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Local models of astrophysical discs

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ABSTRACT

Local models of gaseous accretion discs have been successfully employed for decades to describe an assortment of small scale phenomena, from instabilities and turbulence, to dust dynamics and planet formation. For the most part, they have been derived in a physically motivated but essentially ad hoc fashion, with some of the mathematical assumptions never made explicit nor checked for consistency. This approach is susceptible to error, and it is easy to derive local models that support spurious instabilities or fail to conserve key quantities. In this paper we present rigorous derivations, based on an asympototic ordering, and formulate a hierarchy of local models (incompressible, Boussinesq, and compressible), making clear which is best suited for a particular flow or phenomenon, while spelling out explicitly the assumptions and approximations of each. We also discuss the merits of the anelastic approximation, emphasising that anelastic systems struggle to conserve energy unless strong restrictions are imposed on the flow. The problems encountered by the anelastic approximation are exacerbated by the disk's differential rotation, but also attend non-rotating systems such as stellar interiors. We conclude with a defence of local models and their continued utility in astrophysical research.

Key words: hydrodynamics — methods: analytical — accretion, accretion discs

1 INTRODUCTION

Astrophysical discs exhibit dynamical phenomena on a wide range of scales, from the global (e.g. warps, outbursts, spiral waves, and outflows) to the small-scale (turbulence, planet formation, etc). In order to attack the latter it is convenient, analytically and computationally, to deploy a local model: i.e. to 'zoom in' on a small patch of disc, approximate it as Cartesian, and treat it in isolation of the rest of the system. This technique is commonly employed in the study of celestial mechanics (cf. the Hill equations), galactic dynamics (Goldreich and Lynden-Bell 1965), planetary rings (Wisdom and Tremaine 1988), and gaseous accretion discs (made especially famous by application to the magnetorotational instability, MRI; Hawley et al. 1995).

The derivation of local models for particulate discs is relatively straightforward, but there are subtleties involved when dealing with a gas, and its thermodynamic variables. Depending on the properties of the flow, in particular the sizes of its characteristic lengthscales and Mach number, certain terms in the governing equations are dominant, some negligible; as a consequence, we are led to a number of different local approximations: incompressible, Boussinesq, anelastic, and small and large compressible boxes. Though regularly used, their derivations have been ad hoc for the most part, and though physically motivated it is easy to derive equations that fail to conserve key properties or, even worse, introduce spurious instabilities. These problems arise especially when attempting to incorporate the background thermodynamic gradients, and can be connected to the violation of wave-action conservation.

The main aim of this paper is to highlight these issues while rigorously deriving local approximations that are unambiguously consistent and conservative, hence fit for purpose. The essential assumptions of each model are clearly spelled out so that each may be matched to the appropriate problem. They may then serve as a set of references for researchers in the field. We employ an ordering approach similar in some details to Spiegel and Veronis (1960) and Gough (1969). The different models can then be clearly delineated in terms of a handful of key dimensionless parameters, such as λ/R, λ/H_Z, and $\mathcal{M}$, where λ is a characteristic length scale of the flow, R is disk radius, H_Z is the vertical pressure scale height, and $\mathcal{M}$ is the Mach number of the perturbed flow in the corotating frame.

Starting from a flow in a fully global and fully com-

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pressible disc, we first derive the equations governing the incompressible shearing box, the independent distinguishing features of which are (a) the flow is small-scale, with a characteristic lengthscale much less than the scale height, $\lambda/H_Z \ll 1$, (b) it is subsonic $\mathcal{M} \ll 1$, and (c) the fractional density and pressure perturbations are equally small (note, however, that the density plays no part in the final equations). The Boussinesq box, which we deal with next, also assumes (a) and (b) but instead of (c) lets the fractional density perturbation take significantly larger values than the fractional pressure perturbation (though both must still be $\ll 1$). The last move permits the inclusion of a constant background entropy gradient, and hence buoyancy effects. We show how weak vertical shear may be incorporated consistently, and how potential vorticity manifests in these equations.

Anelastic models generally assume slow flows and small thermodynamic perturbations, as earlier, but permit the flow to range over longer length-scales, up to the scale height, i.e. $\lambda/H_Z \sim 1$. We highlight the inability of this approximation to conserve energy under general conditions, and we derive a consistent set of conservative equations that resemble the so-called 'pseudo-incompressible' approximation (Durran 1989, Vasil et al. 2013), but which requires additional restrictions on the lengthscale of the flow. These additional restrictions issue directly from the strong differential rotation and illustrate the challenges in combining anelasticity and astrophysical disks. Note that even in the absence of strong rotation and shear, as in the stellar context, the anelastic equations remain extremely problematic (see Brown et al. 2012).

Compressible models impose no restriction on the flow speed, and hence the Mach number $\mathcal{M}$ can take any value. We obtain the 'small compressible box' by assuming small scales $\lambda/H_Z \ll 1$, a model that cannot involve any background thermodynamic gradients, and the 'large compressible box', by assuming $\lambda/H_Z \sim 1$, which must include the full vertical structure of the disc. We devote time to exposing the spurious instabilities that appear when the background gradients are included incorrectly, and discuss how they are associated with a breakdown of wave-action conservation.

The paper concludes with a short defence of local models and their importance for research in astrophysical accretion flows. We argue that previous criticisms are overstated, and that global set-ups suffer from problems of a similar or greater magnitude. Our essential point is that, owing to the simpler geometry afforded by local models, researchers can more easily disentangle the salient physical processes and their interconnection, and as a consequence develop a deeper understanding of the overall astrophysical flow. Local model help us work out the conceptual ideas and physical intuition necessary for the interpretation of observations and global simulations, which are usually much more complicated and messy.

The paper is lengthy and technically detailed. In order to ease its readability, each of Sections 3 to 6 are as self-contained as possible and may be read independently. All four section, however, make use of the material in Section 2, in which we present our background global disk (with its key length and timescales) and introduce the thin disk and local approximations, both necessary to derive the equations of any shearing box. Section 3 presents a derivation of the incompressible shearing box, and Section 4 the Boussinesq box, both from this global, compressible starting point. In Section 5, we discuss anelastic models and their problems, while developing a consistent and conservative set of anelastic equations. In Section 6, we present compressible models, purely local ('small box') and vertically stratified ('large box'), emphasising the spurious instabilities that arise if the equations are incorrectly derived. Our conclusions appear in Section 7. For readers who wish to skip the derivations and go straight to the presentation of the final equations associated with each model, the incompressible shearing box equations appear in Section 3.3, the Boussinesq equations in Section 4.3, the consistent anelastic equations in Section 5.2.4, and the compressible equations in Sections 6.1.1 and 6.2.

2 PRELIMINARIES

To begin, we display the equations governing fluid flow in a representative disc alongside its basic global equilibrium. It is upon this quasi-steady equilibrium state that fast small-scale phenomenon (captured by local models) manifests. We subsequently introduce the additional assumptions of a thin disc and small-scales.

2.1 Governing equations

Consider an inviscid astrophysical disc, rotating about a central object. Let us describe it in a rotating cylindrical reference frame with rotation vector $\mathbf{\Omega} = \Omega_0\, \mathbf{e}_z$, where Ω_0 is a constant frequency specified later. We assume that the disk is radially extended, and thus do not consider slender tori or narrow rings, nor their local representations (see for e.g. Goldreich et al. 1986 and Narayan et al. 1987) The equations controlling the flow are

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}, \tag{1}$$

$$\frac{D\mathbf{u}}{Dt} = -2\Omega_0\mathbf{e}_z\times\mathbf{u} - \frac{1}{\rho}\nabla P - \nabla\Phi, \tag{2}$$

$$\frac{DS}{Dt} = \Gamma - \Lambda, \tag{3}$$

where ρ is volumetric mass density, $\mathbf{u}$ is velocity, and P is pressure. The symbol Φ combines the centrifugal potential $-\frac{1}{2}\Omega_0 R^2$, with R cylindrical radius, and the gravitational potential of the star Φ_*. To make life simple we assume that the star possesses a point mass potential. In addition, S denotes the entropy function, Γ represents external heating (from the star or cosmic rays, perhaps), and Λ represents radiative cooling (which may take the form $-\nabla\cdot\mathbf{F}$, where $\mathbf{F}$ is a radiative flux). It is assumed that Γ and Λ are functions of the spatial coordinates, or else of the thermodynamic variables. In what follows, often we combine both cooling and heating into the single function Ξ. Finally, the disc is composed of an ideal gas, so that $P = \mathcal{R}\rho T$, where $\mathcal{R}$ is the gas constant, and T is temperature.

2.2 Equilibrium

A local model's small-scale phenomena take place upon a pre-existing equilibrium state. An assumption, shared by all local models, is that the smaller-scale processes *do not feed back* onto the background equilibrium. The equilibrium remains fixed on the short timescales of interest.

We assume that the disc falls into the steady state

$$\mathbf{u} = \overline{\mathbf{u}} \equiv R\Omega(R,Z)\,\mathbf{e}_\phi, \quad P = \overline{P}(R,Z), \quad \rho = \overline{\rho}(R,Z), \quad (4)$$

where the three functions Ω, $\overline{P}$, and $\overline{\rho}$ may be obtained from the equations:

$$\overline{\mathbf{u}}\cdot\nabla\overline{\mathbf{u}} = -2\Omega_0\mathbf{e}_z\times\overline{\mathbf{u}} - \frac{1}{\overline{\rho}}\nabla\overline{P} - \nabla\Phi, \quad (5)$$

$$\Gamma = \Lambda. \quad (6)$$

Note that Ω is the orbital frequency of the gas in the rotating frame. Because Ω does not appear in the energy balance, we can obtain $\overline{\rho}$ and $\overline{P}$ from the vertical component of Eq. (5) and from Eq. (6). Then the radial component of (5) obtains Ω. For reference, Eqs (5) may be recast as

$$-R(\Omega+\Omega_0)^2 = -\frac{1}{\overline{\rho}}\partial_R\overline{P} - \partial_R\Phi, \quad (7)$$

$$\frac{1}{\overline{\rho}}\partial_Z\overline{P} = -\partial_Z\Phi. \quad (8)$$

We next define three fundamental lengthscales. The equilibrium is taken to vary relatively smoothly with respect to the spatial coordinates and that $\overline{\rho}$, $\overline{P}$, and Ω exhibit well-defined scales of variation. We introduce the radial pressure scale height H_R, and the vertical pressure scale height H_Z. These possess the scalings

$$H_R^{-1} \sim \partial_R \ln\overline{P} \sim \partial_R\ln\overline{\rho}, \quad (9)$$

$$H_Z^{-1} \sim \partial_Z \ln\overline{P} \sim \partial_Z\ln\overline{\rho}. \quad (10)$$

Note that it has been assumed that the density and pressure scale heights are of the same order of magnitude, which is reasonable in most contexts.

An astrophysical disc, perhaps by definition, is rotationally supported, so the scale of Ω's radial variation will be $\sim R$. Its vertical scale of variation we denote by H_Ω, and it satisfies

$$H_\Omega^{-1} \sim \partial_Z \ln\Omega. \quad (11)$$

In fact, this length can be estimated from the thermal wind equation, as shown in the next subsection.

2.3 The local and thin-disc approximations

The equilibrium is disturbed so that $\rho = \overline{\rho} + \rho'$, etc, where a prime denotes a perturbation. The (nonlinear) perturbation equations are

$$\frac{D\rho'}{Dt} = -(\overline{\rho}+\rho')\nabla\cdot\mathbf{u}' - \mathbf{u}'\cdot\nabla\overline{\rho}, \quad (12)$$

$$\frac{D\mathbf{u}'}{Dt} = -\frac{1}{\overline{\rho}+\rho'}\nabla P' + \frac{\nabla\overline{P}}{\overline{\rho}(\overline{\rho}+\rho')}\,\rho' - 2\Omega_0\mathbf{e}_z\times\mathbf{u}' - \mathbf{u}'\cdot\nabla\overline{\mathbf{u}}, \quad (13)$$

$$\frac{DS'}{Dt} = \Gamma - \Lambda - \mathbf{u}'\cdot\nabla\overline{S} \quad (14)$$

where now $D/Dt = \partial_t + (\overline{\mathbf{u}}+\mathbf{u}')\cdot\nabla$, and $\overline{S}$ is the equilibrium entropy distribution.

The perturbations exhibit a characteristic lengthscale λ. We could in fact specify lengthscales in all three directions λ_R, λ_ϕ, and λ_Z, but it suffices to designate only one length, at this stage. We also assume the phenomena exhibits a characteristic velocity scale w, and thus a characteristic timescale of λ/w. The first essential assumption that we make, and which is shared by all local models, is that at any radial location $\lambda \ll R$. The perturbations are small scale relative to radius.

