domain in $\mathbb{C}$. By conformal invariance of Liouville Brownian motion (including its clock process) we can assume without loss of generality that $\mathcal{D}$ is bounded. (See Theorem 1.3 in [Ber15a].)

3.2.1 *Gaussian free field*

We will briefly introduce the Gaussian free field here, mostly to clarify our notation. For more detail see, for example, [She07] or the introduction of [DS11].

Before we can define the GFF we need to define the Dirichlet inner product. For any two smooth, compactly supported functions ϕ and ψ defined on $\mathcal{D}$, we define the Dirichlet inner product as

$$\langle \phi, \psi \rangle_\nabla = \frac{1}{2\pi} \int_{\mathcal{D}} \nabla \phi(z) \cdot \nabla \psi(z) dz.$$

We can now define the Gaussian free field.

Definition 3.2.1. Let $H_0^1(\mathcal{D})$ be the Sobolev space given by the completion under the Dirichlet inner product of smooth, compactly supported functions defined on $\mathcal{D}$. The Gaussian free field is a centered Gaussian process on the space $H_0^1(\mathcal{D})$.

A consequence of the Hilbert space definition given above is that for any two functions $f, g \in H_0^1(\mathcal{D})$, the random variables $\langle h, f\rangle_\nabla$ and $\langle h, g\rangle_\nabla$ are centred Gaussian random variables with covariance

$$\mathrm{Cov}\big(\langle h, f\rangle_\nabla, \langle h, g\rangle_\nabla\big) = \langle f, g\rangle_\nabla .$$

This means that we can define a regularisation of the GFF, and we know about its covariance properties. Fore more details, see the discussion in Section 3.1, in particular Proposition 3.2 in [DS11].

Definition 3.2.2. The average of the GFF h on a circle of radius ε, centred at a point $z \in \{z' \in \mathcal{D} : \mathrm{dist}(z', \partial\mathcal{D}) > \varepsilon\}$ is defined as as

$$h_\varepsilon(z) = \langle h, \xi_\varepsilon^z\rangle_\nabla .$$

The function ξ_ε^z is given by

$$\xi_\varepsilon^z(y) = -\log(|z-y| \vee \varepsilon) + \overline{\phi_\varepsilon^z}(y), \tag{3.2}$$

where $\overline{\phi_\varepsilon^z}$ is harmonic in $\mathcal{D}$ and is equal to $\log(|z-y| \vee \varepsilon)$ for $y \in \partial\mathcal{D}$.

The reason that we think of the above definition as giving the circle average of the GFF is that, as a distribution, we have that $-\Delta\xi_\varepsilon^z = 2\pi\nu_\varepsilon^z$, where ν_ε^z is the uniform distribution on the circle centred at z with radius ε. Therefore, integration by parts gives us

$$\langle h, \xi_\varepsilon^z\rangle_\nabla = \langle h, \nu_\varepsilon^z\rangle ,$$

where $\langle\cdot, \cdot\rangle$ refers to the standard L^2 inner product. We will use a continuous modification of the circle average process $\{h_\varepsilon(z) ;\ \varepsilon > 0\}_{z\in\mathcal{D}}$ throughout. For more detail, see Propositions 3.1 and 3.2 in [DS11].

The following lemma will be useful in Section 3.3.2, as it allows us to use properties of the log function rather than relying on the abstract definition of $\mathbb{E}\left[\, h_\varepsilon(x)h_\eta(y)\,\right]$. It is a previously known result for fairly general log-correlated Gaussian fields, see Section 4.1 of [RV13] for example. We prove now that it holds in the specific case of the Gaussian free field.

Lemma 3.2.3. *Let h be a zero boundary GFF defined on a simply connected domain $\mathcal{D}$. For any subdomain $\tilde{\mathcal{D}}$ which is compactly contained in $\mathcal{D}$, there exists a constant $C > 0$ such that, for all $0 < \varepsilon, \eta \leq dist(\tilde{\mathcal{D}}, \partial\mathcal{D})$ with $\eta \leq \varepsilon$,*

$$\log\frac{1}{|x-y|+\varepsilon} - C \leq \mathbb{E}\left[\, h_\varepsilon(x)h_\eta(y)\,\right] \leq \log\frac{1}{|x-y|+\varepsilon} + C$$

for all $x, y \in \tilde{\mathcal{D}}$.

Proof. First note that, by definition, $\mathbb{E}[h_\varepsilon(x)h_\eta(y)] = \left\langle \xi_\varepsilon^x,\, \xi_\eta^y \right\rangle_\nabla$. Integration by parts lets us write that as $\left\langle \xi_\varepsilon^x,\, \xi_\eta^y \right\rangle_\nabla = \left\langle \xi_\varepsilon^x,\, \nu_\eta^y \right\rangle$. Since $x, y \in \tilde{\mathcal{D}}$ are uniformly bounded away from $\partial\mathcal{D}$, $\mathcal{D}$ is a bounded domain, and $\overline{\phi_\varepsilon^x}$ (defined in (3.2)) is harmonic in $\tilde{\mathcal{D}}$, we know that there exists a constant $\overline{C}$ such that

$$-\overline{C} \leq \overline{\phi_\varepsilon^x}(y) \leq \overline{C} \tag{3.3}$$

for all $x, y \in \tilde{\mathcal{D}}$ and $\varepsilon > 0$. So, to complete the proof, it is sufficient to find bounds on $\left\langle -\log(|x-\cdot| \vee \varepsilon),\, \nu_\eta^y \right\rangle$. To that end, we claim that, for all $x, y \in \mathcal{D}$ and $u \in \mathcal{D}$ such that $|u-y| \leq \eta$, we have

$$\frac{1}{3}\left(|x-y|+\varepsilon\right) \leq |x-u| \vee \varepsilon \leq |x-y|+\varepsilon. \tag{3.4}$$

The right hand inequality follows directly from the triangle inequality. For the left hand inequality, note that

$$\frac{1}{3}\left(|x-y|+\varepsilon\right) \leq \frac{1}{3}|x-u| + \frac{2}{3}\varepsilon$$

by the triangle inequality. Then, if $|x-u| \leq \varepsilon$, we see that

$$\frac{1}{3}|x-u| + \frac{2}{3}\varepsilon \leq \varepsilon = |x-u| \vee \varepsilon,$$

and if $|x-u| > \varepsilon$, we see that

$$\frac{1}{3}|x-u| + \frac{2}{3}\varepsilon \leq |x-u| = |x-u| \vee \varepsilon.$$

Now, the inequalities in (3.4) imply that, for all $x, y \in \tilde{\mathcal{D}}$ and $u \in \partial B(y, \eta)$, there exists some constant $\tilde{C}$ such that

$$-\log(|x-y|+\varepsilon) - \tilde{C} \le -\log\left(|x-u| \vee \varepsilon\right) \le -\log(|x-y|+\varepsilon) + \tilde{C}.$$

When we average over $u \in \partial B(y, \eta)$ therefore, we find that

$$-\log(|x-y|+\varepsilon) - \tilde{C} \le \left\langle -\log(|x-\cdot| \vee \varepsilon),\, \nu_\eta^z \right\rangle \le -\log(|x-y|+\varepsilon) + \tilde{C},$$

which, when we combine it with (3.3), completes the proof. □

One of the properties of the GFF which we will use is the domain Markov property. It roughly states that, given a subdomain $\mathcal{U} \subset \mathcal{D}$, the GFF h on $\mathcal{D}$ can be decomposed as the sum of a zero boundary GFF $\tilde{h}$ on $\mathcal{U}$ and the difference, $h^{har} = h - \tilde{h}$, which is independent of $\tilde{h}$ and harmonic on $\mathcal{U}$. The proof can be found in Section 2.6 of [She07].

Proposition 3.2.4 (Markov property). *Let $\mathcal{U} \subset \mathcal{D}$ be a subdomain of the simply connected domain $\mathcal{D}$. Let h be a GFF on $\mathcal{D}$. Then there exist two random variables $h^{har}, \tilde{h} \in H_0^{-\varepsilon}(\mathcal{D})$ such that we can write $h = h^{har} + \tilde{h}$, and*

1. *h^{har} and $\tilde{h}$ are independent,*
2. *$\tilde{h}$ is a zero boundary GFF on $\mathcal{U}$ and zero on $\mathcal{D} \setminus \mathcal{U}$,*
3. *h^{har} is harmonic on $\mathcal{U}$ and agrees with h on $\mathcal{D} \setminus \mathcal{U}$,*

Note 3.2.5. We will often refer to h^{har} in the decomposition above as "the harmonic projection of h onto $\mathcal{U}$."

We now define the set of α-thick points of the field h. We can think of these as a kind of "level set" of the field. We are interested in how much time the Liouville Brownian motion spends in these points, for $\alpha \in [0, 2)$ in particular.

Definition 3.2.6. The set of α-thick points, $\mathcal{T}_\alpha$ is

$$\mathcal{T}_\alpha = \left\{ z \in \mathcal{D} \,:\, \lim_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{-\log \varepsilon} = \alpha \right\}.$$

3.2.2 Scale invariant Gaussian field

To help with calculations in Section 3.3.1, we introduce a centred Gaussian field Y defined on the whole complex plane, following the presentation of [RV13]. We use this particular log-correlated field because it has the exact stochastic scale invariance property. A great deal more information about log-correlated Gaussian fields and the measures created from them (those of Gaussian multiplicative chaos) can be found in [Kah85, RV10] for example, and more about the scaling relations which log-normal random measures satisfy can be found in [ARV13].

Informally, we define field Y to be a centered Gaussian field on $\mathbb{C}$ with covariance function

$$\mathbb{E}[\, Y(x)Y(y)\,] = \log_+ \frac{T}{|x-y|} + C. \tag{3.5}$$

for positive constants T and C. For simplicity, we will take $T = 1$ and $C = 0$ throughout.

One of the difficulties here is that, with the covariance function (3.5), Y will not be defined pointwise. We therefore need to look at regularisations of Y. We will use is the white noise decomposition of the field Y. We now give a summary of the construction found in [RV10].

Let f be a real, positive definite function on $\mathbb{R}^d$. Then, to construct a Gaussian field X on with a covariance $\mathbb{E}[X(x)X(y)] = f(x-y)$, we can write

$$X(x) = \int_{(0,\infty)\times\mathbb{R}^d} \zeta(x,\xi)\sqrt{\hat{f}(\xi)}g(t,\xi)W(dt,d\xi),$$

where $\zeta(x,\xi) = \cos(2\pi x\cdot\xi) - \sin(2\pi x\cdot\xi)$, $\hat{f}$ is the Fourier transform of f, $W(dt,d\xi)$ is white noise on $(0,\infty)\times\mathbb{R}^d$, and g satisfies $\int_0^\infty g(t,\xi)^2 dt = 1$ for all ξ. With such a construction, we can define the ε-regularisation of the field X by setting

$$X_\varepsilon(x) = \int_{(\varepsilon,\infty)\times\mathbb{R}^d} \zeta(x,\xi)\sqrt{\hat{f}(\xi)}g(t,\xi)W(dt,d\xi).$$

The specific decomposition that we use is also used in [RV13], Appendix C. We give the precise definition:

Definition 3.2.7. Let $(Y_\varepsilon)_{\varepsilon\in(0,1]}$ be the white noise decomposition of the field Y, which has correlation structure

$$\mathbb{E}\left[Y_\varepsilon(x)Y_\varepsilon(y)\right] = \begin{cases} 0 & \text{if } |x-y| > 1 \\ \log\frac{1}{|x-y|} & \text{if } \varepsilon \le |x-y| \le 1 \\ \log\frac{1}{\varepsilon} + 2\left(1 - \frac{|x-y|^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right) & \text{if } |x-y| \le \varepsilon. \end{cases}$$

The following lemma is useful in the study of properties Gaussian multiplicative chaos locally in Y. It is known in the literature (see [RV13], Appendix C, for example), but we include the proof here for completeness.

Lemma 3.2.8 (Stochastic scale invariance)**.** *For all $\lambda < 1$, the field Y satisfies the following scaling relation:*

$$(Y_{\lambda\varepsilon}(\lambda x))_{|x|\le\frac{1}{2}} \overset{d}{=} (Y_\varepsilon(x))_{|x|\le\frac{1}{2}} + \Omega_\lambda, \tag{3.6}$$

where Ω_λ is a centred Gaussian random variable with variance $\log\frac{1}{\lambda}$*, independent of the field Y.*

Proof. We know that both the left and right hand sides of (3.6) are centred Gaussian fields. Therefore, to show the equality in distribution, we need only to show equality in their respective covariance functions.

We start with the right hand side. Let $x, y \in \mathbb{C}$. Then

$$\mathbb{E}\left[(Y_\varepsilon(x)+\Omega_\lambda)(Y_\varepsilon(y)+\Omega_\lambda)\right] = \begin{cases} \log\frac{1}{\lambda} & \text{if } |x-y| > 1 \\ \log\frac{1}{|x-y|} + \log\frac{1}{\lambda} & \text{if } \varepsilon \le |x-y| \le 1 \\ \log\frac{1}{\varepsilon} + 2\left(1 - \frac{|x-y|^{\frac{1}{2}}}{\varepsilon^{\frac{1}{2}}}\right) + \log\frac{1}{\lambda} & \text{if } |x-y| \le \varepsilon. \end{cases}$$

Now, consider the left hand side of (3.6). Again, let $x, y, \in \mathbb{C}$. Then

$$\mathbb{E}\left[Y_{\lambda\varepsilon}(\lambda x)Y_{\lambda\varepsilon}(\lambda y)\right] = \begin{cases} 0 & \text{if } |\lambda x - \lambda y| > 1 \\ \log\frac{1}{|\lambda x - \lambda y|} & \text{if } \lambda\varepsilon \le |\lambda x - \lambda y| \le 1 \\ \log\frac{1}{\lambda\varepsilon} + 2\left(1 - \frac{|\lambda x - \lambda y|^{\frac{1}{2}}}{(\lambda\varepsilon)^{\frac{1}{2}}}\right) & \text{if } |\lambda x - \lambda y| \le \lambda\varepsilon. \end{cases}$$

$$= \begin{cases} 0 & \text{if } |x - y| > 1/\lambda \\ \log\frac{1}{|x-y|} + \log\frac{1}{\lambda} & \text{if } \varepsilon \le |x - y| \le 1/\lambda \\ \log\frac{1}{\varepsilon} + 2\left(1 - \frac{|x-y|^{\frac{1}{2}}}{(\varepsilon)^{\frac{1}{2}}}\right) + \log\frac{1}{\lambda} & \text{if } |x - y| \le \varepsilon. \end{cases}$$

We can see that the covariance functions are equal, provided $|x - y| \le 1$. We want to show equality in distribution only inside the ball centred at the origin with radius $1/2$ and, since $|x - y| \le 1$ for all $x, y \in B(0, 1/2)$, the proof is complete. □

3.2.3 Liouville Brownian motion

The Liouville Brownian motion is defined as a time change of a Brownian motion, with the path chosen independently from the field h. We will start the Brownian motion at the origin (assuming $0 \in \mathcal{D}$), and run it until some a.s. finite stopping time T. The following definition is non-trivial: for more details about the almost sure existence of the limit and other properties, see [Ber15a, GRV13a].

Definition 3.2.9. Let B be a planar Brownian motion, independent of the field h. For $\varepsilon > 0$ and $\gamma \in [0, 2)$, define the regularised time change $F_{\gamma,\varepsilon}$ by

$$F_{\gamma,\varepsilon}(t) = \int_0^{t\wedge T} e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]}ds.$$

The time change F_γ is defined as the limit

$$F_\gamma(t) = \lim_{\varepsilon\to 0} F_{\gamma,\varepsilon}(t).$$

Definition 3.2.10. Using the same Brownian motion B as in Definition 3.2.9, we define the γ-Liouville Brownian motion (γ-LBM for short) Z^γ as

$$Z_t^\gamma = B_{F_\gamma^{-1}(t)}.$$

Two important properties of F_γ, shown in both [Ber15a] and [GRV13a], are that it is continuous and strictly increasing, almost surely. Those properties ensure that the γ-LBM does not get stuck, and is continuous. (See, for example, Proposition 2.8 and Theorem 2.10 in [GRV13a].)

Note 3.2.11. If we call $T_\alpha = \{t \geq 0 : B_t \in \mathcal{T}_\alpha\}$ the set of times that the Brownian motion B spends in α-thick points, the set of times that the γ-LBM Z^γ spends in α-thick points is the image, under the map F_γ, of the times that B spends in them, i.e. $F_\gamma(T_\alpha) = \left\{t \geq 0 : Z_t^\gamma \in \mathcal{T}_\alpha\right\}$.

As the Brownian path B of the Liouville Brownian motion Z^γ is independent of the GFF h, it will be useful to decompose the probability measure $\mathbb{P}$ as

$$\mathbb{P} = \mathbb{P}_B \otimes \mathbb{P}_h.$$

Decomposing $\mathbb{P}$ in this way will let us consider expectations on events which depend only on the field h or the path B.

3.2.4 Hausdorff dimension

We will now recall the definition of the Hausdorff measure of a set, and collect some useful tools for finding upper and lower bounds for the Hausdorff dimension. Since we will be working in either $\mathbb{R}$ or $\mathbb{R}^2$, we will not state the definitions in their full generality. For more detail see, for example, Chapter 4 of [MP10].

Definition 3.2.12. Let $E \subset \mathbb{R}^n$. For $s \geq 0$ and $\delta > 0$ we define

$$\mathcal{H}_\delta^s(E) = \inf\left\{\sum_{i=1}^\infty |E_i|^s \ :\ E \subset \bigcup_{i=1}^\infty E_i \ \text{ and } \ |E_i| < \delta \ \ \forall i \geq 1\right\},$$

where $|E_i| = \sup\{|x-y| : x, y \in E_i\}$ is the diameter of the set E_i. Then the limit

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \mathcal{H}_\delta^s(E)$$

is the s-Hausdorff measure of E.

Definition 3.2.13. The Hausdorff dimension of a set $E \subset \mathbb{R}^n$ is defined as

$$\dim_H(E) = \inf\{s \geq 0 \ : \ \mathcal{H}^s(E) = 0\}.$$

One tool for finding bounds on the Hausdorff dimension of a set is to use Hölder continuity properties of functions. Indeed, if $f : \mathbb{R}^n \to \mathbb{R}^m$ is β-Hölder continuous, then for any set $E \subset \mathbb{R}^n$ we have

$$\dim_H(f(E)) \leq \frac{1}{\beta}\dim_H(E), \tag{3.7}$$

where $f(E) = \{f(x) \ : \ x \in E\}$ is the image of E under f. The assumption of Hölder continuity is too strong for our purpose. We now define what we call a β-Hölder-like function, and show that the property is strong enough that the inequality in (3.7) still holds.

Definition 3.2.14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function, and let $E \subset \mathbb{R}$. We say that f is β-Hölder-like on E if there exist constants $C, R > 0$ such that

$$|f(x) - f(x+r)| \leq Cr^\beta$$

for all $r \in [0, R)$ and $x \in E$.

The following proposition is very similar to the well known result concerning Hausdorff dimension bounds for the image of a set under a Hölder continuous function. We have had to modify the standard proof slightly to ensure that the result still holds with Hölder-like functions.

Proposition 3.2.15. *Let $E \subset \mathbb{R}$, and suppose that $f : \mathbb{R} \to \mathbb{R}$ is increasing and β-Hölder-like on E. Then we have the bound*

$$\dim_H(f(E)) \leq \frac{1}{\beta}\dim_H(E).$$

Proof. Suppose that the radius and multiplicative constant for the Hölder-like property of f are R and C respectively. Let $s > \dim_H(E)$, and let $\varepsilon > 0$. Since $\mathcal{H}^s(E) = 0$, we know that there exists some δ_0 such that $\mathcal{H}^s_\delta(E) \leq \varepsilon$ for all $\delta \in (0, \delta_0)$.

Fix a particular $\delta \in (0, \delta_0 \wedge R)$. Then we can find a cover $\{E_i\}$ of E with $|E_i| < \delta$ for all i such that

$$\sum_{i=1}^{\infty} |E_i|^s < \varepsilon.$$

Without loss of generality, we may assume that the intersection $E \cap E_i$ is non-empty. Therefore, for each E_i we can define an interval $I_i = [a_i, b_i]$, where $a_i = \inf\{E_i \cap E\}$ and $b_i = \sup\{E_i \cap E\}$. Then certainly $|I_i| < \delta$ for all i, the sets $\{I_i\}$ cover E and $\sum |I_i|^s < \varepsilon$.

