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DERIVATIVE OF THE STANDARD p-ADIC L-FUNCTION ASSOCIATED WITH A SIEGEL FORM

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ABSTRACT. In this paper we firstly construct a two-variable p-adic L-function for the standard representation associated with a Hida family of parallel weight genus g Siegel forms, using a method previously developed by Böcherer–Schmidt in one variable. When a form f has weight $g+1$ a non-crystalline trivial zero could appear. In this case, using the two-variable p-adic L-function we have constructed, we can apply the method of Greenberg–Stevens to calculate the first derivative of the p-adic L-function for f and show that it has the form predicted by a conjecture of Greenberg on trivial zeros.

1. INTRODUCTION

Let M be an irreducible motive, pure of weight 0 over $\mathbb{Q}$; suppose that $s = 0$ is a critical integer à la Deligne for M and $L(M, 0) \neq 0$.

We fix a prime number p and let V be the p-adic representation associated with M. We fix once and for all an isomorphism $\mathbb{C} \cong \mathbb{C}_p$. We suppose that we are given a regular submodule D of the (φ, Γ)-module associated with V [1, §0.2]. Conjecturally, there exists a p-adic L-function $L_p(V, D, s)$ which interpolates the special values of the L-function of M twisted by finite-order characters of $1 + p\mathbb{Z}_p$ [14], multiplied by a corrective factor which depends on D. (More precisely, it can be easily compared with the γ-factor associated with D. See [1, §2.3.2] or [3, §6] for the precise definition.) In particular, we expect the following interpolation formula at $s = 0$:

$$L_p(V, D, 0) = E(V, D)\frac{L(M, 0)}{\Omega(M)},$$

for $\Omega(M)$ a complex period, $E(V, D)$ a corrective product of some non-zero numbers and some Euler type factors. (If $E_p(M, s)$ denotes the factor at p defined at the bottom of page 163 in [3] for the regular submodule D denoted by U in *loc. cit.*, then $E(V, D) = \frac{E_p(M,0)}{L_p(M,0)}$ for $L_p(M, 0)$ the Euler factor at p of the motivic L-function.) It may happen that certain of the Euler factors in $E(V, D)$ vanish. In this case the

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connection with what we are interested in, the special values of the L-function, is lost. Motivated by the seminal work of Mazur–Tate–Teitelbaum [13], Greenberg, in the ordinary case [6], and Benois [1] have conjectured the following:

Conjecture 1.1. *[Trivial zeros conjecture] Let e be the number of Euler-type factors of $E_p(V,D)$ which vanish. Then the order of zeros at $s=0$ of $L_p(V,D,s)$ is e and*

$$\lim_{s\to 0}\frac{L_p(V,D,s)}{s^e}=\ell(V,D)E^*(V,D)\frac{L(M,0)}{\Omega(M)} \tag{1.1}$$

where $E^*(V,D)$ is the non-vanishing factors of $E(V,D)$ and $\ell(V,D)$ a non-zero number called the ℓ-invariant as defined in [1].

In this paper we shall study this conjecture for the p-adic standard L-function associated with certain Siegel modular forms. Let f be a genus g Siegel modular form of parallel weight k, level $\Gamma_0(N)$ (see the next section for the precise definition) and trivial Nebentypus. For each Dirichlet character χ we can define a L-function $\mathcal{L}^{(N)}(\mathrm{St}(f),\chi,s)$, defined as an infinite Euler product of factors of degree $2g+1$, which corresponds, up to some Euler factors at bad primes, to the L-function associated with the standard Galois representation constructed in [20]. Its critical points are the integers $1\leq s\leq k-g$ with $(-1)^{s+g}=\chi(-1)$ and $g+1-k\leq s\leq 0$ with $(-1)^{1+s+g}=\chi(-1)$.

We suppose that f is ordinary for the U_p operator, *i.e.* $U_pf=\alpha_pf$ with $|\alpha_p|_p=1$. We know that f can be deformed into a one-variable Hida family of ordinary Siegel forms: we have a finite flat $\mathbb{Z}_p[[1+p\mathbb{Z}_p]]$-algebra $\mathcal{O}(\mathcal{C})$, quotient of the big ordinary Hecke algebra $\mathbb{T}$, which parametrizes systems of Hecke eigenvalues of ordinary Siegel forms of parallel weight. Consequently, we shall denote by $\mathcal{C}$ the corresponding irreducible component of $\mathrm{Spec}(\mathbb{T})$. Sometimes in literature this is called a GL_g-ordinary family.

We say that a point of $\mathrm{Spec}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ is arithmetic if it corresponds to a character

$$z\mapsto\varepsilon(z)z^k,$$

with ε a finite order character. Let w be the projection map

$$w:\mathcal{C}(\mathbb{C}_p)\to\mathrm{Spec}(\mathbb{Z}_p[[\mathbb{Z}_p^\times]]\otimes\mathbb{C}_p).$$

We say that a point $x\in\mathcal{C}(\mathbb{C}_p)$ is arithmetic if $w(x)$ is an arithmetic point of type $z\mapsto z^k$ with $k\geq g+1$ and the corresponding system of eigenvalues of the Hecke algebra is classical. We shall denote by f_x a Siegel eigenform with this system of eigenvalues.

As f_x is ordinary, there is an obvious choice for the regular sub-module D which appears in the definition of the p-adic L-function; we shall hence suppress the dependence from D in the notation. This defines in particular a choice of the Satake parameters for f_x, as we explain in Section 5; we shall denote them by $\beta_i(f_x)$. We define:

$$\begin{aligned}E_1(f_x,t)=&\prod_{i=1}^{g}(1-\beta_i^{-1}(f_x)p^{t-1}),\\E(f_x,t)=&E_1(f_x,t)\prod_{i=1}^{g}\frac{1}{(1-\beta_i(f_x)p^{-t})}.\end{aligned}$$

The first result of the paper is the following:

Theorem 1.2. *Suppose the "multiplicity one for $\mathcal{C}$" hypothesis of Section 5. We have an element $L_p(x,t) \in \mathrm{Frac}(\mathcal{O}(\mathcal{C})\hat{\otimes}\mathbb{Z}_p[[\mathbb{Z}_p^\times]])$ which satisfies, for all points (x,t) arithmetic with $1 \leq t \leq k-g$, $(-1)^{1+k-t}=1$ and with w étale at x, the following interpolation property:*

$$L_p(x,t) = E(f_x, 1-t)\frac{\mathcal{L}^{(Np)}(\mathrm{St}(f), 1-t)}{\Omega(f_x,t)},$$

where $\mathcal{L}^{(Np)}(\mathrm{St}(f),s)$ stands for the L-function without any Euler factor at Np and $\Omega(f_x,t)$ is a complex period involving the Petersson norm of f_x and powers of $2\pi i$.

Here the variable t (which for whole paper shall denote an integer) is related to s in the following way: $t = k-s-g$. See Theorem 5.2 for more details on the factors that appear, for the interpolation formula at points of type (x,ε,s), for twists by characters χ of level prime to p.

This p-adic L-function will be constructed using a two-variable generalization of the method of [2] which expresses $\mathcal{L}^{(N)}(\mathrm{St}(f),s)$ as a double Petersson product of f_x with certain Eisenstein series. The p-adic interpolation of the Fourier coefficients of these series is the key ingredient for the construction of the p-adic L-function.

A more general results, always starting from [2] but using an adelic language, has been given in [12] where a $g+1$-variable p-adic L-function interpolating the other critical values is constructed.

Remark that the p-adic L-function depends on a compatible choice of f_x in the family and a different choice would give a different p-adic L-function. The "p-adic multiplicity one for $\mathcal{C}$" is not really necessary and is used to show that we can make such a compatible choice, allowing us to define a p-adic Petersson product. One can remove this hypothesis using a different construction of a linear form on families of ordinary Siegel forms, namely the one of [12, §7]. As this choice would introduce even more terms in the interpolation formula (terms that are related to other eigenforms in the same Hecke eigenspace of f_x), while not adding much about the conjecture on trivial zeroes, we prefer to keep this extra assumption.

Possible denominators in the first variable are related to the non-étalness of the weight map w and a denominator in the second variable comes from a possible pole of the Kubota–Leopoldt p-adic L-function which appears in the Fourier coefficients of the p-adic Eisenstein series we construct.

We study now when this p-adic L-function presents trivial zeroes.

If $k = g+1$ and p^{-1} appears among the Satake parameters of f_x we have $E_1(f_x,0)=0$. In this case a trivial zero appears and we shall say that such a f_x is $\Gamma_0(p)$-Steinberg (see 6.1 for a precise definition). If f_x comes from a form of level prime to p we have instead

$$E(f_x,1-t)\mathcal{L}^{(Np)}(\mathrm{St}(f),1-t) = E_1(f_x,1-t)(1-p^{t-1})\prod_{i=1}^{g}(1-\beta_i^{-1}p^{t-1})\mathcal{L}^{(N)}(\mathrm{St}(f),1-t).$$

This implies that for $t=1$ the p-adic L-function is identically zero. Moreover, one observes that $\beta_i p^{k-g}$ is an analytic function on $\mathcal{C}^{\mathrm{rig}}$, and this implies that the numerator of $E(f_x, 1-k+g)$ can be p-adically interpolated. Hence, for f which is $\Gamma_0(p)$-Steinberg, these two properties allow one to exploit the strategy of

Greenberg–Stevens to study the trivial zero and calculate the p-adic derivative of $L_p(f,t)$.

