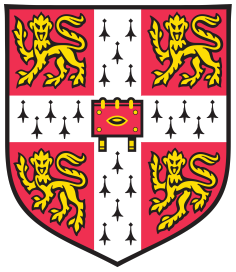
BPS Approaches to Anyons, Quantum Hall States and Quantum Gravity

Carl Peter Turner



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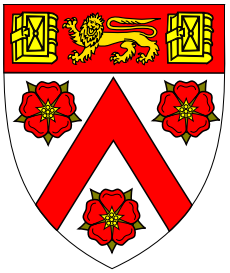

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This dissertation is submitted for the degree of
Doctor of Philosophy

BPS Approaches to Anyons, Quantum Hall States and Quantum Gravity

Carl Peter Turner

We study three types of theories, using supersymmetry and ideas from string theory as tools to gain understanding of systems of more general interest.

Firstly, we introduce non-relativistic Chern-Simons-matter field theories in three dimensions and study their anyonic spectrum in a conformal phase. These theories have supersymmetric completions, which in the non-relativistic case suffices to protect certain would-be BPS quantities from corrections. This allows us to compute one-loop exact anomalous dimensions of various bound states of non-Abelian anyons, analyse some interesting unitarity bound violations, and test some recently proposed bosonization dualities.

Secondly, we turn on a chemical potential and break conformal invariance, putting the theory into the regime of the Fractional Quantum Hall Effect (FQHE). This is illustrated in detail: the theory supports would-be BPS vortices which model the electrons of the FQHE, and they form bag-like states with the appropriate filling fractions, Hall conductivities, and anyonic excitations. This formalism makes possible some novel explicit computations: an analytic calculation of the anyonic phases experienced by Abelian quasiholes; analytic relationships to the boundary Wess-Zumino-Witten model; and derivations of a wide class of QHE wavefunctions from a bulk field theory. We also further test the three-dimensional bosonization dualities in this new setting. Along the way, we accumulate new descriptions of the QHE.

Finally, we turn away from flat space and investigate a problem in (3+1)-dimensional quantum gravity. We find that even as an effective theory, the theory has enough structure to suggest the inclusion of certain gravitational instantons in the path integral. An explicit computation in a minimally supersymmetric case illustrates the principles at work, and highlights the role of a hitherto unidentified scale in quantum gravity. It also is an interesting result in itself: a non-perturbative quantum instability of a flat supersymmetric Kaluza-Klein compactification.

This thesis is dedicated to the memory of my grandmother, Vera Turner

Thank you, Grandma, for the many years of warmth and joy you brought us all

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except as declared in the Preface and specified in the text.

It is not substantially the same as any that I have submitted, or, is being concurrently submitted for a degree or diploma or other qualification at the University of Cambridge or any other University or similar institution except as declared in the Preface and specified in the text. I further state that no substantial part of my dissertation has already been submitted, or, is being concurrently submitted for any such degree, diploma or other qualification at the University of Cambridge or any other University of similar institution except as specified in the text.

This document is mostly adapted from two papers on non-relativistic anyons with conformal symmetry [1, 2]; four papers [3, 4, 5, 6] on quantum Hall physics; and one paper [7] on quantum gravity. These seven papers are broadly reflected in the structure of this document, with superconformal anyons forming Part II, the quantum Hall work comprising Parts III, IV, and V and the quantum gravity work making up Part VI. There is some unpublished material on the bosonization of anyons in Part II. The only section of work for which I was not primarily responsible is the work recapitulated in Chapter 18, for which Nick Dorey was chiefly responsible.

Carl Turner
June 5, 2017

Acknowledgements

Doing a PhD can be challenging, but the last few years have been a very happy time for me, and for that I owe thanks to many people.

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Special thanks go to those who were also kind enough to host visits to several amazing places: Kimyeong Lee at KIAS; Shiraz Minwalla at TIFR; and Shamit Kachru at Stanford. Of course, the people here in the CMS, especially my fellow students, have made being in DAMTP a real pleasure; they get a definite 5/5. I particularly want to thank James, Jack and Eduardo for making the office a welcoming and cheerful place, and providing moral support in the face of adversity, bureaucracy and poorly designed websites.

I must also express my gratitude to my family for all their support; I wish my grandmother, to whose memory this thesis is dedicated, were still here to read this. Most especially I have a great debt to my parents, whose kindness and generosity got me here; their implicit belief has helped me through some tough times. Thank you.

Finally, I must mention the fine people who I have been lucky enough to surround myself with in Cambridge. My old friends from my time as an undergraduate – especially Alex, Amit, Fiona, George, Guy, Hannah, James, Jana, Jonathan, Josh, Rach, Rob, Steph, Tasha and Ted – take a huge amount of credit for making Cambridge such a happy place for me. In more recent years, the *Cambridge University Ceilidh Band* and its satellite outfits have been a constant source of amazing fun, phenomenal music and incredible friends, all of whom have a place in my heart. It also helped me find Alice, to whom I am grateful for laughter, comfort, music, food, and a great many other things.

Thank you all.

Contents

Quantum Hall Notation

For ease of reference, we collect some of the more common symbols used in Parts III-V:

N	Number of electrons (vortices)
ν	Filling fraction
A_μ, B	Background electromagnetic gauge field and its magnetic field
μ	Chemical potential
R	Radius of quantum hall droplet in ground state

Bosonic Chern-Simons theory

p	Rank of gauge group and number of matter flavours
k	Level of non-Abelian part of the gauge group
k'	Level of the Abelian part of the gauge group
ϕ	Bosonic field
$a_\mu, f_{\mu\nu}$	Gauge field and field strength (non-Abelian part from Part IV on)
$\tilde{a}_\mu, \tilde{f}_{\mu\nu}$	Gauge field and field strength (Abelian part from Part IV on)

Matrix model

N	Rank of gauge group (and hence number of electrons or vortices)
p	Number of flavours
Z	Complex $N \times N$ matrix
φ_i	Bosonic scalar with flavours $i = 1, \ldots, p$
α	$U(N)$ gauge field

PART I

Introduction

1 Context and Goals

Most of the interesting problems in theoretical physics today lie in the mysterious depths beyond perturbation theory. There is a veritable zoo of strongly interacting phenomena, challenging our understanding of even the most basic systems, like electrons in the plane. Getting a grip on these exotic effects is one of the main enterprises in modern physics, which explains the sustained interest in dualities that relate hard-to-understand systems to simpler ones, as in holography.

Yet, rather remarkably, certain non-perturbative phenomena turn out to be amenable to direct mathematical calculation. From self-dual spacetimes to vortices in the plane, mathematical physicists have long enjoyed the particular elegance of *BPS states*. These normally sit at very special points of configuration spaces, but are not simply mathematical curiosities. Instead, they can naturally generate significant contributions to physical processes, and enjoy topological protection guaranteeing their stability.

Our perspective on these states is to be slightly different. In this dissertation, we will tackle some hard problems by actively exploiting the fact that we know a lot about such configurations. The strategy is to choose a special version of the problem which should have a nice answer in terms of known topologically protected states, and then see what we learn about the general case from going ahead and solving the special case. We will explore applications of this approach to non-perturbative physics to physical problems in both condensed matter and quantum gravity.

Strongly interacting matter holds much interest from mathematical, physical and experimental perspectives. As we shall shortly outline, we will be interested in theories whose degrees of freedom are *anyonic* in nature. Firstly, we shall study conformal quantum mechanics, using non-relativistic supersymmetry to constrain the spectrum of particles in a harmonic trap. Then we shall turn to quantum Hall states. Here, the magic ingredients are vortices in Chern-Simons theories sat at a critical coupling. We will find these *topological solitons* provide a convenient model for electrons in the plane: we can draw on the understanding of such BPS states which has been accumulated by string theorists in order to understand the behaviour of electrons in impressive detail.

Meanwhile, in quantum gravity, there are long-standing confusions about the general question of what geometries we need to consider as quantum fluctuations of our spacetime. This goes back to the early days of the subject, and the foundational work of

Hawking and contemporaries. In many ways, it is not surprising that we are confused, since we famously do not know what a full theory of quantum gravity should look like. Yet there are concrete, low-energy questions which we are able to ask. We will ask and answer one such question, chosen carefully according to the principles above. This will then lead us to more friendly topological configurations, this time in the form of twisted versions of four-dimensional space. This will clarify some of the confusion about how and when instantons contribute, and reveal some remarkable unappreciated structure inherent to *all* theories of quantum gravity.

It is worth highlighting that although we are using toy models, the problems we will solve are capturing important physics. We will return to this point in more detail throughout the work, and reiterate it during the conclusion in Chapter 23. Our superconformal quantum mechanics of anyons is exactly equivalent to a classic problem in the field. In the context of Hall physics, we will see we are working in the same universality class as the known non-Abelian states we investigate. In the gravitational setting, we shall see that our basic findings about the structure of the theory are insensitive to details like matter content and background.

All that said, let us begin by taking a moment to review the wider context of both anyonic physics in the plane and quantum gravity, and point towards the problems we will address.

The Quantum Hall Effect and Anyonic Physics

The quantum physics of electrons in the plane is remarkably rich. Electrons in the lowest Landau level exhibit an astonishing array of compressible and incompressible states, the latter with both Abelian and non-Abelian topological order. The main goal in the condensed matter section of the dissertation is to examine in a new light one of the many striking effects to be found in these systems: the Quantum Hall Effect.

A Potted History

When a transverse magnetic field $\mathbf{B}$ is applied to a metal, the Lorentz force on electrons gives rise to the classical *Hall effect*, which is easy to understand. Attempting to pass a current through the sample causes electrons to move parallel to the current – but the magnetic force proportional to $\mathbf{v} \times \mathbf{B}$ drives them toward one side of the sample. This induces a voltage orthogonal to the current flow which balances this force. Taking the ratio of the current density and induced electric field allows one to define the *Hall conductivity* $\sigma_H = j_1/E_2$.

However, samples can develop interesting behaviour as we crank up the magnetic field: the graph of the Hall conductivity exhibits striking plateaus. Moreover, these occur at very precisely quantized values of σ_H given essentially by the reciprocals of integers. This is called the *integer quantum Hall effect*.

This counter-intuitive but intriguing phenomenon (which won Klaus von Klitzing the 1985 Nobel Prize) can be understood by considering the fate of localized electron states in the metal. The interaction between the quantization of electron orbits into Landau levels and these localized states gives rise to special phenomena when ν Landau levels are completely filled. One can very elegantly understand the quantization of the Hall conductivity in terms of the integer nature of a topological quantity: the Chern number of a certain Berry-like connection over the space of electron states.

This turned out to be only the tip of the iceberg.

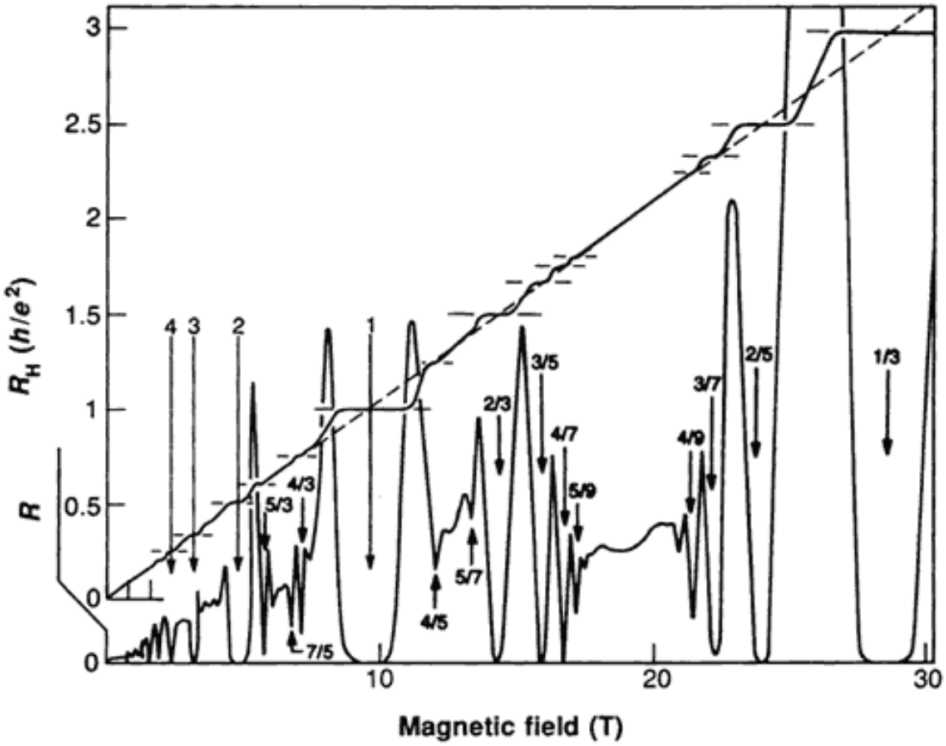


Figure 1.1: Hall resistance (essentially the inverse of the conductivity) is the stepped diagonal line, given as a function of the magnetic field [8]. Filling fractions ν with conspicuous plateaus are labelled. The integer series $\nu = 4, 3, 2, 1$ is also labelled.

The 1998 Nobel Prize was awarded to Laughlin, Störmer and Tsui for the discovery and partial phenomenological explanation of the so-called *fractional* quantum Hall effect. A series of further plateaus were uncovered, corresponding to certain rational filling fractions $\nu = p/q$. These are visible in Figure 1. (In Laughlin's original analysis, he gave a series of explicit trial wavefunctions to describe the electron states at filling $\nu = 1/q$.)

These states of matter have some exotic properties, including quasiholes with fractional charge and *anyonic* statistics. To be concrete, in the $\nu = 1/q$ state, Laughlin showed quasiholes have a charge which is $1/q$ times that of the electron. Later, Arovas, Schrieffer and Wilczek used a plasma analogy for the state to justify Halperin's conjecture that the statistical phase from interchanging two quasiholes is π/q. This is to be contrasted with the more familiar cases of bosons and fermions, which have phase 0 or π respectively. The microscopic mechanism by which these states form is still not very clear.

We are not yet at the bottom of the rabbit hole, however. The right way of understanding the many other observed states remains an open problem. One picture, due to the work of Haldane, Halperin and others, describes the *hierarchy states*. In this picture, one forms Laughlin states from Laughlin quasiparticles, and then repeats this process with the new quasiparticles. Another is the *composite fermion* picture, in which we dress electrons with magnetic flux and form integer quantum Hall states from these new composite objects.

More intriguingly, there is another possible type of state which it seems we may need to worry about. The above theories are *Abelian*, in various senses. For instance: their effective theories are Abelian Chern-Simons field theories, and the algebras describing their quasiholes have trivial fusion rules. But there is a generalization: we must study the predictably named *non-Abelian fractional quantum Hall effect*.

Certain models, developed as putative explanations of unusual states such as the filling fraction $\nu = 5/2$ state, have yet more incredible properties. The most iconic property is famous for its possible applications in quantum computing: quasiholes may have *non-Abelian statistics*. Interchanging a pair of identical quasiholes in such a theory can leave one in a linearly independent state. The dream is that the particles' memory of how they have been braided around each other will one day provide a robust form of quantum data storage.

The mathematics of these states is very rich, and we lack good experimental access to systems realizing them. (Experiments to even verify we are producing these states in the laboratory sit right on the edge of what is achievable.) As a result model-building has dominated the quantum Hall literature, which is very fragmentary. It is filled with different theories which are often the result of brilliant guesswork and intuition – but the interrelations between these models are unclear and subtle.

In this dissertation, we will place these theories on a firmer footing, and work towards making the patchwork into a coherent whole. We will aim to understand the

relationships between these different theories at a deeper level – and see what else we learn along the way.

The Perspective of This Dissertation

We will investigate the many different models of the fractional quantum Hall effect which have emerged over the past three decades, but from a perhaps atypical perspective: that of a high-energy theorist. This will show us new ways to understand the deep and intricate connections between several approaches to understanding quantum Hall states: microscopic wavefunctions [9], low-energy effective Chern-Simons theories [10, 11, 12, 13, 14] and boundary conformal theories [15, 16].

The key idea is to model the electrons as vortices, allowing us unprecedented control over their behaviour. Pleasingly, or possibly confusingly, this means we are assembling topologically ordered states from topologically protected ones.

We will also find ourselves in a position to offer some novel models of these theories, presenting in Part IV both a new *matrix model* description of a wide class of non-Abelian states and a new fermionic Chern-Simons-matter field theory description of the same states.

Matrix models go back to those of Pasquier and Haldane [17] (see also [18]) to describe the compressible state at half-filling. Subsequently, Polychronakos introduced a matrix model for the Laughlin states [19], inspired by earlier work [20]. These elegantly capture the appropriate topological order of certain quantum Hall states. What we will see can be viewed a generalization of these matrix models to a wide class of non-Abelian quantum Hall states.

We will also see that the matrix model gives one more mathematical control over these states, exhibiting in Part III and Appendix C an explicit analytic calculation of the Abelian quasihole charge and statistics, circumventing the need for the plasma analogy.

The fermionic theory, meanwhile, is perhaps reminiscent of some early work on partonic models of the Abelian quantum Hall effect [21, 22]. However, it appears to also be a genuinely new model.

Along the way we will pick up some bonus facts of interest beyond condensed matter: in the investigations of edge theories in Part V, we will find these matrix models offer an alternative description of Wess-Zumino-Witten theories. Also, the work on bosonization in Parts II and IV leads to a generalization of certain three-dimensional dualities which were recently proposed [23], and which are the topic of extensive current research.

Supersymmetry and Conformal Phases

Supersymmetry is a much beloved tool of high energy theorists. Supersymmetric field theories are often tractable, even at strong coupling, yet remain rich enough to exhibit a wide range of interesting dynamics.

In contrast, supersymmetric theories are much less studied in the condensed matter community, even in the limited role of toy models for strongly coupled phenomena. In part this is because supersymmetry typically provides analytic control for relativistic theories at vanishing chemical potential. At finite density, where most problems of interest in condensed matter lie, supersymmetry is usually broken and any advantage it brings is lost.[1] And of course, if supersymmetry is not broken, then most likely it is of questionable use in understanding the real world.

There is, however, a class of theories in $d = 2 + 1$ dimensions which are supersymmetric, yet *non-relativistic* [30]. In these theories, supersymmetry is retained even at finite density. Moreover, the lack of anti-particles means that it is easy to isolate (say) the bosonic sector of the theory and retain most of the power of the supersymmetry.[2] Despite the vast literature on supersymmetric field theories, the quantum dynamics of these models remains relatively unexplored. One of our achievements will be to show that the low-energy physics of these theories is that of the fractional quantum Hall effect. This remarkable fact is what will underlie the success of the story we told above.

However, we will also find it interesting to discuss these theories at zero density, and this will actually be our first port of call after introducing them in Part II. In this regime, the physics will turn out to be scale-invariant, and in fact conformally symmetric. It is natural to be curious about the spectrum of operators in such a theory – moreover, this is equivalent to an interesting and well-known problem, namely studying the spectrum of anyons in a harmonic trap [31, 32, 33, 34, 35].

This, then, shall be our true starting point: we will investigate the spectrum of conformal anyons in a harmonic trap. In doing this, we shall introduce a class of highly symmetric theories which one can also push into a finite density phase. Investigating this phase will then lead us on to the phenomenology of the quantum Hall effect in all its remarkable complexity.

[1] There are a number of notable exceptions, including the role of supersymmetry in disorder [24], the possibility of emergent supersymmetry [25, 26, 27, 28] and the study of supersymmetry protected phases [29].

[2] This can be understood in terms of restrictions on what loop diagrams it is possible to write down in the absence of anti-particles, and hence in the absence of pair production.

Quantum Gravity

Probably the most famous thing about quantum gravity is the fact, mentioned above, that we do not know what it is. Of course, the real meaning of this is that we have not yet identified the right ultra-violet completion of quantum gravity. To put it another way, high-energy questions, such as what happens near the centre of black holes, are beyond us. The answers could be found in string theory [36], loop quantum gravity [37], asymptotic safety [38], or any number of other candidates of varying complexity and plausibility. But this leaves us some room for manoeuvre if we restrict our questions to lower energy processes: we are allowed to ask about infra-red physics [39].

Bearing this idea in mind, let us return to the early days of quantum gravity. It was realized long ago that one can import ideas from normal quantum field theory to formulate gravity in a way at least naively amenable to quantization: the path integral [40].

The idea is very natural for a quantum theorist. Following Feynman, the idea is to take your classical vision of a theory – for gravity, a geometric manifold, with a metric and connection – and define your quantum theory in terms of a sum over all possible configurations in the classical theory. One can happily use this formalism to do many simple computations, such as the computation of Casimir energies [41, 42] in the presence of compact directions.

Yet there is a fundamental and deeply problematic question which has long plagued the subject: *What spacetimes should we include in the gravitational path integral?* This has been an important question ever since Wheeler's vision of spacetime foam [43] and only became more so when gravitational instantons [44] were proposed as non-perturbative corrections to gravitational theories over 30 years ago; and it has never really found a satisfactory answer.

Moreover, calculating the contributions from these instantons has proved surprisingly tricky, despite being studied ever since 1978 when various authors like Hawking [44], Gibbons and Perry [45] drew attention to these corrections.

In spite of these mysteries, physicists have discussed many possible consequences of including topologically distinct spacetimes in the path integral. However, our lack of control over quantum gravity – especially in the high-energy regime – has been apparent in the uncertainty in assessing such proposals. Perhaps the most remarkable of these suggestions is an idea of Sidney Coleman: that summations over wormholes could explain the values taken by the observed constants of nature, most significantly

the cosmological constant [46, 47]. Earlier, Hawking had proposed [48] a slightly different mechanism, also related to wormholes, to constrain the cosmological constant. Various other speculative mechanisms relying on summations over topologies have also been mooted (e.g. [49]), but again rarely with precise computations to support them.

Is there something in ideas like these? It is difficult to be sure. Our progress is impeded by the lack of deep understanding of the theory, and the absence of any way to check our answers. It is a real problem to have no concrete calculations to guide our intuition, and no precise mathematical formulation of the rules of the game. Anyone trying to go beyond the perturbative level in effective quantum gravity is to some extent stumbling around in the dark.

A Concrete Calculation

In this dissertation, we present an attempt to improve the situation a little. Since it would be nice to offer a clean, explicit instanton computation, we carefully choose a theory where everything can be well-controlled.

Our concrete example lies in a minimal supergravity theory. The choice to work with supersymmetry eliminates some unpleasant aspects of the pure gravity theory (like high-order divergences and Casimir energies) and makes the theory amenable to analytic evaluation of various functional determinants. These nice features will allow us to neatly compute one-loop effects around instanton backgrounds. Nonetheless, the underlying physics we are investigating seems insensitive to our special choice of theory.

The other choice we get to make is what background to work in – or to put it another way, what boundary conditions to use. It is convenient to choose a simple, flat topology which we can deform in ways we understand. A 4d flat spacetime with one dimension compactified is a good choice: $\mathbb{R}^{1,2} \times S^1$.

We will think of this (in a happy return to (2+1)-dimensional physics!) as being described by an effective 3d supersymmetric theory in which the radius of the circle R is a dynamical field. We will discover that, just as that 3d theory has Dirac monopoles [50], the full 4d theory has Kaluza-Klein monopoles [51, 52]. These are the famous Taub-NUT spaces, special topological instantons involving twists in the circle factor S^1. These play an important role in the dynamics of the theory – they generate a potential for the radius R.

We will evaluate this potential, overturning in particular a long-standing mistake in Hawking and Pope's original attempt [53] to do this calculation. Along the way we will identify a new scale in quantum gravity, find a sensible set of rules governing instanton

contributions, and discover the remarkable way that the gravitational theory organizes its non-perturbative structure.

2 The Grand Plan

The five main parts of this dissertation are described in this chapter. The four sections on anyonic physics build on each other, but each part of the thesis aims to be fairly independent.

We shall begin the thesis by introducing non-relativistic versions of both supersymmetry and conformal symmetry in Part II, and studying a theory possessing both symmetries together.

Then we shall deform these theories, and get on to the main topic of the thesis: the quantum Hall effect. The main characters of our work on Hall physics are a Chern-Simons field theory and a matrix model quantum mechanics. We are going to explore and derive a web of theories of quantum Hall physics which looks something like what is shown in Figure 2.1.

The five circles here label the five different types of description we will investigate. The links between them are annotated with a brief summary of the way in which we will understand the connection and colour coded: purple edges are those which are already established; red edges indicate new connections to the matrix model which we offer in Parts III, IV and V; and green edges indicate other new relationships due to bosonization as discussed in Part IV.[1]

These relationships are all expanded upon in the following plan of attack.

Part VI, meanwhile, deals with our particular quantum gravity problem, which we study for the insight it offers into quantum gravity theories in general.

Part II

We have two main aims in the opening part of this thesis. One part of our motivation is to study the classic problem of the spectrum of anyons in a harmonic trap. This is a very nice problem, in that even for non-Abelian anyons, a portion of the spectrum can be analytically determined. This spectrum has several rather interesting properties which we will analyse. These range from apparent violations of a unitarity bound (which leads

[1] The grey edge represents a trivial further connection which arises from performing the reduction to the edge theory and the level-rank duality in the other order.

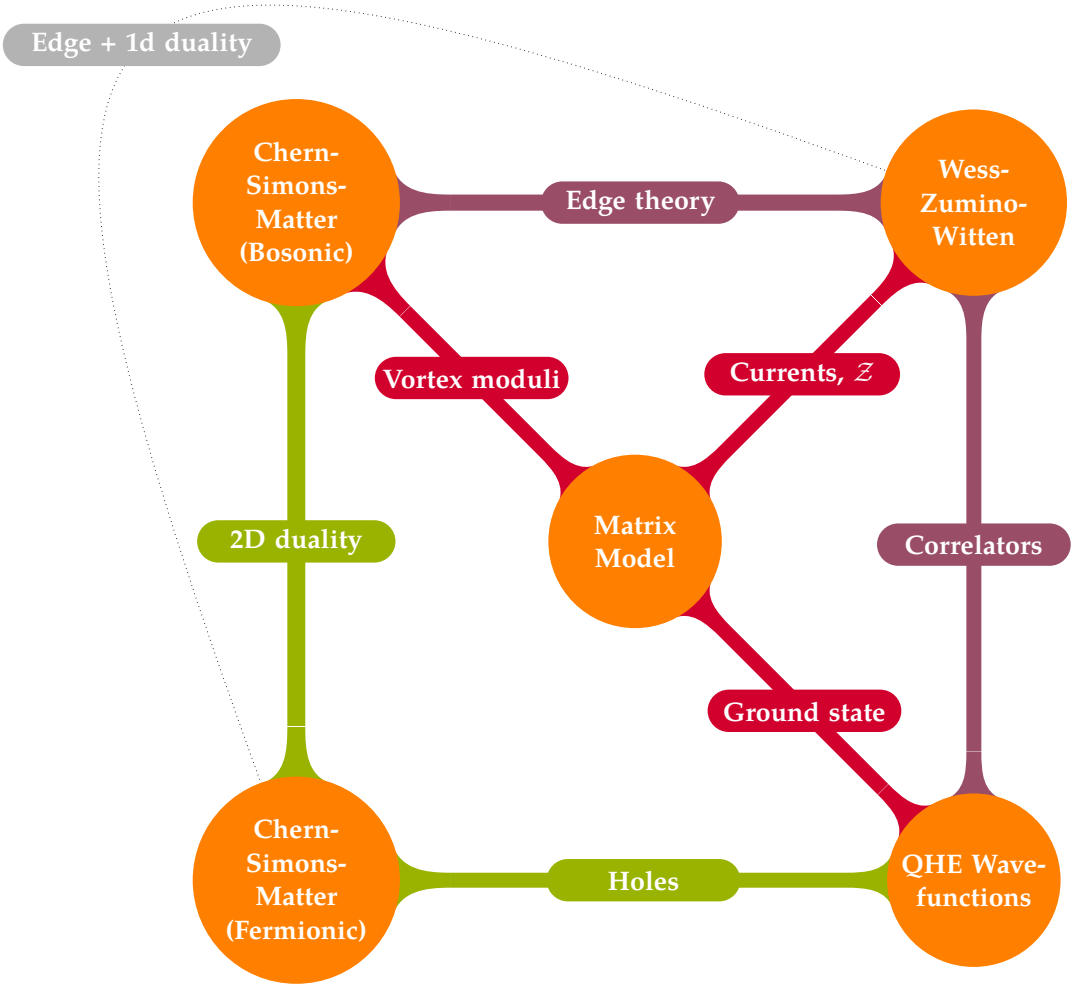


Figure 2.1: The web of ideas in Quantum Hall physics which we will explore.

to an interesting discussion of the role of certain non-topological solitons in the theory) to understanding a duality between bosonic and fermionic versions of the theories we study.

The other ambition we have for the opening chapters is to set up some of the machinery of both non-relativistic field theory and bosonization which we will draw on in our discussion of quantum Hall physics which follows on from it.

In particular, we will introduce a class of non-relativistic $d = 2{+}1$ field theories. These theories are very special, with a very particular choice of their free couplings – they sit at a conformal point, which is the end-point of a renormalization group flow. Moreover, they can even be made supersymmetric, and we shall exploit this in order to make progress with our questions about the spectrum. However, we will also see that any supersymmetric partners always decouple from the questions we ask: supersymmetry is relegated to playing an advisory role.

Part III

Here, the basic ingredients of what is to follow are introduced, and explored in the simplest case: Laughlin physics, or from the field theoretic point of view we shall begin from, Abelian Chern-Simons theories at finite density.

To begin, we show that adding a chemical potential to the theories of Part II leads us to a totally new phase whose low-energy physics exhibits the phenomenology of the Abelian fractional quantum Hall effect. Firstly, we will see that it has supersymmetric (BPS) vortices, and then investigate these in the light of old work on the moduli space of supersymmetric solitons – the upshot is that the dynamics of BPS vortices is governed by the quantum Hall matrix model!

From this, much of the familiar phenomenology of the quantum Hall effect follows. The ground state of multiple vortices is related to the Laughlin wavefunction, while the collective excitations of vortices are chiral edge modes and quasiholes. By explicit computation of the Berry phase, without resorting to the plasma analogy, we can even show that quasiholes have fractional charge and spin.

