# Bounding cohomology for low rank algebraic groups

John Rizkallah

University of Cambridge
Department of Pure Mathematics and Mathematical Statistics
Homerton College

August 2017

This dissertation is submitted for
the degree of Doctor of Philosophy

# Abstract

Let $G$ be a semisimple linear algebraic group over an algebraically closed field of prime characteristic. In this thesis we outline the theory of such groups and their cohomology. We then concentrate on algebraic groups in rank 1 and 2, and prove some new results in their bounding cohomology.

# Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as specified in the text

John Rizkallah
August 1, 2017

# Acknowledgments

Let's call more help
To have them bound again.

*The Comedy of Errors*,
4.4.145-146

Great thanks go to my supervisor, David Stewart, for continually providing me with encouragement, useful advice, and good ideas. Thanks also to Jan Saxl for suggesting the collaboration between David and myself. A special mention must go to two of my fellow students, Julian Brough and Ruadhai Dervan, whose friendship and support was vital. Finally, thanks to Steve Donkin for his contribution to the proofs in the final section.

I acknowledge the financial support of EPSRC.

# Contents

# Introduction

Calculating the spaces $\mathrm{Ext}^q_G(M,N)$, where $G$ is an algebraic group defined over an algebraically closed field of positive characteristic, and $M$ and $N$ are modules for $G$, is a very general problem that has long been of interest to those engaged in the research of the characteristic $p$ representation theory of algebraic groups. Of course it is such a general problem that further restrictions must be imposed if one is to make any progress. For instance, one might suppose that $M$ and $N$ are both Weyl modules, or both simple modules. Assumptions may be made on the algebraic group $G$, insisting that it is simple, semisimple, or simply connected. Often $G$ is taken to be a particular algebraic group; $\mathrm{SL}_2$ being a popular choice as it is the easiest case. But even with some of these restrictions in place, completely determining the space $\mathrm{Ext}^q_G(M,N)$ remains a difficult problem. Thus one may seek not to explicitly calculate the space, but instead to determine bounds for its dimension, or even simply to prove that such bounds exist. This field of research tends to be referred to as *bounding cohomology*.

Robert Guralnick is responsible for some of the first general results in this area. In particular he was interested in determining how large $\dim \mathrm{H}^1(G,V)$ can be, as these cohomology groups are critical to understanding both primitive permutation groups and the module structure of indecomposable modules ([21]). The first result along these lines came in [6]: take $G$ a finite group, $k$ a field of positive characteristic and $V$ a $kG$-module. If $V$ is an irreducible faithful $G$-module over $kG$, then

$$\dim \mathrm{H}^1(G,V) < \dim V.$$

The next result came in [19]: if $V$ is a finite dimensional $kG$-module and $G$ acts faithfully on each composition factor of $V$, then

$$\dim \mathrm{H}^1(G,V) < \frac{2}{3}\dim V.$$

There followed more bounds on $\dim \mathrm{H}^1(G,V)$ in terms of $\dim V$; for instance, in [20], Guralnick and Hoffman show that, if $G$ is quasi-simple, then

$$\dim \mathrm{H}^1(G,V) \leq \frac{1}{2}\dim V.$$

However, it was observed that in many specific cases of single $V$, $\dim \mathrm{H}^1(G,V)$ is very small, meaning that the bounds given above are often far from best possible. This observation led Guralnick to make the following conjecture in [19].

**Guralnick's Conjecture.** *There exists an absolute constant $c$ such that if $G$ is a finite group and $V$ is an irreducible faithful $kG$-module, then*

$$\dim \mathrm{H}^1(G,V) \leq c.$$

In fact, the original statement of this conjecture was with $c = 2$. Whilst this original, rather optimistic version of the conjecture was proven false by Scott in [37], who found examples of 3-dimensional $\mathrm{H}^1(G,V)$, the conjecture about a universal bound proved to be difficult, and for a long time, little progress was made towards it. Then in 2011, using a result of Cline, Parshall and Scott ([14, Thms. 7.3, 7.10]), Guralnick and Tiep were able to prove the existence of a bound dependent on the rank of the underlying algebraic group.

**Theorem 0.1** ([21, Thm. 1.4])**.** *Let $G$ be a finite simple Chevalley group whose underlying algebraic group has rank $r$. Let $V$ be an irreducible $kG$-module where $k$ is an algebraically closed field of arbitrary characteristic. Then there is a constant $c = c(r)$ such that*

$$\dim \mathrm{H}^1(G,V) \leq c.$$

So even if Guralnick's Conjecture is false, studying the growth of $c(r)$ is an important problem.

Parker and Stewart in [31] found a new upper bound for $\dim \mathrm{H}^1(G,V)$, and used this bound, along with the work of Guralnick and Tiep, to prove a growth rate result. Let $\{\gamma_l\}$ be the sequence $\{\max \dim \mathrm{H}^1(G,V)\}$, with the maximum taken over all finite simple groups of Lie type of rank $l$, and over all irreducible modules $V$ of $G$. Then $\log \gamma_l = O(l^3 \log l)$.

Parshall and Scott in [33] proved a result looking more generally at the extension spaces $\mathrm{Ext}^1_G(V,V')$ instead of the cohomology spaces $\mathrm{H}^1(G,V)$.

**Theorem 0.2** ([33, Thm. 5.1])**.** *Let $G$ be a simple simply connected algebraic group with root system $\Phi$ defined over an algebraically closed field of arbitrary positive characteristic. Then there is a constant $c = c(\Phi)$ such that*

$$\dim \mathrm{Ext}^1_G(L,L') \leq c$$

*for any two irreducible rational $G$-modules $L, L'$.*

In the same paper, the authors go further and prove a higher degree version of this result; that is, they consider the spaces $\mathrm{Ext}^m_G(V, V')$ for $m \geq 1$. Here, $X_e$ denotes the set of $p^e$-restricted dominant weights.

**Theorem 0.3** ([33, Thm. 7.1])**.** *Let $m, e$ be non-negative integers. Let $G$ be a semisimple simply connected algebraic group with root system $\Phi$ defined over an algebraically closed field of arbitrary positive characteristic $p$. Then there is a constant $c = c(\Phi, m, e)$ such that, for $\lambda, \nu \in X^+$ with $\lambda \in X_e$,*

$$\dim \mathrm{Ext}^m_G(L(\lambda), L(\nu)) = \dim \mathrm{Ext}^m_G(L(\nu), L(\lambda)) \leq c$$

*for any two irreducible rational $G$-modules $L(\lambda), L(\nu)$.*

As previously, we now have a sequence $\{\max \dim \mathrm{Ext}^m_G(L(\lambda), L(\nu))\}$, with the maximum taken with respect to the rank of the group, whose growth rate we wish to study. Parshall and Scott do just this in their paper [34], and are able to provide both upper and lower bounds for the polynomial growth rate of the sequence in the quantum group case. However, an analogous result in the case of algebraic groups was not forthcoming. The following question was posed.

**Question 0.4** ([34, Question 6.2])**.** Let $\Phi$ be a (finite) root system. Do there exist constants $c = c(\Phi)$ and $f = f(\Phi)$ such that

$$\dim \mathrm{H}^m(G, L) \leq cm^f$$

for all semisimple simply connected groups $G$ with root system $\Phi$ defined over an algebraically closed field (of arbitrary characteristic), and for all irreducible rational $G$-modules $L$?

This question was answered in the negative for the case $G = \mathrm{SL}_2$ by Stewart in [40]. Using a recursive formula of Parker from [30], he was able to show that $\dim \mathrm{Ext}^2_{\mathrm{SL}_2}(L, L)$ can be arbitrarily large for an irreducible module $L$. Furthermore, the cohomological growth rate for this case is proved to be exponential, as $\dim \mathrm{H}^{2m}(\mathrm{SL}_2, L(2^{2m})) \geq 2^{m-1}$ ([40, Thm. 2]).

Replacing the irreducible modules with Weyl modules, Erdmann, Parker and Hannabuss in [17] show that the cohomological growth rate is again exponential; that is, they prove that the sequence

$$(\max\{\dim \mathrm{H}^n(\mathrm{SL}_2, \Delta(m)) : m \in \mathbb{N}\})$$

has an exponential lower bound. They also find an upper bound involving an $n!$. This work is extended by Lux, Ngo and Zhang in [27], who improve the upper bound thusly:

**Theorem 0.5** ([27, Cor. 4.3])**.** *For all $m \geq 0$ and $n \geq 1$,*

$$\dim \mathrm{H}^n(\mathrm{SL}_2, \Delta(m)) \leq (n+1)4^{n+1}e^{2\pi(n+1)/\sqrt{3}}.$$

Let us now return to the more general case of $G$ being any simply connected simple algebraic group. We say a surjective homomorphism $\sigma : G \to G$ is strict if $G(\sigma) = \{g \in G(k) \mid \sigma(g) = g\}$ is finite. Let $\sigma : G \to G$ be a strict endomorphism of $G$ (the most significant example for us of such a map will be the geometric Frobenius endomorphism, defined later - but see [25, I.9.2]), and denote by $G_\sigma$ the scheme-theoretic kernel of $\sigma$, an infinitesimal subgroup of $G$. Scheme-theoretically this is a functor $G_\sigma : k\text{-Alg} \to \mathrm{Gp}$ with $G_\sigma(A) = \{g \in G(A) \mid \sigma(g) = 1_A\}$. The aforementioned results from [33] were extended to apply to both $G(\sigma)$ and $G_\sigma$ by Bendel, Nakano, Parshall, Pillen, Scott and Stewart in [8]. To state these two theorems, note that $X_\sigma \subset X^+$ is the subset of $\sigma$-restricted dominant weights, parametrising the simple $G(\sigma)$-modules; and if $e$ is an integer then $X_e$ is the set of $p^e$-restricted dominant weights, parametrising the simple modules for the $e$th Frobenius kernel $G_e$.

**Theorem 0.6** ([8, Thm. 1.2.1])**.** *Let $m, e$ be non-negative integers. Let $G$ be a simple simply connected algebraic group with finite irreducible root system $\Phi$ defined over an algebraically closed field of arbitrary positive characteristic $p$, and let $\sigma$ be a strict endomorphism of $G$ such that $X_e \subseteq X_\sigma$. Then there is a constant $c(\Phi, m, e)$ such that, for $\lambda \in X_e$ and $\nu \in X_\sigma$, we have*

$$\dim \mathrm{Ext}^m_{G(\sigma)}(L(\lambda), L(\nu)) \leq c(\Phi, m, e).$$

*In particular,*

$$\dim \mathrm{H}^m(G(\sigma), L(\lambda)) \leq c(\Phi, m, 0)$$

*for all $\lambda \in X_\sigma$.*

**Theorem 0.7** ([8, Thm. 1.3.1])**.** *Let $m, e$ be non-negative integers. Let $G$ be a simple simply connected algebraic group with finite irreducible root system $\Phi$ defined over an algebraically closed field of arbitrary positive characteristic $p$, and let $\sigma$ be a strict endomorphism of $G$ such that $X_e \subseteq X_\sigma$. Then there is a constant $c(\Phi, m, e)$ such that, for $\lambda \in X_e$ and $\nu \in X_\sigma$, we have*

$$\dim \mathrm{Ext}^m_{G_\sigma}(L(\lambda), L(\nu)) \leq c(\Phi, m, e).$$

*In particular,*

$$\dim \mathrm{H}^m(G_\sigma, L(\lambda)) \leq c(\Phi, m, 0)$$

*for all $\lambda \in X_\sigma$. Furthermore, there is a constant $c(\Phi)$ such that*

$$\dim \mathrm{Ext}^1_{G_\sigma}(L(\lambda), L(\nu)) \leq c(\Phi)$$

*for all $\lambda, \nu \in X_\sigma$.*

This concludes our overview of the current state of the field of bounding cohomology.

It is our intention in this thesis to present some recent new results bounding the cohomology of low rank algebraic groups. We are interested in proving 'generic' cohomology results, where $p$ is large compared to both the root system and the degree of cohomology. The conclusion of the Borel-Bott-Weil theorem holds for weights which are close to zero; more precisely, if $\lambda$ is in the lowest alcove, then $\mathrm{H}^i(w \cdot \lambda) = 0$ for all $w \in W$, unless $\lambda$ is dominant and $i = l(w)$, in which case $\mathrm{H}^i(w \cdot \lambda) = \mathrm{H}^0(\lambda)$ ([25, II.5.5]). Therefore there is the potential to get concrete results for large $p$. Roughly the first half of this thesis consists of background material pertaining to the original work presented in the second half.

Chapter 1 is a very brief introduction to the theory of linear algebraic groups and affine varieties over an algebraically closed field. We outline how linear algebraic groups arise as special cases of affine varieties, and indicate that all linear algebraic groups can be thought of as groups of matrices. In the final section we construct a Lie algebra associated to each linear algebraic group.

Chapter 2 is an overview of the theory of representations of algebraic groups. We work through the theory of characters, weights and roots, and see Chevalley's classification of simple modules via dominant weights. The particularly useful Tensor Product theorem of Steinberg is introduced, and we give specific examples of all this theory for the groups $\mathrm{SL}_2$ and $\mathrm{SL}_3$. In the final section we state the classification of semisimple algebraic groups by their root data.

Chapter 3 is an introduction to the theory of group cohomology. We begin by describing this in the general setting of derived functors. We then consider some cohomological tools which will be employed extensively in later chapters: the Linkage Principle, the Andersen-Jantzen formula, translation functors, and spectral sequences. Finally we introduce the notion of a filtration of a module.

In Chapter 4 we state and prove some new results in the bounding cohomology of $\mathrm{SL}_2$ with coefficients in an irreducible module. Firstly we use the recursive formula of Parker to determine all the weights which lead to non-trivial cohomology in low (i.e. $\leq 3$) degree. Subsequently we show that under the condition $p > q$ (i.e. the characteristic of the underlying field exceeds the cohomological degree), the dimension of the cohomology space is at most 1, and we describe all of the weights which attain this bound. The contents of this chapter are published in [36].

Chapter 5 contains some new work on the bounding cohomology of rank two algebraic groups, again with coefficients in an irreducible module. We show that in this case, the cohomological dimension can grow arbitrarily large. Specifically, we show that under the condition $p > q$ (see previous paragraph), there is a weight which leads to a cohomological dimension of at least $q/2 - 1$.

# Chapter 1

# Linear Algebraic Groups

In this first chapter we will give a brief introduction to the study of algebraic groups and their Lie algebras. The main references for this material are [28, Ch. 1, 7] and [23, Ch. 7, 9].

## 1.1 First definitions

Let $k$ be an algebraically closed field of positive characteristic.

**Definition 1.1.** An *algebraic set* is a subset of $k^n$ of the form

$$X = X(I) = \{(x_1, \ldots, x_n) \in k^n \mid f(x_1, \ldots, x_n) = 0 \text{ for all } f \in I\}$$

for some ideal $I \lhd k[T_1, \ldots, T_n]$. The *Zariski topology* is given by declaring complements of algebraic sets to be open. An *affine algebraic variety* is an algebraic set together with the induced Zariski topology.

The *radical* $\sqrt{I}$ of an ideal $I \lhd k[T]$ is

$$\{f(T) \in k[T] \mid f(T)^r \in I \text{ for some } r \geq 0\}.$$

We say that $I$ is a radical ideal if it is its own radical. Now let $I(X)$ denote the set of all polynomials vanishing on $X$. Then Hilbert's Nullstellensatz ([23, Theorem 1.1]) states that we have $\sqrt{I} = I(X(I))$, which sets up a one-to-one correspondence between varieties in affine $n$-space and radical ideals of $k[T_1, \ldots, T_n]$. If $I = I(X)$ then the polynomial functions on $X$ can be identified with elements of $k[X] = k[T_1, \ldots, T_n]/I$, and we call this ring the *coordinate ring* of $X$.

In a more general setting, thinking of a variety as a scheme, we can let I be any ideal (not necessarily radical), with the consequence that the scheme may not be smooth. The *coordinate ring* of an affine algebraic variety $X$ is

$$k[X] := k[T_1, \ldots, T_n]/I.$$

A morphism of affine algebraic varieties is a map between varieties which can be defined by polynomial functions in the coordinates. A *linear algebraic group* is an affine algebraic variety with the structure of a group, such that multiplication and inversion are morphisms of varieties.

In this thesis we only consider linear algebraic groups, hence we do not consider the more general class of algebraic groups which includes elliptic curves, for example.

**Example 1.2.** The additive group of $k$ is a linear algebraic group, denoted $\mathbf{G_a}$. Since as a variety $\mathbf{G_a}$ is affine 1-space $k$ with coordinate ring $k[T]$, we have $k[\mathbf{G_a}] \simeq k[T]$. The multiplicative group of $k$ is also a linear algebraic group, denoted $\mathbf{G_m}$, with coordinate ring $k[T_1, T_2]/(T_1T_2 - 1) \simeq k[T_1, T_1^{-1}]$.

To see that $\mathrm{GL}_n = \{A \in k^{n^2} \mid \det A \neq 0\}$ is an algebraic group, identify it with

$$\{(A, y) \in k^{n^2+1} \mid \det(A) \cdot y = 1\}.$$

If $\{T_{i,j} \mid 1 \leq i, j \leq n\} \cup \{S\}$ denotes the set of basic coordinate functions for $k^{n^2+1}$, then the coordinate ring $k[\mathrm{GL}_n] = X(I)$, where $I = (\det T_{i,j} \cdot S - 1)$.

Following the previous example, we see that any closed subgroup $G \leq \mathrm{GL}_n$ is itself a linear algebraic group, as the natural embedding $G \hookrightarrow \mathrm{GL}_n$ is a morphism of linear algebraic groups. Indeed, any linear algebraic group can be thought of this way, as the following theorem shows.

**Theorem 1.3.** *Let $G$ be a linear algebraic group. Then $G$ can be embedded as a closed subgroup into $\mathrm{GL}_n$ for some $n \in \mathbb{N}$.*

Recall that a topological space $X$ is *irreducible* if it cannot be expressed as the union of two proper closed subsets, and is *connected* if it cannot be expressed as the union of two disjoint proper closed subsets. If $X$ is noetherian then it is a union of finitely many maximal irreducible subsets, which we call the *irreducible components* of X. Similarly, $X$ is a union of finitely many maximal connected subsets, the *connected components*. Algebraic sets are noetherian, so they have irreducible and connected components. For a linear algebraic group $G$, the connected components and the irreducible components are the same. There is a unique such component which contains the identity, called the *identity component*, and denoted $G^0$. It is a normal subgroup of $G$ of finite index, and is the maximal connected subgroup of $G$. We say that $G$ is *connected* if $G = G^0$.

The *radical* of $G$, denoted $R(G)$, is the maximal closed connected solvable normal subgroup. This subgroup always exists, and is unique. The *unipotent radical* $R(G)_u$ is the maximal closed connected normal unipotent subgroup of

$G$. We say that $G$ is *reductive* if $R(G)_u = 1$. If, furthermore, $G$ is connected and $R(G) = 1$, then we say $G$ is *semisimple.* Finally, we say that $G$ is (almost) *simple* if the only proper normal subgroups of $G$ are finite.

## 1.2 Actions of Linear Algebraic Groups on Varieties

We will need to consider the actions of linear algebraic groups on varieties. We say $G$ acts on a variety $X$ via $\phi$ if $\phi : G \times X \to X$ is a morphism of varieties, written $\phi(g, x) = g \cdot x$, satisfying $g_1 \cdot (g_2 \cdot x) = (g_1 g_2) \cdot x$ and $e \cdot x = x$.