The next step is to exclusively focus upon a specific location in the disc

$$R = R_0, \quad \phi = \phi_0, \quad Z = Z_0. \quad (15)$$

Then we *choose* the rotation frequency of our frame of reference Ω_0 so that $\Omega(R_0, Z_0) = 0$. This means that at this location the disc's orbital frequency is equal to the frame's rotation rate. We introduce spatial variables centred upon this location

$$x = R - R_0, \qquad y = R_0(\phi-\phi_0), \qquad z = Z - Z_0, \quad (16)$$

and suppose that they take values over a range of order λ. Hence $x, y, z \ll R_0$. For the moment we do not specify the relative sizes of λ and Z_0. It is not hard to see that, to leading order in λ/R_0, the del operator simplifies to

$$\nabla \approx \mathbf{e}_x\partial_x + \mathbf{e}_y\partial_y + \mathbf{e}_z\partial_z, \quad (17)$$

where the new locally Cartesian coordinates x, y, z point in the radial, azimuthal, and vertical directions. All the cylindrical terms are subdominant, because they are $\sim \lambda/R_0 \ll 1$ smaller.

The second essential assumption is that the disc is thin. Mathematically, this corresponds to $H_Z \ll R_0$. Because our local box must be located in the bulk of the disc, we assume that $Z_0 \lesssim H_Z$. Furthermore, the new variable z cannot be much greater than H_Z. The assumption of a thin disc is crucial as it permits us to obtain a dimensional estimate on the equilibrium pressure. From Eq. (8), expanding the point mass potential Φ in small Z/R yields

$$\partial_Z\overline{P} = \overline{\rho}\Omega_0^2 Z.$$

Assuming $Z \sim H_Z$ and $\partial_Z \sim 1/H_Z$ gives the following scaling for the equilibrium pressure

$$\overline{P} \sim \overline{\rho} H_Z^2 \Omega_0^2, \quad (18)$$

one that is unique to astrophysical discs. We then recognise that the sound speed of the gas c scales as $H_Z\Omega_0$.

Finally, we take the curl of (5) and write its ϕ-component as

$$R\partial_Z\Omega^2 = \frac{1}{\overline{\rho}^2}\left(\nabla\overline{P}\times\nabla\overline{\rho}\right)\cdot\mathbf{e}_\phi. \quad (19)$$

This is the 'thermal wind equation'. It helps us assess the amount of vertical shear in the equilibrium. Only baroclinic equilibria, in which pressure depends on both density and another thermodynamic variable, permit a nonzero $\partial_Z\Omega$. Using (18) and the characteristic lengthscales introduced in the previous subsection, we obtain the following estimate:

$$H_\Omega \sim \left(\frac{H_R}{H_Z}\right) R_0 > R_0. \quad (20)$$

Symbols	Definitions
$\overline{\rho}$, $\overline{\mathbf{u}}$, $\overline{P}$, $\overline{S}$	Equilibrium quantities
H_R, H_Z	Eq'm radial and vertical scaleheights
H_Ω	Lengthscale of eq'm vertical shear
$q_R = (\partial \ln\Omega/\partial \ln R)_0$	Dimensionless radial shear rate
$q_Z = (R\partial\ln\Omega/\partial Z)_0$,	Dimensionless vertical shear rate
$\Phi_T = \Omega_0^2 q_R x^2$,	Radial part of the tidal potential
$\Phi_Z = \frac{1}{2}\Omega_0^2 z^2$,	Vertical part of the tidal potential
ρ', $\mathbf{u}'$, P', S'	Perturbations
λ, w	Perturbation length and velocity scales
ρ^*, $\mathbf{u}^*$, P^*, S^*	Dimensionless perturbations
$\boldsymbol{\xi}$	Lagrangian displacement
$\epsilon = \lambda/H_z$	Lengthscale ratio
$\mathcal{M} = w/(H_Z\Omega_0)$	Mach number
$\mathrm{Ro} = w/(\lambda\Omega_0)$	Rossby number
$\ell = u_y + 2\Omega_0 x$	Specific angular momentum
$\boldsymbol{\omega} = \nabla\times\mathbf{u} + 2\Omega_0\mathbf{e}_z$	Vorticity
Θ	Potential vorticity

Table 1. Important symbols, characteristic scales, and definitions.

Thus the characteristic vertical lengthscale of the vertical shear is long, greater than R_0 in fact.

In summary, we have introduced two assumptions, shared by all shearing box models. First: radial locality, i.e. that $\lambda \ll R_0$. Second: the thin disc approximation, i.e. $H_Z \ll R_0$. These assumptions lead to a sequence of local models, their distinguishing features resting on the size of (a) the ratio of lengths

$$\epsilon \equiv \lambda/H_Z, \tag{21}$$

(b) the Mach number of the flow

$$\mathcal{M} \equiv \frac{w}{H_Z\Omega_0}, \tag{22}$$

and (c) the size of the fractional perturbations in pressure and density. In Table I we list the most important parameters, scales, and definitions that appear in the following derivations.

3 INCOMPRESSIBLE DYNAMICS

3.1 Distinguishing assumptions

We begin with the derivation of a purely incompressible model, suitable for slow dynamics: shear instability, convection (both radial and vertical), vortices, and the MRI. First, we assume that $\lambda \ll H_Z$, H_R, which means the parameter $\epsilon \ll 1$. If $Z_0 \lesssim H_Z$, we can expand the equilibrium function Ω in Eqs (12)-(13) in small x and z and only retain the leading order terms. This move is acceptable because Ω's characteristic lengthscales of variation are larger than that of the phenomena in question. In fact,

$$\Omega(R_0 + x,\, Z_0 + z) = (\partial_R\Omega)_0 x + (\partial_Z\Omega)_0 z + \mathcal{O}(x^2, z^2), \tag{23}$$

where the subscript the 0 indicates evaluation at the centre of the box. Note that the leading order constant term in the expansion of Ω is zero. Moreover, if we locate the shearing box at the midplane, $Z_0 = 0$, then $(\partial_Z\Omega)_0 = 0$ because of symmetry: in this special case there is no vertical shear in the box. In general, however, we write

$$\overline{\mathbf{u}} = \Omega_0(q_R\, x + q_Z\, z)\mathbf{e}_y, \tag{24}$$

where

$$q_R \equiv (\partial\ln\Omega/\partial\ln R)_0 \quad \text{and} \quad q_Z \equiv R_0(\partial\ln\Omega/\partial Z)_0. \tag{25}$$

It is worth noting at this point that while typically $q_R \sim 1$, we have $q_Z \sim H_Z/H_R$ and so could be considerably smaller (though not necessarily as small as ϵ) and possible to omit. If $Z_0 = 0$, then $q_Z = 0$ exactly.

What of the equilibrium thermodynamic variables $\overline{P}$ and $\overline{\rho}$? Because we are only interested in very short scales λ, much smaller than the variation in $\overline{P}$ and $\overline{\rho}$, we do not expect $\overline{P}$ and $\overline{\rho}$ to change greatly in our box. In particular, $\overline{\rho} \approx \overline{\rho}_0$, true to leading order in ϵ, where $\overline{\rho}_0 = \overline{\rho}(R_0, Z_0)$, a constant. Similarly $\nabla\overline{P} \approx (\nabla\overline{P})_0$, a constant vector.

The second assumption we make concerns the Mach number of the flow. 'Slow' approximations, such as the incompressible, Boussinesq, and anelastic models, set $\mathcal{M} \ll 1$, and are valuable because they filter out sound waves that may pose numerical challenges and prevent analytical progress. We thus have two small ordering parameters ϵ and $\mathcal{M}$, which are instructive to keep separate (though it is possible to equate them). In fact, if we set $\mathcal{M} \sim \epsilon$ then we are unduly restricting the kinds of phenomena described, ensuring their characteristic timescale is $1/\Omega$, and thus pinned to the orbital period.

The third and final assumption deals with the sizes of the thermodynamic perturbations ρ' and P'. They must remain very small compared to the background but stay the same magnitude as each other. We let

$$\frac{\rho'}{\overline{\rho}} \sim \frac{P'}{\overline{P}} \sim \mathcal{M}^2. \tag{26}$$

It might seem strange to force the pressure perturbations to be small in a nominally incompressible model — but what matters most are the pressure gradients, and because the spatial scales of variation are so small, $\lambda \ll H_Z$, the gradients are potentially huge. The pressure perturbation must be scaled appropriately so the pressure term does not blow up.

3.2 Derivation

We are now in a position to derive our equations, by rescaling all the variables and collecting the leading order terms in ϵ and $\mathcal{M}$. The perturbations may be written as

$$\mathbf{u}' = w\mathbf{u}^*, \quad \rho' = \mathcal{M}^2\overline{\rho}_0\rho^*, \quad P' = \mathcal{M}^2\overline{\rho}_0 H_Z^2\Omega_0^2\, P^*, \tag{27}$$

where the star indicates an order 1 dimensionless variable. The spatial variables are $x = \lambda x^*$, etc, and time is $t = t^*(w/\lambda)$. Acknowledging the scaling Eq. (18), the background equilibrium may be non-dimensionalised as

$$(\partial_Z\overline{P})_0 = \overline{\rho}_0 H_Z\Omega_0^2(\partial_Z\overline{P})_0^*, \quad (\partial_R\overline{P})_0 = \overline{\rho}_0\frac{H_Z^2}{H_R}\Omega_0^2(\partial_R\overline{P})_0^*, \tag{28}$$

$$(\partial_Z\overline{\rho})_0 = \overline{\rho}_0\frac{1}{H_Z}(\partial_Z\overline{\rho})_0^*, \quad (\partial_R\overline{\rho})_0 = \overline{\rho}_0\frac{1}{H_R}(\partial_R\overline{\rho})_0^*, \tag{29}$$

and

$$\overline{\mathbf{u}} = \lambda\Omega_0\overline{\mathbf{u}}^* = \lambda\Omega_0(q_R\,x^* + q_Z\,z^*)\mathbf{e}_y. \quad (30)$$

Expressions (27)-(30) are thrown into the continuity equation Eq. (12). There is only a single order 1 term, with respect to both ϵ and $\mathcal{M}$, and the equation reduces to the incompressibility condition

$$\nabla^*\cdot\mathbf{u}^* = 0. \quad (31)$$

On the other hand, the order 1 terms in the momentum equation are

$$\frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* P^* - 2\mathrm{Ro}^{-1}\mathbf{e}_z\times\mathbf{u}^* - \mathrm{Ro}^{-1}(q_R\,u_x^* + q_Z\,u_z^*)\mathbf{e}_y, \quad (32)$$

where $D/Dt^* = \partial_t^* + (\mathbf{u}^* + \mathrm{Ro}^{-1}\overline{\mathbf{u}}^*)\cdot\nabla$. The background pressure gradient is ϵ smaller than the terms above and is hence dropped. Also, the Rossby number

$$\mathrm{Ro} \equiv \frac{w}{\lambda\Omega_0} \quad (33)$$

appears, which quantifies the importance of the differential rotation. When the characteristic frequency of the phenomenon exceeds Ω_0 then Ro increases and the Coriolis and shear terms become subdominant. In fact, in this limit the shearing box is isotropic and homogeneous. This regime manifests on sufficiently short scales if w does not increase concomitantly with λ. Thus it is to be expected that a typical hydrodynamical turbulent cascade ultimately reaches $\mathrm{Ro}\gg 1$ on a small enough scale, flow on these short scales ignorant of shear and rotation. In some applications (e.g. involving well-coupled dust) it may not be necessary to describe the system with a shearing box at all.

3.3 Final incompressible equations

In dimensional form the final system is

$$\frac{D\mathbf{u}'}{Dt} = -\frac{1}{\overline{\rho}_0}\nabla P' - 2\Omega_0\mathbf{e}_z\times\mathbf{u}' - \Omega_0(q_R\,u_x + q_Z\,u_z)\mathbf{e}_y, \quad (34)$$

$$\nabla\cdot\mathbf{u}' = 0, \quad (35)$$

where $D/Dt = \partial_t + \Omega_0(q_R x + q_Z z)\partial_y + \mathbf{u}'\cdot\nabla$. There is no need for the equation of state, nor the entropy equation.

In terms of total, rather than perturbed, variables, the momentum equation may be written as

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\overline{\rho}_0}\nabla P - 2\Omega_0\mathbf{e}_z\times\mathbf{u} - 2\Omega_0^2(xq_R + zq_Z)\mathbf{e}_x, \quad (36)$$

where the last term is the tidal force. Note that if $q_Z \neq 0$ then this tidal force is not conservative (i.e. cannot be written as the gradient of a scalar).