Indeed, generalizing the paper [18] where the case $g=1$ (the symmetric square of a modular form) has been dealt, one can modify the construction of the Eisenstein family; this allows us to define a second p-adic L-function $L_p^*(x)$. This new p-adic L-function satisfies the following equality of locally analytic functions around f:

$$L_p(x,k-g)=E(f_x,1-k+g)L_p^*(x).$$

Moreover, when $f_x=f$ it precisely interpolates the values

$$\mathcal{L}^{(Np)}(\mathrm{St}(f),0)\prod_{i=1}^{g}(1-\beta_i)^{-1}.$$

We can then prove the main theorem of the paper:

Theorem 1.3. *Let f be a Siegel form of weight $g+1$ and trivial Nebentypus. Suppose that f is $\Gamma_0(p)$-Steinberg and the weight projection w is étale at the corresponding point x. Then we have:*

$$\frac{\mathrm{d}}{\mathrm{d}s}L_p(f,s)_{|s=0}=\ell^{\mathrm{al}}(\mathrm{St}(f))E^*(f)\frac{\mathcal{L}^{(Np)}(\mathrm{St}(f),0)}{\Omega(f)},$$

where $\ell^{\mathrm{al}}(\mathrm{St}(f))$ is the Greenberg–Benois ℓ-invariant as calculated in [19] *and*

$$E^*(f)=\frac{\prod_{i=1}^{g-1}(1-\beta_i^{-1}p^{-1})}{\prod_{i=1}^{g}(1-\beta_i p)}.$$

We conclude the introduction with the remark that this does not prove Conjecture 1.1 as we do not know if $\ell^{\mathrm{al}}(\mathrm{St}(f))\mathcal{L}^{(Np)}(\mathrm{St}(f),0)$ is not vanishing.

Conjecturally the ℓ-invariant should not vanish but practically nothing is known at the moment.

The L-value can indeed vanish as we are interpolating an imprimitive L-function; for certain primes $l\mid N$ we have removed the factor $(1-l^0)=0$. In this case a study of the second derivative should be necessary; to the author, this problem seems very hard.

2. Notation on Siegel Forms

We now recall the basic theory of (parallel weight) Siegel modular forms. We follow closely the notation of [2] and we refer to the first section of *loc. cit.* for more details. Let us denote by $\mathbb{H}_g$ the Siegel space for GSp_{2g}. We have explicitly

$$\mathbb{H}_g=\left\{Z\in\mathbb{C}^{(n,n)}|Z^t=Z\text{ and }\mathrm{Im}(Z)>0\right\}.$$

It has a natural action of $\mathrm{GSp}_{2g}^+(\mathbb{R})$ via fractional linear transformations; for any $\gamma=\begin{pmatrix}A&B\\C&D\end{pmatrix}$ in $\mathrm{GSp}_{2g}^+(\mathbb{R})$ and Z in $\mathbb{H}_g$ we define

$$\gamma(Z)=(AZ+B)(CZ+D)^{-1}.$$

For any function $f:\mathbb{H}_g\to\mathbb{C}$ we define the weight k action

$$f|_k\gamma(Z):=f(\gamma(Z))\det(CZ+D)^{-k}\det(\gamma)^{k/2}.$$

Let $\Gamma = \Gamma_0(M)$ be the congruence subgroup of $\mathrm{Sp}_{2g}(\mathbb{Z})$ of matrices whose lower block C is congruent to 0 modulo M. We consider the space $M_k^{(g)}(M,\phi)$ of Siegel forms of scalar weight k and Nebentypus ϕ:

$$\{f : \mathbb{H}_g \to \mathbb{C} \,|\, f|_k\gamma(Z) = \phi(\gamma)f(Z) \;\; \forall \gamma \in \Gamma, f \text{ holomorphic}\}.$$

When $g = 1$, we require the extra condition that $f|\gamma$ is holomorphic at infinity for all γ in $\mathrm{SL}_2(\mathbb{Z})$. Each f in $M_k^{(g)}(M,\phi)$ admits a Fourier expansion

$$f(Z) = \sum_T a(T)e^{2\pi i \mathrm{tr}(TZ)},$$

where T ranges over all half-integral symmetric matrices T which are positive and semi-definite.

We have two embeddings (of algebraic groups) of Sp_{2g} in Sp_{4g}. For any algebra R, we have:

$$\mathrm{Sp}_{2g}^{\uparrow}(R) = \left\{ \begin{pmatrix} a & 0 & b & 0 \\ 0 & 1_g & 0 & 0 \\ c & 0 & d & 0 \\ 0 & 0 & 0 & 1_g \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(R) \right\},$$

$$\mathrm{Sp}_{2g}^{\downarrow}(R) = \left\{ \begin{pmatrix} 1_g & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 1_g & 0 \\ 0 & c & 0 & d \end{pmatrix} \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}_{2g}(R) \right\}.$$

We can embed $\mathbb{H}_g \times \mathbb{H}_g$ in $\mathbb{H}_{2g}$ in the following way:

$$(z_1, z_4) \mapsto \begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix}.$$

This embedding is compatible with the action of $\mathrm{GSp}_{2g}^+(\mathbb{R})$. For all $\gamma \in \mathrm{GSp}_{2g}^+(\mathbb{R})$ we have:

$$\gamma^{\uparrow}\begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} \gamma(z_1) & 0 \\ 0 & z_4 \end{pmatrix},$$
$$\gamma^{\downarrow}\begin{pmatrix} z_1 & 0 \\ 0 & z_4 \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & \gamma(z_4) \end{pmatrix}.$$

Consequently, evaluation at $z_2 = 0$ gives us the following map:

$$M_k^{(2g)}(M,\phi) \hookrightarrow M_k^{(g)}(M,\phi) \otimes_{\mathbb{C}} M_k^{(g)}(M,\phi).$$

This also induces a closed embedding of two copies of the Siegel variety of genus g in the Siegel variety of genus $2g$. On points, it corresponds to abelian varieties of dimension $2g$ which decompose as the product of two abelian varieties of dimension g.

We define Hecke operators as double cosets operators; let γ be an element of $\mathrm{GSp}_{2g}(\mathbb{Q})$ with positive determinant and Γ_1, Γ_2 two congruences subgroups. We write $\Gamma_1\gamma\Gamma_2 = \sqcup_i \Gamma_1\gamma_i$ and we define the Hecke operator of weight k

$$f|_k[\Gamma_1\gamma\Gamma_2] = \sum_{\gamma_i} f|_k\gamma_i.$$

We denote by τ_N the operator $\left[\Gamma_0(M)\begin{pmatrix} 0 & -1 \\ M & 0 \end{pmatrix}\Gamma_0(M)\right]$. This operator is a generalization of the Atkin–Lehner involution for GL_2 and is sometimes called the Fricke involution. For any Hecke operator T we define the dual operator $T^* = \tau_M^{-1} T \tau_M$. Once we shall define the Petersson scalar product, T^* will indeed be the dual operator of T for this product at level $\Gamma_0(M)$.

Finally, when $p \mid M$, we define the Hecke operator U_p of level $\Gamma_0(M)$ and weight k to be

$$|_k p^{\frac{gk-g(g+1)}{2}} \left[\Gamma_0(M)\begin{pmatrix} 1_g & \\ & p1_g \end{pmatrix}\Gamma_0(M)\right].$$

If $f(Z) = \sum_T a(T) e^{2\pi i \mathrm{tr}(TZ)}$ we have from [10, Proposition 3.5]:

$$U_p f(Z) = \sum_T a(pT) e^{2\pi i \mathrm{tr}(TZ)}.$$

3. Eisenstein series

The aim of this section is to recall certain Eisenstein series which will be used to construct the p-adic L-function as in [2]. In *loc. cit.* the authors consider certain Eisenstein series for GSp_{4g} whose pullback to two embedded copies of the Siegel variety for GSp_{2g} is a **holomorphic** Siegel modular form. We now fix a (parallel weight) Siegel modular form f of genus g. We shall define the standard L-function for f $\mathcal{L}^{(N)}(\mathrm{St}(f), s)$ as an Eulerian product. We shall give an integral expression for this L-function as a double Petersson product between f and these Eisenstein series (see Proposition 3.3). Note that for $g = 1$ the standard L-function of f coincides, up to a twist, with the symmetric square L-function of f.

These Eisenstein series are essentially Maaß–Shimura derivative of classical Siegel Eisenstein series [21] and the integral formulation of the L-function is the classical pullback formula of Garrett and Piatetski-Shapiro–Rallis.

The study of the algebraicity of these differential operators has already been exploited in [7, 8] to show that certain values of the standard L-function, naturally normalized, are algebraic.

The novelty in the approach of Böcherer–Schmidt consists in using a holomorphic differential operator in place of the Maaß–Shimura operators.

The two develop then a *twisting method* which allow to define Eisenstein series whose Fourier coefficients satisfy Kummer's congruences when the character associated with the Eisenstein series varies p-adically. This is the key for their construction of the one-variable (cyclotomic) p-adic L-function and of our two-variable p-adic L-function.

When the character is trivial modulo p there exists a simple relation between the twisted and the not-twisted Eisenstein series [2, §6 Appendix] which introduces certain Euler factors at p in the interpolation formula for the p-adic L-function. Note that the p-adic not-twisted Eisenstein series have only one variable. We shall construct then an *improved* p-adic L-function without these Euler factors (renouncing the cyclotomic variable) interpolating directly the not-twisted Eisenstein series.

3.1. **Some differential operators.** In this section we recall some holomorphic differential operators used by Böcherer–Schmidt. Let k and s be two positive integers; in [2, (1.15)], the authors define a differential operators $\mathring{\mathfrak{D}}^s_{g,k}$. This operator sends a holomorphic function, automorphic of weight k on $\mathbb{H}_{2g}$ to a holomorphic function on $\mathbb{H}_g \times \mathbb{H}_g$ automorphic of weight $k+s$ in both the first and the second variable. A structure theorem for differential operators on nearly automorphic forms [22, Proposition 3.4] ensure us that this operator is a scalar multiple, depending on k and s, of the Maaß–Shimura operator composed with the holomorphic projection.