We will see that this system provides a framework in which one can map the connections between different approaches to the quantum Hall effect, from microscopic many-body physics, to the long-distance effective Chern-Simons theory, to the hydrodynamic non-commutative description.

However, we can push these techniques beyond these Abelian theories.

Part IV

Next, we propose a matrix quantum mechanics for a class of *non-Abelian* quantum Hall states. The model describes electrons which carry an internal $SU(p)$ spin. The ground states of the matrix model include spin-singlet generalizations of the Moore-Read and Read-Rezayi states and, in general, lie in a class previously introduced by Blok and Wen.

How do we understand the way these matrix models arise? The work of Part III points the way: the effective action for these states is a $U(p)$ Chern-Simons theory. We show how the matrix model can be derived from quantization of the vortices in this Chern-Simons theory. Moreover, we explain how the matrix model ground states can be reconstructed as correlation functions in the boundary Wess-Zumino-Witten (WZW) model which comes with that Chern-Simons theory.

We also look beyond the matrix model, and look at what the previously introduced bosonization dualities have to say about our quantum Hall field theories as we replace bosonic matter on one side replaced by fermionic matter on the other.

This means exploring the non-relativistic physics of these theories in the quantum Hall regime. We will have already shown that the bosonic theory lies in a condensed phase and admits vortices which form a non-Abelian quantum Hall state. We ask how this same physics arises in the fermionic theory. We find that a condensed boson corresponds to a fully filled Landau level of fermions, while bosonic vortices map to fermionic holes. We confirm that the ground state of the two theories is indeed described by the same quantum Hall wavefunction.

Part V

The links to the WZW boundary theory are much stronger than outlined above, however. We shall demonstrate that, in the large N limit, our non-Abelian matrix model becomes the chiral WZW conformal field theory.

This represents a very non-trivial generalization of the chiral boson derivation we saw in Part IV. WZW theories are very subtle, and we do not offer a construction of the WZW group-valued field in terms of matrix model degrees of freedom.

Nonetheless, the identification manifests itself clearly in two ways. First, we construct the left-moving Kac-Moody current algebra from matrix degrees of freedom. Secondly, we compute the partition function of the matrix model in terms of Schur and Kostka polynomials and show that, in the large N limit, it coincides with the partition function of the WZW model.

Part VI

The last part of this dissertation turns away from quantum Hall physics to look at instanton solutions in a rather different setting: quantum gravity.[2]

To be concrete, we will study the quantum dynamics of $\mathcal{N} = 1$ supergravity in four dimensions with a compact spatial circle. At a direct level, what we are doing amounts

[2]It is amusing to note that there is another link between the quantum Hall story and our work on quantum gravity, which can be seen by first recalling that quantum Hall states are famously quantum ordered with what is referred to as topological order. The parallel fact is the point made by Hartnoll and Ramirez that quantum gravity on a compact spatial circle actually exhibits another form of quantum order [54] which can be understood through analysing precisely the same instantons which we investigate. The argument is that to exhibit massless modes, or avoid confinement, the compactified theory should have a special property. That property is quantum order, and [54] makes the case for it.

to a quantum mechanical stability analysis of such solutions. Supersymmetry ensures that the perturbative contributions to the Casimir energy on the circle cancel. However, instanton contributions remain. These render this compactification on a circle unstable, even in the presence of supersymmetry, and the background dynamically decompactifies back to four dimensions.

Our interest, however, is ultimately in how the calculation (i) provides a testing ground for some old ideas in Euclidean quantum gravity, and (ii) provides new insight into the hidden structure of effective gravitational theories. In particular, we show that gravitational instantons are associated to a new, infra-red scale which can naturally be exponentially suppressed relative to the Planck scale. This arises from the logarithmic running of the Gauss-Bonnet term, and is generically present in any quantum gravity theory.

There are also interesting technical details to uncover, such as the non-cancellation of bosonic and fermionic determinants around the background of a self-dual gravitational instanton, despite the existence of supersymmetry. It turns out that it is nonetheless possible to complete the calculation of the superpotential capturing this instability, all the way down to factors of $\sqrt{2}$ and π. We will follow this calculation in all its glorious detail.

PART II

Non-Relativistic Anyons

3 Introduction and Summary

The quantum mechanics of multiple, interacting anyons is a wonderfully rich problem. It is simple to state but contains a wealth of interesting physics. Despite several decades of interest, it remains unsolved. The purpose of this part of the thesis is to fail to solve the harder problem of interacting non-Abelian anyons.

In this extended introduction, we will first summarize the story of Abelian anyons. These are particles which, upon an anti-clockwise exchange, pick up a phase $e^{i\theta}$. We will write $\theta = \pi/k$ so that the anyons are bosons when $k = \infty$ and fermions when $k = 1$. In a field theoretic language, anyons are described by a $U(1)$ Chern-Simons theory at level k, coupled to a non-relativistic scalar field.

We will explore the spectrum of n anyons placed in a harmonic trap. (See [31, 32] for early work on this subject, and [33, 34, 35] for reviews.) The trap has potential

$$V = \frac{\omega^2}{2}(x^2 + y^2)\,.$$

To fully specify the Hamiltonian, we also need to describe any interactions between the anyons. It turns out that the problem simplifies tremendously if the particles experience pairwise, contact interactions [55, 56, 57, 58, 59]. The strength of these interactions is determined by seeking a fixed point of an RG flow. However, the sign of the coupling is arbitrary. This leaves us with two options – attractive and repulsive interactions – exhibiting interesting and different physics.

As an aside, we should mention that when these contact interactions are turned on, the quantum mechanics has an $SO(2,1)$ conformal invariance of the type first introduced in [60] and subsequently explored in [61]. This conformal invariance will play an important role in our approach, and indeed the entirety of Chapter 4 is devoted to discussing its nature, but we will not focus on it for the rest of this introduction.

Perhaps the best way to illustrate the physics of anyons is simply to look at the spectrum. Low-lying states were computed numerically for $n = 3$ anyons with repulsive interactions by Sporre, Verbaarschot, and Zahed [62]. Their results are shown in Figure 3. (A similar plot for $n = 4$ anyons can be found in [63].) The energy E is plotted vertically and the statistical parameter $\theta \in [0, \pi]$ is plotted horizontally. The spectrum on the far left coincides with that of free bosons; on the far right it coincides with free fermions. In between, things are more interesting.

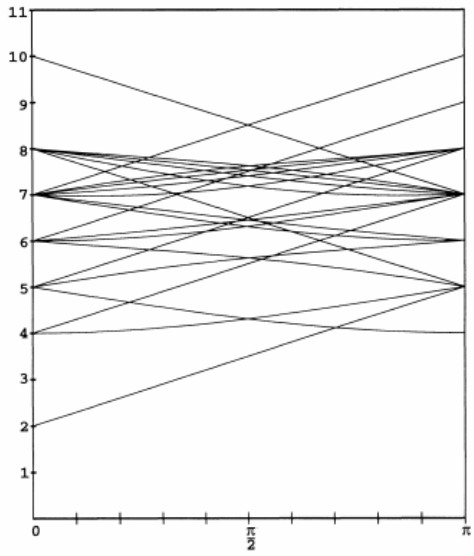


Figure 3.1: Low-lying energy levels of 3 anyons in a harmonic trap. In terms of our conventions, the plot actually shows $E - \omega$, measured in units of ω. Taken from [62].

This plot contains some things that are easy to understand and some things that are hard. Let's start with the hard. The most striking feature is that there is a level crossing of the ground state as θ is increased. Roughly speaking this occurs because the anyons have an intrinsic angular momentum that scales as θ. As we increase θ, we increase both the angular momentum and the energy of the state. For some value of θ, both of these can be lowered if the particles start orbiting in the opposite direction to their intrinsic spin. This is where the ground state level crossing occurs. A similar level crossing is expected for all n, but little is known beyond these numerical results.

Some Simple States

In contrast, some aspects of the spectrum are fairly easy to understand. In particular, there are a number of states whose energy varies linearly with θ. Among these is the small-θ ground state, but not the large-θ ground state which takes over after the level crossing. For obvious reasons, these are sometimes referred to as "linear states" [64, 65]. They persist in the spectrum of n anyons and, in all cases, their wavefunctions and energies are known exactly. For example, in the n anyon quantum mechanics with repulsive interactions, the ground state close to the bosonic end of the spectrum (i.e. for suitably large k) has energy

$$E = \left(n + \frac{n(n-1)}{2k} \right) \omega \,. \tag{3.1}$$

Here the first term is simply the ground state energy of n particles in a two-dimensional harmonic trap (it is $2 \times \frac{1}{2}\hbar\omega$ for each particle, with $\hbar = 1$). The second term can be thought of as a correction due to the inherent angular momentum of the particles.

The fact that some states in the spectrum have such a simple expression for their energy strongly suggests that there is some underlying symmetry that protects them. Indeed there is: it is supersymmetry! This is particularly surprising given that the anyonic quantum mechanics does not have supersymmetry, but is nonetheless true. The reasoning starts with the observation that it possible to write down a supersymmetric theory of two species of anyons whose spins differ by $1/2$ [30]. When restricted to states involving just one species of anyons, this reduces to our problem of interest. Such a statement would not be true in relativistic theories, in which particle-anti-particle pair creation prevents other fields from decoupling at the loop level. However, the lack of anti-particles means that it does hold in our non-relativistic theories. The supersymmetric theory of anyons has short, BPS multiplets whose energies are fixed in terms of their quantum numbers [66, 67]. These BPS states coincide with the "linear states" in the anyon spectrum [1].

It's worth explaining in more detail how this arises. For n anyons, the BPS states have energy given by

$$E = (n - \mathcal{J})\,\omega \tag{3.2}$$

with $\mathcal{J}$ the total angular momentum of n anyons. One of the surprising properties of the angular momentum of anyons is that it does not add linearly. Instead, one finds that $\mathcal{J} \sim n^2$ for large n, together with some sub-leading corrections which are more subtle and depend, even classically, on a choice of regularization procedure [68, 69, 70]. (We will review this in some detail later.) In the present context, a careful analysis shows that

$$\mathcal{J} = -\frac{n(n-1)}{2k}$$

so that the BPS bound (3.2) indeed reproduces the energy spectrum (3.1).

Non-Abelian Anyons

The purpose of this part of the thesis is to rederive the above results, and extend the discussion to non-Abelian anyons. The simplest way to construct such particles is to couple fields to a non-Abelian Chern-Simons theory. For example, in Chapter 5, we will consider an $SU(p)_k$ Chern-Simons theory coupled to scalar fields. Each of these scalar fields transforms in some representation R of $SU(p)$.

Suppose that we place n non-Abelian anyons in a harmonic trap, each labelled by some representation R_i with $i = 1, \ldots, n$. We once again tune the contact interactions so that the theory sits at an RG fixed point. Our goal is to understand the energy spectrum.

We will fall short of this goal. As with Abelian anyons, there are many questions that we are unable to answer analytically, such as those about possible level crossings in the ground state of the system. We will, however, show that there are states in the spectrum analogous to (3.1) whose energy can be determined exactly. We show that the energy of these states again takes the form $E = (n - \mathcal{J})\omega$ but this still leaves open the problem of determining the angular momentum $\mathcal{J}$ of n non-Abelian anyons. This is determined by some simple group theory.

Suppose, for example, that we place $n = 2$ anyons in a trap with representations R_1 and R_2. The possible representations of the resulting bound states are determined by the decomposition of the tensor product $R_1 \otimes R_2$. The angular momentum of the bound state in the irreducible representation $R \subset R_1 \otimes R_2$ turns out to be

$$\mathcal{J} = -\frac{C_2(R) - C_2(R_2) - C_2(R_1)}{2k} \tag{3.3}$$

where $C_2(R)$ is the quadratic Casimir of the representation R. This, in turn, determines the energy of this state using (3.2). We will see that there is a straightforward generalization of this result to n anyons, each of which sits in a different representation.

Our work on anyons is primarily devoted to telling the story above and providing a number of examples. The tools we will use are those of non-relativistic field theory, rather than non-relativistic quantum mechanics. In Chapter 4, we review the properties of field theories that enjoy a non-relativistic $SO(2,1)$ conformal symmetry. This conformal extension of the Galilean symmetry is known as the Schrödinger symmetry. The state-operator map in such theories allows us to translate the problem of the spectrum of anyons in a harmonic trap to the problem of computing the scaling dimension of certain operators.

In Chapter 5, we consider a bosonic Chern-Simons matter theory. Much of this chapter is devoted to proving the result (3.3) for the angular momentum of two anyons, as well as its generalization to n anyons. We use this to determine the energy of these states, and confirm our results with explicit one-loop computations. In Chapter 6 we repeat this story for fermionic Chern-Simons-matter theories, and discuss how bosonization relates the results to those of the bosonic theory.

4 Non-Relativistic Conformal Invariance

We wish to investigate the spectrum of non-Abelian anyons in a harmonic trap. The most natural setting to address this problem is Chern-Simons theory, where flux attachment and the associated Aharonov-Bohm effect give rise to the desired non-Abelian statistics.

The theories we will study have a non-relativistic conformal invariance. We will describe these theories in some detail below. In this chapter, we start by reviewing some basic aspects of conformal invariance in non-relativistic field theories, following the seminal work of Nishida and Son [71].

For high-energy theorists, used to studying relativistic conformal field theories, some aspects of their non-relativistic counterparts can be a little counter-intuitive. In an attempt to reorient these readers, we begin by stating the blindingly obvious. First, non-relativistic field theories, conformal or otherwise, describe the dynamics of massive particles. Second, these theories do not have anti-particles. This means that much of the subtlety of relativistic quantum field theory disappears. Indeed, if we choose to focus on a sector of a non-relativistic theory with a fixed particle number, then the theory reduces to quantum mechanics. Nonetheless, the field theoretic description is often more useful and, despite the very obvious differences described above, there are ultimately similarities between relativistic and non-relativistic conformal theories.

For simplicity, suppose that all particles have the same mass m. We introduce the particle density $\rho(\mathbf{x})$ and momentum density $\mathbf{j}(\mathbf{x})$, where we are working in the Schrödinger picture so that field theoretic operators do not depend on time. From these we can build the familiar conserved charges corresponding to particle number $\mathcal{N}$, total momentum $\mathbf{P}$ and angular momentum $\mathcal{J}$:

$$\mathcal{N} = \int \mathrm{d}^2x\, \rho(\mathbf{x}) \quad , \quad \mathbf{P} = \int \mathrm{d}^2x\, \mathbf{j}(\mathbf{x}) \quad , \quad \mathcal{J} = \int \mathrm{d}^2x\, \mathbf{x} \times \mathbf{j}(\mathbf{x}) + \Sigma$$

where Σ is the spin of the fields. (For us, it will be half the fermion number.) As in any quantum system, time evolution is implemented by the Hamiltonian H. The continuity equation then reads

$$i[H, \rho] + \nabla \cdot \mathbf{j} = 0\,.$$

In a conformal field theory, there are three further, less familiar, generators that we can also build from ρ and $\mathbf{j}$. These are the generators of Galilean boosts $\mathbf{G}$, the dilatation operator D and the special conformal generator C, defined as

$$\mathbf{G} = \int \mathrm{d}^2x\, \mathbf{x}\,\rho(\mathbf{x}) \quad , \quad D = \int \mathrm{d}^2x\, \mathbf{x}\cdot\mathbf{j}(\mathbf{x}) \quad , \quad C = \frac{m}{2}\int \mathrm{d}^2x\, \mathbf{x}^2\rho(\mathbf{x})\,. \tag{4.1}$$

To these we should add the Hamiltonian H. In a conformal field theory, these generators obey the algebra

$$\begin{aligned} &i[D,\mathbf{P}] = -\mathbf{P} \quad , \quad i[D,\mathbf{G}] = +\mathbf{G} \quad , \quad i[D,H] = -2H \quad , \quad i[D,C] = +2C \\ &i[C,\mathbf{P}] = -\mathbf{G} \quad , \quad [H,\mathbf{G}] = -i\mathbf{P} \quad , \quad [H,C] = -iD \quad , \quad [P_p,G_q] = -im\mathcal{N}\delta_{pq} \end{aligned} \tag{4.2}$$

with all other commutators that don't involve $\mathcal{J}$ vanishing. This is sometimes referred to as the Schrödinger algebra. The triplet of operators H, D and C form an $SO(2,1)$ subgroup. The commutators $[\mathcal{J},\mathbf{P}]$ and $[\mathcal{J},\mathbf{G}]$ are non-zero and tell us that $\mathbf{P}$ and $\mathbf{G}$ transform as vectors.

4.1 States and Operators

In such theories, the spectrum of the Hamiltonian is necessarily continuous. Instead, as with their relativistic counterparts, the interesting questions lie in the spectrum of the dilatation operator D.

We consider local operators, evaluated at the origin: $\mathcal{O} = \mathcal{O}(\mathbf{x}=0)$. These operators can be taken to have fixed particle number $n_\mathcal{O}$ and angular momentum $j_\mathcal{O}$, defined by

$$[\mathcal{J},\mathcal{O}] = j_\mathcal{O}\mathcal{O} \quad , \quad [\mathcal{N},\mathcal{O}] = n_\mathcal{O}\mathcal{O}\,.$$

Unitarity restricts $n_\mathcal{O} \geq 0$. This is the statement that there are no anti-particles in the theory.

More interesting are the transformations under dilatations. We say that the operators have *scaling dimension* $\Delta_\mathcal{O}$ if they obey

$$i[D,\mathcal{O}] = -\Delta_\mathcal{O}\mathcal{O}\,.$$

If we find one operator $\mathcal{O}$ with definite scaling dimension, then the algebra (4.2) allows us to construct an infinite tower of further operators with the same property. Both H and $\mathbf{P}$ act as raising operators: $[H,\mathcal{O}]$ has scaling dimension $\Delta_\mathcal{O}+2$ and $[\mathbf{P},\mathcal{O}]$ has scaling dimension $\Delta_\mathcal{O}+1$. In contrast, both C and $\mathbf{G}$ act as lowering operators: $[C,\mathcal{O}]$ has scaling dimension $\Delta_\mathcal{O}-2$ while $[\mathbf{G},\mathcal{O}]$ has scaling dimension $\Delta_\mathcal{O}-1$.

The spectrum of D must be bounded below. Indeed, a simple unitarity argument [72] shows that

$$\Delta_{\mathcal{O}} \geq 1\,. \tag{4.3}$$

This means that there must be operators sitting at the bottom of the tower which obey

$$[\mathbf{G}, \mathcal{O}] = [C, \mathcal{O}] = 0\,.$$

Such operators are called *primary* [71, 73]. The other operators in the tower are called *descendants*; they can be constructed by acting with H and $\mathbf{P}$. The full tower built in this way is an irreducible representation of the Schrödinger algebra.

4.1.1 The State-Operator Map

One of the most beautiful aspects of relativistic conformal field theories is the state operator map. This equates the spectrum of the dilatation operator on the plane to the spectrum of the Hamiltonian when the theory is placed on a sphere.

There is also such a map in non-relativistic conformal field theories which, if anything, is even more simple. First, the algebra: we define a modified Hamiltonian

$$L_0 = H + C\,. \tag{4.4}$$

For each local, primary operator $\mathcal{O}(0)$, we define the state $|\Psi_{\mathcal{O}}\rangle = e^{-H}\mathcal{O}(0)|0\rangle$. Then it is simple to check that

$$L_0|\Psi_{\mathcal{O}}\rangle = \Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle\,. \tag{4.5}$$

Further, $\mathcal{J}|\Psi_{\mathcal{O}}\rangle = j_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$ and $\mathcal{N}|\Psi_{\mathcal{O}}\rangle = n_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$.

Now the physics: we view L_0 as a new Hamiltonian, with a very simple interpretation. This follows from the definition of C in (4.1) which tells us that we have taken the original theory, defined by H, and placed it in a harmonic trap. (We have used conventions where the strength of the harmonic trap is $\omega = 1$.) The spectrum of particles in this harmonic trap is equal to the spectrum of the dilatation operator. This was first pointed out for field theories in [71], although the analogous statement in quantum mechanics can be traced back to the earliest work on conformal invariance [60].

In relativistic theories, we are very used to the state-operator map holding only for local operators. This limitation is usually thought to also hold in the non-relativistic framework considered here. However, in Chapter 5, we will see that we can also apply this map to certain Wilson line operators.

The tower of descendant operators maps into a tower of higher energy states in the trap. There are two ways to raise the energy. The first is to construct states which sit further out in the trap. This is achieved by constructing the raising and lowering operators

$$L_\pm = H - C \pm iD \quad \Rightarrow \quad \begin{cases} \left[L_0, L_\pm\right] = \pm 2 L_\pm \\ \left[L_+, L_-\right] = -4 L_0 \end{cases}$$

from our algebra. The second way is to take a given state and make it oscillate backwards and forwards. This is achieved by introducing an unusual sort of complexified momentum

$$\tilde{\mathbf{P}} = \mathbf{P} + i\mathbf{G} \quad \Rightarrow \quad \begin{cases} \left[L_0, \tilde{\mathbf{P}}\right] = \tilde{\mathbf{P}} \\ \left[L_0, \tilde{\mathbf{P}}^\dagger\right] = -\tilde{\mathbf{P}}^\dagger \end{cases}$$

which is built from the momentum and boost generators of the theory. The primary states sit at the bottom of this tower and obey $L_-|\Psi_\mathcal{O}\rangle = \tilde{\mathbf{P}}^\dagger|\Psi_\mathcal{O}\rangle = 0$. Acting on these primary states with L_+ and $\tilde{\mathbf{P}}$ raises the energy, filling out the representation of the Schrödinger algebra.

4.1.2 Unitarity Bounds and Anti-Particles

The algebra alone is enough to force additional constraints of positivity on the physical states of any non-relativistic conformal theory: these are referred to as *unitarity bounds*. The first result follows from considering the action of $\tilde{\mathbf{P}}$ upon an arbitrary state, and is particularly simple:

$$\left|\left|\, \tilde{\mathbf{P}}|\Psi_\mathcal{O}\rangle \,\right|\right|^2 \geq 0 \quad \Rightarrow \quad n_\mathcal{O} \geq 0\,.$$

This is the result that the theory contains no anti-particles, which one should of course expect in a non-relativistic theory: by definition, one works at energies much lower than those required for anti-particles to be relevant. This simple observation will be crucial for much of what follows. In particular, it massively reduces the number of potential loop corrections, and means that one may safely restrict to sectors of fixed particle number (reducing the field theory to quantum mechanics). We will return to these points later.

The other unitarity bound is obtained by considering also $L_\pm$. Assuming that we are not in the vacuum state, so $n_\mathcal{O} \neq 0$, we have [72]

$$\left|\left|\, 2m\mathcal{N}L_+ - \tilde{\mathbf{P}}^2|\Psi_\mathcal{O}\rangle \,\right|\right|^2 \geq 0 \quad \Rightarrow \quad \Delta_\mathcal{O} \geq 1\,. \tag{4.6}$$

Further, states that saturate this bound obey the equation

$$L_+ |\Psi_{\mathcal{O}}\rangle = \frac{1}{2mn_{\mathcal{O}}} \tilde{\mathbf{P}}^2 |\Psi_{\mathcal{O}}\rangle \,. \tag{4.7}$$

This looks, formally, like the Schrödinger equation for a free particle. (Recall that in relativistic theories, saturation of a unitarity bound indicates that the operator is free.) In Section 5.5, we will find that this bound is naively violated by some simple operators, which leads to some surprising physics, and these ideas will help us come up with a possible interpretation.

4.2 Supersymmetry

We have seen how to form an algebra with conformal symmetry in a non-relativistic theory, but it will be extremely useful to us to look at a supersymmetric extension of the algebra. This is called the super-Schrödinger algebra; some early papers on these sorts of structures include [74, 30, 75].

4.2.1 The Superconformal Algebra

Firstly, we can consider two fermionic charges Q_1, Q_2. We will call them respectively the *kinematical* and *dynamic* supercharges. We will take them to satisfy the algebra

$$\{Q_1, Q_1^\dagger\} = \frac{m}{2}\mathcal{N} \quad , \quad \{Q_2, Q_2^\dagger\} = H \quad , \quad \{Q_1, Q_2^\dagger\} = P \tag{4.8}$$

with other terms vanishing. (Here we complexify $P = \frac{1}{2}(P_1 - iP_2)$.) The charges both commute with H, P and $\mathcal{N}$. (The possibility of the non-relativistic supersymmetry generator Q_1 seems to have been first raised in [76] where it is also pointed out that this generator is spontaneously broken in any vacuum with non-vanishing particle number.)

The fermionic charges form a spinor, which means they are expected to have half-integer spin such that

$$[\mathcal{J}, Q_1] = -\frac{1}{2}Q_1 \quad \text{and} \quad [\mathcal{J}, Q_2] = \frac{1}{2}Q_2 \tag{4.9}$$

as well as a particular behaviour under Galilean boosts $G = \frac{1}{2}(G_1 - iG_2)$ [30], namely

$$i[G, Q_1] = 0 \quad \text{and} \quad i[G, Q_2] = -Q_1 \,.$$

So far we have considered the structure of the supersymmetric theory without the conformal parts of the algebra, namely C and D. Introducing these, we find that the

dilatation operator does not introduce anything new since

$$i[D, Q_1] = 0 \quad \text{and} \quad i[D, Q_2] = -Q_2\,,$$

but the special conformal operator does:

$$i[C, Q_1] = 0 \quad \text{but} \quad i[C, Q_2] = S\,.$$

This third fermionic charge S is the superconformal generator of the theory. It extends the SUSY algebra with three new relations,

$$\{S, S^\dagger\} = C \quad , \quad \{Q_1, S^\dagger\} = -G \quad , \quad \{Q_2, S^\dagger\} = \frac{i}{2}\left(iD - \mathcal{J} + \mathcal{R}\right)$$

where $\mathcal{R}$ is an R-charge under which the SUSY generators are charged as

$$[\mathcal{R}, Q_1] = \frac{3}{2}Q_1 \quad , \quad [\mathcal{R}, Q_2] = \frac{3}{2}Q_2 \quad , \quad [\mathcal{R}, S] = \frac{3}{2}S\,.$$

We will see this R-charge is roughly the difference between the number of bosons and number of fermions in the states of the supersymmetric theory. The factor of $3/2$ appearing here is sometimes absorbed into the definition of $\mathcal{R}$.

The remaining non-trivial commutators between bosonic and fermionic generators are

$$i[D, S] = S \quad , \quad i[H, S] = -Q_2 \quad , \quad i[P, S] = -Q_1\,. \tag{4.10}$$

4.2.2 The State-Operator Map Revisited

It is helpful to also adapt the superconformal algebra we have discussed to the situation where the Hamiltonian is $L_0 = H + C$ as discussed above. This is easy enough; one simply defines

$$\mathcal{Q} = Q_2 - iS \quad \text{and} \quad \mathcal{S} = Q_2 + iS\,. \tag{4.11}$$

These obey the algebra

$$\begin{aligned} \{\mathcal{Q}, \mathcal{Q}^\dagger\} &= L_0 + (\mathcal{J} - \mathcal{R}) \quad , \quad \{\mathcal{Q}, \mathcal{S}^\dagger\} = L_+, \\ \{\mathcal{S}, \mathcal{S}^\dagger\} &= L_0 - (\mathcal{J} - \mathcal{R}) \quad , \quad \{\mathcal{Q}^\dagger, \mathcal{S}\} = L_-\,. \end{aligned} \tag{4.12}$$

Their commutators with the generators L_0 and $L_\pm$ are given by

$$\begin{aligned} &[L_0, \mathcal{Q}] = \mathcal{Q} \quad , \quad [L_0, \mathcal{Q}^\dagger] = -\mathcal{Q}^\dagger \quad , \quad [L_0, \mathcal{S}] = -\mathcal{S} \quad , \quad [L_0, \mathcal{S}^\dagger] = \mathcal{S}^\dagger, \\ &[L_+, \mathcal{Q}] = 0 \quad , \quad [L_-, \mathcal{S}] = 0 \quad , \quad [L_-, \mathcal{Q}] = 2\mathcal{S} \quad , \quad [L_+, \mathcal{S}] = -2\mathcal{Q}\,. \end{aligned} \tag{4.13}$$

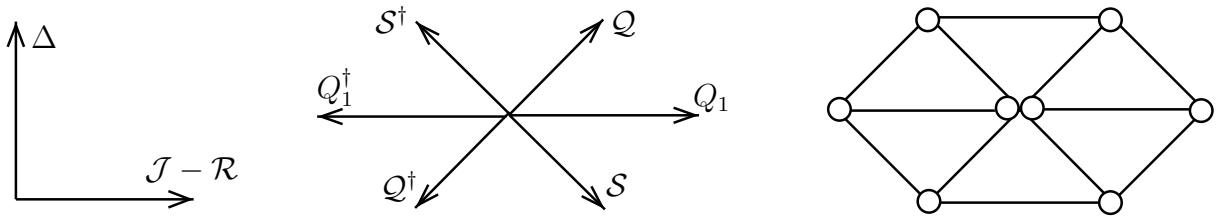


Figure 4.1: A generic supersymmetric multiplet, following [67].

We see that, acting on an eigenstate of L_0, the operators $\mathcal{Q}$ and $\mathcal{S}^\dagger$ raise the energy, while $\mathcal{Q}^\dagger$ and $\mathcal{S}$ lower the energy. The upshot is that a superconformal primary operator gives rise to a superconformal primary state, sitting at the bottom of a tower and obeying

$$L_-|\Psi_\mathcal{O}\rangle = \mathcal{Q}^\dagger|\Psi_\mathcal{O}\rangle = \mathcal{S}|\Psi_\mathcal{O}\rangle = 0\,. \tag{4.14}$$

Representations of the super-Schrödinger algebra sit in supersymmetric multiplets, built on these superconformal primary states [66, 67]. There is a unique trivial multiplet: the vacuum state, which is annihilated by all supercharges and, in our theory, has quantum numbers $\Delta = n = j = r = 0$.

A generic excited state sits in a long multiplet. This contains 8 primary states. The action of the supercharges Q_1, $\mathcal{Q}$ and $\mathcal{S}$ on these states is, following [67], shown in Figure 4.1.