## 1.3 The Lie Algebra of a Linear Algebraic Group

Let $A$ be an associative $k$-algebra. A *derivation* of $A$ is a $k$-linear map $D : A \to A$ such that $D(fg) = fD(g) + D(f)g$ for all $f, g \in A$. The space of derivations is denoted $\mathrm{Der}_k(A)$. We can make $\mathrm{Der}_k(A)$ into a Lie algebra with the following Lie bracket:

$$[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1.$$

Let $G$ be a linear algebraic group, and take $A = k[G]$, the coordinate ring. For $g \in G$, define $\lambda_g : A \to A$ by

$$(\lambda_g \cdot f)(x) := f(g^{-1}x)$$

for each $f \in A$ and each $x \in G$.

**Definition 1.4.** The *Lie algebra of G* is

$$\mathfrak{g} = \mathrm{Lie}(G) := \{D \in \mathrm{Der}_k(k[G]) \mid D\lambda_g = \lambda_g D \text{ for all } g \in G\}.$$

One important use of the Lie algebra is in defining the *adjoint representation* of $G$. Consider, for each $x \in G$, the *inner automorphism* $\mathrm{Int}_x : G \to G$, given by $\mathrm{Int}_x(y) = xyx^{-1}$. The differential $d\,\mathrm{Int}_x : \mathfrak{g} \to \mathfrak{g}$ is a Lie algebra automorphism; we write $\mathrm{Ad}\, x$ in place of $d\,\mathrm{Int}_x$. This allows us to define the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g}),\ x \mapsto \mathrm{Ad}\, x.$$

# Chapter 2

# Representations of Algebraic Groups

In this chapter we work towards a definition of the modules which will be the objects of our study. We also present the classification of semisimple algebraic groups. The main references are again [28], [23] and [25]; more specific references will be given throughout the chapter.

## 2.1 Characters, Weights, Roots

Let $G$ be a linear algebraic group. A *Borel subgroup* $B$ of $G$ is a maximal closed connected solvable subgroup.

A variety $V$ is said to be *projective* if it can be embedded as a closed subvariety of $\mathbb{P}^n$ for some $n$, and $V$ is a $G$-variety if there is an action of $G$ on $V$, that is a morphism $\phi : G \times V \to V$ of varieties such that $\phi(gh, v) = \phi(g, \phi(h, v))$ for all $g, h \in G$ and $v \in V$.

The following two theorems may be found in [28, Ch. 6] and [23, Ch. 21].

**Theorem 2.1** (Borel fixed point theorem)**.** *Let $G$ be a connected, solvable linear algebraic group acting on $X$, a non-empty projective $G$-variety. Then there is a point in $X$ which is fixed by every element of $G$.*

**Theorem 2.2.** *If $G$ is a linear algebraic group, then all Borel subgroups of $G$ are conjugate.*

A linear algebraic group $T$ is a *torus* if it is isomorphic to a direct product of copies of $\mathbf{G_m}$. Equivalently, $T$ is a torus if it is isomorphic to the subgroup of diagonal matrices of $\mathrm{GL}_n$, for some $n$. The *rank* of a torus will simply be its dimension; the rank of any reductive group is the dimension of a maximal torus. It is clear from the second definition that all tori are commutative. Thus they are solvable, and so every maximal torus of $G$ lies in some Borel subgroup. But all maximal tori are conjugate in solvable groups. One of the main applications of Theorem 2.2 is to show that all maximal tori of $G$ are conjugate [23, 21.3 Cor. A].

Let $G$ be a connected reductive algebraic group over an algebraically closed field $k$ of characteristic $p > 0$. Let $\mathfrak{g}$ be the Lie algebra of $G$, and $B$ a Borel subgroup of $G$ containing a maximal torus $T$ of rank $r$.

**Definition 2.3.** The set of *characters* of $T$ is

$$X(T) = \mathrm{Hom}(T, \mathbf{G_m}).$$

The set of *cocharacters* of $T$ is

$$Y(T) = \mathrm{Hom}(\mathbf{G_m}, T).$$

Since $\mathrm{Hom}(\mathbf{G_m}, \mathbf{G_m}) \simeq \mathbb{Z}$ and $T \simeq \mathbf{G_m}^r$, we get that $X(T) \simeq \mathbb{Z}^r$, a free abelian group of rank $r$. Similarly, $Y(T) \simeq \mathbb{Z}^r$. Given $\chi \in X(T)$ and $\gamma \in Y(T)$, we have $\chi \circ \gamma \in \mathrm{End}(\mathbf{G_m}) \simeq \mathbb{Z}$, so there is $\langle \lambda, \gamma \rangle \in \mathbb{Z}$ such that $\chi \circ \gamma$ acts as $c \mapsto c^{\langle \lambda, \gamma \rangle}$ for all $c \in \mathbf{G_m}$. Thus we obtain a bilinear map $\langle \ , \ \rangle : X(T) \times Y(T) \to \mathbb{Z}$, called a *perfect pairing.*

Let $V$ be a vector space such that $G$ acts by linear maps, that is

$$g \cdot (av + bw) = ag \cdot (v) + bg \cdot (w)$$

for all $g \in G$, $v, w \in V$ and $a, b \in k$. Take a basis $\{v_i \mid i \in I\}$ of $V$. Then $V$ is said to be a (rational) $G$-module if for all $i, j \in I$, there is $f_{ji} \in k[G]$ such that $g \cdot v_i = \sum_j f_{ji} g \cdot v_j$ for all $g \in G$, with co-finitely many of the $f_{ji}$ being zero. The category of all $G$-modules is denoted $\mathrm{Mod}(G)$.

Let $V$ be a $G$-module, and let $\psi : G \to \mathrm{GL}(V)$ be a representation. We say that $\lambda \in X(T)$ is a *weight* of $V$ (with respect to $T$) if the weight space

$$V_\lambda = \{v \in V \mid \psi(t)v = \lambda(t)v \ \forall \ t \in T\}$$

is nonzero. Now $V$ is the direct sum of the nonzero weight spaces, since the image of $T$ is a diagonalisable subgroup of $\mathrm{GL}(V)$.

In the case $V = \mathfrak{g}$ (so $G$ acts through the adjoint representation), the non-zero weights are called the *roots* of $G$ with respect to $T$, and the set of roots is denoted $\Phi(G)$. The Lie algebra $\mathfrak{g}$ can be expressed as the vector space direct sum

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \Big( \bigoplus_{\alpha \in \Phi(G)} \mathfrak{g}_\alpha \Big),$$

where $\mathfrak{g}_0$ is the fixed point subspace of the action of $T$. The $\mathfrak{g}_\alpha$ for $\alpha \neq 0$ are called the *root spaces* of $\mathfrak{g}$.

A choice of Borel subgroup defines a partition of the roots $\Phi(G) = \Phi^+ \dot{\cup} \Phi^-$ into positive and negative roots. The weights inherit a partial order from this, via $\lambda \succ \mu$ whenever $\lambda - \mu$ is a linear combination of positive roots with non-negative integer coefficients.

Returning to the general case where $V$ is any $G$-module, if there is a weight $\lambda$ for $T$ in $V$ which is maximal subject to the partial order then we say $\lambda$ is a *highest weight* for $V$, and if $0 \neq v \in V_\lambda$ then $v$ is a *highest weight vector* in $V$.

A subset $\Delta \subseteq \Phi(G)$ is a *base* of $\Phi(G)$ if any $\beta \in \Phi(G)$ can be written in the form

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with either all $c_\alpha$ non-negative integers, or all $c_\alpha$ non-positive integers. We call the elements of $\Delta$ *simple roots*.

For each $\alpha \in \Phi(G)$, there is a *root homomorphism* $x_\alpha : \mathbf{G_a} \to G$ such that

$$tx_\alpha(c)t^{-1} = x_\alpha(\alpha(t)c)$$

for all $t \in T, c \in k$. Such a root homomorphism is unique (up to multiplication by a non-zero element in $k$). It induces an isomorphism from $\mathbf{G_a}$ to its image. We denote the image $U_\alpha$, and call it the *root subgroup* of $G$ corresponding to $\alpha$.

For each $\alpha \in \Phi(G)$ there is a corresponding *coroot* $\alpha^\vee \in Y(T)$, defined by

$$n_\alpha(c) := x_\alpha(c)x_{-\alpha}(-c^{-1})x_\alpha(c) \in N_G(T) \text{ for all } c \in k$$

and

$$\alpha^\vee(c) := n_\alpha(c)n_\alpha(1)^{-1} \in T \text{ for all } c \in k.$$

For each $\alpha \in \Phi(G)$, we denote by $s_\alpha$ the corresponding reflection on $X(T)$

$$s_\alpha \lambda = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha.$$

Now $W := \langle s_\alpha \mid \alpha \in \Phi(G) \rangle$ is the *Weyl group* of $\Phi(G)$. This group is always finite, and is in fact generated only by the $s_\alpha$ with $\alpha \in \Delta$ a base.

The reflections above define an action of $W$ on $X(T)$. Extending $X(T)$ to $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, we may define a second action of $W$ on $X(T)$, called the *dot action*, by

$$w \cdot \lambda = w(\lambda + \rho) - \rho$$

where $\rho \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ is half the sum of the positive roots.

## 2.2 Simple Modules

**Definition 2.4.** A weight $\lambda \in X(T)$ is called *dominant* if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Delta$, where $\Delta$ is a base for $\Phi(G)$. The subset of dominant weights will be denoted $X(T)^+ \subseteq X(T)$.

We now need to define, for each dominant weight $\lambda \in X(T)$, the modules $L(\lambda)$, $\mathrm{H}^0(\lambda)$ and $\Delta(\lambda)$. To do this, we define an induction functor $\mathrm{ind}_H^G$ which maps the set of $H$-modules to the set of $G$-modules, where $H$ is a closed subgroup of $G$. Let $M$ be an $H$-module, and consider $\mathrm{Mor}(G, M) := M \otimes k[G]$, where we regard this as a subset of the set of all functions $G \to M$ via $(m \otimes f)g = f(g) \cdot m$, extending linearly. Then we define

$$\mathrm{ind}_H^G M = \{f \in \mathrm{Mor}(G, M) \mid f(gh) = h^{-1} f(g) \text{ for all } g \in G, h \in H\}$$

and $G$ acts by left translation, that is $gf(-) = f(g^{-1}-)$.

When the coset variety $G/H$ is affine, $\mathrm{ind}_H^G M$ can be infinite dimensional, even when $M$ is finite dimensional. By contrast, when $G/H$ is projective, $\mathrm{ind}_H^G M$ is finite dimensional (see [25, II.5.2]).

An important case is when $H = B$ is a Borel subgroup. We choose $B$ corresponding to the negative roots, in order to ensure that the highest weight of $\mathrm{ind}_B^G(\lambda)$ is $\lambda$. Then, by the Lie-Kolchin theorem, the irreducible representations of $B$ are all 1-dimensional (since $B$ is solvable). For $\lambda \in X(T)$, we can extend $\lambda$ to a representation of $B$, and then induce this up to $G$. The resulting module we shall denote $\mathrm{H}^0(\lambda)$. If $\lambda$ is not dominant, then $\mathrm{H}^0(\lambda) = 0$ (see [25, II.2.6]). But if $\lambda \in X(T)$ is dominant, then $\mathrm{H}^0(\lambda) \neq 0$, and one can show that $\mathrm{H}^0(\lambda)$ has a simple socle which is the unique simple submodule; we denote this socle by $L(\lambda)$. The modules $L(\lambda)$ are significant as they form a set of representatives for the isomorphism classes of simple $G$-modules.

**Proposition 2.5** (Chevalley)**.** *Any simple $G$-module is isomorphic to exactly one $L(\lambda)$, with $\lambda \in X(T)$ a dominant weight.*

Given $\lambda \in X(T)^+$, we define the *Weyl module of weight* $\lambda$ by $\Delta(\lambda) := H^0(-w_0\lambda)^*$, where $w_0$ is the longest element of the Weyl group. As $\mathrm{H}^0(\lambda)$ has a simple socle it follows from duality that $\Delta(\lambda)$ has a simple head.

## 2.3 Steinberg's Tensor Product Theorem

Let $G$ be a semisimple algebraic group defined over an algebraically closed field $k$ of characteristic $p > 0$. A reductive group is said to be split if it has a maximal torus isomorphic to a direct product of copies of $\mathbf{G_m}$. The classification of split algebraic groups implies that they are all defined over $\mathbb{Z}$, hence also defined over $\mathbb{F}_p$. Since me may choose a maximal torus $T$ defined over $\mathbb{F}_p$ this additionally implies that $T$ is $F$-stable. But then each root subgroup $U_\alpha$ is also $F$-stable, so taking a Borel subgroup $B$ to be the subgroup generated by $T$ and the $U_\alpha$ for $\alpha$ a negative root, we arrange that $B$ is $F$-stable as well. Thus, the morphism $k[G] \to k[G]$, $f \otimes a \mapsto f^p \otimes a$ ([25, I.9.2]) gives rise to an endomorphism $F : G \to G$ which we shall call the *Frobenius endomorphism* on $G$. This map has a scheme-theoretic kernel which will be of interest to us later on (see §3.3).

Now, given a representation $\psi : G \to GL(V)$, we can create new representations $\psi^{[i]} := \psi \circ F^i$. The corresponding module is denoted $V^{[i]}$, and called the $i$th *Frobenius twist* of $V$. The action is given by $\psi^{[i]}(g)v = \psi(F^i(g))v$. For the simple modules $L(\lambda)$, we have that $L(\lambda)^{[1]} \simeq L(p\lambda)$ (see [25, III.3.16]). This fact motivates the definition of the *p-restricted weights*, that is the dominant weights $\lambda \in X(T)$ satisfying $\langle \lambda, \alpha^\vee \rangle < p$ for all $\alpha \in \Phi(G)$, and leads to the following theorem of Steinberg.

**Theorem 2.6** (Steinberg). *Let $G$ be a semisimple simply connected linear algebraic group, and let $\lambda \in X(T)$ be a dominant weight. Write $\lambda = \lambda_0 + p\lambda_1 + \cdots + p^n\lambda_n$, where the $\lambda_i$ are p-restricted weights. Then*

$$L(\lambda) \simeq L(\lambda_0) \otimes L(\lambda_1)^{[1]} \otimes \cdots \otimes L(\lambda_n)^{[n]}$$

*as $kG$-modules.*

This theorem is very useful as it allows calculations involving arbitrary simple modules to be reduced to the case of simple modules with a $p$-restricted highest weight, of which there are finitely many for any given $p$.

**Example 2.7.** Take $G = \mathrm{SL}_2 = \{A \in M_2(k) \mid \det A = 1\}$. We take $B$ to be the subgroup of upper triangular matrices, and $T$ the diagonal matrices. We have $X(T) \simeq \mathbb{Z}$, since any homomorphism $T \to \mathbf{G_m}$ must be of the form $\left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right) \mapsto c^n$ for some $n \in \mathbb{Z}$. Let $\lambda$ be the map $\left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right) \mapsto c$. Then the root system $\Phi(G) = \{2\lambda, -2\lambda\}$. We will write $\alpha$ for $2\lambda$ subsequently. Take $V$ to be the natural module, with natural basis $\{x, y\}$. Then $B$ leaves $\langle x \rangle$ invariant, and $\left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right)x = cx = \lambda\left(\begin{smallmatrix} c & \\ & c^{-1} \end{smallmatrix}\right)x$, so $V$ is of highest weight $\lambda$. In fact, suppressing the $\lambda$, we can say $V = L(1)$. The root homomorphism $x_\alpha : \mathbf{G_a} \to G$ is given by $c \mapsto \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)$, and $x_{-\alpha} : \mathbf{G_a} \to G$ is $c \mapsto \left(\begin{smallmatrix} 1 & 0 \\ c & 1 \end{smallmatrix}\right)$. Now

$$n_\alpha(c) = \left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 0 \\ -c^{-1} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & c \\ 0 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & c \\ -c^{-1} & 0 \end{smallmatrix}\right)$$

and

$$\alpha^\vee(c) = \left(\begin{smallmatrix} 0 & c \\ -c^{-1} & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)^{-1} = \left(\begin{smallmatrix} c & 0 \\ 0 & c^{-1} \end{smallmatrix}\right).$$

If $\chi \in X(T)$ then $\langle \chi, \alpha^\vee \rangle = \chi$, so the set of dominant weights $X(T)^+$ is isomorphic to the set of non-negative integers as a monoid. So by the above theorem of Chevalley, the simple $G$-modules are $L(\mu)$ for $\mu \in \mathbb{Z}_{\geq 0}$. The reflection corresponding to $\alpha$ is $s_\alpha : \chi \mapsto \chi - \chi\alpha = -\chi$, and $s_{-\alpha}$ is also $\chi \mapsto -\chi$, so the Weyl group $W$ is isomorphic to the cyclic group of order 2. The $p$-restricted weights are the integers $0, \ldots, p-1$, and the Frobenius morphism is given by:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a^p & b^p \\ c^p & d^p \end{pmatrix}.$$

Finally we record the dot action: if $w_0$ is the non-identity element of $W$ and $\mu$ is a weight, then $w_0 \cdot \mu = -\mu - 2$.

**Example 2.8.** Take $G = \mathrm{SL}_3$. As in the previous example, we take $B$ to be the subgroup of upper triangular matrices, and $T$ the diagonal matrices. Since $T \simeq \mathbf{G}_\mathbf{m}^2$, we have $X(T) \simeq \mathbb{Z}^2$ and $X(T)^+ \simeq (\mathbb{Z}_{\geq 0})^2$. If $\alpha$ and $\beta$ are simple roots forming a base, then $\Phi(G) = \{\pm\alpha, \pm\beta, \pm(\alpha+\beta)\}$. The $p$-restricted weights are $(a, b)$ where $0 \leq a, b < p$, and the reflections $s_\alpha$ and $s_\beta$ are given by $s_\alpha \cdot (a, b) = (-a-2, a+b+1)$ and $s_\beta \cdot (a, b) = (a+b+1, -b-2)$, where $(a, b)$ is any weight. One can check that $s_\alpha$ and $s_\beta$ generate a copy of the symmetric group $S_3$, so that the Weyl group

$$W = \{1, s_\alpha, s_\beta, s_\alpha s_\beta, s_\beta s_\alpha, \omega_0 = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta\} \simeq S_3.$$

We now record the complete dot action of $W$ on the weights.

| $w$ | $l(w)$ | $w \cdot (a, b)$ |
|---|---|---|
| 1 | 0 | $(a, b)$ |
| $s_\alpha$ | 1 | $(-a-2, a+b+1)$ |
| $s_\beta$ | 1 | $(a+b+1, -b-2)$ |
| $s_\beta s_\alpha$ | 2 | $(b, -a-b-3)$ |
| $s_\alpha s_\beta$ | 2 | $(-a-b-3, a)$ |
| $\omega_0$ | 3 | $(-b-2, -a-2)$ |

## 2.4 The Classification of Semisimple Algebraic Groups

The goal of this section is to state the classification of semisimple algebraic groups by their root data. Proof of this classification can be found in, for example, [28, Ch. 9] and [23, Ch. 32]. We begin by recording some properties of the root data in the following definition.