For each symmetric matrix $I = \begin{pmatrix} T_1 & T_2 \\ T_2^t & T_4 \end{pmatrix}$ of size $2g$, they define a number $\mathfrak{b}_k^s(I)$ satisfying:

$$\mathring{\mathfrak{D}}_{g,k}^s(e^{\mathrm{tr}(IZ)}) = \mathfrak{b}_k^s(I)e^{\mathrm{tr}(T_1z_1+T_4z_4)}.$$

The $\mathfrak{b}_k^s(I)$'s are polynomials in the entries of I, homogeneous of degree sg. If $\frac{1}{L}T_1$ and $\frac{1}{L}T_4$ are both half-integral then we have the congruence [2, (1.34)]:

$$4^{gs}\mathfrak{b}_k^s(I) \equiv 2^{gs}c_{g,k}^s \det(2T_2)^s \bmod L, \tag{3.1}$$

where $c_{g,k}^s$ is as in [2, (1.21)].

This relation (applied for $L = p^n$) tell us that $(U_p^n \otimes U_p^n)\mathring{\mathfrak{D}}_{g,k}^s$ is, as a differential operator, modulo p very similar to $(U_p^n \otimes U_p^n)\partial_2^s$, where we denote by

$$\partial_2 := \det\left(\frac{1}{2}\frac{\partial}{\partial z_{i,j}}\right)_{i,j} \text{ for } i = 1,\dots,g \text{ and } j = g+1,\dots,2g.$$

This relation will be the key to interpolate p-adically the ordinary projection of the Eisenstein series we are about to introduce.

3.2. **Some Eisenstein series.** We fix a weight k, an integer N prime to p and a Nebentypus ϕ. Let f be an eigenform in $M_k^{(g)}(Np,\phi)$. Let R be an integer coprime with N and p and N_1 a positive integer such that $N_1 \mid N$. We fix a Dirichlet character χ modulo N_1Rp which we write as $\chi_1\chi'\varepsilon_1$, with χ_1 defined modulo N_1, χ' primitive modulo R and ε_1 defined modulo p. Let us denote by S the product of all prime dividing Rp.

The key construction is what is called the twisting method in [2, (2.18)] and that we now recall. Given a Dirichlet character η defined modulo D (not necessarily primitive) and a modular form f on $\mathbb{H}_{2g}$ which is invariant for the group $\begin{pmatrix} 1_{2g} & \mathbb{Z}^{(2g,2g)} \\ 0_{2g} & 1_{2g} \end{pmatrix}$, Böcherer and Schmidt defined:

$$f^{(\eta)} := \sum_{X\in\mathbb{Z}^{(g,g)} \bmod D} \eta(\det(X))f|\begin{pmatrix} 1_{2g} & S(X/D) \\ 0_{2g} & 1_{2g} \end{pmatrix}$$

for $S(X)$ the $2g\times 2g$ antidiagonal (in blocks) matrix with X and X^t on the antidiagonal.

Let $t \geq 1$ be an integer and let $\mathbb{F}^{t+g}\left(Z, R^2N^2p^{2n}, \phi, u\right)^{(\chi)}$ be the twisted Eisenstein series of [2, (5.3)], with $Z \in \mathbb{H}_{2g}$.

When ε_1 is not trivial, we define $\mathcal{H}'_{L,k,t,\chi}(z_1,z_4)$ to be

$$\mathcal{L}(\phi\chi, t+g+2u)\mathring{\mathfrak{D}}_{t+g}^s\left(\mathbb{F}^{t+g}\left(\begin{pmatrix} z_1 & z_2 \\ {}^tz_2 & z_4 \end{pmatrix}, R^2N^2p^{2n}, \phi, u\right)^{(\chi)}\right)_{|u=\frac{1}{2}-t, z_2=0} |^{z_1}U_{L^2}\, |^{z_4}U_{L^2}$$

for s a non-negative integer, $p^n \mid L$, with L a p-power and $\mathcal{L}(\phi\chi, t+g+2u)$ the product of Dirichlet L-functions defined in [2, Theorem 3.1]. It is a form of level $\Gamma_0(N^2R^2p)\times\Gamma_0(N^2R^2p)$ and weight $k = t+g+s$. A priori this Eisenstein series depends also on ϕ, but as this character will not change we prefer to omit it to lighten the notation. One can also choose $L = 1$ and in this case the level is $N^2R^2p^{2n}$.

When ε_1 is trivial one define $\mathcal{H}'_{L,k,t,\chi}(z_1, z_4)$ as:

$$\mathcal{L}(\phi\chi, t+g+2u)\mathring{\mathfrak{D}}^s_{t+g}\mathbb{F}^{t+g}\left(-, R^2N^2p^2, \phi, u\right)^{(\chi_1\chi')} \mid \left| \left(\sum_{i=0}^{g}(-1)^i p^{\frac{i(i-1)}{2}}p^{-ig}\sum_j \begin{pmatrix} 1_{2g} & S(g_{i,j}) \\ 0_{2g} & 1_{2g}\end{pmatrix}\right)\right|_{u=\frac{1}{2}-t, z_2=0} |^{z_1}U_{L^2}\, |^{z_4}U_{L^2} \tag{3.2}$$

where $g_{i,j}$ runs along the representative of the double cosets in [2, (2.38)]. It is proved in [2, Theorem 8.5] that (small modifications of) these functions $\mathcal{H}'_{L,k,t,\chi}(z_1, z_4)$ satisfy Kummer's congruences when ε_1 varies p-adically.

These Eisenstein series will be p-adically interpolated and shall be used for the construction of the two-variable p-adic L-function.

Whenever ε_1 is trivial, we define another Eisenstein series, the one appearing inside formula (2.25') of [2], namely:

$$\mathcal{H}'^{*}_{L,k,t,\chi_1\chi'}(z_1, z_4) := \mathcal{L}(\phi\chi, t+g+2u)\mathring{\mathfrak{D}}^s_{t+g}\left(\mathbb{F}^{t+g}\left(\left(\begin{array}{cc} z_1 & z_2 \\ {}^t z_2 & z_4\end{array}\right), R^2N^2p^2, \phi, u\right)^{(\chi_1\chi')}\right)_{|u=\frac{1}{2}-t, z_2=0}.$$

The difference between this Eisenstein series and the one in (3.2) is that the twist by

$$\left(\sum_{i=0}^{g}(-1)^i p^{\frac{i(i-1)}{2}}p^{-ig}\sum_j \begin{pmatrix} 1_{2g} & S(g_{i,j}) \\ 0_{2g} & 1_{2g}\end{pmatrix}\right)$$

is omitted. These series will be p-adically interpolated along the variable k and used to construct the improved p-adic L-function, in one variable, which shall not bring the trivial zero. While constructing this improved p-adic L-function, we will have $t = k - g$, *i.e.* $s = 0$, meaning that no differential operator is needed to construct the improved p-adic L-function.

For any prime number q and matrices I as in the previous section, let $B_q(X, I)$ be the polynomial of degree at most $2g-1$ of [2, Proposition 5.1]. Let χ be a Dirichlet character and T_2 a semi-positive definite matrix of size $g \times g$. For a positive integer t we define:

$$\begin{aligned} B_{2g}(t) &= (-1)^{g(g+t)}\frac{2^{g+2gt}}{\Gamma_{2g}\left(g+\frac{1}{2}\right)}\pi^{g+2g^2}, \\ \Gamma_g(s) &= \pi^{\frac{g(g+1)}{4}}\prod_{i=1}^{g}\Gamma\left(s - \frac{i-1}{2}\right), \\ G_g(T_2, N, \chi) &= \sum_{X \in M_g(\mathbb{Z}) \bmod N}\chi(\det X)e^{2\pi i \mathrm{tr}\left(\frac{1}{L}T_2X\right)}. \end{aligned}$$

When χ is of conductor C we have [2, Proposition 6.1]:

$$G_g(T_2, C, \chi) = C^{\frac{1}{2}g(g-1)}\chi^{-1}(\det(T_2))G(\chi)^g.$$

We deduce easily from [2, (7.3),(7.14)] the following theorem:

Theorem 3.1. *The Eisenstein series defined above have the following Fourier development:*

$$\begin{aligned}\mathcal{H}'_{L,k,t,\chi}(z_1,z_4) =&(Rp^n)^{\frac{g(g-1)}{2}} B_{2g}(t)(2\pi i)^{sg} G\left(\chi'\varepsilon_1\right)^g \sum_{T_1\geq 0}\sum_{T_4\geq 0}\\ &\left(\sum_I \mathfrak{b}^s_{g,t+g}(I) G_g(2T_2,N,\chi_1)(\chi'\varepsilon_1)^{-1}(\det(2T_2))\right.\\ &\sum_{G\in \mathrm{GL}_2(\mathbb{Z})\backslash \mathbf{D}(I)} (\phi\chi)^2(\det(G))|\det(G)|^{2t-1}\\ &\left.\times L^{(\det(2I)SN)}(\sigma_{\det(2I)}\phi\chi,1-t)\prod_{q|\det(2G^{-t}IG^{-1})} B_q\left(\chi\phi(q)q^{t-g-1},G^{-t}IG^{-1}\right)\right)\\ &e^{2\pi i \mathrm{tr}(T_1z_1+T_4z_4)},\end{aligned}$$

where the sum over I runs along the matrices $\begin{pmatrix} L^2T_1 & T_2\\ T_2 & L^2T_4\end{pmatrix}$ positive definite and with $2T_2\in M_g(\mathbb{Z})$, $\chi(M)=0$ if $(M,S)\neq 1$, σ_c is the (non primitive) quadratic Dirichlet character $\left(\frac{(-1)^g c}{\bullet}\right)$, associated with the quadratic extension $\mathbb{Q}(\sqrt{(-1)^g c})/\mathbb{Q}$, and

$$\mathbf{D}(I)=\left\{G\in M_{2g}(\mathbb{Z})|G^{-t}IG^{-1} \text{ is a half-integral symmetric matrix}\right\}.$$