In summary: to obtain these equations we have adopted (a) radial locality, (b) the thin disc approximation, as in all shearing box models, and then assumed that (c) the characteristic scales are much smaller than the scale heights, i.e. $\lambda \ll H_Z, H_R$, (d) the phenomena are very slow, so that the Mach number is $\mathcal{M}\ll 1$, and (e) the fractional thermodynamic perturbations are both $\sim\mathcal{M}^2$. Note that the last two conditions are not enforced by the model: simulations that exhibit extremely strong flows could be violating these assumptions. Runaway MRI channel flows in incompressible boxes are an example that come to mind (Lesaffre et al. 2009). Given, however, that formally the soundspeed is infinity, it is difficult to judge a posteriori whether restriction (d) is violated from the simulation data itself.

3.4 Conservation laws

In order to produce physically acceptable dynamics, the governing equations must conserve certain quantities. Energy, angular momentum, and vorticity are the most important.

3.4.1 Kinetic energy

We derive an equation for the specific kinetic energy by taking the scalar product of Eq. (36) with $\mathbf{u}$ and making repeated use of the incompressibility condition. We find

$$\partial_t(\tfrac{1}{2}u^2) + \nabla\cdot\left[(\tfrac{1}{2}u^2 + h + \tfrac{1}{2}\Omega_0^2 q_R x^2)\mathbf{u}\right] = -\Omega_0^2 q_Z z\,u_x, \quad (37)$$

where $h = P/\rho_0$ is the pseudo-enthalpy. Note the source term on the right hand side: when vertical shear is included in the model, i.e. $q_z\neq 0$, then energy is *not* conserved in the box.

The source term, however, is physical and comes from the rate of doing PdV work in a baroclinic flow. In a barotropic fluid, energy is conserved on a closed stream tube. But this need not be true if the fluid were baroclinic: energy can be input when material flows from a high density region to a low density region and back via a different path, even when that flow is incompressible. In Appendix A, the origin and form of this term is discussed in greater detail.

3.4.2 Angular momentum

Next we turn to angular momentum, which in the shearing sheet is represented by the specific canonical y-momentum

$$\ell \equiv u_y + 2\Omega_0 x. \quad (38)$$

Rearranging the y-component of (36) yields

$$\partial_t\ell + \mathbf{u}\cdot\nabla\ell = -\partial_y P. \quad (39)$$

Thus the angular momentum of a fluid blob can only change due to azimuthal accelerations from the pressure gradient. It follows that angular momentum is materially conserved in axisymmetric flow, and there can be no accretion in this case; fluid blobs (or rather rings) cannot exchange angular momentum because they cannot azimuthally accelerate one another (also see Stone and Balbus 1996). Finally, if we integrate Eq. (39) over the box, use the incompressibility condition, and impose periodic boundaries in y, we observe that the total angular momentum of the system is constant.

3.4.3 Vorticity

Finally, we exhibit the vorticity equation. By taking the curl of (36) and applying standard vector identities, one obtains

$$\partial_t\boldsymbol{\omega} + \mathbf{u}\cdot\nabla\boldsymbol{\omega} - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = -\Omega_0^2 q_Z\mathbf{e}_y, \quad (40)$$

where

$$\boldsymbol{\omega} = \nabla \times \mathbf{u} + 2\Omega_0 \mathbf{e}_z \quad (41)$$

is the vorticity in the shearing box. Note the constant source term on the right side of Eq. (40); it is zero only in the absence of vertical shear. The source term is result of the baroclinicity of the flow, and in fact, is the local manifestation of the $\nabla\overline{\rho} \times \nabla\overline{P}$ term in the thermal wind balance (19).

The existence of a constant injection of vorticity possibly causes problems in simulations of the flow. It comes about essentially because of the local dynamics' inability to react back on the equilibrium conditions. This might be reasonable when dealing with the radial shear, but the vertical shear might get smoothed out effectively on short times by the Goldreich-Schubert-Fricke instability (Goldreich and Schubert 1967, Fricke 1968, Nelson et al. 2013) unless it is forcibly maintained (by powerful stellar radiation, for instance; Barker and Latter 2015).

4 THE BOUSSINESQ APPROXIMATION

4.1 Distinguishing assumptions

This class of model differs from the previous incompressible case in its treatment of the density perturbation. As before, we assume that the phenomena of interest exhibit lengthscales much less than the scale heights of the disk, $\lambda \ll H_Z, H_R$ so that $\epsilon \ll 1$. As a result the background fields $\overline{\mathbf{u}}$, $\overline{P}$ and $\overline{\rho}$ and their derivatives may be expanded in small x and z. The former is hence linear in these variables, taking the form of Eq. (24), while the latter two are constant. In addition, the flow is presumed to be subsonic, so that $\mathcal{M} \ll 1$. However, we assume that $\rho'/\overline{\rho}$ is much larger than $P'/\overline{P}$, though both remain $\ll 1$. More precisely

$$\frac{\rho'}{\overline{\rho}} \sim \frac{\mathcal{M}^2}{\epsilon}, \qquad \frac{P'}{\overline{P}} \sim \mathcal{M}^2, \quad (42)$$

with the additional requirement that $\mathcal{M}^2 \ll \epsilon$. We could just set $\mathcal{M} \sim \epsilon$ but this is unnecessary, and in fact imposes an additional restriction that is undesirable.

4.2 Derivation

As in Section 3.2 we proceed by introducing dimensionless variables. The equilibrium fields are expressed as in Eqs (28)-(30), while the independent variables may be written as $x = \lambda x^*$, etc, and $t = (w/\lambda)t^*$, where a star indicates a dimensionless order 1 quantity. The perturbations are

$$\mathbf{u}' = w\mathbf{u}^*, \;\; \rho' = (\mathcal{M}^2/\epsilon)\overline{\rho}_0\rho^*, \;\; P' = \mathcal{M}^2\overline{\rho}_0 H_Z^2\Omega_0^2\, P^*. \quad (43)$$

Throwing these into the continuity equation yields to order 1,

$$\nabla^* \cdot \mathbf{u}^* = 0, \quad (44)$$

the incompressibility condition again. The momentum equation is more interesting. We obtain

$$\frac{D\mathbf{u}^*}{Dt^*} = -\nabla^* P^* - 2\mathrm{Ro}^{-1}\mathbf{e}_z \times \mathbf{u}^* - \mathrm{Ro}^{-1}(q_R\, u_x^* + q_Z\, u_z^*)\mathbf{e}_y + \left[\frac{H_Z}{H_R}\left(\partial_R\overline{P}\right)_0^*\mathbf{e}_x + (\partial_Z\overline{P})_0^*\mathbf{e}_z\right]\rho^*, \quad (45)$$

where Ro$= w/(\lambda\Omega_0)$ is the Rossby number, and $D/Dt^* = \partial_t^* + (\mathbf{u}^* + \mathrm{Ro}^{-1}\overline{\mathbf{u}}^*)\cdot\nabla$. Note the buoyancy term on the right hand side in square brackets, absent in the incompressible derivation in Section 3.2. It prompts a number of comments.

First, if our sheet is located on the midplane, i.e. $Z_0 = 0$, then $(\partial_Z\overline{P})_0 = 0$ by symmetry and hence there can be no vertical buoyancy contribution.

Second, the radial buoyancy term is multiplied by a factor H_Z/H_R. We have been careful, so far, not to specify explicitly the relative magnitudes of the two scale heights. If $H_R \sim R_0$, as we might expect for a very smooth thin disc, then the radial buoyancy term is ϵ smaller than the other terms and should be dropped. If however, the disc exhibits more abrupt radial structure, so that $H_Z \lesssim H_R$, then the term should be retained. Dead-zone edges, gaps opened by planets, or icelines in protoplanetary discs are examples of structures that could give rise to the latter.

Third, the new buoyancy term brings in the variable ρ^*, in addition to P^* and $\mathbf{u}^*$, and so another equation is required. From the definition of the entropy function $S \propto \ln(P\rho^{-\gamma})$, we have

$$S^* = -\gamma\frac{\rho'}{\overline{\rho}} = -(\mathcal{M}^2/\epsilon)\,\gamma\rho^*, \quad (46)$$

to leading order, according to the scalings (42). Next, in Eq. (14) the equilibrium entropy $\nabla\overline{S}$ is expanded in small x and z, and is, to leading order, an order 1 constant vector. The dominant terms in the entropy equation are then

$$\frac{D\rho^*}{Dt^*} = -\frac{1}{\gamma}\mathrm{Ro}^{-2}\mathbf{u}^*\cdot\left[\frac{H_Z}{H_R}(\partial_R\overline{S})_0\mathbf{e}_x + (\partial_Z\overline{S})_0\mathbf{e}_z\right] + \Xi^*(\rho^*), \quad (47)$$

where the perturbed non-adiabatic contributions have been packaged into the linear function/operator Ξ. An important case is when the external heating does not depend on ρ and radiative cooling can be represented using the diffusion approximation. Then $\Xi(\rho^*) = \eta\nabla^2\rho^*$, with η the thermal diffusivity. An alternative is a cooling law, such as $\Xi \propto -\rho^*$. Whatever form Ξ takes, however, it should be linear. Note again the coefficient H_Z/H_R, which may be small, but the term is retained, as explained earlier.

Finally, the entropy gradient term in the ρ^*-equation (47) is multiplied by the inverse squared Rossby number. In a typical hydrodynamic turbulent cascade, the flow enters the Ro $\gg 1$ regime on sufficiently short scales; then, not only does the differential rotation drop out of the problem, so does the background entropy gradient. As a result, the ρ^* perturbation is controlled by simple advection and the heating/cooling physics embodied in Ξ. Certainly the latter we expect to return ρ^* to zero and the thermal dimension of the problem likely drops out entirely.

4.3 Final Boussinesq equations

When we return to dimensional variables it is convenient to introduce the following notation. Define the radial and vertical buoyancy frequencies by

$$N_R^2 \equiv -\frac{1}{\gamma\overline{\rho}_0}(\partial_R\overline{P})_0(\partial_R\overline{S})_0, \quad N_Z^2 \equiv -\frac{1}{\gamma\overline{\rho}_0}(\partial_Z\overline{P})_0(\partial_Z\overline{S})_0. \tag{48}$$

And now formally set the radial and vertical stratification lengths to

$$\frac{1}{H_R} \equiv \frac{1}{\gamma}(\partial_R\overline{S})_0, \qquad \frac{1}{H_Z} \equiv \frac{1}{\gamma}(\partial_Z\overline{S})_0. \tag{49}$$

Then the dimensional Boussinesq equations may be written as

$$\frac{D\mathbf{u}'}{Dt} = -\frac{1}{\overline{\rho}_0}\nabla P - 2\Omega_0\mathbf{e}_z\times\mathbf{u} - \Omega_0(q_R\,u_x + q_Z\,u_z)\mathbf{e}_y + \left(H_R N_R^2\mathbf{e}_x + H_Z N_Z^2\mathbf{e}_z\right)\left(\frac{\rho'}{\overline{\rho}_0}\right), \tag{50}$$

$$\nabla\cdot\mathbf{u}' = 0, \tag{51}$$

$$\frac{D}{Dt}\left(\frac{\rho'}{\overline{\rho}_0}\right) = \frac{1}{H_R}u_x' + \frac{1}{H_Z}u_z' + \Xi[\rho']. \tag{52}$$

Usually, we choose stratification in one direction only (either radial or vertical), and then it is convenient to introduce the buoyancy variable $\theta = H\rho'/\overline{\rho}_0$ (with units of length), where H can be either H_R or H_Z. As a consequence, the system only depends on the buoyancy frequency and the rotation law.

The momentum equation in terms of total, rather than perturbed, velocity, is

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\overline{\rho}_0}\nabla P - 2\Omega_0\mathbf{e}_z\times\mathbf{u} - 2\Omega_0^2(xq_R + zq_Z)\mathbf{e}_x + \left(H_R N_R^2\mathbf{e}_x + H_Z N_Z^2\mathbf{e}_z\right)\left(\frac{\rho'}{\overline{\rho}_0}\right). \tag{53}$$

Finally, if we want the temperature perturbation then we turn to the equation of state, which to leading order gives $\rho'/\overline{\rho}_0 = -T'/\overline{T}_0$.