**Definition 2.9.** A subset $\Phi$ of a finite dimensional Euclidean space $E$ is an *(abstract) root system* if the following hold:

(i) $\Phi$ is finite and spans $E$,

(ii) if $c \in \mathbb{R}$ and $\alpha, c\alpha \in \Phi$ then $c = \pm 1$,

(iii) for each $\alpha \in \Phi$, the reflection $s_\alpha \in \mathrm{GL}(E)$ along $\alpha$ stabilizes $\Phi$,

(iv) for all $\alpha, \beta \in \Phi$, $s_\alpha \cdot \beta - \beta = n\alpha$ for some $n \in \mathbb{Z}$.

Let $\Phi$ be an abstract root system in $E$. A *base* of $\Phi$ is a subset $\Delta \subseteq \Phi$ with the property that any $\beta \in \Phi$ can be written as

$$\beta = \sum_{\alpha \in \Delta} c_\alpha \alpha$$

with either all $c_\alpha$ non-negative integers, or all $c_\alpha$ non-positive integers. The subset of $\Phi$ consisting of the non-negative integral combinations of elements of $\Delta$ is denoted $\Phi^+$.

A root system can be recovered from a base (see [28, 9.4 (c)]), so to describe a root system we just need to describe a base. A base can in turn be described by its associated *Dynkin diagram.* This diagram has an underlying graph with vertex set equal to the base $\Delta$, and $\alpha, \beta \in \Delta$ are connected by an edge of multiplicity $m_{\alpha,\beta}$, where

$$m_{\alpha,\beta} = \begin{cases} 0 & \text{if } |(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+| = 2, \\ 1 & \text{if } |(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+| = 3, \\ 2 & \text{if } |(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+| = 4, \\ 3 & \text{if } |(\mathbb{Z}\alpha + \mathbb{Z}\beta) \cap \Phi^+| = 6. \end{cases}$$

Additionally, if $\alpha$ and $\beta$ are adjacent and of different lengths, the edge joining them is given an arrow pointing to the shorter one. Now, two root systems are isomorphic if and only if they have the same Dynkin diagram.

Take $\Phi$ a root system with base $\Delta$. If $\Delta$ can be written as $\Delta = \Delta_1 \sqcup \Delta_2$ with $\Delta_1$ and $\Delta_2$ mutually orthogonal, then we say $\Phi$ is *decomposable.* If no such decomposition of $\Delta$ exists then $\Phi$ is said to be *indecomposable.* A root system is indecomposable if and only if the corresponding Dynkin diagram is connected.

It can be shown (see [23, Cor. 27.5]) that an algebraic group is simple if and only if its root system is indecomposable. Thus, we need to classify indecomposable root systems (see [28, Thm. 9.6]):

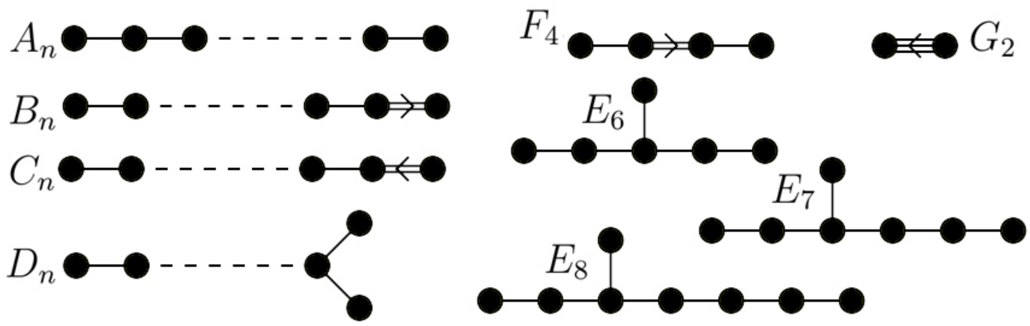


Figure 2.1: Dynkin diagrams of indecomposable root systems

**Theorem 2.10.** *Let $\Phi$ be an indecomposable root system in a real vector space $E = \mathbb{R}^m$. Then $\Phi$ is (isomorphic to) one of the following types:*

$$A_n(n \geq 1), B_n(n \geq 2), C_n(n \geq 3), D_n(n \geq 4), E_6, E_7, E_8, F_4, G_2,$$

*with associated Dynkin diagrams as shown in Figure 2.1.*

With this classification of indecomposable root systems in place, we are now ready to state the classification of semisimple algebraic groups by their root data. The following definition of a root datum is intended to collect together all the information necessary to uniquely identify an algebraic group.

**Definition 2.11.** A quadruple $(X, \Phi, Y, \Phi^\vee)$ is a *root datum* if the following four properties hold:

(i) $X \cong Y \cong \mathbb{Z}^n$, and there is a perfect pairing $\langle\ ,\ \rangle : X \times Y \to \mathbb{Z}$,

(ii) $\Phi \subseteq X$ is an abstract root system in $\mathbb{Z}\Phi \otimes_\mathbb{Z} \mathbb{R}$ and $\Phi^\vee \subseteq Y$ is an abstract root system in $\mathbb{Z}\Phi^\vee \otimes_\mathbb{Z} \mathbb{R}$,

(iii) there exists a bijection $\Phi \to \Phi^\vee$ such that $\langle \alpha, \alpha^\vee \rangle = 2$,

(iv) the reflections $s_\alpha \in \Phi$ and $s_\alpha^\vee \in \Phi^\vee$ are given by

$$s_\alpha \cdot \chi = \chi - \langle \chi, \alpha^\vee \rangle \alpha \text{ for all } \chi \in X$$

and

$$s_{\alpha^\vee} \cdot \gamma = \gamma - \langle \alpha, \gamma \rangle \alpha^\vee \text{ for all } \gamma \in Y$$

respectively.

**Theorem 2.12** (Chevalley Classification Theorem)**.** *Two semisimple linear algebraic groups are isomorphic if and only if they have isomorphic root data. For each root datum there exists a corresponding semisimple linear algebraic group. This group is simple if and only if its root system is indecomposable.*

# Chapter 3

# Cohomology of Algebraic Groups

In this chapter we will introduce the cohomology spaces $\mathrm{H}^n(G, M)$.

## 3.1 Cohomology

We begin this section by describing the notion of *derived functors* in their most general setting. Suppose we have abelian categories $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$ has *enough injectives*, meaning that for any object $X$ in $\mathbf{A}$ there exists a monomorphism $A \to I$, where $I$ is an injective object of $\mathbf{A}$. Further suppose we have a left exact functor $F : \mathbf{A} \to \mathbf{B}$, meaning that if

$$0 \to A \to B \to C \to 0$$

is a short exact sequence in $\mathbf{A}$, then

$$0 \to F(A) \to F(B) \to F(C) \tag{3.1}$$

is an exact sequence in $\mathbf{B}$. Now, for any object $X$ in $\mathbf{A}$, we can construct an *injective resolution* of $X$, that is a long exact sequence of the form

$$0 \to X \to I^0 \to I^1 \to I^2 \to \dots$$

where the $I^i$ are all injective. If we apply $F$ to this sequence, we obtain the chain complex

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to \dots$$

We can now define the $i$th right derived functor of $F$, $\mathrm{R}^i F$, by

$$\mathrm{R}^i F(X) = \mathrm{Ker}(F(I^i) \to F(I^{i+1})) / \mathrm{Im}(F(I^{i-1}) \to F(I^i)).$$

The reason for this definition is that the derived functors now form a continuation of the exact sequence 3.1 above:

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1 F(A) \to \mathrm{R}^1 F(B) \to \mathrm{R}^1 F(C) \\ \to \mathrm{R}^2 F(A) \to \dots$$

Note that $F$ is exact if and only if $\mathrm{R}^1 F$ is zero; in this sense, the right derived functors of $F$ are measuring how far $F$ is from being exact.

Now for any linear algebraic group $G$, the rational $G$-modules form an abelian category with enough injectives ([25, I.3.9]). Thus we may consider the right derived functors of any left exact functor $F$ with domain $\mathrm{Mod}(G)$. In the case $F = \mathrm{ind}_B^G$, we write $\mathrm{R}^i F = \mathrm{H}^i$. Similarly, for any rational $G$-module $M$, the functor $\mathrm{Hom}_G(M, -)$ of $G$-modules is left exact. Thus it has right derived functors, which we denote by $\mathrm{Ext}_G^n(M, -)$. We define the *cohomology functors*

$$\mathrm{H}^n(G, -) := \mathrm{Ext}_G^n(k, -).$$

The space $\mathrm{H}^n(G, M)$ is called the *$n$th cohomology group of $G$ with coefficients in $M$*. Henceforth, we will drop the subscript $G$ and simply write $\mathrm{Ext}^n(-, -)$.

One way to think of these objects is as sets of equivalence classes of extensions. For instance, if $G$ is any group and $M$ is a $G$-module, then it is well known (see [22]) that the second cohomology group $\mathrm{H}^2(G, M)$ is in one-to-one correspondence with the set of equivalence classes of extensions $E$ of $M$ by $G$

$$0 \to M \xrightarrow{\tau} E \to G \to 1$$

where $\tau$ is a homomorphism from the additive group of $M$ to the multiplicative group of $E$, and $E$ induces the module action of $G$ on $M$ by conjugation. Two extensions $E$ and $D$ are considered equivalent if there is an isomorphism $\phi : E \to D$ which induces the identity on both $M$ and $G$. Furthermore, a binary operation on the extensions called the Baer sum may be defined, which makes this correspondence into a group homomorphism. We note that there are ways of interpreting the higher cohomology groups $\mathrm{H}^n(G, M)$ which have a group-theoretic significance; see [22] for details.

These cohomology spaces can also be defined in terms of *cochain complexes*. Let $C^n = M \otimes k[G]^n$, where an element on the right hand side is interpreted as a function $\phi : G^{n+1} \to M$ via $m \otimes (f_1, \dots, f_n)(g) = f_1(g) \dots f_n(g)m$. The elements of $C^n$ which satisfy

$$\phi(gg_0, gg_1, \dots, gg_n) = g\phi(g_0, g_1, \dots, g_n)$$

for all $g, g_0, \dots, g_n \in G$ are known as *$n$-cochains*. The *coboundary* of an

$n$-cochain $\phi$ is the $(n+1)$-cochain $\delta\phi$ given by

$$(\delta\phi)(g_0, \ldots, g_{n+1}) = \sum_{0 \le i \le n+1} (-1)^i \phi(g_0, \ldots, g_{i-1}, g_{i+1}, \ldots, g_{n+1}).$$

It is straightforward to check that $\delta^2 = 0$, and thus we have a cochain complex

$$\ldots \xleftarrow{\delta} C^2 \xleftarrow{\delta} C^1 \xleftarrow{\delta} C^0 \leftarrow 0.$$

Then the $n$th cohomology group of this complex is

$$\mathrm{H}^n(G, M) = \mathrm{Z}^n(G, M) / \mathrm{B}^n(G, M)$$

where $Z^n(G, M) = \mathrm{Ker}(\delta : C^n \to C^{n+1})$ is the group of *cocycles*, and $B^n(G, M) = \mathrm{Im}(\delta : C^{n-1} \to C^n)$ is the group of *coboundaries*. To see that this method does indeed compute the derived functors as defined earlier, see [25, I.4.16].

We will be interested in calculating the spaces $\mathrm{H}^n(G, M)$ where $G$ is a semisimple algebraic group and $M$ is a simple $G$-module.

## 3.2 The Linkage Principle

Recall the dot action of $W$ on $X(T)$ from §2.1. We wish to extend this action, so that it includes translations by elements of $p\mathbb{Z}\Phi(G)$. Thus we have the *affine Weyl group* $W_p$, given by $W_p \simeq p\mathbb{Z}\Phi(G) \rtimes W$, acting on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. This action is important, as there is a relationship between the $W_p$-orbits on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$ and extensions between modules. This relationship is called the *Linkage Principle*. In a 'weak' form, it can be stated as:

**Proposition 3.1** (The Linkage Principle)**.** *[2] Take $\lambda$ and $\mu$ to be dominant weights. If $\mathrm{Ext}^n(L(\lambda), L(\mu)) \neq 0$ for any $n \geq 0$, then $\lambda \in W_p \cdot \mu$.*

In other words, if one simple module exists as an extension of another simple module, then their highest weights are in the same $W_p$-orbit. In this case, we say that the weights $\lambda$ and $\mu$ are *linked*. In particular, by taking $\lambda = 0$ in the statement of the proposition, we get that if $\mathrm{H}^n(G, L(\mu)) \neq 0$, then $\mu$ is linked to 0.

The Linkage Principle also holds if the simple modules are replaced with Weyl modules. Furthermore, it holds between Weyl modules and simple modules, that is, if $\mathrm{Ext}^n(\Delta(\lambda), L(\mu)) \neq 0$ for any $n \geq 0$, then $\lambda$ and $\mu$ are linked.

**Example 3.2.** Let us consider the case $G = \mathrm{SL}_2$. The affine Weyl group is given by $W_p \simeq p\mathbb{Z}\Phi(G) \rtimes W \simeq 2p\mathbb{Z} \rtimes C_2$. Let $\mu_1$ and $\mu_2$ be dominant weights. We want to determine when $\mu_1$ is linked to $\mu_2$. If $w \in W_p$ then

$$w \cdot \mu_2 = w(\mu_2 + 1) - 1 = \begin{cases} \mu_2 + 2n(\mu_2 + 1)p & \text{if } w = (2np, 1) \\ -\mu_2 - 2 + 2n(\mu_2 + 1)p & \text{if } w = (2np, w_0) \end{cases}$$

Thus, $\mu_1 \in W_p \cdot \mu_2$ if and only if either $\mu_1 + \mu_2 \equiv -2 \bmod 2p$ or $\mu_1 \equiv \mu_2 \bmod 2p$. In particular, we have the following special case of the Linkage Principle for $\mathrm{SL}_2$:

If $\mathrm{H}^i(\mathrm{SL}_2, L(\lambda)) \neq 0$ then either $\lambda \equiv -2 \bmod 2p$ or $\lambda \equiv 0 \bmod 2p$.

**Example 3.3.** For $G = \mathrm{SL}_3$, the best way to understand the linkage principle is to draw some pictures. Let us choose the specific example when $p = 5$. Computing the dot action of $W$ on the weight $(0, 0)$ (see Example 2.8) we see that $(3, 6), (6, 3), (2, 5), (5, 2)$ and $(3, 3)$ are all linked to $(0, 0)$. These weights are marked on the weight lattice shown in Figure 3.1. Additional weights linked to zero are obtained as translations of the previous weights by elements of $p\mathbb{Z}\Phi(G)$. In this case, elements of $p\mathbb{Z}\Phi(G)$ are just integer multiples of $5\alpha, 5\beta$ and $5(\alpha + \beta)$. The roots $\alpha$ and $\beta$ are indicated near the bottom of Figure 3.1.

## 3.3 Frobenius Kernels

Recall the Frobenius morphism introduced in §2.3, $F : G \to G$. It has an infinitesimal scheme-theoretic kernel, denoted $G_1$, and called the *first Frobenius kernel* of $G$. Similarly, the kernel of the iterated map $F^r$ is denoted $G_r$, and called the $r$th Frobenius kernel of $G$.

Much research has been done calculating the cohomology for Frobenius kernels of simple algebraic groups. The first result in this area came from Friedlander and Parshall in [18], where they compute the cohomology ring $\mathrm{H}^\bullet(G_1, k)$ for $p \geq 3(h-1)$. Their result was later generalized by Andersen and Jantzen to the spaces $\mathrm{H}^i(G_1, \mathrm{H}^0(\lambda))$ for $p \geq h$, although they still had some restrictions on the underlying root system. These restrictions were removed by Kumar, Lauritzen and Thomsen in [26], using Frobenius splittings. The full result follows, known as the Andersen-Jantzen formula (see [25, II.12.15]):

**Theorem 3.4.** *Let $G$ be a semisimple simply connected algebraic group. Assume $p > h$ and $\mu \in X(T)^+$ is of the form $\mu = w \cdot 0 + p\lambda$, with $w \in W$ and $\lambda \in X(T)$. Then*

$$\mathrm{H}^i(G_1, \mathrm{H}^0(\mu))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i - l(w)}{2}}(\mathfrak{u}^*) \otimes k_\lambda).$$

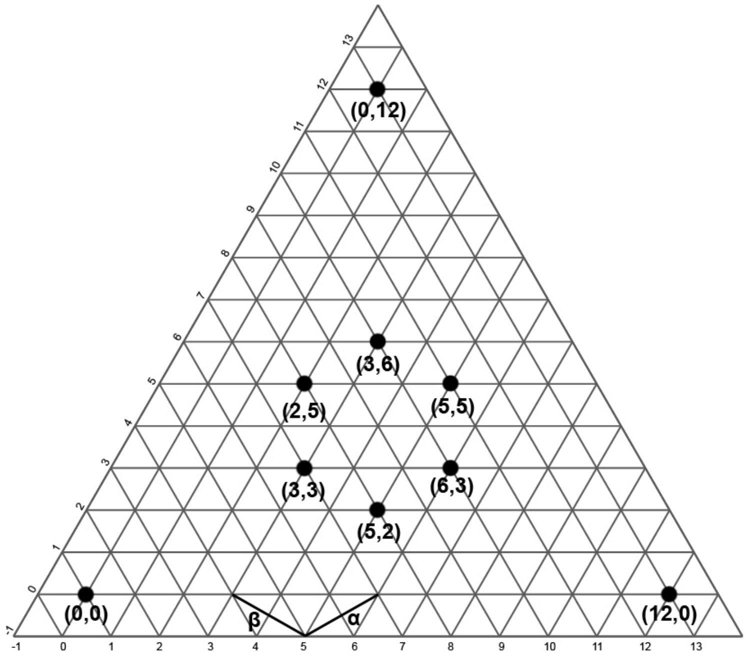


Figure 3.1: $\mathrm{SL}_3$ weights linked to zero, $p = 5$.

This work was taken further by Bendel, Nakano and Pillen in their papers [9], [10] and [11], in which they compute the spaces $\mathrm{H}^i(G_r, \mathrm{H}^0(\mu))^{[-r]}$ for $p \geq 3$ and $r \geq 1$, for $i = 1$, 2 and 3 respectively. The general strategy is first to reduce the problem to computing the cohomology of the Frobenius kernel $B_r$ by proving the isomorphism

$$\mathrm{H}^i(G_r, \mathrm{H}^0(\mu))^{[-r]} \simeq \mathrm{H}^0(\mathrm{H}^i(B_r, k_\mu)^{[-r]}).$$

Now, using the Lyndon-Hochschild-Serre spectral sequence for $B_1 \subset B_r$, the $B_r$ cohomology can be determined from the $B_1$ cohomology. In the case $i = 1$, the $B_1$ cohomology had been calculated by Jantzen in [24]. For $i = 2$ or 3, the problem is reduced still further, to computing the space $\mathrm{H}^i(\mathfrak{u}, k)$, which amounts to a calculation with some root elements. The case $i = 2, p = 2$ was done by Wright in [41].

## 3.4 Translation Functors

Another tool we will make use of is a certain class of functors from $\mathrm{Mod}(G)$ to itself, called *translation functors.*

Consider, for each weight $\lambda \in X(T)$, the category $\mathcal{M}_\lambda$ consisting of all $G$-modules whose only composition factors are of the form $L(\mu)$, with $\mu \in W_p \cdot \lambda$.