We also have

$$\begin{aligned}\mathcal{H}'^{*}_{L,k,t,\chi_1\chi'}(z_1,z_4) =&(Rp^n)^{\frac{g(g-1)}{2}} B_{2g}(t)(2\pi i)^{sg} G\left(\chi'\right)^g \sum_{T_1\geq 0}\sum_{T_4\geq 0}\\ &\left(\sum_I \mathfrak{b}^s_{g,t+g}(I) G_g(2T_2,N,\chi_1)(\chi')^{-1}(\det(2T_2))\right.\\ &\sum_{G\in \mathrm{GL}_2(\mathbb{Z})\backslash \mathbf{D}(I)} (\phi\chi_1\chi')^2(\det(G))|\det(G)|^{2t-1}\\ &\times L^{(\det(2I)RN)}(\sigma_{\det(2I)}\phi\chi_1\chi',1-t)\\ &\left.\times\prod_{q|\det(2G^{-t}IG^{-1})} B_q\left(\chi_1\chi'\phi(q)q^{t-g-1},G^{-t}IG^{-1}\right)\right)\\ &e^{2\pi i \mathrm{tr}(T_1z_1+T_4z_4)}.\end{aligned}$$

Proof. The only difference from *loc. cit.* is that we do not apply $\left|\begin{pmatrix} 1 & 0\\ 0 & N^2S\end{pmatrix}\right.$. $\square$

Remark 3.2. The difference between the Fourier expansion of $\mathcal{H}'^{*}_{L,k,t,\chi_1\chi'}$ and $\mathcal{H}'_{L,k,t,\chi}$ is that in the latter the terms for I with $(\det(T_2),p)\neq 1$ are always 0. This is due to the twist by

$$\left|\left(\sum_{i=0}^{g}(-1)^i p^{\frac{i(i-1)}{2}}p^{-ig}\sum_j\begin{pmatrix} 1_{2g} & S(g_{i,j})\\ 0_{2g} & 1_{2g}\end{pmatrix}\right)\right.$$

in (3.2) and it is necessary to obtain p-adic interpolation in two variables.

Note that each sum over I is finite because I must have positive determinant. Moreover, we can rewrite it as a sum over T_2, with $(2\det(T_2), N_1Rp) = 1$ and $T_4 -{}^t T_2 T_1^{-1} T_2 > 0$.

Let $f \in M_k^{(g)}(\Gamma_0(Np), \phi)$ as before and $\mathbb{T}_{Np}$ be the abstract Hecke algebra for $\Gamma_0(Np)$ as defined in [2, §3], where it is denoted by $\mathfrak{H}^\circ$. We can decompose $\mathbb{T}_{Np}$ as a restricted product $\bigotimes'_q \mathbb{T}_{Np,q}$. We suppose that f is an eigenform for $\mathbb{T}_{Np}$. If $q \nmid Np$, let us denote by $\alpha_{q,1}^{\pm 1}, \ldots, \alpha_{q,g}^{\pm}$ the Satake parameters associated with f; they are well defined up to the action of the Weyl group of GSp_{2g}. Also when $q \mid Np$ the q-component of the Hecke algebra is commutative; we denote by $\beta_{q,1}, \ldots, \beta_{q,g}$ the "Satake parameters" associated with f, well defined up to permutation (see [2, Corollary 3.2]).

For each Dirichlet character χ we define the (incomplete) standard L-function $\mathcal{L}^{(Np)}(\mathrm{St}(f), \chi, u)$ associated with f and χ as the infinite product:

$$\mathcal{L}^{(Np)}(\mathrm{St}(f), \chi, u) = \prod_{q \nmid N} D_q(\chi(q) q^{-u})^{-1},$$

where

$$D_q(T) = (1 - \phi(q)T) \prod_{i=1}^{g} (1 - \phi(q)\alpha_{q,i}^{-1} T)(1 - \phi(q)\alpha_{q,i} T).$$

It is called the standard L-function because it corresponds to the standard representation of GSpin_{2g+1}, the Langlands dual of GSp_{2g}.
We also define, for $q \mid Np$, the factor:

$$D_q(T) = \prod_{i=1}^{g} (1 - \beta_{q,i} T).$$

This L-function differs from the (conjectural) motivic/automorphic L-function $L(\mathrm{St}(f) \otimes \chi\phi, u)$ by a finite number of Euler factors at primes dividing N. See [17, Table A.10] for these local factors when $g = 2$. (Note that in general $D_q(T)$ is not the local factor of the automorphic L-function.)

In what follows, for two forms f and g of level Γ and weight k, we shall denote by $\langle f, g \rangle$ the normalized Petersson product:

$$\langle f, g \rangle := \int_{\mathbb{H}_g/\Gamma} f(Z) \overline{g(-\overline{Z})} \det(\mathrm{Im}(Z))^k \frac{\mathrm{d}Z \mathrm{d}\overline{Z}}{\det(\mathrm{Im}(Z))^{g+1}}.$$

We conclude with the integral formulation of $\mathcal{L}^{(N)}(\mathrm{St}(f), \chi, u)$ [2, Theorem 3.1, Proposition 7.1 case (7.13)].

Proposition 3.3. *Let f be a form of weight k, Nebentypus ϕ. We put $t + s = k - g$ and $\mathcal{H}' = \mathcal{H}'_{1,k,t,\chi}(z_1, z_4)$. We have*

$$\begin{aligned} \left\langle f(z_4) \left|\tau_{N^2 p^{2n} R^2}\right., \mathcal{H}' \right\rangle = & \frac{\Omega_{k,s}(\frac{1}{2} - t) p_{\frac{1}{2} - t}(t+g)}{(-1)^{gk} \chi(-1)^{g-1} d_{\frac{1}{2}-t}(t+g)} (R^2 N^2 p^{2n})^{\frac{g(g+1)-gk}{2}} (N_1 R p^n)^{g(k-t)} \\ & \times \prod_{i=1}^{g} (1 - \beta_{p,i}^{-1} \chi^{-1}(p) p^{-t}) \frac{\mathcal{L}^{(Np)}(\mathrm{St}(f), \chi^{-1}, 1-t)}{\prod_{q \mid Np, q \nmid N_1} D_q(\chi^{-1}(q) q^{t-1})} f(z_1) | U_{N^2/N_1^2} \end{aligned}$$

where the Petersson norm is for forms of level $\Gamma_0(N^2p^{2n}R^2)$ *and* $\frac{p_{\frac{1}{2}-t}(t+g)}{d_{\frac{1}{2}-t}(t+g)}\Omega_{k,s}\left(\frac{1}{2}-t\right)$ *is an explicit products of* Γ*-functions (see Proof of Theorem 5.2).*

If $\chi = \chi'\chi_1$ *is trivial modulo* p*, we let* $\mathcal{H}'^{*} = \mathcal{H}'^{*}_{1,k,t,\chi'\chi_1}(z_1,z_4)$ *and we have:*

$$\left\langle f(z_4)\left|\tau_{N^2p^2R^2}\right., \mathcal{H}'^{*}\right\rangle = \frac{\Omega_{k,s}(\frac{1}{2}-t)p_{\frac{1}{2}-t}(t+g)}{(-1)^{gk}\chi(-1)^{g-1}d_{\frac{1}{2}-t}(t+g)}(R^2N^2p^2)^{\frac{g(g+1)-gk}{2}}(N_1Rp)^{g(k-t)}$$
$$\times \frac{\mathcal{L}^{(Np)}(\mathrm{St}(f),\chi^{-1},1-t)}{\prod_{q\mid Np, q\nmid N_1} D_q(\chi^{-1}(q)q^{t-1})} f(z_1)|U_{N^2/N_1^2}.$$

In particular

$$p^{-\frac{g(g+1)}{2}}\prod_{i=1}^{g}(1-\beta_{p,i}^{-1}\chi^{-1}(p)p^{-t})\left\langle f(z_4)\left|\tau_{N^2p^2R^2}\right., \mathcal{H}'^{*}\right\rangle = \left\langle f(z_4)\left|\tau_{N^2p^2R^2}\right., \mathcal{H}'\right\rangle.$$

Proof. With the notation of [2, Theorem 3.1] we have $M = R^2N^2p^{2n}$ and $N = N_1Rp^n$. We have that $\frac{d_{\frac{1}{2}-t}(t+g)}{p_{\frac{1}{2}-t}(t+g)}\mathcal{H}'$ is the holomorphic projection of the Eisenstein series of [2, Theorem 3.1] (see [2, (1.30),(2.1),(2.25)]) and the factor $(-1)^{gk}$ comes from the Atkin–Lehner involution $\tau_{N^2p^{2n}R^2}$.

The second formula is [2, §3 Appendix]. □

Remark 3.4. This is essentially a reformulation of the classical pull-back formula of Garrett and Piatetski-Shapiro–Rallis [5, Part A].

If $U_N f \neq 0$ we can then take $N_1 = 1$.

4. Families of Eisenstein series

4.1. **Families of ordinary Siegel forms.** We recall Hida's theory for parallel weight Siegel forms, following [16, §4]. Let $\Gamma = \Gamma_0(Np)$. For $i = 0,\dots,g$ we define the Hecke operators:

$$U_{p,i} = \left[\Gamma\begin{pmatrix} 1_{g-i} & & & \\ & p^{-1}1_i & & \\ & & 1_{g-i} & \\ & & & p1_i\end{pmatrix}\Gamma\right],$$

and $U_p = p^{\frac{gk-g(g+1)}{2}}\left[\Gamma\begin{pmatrix} 1_g & \\ & p1_g\end{pmatrix}\Gamma\right]$. Note that $U_p^2 = p^{gk-g(g+1)}U_{p,g}$. This normalization is optimal if one wants the U_p operator to preserve both p-integrality and non-vanishing modulo p. We let $V_{l,\infty}$ be the set of p-adic modular forms of tame level N with $\mathbb{Z}/p^l\mathbb{Z}$-coefficients (of any weight); it is equipped with an action of $\mathrm{GL}_g(\mathbb{Z}_p)$. We define $V_\infty = (\varinjlim_l V_{l,\infty})^{\mathrm{SL}_g(\mathbb{Z}_p)}$, which should be thought as a space of p-adic Siegel forms with $\mathbb{Q}_p/\mathbb{Z}_p$ coefficients. We have a U_p-operator acting on V_∞ and we define $\mathbb{V} = \mathrm{Hom}_{\mathbb{Z}_p}(e_{\mathrm{GL}_g}V_\infty, \mathbb{Q}_p/\mathbb{Z}_p)$ for $e_{\mathrm{GL}_g} = \lim_n U_p^{n!}$. This is the ordinary projector "adapted to parallel weight" of [16, Définition 5.1].