In summary, the Boussinesq approximation requires that $\epsilon \ll 1$ and $\mathcal{M} \ll 1$, as in the incompressible model, but it differs in the permitted magnitude of the fractional density perturbation. It assumes $P'/\overline{P} \ll \rho'/\overline{\rho} \ll 1$. The larger density fluctuation gives rise to buoyancy terms in both the radial and vertical direction. Note that if the box is placed at the midplane there is no vertical buoyancy force. Note also that the radial buoyancy force is of order H_Z/H_R smaller than the other terms and could be neglected in certain circumstances. The equations do not guarantee that ρ' remains small in the way defined: it is possible for the system to evolve away from its domain of validity, just as in the incompressible box.

Finally, these equations are what Umurhan and Regev (2004, 2008) refer to as the 'small shearing box', which they derive in an alternative manner. Because the equation for their buoyancy variable arises from the continuity equation, however, their system cannot incorporate background entropy gradients (hence convection) nor diabatic effects such as cooling or thermal diffusion, and is hence far more restrictive.

4.4 Conservation laws

As in Section 3.1, we derive a number of conservation laws that the flow obeys. To make life simpler, we set $q_Z = 0$ and assume stratification in only one direction, the z direction. Thus $N_R = 0$. The more general case of x and z stratification can be found in Appendix B. In addition, the fluid is taken to be adiabatic so that $\Xi = 0$. Finally, we do not treat angular momentum, as the result is identical to that appearing in the incompressible case (Section 3.4.2).

4.4.1 Energy

By taking scalar products and manipulating, we obtain the energy result

$$\partial_t(\tfrac{1}{2}u^2 + \tfrac{1}{2}N_Z^2\theta^2) + \nabla\cdot\left[(\tfrac{1}{2}u^2 + h + \tfrac{1}{2}\Omega_0^2x^2q_R + \tfrac{1}{2}N_Z^2\theta^2)\mathbf{u}\right] = 0. \tag{54}$$

The specific 'thermal energy' in the Boussinesq box is thus $\frac{1}{2}N_Z^2\theta^2$.

4.4.2 Potential vorticity

We introduce the convenient 'total' buoyancy variable $\theta_z = H_Z\rho'/\overline{\rho}_0 + z$, which transforms the entropy equation into

$$\frac{D\theta_z}{Dt} = 0. \tag{55}$$

Taking the curl of (53) obtains an equation for the vorticity in the shearing box:

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} - N_Z^2\nabla\theta_z\times\mathbf{e}_z\,, \tag{56}$$

where $\boldsymbol{\omega}$ is the vorticity, defined in Eq. (41). This equation is saying that vorticity can be generated by the buoyancy term, but only in a direction perpendicular to both $\mathbf{e}_z$ (the direction of the background entropy gradient) and to the gradient in θ_z, which suggest ways to construct conserved quantities.

We take the inner product of (56) with $\mathbf{e}_z$ and obtain:

$$\frac{D\omega_z}{Dt} = \boldsymbol{\omega}\cdot\nabla u_z,$$

which, with incompressibility, can then be transformed into the conservation law:

$$\frac{\partial\omega_z}{\partial t} + \nabla\cdot(\omega_z\,\mathbf{u} - u_z\,\boldsymbol{\omega}) = 0. \tag{57}$$

The component of the vorticity in the direction of the stratification is always conserved.

We now consider the second direction and take the inner product of (56) with $\nabla\theta_z$, giving

$$\nabla\theta_z\cdot\frac{D\boldsymbol{\omega}}{Dt} = \nabla\theta_z\cdot\left[(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\right]. \tag{58}$$

Next we take the gradient of (55) and find

$$\frac{D\nabla\theta_z}{Dt} = -(\nabla\mathbf{u})\cdot\nabla\theta_z. \tag{59}$$

Putting these two together obtains a second conservation law:

$$\frac{\partial\Theta}{\partial t} + \nabla\cdot(\Theta\,\mathbf{u}) = 0, \tag{60}$$

where the conserved quantity is defined by $\Theta \equiv \boldsymbol{\omega}\cdot\nabla\theta_z$. This we regard as the *potential vorticity* in the Boussinesq shearing box. Moreover, because of incompressibility we have $D\Theta/Dt = 0$, and Θ is materially conserved. Of course, Eq. (60) is nothing but Ertel's theorem in the context of Boussinesq hydrodynamics (Müller, 1995).

Note that a conservation law is possible even when there is some kind of 'frictional force', $\mathbf{G}$, in (53) (see Haynes and McIntyre 1987). This could be viscosity or hyperviscosity. The vorticity equation now picks up a term $\nabla\times\mathbf{G}$ on its right hand side. Taking the inner product of the new term with $\nabla\rho$ yields

$$\nabla\rho\cdot(\nabla\times\mathbf{G}) = \nabla\cdot(\mathbf{G}\times\nabla\rho), \tag{61}$$

where we have used the fact that a curl of a gradient is zero. The modified conservation law for Θ is then

$$\frac{\partial\Theta}{\partial t} + \nabla\cdot(\Theta\,\mathbf{u} + \mathbf{G}\times\nabla\rho) = 0. \tag{62}$$

5 THE ANELASTIC APPROXIMATION

The next model in our hierarchy still requires the motions to be slow and the Mach number and thermodynamic perturbations small. However, it extends the size of the box to cover the disc's vertical scale height. In its strongest form it also permits λ to reach H_Z. This is the anelastic model (Ogura and Phillips 1962, Gough 1969), used with some success in modelling solar convection, though less frequently in the disc context (see Barranco and Marcus 2005, 2006). Basically, an anelastic shearing box is a vertically stratified shearing box in which the sound waves and other compressible dynamics have been filtered out.

As has been discussed elsewhere, it is not straightforward to formally derive an anelastic model that satisfies the conservation laws one would want it to. This can lead to the advent of spurious instabilities, amongst other problems (e.g. Bannon 1996, Brown et al. 2012, Vasil et al. 2013). One response has been to make ad hoc tweaks to the derived equations in order that the conservation laws are preserved. Of course, the relationship of the new equations to the original set is unclear, and even then the new systems are not conservative in general.

In this section we revisit the problems the analestic model faces. We then show that anelastic models with additional restrictions can be derived that possess the correct properties.

5.1 Classical anelastic equations

In this section we demonstrate how to derive the 'classical anelastic equations' as presented in Gough (1969) but in the context of an astrophysical disc (not a star). The model permits the characteristic lengthscale of phenomena to reach the scale height, i.e. $\lambda \sim H_Z$ (thus $\epsilon\sim 1$), but for simplicity we set $H_Z \ll H_R$. The fractional perturbations in density and pressure are assumed small, so that

$$\frac{\rho'}{\overline{\rho}} \sim \frac{P'}{\overline{P}} \sim \mathcal{M}^2 \ll 1, \tag{63}$$

where we assume that the Mach number of the flow is also small.

5.1.1 Derivation and governing equations

The first thing to note is that the background thermodynamic variables $\overline{\rho}$ and $\overline{P}$ can no longer be expanded in z and then truncated at leading order. They may still, however, be expanded and truncated in small x, as we assume that $\lambda \ll H_R$. In contrast, the rotation profile Ω may be expanded in z because its characteristic scale of variation is $\sim (H_R/H_Z)R_0 \gg \lambda$. The natural location to anchor an anelastic shearing box is at the midplane, and thus $Z_0 = 0$. But then this means $(\partial_z\Omega)_0 = 0$ and vertical shear drops out of the problem; the quadratic terms in z are at most $(H_Z/R_0)^2$ smaller than the leading order terms. To retain vertical shear, additional scaling assumptions are required such as radial geostrophic balance (see Nelson et al. 2013).

We now rescale the variables. First, $x = H_Z x^*$, etc, and $t = (H_Z/w)t^*$, and the perturbations follow

$$\mathbf{u}' = w\mathbf{u}^*, \quad \rho' = \mathcal{M}^2\overline{\rho}_0\rho^* \quad P' = \mathcal{M}^2\overline{\rho}_0 H_Z^2\Omega_0^2 P^*, \tag{64}$$

where now $\overline{\rho}_0$ should be understood as the equilibrium density at the midplane $Z = Z_0 = 0$ and at $R = R_0$. The equilibrium quantities are written as

$$\overline{\rho} = \overline{\rho}_0\overline{\rho}^*(z), \quad \overline{P} = \rho_0 H_Z^2\Omega_0^2\overline{P}^*(z), \tag{65}$$

$$\partial_R\overline{\rho} = \overline{\rho}_0\frac{1}{H_R}(\partial_R\overline{\rho})^*(z), \quad \partial_R\overline{P} = \overline{\rho}_0\frac{H_Z^2}{H_R}(\partial_R P)^*(z), \tag{66}$$

where it is understood that the starred quantities are the dimensionless leading-order terms in an expansion in x around $R = R_0$. They do not depend on x but they do depend on z, as indicated. Finally, $\overline{\mathbf{u}} = \lambda\Omega_0 q_R x^*\mathbf{e}_y$, where q_R is defined in (25).

These scaling are substituted into the governing equations. The continuity equation at order 1 becomes

$$\nabla^*\cdot(\overline{\rho}^*\mathbf{u}^*) = 0. \tag{67}$$

The other terms are either $\mathcal{M}^2$ or H_Z/H_R smaller. The momentum equation to leading order is

$$\frac{D\mathbf{u}^*}{Dt^*} = -\frac{1}{\overline{\rho}^*}\nabla^* P^* + \frac{\partial_z^*\overline{P}^*}{\overline{\rho}^*}\left(\frac{\rho^*}{\overline{\rho}^*}\right)\mathbf{e}_z - 2\mathrm{Ro}^{-1}\mathbf{e}_z\times\mathbf{u}^* - \mathrm{Ro}^{-1}q_R\,u_x^*\mathbf{e}_y, \tag{68}$$

where Ro$= w/(\lambda\Omega_0)$ is the Rossby number. Again we have dropped a term a factor H_Z/H_R smaller than the others (the radial gradient of the background pressure). Also, $(\partial_z^*\overline{P}^*/\overline{\rho}^*)$ may be replaced by z^*, on using the vertical hydrostatic balance equation.

The entropy equation can be tackled by first linearising S in P^* and ρ^*, which sets the entropy perturbation as

$$S^* = \frac{P^*}{\overline{P}^*} - \gamma\frac{\rho^*}{\overline{\rho}^*}. \tag{69}$$

Likewise, the heating and cooling terms can be expanded. The resulting entropy equation is written in terms of density and pressure or, more compactly, as

$$\frac{DS^*}{Dt^*} = -u_z^*\partial_z^*\overline{S}^* + (\partial_\rho\Xi)^*\rho^* + (\partial_P\Xi)^*P^*, \tag{70}$$

where we have packaged heating and cooling into the single function $\Xi(\rho, P)$. Returning to total velocity variables and putting things in dimensional form, we get the set:

$$\nabla \cdot (\overline{\rho}\mathbf{u}) = 0, \tag{71}$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\overline{\rho}}\nabla^* P' - z\Omega_0^2 \left(\frac{\rho'}{\overline{\rho}}\right) \mathbf{e}_z - 2\Omega_0 \mathbf{e}_z \times \mathbf{u} - \nabla \Phi_T, \tag{72}$$

$$\frac{DS'}{Dt} = -u_z \partial_z \overline{S} + (\partial_\rho \Xi)\rho' + (\partial_P \Xi) P', \tag{73}$$

$$S' = \frac{P'}{\overline{P}} - \gamma \frac{\rho'}{\overline{\rho}}. \tag{74}$$

where $\Phi_T = \Omega_0^2 q_R x^2$.

5.1.2 *Conservation issues*

As has been pointed our for some time, the anelastic approximation struggles with conserving energy. See Brown et al. (2012) for some discussion of this issue. Certainly, as they stand, equations (71)-(74) do not conserve total energy.