It then follows from the linkage principle that the category of all $G$-modules is a direct product of all the different $\mathcal{M}_\lambda$, that is, of the $\mathcal{M}_\lambda$ with $\lambda$ running over a system of representatives for the $W_p$-orbits in $X(T)$. We will normally take our system of representatives to be $\overline{C}_\mathbb{Z}$, the lowest alcove (see §5.2). We are going to introduce translation functors $T_\lambda^\mu : \mathcal{M}_\lambda \to \mathcal{M}_\mu$ for all $\lambda, \mu \in \overline{C}_\mathbb{Z}$, although in practice we will regard them as functors from $\mathrm{Mod}(G)$ to itself.

For any $G$-module $V$, and any weight $\lambda \in X(T)$, we define $\mathrm{pr}_\lambda V$ to be the sum of all submodules of $V$ such that all its composition factors have highest weight linked to $\lambda$. Then $\mathrm{pr}_\lambda V$ is the largest submodule with this property. We have (see [25, II.7.3])

$$V = \bigoplus_{\lambda \in Z} \mathrm{pr}_\lambda V$$

where $Z$ is a system of representatives for the $W_p$-orbits in $X(T)$. Note that, for all $\mu \in X(T)^+$, we have

$$\mathrm{pr}_\lambda L(\mu) = \begin{cases} L(\mu) & \text{if } \mu \in W_p \cdot \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Now, take $\lambda, \mu \in \overline{C}_\mathbb{Z}$. There is a unique $\nu \in X(T)^+ \cap W(\mu - \lambda)$. We now define the *translation functor* $T_\lambda^\mu$ from $\lambda$ to $\mu$ by

$$T_\lambda^\mu V = \mathrm{pr}_\mu(L(\nu) \otimes \mathrm{pr}_\lambda V)$$

for each $G$-module $V$. These functors are exact, and $T_\lambda^\mu$ and $T_\mu^\lambda$ are adjoint to each other (see [25, II.7.6]). Thus, for any $G$-module $V$, we have an isomorphism of functors

$$\mathrm{Hom}_G(V, -) \circ T_\mu^\lambda \simeq \mathrm{Hom}_G(T_\lambda^\mu V, -),$$

and therefore we also have isomorphisms of their derived functors

$$\mathrm{Ext}^n(V, -) \circ T_\mu^\lambda \simeq \mathrm{Ext}^n(T_\lambda^\mu V, -).$$

Then, finally, for each $G$-module $V'$ and each $n \in \mathbb{N}$, we have the isomorphism

$$\mathrm{Ext}^n(V, T_\mu^\lambda V') \simeq \mathrm{Ext}^n(T_\lambda^\mu V, V').$$

**Example 3.5.** By way of example, we record the values of $\mathrm{T}_{(0,0)}^\mu L(\lambda)$ for various $\mu, \lambda$ in the case $G = \mathrm{SL}_3$, $p > 5$.

| $\mathrm{T}^{\mu}_{(0,0)}\, L(\lambda)$ | $\mu$: $(0,1)$ | $(1,0)$ | $(1,1)$ |
|---|---|---|---|
| $\lambda$: | | | |
| $(0,0)$ | $L(0,1)$ | $L(1,0)$ | $L(1,1)$ |
| $(p-3,0)$ | $L(p-4,0)$ | $L(p-4,1)$ | $L(p-5,1)$ |
| $(0,p-3)$ | $L(1,p-4)$ | $L(0,p-4)$ | $L(1,p-5)$ |
| $(p-2,1)$ | $L(p-3,2)$ | $L(p-2,2)$ | $L(p-3,3)$ |
| $(1,p-2)$ | $L(2,p-2)$ | $L(2,p-3)$ | $L(3,p-3)$ |
| $(p-2,p-2)$ | $L(p-2,p-3)$ | $L(p-3,p-2)$ | $L(p-3,p-3)$ |

| $\mathrm{T}^{\mu}_{(0,0)}\, L(\lambda)$ | $\mu$: $(0,2)$ | $(2,0)$ | $(1,2)$ |
|---|---|---|---|
| $\lambda$: | | | |
| $(0,0)$ | $L(0,2)$ | $L(2,0)$ | $L(1,2)$ |
| $(p-3,0)$ | $L(p-5,0)$ | $L(p-5,2)$ | $L(p-6,1)$ |
| $(0,p-3)$ | $L(2,p-5)$ | $L(0,p-5)$ | $L(2,p-6)$ |
| $(p-2,1)$ | $L(p-4,3)$ | $L(p-2,3)$ | $L(p-4,4)$ |
| $(1,p-2)$ | $L(3,p-2)$ | $L(3,p-4)$ | $L(4,p-3)$ |
| $(p-2,p-2)$ | $L(p-2,p-4)$ | $L(p-4,p-2)$ | $L(p-3,p-4)$ |

| $\mathrm{T}^{\mu}_{(0,0)}\, L(\lambda)$ | $\mu$: $(2,1)$ | $(2,2)$ |
|---|---|---|
| $\lambda$: | | |
| $(0,0)$ | $L(2,1)$ | $L(2,2)$ |
| $(p-3,0)$ | $L(p-6,2)$ | $L(p-7,2)$ |
| $(0,p-3)$ | $L(1,p-6)$ | $L(2,p-7)$ |
| $(p-2,1)$ | $L(p-3,4)$ | $L(p-4,5)$ |
| $(1,p-2)$ | $L(4,p-4)$ | $L(5,p-4)$ |
| $(p-2,p-2)$ | $L(p-4,p-3)$ | $L(p-4,p-4)$ |

## 3.5 Spectral Sequences

Let $V$ be a $G$-module, and consider $M = \mathrm{H}^i(G_1, V)$, the cohomology of the Frobenius kernel $G_1$. It has the structure of a $G/G_1$-module, and so also of a $G$-module. Since $G_1$ acts trivially on this module, there is a Frobenius untwist $M^{[-1]}$ of $M$, and $G/G_1$ acts on $M$ as $G$ acts on $M^{[-1]}$. Applying this to the Lyndon-Hochschild-Serre spectral sequence (see [25, I.6.6 (3)]) gives the following result.

**Proposition 3.6.** *For each $G$-module $V$ there is a spectral sequence*

$$E_2^{i,j} = \mathrm{H}^i(G, \mathrm{H}^j(G_1, V)^{[-1]}) \Rightarrow \mathrm{H}^{i+j}(G, V).$$

We will now summarize the principal features of spectral sequences. A spectral sequence consists of a three dimensional array of spaces $E_q^{i,j}$. The lower index refers to the page of the spectral sequence, and the upper indices refer to the position on the page. In Proposition 3.6, only the second page is explicitly defined. Furthermore, it is defined only for the first quadrant, i.e. for $i, j \geq 0$. Outside this quadrant we have $E_2^{i,j} = 0$ (not all spectral sequences have this first quadrant property, but the only spectral sequences we will use are indeed first quadrant). On the second page, through each point $E_2^{i,j}$ there are the maps

$$... \to E_2^{i-2,j+1} \xrightarrow{\rho} E_2^{i,j} \xrightarrow{\sigma} E_2^{i+2,j-1} \to ...$$

which form a complex, i.e. $\sigma\rho = 0$. The points on the next page are formed by taking the cohomology of the previous page at the same point, i.e. $E_3^{i,j} = \ker\sigma/\mathrm{im}\rho$. This process iteratively defines the points on every page. In general, maps on the $q$th page of the spectral sequence go

$$... \to E_q^{i-q,j+q-1} \xrightarrow{\rho} E_q^{i,j} \xrightarrow{\sigma} E_2^{i+q,j-q+1} \to ...$$

When $q$ is big enough compared to $i$ and $j$, the map $\sigma$ begins outside the first quadrant, and $\rho$ ends outside the first quadrant. So in this situation, calculating the cohomology doesn't change anything, i.e. $E_q^{i,j} = E_{q+1}^{i,j}$. So each point of the spectral sequence eventually stabilises, and this stable value is denoted $E_\infty^{i,j}$. Finally, we have

$$\mathrm{H}^n(G, V) = \bigoplus_{i+j=n} E_\infty^{i,j}$$

which is the meaning of the notation '$\Rightarrow \mathrm{H}^{i+j}(G, V)$' in Proposition 3.6.

## 3.6 Filtrations

We will make use of *filtrations* in the final section of this thesis. The main reference for this section is [25, II.4.16].

**Definition 3.7.** Let $V$ be a $G$-module. An ascending chain

$$0 = V_0 \subset V_1 \subset V_2 \subset \dots$$

of submodules of $V$ is a *good filtration* of $V$ if

- $V = \bigcup_{i \geq 0} V_i$, and
- each $V_i/V_{i-1}$ is either 0 or isomorphic to $\mathrm{H}^0(\lambda_i)$ for some $\lambda_i \in X(T)^+$ for each $i \geq 1$.

*Remark* 3.8. If $V$ is a $B$-module such that all its weights are dominant, and $\mu$ is a (1-dimensional) $B$-submodule, then applying the functor $\mathrm{ind}_B^G$ to

$$0 \to \mu \to V \to V/\mu \to 0$$

gives an exact sequence

$$0 \to \mathrm{H}^0(\mu) \to \mathrm{ind}_B^G(V) \to \mathrm{ind}_B^G(V/\mu) \to 0$$

since $\mathrm{H}^1(\mu) = 0$ by Kempf's Vanishing Theorem. So, by induction on dimension, $\mathrm{ind}_B^G(V)$ has a good filtration.

# Chapter 4

# Cohomology of $\mathrm{SL}_2$

In this chapter, we will prove some new results calculating the cohomology spaces for $\mathrm{SL}_2$ with simple coefficients, $\mathrm{H}^n(\mathrm{SL}_2, L(\lambda))$.

## 4.1 Statement of results

In [33], the authors prove that for any semisimple simply connected algebraic group $G$ with root system $\Phi$ over an algebraically closed field $k$ of positive characteristic, there is a constant $c = c(\Phi, q)$ such that $\dim \mathrm{H}^q(G, L(\lambda)) \leq c$ for any dominant weight $\lambda$. Note that this constant can be chosen independently of the characteristic $p$ of $k$. However, one cannot drop the dependence of $c$ on $q$, even for the case $G = \mathrm{SL}_2$: in [40], Stewart shows that, for any fixed $p$, the sequence

$$\{\gamma_q\} = \{\max_{\lambda} \dim \mathrm{H}^q(\mathrm{SL}_2, L(\lambda))\}$$

grows exponentially in $q$. However, for $p$ sufficiently large compared to the degree of the cohomology, we show that in the case $G = \mathrm{SL}_2$ the constant $c$ can indeed be chosen independently of $q$. Specifically, we show that, if $p > q$, then $\dim \mathrm{H}^q(\mathrm{SL}_2, L(\lambda)) \leq 1$. In order to prove this result we develop, using a theorem of Parker, a method for finding the weights $\lambda$ such that the space $\mathrm{H}^q(\mathrm{SL}_2, L(\lambda))$ is non-trivial. This method also produces a closed-form description of these weights which is uniform in $p$. We demonstrate the method in the cases $q = 1, 2, 3$. The first case recovers a special case of a result of Cline in [13], the second case recovers a result of Stewart in [38]. The third case is a new result:

**Theorem A.** *Suppose $p > 2$ and $M$ is isomorphic to any Frobenius twist (possibly trivial) of one of the following simple modules:*

*(i)* $L(4p-2)$

*(ii)* $L(2p^2-4p)$ $\quad (p > 3)$

*(iii)* $L(p^n(2p-2)+2p)$ $\quad (n > 1)$

*(iv)* $L(2p^2+2p-2)$

*(v)* $L(p^n(2p-2)+2p^2-2p-2)$ $\quad (n > 2)$

*(vi)* $L(2p^3-2p^2-2p-2)$

*(vii)* $L(p^n\lambda_2+2p-2)$ $\quad$ ($n > 1$, $\lambda_2$ *satisfying* $\mathrm{H}^2(\mathrm{SL}_2, L(\lambda_2)) \neq 0$*).*

*Then* $\mathrm{H}^3(\mathrm{SL}_2, M) = k$*. For all other simple* $\mathrm{SL}_2$*-modules* $M$*,* $\mathrm{H}^3(\mathrm{SL}_2, M) = 0$*.*

*Suppose $p = 2$ and $M$ is isomorphic to any Frobenius twist (possibly trivial) of one of the following simple modules:*

*(i)* $L(6)$

*(ii)* $L(8)$

*(iii)* $L(2^n+2)$ $\quad (n > 3)$

*(iv)* $L(2^n+4)$ $\quad (n > 3)$

*(v)* $L(2^n+10)$ $\quad (n > 4)$*.*

*Then* $\mathrm{H}^3(\mathrm{SL}_2, M) = k$*, except when $M = L(2^n+4)$ and $n > 4$, in which case we have* $\dim \mathrm{H}^3(\mathrm{SL}_2, M) = 2$*. For all other simple* $\mathrm{SL}_2$*-modules* $M$*,* $\mathrm{H}^3(\mathrm{SL}_2, M) = 0$*.*

In §4.3, we apply our method to get generic results when $p > q$ for $q$ the degree of cohomology. For this we use Parker's formula together with Jantzen's translation functors; ultimately, we get a closed form description of certain basic weights, a set we denote by $\Omega_q$.

In order to express the set $\Omega_q$, we give the following two definitions.

**Definition 4.1.** Given a weight $\lambda \in \mathbb{N}$ with $\lambda \equiv 0 \bmod 2p$ or $\lambda \equiv -2 \bmod 2p$, and $n \in \mathbb{N}$, define $\lambda$ *shifted by* $n$, denoted $\lambda \parallel n$, as follows:

(i) If $\lambda \equiv 0 \bmod 2p$ and $n$ is even, $\lambda \parallel n = p(\lambda + n)$.

(ii) If $\lambda \equiv 0 \bmod 2p$ and $n$ is odd, $\lambda \parallel n = p(\lambda + n) + p - 2$.

(iii) If $\lambda \equiv -2 \bmod 2p$ and $n$ is even, $\lambda \parallel n = p(\lambda - n)$.

(iv) If $\lambda \equiv -2 \bmod 2p$ and $n$ is odd, $\lambda \parallel n = p(\lambda - n) + p - 2$.

**Definition 4.2.** We say that a weight $\lambda$ is $(q, n)$-*cohomological* if $\mathrm{Ext}^q(\Delta(n), L(\lambda)) \neq 0$. In particular, we say that $\lambda$ is $q$-*cohomological* if it is $(q, 0)$-cohomological, i.e. if $\mathrm{Ext}^q(\Delta(0), L(\lambda)) = \mathrm{H}^q(\mathrm{SL}_2, L(\lambda)) \neq 0$.

For each $q \geq 1$, the set of all weights $\lambda$ such that

$$\mathrm{H}^q(\mathrm{SL}_2, L(\lambda)) \neq 0 \text{ and } \mathrm{H}^q(\mathrm{SL}_2, L(\lambda)^{[-1]}) = 0$$

will be denoted $\Omega_q$. We refer to the elements of $\Omega_q$ as the *maximally untwisted* $q$-cohomological weights. We will take $\Omega_0 = \{0\}$, as this makes the statement of Proposition 4.10 neater.

The main result is

**Theorem B.** *Assume $p > q \geq 1$. Then*

$$\Omega_q = \{(p^n \lambda_i) \parallel (q - i) \mid n \geq 0, \lambda_i \in \Omega_i, i = 0, \dots, q - 1\}.$$

*Moreover, for any $\lambda \in \Omega_q$ and any $m \geq 0$, we have* $\dim \mathrm{H}^q(\mathrm{SL}_2, L(\lambda)^{[m]}) = 1$.

In order to prove our Theorem A and Theorem B, we will make use of the following theorem of Parker. It essentially contains all the information necessary to construct the algorithm we use in the next section.

**Theorem 4.3** ([30, Theorem 6.1])**.** *Suppose that $p > 2$, $b \in \mathbb{N}$, $0 \leq i \leq p-2$, and $M$ is a finite-dimensional rational $G$-module. Then*

$$\mathrm{Ext}^q(\Delta(pb + i), M^{[1]} \otimes L(i)) \simeq \bigoplus_{\substack{n=0 \\ n \text{ even}}}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n + b), M) \tag{4.1}$$

$$\mathrm{Ext}^q(\Delta(pb + i), M^{[1]} \otimes L(p - 2 - i)) \simeq \bigoplus_{\substack{n=0 \\ n \text{ odd}}}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n + b), M) \tag{4.2}$$

We will also use the following, a consequence of the equivalence of categories of $G$-modules arising from the functor $F : V \mapsto \mathrm{St} \otimes V^{[1]}$ from $\mathrm{Mod}(G)$ to $F\,\mathrm{Mod}(G)$ (see [25, II.10.5] and [30, §3]). We have

$$\mathrm{Ext}^q(\Delta(pb + p - 1, M^{[1]} \otimes L(p - 1)) \cong \mathrm{Ext}^q(\Delta(b), M) \tag{4.3}$$

With the help of the Linkage Principle and Steinberg's Tensor Product Theorem, these formulae can be used to find closed-form descriptions of the sets $\Omega_q$ for $q \geq 1$. This is the method we develop in this chapter.

## 4.2 Closed form descriptions of $\dim \mathrm{H}^q(G, M)$ for fixed primes

We demonstrate an algorithm which uses Theorem 4.3 to give, for fixed $p$, a list of the $(q,r)$-cohomological weights $\lambda$, and the dimensions of $\mathrm{Ext}^q(\Delta(r), L(\lambda))$, provided that

(i) all the $(s,t)$-cohomological weights and the associated dimensions are known for all $s < q$, where $s + t = q + r'$; and

(ii) all the $(q,r')$-cohomological weights and the associated dimensions are known;

where $r = pr' + r_0$ and $0 \leq r_0 < p$.

Certainly the $(0,r)$-cohomological weights are known: if $\mathrm{Ext}^0(\Delta(r), L(\lambda)) \neq 0$ then $\lambda = r$, and the dimension of the Ext-group is 1. We aim to continue by induction, producing iteratively a list of all $q$-cohomological weights for any fixed $q \in \mathbb{N}$.

Fix $q \in \mathbb{N}$ and let $\lambda, r \in \mathbb{Z}_{\geq 0}$ be fixed but arbitrary with $\lambda = p\lambda' + \lambda_0$ and $r = pr' + r_0$, where $0 \leq r_0, \lambda_0 < p$. Assume that $\mathrm{Ext}^q(\Delta(r), L(\lambda)) \neq 0$ and also assume that (i) and (ii) hold. If $r_0 = p - 1$ then by Equation (4.3) we have

$$\mathrm{Ext}^q(\Delta(r), L(\lambda)) = \mathrm{Ext}^q(\Delta(r'), L(\lambda')).$$

Now by (ii) we know the non-zero dimensions of $\mathrm{Ext}^q(\Delta(r'), L(\lambda'))$, so we may write down a list of all the $(q,r)$-cohomological weights in this case. Specifically, suppose $\lambda'$ is a $(q,r')$-cohomological weight with associated dimension $d$; then $p\lambda' + p - 1$ is a $(q,r)$-cohomological weight with associated dimension $d$, and every $(q,r)$-cohomological weight arises this way. This deals with the case $r_0 = p - 1$.