We let $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^\times]]$, it has a natural $\mathbb{Z}_p$-linear action on V_∞: $z \in \mathbb{Z}_p^\times$ acts via any matrix $g \in \mathrm{GL}_g(\mathbb{Z}_p)$ of determinant z. We shall write $z \in \mathbb{Z}_p^\times$ as $\omega(z)\langle z\rangle$, being ω the Teichmüller character.

Hence we can consider V_∞ and $\mathbb{V}$ as Λ-modules and we define

$$\begin{aligned} M_\infty :=& \mathrm{Hom}_\Lambda(\mathrm{Hom}_{\mathbb{Z}_p}(V_\infty, \mathbb{Q}_p/\mathbb{Z}_p), \Lambda),\\ \mathbb{M} :=& \mathrm{Hom}_\Lambda(\mathbb{V}, \Lambda). \end{aligned}$$

The module $\mathbb{M}$ is the (free of finite type) module of Λ-adic ordinary forms. As shown in [12], each element of M_∞ or $\mathbb{M}$ admits a q-expansion with Λ coefficients; indeed, each element in $\mathbb{V}$ has a q-expansion in $\mathbb{Q}_p/\mathbb{Z}_p$ and each T defines a map $\tilde{\lambda}_T : \mathbb{V} \to \mathbb{Q}_p/\mathbb{Z}_p$. Applying duality twice to $\tilde{\lambda}_T$ we obtain a map $\lambda_T : \mathbb{M} \to \Lambda$ which defines the q^T-th coefficient of an element of $\mathbb{M}$ and commute with specialization at any point of Λ.

We have an action of the abstract Hecke algebra $\mathbb{T}$ on $\mathbb{M}$; let $\mathbb{T}_\Lambda$ be the image of $\mathbb{T}$ in $\mathrm{End}_\Lambda(\mathcal{M})$. Then $\mathbb{M}$ is a finite module over $\mathbb{T}_\Lambda$. Moreover $\mathbb{T}_\Lambda$ is finite over Λ. We shall call an irreducible component $\mathcal{C}$ of $\mathrm{Spec}(\mathbb{T}_\Lambda)$ a Hida family of Siegel eigensystem. To each $\mathcal{C}$ we can associate the corresponding eigenspace on $\mathbb{M}$ consisting of $\mathcal{O}(\mathcal{C})$-adic q-expansions.

The same theory can be developed for Siegel form with Nebentypus outside p [16, Théorème 7.2]. In what follows we shall, if not clear from the context, emphasize the dependence from the tame level N by writing $\mathbb{M}_N$ and $\mathbb{T}_{\Lambda,N}$.

We conclude this section recalling this important theorem, originally due to Hida, which in this form is due Pilloni:

Theorem 4.1. *Let k be an integer, i an integer, $0 \leq i < p-1$ and P_k the kernel of the map $[i,k] : \Lambda \to \mathbb{Z}_p$ sending $z \in \mathbb{Z}_p^\times$ to $\omega^i(z)\langle z\rangle^k \in \mathbb{Z}_p$. For k big enough we have*

$$\mathbb{M} \otimes \Lambda/P_k \cong M_k^{(g)}(\Gamma_0(Np), \chi\omega^{-i})^{\mathrm{ord}},$$

where the subscript "ord" refers to ordinarity for U_p.

Remark 4.2. Conjecturally, the optimal k for the theorem to hold is $g+1$ and this is proven for $g = 1$. For $g = 2$, the best bound is 4 [15].

We want to point out the Hecke action on the space of Siegel forms in [16, 10] and in this paper differ by a character. Namely, using the complex uniformization of the Siegel variety, the action in *loc. cit.* would be given by

$$f|_k M(Z) = f(M(Z))\det(CZ + D)^{-k}.$$

Recall the "Satake parameters" $\beta_{p,j}$ for a form f of level $\Gamma_0(Np)$ introduced in the previous section. If we denote by λ_i the eigenvalue for $U_{p,i}$ on f the we have the relation:

$$\prod_{j=1}^{g}(1 - \beta_{p,j}X) = \sum_i (-1)^i p^{\frac{i(i+1)}{2} - i(g+1)}\lambda_i X^i. \tag{4.1}$$

In particular, f is ordinary if $|\prod_i \beta_{p,j}|_p = |p^{\frac{g(g+1)}{2} - gk}|_p$.

4.2. **The Eisenstein family.** We now use the above defined Eisenstein series to give examples of p-adic families. More precisely, we shall construct a two-variable measure (which will be used for the two-variable p-adic L-function) and a one-variable measure (which will be used to construct the improved onevariable L-function).
Let us fix $\chi = \chi_1\chi'\varepsilon_1$ as before. We suppose χ even. We recall the Kubota-Leopoldt p-adic L-function:

Theorem 4.3. *Let η be a even Dirichlet character modulo C. There exists a p-adic L-function $L_p(\eta,\kappa)$ satisfying the following interpolation formula for any integer $t \geq 1$ and finite-order character ε of $1+2p\mathbb{Z}_p$:*

$$L_p(\eta, \varepsilon(u)u^t - 1) = L^{(Cp)}(\varepsilon\omega^{-t}\eta, 1-t),$$

where η_0 stands for the primitive character associated with η and $L^{(C)}(\eta,u)$ stands for the L-function without Euler factors at primes dividing C. If η is not trivial then $L_p(\eta,\kappa)$ is holomorphic. Otherwise, it has a simple pole at $t=0$.

Note that for the characters in the proposition the parity condition $\varepsilon\omega^{-t}\eta(-1) = (-1)^t$, that ensures that $1-t$ is Deligne critical, is always satisfied. If this condition is not satisfied, then the p-adic L-function is identically 0.

We can consequently define p-adic analytic functions interpolating the Fourier coefficients of the Eisenstein series defined in the previous section; for any z in $\mathbb{Z}_p^*$, we define $l_z = \frac{\log_p(z)}{\log_p(u)}$. For an integer k, we denote by $[k]$ the weight corresponding to $z \mapsto z^k$. For T_1 and T_4 two positive semi-definite matrices we define the function:

$$\begin{aligned} a_{T_1,T_4,L}(\kappa,\kappa') = \Bigg(& \sum_I \kappa\kappa'^{-1}[-g](u^{l_{\det(2T_2)}})\chi'^{-1}(\det(2T_2)) \times \\ & \times \sum_{G\in \mathrm{GL}_{2g}(\mathbb{Z})\backslash \mathbf{D}(I)} (\phi\chi_1\chi')^2(\det(G))|\det(G)|^{-1}\kappa'^2(u^{l_{|\det(G)|}}) \\ & L_p(\sigma_{-\det(2I)}\phi\chi,\kappa') \prod_{q|\det(2G^{-t}IG^{-1})} B_q\left(\phi(q)\kappa'(u^{l_q})q^{-g-1}, G^{-t}IG^{-1}\right)\Bigg) \end{aligned}$$

for I as in Theorem 3.1. (Here κ is the weight variable and κ' is the variable for t. Recall that $s = k-g-t$.)

Note that the terms for I with $(\det(T_2),p) \neq 1$ are always 0.
We recall that if $p^{-j}T_1$ and $p^{-j}T_4$ are half integral we have [2, (1.21, 1.34)]:

$$4^{gs}\mathfrak{b}^s_{g,t+g}(I) \equiv (-1)^s 2^{gs} c^s_{g,t+g} \det(2T_2)^s \bmod p^j,$$

for $s = k-t-g$. Consequently, if we define

$$\mathcal{H}_L(\kappa,\kappa') = \sum_{T_1\geq 0}\sum_{T_4\geq 0} a_{T_1,T_4,L}(\kappa,\kappa')q_1^{T_1}q_2^{T_4}$$

from the congruence (3.1) we have:

$$\mathcal{H}_{p^j}([k],\varepsilon[t]) \equiv \frac{(-1)^s}{2^s c^s_{g,t+g}A}\mathcal{H}'_{p^j,k,t,\chi\varepsilon\omega^{-t}}(z_1,z_4) \bmod p^j, \tag{4.2}$$

with

$$A = A(k,t,\varepsilon) = (Rp^n)^{\frac{g(g-1)}{2}} B_{2g}(t)(2\pi i)^{sg} G\left(\chi\varepsilon\omega^{-t}\right)^g.$$

We have the following lemma:

Lemma 4.4. *There exists a projector*

$$e^{\mathrm{ord}}_{g,g} : M_\infty \otimes M_\infty \to \mathbb{M} \otimes \mathbb{M}$$

such that, for i, j big enough the following holds:

$$(U_p^{\otimes^2})^{-2i} e^{\mathrm{ord}}_{g,g} (U_p^{\otimes^2})^{2i} f(\kappa) = (U_p^{\otimes^2})^{-2j} e^{\mathrm{ord}}_{g,g} (U_p^{\otimes^2})^{2j} f(\kappa).$$

Proof. We define $e^{\mathrm{ord}}_{g,g} := e^{\otimes^2}_{\mathrm{GL}_g}$. The rest is an immediate consequence of the fact that U_p and e_{GL_g} commutes and that U_p is invertible on the ordinary part. $\square$