4.2.3 Chiral Primary Operators and Another Unitarity Bound

There are also short multiplets in which the dimension of the superconformal primary is fixed by the superconformal algebra. These are the states that interest us here. A chiral primary operator gives rise to a chiral primary state obeying, in addition to (4.14),

$$[Q_2, \mathcal{O}]_\pm = 0 \quad \Leftrightarrow \quad \mathcal{Q}|\Psi_\mathcal{O}\rangle = 0\,.$$

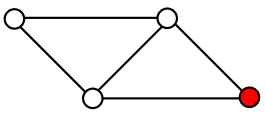


Figure 4.2: A chiral multiplet.

(The brackets on the left here are commutators or anti-commutators according to whether $\mathcal{O}$ is bosonic or fermionic. Note that the shift from Q_2 on operators to $\mathcal{Q}^\dagger$ acting on states works due to the factor of e^{-H} in the state-operator map, and the fact $[H, S] \sim Q_2$.) The associated multiplet contains four primary states, as shown in the figure. Of these, one is special, denoted by the red dot; its quantum numbers are dictated by the algebra (4.12) and satisfy

$$\Delta_\mathcal{O} = -\left(j_\mathcal{O} - r_\mathcal{O}\right)\,. \tag{4.15}$$

We will see some simple examples of chiral primary operators shortly, when we look at

the class of theories we are interested in.

An anti-chiral primary operator gives rise to an anti-chiral primary state which obeys, in addition to (4.14),

$$[Q_2^\dagger, \mathcal{O}] = 0 \quad \Leftrightarrow \quad \mathcal{S}^\dagger |\Psi_\mathcal{O}\rangle = 0\,.$$

There are again four primary states in the multiplet, as shown in the figure. One of these, denoted by the red dot, obeys

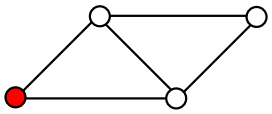


Figure 4.3: A anti-chiral multiplet.

$$\Delta_\mathcal{O} = +\left(j_\mathcal{O} - r_\mathcal{O}\right)\,. \tag{4.16}$$

Finally, we note that the supersymmetric structure of the theory also introduces an additional unitarity bound. It is easy to see from the algebra that

$$\left.\begin{array}{l} \langle\Psi_\mathcal{O}|\{\mathcal{Q}, \mathcal{Q}^\dagger\}|\Psi_\mathcal{O}\rangle \geq 0 \\ \langle\Psi_\mathcal{O}|\{\mathcal{S}, \mathcal{S}^\dagger\}|\Psi_\mathcal{O}\rangle \geq 0 \end{array}\right\} \quad \Rightarrow \quad \Delta_\mathcal{O} \geq |j_\mathcal{O} - r_\mathcal{O}|$$

where this bound is saturated by (anti-)chiral primary states. The relative sign of $j_\mathcal{O} - r_\mathcal{O}$ depends upon whether the state is anti-chiral or chiral.

4.3 A Non-Relativistic Superconformal Action

There is a very natural class of actions which satisfy the full superconformal symmetry described in the previous section. Starting from Chern-Simons theories coupled to gapped, relativistic matter, one may take a non-relativistic limit in which anti-particles decouple but particles remain. Surprisingly, supersymmetry not only survives this limit but is enhanced to a non-relativistic superconformal (or super-Schrödinger) symmetry. The first construction of this type was presented in [30] for an $\mathcal{N} = 2$ Abelian Chern-Simons theory coupled to a single chiral multiplet. Subsequent generalizations to other gauge groups, and different amounts of supersymmetry, were described in [67, 77, 78, 79, 80].

The process of taking a non-relativistic limit is outlined in Appendix A, if the reader is interested in seeing how this is done.

We will be specifically interested in the cases of $SU(p)_k$, $U(1)_k$ and their $U(p)$ products. The Chern-Simons action takes the familiar form

$$S_{CS} = -\frac{k}{4\pi}\int \mathrm{d}^3x \ \mathrm{Tr}\ \epsilon^{\mu\nu\rho}\left(a_\mu\partial_\nu a_\rho - \frac{2i}{3}a_\mu a_\nu a_\rho\right)\,. \tag{4.17}$$

Of course, we are interested in coupling this to non-relativistic matter. This will take the form of N_f multiplets (ϕ_i, ψ_i), with $i = 1, \ldots, N_f$, each consisting of a scalar field and a fermion. Each of them transforms in some representation R_i under the gauge group. We will denote the corresponding generators as $t^\alpha[R_i]$ (where for $SU(p)$ for example $\alpha = 1, \ldots, p^2 - 1$) and we have suppressed the matrix indices. We will often simplify $t^\alpha[R_i] \to t^\alpha$ where the representation may be inferred from contractions. The generators in the fundamental representation are normalized such that $\mathrm{Tr}\ t^\alpha t^\beta = \delta^{\alpha\beta}$.

Each field is endowed with a non-relativistic kinetic term – the key difference for scalars is that they are first order in time; for fermions the key point is that they carry no spinor index. For simplicity, we give each particle the same mass m. The action is given by

$$S = S_{CS} + \int \mathrm{d}t\mathrm{d}^2x \left\{ i\phi_i^\dagger \mathcal{D}_0 \phi_i + i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{1}{2m}\left(\vec{\mathcal{D}}\phi_i^\dagger \vec{\mathcal{D}}\phi_i + \vec{\mathcal{D}}\psi_i^\dagger \vec{\mathcal{D}}\psi_i - \psi_i^\dagger f_{12} \psi_i \right) \right. \tag{4.18}$$
$$\left. - \frac{\pi}{mk}\left((\phi_i^\dagger t^\alpha \phi_i)(\phi_j^\dagger t^\alpha \phi_j) + (\phi_i^\dagger t^\alpha \phi_i)(\psi_j^\dagger t^\alpha \psi_j) + 2(\psi_i^\dagger t^\alpha \phi_i)(\phi_j^\dagger t^\alpha \psi_j) \right) \right\} .$$

The quartic terms give rise to carefully tuned delta-function interactions between particles, as we will discuss later. Their special nature is especially clear when one writes the Hamiltonian in complex coordinates. If one uses Gauss's law, the Hamiltonian may be written as

$$H = \frac{2}{m} \int \mathrm{d}^2x \ |\mathcal{D}_z \phi_i|^2 + |\mathcal{D}_{\bar{z}} \psi_i|^2 + \frac{\pi}{k} (\psi_i^\dagger t^\alpha \phi_i)(\phi_j^\dagger t^\alpha \psi_j) \, . \tag{4.19}$$

In this theory, the generators of the conformal algebra are constructed from the particle density and momentum current

$$\rho = \phi_i^\dagger \phi_i + \psi_i^\dagger \psi_i \quad \text{and} \quad \mathbf{j} = -\frac{i}{2}\left(\phi_i^\dagger \vec{\mathcal{D}} \phi_i - (\vec{\mathcal{D}}\phi_i^\dagger)\phi_i + \psi_i^\dagger \vec{\mathcal{D}} \psi_i - (\vec{\mathcal{D}}\psi_i^\dagger)\psi_i \right)$$

together with the spin $\Sigma = \frac{1}{2}\int \mathrm{d}^2x \, \psi_i^\dagger \psi_i$.

The generator which will be of most interest to us is the angular momentum, which we write as

$$\mathcal{J} = \int \mathrm{d}^2x \left(\phi_i^\dagger (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})\phi_i + \psi_i^\dagger (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})\psi_i + \frac{1}{2}\psi_i^\dagger \psi_i \right) . \tag{4.20}$$

The last term here shows we have indeed given the fermions spin $+\frac{1}{2}$.

The supercharges are extremely simple to write down:

$$Q_1 = i\sqrt{\frac{m}{2}} \int \mathrm{d}^2x \, \phi_i^\dagger \psi_i \quad \text{and} \quad Q_2 = \sqrt{\frac{2}{m}} \int \mathrm{d}^2x \, \phi_i^\dagger \mathcal{D}_{\bar{z}} \psi_i \, .$$

Note that since a_0 is not a dynamical field, the supercharges do not specify the transformation of it under the two supersymmetries; we simply choose it to make the action invariant. This is of no consequence provided we impose Gauss's law of course, since this is what a_0 multiplies in the action. However, subtleties to do with Gauss's law will return to bite us in Part III.

The kinematical supersymmetry is the simpler of the two. To be explicit, the variation it generates with parameter $\sqrt{2/m}\epsilon_1$ is

$$\delta_1\phi_i = i\epsilon_1^\dagger\psi_i \quad , \quad \delta_1\psi_i = -i\epsilon_1\phi_i \quad , \quad \delta_1 a_z = 0 \quad , \quad \delta_1 a_0^\alpha = \frac{\pi i}{mk'}\left(\epsilon_1\psi^\dagger t^\alpha\phi_i - \epsilon_1^\dagger\phi^\dagger t^\alpha\psi_i\right) .$$

(This structure, especially for fundamental matter, is reminiscent of the Green-Schwarz spacetime supersymmetry on the string worldsheet.)

Under the dynamical supersymmetry, the fields transform as

$$\begin{gathered}\delta_2\phi_i = \epsilon_2^\dagger\mathcal{D}_{\bar{z}}\psi_i \quad , \quad \delta_2\psi_i = \epsilon_2\mathcal{D}_z\phi_i \quad , \quad \delta_2 a_z^\alpha = -\tfrac{i\pi}{k'}\epsilon_2^\dagger\phi_i^\dagger t^\alpha\psi_i, \\ \delta_2 a_0^\alpha = \tfrac{i\pi}{mk'}\left(\epsilon_2^\dagger\,\phi_i^\dagger t^\alpha(\mathcal{D}_{\bar{z}}\psi_i) - \epsilon_2\,(\mathcal{D}_z\psi_i^\dagger)t^\alpha\phi_i\right) .\end{gathered}$$

The numbers of bosons and fermions in this theory are individually conserved, with the corresponding Noether charges being simply

$$\mathcal{N}_B = \int \mathrm{d}^2x\ \phi_i^\dagger\phi_i \quad \text{and} \quad \mathcal{N}_F = \int \mathrm{d}^2x\ \psi_i^\dagger\psi_i\,. \tag{4.21}$$

The total particle number is of course $\mathcal{N} = \mathcal{N}_B + \mathcal{N}_F$, with the combination $\mathcal{R} = \mathcal{N}_B - \frac{1}{2}\mathcal{N}_F$ then playing the role of the R-symmetry in the supersymmetry algebra. Note that the naive $U(1)_R$ charge one would expect from the relativistic theory is mixed with the additional Σ charge, which is possible since particle number and hence spin is conserved in the non-relativistic theory.

One may work out the expression for the superconformal generator using the relation $[C, Q_2] = -iS$. It is simply

$$S = i\sqrt{\frac{m}{2}}\int \mathrm{d}^2x\ z\phi_i^\dagger\psi_i\,.$$

5 The Bosonic Theory

In this section we study a class of $d = 2 + 1$ bosonic Chern-Simons-matter theories given simply by the bosonic sectors of the theories introduced in Section 4.3. These will principally be $SU(p)_k$ Chern-Simons theory together with N_f non-relativistic scalar fields ϕ_i with mass m.

The action we are interested in is generally given by

$$S = S_{CS} + \int \mathrm{d}t\, \mathrm{d}^2x \left\{ i\phi_i^\dagger \mathcal{D}_0 \phi_i - \frac{1}{2m} \vec{\mathcal{D}}\phi_i^\dagger \vec{\mathcal{D}}\phi_i - \lambda(\phi_i^\dagger t^\alpha [R_i]\phi_i)\,(\phi_j^\dagger t^\alpha [R_j]\phi_j) \right\}. \tag{5.1}$$

As mentioned above, the quartic term which we previously fixed by supersymmetry gives rise to a delta-function interaction between particles. In non-supersymmetric theories, the coupling λ is marginal and is known to run logarithmically. There are two fixed points given by [56, 57]

$$\lambda = \pm \frac{\pi}{mk}$$

where $\lambda > 0$ fixed point is stable; the $\lambda < 0$ fixed point is unstable. In what follows, we choose to set

$$\lambda = +\frac{\pi}{mk} \tag{5.2}$$

agreeing with the supersymmetric choice of (4.18). In not fixing the sign of the Chern-Simons coupling k, we still allow λ to take either sign, so this choice includes both stable ($k > 0$) and unstable ($k < 0$) fixed points.

This fixed point also exists in the $U(1)$ theory, where $\lambda > 0$ corresponds to repulsive interactions between particles and $\lambda < 0$ corresponds to attractive interactions. In the non-Abelian theory, this classification is not so simple because, for a fixed sign of λ, interactions in channels for different irreducible representations $R \subset R_1 \otimes R_2$ can be either attractive or repulsive. (Such behaviour also holds in classical Yang-Mills theory. For example, a quark and anti-quark attract in the singlet channel, but repel in the adjoint channel.)

From what we saw in Chapter 4, it should be obvious that (5.2) exhibits an enhanced non-relativistic conformal invariance [32]. This parallels the more familiar relativistic situation where we find conformal theories at endpoints of RG flows.

The Hamiltonian of this theory is easily seen from (4.19) to be

$$H = \frac{2}{m} \int \mathrm{d}^2x \ |\mathcal{D}_z \phi_i|^2 \,. \tag{5.3}$$

However, our interest now lies in the spectrum of non-Abelian anyons when placed in a harmonic trap. In the present context, this means that we want the spectrum of $L_0 = H + C$. As we explained in Chapter 4, this is equivalent to determining the spectrum of the dilatation operator D. It turns out that this latter formulation of the problem is somewhat simpler to work with.

5.1 Gauge Invariant Operators

The first thing to do is to identify the operators of interest. As always, we must talk about gauge invariant operators. A particularly simple way of seeing this is to observe that it is required by Gauss's law, which for our bosonic theory reads

$$f_{12}^{\alpha} = \frac{2\pi}{k} \sum_i \phi_i^{\dagger} t^{\alpha} \phi_i \tag{5.4}$$

where f_{12}^{α} is the non-Abelian magnetic field. The left-hand side generates gauge transformations of the a field, and the right-hand side those of our matter.

In the case of an Abelian Chern-Simons theory, it is standard to dress matter with a monopole operator $e^{i\sigma}$, given by the exponential of the dual photon σ (here taken to have periodicity 2π). A Chern-Simons term of level k imbues this with a charge $-k$. Therefore a composite, gauge-invariant operator may be defined using $\Phi = e^{-i\sigma/k}\phi$. Notice that this has fractional monopole charge, which can necessarily be detected at long distances, so Φ is not strictly an honest local operator.

This generalizes fairly straightforwardly to the non-Abelian case. We will construct gauge invariant operators simply by attaching Wilson lines stretching out to infinity. Thus we define

$$\Phi_i(\mathbf{x}) = \mathcal{P} \exp\left(i \int_{\infty}^{\mathbf{x}} a^{\alpha} \, t^{\alpha}[R_i] \right) \phi_i(\mathbf{x}) \,. \tag{5.5}$$

This all clearly requires some explanation. $\Phi(\mathbf{x})$ is not a local operator; it depends on the value of the gauge field along a line stretching to infinity. Meanwhile, the state-operator map described in the previous section is usually taken to hold only for local operators. However, closer inspection of the argument leading to (4.5) shows that we require only that the operator $\mathcal{O}(0)$ has a well defined scaling dimension. It is simple to check that the Wilson line does not affect this property of Φ. (This isn't too surpris-

ing: notice that $\mathcal{P}\exp(i\int_\infty^0 a)$ is a covariant quantity characterized only by its endpoints, which are invariant under dilatations centred at the origin.)

Under the state operator map, the state $|\Phi_i^\dagger\rangle$ describes a single anyon, transforming in the representation R_i, sitting in a harmonic trap. The particle retains the attached Wilson line and is entirely analogous to the correct description of a physical electron in QED. Importantly, the $SU(p)_k$ Chern-Simons theory does not confine and so this particle has finite energy. We will compute this energy explicitly below.

It's worth pausing to comment that the situation differs from that in relativistic conformal theories, where the state-operator map is restricted to local operators. Indeed, in the relativistic context the states are considered on a spatial sphere where there is no option to attach a Wilson line that stretches to infinity. Instead, in Chern-Simons theories Gauss's law requires that charged states are accompanied by monopole operators, which places further constraints on the possible electric excitations. At least for this aspect of the physics, thinking about Chern-Simons-matter theories with relativistic conformal invariance does not appear to be a good guide to the non-relativistic theories.

Now we can discuss the kinds of operators that we are interested in. In the n-particle sector, we will look at operators of the form

$$\mathcal{O} \sim \prod_{a=1}^{n} (\partial^{l_a} \bar{\partial}^{m_a} \Phi_{i_a}^\dagger) \tag{5.6}$$

where we have introduced (anti)-holomorphic spatial derivatives $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. The primary operators are those which cannot be written as a total derivative.

Before we proceed, a comment is in order. The operators written above are not the most general and, indeed, do not necessarily have fixed scaling dimension. This is because there's nothing to stop these mixing with operators of the form $(\Phi^\dagger)^{n+l}\Phi^l$, possibly with derivatives attached too. However, because non-relativistic theories contain no anti-particles, these additional operators annihilate the vacuum $|0\rangle$ and so result in the same state $|\mathcal{O}\rangle$ under the state-operator map. Since our real interest lies in the theory with the harmonic trap, for many purposes it will suffice to use (5.6) as a way to characterize the operators.

It is not a totally trivial task to list the primary operators from (5.6). The only one that is simple to write down has no derivatives:

$$\mathcal{O}_{i_1\ldots i_n} = \Phi_{i_1}^\dagger \ldots \Phi_{i_n}^\dagger \,. \tag{5.7}$$

(Here i_a are flavour indices. We have suppressed colour indices.) The n-particle ground state is expected to take such a form for suitably large k; we will compute its energy shortly.

To highlight how other primary operators arise, it will be useful to look at a simple example. We take $U(1)_k$ with a single field ϕ of charge $+1$. (This was the case discussed in the introduction.) To make contact with the introduction and, in particular, the numerical spectrum of [62], let us look at the case $n = 3$. As we mentioned above, the large k ground state is simply the state corresponding to $(\Phi^\dagger)^3$, as we will see shortly through explicit computation. What about higher states? Any state with a single derivative can be written as a total derivative and so is a descendant. This explains the gap between the ground state and the first excited state seen in Figure 1. The next primary operator will contain two derivatives. There are six such operators: $\partial\Phi^\dagger\partial\Phi^\dagger\Phi^\dagger$, $\partial\Phi^\dagger\bar\partial\Phi^\dagger\Phi^\dagger$, $\bar\partial\Phi^\dagger\bar\partial\Phi^\dagger\Phi^\dagger$, $\partial^2\Phi^\dagger\Phi^{\dagger\,2}$, $\partial\bar\partial\Phi^\dagger\Phi^{\dagger\,2}$ and $\bar\partial^2\Phi^\dagger\Phi^{\dagger\,2}$. However, four linear combinations of these can be written as total derivatives of the form $\partial(\partial\Phi^\dagger\,\Phi^{\dagger\,2})$, where either derivative could also be $\bar\partial$. The upshot is that there are two primary states with two derivatives. This agrees with the spectrum shown in Figure 1.

We can play a similar game with operators that contain three derivatives. It is simple to check that one can write down 13 such operators, 10 of which turn out to be descendants. The upshot is that there are 3 primary operators that contain 3 derivatives. (The obvious pattern does not persist!) From Figure 1, we learn that one of these will become the ground state at small k.

5.2 The Spectrum

Next comes the question that we initially set out to answer: what is the spectrum of the states (5.6)? As we stressed in the introduction, this is a difficult and unsolved question, even for Abelian anyons. Here we offer two approaches.

In Section 5.4 we explain how one can compute the spectrum of these operators for Chern-Simons theory with scalars perturbatively in $1/k$. We present the results only at one-loop.

However, before we do this, there is a special class of operators for which the result simplifies tremendously. These are the "linear states" referred to in the introduction. They correspond to the *chiral primary operators* we defined in Section 4.2.3, and their descendants. These are the operators which have no antiholomorphic derivatives,

$$\mathcal{O} \sim \prod_{a=1}^{n} (\partial^{m_a}\Phi^\dagger_{i_a})\,. \tag{5.8}$$

The simplest such operator is $\mathcal{O}_{i_1\ldots i_n}$ in (5.7). Since these states saturate the unitarity bound (4.15), the scaling dimension of any such operator is fixed by its angular momentum $\mathcal{J}$:

$$\Delta = n - \mathcal{J}\,. \tag{5.9}$$

Note that each derivative ∂ decreases the angular momentum by one. Correspondingly, the dimension of a chiral operator (5.8) is given by

$$\Delta_{\mathcal{O}} = n + \sum_{a=1}^{n} m_a - \mathcal{J}_0$$

where $\mathcal{J}_0$ is the angular momentum of $\mathcal{O}_{i_1\ldots i_n}$.

5.3 Angular Momentum

From the discussion above, we learn that the dimension of $\mathcal{O}_{i_1\ldots i_n}$ and other chiral operators (5.8) is entirely determined by the angular momentum $\mathcal{J}$, which from (4.20) in this theory is simply

$$\mathcal{J} = \int \mathrm{d}^2x\; \phi_i^\dagger (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})\phi_i\,. \tag{5.10}$$

But how do we compute this angular momentum?

The tensor product of representations $\otimes_{I=1}^{n} R_{i_a}$ is decomposed into irreps. (Note that $\otimes$ denotes the tensor product here. However, in the presence of a Chern-Simons term at finite level one should be careful about which representations one includes – we will mention the role of *fusion rules* in this theory below.) When the operator $\mathcal{O}$ sits in the representation R, its angular momentum is given by

$$\mathcal{J}_0 = -\frac{C_2(R) - \sum_a C_2(R_{i_a})}{2k} \tag{5.11}$$

where C_2 is the quadratic Casimir, defined by

$$\sum_\alpha t^\alpha[R] t^\alpha[R] = C_2(R)\mathbf{1}\,.$$

Note that because $\mathcal{O}_{i_1\ldots i_n}$ is built out of commuting scalar fields Φ, in the absence of any derivatives it must transform in the fully symmetrized representation

$$R_{\,\mathrm{sym}} = \mathrm{Sym}\left[\otimes_{a=1}^{n} R_{i_a}\right]\,.$$

However, the more general operators (5.8) can transform in other representations.

It is worth pointing out that the expression (5.11) is the difference of some expressions which are well-known in all the physical contexts in which affine Lie algebras crop up. The *trace anomaly* associated to the representation R is given by [81]

$$h_R = \frac{C_2(R)}{2k}$$

if the underlying algebra has the level $k - p$, where p is the dual Coxeter number of the group. (This is correct for what we have referred to as $SU(p)$ with Chern-Simons term proportional to k, due to a one-loop shift discussed in Chapter 6.) This quantity appears in the energy-momentum tensor of Wess-Zumino-Witten models (where it emerges nicely in the Sugawara construction) [82], and – most famously and relevantly – in the context of pure Chern-Simons theory, where it plays a crucial role in correlation functions and statistics [83], much as it shall for us.

Of course, (5.11) is just the particular definition of $\mathcal{J}$ which appears in our algebra. It differs by only a central charge (or choice of regularization) from the more concise $\mathcal{J}' = -C_2(R)/2k = -h_R$ which is familiar in Wess-Zumino-Witten and Chern-Simons theory. Importantly, $\mathcal{J}'$ is the angular momentum appearing in the spin-statistics relation. (For example, flux attachment dictates that a boson turns into a fermion in an Abelian Chern-Simons theory at level 1 – and indeed we see here that it carries an angular momentum $\mathcal{J}' = -1/2$.) We will have use for this angular momentum later on, when we compare the bosonic and fermionic theories in Section 6.2.

5.3.1 Examples

The purpose of this section is to prove the result (5.11). Before we do this, we will first look at some examples.

Example: $U(1)$

We start with an Abelian gauge theory $U(1)_k$, where representations are labelled by charge $q \in \mathbb{Z}$. The quadratic Casimir in this case is simply $C_2(q) = q^2$. The result (5.11) says that the angular momentum of n anyons, each of charge 1, is given by

$$\mathcal{J} = -\frac{n(n-1)}{2k}\,. \tag{5.12}$$

This is indeed the angular momentum of n anyons. (Moreover, when substituted into (5.9), it gives us the correct answer for the dimension of the n anyon operator; this is the result quoted in (3.1).)

This demonstrates that the angular momentum of n anyons has the unusual property, first discovered in [68, 69], that it scales as n^2 rather than n. This fact will play an im-

portant role in our analysis, and will help us prove the general result, so we pause here to review the underlying classical physics. (More details can be found, for example, in the book [33]. For a derivation in the quantum theory see, for example, [70].)

The important term is the gauge field buried in the covariant derivatives in the expression (5.10). Picking a configuration with no traditional orbital angular momentum, we're still left with

$$\mathcal{J} = -\int \mathrm{d}^2x\, \epsilon^{pq} x_p a_q\, \phi^\dagger\phi\,.$$

(We work with real coordinates labelled by $p, q = 1, 2$ for brevity.) The gauge field is determined by Gauss's law (5.4). Choosing the gauge $\partial_p a^p = 0$, we can solve (5.4) for the vector field, giving

$$a^p(\mathbf{x}) = -\frac{2\pi}{k}\epsilon^{pq}\partial_q \int \mathrm{d}^2x'\, G(\mathbf{x}-\mathbf{x}')\rho_B(\mathbf{x}') \tag{5.13}$$

with $G(\mathbf{x}-\mathbf{x}') = \frac{1}{2\pi}\log|\mathbf{x}-\mathbf{x}'|$ being the usual Green's function for the Laplacian in the plane. This gives the following contribution to the angular momentum:

$$\mathcal{J} = -\frac{2\pi}{k}\int \mathrm{d}^2x\, \mathrm{d}^2x'\, \rho(\mathbf{x})\rho(\mathbf{x}')x^p\partial_p G(\mathbf{x}-\mathbf{x}')\,.$$

We take the charge distribution to be a sum of delta-functions at n distinct points $\mathbf{r}_a$,

$$\rho_B = \sum_{a=1}^{n}\delta^2(\mathbf{x}-\mathbf{r}_a(t))$$

and so the orbital angular momentum becomes

$$\mathcal{J} = -\frac{2\pi}{k}\sum_{a,b}\mathbf{r}_b\cdot\frac{\partial}{\partial\mathbf{r}_b}G(\mathbf{r}_a-\mathbf{r}_b)\,.$$

At this point we need a procedure to deal with the fact that this expression is ill-defined when $\mathbf{r}_a = \mathbf{r}_b$. Any regularization which preserves antisymmetry under reflection gives $\lim_{x\to x'}\partial_p G(\mathbf{x}-\mathbf{x}') = 0$. With this choice, the sum is over pairs of particles only and we have

$$\mathcal{J}_0 = -\frac{2\pi}{k}\sum_{a=1}^{n}\sum_{b\neq a}\mathbf{r}_a\cdot\frac{\partial}{\partial\mathbf{r}_a}G(\mathbf{r}_a-\mathbf{r}_b) = -\frac{n(n-1)}{2k} \tag{5.14}$$

as promised above. Note that, in general, this is not the lowest angular momentum of n anyons: in certain cases, one can decrease the spin by giving the individual particles additional relative orbital angular momentum. The result (5.14) is, however, the angular momentum that one gets when adiabatically increasing the statistical parameter of

n bosons.

Example: $SU(2)_k$

Next, consider $SU(2)_k$. Representations of $SU(2)$ are labelled by a spin $s \in \frac{1}{2}\mathbb{N}_0$. Suppose we consider several spins s_{i_a} coming together into a final bound state of spin $S = \sum_a s_{i_a}$. In this case, the angular momentum is given by

$$\mathcal{J} = -\frac{S(S+1) - \sum_a s_{i_a}(s_{i_a}+1)}{k}$$

from the standard expression for $SU(2)$ Casimirs.

Example: $SU(p)_k$**,** $U(p)_k$

This has a simple generalization to n anyons, each of which sits in the fundamental representation of $SU(p)$. We have $C_2(\mathbf{p}) = (p^2-1)/p$. The bound state transforms in the n^{th} symmetric representation of $SU(p)$, with $C_2(\mathrm{Sym}^n(\mathbf{p})) = n(p-1)(p+n)/p$. We have

$$\mathcal{J} = -\frac{n(n-1)}{2k} \times \frac{p-1}{p}\,. \tag{5.15}$$

For a more general operator, it is helpful to characterize the representation by its highest weight, or equivalently a partition λ. (For a technical review of such matters, see Section 18.1.) Consider a general representation of $SU(p)$ whose Young diagram has rows of length[1] $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$. Such a state has a quadratic Casimir given by the formula $C_2(\lambda) = \langle \lambda, \lambda + 2\rho \rangle$, where λ is the highest weight and ρ is the Weyl vector. In particular, we have

$$C_2(\lambda) = \begin{cases} \sum_{i=1}^p \left[\lambda_i^2 + (p+1-2i)\lambda_i\right] & \text{for } U(p) \\ \sum_{i=1}^p \left[\lambda_i^2 + (p+1-2i)\lambda_i\right] - \frac{1}{p}\left(\sum_{i=1}^p \lambda_i\right)^2 & \text{for } SU(p) \end{cases}.$$

This translates into the following results for n fundamental anyons brought together into the representation λ:

$$\mathcal{J} = -\frac{\sum_{i=1}^p \left[\lambda_i^2 - (2i-1)\lambda_i\right]}{2k} \quad \text{for } U(p)_k$$

and

$$\mathcal{J} = -\frac{\sum_{i=1}^p \left[\lambda_i^2 - (2i-1)\lambda_i\right] - n(n-1)/p}{2k} \quad \text{for } SU(p)_k\,.$$

[1]Note that by including the possibility of unreduced diagrams with $\lambda_p \neq 0$, this formula works even for states containing factors which are $SU(p)$ singlets. λ_p has the interpretation of the number of these "baryons" in the state.

As an aside which will be of some use later, it is pleasing to note that the above expression for $\mathcal{J}$ in $U(p)$ has a nice interpretation in terms of the Young diagram of the representation λ. We write

$$\sum_{i=1}^{p}\left[\lambda_i^2-(2i-1)\lambda_i\right]=2\sum_{i=1}^{p}\left[\frac{1}{2}\lambda_i(\lambda_i-1)-(i-1)\lambda_i\right].$$

Then the first term inside the brackets counts the number of pairs of boxes lying in each row of the Young diagram; and the second subtracts off the number of pairs lying in each column.