First by taking the scalar product of (72) with $\mathbf{u}$, one obtains an equation for the kinetic energy. With a bit of manipulation this is

$$\begin{aligned}\partial_t(\tfrac{1}{2}\overline{\rho}u^2) + \nabla \cdot \left[\mathbf{u}\left(\tfrac{1}{2}\overline{\rho}u^2 + \overline{\rho}\Phi_T + P'\right)\right] &= -P'\nabla \cdot \mathbf{u} + \frac{\rho'}{\overline{\rho}}\mathbf{u} \cdot \nabla \overline{P} \\ &= -P'\overline{P}^{-1/\gamma}\nabla \cdot (\overline{P}^{1/\gamma}\mathbf{u}) - \frac{S'}{\gamma}\mathbf{u} \cdot \nabla \overline{P}. \end{aligned} \tag{75}$$

For demonstration purposes, we assume that the background state is barotropic so that $\overline{P} = \overline{P}(\overline{S})$ and thus $\nabla \overline{P} = (d\overline{P}/d\overline{S})\nabla \overline{S}$. Using this and (74) we obtain the total energy equation

$$\begin{aligned}&\partial_t(\tfrac{1}{2}\overline{\rho}u^2 + \mathcal{V}) + \nabla \cdot \left[\mathbf{u}\left(\tfrac{1}{2}\overline{\rho}u^2 + \overline{\rho}\Phi_T + \mathcal{V} + P'\right)\right] \\ &= \frac{-P'}{\overline{P}^{1/\gamma}}\nabla \cdot (\overline{P}^{1/\gamma}\mathbf{u}) - \frac{S'}{\gamma}\frac{d\overline{P}}{d\overline{S}}\left[(\partial_\rho \Xi)\rho' + (\partial_P \Xi)P'\right] \\ &\quad - \frac{\overline{\rho}S'^2}{2\gamma}\frac{d}{d\overline{S}}\left(\frac{1}{\overline{\rho}}\frac{d\overline{P}}{d\overline{S}}\right)\mathbf{u} \cdot \nabla \overline{S}. \end{aligned} \tag{76}$$

Here $\mathcal{V} \equiv -S'^2(d\overline{P}/d\overline{S})/(2\gamma)$ plays the role of thermal energy. We remark that for linear adiabatic perturbations $S' = -\xi_z d\overline{S}/dz$, where ξ_z is the vertical component of the Lagrangian displacement, and accordingly

$$\mathcal{V} = \frac{1}{2}\overline{\rho}\xi_z^2 N_Z^2, \tag{77}$$

which being a quadratic form in the perturbation, represents the potential energy associated with linear buoyant motions.

Each of the three terms on the right hand side of (76) destroys conservation. The first term is the most serious as it is non-zero when the background entropy gradient is non-zero, and thus only vanishes for homentropic equilibria. The term is associated with a failure to conserve energy properly even for adiabatic linear waves, and thus wave-action conservation for those waves is incorrectly represented. The second term vanishes for adiabatic motions, and is hence unproblematic, while the third term is third order in the amplitude of the perturbations. This third term does not affect energy conservation for linear perturbations but could cause departures on a time scale that scales inversely with the amplitude of those perturbations.

Similar problems afflict most formulations of the anelastic equations. For example, the anelastic scheme introduced by Bannon (1996) and employed by Barranco & Marcus (2005, 2006), replaces the entropy perturbation Eq. (69) with an ad hoc prescription that is only strictly true for a vertically homentropic background. Furthermore, they show even then that energy conservation only holds in the special case of an *isothermal* background. We are aware of no anelastic system that is conservative in general. If indeed conservative anelastic models are confined to isothermality, their utility is greatly reduced; they can no longer reliably describe convection, nor the influence of stratification on wave propagation, dynamo action, etc.

In the next subsection, we derive a set of conservative anelastic equations that retains diabaticity. The price to be paid, however, is a restriction on the characteristic wavelength of the phenomena, which must be significantly less than H_Z (as in the Boussinesq approximation), yet we permit the flow to range over a domain of size H_Z. Physical problems well suited to this model include: wave packets propagating upward from the disc midplane to the surface, where they may suffer refraction, wave channelling, or breaking (Bate et al. 2002), and small-scale disc turbulence localised to certain altitudes, such as MRI in the disc surface (Fleming and Stone 2003), or convection in a limited range of convectively unstable layers (Stone and Balbus 1996). The system derived below bears a close similarity to the 'pseudo-incompressible' equations, first presented by Durran (1989) in the context of atmospheric dynamics. We, however, must enforce the additional lengthscale restriction because of the strong differential rotation exhibited by astrophysical disks (absent in most atmospheric and planetary applications). This dynamical feature is uniquely dominant in disks, and presents a considerable obstacle to the derivation of consistent anelastic models in that context.

5.2 A conservative anelastic model

We assume that the phenomena are spread across a domain of spatial extent $\sim H_Z$, meaning that $-\mathcal{O}(H_Z) < (x, y, z) < \mathcal{O}(H_Z)$. However, we impose the condition that the characteristic radial and vertical length scales of the phenomena under study are much smaller, so that $\lambda \ll H_Z$, or in other words $\epsilon \ll 1$. The scale of these motions in the direction of shear, y, on the other hand, we let equal the vertical scale height which makes the motions almost axisymmetric (the tight-winding approximation). It is necessary to enforce short scales, radially at the very least, because across $\sim H_Z$ the background shear velocity is supersonic, and large-scale disturbances rapidly wind up into smaller scale structures. A related issue is the scaling of the background pressure, Eq. (18), which differs from that appearing in planetary and stellar contexts.

In contrast to the classical anelastic approximation, we scale the fractional thermodynamic perturbations according to

$$\frac{\rho'}{\overline{\rho}} \sim \epsilon, \qquad \frac{P'}{\overline{P}} \sim \mathcal{M}^2. \tag{78}$$

With this scaling $P'/\overline{P} \sim \epsilon^2 \mathrm{Ro}^2$, so that for Rossby numbers of order unity we are assured $P'/\overline{P} \ll \rho'/\overline{\rho}$.

We introduce dimensionless independent variables, $x = \lambda x^*$, $z = \lambda z^*$, and $t = (\lambda/w)t^*$, and scale the dependent variables through

$$\mathbf{u}' = w\mathbf{u}^*, \quad \rho' = \epsilon\overline{\rho}_0\rho^*, \quad P' = \mathcal{M}^2\overline{\rho}_0 H_Z^2\Omega^2 P^*, \tag{79}$$

where now $\overline{\rho}_0$ should be understood as the equilibrium density at the midplane and at $R = R_0$. The equilibrium quantities are written as earlier in Eqs (65) and (66).

5.2.1 *The continuity equation*

These scaling are inserted into the continuity equation which becomes, without approximation,

$$\epsilon\left(\frac{\partial\rho^*}{\partial t^*} + \frac{q_R x}{\mathrm{Ro}}\frac{\partial\rho^*}{\partial y} + \nabla^*\cdot(\rho^*\mathbf{u}^*)\right) + \nabla^*\cdot(\overline{\rho}^*\mathbf{u}^*) = 0. \tag{80}$$

Here $\nabla^* \equiv \mathbf{e}_x\partial/\partial x^* + \mathbf{e}_z\partial/\partial z^* + \mathbf{e}_y\lambda\partial/\partial y$. We have been careful to retain dimensional x where it appears in equation (80) because the radial size of the domain may greatly exceed λ. In addition, the y derivative in the del operator is kept dimensional for the moment. Note that the term proportional to ϵ in Eq. (80) is small and one might expect to be able to neglect it. This is appropriate if disturbances with scales $\sim \lambda$ are considered in a domain of comparable size. However, if we wish to consider disturbances propagating over scales $H_Z \gg \lambda$, doing so may be invalid and this term is retained for now. The contribution $\mathbf{e}_y\lambda\partial/\partial y$ in ∇^* can be argued to be small for disturbances of scale H_Z in the y direction. However, on account of the second term in (80), the dependence on y may not be neglected except for axisymmetric disturbances so we shall also retain this term.

By making use of the identity

$$\nabla^*\cdot(\rho^*\mathbf{u}^*) = \frac{\rho^*}{\overline{\rho}^*}\nabla^*\cdot(\overline{\rho}^*\mathbf{u}^*) + \overline{\rho}^*\mathbf{u}^*\cdot\nabla^*\left(\frac{\rho^*}{\overline{\rho}^*}\right) \tag{81}$$

equation (80) may be rewritten in the form

$$\frac{\lambda\overline{\rho}^2}{\rho\overline{\rho}_0 H_Z}\left(\frac{\partial}{\partial t^*} + \frac{q_R x}{\mathrm{Ro}}\frac{\partial}{\partial y} + \mathbf{u}^*\cdot\nabla^*\right)\left(\frac{\rho^*}{\overline{\rho}^*}\right) + \nabla^*\cdot(\overline{\rho}^*\mathbf{u}^*) = 0, \tag{82}$$

where we recall that $\rho = (\epsilon\rho^* + \overline{\rho}^*)\overline{\rho}_0$.

5.2.2 *The entropy equation*

We begin by recognising that, for a simple ideal gas, the density, pressure and entropy are related by

$$\rho = P^{1/\gamma}\exp(-S/\gamma). \tag{83}$$

Writing $S = \overline{S} + S'$, where $\overline{S}$ is the equilibrium entropy and S' is the perturbation, we have

$$1 + \frac{\rho'}{\overline{\rho}} = \left(\frac{\overline{P} + P'}{\overline{P}}\right)^{1/\gamma}\exp(-S'/\gamma). \tag{84}$$

In our ordering scheme the pressure perturbation is of higher order than the relative density perturbation. For this reason it will be neglected. The removal of pressure fluctuations in this way filters out sound waves and thus leads us to an anelastic approximation. Adopting dimensionless variables, the entropy perturbation is then related to the density perturbation through

$$\exp(-S^*/\gamma) = 1 + \epsilon\frac{\rho^*}{\overline{\rho}^*}, \tag{85}$$

where we have written $S^* \equiv S'$.

Neglecting the pressure perturbation therein, the heating/cooling term is expressed in terms of $\rho^*/\overline{\rho}^*$ only. The entropy equation will then only contain the density perturbation, taking the form

$$\left(\frac{\partial}{\partial t^*} + \frac{q_R x}{\mathrm{Ro}}\frac{\partial}{\partial y} + \mathbf{u}^*\cdot\nabla^*\right)S^* = -u_z^*\partial_z^*\overline{S} + (\partial_\rho\Xi)^*\rho^*. \tag{86}$$

We emphasise that the only approximations made up to now are the neglect of the pressure perturbation together with the linearization of the heating/cooling term . We go on to use (85) to eliminate ρ^* in equation (82); the latter then takes the form

$$\frac{\overline{\rho}}{\gamma\overline{\rho}_0}\left(\frac{\partial}{\partial t^*} + \frac{q_R x}{\mathrm{Ro}}\frac{\partial}{\partial y} + \mathbf{u}^*\cdot\nabla^*\right)S^* = \nabla^*\cdot(\overline{\rho}^*\mathbf{u}^*). \tag{87}$$

We remark that, on account of the smallness of the neglected pressure perturbation on the left hand side, the above equation incorporates all corrections of order ϵ to the dominant term on the right hand side.

5.2.3 *The momentum equation*

The only approximation made in the momentum equation is the neglect of the pressure perturbation except where it appears as a gradient. We shall work directly with the unscaled nonlinear equation, which takes the form

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P - 2\rho\Omega_0\mathbf{e}_z\times\mathbf{u} - \rho\nabla\Phi, \tag{88}$$

where the potential $\Phi = \Phi_Z + \Phi_T$, with $\Phi_Z = \Omega_0^2 z^2/2$ and $\Phi_T = q_R\Omega_0^2 x^2$. We set $P = \overline{P} + P'$ with $d\overline{P}/dz = \overline{\rho}d\Phi_Z/dz$ which is just the condition of vertical hydrostatic equilibrium. Equation (88) then becomes

$$\rho\frac{D\mathbf{u}}{Dt} = -\nabla P' - 2\rho\Omega_0\mathbf{e}_z\times\mathbf{u} - \rho\nabla\Phi_T - (\rho - \overline{\rho})\nabla\Phi_Z, \tag{89}$$

Anticipating what is needed to obtain a system that conserves energy, we *modify* equation (89) so that it becomes

$$\rho\frac{D\mathbf{u}}{Dt} = -\overline{P}^{1/\gamma}\nabla\left(\frac{P'}{\overline{P}^{1/\gamma}}\right) - 2\rho\Omega_0\mathbf{e}_z\times\mathbf{u} - \rho\nabla\Phi_T - (\rho - \overline{\rho})\nabla\Phi_Z. \tag{90}$$

Note that (89) and (90) differ by a term on the RHS that is $\propto P'$ and is of order ϵ smaller than the term involving the gradient of P'. Naively, the two equations may be considered identical, to leading order. An objection one may raise, however, is that this procedure permits the addition of arbitrary terms of the same order and we end up with a set of equations lacking uniqueness. This is not a problem here. The term added to the right hand side is

$$\frac{P'}{\gamma\overline{P}}\nabla\overline{P} = -\frac{P'\overline{\rho}}{\gamma\overline{P}}\nabla\Phi_Z, \tag{91}$$

which can be viewed as supplying an additional contribution to $\rho - \overline{\rho}$. This contribution turns out to be $\propto P'$ and was in fact dropped when approximating equation (84) by (85). The procedure hence reincorporates this contribution correct to linear order in the density and pressure perturbations. This justifies the modification.