Otherwise, $r_0 \leq p - 2$. By the Linkage Principle, we have either $\lambda_0 = r_0$ or $\lambda_0 = p - 2 - r_0$. In the first case, putting $M = L(\lambda')$, $b = r'$ and $i = \lambda_0$ into Parker's formula (4.1) gives us that

$$\mathrm{Ext}^q(\Delta(r), L(\lambda)) = \bigoplus_{\substack{n=0 \\ n \text{ even}}}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n + r'), L(\lambda')).$$

We analyse the direct summands in turn. The zeroth direct summand is $\mathrm{Ext}^q(\Delta(r'), L(\lambda'))$. If $r > 0$ we have $r' < r$, and so again by (ii) we have that $\mathrm{Ext}^q(\Delta(r'), L(\lambda'))$ is known, so we may pass to the next summand. If $r = 0$, this first summand is $\mathrm{Ext}^q(\Delta(0), L(\lambda'))$, which we may assume is also known, this time by induction on $\lambda$, since $\mathrm{Ext}^q(\Delta(0), L(0)) = 0$ for any

$q > 0$. So we may pass to the next summand. The remaining summands are of the form $\mathrm{Ext}^{q-n}(\Delta(n+r'), L(\lambda'))$ with $n \neq 0$. Now by (i) these values are assumed to be known, so we are done. More specifically, one obtains all the $(q, r)$-cohomological weights from the union of the $(q, r')$-cohomological weights together with the $(q-n, n+r')$-cohomological weights, where $n$ is even. To calculate the associated dimension of a $(q, r)$-cohomological weight $\lambda$, we note how many times $\lambda'$ appears as a $(q-n, n+r')$-cohomological weight over all even values of $n$, which is achieved by induction on $\lambda$.

The second case (where $\lambda_0 = p - 2 - r_0$) is very similar to the first, using formula (4.2) instead of formula (4.1).

By way of example, we use the above procedure to prove the following theorem. The first part of this theorem confirms a result of Cline in [13]. The second part confirms a result of Stewart in [38].

**Theorem 4.4.** *(i) Suppose $M$ is isomorphic to any Frobenius twist (possibly trivial) of $L(2p-2)$. Then $\mathrm{H}^1(\mathrm{SL}_2, M) \neq 0$, and for all other simple $\mathrm{SL}_2$-modules $M$, $\mathrm{H}^1(\mathrm{SL}_2, M) = 0$.*

*(ii) Suppose $M$ is isomorphic to any Frobenius twist (possibly trivial) of one of the following:*

- $L(2p)$
- $L(2p^2 - 2p - 2) \qquad (p > 2)$
- $L(p^n(2p-2) + 2p - 2) \qquad (n > 1)$.

*Then $\mathrm{H}^2(\mathrm{SL}_2, M) \neq 0$, and for all other simple $\mathrm{SL}_2$-modules $M$, $\mathrm{H}^2(\mathrm{SL}_2, M) = 0$.*

*(iii) Suppose $p > 2$ and $M$ is isomorphic to any Frobenius twist (possibly trivial) of one of the following:*

- $L(4p-2)$
- $L(2p^2 - 4p) \qquad (p > 3)$
- $L(p^n(2p-2) + 2p) \qquad (n > 1)$
- $L(2p^2 + 2p - 2)$
- $L(p^n(2p-2) + 2p^2 - 2p - 2) \qquad (n > 2)$
- $L(2p^3 - 2p^2 - 2p - 2)$
- $L(p^n\lambda_2 + 2p - 2) \qquad$ *($n > 1$, $\lambda_2$ any 2-cohomological weight (see Definition 4.2)).*

*Then* $\mathrm{H}^3(\mathrm{SL}_2, M) \neq 0$*, and for all other simple* $\mathrm{SL}_2$*-modules* $M$*,* $\mathrm{H}^3(\mathrm{SL}_2, M) = 0$.

*Suppose* $p = 2$ *and* $M$ *is isomorphic to any Frobenius twist (possibly trivial) of one of the following:*

- $L(6)$
- $L(8)$
- $L(2^n + 2)$ $\quad (n > 3)$
- $L(2^n + 4)$ $\quad (n > 3)$
- $L(2^n + 10)$ $\quad (n > 4)$

*Then* $\mathrm{H}^3(\mathrm{SL}_2, M) \neq 0$*, and for all other simple* $\mathrm{SL}_2$*-modules* $M$*,* $\mathrm{H}^3(\mathrm{SL}_2, M) = 0$.

*Proof.* *(i)* Write $\lambda = p\lambda' + \lambda_0$. First let us assume that $\lambda_0 = p - 2$. Then by (4.2)

$$\mathrm{Ext}^1(\Delta(0), L(\lambda)) = \mathrm{Ext}^0(\Delta(1), L(\lambda')).$$

This is nonzero if and only if $\lambda' = 1$, and so

$$\dim \mathrm{Ext}^1(\Delta(0), L(\lambda)) = 1 \text{ when } \lambda = 2p - 2.$$

If we assume that $\lambda_0 = 0$ then by (4.1)

$$\mathrm{Ext}^1(\Delta(0), L(\lambda)) = \mathrm{Ext}^1(\Delta(0), L(\lambda')).$$

This tells us that $\lambda$ is 1-cohomological if and only if $\lambda'$ is 1-cohomological. Since $\lambda = p\lambda'$, this is just the statement that the twist of a cohomological weight is itself cohomological. By the Linkage Principle, if $\lambda_0$ is neither 0 nor $p - 2$ then $\mathrm{Ext}^1(\Delta(0), L(\lambda))$ is zero. Thus we conclude that the only 1-cohomological weights are $p^n(2p - 2)$ for $n \geq 0$, i.e. $\Omega_1 = \{2p - 2\}$, and that in this case the cohomological dimension is 1.

*(ii)* Write $\lambda = p\lambda' + \lambda_0$. First let us assume that $\lambda_0 = 0$. Then by (4.1)

$$\mathrm{Ext}^2(\Delta(0), L(\lambda)) = \mathrm{Ext}^2(\Delta(0), L(\lambda')) \oplus \mathrm{Ext}^0(\Delta(2), L(\lambda')).$$

In order for the $\mathrm{Ext}^0$ term to be nonzero we must have $\lambda' = 2$. This forces the $\mathrm{Ext}^2$ term to be zero, and so we conclude that

$$\dim \mathrm{Ext}^2(\Delta(0), L(\lambda)) = 1 \text{ when } \lambda = 2p.$$

Now let's assume that $\lambda_0 = p - 2$. Then by (4.2)

$$\mathrm{Ext}^2(\Delta(0), L(\lambda)) = \mathrm{Ext}^1(\Delta(1), L(\lambda')).$$

We now need to determine the $(1,1)$-cohomological weights. So we write $\lambda' = p\lambda'' + \lambda'_0$, and first assume that $\lambda'_0 = p - 3$. Then by (4.2)

$$\mathrm{Ext}^1(\Delta(1), L(\lambda')) = \mathrm{Ext}^0(\Delta(1), L(\lambda'')).$$

This is nonzero if and only if $\lambda'' = 1$, and so

$$\dim \mathrm{Ext}^2(\Delta(0), L(\lambda)) = 1 \text{ when } \lambda = 2p^2 - 2p - 2.$$

If we assume that $\lambda'_0 = 1$, then by (4.1)

$$\mathrm{Ext}^1(\Delta(1), L(\lambda')) = \mathrm{Ext}^1(\Delta(0), L(\lambda'')).$$

This is nonzero if and only if $\lambda'' = p^n(2p - 2)$, and so

$$\dim \mathrm{Ext}^2(\Delta(0), L(\lambda)) = 1 \text{ when } \lambda = 2p - 2 + p^n(2p - 2) \text{ for some } n \geq 2.$$

So $\Omega_2 = \{2p, 2p^2 - 2p - 2, 2p - 2 + p^n(2p - 2) \mid n \geq 2\}$.

*(iii)* Write $\lambda = p\lambda' + \lambda_0$. First let us assume that $\lambda_0 = 0$. Then by (4.1)

$$\mathrm{Ext}^3(\Delta(0), L(\lambda)) = \mathrm{Ext}^3(\Delta(0), L(\lambda')) \oplus \mathrm{Ext}^1(\Delta(2), L(\lambda')).$$

Write $\lambda' = p\lambda'' + \lambda'_0$ and assume $\lambda'_0 = 2$. Then by (4.1)

$$\mathrm{Ext}^1(\Delta(2), L(\lambda')) = \mathrm{Ext}^1(\Delta(0), L(\lambda'')).$$

So $\lambda'' = p^n(2p - 2)$, giving $\lambda = p^n(2p - 2) + 2p$ for some $n \geq 2$.
If we assume that $\lambda'_0 = p - 4$ then by (4.2)

$$\mathrm{Ext}^1(\Delta(2), L(\lambda')) = \mathrm{Ext}^0(\Delta(1), L(\lambda''))$$

which gives $\lambda = 2p^2 - 4p$.
Now assume that $\lambda_0 = p - 2$. Then by (4.2)

$$\mathrm{Ext}^3(\Delta(0), L(\lambda)) = \mathrm{Ext}^2(\Delta(1), L(\lambda')) \oplus \mathrm{Ext}^0(\Delta(3), L(\lambda')).$$

The $\mathrm{Ext}^0$ term is nonzero if and only if $\lambda' = 3$, giving $\lambda = 4p - 2$.
For the $\mathrm{Ext}^2$ term, write $\lambda' = p\lambda'' + \lambda'_0$ and assume $\lambda'_0 = 1$. Then by (4.1)

$$\mathrm{Ext}^2(\Delta(1), L(\lambda')) = \mathrm{Ext}^2(\Delta(0), L(\lambda'')) \oplus \mathrm{Ext}^0(\Delta(2), L(\lambda'')).$$

The $\mathrm{Ext}^0$ term is nonzero if and only if $\lambda'' = 2$, giving $\lambda = 2p^2 + 2p - 2$. The $\mathrm{Ext}^2$ term is nonzero if and only if $\lambda = p^2(\lambda_2) + 2p - 2$, where $\lambda_2$ denotes any 2-cohomological weight.
If we assume that $\lambda'_0 = p - 3$ then by (4.2)

$$\mathrm{Ext}^2(\Delta(1), L(\lambda')) = \mathrm{Ext}^1(\Delta(1), L(\lambda'')).$$

So now we have to write $\lambda'' = p\lambda''' + \lambda''_0$. If $\lambda''_0 = 1$ then by (4.1)

$$\mathrm{Ext}^1(\Delta(1), L(\lambda'')) = \mathrm{Ext}^1(\Delta(0), L(\lambda'''))$$

which is nonzero if and only if $\lambda''' = p^n(2p-2)$, giving $\lambda = p^n(2p-2) + 2p^2 - 2p - 2$ for some $n \geq 3$.
If $\lambda''_0 = p - 3$ then by (4.2)

$$\mathrm{Ext}^1(\Delta(1), L(\lambda'')) = \mathrm{Ext}^0(\Delta(1), L(\lambda'''))$$

which is nonzero if and only if $\lambda''' = 1$, giving $\lambda = 2p^3 - 2p^2 - 2p - 2$.

For the $p = 2$ result, we use the following formula, taken from [30, Theorem 6.1]:

$$\mathrm{Ext}^q(\Delta(2b), L(\lambda)) = \bigoplus_{n=0}^{n=q} \mathrm{Ext}^{q-n}(\Delta(b+n), L(\lambda'))$$

where $\lambda = 2\lambda'$. So with $b = 0$ and $q = 3$, we have

$$\begin{aligned}\mathrm{Ext}^3(\Delta(0), L(\lambda)) = &\mathrm{Ext}^3(\Delta(0), L(\lambda')) \oplus \mathrm{Ext}^2(\Delta(1), L(\lambda')) \\ &\oplus \mathrm{Ext}^1(\Delta(2), L(\lambda')) \oplus \mathrm{Ext}^0(\Delta(3), L(\lambda')).\end{aligned}$$

The $\mathrm{Ext}^0$ term is nonzero if and only if $\lambda' = 3$, giving $\lambda = 6$.
Applying the formula to the $\mathrm{Ext}^1$ term gives

$$\mathrm{Ext}^1(\Delta(2), L(\lambda')) = \mathrm{Ext}^1(\Delta(1), L(\lambda'')) \oplus \mathrm{Ext}^0(\Delta(2), L(\lambda'')).$$

The $\mathrm{Ext}^0$ term is nonzero if and only if $\lambda'' = 2$, giving $\lambda = 8$.
We have $\mathrm{Ext}^1(\Delta(1), L(\lambda'')) = \mathrm{Ext}^1(\Delta(0), L(\lambda'''))$, where $\lambda'' = 2\lambda''' + 1$. Taking $\lambda''' = 2^n$ for $n > 0$ gives $\lambda = 2^n + 4$ for $n > 3$.
The next summand to analyse is $\mathrm{Ext}^2(\Delta(1), L(\lambda'))$. We have

$$\mathrm{Ext}^2(\Delta(1), L(\lambda')) = \mathrm{Ext}^2(\Delta(0), L(\lambda''))$$

where $\lambda' = 2\lambda'' + 1$. If this is non-zero then either $\lambda'' = 2^n$ with $n > 1$ or $\lambda'' = 2^n + 2$ with $n > 2$. These give $\lambda = 2^n + 2$ with $n > 3$ and $\lambda = 2^n + 10$ with $n > 4$ respectively. $\square$

## 4.3 Generic results for large primes

From the examples in the previous section, one sees that when $p \geq 5$, the list of cohomological weights is uniform with $p$. Indeed, by inspecting examples for low values of $q$, one may observe that when $p > q$, the list of $q$-cohomological weights is uniform with $p$. In this section we will prove that this is always true. Henceforth we shall assume that $p > q$.

Suppose $\lambda$ is $q$-cohomological, so $H = \mathrm{H}^q(G, L(\lambda)) = \mathrm{Ext}^q(\Delta(0), L(\lambda)) \neq 0$. Then it follows from the Linkage Principle that either $\lambda = p\lambda'$ for some even $\lambda'$ or $\lambda = p\lambda' + p - 2$ for some odd $\lambda'$. Thus, by Parker's theorem,

$$H = \bigoplus_{n=0}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n), L(\lambda'))$$

where the sum is taken over either even or odd numbers only. Clearly if $H$ is nonzero then one of the summands $\mathrm{Ext}^{q-i}(\Delta(i), L(\lambda'))$ is nonzero. So, Parker's formulae have reduced the computation of $q$-cohomological weights to the computation of $(q-n, n)$-cohomological weights, for all $0 < n \leq q$. Furthermore, every maximally untwisted $q$-cohomological weight arises from a $(q-n, n)$-cohomological weight in this way.

The next result shows how the $(q-n, n)$-cohomological weights themselves arise from $(q-n)$-cohomological weights.

**Lemma 4.5.** *Suppose $\lambda \in X(T)^+$ is linked to zero, and let $0 \leq n \leq p-2$ be an integer.*

*(i) If $\lambda \equiv 0 \bmod 2p$ then* $\mathrm{Ext}^i(\Delta(0), L(\lambda)) = \mathrm{Ext}^i(\Delta(n), L(\lambda + n))$.

*(ii) If $\lambda \equiv -2 \bmod 2p$ then* $\mathrm{Ext}^i(\Delta(0), L(\lambda)) = \mathrm{Ext}^i(\Delta(n), L(\lambda - n))$.

*Proof.* In [25, II.7.6], Jantzen gives the isomorphisms

$$\mathrm{Ext}^i_G(V, T_\mu^\lambda V') \simeq \mathrm{Ext}^i_G(T_\lambda^\mu V, V')$$

for all $i$, where $V$ and $V'$ are $G$-modules and $T_\lambda^\mu$ is a translation functor. Setting $\mu = \lambda + n$ we can apply this to get

$$\begin{aligned} \mathrm{Ext}^i(\Delta(0), L(\lambda)) &= \mathrm{Ext}^i(\Delta(0), T_\mu^\lambda L(\mu)) \\ &\simeq \mathrm{Ext}^i(T_\lambda^\mu \Delta(0), L(\mu)) \\ &= \mathrm{Ext}^i(\Delta(n), L(\mu)) \end{aligned}$$

which proves *(i)*. Part *(ii)* is similar. $\square$

*Remark* 4.6. Alternatively, this Lemma can be proved directly from Parker's formulae: For part *(i)*, if we expand $\mathrm{Ext}^i(\Delta(0), L(\lambda))$ and $\mathrm{Ext}^i(\Delta(n), L(\lambda + n))$ using formula (4.1), we see that they are both equal to

$$\mathrm{Ext}^i(\Delta(0), L(\lambda)^{[-1]}) \oplus \mathrm{Ext}^{i-2}(\Delta(2), L(\lambda)^{[-1]}) \oplus \cdots \oplus \mathrm{Ext}^a(\Delta(i-a), L(\lambda)^{[-1]})$$

where $a = 1$ if $i$ is odd, $a = 0$ otherwise. Part *(ii)* is proved similarly with formula (4.2).

Therefore, all $q$-cohomological weights arise from $(q-n)$-cohomological weights via the method described above.

Lemma 4.5 motivates Definition 4.1 which the reader should recall now.

**Corollary 4.7.** *If $\lambda$ is $(q-n)$-cohomological, then $\lambda \parallel n$ is $q$-cohomological.*

*Proof.* There are four separate cases to consider, which, following Definition 4.1, arise from the parity of $q - n$ and the parity of $\lambda$. We will only prove the case where $q - n$ is even and $\lambda \equiv 0 \bmod 2p$; the remaining cases can be proved with similar arguments. We have:

$$\begin{aligned}
\dim \mathrm{Ext}^q(\Delta(0), L(\lambda \parallel n)) &= \dim \mathrm{Ext}^q(\Delta(0), L(p(\lambda + n))) \\
&\geq \dim \mathrm{Ext}^{q-n}(\Delta(n), L(\lambda + n)) \quad &&\text{(by formula (4.1))} \\
&= \dim \mathrm{Ext}^{q-n}(\Delta(0), L(\lambda)) &&\text{(by Lemma 4.5)} \\
&> 0. &&\text{(by assumption)}
\end{aligned}$$

□

**Lemma 4.8.** *Assume $p > q \geq 1$. If $\lambda$ is linked to zero then* $\mathrm{H}^q(G, L(\lambda)) = \mathrm{H}^q(G, L(\lambda)^{[1]})$.

*Proof.* We have:

$$\begin{aligned}
\mathrm{H}^q(G, L(\lambda)^{[1]}) &= \mathrm{Ext}^q(\Delta(0), L(\lambda)^{[1]}) \\
&= \bigoplus_{\substack{n=0 \\ n \text{ even}}}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n), L(\lambda)) \qquad &&\text{(by formula (4.1))} \\
&= \mathrm{Ext}^q(\Delta(0), L(\lambda)) &&\text{(by the Linkage Principle,} \\
&= \mathrm{H}^q(G, L(\lambda)). &&\text{since } n \leq q < p.)
\end{aligned}$$

□

The following is an immediate consequence of the lemma, using the injective map $\mathrm{H}^m(G, M) \to \mathrm{H}^m(G, M^{[1]})$ induced by the Frobenius twist.

**Corollary 4.9.** *Assume $p > q \geq 1$. If $\lambda$ is $q$-cohomological then $p^n\lambda$ is $q$-cohomological for all $n > 0$.*

The next result gives a closed-form description of the set $\Omega_q$.

**Theorem 4.10.** *Assume $p > q \geq 1$. Then*

$$\Omega_q = \{(p^n\lambda_i) \parallel (q-i) \mid n \geq 0, \lambda_i \in \Omega_i, i = 0 \dots q-1\}.$$

*Proof.* If $\lambda \in \Omega_q$ then by Theorem 4.3 there is an integer $n$ such that $1 \leq n \leq q$ and the space $\mathrm{Ext}^{q-n}(\Delta(n), L(\lambda'))$ is nonzero (where $\lambda = p\lambda' + \lambda_0$ and $0 \leq \lambda_0 < p$). By Lemma 4.5, either $\lambda' + n$ or $\lambda' - n$ is $(q-n)$-cohomological. But then either $\lambda = (\lambda' + n) \parallel n$ or $\lambda = (\lambda' - n) \parallel n$, and so $\lambda \in \{(p^n\lambda_i) \parallel (q-i) \mid n \geq 0, \lambda_i \in \Omega_i, i = 0 \dots q-1\}$.