The striking thing about this form of the answer is that it guarantees that upon transposition of the Young diagram, this only changes by a sign. This suggestive fact will give rise to a bosonization duality in the system, as we shall see later.

5.3.2 Deriving the Angular Momentum

We now return to prove the result (5.11) for the angular momentum. We insert n anyons in various representations R_{i_a} of the group G at level k, such that they collectively transform in the irrep $R\subset\otimes_a R_{i_a}$.

There are two issues which we need to explain. The first is that angular momentum of this state, inserted at the origin, is related to the quadratic Casimir $C_2(R)$. The second is that there are some ambiguities to do with regulators, but that the correct choice of angular momentum for our purposes is the one given above:

$$\mathcal{J}=-\frac{C_2(R)-\sum_a C_2(R_{i_a})}{2k}.$$

It will be helpful to first develop some intuition for how this quadratic behaviour arises. We have already seen that that pairwise contributions arise naturally from the $U(1)$ example, but it can be understood more clearly by considering the phase of the wavefunction for our n anyons under rotations. To see this, place each anyon at a different distance from the origin. Now rotate the configuration by 2π. In doing this, each anyon encircles all the others which are closer to the origin than itself, accumulating an Aharonov-Bohm phase per pair of particles. We additionally pick up a phase due to the inherent spin of each individual anyon. As we scale the configuration towards the origin, these are the only phases contributing to the behaviour of the wavefunction.

This decomposition into two phases is very similar to the usual decomposition of angular momentum into orbital and spin parts. We will find that the $\mathcal{J}$ arising in the conformal (and superconformal) algebra is the one without intrinsic spins.

We begin our computation in exactly the same way that we approached the classical Abelian calculation. Recall that the angular momentum used in our algebra is given by equation (5.10), reprinted here for convenience:

$$\mathcal{J} = \int \mathrm{d}^2 z \, \sum_a \phi_a^\dagger (z\mathcal{D}_z - \bar{z}\mathcal{D}_{\bar{z}})\phi_a \, .$$

Again, since we are going to place all particles at the origin, we can ignore the normal orbital angular momentum terms[2] $\phi^\dagger(z\partial_z - \bar{z}\partial_{\bar{z}})\phi$. However, we will use Gauss's law (5.4), which relates particle density to the magnetic field f_{12}^α, in a new way: instead of solving it for the gauge field, we will use it to eliminate the particle density. Putting these ideas together, we obtain

$$\mathcal{J} = \frac{ik}{2\pi} \int \mathrm{d}^2 z \, (\bar{z} a_{\bar{z}}^\alpha - z a_z^\alpha) f_{12}^\alpha$$

which holds when acting on any state satisfying Gauss's law.

The reason for doing this is that this expression is now only sensitive to the Wilson line in (5.5). Let us give this a name: pick some representation t^α, and let

$$W(\mathbf{x}) = \left[\mathcal{P} \exp\left(i \int_\infty^{\mathbf{x}} a^\alpha \, t^\alpha \right) \right]^\dagger .$$

If Gauss's law (5.4) holds for the object $\Phi = W^\dagger \phi$, then it is straightforward to show that[3]

$$\left[\frac{k}{2\pi} f_{12}^\alpha(\mathbf{x}), W(\mathbf{x}') \right] = W(\mathbf{x}') \, t^\alpha \, \delta^{(2)}(\mathbf{x} - \mathbf{x}') \, . \tag{5.16}$$

This is enough to start computing the action of $\mathcal{J}$ on a state containing Wilson lines. We take the Wilson lines to be $W_a = W(z_a, \bar{z}_a) \, |_{t=t_a}$, where we allow each W_a to sit in a

[2] This is only strictly true within any reflection-invariant regularization. Similarly, were one to integrate by parts and find an expression for $\mathcal{J}$ which did not explicitly change sign under a reflection, this would not hold.

[3] One can prove this using the commutation relation $[a_1^\alpha(\mathbf{x}), a_2^\beta(\mathbf{x}')] = -\frac{2\pi i}{k} \, \delta^{\alpha\beta} \, \delta^{(2)}(z - z')$ arising from the term $-\frac{k}{4\pi} \, \mathrm{Tr} \, \epsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho$ in the Chern-Simons action S_{CS}. Notice that $ikf_{12}^\alpha/2\pi$ generates spatial gauge transformations: for any function $h^\alpha(\mathbf{x})$

$$\left[\int \mathrm{d}^2 x \, \frac{ik}{2\pi} f_{12}^\alpha h^\alpha, a_m^\beta(\mathbf{x}') \right] = \mathcal{D}_m h^\beta(\mathbf{x}') \, .$$

But the Wilson line is charged only at its endpoints, and for compactly supported h it transforms at the $\mathbf{x}$ end so that $\left[\int \mathrm{d}^2 x \, \frac{ik}{2\pi} f_{12}^\alpha h^\alpha, W(\mathbf{x}') \right] = +iW(\mathbf{x}') \, t^\alpha h^\alpha(\mathbf{x}')$. Setting h to be a delta function, we obtain the above result.

different representation R_{i_a} whose generators are $t^\alpha_{i_a}$. Explicitly,

$$\begin{aligned}\mathcal{J}\, W_1 \otimes \cdots \otimes W_n \,|0\rangle &= \frac{ik}{2\pi}\int \mathrm{d}^2 z\, (\bar{z} a^\alpha_{\bar{z}} - z a^\alpha_z) f^\alpha_{12}\, W_1 \otimes \cdots \otimes W_n \,|0\rangle \\ &= i\int \mathrm{d}^2 z\, (\bar{z} a^\alpha_{\bar{z}} - z a^\alpha_z) \Big[\, W_1 \otimes \cdots \otimes W_n \,\Big] \sum_{a=1}^{n} t^\alpha_{i_a} \delta^{(2)}(z - z_a)\,|0\rangle\end{aligned}$$

where t_{i_a} is understood to act only on the a^{th} factor of the product to the left.

Now we set $\mathbf{x}' = 0$ in (5.16) and use the complex coordinate $z = x_1 + i x_2 = r e^{i\theta}$. If we integrate over a disc of radius r, the integral reduces to a boundary term, and

$$\frac{1}{2\pi}\int \mathrm{d}\theta\, [\bar{z} a^\alpha_{\bar{z}}(z,\bar{z}) - z a^\alpha_z(z,\bar{z}), W(0)] \;=\; \frac{i}{k} W(0)\, t^\alpha\,.$$

That is, evaluating this quantity around a circle gives this particular non-zero contribution if the Wilson line ends inside that circle; by contrast, it is zero if the end is outside that circle. This shows why we need to be careful with regularization.

To regularize, let us proceed as above by smearing each W_i around progressively smaller circles, of radius $|z_1| > |z_2| > \cdots > |z_n|$, and then taking the smallest one to zero first. In this manner, we find that we get only *one* contribution to the result per distinct pair (a, b). However, it is not yet clear what happens when both terms in $\mathcal{J}$ hit the same Wilson line.

To address this last case, we need one final argument. The simplest line of reasoning is that any translationally invariant regularization of terms like $[a_z(z_a, \bar{z}_a), W(z_a, \bar{z}_a)]$ must vanish if we multiply it by z_a and then take $z_a \to 0$.[4]

Now we have the result

$$\mathcal{J}\, W_1 \otimes \cdots \otimes W_n \,|0\rangle \;=\; -\frac{1}{k} W_1 \otimes \cdots \otimes W_n \left[\sum_{a<b} t^\alpha_{i_a} \otimes t^\alpha_{i_b}\right] |0\rangle$$

where as above, t_{i_a} is understood to act only on the a^{th} factor of the product to the left.

All that remains is to relate this to the quadratic Casimir. But notice that the product representation R in which the particles sit has the generators $T^\alpha = \sum t^\alpha_{i_a}$, and hence the quadratic Casimir is given by

$$C_2(R) = T^\alpha T^\alpha = \sum_{a,b} t^\alpha_{i_a} \otimes t^\alpha_{i_b} = 2\sum_{a<b} t^\alpha_{i_a} \otimes t^\alpha_{i_b} + \sum_a C_2(R_{i_a})\,.$$

[4]For a more careful approach, one could complexify and work in holomorphic gauge $a_{\bar{z}} = 0$, and then solve explicitly for a_z as an integral of f_{12}. This generalizes the approach we took in Coulomb gauge for the Abelian case.

This completes our proof. We have

$$\mathcal{J} = -\frac{C_2(R) - \sum_a C_2(R_{i_a})}{2k}$$

as claimed.

5.3.3 OPEs and Branch Cuts

The above computation precisely measures the anomalous angular momentum $\mathcal{J}$ of renormalized compound operators formed by products of Φ. There is essentially the same as what one discovers in computing the Operator Product Expansion (OPE) of chiral operators in a pure Chern-Simons or Wess-Zumino-Witten CFT. Suppose each operator $\mathcal{O}_{\mathcal{R}}$, in the representation R, has the spin $\mathcal{J}'_R = -C_2(R)/2k$; this is a familiar result, mentioned above. Then if $\mathcal{J} = \mathcal{J}'_R - \mathcal{J}'_{R_1} - \mathcal{J}'_{R_2}$ one finds the OPE

$$\mathcal{O}_{R_1}(z_1)\mathcal{O}_{R_2}(z_2) \sim (z_1 - z_2)^{\mathcal{J}}\mathcal{O}_R$$

holds in the set of chiral operators in the theory. (Of course, the above implicitly makes some assumptions about what one does with the index structure as the fields approach. In general, there is a linear combination of the various possible representations.)

The anyonic nature of the underlying excitations is manifested in the fractional powers which appear here in the form of $\mathcal{J}$. A particularly notable feature of these terms is that they introduce branch cuts. One might quite reasonably ask: *Which branch do I choose*? This betrays something we have not been careful about, namely where precisely all our Wilson lines go.

In order to fix this ambiguity, one needs to specify precisely how the Wilson lines reach out to infinity. If we rearrange them so that they intersect each other, for instance, the resulting state may be distinct. This leads to consideration of the *braid group*, often associated with models possessing affine Lie algebra structure, whose anyonic representations we are here to study.

Note that there is a lot of richness here which we will not investigate, forming a story famously told in [83]. We will return to some ideas related to this picture at the end of Chapter 6, but our focus will mainly remain on simple properties of the spectrum of the Chern-Simons-matter theory. Nonetheless, one important issue deserves mention now: *fusion rules*.

5.3.4 Fusion Rules

The most famous incarnation of the fusion rules is in the context of Wess-Zumino-Witten theory, where at a given level $\hat{k}$ they imply that all physical operators are projected onto representations with fewer than $\hat{k}$ symmetrizations.[5] (See, for example, the textbook [82], for a detailed description of how the rules work.[6]) More specifically, the *integrable* representations of the $SU(p)$ algebra at level $\hat{k}$ are those whose Young diagrams have at most $\hat{k}$ columns.

One can show that non-integrable representations decouple from all correlation function in Wess-Zumino-Witten theory, provided one has at least one field in an integrable representation, by a sort of integration by parts [82]. Let us quickly sketch that argument. Suppose that E^{θ}_{-1} is a generator of an affine algebra such that $(E^{\theta}_{-1})^{\hat{k}+1}$ annihilates the identity field $\mathbb{I}$. (Here, θ is the highest root.) Then inserting $(E^{\theta}_{-1})^{\hat{k}+1}\mathbb{I}(z)$ into arbitrary correlation functions $\langle \mathcal{O}_1(z_1)\cdots\mathcal{O}_n(z_n)\rangle$ makes them vanish. Writing each factor as

$$E^{\theta}_{-1} = \frac{1}{2\pi i}\oint \frac{\mathrm{d}\zeta}{\zeta - z}E^{\theta}(\zeta)$$

and deforming each contour to encircle the z_a, one finds that the correlation function may be expressed in terms of various OPEs of the $\mathcal{O}_a$ operators with E^{θ}. Because the location z was arbitrary, each term in the resulting expansion must vanish independently. In particular, $\langle \left[(E^{\theta})^{k+1}\mathcal{O}_1\right](z_1)\cdots\mathcal{O}_n(z_n)\rangle = 0$. Therefore any correlator involving $(E^{\theta}_0)^{k+1}\mathcal{O}_1$ must vanish. But E^{θ} is in a suitable sense invertible acting upon non-integrable representations, and so any correlator containing a non-integrable field vanishes.

However, these principles raise themselves in Chern-Simons theory too. Pure Chern-Simons theory on an arbitrary manifold M has an empty Hilbert space if one inserts Wilson lines in non-integrable representations [83].

What about our theory? We have been rather cavalier in writing down objects like $\Phi^{\dagger n}_{a_i}$ which we are taking to transform in arbitrarily large symmetric representations. (Let us focus on fundamental matter fields.) But these states are essentially pure Chern-Simons states in the presence of marked endpoints where Wilson lines are attached (both at the particle locations, and infinity). Hence something should go wrong when we study these objects with many symmetrizations, by the usual arguments: the Hilbert space

[5] As we will mention again below, there is a good reason for using the distinct quantity $\hat{k}$ to refer to the level of the current algebra, since there is a one-loop shift relative to what we have been calling the level.

[6] Also, a Mathematica code written by the author for computing $SU(p)_k$ fusion rules can be downloaded at http://blog.suchideas.com/2mBUW.

of a Chern-Simons theory (with sources) is the space of conformal blocks of a Wess-Zumino-Witten theory (with certain operator insertions), or the space of holomorphic line sections of a certain bundle. In the interests of building our intuition a little, we will make a little detour and give a more physically intuitive direct argument here, based around the computation of $\mathcal{J}$ above.

It helps to discuss this at a heuristic level first. Notice that attaching Wilson lines associates an angular momentum $\mathcal{J}'$ to each Φ, and this leads to a phase upon interchanging particles by a standard sort of spin-statistics principle. (There are various ways of looking at this. For example, it may be thought of in terms of crossing branch cuts or performing transformations of knotted Wilson lines.)

For the case of fermions, this is very familiar. An intrinsic half-integer spin leads to a phase $\exp(\frac{1}{2} \times 2\pi i)$ when we swap the particles. An Abelian theory of bosons coupled to Chern-Simons theory at level 1 behaves identically, as we will discuss in detail in the next chapter. A conspicuous consequence of this is that two fermions cannot be symmetrized together.

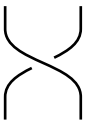
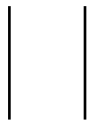
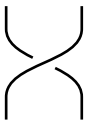
We, of course, are assembling Wilson lines carrying gauge charges and fractional spin that can exist in bound states with various anomalous angular momenta. One might wonder if somehow the basic property of fermions carries across to these more exotic entities, and a theory at a finite level can sustain only a limited number of symmetrizations of these Wilson lines before vanishing. This is indeed so; in fact, essentially one finds that if you restrict to only symmetric representations, the theory behaves like a parafermionic one [84, 85, 86]. Parafermions interpolate between fermions (two symmetric particles annihilate each other) and bosons (infinitely many particles may be symmetrized together) in a natural way.

However, since this it is not our main focus, but more an interesting diversion, so we will restrict ourselves to outlining a case that captures the main features of the phenomenon.

Braiding in Chern-Simons Theory

The key observation is that, in $SU(p)$ Chern-Simons theory, Wilson lines obey a simple relation that restricts the number of distinct states which can be formed by braiding. This should be thought of as a generalization of the idea that any pair of indices in an $SU(p)$ tensor can be written as a linear combination of one which is symmetrized or antisymmetrized.

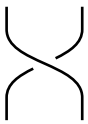
This relation is called the *skein relation*. Such relations are ubiquitous in knot theory and relate links (for us braids) which differ only in whether one line passes over or beneath the other. With such a prescription, one can define for instance a knot polynomial, by reducing all knots to a base case. They take the form

$$\alpha$$  $$+\beta$$  $$+\gamma$$  $$= 0\,.$$ (5.17)

Note that in the interest of simplicity, we have not specified the *framing* of the above knots. Secretly, each Wilson line has a notion of whether or not it is twisted, which may be tracked by drawing them as ribbons rather than lines. This will be important below.

In Chern-Simons theory, we can view the skein relation as defining an equivalence relation on the physical states of the theory: the quotient of the full braid group by the above relation leaves one with a finite-dimensional space of potential states containing Wilson lines connecting the given end points. Such a relation must exist if we want a standard picture of the theory in which two fundamental Wilson lines can fall into either symmetric or antisymmetric representations, since in such a situation, there must be a two-dimensional Hilbert space of braids when we have Wilson lines – and hence the three states constructed using the above diagram must be linearly dependent!

To derive values for α, β, γ, consider the symmetric and antisymmetric representations we require the pair of lines to fall into. Call these $|S\rangle$ and $|A\rangle$ respectively. Now define the operation B which twists the two strands of the braids depicted in (5.17) round, mapping

$$\cdots \longrightarrow$$  $$\longrightarrow$$ $$\longrightarrow$$  $$\longrightarrow \cdots$$

and so on. This is clearly the action of a rotation by π, up to issues with the framing, but since we know the angular momentum $\mathcal{J}$ of the symmetric and antisymmetric states, we know that in $SU(p)$ with statistical parameter $1/k$

$$B\,|S\rangle = \exp\left[\pi i\left(\frac{1-p}{kp}\right)\right]|S\rangle \qquad \text{and} \qquad B\,|A\rangle = -\exp\left[\pi i\left(\frac{1+p}{kp}\right)\right]|A\rangle\ .$$

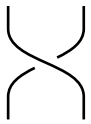
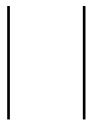
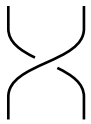
Now by the Cayley-Hamilton theorem, we know that $B^2 - \mathrm{Tr}\,[B]\,B + \det[B] = 0$, where both trace and determinant have simple expressions in terms of these eigenvalues. Applying the left-hand side of this equation to the braid on the left in the above sequence, we therefore derive a linear relation of the form of (5.17)!

The only detail remaining is to determine the framing of the above braids. We wish to obtain (5.17) with the "blackboard" framing, in which the ribbons look flat from our viewpoint. However, if B is implemented using a rotation by π, then the ribbons get twisted as it maps between them. The change of framing corresponds to a rotation of an individual Wilson line's internal structure, which is associated with the spin $\mathcal{J}'$ of each line.

Putting these ingredients together, and defining

$$q = \exp\left(\frac{2\pi i}{k}\right)$$

we obtain

$$-q^{\frac{1-p}{2p}} q^{\frac{1+p}{2p}} \times q^{-\left(\frac{1-p^2}{2p}\right)}$$  $$- \left(q^{\frac{1-p}{2p}} - q^{\frac{1+p}{2p}}\right)$$  $$+ q^{+\left(\frac{1-p^2}{2p}\right)}$$  $$= 0\,.$$

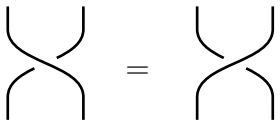
Consequently, multiplying everything by $q^{-1/2p}$ for convenience, we may take

$$\alpha = -q^{p/2} \qquad , \qquad \beta = q^{1/2} - q^{-1/2} \qquad , \qquad \gamma = q^{-p/2}\,.$$

This is the canonical form of the skein relation in an $SU(p)$ theory.

One may check straightforwardly that these results for α, β, γ are invariant when one gauges the $U(1)$ part. Thus the skein relation is only a property of the $SU(p)$ level, as it should be.

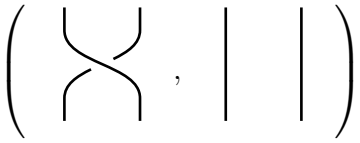
It is nice to perform some a simple check of this result by taking the $k \to \infty$ limit, in which the Chern-Simons theory should decouple. In this limit, $q \to 1$, and so the skein relation reduces to the identity



which is indeed exactly what one expects in this limit!

However, our interest lies at finite k, where the effects of the Chern-Simons level are much more interesting. Let us focus on the two-particle case again, and try to identify

the symmetric $|S\rangle$ state. In the

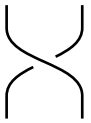
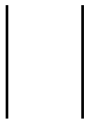


basis, B takes the form

$$B = q^{\frac{1-p^2}{2p}} \begin{pmatrix} 0 & q^p \\ 1 & q^{\frac{p-1}{2}} - q^{\frac{p+1}{2}} \end{pmatrix} .$$

Again, in the $q \to 1$ limit we clearly recover the familiar $(1,1)$ and $(1,-1)$ eigenstates with the usual $+1, -1$ eigenvalues respectively. But something striking happens when $q^{\frac{p+1}{2}} = -1$.

At this point, the *symmetric* representation corresponds to the eigenvector $(1,-1)$, what one would naively have thought was a antisymmetric object,

$$|S\rangle \quad \propto$$  $$-$$  .

What is the significance of this? Suppose we insert a pair of Wilson lines, anchored to infinity, and consider them to fall in a symmetric representation. Then the object is an antisymmetric function of its dangling endpoints, and in particular we take it to vanish when those endpoints are brought together. In other words, this is a null state in the theory, and it decouples. Thus one cannot symmetrize two Wilson lines in such a theory. (By contrast, the antisymmetric representation at $k = \infty$ is actually symmetric between its endpoints, since there are also antisymmetrized gauge indices there.)

But notice that $q^{\frac{p+1}{2}} = -q^{\frac{1-(k-p)}{2}}$, so with the identification $\hat{k} = k - p$, which we shall see later is the right thing to do, the above describes the correct $SU(p)$ fusion rule at level $\hat{k} = 1$.

To derive the analogous result for higher $\hat{k}$ would be a distraction, so let us leave this and just bear in mind that on general grounds, we should restrict ourselves to states obeying the fusion rules.

5.4 Perturbation Theory

The field theoretic approach to non-relativistic anyons comes equipped with the powerful methods developed for relativistic field theories. In particular, we can use Feynman diagrams to compute quantum corrections order by order in perturbation theory.

The diagrammatic method applies in much the same way as for relativistic theories with one crucial difference: there are no anti-particles, not even in loops! The absence of propagating anti-particles drastically reduces the number of diagrams, rendering the loop computations tractable.

It also follows that, in theories with multiple flavours, one type of species does not affect the dynamics of the other types unless it is present as an external particle. The utility of this was shown in [1] – the supersymmetry algebra shows that the scaling dimensions of certain chiral operators involving only bosons are one-loop exact, a property which then survives even when the fermions are thrown away!

Our goal in this section is to determine the one-loop corrections to the dimensions of operators of the form (5.6). We will use this to confirm our previous, algebraic results (5.9) and (5.11).

One-loop Corrections

The Feynman rules for (5.1) are a simple generalization of those presented in [1]. To avoid clutter we only discuss the case with only one species of scalars living in some representation R of the gauge group. The generalization to multiple species living in different representations is straightforward. Colour indices will be denoted by Greek letters $\rho, \sigma = 1, \ldots, \dim(R)$.

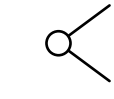
Since the interactions are at most quartic, all corrections arise from pairwise diagrams. Thus it suffices to compute the one-loop correction to the two-anyon operator

$$\partial^{n_1}\bar{\partial}^{m_1}\Phi^\dagger_{\rho_1}\partial^{n_2}\bar{\partial}^{m_2}\Phi^\dagger_{\rho_2}\,. \tag{5.18}$$

The one-loop corrections to (5.18) are encoded in the correlation function

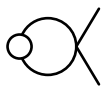
$$\langle \Phi_{\sigma_1}(p_1)\Phi_{\sigma_2}(p_2)\;\partial^{n_1}\bar{\partial}^{m_1}\Phi^\dagger_{\rho_1}\partial^{n_2}\bar{\partial}^{m_2}\Phi^\dagger_{\rho_2}\rangle\,. \tag{5.19}$$

At tree level we schematically denote this correlation function by this diagram:



$$\begin{aligned} &= \delta^{\rho_1}_{\sigma_1}\delta^{\rho_2}_{\sigma_2}(-ip_{1z})^{n_1}(-ip_{1\bar{z}})^{m_1}(-ip_{2z})^{n_2}(-ip_{2\bar{z}})^{m_2} \\ &\quad + \delta^{\rho_1}_{\sigma_2}\delta^{\rho_2}_{\sigma_1}(-ip_{1z})^{n_2}(-ip_{1\bar{z}})^{m_2}(-ip_{2z})^{n_1}(-ip_{2\bar{z}})^{m_1} \end{aligned} \tag{5.20}$$

Henceforth we shall suppress the diagram labels. At one-loop order the four-point function (5.19) is corrected by the diagrams



$$= -\frac{1}{2k}\log\frac{\Lambda}{\mu}\;\left(t^\alpha_{\rho_1\sigma_1}t^\alpha_{\rho_2\sigma_2} + t^\alpha_{\rho_1\sigma_2}t^\alpha_{\rho_2\sigma_1}\right)\left(-\frac{i}{2}P^+_z\right)^{n_1+n_2}\left(-\frac{i}{2}P^+_{\bar{z}}\right)^{m_1+m_2} + \mathcal{O}(\Lambda^2)$$

with $P^{\pm} = p_1 \pm p_2$, and

$$= \frac{1}{2k} \log \frac{\Lambda}{\mu} \left(t^{\alpha}_{\rho_1\sigma_1} t^{\alpha}_{\rho_2\sigma_2} K(p_1, p_2) + t^{\alpha}_{\rho_1\sigma_2} t^{\alpha}_{\rho_2\sigma_1} K(p_2, p_1) \right) + \mathcal{O}(\Lambda^2) \tag{5.21}$$

where $K(p_1, p_2)$ is given by the following integral over the Feynman parameter x:

$$K(p_1,p_2) = (-i)^l \int_0^1 \mathrm{d}x \Bigg\{ \prod_{i=1}^{2} \left(\frac{P_z^+}{2} - \frac{(-1)^i x P_z^-}{2} \right)^{n_i} \frac{\partial}{\partial x} \left[\prod_{i=1}^{2} \left(\frac{P_{\bar{z}}^+}{2} - \frac{(-1)^i x P_{\bar{z}}^-}{2} \right)^{m_i} \right] - \frac{\partial}{\partial x} \left[\prod_{i=1}^{2} \left(\frac{P_z^+}{2} - \frac{(-1)^i x P_z^-}{2} \right)^{n_i} \right] \prod_{i=1}^{2} \left(\frac{P_{\bar{z}}^+}{2} - \frac{(-1)^i x P_{\bar{z}}^-}{2} \right)^{m_i} \Bigg\} \tag{5.22}$$

with $l = n_1 + n_2 + m_1 + m_2$.

Note that we only need the logarithmic correction to extract the contribution to the anomalous dimension. Moreover, the results above are all one needs to evaluate the anomalous dimension of operators of the form (5.6) at one-loop.

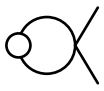
We remark that the operators of the form (5.6) may not have a well-defined dimension at one-loop.[7] Nonetheless for a special class of such operators it is an easy task to find operators with well-defined scaling dimensions. These are the chiral operators discussed earlier (5.8) which only include holomorphic derivatives. Below we consider a few examples of such operators.

Examples

As a warm-up, consider the following operator in an Abelian theory with a single species of unit charge scalars:

$$\mathcal{O}_n = \Phi^{\dagger\, n} \,.$$

This is the n-anyon operator with no derivatives. The kinematical factor in (5.21) vanishes and we only have the bubble diagrams (5.21) to sum over. As was discussed earlier, there is one such diagram for each pair, and each yields the same contribution



$$= -\frac{1}{k} \log \frac{\Lambda}{\mu} + \mathcal{O}(1) \,.$$

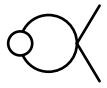
This results in the anomalous dimension

$$\Delta - n = \frac{n(n-1)}{2k} = -\mathcal{J} \tag{5.23}$$

[7] As is evident from the results above, a two-anyon operator with fixed n_i and m_i mixes at one-loop with operators with the same number of derivatives. Furthermore the one-loop diagrams above have polynomial divergences which need to be removed by counter-terms with fewer derivatives.

in agreement with (5.12).

Now consider the operator $\mathcal{O}_{\rho_1\ldots\rho_n} = \Phi^\dagger_{\rho_1}\ldots\Phi^\dagger_{\rho_n}$ in an $SU(p)$ Chern-Simons theory coupled to a single species of scalars in the fundamental representation of the gauge group. (Note that the ρ_a here are the hitherto suppressed gauge group indices; there are no flavour indices i_a.) As in the previous examples the absence of derivatives implies that we only need to evaluate the bubble diagram:



$$= -\frac{p-1}{2pk}\log\frac{\Lambda}{\mu}\,\left(\delta^{\rho_1}_{\sigma_1}\delta^{\rho_2}_{\sigma_2}+\delta^{\rho_1}_{\sigma_2}\delta^{\rho_2}_{\sigma_1}\right)+\mathcal{O}(1)$$

where we have used the following identity satisfied by the generators of $SU(p)$ in the fundamental representation

$$t^\alpha_{\rho_1\sigma_1}t^\alpha_{\rho_2\sigma_2} = \delta^{\rho_1}_{\sigma_2}\delta^{\rho_2}_{\sigma_1} - \frac{1}{p}\delta^{\rho_1}_{\sigma_1}\delta^{\rho_2}_{\sigma_2}\,. \tag{5.24}$$

Taking the contribution from each pair into account we obtain the anomalous dimension

$$\Delta - n = \frac{n(n-1)(p-1)}{2pk}$$

reproducing (5.15).

As our last example, consider the $SU(p)$ theory with one species of scalars in the fundamental representation. This time we look at the operator $\mathcal{O} = \Phi^\dagger_{[\rho_1}\partial\Phi^\dagger_{\rho_2}\ldots\partial^{n-1}\Phi^\dagger_{\rho_n]}$. Contrary to the previous examples, the corrections to this operator only arise from the gluon exchange diagrams:

$$= \frac{p+1}{2pk}\log\frac{\Lambda}{\mu}\,\delta^{\rho_a}_{[\sigma_a}\delta^{\rho_b}_{\sigma_b]}\left[(-ip_{Iz})^{n_a}(-ip_{Jz})^{n_b}-(-ip_{Iz})^{n_b}(-ip_{Jz})^{n_a}\right]+\mathcal{O}(\Lambda^2)\,.$$

The contribution is the same for every pair yielding the one-loop corrected dimension

$$\Delta_n = \frac{n(n+1)}{2} - \frac{n(n-1)}{2}\frac{p+1}{kp}$$

and in particular the dimension of the "baryonic" operator which one forms by anti-symmetrizing $n = p$ fundamentals together is given by

$$\Delta_p = \frac{p(p+1)}{2} - \frac{p^2-1}{2k}\,.$$

Evaluating the angular momentum, remembering to include a contribution of -1 per ∂ derivative, yields the same result.