We shall now construct the improved Eisenstein family. We let $\chi = \chi_1\chi'$ be a Dirichlet character modulo $N_1 R$ and (as before) we see ϕ as a Dirichlet character modulo Np. We define

$$a^*_{T_1,T_4}(\kappa) = \left(\sum_I (\chi'\chi_1)^{-1}(2T_2) \sum_{G \in \mathrm{GL}_2(\mathbb{Z})\backslash \mathbf{D}(I)} (\phi\chi'\chi_1)^2(\det(G))|\det(G)|^{-1-2g} \kappa(u^{l_{2|\det(G)|}}) \times \right.$$
$$\left. \times L_p(\sigma_{-\det(2I)}\phi\chi'\chi_1, \kappa[-g]) \prod_{q|\det(2G^{-t}IG^{-1})} B_q\left(\phi\chi'\chi_1(q)\kappa(u^{l_q})q^{-1}, G^{-t}IG^{-1}\right) \right).$$

Note that the terms for I with $(\det(T_2), p) \neq 1$ are not necessarily 0 contrary to the previous construction. We now construct another p-adic family of Eisenstein series:

$$\mathcal{H}^*(\kappa) = \sum_{T_1 \geq 0} \sum_{T_4 \geq 0} a^*_{T_1,T_4}(\kappa[1-k_0]) q_1^{T_1} q_2^{T_2}.$$

We can construct a two-variable family of Eisenstein series, generalizing [2, Theorem 8.6]:

Proposition 4.5. *We have a p-adic measure $\mathcal{H}(\kappa,\kappa')$ on $\mathbb{Z}_p^\times \times \mathbb{Z}_p^\times$ with values in $\mathbb{M} \otimes \mathbb{M}$ such that for all k,t,ε arithmetic, with ε of conductor p^n, we have*

$$\mathcal{H}([k],\varepsilon[t]) = \frac{1}{(-1)^s c^s_{g,t+g} A} e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}}(z_1,z_4),$$

for any $i \geq 2n$.

We have a one-variable measure $\mathcal{H}^*(\kappa)$ on $\mathbb{Z}_p^\times$ with values in $\mathbb{M} \otimes \mathbb{M}$ such that

$$\mathcal{H}^*([k]) = A^{*-1} e^{\mathrm{ord}}_{g,g} \mathcal{H}'^*_{1,k,k-g,\chi_1\chi'}(z_1,z_4),$$

for

$$A^* = A^*(k) = R^{\frac{g(g-1)}{2}} B_{2g}(k-g)(2\pi i)^{sg} G\left(\chi'\right)^g.$$

Proof. Note that from its own definition we have $(U_p^{\otimes^2})^{2j} \mathcal{H}'_{1,k,t,\chi}(z_1,z_4) = \mathcal{H}'_{p^j,k,t,\chi}(z_1,z_4)$. We define

$$\mathcal{H}(\kappa,\kappa') = e^{\mathrm{ord}}_{g,g} \mathcal{H}_1(\kappa,\kappa'). \tag{4.3}$$

The key congruence (4.2) give us

$$\begin{aligned} A(-1)^s 4^s c^s_{g,t+g} e^{\mathrm{ord}}_{g,g} \mathcal{H}_1([k],\varepsilon[t]) =& A(-1)^s 4^s c^s_{g,t+g} U_p^{\otimes^{2-2j}} e^{\mathrm{ord}}_{g,g} \mathcal{H}_{p^j}([k],\varepsilon[t]) \\ \equiv& 4^s U_p^{\otimes^{2-2j}} e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{p^j,k,t,\chi\varepsilon\omega^{-t}}(z_1,z_4) \\ =& 4^s e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}}(z_1,z_4) \end{aligned}$$

as U_p acts on the ordinary part with norm 1 hence it preserves the q-expansion norm (which induces the sup-norm on the ordinary locus). Taking the limit over j gives the desired result. Similarly we define

$$\mathcal{H}^*(\kappa) = e^{\mathrm{ord}}_{g,g} \mathcal{H}^*(\kappa,\kappa'). \tag{4.4}$$

□

We want to explain briefly why the construction above works in the ordinary setting and not in the finite slope one.

It is slightly complicated to explicitly calculate the polynomial $\mathfrak{b}^s_{t+g}(I)$ and in particular to show that they vary p-adically with s. But we know that $\mathring{\mathfrak{D}}^s_{g,k-2s}$ is a homogeneous polynomial in $\frac{\partial}{\partial z_{i,j}}$ of degree gs. Consider the embedding $\mathbb{H}_g \times \mathbb{H}_g \hookrightarrow \mathbb{H}_{2g}$; we have a single monomial of $\mathring{\mathfrak{D}}^s_{g,k}$ which does belong to the normal bundle, namely $c^s_{g,t+g}\partial^s_2$, and this is the term that does not involve a partial derivative $\frac{\partial}{\partial z_{i,j}}$ in a variable $z_{i,j}$ of $\mathbb{H}^2_g$. Consequently, in $\mathfrak{b}^s_{t+g}(I)$ there is a single monomial without T_1 and T_4.

When the entries on the diagonal of I are divisible by p^i, $\mathfrak{b}^s_{t+g}(I)$ reduces to c^s_{t+g} modulo p^i. Hence applying $U_p^{\otimes^2}$ many times we approximate $\mathring{\mathfrak{D}}^s_{g,k}$ by ∂^s (multiplied by a constant). The more times we apply $U_p^{\otimes^2}$, the better we can approximate p-adically $\mathring{\mathfrak{D}}^s_l$ by ∂^s_2. At the limit, we obtain equality. We note that p-adic differential operators for unitary groups have been studied in [4].

The same method does not work for finite slope forms, as the finite slope projector is very different from Hida's ordinary projector. To construct p-adic L-functions for families of finite slope forms one should then be able to estimate the overconvergent norm of these derivatives and show that they satisfy certain distribution relations. The reader interest in overconvergent Maaß–Shimura operators should read [11].

5. p-ADIC L-FUNCTIONS

We now construct two p-adic L-functions using the above Eisenstein measures: the two-variable one and the improved. We fix a tame level N and two characters: χ_1 modulo N_1 for $N_1 \mid N$, and χ' modulo R, with $(R, Np) = 1$. Let R_0 be the product of all the primes dividing R, $S = R_0 p$ and $\chi = \chi'\chi_1$. Consider an irreducible component $\mathcal{C}$ of the ordinary Hecke algebra $\mathbb{T}_{\Lambda,NR_0}$; remember that all the classical specializations of $\mathcal{C}$ have level $\Gamma_0(p)$ at p.

We make the following hypothesis:

Multiplicity one for $\mathcal{C}$: The generalized Hecke eigenspace on $\mathbb{M} \otimes_\Lambda \mathrm{Frac}(\Lambda)$ associated with $\mathcal{C}$ is one-dimensional.

This hypothesis is not really necessary but simplify the definition and the evaluation of the p-adic L-function. The reader interested in removing this hypothesis is referred to the construction given in [12, §7].

Let us decompose

$$\mathbb{T}^{\mathrm{ord}} \otimes_{\Lambda} \mathrm{Frac}(\Lambda) = \mathbb{I}_{\mathcal{C}} \oplus \mathbb{B}$$

and let $1_{\mathcal{C}}$ be the corresponding idempotent. Let us denote by $\mathbf{F}$ a Siegel form in $1_{\mathcal{C}}\mathcal{M}^{\mathrm{ord}} \otimes_{\Lambda} \mathrm{Frac}(\Lambda)$.

We define a twisted trace operator, following ideas of Perrin-Riou and Hida [9, 1.VI]. Let N and L be two integers, we define the operator $[N/L]$ to be

$$\begin{array}{cccc} [N/L] & : & M_k^{(g)}(\Gamma_0(L), \phi) & \longrightarrow & M_k^{(g)}(\left(\begin{smallmatrix} 1_g & 0 \\ 0 & N/L \cdot 1_g \end{smallmatrix}\right)\Gamma_0(L)\left(\begin{smallmatrix} 1_g & 0 \\ 0 & L/N \cdot 1_g \end{smallmatrix}\right), \phi) \\ & & f & \mapsto & f|_k \begin{pmatrix} 1_g & 0 \\ 0 & L/N \cdot 1_g \end{pmatrix} \end{array}$$

Let now $N \mid L$ and define the twisted trace

$$T_{L/N} := \mathrm{Tr}_{L/N} \circ [N/L],$$

for $\mathrm{Tr}_{L/N}$ the trace from $M_k^{(g)}(\left(\begin{smallmatrix} 1_g & 0 \\ 0 & N/L \cdot 1_g \end{smallmatrix}\right)\Gamma_0(L)\left(\begin{smallmatrix} 1_g & 0 \\ 0 & L/N \cdot 1_g \end{smallmatrix}\right), \phi)$ to $M_k^{(g)}(\Gamma_0(N), \phi)$. If L is coprime with p, then this operator can be p-adically interpolated over the weights space. Let R_0 be the product of all prime factors of R. We hence define $L_p(\kappa, \kappa')$ to be such that

$$1_{\mathcal{C}} \otimes 1_{\mathcal{C}}(\mathrm{Id} \otimes T_{N^2R^2/NR_0}\mathcal{H}(\kappa, \kappa')) = L_p(\kappa, \kappa')\mathbf{F} \otimes \mathbf{F}.$$

Remark 5.1. It is clear that this p-adic L-function depends from the choice of $\mathbf{F}$. We do not have a duality between modular form and Hecke algebra so there is no clear choice for such $\mathbf{F}$. In contrast, for $g = 1$, there is a duality given by the first Fourier coefficient $a(1, f|T)$, as $a(1, T_n f) = a(n, f)$.