As f is increasing, we know that

$$|f(I_i)| = |f(a_i) - f(b_i)|.$$

Now, a_i is a limit point of $E_i \cap E$, so we can find a sequence $\{x_n\} \subset E_i \cap E$ such that $x_n \to a_i$ as $n \to \infty$. Since each $x_n \in E$, the β-Hölder-like property of f, tells us that

$$|f(I_i)| \leq |f(a_i) - f(x_n)| + |f(x_n) - f(b_i)| \leq |f(a_i) - f(x_n)| + C|x_n - b_i|^{\beta}.$$

So, letting $n \to \infty$, and recalling that f is continuous (by assumption), we see that

$$|f(I_i)| \leq C|a_i - b_i|^{\beta} = C|I_i|^{\beta}.$$

Therefore, we can deduce that

$$\sum_{i=1}^{\infty} |f(I_i)|^{\frac{s}{\beta}} \leq C \sum_{i=1}^{\infty} |I_i|^s < C\varepsilon.$$

Since $\{f(I_i)\}$ covers $f(E)$, we have shown that

$$\mathcal{H}_{\delta}^{\frac{s}{\beta}}(f(E)) < C\varepsilon.$$

We may now let $\delta \downarrow 0$ and then $\varepsilon \downarrow 0$ to see that $\mathcal{H}^{\frac{s}{\beta}}(f(E)) = 0$, and hence

$$\dim_H(f(E)) \leq \frac{s}{\beta}.$$

Now letting $s \downarrow \dim_H(E)$ gives the desired result. □

3.3 Proofs of the main theorems

One of the tools we use in the proof of the lower bound is the exact stochastic scale invariance of the auxiliary field Y as detailed in Lemma 3.2.8. However, Lemma 3.2.8 holds inside the ball $B(0, \frac{1}{2})$, and so we need to ensure that we consider times when the Brownian motion B does not stray too far from the origin. Therefore, we define the stopping time $\tau = \inf\left\{t \geq 0 \,:\, B_t \notin B(0, \frac{1}{2})\right\}$, where $B(0, \frac{1}{2})$ is the ball of radius $\frac{1}{2}$ centred at the origin. For simplicity, we will assume that our domain contains the ball of radius $\frac{1}{2}$, $B(0, \frac{1}{2}) \subset \mathcal{D}$.

3.3.1 *Moments of F_γ around a thick point*

We first need to obtain estimates on the moments of the time change F_γ around α-thick points of the free field. We will use these bounds in Section 3.3.3 to derive Hölder-like properties of F_γ.

Since the law of the GFF conditional on the origin being a thick point is that of an independent zero boundary GFF plus a log singularity, $h(z) \stackrel{d}{=} \tilde{h}(z) - \alpha \log|z|$, the effect on the measure is to divide by $|z|^{\alpha\gamma}$. That is why we are thinking the results in this section as results about F_γ close to thick points.

Proposition 3.3.1 (A positive moment is bounded). *Let $\alpha, \gamma \in [0, 2)$. Then, for some $p > 0$, there exists a finite constant C_p such that*

$$\sup_{\varepsilon \in [0,1)} \mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p\right] \leq C_p.$$

Proof. By Kahane's convexity inequality (Lemma 2 in [Kah85]), taking the measure $\nu(ds) = \frac{ds}{(|B_s|+\varepsilon)^{\alpha\gamma}}$, it is sufficient to prove the proposition for the scale invariant field Y.

Let σ be the first time that B leaves the disc of radius $\frac{1}{2\sqrt{2}}$. Then, by subadditivity of $x \mapsto x^p$

for $p \in (0,1)$, we know that

$$\mathbb{E}\left[\left(\int_0^\tau \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p\right] \le \mathbb{E}\left[\left(\int_0^\sigma \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p\right] + \mathbb{E}\left[\left(\int_\sigma^\tau \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p\right], \tag{3.8}$$

and it is sufficient to find a uniform upper bound for the right hand side.

We will first find a uniform bound for the second term on the right hand side of (3.8). Let $R < \frac{1}{2\sqrt{2}}$ be fixed, which we will choose later, and define the time $\tau_R = \inf\{t > \sigma : |B_t| \le R\}$ that the Brownian motion returns to the ball of radius R after it reaches the circle of radius $\frac{1}{2\sqrt{2}}$.

On the event $\{\tau_R > \tau\}$, we know that $|B_t| > R$ for all $t \in (\sigma, \tau)$. Therefore we find the bound

$$\begin{aligned}\mathbb{E}&\left[\left(\int_\sigma^\tau \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p \mathbb{1}_{\{\tau_R>\tau\}}\right] \le \\ &\le R^{-\alpha\gamma p}\mathbb{E}\left[\left(\int_\sigma^\tau e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}ds\right)^p \mathbb{1}_{\{\tau_R>\tau\}}\right] \\ &\le R^{-\alpha\gamma p}\mathbb{E}\left[\left(\int_\sigma^\tau e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}ds\right)^p\right]\end{aligned} \tag{3.9}$$

Now, the L^1 norm of the regularised change of time process is uniformly bounded in ε and, since $p < 1$, the expectation of the p^{th} power of it on the event must also be uniformly bounded in ε. We will call the uniform bound M.

On the event $\{\tau_R < \tau\}$ we will split up the interval (σ, τ) into (σ, τ_R) and (τ_R, τ), using the sub-additivity of $x \mapsto x^p$ as before to find that

$$\begin{aligned}\mathbb{E}&\left[\left(\int_\sigma^\tau \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p \mathbb{1}_{\{\tau_R<\tau\}}\right] \le \\ &\le \mathbb{E}\left[\left(\int_\sigma^{\tau_R} \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p \mathbb{1}_{\{\tau_R<\tau\}}\right] + \\ &+ \mathbb{E}\left[\left(\int_{\tau_R}^\tau \frac{e^{\gamma Y_\varepsilon(B_s)-\frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}}ds\right)^p \mathbb{1}_{\{\tau_R<\tau\}}\right],\end{aligned} \tag{3.10}$$

Consider the first term on the right hand side of (3.10). Similarly to before, we know that $|B_t| > R$ for all $t \in (\sigma, \tau_R)$, and so we can bound the expectation uniformly by

$$\mathbb{E}\left[\left(\int_\sigma^{\tau_R} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}}\right] \le$$

$$\begin{aligned} &\le R^{-\alpha\gamma p}\mathbb{E}\left[\left(\int_\sigma^{\tau_R} e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}}\right] \\ &\le R^{-\alpha\gamma p}\mathbb{E}\left[\left(\int_\sigma^{\tau} e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]} ds\right)^p\right] \\ &\le R^{-\alpha\gamma p} M. \end{aligned} \tag{3.11}$$

To deal with the interval (τ_R, τ), let W be another Brownian motion, with $W_0 = B_{\tau_R}$ and which, for $t > 0$, evolves independently of B. Let $T = \inf\left\{t > 0 \ : \ W_t \notin B(0, \frac{1}{2})\right\}$. Then, by the strong Markov property of Brownian motion, we see that

$$\left(\int_{\tau_R}^{\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}} \overset{d}{=} \left(\int_0^{T} \frac{e^{\gamma Y_\varepsilon(W_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(W_s)^2]}}{(|W_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}},$$

and that $\int_0^T \frac{e^{\gamma Y_\varepsilon(W_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(W_s)^2]}}{(|W_s|+\varepsilon)^{\alpha\gamma}} ds$ is independent of $\mathbb{1}_{\{\tau_R < \tau\}}$. Therefore, we see that

$$\mathbb{E}\left[\left(\int_{\tau_R}^{\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}}\right] =$$

$$= \mathbb{E}\left[\left(\int_0^{T} \frac{e^{\gamma Y_\varepsilon(W_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(W_s)^2]}}{(|W_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p\right] \mathbb{P}[\, \tau_R < \tau \,]. \tag{3.12}$$

Because we started W closer to the boundary of $B(0, \frac{1}{2})$ we know that, on average, it has a shorter lifespan than B, and so

$$\mathbb{E}\left[\left(\int_0^{T} \frac{e^{\gamma Y_\varepsilon(W_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(W_s)^2]}}{(|W_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p\right] \le \mathbb{E}\left[\left(\int_0^{\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p\right]. \tag{3.13}$$

Combining (3.12) and (3.13) gives us the bound

$$\mathbb{E}\left[\left(\int_{\tau_R}^{\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p \mathbb{1}_{\{\tau_R < \tau\}}\right] \le$$

$$\le \mathbb{E}\left[\left(\int_0^{\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds\right)^p\right] \mathbb{P}[\, \tau_R < \tau \,]. \tag{3.14}$$

Now, consider the first term on the right hand side of (3.8). Scaling time by a factor of $\frac{1}{2}$ and space by a factor of $\frac{1}{\sqrt{2}}$ gives

$$\begin{aligned}
\int_0^{\sigma} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}} ds &= 2^{-1} \int_0^{2\sigma} \frac{e^{\gamma Y_\varepsilon(B_{u/2}) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_{u/2})^2]}}{\left(|B_{u/2}|+\varepsilon\right)^{\alpha\gamma}} du \\
&\overset{d}{=} 2^{-(1-\frac{\alpha\gamma}{2})} \int_0^{\tilde{\tau}} \frac{e^{\gamma Y_\varepsilon(\frac{1}{\sqrt{2}}\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(\frac{1}{\sqrt{2}}\tilde{B}_u)^2]}}{\left(|\tilde{B}_u|+\varepsilon\sqrt{2}\right)^{\alpha\gamma}} du \\
&\overset{d}{=} 2^{-(1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4})} e^{\gamma\Omega_{\sqrt{2}}} \int_0^{\tilde{\tau}} \frac{e^{\gamma Y_{\varepsilon\sqrt{2}}(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[Y_{\varepsilon\sqrt{2}}(\tilde{B}_u)^2]}}{\left(|\tilde{B}_u|+\varepsilon\sqrt{2}\right)^{\alpha\gamma}} du,
\end{aligned}$$

the last line coming from Lemma 3.2.8, where $\tilde{B}$ is an independent Brownian motion, $\tilde{\tau}$ is the time that $\tilde{B}$ leaves the disc of radius $\frac{1}{2}$, and $\Omega_{\sqrt{2}}$ is a centred Gaussian random variable with variance $\log\sqrt{2}$. Therefore, when we take the pth moment, we find

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^{\sigma} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_\varepsilon(B_s)^2]}}{(|B_s|+\varepsilon)^{\alpha\gamma}} ds\right)^p\right] = \\
&\quad = 2^{\frac{\gamma^2}{4}p^2 - (1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4})p} \mathbb{E}\left[\left(\int_0^{\tilde{\tau}} \frac{e^{\gamma Y_{\varepsilon\sqrt{2}}(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[Y_{\varepsilon\sqrt{2}}(\tilde{B}_u)^2]}}{\left(|\tilde{B}_u|+\varepsilon\sqrt{2}\right)^{\alpha\gamma}} du\right)^p\right]. \qquad (3.15)
\end{aligned}$$

Let us define a sequence of scales by setting $\varepsilon_n = 2^{-\frac{n}{2}}$, for $n \in \mathbb{N}$, and call the expectation

$$E_n = \mathbb{E}\left[\left(\int_0^{\tau} \frac{e^{\gamma Y_{\varepsilon_n}(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y_{\varepsilon_n}(B_s)^2]}}{(|B_s|+\varepsilon_n)^{\alpha\gamma}} ds\right)^p\right].$$

Using this notation, we substitute the scaling relation in (3.15), the uniform bounds in (3.9) and (3.11), and the inequality in (3.14) into (3.8), to see that, for $n \geq 1$,

$$E_n \leq 2^{\frac{\gamma^2}{4}p^2 - (1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4})p} E_{n-1} + 2R^{-\alpha\gamma p} M + E_n \mathbb{P}[\tau_R < \tau]. \qquad (3.16)$$

Upon re-arrangement, the inequality in (3.16) becomes

$$E_n \leq \left(\frac{2^{\frac{\gamma^2}{4}p^2 - (1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4})p}}{1-\mathbb{P}[\tau_R < \tau]}\right) E_{n-1} + \frac{2R^{-\alpha\gamma p} M}{1-\mathbb{P}[\tau_R < \tau]}.$$

By choosing $p > 0$ first and then $R > 0$ fixed and small enough, we can ensure that the factor multiplying E_{n-1}, $\frac{2^{\frac{\gamma^2}{4}p^2 - (1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4})p}}{1-\mathbb{P}[\tau_R<\tau]}$, is less than 1. When we have done this, what we have shown is that, for some $\rho \in (0,1)$ and some constant $\tilde{M}$, we have $E_n < \rho E_{n-1} + \tilde{M}$, which implies that the sequence $\{E_n\}_{n\in\mathbb{N}}$ is bounded. □

Proposition 3.3.2 (A negative moment is bounded). *Let $\alpha, \gamma \in [0,2)$. Then there exists a finite constant C_{-1} such that*

$$\sup_{\varepsilon\in[0,1)} \mathbb{E}\left[\left(\int_0^{1\wedge\tau} \frac{e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}}\right)^{-1}\right] \le C_{-1}.$$

Proof. Again, using Kahane's convexity inequality (Lemma 2 of [Kah85]), it is sufficient to prove this result for the scale invariant field Y. We know that $|B_s| + \varepsilon \le \frac{3}{2}$ for all $s \in (0,\tau)$ and for all $\varepsilon \in [0,1)$, and so we find the bound

$$\sup_{\varepsilon\in[0,1)} \mathbb{E}\left[\left(\int_0^{1\wedge\tau} \frac{e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}}\right)^{-1}\right] \le$$
$$\le \left(\frac{3}{2}\right)^{\alpha\gamma} \sup_{\varepsilon\in[0,1)} \mathbb{E}\left[\left(\int_0^{1\wedge\tau} e^{\gamma Y_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[Y(B_s)^2]}\right)^{-1}\right]. \tag{3.17}$$

By Lemmas 2.13 and 2.14 of [GRV13a], the right hand side of (3.17) is finite, and so we are done. □

The following corollaries will be useful in Section 3.3.3.

Corollary 3.3.3. *For any power q, we have a polynomial bound on the probability*

$$\mathbb{P}\left[\int_0^{1\wedge\tau} \frac{e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds \le r^q\right] \le C_{-1} r^q$$

for any $r > 0$.

Corollary 3.3.4. *There exists some $p > 0$ such that, for any q, we have a polynomial bound on the probability*

$$\mathbb{P}\left[\int_0^{\tau} \frac{e^{\gamma h_\varepsilon(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]}}{(|B_s| + \varepsilon)^{\alpha\gamma}} ds \ge r^{-q}\right] \le C_p r^{pq}$$

for any $r > 0$.

The proof of both of these are simple applications of Markov's inequality.

3.3.2 Scaling of a Gaussian free field

As well as the polynomial behaviour of the tails of F_γ around thick points that we saw in Corollaries 3.3.3 and 3.3.4, we also need a bound on the tail behaviour on the supremum (or infimum) of the harmonic projection of a GFF on a disc of radius $\sqrt{r}$.

We were able to use the scale invariant field Y throughout Section 3.3.1 because the moments we were trying to bound were convex (or concave) functions of a multiplicative chaos measure, and so Kahane's convexity theorem let us change between fields. In Section 3.3.3, however, we need to consider moments of certain integrals of the field, weighted by indicator functions depending on the field itself. Kahane's convexity theorem therefore no longer applies, and we have to work directly with the GFF. So, we need an analogue of the scaling property that the field Y has.

Lemma 3.3.5. *Let $\mathcal{D} \subset \mathbb{C}$ be a bounded proper domain, and let $\tilde{\mathcal{D}} \subset \mathcal{D}$ be a compactly contained subdomain of $\mathcal{D}$. Let h be a zero-boundary condition GFF on $\mathcal{D}$. Now, using the Markov property (Proposition 3.2.4), let us write*

$$h = h^{har} + \tilde{h},$$

where h^{har} is the harmonic projection of h onto the disc of radius $2\sqrt{r}$ and centred at $x \in \tilde{\mathcal{D}}$, and $\tilde{h}$ is an independent, zero-boundary condition GFF on the disc of radius $2\sqrt{r}$ and centred at $x \in \tilde{\mathcal{D}}$. Let

$$\Omega^x_{\sqrt{r}} = \sup_{z \in B(x,\sqrt{r})} h^{har}(z)$$

be the supremum of h^{har} on $B(x,\sqrt{r})$. Then there exist constants $C, p > 0$ such that, for all $r > 0$ small enough,

$$\sup_{x \in \tilde{\mathcal{D}}} \mathbb{P}\left[\Omega^x_{\sqrt{r}} > -\log r\right] \leq C r^p.$$

Proof. The proof is essentially the same as the proof of Kolmogorov's continuity criterion. Instead of taking limits to obtain almost sure results, however, we obtain quantitative estimates which hold with high probability. First, let us fix $x \in \tilde{\mathcal{D}}$. From the proof of Proposition 3.1 in [DS11], we know that there exists some constant K_2 such that, for $z, w \in \mathcal{D}$ and $\varepsilon > 0$,

$$\mathbb{E}\left[|h_{\varepsilon\sqrt{r}}(z) - h_{\varepsilon\sqrt{r}}(w)|^2\right] \leq K_2|z-w|. \tag{3.18}$$

Now, recall that h^{har} is harmonic on $B(x, 2\sqrt{r})$ and so, using the mean value property of harmonic functions, we see that the circle average regularisation of h^{har} is equal to h^{har} on sets inside $B(x, 2\sqrt{r})$, i.e. $h^{har}_{\varepsilon\sqrt{r}}(z) = h^{har}(z)$ for ε small enough and $z \in B(x, \sqrt{r})$. Therefore, we know that

$$h_{\varepsilon\sqrt{r}}(z) = h^{har}(z) + \tilde{h}_{\varepsilon\sqrt{r}}(z). \tag{3.19}$$

Equation (3.19) and the independence of h^{har} and $\tilde{h}$, when combined with (3.18) shows us that

$$\mathbb{E}\left[|h^{har}(z) - h^{har}(w)|^2\right] \leq \mathbb{E}\left[|h_{\varepsilon\sqrt{r}}(z) - h_{\varepsilon\sqrt{r}}(w)|^2\right] \leq K_2|z-w| \tag{3.20}$$

for $z, w \in B(x, \sqrt{r})$. Because the field h does not depend on our choice of x, we can see that the middle term in (3.20) is independent of x, and therefore so is the constant K_2.

Because $h^{har}(z) - h^{har}(w)$ is a Gaussian random variable, (3.20) implies that for any $\eta > 0$, there exists a constant K_η such that

$$\mathbb{E}\left[|h^{har}(z) - h^{har}(w)|^\eta\right] \leq K_\eta|z-w|^{\eta/2} \tag{3.21}$$

for $z, w \in B(x, \sqrt{r})$. Again, K_η is independent of x by the independence of K_2 from x.

Because the supremum of a harmonic function on a domain is attained at the boundary of that domain, we need only consider $z \in \partial B(0, \sqrt{r})$. Therefore, let us set

$$X_t = h^{har}\left(x + \sqrt{r}e^{2\pi i t}\right),$$

for $t \in [0,1]$. From the inequality in (3.21), we deduce that for $s, t \in [0,1]$

$$\begin{aligned}\mathbb{E}\left[|X_s - X_t|^\eta\right] &\leq K_\eta|\sqrt{r}\left(e^{2\pi i s} - e^{2\pi i t}\right)|^{\eta/2} \\ &\leq \tilde{K}_\eta r^{\eta/2}|s-t|^{\eta/2}.\end{aligned}$$

The bound on the covariance structure of X clearly does not depend on the choice of x.

Now that we have a bound on the η-moment of the increments, we can use Markov's inequality to say things about the probability that the process X is irregular. So, let $D_n = \{k2^{-n} : k = 0, 1, \ldots 2^n\}$ be the set of dyadic points in the unit interval at level n. For some power p, to be chosen later, we have

$$\begin{aligned}\mathbb{P}\left[|X_{k2^{-n}} - X_{(k+1)2^{-n}}| > 2^{-np}r^{1/4}\right] &\leq 2^{np\eta}r^{-\eta/4}\mathbb{E}\left[|X_{k2^{-n}} - X_{(k+1)2^{-n}}|^{\eta}\right] \\ &\leq \tilde{K}_{\eta}r^{\eta/4}2^{np\eta}|k2^{-n} - (k+1)2^{-n}|^{\eta/2} \\ &\leq \tilde{K}_{\eta}r^{\eta/4}2^{-n(\frac{1}{2}\eta - p\eta)},\end{aligned}$$

for $k = 0, 1, \ldots, 2^n - 1$. Therefore, a simple union bound shows that

$$\mathbb{P}\left[\sup_k |X_{k2^{-n}} - X_{(k+1)2^{-n}}| > 2^{-np}r^{1/4}\right] \leq \tilde{K}_{\eta}r^{\eta/4}2^{-n(\frac{1}{2}\eta - p\eta - 1)}.$$

If we choose $0 < p < \frac{1}{2}$ and η sufficiently large, we find that $q := \frac{1}{2}\eta - p\eta - 1 > 0$. Because we have ensured that $q > 0$, we can again use the union bound to find that

$$\begin{aligned}\mathbb{P}\left[\sup_{n\geq 0}\sup_k |X_{k2^{-n}} - X_{(k+1)2^{-n}}| > 2^{-np}r^{1/4}\right] &\leq \tilde{K}_{\eta}r^{\eta/4}\sum_{n\geq 0}2^{-nq} \\ &= \tilde{K}_{\eta}r^{\eta/4}\left(\frac{1}{1-2^{-q}}\right) \\ &= \overline{K}_{\eta}r^{\eta/4}.\end{aligned}$$

So, we see that the event

$$A := \left\{X_t \text{ is } p\text{-Hölder continuous with constant } r^{1/4}\right\}$$

occurs with probability greater than $1 - \overline{K}_{\eta}r^{\eta/2}$. On that event we can see that

$$|\sup_t X_t - \inf_t X_t| \leq r^{1/4}. \tag{3.22}$$

Using (3.22), we can find a bound for $\Omega^x_{\sqrt{r}}$ in terms of objects we have good control over. Specifically, we have

$$\Omega^x_{\sqrt{r}} \leq |\sup_t X_t - \inf_t X_t| + |\overline{X}|,$$

where $\overline{X}$ is the mean value of the process X_t. Let us consider that second term. Since X is really just h^{har} on a circle, and h^{har} is harmonic, we can use the mean value theorem to see that $\overline{X} = h^{har}(x)$. But again, we can apply the mean value theorem to see that $h^{har}(x)$ is really just the average of h on $\partial B(x, 2\sqrt{r})$, i.e. $h^{har}(x) = h_{2\sqrt{r}}(x)$. So, we have the inequality

$$\mathbb{P}\left[\Omega_{\sqrt{r}} \geq -\log r\right] \leq \mathbb{P}\left[|\sup_t X_t - \inf_t X_t| \geq -\frac{1}{2}\log r\right] + \mathbb{P}\left[h_{2\sqrt{r}}(x) \geq -\frac{1}{2}\log r\right].$$

Because $h_{2\sqrt{r}}(x) \sim N(0, -\log 2\sqrt{r} + \log C(x, \mathcal{D}))$, we know that the second term on the right hand side decays polynomially in r as $r \to 0$. Furthermore, since the conformal radius, $C(x, \mathcal{D})$, is bounded for $x \in \tilde{\mathcal{D}}$, the coefficients we choose in the polynomial bound can be chosen to hold uniformly for all $x \in \tilde{\mathcal{D}}$.