5.5 Operators at the Unitarity Bound

Let us focus for a little while on an intriguing aspect of the Abelian anyon spectrum with a single flavour. (We will generalize the discussion to non-Abelian physics in Section 5.5.3.) There is a rather striking difference between repulsive interactions, with $k > 0$, and attractive interactions, with $k < 0$. In both cases, the dimension of the BPS state $\mathcal{O} \sim \Phi^{\dagger\, n}$ is given by

$$\Delta = n + \frac{n(n-1)}{2k} \,.$$

For $k > 0$, this is the spectrum of anyons discussed in the introduction. For $k < 0$, there is a new twist to the story because, for $n > |2k|$, this operator appears to violate the unitarity bound $\Delta \geq 1$.[8] What is going on?

5.5.1 Quantum Mechanics of Abelian Anyons

To understand why these states violate the unitarity bound, we turn to the quantum mechanical description of the problem. Such a formulation exists because there are no anti-particles in the action (5.1) and, moreover, the dynamics of the gauge field whose kinetic terms are given by (4.17) is tied to that of the particles. This means that there can be no fluctuation of particle number so, if you fix the number of bosons and fermions in the problem, then the field theory reduces to the quantum mechanics of a finite number of degrees of freedom. A simple derivation of how to move from the field theory language to the quantum mechanics can be found, for example, in the book [33].

Here we consider the sector with n particles. Each particle has position x_a^α, with $p = 1, 2$ the spatial index and $a = 1, \ldots, n$ labelling the particle. The quantum mechanics Hamiltonian is

$$H = -\frac{1}{2m}\sum_{a=1}^{n}\left(\partial_p^a + \frac{i}{k}\epsilon_{pq}\,\partial_q^b \sum_{b\neq a}\log|\mathbf{x}_a - \mathbf{x}_b|\right)^2 + \frac{2\pi}{mk}\sum_{a<b}\delta^2(\mathbf{x}_a - \mathbf{x}_b)\,. \tag{5.25}$$

Here the log term arises from the gauge field which is given by (5.13); this is the term which imposes anyonic statistics on the particles which now pick up a phase π/k when exchanged. The delta-functions arise from the $|\phi|^4$ interactions in the Lagrangian, as was previously highlighted. These contact interactions are repulsive for $k > 0$ and attractive for $k < 0$. We should ultimately add to the Hamiltonian (5.25) the harmonic potential. We'll do this below, but it won't be important for our immediate discussion.

[8] We will not worry at all about the fusion rules for our studies of Abelian theories; indeed since, they may be defined for arbitrary k, Abelian theories must make sense in the absence of a truncation in the available states at special values of n.

For us, the role played by the delta-function contact interactions is key. These arise naturally from the field theory and endow the quantum mechanics with a number of nice features. Indeed, as we review below, they are necessary for the quantum mechanics to exhibit scale invariance. For now, their main purpose is to impose boundary conditions on the wavefunction[9] $\Psi(\mathbf{x}_a)$ as anyons get close to each other. For two particles, their s-wave state has boundary condition

$$\Psi(\mathbf{x}_1, \mathbf{x}_2) \sim |\mathbf{x}_1 - \mathbf{x}_2|^{1/k} \quad \text{as} \;\; \mathbf{x}_1 \to \mathbf{x}_2 \tag{5.26}$$

with a pairwise generalization to multiple particles[10].

For repulsive contact interactions, i.e. $k > 0$, the wavefunction (5.26) vanishes as the particles approach; it is equivalent to imposing a hard-core boundary condition.

In contrast, with an attractive contact interaction, corresponding to $k < 0$, the wavefunction diverges as the two particles approach. For two particles, this is not problematic because the wavefunction (5.26) is normalizable as long as $|k| > 1$. But this divergence becomes more serious when we add too many particles, as we now describe.

The wavefunction for n particles in which each pair sits in the s-wave is

$$\Psi_0 = \prod_{a<b} |\mathbf{x}_a - \mathbf{x}_b|^{-1/|k|} \,. \tag{5.27}$$

This corresponds to the operator $(\Phi^\dagger)^n$. (We will make the connection between operators and wavefunctions more precise below.) One can check that Ψ_0 is a zero-energy eigenstate of the Hamiltonian (5.25). As discussed in the next section, there is actually a large degeneracy at zero-energy, so one might reasonably ask what is special about this eigenstate. The answer is that it is the one adiabatically connected to the ground state (5.31) in the presence of a trap breaking this degeneracy. (The connection to $(\Phi^\dagger)^n$ makes this no surprise.)

When all n particles coincide, one finds a divergence from each of the $n(n-1)/2$ pairs of particles. This means that the wavefunction takes the schematic form $\Psi_0 \sim r^{-n(n-1)/2|k|}$ where r measures the "radial" relative distance from the coincident point.

[9] In this section, we will use Ψ to denote the wavefunction of bosonic anyons. It is not to be confused with the composite operator introduced later to denote fermionic anyons.

[10] There is an alternative way to view these boundary conditions. One could exclude from the configuration space the points where particles meet and propose a self-adjoint extension of the quantum mechanics [55]. There are precisely two such extensions which are compatible with scale invariance, corresponding to (5.26) with $k > 0$ and $k < 0$. We will see this again in Section 6.2.3.

The normalization is

$$\int \prod_{a=1}^{n} \mathrm{d}^2 x_a \, |\Psi_0|^2 \sim \int \mathrm{d}^2 X \int \mathrm{d}r \, r^{2n-3} |\Psi_0|^2 \sim \int \mathrm{d}^2 X \int \mathrm{d}r \, \frac{r^{2n-3}}{r^{n(n-1)/|k|}} \tag{5.28}$$

where X is the centre of mass. We see that the norm is UV finite if and only if

$$2n - 3 - \frac{n(n-1)}{|k|} > -1 \qquad \Longleftrightarrow \qquad n < 2|k| \, .$$

The wavefunction is normalizable only when $n < 2|k|$. This, of course, coincides with the threshold that we found from the unitarity bound.

This, then, is the answer to our puzzle: the operators $(\Phi^\dagger)^n$ which violate the unitarity bound correspond to wavefunctions in the quantum mechanics which are non-normalizable. Note, in particular, that the wavefunction with $n = 2|k|$ particles is also (logarithmically) non-normalizable, despite the fact that the operator saturates the unitarity bound. The relationship between violations of the unitarity bound and the non-normalizability of the wavefunction was previously noted in a different context in [87].

Mapping Between Operators and Wavefunctions

We have learned that the operator $(\Phi^\dagger)^n$ corresponds to a non-normalizable state for $n \geq 2|k|$. This leaves open the simple question: what is the ground state of $n \geq 2|k|$ anyons in a trap? In general, there is no reason to believe that the ground state lies in a chiral multiplet. This makes the question difficult. We can, however, answer the simpler question: what is the lowest energy chiral state for $n \geq 2|k|$?

To answer this question, we will extend the correspondence

$$\mathcal{O} = (\Phi^\dagger)^n \quad \longleftrightarrow \quad \Psi_0 = \prod_{a<b} |\mathbf{x}_a - \mathbf{x}_b|^{1/k}$$

to other chiral operators. This will allow us to determine which operators $\mathcal{O}$ correspond to normalizable wavefunctions.

To proceed, we rewrite our Hamiltonian (5.25) in a way which eliminates the delta-functions. Indeed, there is independently good reason to do this, related to the scale invariance of the theory. In two dimensions, the delta-function has the same scaling as the Laplacian ∇^2, which means that a Hamiltonian of the form (5.25) would appear to be scale invariant for any coefficient of the delta-function interaction. This is misleading. Delta-functions in quantum mechanics require a regularization and this typically breaks scale invariance, resulting in a simple example of an anomaly in a quantum mechanical setting [88, 89, 90]. A similar effect also arises from the log term in (5.25). For

the choice of coefficient in front of the delta-function in (5.25), these two effects cancel. This, of course, mimics the field theory analysis of [56, 57].

To see the cancellation explicitly, we can rewrite the Hamiltonian by introducing complex coordinates $z_a = x_a^1 + ix_a^2$ for each particle. For the specific coefficient of the delta-function interaction given in (5.25), we have

$$H = -\frac{2}{m}\sum_{a=1}^{n} \Psi_0 \left(\partial_{\bar{z}_a}\partial_{z_a} + \frac{1}{k}\sum_{b\neq a} \frac{\partial_{z_a}}{\bar{z}_a - \bar{z}_b} \right) \Psi_0^{-1} \tag{5.29}$$

where Ψ_0 is given in (5.27). The ∂ operators in (5.29) are understood to act on everything to the right including, ultimately, the wavefunction. (Note that, as one might expect, there is a close relationship between the form of (5.29) and (15.3), the Knizhnik-Zamolodchikov equation, which we will use later in our analysis of correlation functions in Wess-Zumino-Witten theories.)

This form of the Hamiltonian (5.29) has no delta-functions and, correspondingly, no need for regularization: it provides a manifestly scale invariant description of the dynamics. Further, it is immediately clear that the wavefunction Ψ_0 obeys $H\Psi_0 = 0$ as previously claimed. It is also easy to write down a large class of eigenfunctions, given by

$$\Psi = \bar{f}(\bar{z}_1, \ldots, \bar{z}_n) \prod_{i<j} |z_a - z_b|^{1/k} \tag{5.30}$$

where $\bar{f}(\bar{z})$ is an antiholomorphic function, symmetric in its arguments $\bar{z}_a$. We propose that this class of wavefunctions is equivalent to the set of chiral operators of the conformal field theory, with the mapping given up to normalization by

$$\mathcal{O} = \partial^{p_1}\Phi^\dagger \cdots \partial^{p_n}\Phi^\dagger \qquad \longleftrightarrow \qquad f = z_1^{p_1}\cdots z_n^{p_n} + \text{permutations}\,.$$

In particular we see that descendants in the CFT, which are obtained by total derivatives, correspond to choices of f with factors of $\sum z_a$:

$$\tilde{\mathcal{O}} = \partial\mathcal{O} \qquad \longleftrightarrow \qquad \tilde{f} = \left(\sum_{a=1}^{n} z_a\right) f\,.$$

Hence, if we exclude the descendants we are left to form f by symmetrizing products of terms like $(z_a - z_b)^2$ which respect the bosonic properties of the particles. This provides a useful way to describe and enumerate all chiral primaries. For example, there are no chiral primaries with just a single derivative while, at the two derivative level, we find

$$\mathcal{O} = \left(\partial^2\Phi^\dagger\Phi^\dagger - \partial\Phi^\dagger\partial\Phi^\dagger\right)\Phi^\dagger\cdots\Phi^\dagger \qquad \longleftrightarrow \qquad f = (z_1 - z_2)^2 + \text{other pairs}\,.$$

In this manner, we see that chiral primary operators arise by giving pairs of particles extra relative angular momentum. Correspondingly, the angular momentum of chiral primaries is spaced in even-integer steps, since we always need polynomials of even order in z. In contrast, the angular momentum of descendants is spaced in integer steps.

Now we can ask which of these wavefunctions lie in the Hilbert space. The divergences of the chiral wavefunctions (5.30) are softer than those of Ψ_0. Heuristically, this is because the addition of $(z_a - z_b)^2$ factors increases the relative angular momentum of a pair of particles, so that their wavefunction is damped where they meet. However, we need to determine what form of f is sufficient to render the wavefunctions normalizable as the particles converge.

Let us suppose that f is a polynomial of order $2m$. The simplest criterion on the wavefunction arises from the situation where all n particles converge to a point. In this case, repeating the calculation (5.28), the requirement for normalizability is

$$2n - 3 - \frac{n(n-1)}{|k|} + 4m > -1 \qquad \Longleftrightarrow \qquad 2m > (n-1)\left(\frac{n}{2|k|} - 1\right) .$$

This coincides with the requirement that the dimension of the corresponding operator, which is schematically of the form $\mathcal{O} \sim \partial^{2m}(\Phi^\dagger)^n$, sits strictly above the unitarity bound $\Delta > 1$. This agreement is reassuring but it is not the end of the story.

Suppose that we instead bring some subset of $q < n$ particles together. Without loss of generality, we can pick particles $a = 1, \ldots, q$. The wavefunction (5.30) diverges as $r^{2m_q - q(q-1)/2|k|}$ where m_q is the *smallest* number of relative angular momentum terms $(z_a - z_b)^2$ with $a, b = 1, \ldots, q$ that appears in the expansion of f. Clearly when we include all particles we include all winding terms, so $m_n = m$.

This is perhaps best illustrated with an example. Consider $n = 4$, with $f \sim (z_1 - z_2)^2(z_1 - z_3)^2 + \cdots$. We see that $m_4 = 2$ is the total number of angular momentum terms. However, $m_3 = 0$ because f remains of order 1 if particle 1 is separated while particles 2, 3 and 4 are brought together. Thus the additional angular momentum in f has not helped convergence at $q = 3$.

The significance of this is that there are additional constraints at each order q on the form of f and, correspondingly, on the possible chiral operators $\mathcal{O}$ in the theory. These constraints are equivalent to imposing that $\Delta > 1$ not only for the operator $\mathcal{O}$ itself, but for every 'channel' of $\mathcal{O}$: roughly speaking, if we can express $\mathcal{O} = \mathcal{O}_1\mathcal{O}_2$ then we need $\Delta_{\mathcal{O}_i} > 1$ as well.

Nonetheless, by including enough angular momentum, one may see that it is in fact always possible to find UV-normalizable chiral states in the theory. The relative angular momentum forms a barrier, supporting the wavefunction away from the origin so that the wavefunction survives in the Hilbert space.

Solutions in the Trap

Finally, for completeness we observe that we can also find explicit chiral wavefunctions in a trap. The Hamiltonian is now

$$L_0 = H + \frac{m}{2}\sum_a |\mathbf{x}_a|^2 \,.$$

It is again more convenient to express this in complex coordinates in a manner analogous to (5.29). It is

$$L_0 = \tilde{\Psi}_0 \left[\sum_{a=1}^{n}\left(-\frac{2}{m}\partial_{\bar{z}_a} - \frac{2}{mk}\sum_{b\neq a}\frac{1}{\bar{z}_a - \bar{z}_b} + z_a\right)\partial_{z_a} + n + \frac{n(n-1)}{2k} + \sum_{a=1}^{n}\bar{z}_a\partial_{\bar{z}_a}\right]\tilde{\Psi}_0^{-1}$$

where $\tilde{\Psi}_0$ is the ground state wavefunction in the trap,

$$\tilde{\Psi}_0 = \prod_{a<b}|z_a - z_j|^{1/k}\exp\left(-\frac{m}{2}\sum_{a=1}^{n}|z_a|^2\right) \,. \tag{5.31}$$

Hence writing $\Psi = \bar{f}(\bar{z})\tilde{\Psi}_0$ with f any symmetric degree d polynomial, we analytically find a class of wavefunctions with energies

$$\Delta = n + \frac{n(n-1)}{2k} + d \,.$$

Using the expression (5.12) for the angular momentum in the Abelian theory, one easily observes that this coincides with the chiral bound (4.15).

5.5.2 Relationship to Jackiw-Pi Vortices

The result that $n = 2|k|, k < 0$ is a special point in the Abelian theory leads us to a tantalizing conjecture. Let us look at the form of $Q_2 \propto \int \mathrm{d}^2x\phi^\dagger\mathcal{D}_{\bar{z}}\psi$ in the supersymmetric theory. One obtains from this a condition for BPS solutions to the theory, $\mathcal{D}_z\phi = 0$. Classical solutions to this equation clearly make the Hamiltonian (5.3) vanish and hence solve the equations of motion of the bosonic theory. Of course, we must supplement this with Gauss's law:

$$f_{12} = \frac{2\pi}{k}|\phi|^2 \quad , \quad \mathcal{D}_z\phi = 0 \,. \tag{5.32}$$

These equations are somewhat unusual: they admit non-topological but nonetheless quantized vortex solutions with a non-trivial moduli space, named for their discovers, Jackiw and Pi [91, 32]. The role of these solitons in the quantum theory seems to be poorly understood; in this section we will suggest that they are related to the divergent states analysed above. Let us begin by briefly reviewing these *Jackiw-Pi vortices*. (See for instance [92, 93] for more detailed expositions.)

Solutions to the system (5.32) exist only for $k < 0$. An explicit form for the general solution may be straightforwardly obtained. We decompose the scalar field as

$$\phi = \sqrt{\rho} e^{i\chi} \tag{5.33}$$

where $\rho = \phi^\dagger \phi$ is the matter density. The gauge field is determined by the second equation in (5.32) to be

$$a_z = \partial_z \chi - \frac{i}{2} \partial_z \log \rho \,. \tag{5.34}$$

Substituting this into Gauss's law constraint reveals that ρ satisfies the Liouville equation,

$$\nabla^2 \log \rho = \frac{4\pi}{k} \rho \,. \tag{5.35}$$

The general solution to (5.35) for $k < 0$ can be written in terms of a holomorphic function $u(z)$,

$$\rho = \frac{k}{2\pi} \nabla^2 \log \left(1 + |u(z)|^2\right) \,. \tag{5.36}$$

It is illuminating to look at the axially symmetric solutions, with $u(z) = (z_0/z)^q$. These take the form

$$\rho = \frac{2|k| q^2 r_0^{2q}}{\pi} \frac{r^{2(q-1)}}{(r_0^{2q} + r^{2q})^2} \,. \tag{5.37}$$

Asymptotically, the matter density scales as $\rho \sim r^{-2(q+1)}$ and normalizability requires that $q > 0$. Meanwhile, at the origin, the matter density scales as $\rho \sim r^{2(q-1)}$. To ensure that the gauge field (5.34) is non-singular, the phase of χ must wind accordingly. This requires q to be integer, with the scalar field profile given by

$$\phi = \sqrt{\frac{2|k| q^2 r_0^{2q}}{\pi}} \frac{r^{(q-1)}}{r_0^{2q} + r^{2q}} e^{-i(q-1)\theta} \,.$$

This means that, although there is no topology in the vacuum manifold supporting these solitons, their charge is nonetheless quantized. The integral of the matter density

is

$$n = \mathcal{N} = \int \mathrm{d}^2x\, \rho = 2|k|q \tag{5.38}$$

and the corresponding flux is $\int f = -4\pi q$. Note that the minimal flux carried by the vortices is twice that usually required by flux quantization. This is a well known, if rather peculiar, feature of classical Jackiw-Pi vortices.

Although we derived (5.38) for axially symmetric solutions, it continues to hold for the most general solution. For separated vortices, we may take

$$u(z) = \sum_{a=1}^{p} \frac{c_a}{z - z_a}\,. \tag{5.39}$$

This describes p vortices at positions z_a, with c_a providing a scale size and phase for each vortex. (This solution needs amending as the vortices coincide.) The collective coordinates z_a and c_a parametrize the moduli space $\mathcal{M}_p$ of Jackiw-Pi vortices which has dimension $\dim \mathcal{M}_p = 4p$.

There is something striking about the result (5.38): setting $p = 1$, we see the single vortex has $n = 2|k|$, which is exactly the same point where the operator $(\Phi^\dagger)^n$ hits the unitarity bound $\Delta = 1$! An operator at the unitarity bound should describe a single, free excitation. It is therefore natural to conjecture that semi-classically, a suitably regularized $(\Phi^\dagger)^n$ operator creates a Jackiw-Pi vortex. More precisely, one might imagine that it is necessary to introduce a length scale to regularize the aforementioned operator, and then that this length scale would become the parameter c providing the scale size of the vortex.

One might then hope that the quantum numbers of this operator agree with those of the Jackiw-Pi configuration. With that in mind, let us look at the angular momentum as defined in (5.10). Evaluating this classically gives us $\mathcal{J} = \mathcal{N} = 2|k|p$ when evaluated on vortices. (Note that this is linear in p, rather than quadratic.) This angular momentum is greater than that of any chiral primary operator of the corresponding $\mathcal{N}$ eigenvalue. In particular, recall that $(\Phi^\dagger)^{2|k|}$ has $\mathcal{J} = 2|k| - 1$, while including P within an operator decreases the angular momentum. Similarly, the classical configuration has $D = 0$ whilst the quantum one has $\Delta = 1$. This argues against the conjecture.

On the other hand, it is not clear that evaluating a classical generator and computing the commutator with the quantum one are truly analogous. (For instance, due to subtleties in the quantum theory, it is iD that has real eigenvalues, even though it is D which is a real operator.)

Moreover, notice that in introducing a new scale c, one necessarily ends up with a configuration which is not scale invariant unless one also scales the parameter c – in other words, the Jackiw-Pi vortices fill out a representation of D rather than diagonalizing it.

It seems possible that subtleties in going between the classical and quantum pictures here may explain the discrepancies we are seeing.

It is worth noting that the proposal being made here is analogous to what is believed to happen in similar situations in certain relativistic theories in three and four dimensions [94, 95]. The general idea is that one starts with a microscopic theory, and then makes a naive guess at the structure of its IR limit: a non-trivial superconformal field theory. However, one finds that there are operators (monopole operators in three dimensions, mesons in four) which would apparently saturate or violate a unitarity bound. The conclusion is that the proposed description is wrong – instead, it seems that the moduli space of vacua contains directions associated to a free field theory.

This fits in well with the situation here. We observe that, since JP vortices have vanishing Hamiltonian for arbitrary values of their moduli, they represent degenerate choices of vacua for a given $\mathcal{N}$. So instead of thinking about the operator $(\Phi^\dagger)^{2|k|}$ and its descendants, one should think about quantizing the moduli of the Jackiw-Pi vortex instead. Nonetheless, it seems like one should be able to find a clear correspondence between the quantum numbers of these modes.

(Note that in [1], solitons in the presence of a harmonic trap were discussed. This is motivated by how much easier it is to compare solitons to the extended states in the trap than to local operators. However, these have different properties to Jackiw-Pi vortices and seemed there not to offer any immediate insight into the problem. A proposal for a matrix model description of these vortices was also made.)

Let us set this aside and discuss the non-Abelian analogue of the issues we have discussed.

5.5.3 Non-Abelian Generalization

In the above, we saw that the presence of attractive delta-function interactions between Abelian anyons forces wavefunctions to diverge as particles come close. For sufficiently many particles, this divergence becomes logarithmically non-normalizable and this state is no longer part of the Hilbert space. The true chiral ground state requires extra orbital angular momentum, softening the divergence.

This same behaviour also occurs in the non-Abelian theory. Roughly speaking, symmetrized representations have anomalous dimensions that scale as $+1/k$, while those of antisymmetrized representations scale as $-1/k$. When $k < 0$, it is simple to see that placing too many anyons together in a symmetrized representation will violate the unitarity bound. There is now, however, an interesting question about $k > 0$. Perhaps the simplest example of an operator that might violate the unitarity bound for $k > 0$ arises in the case of $SU(p)$ with $N_f = p$ different species of scalar, ϕ_i, each in the fundamental representation. We can then build a baryon operator without the need to add any derivatives: $B = \epsilon_{i_1 \ldots i_p} \Phi_1^{i_1} \ldots \Phi_p^{i_p}$. Using the methods above, the dimension of this operator is

$$\Delta_B = p - \frac{p^2 - 1}{2k} \,.$$

This violates the unitarity bound $\Delta_B \geq 1$ for $k < (p+1)/2$. Note that here the bound constrains the rank of the gauge group, p. Presumably, this can once again be traced to the non-normalizability of the quantum mechanical wavefunction.

Interestingly, however, non-relativistic theories describing the low energy dynamics of massive relativistic theories always satisfy $|k| > p$ due to quantum shifts of the level. (We will discuss these issues in Chapter 6.) This means that in these theories, the baryon B never violates the bound for any k. In fact, one can check fairly straightforwardly that $\Delta > 1$ for all $SU(p)$ representations built from fundamental matter with more than one particle, provided $k > p$.

It is natural to ask if the suggested connection to Jackiw-Pi vortices highlighted in Section 5.5.2 above can be extended to the non-Abelian situation. The answer is yes. To begin, let us write down the relevant BPS equations:

$$f_{12}^{\alpha} = \frac{2\pi}{k} \phi_{\rho}^{\dagger} t_{\rho\sigma}^{\alpha} \phi_{\sigma} \quad , \quad \mathcal{D}_z \phi_{\rho} = 0 \,. \tag{5.40}$$

Here, α labels the generators of some unitary group, and ρ, σ label the weight vectors in the representation whose generators are t^{α}.

In general, much less is known about the classical solutions to the BPS equations in non-Abelian theories (see e.g. [92, 96] for some discussions), but there is one particularly simple class of solutions which we can analyse with little extra effort.

The idea is to only turn on one component of the non-Abelian object ϕ_{ρ}. The advantage in this is that $\phi_{\rho}^{\dagger} t_{\rho\sigma}^{\alpha} \phi_{\sigma}$ will then only have components for values of α corresponding to Cartan elements; hence it suffices to turn on only Cartan components of the gauge field a_z^{α}, and the awkward non-Abelian structure of the equations can be avoided.

Concretely, let us suppose that the h^A span the Cartan subalgebra of the group. Then if μ^A is the weight associated to the weight vector ρ which we have turned on, the second equation in (5.40) becomes

$$\partial_z \log \phi_\rho = i\mu^A a_z^A \,.$$

Meanwhile, Gauss's law for a purely Cartan gauge field is

$$2i(\partial_{\bar{z}} a_z^A - \partial_z a_{\bar{z}}^A) = \frac{2\pi}{k}\mu^A |\phi_\rho|^2 \,.$$

Then taking the further ansatz $a_z^A \propto \mu^A$ reduces the equations to

$$\partial_{\bar{z}}\partial_z \phi_\rho = \frac{\pi|\mu|^2}{k}|\phi_\rho|^2$$

which is identical to the Liouville equation (5.35) which we found in the Abelian case, except for the dependence upon $|\mu|^2$, which is new.

It follows that the solutions obey a modified quantization condition, with

$$\mathcal{N} = \frac{2|k|q}{|\mu|^2}, \qquad q \in \mathbb{N} \,. \tag{5.41}$$

We still require negative k.

Now clearly, if these states are to be compared with an operator in the quantum theory, it should be $\mathcal{O} = (\Phi_\rho^\dagger)^n$. This operator, of course, is in general not associated to a single representation of the gauge group and so outside the scope of what we have discussed. However, there is one easy case which is encouraging. Suppose ρ corresponds to a highest weight of the representation R. Then this state is a representative of the totally symmetric product $\mathrm{Sym}[R^n]$.

It is now easy to compute $\mathcal{J}$ for the n^{th} symmetric product of the representation R whose highest weight is μ. (We calculated this as an example earlier.) From $C_2(\mu) = \langle \mu, \mu + 2\rho \rangle$ we have

$$\mathcal{J} = -\frac{C_2(n\mu) - nC_2(\mu)}{2k} = -\frac{n(n-1)\,\langle \mu, \mu \rangle}{2k} \,.$$

Then the unitarity bound is hit at $1 = n - \mathcal{J}$ or

$$n = -\frac{2k}{|\mu|^2}$$

in perfect agreement with (5.41).

It is possible to generalize this to other weights, by decomposing the symmetric product carefully into its irreps, though there seems little further insight to be gained from this process.

Notice that again we find, as one always must, that the classical angular momentum is $\mathcal{J} = \mathcal{N}$, whilst the quantum operator has $\mathcal{J} = \mathcal{N} - 1$. (Similarly, $D = 0$ whilst $\Delta = 1$.) Clearly the same issues as in the Abelian case are at work here. It would be nice to understand what the true connection to these non-topological vortices is.

6 Bosonization and The Fermionic Theory

In this section, we give another description of anyons, this time using non-relativistic fermions as the starting point. We will couple these fermions to an $SU(p)_k$ Chern-Simons theory. The real goal in introducing these theories is to explore the role of the exciting *bosonization dualities* which have seen so much recent interest; we will postpone discussion of these phenomena until Section 6.2, after we have outlined the fermionic theory.

The matter consists of N_f complex, Grassmann-valued fields ψ_i, each transforming in some representation R_i of $SU(p)$. These fields have non-relativistic kinetic terms, with the action given by

$$S = S_{CS} + \int \mathrm{d}t\, \mathrm{d}^2x \left\{ i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{1}{2m} \vec{\mathcal{D}}\psi_i^\dagger \, \vec{\mathcal{D}}\psi_i - \frac{1}{2m} \psi_i^\dagger f_{12}^\alpha \, t^\alpha[R_i] \psi_i \right\} . \tag{6.1}$$

The coupling to the non-Abelian magnetic field f_{12} plays an analogous role to the quartic interactions in the bosonic Lagrangian (5.1). (This is particularly apparent from the expression for f_{12} which Gauss's law furnishes us with.)

Like its bosonic counterpart, this theory also exhibits conformal invariance. The various symmetry generators can be constructed from the number density and momentum current, which are given by

$$\rho = \psi_i^\dagger \psi_i \quad \text{and} \quad \mathbf{j} = -\frac{i}{2}\Big(\psi_i^\dagger \vec{\mathcal{D}}\psi_i - (\vec{\mathcal{D}}\psi_i^\dagger)\psi_i \Big) .$$

The Hamiltonian is given by

$$H = \int \mathrm{d}^2x \, \frac{2}{m} \mathcal{D}_z \psi_i^\dagger \, \mathcal{D}_{\bar{z}} \psi_i \, .$$

As explained in Chapter 5, we can construct gauge invariant operators by attaching a semi-infinite Wilson line to each particle,

$$\Psi_i(\mathbf{x}) = \mathcal{P} \exp\left(i \int_\infty^{\mathbf{x}} a^\alpha \, t^\alpha[R_i] \right) \psi_i(\mathbf{x}) \, . \tag{6.2}$$

As before, our interest lies in the spectrum of n anyons in a trap. The most general operator takes the form

$$\mathcal{O} \sim \prod_{a=1}^{n} (\partial^{l_a} \bar{\partial}^{m_a} \Psi^\dagger_{i_a}) \tag{6.3}$$

where, again, primary operators are those which cannot be written as a total derivative.