It then follows from Corollaries 4.9 and 4.7 that $\Omega_q$ contains this set. □

*Proof of Theorem B.* If $\mathrm{H}^q(G, L(\lambda)) \neq 0$ then by Lemma 4.8 we may assume that $\lambda \in \Omega_q$. So it remains to show that if $\lambda \in \Omega_q$ then $\dim \mathrm{H}^q(G, L(\lambda)) = \dim \mathrm{Ext}^q(\Delta(0), L(\lambda)) = 1$. We will use induction on $q$, and restrict our attention to the case where $q - i$ is even and $\lambda_i \equiv 0 \bmod 2p$; as before, the remaining cases are dealt with similarly. Let $\lambda_i \in \Omega_i$, so that $\lambda = \lambda_i \parallel (q-i) \in \Omega_q$. Then

$$\mathrm{Ext}^q(\Delta(0), L(\lambda)) \simeq \bigoplus_{\substack{n=0 \\ n \text{ even}}}^{n=q} \mathrm{Ext}^{q-n}(\Delta(n), L(\lambda_i + q - i)).$$

Since $n \leq q < p$, then by the Linkage Principle, the only summand that can be nonzero corresponds to $n = q - i$. Thus

$$\begin{aligned} \mathrm{Ext}^q(\Delta(0), L(\lambda)) &= \mathrm{Ext}^i(\Delta(q-i), L(\lambda_i + q - i)) \\ &= \mathrm{Ext}^i(\Delta(0), L(\lambda_i)) \qquad \text{(by Lemma 4.5)} \end{aligned}$$

whose dimension is 1 by induction. □

*Proof of Theorem A.* The only part of Theorem A that remains to be shown is the statement that $\dim \mathrm{H}^3(G, L(\lambda)) = 2$ when $p = 2$ and $\lambda = 2^n + 4$ for $n > 4$. The proof of this is a calculation, using the formula for $p = 2$ given in [30, Theorem 6.1]. Let $H = \mathrm{H}^3(G, L(2^n + 4))$. Then

$$\begin{aligned} H =& \mathrm{Ext}^3(\Delta(0), L(2^{n-1} + 2)) \oplus \mathrm{Ext}^2(\Delta(1), L(2^{n-1} + 2)) \\ &\oplus \mathrm{Ext}^1(\Delta(2), L(2^{n-1} + 2)) \oplus \mathrm{Ext}^0(\Delta(3), L(2^{n-1} + 2)). \end{aligned}$$

We will consider each of these summands in turn.

The $\mathrm{Ext}^2$ and $\mathrm{Ext}^0$ summands are zero. Expanding the $\mathrm{Ext}^3$ summand, we get:

$$\begin{aligned}
\mathrm{Ext}^3(\Delta(0), L(2^{n-1}+2)) &= \mathrm{Ext}^3(\Delta(0), L(2^{n-2}+1)) \\
&\quad \oplus \mathrm{Ext}^2(\Delta(1), L(2^{n-2}+1)) \\
&\quad \oplus \mathrm{Ext}^1(\Delta(2), L(2^{n-2}+1)) \\
&\quad \oplus \mathrm{Ext}^0(\Delta(3), L(2^{n-2}+1)) \\
&= \mathrm{Ext}^2(\Delta(1), L(2^{n-2}+1)) \\
&\quad \oplus \mathrm{Ext}^0(\Delta(3), L(2^{n-2}+1)) \\
&= \mathrm{Ext}^2(\Delta(0), L(2^{n-3})) \oplus \mathrm{Ext}^0(\Delta(1), L(2^{n-3})) \\
&= \mathrm{Ext}^2(\Delta(0), L(2^{n-3})).
\end{aligned}$$

Now expanding the $\mathrm{Ext}^1$ summand, we get:

$$\begin{aligned}
\mathrm{Ext}^1(\Delta(2), L(2^{n-1}+2)) &= \mathrm{Ext}^1(\Delta(1), L(2^{n-2}+1)) \\
&\quad \oplus \mathrm{Ext}^0(\Delta(2), L(2^{n-2}+1)) \\
&= \mathrm{Ext}^1(\Delta(0), L(2^{n-3})).
\end{aligned}$$

Therefore

$$\begin{aligned}
H &= \mathrm{Ext}^2(\Delta(0), L(2^{n-3})) \oplus \mathrm{Ext}^1(\Delta(0), L(2^{n-3})) \\
&= \mathrm{H}^2(G, L(2^{n-3})) \oplus \mathrm{H}^1(G, L(2^{n-3})) \\
&= k \oplus k
\end{aligned}$$

where the last equality follows from Theorem 4.4. $\square$

# Chapter 5

# Cohomology of $\mathrm{SL}_3$

In the previous chapter, we proved that, under the assumption $p > n$, the dimension of the cohomology space $\mathrm{H}^n(\mathrm{SL}_2, M)$ is at most 1, for any simple module $M$. This raises the question of whether or not a similar result holds in the case of other algebraic groups.

**Question 5.1.** Does there exist a constant $p_0 = p_0(n)$ such that, if $p > p_0$ and $G$ is a semisimple algebraic group with root system $\Phi$ over an algebraically closed field of characteristic $p$, then there is a constant $c = c(\Phi)$ such that $\dim \mathrm{H}^n(G, M) \leq c$ for any simple $G$-module $M$?

In this chapter we answer this question in the negative for the case $G = \mathrm{SL}_3$. The main result is:

**Theorem C.** *Let $G$ be the simple algebraic group $\mathrm{SL}_3$ defined over an algebraically closed field of characteristic $p$. Let $n$ be a positive integer, and assume that $p > 2n$. Then there is a dominant weight $\lambda$ such that*

$$\dim \mathrm{H}^{2n}(G, L(\lambda)) \geq n - 1.$$

This answers Question 5.1 in the negative because it shows that for any choice of $c(\Phi)$ and $p_0(n)$ we may put $n = c(\Phi) + 2$ into the theorem. Then the theorem guarantees a sufficiently large $p > p_0(n)$ and some $\lambda$ such that $\dim \mathrm{H}^{2n}(G, L(\lambda)) > c(\Phi)$.

The main tool used in the proof of this result is the Lyndon-Hochschild-Serre spectral sequence, applied to a particular simple module constructed using translation functors. We also make use of a new result calculating the module $\mathrm{ind}_B^G(\mathrm{S}^2 \mathfrak{u}^*)$.

## 5.1 Preliminary results

Recall the Lyndon-Hochschild-Serre spectral sequence from Proposition 3.6:

$$E_2^{i,j} = \mathrm{H}^i(G, \mathrm{H}^j(G_1, V)^{[-1]}) \Rightarrow \mathrm{H}^{i+j}(G, V).$$

Before we can make use of this spectral sequence, we will need to understand the spaces $\mathrm{H}^j(G_1, V)^{[-1]}$, where $V$ is a simple module. For small values of $j$, these spaces have been described, and we record these results below. For $j = 1$, this was done in [42]; for $j = 2$, see [39, Proposition 2.5]; for $j = 3$ see [15, Proposition 1]. Of course, when $j = 0$, we have that $\mathrm{H}^0(G_1, L(\lambda_0))^{[-1]} = \mathrm{Hom}_{G_1}(k, L(\lambda_0)) \neq 0$ if and only if $\lambda_0 = (0, 0)$.

**Proposition 5.2.** *Assume $p > 3$.*

$$\mathrm{H}^1(G_1, L(\lambda_0))^{[-1]} = \begin{cases} L(1,0) & \textit{if } \lambda_0 = (p-2,1) \\ L(0,1) & \textit{if } \lambda_0 = (1,p-2) \\ k & \textit{if } \lambda_0 = (p-2,p-2) \\ 0 & \textit{otherwise.} \end{cases}$$

**Proposition 5.3.** *Assume $p > 3$.*

$$\mathrm{H}^2(G_1, L(\lambda_0))^{[-1]} = \begin{cases} L(1,0) & \textit{if } \lambda_0 = (p-3,0) \\ L(0,1) & \textit{if } \lambda_0 = (0,p-3) \\ L(1,1) & \textit{if } \lambda_0 = (0,0) \\ 0 & \textit{otherwise.} \end{cases}$$

**Proposition 5.4.** *Assume $p > 3$.*

$$\mathrm{H}^3(G_1, L(\lambda_0))^{[-1]} = \begin{cases} L(1,0) \oplus L(0,2) \oplus L(2,1) & \textit{if } \lambda_0 = (p-2,1) \\ L(0,1) \oplus L(2,0) \oplus L(1,2) & \textit{if } \lambda_0 = (1,p-2) \\ L(1,1) \oplus L(1,1) & \textit{if } \lambda_0 = (p-2,p-2) \\ 0 & \textit{otherwise.} \end{cases}$$

We need only consider the restricted weights which are in the $G_1$-linkage class of $(0, 0)$, these are

(i) $(0, 0)$

(ii) $(p-3, 0) = p(1, 0) + s_\alpha s_\beta \cdot (0, 0)$

(iii) $(0, p-3) = p(0, 1) + s_\beta s_\alpha \cdot (0, 0)$

(iv) $(p-2, 1) = p(1, 0) + s_\alpha \cdot (0, 0)$

(v) $(1, p-2) = p(0,1) + s_\beta \cdot (0,0)$

(vi) $(p-2, p-2) = p(1,1) + \omega_0 \cdot (0,0)$.

Thus, by the Andersen-Jantzen formula, we have for $i$ even

(i) $\mathrm{H}^i(G_1, \mathrm{H}^0(0,0))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i}{2}}(\mathfrak{u}^*) \otimes k_{(0,0)})$

(ii) $\mathrm{H}^i(G_1, \mathrm{H}^0(p-3,0))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-2}{2}}(\mathfrak{u}^*) \otimes k_{(1,0)})$

(iii) $\mathrm{H}^i(G_1, \mathrm{H}^0(0,p-3))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-2}{2}}(\mathfrak{u}^*) \otimes k_{(0,1)})$

and for $i$ odd

(iv) $\mathrm{H}^i(G_1, \mathrm{H}^0(p-2,1))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-1}{2}}(\mathfrak{u}^*) \otimes k_{(1,0)})$

(v) $\mathrm{H}^i(G_1, \mathrm{H}^0(1,p-2))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-1}{2}}(\mathfrak{u}^*) \otimes k_{(0,1)})$

(vi) $\mathrm{H}^i(G_1, \mathrm{H}^0(p-2,p-2))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-3}{2}}(\mathfrak{u}^*) \otimes k_{(1,1)})$.

To be clear, the above means that if $i$ is even, then $\mathrm{H}^i(G_1, \mathrm{H}^0(\lambda_0))^{[-1]}$ vanishes for $\lambda_0 = (p-2,1), (1,p-2)$ or $(p-2,p-2)$. Similarly, if $i$ is odd, then $\mathrm{H}^i(G_1, \mathrm{H}^0(\lambda_0))^{[-1]}$ vanishes for $\lambda_0 = (0,0), (p-3,0)$ or $(0,p-3)$.

However, we are interested in $\mathrm{H}^i(G_1, L(\lambda))^{[-1]}$, rather than $\mathrm{H}^i(G_1, \mathrm{H}^0(\lambda))^{[-1]}$. The following proposition provides the information we will need to move between the two.

**Proposition 5.5.** *Assume $p > 3$. Let $\lambda$ be a restricted dominant weight which is $G_1$-linked to $(0,0)$. Then the structure of $\mathrm{H}^0(\lambda)$ is as follows:*

| $\lambda$ | $\mathrm{H}^0(\lambda)$ |
|---|---|
| $(0,0)$ | $k$ |
| $(p-3,0)$ | $L(p-3,0)$ |
| $(0,p-3)$ | $L(0,p-3)$ |
| $(p-2,1)$ | $L(p-3,0) \mid L(p-2,1)$ |
| $(1,p-2)$ | $L(0,p-3) \mid L(1,p-2)$ |
| $(p-2,p-2)$ | $k \mid L(p-2,p-2)$ |

*where $L(\lambda)|L(\mu)$ indicates a uniserial module of length 2 with head $L(\lambda)$ and socle $L(\mu)$.*

*Proof.* This is well known, see for example [39, Lemma 2.4]. □

Thus we can write down this general proposition.

**Proposition 5.6.** *Assume $p > 3$. Then*

*(i)* $\mathrm{H}^i(G_1, L(0,0))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i}{2}}(\mathfrak{u}^*))$

*(ii)* $\mathrm{H}^i(G_1, L(p-3,0))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-2}{2}}(\mathfrak{u}^*) \otimes k_{(1,0)})$

*(iii)* $\mathrm{H}^i(G_1, L(0,p-3))^{[-1]} = \mathrm{H}^0(\mathrm{S}^{\frac{i-2}{2}}(\mathfrak{u}^*) \otimes k_{(0,1)})$.

In particular, we have

**Proposition 5.7.** *Assume $p > 3$. Then*

$$\mathrm{H}^4(G_1, L(\lambda_0))^{[-1]} = \begin{cases} \mathrm{H}^0(\mathrm{S}^2(\mathfrak{u}^*)) & \text{if } \lambda_0 = (0,0) \\ L(2,1) \oplus L(0,2) & \text{if } \lambda_0 = (p-3,0) \\ L(1,2) \oplus L(2,0) & \text{if } \lambda_0 = (0,p-3) \\ 0 & \text{otherwise.} \end{cases}$$

Finally, our proof of Theorem C will make use of the following well-known fact.

**Lemma 5.8.** *Let $N \trianglelefteq G$. For all $G$-modules $V$, $W$ and $E$ such that $E|_N$ is trivial, we have*

$$\mathrm{Ext}^\bullet_N(V, W \otimes E) \simeq \mathrm{Ext}^\bullet_N(V, W) \otimes E.$$

*Proof.* We first observe that $\mathrm{Hom}_N(V, W \otimes E) \simeq \mathrm{Hom}_N(V, W) \otimes E$, which can be seen by direct calculations. Since $(- \otimes E)$ takes injective modules to acyclic modules, we may apply Grothendieck's spectral sequence ([25, I.4.1]) to calculate the cohomology of the composition $\mathrm{Hom}_N(V, -) \circ (- \otimes E)$, which is $\mathrm{Hom}_N(V, -) \otimes E$ by our observation. Since $(- \otimes E)$ is exact, the spectral sequence collapses to yield isomorphisms

$$\mathrm{Ext}^\bullet_N(V, W \otimes E) \simeq \mathrm{Ext}^\bullet_N(V, W) \otimes E.$$

□

## 5.2 Main results

The next proposition, which follows from [32, Main Theorem], will allow us to get lower bounds for the cohomological dimension since we can estimate the terms on the right hand side of the equation.

**Proposition 5.9.** *Let $V$ be a simple $G$-module. Then the Lyndon-Hochschild-Serre spectral sequence stabilises in the region $i+j<p$, i.e. $E_2^{i,j}=E_\infty^{i,j}$ whenever $i+j<p$. Thus, if $p>n$ then*

$$\mathrm{H}^n(G,V)=\bigoplus_{i+j=n}E_2^{i,j}.$$

The following two lemmas will play small but crucial roles in our later proofs. We first define some notions of alcove geometry from [25, II.6.2]. Recall the dot action of the affine Weyl group $W_p$ on $X(T)\otimes_{\mathbb{Z}}\mathbb{R}$ (§3.2). A *facet* $F$ (for $W_p$) is a non-empty subset of the form

$$\begin{aligned}F=\{\lambda\in X(T)\otimes_{\mathbb{Z}}\mathbb{R}\mid{}&\langle\lambda+\rho,\alpha^\vee\rangle=n_\alpha p\text{ for all }\alpha\in\Phi_0^+(F),\\(n_\alpha-1)p<{}&\langle\lambda+\rho,\alpha^\vee\rangle<n_\alpha p\text{ for all }\alpha\in\Phi_1^+(F)\}\end{aligned}$$

for suitable integers $n_\alpha\in\mathbb{Z}$ and for a disjoint decomposition

$$\Phi^+=\Phi_0^+(F)\dot\cup\Phi_1^+(F).$$

For a facet $F$, we define the *closure* $\overline{F}$ of $F$ and the *upper closure* $\widehat{F}$ of $F$ as follows:

$$\begin{aligned}\overline{F}=\{\lambda\in X(T)\otimes_{\mathbb{Z}}\mathbb{R}\mid{}&\langle\lambda+\rho,\alpha^\vee\rangle=n_\alpha p\text{ for all }\alpha\in\Phi_0^+(F),\\(n_\alpha-1)p\le{}&\langle\lambda+\rho,\alpha^\vee\rangle\le n_\alpha p\text{ for all }\alpha\in\Phi_1^+(F)\}\end{aligned}$$

$$\begin{aligned}\widehat{F}=\{\lambda\in X(T)\otimes_{\mathbb{Z}}\mathbb{R}\mid{}&\langle\lambda+\rho,\alpha^\vee\rangle=n_\alpha p\text{ for all }\alpha\in\Phi_0^+(F),\\(n_\alpha-1)p<{}&\langle\lambda+\rho,\alpha^\vee\rangle\le n_\alpha p\text{ for all }\alpha\in\Phi_1^+(F)\}.\end{aligned}$$

A facet $F$ is called an *alcove* if $\Phi_0^+(F)=\emptyset$. There is a 'standard alcove' which we shall be concerned with. Set

$$C=\{\lambda\in X(T)\otimes_{\mathbb{Z}}\mathbb{R}\mid 0<\langle\lambda+\rho,\alpha^\vee\rangle<p\text{ for all }\alpha\in\Phi^+\}.$$

Now we have the *lowest alcove* $\overline{C}_{\mathbb{Z}}$, defined by $\overline{C}_{\mathbb{Z}}=\overline{C}\cap X(T)$.

**Lemma 5.10.** *If $\lambda\in\overline{C}_{\mathbb{Z}}\cap X(T)^+$ then $\mathrm{H}^{l(w)}(w\cdot\lambda)\cong\mathrm{H}^0(\lambda)$ for all $w\in W$.*

*Proof.* See [25, II.5.5]. □

**Lemma 5.11.** *Let $\lambda,\mu\in\overline{C}_{\mathbb{Z}}$ such that $\mu$ belongs to the closure of the facet containing $\lambda$. Let $w\in W_p$ with $w\cdot\lambda\in X(T)^+$, and denote by $F$ the facet with $w\cdot\lambda\in F$. Then*

$$T_\lambda^\mu L(w\cdot\lambda)\simeq\begin{cases}L(w\cdot\mu) & \textit{if } w\cdot\mu\in\widehat{F}\\ 0 & \textit{otherwise.}\end{cases}$$

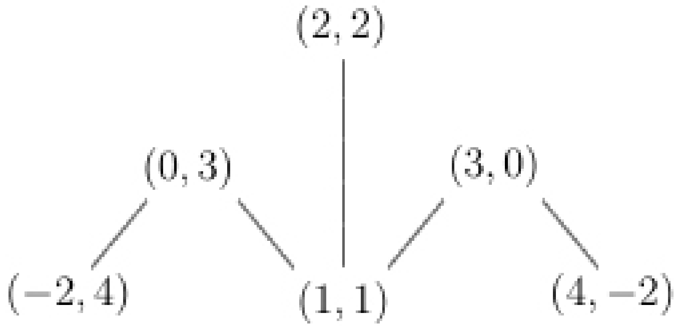


Figure 5.1: $B$-module structure of $\mathrm{S}^2\,\mathfrak{u}^*$.