5.2.4 *Final anelastic governing equations*

Putting the equations in dimensional form and eliminating S' from (86) and (87), obtains the set:

$$\nabla\cdot(\overline{\rho}\mathbf{u}) = -\frac{\overline{\rho}}{\gamma}\left(u_z\partial_z\overline{S} - (\partial_\rho\Xi)(\rho - \overline{\rho})\right), \tag{92}$$

$$\rho\frac{D\mathbf{u}}{Dt} = -\overline{P}^{1/\gamma}\nabla\left(\frac{P'}{\overline{P}^{1/\gamma}}\right) - 2\rho\Omega_0\mathbf{e}_z\times\mathbf{u} - \rho\nabla\Phi_T - (\rho-\overline{\rho})\nabla\Phi_Z. \tag{93}$$

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}. \tag{94}$$

These yield five equations for the three velocity components, ρ and P'. The pressure perturbation, P', is determined after applying (92) as a constraint condition on the velocities in the same way as would be done in an incompressible model. In this case the constraint (92) replaces the condition $\nabla\cdot\mathbf{u} = 0$ in the incompressible case, and $\nabla\cdot(\overline{\rho}\mathbf{u}) = 0$ in the classical anelastic model.

An important aspect of the new equations is that they yield conservation laws for entropy, potential vorticity and total energy when heating and/or cooling is absent. In this case, the constraint equation (92) becomes

$$\nabla\cdot(\overline{P}^{1/\gamma}\mathbf{u}) = 0, \tag{95}$$

which is in a form familiar from the pseudo-incompressible approximation (Durran 1989, Vasil et al. 2013).

5.2.5 *Conservation of energy*

Taking the scalar product of (90) with $\mathbf{u}$, while making use of (95) and vertical hydrostatic equilibrium, gives us the conservation law for the energy of the system, in the form

$$\partial_t\left(\rho\left(\tfrac{1}{2}u^2 + \Phi_T + \Phi_Z + \frac{\overline{P}}{\rho(\gamma-1)}\right)\right) + \nabla\cdot\left[\mathbf{u}\left(\tfrac{1}{2}\rho u^2 + \rho(\Phi_T+\Phi_Z) + P' + \frac{\gamma\overline{P}}{\gamma-1}\right)\right] = 0. \tag{96}$$

The quantity $\overline{P}/[(\gamma-1)\rho]$ on the left hand plays the role of the internal energy per unit mass. It involves $\overline{P}$ rather than the usual P on account of the neglect of P' in the entropy equation. Owing to the cancellation of ρ it ultimately makes no contribution. However, we have included it in order to relate to the general case.

5.2.6 *Conservation of entropy*

Combining (92) with (95) we obtain

$$\frac{D}{Dt}\left(\frac{\rho}{\overline{P}^{1/\gamma}}\right) = 0. \tag{97}$$

This is a statement of the conservation of entropy. But note that $\overline{P}^{1/\gamma}$ occurs rather than the expected $P^{1/\gamma}$. This is because the Eulerian pressure perturbation is assumed to be negligible in the anelastic model.

5.2.7 *Conservation of potential vorticity*

Dividing (90) by ρ, taking the curl and making use of the continuity equation (94) we obtain

$$\rho\frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) - \boldsymbol{\omega}\cdot\nabla\mathbf{u} = \nabla\left(\overline{P}^{1/\gamma}/\rho\right)\times\left(\frac{\overline{\rho}}{\overline{P}^{1/\gamma}}\nabla\Phi_Z - \nabla(P'/\overline{P}^{1/\gamma})\right), \tag{98}$$

where $\boldsymbol{\omega} = \nabla\times\mathbf{u} + 2\Omega_0\mathbf{e}_z$ is the absolute vorticity. Making use of (97) we obtain the conservation of potential vorticity

$$\frac{D}{Dt}\left[\left(\frac{\boldsymbol{\omega}}{\rho}\right)\cdot\nabla\left(\frac{\rho}{\overline{P}^{1/\gamma}}\right)\right] = 0 \tag{99}$$

Note that this conservation law survives the introduction of viscous forces as indicated in Section 4.4.2.

6 COMPRESSIBLE DYNAMICS

Finally we deal with local models that fully incorporate compressible motions, and thus describe fast phenomena such as sound waves and transonic turbulence. In the previous 'slow' approximations we set the Mach number to be small, but now suppose $\mathcal{M}$ can take any value. In addition, we do not initially specify the length scales over which the dynamics will manifest, and thus $\epsilon = \lambda/H_Z$ will be free for the time being. The thermodynamic perturbations are assumed to be such that

$$\frac{\rho'}{\overline{\rho}} \sim 1, \qquad \frac{P'}{\overline{P}} \sim 1. \tag{100}$$

and now the equations are set up to capture large fluctuations in density and pressure.

Enforcing radial locality, we expand the background in small x and retain the leading order terms. The equilibrium, however, retains its full dependence on z at first. We next scale the background fields according to Eqs (30) and (65)-(66). Perturbation scalings are given by (64) but with $\mathcal{M}$ set equal to unity. Finally, $x = \lambda x^*$, etc, and $t = (\lambda/w)t^*$ where a star indicates a dimensionless order 1 quantity.

We insert the scaled form of the dependent and independent variables into the governing equations (12)-(14). The continuity equation becomes

$$\frac{D\rho^*}{Dt^*} = -(\overline{\rho}^* + \rho^*)\nabla^*\cdot\mathbf{u}^* - \epsilon u_z^*\partial_z^*\overline{\rho}^* - \epsilon\frac{H_Z}{H_R}u_x^*(\partial_R\overline{\rho})^*. \tag{101}$$

Here we have made use of (28) and (29). Similarly for the momentum equation we have

$$\frac{D\mathbf{u}^*}{Dt^*} = -\frac{\mathcal{M}^{-2}}{\overline{\rho}^* + \rho^*}\nabla^* P^* - 2\mathrm{Ro}^{-1}\mathbf{e}_z \times \mathbf{u}^* \\ - \mathrm{Ro}^{-1}(q_R\, u_x^* + q_Z\, u_z^*)\mathbf{e}_y \\ + \epsilon\mathcal{M}^{-2}\frac{\rho^*}{\overline{\rho}^*(\overline{\rho}^* + \rho^*)}\left(\partial_z^*\overline{P}^*\mathbf{e}_z + \frac{H_Z}{H_R}\partial_R^*\overline{P}^*\mathbf{e}_x\right), \quad (102)$$

where again use has been made of (28) and (29). In the above, Ro= $w/(\lambda\Omega_0)$ is the Rossby number. We next analyse these equations in the two relevant limits for compressible flow.

6.1 The 'small' compressible shearing box

The next assumption we make is that the compressible phenomena of interest takes place on small scales, so that $\lambda \ll H_Z$, i.e. $\epsilon \ll 1$. We may then expand the background thermodynamic variables in small z, and to leading order they become constant. The continuity equation is, to leading order in ϵ,

$$\frac{D\rho^*}{Dt^*} = -(\overline{\rho}^* + \rho^*)\nabla^* \cdot \mathbf{u}^*, \quad (103)$$

while the momentum equation is obtained by making use of (2), (4) and (24):

$$\frac{D\mathbf{u}^*}{Dt^*} = -\frac{\mathcal{M}^{-2}}{\overline{\rho}^* + \rho^*}\nabla^* P^* - 2\mathrm{Ro}^{-1}\mathbf{e}_z \times \mathbf{u}^* \\ - \mathrm{Ro}^{-1}(q_R\, u_x^* + q_Z\, u_z^*)\mathbf{e}_y. \quad (104)$$

We have not explicitly constrained the velocity scale w yet. If we insist that it is of order the sound speed then $\mathcal{M} \sim 1$, and $w \sim H_Z\Omega_0$, but this immediately means that $\mathrm{Ro}^{-1} \sim \epsilon$ and hence the shear and rotation terms drop out! Small-scale fast disturbances are unaware they exist in a shearing box. As the compressible model may also describe slower phenomena, in addition to transonic flow, and the only approximation made so far is that $\epsilon \ll 1$, we keep all the terms in (104) for the moment. Finally, the entropy equation is simply $DS^*/Dt^* = \Xi^*$, where heating and cooling have been incorporated into the single function Ξ, and the background gradients in the entropy are subdominant and do not appear.

6.1.1 Final compressible equations

In dimensional form and using total variables rather than perturbations, we have the set

$$\frac{D\rho}{Dt} = -\rho\nabla\cdot\mathbf{u}, \quad (105)$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P - 2\Omega_0\mathbf{e}_z \times \mathbf{u} - \nabla\Phi - (2\Omega_0^2 q_Z)z\mathbf{e}_x, \quad (106)$$

$$\frac{DS}{Dt} = \Xi, \quad (107)$$

where the tidal potential is $\Phi = \Omega_0^2 q_R x^2$. We note that to maintain consistency in the ordering scheme, the background density and pressure are taken to be constants and there is no vertical stratification. This is because terms involving the gradient of the background state variables have vanished as a result of the assumption $\epsilon \ll 1$.

A few comments about this 'small box' compressible model. First, especially in numerical simulations, these equations are frequently employed to describe phenomena on length scales of order the vertical scale height H_Z, which is strictly outside its range of validity. Indeed, many of the seminal MRI simulations were undertaken in computational domains equal to H_Z or larger (Hawley et al. 1995, etc). If we are tracking phenomena with length scales $\sim H_Z$, then we should avoid using this approximation. However, the failure to do so is harmless for many applications, and the results obtained still instructive.

Second, in a number of simulations, especially those studying the subcritical baroclinic instability, background gradients in pressure are included in the momentum equation (the last terms in (102)), despite $\epsilon \ll 1$. Doing so turns out to be far from harmless, however, as they give rise to spurious overstabilities. This is discussed further in Section 6.4.

6.2 The 'large' (vertically stratified) shearing box

We next allow λ to be of order H_Z and accordingly set $\epsilon = 1$. In addition, the velocity scale is set to be $\Omega_0 H_Z$ so that $\mathcal{M} = 1$. As in the anelastic model earlier, our shearing box sits at the midplane, so that $Z_0 = 0$ and hence $q_z = 0$. When we expand the background flow, we obtain $\overline{\mathbf{u}} = \Omega_0 q_R x\mathbf{e}_y$ to leading order. Vertical shear is difficult to justify, as the next order term (quadratic in z) in $\overline{\mathbf{u}}$ is $(H_Z/R_0)^2$ smaller than the other terms (and even smaller than the omitted cylindrical terms). Unsurprisingly, Lin and Youdin (2015) report the emergence of spurious instabilities when it is included that may be traced back to this inconsistency.

Because z ranges over H_Z, we retain the explicit variation of $\overline{\rho}$ and $\overline{P}$ with z, but Taylor expand them to leading order in x. In order to perform the latter operation we must assume that $H_R \gg H_Z$ as $x \sim H_Z$. As a consequence, the equilibrium radial-gradient terms in (101) and (102) must be dropped to maintain consistency: unlike the Boussinesq approximation, there is no flexibility when it comes to incorporating *constant* background radial gradients. If radial gradients are sufficiently sharp, their full x structure must be retained, as is done in local models of slender tori, narrow rings, and localised density bumps (not treated in this paper; see Goldreich et al. 1986 and Narayan et al. 1987).

The continuity equation (101) is then

$$\frac{D\rho^*}{Dt^*} = -\left[\overline{\rho}^*(z) + \rho^*\right]\nabla^*\cdot\mathbf{u}^* - u_z^*\partial_z^*\overline{\rho}^*, \quad (108)$$

where we have dropped terms of order H_Z/H_R. The momentum equation becomes

$$\frac{D\mathbf{u}^*}{Dt^*} = -\frac{1}{\overline{\rho}^* + \rho^*}\nabla^* P^* - 2\mathrm{Ro}^{-1}\mathbf{e}_z \times \mathbf{u}^* \\ - \mathrm{Ro}^{-1} q_R\, u_x^*\,\mathbf{e}_y + \frac{\rho^*}{\overline{\rho}^*(\overline{\rho}^* + \rho^*)}\partial_z^*\overline{P}^*\mathbf{e}_z, \quad (109)$$

where again we have dropped the radial gradient in the background pressure as subdominant. Finally, these equations in dimensional form and in total variables are identical to Eqs (105)-(107), but with $q_z = 0$, the tidal potential

$\Phi = \Omega_0^2 q_R x^2 + \frac{1}{2}\Omega_0^2 z^2$, and the background no longer uniform but dependant on z. We point out that this model corresponds to what Umurhan and Regev (2008) call the 'large shearing box'.