However, the anti-commuting nature of ψ_i means that the simplest operators are rather different to those in the bosonic case. Consider, for example, the situation where we have a single species of fermion Ψ transforming in the $\mathbf{p}$ representation of $SU(p)$. Now the operator

$$\mathcal{O} = \Psi^{\dagger\, n} \tag{6.4}$$

is non-vanishing only for $n \leq p$ and transforms in the n^{th} antisymmetric representation. If we wish to place $n > p$ anyons in a trap, the different operators must be dressed with derivatives. To illustrate this, let's revert to Abelian anyons, charged under a $U(1)$ gauge field. Now there is no operator of the form (6.4) with $n > 1$. Instead, the operator with the lowest number of derivatives takes the form

$$\mathcal{O} = \Psi^\dagger\, \partial\Psi^\dagger\, \bar{\partial}\Psi^\dagger\, \partial^2\Psi^\dagger\, \partial\bar{\partial}\Psi^\dagger \ldots .$$

This operator has $\sim n^{3/2}$ derivatives. At large k, this is the ground state of the n anyon system, with $\Delta \sim n^{3/2}$. However, at smaller k, the ground state is expected to undergo level crossing.

Computing the spectrum in the fermionic case is no easier than for bosons. Once again, there are two approaches that we can take. The first is the brute force, perturbative approach, valid for large k. We describe this below in Section 6.1. However, once again there is a class of operators whose spectrum is constrained by their angular momentum. These are of course the *anti-chiral operators* and have only antiholomorphic derivatives

$$\mathcal{O} = \prod_{a=1}^{n} (\bar{\partial}^{m_a} \Psi^\dagger_{i_a}) \,.$$

For these operators, the dimension is fixed in terms of their angular momentum as

$$\Delta = \frac{n}{2} + \mathcal{J} \,. \tag{6.5}$$

Note the opposite minus sign and factor of two compared to (5.9), which can be traced back to (4.16), and the expression for the R-symmetry in the supersymmetric theory.

In the context of supersymmetry, the difference is that these should be thought of as anti-BPS states rather than BPS states.

Examples

The simplest example we can consider is a single fermion ψ coupled to an Abelian $U(1)_k$ Chern-Simons theory. The simplest anti-chiral n-particle operator is

$$\mathcal{O}_n = \Psi^\dagger \, \bar{\partial}\Psi^\dagger \, \bar{\partial}^2\Psi^\dagger \ldots \bar{\partial}^{n-1}\Psi^\dagger \, .$$

This operator has $n(n-1)/2$ derivatives, each of which contributes $+1$ to the total angular momentum, and n spin $1/2$ fermions. Meanwhile, the angular momentum from the Wilson lines is given by (5.12) as for the bosonic theory. We have

$$\mathcal{J} = \frac{n(n-1)}{2} + \frac{n}{2} - \frac{n(n-1)}{2k} \quad \Rightarrow \quad \Delta = \frac{n(n+1)}{2} - \frac{n(n-1)}{2k} \, . \tag{6.6}$$

Notice that there is something interesting about this dimension: if we replace $1/k \to 1 - 1/k$, then the dimensions of (6.6) trace out the same spectrum as (3.1) – the bosonic operators $(\Phi^\dagger)^n$ have the same spectrum! Indeed, by standard flux attachment considerations, at $k = 1$, the Φ excitations are fermions and the Ψ excitations are bosons.

This can be easily understood if one repeats the quantum mechanical analysis of Section 5.5.1: the ground state wavefunctions for fermions differ from those of bosons simply by the requirement that they be antisymmetric, and hence there is always an overall factor like $\prod_{a<b}(z_a - z_b)$ in the wavefunction.[1] This conspires with the "1" of $1 - 1/k$ to leave everything unchanged. (We will explore the quantum mechanics in some detail for the case of two anyons shortly.)

Moreover, one can easily verify that the alternative angular momentum $\mathcal{J}'$ which was introduced in Section 5.3, in which one does not subtract off the contributions $C_2(R_a)$ of the individual particles, agrees across the two sides up to an overall sign, since for the bosons $\mathcal{J}' = -n^2/2k$ whilst for the fermions

$$\mathcal{J}' = \frac{n(n-1)}{2} + \frac{n}{2} - \frac{n^2}{2k} = \frac{n^2}{2}\left(1 - \frac{1}{k}\right) \, .$$

Note that the relative sign might not be so surprising: we know that on the bosonic side of the duality, the operator ∂ preserves the chiral nature of the operator, whilst on the fermionic side it is $\bar{\partial}$. This might hint that the theories differ, but only by a parity transformation.

[1] This can also be rewritten as $\epsilon_{a_1 \ldots a_n} z_{a_1}^0 z_{a_2}^1 \cdots z_{a_n}^{n-1}$, making contact with the form of (6.12). We will see many more identities like this when we discuss the quantum Hall effect.

This is the first hint of bosonization in this theory. We will discuss this at length in Section 6.2.

In $SU(p)$ gauge theories, the simplest operator (6.4) sits in the n^{th} antisymmetric representation. We have $C_2(\text{Antisym}^n(\mathbf{p})) = n(p-n)(p+1)/p$ and, correspondingly,

$$\mathcal{J} - \frac{n}{2} = \frac{n(n-1)(p+1)}{2pk}\,. \tag{6.7}$$

Recall that for the bosonic case, when $k > 0$ the symmetrized representations increased the dimension of the operator whilst antisymmetrized ones decreased it. Because of the different sign in (6.5) relative to (5.9), this is reversed for fermions.

In the bosonic theories, we saw that certain states violate the unitarity bound. These do not arise in the Abelian fermionic theories, nor in the non-Abelian theories with $k > 0$. However, there are such states in the non-Abelian fermionic theories with $k < 0$, with the baryon the obvious example.

6.1 Perturbation Theory with Fermions

Non-relativistic conformal fermions with Chern-Simons interactions can be studied perturbatively in much the same way as the scalars in Section 5.4. Here we restrict the analysis to one-loop order.

One-loop Corrections

Similar to the theory with scalars, all one-loop corrections to the operators of the form (6.3) arise from pairwise diagrams. Therefore to extract the anomalous dimension of such operators we need only to compute the logarithmic correction to the two-anyon operator

$$\partial^{n_1}\bar{\partial}^{m_1}\Psi^\dagger_{\rho_1}\partial^{n_2}\bar{\partial}^{m_2}\Psi^\dagger_{\rho_2} \tag{6.8}$$

with the Greek letters $\rho, \sigma = 1, \ldots, \dim R$ denoting the colour indices. We restrict the analysis to a single flavour of fermions living in the representation R of the gauge group but the generalization to multiple flavours is straightforward. As in the bosonic case, we focus on the correlation function

$$\langle \Psi_{\sigma_2}(p_2)\Psi_{\sigma_1}(p_1)\; \partial^{n_1}\bar{\partial}^{m_1}\Psi^\dagger_{\rho_1}\partial^{n_2}\bar{\partial}^{m_2}\Psi^\dagger_{\rho_2}\rangle\,. \tag{6.9}$$

At tree level, we schematically denote this correlation function by the following diagram:

$$\begin{aligned} &= \delta^{\rho_1}_{\sigma_1}\delta^{\rho_2}_{\sigma_2}(-ip_{1z})^{n_1}(-ip_{1\bar{z}})^{m_1}(-ip_{2z})^{n_2}(-ip_{2\bar{z}})^{m_2} \\ &\quad - \delta^{\rho_1}_{\sigma_2}\delta^{\rho_2}_{\sigma_1}(-ip_{1z})^{n_2}(-ip_{1\bar{z}})^{m_2}(-ip_{2z})^{n_1}(-ip_{2\bar{z}})^{m_1}\,. \end{aligned} \tag{6.10}$$

The only correction this correlation function receives at one-loop arises from the gluon exchange diagram

$$\begin{aligned} &= \frac{1}{2k}\log\frac{\Lambda}{\mu}\Bigg[\left(t^\alpha_{\rho_1\sigma_1}t^\alpha_{\rho_2\sigma_2} - t^\alpha_{\rho_1\sigma_2}t^\alpha_{\rho_2\sigma_1}\right)\left(-\frac{i}{2}P^+_z\right)^{n_1+n_2}\left(-\frac{i}{2}P^+_{\bar{z}}\right)^{m_1+m_2} \\ &\qquad + t^\alpha_{\rho_1\sigma_1}t^\alpha_{\rho_2\sigma_2}K(p_1,p_2) - t^\alpha_{\rho_1\sigma_2}t^\alpha_{\rho_2\sigma_1}K(p_2,p_1)\Bigg] + \mathcal{O}(\Lambda^2) \end{aligned} \tag{6.11}$$

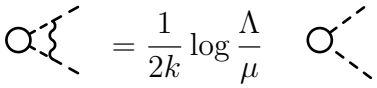
with $P^\pm = p_1 \pm p_2$. The function $K(p_1, p_2)$ is the same function (5.22) we encountered in the perturbative study of scalars. The above diagram is sufficient to evaluate the anomalous dimension of operators of the form (6.3) at one-loop.

Examples

Let us start by considering the $U(1)$ theory with a single flavour of fermion. The simplest operator of the form (6.3) is

$$\mathcal{O}_n = \Psi^\dagger\bar{\partial}\Psi^\dagger\ldots\bar{\partial}^{n-1}\Psi^\dagger\,. \tag{6.12}$$

In the supersymmetric theory this is an anti-chiral primary operator and is therefore one-loop exact. This holds true even in the non-supersymmetric theory and the operator is only corrected by the pairwise diagrams correcting $\bar{\partial}^{m_1}\Psi^\dagger\bar{\partial}^{m_2}\Psi^\dagger$ which evaluate to



As this is independent of the number of derivatives m_i the dimension of $\mathcal{O}_n$ is simply

$$\Delta = \frac{n(n+1)}{2} - \frac{n(n-1)}{2k} \tag{6.13}$$

as derived earlier, in (6.6).

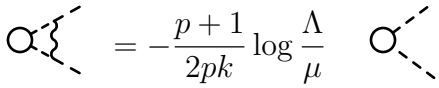
Another important example is the baryon operator in $SU(p)$ Chern-Simons theory

$$B = \Psi^\dagger_1\ldots\Psi^\dagger_p\,. \tag{6.14}$$

More generally, we can consider the operators

$$\mathcal{O}_{\rho_1\ldots\rho_n} = \Psi^\dagger_{\rho_1}\ldots\Psi^\dagger_{\rho_n} \tag{6.15}$$

with $B = \mathcal{O}_{1\ldots p}$. The pairwise diagrams that contribute to the anomalous dimension of these operator evaluate to



$$= -\frac{p+1}{2pk}\log\frac{\Lambda}{\mu}$$

$$+\mathcal{O}(\Lambda^2)\,.$$

The dimension of $\mathcal{O}_{\rho_1\ldots\rho_n}$ therefore evaluates to

$$\Delta = n + \frac{n(n-1)(p+1)}{2pk} \tag{6.16}$$

which is consistent with (6.7).

6.2 Bosonization Dualities

We have studied Chern-Simons theories coupled to both bosons and fermions. Yet, in both cases, the resulting particles actually interpolate between these statistics: they are anyons. This motivates the possibility that the theory of bosonic and fermionic theories are actually equivalent.

One simple example of this was seen above: the bosonic and fermionic theory are naturally interchanged under the map $1/k \to 1 - 1/k$. As we will soon see, this is the tip of the iceberg.

6.2.1 Introduction to Bosonization

We have discussed how by exploiting the structure of special Chern-Simons theories, we can calculate exact quantities even in a strongly interacting theory. Remarkably, seems that supersymmetry really provides only an informative role in choosing the right non-supersymmetric theory to investigate, and what calculations to do. This is a theme we will pick up again when we turn to discuss quantum Hall physics: we will open up a web of quantum Hall dualities, and directly derive strong results about these theories with no supersymmetric partners in sight.

Recent years have seen great progress in our understanding of dualities in (2+1)-dimensional quantum field theories as we have again managed to shrug off the holomorphic comfort blanket of supersymmetry. These developments have arisen from a wonderfully disparate array of topics, including the study of holography, the non-Fermi liquid state of the half-filled Landau level, and the surface physics of topological insulators.

Underlying many of these results is the idea of *bosonization*. Roughly speaking, this states that theories of scalars interacting with $U(p)_k$ Chern-Simons theories are equiv-

alent to theories of fermions interacting with $U(k)_p$ Chern-Simons theories. (More precise statements will be made shortly.) These dualities were originally conjectured in the limit of large p and k [97, 98, 99], motivated in part by their connection to higher spin theories in AdS_4 (recently reviewed in [100]). They have subsequently been subjected to a battery of very impressive tests [101, 102, 103, 104].

Versions of these dualities are also believed to hold for finite p and k. The first arguments in favour of their existence were given in [105], and the first precise dualities were described by Aharony [23] by piecing together evidence from level-rank dualities [106], known supersymmetric dualities [107, 108, 109, 110, 111, 112], and the map between monopole and baryon operators [113].

When extrapolated to $p = 1$, the dualities imply relationships between Abelian gauge theories, some of which had been previously proposed [114]. An example of such a duality equates a theory of bosons, coupled to a Chern-Simons gauge field, to a free fermion. (Closely related conjectures, which differ in some details, have long been a staple of the condensed matter literature – see, for example, [115, 116, 117, 118, 119].) Recently it was shown that these Abelian bosonization dualities can be used to derive a whole slew of further dualities [120, 121], including the familiar bosonic particle-vortex duality [122, 123], as well as its more novel fermionic version [124, 125, 126]. The upshot is that there is a web of $d = 2 + 1$ Abelian dualities, with bosonization lying at its heart.

For this dissertation, our interest lies in the generalized class of non-Abelian versions of the bosonization dualities. For these, it is a little too quick to say that they relate $U(p)_k$ bosons to $U(k)_p$ fermions since there are subtleties in identifying the levels of the $U(1)$ factors on both sides. These subtleties were largely addressed in [23] and, more recently, in [127]. Before proceeding, we review these results and provide a slight generalization.

Theory A

We start by describing the bosonic theory. This consists of N_f scalar fields with quartic couplings, transforming in the fundamental representation of the gauge group

$$U(p)_{k,\,k'} = \frac{U(1)_{k'p} \times SU(p)_k}{\mathbb{Z}_p}\,. \tag{6.17}$$

Here k and $k'p$ denote the levels of the $SU(p)$ and $U(1)$ Chern-Simons terms respectively, so that the action governing the gauge fields is given by

$$\mathcal{L}_A = \frac{k}{4\pi}\,\mathrm{Tr}\;\epsilon^{\mu\nu\rho}(a_\mu\partial_\nu a_\rho - \frac{2i}{3}a_\mu a_\nu a_\rho) + \frac{k'p}{4\pi}\epsilon^{\mu\nu\rho}\tilde{a}_\mu\partial_\nu\tilde{a}_\rho \tag{6.18}$$

with a the $SU(p)$ gauge field and $\tilde{a}$ the $U(1)$ gauge field. Regularization of each Chern-Simons theory is required; this may be done with a small Yang-Mills term or another technique such as dimensional regularization. We will state the dualities for both types of regularization.

The discrete quotient in (6.17) restricts the allowed values of k' to take the form

$$k' = k + np \quad \text{with } n \in \mathbb{Z}\,.$$

A simple way to see this is to construct the $u(p)$-valued gauge field $a_{u(p)} = a + \tilde{a}\mathbf{1}_p$; the action (6.18) becomes a Chern-Simons action for $a_{u(p)}$ at level k, which we denote as $U(p)_k$, together with an Abelian Chern-Simons action for $\mathrm{Tr}\ a_{u(p)}$ at level n.

The dual of Theory A depends on the choice of Abelian Chern-Simons level k' or, equivalently, on n. For $n = 0, 1$ and ∞, the duals were first proposed by Aharony [23]. More recently, Hsin and Seiberg described the dual for the choice $n = -1$ [127]. Although not explicitly stated by the authors, the techniques of [127] allow for a straightforward generalization[2] to any n, which we now describe.

Theory B: Yang-Mills Regularization

This consists of N_f fermionic fields, transforming under the fundamental representation of the gauge group $U(k)_{-p+N_f/2}$. The $U(1) \subset U(k)$ gauge field also interacts through a minimal BF coupling with a further $U(1)_n$ Chern-Simons theory. The resulting action for the gauge fields is

$$\begin{aligned}\mathcal{L}_B \;=\; & \frac{-p+N_f/2}{4\pi}\left[\mathrm{Tr}\ \epsilon^{\mu\nu\rho}(c_\mu\partial_\nu c_\rho - \frac{2i}{3}c_\mu c_\nu c_\rho) + k\,\epsilon^{\mu\nu\rho}\tilde{c}_\mu\partial_\nu\tilde{c}_\rho\right] \\ & + \frac{k}{2\pi}\epsilon^{\mu\nu\rho}\tilde{c}_\mu\partial_\nu b_\rho + \frac{n}{4\pi}\epsilon^{\mu\nu\rho}b_\mu\partial_\nu b_\rho\end{aligned} \tag{6.19}$$

with c the $SU(k)$ gauge field and $\tilde{c}, b$ both $U(1)$ gauge fields.

For certain values of n, we can integrate out the auxiliary gauge field b. These values give the following dualities:

$$\begin{array}{lrcl} n=\infty: & N_f \text{ scalars with } SU(p)_k & \longleftrightarrow & N_f \text{ fermions with } U(k)_{-p+N_f/2} \\ n=0: & N_f \text{ scalars with } U(p)_k & \longleftrightarrow & N_f \text{ fermions with } SU(k)_{-p+N_f/2} \\ n=\pm 1: & N_f \text{ scalars with } U(p)_{k,\,k\pm p} & \longleftrightarrow & N_f \text{ fermions with } U(k)_{-p+N_f/2,\,-p\mp k+N_f/2} \end{array}$$

[2] This generalization was also noticed by Ofer Aharony – the author is grateful to him for extensive discussions on this issue.

These are the dualities previously described in [23] (for $n = 0, 1$ and ∞) and in [127] (for $n = -1$). For general n, we cannot integrate out b without generating fractional Chern-Simons levels. In this case, the correct form of the duality is (6.19).

These dualities are essentially level-rank dualities of the underlying non-Abelian algebras, dressed up with carefully chosen $U(1)$ factors. The other visible oddities are the $N_f/2$ shifts of the levels in the fermionic theories. The need for some such term is clear for $N_f = 1$, where without a half-integer CS level, the fermionic theory would be anomalous.

Theory B: The Other Regularization

However, there is one further subtlety which can arise, according to how one regularizes computations in the theory. If one has to compute a one-loop renormalization of the gluon propagator as in Yang-Mills regularization, then the non-Abelian level for $SU(p)$ receives a finite renormalization. The quantity appearing in the dressed propagator is shifted from the bare value $\hat{k}$ in the Lagrangian, replacing it with $k = \hat{k}+\mathrm{sgn}(\hat{k})p$ [23]. If one does not wish to include such a shift in loop computations, one should use the dressed propagator, or equivalently the theory with the non-Abelian Chern-Simons level k instead.

A prototypical example of a regularization one might use which falls into this category is dimensional regularization.

The theory is identical to that above, except that we and shift the levels of the non-Abelian groups, giving a set of dualities of the following form, for positive $k > 0$:

$$N_f \text{ scalars and } U(p)_{\hat{k}+p,\hat{k}+np} \quad \longleftrightarrow \quad N_f \text{ fermions and } U(\hat{k})_{-\hat{k}-p+N_f/2,-p+N_f/2} \times U(1)_n$$

In terms of the variable $k = \hat{k} + p$, they become the following instead:

$$N_f \text{ scalars and } U(p)_{k,k+(n-1)p} \quad \longleftrightarrow \quad N_f \text{ fermions and } U(k-p)_{-k+N_f/2,-p+N_f/2} \times U(1)_n$$

Again, special forms are possible for the values of n found above.

6.2.2 Non-Relativistic Limits

It is interesting to ask whether there is a non-relativistic counterpart of these dualities. The answer will turn out to be yes, and as well as having relevance for the superconformal theories we have thus far considered, evidence for the equivalence of quantum Hall states in such pairs of theories will be covered in Chapter 14 when $N_f = p$.

Of course, as we discussed earlier, to access this regime from the relativistic theories we will need to deform both sides of the duality. To recap, this is achieved by first turning on mass deformations so that the theories sit in a gapped phase. We then we take the non-relativistic limit by integrating out anti-particles, leaving us in a theory with fixed particle number.

The retreat to a non-relativistic corner of the theories throws away much of the dynamics that makes bosonization dualities non-trivial. Indeed, here the dualities are souped-up version of flux attachment, which is used to transmute the statistics of particles in quantum mechanics [128]. Nonetheless, there remains a lot of interesting physics to extract in this limit and (especially in our later discussions of Hall physics) a number of conceptual issues must be understood before we will ultimately find agreement between the two theories.

The first thing we must do is establish what regularization convention we are adhering to. In our calculations up to this point, we have not included any loop effects renormalizing the gluon propagator as would be necessary in Yang-Mills regularization; thus we should think of this as a non-relativistic limit of the dimensionally regularized theory.[3] With that in mind, we can write down the theories:

Bosonic Theory: $U(p)_{k,\,k+(n-1)p}$ coupled to N_f fundamental scalars. The bare non-Abelian level is $\hat{k} = k - p$.

Fermionic Theory: $U(k-p)_{-k,-p}$ coupled to N_f fundamental fermions and, through a BF coupling, to $U(1)_n$. The bare non-Abelian level is $-p$.

Note the $N_f/2$ shift in the Chern-Simons level of the fermionic theory has gone away again; this arises because taking the non-relativistic limit involves integrating out the Dirac sea of filled fermionic states.

6.2.3 The Duality in Action

We need to get some feel for how these dualities manifest in our theories. To get a sense for the role of the extra $U(1)$ factors, and as a first check on our conventions, we will look at the purely Abelian case $p = k - p = 1$ first. Then we will see the basic mechanism by which the non-Abelian duality works.

[3]Note that in particular, in our conventions we would always have $|k| > p$, which is the inequality we saw previously ensured that states never violate the unitarity bound for $k > 0$.

The Abelian Case

Consider these theories, respectively coupled to the currents of a single boson and a single fermion:

$$\begin{aligned}\mathcal{L}_B &= \frac{n+1}{4\pi} a \wedge \mathrm{d}a + a_\mu J_B^\mu \,, \\ \mathcal{L}_F &= -\frac{1}{4\pi} b \wedge \mathrm{d}b + \frac{1}{2\pi} b \wedge \mathrm{d}c + \frac{n}{4\pi} c \wedge \mathrm{d}c + b_\mu J_F^\mu \,.\end{aligned}$$

Now suppose one naively integrates out c in the theory $\mathcal{L}_F$; its equation of motion is $nc = -b$ and hence we get

$$\mathcal{L}'_F = b_\mu J_F^\mu - \left(\frac{1}{1 - \frac{1}{n+1}} \right) \frac{1}{4\pi} b \wedge \mathrm{d}b \,.$$

Letting $\kappa = n + 1$, we learn that the bosonic theory with the inverse level $1/\kappa$ should be dual to the fermionic theory with $1 - 1/\kappa$, up to a parity transformation. This is indeed what we have found, right down to the parity issue – recall that on one side of the duality, we added ∂_z derivatives, whilst on the other we added $\partial_{\bar{z}}$ derivatives. This is our first confirmation of these dualities!

Furthermore, as we mentioned above, the spins of Φ and Ψ, as measured by $\mathcal{J}'$ (see Section 5.3), are respectively

$$\mathcal{J}'_\Phi = -\frac{1}{2\kappa} \qquad \text{and} \qquad \mathcal{J}'_\Psi = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{\kappa}\right) = +\frac{1}{2\kappa} \,,$$

so in particular $\mathcal{J}'_\Psi = -\mathcal{J}'_\Phi$.

Two Particle Abelian Wavefunctions and Spectral Flow

To get a better sense of what is going on, it is very helpful to look at the case of two Abelian anyons in a harmonic trap, which is exactly solvable [129]. We could do this by following the approach of Section 5.5.1, but it is helpful to instead consider the problem from first principles.

Consider the configuration space of two identical particles in two dimensions. The space is given by $\mathbb{R}^2 \times \mathbb{R}^2/\mathbb{Z}_2$. Let us focus on the relative degrees of freedom, $\mathbb{R}^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ quotient identifies the points $\mathbf{x} \sim -\mathbf{x}$. We can create a Hilbert space for these particles by fibering a one-dimensional complex line over this space, so let us start by doing this. However, it is clear that there is a singular point in this space, namely at the origin, and as a result we have a choice as to precisely what gluing of these lines we make: there can be monodromy around the origin.

Concretely, if we consider a wavefunction which is section of this bundle, $\chi(r,\theta)$, then it may behave like $\chi(r,\theta+\pi) = U\chi(r,\theta)$ for some unitary operator U. We may freely choose the operator U which sits here; each choice clearly defines a superselection rule in the theory. Here, since the Hilbert space is one-dimensional, $U = \exp(i\pi/k)$ is simply a phase. It is, of course, the *statistical phase* of the particles. For bosons, $1/k = 0$ and for fermions $1/k = 1$.

Suppose the particles are free, except that we put the theory in a harmonic trap. In this case, the Hamiltonian would be

$$H = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta}\right)^2\right) + \frac{1}{2}m\omega^2 r^2\,.$$

However, it is convenient to change to work with a single-valued function $\hat{\chi}(r,\theta) = \exp(-i\theta/k)\chi(r,\theta)$, and then the Hamiltonian becomes

$$\hat{H} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\left(\frac{\partial}{\partial\theta} + \frac{i}{k}\right)^2\right) + \frac{1}{2}m\omega^2 r^2\,.$$

This takes exactly the form of the standard two-dimensional simple harmonic oscillator, with solutions $\hat{\chi}(r,\theta) = \exp(il\theta)R_n(r)$, except that the appearances of the angular momenta l are shifted by $1/k$.

This means that almost all of the spectrum of the relative degrees of freedom is immediately obvious from the standard approach to the harmonic oscillator. Assuming that $|1/k| \leq 1$, the spectrum for angular momenta $l \neq 0$ is

$$\frac{E}{\omega} = 1 + 2q + \left|l + \frac{1}{k}\right| \quad \text{where} \quad q = 0,1,2,\ldots \quad \text{and} \quad l = \pm 2, \pm 4, \ldots\,. \tag{6.20}$$

However, the $l = 0$ sector is slightly subtle, since in fact there is a continuum of energy eigenvalues associated to square-integrable wavefunctions unless one imposes more precise boundary conditions on the behaviour at the origin. As we discussed in some detail back in Section 5.5.1, our choice for the bosonic theory is equivalent to making $\hat{\chi} \sim r^{1/k}$ at the origin for these wavefunctions. With this convention, the remaining states have energy

$$\frac{E}{\omega} = 1 + 2q + \frac{1}{k} \quad \text{where} \quad q = 0,1,2,\ldots \quad \text{for} \quad l = 0,\ldots\,. \tag{6.21}$$

Some nice features of this spectrum are now visible. Firstly, suppose we smoothly increase $1/k$ from 0 to 1. We see that the resulting spectrum smoothly interpolates between the spectrum of a boson and of a fermion, where the fermion state which matches

a boson state always has its l eigenvalue one larger. Thus the picture of flux attachment works very neatly in this case.

Suppose we instead decreased $1/k$ from 0 to -1. Now something a little odd happens: the $l = 0$ states described by (6.21) go the wrong way. In particular, although almost all of them end up lining up with a free fermion state as they should, the $q = 0, l = 0$ state heads down to $E/\omega = 0$. At this point, of course, the state then becomes logarithmically non-normalizable, and a Jackiw-Pi vortex appears. We are not going to pursue this line of reasoning again, however.

We are really here to see what form bosonization takes from this perspective. It is easy enough to capture. Suppose we consider fermions with a phase shift of $1/k - 1$. We can read off their spectrum from (6.20) simply by looking at odd l, as noted above. But then the -1 within the phase shift conspires with the summation over odd integers to produce a summation over even integers at the phase shift $1/k$. In other words, we obtain perfect agreement with a bosonic spectrum at phase shift $1/k$.

We also see that the bosonic state of angular momentum l arises from a fermionic one of angular momentum $l + 1$. This ties in with what we have seen in terms of chiral operators; the fermionic dual of $\Phi^\dagger\Phi^\dagger$ was $\Psi^\dagger\partial\Psi^\dagger$. The above shows that this extends in an elegant way to the rest of the spectrum.

The Basic Non-Abelian Case

One can perform the same manipulations as we did for $U(1)$ for the more general case of

$$N_f \text{ bosons} + U(p)_{k,k-p+np} \longleftrightarrow N_f \text{ fermions} + U(k-p)_{-k,-p} \times U(1)_n \tag{6.22}$$

provided one is not too bothered about having a fractional $U(1)$ level. This duality then becomes the following:

$$N_f \text{ bosons} + U(p)_{k,k-p+np} \longleftrightarrow N_f \text{ fermions} + U(k-p)_{-k,-p-(k-p)/n}$$

Conveniently, even though the $U(1)$ level here is not necessarily an integer, substituting it naively into our formulae gives the same answer as working in the theory with the extra gauge fields and extending our previous analysis to cover this situation.

As we saw for the $U(1)$ case, the fact that we have investigated chiral states on one side and anti-chiral states on the other implies that in fact we have been looking at the left-hand theory together with a parity inversion of the right-hand theory. This parity inversion leaves the dimensions invariant, so we need not worry about it further

– we will quote results from our previous work but apply them to the true pair of dual theories. With all this in mind, let us see if we can verify the non-Abelian duality (6.22).