So now let us consider the first term.

$$\begin{aligned}
\mathbb{P}\left[|\sup_t X_t - \inf_t X_t| \geq -\frac{1}{2}\log r\right] &= \mathbb{P}\left[\left\{|\sup_t X_t - \inf_t X_t| \geq -\frac{1}{2}\log r\right\} \cap A\right] \\
&\quad + P\left[\left\{|\sup_t X_t - \inf_t X_t| \geq -\frac{1}{2}\log r\right\} \cap A^c\right] \\
&\leq \mathbb{P}\left[r^{1/4} \geq -\frac{1}{2}\log r\right] + \mathbb{P}\left[A^c\right] \\
&\leq 0 + \overline{K}_\eta r^{\eta/2},
\end{aligned}$$

for r small enough. Since $\overline{K}_\eta$ does not depend on $x \in \tilde{\mathcal{D}}$, we have the desired result. □

3.3.3 *Hölder-like properties of F_γ*

We will now show the required regularity properties of the time change function F_γ. It will be convenient to introduce the following measures.

Definition 3.3.6. Let μ_γ be the measure on the interval $[0, \tau]$ defined by $\mu_\gamma = \mathcal{L} \circ F_\gamma^{-1}$, where $\mathcal{L}$ is Lebesgue measure on the interval $[0, F_\gamma(\tau)]$. In other words, for $s, t \in [0, \tau]$ with $s \leq t$, we set

$$\mu_\gamma([s, t]) = F_\gamma(t) - F_\gamma(s).$$

Define the measure μ_α in a similar way, for $\alpha \in [0, 2)$.

Remark 3.3.7. To get some intuition behind the next few results, let us think of the measures μ_α and μ_γ as probability measures for a moment. Then, if we sample a time $t \in [0, \tau]$ according to μ_α, it will almost surely be such that the Brownian motion B is in an α-thick point, i.e. $t \in T_\alpha$ almost surely, because the α-LMB, Z^α, spends Lebesgue-almost all of its time in α-thick points. Similar statements hold if we sample a time from μ_γ.

Proposition 3.3.8. *For all $\alpha \in [0,2)$, and $\gamma \in [0,2)$, fix $\delta > 0$, and let $\beta = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}$. Define the set of times*

$$L_\gamma^N = \left\{ t \in [0,\tau] \,:\, \mu_\gamma([(t,(t+r)\wedge\tau]) \geq r^{\beta+\delta} \quad \forall r \in [0, 2^{-N}) \right\}.$$

Then for all $\Delta > 0$, which may be random and may depend on $\mu_\alpha([0,\tau])$, there exists some random but almost surely finite $N \in \mathbb{N}$ such that

$$\mu_\alpha\left(L_\gamma^N\right) \geq \mu_\alpha\left([0,\tau]\right) - \Delta.$$

This proposition is essentially saying that if t is an α-thick time, then the μ_γ mass of an interval of length r, starting at t, decays more slowly than $r^{\beta+\delta}$. It is *almost* like saying that around α-thick points, the map function F_γ^{-1} is $\frac{1}{\beta+\delta}$-Hölder continuous.

The proof will rely on the following lemma:

Lemma 3.3.9. *As before, fix $\delta > 0$ and let $\beta = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}$, and let $E > 0$ be some positive constant. Then there exist two constants $D > 0$ and $q > 0$ such that*

$$\mathbb{E}\Big[\, \mu_\alpha\left(\left\{ t \in [0,\tau] \,:\, \mu_\gamma([t,(t+r)\wedge\tau]) < E r^{\beta+\delta} \right\}\right) \Big] \leq D r^q.$$

for all $r \in (0,1)$.

Proof. To ease notation, we will prove the case $E = 1$. The reader will be able to see that the same argument works for any positive E, with possibly different constants D and q.

Let $r \geq 0$ and fix $\varepsilon, \varepsilon' > 0$ so that $\varepsilon' < \varepsilon\sqrt{r}$. Then by Girsanov's change of measure theorem, we get

$$\mathbb{E}_B\mathbb{E}_h\Big[\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}} e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}\mathbb{1}_{\{\tau>t\}}\Big] =$$

$$= \mathbb{E}_B\left[\mathbb{1}_{\{\tau>t\}}\mathbb{P}_h\left[\int_t^{(t+r)\wedge\tau} e^{\gamma\left(h_{\varepsilon\sqrt{r}}(B_s)+\alpha\mathbb{E}_h\left[h_{\varepsilon\sqrt{r}}(B_s)h_{\varepsilon'}(B_t)\right]\right)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}ds < r^{\beta+\delta}\right]\right]$$
$$= \mathbb{E}_B\left[\mathbb{1}_{\{\tau>t\}}\mathbb{E}_B\left[\mathbb{P}_h\left[\int_t^{(t+r)\wedge\tau} e^{\gamma\left(h_{\varepsilon\sqrt{r}}(B_s)+\alpha\mathbb{E}_h\left[h_{\varepsilon\sqrt{r}}(B_s)h_{\varepsilon'}(B_t)\right]\right)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}ds < r^{\beta+\delta}\right]\middle|\mathcal{F}_t\right]\right] \tag{3.23}$$

where $\mathcal{F}_t = \sigma(B_s;\ s \le t)$ is the natural filtration for B. Now, using Lemma 3.2.3, we know that almost surely on the event $s, t < \tau$, there is a constant C such that

$$\mathbb{E}_h\left[h_{\varepsilon\sqrt{r}}(B_s)h_{\varepsilon'}(B_t)\right] \ge \log\frac{1}{|B_s - B_t| + \varepsilon\sqrt{r}} - C,$$

and so we can bound the integral in (3.23) from below by

$$\int_t^{(t+r)\wedge\tau} e^{\gamma\left(h_{\varepsilon\sqrt{r}}(B_s)+\alpha\mathbb{E}_h\left[h_{\varepsilon\sqrt{r}}(B_s)h_{\varepsilon'}(B_t)\right]\right)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}ds \ge$$
$$\ge e^{-\alpha\gamma C}\int_t^{(t+r)\wedge\tau}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}}{(|B_s - B_t| + \varepsilon\sqrt{r})^{\alpha\gamma}}ds \tag{3.24}$$

Now, by changing variables and using the scaling properties of Brownian motion we see that the right hand side of (3.24) becomes

$$\begin{aligned}
\int_t^{(t+r)\wedge\tau}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}}{(|B_s - B_t| + \varepsilon\sqrt{r})^{\alpha\gamma}}ds &= \int_0^{r\wedge(\tau-t)}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(B_{t+s})-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_{t+s})^2\right]}}{(|B_{t+s} - B_t| + \varepsilon\sqrt{r})^{\alpha\gamma}}ds \\
&= r\int_0^{1\wedge((\tau-t)/r)}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(B_{t+ru})-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_{t+ru})^2\right]}}{(|B_{t+ru} - B_t| + \varepsilon\sqrt{r})^{\alpha\gamma}}du \\
&\overset{d}{=} r\int_0^{1\wedge\tau'}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)^2\right]}}{(|\sqrt{r}\tilde{B}_u| + \varepsilon\sqrt{r})^{\alpha\gamma}}du \\
&= r^{1-\frac{\alpha\gamma}{2}}\int_0^{1\wedge\tau'}\frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)^2\right]}}{(|\tilde{B}_u| + \varepsilon)^{\alpha\gamma}}du,
\end{aligned}$$

where $\tilde{B}$ is an independent Brownian motion started at the origin, and

$$\tau' = \inf\left\{u > 0 : |\sqrt{r}\tilde{B}_u + B_t| = \frac{1}{2}\right\}$$

is the first time $\sqrt{r}\tilde{B}_u + B_t$ exits the disc of radius 1. The equality in distribution holds $\mathbb{P}_h$-almost surely. In order to use the scaling property of the GFF h, (Lemma 3.3.5), we also need to make sure that $\tilde{B}$ stays bounded. So, let

$$\tilde{\tau} = \tau' \wedge \inf\left\{u > 0 : |\tilde{B}_u| = \frac{1}{2}\right\}.$$

Then we certainly know that

$$\begin{aligned} r^{1-\frac{\alpha\gamma}{2}} \int_0^{1\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \geq \\ \geq r^{1-\frac{\alpha\gamma}{2}} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du. \end{aligned} \tag{3.25}$$

Now let us use the fact that we are conditioning on $\mathcal{F}_t$ and the Markov property of h to write $h = h^{har} + \tilde{h}$, where h^{har} is the harmonic projection of h onto the disc of radius $2\sqrt{r}$, centred at B_t, and $\tilde{h}$ has the law of a zero-boundary GFF on the disc of radius $2\sqrt{r}$, centred at B_t. If we write $\Omega_{\sqrt{r}} = \inf_{z\in B(B_t,\sqrt{r})} h^{har}(z)$, we know that $h \geq \Omega_{\sqrt{r}} + \tilde{h}$ inside the disc $B(B_t, \sqrt{r})$, and so we can continue from (3.25) to see that

$$\begin{aligned} r^{1-\frac{\alpha\gamma}{2}} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \geq \\ \geq r^{1-\frac{\alpha\gamma}{2}} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)-\frac{\gamma^2}{2}(-\log(\varepsilon\sqrt{r})+C)}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \\ = r^{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}} e^{-\frac{\gamma^2}{2}C} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma h_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)+\frac{\gamma^2}{2}\log\varepsilon}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \\ \geq r^{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}} e^{-\frac{\gamma^2}{2}C} e^{\gamma\Omega_{\sqrt{r}}} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma \tilde{h}_{\varepsilon\sqrt{r}}(\sqrt{r}\tilde{B}_u+B_t)+\frac{\gamma^2}{2}\log\varepsilon}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \\ \geq r^{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}} e^{-\gamma^2 C} e^{\gamma\Omega_{\sqrt{r}}} \int_0^{1\wedge\tilde{\tau}} \frac{e^{\gamma h'_\varepsilon(\tilde{B}_u)-\frac{\gamma^2}{2}\mathbb{E}\left[h'_\varepsilon(\tilde{B}_u)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}} du \end{aligned} \tag{3.26}$$

where h' is a zero boundary GFF on the disc of radius $\frac{1}{2}$. Substituting the last expression of

(3.26) back into (3.23) (and noticing that the exponent of r is in fact β) lets us see that

$$\mathbb{P}\left[\int_t^{(t+r)\wedge\tau} e^{\gamma\left(h_{\varepsilon\sqrt{r}}(B_s)+\alpha\mathbb{E}_h\left[h_{\varepsilon\sqrt{r}}(B_s)h_{\varepsilon'}(B_t)\right]\right)-\frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{r}}(B_s)^2\right]}ds < r^{\beta+\delta}\middle|\mathcal{F}_t\right] \leq$$

$$\leq \mathbb{P}\left[e^{-(\alpha\gamma+\gamma^2)C}r^{\beta}e^{\gamma\Omega_{\sqrt{r}}}\int_0^{1\wedge\tilde{\tau}}\frac{e^{\gamma h'_\varepsilon(\tilde{B}_u)-\frac{\gamma^2}{2}\mathbb{E}\left[h'_\varepsilon(\tilde{B}_u)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}}du < r^{\beta+\delta}\middle|\mathcal{F}_t\right]$$

$$\leq \mathbb{P}\left[e^{-(\alpha\gamma+\gamma^2)C}e^{\gamma\Omega_{\sqrt{r}}} < r^{\frac{\delta}{2}}\middle|\mathcal{F}_t\right]+\mathbb{P}\left[\int_0^{1\wedge\tilde{\tau}}\frac{e^{\gamma h'_\varepsilon(\tilde{B}_u)-\frac{\gamma^2}{2}\mathbb{E}\left[h'_\varepsilon(\tilde{B}_u)^2\right]}}{(|\tilde{B}_u|+\varepsilon)^{\alpha\gamma}}du < r^{\frac{\delta}{2}}\right]. \qquad (3.27)$$

The first term decays polynomially in r as $r \to 0$ uniformly in B_t, by Lemma 3.3.5, and the second term decays polynomially in r as $r \to 0$ by Corollary 3.3.3. Therefore, looking back at (3.23) again, there certainly exist some positive constants D and q such that

$$\mathbb{E}\left[\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau}e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}\mathbb{1}_{\{\tau>t\}}\right] \leq$$

$$\leq \mathbb{E}\left[Dr^q\mathbb{1}_{\{\tau>t\}}\right]$$

$$= Dr^q\mathbb{P}[\tau > t]. \qquad (3.28)$$

When we integrate (3.28) over $t > 0$, we find that

$$\mathbb{E}\left[\int_0^{\tau}\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau}e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}dt\right] \leq \mathbb{E}[\tau]\,Dr^q. \qquad (3.29)$$

Proposition 2.8 of [GRV13a] tells us that, almost surely in B and h, the measure defined by $\mu_\alpha^{\varepsilon'}(dt) = e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}dt$ converges weakly to the measure we have called μ_α. Therefore, as the set in the indicator function is open, we may use the portmanteau lemma and Fatou's lemma to see that

$$\mathbb{E}\left[\int_0^{\tau}\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau}e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}\mu_\alpha(dt)\right] \leq$$

$$\leq \mathbb{E}\left[\liminf_{\varepsilon'\to 0}\int_0^{\tau}\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau}e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}dt\right]$$

$$\leq \liminf_{\varepsilon'\to 0}\mathbb{E}\left[\int_0^{\tau}\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau}e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}\left[h_{\varepsilon'}(B_t)^2\right]}dt\right].$$

Since D and q from (3.29) are independent of ε', we can therefore see that

$$\mathbb{E}\left[\int_0^\tau \mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}\mu_\alpha(dt)\right] \le \mathbb{E}[\,\tau\,]\,Dr^q.$$

We then use Fatou's lemma twice to conclude

$$\begin{aligned}\mathbb{E}[\,\mu_\alpha\left(\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r)\wedge\tau])<r^{\beta+\delta}\}\right)\,] &= \\ &= \mathbb{E}\left[\int_0^\tau \mathbb{1}_{\left\{\mu_\gamma([t,(t+r)\wedge\tau])<r^{\beta+\delta}\right\}}\mu_\alpha(dt)\right] \\ &= \mathbb{E}\left[\int_0^\tau \liminf_{\varepsilon\to0}\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}\mu_\alpha(dt)\right] \\ &\le \liminf_{\varepsilon\to0}\mathbb{E}\left[\int_0^\tau \mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds<r^{\beta+\delta}\right\}}\mu_\alpha(dt)\right] \\ &\le \mathbb{E}[\,\tau\,]\,Dr^q,\end{aligned}$$

and we are done, since $\mathbb{E}[\,\tau\,]<\infty$ (as it has exponentially decaying tails). □

Proof of Proposition 3.3.8. Using Lemma 3.3.9 (taking $E=2^{\beta+\delta}$) and Markov's inequality, we can bound the upper tail of the μ_α-measure of the set of times when μ_γ decays unusually fast by

$$\begin{aligned}\mathbb{P}\left[\mu_\alpha\left(\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r)\wedge\tau])<2^{\beta+\delta}r^{\beta+\delta}\right\}\right)\ge r^{q/2}\right] &\le \\ &\le r^{-q/2}\mathbb{E}\left[\mu_\alpha\left(\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r)\wedge\tau])<2^{\beta+\delta}r^{\beta+\delta}\right\}\right)\right] \\ &\le Dr^{q/2}.\end{aligned}$$

So, taking a sequence of scales $r_n=2^{-n}$, we see that the events

$$\left\{\mu_\alpha\left(\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r_n)\wedge\tau])<2^{\beta+\delta}r_n^{\beta+\delta}\right\}\right)\ge r_n^{q/2}\right\}_{n\in\mathbb{N}}$$

occur only finitely often almost surely, by Borel-Cantelli. Therefore, for all $\Delta>0$ (which may be random and depend on $\mu_\alpha([0,\tau])$), we can find a random but almost surely finite $N\in\mathbb{N}$ such that

$$\mu_\alpha\left(\bigcup_{n\ge N}\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r_n)\wedge\tau])<2^{\beta+\delta}r_n^{\beta+\delta}\right\}\right)\le\sum_{n\ge N}2^{-qn/2}\le\Delta,$$

and hence

$$\mu_\alpha\left(\bigcap_{n\geq N}\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r_n)\wedge\tau])\geq 2^{\beta+\delta}r_n^{\beta+\delta}\right\}\right)\geq\mu_\alpha([0,\tau])-\Delta. \tag{3.30}$$

We now need to infer the result for all $r\in(0,2^{-N})$ from the discrete set of radii we have it for in (3.30). So, let $t\in\bigcap_{n\geq N}\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r_n)\wedge\tau])\geq r_n^{\beta+\delta}\right\}$, take $r\in(0,2^{-N})$, and suppose n is such that $r_{n+1}<r\leq r_n$. Then

$$\mu_\alpha([t,(t+r)\wedge\tau])\geq\mu_\alpha([t,(t+r_n)\wedge\tau])\geq 2^{\beta+\delta}r_{n+1}^{\beta+\delta}=r_n^{\beta+\delta}\geq r^{\beta+\delta},$$

which implies that the discrete radii event is a subset of the continuous radii event:

$$\bigcap_{n\geq N}\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r_n)\wedge\tau])\geq 2^{\beta+\delta}r_n^{\beta+\delta}\right\}\subset$$
$$\subset\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r)\wedge\tau])\geq r^{\beta+\delta}\quad\forall r\in[0,2^{-N})\right\}=L_\gamma^N,$$

and so we can conclude that

$$\mu_\alpha(L_\gamma^N)\geq\mu_\alpha([0,\tau])-\Delta.$$

□

We now state and prove a result which is essentially a "converse" to Proposition 3.3.8.

Proposition 3.3.10. *Fix $\delta>0$, and let $\beta=1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}$. Define the set of times*

$$U_\gamma^N=\left\{t\in[0,\tau]\,:\,\mu_\gamma([t,(t+r)\wedge\tau])\leq r^{\beta-\delta}\quad\forall r\in[0,2^{-N})\right\}$$

Then for all $\Delta>0$, which may be random and depend on $\mu_\alpha([0,\tau])$, there exists some random but almost surely finite $N\in\mathbb{N}$ such that

$$\mu_\alpha\left(U_\gamma^N\right)\geq\mu_\alpha\left([0,\tau]\right)-\Delta.$$

This proposition is essentially saying that around α-thick points, the map function F_γ is $(\beta-\delta)$-Hölder continuous. To prove it, we need a lemma that is the equivalent of Lemma 3.3.9.

Lemma 3.3.11. *Fix $\delta > 0$ and let $\beta = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}$, and let $E > 0$ be some positive constant. Then there exist two constants $D > 0$ and $\eta > 0$ such that*

$$\mathbb{E}\Big[\, \mu_\alpha \Big(\big\{ t \in [0, \tau] \;:\; \mu_\gamma([t, (t+r) \wedge \tau]) > E r^{\beta - \delta} \big\} \Big) \,\Big] \le D r^{\eta}.$$

for all $r > 0$.

The introduction of long range correlations by Brownian motion is much more apparent in this proof than the proof of Lemma 3.3.9. Instead of re-scaling time by a factor of r and space by a factor of $\sqrt{r}$ as we did previously, we will need to allow a bit of extra wiggle room. This is essentially due to the modulus of continuity of Brownian motion around time $r \downarrow 0$ being $\sqrt{2r \log \frac{1}{r}}$; we need a slightly lower power of r to account for the log correction. We will introduce the radius R, which we will use as our scaling radius, and calculate what it needs to be closer to the end of the proof.