Similarly we have a one-variable p-adic L-function $L_p^*(\kappa)$ defined as

$$1_{\mathcal{C}} \otimes 1_{\mathcal{C}}(\mathrm{Id} \otimes T_{N^2R^2/NS'}\mathcal{H}^*(\kappa)) = L_p^*(\kappa)\mathbf{F} \otimes \mathbf{F}.$$

For a form f_x in the family $\mathbf{F}$ we denote by $\beta_i(f_x)$ the Satake parameters at p for f_x. On the line spanned by f_x we have $\prod_i \beta_i(f_x) = p^{\frac{g(g+1)}{2} - gk}U_p^2$. Hence the right hand side of (4.1), evaluated at $p^k X$, is a polynomial in $\mathcal{O}(\mathcal{C})[1/p]$ with leading coefficient a unit (by the ordinarity assumption). In particular, the eigenvalue for $p^{ik}U_{p,i}$ on $\mathbf{F}$ is an element of $\mathcal{O}(\mathcal{C})[1/p]$. As the numbers $\beta_i^{-1}p^{-k}$ are the roots of this Hecke polynomial, they also define analytic functions in $\mathcal{O}(\mathcal{C}')[1/p]$, where $\mathcal{C}'$ is finite over $\mathcal{C}$. We shall denote by $\mathbb{B}_i^{-1} := \mathbb{B}_i^{-1}(x)$ the corresponding element. (May the reader note that our normalisation of $U_{p,i}$ is slightly different from the standard one used in Hida theory, such as in [10, 16]. This is why right above we have inverted p.)

We define

$$\begin{aligned} E_1(f_x, \varepsilon, t) = & \prod_{i=1}^{g} (1 - (\chi\varepsilon\omega^{t-1})(p)\beta_i^{-1}p^{t-1}), \\ E_2(f_x, \varepsilon, t) = & \prod_{i=1}^{g} \frac{1}{(1 - (\chi^{-1}\varepsilon^{-1}\omega^{t})(p)\beta_i p^{-t})}, \\ E(f_x, \varepsilon, t) = & E_1(f_x, \varepsilon, t)E_2(f_x, \varepsilon, t). \end{aligned}$$

We shall write $w : \mathcal{C}^{\mathrm{rig}} \to \mathcal{W} := \mathrm{Spf}(\Lambda)^{\mathrm{rig}}$. We shall say that a point $x \in \mathcal{C}(\mathbb{Q}_p)$ is étale if w is étale at x. Recall that for a character ε we write its conductor as p^n;

recall also that with our previous notation $s = k - g - t$. The main theorem of the section is the following:

Theorem 5.2. *For points (x, κ') of type (k, t, ε) such that x is étale, $1 \leq t \leq k - g$ we have the following interpolation formula:*

$$\begin{aligned}L_p(x, \varepsilon[t]) =& 2^{1-g-gs}\alpha_{N^2/N_1^2}(f_x)R^{(1-t)g+\frac{g(g+1)}{2}}N^{g(g+1-k)}N_1^{g(k-t)}\chi(-1)^g(-1)^{\frac{gk}{2}+gs+tg}\\ &\times \frac{\prod_{i=1}^g (s+g-i)!}{(2\pi i)^{sg}\pi^{\frac{g(g+1)}{2}}}\left(\prod_{q|N, q\nmid N_1} D_q(\chi^{-1}(q)q^{t-1})^{-1}\right)\\ &\times \frac{p^{ng(1-t)}E(f_x, \varepsilon, 1-t)\mathcal{L}^{(Np)}(\mathrm{St}(f), \chi^{-1}\varepsilon^{-1}\omega^t, 1-t)}{G(\chi'\varepsilon\omega^{-t})^g\left(p^{\frac{g(g+1)}{2}-gk}\alpha_p(f_x)^2\right)^n\left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ NS & 0\end{pmatrix}, f_x\right\rangle_{NS}}.\end{aligned}$$

Moreover we have a function $L_p^*(x)$ such that the following equality of locally analytic functions around étale points $x \in \mathcal{C}^{\mathrm{rig}}$ holds:

$$L_p(x, [k-g]) = E_1(f_x, 1, 1+g-k)L_p^*(x), \tag{5.1}$$

with

$$\begin{aligned}L_p^*(x) =& 2^{1-g}\alpha_{N^2/N_1^2}(f_x)R^{(1-t)g+\frac{g(g+1)}{2}}N^{g(g+1-k)}N_1^{g(k-t)}\chi(-1)^g(-1)^{\frac{gk}{2}+tg}\\ &\times \prod_{q|N, q\nmid N_1} D_q(\chi^{-1}(q)q^{k-g-1})^{-1}\frac{\prod_{i=1}^g (g-i)!}{\pi^{\frac{g(g+1)}{2}}}\\ &\times \frac{p^{g(1+g-k)}E_2(f_x, \varepsilon, 1+g-k)\mathcal{L}^{(Np)}(\mathrm{St}(f), \chi^{-1}\varepsilon^{-1}\omega^{k-g}, 1+g-k)}{G(\chi'\varepsilon\omega^{g-k})^g\left(p^{\frac{g(g+1)}{2}-gk}\alpha_p(f_x)^2\right)\left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ NS & 0\end{pmatrix}, f_x\right\rangle_{NS}}\end{aligned}$$

for all x arithmetic and étale.

Remark 5.3. Let us denote by $\rho_{f,\mathrm{Sta}}$ the standard Galois representation constructed in [20] and by $\mathcal{D}_{\mathrm{St}}(\rho_{f,\mathrm{Sta}})$ the semi-stable (φ, N)-module associated with it. The factor

$$\frac{p^{ng(1-t)}}{G(\varepsilon\omega^{-t})^g\left(p^{\frac{g(g+1)}{2}-gk}\alpha_p(f_x)^2\right)^n}E(f_x, \varepsilon, 1-t)$$

is the γ-factor of the Weil–Deligne representation associated with the (φ, N)-submodule spanned by the eigenvectors of eigevanlues $\beta_i(f_x)$. Moreover, it is the factor at p of the p-adic L-function predicted by Coates [3, §6]. Note moreover that this product does not depend on the monodromy of $\mathcal{D}_{\mathrm{St}}(\rho_{f,\mathrm{Sta}})$.

Remark 5.4. The key to obtain the factorization above is that the factor E_1 which brings the trivial zero for forms $\Gamma_0(p)$-Steinberg is an analytic function of x:

$$E_1(f_x, 1, 1-k+g) = \prod_{i=1}^g (1 - \mathbb{B}_i^{-1}(x)p^g) \in \mathcal{O}(\mathcal{C}).$$

Note that it belongs to $\mathcal{O}(\mathcal{C})$ as it is symmetric in the $\mathbb{B}_i^{-1}(x)$, the analytic functions interpolating the Satake parameters at p which have been defined after Remark 5.1.

Proof. Let f_x be the evaluation of $\mathbf{F}$ at x. We need to calculate the coefficient at $f_x \otimes f_x$ of $\frac{1}{(-1)^{gs} c^s_{t+g} A} e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}}(z_1, z_4)$. We begin from the coefficient of $\mathrm{Id} \otimes T_{N^2R^2/NR_0} e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}}$ which is

$$\frac{\left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ NR_0p & 0 \end{pmatrix}, \left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ NR_0p & 0 \end{pmatrix}, (\mathrm{Id} \otimes T_{N^2R^2/NR_0}) e^{\mathrm{ord}}_{g,g} \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}} \right\rangle_{NR_0p} \right\rangle_{NR_0p}}{\left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ NR_0p & 0 \end{pmatrix}, f_x \right\rangle^2_{NR_0p}},$$

as the Hecke operators are self-dual for the normalized Petersson product $\left\langle f_x \middle| \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}, f_x \right\rangle$. The proof of [16, Proposition 5.3] tells us that the U_p operator on the right-hand-side can be written as $p^{m\frac{gk-g(g+1)}{2}} \left[\Gamma_0(Np^m) \begin{pmatrix} 1_g & 0 \\ 0 & p^m 1_g \end{pmatrix} \Gamma_0(Np) \right]$. We know the relation

$$\left|_k \begin{pmatrix} 0 & -1 \\ Np & 0 \end{pmatrix} \right|_k \begin{pmatrix} p^m & 0 \\ 0 & 1 \end{pmatrix} = \left|_k \begin{pmatrix} 0 & -1 \\ Np^{m+1} & 0 \end{pmatrix}, \right.$$

$$\left|_k \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \right|_k \begin{pmatrix} 1 & 0 \\ 0 & N/L \end{pmatrix} = \left|_k \frac{N}{L} \begin{pmatrix} 0 & -1 \\ L & 0 \end{pmatrix}. \right.$$

We use Lemma 4.4 and [2, Lemma 4.1] to see that the numerator is

$$\alpha_p(f_x)^{-2n} p^{n(gk-g(g+1))} \left\langle f_x \left| \tau_{NS} \right., \left\langle f_x \left| \tau_{N^2R^2p^{2n+1}} \right., \mathcal{H}'_{1,k,t,\chi\varepsilon\omega^{-t}} \right\rangle_{N^2R^2p^{2n}} \right\rangle_{NR_0p^{2n}},$$

where we recall $S = R_0 p$.