6.3 Conservation laws

In both compressible shearing boxes the conservation laws of energy and potential vorticity for inviscid adiabatic flows are relatively familiar (see e.g. Ogilvie 2016). Introducing the specific internal energy $e = P/(\gamma - 1)$, the former may be expressed as

$$\partial_t(\tfrac{1}{2}\rho u^2 + \rho\, e) + \nabla\cdot\left[\rho\mathbf{u}\left(\tfrac{1}{2}u^2 + \Phi + e\right) + P\mathbf{u}\right] = 0. \quad (110)$$

where the potential Φ is either $\Omega_0^2 q_R x^2$ or $\Omega_0^2 q_R x^2 + \frac{1}{2}\Omega_0^2 z^2$.

If $\mathcal{A}$ is any materially conserved quantity that is a function of the thermodynamic variables (such as the entropy), i.e. it satisfies $D\mathcal{A}/Dt = 0$, then it follows that the quantity $\Theta = (\boldsymbol{\omega}\cdot\nabla\mathcal{A})/\rho$ also satisfies $D\Theta/Dt = 0$, and is thus a generalisation of the potential vorticity.

6.4 Spurious instabilities

In the astrophysical literature one can find examples of small compressible shearing boxes that include a constant gradient in the thermodynamic background; this is in order to either drive radial or vertical convection or in a misguided attempt at completeness. Such terms are formally subdominant, but when included these terms are dangerous, as we now explore.

Consider a small box model of an isothermal gas with perturbation equations

$$\frac{D\rho'}{Dt} = -(\overline{\rho}_0 + \rho')\nabla\cdot\mathbf{u}', \quad (111)$$

$$\frac{D\mathbf{u}'}{Dt} = -\frac{\nabla P'}{\overline{\rho}_0 + \rho'} - 2\Omega_0\mathbf{e}_z\times\mathbf{u}' - \Omega_0 q_R u_x'\mathbf{e}_y + \frac{\rho'}{\overline{\rho}_0(\overline{\rho}_0 + \rho')}(\partial_Z\overline{P})_0\mathbf{e}_z. \quad (112)$$

Here $\overline{\rho}_0$ and $(\partial_z\overline{P})_0$ are constants, and $\rho = c^2 P$, with c the isothermal sound speed. These are essentially Eqs (108) and (109) but with an extra constant term (formally subdominant) involving a background vertical pressure gradient so as to allow for possible buoyancy effects.

If we linearise these equations and seek normal modes of the type $\mathrm{e}^{\mathrm{i}k_z z - \mathrm{i}\omega t}$, where ω is a possibly complex frequency and k_z is a real wavenumber, then the vertical and horizontal oscillations decouple. The former possess the dispersion relation

$$\omega^2 = k_z^2 c^2 + \mathrm{i}k_z\frac{(\partial_Z\overline{P})_0}{\overline{\rho}_0} \quad (113)$$

and result in growing sound waves, on account of the second imaginary term on the right side. This term arises explicitly from inclusion of the constant (and subdominant) vertical gradient in pressure. Note that the instability occurs for *all* vertical wave numbers.

Let us next include a constant *radial* gradient, but no vertical gradient, and replace the last term in (112) with $\rho'/[\overline{\rho}_0(\overline{\rho}_0 + \rho')](\partial_R\overline{P})_0\mathbf{e}_x$. Again we linearise but now examine modes $\propto \mathrm{e}^{\mathrm{i}k_x x - \mathrm{i}\omega t}$, where k_x is a real radial wavenumber. The ensuing dispersion relation is

$$\omega^2 = \kappa^2 + \mathrm{i}k_x\frac{(\partial_R\overline{P})_0}{\overline{\rho}_0} + k_x^2 c^2, \quad (114)$$

where the epicyclic frequency is defined as $\kappa^2 = 2\Omega_0^2(2+q_R)$. Equation (114) describes classical density waves, but these are growing because of the imaginary second term, which issues from the (subdominant) background radial gradient. As above, the instability attacks *all* radial scales.

It is straightforward to show these two instabilities are spurious by simply examining the Høiland criteria (e.g. Ogilvie 2016). We think it useful instead to tackle the two linear problems directly in semi-global geometries, thereby demonstrating instabilites fail to appear when background gradients are incorporated in full.

To deal with the first example we turn to the large compressible shearing box, which exhibits the disk's full vertical structure. In this case, the background equilibrium is $\overline{\rho} = \overline{\rho}_0\exp\left[-z^2/(2H^2)\right]$, where $H = c/\Omega_0$. We perturb this with modes $\propto \mathrm{e}^{-\mathrm{i}\omega t}$ and depending only on z. The problem reduces to a single equation for the perturbed enthalpy, $h = c^2\rho'/\overline{\rho}$:

$$\frac{d^2h}{dz^2} - \frac{z}{H^2}\frac{dh}{dz} + \frac{\omega^2}{c^2}h = 0. \quad (115)$$

This is the Hermite equation. For sensible solutions at $z = \pm\infty$, the dispersion relation is $\omega^2 = n\Omega_0^2$, where n is a positive integer. This equation should be compared with Eq. (113). With the identification $k_z H = \sqrt{n}$, the first terms in each expression agree, but not the second: when the problem is done without approximation there is no instability.

It is a little more involved to show that the radial instability is spurious. To do so we investigate the modes of a global cylindrical disc with background structure $\overline{\rho} = \overline{\rho}(R)$, $\overline{P} = \overline{P}(R)$. We take the modes to be isothermal, to depend only on R, and to be $\propto \mathrm{e}^{-\mathrm{i}\omega t}$. The governing linearised equations are

$$-\mathrm{i}\omega u_R' = -\frac{c^2}{\overline{\rho}}\partial_R\rho' - 2\Omega u_\phi' + \frac{\partial_R\overline{P}}{\overline{\rho}^2}\rho', \quad (116)$$

$$-\mathrm{i}\omega u_\phi' = \frac{\kappa^2}{2\Omega}u_R', \qquad -\mathrm{i}\omega\rho' = -\frac{1}{R}\partial_R(R\overline{\rho}u_R'), \quad (117)$$

where the epicyclic frequency in the global problem is $\kappa^2 = (2\Omega/R)d(R^2\Omega)/dR$. These equations may be manipulated into a single ODE for the dependent variable $W = R\overline{\rho}u_R'$,

$$\overline{\rho}\frac{d}{dR}\left[\frac{c^2}{\overline{\rho}R}\frac{dW}{dR}\right] + \frac{1}{R}(\omega^2 - \kappa^2)W = 0, \quad (118)$$

and assume the innocuous condition that $W = 0$ at the radial boundaries of the disc. Multiplying (118) by the complex conjugate of $W/\overline{\rho}$ and integrating over radius, one can obtain

$$\omega^2 = \frac{\int\kappa^2 f|W|^2 dR}{\int f\,|W|^2 dR} + c^2\frac{\int f|dW/dR|^2 dR}{\int f\,|W|^2 dR}, \quad (119)$$

where $f = 1/(\overline{\rho}R)$. Because f is always positive, both terms in (119) must also be positive. They, in fact, correspond to the first and third terms in Eq. (114); there is no equivalent to the imaginary term. It follows that ω is real and no instability occurs, in contradiction to Eq. (114).

These demonstrations show clearly that when the problem is done correctly in a global or quasi-global setting (in which the background equilibrium is accounted for without approximation) the local instabilities vanish. They are spurious. The conclusion is that it is not always safe to include subdominant terms: they may not always offer small corrections to the outcome but something radically incorrect that on intermediate to long times may overwhelm a simulation.

6.5 Wave action and energy conservation for small perturbations

The problems discussed above issue from the fact that the fundamental conservation laws for wave action and energy, applicable to small perturbations, break down when spatial variations in the background state are incorporated incorrectly. In fact, these conservation laws exclude instabilities of the kind presented in the previous subsection.

In the absence of heating and cooling, conservation laws issue from the fact that the equations of motion can be derived from a stationary action principle. The principle is applied to the action

$$\mathcal{S}[\mathbf{r}] = \int \mathcal{L}(\mathbf{r}) d^3\mathbf{r} dt, \qquad (120)$$

in which we have employed a Lagrangian description and $\mathbf{r}$ represents the coordinates of a fluid element. The action is a function of the initial coordinate positions $\mathbf{r}_0$ and t, and the Lagrangian density for the compressible system is given by

$$\mathcal{L} = \rho\left(\frac{1}{2}\mathbf{u}^2 + \Omega_0(\mathbf{e}_z\times\mathbf{r})\cdot\mathbf{u} - \Phi_T - \frac{P}{(\gamma-1)\rho} - \frac{\mathbf{B}^2}{8\pi}\right). \qquad (121)$$

Apart from the second term, arising from the rotating frame, this expression is identical to that presented by Ogilvie (2016). An expression incorporating the second term but without a magnetic field, $\mathbf{B}$, applicable to a 2D shearing sheet, has been formulated by Goldreich et al. (1987). For the incompressible and anelastic cases, the fourth thermal energy term is absent and instead one must apply a conservation constraint through the use of a Lagrange multiplier (see Vasil et al. 2013). In the incompressible case, the conserved quantity is ρ and in the anelastic case it is $\rho/\overline{P}^{1/\gamma}$.

One may suppose that there is a steady state background flow and consider perturbations to the particle positions such that $\mathbf{r}\to\mathbf{r}+\boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the Lagrangian displacement, related to the Eulerian velocity perturbation via

$$\mathbf{u}' = \frac{\partial\boldsymbol{\xi}}{\partial t} + \overline{\mathbf{u}}\cdot\nabla\boldsymbol{\xi}. \qquad (122)$$

When substituted into (120), the action integral is quadratic in the perturbations, on account of it being stationary in the background state. Thus the action converts into

$$\mathcal{S}[\boldsymbol{\xi}] = \int \mathcal{L}(\boldsymbol{\xi}, \partial\boldsymbol{\xi}/\partial t, \nabla\boldsymbol{\xi}) d^3\mathbf{r} dt, \qquad (123)$$

(see Ogilvie 2016, Goldreich et al. 1987 for specific evaluations). Wave-action conservation follows by noting that the background is independent of y and so expresses translational symmetry in that direction. Noether's theorem then yields a wave-action conservation law

$$\frac{\partial(\overline{\rho}Q)}{\partial t} + \nabla\cdot\mathbf{J} = 0, \qquad (124)$$

(see Goldreich et al. 1987), where the wave-action density is

$$\overline{\rho}Q = \frac{\partial\mathcal{L}}{\partial(\partial\boldsymbol{\xi}/\partial t)}\cdot\frac{\partial\boldsymbol{\xi}}{\partial y}, \qquad (125)$$

and the wave-action density flux is

$$\mathbf{J} = \frac{\partial\mathcal{L}}{\partial(\partial\boldsymbol{\xi}/\partial\mathbf{r})}\cdot\frac{\partial\boldsymbol{\xi}}{\partial y} - \mathcal{L}\mathbf{e}_y. \qquad (126)$$

The scalar products in these equations are between the two occurrences of $\boldsymbol{\xi}$.

One may also obtain an energy conservation law for perturbations by replacing $\partial\boldsymbol{\xi}/\partial y$ by $\partial\boldsymbol{\xi}/\partial t$ in (125) and (126) and then adding $-\mathcal{L}$ to (125) while removing the term $\propto \mathbf{e}_y$ from (126). This is useful for discussing disturbances that do not depend on y. However, for the sake of brevity we shall focus here on the conservation of wave action.

Alternatively, we can derive conservation laws from the equations governing the linear perturbations in the Lagrangian formulation (see eg. Lynden-Bell and Ostriker 1967). Note that the wave-action conservation law may be rescaled so as to represent the conservation of y-momentum (and then angular momentum) through an additional multiplication by R_0 (the radial location of the shearing box in the disc). This scaling can be determined by incorporating an external forcing potential into the linearized equations of motion and considering the consequent injection of energy and momentum in the modified conservation laws. In this way $-\rho Q$ can be interpreted as the wave y-momentum density.

Wave-action conservation can tell us how the amplitude of a wave varies as it propagates through a variable background, by computing the 'instantaneous' wave's properties in the local limit. As an illustrative example we consider the anelastic model governed by equations (90), (92) and (94) and also the compressible model governed by (105), (106) and (107) allowing for the incorporation of vertical stratification if needed, all with no cooling. Extension to the incompressible case follows by taking the limit $\gamma\to\infty$.