Proof. Again, we will prove this only in the case that $E = 1$. Let $r, R > 0$, and fix $\varepsilon, \varepsilon' > 0$ so that $\varepsilon' < \varepsilon\sqrt{R}$. Using Girsanov's change of measure theorem and Lemma 3.2.3 as we did in Lemma 3.3.9, we see that

$$\mathbb{E}_B \mathbb{E}_h \Big[\mathbb{1}_{\left\{ \int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s) - \frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]} ds > r^{\beta-\delta} \right\}} e^{\alpha h_{\varepsilon'}(B_t) - \frac{\alpha^2}{2}\mathbb{E}[\, h_{\varepsilon'}(B_t)^2\,]} \mathbb{1}_{\{\tau > t\}} \Big] =$$

$$= \mathbb{E}_B \left[\mathbb{1}_{\{\tau > t\}} \mathbb{P} \left[e^{\alpha\gamma C} \int_t^{(t+r)\wedge\tau} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(B_s) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(B_s)^2\right]}}{\left(|B_s - B_t| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} ds > r^{\beta-\delta} \,\middle|\, \mathcal{F}_t \right]\right], \tag{3.31}$$

where $\mathcal{F}_t = \sigma(B_s;\ s \le t)$ is the natural filtration for B. Now, consider the integral in (3.31). We are looking for upper bounds on it, this time, to find an upper bound on the probability in

(3.31). First of all, we apply a simple change of time, first $s \mapsto s - t$ and then $s = Ru$ to see that

$$\begin{aligned}\int_t^{(t+r)\wedge\tau} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(B_s) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(B_s)^2\right]}}{\left(|B_s - B_t| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} ds &= \int_0^{r\wedge(\tau - r)} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(B_{t+s}) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(B_{t+s})^2\right]}}{\left(|B_{t+s} - B_t| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} ds \\ &= R\int_0^{\frac{r}{R}\wedge\frac{\tau - r}{R}} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(B_{t+Ru}) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(B_{t+Ru})^2\right]}}{\left(|B_{t+Ru} - B_t| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} du \\ &\overset{d}{=} R\int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u + B_t) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u + B_{t+Ru})^2\right]}}{\left(|\sqrt{R}\tilde{B}_u| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} du \\ &= R^{1-\frac{\alpha\gamma}{2}}\int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u + B_t) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u + B_{t+Ru})^2\right]}}{\left(|\tilde{B}_u| + \varepsilon\right)^{\alpha\gamma}} du\end{aligned}$$

where $\tilde{B}$ is an independent Brownian motion started at zero, and

$$\tau' = \inf\left\{u > 0 \ : \ |\sqrt{R}\tilde{B}_u + B_t| = \frac{1}{2}\right\}.$$

The equality in distribution holds $\mathbb{P}_h$-almost surely.

We now want to use the scaling properties of the field h, from Lemma 3.3.5. So, as before, we use the Markov property of the GFF to write $h = h^{har} + \tilde{h}$, where h^{har} is the harmonic projection of h onto the disc of radius $2\sqrt{R}$, centred at B_t, and $\tilde{h}$ has the law of a zero-boundary GFF on the disc of radius $2\sqrt{R}$, centred at B_t. If we write $\Omega_{\sqrt{R}} = \sup_{z\in B(B_t,\sqrt{R})} h^{har}(z)$, we know that $h \le \Omega_{\sqrt{R}} + \tilde{h}$ inside the disc $B(B_t, \sqrt{R})$. In order to use Lemma 3.3.5, we need to make sure that the $\tilde{B}$ does not move far from its starting point. So, let $\tilde{\tau}$ be the exit time of $\tilde{B}$

from the unit disc. Then, on the event $\left\{\tilde{\tau} > \frac{r}{R}\right\}$ we can see that

$$
\begin{aligned}
R^{1-\frac{\alpha\gamma}{2}} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u)^2\right]}}{\left(|\tilde{B}_u| + \varepsilon\right)^{\alpha\gamma}} du \leq \\
&\leq R^{1-\frac{\alpha\gamma}{2}} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u) - \frac{\gamma^2}{2}(-\log(\varepsilon\sqrt{R}) - C)}}{(|\tilde{B}_u| + \varepsilon)^{\alpha\gamma}} du \\
&= R^{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}} e^{\frac{\gamma^2}{2}C} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u) + \frac{\gamma^2}{2}\log(\varepsilon)}}{(|\tilde{B}_u| + \varepsilon)^{\alpha\gamma}} du \\
&\leq R^{1-\frac{\alpha\gamma}{2}+\frac{\gamma^2}{4}} e^{\frac{\gamma^2}{2}C} e^{\gamma\Omega_{\sqrt{R}}} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma \tilde{h}_{\varepsilon\sqrt{R}}(\sqrt{R}\tilde{B}_u) + \frac{\gamma^2}{2}\log(\varepsilon)}}{(|\tilde{B}_u| + \varepsilon)^{\alpha\gamma}} du \\
&\leq e^{\gamma^2 C} R^{\beta} e^{\gamma\Omega_{\sqrt{R}}} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h'_\varepsilon(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[h'_\varepsilon(\tilde{B}_u)^2]}}{\left(|\tilde{B}_u| + \varepsilon\right)^{\alpha\gamma}} du, \qquad (3.32)
\end{aligned}
$$

where h' is a zero boundary GFF on the unit disc. Therefore, we can use the right hand side of (3.32) to bound the probability in (3.31) by

$$
\begin{aligned}
&\mathbb{P}\left[e^{\alpha\gamma C} \int_t^{(t+r)\wedge\tau} \frac{e^{\gamma h_{\varepsilon\sqrt{R}}(B_s) - \frac{\gamma^2}{2}\mathbb{E}\left[h_{\varepsilon\sqrt{R}}(B_s)^2\right]}}{\left(|B_s - B_t| + \varepsilon\sqrt{R}\right)^{\alpha\gamma}} ds > r^{\beta-\delta} \middle| \mathcal{F}_t \right] \\
&\leq \mathbb{P}\left[e^{(\alpha\gamma+\gamma^2)C} R^{\beta} e^{\gamma\Omega_{\sqrt{R}}} \int_0^{\frac{r}{R}\wedge\tau'} \frac{e^{\gamma h'_\varepsilon(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[h'_\varepsilon(\tilde{B}_u)^2]}}{(|\tilde{B}_u| + \varepsilon)} du > r^{\beta-\delta}; \tilde{\tau} > \frac{r}{R} \middle| \mathcal{F}_t \right] + \mathbb{P}\left[\tilde{\tau} < \frac{r}{R}\right] \\
&\leq \mathbb{P}\left[e^{(\alpha\gamma+\gamma^2)C} R^{\beta} e^{\gamma\Omega_{\sqrt{R}}} \int_0^{\tilde{\tau}\wedge\tau'} \frac{e^{\gamma h'_\varepsilon(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[h'_\varepsilon(\tilde{B}_u)^2]}}{(|\tilde{B}_u| + \varepsilon)} du > r^{\beta-\delta} \middle| \mathcal{F}_t \right] + \mathbb{P}\left[\tilde{\tau} < \frac{r}{R}\right] \\
&\leq \mathbb{P}\left[e^{(\alpha\gamma+\gamma^2)C} e^{\gamma\Omega_{\sqrt{R}}} > \left(\frac{r}{R}\right)^{\frac{\beta}{2}} r^{-\frac{\delta}{2}} \middle| \mathcal{F}_t \right] + \\
&\qquad + \mathbb{P}\left[\int_0^{\tilde{\tau}} \frac{e^{\gamma h_\varepsilon(\tilde{B}_u) - \frac{\gamma^2}{2}\mathbb{E}[h_\varepsilon(\tilde{B}_u)^2]}}{(|\tilde{B}_u| + \varepsilon)} du > \left(\frac{r}{R}\right)^{\frac{\beta}{2}} r^{-\frac{\delta}{2}} \right] + \mathbb{P}\left[\tilde{\tau} < \frac{r}{R}\right]. \qquad (3.33)
\end{aligned}
$$

We are now in a position to see what choice we should make for the radius R. We want $\frac{r}{R} \to 0$ as $r \to 0$ polynomially in r, so that the third term in (3.33) decays polynomially. We also want $\left(\frac{r}{R}\right)^{\frac{\beta}{2}} r^{-\frac{\delta}{2}}$ to converge to infinity, polynomially in r, so that the other terms in (3.33) also decay polynomially: see below. The choice $R = r^{1-\frac{\delta^2}{2}}$ works for δ small enough, since then we certainly have $\frac{r}{R} = r^{\delta^2} \to 0$, and also $\left(\frac{r}{R}\right)^{\frac{\beta}{2}} r^{-\frac{\delta}{2}} = r^{\frac{\beta}{2}\delta^2 - \frac{\delta}{2}} \to \infty$. The exponent $\frac{\beta}{2}\delta^2 - \frac{1}{2}\delta$ is negative for δ small enough, and so we have the desired properties. With this choice of R, the first term on the right hand side of (3.33) decays polynomially by Lemma 3.3.5, the second term decays polynomially by Corollary 3.3.4. We can bound the third term above by

$$\mathbb{P}\left[\tilde{\tau} < \frac{r}{R}\right] \leq \mathbb{P}\left[T < \frac{r}{R}\right],$$

where T is the exit time of a one dimensional Brownian motion from the interval $[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$. The stopping time T has exponentially decaying tails, and so we can see that the third term decays polynomially as well.

Therefore, going all the way back to (3.31), there certainly exist some constants $D > 0$ and $\eta > 0$ such that, for $t > 0$,

$$\mathbb{E}\left[\mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds > r^{\beta-\delta}\right\}} e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}[\,h_{\varepsilon'}(B_t)^2\,]}\mathbb{1}_{\{\tau>t\}}\right] \leq Dr^{\eta}\mathbb{P}[\,\tau > t\,]\,.$$

Integrating over $t > 0$ gives

$$\mathbb{E}\left[\int_0^{\tau} \mathbb{1}_{\left\{\int_t^{(t+r)\wedge\tau} e^{\gamma h_{\varepsilon\sqrt{r}}(B_s)-\frac{\gamma^2}{2}\mathbb{E}[h_{\varepsilon\sqrt{r}}(B_s)^2]}ds > r^{\beta-\delta}\right\}} e^{\alpha h_{\varepsilon'}(B_t)-\frac{\alpha^2}{2}\mathbb{E}[\,h_{\varepsilon'}(B_t)^2\,]}dt\right] \leq Dr^{\eta}\mathbb{E}[\,\tau\,]\,. \tag{3.34}$$

Now, note that (3.34) is almost identical to (3.29). We use the same arguments to let ε' and ε converge to zero, and conclude that

$$\mathbb{E}\left[\,\mu_\alpha\left(\left\{t \in [r,\tau] \,:\, \mu_\gamma([(t)\vee 0, (t+r)\wedge\tau]) > Er^{\beta-\delta}\right\}\right)\,\right] \leq Dr^{\eta}\mathbb{E}[\,\tau\,]\,,$$

which completes the proof, as $\mathbb{E}[\,\tau\,] < \infty$. □

Proof of Proposition 3.3.10. Proposition 3.3.10 follows from Lemma 3.3.11 in exactly the same way that Proposition 3.3.8 followed from Lemma 3.3.9.

□

3.3.4 Proof of Theorem 3.1.1 and Corollary 3.1.2

We now have all of the tools ready to prove Theorem 3.1.1, which we re-state here in more detail.

Theorem 3.3.12. *Let B be the Brownian motion used to construct the LBM time changes F_α and F_γ. Call $T_\alpha = \{t > 0 : B_t \in \mathcal{T}_\alpha\}$ the set of times that the Brownian motion B is in an α-thick point. Then*

$$\dim_H(F_\gamma(T_\alpha)) \geq \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}}$$

where, by the definition of the change of time, $F_\gamma(T_\alpha)$ is the set of times the γ-LBM is in α-thick points.

Proof. We will in fact prove the lower bound for times only for the stopping time τ when the Brownian motion leaves the disc of radius $\frac{1}{2}$. To that end, we will abuse notation slightly and re-define the set of times T_α as

$$T_\alpha = \{t \in [0, \tau] : B_t \in \mathcal{T}_\alpha\}.$$

Because $T_\alpha \cap L_\gamma^N \subset T_\alpha$, where L_γ^N is defined as in Proposition 3.3.8, we know that

$$\dim_H(F_\gamma(T_\alpha \cap L_\gamma^N)) \leq \dim_H(F_\gamma(T_\alpha)).$$

But, because F_γ^{-1} is a $\frac{1}{\beta+\delta}$-Hölder-like function, in the sense of Definition 3.2.14, on intervals starting at times in the image $F_\gamma(L_\gamma^N)$, Proposition 3.2.15 implies that

$$\dim_H(T_\alpha \cap L_\gamma^N) \leq (\beta + \delta) \dim_H(F_\gamma(T_\alpha \cap L_\gamma^N)), \tag{3.35}$$

and so to get a lower bound on $\dim_H(F_\gamma(T_\alpha))$, we want to find a lower bound for $\dim_H(T_\alpha \cap L_\gamma^N)$. We now use Propositions 3.3.8 and 3.3.10 to see that we can take N large enough to ensure that $T_\alpha \cap L_\gamma^N \cap U_\alpha^N$ has positive μ_α-measure (taking $\gamma = \alpha$ in Proposition 3.3.10, and $\Delta = \frac{1}{4}\mu_\alpha([0, \tau])$ for example). Since we also know that $\mu_\alpha[0, \tau] < \infty$ almost surely, the measure μ_α defines a mass distribution on the set $T_\alpha \cap L_\gamma^N \cap U_\alpha^N$, and by the definition of U_α^N, we know that

$$\mu_\alpha([t, t + r]) \leq r^{1 - \frac{\alpha^2}{4} - \delta}$$

for all $r \in [0, 2^{-N})$ and $t \in T_\alpha \cap L_\gamma^N \cap U_\alpha^N$. So, by the mass distribution principle (Theorem 4.19 of [MP10] for example), we find that

$$\dim_H(T_\alpha \cap L_\gamma^N \cap U_\alpha^N) \geq 1 - \frac{\alpha^2}{4} - \delta.$$

Therefore we certainly have the bound $\dim_H(T_\alpha \cap L_\gamma^N) \geq 1 - \frac{\alpha^2}{4} - \delta$, which we can substitute into (3.35) and re-arrange to find

$$\dim_H(F_\gamma(T_\alpha)) \geq \frac{1 - \frac{\alpha^2}{4} - \delta}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4} + \delta}.$$

Since δ was arbitrary we can take the limit $\delta \to 0$, and we have shown the result. □

And now we re-state Corollary 3.1.2 and prove it:

Corollary 3.3.13. *Let B be the Brownian motion used to construct the LBM time changes F_α and F_γ. Call $T_\alpha = \{t \in [0,T] : B_t \in \mathcal{T}_\alpha\}$ the set of times that the Brownian motion B is in an α-thick point. Then*

$$\dim_H(F_\gamma(T_\alpha)) = \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}}$$

where, by the definition of the change of time, $F_\gamma(T_\alpha)$ is the set of times the γ-LBM is in α-thick points.

Proof. First, following from Theorem 1.4 of [Ber15a], we introduce the sets

$$\mathcal{T}_\alpha^- = \left\{ z \in \mathbb{C} : \liminf_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log \frac{1}{\varepsilon}} \geq \alpha \right\},$$
$$\mathcal{T}_\alpha^+ = \left\{ z \in \mathbb{C} : \limsup_{\varepsilon \to 0} \frac{h_\varepsilon(z)}{\log \frac{1}{\varepsilon}} \leq \alpha \right\}$$

We will call $T_\alpha^- = \{t \in [0,\tau] : B_t \in \mathcal{T}_\alpha^-\}$, and similarly define T_α^+ from $\mathcal{T}_\alpha^+$. We know, from [Ber15a], that for $\alpha > \gamma$ we have the upper bound

$$\dim_H(F_\gamma(T_\alpha^-)) \leq \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}},$$

and the same result holds when we have $\alpha < \gamma$ and we replace T_α^- with T_α^+.

Let us consider the case $\alpha > \gamma$. We know that $T_\alpha \subset T_\alpha^-$, and so we have

$$\frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}} \leq \dim_H(F_\gamma(T_\alpha)) \leq \dim_H(F_\gamma(T_\alpha^-)) \leq \frac{1 - \frac{\alpha^2}{4}}{1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}},$$

showing us the equality. We can show equality in the case $\alpha < \gamma$ in the same way. □

3.3.5 Proofs of regularity properties

We can now state Theorem 3.1.3 again, and give the proof.

Theorem 3.3.14. *For μ_α-almost every $t \geq 0$, the change of time F_γ has the following growth rate:*

$$\lim_{r \to 0} \frac{\log |F_\gamma(t) - F_\gamma(t+r)|}{\log |r|} = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}, \tag{3.36}$$

almost surely.

Before we start the proof, we would like to explain the intuition behind μ_α-almost every $t \in T_\alpha$. Suppose we have our GFF h, and the Brownian motion B which is the path of our Liouville Brownian motion. We now run an α-LBM, Z^α, along the path B, using h to calculate the time change F_α. At some time $\mathbf{t}$, chosen uniformly at random from the lifetime of Z^α, we inspect the point in the plane occupied by $Z^\alpha_{\mathbf{t}}$. Because an α-LBM spends Lebesgue-almost all of its time in α-thick points, we know that the point chosen by $Z^\alpha_{\mathbf{t}}$ is an α-thick point, almost surely. We also know that it is on the path of the Brownian motion B. If we call the time B passes through this point t, i.e. $t = F_\alpha^{-1}(\mathbf{t})$, we know that, around this time, the γ-time change, F_γ, has the regularity property given in (3.36).

To prove Theorem 3.3.14, we will first prove it while taking the limit $r \downarrow 0$, i.e. as r approaches 0 from above.

Lemma 3.3.15. *For μ_α-almost every $t \geq 0$, the change of time F_γ has the following growth rate:*

$$\lim_{r \downarrow 0} \frac{\log |F_\gamma(t) - F_\gamma(t+r)|}{\log r} = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4},$$

almost surely.

Proof. Most of the work for this proof has been done in Propositions 3.3.8 and 3.3.10. Recall that for some arbitrary $\delta > 0$ we defined

$$L_\gamma^N = \left\{ t \in [0, \tau] \, : \, \mu_\gamma([t, (t+r) \wedge \tau]) \geq r^{\beta+\delta} \quad \forall r \in [0, 2^{-N}) \right\},$$

and

$$U_\gamma^N = \left\{t \in [0,\tau] \,:\, \mu_\gamma([t,(t+r)\wedge\tau]) \le r^{\beta-\delta} \quad \forall r \in [0,2^{-N})\right\}.$$

Now let us define

$$L_\gamma = \bigcup_N L_\gamma^N = \left\{t \in [0,\tau) \,:\, \limsup_{r\to 0} \frac{\log \mu_\gamma([t,t+r]))}{\log r} \le \beta+\delta\right\},$$

and similarly define $U_\gamma = \bigcup_N U_\gamma^N$.

We showed in Propositions 3.3.8 and 3.3.10 that for any $\Delta > 0$, we could find N large enough that

$$\mu_\alpha(L_\gamma^N) \ge \mu_\alpha([0,\tau]) - \Delta,$$

and

$$\mu_\alpha(U_\gamma^N) \ge \mu_\alpha([0,\tau]) - \Delta.$$

Since Δ was arbitrary and L_γ^N, U_γ^N are increasing sets, we find that

$$\mu_\alpha(L_\gamma \cap U_\gamma) = \mu_\alpha([0,\tau]).$$

Because δ was arbitrary, and we defined $\mu_\gamma([t,t+r]) := F(t+r) - F(t)$, we have shown the result. □

We now need a lemma which allows us to "reverse time" in some way, and extend the result from Lemma 3.3.15 to the statement in Theorem 3.3.14.

Lemma 3.3.16. *Let B be a Brownian motion started at zero, and let $t > 0$. Define two stochastic processes, conditional on B_t, by setting*

$$W_s^+ = B_{t+s}$$

for $s \ge 0$, and

$$W_s^- = B_{t-s}$$

for $s \in [0,t]$. Now, let $\varepsilon < t$. Then, conditional on the event $\{\tau > t\}$ (where τ is the first exit time of B from the disc of radius $\frac{1}{2}$), the laws of the restricted processes $(W_s^+)_{s\in[0,\varepsilon]}$ and $(W_s^-)_{s\in[0,\varepsilon]}$ are absolutely continuous with respect to each other.

Proof. Conditional on $B_t = z$ and the event $\{\tau > t\}$, the law of the Brownian motion B is that of a Brownian bridge of duration t, joining the origin and z, conditioned to stay inside the disc of radius $\frac{1}{2}$, followed by an independent Brownian motion started at z. Because the event that the maximum modulus of this Brownian bridge is less than $\frac{1}{2}$ has positive probability, it does not affect the absolute continuity of measures. So for the rest of the proof, we may ignore the fact that we are conditioning on that event.