Let us begin on the left-hand side of (6.22). Consider a bosonic operator transforming in the representation R of $U(p)$. This has an associated Young diagram λ, and is composed of $|\lambda|$ fundamental scalars with m derivatives. Then by what we have seen, this has the dimension

$$\Delta^B = |\lambda| + m + \frac{\sum \left[\frac{1}{2}\lambda_i(\lambda_i - 1) - (i-1)\lambda_i\right]}{k} + \frac{1}{2}|\lambda|(|\lambda| - 1) \times \frac{1}{p}\left[\frac{1}{k - p + np} - \frac{1}{k}\right].$$

Assume for the moment that R is such that λ has at most $\hat{k}$ columns. Then we can consider a fermionic operator in the representation $\tilde{R}$ associated to the Young diagram λ^T, living on the right-hand side of (6.22). Firstly, notice that because of the change in statistics and the change in representation, one can leave the derivatives in exactly the same place without causing any problems with vanishing symmetrizations or the like. Further, this means that the operator transforms under the $SU(N_f)$ global symmetry in the same representation as the bosonic one did. But now we can compute the dimension of this operator. We find that

$$\begin{aligned}
\Delta^F &= |\lambda^T| + m + \frac{\sum \left[\frac{1}{2}\lambda_i^T(\lambda_i^T - 1) - (i-1)\lambda_i^T\right]}{-k} \\
&\quad + \frac{1}{2}|\lambda^T|(|\lambda^T| - 1) \times \frac{1}{k - p}\left[\frac{1}{-p - (k-p)/n} - \frac{1}{-k}\right] \\
&= |\lambda| + m + \frac{\sum \left[(i-1)\lambda_i - \frac{1}{2}\lambda_i(\lambda_i - 1)\right]}{-k} + \frac{1}{2}|\lambda|(|\lambda| - 1) \times \frac{1 - n}{k(k - p + np)} \\
&= \Delta^B
\end{aligned}$$

confirming that the dimensions of these BPS operators agree exactly!

One may easily verify that the angular momenta $\mathcal{J}'$ also agree between these two operators. This is a consequence of the fact that the fundamental representation of $U(p)$ has the quadratic Casimir p and the identity

$$\frac{p}{2(\hat{k} + p)} = \frac{1}{2} - \frac{\hat{k}}{2(\hat{k} + p)}$$

which shows that the true spin of an isolated boson in a $U(p)$ theory is identical to that of a fermion in a $U(\hat{k})$ theory whose bare spin is $1/2$.

6.2.4 Fusion Rules and Baryons

There are some subtleties associated with operators which cannot be expressed purely in terms of a single Young diagram λ with at most $\hat{k}$ columns, however. Whenever there

are "too many symmetrizations" on the bosonic side – more than there are colours of fermion – we cannot write down a large enough antisymmetrization to form the transpose of the Young diagram.

The underlying reason for these subtleties is of course the fusion rules, introduced in Section 5.3.4.

For instance, consider a bosonic theory with $SU(2)$ at level $\hat{k} = 1$ and only one flavour. Then one might ask about operators of the form $S_{\sigma\tau} = \Phi_\sigma \Phi_\tau$. This vanishes under antisymmetrization on σ, τ, and hence has no component which is non-vanishing under implementation of the fusion rules. Correspondingly, the would-be dual is $\Psi\Psi = 0$. By contrast, $T_{\sigma\tau} = \Phi_\sigma \bar{\partial} \Phi_\tau$ is a valid operator when antisymmetrized over σ, τ; this singlet is then dual to $\Psi\bar{\partial}\Psi \neq 0$. Also, in an $SU(2)$ theory with two flavours, one can form $S_{\sigma\tau\mu\nu} = \epsilon_{ij}\epsilon_{kl}\Phi^i_\sigma \Phi^j_\tau \Phi^k_\mu \Phi^l_\nu$ which does not vanish under any antisymmetrizations.

But this last example brings us back to a subtlety we had previously swept under the rug. Recall that in Section 5.3, we allowed ourselves to include a p^{th} row in an $SU(p)$ Young diagram to keep track of the number of singlets (baryons) we had formed. This conveniently gave the correct results for the anomalous dimensions. But all p-high columns should be removed from a diagram for an $SU(p)$ representation, and only the remaining, *reduced* diagram need have $\leq \hat{k}$ columns. So we are always permitted to add an arbitrary number of baryons without violating the fusion rules (providing we have enough flavours to form them).

This creates a problem: what are the duals of these states? If one naively follows the prescription above, then one immediately runs into problems. For example, consider a product of two $U(p)$ baryons in a bosonic theory with $\hat{k} = 1$, taking the number of flavours to be $N_f = p$ for convenience:

$$\mathcal{O} = (\epsilon_{i_1 \cdots i_p} \epsilon^{\rho_1 \cdots \rho_p} \Phi^{i_1}_{\rho_1} \cdots \Phi^{i_p}_{\rho_p})^2 \,.$$

Then the dual description must live in a $U(1)$ fermionic theory; but the operator $\tilde{\mathcal{O}} = (\epsilon_{i_1 \cdots i_p} \Psi^{i_1} \cdots \Psi^{i_p})^2$ trivially vanishes because it contains two copies of each fermion field! (Moreover, even if it did not vanish, the dimension and angular momentum would not agree with the above operator.) What can we do?

For inspiration, we turn back to the simple example of dual operators in the $U(1)$ theories discussed above: Φ^q is dual to $\Psi\bar{\partial}\Psi \cdots \bar{\partial}^{q-1}\Psi$. (Note that, although there is apparently no $SU(1)$ level, by comparison to the general form of the dualities, we should think of this as the $p = \hat{k} = 1$ case. Also note that we are using the parity-reversed theory.) It seems like adding derivatives is the right thing to do.

Indeed, we find

$$\tilde{\mathcal{O}} = (\epsilon_{i_1\cdots i_p}\Psi^{i_1}\cdots\Psi^{i_p})(\epsilon_{i_1\cdots i_p}\bar{\partial}\Psi^{i_1}\cdots\bar{\partial}\Psi^{i_p})$$

has an anomalous dimension which agrees with that of $\mathcal{O}$. This now involves a conspiracy between the number of derivatives and the group theoretic terms.

This generalizes straightforwardly. For $\hat{k} = 1$, we always have a $U(1)$ fermionic theory, and we keep adding one more derivative to every additional $SU(N_f)$ baryon. When $\hat{k} > 1$, one first forms singlets amongst the fermion flavours, then moves on to add derivatives, and so forth.

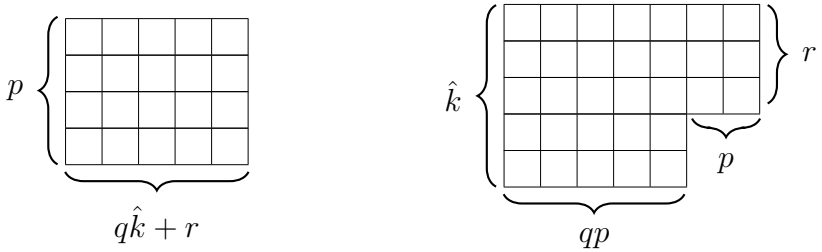
As an example of this, consider a $U(p)$ theory with a general $\hat{k} > 0$. Suppose that we have $N_f = p$ flavorus; we will suppress flavour indices for brevity, but both operators will transform in the same way under the flavour group. Consider

$$\mathcal{O} = (\epsilon^{\rho_1\cdots\rho_p}\Phi_{\rho_1}\cdots\Phi_{\rho_p})^{q\hat{k}+r}\,.$$

The dual of this operator is (up to gauge rotations)

$$\tilde{\mathcal{O}} = \prod_{t=1}^{p}\left[\prod_{s=0}^{q-1}(\epsilon^{\sigma_1\cdots\sigma_{\hat{k}}}\bar{\partial}^s\Psi_{\sigma_1}\cdots\bar{\partial}^s\Psi_{\sigma_{\hat{k}}})\times\prod_{\sigma=1}^{r}\bar{\partial}^q\Psi_\sigma\right]$$

where each term in the product over $t = 1,\ldots,p$ comes from one flavour in $\mathcal{O}$. These transform, in our extended notation for gauge group representations, as the pair of diagrams shown here.



(6.23)

The quantum dimensions of these operators, in the $U(p)_{\hat{k}+p,\hat{k}+p} \longleftrightarrow U(\hat{k})_{-\hat{k}-p,-\hat{k}-p}$ theories respectively, are

$$\Delta = (q\hat{k}+r)p + \frac{p(q\hat{k}+r)(q\hat{k}+r-1) - (q\hat{k}+r)p(p-1)}{2(p+\hat{k})}$$

and

$$\begin{aligned}\tilde{\Delta} &= (q\hat{k}+r)p + \frac{1}{2}q(q-1)\hat{k}p + rqp \\ &\quad + \frac{r(q+1)p((q+1)p-1) + (\hat{k}-r)qp(qp-1) - qp\hat{k}(\hat{k}-1) - pr(r-1)}{-2(p+\hat{k})}\end{aligned}$$

and a little algebra indeed verifies that $\Delta = \tilde{\Delta}$.

A general proof that this works goes as follows: any valid operator in the bosonic theory $SU(p)$ with bare level $\hat{k}$ takes the form of some number $q\hat{k}+r$ of baryons followed by a reduced diagram μ for an integrable representation (one with at most $\hat{k}$ columns). The transpose of this diagram is not generically a valid $SU(\hat{k})$ diagram, since it has too many rows. However, this can be fixed in the following way. To begin with, ignore the q separate $p \times \hat{k}$ rectangular blocks. This leaves us with are r horizontal strips of p blocks above μ^T. If this has at most $\hat{k}$ rows, then this is already a valid diagram. If not, take the portion of the diagram below the $\hat{k}^{\text{th}}$ row and place it at the top right of the diagram. This is now a valid diagram: in particular, the excess height beyond $\hat{k}$ was at most r. Finally place the q large rectangles we ignored in a long row to the left of the diagram.

How much does this change the $\mathcal{J}$ eigenvalue of the diagram? Well, the movement of the rectangular blocks reduces the number of box-pairs in the same column by $q(q-1)p\hat{k}^2/2$ and increases the number in the same row by $q(q-1)p^2\hat{k}/2$. The movement of the $c = |\mu| + pr$ other bits relative to these rectangles shifts the column count down by $q\hat{k}c$ and the row count up by qpc. Finally, the chopping and changing of the remaining d cells moves the counts $-\hat{k}d$ and pd further.

In particular, because only moves sets of cells from columns with a multiple of $\hat{k}$ other cells to rows with a multiple of p, one only shifts $\mathcal{J} \sim (\text{column pairs} - \text{row pairs})/(\hat{k}+p)$ by an integer. To be precise, suppose one labels the $p \times \hat{k}$ regions $I = 1, 2, \ldots$, including any partially occupied rectangles beyond the q filled ones. Then the movement of each cell in the I^{th} region shifts $\mathcal{J} \to \mathcal{J} - (I-1)$.

This tells us exactly what we should include in order to make sure that the resulting diagram does not vanish when it is built out of fermions: each cell in the I^{th} region should be accompanied by $(I-1)$ extra derivatives (on top of whatever the bosons required), which indeed then ensures that it can be safely symmetrized with otherwise identical fermionic terms without vanishing. Then, once we are done, we are guaranteed to end up with an operator whose $\mathcal{J}'$ and Δ eigenvalues agree perfectly.

However, if we now reduce the diagrams – for instance those in (6.23) – it is clear that these diagrams are not related by something as simple as transposition as in the cases

we initially discussed. In order to understand what is going on here, we need to know a little more about the nature of level-rank duality.

A key fact is that it does not relate representations of the dual algebras directly by transposition. Instead, it relates representations modulo the outer automorphisms of the algebra. (Equivalently, this is the centre of the algebra.) Let us explain what this means.

The *outer automorphism group* of $SU(p)$ is $\mathbb{Z}_p$. It is generated by the basic outer automorphism operator σ which obeys $\sigma^p = 1$. This has an action on representations of the algebra with (bare) level $\hat{k}$ which can be nicely explained using reduced Young diagrams. We start with a given Young diagram λ. Then $\sigma(\lambda)$ is a second Young diagram which we construct using the following procedure: first, add a row of length $\hat{k}$ to the top of λ; next remove any columns of length p to obtain a suitably reduced Young diagram. One may easily verify that this procedure gives $\sigma^p(\lambda) = \lambda$ for any λ.

It is self-evident that the transposition of the singlet is another singlet, and that the orbit of a singlet under the outer automorphism group is rectangular diagrams with the maximum number of columns, $\hat{k}$. Applying this logic with $p \leftrightarrow \hat{k}$ shows that in fact the pair of diagrams in (6.23) belong to dual orbits, tying this story together nicely.

In fact, it is easy to see that in general the above algorithm applied to a reduced diagram λ for $SU(p)$ gives a reduced diagram in $SU(\hat{k})$ corresponding to $\sigma^r(\lambda^T)$.

6.2.5 Puzzles

There are some puzzles which remain, however. Here are two:

- Suppose that you have a $U(1)_{-2}$ fermion theory and two flavours. What is the $U(1)_2$ bosonic dual of $\Psi^{[i}\Psi^{j]}$? It is easy to check that this has a dimension $\Delta = \frac{3}{2}$ lower than any chiral state containing with two bosons, since for such states $\Delta = 2 + m + \frac{1}{2}$ where m is the number of $\bar{\partial}$ insertions.

- Consider $(\epsilon_{\mu\nu}\Phi_\mu\bar{\partial}\Phi_\nu)^2$ in an $U(2)_3$ theory with one flavour. This should be dual to a $U(1)_{-3}$ state, but one needs a four fermion operator with four chiral derivatives to match, and such states always vanish.

The first sort of puzzle could potentially be addressed simply by restricting the number of flavours to be at most the number of gauge degrees of freedom. This is reminiscent of suggestions in the literature (motivated by other concerns) that the duality may hold only for $N_f \leq p$ [127].

However, it seems likely that the road to understanding the second puzzle is to think more carefully about the meaning of the many operators we have been happily writing down. We already highlighted, in Section 5.3, how non-trivial statistics and branch cuts complicate the nature of these operators.

A possible explanation for the second point raised above, for example, would be that the Wilson lines – which at level 1 may only fuse into baryons when brought together in pairs – somehow require the state to be antisymmetric over *any* pair of indices at infinity. Another would be that we are dismissing certain "fermion" states as necessarily vanishing when they need not, since they are really anyons. This needs further investigation.

A possible concern one might have about the duality is the following: the dimensions of the representations we have claimed are dual to each other are drastically different. This goes hand-in-hand with the clear fact that the global symmetry groups of the two theories are $SU(p)$ and $SU(\hat{k})$, whose representations are unrelated to each other. Indeed, one might simply dismiss this issue with the observation that these dimensions are not (globally) gauge invariant observables, so who cares?

But in anyonic theories, there is a notion of dimension which is gauge invariant and which is indeed preserved by the duality. One should count *fusion channels* between anyonic fields (or more crudely, perhaps, measure the quantum dimension of anyonic fields). Concretely, one identifies each distinct integrable representation (modulo outer automorphisms) as a distinct species of anyon. The fusion rules then dictate how many different anyons can be formed when two anyons are brought together, and in how many ways [130]. These numbers, it turns out, are indeed preserved by level-rank duality [82].

PART III

Vortices as Electrons

7 Introduction and Summary

The fractional quantum Hall effect is one of the most studied topics in physics over the past three decades. As we mentioned in the introduction, the theory rests on a beautiful and intricate web of ideas involving microscopic wavefunctions [9], low-energy effective Chern-Simons theories [10, 11, 12, 13, 14] and boundary conformal theories [15, 16].

In this part of the thesis, we will analyse different aspects of a non-relativistic supersymmetric model – a simple deformation of the conformal theory described up to this point – and show that its low-energy physics is precisely that of the quantum Hall effect. The idea is that if one throws away the fermionic matter and restricts attention to the bosonic sector of the theory (which, as before, we will see is perfectly reasonable) then this is an excellent toy model for exploring some of the links between these different approaches to the quantum Hall effect.

In the rest of this introduction, we describe our model in more detail and explain what it's good for. It is an Abelian Chern-Simons theory, coupled to a non-relativistic bosonic matter field. It has a supersymmetric completion with a fermionic field; however these will play essentially no role in our discussion and are included only for completeness. In this manner, it is an amalgamation of effective theories of [12] and [13]. The model has vortices and these are viewed as the "electrons". The vortices are "BPS objects" [131]: this means that they experience no classical static forces. It also means that they are protected by supersymmetry in a simple way which we describe in the main text. This property allows us to perform an explicit quantization of the vortex dynamics. We show that the ground state wavefunction of the vortices lies in the same universality class as the Laughlin wavefunction. It has the same long range correlations, but differs on short distance scales.

We also describe the excitations of a droplet of vortices. There are gapless, chiral edge excitations which, we show, are governed by the usual action for a chiral boson [132], suitably truncated due to the presence of a finite number of vortices. Finally, we construct the quasihole excitations in this model and compute their Berry phase. This is, of course, a famous computation for the Laughlin wavefunctions [133]. However the usual analysis relies on the plasma analogy [9], and the (admittedly well justified) assumption that the classical 2d plasma exhibits a screening phase. In contrast, here we are able to perform the relevant overlap integrals analytically, at finite electron number, to show that the quasiholes have the expected fractional charge and statistics.

Many of the properties of vortices described above follow from the fact that their dynamics is governed by a quantum mechanical matrix model, which was introduced by Polychronakos to describe quantum Hall physics [19] and further studied in a number of works [134, 135, 136, 137]. We will show how this matrix model is related to more familiar effective field theories of the quantum Hall effect.

This part of the dissertation is organized as follows. In Chapter 8 we introduce the non-relativistic, supersymmetric theory. After a fairly detailed description of the symmetries of the theory, we discuss its two different phases and its spectrum of excitations. Chapter 9 is devoted to a study of BPS vortices and contains the meat of Part III. We will show that the low-energy dynamics of vortices is governed by the matrix model introduced in [19]. We review a number of results about this matrix model and derive some new ones. Finally, in Chapter 10 we look at where this all leaves us. A number of calculations are relegated to appendices.

Then, equipped with a clear picture of the Abelian case, in Parts IV and V, we will widen our scope to cover non-Abelian theories.

8 Non-Relativistic Chern-Simons-Matter Theories

We start by introducing the $d = 2 + 1$ non-relativistic, supersymmetric Chern-Simons theory which we are going to study. The theory consists of an Abelian gauge field a_μ, coupled to complex scalar field ϕ and a complex fermion ψ. The action is

$$S = \int \mathrm{d}t\, \mathrm{d}^2x \quad \left\{ i\phi^\dagger \mathcal{D}_0\phi + i\psi^\dagger \mathcal{D}_0\psi - \frac{1}{2m}\mathcal{D}_p\phi^\dagger\, \mathcal{D}_p\phi - \frac{1}{2m}\mathcal{D}_p\psi^\dagger\, \mathcal{D}_p\psi - \frac{k'}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \right.$$
$$\left. -\mu a_0 + \frac{1}{2m}\psi^\dagger f_{12}\psi - \frac{\pi}{mk'}\left(|\phi|^4 - \mu|\phi|^2 + 3|\phi|^2|\psi|^2\right) \right\} . \tag{8.1}$$

A refresher of our conventions: the subscripts $\mu, \nu, \rho = 0, 1, 2$ run over both space and time indices, while $p = 1, 2$ runs over spatial indices only. The fermion carries no spinor index. Both ϕ and ψ are assigned charge 1, so the covariant derivatives read $\mathcal{D}_\mu\phi = \partial_\mu\phi - ia_\mu\phi$ and similarly for ψ. The magnetic field is $f_{12} = \partial_1 a_2 - \partial_2 a_1$. Finally $|\psi|^2 = \psi^\dagger\psi = -\psi\psi^\dagger$.

This is almost exactly the same as the Abelian theory defined in Section 4.3, except for the presence of the chemical potential term μa_0, and its supersymmetric completion (which is simply proportional to the conserved charge $\mathcal{N}_B$).

Including this new term, there are now three parameters in the Lagrangian: the Chern-Simons level $k' \in \mathbb{N}$, the mass m of both bosons and fermions, and the chemical potential μ. As we will see later, the chemical potential μ can be more fruitfully thought of as a background magnetic field for vortices. (The reason for using k' to refer to the level rather than k is that the latter will be reserved for the *non-Abelian* level introduced in Part IV.)

The first order kinetic terms mean that the action (8.1) describes both bosonic and fermionic particles, but no anti-particles. The quartic potential terms correspond to delta function contact interactions between these particles. In the condensed matter context, the gauge field is considered to be emergent. One of its roles is to attach flux to particles through the Gauss's law constraint, which arises as the equation of motion for a_0,

$$f_{12} = \frac{2\pi}{k'}\left(|\phi|^2 + |\psi|^2 - \mu\right) . \tag{8.2}$$

We'll learn more about the importance of this relation later.

As with the superconformal theory, the action (8.1) can be constructed by starting from a relativistic Chern-Simons theory with $\mathcal{N} = 2$ supersymmetry and taking a limit in which the anti-particles decouple, and we illustrate this procedure in Appendix A. This supersymmetric theory with $\mu \neq 0$ seems to not have been constructed prior to the author's work [6], although the bosonic sector is similar, but not identical, to a model studied by Manton [138] which shares the same vortices as (8.1). We will describe these vortices in some detail in Chapter 9.

8.1 Deformed Symmetries

The action (8.1), being very closely related to the superconformal theory of Section 4.3, is invariant under a similar symmetry algebra. Importantly, however, the chemical potential deformation leads to two key differences between the two cases. The first is rather obvious, given that μ carries the dimensions of inverse length squared – the conformal invariance is spoiled. The second, however, is a little more subtle, and related to the fact that magnetic fields want to replace translations with *magnetic translations*.

Since these subtleties will play a critical role in the following work, we will take a moment to explain how the algebra of symmetries is altered by the deformation.

Bosonic Symmetries

Invariance under time translations gives rise to the Hamiltonian. After imposing the new Gauss's law constraint (8.2), this still takes the concise form

$$H = \frac{2}{m}\int \mathrm{d}^2x \;\; |\mathcal{D}_z\phi|^2 + |\mathcal{D}_{\bar{z}}\psi|^2 + \frac{\pi}{k'}|\phi|^2|\psi|^2 \tag{8.3}$$

where $z = x^1 + ix^2$ and $\bar{z} = x^1 - ix^2$. Correspondingly, $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\partial_{\bar{z}} = \frac{1}{2}(\partial_1 + i\partial_2)$.

Invariance under spatial translations gives rise to the complex momentum, $P = \frac{1}{2}(P_1 - iP_2)$, which we write as

$$P = \hat{P} - \frac{\mu}{2}\int \mathrm{d}^2x \; \bar{z} f_{12} \quad \text{with} \quad \hat{P} = \int \mathrm{d}^2x \; \phi^\dagger \mathcal{D}_z\phi - \mathcal{D}_z\psi^\dagger\psi \,. \tag{8.4}$$

The $\hat{P}$ contribution is the standard Noether charge for spatial translations. The second term, proportional to the chemical potential μ, requires some explanation. As shown in [139], it arises because a translation is necessarily accompanied by a shift of the gauge field. (It is most natural to choose this so that, for example, $\delta_i\phi = \mathcal{D}_i\phi$.) The presence of the chemical potential term μa_0 in the action then means that the naive Noether charge

for translations is not gauge invariant. This is remedied by the addition of a total derivative, resulting in the improved, gauge invariant momentum above. Note, however, that the resulting momentum P is not itself translationally invariant. We shall comment further on this below.

A similar subtlety occurs for rotations. The conserved angular momentum is given by

$$\mathcal{J} = \int \mathrm{d}^2x \left(z\phi^\dagger \mathcal{D}_z\phi + \bar{z}\mathcal{D}_{\bar{z}}\phi^\dagger\,\phi + z\psi^\dagger\mathcal{D}_z\psi + \bar{z}\mathcal{D}_{\bar{z}}\psi^\dagger\,\psi + \frac{1}{2}\psi^\dagger\psi - \frac{\mu}{2}|z|^2 f_{12}\right). \tag{8.5}$$

The first five terms agree with the superconformal definition (4.20), except that we have done an integration by parts and then shifted the definition by the central charge $\mathcal{N}$ – this is simply because this term will diverge in our chosen ground state. The final term again arises as an improvement term in the Noether procedure which ensures that the resulting angular momentum is gauge invariant [139].

The number of bosons and fermions in this model remain individually conserved as before.

The presence of the anomalous term in the expression for the momentum (8.4) has an interesting effect on the commutation relations. (Here we describe the quantum commutation relations rather than classical Poisson brackets.) We find

$$[H, \hat{P}] = -\frac{2\pi\mu}{mk'}\,\hat{P} \quad \text{and} \quad [H, P] = 0\,. \tag{8.6}$$

So the Noether charge P is conserved, but the translationally invariant momenta $\hat{P}^\dagger$ and $\hat{P}$ act as raising and lowering operators for the spectrum. Further, the conserved momenta do not commute. We have

$$[P, P^\dagger] = -\frac{\pi\mu}{k'}\mathcal{N}\,. \tag{8.7}$$

Both (8.6) and (8.7) are similar to the commutation relations in quantum mechanics for momenta in a magnetic field. This is because, as we will describe in more detail below, μ acts like an effective magnetic field for vortices while Gauss's law constrains all excitations to carry some vortex charge.

We have of course lost the dilatation operator D, special conformal generator C and superconformal generator S. But note that the Galilean boost symmetry is also broken by the presence of a chemical potential (again, if one thinks of this as a background magnetic field, this is no surprise).

Supersymmetries

As promised, the action (8.1) is supersymmetric. We will shortly discuss how the algebraic structure of this supersymmetry is realised. However, firstly a few words on our motivation in using this supersymmetry.

As we saw in Part II, the key advantage of working at this supersymmetric point is essentially nothing to do with the fermionic content. The magic is that it will aid our understanding of the (purely bosonic) solitonic modes in the system. For instance, they obey first-order BPS equations; currents which are supersymmetric under the preserved supersymmetry are good quantum numbers for solitons; they experience no relative forces; there are known constructions of the moduli space of solitons; and so forth. We will revisit all of these issues along the way. For now, let us return to the supersymmetries.

Our action (8.1) continues to enjoy two complex supersymmetries, the same *kinematical* and *dynamical* supersymmetries introduced previously:

$$Q_1 = i\sqrt{\frac{m}{2}}\int \mathrm{d}^2x\ \phi^\dagger\psi \tag{8.8}$$

and

$$Q_2 = \sqrt{\frac{2}{m}}\int \mathrm{d}^2x\ \phi^\dagger\mathcal{D}_{\bar{z}}\psi\,. \tag{8.9}$$

As previously emphasized, no transformation for a_0 is intrinsically specified by these supercharges, since it is a Lagrange multiplier for a constraint, which does no harm as long as we allow ourselves to impose Gauss's law. We will see the implications of this below.

The supersymmetry algebra is almost exactly the same as for the superconformal theory, with the subtle distinction that although $\{Q_1, Q_2^\dagger\}$ still generates the translationally invariant momentum, this is no longer equal to the conserved momentum P:

$$\{Q_1, Q_1^\dagger\} = \frac{m}{2}\mathcal{N} \quad , \quad \{Q_2, Q_2^\dagger\} = H \quad , \quad \{Q_1, Q_2^\dagger\} = \hat{P}\,. \tag{8.10}$$

There is also a mild surprise in the commutators of bosonic and fermionic charges, in particular

$$[H, Q_1] = -\frac{2\pi\mu}{mk'}\,Q_1 \tag{8.11}$$

This means that although the kinematic supersymmetries leave the action invariant, when $\mu \neq 0$ they do not result in a symmetry of the spectrum. This can be traced to the fact that Gauss's law was required, both in the construction of the Hamiltonian (8.3)

and in the derivation of the commutators (8.11). Other commutators follow from Jacobi identities and give $[Q_2, H] = [Q_1, \hat{P}] = [Q_1^\dagger, \hat{P}] = 0$ while $[Q_2, \hat{P}] = [H, Q_1]$.

Finally, the commutators of the angular momentum will also be important for our story. The anomalous term in $\mathcal{J}$ and the change to Gauss's law come together to leave the results unchanged:

$$[\mathcal{J}, Q_1] = -\frac{1}{2}Q_1 \quad \text{and} \quad [\mathcal{J}, Q_2] = \frac{1}{2}Q_2\,.$$

The means that $\mathcal{J}$ is almost supersymmetric; specifically,

$$[\mathcal{J} + \frac{1}{2}\mathcal{N}_F, Q_2] = 0\,. \tag{8.12}$$

This fact will be important in Section 9.2.

8.2 The Vacuum, The Hall Phase, and Excitations

Let us now describe some basic features of the dynamics of our model. Because non-relativistic field theories have no anti-particles, the theory decomposes into sectors labelled by the conserved particle numbers which, in our case, are $\mathcal{N}_B$ and $\mathcal{N}_F$. To solve the theory, we need to determine the energy spectrum in each of these sectors.

One way to organize these sectors is to start with the $\mathcal{N} = 0$ Hilbert space and build up by adding successive particles. Instead, we will take a dual perspective. Our theory enjoys a conserved topological current,

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho\,. \tag{8.13}$$

The associated particles are vortices. We will view these vortices as the "electrons" of our theory.

Our theory has two translationally invariant ground states consistent with Gauss's law (8.2), both of which have $H = 0$. We call these the *vacuum* and the *Hall Phase*. They are defined as follows:

$$\underline{\text{The Vacuum:}} \qquad |\phi|^2 = \mu \quad \text{and} \quad f_{12} = 0\,. \tag{8.14}$$

$$\underline{\text{The Hall Phase}}\text{:} \qquad |\phi|^2 = 0 \quad \text{and} \quad f_{12} = -\frac{2\pi\mu}{k'}\,. \tag{8.15}$$

The vacuum state contains no vortices, $\int \mathrm{d}^2x\ J^0 = 0$. However, the bosons have condensed which means that the particle number is $\mathcal{N} = \infty$. In contrast, the Hall phase

has vanishing particle number but infinite vortex number, $\int \mathrm{d}^2x\ J^0 = \infty$.

We shall seek to understand what happens as we inject vortices into the vacuum. For any finite number of vortices, the system breaks translational invariance. But, as we fill the plane with vortices, the Hall phase emerges. In Chapter 9, we tell both the classical and quantum versions of this story in some detail. First, however, we describe some simple properties of excitations above each of these ground states.

The Vacuum

The key feature of the vacuum state is that $U(1)$ gauge symmetry is broken. This ensures that the theory admits topological, localized vortex solutions. These vortices will be the main focus of our work, and we postpone a more detailed discussion of them until Chapter 9. For now, we shall just summarize their three main properties:

- Vortices are gapless. States with an arbitrary number of vortices exist with $H = 0$.
- Vortices have statistical phase $\pi k'$. This means that the vortices are bosons when k' is even and fermions when k' is odd.
- Vortices are singlets under supersymmetry.