We determine the relevant conservation law for wave action by first averaging over the y direction. Then

$$Q = \left\langle \frac{\partial\boldsymbol{\xi}}{\partial t}\cdot\frac{\partial\boldsymbol{\xi}}{\partial y} + \overline{u}_y\frac{\partial\boldsymbol{\xi}}{\partial y}\cdot\frac{\partial\boldsymbol{\xi}}{\partial y} - \Omega_0\mathbf{e}_z\cdot\left(\frac{\partial\boldsymbol{\xi}}{\partial y}\times\boldsymbol{\xi}\right)\right\rangle \qquad (127)$$

and

$$\mathbf{J} = \left\langle P'\frac{\partial\boldsymbol{\xi}}{\partial y}\right\rangle, \qquad (128)$$

where the angle brackets denote an integral mean over the y domain under the assumption that periodic boundary conditions apply. By integrating (124) over the volume of the box V, given that fresh wave-action density cannot enter through the boundaries, one discovers that the total integrated wave-action density,

$$\int_V \overline{\rho}Q d^3\mathbf{r}, \qquad (129)$$

is a constant, which simply expresses conservation of the total angular momentum associated with the disturbance. It must be stressed that when the background's variation is retained in an inconsistent manner, the correct action conservation cannot be obtained.

We next apply expression (129) to the question of instability. It is clear that if a small disturbance $\boldsymbol{\xi}$ is to grow (i.e. there exists an instability), the wave-action density must change sign somewhere in the domain V: the only way a disturbance can grow exponentially somewhere in the box is if there is a cancellation with an exponentially growing disturbance with the *opposite* sign of wave action density elsewhere. This is precisely the mechanism that drives corotation-type instabilities (Papaloizou & Pringle 1984), which work via the interaction of disturbances with opposite signs of angular momentum. To understand these instabilities a global treatment is usually required (though see the slender torus models of Goldreich et al. 1986, Narayan et al. 1987, and Latter and Balbus 2009).

On the other hand, Eq. (129) tells us highly localised disturbances with wave-action density of a well defined sign cannot grow while they propagate. This precludes instabilities of the type explored in the previous subsection. Note that this principle allows velocity amplitudes to increase as the wave enters a low density region, but it is incorrect to view this as localised exponential growth (and hence instability). For perturbations such as these, which vary harmonically in time, the amplitude of a propagating wave packet is governed by $\nabla\cdot\mathbf{J} = 0$, with a further time average applied to $\mathbf{J}$. Its properties can be determined, apart from a constant amplitude, in the local limit for which variations in the background variables are neglected. Then the above equation constrains, and may completely determine, the variation of the amplitude as it propagates.

7 CONCLUSION

In this paper we start from a fully global and fully compressible disk and derive a sequence of local shearing box models that describe small regions of the disk under various assumptions. These approximations are consistent with the original equations via a well defined ordering scheme and satisfy key conservation laws (energy, potential vorticity, entropy, etc). We stress that one must be careful in deriving local models: various terms and background gradients cannot be thrown in or removed arbitrarily. Spurious instabilities or other undesirable features may arise.

Slow phenomena on short scales can be described by incompressible or Boussinesq equations, the two models only differing in the relative sizes of the fractional density and pressure perturbations. We show how vertical shear may be incorporated in both, but that only the Boussinesq system involves a background gradient in entropy.

We reiterate the problems inherent in anelastic models; commonly used versions of these equations fail to conserve energy. When ad hoc adjustments are made, conservation is assured only in special cases, such as isothermality, which precludes the treatment of convection. A set of conservative anelastic equations is derived that permits diabaticity, but restricts the characteristic length scale of phenomena, though not the domain over which the phenomena can range. Admittedly, the restriction to small-scales, necessitated by the strong shear in a thin Keplerian disc, is a strong imposition, but it does illustrate the challenges posed by the anelastic approximation. Applications of this conservative set include magnetorotational or convective turbulence localised to certain layers, or the propagation and refraction of wave packets upward in the disc.

We derive two forms of compressible model, one in which the characteristic length scales are much less than the scale height (the 'small compressible shearing box') and one in which they are of order the scale height (the 'large' or vertically stratified box). Emphasis is put on the problems that arise when the background gradients are included in the former, especially the generation of spurious acoustic overstabilities. We discuss how these relate to the breaking of wave-action conservation: in particular, if wave action is properly conserved in the model, localised disturbances with a well defined sign for the associated action density do not undergo such overstabilities.

Finally, we point out that in a typical hydrodynamical cascade, as energy tumbles to smaller and smaller scales, the flow becomes more and more incompressible and rotation and shear less and less important. Ultimately, on some scale above the viscous dissipation length, the flow may be approximated within a non-shearing incompressible local model. This should be kept in mind when studying the microscales in discs, such as the interactions between dust and turbulent eddies. If the fluid is sufficiently ionised, and a similar cascade is functioning, then any imposed magnetic field (no matter how weak) will ultimately dominate on some small scale. In this case, the flow can be modelled by the equations of reduced MHD (Biskamp 1993), and indeed recent simulations of the MRI show evidence of this regime (Zhdankin et al. 2017).

The derivations in this paper were purely hydrodynamic, as the main issues and problems issue from the thermodynamics, but it is straightforward to generalise these derivations to MHD: the induction equation and Lorentz force pose no additional complications. On the other hand, it is extremely difficult to extend the 'locality' of the shearing box, either by including higher order terms arising from the disc's cylindrical geometry (Pessah and Psaltis 2005), or by relaxing the assumption that $z \lesssim H_Z$ (McNally and Pessah 2015). The former case seeds spurious modes on arbitrarily small-scales, while the latter encounters conservation difficulties, as pointed out in detail by McNally and Pessah (2015). In an interesting contrast, the local manifestation of global phenomena such as warps and eccentricities can be described consistently in a suitably modified shearing box (Ogilvie & Latter 2013, Ogilvie & Barker 2014).

Though it is possible to generalise the shearing box to relativistic flow (Heinemann, private communication) one must be careful with the radial boundary conditions. Shearing periodic boundaries, as employed in numerical realisations (Riquelme et al. 2012, Hoshino 2013, 2015), are inconsistent with Lorentz invariance (see Peters 1983), and produce spurious effects such as 'run-away' particles (Kimura et al. 2016). The infinite relativistic shearing sheet, however, may offer a useful platform to undertake purely theoretical work.

Finally, as is well known, two-dimensional razor-thin or vertically integrated shearing boxes cannot be rigorously derived from three-dimensional equations on account of the quadratic velocity nonlinearity in the momentum equation. However, if it is assumed that the planar velocities exhibit little to no vertical variation, and that phenomena possesses planar scales much greater than H_Z, it is possible to well-motivate two-dimensional vertically integrated shearing box equations (e.g. Shu and Stewart 1985, Stehle and Spruit 1999): a 2D incompressible model from the classical anelastic equations, and a 2D compressible model from the large compressible box.

We finish by stressing the continuing value of the local shearing box model in understanding disc phenomena. On account of its simpler geometry, problems are analytically and numerically easier; they hence permit researchers to disentangle the salient physical effects and their relationships, and hence make real progress in our understanding. The resolution obtained in numerical simulations is also an advantage local models wield over global set-ups, where typically one is more concerned with obtaining a reasonable scale separation between radius and the vertical scale height, rather than between the input and dissipation scales in turbulence. In fact, only in local models can any kind of turbulent inertial range be simulated adequately; at present 'turbulence' in global models resembles more a monoscale chaotic flow. Of course, local models have their deficiencies, some of them outlined in Umurhan and Regev (2008), though we feel most of their criticisms are overstated. For instance, problems issuing from symmetries, boundary conditions, and enhanced fluctuations can be ameliorated by simply varying the boundary conditions and/or taking bigger boxes (admittedly certain problems do pose special difficulties on this count; e.g. Fromang et al. 2013). Global models also struggle with boundary conditions, which are often ambiguous, unrealistic, or numerically problematic. No model is perfect, and each has its strengths and weaknesses. If we are alert to these, shearing boxes remain valuable tools in helping us understand the complicated astrophysical flows around planets, stars, and black holes.

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APPENDIX A: ENERGY CONSERVATION WITH VERTICAL SHEAR

The origin of the energy source term on the right hand of (37) can be understood more generally by considering the basic equation of motion (2). From this we may obtain without assuming $\nabla \cdot \mathbf{u} = 0$,

$$\partial_t(\rho(\tfrac{1}{2}u^2 + \Phi)) + \nabla \cdot \left[(\tfrac{1}{2}u^2 + h + \Phi)\rho\mathbf{u}\right] = -\frac{P}{\rho}\frac{D\rho}{Dt}, \quad \text{(A1)}$$

where now we have $h = P/\rho$. If ρ were constant (A1) would express conservation of energy. But in order to be consistent with $q_Z \neq 0$, this cannot be case and the rate of doing PdV work provides a source.

Given that ρ' is small, to lowest order in ϵ, we may set $D\rho/Dt = \mathbf{u}\cdot\nabla\rho$. We also replace P by $P - \overline{P}_0$ thus measuring it relative to the constant value in the centre of the box. In addition, to lowest order in ϵ we neglect P' and ρ'. We can then set $\rho = \overline{\rho}$ and $P = \overline{P} - \overline{P}_0$ and perform a first order Taylor expansion about the centre of the box to obtain the latter quantity and hence the rate of doing PdV work. To first order in ϵ, we obtain

$$\partial_t(\rho(\tfrac{1}{2}u^2 + \Phi)) + \nabla\cdot\left[(\tfrac{1}{2}u^2 + h + \Phi)\rho\mathbf{u}\right] = -\frac{\mathbf{u}\cdot(\nabla\overline{\rho})_0}{\rho_0}\left((\partial_R\overline{P})_0 x + (\partial_Z\overline{P})_0 z\right). \tag{A2}$$

After some manipulations and use of (19) this can be written as

$$\partial_t(\rho(\tfrac{1}{2}\rho u^2 + \Phi)) + \nabla\cdot\left[(\tfrac{1}{2}u^2 + h + \Phi + \mathcal{Q})\rho\mathbf{u}\right] = -2\overline{\rho}_0\Omega_0^2 q_Z z\, u_x + \mathcal{Q}\rho\nabla\cdot\mathbf{u} \tag{A3}$$

where

$$\mathcal{Q} = \frac{(\partial_R\overline{\rho})_0(\partial_R\overline{P})_0 x^2 + (\partial_Z\overline{\rho})_0(\partial_Z\overline{P})_0 z^2 + 2(\partial_R\overline{\rho})_0(\partial_Z\overline{P})_0 zx}{2\rho\overline{\rho}_0}. \tag{A4}$$

When $\nabla\cdot\mathbf{u}$ is neglected, the source terms in (37) and (A3) are seen to be identical.

APPENDIX B: POTENTIAL VORTICITY CONSERVATION IN GENERAL BOUSSINESQ SYSTEMS

In Section 3 we derived conservation laws in the more straightforward case of vertical stratification. Now we examine the general barotropic case, for which $N_R^2 H_R^2 = N_Z^2 H_Z^2$ and $q_Z = 0$. We introduce the convenient 'total' buoyancy variable $\theta_s = H_Z\rho'/\overline{\rho}_0 + z + xH_Z/H_R$, which transforms the entropy equation into $D\theta_s/Dt = 0$. Next we take the curl of (53) and obtain an equation for the vorticity in the shearing box:

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} - N_Z^2\nabla\theta_s\times(\mathbf{e}_z + \mathbf{e}_x H_Z/H_R)\,. \tag{B1}$$

We now take the inner product of (B1) and $\nabla\theta_s$, which gives

$$\nabla\theta_s\cdot\frac{D\boldsymbol{\omega}}{Dt} = \nabla\theta_s\cdot\left[(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}\right]. \tag{B2}$$

The gradient of the entropy equation is

$$\frac{D\nabla\theta_s}{Dt} = -(\nabla\mathbf{u})\cdot\nabla\theta_s. \tag{B3}$$

Finally, we examine the total derivative of $\nabla\theta_s\cdot\boldsymbol{\omega}$, and see that

$$\frac{D(\nabla\theta_s\cdot\boldsymbol{\omega})}{Dt} = 0 \tag{B4}$$

The final result is:

$$\frac{\partial\Theta}{\partial t} + \nabla\cdot(\Theta\,\mathbf{u}) = 0, \tag{B5}$$

where the conserved quantity is $\Theta = \nabla\theta_s\cdot\boldsymbol{\omega}$. This we regard as the potential vorticity in the Boussinesq shearing box for a barotropic flow with general stratification.