By reversibility of Brownian bridges, the process W^- has the law of a Brownian bridge of duration t, connecting z and the origin. And, as stated above, W^+ has the law of a Brownian motion started at z. So, by (6.28) of [KS91] (or, slightly more explicitly, Lemma 3.1 of [BGRV14]), we see that the laws of a Brownian bridge of duration t and a Brownian motion, with a common starting point, are absolutely continuous with respect to each other on intervals shorter than t. □

Proof of Theorem 3.3.14. Let T be an exponential random variable with mean 1, independent of the GFF h and the Brownian motion B. Recall that the measure μ_α is defined by

$$\mu_\alpha([a,b]) = F_\alpha(b) - F_\alpha(a),$$

which can also be written as $\mu_\alpha = \mathcal{L}eb \circ F_\alpha$. Now, because the law of T is absolutely continuous with respect to Lebesgue measure and F_α is a bijection, the law of $F_\alpha^{-1}(T)$ is absolutely continuous with respect to μ_α. Therefore, by Lemma 3.3.15, we see that

$$F_\alpha^{-1}(T) \in \left\{ t > 0 \,:\, \lim_{r \to 0} \frac{\log |F_\gamma(t) - F_\gamma(t+r)|}{\log r} = \beta \right\},$$

almost surely. It therefore follows from Lemma 3.3.16 that we also have

$$F_\alpha^{-1}(T) \in \left\{ t > 0 \,:\, \lim_{r \to 0} \frac{\log |F_\gamma(t) - F_\gamma(t-r)|}{\log r} = \beta \right\},$$

almost surely. Finally, by absolute continuity of the law of $F_\alpha^{-1}(T)$ and the measure μ_α again, we deduce that for μ_α-almost every t we have

$$\lim_{r \to 0} \frac{\log |F_\gamma(t) - F_\gamma(t+r)|}{\log |r|} = \beta$$

almost surely, completing the proof. □

We can use the regularity property of F_γ from Theorem 3.3.14 that we have just shown to find a bound on the growth rate of LBM around thick points of different levels. We first prove a lemma about the growth rate of LBM given a lot of control on how we choose the time we consider. We will then extend that to the more general statement given in Corollary 3.1.4.

Lemma 3.3.17. *Let $t \geq 0$ be such that*

$$\lim_{r\to 0} \frac{\log |F_\gamma(t) - F_\gamma(t+r)|}{\log |r|} = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}.$$

Then

$$\limsup_{u\to 0} \frac{\log |Z_{F_\gamma(t)} - Z_{F_\gamma(t)+u}|}{\log |u|} = \frac{1}{2 - \alpha\gamma + \frac{\gamma^2}{2}},$$

almost surely.

Note 3.3.18. In Lemma 3.3.17, we have let $r \to 0$ and $u \to 0$ from above and below. In the proof of Corollary 3.3.19, only the result as $r \downarrow 0$ and $u \downarrow 0$ are used, but the distinction is important for the proof of Corollary 3.3.21.

Proof. Let $\delta > 0$. Then by Lévy's modulus of continuity of Brownian motion, we know that, almost surely, there exists some $S < \infty$ such that

$$|B_t - B_{t+s}| \leq s^{\frac{1}{2}-\delta} \tag{3.37}$$

for all $s \in [-S, S]$, and for all $\varepsilon > 0$ there exists some $s \in [-\varepsilon, \varepsilon]$ such that

$$|B_t - B_{t+s}| \geq s^{\frac{1}{2}+\delta}. \tag{3.38}$$

Now, let us write $\beta = 1 - \frac{\alpha\gamma}{2} + \frac{\gamma^2}{4}$. Then by assumption, there exists some $R < \infty$ such that

$$r^{\beta+\delta} \leq F_\gamma(t+r) - F_\gamma(t) \leq r^{\beta-\delta}$$

for all $r \in [-R, R]$. Since F_γ^{-1} is well defined, this in turn implies that, for all u with $|u|$ small enough,

$$u^{\frac{1}{\beta-\delta}} \leq F_\gamma^{-1}(F_\gamma(t) + u) - t \leq u^{\frac{1}{\beta+\delta}}. \tag{3.39}$$

Recalling the definition $Z_t^\gamma = B_{F_\gamma^{-1}(t)}$ and combining (3.37) and (3.39) shows us that

$$|Z_{F_\gamma(t)} - Z_{F_\gamma(t)+u}| \le \left(|u|^{\frac{1}{\beta+\delta}}\right)^{\frac{1}{2}-\delta} \tag{3.40}$$

for all $|u|$ small enough. Furthermore, combining (3.38) and (3.39) shows us that, for any $\varepsilon' > 0$ there exists some $u \in [-\varepsilon', \varepsilon']$ such that

$$|Z_{F_\gamma(t)} - Z_{F_\gamma(t)+u}| \ge \left(|u|^{\frac{1}{\beta-\delta}}\right)^{\frac{1}{2}+\delta}.$$

Taking logs then implies that

$$\frac{1-\delta}{2(\beta+\delta)} \le \limsup_{u\to 0} \frac{\log|Z_{F_\gamma(t)} - Z_{F_\gamma(t)+u}|}{\log|u|} \le \frac{1+\delta}{2(\beta-\delta)}$$

almost surely. Therefore, letting $\delta \to 0$ along a countable sequence shows us that the limsup equals $\frac{1}{2\beta}$ almost surely, as claimed. □

We can now use the results from Theorem 3.3.14 and Lemma 3.3.17 to prove Corollary 3.1.4, which we restate here.

Corollary 3.3.19. *Suppose that the starting point of a γ-Liouville Brownian motion is chosen according to M_α, i.e. $Z_0^\gamma \sim M_\alpha$. Then*

$$\limsup_{t\downarrow 0} \frac{\log|Z_t^\gamma|}{\log t} = \frac{1}{2-\alpha\gamma+\frac{\gamma^2}{2}},$$

almost surely.

Proof. Let T be an exponential random variable with mean 1, which is independent of the GFF h and the Brownian motion B.

By the same reasoning as that used in the proof of Theorem 3.3.14, we see that

$$F_\alpha^{-1}(T) \in \left\{ t \ge 0 \;:\; \lim_{r\to 0} \frac{\log|F_\gamma(t) - F_\gamma(t+r)|}{\log r} = \beta \right\},$$

almost surely. (If $T > \sup_t F_\alpha(t)$, we set $F_\alpha^{-1}(T) = \varnothing$, and claim that the equality below holds, vacuously.) Therefore if we write $T' = F_\gamma(F_\alpha^{-1}(T))$, Lemma 3.3.17 tells us that

$$\limsup_{u\to 0} \frac{\log|Z_{T'}^\gamma - Z_{T'+u}^\gamma|}{\log u} = \frac{1}{2\beta}.$$

Let $\mathcal{H}$ be the sigma algebra generated by the GFF h, i.e.

$$\mathcal{H} = \sigma\left(\langle h, f\rangle_\nabla \,:\, f \in H_0^1(\mathcal{D})\right).$$

Now consider the filtration defined by

$$\begin{aligned}\mathcal{G}_t &= \sigma(Z_s^\gamma \,:\, s < t) \vee \mathcal{H} \\ &= \sigma(B_s \,:\, s < F_\gamma^{-1}(t)) \vee \mathcal{H}.\end{aligned}$$

The process Z^γ is certainly $\mathcal{G}_t$-adapted, and T' is a $\mathcal{G}_t$-stopping time since

$$\{T' > t\} = \left\{F_\gamma(F_\alpha^{-1}(T)) > t\right\} = \left\{T > F_\alpha(F_\gamma^{-1}(t))\right\},$$

and

$$F_\alpha(F_\gamma^{-1}(t)) = \lim_{\varepsilon\to 0}\int_0^{F_\gamma^{-1}(t)} e^{\alpha h_\varepsilon(B_s) - \frac{\alpha^2}{2}\mathbb{E}[h_\varepsilon(B_s)^2]} ds$$

is $\mathcal{G}_t$-measurable.

We can therefore use the strong Markov property of Z^γ to deduce that

$$\limsup_{t\downarrow 0} \frac{\log|Z_t^\gamma|}{\log t} = \frac{1}{2\beta} \tag{3.41}$$

whenever Z_0^γ is chosen according to $\mathcal{P}_T^\alpha$, the law of Z_T^α.

From Theorem 2.5 in [GRV13b], we know that, for a fixed $t \geq 0$, the law of Z_t^α is absolutely continuous with respect to the Liouville measure M^α, with Radon-Nikodym derivative

$$\frac{d\mathcal{P}_t^\alpha}{dM^\alpha}(y) = p_t^\alpha(0, y) \geq 0.$$

We can therefore write

$$\frac{d\mathcal{P}_T^\alpha}{dM^\alpha}(y) = \int_0^\infty e^{-t} p_t^\alpha(0, y) dt.$$

Theorem 2.5 of [GRV13b] also implies that, for M^α-almost every $y \in \mathcal{D}$, the transition density $p_t^\alpha(0, y)$ is strictly positive for all t in a measurable set with positive Lebesgue measure. (This fact was noted in an earlier version of their paper.) But that implies that

$$\frac{d\mathcal{P}_T^\alpha}{dM^\alpha}(y) > 0$$

for M^α-almost every $y \in \mathcal{D}$, i.e. the Liouville measure M^α and $\mathcal{P}_T^\alpha$ are absolutely continuous with respect to each other. Therefore, since (3.41) holds almost surely whenever Z_0^γ was chosen according to $\mathcal{P}_T^\alpha$, we deduce that it also holds almost surely with Z_0^γ is chosen according to M^α. □

Remark 3.3.20. The exponential time T in the proof above can be replacecd with a deterministic time t provided we know the existence of a continuous version of the transition density, for which $p_t(x, y) > 0$ for all $x, y \in \mathcal{D}$ and all $t > 0$. This is known in the case of a torus [MRVZ14], and similar arguments probably work in the planar case as well. We have made no attempt to check this, however.

We now restate and prove Corollary 3.1.6:

Corollary 3.3.21. *Let $\gamma \in (\sqrt{2}, 2)$. Then the γ-Liouville Brownian motion Z^γ is Lebesgue-almost everywhere differentiable with derivative zero, almost surely.*

Proof. By taking $\alpha = \gamma$ in Theorem 3.3.14, we know that for μ_γ-almost every $t \geq 0$, the change of time F_γ has the following growth rate:

$$\lim_{r\to 0} \frac{\log|F_\gamma(t) - F_\gamma(t+r)|}{\log|r|} = 1 - \frac{\gamma^2}{4}.$$

Now, let $\delta \in (0, \frac{1}{2-\frac{\gamma^2}{2}} - 1)$. We can apply Lemma 3.3.17, or specifically (3.40) in the proof of Lemma 3.3.17, to see that for μ_γ-almost every $t \geq 0$ we have

$$|Z^\gamma_{F_\gamma(t)} - Z^\gamma_{F_\gamma(t)+r}| \leq |r|^{1/(2-\frac{\gamma^2}{2})-\delta},$$

for all r with $|r|$ small enough. But, by the definition of μ_γ, the F_γ image of a set with full μ_γ measure has full Lebesgue measure. Therefore, we can see that for Lebesgue-almost every $t \geq 0$ we have

$$|Z^\gamma_t - Z^\gamma_{t+r}| \leq |r|^{1/(2-\frac{\gamma^2}{2})-\delta}$$

for all r with $|r|$ small enough. Therefore, we have

$$\lim_{r\to 0} \frac{|Z^\gamma_t - Z^\gamma_{t+r}|}{|r|} \leq \lim_{r\to 0} |r|^{1/(2-\frac{\gamma^2}{2})-\delta-1} = 0$$

where the final inequality is because we have chosen δ to ensure that $\frac{1}{2-\frac{\gamma^2}{2}} - \delta - 1 > 0$. So, we certainly have differentiability for Z^γ, for Lebesgue-almost every $t \geq 0$, and the derivative is equal to zero. □

4 Rohde-Schramm Theorem Through Coupling

This chapter gives a novel proof of the Rohde-Schramm theorem, first shown in [RS11]. Our proof uses the coupling of the reverse SLE with the Neumann boundary GFF to bound the derivative of the inverse of the Loewner flow close to the origin. We are able to write the absolute value of the derivative of the reverse flow in terms of the exponential of two Gaussian random fields; one of them is a Neumann boundary GFF and the other is very closely related. We can use our knowledge of the structure of the GFF to find bounds which ensure continuity of the SLE trace.

4.1 Introduction

We will provide a new proof for the Rohde-Schramm theorem in the case $\kappa \neq 8$. The theorem roughly states that "a Schramm-Loewner evolution is generated by a path". It was first proved for $\kappa \neq 8$ in [RS11] and for $\kappa = 8$ in [LSW11].

In order to state the Rohde-Schramm theorem properly we will first recall some of the notation that we met in Chapter 1. We will do this briefly now, and in more detail in Sections 4.2.1 and 4.2.2

Let (ξ_t) be a real valued, continuous process defined for $t > 0$. Define the family of conformal maps (g_t) as the maximal solution to Loewner's equation:

$$dg_t(z) = \frac{2}{g_t(z) - \xi_t} dt \tag{4.1}$$

for each $z \in \mathbb{H}$. For each $z \in \mathbb{H}$, we will call the lifetime of the solution $\tau(z)$. For a given time $t > 0$, we call the subset of $\mathbb{H}$ for which the solution still exists H_t. We can write it more precisely as

$$H_t = \{z \in \mathbb{H} \ : \ t < \tau(z)\} .$$

The complement of H_t in the upper half plane, $K_t = \mathbb{H} \setminus H_t$, is the region which g_t "maps out," i.e. $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$. Note also that we can see

$$K_t = \{z \in \mathbb{H} \ : \ t \geq \tau(z)\} .$$

The driving function (ξ_t) is called the Loewner transform of the hulls (K_t).

We say that the hulls (K_t) are generated by a curve if there exists a curve $(\gamma_t) \subset \mathbb{H}$ such that, for all $t > 0$, the set $H_t = \mathbb{H} \setminus K_t$ is the unique unbounded component of $\mathbb{H} \setminus \gamma[0, t]$.

The following, which appears as Theorem 4.1 in [RS11], gives a condition for the hulls (K_t) to be generated by a curve.

Theorem 4.1.1. *Let $\xi : [0, \infty) \to \mathbb{R}$ be continuous, and let g_t be the corresponding solution of*

Loewner's equation, (4.1)*. Assume that*

$$\gamma(t) := \lim_{y\to 0} g_t^{-1}(iy + \xi_t)$$

exists for all $t \in [0,\infty)$ *and is continuous. Then* g_t^{-1} *extends continuously to* $\overline{\mathbb{H}}$ *and* H_t *is the unbounded connected component of* $\mathbb{H} \setminus \gamma[0,t]$*, for every* $t \in [0,\infty)$.

We are particularly interested in the case that $\xi_t = \sqrt{\kappa} B_t$, where $\kappa > 0$ and (B_t) is a standard Brownian motion. Then the hulls (K_t) form an SLE$_\kappa$ process. For the rest of the chapter, we will assume that (ξ_t) has this form.

Throughout the rest of the chapter, we will also use the notation

$$f_t := g_t^{-1} \qquad \text{and} \qquad \hat{f}_t(z) := f_t(z+\xi_t) = g_t^{-1}(z+\xi_t). \tag{4.2}$$

The following, which appears as Theorem 3.6 in [RS11], shows that the assumption in Theorem 4.1.1 is true when (ξ_t) is the driving function for an SLE process.

Theorem 4.1.2. *Let* (ξ_t) *be a driving function given by* $\xi_t = \sqrt{\kappa} B_t$ *for some* $\kappa > 0$*, where* (B_t) *is a standard Brownian motion. Let* (g_t) *be the corresponding solution to Loewner's equation,* (4.1)*, and let* $(\hat{f}_t)$ *be the centred inverse as defined in* (4.2)*. Define*

$$H(y,t) := \hat{f}_t(iy) \quad \textit{for} \quad (y,t) \in (0,\infty)\times[0,\infty).$$

If $\kappa \neq 8$*, then almost surely* $H(y,t)$ *extends continuously to* $[0,\infty)\times[0,\infty)$.

As we saw in Chapter 1, the proof of Theorem 4.1.2 requires control over the modulus of the derivative $|\hat{f}_t'|$. It is this control that we concentrate on in this chapter.

4.1.1 Statement of results

We will now prove the following result on the tail behaviour of $|\hat{f}_t'|$. The result was known in [RS11], and a similar result appears as Lemma 4.32 in [Law08]. Our proof takes a completely different approach from those; we use the coupling between SLE processes and the Neumann boundary GFF from [She10].

In spite of our different approach, however, the polynomial decay in the tail that we find is, up to logarithmic corrections, exactly the same as that found in previous proofs.

Theorem 4.1.3. *Let $\kappa \neq 8$, and let $\hat{f}_t$ be the centred inverse of the Loewner flow as defined in* (4.2). *Then there exist constants $\varepsilon > 0$, $\delta > 0$ and $C > 0$ such that*

$$\mathbb{P}\left[|\hat{f}_t'(iy)| > y^{-(1-\varepsilon)}\right] \leq Cy^{2+\delta}$$

for all $t \in [0,1]$ and $y \in (0,1)$.

4.2 Setup

We will now start to formally define all of the objects we will need for our proof.

4.2.1 *Forward SLE*

Chordal SLE_κ are the one parameter family of conformally invariant, non-self-intersecting curves which connect two marked points on the boundary of a complex domain. They were introduced in [Sch00] as the scaling limit of loop-erased random walks but, as we will see soon, it was not obvious that the objects defined actually produced curves.

We will take our domain to be the upper half plane, $\mathbb{H}$, and our SLE_κ will connect the boundary points 0 and ∞. In fact, when we introduce the idea of the SLE_κ process evolving through time, we will think of them as growing from 0 towards ∞. The SLE_κ process is defined through a family of conformal maps, (g_t), called the Loewner flow, and a driving function (ξ_t).

Definition 4.2.1 (Chordal SLE)**.** Fix $\kappa \geq 0$. Let B_t be a Brownian motion started at zero and set $\xi_t = \sqrt{\kappa}B_t$. For each $z \in \overline{\mathbb{H}} \setminus \{0\}$ let $g_t(z)$ be the maximal solution to

$$dg_t(z) = \frac{2}{g_t(z) - \xi_t}dt, \tag{4.3}$$

with $g_0(z) = z$. Let $\tau(z)$ be the lifetime of the solution. We will call the family of maps (g_t) an SLE_κ.

Let us define two families of subsets of the upper half plane. For $t \geq 0$, set

$$H_t := \{z \in \mathbb{H} \ : \ t < \tau(z)\}, \qquad K_t := \{z \in \mathbb{H} \ : \ t \geq \tau(z)\}.$$

The set H_t is those points in $\mathbb{H}$ for which the solution to (4.3) is still defined at time t. For each t, the function g_t is a conformal map from H_t to $\mathbb{H}$. The sets $K_t = \mathbb{H} \setminus H_t$ are called the SLE hulls.

Definition 4.2.1 gives us the set of SLE hulls $(K_t)_{t\geq 0}$ parameterised in "half-plane capacity" time. To explain exactly what we mean by that, we first need the definition of half-plane capacity.

Definition 4.2.2 (Half-plane capacity)**.** Let K be a compact $\mathbb{H}$-hull. The half-plane capacity of K is

$$\operatorname{hcap}(K) := \lim_{y\to\infty} y\mathbb{E}_{iy}[\Im(B_{\tau_K})],$$

where $\tau_K = \inf\{t \geq 0 \ : \ B_t \in K \cup \mathbb{R}\}$ is the first time the Brownian motion hits the hull K or the real line.

The half-plane capacity is, in some sense, the height of the set as viewed from infinity. For some concrete examples, the half-plane capacity of the unit disc in the upper half plane is $\operatorname{hcap}(\overline{\mathbb{D}} \cap \mathbb{H}) = 1$, and the half-plane capacity of a vertical line of length 1 in the upper half plane is $\operatorname{hcap}((0, i]) = 1/2$.

We said before that the SLE hulls $(K_t)_{t\geq 0}$ are parameterised in terms of half-plane capacity. What we mean by that is that $\operatorname{hcap}(K_t) = 2t$ for all $t \geq 0$.

In Section 4.4, we will use the following definition and bound of the height of a compact hull. It is a previously known result, but we include the proof here for completeness.

Definition 4.2.3 (Height)**.** Let K be a compact $\mathbb{H}$-hull. Its height is defined to be

$$\operatorname{height}(K) := \sup\{\Im(z) \ : \ z \in K\}.$$

Lemma 4.2.4. *Let K be a compact $\mathbb{H}$-hull with* $\text{height}(K) > 2\sqrt{\alpha}$. *Then* $\text{hcap}(K) \geq \alpha$.

Proof. Since both the height of a hull and its half-plane capacity are translation invariant along the real line, and $\text{height}(K) > 2\sqrt{\alpha}$, we can assume that $2i\sqrt{\alpha} \in K$.

Let K' be the reflection of K in the imaginary axis. Further, let $\tilde{K}$ be the complement of the connected component of $\mathbb{H} \setminus (K \cup K')$ which contains infinity. (See Figure 4.1.)

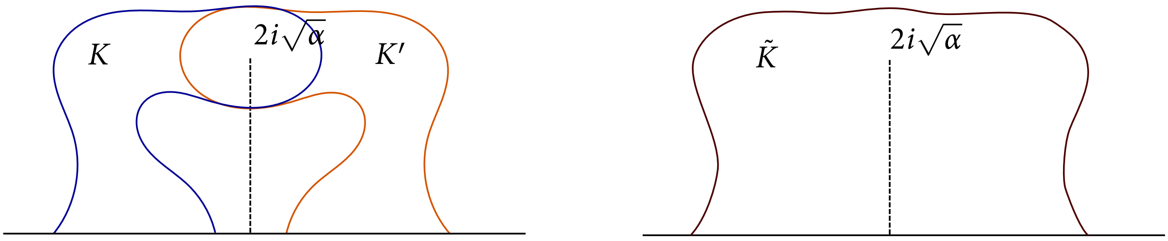


Figure 4.1: Left: The hull K and its reflection K'. Right: The joined and filled $\tilde{K}$.

By construction, $\tilde{K}$ contains the line segment $(0, 2i\sqrt{\alpha}]$ and therefore we know that

$$\text{hcap}(\tilde{K}) \geq 2\alpha.$$

To find our bound on the half-plane capacity, we need to consider the limit given in Definition 4.2.2. We will de-construct the expectation in that limit.