*Proof.* See [25, II.7.15]. □

Recall ([25, II.5.7]) the Euler characteristic, defined for each finite dimensional $B$-module $M$:

$$\chi(M) = \sum_{i\geq 0}(-1)^i \operatorname{ch} \mathrm{H}^i(M).$$

**Proposition 5.12.** *If $p > 5$, then* $\mathrm{H}^0(\mathrm{S}^2(\mathfrak{u}^*)) \simeq L(2,2) \oplus L(1,1)$.

*Proof.* In this proof we will write $(a,b)$ in place of $k_{(a,b)}$.

Let $M = \mathrm{S}^2\,\mathfrak{u}^*$. Its structure as a $B$-module is shown in the Alperin diagram (see [1]) in Figure 5.1.

There is a short exact sequence

$$0 \to Q \to M \to (2,2) \to 0$$

and we have $\chi(M) = \chi(Q) + \chi(2,2) = \chi(Q) + \operatorname{ch} \mathrm{H}^0(2,2)$. There is a short exact sequence

$$0 \to (1,1) \to Q \to Q_2 \to 0$$

and we have $\chi(Q) = \chi(1,1) + \chi(Q_2) = \operatorname{ch} \mathrm{H}^0(1,1) + \chi(Q_2)$, where

$$Q_2 = (0,3)|(-2,4) \oplus (3,0)|(4,-2).$$

Let $N = (0,3)|(-2,4)$. There is a short exact sequence

$$0 \to (-2,4) \to N \to (0,3) \to 0$$

so

$$\chi(N) = \chi(0,3) + \chi(-2,4) = \operatorname{ch} \mathrm{H}^0(0,3) - \operatorname{ch} \mathrm{H}^1(-2,4) = 0$$

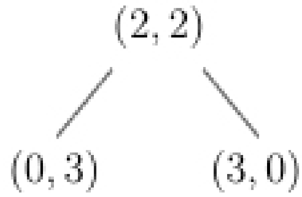


Figure 5.2: $B$-module structure of $S$.

where the second equality is due to the definition of the Euler characteristic, and the third follows since $\mathrm{H}^1(-2,4) \simeq \mathrm{H}^0(0,3)$ by Lemma 5.10.
By symmetry, $\chi((3,0)|(4,-2)) = 0$, so $\chi(Q_2) = 0$. Therefore

$$\chi(M) = \operatorname{ch} \mathrm{H}^0(2,2) + \operatorname{ch} \mathrm{H}^0(1,1).$$

Let $\mu = (-2,4) \oplus (1,1) \oplus (4,-2)$, a submodule of $M$. We now have an exact sequence

$$\mathrm{H}^0(\mu) \to \mathrm{H}^0(M) \to \mathrm{H}^0(S) \to \mathrm{H}^1(\mu) \tag{5.1}$$

where $S$ is as shown in Figure 5.2.

By Kempf's Vanishing Theorem

$$\operatorname{ch} \mathrm{H}^0(S) = \operatorname{ch} \mathrm{H}^0(2,2) + \operatorname{ch} \mathrm{H}^0(0,3) + \operatorname{ch} \mathrm{H}^0(3,0).$$

But now, since $p > 5$, the linkage principle implies that

$$\mathrm{H}^0(S) = \mathrm{H}^0(2,2) \oplus \mathrm{H}^0(0,3) \oplus \mathrm{H}^0(3,0).$$

Again by Kempf's Vanishing Theorem, and by Lemma 5.10, we get that $\mathrm{H}^1(\mu) = \mathrm{H}^0(0,3) \oplus \mathrm{H}^0(3,0)$. We also know that $\mathrm{H}^0(\mu) = \mathrm{H}^0(1,1)$. So now our exact sequence 5.1 is

$$\mathrm{H}^0(1,1) \to \mathrm{H}^0(M) \xrightarrow{\tau} \mathrm{H}^0(2,2) \oplus \mathrm{H}^0(0,3) \oplus \mathrm{H}^0(3,0) \xrightarrow{\upsilon} \mathrm{H}^0(0,3) \oplus \mathrm{H}^0(3,0).$$

Hence $\mathrm{H}^0(2,2)$ is in the kernel of $\upsilon$, and thus in the image of $\tau$. So $\mathrm{H}^0(2,2)$ is a quotient of $\mathrm{H}^0(M)$.

We now claim that the image of $\tau$ is precisely $\mathrm{H}^0(2,2)$. Clearly the image is a submodule of $\mathrm{H}^0(2,2) \oplus \mathrm{H}^0(0,3) \oplus \mathrm{H}^0(3,0)$. But $\chi(M) = \operatorname{ch} \mathrm{H}^0(2,2) + \operatorname{ch} \mathrm{H}^0(1,1)$. Thus $M$ can contain no composition factor isomorphic to $\mathrm{H}^0(0,3)$ or $\mathrm{H}^0(3,0)$, since otherwise our formula for $\chi(M)$ would be wrong. This proves the claim.

□

Recall from §3.4 that for any two weights $\lambda, \mu \in \overline{C}_{\mathbb{Z}}$, there is an exact functor $T_\lambda^\mu$ of $G$-modules with the property that, for $V_1, V_2$ any two $G$-modules,

$$\operatorname{Ext}^i_G(V_1, T_\mu^\lambda V_2) \cong \operatorname{Ext}^i_G(T_\lambda^\mu V_1, V_2).$$

We are now ready to prove the main result.

*Proof of Theorem C.* The proof will use induction on $n$. The induction statement is:

$$\begin{aligned} \text{there exists } \lambda \in X(T)^+ \text{ such that } &\dim \mathrm{H}^{2n}(G, L(\lambda)) \geq n-1 \\ \text{and } &\dim \mathrm{H}^{2(n-1)}(G, L(\lambda)) \geq 1. \end{aligned}$$

For $n = 1$, this statement is true; take $\lambda = (0,0)$. For $n = 2$, take $\lambda = (p,p)$.

Now suppose that $V$ satisfies the statement for $n = k-1$, i.e. $V$ is such that $\dim \mathrm{H}^{2(k-1)}(G, V) \geq k-2$ and $\dim \mathrm{H}^{2(k-2)}(G, V) \geq 1$.

Note that (the highest weight of) $V$ is linked to zero. Since $n > 1$ and $p > 2n$ (by hypothesis), the weight $(1,1)$ is in the lowest alcove. Thus $T_{(0,0)}^{(1,1)} V$ is simple by Lemma 5.11. Now let $L(\lambda) = (T_{(0,0)}^{(1,1)} V)^{[1]}$.

Consider the Lyndon-Hochschild-Serre spectral sequence applied to $L(\lambda)$:

$$E_2^{i,j} = \mathrm{H}^i(G, \mathrm{H}^j(G_1, L(\lambda))^{[-1]}) \Rightarrow \mathrm{H}^{i+j}(G, L(\lambda)).$$

We are assuming that $p > 2k$, so we know that this spectral sequence stabilises in the region $i + j \leq 2k$ by Proposition 5.9. Thus we have

$$\mathrm{H}^{2k}(G, L(\lambda)) = \bigoplus_{i=0}^{2k} \mathrm{H}^{2k-i}(G, \mathrm{H}^i(G_1, L(\lambda))^{[-1]}).$$

Restricting our attention to the terms corresponding to $i = 2$ and $i = 4$ gives:

$$\begin{aligned} \dim \mathrm{H}^{2k}(G, L(\lambda)) \geq &\dim \mathrm{H}^{2(k-1)}(G, \mathrm{H}^2(G_1, L(\lambda))^{[-1]}) \\ &+ \dim \mathrm{H}^{2(k-2)}(G, \mathrm{H}^4(G_1, L(\lambda))^{[-1]}). \end{aligned} \tag{5.2}$$

Since $L(\lambda) = (T_{(0,0)}^{(1,1)} V)^{[1]}$, we know that

$$\begin{aligned} \mathrm{H}^2(G_1, L(\lambda))^{[-1]} &= \mathrm{H}^2(G_1, k)^{[-1]} \otimes T_{(0,0)}^{(1,1)} V \\ &= \operatorname{ind}_B^G(\mathfrak{u}^*) \otimes T_{(0,0)}^{(1,1)} V \\ &= L(1,1) \otimes T_{(0,0)}^{(1,1)} V \end{aligned}$$

and

$$\begin{aligned}\mathrm{H}^4(G_1, L(\lambda))^{[-1]} &= \mathrm{H}^4(G_1, k)^{[-1]} \otimes T_{(0,0)}^{(1,1)} V \\ &= \operatorname{ind}_B^G(\mathrm{S}^2\,\mathfrak{u}^*) \otimes T_{(0,0)}^{(1,1)} V \\ &= (L(1,1) \oplus L(2,2)) \otimes T_{(0,0)}^{(1,1)} V.\end{aligned}$$

Substituting these into (5.2) yields

$$\begin{aligned}\dim \mathrm{H}^{2k}(G, L(\lambda)) \geq{}& \dim \mathrm{Ext}^{2(k-1)}(L(1,1), T_{(0,0)}^{(1,1)} V) \\ &+ \dim \mathrm{Ext}^{2(k-2)}(L(1,1), T_{(0,0)}^{(1,1)} V).\end{aligned}$$

Now apply the translation functor $T_{(1,1)}^{(0,0)}$ to both arguments of the Ext terms on the right hand side to get

$$\begin{aligned}\dim \mathrm{H}^{2k}(G, L(\lambda)) &\geq \dim \mathrm{H}^{2(k-1)}(G, V) + \dim \mathrm{H}^{2(k-2)}(G, V) \\ &\geq k-1.\end{aligned}$$

We have proved the first part of the induction statement. The second part is proved similarly:

$$\begin{aligned}\dim \mathrm{H}^{2(k-1)}(G, L(\lambda)) &= \sum_{i=0}^{2k-2} \dim \mathrm{H}^{2(k-1)-i}(G, \mathrm{H}^i(G_1, L(\lambda))^{[-1]}) \\ &= \sum_{i=0}^{2k-2} \dim \mathrm{H}^{2(k-1)-i}(G, \operatorname{ind}_B^G(\mathrm{S}^{i/2}\,\mathfrak{u}^*) \otimes T_{(0,0)}^{(1,1)} V) \\ &\geq \dim \mathrm{H}^{2(k-2)}(G, L(1,1) \otimes T_{(0,0)}^{(1,1)} V) \\ &= \dim \mathrm{Ext}^{2(k-2)}(L(1,1), T_{(0,0)}^{(1,1)} V) \\ &= \dim \mathrm{Ext}^{2(k-2)}(T_{(1,1)}^{(0,0)} L(1,1), V) \\ &= \dim \mathrm{H}^{2(k-2)}(G, V) \\ &\geq 1.\end{aligned}$$

□

### 5.2.1 A Generalisation

The notation in this section is taken to be consistent with [15].

The proof of Theorem C given above relies on the fact that both $\mathrm{H}^2(G_1, k)^{[-1]}$ and $\mathrm{H}^4(G_1, k)^{[-1]}$ contain $L(1,1)$ as a composition factor. This fact is used to construct an inductive argument. Note that the same inductive argument could have been constructed from a different fact, namely that both $\mathrm{H}^1(G_1, L(p-2,1))^{[-1]}$ and $\mathrm{H}^3(G_1, L(p-2,1))^{[-1]}$ contain $L(1,0)$ as a composition factor. Crucially, this observation generalises to the other rank two algebraic groups, $\mathrm{Sp}_4$ and $G_2$. So, for $G = \mathrm{Sp}_4$, we have that both $\mathrm{H}^1(G_1, L(\lambda_0))^{[-1]}$ ([15, Lem. 2]) and $\mathrm{H}^3(G_1, L(\lambda_0))^{[-1]}$ ([15, Prop. 1(b)]) contain $L(1,0)$ as a composition factor, where $\lambda_0 = \omega_1(p-4,0)$. Similarly, for $G = G_2$, both $\mathrm{H}^1(G_1, L(\lambda_0))^{[-1]}$ ([15, Lem. 3]) and $\mathrm{H}^3(G_1, L(\lambda_0))^{[-1]}$ ([15, Prop. 1(c)]) contain $L(1,0)$ as a composition factor, where $\lambda_0 = \omega_4(0)$. So the result for $\mathrm{SL}_3$ will also hold for $\mathrm{Sp}_4$ and $G_2$, as long as the base case of the inductive argument holds, that is, as long as there exists a simple module $L(\lambda)$ such that $\dim \mathrm{H}^1(G, L(\lambda)) \geq 1$ and $\dim \mathrm{H}^3(G, L(\lambda)) \geq 1$. Put differently, the base case asks for the existence of a weight which is both 1-cohomological and 3-cohomological. Such a weight does indeed exist in both cases: for $\mathrm{Sp}_4$, $\lambda = \omega_2(0) + p(0,1)$ is 1-cohomological ([15, Lem. 5]) and 3-cohomological ([15, Lem. 18(b)]), and for $G_2$, $\lambda = \delta_3(0)$ is 1-cohomological ([15, Lem. 6]) and 3-cohomological ([15, Lem. 18(c)]).

## 5.3 Other results

The set of weights of $\mathfrak{u}^*$ is $\{(1,1), (2,-1), (-1,2)\}$. The weights of $\mathrm{S}^n(\mathfrak{u}^*)$ are formed as sums of $n$ of the weights of $\mathfrak{u}^*$, i.e. the set of weights of $\mathrm{S}^n(\mathfrak{u}^*)$ is

$$\{\ (a+2b-c, a+2c-b)\ \mid\ a+b+c = n,\ \ 0 \leq a, b, c \leq n\ \}.$$

We will say that $(a,b,c)$ is a *generating triple* corresponding to the weight $(a+2b-c, a+2c-b)$.

For all $j$ such that $0 \leq j < n$ and $j \equiv n \bmod 2$, we have

$$s_\alpha \cdot \Big(j, \frac{3n-j}{2}\Big) = \Big(-j-2, \frac{3n+j+2}{2}\Big)$$

where the generating triples for these weights are

$$\Big(\frac{n+j}{2}, 0, \frac{n-j}{2}\Big) \text{ and } \Big(\frac{n-j-2}{2}, 0, \frac{n+j+2}{2}\Big)$$

respectively. Dually, for all $j$ such that $0 \leq j < n$ and $j \equiv n \bmod 2$, we have

$$s_\beta \cdot \Big(\frac{3n-j}{2}, j\Big) = \Big(\frac{3n+j+2}{2}, -j-2\Big)$$

where the generating triples for these weights are

$$\Big(\frac{n+j}{2}, \frac{n-j}{2}, 0\Big) \text{ and } \Big(\frac{n-j-2}{2}, \frac{n+j+2}{2}, 0\Big)$$

respectively. The consequence of these equalities is that the contributions to the Euler characteristic of these weights cancel each other out pairwise. This leads us to the following proposition.

**Proposition 5.13.**

$$\begin{aligned}\{\textit{weights of } \mathrm{S}^n(\mathfrak{u}^*)\} = \ & \{(n,n)\} \\ & \cup \ \{\textit{weights not contributing to the Euler characteristic}\} \\ & \cup \ \{\textit{weights of } \mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{(1,1)}\}.\end{aligned}$$

*Proof.* Suppose $\lambda$ is a weight of $\mathrm{S}^n(\mathfrak{u}^*)$ with corresponding generating triple $(a,b,c)$. So $\lambda = (a+2b-c, a+2c-b)$ and $a+b+c=n$.

If $b=c=0$ then $\lambda = (n,n)$.

If $b=0$ and $c \neq 0$, or if $b \neq 0$ and $c=0$, then $\lambda$ is one of the weights described above that do not contribute to the Euler characteristic.

If $b \neq 0$ and $c \neq 0$ then

$$\lambda = (a+2(b-1)-(c-1), a+2(c-1)-(b-1)) + (1,1)$$

which, it is straightforward to check, is a weight of $\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{(1,1)}$. $\square$

This proposition is improved upon in the following two theorems. Theorem 5.15 is a more general version of Theorem 5.14. It follows as a consequence of Corollary 5.20.

**Theorem 5.14.** *Let $n \in \mathbb{N}$. Then if $p$ is sufficiently large, we have*

$$\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*)) \simeq L(n,n) \oplus \mathrm{H}^0(\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{(1,1)}).$$

**Theorem 5.15.** *Let $\lambda = (x,y) \in X(T)$ be a dominant weight, and let $n \in \mathbb{N}$. Then if $p$ is sufficiently large, we have*

$$\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*) \otimes k_\lambda) \simeq \bigoplus_{i=0}^{x+y} M_i \oplus \mathrm{H}^0(\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{\lambda+(1,1)})$$

*where the $M_i$ are simple modules. The set of all $M_i$ is given in the below table for some small values of $\lambda$.*

| $\lambda$ | $\{M_i\}$ |
|---|---|
| $(0,0)$ | $L(n,n)$ |
| $(0,1)$ | $L(n,n+1), L(n+1,n-1)$ |
| $(1,0)$ | $L(n+1,n), L(n-1,n+1)$ |
| $(1,1)$ | $L(n+1,n+1), L(n-1,n+2), L(n+2,n-1)$ |

## 5.4 Proof of Theorems

The theorems follow by induction (see below). We first show that the modules $\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*)\otimes k_{(x,y)})$ have a good filtration with bounded weights. Then, taking $p$ large, we show that they all have dominant weights in the lowest alcove and so they are semisimple with simple summands described by the filtration multiplicities.

Recall that if $V$ is a $k$-vector space with basis $v_1, \ldots, v_n$ then, putting $V_1 = kv_1$, the natural map between symmetric algebras $\mathrm{S}(V) \to \mathrm{S}(V/V_1)$ is surjective and has kernel $\mathrm{S}(V)v_1$. We get a short exact sequence

$$0 \to \mathrm{S}^{n-1}(V)v_1 \to \mathrm{S}^n(V) \to \mathrm{S}^n(V/V_1) \to 0$$

in each degree $n \geq 1$. If $V$ is a rational $B$-module and $V_1$ is a submodule with weight $\nu$ this gives a short exact sequence of $B$-modules

$$0 \to \mathrm{S}^{n-1}(V) \otimes k_\nu \to \mathrm{S}^n(V) \to \mathrm{S}^n(V/V_1) \to 0.$$

Set $\alpha = (2,-1)$, $\beta = (-1,2)$ and let $P_\alpha$, $P_\beta$ be the corresponding parabolic subgroups. Let $\lambda = (a,b) \in X(T)$. If $a \geq 0$ we put $\nabla_\alpha(\lambda) = \mathrm{ind}_B^{P_\alpha} k_\lambda$, and if $b \geq 0$ we put $\nabla_\beta(\lambda) = \mathrm{ind}_B^{P_\beta} k_\lambda$. The unipotent radical $\mathrm{R}_u(P_\alpha)$ acts trivially on $\nabla_\alpha(\lambda)$ and one sees from $\mathrm{SL}_2$ theory that, for each $n \geq 0$, $\nabla_\alpha(n,n) = \mathrm{S}^n \nabla_\alpha(1,1)$, that is the $n$th symmetric power of the two dimensional module $\nabla_\alpha(1,1)$ with weights $(1,1)$ and $(-1,2)$. Similarly, we have $\nabla_\beta(n,n) = \mathrm{S}^n \nabla_\beta(1,1)$.