From Proposition 3.3 we have a term $\frac{\Omega_{k,s}(\frac{1}{2}-t) p_{\frac{1}{2}-t}(t+g)}{d_{\frac{1}{2}-t}(t+g)}$ appearing; we are left to evaluate

$$\begin{aligned}
&\frac{\Omega_{k,s}(\frac{1}{2}-t) p_{\frac{1}{2}-t}(t+g)}{B_{2g}(t)(-1)^{gs} c^s_{g,t+g} d_{\frac{1}{2}-t}(t+g)} \\
=&\frac{\Omega_{k,s}(\frac{1}{2}-t)}{B_{2g}(t)(-1)^{gs} c^s_{g,g+\frac{1}{2}}} \\
=&\frac{(-1)^{g(g+t+s)} \Gamma_{2g}\left(g+\frac{1}{2}\right)}{2^{g(1+2t)} \pi^{g+2g^2}} \prod_{j=1}^{s} \frac{\Gamma_g(s+\frac{g}{2}-\frac{j}{2})}{\Gamma_g(s+\frac{g}{2}+1-\frac{j}{2})} \\
&\times (-1)^{\frac{gk}{2}} 2^{1+\frac{g(g+1)}{2}-g+2gt} \pi^{\frac{g(g+1)}{2}} \frac{\Gamma_g(k-t+\frac{1-g}{2}) \Gamma_g(k-t-\frac{g}{2})}{\Gamma_g(k-s+\frac{1}{2}-t) \Gamma_g(k-s+\frac{1-g}{2}-t)} \\
=&(-1)^{\frac{3gk}{2}} 2^{1+\frac{g(g+1)}{2}-2g} \pi^{-\frac{g(3g+1)}{2}} \frac{\Gamma_{2g}(g+\frac{1}{2})}{\Gamma_g(g+\frac{1}{2}) \Gamma_g(\frac{g+1}{2})} \Gamma_g(s+\frac{g-s}{2}+\frac{1}{2}) \Gamma_g(s+\frac{g-s}{2})
\end{aligned}$$

and to conclude we use that:

$$\Gamma_g(s) \Gamma_g\left(s+\frac{1}{2}\right) = \pi^{\frac{g(g-1)}{2}} 2^{\frac{g(g+1)}{2}-2gs} \pi^{\frac{g}{2}} \prod_{i=1}^{g} \Gamma(2s-i+1),$$

$$\frac{\Gamma_{2g}(g+\frac{1}{2})}{\Gamma_g(g+\frac{1}{2}) \Gamma_g(\frac{g+1}{2})} = \pi^{\frac{g^2}{2}}.$$

The proof of second part of the theorem is very similar. Comparing [2, (7.13), (7.13)'] we see that the only difference with the previous calculation is that we have to remove the factor

$$p^{-\frac{g(g+1)}{2}}\prod_{i=1}^{g}(1-\beta_i^{-1}(f_x)\chi^{-1}(p)p^{-t}).$$

Here the power of p compensate the missing power of p in the term A^* which appears in the interpolation formula (4.4). □

6. A formula for the derivative

We now fix a form f of weight $g+1$ and we suppose that f has a Satake parameter, let us say β_g, equal to p^{-1}. Conjecturally, this should imply that π_f has not spherical level at p (as otherwise the β_i's should all be Weil number of weight zero). In particular, if $\varphi=(\rho,N)$ is the L-parameter (with values in GSpin_{2g+1}) of $\pi_{f,p}$ then N should have a 1 in the $g,g+1$ entry.

Definition 6.1. We say that f is $\Gamma_0(p)$-Steinberg at p if $\beta_g=p^{-1}$ and the $g,g+1$ entry of N is not zero.

This condition should conjecturally ensure us that the trivial zero is brought by E_1 and not by the missing factor $(1-p^{-s})$ of the completed L-function for $\mathrm{St}(f)$. We remark that $\Gamma_0(p)$-Steinberg points are *isolated* in $\mathrm{Spec}(\mathbb{T})$ in the sense that only finitely many forms in $\mathrm{Spec}(\mathbb{T})$ satisfy this condition (as $p^{k-g}\beta_g$ must have fixed p-adic valuation).
Given a modular form f of weight k_f corresponding to a point $x_f\in\mathcal{C}$, we define $L_p(\mathrm{St}(f),s):=L_p(x_f,[k_f-g-s])$. The utility of the shift from the variable t to the variable s will become clear in the proof of the main theorem of the paper that we now recall:

Theorem 6.2. *Let f be a Siegel form of weight $g+1$ and trivial Nebentypus; suppose that f is $\Gamma_0(p)$-Steinberg and the corresponding point on $\mathcal{C}^{\mathrm{rig}}$ is étale, then*

$$\frac{\mathrm{d}}{\mathrm{d}s}L_p(\mathrm{St}(f),s)_{|s=0}=\ell^{\mathrm{al}}(\mathrm{St}(f))E^*(f)\frac{\mathcal{L}^{(Np)}(\mathrm{St}(f),0)}{\Omega(f)\prod_{q|N}D_q(q^{t-1})},$$

for $\Omega(f)$ a suitable complex period from 5.2 and

$$E^*(f)=\frac{\prod_{i=1}^{g-1}(1-\beta_i^{-1}p^{-1})}{\prod_{i=1}^{g}(1-\beta_i p)}.$$

Proof. By hypothesis we have that f corresponds to a point x which is étale over $\mathcal{W}=\mathrm{Spf}(\Lambda)^{\mathrm{rig}}$. Let us write $t_0=(z\mapsto\omega^{-g-1}(z)z^{g+1})$; t_0 is a local uniformizer in $\mathcal{O}_{\mathcal{W},[g+1]}$ and, by étalness, t_0 is also a local uniformizer for $\mathcal{O}_{\mathcal{C}^{\mathrm{rig}},x}$. We have an isomorphism between the tangent spaces and this induces an isomorphism on derivations:

$$\mathrm{Der}_K(\mathcal{O}_{\mathcal{W},[g+1]},\mathbb{C}_p)\cong\mathrm{Der}_K(\mathcal{O}_{\mathcal{C}^{\mathrm{rig}},x},\mathbb{C}_p).$$

The isomorphism is made explict by fixing a common basis $\frac{\partial}{\partial T}$. We consider the two-variable p-adic L-function of Theorem 5.2 with $\chi=1$ and $N_1=1$. We shall now write $L_p(k,t)$ for $L_p(x,t)$ (here $k=w(x)$).

If $w(x)$ is big enough, we know that f_x must be a classical form of level prime to p [16, Théorème 1.1 (6)]. Then we know that the Euler factor at p of $L(\mathrm{St}(f), s)$ is

$$(1-\phi(p)p^{-s})\prod_{i=1}^{g}(1-\phi(p)\beta_{p,i}^{-1}p^{-s})(1-\phi(p)\beta_{p,i}p^{-s}).$$

Hence we can rewrite

$$E(f_x,t)\mathcal{L}^{(Np)}(\mathrm{St}(f),1-t) = E_1(f_x,t)(1-p^{t-1})\prod_{i=1}^{g}(1-\beta_i^{-1}p^{t-1})\mathcal{L}^{(N)}(\mathrm{St}(f),1-t).$$

This means that the two-variable p-adic L-function vanishes on $t=1$. In what follows we shall use s as a variable rather than t; remember that $t=k-g-s$. The following formula is a straightforward consequence of the vanishing along $s=k-g-1$:

$$\frac{\mathrm{d}}{\mathrm{d}s}L_p(k,s)_{|s=0,k=g+1} = -\frac{\mathrm{d}}{\mathrm{d}k}L_p(k,s)_{|s=0,k=g+1}.$$

From the factorization in Theorem 5.2 we are left to calculate the derivative at $k=g+1$ of $E_1(f_x,1,1+g-k)$. Using the factorization given in Remark 5.4 we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}k}L_p(k,k-g)_{|k=g+1} = \frac{\mathrm{d}\mathbb{B}_g(k)}{\mathrm{d}k}_{|k=g+1}\prod_{i=1}^{g-1}(1-\mathbb{B}_i^{-1}(g+1)p^g)L_p^*(g+1).$$

To conclude we use [19, Theorem 1.3]:

$$\ell^{\mathrm{al}}(\mathrm{St}(f)) = -\frac{\mathrm{d}\mathbb{B}_g(k)}{\mathrm{d}k}_{|k=g+1}.$$

$\square$

6.1. **Some examples.** In this last section we want to present some examples of the automorphic representation of $\mathrm{GSp}_4(\mathbb{Q}_p)$ which could be associated with a Siegel modular form of level $\Gamma_0(p)$ and check whether our theorem applies or not.

We now consider an automorphic representation π which at p is isomorphic to the one labelled IIIa in [17, Table A.1]. We see that it admits exactly two vectors invariant for $\Gamma_0(p)$ ([17, Table A.15], where $\Gamma_0(p)$ is called $\mathrm{Si}(\mathfrak{p})$) and the corresponding Satake parameters are respectively $\chi(p), p^{-1}$ and $\chi(p), p$; only one corresponds to a U_p-ordinary Siegel form. The monodromy of the corresponding Weil–Deligne representation is given by the following matrix [17, Table A.7, §A.7]:

$$\begin{pmatrix} 0 & & & & \\ & 0 & -1 & -1/2 & \\ & & 0 & 1 & \\ & & & 0 & \\ & & & & 0 \end{pmatrix}.$$

From [17, Table A.10] we see that the factor E_1 vanishes and the main theorem applies. Note that the corresponding automorphic representation comes from the induction from the Klingen parabolic of a representation whose GL_2 part is a twist of the Steinberg representation. As the family of twists of the Steinberg representation for GL_2 has non zero Plancherel measure we can then use [23, Theorem

5.7] to deduce that there exists a weight 3 Siegel form with this local representation.

We now consider an automorphic representation π which at p is isomorphic to the one labelled IIa in [17, Table A.1]; we see that it admits exactly one vector invariant for $\Gamma_0(N)$. The monodromy of the corresponding Weil–Deligne representation is given by

$$\begin{pmatrix} 0 & 1 & & & \\ & 0 & & & \\ & & 0 & & \\ & & & 0 & -1 \\ & & & & 0 \end{pmatrix}.$$

We see that the trivial zero here comes from the factor E_2 and we can not deal with it.

In a third case, the one labelled IVb, the monodromy of the corresponding Weil–Deligne representation is given by the same matrix as IIIa; hence our theorem would apply if a form f with this local representation exists (the Plancherel measure of this representation is 0 so the aforementioned theorem of Shin does not apply).

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