We can ignore the event that the Brownian motion exits through the real line since that does not contribute to the expectation, i.e.

$$\mathbb{E}_{iy}[\Im(B_{\tau_{\tilde{K}}})] = \mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial\tilde{K}\}\right].$$

Since $\partial\tilde{K} \subset \partial K \cup \partial K'$, we know that $\left\{B_{\tau_{\tilde{K}}} \in \partial\tilde{K}\right\} \subset \left\{B_{\tau_{\tilde{K}}} \in \partial K\right\} \cup \left\{B_{\tau_{\tilde{K}}} \in \partial K'\right\}$. Therefore, we can bound the half-plane capacity of $\tilde{K}$ as follows:

$$\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial\tilde{K}\}\right] \leq \mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K\}\right] + \mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K'\}\right] \quad (4.4)$$

By symmetry, we know that

$$\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K\}\right] = \mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K'\}\right],$$

and so, substituting this into (4.4) we find

$$\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial\tilde{K}\}\right] \leq 2\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K\}\right]. \tag{4.5}$$

Since we know that, if a Brownian motion stopped at $\tau_{\tilde{K}}$ exits through K, then the same Brownian motion stopped at τ_K also exits through K, we see that $\{B_{\tau_{\tilde{K}}} \in \partial K\} \subset \{B_{\tau_K} \in \partial K\}$. Also, on the event $\{B_{\tau_{\tilde{K}}} \in \partial K\}$, the stopping times τ_K and $\tau_{\tilde{K}}$ are equal which means, in particular, that $\Im(B_{\tau_{\tilde{K}}}) = \Im(B_{\tau_K})$. Using these facts in conjunction with (4.5), we find

$$\begin{aligned}
\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial\tilde{K}\}\right] &\leq 2\mathbb{E}_{iy}\left[\Im(B_{\tau_{\tilde{K}}})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K\}\right] \\
&= 2\mathbb{E}_{iy}\left[\Im(B_{\tau_K})\mathbb{1}\{B_{\tau_{\tilde{K}}} \in \partial K\}\right] \\
&\leq 2\mathbb{E}_{iy}\left[\Im(B_{\tau_K})\mathbb{1}\{B_{\tau_K} \in \partial K\}\right] \\
&= 2\mathbb{E}_{iy}\left[\Im(B_{\tau_K})\right].
\end{aligned}$$

Multiplying by y and taking limits, we find that

$$\mathrm{hcap}(\tilde{K}) \leq 2\mathrm{hcap}(K),$$

which, combined with the fact that $\mathrm{hcap}(\tilde{K}) \geq 2\alpha$, gives us the result that

$$\mathrm{hcap}(K) \geq \alpha.$$

□

Corollary 4.2.5. *Let K be a compact $\mathbb{H}$-hull with half place capacity* $\mathrm{hcap}(K) = \alpha$. *Then we can bound the height of K by*

$$\mathrm{height}(K) \leq 2\sqrt{\alpha}.$$

Proof. By Lemma 4.2.4 we know that, if $\mathrm{height}(K) > 2\sqrt{\alpha}$, then $\mathrm{hcap}(K) \geq \alpha$. Therefore, since $\mathrm{hcap}(K) = \alpha$, we must have $\mathrm{height}(K) \leq 2\sqrt{\alpha}$. □

4.2.2 Reverse SLE

The time reversibility of Brownian motion allows us to give meaning to the idea of a "reverse SLE" as well as the forward one defined in Section 4.2.1. The reverse SLE is the one we will need to use for the coupling between the Gaussian free field and SLE which we use, first shown in [She10]. We will explain more about this coupling in Section 4.2.5.

In order to follow the notation of [She10] we need to use a slightly different definition for the reverse SLE; it needs to be centered in some way so that the growth of the SLE is at the origin. (The intuition for this is much easier if we assume that the Rohde-Schramm theorem is true and the SLE hull is generated by a curve.) We will explain how the different normalisation relates to that given in Definition 4.2.1 and the maps $(\hat{f}_t)$ used in Lemma 4.2.7.

Definition 4.2.6 (Reverse SLE)**.** Fix $\kappa \geq 0$ and let B_t be a Brownian motion started at zero. For each $z \in \mathbb{H}$ let $f_t(z)$ be the solution to

$$df_t(z) = -\frac{2}{f_t(z)}dt - \sqrt{\kappa}dB_t, \tag{4.6}$$

with $f_0(z) = z$. We will call the collection of maps (f_t) a reverse SLE_κ.

We now show how Definitions 4.2.1 and 4.2.6 of the forward and reverse SLE flow relate to each other and to the maps $(\hat{f}_t)$ defined in (4.2). This is a known result (see Section 4 in [Law08], for example), but we check it here with our specific combination of centred reverse flow and un-centred forward flow.

Lemma 4.2.7. *Fix a time $T \geq 0$ and some value $\kappa \geq 0$. Let (g_t) be an SLE_κ with driving function (ξ_t), (f_t) a reverse SLE_κ and define*

$$\hat{f}_t(z) := g_t^{-1}(z + \xi_t).$$

Then

$$\hat{f}_T \overset{d}{=} f_T.$$

Note that the equality in distribution holds for a single fixed time T, not for the range of times $t \in [0, T]$.

Proof. The definition for the time reversed SLE flow, for fixed $T > 0$ is

$$dr_t(z) = -\frac{2}{r_t(z) - \xi_{T-t}} dt, \qquad t \in [0, T],$$

with $r_0(z) = z$. This is simply a time change of (4.3) given by $t \mapsto T - t$. The reverse flow defined in this way has the property that $r_T = g_T^{-1}$ (see, for example, Section 4 in [Law08]).

If we now define $\hat{r}_t(z) = r_t(z) - \xi_{T-t}$, we find that

$$d\hat{r}_t(z) = -\frac{2}{\hat{r}_t(z)} - d\xi_{T-t} \tag{4.7}$$

which, by reversibility of Brownian motion, can be written as

$$d\hat{r}_t(z) = -\frac{2}{\hat{r}_t(z)} - \sqrt{\kappa} dB_t. \tag{4.8}$$

The difference between (4.6) and (4.8) is the initial conditions, which are often only implicitly defined. All of the standard definitions, including (4.6), give the initial condition that the map in question at time zero should be equal to the identity. As this is true for (4.7), i.e. $r_0(z) = z$, we see that the initial condition for $\hat{r}$ is

$$\hat{r}_0(z) = z - \xi_T.$$

We therefore see that

$$f_0(z) = \hat{r}_0(z + \xi_T)$$

and, since they are driven by the same SDE, this equality will continue through time. Therefore, for time T we see that

$$\begin{aligned} f_T(z) &= \hat{r}_T(z + \xi_T) \\ &= r_T(z + \xi_T) - \xi_{T-T} \\ &= r_T(z + \xi_T) \\ &= g_T^{-1}(z + \xi_T). \end{aligned}$$

This means that, for a single fixed time T, the reverse map f from Definition 4.2.6 is the same as the map $\hat{f}$. □

The following (well known) bounds will be useful for us later on.

Lemma 4.2.8. *Let $\kappa \geq 0$ and let (f_t) be a reverse SLE$_\kappa$. Then, for any fixed $y > 0$, the imaginary part of $f_t(iy)$ is increasing but bounded above for all $t \geq 0$ by*

$$\Im(f_t(iy)) \leq \sqrt{4t + y^2}.$$

Proof. Fix $y > 0$ and, for ease of notation, let $f_t(iy) = Z_t = X_t + iY_t$. Then we know that Z_t satisfies the SDE (4.6) with $Z_0 = iy$. Looking at the imaginary part of the equation shows that the imaginary part of Z_t, Y_t, satisfies the SDE

$$\begin{aligned} dY_t &= -\Im\left(\frac{2}{Z_t}\right)dt \\ &= -\Im\left(\frac{2(X_t - iY_t)}{|Z_t|^2}\right)dt \\ &= \frac{2Y_t}{X_t^2 + Y_t^2}dt, \end{aligned} \tag{4.9}$$

with $Y_0 = y$. Since $y > 0$, we can see that the right hand side of (4.9) is greater than zero and stays positive. This tells us that Y_t is increasing, finishing the first part of the proof.

The fact that Y_t is increasing also shows us that Z_t is bounded away from 0, since $\Im(Z_t) \geq y_0 > 0$. That lets us see that the coefficients of the SDE which Z satisfies,

$$dZ_t = -\frac{2}{Z_t}dt - \sqrt{\kappa}dB_t,$$

are Lipschitz in space. Therefore, Z_t exists as a strong solution. See Theorem 2.5 of [KS91], for example.

For the second part, we use the fact that $X_t^2 \geq 0$ to see the bound

$$\begin{aligned} dY_t &= \frac{2Y_t}{X_t^2 + Y_t^2}dt \\ &\leq \frac{2Y_t}{Y_t^2}dt \\ &= \frac{2}{Y_t}dt. \end{aligned}$$

The solution to the equation $x' = 2/x$ and the existence of strong solutions gives us our upper bound. $\square$

Corollary 4.2.9. *Let $\kappa \geq 0$ and let (f_t) be a reverse SLE$_\kappa$. Then, for any fixed $y > 0$, the absolute value of the derivative, $|f_t'(iy)|$, is bounded above by*

$$|f_t'(iy)| \leq \frac{4}{y}\sqrt{4t + y^2}.$$

Proof. As f_t is a reverse SLE flow we know that $f_t : \mathbb{H} \to H_t$ for $H_t = \mathbb{H} \setminus K_t$. Koebe's 1/4 Theorem tells us that

$$|f_t'(iy)|\text{dist}(iy, \partial\mathbb{H}) \leq 4 \cdot \text{dist}(f_t(iy), \partial H_t). \tag{4.10}$$

We know that $\text{dist}(iy, \partial\mathbb{H}) = y$ and

$$\begin{aligned}\text{dist}(f_t(iy), \partial H_t) &\leq \text{dist}(f_t(iy), \partial\mathbb{H}) \\ &= \Im(f_t(iy)).\end{aligned}$$

Therefore we can use the bound from Lemma 4.2.8 along with (4.10) to see

$$|f_t'(iy)|y \leq 4\sqrt{4t + y^2}.$$

Re-arranging gives the result. $\square$

Lemma 4.2.10. *Let $\kappa \geq 0$, let (f_t) be a reverse SLE$_\kappa$, and let (ξ_t) be its driving process. Then, for any fixed $y > 0$, the absolute value of the real part of $f_t(iy)$ is bounded above by the absolute value of ξ_t, i.e.*

$$|\Re(f_t(iy))| \leq |\xi_t|.$$

Proof. As before, we write $f_t(iy) = Z_t = X_t + iY_t$ so that $\Re(f_t(iy)) = X_t$. By taking real parts of (4.6), we know that X_t satisfies the SDE

$$\begin{aligned}dX_t &= -\Re\left(\frac{2}{Z_t}\right)dt - \sqrt{\kappa}dB_t \\ &= -\Re\left(\frac{2(X_t - iY_t)}{|Z_t|^2}\right)dt - \sqrt{\kappa}dB_t \\ &= -\frac{2X_t}{X_t^2 + Y_t^2}dt - \sqrt{\kappa}dB_t,\end{aligned}$$

with $X_0 = 0$. The drift term is always towards the origin and, as we saw in Lemma 4.2.8, strong solutions to this SDE exist. Therefore, we see that $|X_t| \le \sqrt{\kappa}|B_t|$, finishing the proof. □

4.2.3 *Zero boundary Gaussian free field*

We now briefly introduce the zero boundary Gaussian free field as a building block to the Neumann boundary Gaussian Free Field of Subsection 4.2.4. We also collect some of the known results of the GFF that we rely on.

In order to define the zero boundary GFF, we first need to define the Dirichlet inner product.

Definition 4.2.11 (Dirichlet Inner Product)**.** Let $D \subset \mathbb{C}$ be a bounded domain. Let $C_0^\infty(D)$ be the set of smooth, compactly supported functions on D. For f, g in $C_0^\infty(D)$, define the Dirichlet inner product as

$$\langle f, g\rangle_\nabla = \frac{1}{2\pi}\int_D \nabla f(z) \cdot \nabla g(z) dz.$$

The zero boundary GFF can be viewed as a standard Gaussian on the Hilbert space completion of $C_0^\infty(D)$ under the Dirichlet inner product, which we denote $H_0(D)$. There are several equivalent ways of defining the GFF. We give the definition used in [She07].

Definition 4.2.12 (Zero boundary Gaussian free field)**.** We say that h is a zero boundary Gaussian free field if, for any $f, g \in H_0(D)$, the random variables $\langle h, f\rangle_\nabla$ and $\langle h, g\rangle_\nabla$ are centred Normal random variables with covariance equal to

$$\mathbb{E}\left[\langle h, f\rangle_\nabla \langle h, g\rangle_\nabla\right] = \langle f, g\rangle_\nabla .$$

One of the difficulties with working with the GFF is that it not defined pointwise, and so we cannot treat it as a Gaussian field on D. It does not exist as a random variable in $H_0(D)$. A Weyl asymptotics argument can be used to show that it is a member of the dual space $H_0^{-1}(D)$, and there is a version which exists as a member of $\mathcal{D}'(D)$. For details, see the discussion in Section 3.3 of [She10], for example.

We can use regularisations to talk about approximations of the field pointwise. The circle average regularisation, in particular, will be useful for us.

Definition 4.2.13 (Circle Average)**.** Let D be a domain in $\mathbb{C}$, and let $\tilde{h}$ be a zero boundary GFF in D. Let $\varepsilon > 0$ and $z \in \{z' \in D \,:\, \operatorname{dist}(z', \partial D) > \varepsilon\}$. Define the function ξ_ε^z by

$$\xi_\varepsilon^z(y) = -\log(|z-y| \vee \varepsilon) + \phi^z(y), \tag{4.11}$$

where ϕ^z is harmonic in D and equal to $\log(|z-y|)$ for $y \in \partial D$. Then average of $\tilde{h}$ on the circle of radius ε centred at z is defined by

$$\tilde{h}_\varepsilon(z) := \left\langle \tilde{h},\, \xi_\varepsilon^z \right\rangle_\nabla . \tag{4.12}$$

The circle average process appears in Section 3.1 of [DS11]. Essentially, we can think of (4.12) as a circle average because we know that $-\Delta \xi_\varepsilon^z = 2\pi \nu_\varepsilon^z$, where ν_ε^z is the uniform measure on the circle of radius ε, centred at z. Therefore, a formal integration by parts lets us say that

$$\left\langle \tilde{h},\, \xi_\varepsilon^z \right\rangle_\nabla = \left\langle \tilde{h},\, \nu_\varepsilon^z \right\rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard L^2 inner product. For more details about the justification of the integration by parts, and viewing the GFF as it acts on measures, see Section 2.6 of [She07].

Definition 4.2.17 and the form of ξ_ε^z allows us to calculate the variance of the circle averages. The following appears as Proposition 3.2 in [DS11].

Lemma 4.2.14. *Let D be a domain in $\mathbb{C}$, and let $\tilde{h}$ be a zero boundary GFF on D. Fix $\varepsilon > 0$, and let $z \in \mathbb{D}$ be further than ε from the boundary. Then the circle average, $\tilde{h}_\varepsilon(z)$, is a centred Gaussian random variable with variance*

$$\mathbb{E}\left[\tilde{h}_\varepsilon(z)^2\right] = -\log \varepsilon + \log R(z; D),$$

where $R(z; D)$ is the conformal radius of D viewed from z.

We will also need to use domain Markov property of the GFF. A thorough explanation can be found in Section 2.6 of [She07].

Theorem 4.2.15 (Domain Markov Property)**.** *Let D be a domain, and $U \subset D$ a subdomain. Let h be a GFF on D. Then we can write $h = h_1 + h_2$, where*

- *h_1 and h_2 are independent,*
- *h_2 is a GFF on U and zero on $D \setminus U$,*
- *$h_1, h_2 \in H_0^{-1}(D)$, and*
- *h_1 is harmonic on U and agrees with h on $D \setminus U$.*

Finally, we rely on the conformal invariance of the GFF. It is a property inherited from the conformal invariance of the Dirichlet inner product. See, for example, Section 2.1 of [She07].

Theorem 4.2.16 (Conformal Invariance)**.** *Let D and D' be two domains in $\mathbb{C}$, and let ψ be a bijective, conformal map from D to D'. Further, let h be a GFF on D, and h' a GFF on D'. Then $h \stackrel{d}{=} h' \circ \psi$ in the sense that, for any function $f \in H_0(D)$, we have*

$$\langle h, f \rangle_\nabla \stackrel{d}{=} \left\langle h', f \circ \psi^{-1} \right\rangle_\nabla .$$

4.2.4 Neumann boundary Gaussian free field

In order to couple the reverse SLE of Section 4.2.2 with a Gaussian free field, as in [She10], we need to understand the Neumann (or free) boundary Gaussian free field. We give a brief introduction here. Much more information can be found in, for example, Section 4.2.4 of [MS13].

The Dirichlet inner product defined in Definition 4.2.11 can be extended from the set of smooth, compactly supported functions to $f, g \in C^\infty(D)$, the set of smooth functions on D. It is no longer strictly an inner product, as all constant functions will have Dirichlet inner product equal to zero. To get around this, we define the space $\overline{H}(D)$ as the Hilbert space completion under the Dirichlet inner product of the functions $f \in C^\infty(D)$ which we view modulo additive constants.

Note that, since we only ever look at test functions defined up to an additive constant, the Neumann boundary GFF is defined as a distribution only up to an additive constant. Later, we will need to fix this additive constant.

Definition 4.2.17 (Neumann boundary Gaussian free field)**.** We say that h is a Neumann boundary Gaussian free field if, for any $f, g \in \overline{H}(D)$, the random variables $\langle h, f \rangle_\nabla$ and $\langle h, g \rangle_\nabla$ are centred Normal random variables with covariance equal to

$$\mathbb{E}\left[\langle h, f \rangle_\nabla \langle h, g \rangle_\nabla\right] = \langle f, g \rangle_\nabla .$$

As in the zero boundary case, the Neumann boundary GFF is not defined pointwise, but it exists as a continuous linear functional. See Section 3.3 of [She10]. Also, it inherits a conformal invariance property from the conformal invariance of the Dirichlet inner product. More information about how the Neumann boundary GFF can be defined on measures, using the Green function for Laplace's equation with Neumann boundary conditions, can be found in [Ber15c].

In order to use the conformal invariance of the field, and to understand the statement of Theorem 4.2.25 in Section 4.2.5, we need to be clear what is means by $h \circ \psi$, where h is a Neumann boundary GFF on D and $\psi : U \to D$ is a bijective, conformal map. We simply mean that, for any function $f \in \overline{H}(U)$,

$$\langle h \circ \psi, f \rangle_\nabla := \left\langle h, f \circ \psi^{-1} \right\rangle_\nabla .$$

We will often talk about the "harmonic extension" of the Neumann boundary GFF from the boundary of some subdomain $U \subset D$ to the domain U. The following lemma, which appears in Section 4.2.5 of [MS13], allows us to do that.

Lemma 4.2.18. *Let h be a Neumann boundary GFF on a domain D. Then we can write*

$$h = \tilde{h} + \mathrm{Harm}_D(h),$$

where $\tilde{h}$ is a zero boundary GFF on D and $\mathrm{Harm}_D(h)$ *is an independent harmonic function on D, defined up to an additive constant.*

The proof relies on the fact that the space of functions $H_0(D)$ is orthogonal to the space of harmonic functions on D under the Dirichlet inner product.

Lemma 4.2.18 lets us define the harmonic extension of the Neumann boundary GFF as follows.

Definition 4.2.19 (Harmonic extension)**.** Let $U \subset D$ be a subdomain, and let h be a Neumann boundary GFF on D with decomposition

$$h = \tilde{h} + \mathrm{Harm}_D(h),$$

where $\tilde{h}$ is a zero boundary GFF on D and $\mathrm{Harm}_D(h)$ is an independent harmonic function defined up to an additive constant, as in Lemma 4.2.18. Using the Markov property for $\tilde{h}$, we can write

$$\tilde{h} = h' + u,$$

where h' is a zero boundary GFF on U and u is harmonic on U. Then the harmonic extension of h from ∂U to U is defined to be

$$\mathrm{Harm}_U(h) = \mathrm{Harm}_D(h)|_U + u|_U.$$

When we fix the additive constant that is still free in the definition of the Neumann boundary GFF, it can be the case that $\tilde{h}$ and $\mathrm{Harm}_D(h)$ are no longer independent. For example, if we normalise the GFF h by specifying that its average value on some set is equal to zero, there will be some interaction between $\tilde{h}$ and $\mathrm{Harm}_D(h)$. This is because the zero boundary GFF $\tilde{h}$ is fully specified, so the function $\mathrm{Harm}_D(h)$ will have to shift by a constant to compensate for the average value of $\tilde{h}$ on the set in question.