Now take $V = \mathfrak{u}^*$. Then $V$ has weights $(1,1)$, $(2,-1)$ and $(-1,2)$ and we have short exact sequences of $B$-modules

$$0 \to k_{(2,-1)} \to V \to \nabla_\alpha(1,1) \to 0 \tag{5.3}$$

and

$$0 \to k_{(-1,2)} \to V \to \nabla_\beta(1,1) \to 0. \tag{5.4}$$

Hence we also get exact sequences of $B$-modules

$$0 \to \mathrm{S}^{n-1} V \otimes k_{(2,-1)} \to \mathrm{S}^n V \to \nabla_\alpha(n,n) \to 0 \tag{5.5}$$

and

$$0 \to \mathrm{S}^{n-1} V \otimes k_{(-1,2)} \to \mathrm{S}^n V \to \nabla_\beta(n,n) \to 0 \tag{5.6}$$

for $n \geq 1$.

Thus, by (5.4), for $n \geq 1$, the module $\mathrm{S}^n V$ has a filtration with sections (from bottom to top) $\mathrm{S}^{n-1} V \otimes k_{(2,-1)}$ and $\nabla_\alpha(n,n)$. Now, applying (5.6) with $n$ replaced by $n-1$ and tensored by $k_{(2,-1)}$, we get the following.

**Lemma 5.16.** *For $n \geq 2$, the module $\mathrm{S}^n V$ has a $B$-module filtration with sections $\mathrm{S}^{n-2} V \otimes k_{(1,1)}$, $\nabla_\beta(n-1, n-1) \otimes k_{(2,-1)}$ and $\nabla_\alpha(n,n)$.*

We shall see that inducing $\mathrm{S}^n V \otimes k_\lambda$, where $\lambda$ is dominant, gives a good filtration with characteristic free multiplicities.

**Lemma 5.17.** *Let $\lambda = (a,b), \mu = (c,d) \in X^+$.*

*(i) The induced module $\operatorname{ind}_B^G(\nabla_\alpha(\lambda)\otimes\mu)$ (respectively, $\operatorname{ind}_B^G(\nabla_\beta(\lambda)\otimes\mu)$) has a good filtration with sections $\nabla(\lambda+\mu)$, $\nabla(\lambda+\mu-\alpha)$, …, $\nabla(\lambda+\mu-m\alpha)$, where $m$ is the minimum of $a$ and $c$ (respectively, of $b$ and $d$).*

*(ii) We have*

$$\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\alpha(\lambda) \otimes \mu) = \mathrm{R}^i \operatorname{ind}_B^G(\nabla_\beta(\lambda) \otimes \mu) = 0$$

*for all $i > 0$.*

*Proof.* We first deal with $\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\alpha(\lambda) \otimes \mu)$. The weights of $\nabla_\alpha(\lambda)$ are $\lambda$, $\lambda - \alpha$, …, $\lambda - a\alpha = (-a, b+a)$. If $a \leq c$ then all the weights of $\nabla_\alpha(\lambda) \otimes \mu$ are dominant, so $\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\alpha(\lambda) \otimes \mu) = 0$ for $i > 0$, by Kempf's Vanishing Theorem, and $\operatorname{ind}_B^G(\nabla_\alpha(\lambda)\otimes\mu)$ has a filtration with sections $\operatorname{ind}_B^G(k_{(a+c,b+d)})$, $\operatorname{ind}_B^G(k_{(a+c,b+d)-\alpha})$, …, $\operatorname{ind}_B^G(k_{(a+c,b+d)-a\alpha})$, that is, $\nabla(\lambda+\mu)$, $\nabla(\lambda+\mu-\alpha)$, …, $\nabla(\lambda+\mu-a\alpha)$.

But, using the factorisation $\operatorname{ind}_B^G = \operatorname{ind}_{P_\alpha}^G \circ \operatorname{ind}_B^{P_\alpha}$, we have

$$\begin{aligned}
\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\alpha(\lambda) \otimes \mu) &= \mathrm{R}^i \operatorname{ind}_{P_\alpha}^G \operatorname{ind}_B^{P_\alpha}(\nabla_\alpha(\lambda) \otimes \mu) \\
&= \mathrm{R}^i \operatorname{ind}_{P_\alpha}^G(\nabla_\alpha(\lambda) \otimes \nabla_\alpha(\mu)) \\
&= \mathrm{R}^i \operatorname{ind}_B^G(\lambda \otimes \nabla_\alpha(\mu))
\end{aligned}$$

so we may interchange $\lambda$ and $\mu$ and thus get the result also for $a > c$. □

*Remark* 5.18. If $\lambda = (a,b) \in X(T)$ and $a = -1$ or $b = -1$, then $\mathrm{R}^i \operatorname{ind}_B^G k_\lambda = 0$ for all $i \geq 0$. (See, for example, [25, II.5.2(b)].)

**Proposition 5.19.** *Let $\lambda \in X^+$ and $n \geq 0$.*

*(i)* $\mathrm{R}^i \operatorname{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) = 0$ *for all $i > 0$.*

*(ii) For $n \geq 1$ we have short exact sequences*

$$0 \to \operatorname{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_{\lambda+\alpha}) \to \operatorname{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) \to \operatorname{ind}_B^G(\nabla_\alpha(n,n) \otimes k_\lambda) \to 0$$

*and*

$$0 \to \operatorname{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_{\lambda+\beta}) \to \operatorname{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) \to \operatorname{ind}_B^G(\nabla_\beta(n,n) \otimes k_\lambda) \to 0$$

*of $G$-modules.*

*Proof.* For $n = 0$ we have (i) by Kempf's Vanishing Theorem. For $n \geq 1$ we have the short exact sequence

$$0 \to \mathrm{S}^{n-1} V \otimes k_{\lambda+\alpha} \to \mathrm{S}^n V \otimes k_\lambda \to \nabla_\alpha(n,n) \otimes k_\lambda \to 0 \tag{5.7}$$

coming from (5.5).

Suppose $n = 1$. Thus we have a short exact sequence

$$0 \to k_{\lambda+\alpha} \to V \otimes k_\lambda \to \nabla_\alpha(1,1) \otimes k_\lambda \to 0.$$

Let $\lambda = (a, b)$. If $b = 0$ then $k_{\lambda+\alpha} = k_{(a+2,-1)}$, and so by Remark 5.18, $\mathrm{R}^i \operatorname{ind}_B^G k_{\lambda+\alpha} = 0$ for all $i \geq 0$. If $b \neq 0$ then $\mathrm{R}^i \operatorname{ind}_B^G k_{\lambda+\alpha}$ has a good filtration for $i = 0$, and is 0 for $i > 0$, by Kempf's Vanishing Theorem. Moreover, we have that $\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\alpha(1,1) \otimes k_\lambda)$ has a good filtration for $i = 0$, and is 0 for $i > 0$, by Lemma 5.17. Hence, from the long exact sequence obtained by inducing the above short exact sequence, we get another short exact sequence:

$$0 \to \operatorname{ind}_B^G(k_{\lambda+\alpha}) \to \operatorname{ind}_B^G(V \otimes k_\lambda) \to \operatorname{ind}_B^G(\nabla_\alpha(1,1) \otimes k_\lambda) \to 0.$$

Now suppose that $n \geq 2$. We need to show that $\mathrm{R}^i \operatorname{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_\lambda)$ is 0 for $i > 0$. By using (5.6), with $n$ replaced by $n-1$ and tensored by $k_{\alpha+\lambda}$, we have the short exact sequence

$$0 \to \mathrm{S}^{n-2} V \otimes k_{(1,1)} \otimes k_\lambda \to \mathrm{S}^{n-1} V \otimes k_{\lambda+\alpha} \to \nabla_\beta(n-1, n-1) \otimes k_\lambda \to 0.$$

It follows by induction that

$$\mathrm{R}^i \operatorname{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_\lambda) = 0$$

for all $i > 0$, since $\mathrm{R}^i \operatorname{ind}_B^G(\nabla_\beta(n-1, n-1) \otimes k_\lambda) = 0$ by Lemma 5.17, and $\mathrm{R}^i \operatorname{ind}_B^G(\mathrm{S}^{n-2} V \otimes k_{(1,1)} \otimes k_\lambda) = 0$ by the induction hypothesis.

Now inducing (5.7) we obtain the short exact sequence

$$0 \to \operatorname{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_{\lambda+\alpha}) \to \operatorname{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) \to \operatorname{ind}_B^G(\nabla_\alpha(n,n) \otimes k_\lambda) \to 0$$

which completes the proof. □

For a module $M$ with a good filtration and $\mu \in X^+$ we write $(M : \nabla(\mu))$ for the filtration multiplicity.

**Corollary 5.20.** *Let $\lambda \in X^+$ and $n \geq 0$.*

*(i) The module $\operatorname{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda)$ has a good filtration.*

(ii) *For $n \geq 2$ the filtration multiplicities in $\mathrm{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda)$ are given by*

$$\begin{aligned}(\mathrm{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) : \nabla(\mu)) = {} & (\mathrm{ind}_B^G(\mathrm{S}^{n-2} V \otimes k_{\lambda+(1,1)}) : \nabla(\mu)) \\ & + (\mathrm{ind}_B^G(\nabla_\alpha(n,n) \otimes k_\lambda) : \nabla(\mu)) \\ & + (\mathrm{ind}_B^G(\nabla_\beta(n-1,n-1) \otimes k_{\lambda+\alpha}) : \nabla(\mu))\end{aligned}$$

*for $\mu \in X^+$.*

*Proof.* (i) For $n = 0$ this is clear. For $n \geq 1$ it follows by induction from Lemma 5.17 and Proposition 5.19 (ii) applied twice, once with $\alpha$ and once with $\beta$.

(ii) Using Proposition 5.19 (ii) twice, we have

$$\begin{aligned}(\mathrm{ind}_B^G(\mathrm{S}^n V \otimes k_\lambda) : \nabla(\mu)) = {} & (\mathrm{ind}_B^G(\mathrm{S}^{n-1} V \otimes k_{\lambda+\alpha}) : \nabla(\mu)) \\ & + (\mathrm{ind}_B^G(\nabla_\alpha(n,n) \otimes k_\lambda) : \nabla(\mu)) \\ = {} & (\mathrm{ind}_B^G(\mathrm{S}^{n-2} V \otimes k_{\lambda+(1,1)}) : \nabla(\mu)) \\ & + (\mathrm{ind}_B^G(\nabla_\alpha(n,n) \otimes k_\lambda) : \nabla(\mu)) \\ & + (\mathrm{ind}_B^G(\nabla_\beta(n-1,n-1) \otimes k_{\lambda+\alpha}) : \nabla(\mu)).\end{aligned}$$

□

*Remark* 5.21. The multiplicities may now be calculated recursively using the above and Lemma 5.17.

*Proof of Theorem 5.14.* Taking $\lambda = 0$ in Corollary 5.20 (i), we see that $\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*))$ has a good filtration. Furthermore, we know it is semisimple with simple summands described by the filtration multiplicities; by Corollary 5.20 (ii), these are

$$\mathrm{H}^0(\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{(1,1)}), \ \ \mathrm{H}^0(\nabla_\alpha(n,n)), \ \text{ and } \ \mathrm{H}^0(\nabla_\beta(n-1,n-1) \otimes k_\alpha).$$

Now by Lemma 5.17 (i), $\mathrm{H}^0(\nabla_\alpha(n,n))$ has a good filtration with one section, $\nabla(n,n)$, whilst $\mathrm{H}^0(\nabla_\beta(n-1,n-1) \otimes k_\alpha)$ is zero by Remark 5.18. Thus $\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*)) \simeq L(n,n) \oplus \mathrm{H}^0(\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{(1,1)})$, and we are done by induction. □

*Proof of Theorem 5.15.* By Corollary 5.20, we know that $\mathrm{H}^0(\mathrm{S}^n(\mathfrak{u}^*) \otimes k_\lambda)$ has a good filtration, and we know it is semisimple with simple summands:

$$\mathrm{H}^0(\mathrm{S}^{n-2}(\mathfrak{u}^*) \otimes k_{\lambda+(1,1)}), \ \ \mathrm{H}^0(\nabla_\alpha(n,n) \otimes k_\lambda), \ \text{ and } \ \mathrm{H}^0(\nabla_\beta(n-1,n-1) \otimes k_{\lambda+\alpha}).$$

Now we may use Lemma 5.17 (i) to check the multiplicities of the last two summands; for instance, when $\lambda = (0,1)$, $\mathrm{H}^0(\nabla_\alpha(n,n) \otimes k_\lambda)$ has just one section, $\nabla(n,n+1)$, and likewise $\mathrm{H}^0(\nabla_\beta(n-1,n-1) \otimes k_{\lambda+\alpha})$ has just $\nabla(n+1,n-1)$. Applying this recursively, we get the statement of the theorem. □

# Bibliography

[1] J. L. Alperin. Diagrams for modules. *J. Pure and Applied Algebra*, 16:111–119, 1980.

[2] H. H. Andersen. The strong linkage principle. *J. reine angew. Math.*, 315:53–59, 1980.

[3] H. H. Andersen. Extensions of modules for algebraic groups. *Amer. J. Math.*, 106:489–504, 1984.

[4] H. H. Andersen. Extensions of simple modules for finite Chevalley groups. *J. Algebra*, 111:388–403, 1987.

[5] H. H. Andersen and J. C. Jantzen. Cohomology of induced representations for algebraic groups. *Math. Ann.*, 269:487–525, 1984.

[6] M. Aschbacher and R. M. Guralnick. Some Applications of the First Cohomology Group. *J. Algebra*, 90:446–460, 1984.

[7] S. Baier, S. Lamzin, and V. Miemietz. Dimensions of Ext-groups of Weyl modules for $GL_2$. arXiv:1305.5947, 2013.

[8] C. P. Bendel, D. K. Nakano, B. J. Parshall, C. Pillen, L. L. Scott, and D. I. Stewart. Bounding cohomology for finite groups and Frobenius kernels. *Algebras and Representation Theory*, 18:739–760, 2015.

[9] C. P. Bendel, D. K. Nakano, and C. Pillen. Extensions for Frobenius kernels. *J. Algebra*, 272:476–511, 2004.

[10] C. P. Bendel, D. K. Nakano, and C. Pillen. Second cohomology groups for Frobenius kernels and related structures. *Adv. Math.*, 209:162–197, 2007.

[11] C. P. Bendel, D. K. Nakano, and C. Pillen. Third cohomology for Frobenius kernels and related structures. In *Proc. Symposia in Pure Math.*, volume 92, pages 81–118. Amer. Math. Soc., 2016.

[12] C. D. Bowman, S. R. Doty, and S. Martin. Decomposition of tensor products of modular irreducible representations for $SL_3$ (With an Appendix by C.M.Ringel). *Int. Electron. J. Algebra*, 9:177–219, 2011.

[13] E. T. Cline. $\mathrm{Ext}^1$ for $\mathrm{SL}_2$. *Comm. Algebra*, 7:107–111, 1979.

[14] E. T. Cline, B. J. Parshall, and L. L. Scott. Reduced standard modules and cohomology. *Trans. Amer. Math. Soc.*, 361:5223–5261, 2009.

[15] A. S. Dzhumadil'daev and S. S. Ibraev. On the third cohomology of algebraic groups of rank two in positive characteristic. *Math. Sbornik*, 205:41–82, 2014.

[16] K. Erdmann. $\mathrm{Ext}^1$ for Weyl modules of $\mathrm{SL}_2(\mathrm{K})$. *Math. Z.*, 218:447–459, 1995.

[17] K. Erdmann, K. C. Hannabuss, and A. E. Parker. Bounding and unbounding higher extensions for $SL_2$. *J. Algebra*, 389:98–118, 2013.

[18] E. M. Friedlander and B. J. Parshall. Cohomology of Lie algebras and algebraic groups. *Amer. J. Math.*, 108:235–253, 1986.

[19] R. M. Guralnick. The dimension of the first cohomology group. In *Lecture Notes in Math.*, volume 1178, pages 94–97. Springer, 1986.

[20] R. M. Guralnick and C. G. Hoffman. The first cohomology group and generation of simple groups. In *Trends Math.*, pages 81–89. Birkhäuser, Basel, 1998.

[21] R. M. Guralnick and P. H. Tiep. First cohomology groups of Chevalley groups in cross characteristic. *Ann. Math.*, 174:543–559, 2011.

[22] D. F. Holt. An interpretation of the cohomology groups $H^n(G, M)$. *J. Algebra*, 60:307–320, 1979.

[23] J. E. Humphreys. *Linear Algebraic Groups*. Springer-Verlag, 1975.

[24] J. C. Jantzen. First cohomology groups for classical Lie algebras. In *Progr. Math.*, volume 95, pages 289–315. Birkhäuser, 1991.

[25] J. C. Jantzen. *Representations of Algebraic Groups*. Amer. Math. Soc., second edition, 2003.

[26] S. Kumar, N. Lauritzen, and J. F. Thomsen. Frobenius splitting of cotangent bundles of flag varieties. *Invent. Math.*, 136:603–621, 1999.

[27] K. M. Lux, N. V. Ngo, and Y. Zhang. Cohomology of $SL_2$ and related structures. arXiv:1508.05534, 2015.

[28] G. M. Malle and D. M. Testerman. *Linear Algebraic Groups and Finite Groups of Lie Type*. Cambridge University Press, 2011.

[29] G. McNinch. The Second Cohomology of Small Irreducible Modules for Simple Algebraic Groups. *Pacific J. Math.*, 204:459–472, 2002.

[30] A. E. Parker. Higher extensions between modules for $SL_2$. *Adv. Math.*, 209:381–405, 2007.

[31] A. E. Parker and D. I. Stewart. First cohomology groups for finite groups of Lie type in defining characteristic. *Bull. London Math. Soc.*, 46:227–238, 2014.

[32] A. E. Parker and D. I. Stewart. Stabilisation of the LHS spectral sequence for algebraic groups. *J. Lie Theory*, 25:807–813, 2015.

[33] B. J. Parshall and L. L. Scott. Bounding Ext for Modules for Algebraic Groups, Finite Groups and Quantum Groups. *Adv. Math.*, 226:2065–2088, 2011.

[34] B. J. Parshall and L. L. Scott. Cohomological growth rates and Kazhdan-Lusztig polynomials. *Israel J. Math.*, 191:85–110, 2012.

[35] B. J. Parshall, L. L. Scott, and D. I. Stewart. Shifted Generic Cohomology. *Compositio Mathematica*, 149:1765–1788, 2013.

[36] J. Rizkallah. On the Cohomology of $SL_2$ with coefficients in a simple module. *Comm. Algebra*, 45:1785–1792, 2016.

[37] L. L. Scott. Some new examples in 1-cohomology. *J. Algebra*, 260:416–425, 2003.

[38] D. I. Stewart. The Second Cohomology of Simple $SL_2$-modules. *Proc. Amer. Math. Soc.*, 138:427–434, 2010.

[39] D. I. Stewart. The Second Cohomology of Simple $SL_3$-modules. *Comm. Algebra*, 40:4702–4716, 2012.

[40] D. I. Stewart. Unbounding Ext. *J. Algebra*, 365:1–11, 2012.

[41] C. B. Wright. Second cohomology groups for algebraic groups and their Frobenius kernels. *J. Algebra*, 330:60–75, 2011.

[42] S. E. B. Yehia. *Extensions of simple modules for the Chevalley groups and its parabolic subgroups.* PhD thesis, University of Warwick, 1982.