However, we will always normalise by specifying the value of $\mathrm{Harm}_D(h)$ at a point. This will ensure that we still have independence:

Lemma 4.2.20. *Let h be a Neumann boundary GFF on a domain D, and let $z \in D$. If we specify that* $\mathrm{Harm}_D(h)(z) = 0$ *in the decomposition of Lemma 4.2.18, we know that $\tilde{h}$ and* $\mathrm{Harm}_D(h)$ *are still independent.*

Proof. In Lemma 4.2.18, $\mathrm{Harm}_D(h) \in \overline{\mathrm{Harm}}(D)$ represents an equivalence class of harmonic functions, $[\mathrm{Harm}_D(h)] \subset \mathrm{Harm}(D)$, where two functions are equivalent if they agree up to an additive constant. We can choose the unique representative of that equivalence class which equals 0 at z, and call that function $\mathrm{Harm}_D(h)$. Since $\mathrm{Harm}_D(h)$ depends only on the equivalence class $[\mathrm{Harm}_D(h)]$, and the equivalence class is independent of $\tilde{h}$, then $\mathrm{Harm}_D(h)$ is also independent of $\tilde{h}$. □

We will need some quantitative bounds on the variance of the harmonic part of the Neumann boundary GFF when it is pinned at a certain point. We will use the following, which appears as Lemma 2.9 in [GMS14]:

Lemma 4.2.21. *Let* $\mathrm{Harm}_{\mathbb{D}}(h)$ *be the harmonic part of a Neumann boundary GFF on the unit disc* $\mathbb{D}$*, normalised so that* $\mathrm{Harm}_{\mathbb{D}}(h)(0) = 0$*. Then, for any* $z, w \in \mathbb{D}$, $\mathrm{Harm}_{\mathbb{D}}(h)(z)$ *and* $\mathrm{Harm}_{\mathbb{D}}(h)(w)$ *are jointly Gaussian with mean zero and covariance*

$$\mathbb{E}\left[\mathrm{Harm}_{\mathbb{D}}(h)(z)\mathrm{Harm}_{\mathbb{D}}(h)(w)\right] = -2\log|1 - z\overline{w}|.$$

We can use a coordinate change from the upper half plane to the unit disc, along with conformal invariance of Gaussian free field, to get the following bound on the variance of the harmonic part of the Neumann boundary GFF on the upper half plane.

Lemma 4.2.22. *Let* $\mathrm{Harm}_{\mathbb{H}}(h)$ *be the harmonic part of a Neumann boundary GFF on the upper half plane* $\mathbb{H}$*, normalised so that* $\mathrm{Harm}_{\mathbb{H}}(h)(iy_0) = 0$ *for some (fixed)* $y_0 > 0$*. Then, for any* $z = x + iy \in \mathbb{H}$ *we see that*

$$\mathbb{E}\left[\mathrm{Harm}_{\mathbb{H}}(h)(z)^2\right] = -2\log y + 2\log\left(x^2 + (y + y_0)^2\right) - 2\log 4y_0.$$

Proof. Let h' be a Neumann boundary GFF on the unit disc, $\mathbb{D}$, normalised so that

$$\mathrm{Harm}_{\mathbb{D}}(h')(0) = 0,$$

and let m_{y_0} be the Möbius transformation

$$m_{y_0}(z) = \frac{z - iy_0}{z + iy_0}.$$

Then $m_{y_0} : \mathbb{H} \to \mathbb{D}$ so that $m_{y_0}(iy_0) = 0$. Therefore, if we set

$$h = h' \circ m_{y_0},$$

then h is a Neumann boundary GFF on $\mathbb{H}$ and, by conformal invariance of the GFF and harmonic extensions, we see that for $z \in \mathbb{H}$,

$$\begin{aligned} \mathrm{Harm}_{\mathbb{H}}(h)(z) &= \mathrm{Harm}_{\mathbb{H}}(h' \circ m_{y_0})(z) \\ &= \mathrm{Harm}_{\mathbb{D}}(h')(m_{y_0}(z)). \end{aligned}$$

Therefore, with $z = iy_0$, we see that $\mathrm{Harm}_{\mathbb{H}}(h)(iy_0) = \mathrm{Harm}_{\mathbb{D}}(h')(m_{y_0}(iy_0)) = 0$. So, h is normalised in the way that we want. We can now calculate its variance using the coordinate change and Lemma 4.2.21, as follows.

Let $z = x + iy \in \mathbb{H}$ with $y \in (0, y_0)$. Then, by Lemma 4.2.21, we see

$$\begin{aligned} \mathbb{E}\left[\mathrm{Harm}_{\mathbb{H}}(h)(x+iy)^2\right] &= \mathbb{E}\left[\mathrm{Harm}_{\mathbb{D}}(h')(m_{y_0}(x+iy))\right] \\ &= -2\log\left|1 - |m_{y_0}(x+iy)|^2\right|. \end{aligned} \tag{4.13}$$

Expanding the term inside the log in (4.13), we find

$$\begin{aligned} 1 - |m_{y_0}(x+iy)|^2 &= 1 - \frac{x^2 + (y-y_0)^2}{x^2 + (y+y_0)^2} \\ &= \frac{(y+y_0)^2 - (y-y_0)^2}{x^2 + (y+y_0)^2} \\ &= \frac{4y_0 y}{x^2 + (y+y_0)^2}. \end{aligned} \tag{4.14}$$

Substituting (4.14) into (4.13), we see that

$$\begin{aligned} \mathbb{E}\left[\mathrm{Harm}_{\mathbb{H}}(h)(x+iy)^2\right] &= -2\log\left(\frac{4y_0 y}{x^2 + (y+y_0)^2}\right) \\ &= -2\log y + 2\log\left(x^2 + (y+y_0)^2\right) - 2\log 4y_0, \end{aligned}$$

completing the proof. □

We should note that, if we restrict to the case where $z = x + iy \in \mathbb{H}$ with $y \in (0, y_0)$, Lemma 4.2.22 easily gives us the slightly simpler inequality that

$$\mathbb{E}\left[\mathrm{Harm}_{\mathbb{H}}(h)(z)^2\right] \leq -2\log y + 2\log\left(x^2 + 4y_0^2\right) - 2\log 4y_0, \tag{4.15}$$

because $(y + y_0)^2 \leq 4y_0^2$.

Corollary 4.2.23. *Let* $\mathrm{Harm}_{\mathbb{H}}(h)$ *be the harmonic part of a Neumann boundary GFF on the upper half plane* $\mathbb{H}$*, normalised so that* $\mathrm{Harm}_{\mathbb{H}}(h)(iy_0) = 0$ *for some (fixed)* $y_0 > 0$*. Then, for any purely imaginary* $z = iy \in \mathbb{H}$ *with* $y \in (0, y_0)$*, we see that*

$$\mathbb{E}\left[\mathrm{Harm}_{\mathbb{H}}(h)(iy)^2\right] \leq 2\log y_0 - 2\log y.$$

Proof. This is simply the case of $x = 0$ in (4.15). □

Lemma 4.2.24. *Let* K *be a compact hull, and let* $H = \mathbb{H} \setminus K$*. Let* h *be a Neumann boundary GFF on* $\mathbb{H}$ *normalised so that its harmonic part vanishes at* iy_0 *for some* $y_0 > 0$*. Let* $\mathrm{Harm}_H(h)$ *be the harmonic extension of* h *from* ∂H *to* H*. Then for any* $z = x + iy \in H$ *with* $y < y_0$*, we know that* $\mathrm{Harm}_H(h)(z)$ *is a centred Gaussian random variable with variance*

$$\mathbb{E}\left[\mathrm{Harm}_H(h)(z)^2\right] \leq -3\log(\mathrm{dist}(z, \partial H)) + 2\log(x^2 + 4y_0^2) + C,$$

where C *is a constant which depends on* y_0 *only.*

Proof. We know from Lemmas 4.2.18 and 4.2.20 that we can write $h = \tilde{h} + \mathrm{Harm}_{\mathbb{H}}(h)$, where $\tilde{h}$ is a zero boundary GFF on $\mathbb{H}$ and $\mathrm{Harm}_{\mathbb{H}}(h)$ is an independent harmonic function, determined uniquely. Using the Markov property of the zero boundary GFF we can also say that, restricting to H, we have

$$\tilde{h}|_H = h' + u,$$

where h' is a zero boundary GFF on H and u is an independent harmonic function. We can therefore see that

$$\begin{aligned} h|_H &= h' + u + \mathrm{Harm}_{\mathbb{H}}(h)|_H \\ &= h' + \mathrm{Harm}_H(h). \end{aligned}$$

Because u is independent of h' and $\mathrm{Harm}_{\mathbb{H}}(h)$ is independent of $\tilde{h} = h' + u$, we see that h' and $\mathrm{Harm}_H(h) = u + \mathrm{Harm}_{\mathbb{H}}(h)$ are independent.

Let $\varepsilon < \mathrm{dist}(z, \partial H)$. Then we can take the circle average of the GFFs to see that

$$h'_\varepsilon(z) + \mathrm{Harm}_H(h)(z) = \tilde{h}_\varepsilon(z) + \mathrm{Harm}_{\mathbb{H}}(h)(z).$$

By independence of the terms on both the left hand side and the right hand side, we can say that

$$\mathrm{Var}(\mathrm{Harm}_H(h)(z)) \leq \mathrm{Var}(\tilde{h}_\varepsilon(z)) + \mathrm{Var}(\mathrm{Harm}_{\mathbb{H}}(h)(z)). \tag{4.16}$$

We can use Lemmas 4.2.14 and 4.2.22 to bound the terms on the right hand side of (4.16) to get

$$\mathrm{Var}(\mathrm{Harm}_H(h)(z)) \leq -\log\varepsilon + \log R(z;\mathbb{H}) - 2\log y + 2\log(x^2 + 4y_0^2) - 2\log 4y_0.$$

We know that $\log R(z;\mathbb{H})$ is uniformly bounded above for all $z = x + iy$ with $y \in [0, y_0]$. Let us write this upper bound along with the $-2\log 4y_0$ term as the constant C, so

$$\mathrm{Var}(\mathrm{Harm}_H(h)(z)) \leq -\log\varepsilon - 2\log y + 2\log(x^2 + 4y_0^2) + C.$$

Now let us take ε as large as we can, i.e. $\varepsilon = \mathrm{dist}(z, \partial H)$. Noting also that, as H is a subset of $\mathbb{H}$, $\mathrm{dist}(z, \partial\mathbb{H}) = y \geq \mathrm{dist}(z, \partial H)$, we find

$$\mathrm{Var}(\mathrm{Harm}_H(h)(z)) \leq -3\log\mathrm{dist}(z, \partial H) + 2\log(x^2 + 4y_0^2) + C,$$

as we wanted. □

4.2.5 *Coupling reverse SLE and Neumann boundary GFF*

We now set out the coupling between reverse SLE and the Neumann boundary GFF, first proved in [She10]. Throughout, fix $\kappa > 0$ and $Q = \frac{2}{\sqrt{\kappa}} + \frac{\sqrt{\kappa}}{2}$.

Theorem 4.2.25 (GFF SLE coupling). *Let (f_t) be the reverse SLE$_\kappa$ flow as defined by (4.6), with hulls (K_t). Let h be a Neumann boundary GFF on $\mathbb{H}$, independent of (f_t), normalised so that* *the harmonic part vanishes at iy_0 for some $y_0 > 0$ to be specified later. For $t > 0$, let*

$$h_t = h \circ f_t + \frac{2}{\sqrt{\kappa}} \log|f_t|. \tag{4.17}$$

Then

$$h_t + Q\log|f_t'| \overset{d}{=} h_0$$

modulo a global additive constant.

Corollary 4.2.26. *In the same setting as Theorem 4.2.25, there exists a random constant (in space) b_t such that*

$$|f_t'(iy)| = \left(\frac{y}{|f_t(iy)|}\right)^{\frac{2}{Q\sqrt{\kappa}}} \exp\left(\frac{1}{Q}\left(\mathrm{Harm}_{\mathbb{H}}(h')(iy) - \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy) + b_t\right)\right),$$

where h' is also a Neumann boundary GFF, normalised so that the harmonic part vanishes at iy_0.

We should note that the joint distribution of h' and $h \circ f_t$ will not matter anywhere in the rest of this chapter.

Proof. Theorem 4.2.25 tells us that $h_t + Q\log|f_t'| \overset{d}{=} h_0$ modulo an additive constant. Let b_t be this constant, so that

$$h_t + Q\log|f_t'| - b_t \overset{d}{=} h_0. \tag{4.18}$$

Let

$$h_0' = h_t + Q\log|f_t'| - b_t \tag{4.19}$$

so that $h_0' \overset{d}{=} h_0$, i.e. h_0' is a Neumann boundary GFF on $\mathbb{H}$, h', plus a log singularity, $\frac{2}{\sqrt{\kappa}}\log|z|$.

Re-arranging (4.19) gives us

$$\begin{aligned} Q\log|f_t'(iy)| &= (h_0' - h_t)(iy) + b_t \\ &= (h' - h \circ f_t)(iy) + \frac{2}{\sqrt{\kappa}}\log|iy| - \frac{2}{\sqrt{\kappa}}\log|f_t(iy)| + b_t \\ &= (h' - h \circ f_t)(iy) + \frac{2}{\sqrt{\kappa}}\log\frac{y}{|f_t(iy)|} + b_t. \end{aligned} \tag{4.20}$$

The left hand side of (4.20) is harmonic in $\mathbb{H}$, as are the last two terms on the right hand side. The terms h' and $h \circ f_t$ are not defined pointwise, but their difference is. Furthermore, since all other terms are harmonic, the difference $h' - h \circ f_t$ must also be harmonic.

Integrating with respect to the Poisson kernel on the real line lets us look at the harmonic extensions of these function from $\mathbb{R}$ to $\mathbb{H}$. As both sides of the equation are harmonic already, this has no effect on the functions. However, linearity of integration against the Poisson kernel lets us say that

$$Q \log |f_t'(iy)| = \mathrm{Harm}_{\mathbb{H}}(h')(iy) - \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy) + \frac{2}{\sqrt{\kappa}} \log \frac{y}{|f_t(iy)|} + b_t, \tag{4.21}$$

which rearranges further to give

$$|f_t'(iy)| = \left(\frac{y}{|f_t(iy)|}\right)^{\frac{2}{Q\sqrt{\kappa}}} \exp\left(\frac{1}{Q}\left(\mathrm{Harm}_{\mathbb{H}}(h')(iy) - \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy) + b_t\right)\right).$$

□

4.3 Proof of Theorem 4.1.3

We restate the theorem:

Theorem 4.3.1. *Let $\kappa \neq 8$, and let $\hat{f}_t$ be the centred inverse of the Loewner flow as defined in* (1.6). *Then there exist constants $\varepsilon > 0$, $\delta > 0$ and $C > 0$ such that*

$$\mathbb{P}\left[|\hat{f}_t'(iy)| > y^{-(1-\varepsilon)}\right] \leq C y^{2+\delta} \tag{4.22}$$

for all $t \in [0,1]$ and $y \in (0,1)$.

Thanks to the distributional equality between the reverse flow, f_t, and the centered inverse, $\hat{f}_t$, shown in Lemma 4.2.7, we need only show condition (4.22) for the reverse flow.

Proposition 4.3.2. *Let (f_t) be a reverse SLE_κ for $\kappa > 0$ and $\kappa \neq 8$, coupled with a Neumann boundary GFF as in Corollary 4.2.26 and let $(b_t)_{t \geq 0}$ be the coupling constants. Finally, let $(\xi_t)_{t \geq 0}$ be the driving function of the reverse SLE f.*

Then there exists some $\varepsilon > 0$, $\delta > 0$ *and* $C > 0$ *such that, for all* $t \in [0,1]$ *and* $y \in [0,1]$ *we have*

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)}\right] \le Cy^{2+\delta}.$$

Proof. We need to break the probability up into events which are relatively easy to deal with. A simple union bound lets us say that

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)}\right] \le \mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \le -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \le y^{-\varepsilon}\right] + \\ + \mathbb{P}\left[b_t > -\varepsilon \log y\right] + \mathbb{P}\left[\sup_{t\in[0,1]} \xi_t > y^{-\varepsilon}\right].$$

The fact that the first term on the right hand side has the correct polynomial decay is the subject of Section 4.3.1, Proposition 4.3.3 in particular. We show that the coupling constant b_t has sub-exponential decay in Section 4.4, giving us the arbitrary polynomial decay that we need here. Finally, it is well known that the supremum of a Brownian motion over a finite time interval has sub-exponential tails, and so the third term decays faster than any polynomial as $y \to 0$. □

The details of this proof are given through the rest of the chapter.

4.3.1 Bounding the probability on the good event

The good event that we bound the probability on introduces constraints on the supremum of the driving function of the SLE and a bound on the coupling constant b_t introduced in Corollary 4.2.26. We show that these constraints both hold with very high probability in Section 4.4.

Throughout, we need to fix a point iy_0 for some $y_0 > 0$ that we use to normalise the Neumann boundary GFF used in the coupling arguments. In order to apply Lemma 4.2.24 we need to ensure that any complex point that we consider, especially those of the form $f_t(iy)$, have imaginary parts smaller than y_0. Happily, we need consider only times $t \in [0,1]$ and the starting points iy with $y \in [0,1]$. Therefore, Lemma 4.2.8 guarantees that

$$\Im(f_t(iy)) \leq \sqrt{4t + y^2} \leq \sqrt{5},$$

for all $t \in [0,1]$ and $y \in [0,1]$. Furthermore, it will be important in Section 4.4 that

$$y_0 > 2\sqrt{2}.$$

So we fix a point iy_0, with

$$y_0 > \max\left(\sqrt{5}, 2\sqrt{2}\right) = 2\sqrt{2}, \tag{4.23}$$

which we will use as the pinned point for the remainder of this chapter.

Proposition 4.3.3. *Let (f_t) be a reverse SLE$_\kappa$ for $\kappa > 0$ and $\kappa \neq 8$, coupled with a Neumann boundary GFF as in Corollary 4.2.26 and let $(b_t)_{t\geq 0}$ be the coupling constants. Finally, let $(\xi_t)_{t\geq 0}$ be the driving function of the reverse SLE f.*

Then there exists some $\varepsilon > 0$, $\delta > 0$ and $C > 0$ such that, for all $t \in [0,1]$ and $y \in [0,1]$ we have

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \leq y^{-\varepsilon}\right] \leq C y^{2+\delta}.$$

The argument begins with a slight modification to Markov's inequality. We start with the following lemma.

Lemma 4.3.4. *Fix $\varepsilon > 0$, $t > 0$ and $y > 0$. Let $\{f_s\}$ be the reverse SLE flow as defined by* (4.6)*. Then, on the event that $\{|f_t'(iy)| > y^{-(1-\varepsilon)}\}$, we know that* $\operatorname{dist}(f_t(iy), \partial H_t) > y^\varepsilon/4$.

Proof. As f_t is a reverse SLE flow we know that $f_t : \mathbb{H} \to H_t$ for $H_t = \mathbb{H} \setminus K_t$. Koebe's 1/4 Theorem tells us that

$$|f_t'(iy)| \operatorname{dist}(iy, \partial\mathbb{H}) \leq 4 \cdot \operatorname{dist}(f_t(iy), \partial H_t). \tag{4.24}$$

Since $\operatorname{dist}(iy, \partial\mathbb{H}) = y$ we know that, on the event $\{|f_t'(iy)| > y^{-(1-\varepsilon)}\}$,

$$|f_t'(iy)| \operatorname{dist}(iy, \partial\mathbb{H}) > y^\varepsilon. \tag{4.25}$$

Combining the inequalities in (4.24) and (4.25) gives the result. □

Corollary 4.3.5. *Fix* $\varepsilon > 0$, $t > 0$ *and* $y > 0$. *Let* $\{f_s\}_{s>0}$ *be the reverse SLE flow as defined by* (4.6). *Then we have that*

$$\left\{|f_t'(iy)| > y^{-(1-\varepsilon)}\right\} \subseteq \left\{|f_t(iy)| > \frac{1}{4}y^{\varepsilon}\right\}.$$

Proof. Since $H_t \subset \mathbb{H}$ we can see that

$$\begin{aligned}\operatorname{dist}(f_t(iy), \partial H_t) &\le \operatorname{dist}(f_t(iy), \partial\mathbb{H})\\ &= \Im(f_t(iy))\\ &\le |f_t(iy)|.\end{aligned}$$

Combining this with Lemma 4.3.4 finishes the proof. □

We can use Lemma 4.3.4 to get a good bound on the variance of $\operatorname{Harm}_{\mathbb{H}}(h \circ f_t)$ on the event that $\{|f_t'(iy)| > y^{-(1-\varepsilon)}\}$.

Lemma 4.3.6. *Let* (f_s) *be an* SLE_κ *process with driving function* (ξ_t). *Let* h *be an independent Neumann boundary GFF, normalised so that its harmonic part vanishes at the point* iy_0. *Write* $\mathbb{E}_h$ *for the expectation with respect to the law of* h *conditional on* (f_s). *Then, for fixed* $\varepsilon > 0$ *and all* $t \in [0,1]$, *and* $y \in [0,1]$,

$$\mathbb{E}_h\left[\operatorname{Harm}_{\mathbb{H}}(h \circ f_t)(iy)^2 \mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)}\}}\mathbb{1}_{\{\sup_{t\in[0,1]}\xi_t \le y^{-\varepsilon}\}}\right] \le C' - 7\varepsilon\log y,$$

where C' *is a constant depending on the pinned point* y_0 *only.*

Proof. Conformal invariance of solutions to Laplace's equation lets us say that

$$\operatorname{Harm}_{\mathbb{H}}(h \circ f_t)(\cdot) = \operatorname{Harm}_{H_t}(h)(f_t(\cdot)).$$

Since we chose the pinned point iy_0 high enough in (4.23), we can apply Lemma 4.2.24 to see that

$$\begin{aligned}&\mathbb{E}_h\left[\operatorname{Harm}_{\mathbb{H}}(h \circ f_t)(iy)^2 \mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)}\}}\mathbb{1}_{\{\sup_{t\in[0,1]}\xi_t \le y^{-\varepsilon}\}}\right] \le\\ &\quad\le \left(-3\log(\operatorname{dist}(f_t(iy), \partial H_t)) + 2\log(\Re(f_t(iy))^2 + 4y_0^2) + C\right)\mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)}\}},\end{aligned}$$

which, after applying Lemma 4.3.4 to the first term and Lemma 4.2.10, gives us the bound

$$\mathbb{E}_h \left[\mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy)^2 \mathbb{1}_{\{|f_t'(iy)| > y^{-(1-\varepsilon)}\}} \mathbb{1}_{\{\sup_{t\in[0,1]} \xi_t \le y^{-\varepsilon}\}} \right] \le$$
$$\le -3 \log\left(\frac{y^{\varepsilon}}{4}\right) + 2\log\left(y^{-2\varepsilon} + 4y_0^2\right) + C. \tag{4.26}$$

We know that $y^{-2\varepsilon} \ge 1$ and $y_0 > 1$, and so we can see that

$$\log(y^{-2\varepsilon} + 4y_0^2) \le \log(5y^{-2\varepsilon}) + \log(5y_0^2).$$

Therefore, we can simplify the bound in (4.26) to

$$\mathbb{E}_h \left[\mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy)^2 \mathbb{1}_{\{|f_t'(iy)| > y^{-(1-\varepsilon)}\}} \mathbb{1}_{\{\sup_{t\in[0,1]} \xi_t \le y^{-\varepsilon}\}} \right] \le -7\varepsilon \log y + C',$$

where C' depends only on y_0, as we wanted. □

Lemma 4.3.7. *Let (f_s) be a reverse SLE$_\kappa$ with driving function (ξ_s). Fix $t \in [0,1]$, $\varepsilon > 0$ and $y > 0$. Let $a > 1$ and $b = a/(a-1)$ be its Hölder conjugate. Let h be a Neumann boundary GFF independent of (f_s) and write $\mathbb{E}_h$ for expectation with respect to the law of h, conditionally on (ξ_s). Finally, let h' be a Neumann boundary GFF as defined by the coordinate change in Corollary 4.2.26. Define the event A by*

$$A := \{|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \le -\varepsilon \log y,\ \sup_{t\in[0,1]} |\xi_t| \le y^{-\varepsilon}\}.$$

Then

$$|f_t'(iy)|^a \mathbb{1}_A \le \tilde{C} y^{\frac{2a}{Q\sqrt{\kappa}} - \hat{C}\varepsilon} \mathbb{E}_h \left[\exp\left(\frac{a}{Q} \left(\mathrm{Harm}_{\mathbb{H}}(h')(iy)\right)\right)\right],$$

where the constants $\tilde{C}$ and $\hat{C}$ depend only on κ, the power a and the pinned point iy_0 used in the coupling.

Proof. We start with the relation found in Corollary 4.2.26, with the small addition of the indicator function, that

$$|f_t'(iy)| \mathbb{1}_A = \left(\frac{y}{|f_t(iy)|}\right)^{\frac{2}{Q\sqrt{\kappa}}} \exp\left(\frac{1}{Q}\left(\mathrm{Harm}_{\mathbb{H}}(h')(iy) - \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy) + b_t\right)\right) \mathbb{1}_A. \tag{4.27}$$

Corollary 4.3.5 tells us that $\mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)}\}} = \mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)}\}}\mathbb{1}_{\{|f_t(iy)|>y^{\varepsilon}/4\}}$. So, we can tell straight away that

$$\frac{y}{|f_t(iy)|}\mathbb{1}_A \le 4y^{1-\varepsilon}\mathbb{1}_A. \tag{4.28}$$

Furthermore we have ensured, by the definition of the event A, that the coupling constant b_t is bounded above by $b_t \le -\varepsilon \log y$. We therefore know that

$$\exp\left(\frac{1}{Q}b_t\right)\mathbb{1}_A \le y^{-\varepsilon/Q}\mathbb{1}_A. \tag{4.29}$$

Substituting (4.28) and (4.29) into (4.27) gives the inequality

$$|f_t'(iy)|\mathbb{1}_A \le (4y^{1-\varepsilon})^{\frac{2}{Q\sqrt{\kappa}}} \cdot y^{-\varepsilon/Q}\exp\left(\frac{1}{Q}(\mathrm{Harm}_{\mathbb{H}}(h')(iy) - \mathrm{Harm}_{\mathbb{H}}(h\circ f_t)(iy))\right)\mathbb{1}_A.$$

We can now take expectation with respect to the law of h conditional on (ξ_s) and use Hölder's inequality, taking care where we put the indicator function, to say that

$$\begin{aligned}|f_t'(iy)|\mathbb{1}_A &\le \\ &\le \left(4y^{1-\varepsilon\left(1+\frac{\sqrt{\kappa}}{2}\right)}\right)^{\frac{2}{Q\sqrt{\kappa}}}\mathbb{E}_h\left[\exp\left(\frac{a}{Q}(\mathrm{Harm}_{\mathbb{H}}(h')(iy))\right)\right]^{\frac{1}{a}}\mathbb{E}_h\left[\exp\left(-\frac{b}{Q}\mathrm{Harm}_{\mathbb{H}}(h\circ f_t)(iy)\right)\mathbb{1}_A\right]^{\frac{1}{b}}.\end{aligned} \tag{4.30}$$

Now we know that, conditionally on (ξ_s), $\mathrm{Harm}_{\mathbb{H}}(h\circ f_t)(iy)$ is Gaussian, and we know its conditional variance from Lemma 4.3.6 . So we can bound the final expectation in (4.30) as follows:

$$\begin{aligned}\mathbb{E}_h\left[\exp\left(-\frac{b}{Q}\mathrm{Harm}_{\mathbb{H}}(h\circ f_t)(iy)\right)\mathbb{1}_A\right] &\le \exp\left(\frac{b^2}{2Q^2}\cdot(C' - 7\varepsilon\log y)\right) \\ &= e^{\frac{b^2C'}{2Q^2}}y^{-\varepsilon\frac{7b^2}{2Q^2}}.\end{aligned} \tag{4.31}$$

Substituting (4.31) into (4.30) gives

$$\begin{aligned}|f_t'(iy)|\mathbb{1}_A &\le \left(4y^{1-\varepsilon\left(1+\frac{\sqrt{\kappa}}{2}\right)}\right)^{\frac{2}{Q\sqrt{\kappa}}}e^{\frac{bC'}{2Q^2}}y^{-\varepsilon\frac{7b}{2Q^2}}\mathbb{E}_h\left[\exp\left(\frac{a}{Q}(\mathrm{Harm}_{\mathbb{H}}(h')(iy))\right)\right]^{\frac{1}{a}} \\ &= 4^{\frac{2}{Q\sqrt{\kappa}}}e^{\frac{bC'}{2Q^2}}y^{\frac{2}{Q\sqrt{\kappa}}-\varepsilon\left(\frac{2}{Q\sqrt{\kappa}}+\frac{1}{Q}+\frac{7b}{2Q^2}\right)}\mathbb{E}_h\left[\exp\left(\frac{a}{Q}(\mathrm{Harm}_{\mathbb{H}}(h')(iy))\right)\right]^{\frac{1}{a}}\end{aligned}$$

We finish the proof by raising everything to the power a. □

Corollary 4.3.8. *In the same setting as Lemma 4.3.7, we have the following bound on the probability of interest:*

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \leq y^{-\varepsilon}\right] \leq C y^{a\left(1+\frac{2}{Q\sqrt{\kappa}}-\frac{a}{Q^2}\right)-\overline{C}\varepsilon},$$

where C *and* $\overline{C}$ *are constants which depend only on the Hölder exponent* a *and the pinned point* iy_0.

Proof. First, recall the definition of the set A from Lemma 4.3.7, that

$$A := \{|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \leq y^{-\varepsilon}\}.$$

The proof begins as a slight modification to Markov's inequality, and then an application of Lemma 4.3.7:

$$\begin{aligned}\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \leq y^{-\varepsilon}\right] &= \mathbb{E}\left[\mathbb{1}_{\{|f_t'(iy)|>y^{-(1-\varepsilon)},\, b_t\leq-\varepsilon\log y,\, \sup_{t\in[0,1]}\xi_t\leq y^{-\varepsilon}\}}\right] \\ &\leq \mathbb{E}\left[y^{a(1-\varepsilon)}|f_t'(iy)|^a \mathbb{1}_A\right] \\ &\leq \mathbb{E}\left[y^{a(1-\varepsilon)}\tilde{C}y^{\frac{2a}{Q\sqrt{\kappa}}-\hat{C}\varepsilon}\mathbb{E}_h\left[\exp\left(\frac{a}{Q}\mathrm{Harm}_{\mathbb{H}}(h')(iy)\right)\right]\right] \\ &= \tilde{C}y^{a\left(1+\frac{2}{Q\sqrt{\kappa}}\right)-\varepsilon(\hat{C}+a)}\mathbb{E}\left[\exp\left(\frac{a}{Q}\mathrm{Harm}_{\mathbb{H}}(h')(iy)\right)\right]. \end{aligned} \tag{4.32}$$

Now, we know a bound on the variance of $\mathrm{Harm}_{\mathbb{H}}(h')(iy)$ thanks to Corollary 4.2.23, so we can find a bound for the expectation in (4.32), as follows:

$$\begin{aligned}\mathbb{E}\left[\exp\left(\frac{a}{Q}\mathrm{Harm}_{\mathbb{H}}(h')(iy)\right)\right] &\leq \exp\left(\frac{a^2}{2Q^2}\left(2\log y_0 - 2\log y\right)\right) \\ &= y_0^{\frac{a^2}{Q^2}} y^{-\frac{a^2}{Q^2}}. \end{aligned} \tag{4.33}$$

So, substituting (4.33) into (4.32) we find

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t\in[0,1]} \xi_t \leq y^{-\varepsilon}\right] \leq \tilde{C} y_0^{\frac{a^2}{Q^2}} y^{a\left(1+\frac{2}{Q\sqrt{\kappa}}-\frac{a}{Q^2}\right)-\varepsilon(\hat{C}+a)},$$

as we wanted.

□

We have done essentially all the work for Proposition 4.3.3 now. The last thing left to check is what happens when we maximise the exponent we found over the value of a.

Proof of Proposition 4.3.3. Proposition 4.3.3 will follow from Corollary 4.3.8 if we can show that the exponent of y is greater than 2. As we can choose ε however we like, let us ignore it for now and concentrate on the part of the exponent which is independent of ε. We then get a quadratic in a:

$$a\left(1+\frac{2}{Q\sqrt{\kappa}}-\frac{a}{Q^2}\right). \tag{4.34}$$

It has roots at $a = 0$ and $a = Q^2\left(1+\frac{2}{Q\sqrt{\kappa}}\right)$, and so achieves its maximum at the average of these two: at $a_{max} = \frac{Q^2}{2}\left(1+\frac{2}{Q\sqrt{\kappa}}\right)$. Substituting a_{max} into (4.34) gives

$$a_{max}\left(1+\frac{2}{Q\sqrt{\kappa}}-\frac{a_{max}}{Q^2}\right)=\frac{Q^2}{4}\left(1+\frac{2}{Q\sqrt{\kappa}}\right)^2. \tag{4.35}$$

We now want to look at the exponent $\frac{Q^2}{4}\left(1+\frac{2}{Q\sqrt{\kappa}}\right)^2$ to make sure that it is always greater than 2. First of all, note that

$$Q=\frac{2}{\sqrt{\kappa}}+\frac{\sqrt{\kappa}}{2}, \qquad \frac{Q}{\sqrt{\kappa}}=\frac{2}{\kappa}+\frac{1}{2}, \qquad Q^2=\frac{4}{\kappa}+\frac{\kappa}{4}+2.$$

Therefore, expanding out the exponent, we find

$$\begin{aligned}\frac{Q^2}{4}\left(1+\frac{2}{Q\sqrt{\kappa}}\right)^2 &= \frac{Q^2}{4}\left(1+\frac{4}{Q\sqrt{\kappa}}+\frac{4}{Q^2\kappa}\right)\\ &= \frac{Q^2}{4}+\frac{Q}{\sqrt{\kappa}}+\frac{1}{\kappa}\\ &= \frac{1}{\kappa}+\frac{\kappa}{16}+\frac{1}{2}+\frac{2}{\kappa}+\frac{1}{2}+\frac{1}{\kappa}\\ &= \frac{4}{\kappa}+\frac{\kappa}{16}+1.\end{aligned}$$

Differentiating to find a minimum, we find

$$\frac{\partial}{\partial\kappa}\left(\frac{4}{\kappa}+\frac{\kappa}{16}+1\right)=-\frac{4}{\kappa^2}+\frac{1}{16} \qquad \text{and so} \qquad \frac{2}{\kappa}=\frac{1}{4}.$$

Therefore, the exponent is minimised at $\kappa = 8$, at which point the exponent takes the value

$$\frac{4}{8}+\frac{8}{16}+1=2.$$

So we see that, for $\kappa \neq 8$, we can choose $\varepsilon > 0$ small enough (depending on κ but not y) that there exists some $\delta > 0$ such that

$$\mathbb{P}\left[|f_t'(iy)| > y^{-(1-\varepsilon)},\ b_t \leq -\varepsilon \log y,\ \sup_{t \in [0,1]} \xi_t \leq y^{-\varepsilon}\right] \leq C y^{a_{max}\left(1+\frac{2}{Q\sqrt{\kappa}}-\frac{a_{max}}{Q^2}\right)-\overline{C}\varepsilon}$$
$$= C y^{2+\delta},$$

finishing the proof.

□

4.4 Bounding the coupling constant

The aim of this section is to prove the upper bound that we need for the coupling constant b_t that was introduced in Corollary 4.2.26. We want to show that it has the following polynomial upper tail:

Proposition 4.4.1. *The constant b_t from Corollary 4.2.26 has sub-exponential decay i.e. for any $\lambda > 0$ there exists some constant C_λ such that, for all $x > 0$ we have:*

$$\mathbb{P}[b_t > x] \leq C_\lambda e^{-\lambda x}.$$

The constant C_λ can be chosen uniformly for $t \in [0,1]$.

Proof. Let $\lambda > 0$. We want to use Markov's inequality to say that

$$\mathbb{P}[b_t > x] = \mathbb{P}\left[e^{\lambda b_t} > e^{\lambda x}\right]$$
$$\leq e^{-\lambda x}\mathbb{E}\left[e^{\lambda b_t}\right].$$

We need to check that $\mathbb{E}\left[e^{\lambda b_t}\right]$ is finite for all $\lambda > 0$, with a uniform bound for $t \in [0,1]$. We can rearrange (4.21) to see that, for $y > 0$,

$$b_t = \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy) - \mathrm{Harm}_{\mathbb{H}}(h')(iy) + Q \log|f_t'(iy)| + \frac{2}{\sqrt{\kappa}} \log \frac{|f_t(iy)|}{y}.$$

In fact, if we look at the pinned point iy_0, the point where $\mathrm{Harm}_{\mathbb{H}}(h')$ vanishes (which is how b_t is chosen), we can see that

$$b_t = \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0) + Q \log |f_t'(iy_0)| + \frac{2}{\sqrt{\kappa}} \log \frac{|f_t(iy_0)|}{y_0}.$$

Exponentiating and taking expectation with respect to the free field h which, the reader will recall, is independent of (f_t), we find that

$$\mathbb{E}_h\left[e^{\lambda b_t}\right] = |f_t'(iy_0)|^{\lambda Q} \left(\frac{|f_t(iy_0)|}{y_0}\right)^{\frac{2\lambda}{\sqrt{\kappa}}} \mathbb{E}_h\left[\exp\left(\lambda \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0)\right)\right]. \tag{4.36}$$

We can use Lemma 4.2.24 to bound the expectation $\mathbb{E}_h\left[\exp\left(\lambda \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0)\right)\right]$, using the fact that $\mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0) = \mathrm{Harm}_H(h)(f_t(iy_0))$, as follows:

$$\begin{aligned} &\mathbb{E}_h\left[\exp\left(\lambda \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0)\right)\right] \leq \\ &\qquad \leq \exp\left(\frac{\lambda^2}{2}\left(-3\log(\mathrm{dist}(f_t(iy_0), \partial H_t)) + 2\log(\Re(f_t(iy_0))^2 + 4y_0^2) + C\right)\right). \end{aligned} \tag{4.37}$$

Now we can use Corollary 4.2.5 and the fact that $\Im(f_t(iy_0))$ is increasing to say that

$$\begin{aligned} \mathrm{dist}(f_t(iy_0), \partial H_t) &\geq \Im(f_t(iy_0)) - \mathrm{height}(H_t) \\ &\geq y_0 - 2\sqrt{2t} \\ &\geq y_0 - 2\sqrt{2}, \end{aligned} \tag{4.38}$$

with our choice of y_0 in (4.23) ensuring that this is strictly positive. Using the inequality (4.38) in (4.37) we get

$$\mathbb{E}_h\left[\exp\left(\lambda \mathrm{Harm}_{\mathbb{H}}(h \circ f_t)(iy_0)\right)\right] \leq (y_0 - 2\sqrt{2})^{-\frac{3\lambda^2}{2}} (\Re(f_t(iy_0))^2 + 4y_0^2)^{\lambda^2} e^{\frac{\lambda^2 C}{2}}. \tag{4.39}$$

Substituting (4.39) into (4.36) gives us the inequality

$$\mathbb{E}_h\left[e^{\lambda b_t}\right] \leq |f_t'(iy_0)|^{\lambda Q} \left(\frac{|f_t(iy_0)|}{y_0}\right)^{\frac{2\lambda}{\sqrt{\kappa}}} (y_0 - 2\sqrt{2})^{-\frac{3\lambda^2}{2}} (\Re(f_t(iy_0))^2 + 4y_0^2)^{\lambda^2} e^{\frac{\lambda^2 C}{2}}. \tag{4.40}$$

Corollary 4.2.9 lets us bound the first term in (4.40) by

$$|f_t'(iy_0)| \leq \frac{4}{y_0}\sqrt{y_0^2 + 4}.$$

Lemma 4.2.8 lets us bound the second term in (4.40) by

$$|f_t(iy_0)| \leq \sqrt{\Re(f_t(iy_0))^2 + 4t + y_0^2}$$
$$\leq \sqrt{\Re(f_t(iy_0))^2 + 4y_0^2},$$

where the final inequality comes from the fact that we know $4t \leq 4 \leq 3y_0^2$. It simplifies the result of substituting the two previous bounds into (4.40), which is

$$\mathbb{E}_h\left[e^{\lambda b_t}\right] \leq 4^{\lambda Q} y_0^{-\lambda Q - \frac{2\lambda}{\sqrt{\kappa}}} e^{\frac{\lambda^2 C}{2}} \frac{(y_0+4)^{\lambda Q/2}}{(y_0 - 2\sqrt{2})^{3\lambda^2/2}} \left(\Re(f_t(iy_0))^2 + 4y_0^2\right)^{\lambda^2 + \frac{\lambda}{\sqrt{\kappa}}}.$$

We now take expectations with respect to the law of (f_t) to see that

$$\begin{aligned}
\mathbb{E}[e^{\lambda b_t}] &\leq 4^{\lambda Q} y_0^{-\lambda Q - \frac{2\lambda}{\sqrt{\kappa}}} e^{\frac{\lambda^2 C}{2}} \frac{(y_0+4)^{\lambda Q/2}}{(y_0 - 2\sqrt{2})^{3\lambda^2/2}} \mathbb{E}\left[\left(\Re(f_t(iy_0))^2 + 4y_0^2\right)^{\lambda^2 + \frac{\lambda}{\sqrt{\kappa}}}\right] \\
&\leq 4^{\lambda Q} y_0^{-\lambda Q - \frac{2\lambda}{\sqrt{\kappa}}} e^{\frac{\lambda^2 C}{2}} \frac{(y_0+4)^{\lambda Q/2}}{(y_0 - 2\sqrt{2})^{3\lambda^2/2}} \mathbb{E}\left[\left(\kappa B_t^2 + 4y_0^2\right)^{\lambda^2 + \frac{\lambda}{\sqrt{\kappa}}}\right] \\
&\leq 4^{\lambda Q} y_0^{-\lambda Q - \frac{2\lambda}{\sqrt{\kappa}}} e^{\frac{\lambda^2 C}{2}} \frac{(y_0+4)^{\lambda Q/2}}{(y_0 - 2\sqrt{2})^{3\lambda^2/2}} \mathbb{E}\left[\left(\kappa B_1^2 + 4y_0^2\right)^{\lambda^2 + \frac{\lambda}{\sqrt{\kappa}}}\right], \qquad (4.41)
\end{aligned}$$

where the final line comes from bound in Lemma 4.2.10. Since B_1 is a Gaussian random variable it has moments of all orders, so we know that the expectation in (4.41) is finite. Therefore, the right hand side of (4.41) is our constant C_λ. It does not depend on t, so we have found a uniform constant for $t \in [0,1]$ such that

$$\mathbb{P}[b_t > x] \leq C_\lambda e^{-\lambda x}.$$

□

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