There are further excitations above the vacuum arising from the fundamental fields ϕ and ψ. These excitations are both gapped, with an excitation energy $2\pi\mu/mk'$. These excitations can be generated from the vacuum by using the raising operators $\hat{P}^\dagger$ and $Q_1^\dagger$, together with the supercharges Q_2 and $Q_2^\dagger$.

The Hall Phase

The Hall phase has an unbroken $U(1)$ gauge symmetry and the long-distance physics is dominated by the Chern-Simons term. It is well known that such theories capture the essential properties of the fractional quantum Hall effect. We now take the opportunity to review this standard material (see, for example, [140, 141] for reviews).

To describe quantum Hall physics, it is not enough to specify the Lagrangian; we need to know how electromagnetism couples to the theory.[1] (Recall that the Abelian gauge field a_μ in the Lagrangian (8.1) should be thought of as an emergent, statistical gauge field, not the electromagnetic field.) Since we wish to treat the vortices as the

[1]There are two, dual, descriptions of the long-wavelength quantum Hall physics in terms of Chern-Simons theories. In one description, the Chern-Simons level is equal to ν, the filling fraction [10, 11], the electrons are the fundamental excitations and the vortices the fractionally charged quasiparticles. Here we are interested in the dual description, related by a particle-vortex duality transformation, where the Chern-Simons coefficient is $1/\nu$ and the electrons are vortices.

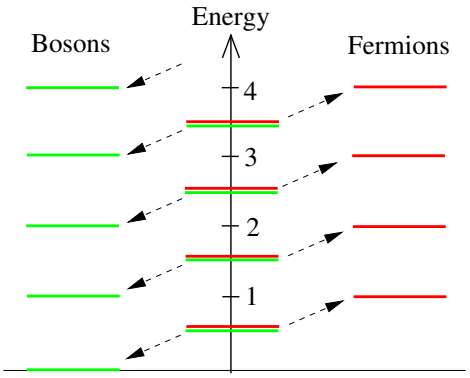


Figure 8.1: Fundamental excitations

"electrons" of the theory, the background electromagnetic field A_μ must couple to the topological current (8.13),

$$\mathcal{L}_{\text{Hall}} = \frac{k'}{4\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho + eA_\mu J^\mu + \dots .$$

Here e denotes the electron charge, while $\dots$ includes the rest of the Lagrangian (8.1), as well as the (3+1)-dimensional Maxwell term for A_μ.

We momentarily ignore the fundamental fields ϕ and ψ. Integrating out a_μ, the quadratic Lagrangian for the background field is given by

$$\mathcal{L}_{\text{Hall}} = -\frac{e^2}{4\pi k'}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho + \dots .$$

The effective action $S_{\text{eff}}[A] = \int \mathrm{d}^3x\,\mathcal{L}_{\text{Hall}}$, is now a functional of the non-dynamical, background electromagnetic field. Its role is to tell us how the system responds to an applied electromagnetic field through the relation $\langle J^\mu\rangle = \partial\mathcal{S}_{\text{eff}}/\partial A_\mu$. The result is a Hall conductivity

$$\sigma_H = \frac{e^2}{2\pi k'}\,. \tag{8.16}$$

This is the response of a fractional quantum Hall fluid at filling fraction $\nu = 1/k'$. (From now on, we will set $e = 1$ for brevity.)

Let us now return to the fundamental fields ϕ and ψ. Each of these experiences a magnetic field $f_{12} = -2\pi\mu/k'$ and forms Landau levels. The usual Landau level quantization results in a spectrum

$$E_{LL} = \frac{|f_{12}|}{m}(l + 1/2)$$

with $l = 0, 1, \ldots$. However, the Lagrangian (8.1) also includes extra terms which shift the overall energy of these states. The shift is down for bosons and up for fermions, as shown in Figure 8.1. The net result is that the energies of the Landau levels, at leading order, are given by

$$E = \frac{2\pi\mu l}{mk'} \quad \begin{cases} l = 0, 1, 2, \ldots & \text{for } \phi \\ l = 1, 2, \ldots & \text{for } \psi \end{cases} .$$

The gapped states ($l \geq 1$) arising from ϕ have spin $1/2k'$; those arising from ψ have spin $(1+k')/2k'$. Gauss's law (8.2) ensures that, when coupled to a background electromagnetic field, each of these carries charge $-1/k'$. These are the quasiparticle excitations of our supersymmetric quantum Hall fluid. The supercharges Q_2 and $Q_2^\dagger$ map between the fermionic and bosonic gapped Landau levels.

The system also has an a gapless band of quasiparticles, arising from the lowest Landau level of ϕ. These modes are not free: they interact through the ϕ^4 potential in (8.1). Nonetheless, supersymmetry ensures that these states have vanishing energy at all orders of perturbation theory. This is because the commutation relations for Q_2 require that any excitation with $H > 0$ must be paired with an excitation that differs by spin $1/2$. Yet the states in lowest Landau level have no partners and must, therefore, remain at zero energy. In essence, the theory has an infinite Witten index $\mathrm{Tr}(-1)^F$. If we start from the lowest Landau level, we can build up to higher levels by acting with $\hat{P}^\dagger$ and $Q_1^\dagger$.

Note that although we normally think of supersymmetry as protecting these states, in fact the fermionic field ψ in the theory cannot run in any loops to shift their energy (since there are no anti-particles, and these states have no fermions in them to begin with).

At this point, we return to our claim that including the supersymmetric partner ψ of ϕ is more a mathematical tool than an important part of our physical model. Notice that the fermionic fields have decoupled entirely from the lowest Landau level for bosons. If we are interested only in low-energy, lowest Landau level physics, then one might suspect fermions are irrelevant. We will see a different (and for our purposes stronger) version of this result below when we point out the lack of fermionic zero modes for vortices.

Meanwhile, the presence of a gapless Landau level may appear to contradict our claim that this system describes quantum Hall physics. After all, one of the defining features of a quantum Hall state is that it is gapped and incompressible. We will resolve this in Chapter 9 by studying how the Hall phase emerges from vortices when placed

in a confining potential. We will show that, for any finite number of vortices, there is a unique incompressible droplet of lowest angular momentum. However, in the absence of a confining potential, this droplet has zero energy edge modes and zero energy quasihole excitations. The gapless Landau level describes these degrees of freedom for an infinite number of BPS vortices, an interpretation recently suggested in a different context in [142]. We will revisit this in Section 9.4 in the context of the non-commutative approach to quantum Hall physics.

It is worth mentioning that this situation is not unusual in quantum Hall systems. The special, ultra-local Hamiltonians (such as Haldane pseudo-potentials) commonly used as models of quantum Hall physics also have zero energy edge modes and zero energy quasihole excitations for finite droplets. See, for example, [143, 144] for related discussions.

9 A Quantum Hall Fluid of Vortices

We would like to understand how to interpolate from the vacuum to the Hall phase. We do this by injecting vortices. These vortices are BPS which, in this context, means that they have $H = 0$ and lie in a protected sector of the theory. From the form of the Hamiltonian (8.3) and Gauss's law (8.2), it is clear that solutions with vanishing energy, $H = 0$, can be constructed by solving the equations

$$\mathcal{D}_z\phi = 0 \quad \text{and} \quad f_{12} = \frac{2\pi}{k'}(|\phi|^2 - \mu) \tag{9.1}$$

with the fermions set to zero: $\psi = 0$. (Abelian BPS vortices also appeared in the context of quantum Hall physics in [145].)

The vortex equations (9.1) are well studied. Solutions are labelled by the integer winding of the scalar field ϕ or, equivalently, by the magnetic flux

$$N = -\frac{1}{2\pi}\int \mathrm{d}^2x\, f_{12} \;\in\; \mathbb{Z}_{\geq 0}\,. \tag{9.2}$$

In the sector with winding N, the most general solution to (9.1) has $2N$ real parameters [146, 147]. These parameters are referred to as *collective coordinates* or, in the string theory literature, *moduli*. When vortices are well separated, these correspond to N positions on the complex plane. The existence of these moduli reflects the fact that the coefficient of the quartic interaction in (8.1) has been tuned to the critical value, ensuring that there are neither attractive nor repulsive forces between the vortices.

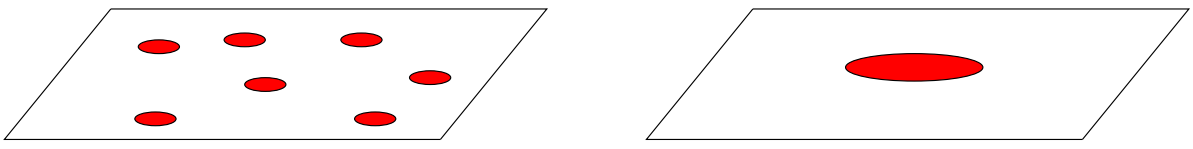


Figure 9.1: Two points in the moduli space of $N = 7$ vortices

As vortices coalesce, they lose their individual identities and the interpretation of these moduli changes. It is tempting to label the vortex by the point at which the Higgs field vanishes, but this does not provide an accurate description of what the vortex profile looks like. Instead, as we show in Section 9.4, in this regime it is better to think of the $2N$ moduli as describing the edge modes of a large, incompressible fluid.

Why do Vortices Form a Fractional Quantum Hall State?

The rest of this chapter is devoted to a detailed analysis of the quantum dynamics of vortices. We will ultimately show that their ground state is given by the Laughlin wavefunction. But here we first provide a hand-waving argument for why we expect the vortices to form a quantum Hall fluid.

We first note that the chemical potential term μa_0, present in the Lagrangian (8.1), can be viewed as a background magnetic field for vortices. It can be written as

$$-\int \mathrm{d}^3x\ \mu a_0 = \int \mathrm{d}^3x\ J^\mu[a]A_\mu$$

where J^μ is the topological current (8.13) and

$$A_p = -\frac{B}{2}\epsilon_{pq}x^q \quad \text{with} \quad B = 2\pi\mu\,. \tag{9.3}$$

This means that we expect the dynamics of vortices to correspond to particles moving in a background magnetic field. Nonetheless, it may be rather surprising that the vortices form a Hall state because, as we have seen, there is no force between the vortices. Yet the key physics underlying the fractional quantum Hall effect is the repulsive interactions between electrons, opening up a gap in the partially filled Landau level.

Although there is no force between vortices, they are not point particles. Instead, they are solitons obeying non-linear equations and, as they approach, the solutions deform. Indeed, when the vortices are as closely packed as they can be, they form a classically incompressible fluid as shown in the right-hand side of Figure 9.1. The scalar field ϕ has an N^{th} order zero in the centre of the disc and numerical studies show that the solution is well approximated as a disc of magnetic flux in which $\phi = 0$ and $f_{12} = -2\pi\mu/k'$. This motivated the "bag model" of vortices in [148, 149]. For us, it means that the vortex is a droplet of what we have called the "Hall phase".

When N vortices coalesce, the radius R of the resulting droplet can be estimated using the flux quantization (9.2) to be

$$R \approx \sqrt{\frac{k'N}{\pi\mu}}\,. \tag{9.4}$$

Now we can do a back-of-the-envelope calculation. In a magnetic field B, the number of states per unit area in the lowest Landau level of a charge 1 particle is $B/2\pi = \mu$. In an area $A = \pi R^2 = Nk'/\mu$, the lowest Landau level therefore admits $BA/2\pi = Nk'$

states. We've placed N vortices in this region, so the filling fraction is

$$\nu = \frac{1}{k'} \,.$$

This, of course, is the expected filling fraction in the Hall phase with conductivity (8.16).

9.1 The Dynamics of Vortices

We now turn to a more detailed description of the dynamics of vortices. We first introduce the *vortex moduli space*, $\mathcal{M}_N$. This is space of solutions to the vortex equations (9.1) with winding number N. As we have already mentioned,

$$\dim(\mathcal{M}_N) = 2N \,.$$

The coordinates X^a, $a = 1, \ldots, 2N$, parametrizing $\mathcal{M}_N$ are the collective coordinates of vortex solutions: $\phi(x; X)$ and $a_p(x; X)$.

The standard approach to soliton dynamics is to assume that, at low energies, motion can be modelled by restricting to the moduli space [150]. This is usually applied in relativistic theories where the action is second order in time derivatives and typically provides an accurate approximation to the real dynamics. Here we have a non-relativistic theory, first order in time derivatives, and this results in a number of differences which we now explain. One ultimate surprise – which we will get to in Section 9.2 – is that there is no approximation involved in the moduli space dynamics in this system; instead it is exact.

The first, and most important difference, is associated to the meaning of the space $\mathcal{M}_N$. In relativistic theories, $\mathcal{M}_N$ is the configuration space of vortices and the dynamics is captured by geodesic motion on $\mathcal{M}_N$ with respect to a metric $g_{ab}(X)$. It is known that $\mathcal{M}_N$ is a complex manifold, with complex structure J, and the metric $g_{ab}(X)$ is Kähler. For completeness, we explain how to construct this metric in Appendix B.

In our non-relativistic context, it is no longer true that $\mathcal{M}_N$ is the configuration space of vortices. Instead, it is the *phase space*. The dynamics of the vortices is described by a quantum mechanics action of the form

$$S_{\text{vortex}} = \int \mathrm{d}t \; \mathcal{F}_a(X)\dot{X}^a \tag{9.5}$$

where $\mathcal{F}(X)$ is a one-form over $\mathcal{M}_N$. Our goal is determine this one-form.

In fact, this problem has already been solved in the literature. A model which shares its vortex dynamics with ours was previously studied by Manton [138] and subsequently, in more geometric form, in [151, 152]. The main result of these papers is that $\mathcal{F}$ is an object known as the symplectic potential. It has the property that

$$\mathrm{d}\mathcal{F} = \Omega \tag{9.6}$$

where Ω is the Kähler form on $\mathcal{M}_N$, compatible with the metric g_{ab} and the complex structure J.

For a single vortex, the moduli space is simply the plane $\mathbb{C}$ and the Kähler form is

$$\Omega = \frac{\pi\mu}{2}\,\mathrm{d}z \wedge \mathrm{d}\bar{z}\,.$$

For $N \geq 2$ vortices the Kähler form is more complicated. We describe the construction of Ω in Appendix B. Explicit expressions are only known for well-separated vortices [152].

The derivation of (9.6) given in [138, 151, 152] relies on a parametrization of the vortex moduli space introduced earlier in [153]. The use of these coordinates means that the calculation is not entirely straightforward. For this reason, in Appendix B, we present a simpler derivation of (9.6) which does not rely on any choice of coordinates. (For a different approach to particle dynamics appropriate for vortices, see [154].)

There are No Fermion Zero Modes

The vortices are BPS states: they are annihilated by the supercharge Q_2. In the context of first order dynamics, this means that the collective coordinates X do not transform under Q_2. In particular, there are no accompanying Grassmann collective coordinates. Indeed, it is simple to check explicitly that there are no fermionic zero modes in the background of the vortex.

This fits in nicely from the picture suggested by Figure 8.1: fermionic excitations are gapped by a scale $\sim \mu/m$ from the lowest Landau level physics of bosons, which is where vortices live.

The upshot is that the vortices themselves are supersymmetric singlets. The role of supersymmetry in the vortex dynamics (9.5) is to tune the vortices to have strictly vanishing energy, $H = 0$, even in the full quantum theory. But instead, we could in fact simply focus on the bosonic sector, which is essentially what we will do from this point onwards.

The fact that the BPS solitons have no fermion zero modes may come as something of a surprise. Indeed, it is rather different from what happens for BPS solitons in relativistic field theories or in string theory. It is worth pausing to explain this difference. In more familiar relativistic theories, if a soliton is invariant under a given supercharge $\mathcal{Q}$ then that supercharge will descend to the worldvolume theory, relating bosonic and fermionic zero modes on the worldvolume. However, when we say that a soliton is invariant under $\mathcal{Q}$, we mean that the static configuration is invariant: when the soliton moves, the supercharge $\mathcal{Q}$ typically acts and generates a fermionic zero mode. This means that while $\mathcal{Q}$ does not act on the bosonic configuration space of the soliton, it does act on the phase space.

In our non-relativistic theory, the statement that Q_2 annihilates the soliton is stronger: it means that Q_2 does not act on the soliton phase space. This is the reason that there are no associated fermionic zero modes.

9.2 Introducing a Harmonic Trap

We have derived a low-energy effective action (9.5) for the vortex dynamics. However, this dynamics is boring. The equation of motion arising from (9.5) is

$$\Omega_{ab}\dot{X}^b = 0 \quad \Rightarrow \quad \dot{X}^a = 0\,.$$

The vortices don't move. They are pinned in place.

The lack of dynamics follows because there is no force between vortices and, in a first order system, we don't have the luxury of giving the vortices an initial velocity. To get something more interesting, we impose an external force on the vortices. We will do so by introducing a harmonic trap. We want this trap to be compatible with supersymmetry. We can do this by choosing the new Hamiltonian

$$H_{\text{new}} = H + \omega\left(\mathcal{J} + \frac{1}{2}\mathcal{N}_F\right)$$

where $\mathcal{J}$ is the angular momentum (8.5), $\mathcal{N}_F$ the fermion number operator (4.21) and ω dictates the strength of the trap. From (8.12), we see that this Hamiltonian remains invariant under Q_2, although not Q_1. When evaluated on BPS vortices, the Hamiltonian is simply

$$H_{\text{new}} = -\frac{\mu\omega}{2}\int \mathrm{d}^2x\, |z|^2 f_{12}\,. \tag{9.7}$$

This new Hamiltonian is the angular momentum of a given BPS vortex configuration: it preserves the BPS nature of vortices while shifting their energy. Evaluating (9.7) on a

vortex configuration provides a function $\mathcal{J}(X)$ over the vortex moduli space $\mathcal{M}_N$ which governs the their low-energy dynamics,

$$S_{\text{vortex}} = \int \mathrm{d}t \, \Big(\mathcal{F}_a(X)\dot{X}^a - \omega \mathcal{J}(X) \Big) \, . \tag{9.8}$$

We will now look at some examples of the classical dynamics described by this action.

Classical Motion in the Trap

The harmonic trap (9.7) favours those vortex solutions that are clustered towards the origin. The lowest energy configuration now has all vortices coincident at the origin, as in the right-hand picture in Figure 9.1. As we have seen, the size of this coalesced vortex is given by (9.4), so the angular momentum of this state is

$$\mathcal{J}_0 \approx -\frac{\mu}{2} \int_0^R \mathrm{d}r \; 2\pi r^3 f_{12} = \frac{k' N^2}{2} \, . \tag{9.9}$$

This is the only static configuration. All other solutions evolve through the equation of motion

$$\Omega_{ab} \dot{X}^b = \omega \frac{\partial \mathcal{J}}{\partial X^a} \, . \tag{9.10}$$

In particular, a single vortex displaced a distance $r \gg \sqrt{1/\mu}$ from the origin, will have angular momentum $\mathcal{J} \approx \pi \mu r^2$. This vortex orbits around the origin with frequency ω.

There is something rather surprising about the moduli space approximation for this first order dynamics: it is exact! The solutions to the equation of motion in the presence of the trap are simply time dependent rotations of the static solutions so, for example, $\phi = \phi(x; X(t))$, with $X(t)$ obeying (9.10). This a property of any first order system with a Hamiltonian, such as $H = \mathcal{J}$, which acts as a symmetry generator on the moduli space.

9.3 The Quantum Hall Matrix Model

The description of the vortex dynamics (9.8) is, unfortunately, rather abstract. For $N \geq 2$ vortices, we have only implicit definitions of the Kähler form Ω and the angular momentum $\mathcal{J}$ on the vortex moduli space. It seems plausible that one could make progress using the parametrization of the vortex moduli space introduced in [153]. Here, however, we take a different approach.

An alternative description of the vortex moduli space is provided by D-branes in string theory [155]. This is analogous to the ADHM construction of the instanton mod-

uli space. The vortex moduli space $\mathcal{M}_N$ is parametrized by:

- An $N \times N$ complex matrix Z
- A N-component complex vector φ

These provide $N(N+1)$ complex degrees of freedom. We will identify configurations related by the $U(N)$ action

$$Z \to UZU^\dagger \quad \text{and} \quad \varphi \to U\varphi \qquad \text{with } U \in U(N)\,. \tag{9.11}$$

We further require that Z and φ satisfy the matrix constraint[1],

$$\pi\mu\,[Z, Z^\dagger] + \varphi\varphi^\dagger = k'\,\mathbf{1}_N\,. \tag{9.12}$$

This constraint is the moment map for the action (9.11) with level k'. We define the moduli space $\tilde{\mathcal{M}}_N$ through the symplectic quotient

$$\tilde{M}_N = \left\{Z, \varphi \text{ such that } \pi\mu[Z, Z^\dagger] + \varphi\varphi = k'\right\}/U(N)\,.$$

This space has real dimension $\dim(\tilde{\mathcal{M}}_N) = 2N$. The string theory construction of [155] shows that this space is related to the vortex moduli space

$$\tilde{\mathcal{M}}_N \cong \mathcal{M}_N\,.$$

These spaces are conjectured to be isomorphic as complex manifolds, and have the same Kähler class. The author is not aware, of a direct proof of this conjecture beyond the string theory construction provided in [155].

The matrix description provides a different parametrization of the vortex moduli space. When the vortices are well separated, Z is approximately diagonal. The positions of the vortices are described by these N diagonal elements. (The normalization of $\pi\mu$ in (9.12) is associated to the magnetic length which is of the same order as the vortex size.) However, as the vortices approach, Z is no longer approximately diagonal, reflecting the fact it is better to think of the locations of the vortices as fuzzy, spread out over a disc of radius (9.4). This feature is captured by the matrix description of the vortex moduli space.

The moduli space $\tilde{M}_N$ inherits a natural metric through the quotient construction described above. This does *not* coincide with the metric on the vortex moduli space $\mathcal{M}_N$

[1] As an aside: for relativistic vortices, the right-hand side of (9.12) is $2\pi/e^2$, where e^2 is the gauge coupling constant. Comparing the vortex equations (9.1) to their relativistic counterparts shows that this becomes k' in the non-relativistic context. The fact that this is integer valued for vortices in the Chern-Simons theory will prove important below.

described in Appendix B. Nonetheless, there are now a number of examples in which computations of BPS quantities using $\tilde{\mathcal{M}}_N$ coincide with those of computed from the vortex moduli space $\mathcal{M}_N$ because they are insensitive to the details of the metric (see, for example, [156, 157, 158, 159]). Here we will ultimately be interested in holomorphic wavefunctions over the vortex moduli space. Assuming the conjectured equivalence of the spaces as complex manifolds, it will suffice to work with the matrix model description of the vortex moduli space.

The Matrix Model Action

It is now a simple matter to write the vortex dynamics in terms of these new fields. We introduce a $U(N)$ gauge field, α, on the worldline of the vortices. In the absence of a harmonic trap, the low-energy vortex dynamics is governed by the $U(N)$ gauged quantum mechanics

$$S_{\text{vortex}} = \int \mathrm{d}t\; i\pi\mu\ \mathrm{Tr}\ \left(Z^\dagger \mathcal{D}_0 Z\right) + i\varphi^\dagger \mathcal{D}_0 \varphi - k'\ \mathrm{Tr}\ \alpha \tag{9.13}$$

where $\mathcal{D}_0 Z = \partial_0 Z - i[\alpha, Z]$ and $\mathcal{D}_0\varphi = \partial_0\varphi - i\alpha\varphi$. The quantum mechanical Chern-Simons term ensures that Gauss's law for the matrix model coincides with (9.12). This means that this action describes the same physics as (9.5).

The action (9.13) is the *quantum Hall matrix model*, previously proposed as a description of the fractional quantum Hall effect by Polychronakos [19] and further explored in [134, 135, 160, 161, 162, 163]. The connection to first order vortex dynamics was noted earlier in [137].

We note in passing that we've used the D-brane construction of [155] in a fairly indirect way to derive the quantum Hall matrix model. A more direct D-brane derivation of the matrix model was provided previously in [164]. It would be interesting to see how this work, or the string theory construction of [165], is related to the present set-up.

We would also like to add the harmonic trap to the matrix model. This too was explained in [19]. Spatial rotation within the matrix model acts as $Z \to e^{i\theta} Z$, with the associated charge $\mathcal{J} = \pi\mu\,\mathrm{Tr}\ Z^\dagger Z$. Adding this to the action, we get the matrix model generalization of (9.8),

$$S_{\text{vortex}} = \int \mathrm{d}t\; i\pi\mu\ \mathrm{Tr}\ \left(Z^\dagger \mathcal{D}_t Z\right) + i\varphi^\dagger \mathcal{D}_t \varphi - k'\ \mathrm{Tr}\ \alpha - \omega\pi\mu\ \mathrm{Tr}\left(Z^\dagger Z\right)\ . \tag{9.14}$$

In the rest of this chapter, we describe the properties of this matrix model. Much of this is review of earlier work, in particular [19] and [134, 135]. However, we also make

a number of new observations about the matrix model, most notably the computation of the charge and statistics of quasihole excitations.

The Classical Ground State

In the presence of the harmonic trap, the classical equations of motion comprise the constraint (9.12) and a classical equation of motion for Z:

$$i\mathcal{D}_t Z = \omega Z\,. \tag{9.15}$$

There is a unique time independent solution, with $\dot{Z} = 0$, obeying $[\alpha, Z] = \omega Z$. (This equation can also be viewed as the statement that rotating the phase of Z is equivalent to a gauge transformation.) This solution was given in [19], and takes the form

$$Z_0 = \sqrt{\frac{k'}{\pi\mu}} \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & & \ddots & \\ & & & 0 & \sqrt{N-1} \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \varphi_0 = \sqrt{k'} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix} \tag{9.16}$$

with $\alpha = \omega\,\mathrm{diag}(N-1, N-2, \ldots, 2, 1, 0)$.

As promised, Z_0 is not approximately diagonal. This reflects the fact that individual vortices do not have well-defined positions. Nonetheless, we can reconstruct a number of simple properties of the vortex solution from this matrix. The radius-squared of the disc can be thought of as the maximum eigenvalue of $Z_0^\dagger Z_0$ [19]. To leading order in the vortex number N, this gives

$$R^2 \approx \frac{k'N}{\pi\mu}$$

which agrees with our the radius of the classical vortex solution (9.4). Meanwhile, the angular momentum of a given solution is $\mathcal{J} = \mathrm{Tr}\ Z^\dagger Z$. The angular momentum of the ground state is

$$\mathcal{J}_0 = \pi\mu\ \mathrm{Tr}\ \left(Z_0^\dagger Z_0\right) = \frac{k'N(N-1)}{2} \tag{9.17}$$

which, to leading order in $1/N$, agrees with the angular momentum of the classical vortex solution (9.9).

The Quantum Ground State

The quantization of the matrix model (9.14) was initiated in [19] and explored in some detail in [134] and [135]. The individual components of the matrix Z and vector φ are promoted to quantum operators, with commutation relations

$$\pi\mu\,[Z_{ab}, Z^\dagger_{cd}] = \delta_{ad}\delta_{bc} \quad \text{and} \quad [\varphi_a, \varphi^\dagger_b] = \delta_{ab}\,.$$

We choose the vacuum state $|0\rangle$ such that $Z_{ab}|0\rangle = \varphi|0\rangle = 0$. However this does not, in general, correspond to the ground state of the theory because the physical Hilbert space must obey the quantum version of Gauss's law (9.12). It is useful to view the trace and traceless part of this constraint separately. The trace constraint reads

$$\sum_{a=1}^{N} \varphi_a\varphi^\dagger_a = k'N \quad \Rightarrow \quad \sum_{a=1}^{N} \varphi^\dagger_a\varphi_a = (k'-1)N\,. \tag{9.18}$$

We now introduce k via $k' \equiv k+1$; this coincides with the value it will take in non-Abelian theories. This means that physical states must have kN φ-excitations. Note that the ordering of the original constraint has resulted in a shift $k' \to k$. This will prove important below.

Meanwhile, the traceless part of the constraint (9.12) tells us that physical states must be $SU(N) \subset U(N)$ singlets. We can form such singlet operators out of $Z^\dagger$ and $\varphi^\dagger$ either from baryons or from traces. The baryonic operators are

$$\epsilon^{a_1\cdots a_N}(\varphi^\dagger Z^{\dagger\,p_1})_{a_1}\cdots(\varphi^\dagger Z^{\dagger\,p_N})_{a_N}$$

where $p_1, \ldots, p_N$ are, necessarily distinct, integers. The trace operators are

$$\mathrm{Tr}(Z^{\dagger\,p})\,.$$

There can be complicated relations between the baryonic and trace operators; explicit descriptions for low numbers of vortices were given in [166].

The trace constraint (9.18) means that physical states contain exactly k baryonic operators. The harmonic trap endows these with an energy proportional to the number of $Z^\dagger$ excitations,

$$H = \omega\mathcal{J} = \omega\pi\mu\sum_{a,b=1}^{N} Z^\dagger_{ab}Z_{ba}\,.$$

To minimize this energy, we must act with k baryonic operators, each with $p_i = i - 1$. This results in the ground state

$$|\text{ground}\rangle_k = \left[\epsilon^{a_1 \cdots a_N} \varphi^\dagger_{a_1} (\varphi^\dagger Z^\dagger)_{a_2} \cdots (\varphi^\dagger {Z^\dagger}^{N-1})_{a_N}\right]^k |0\rangle \,. \tag{9.19}$$

The angular momentum of this ground state coincides with that of the classical ground state (9.17) up to a quantum shift $k' \to k$.

There is a close resemblance between these ground states and the Laughlin states [9] for N electrons at filling fraction $\nu = 1/k'$,

$$|\text{Laughlin}\rangle_{k'} = \prod_{a<b} (z_a - z_b)^{k'} \, e^{-\frac{B}{4}\sum |z_a|^2} = \left[\epsilon^{a_1 \cdots a_N} z^0_{a_1} z_{a_2} \cdots z^{N-1}_{a_N}\right]^{k'} e^{-\frac{B}{4}\sum |z_a|^2} \,. \tag{9.20}$$

A formal map between the states was suggested in [134]. However, this similarity can be misleading: the operators $Z^\dagger$ and $\varphi^\dagger$ are very different objects from the holomorphic position variables z_a. To make this connection precise, we need to be more careful about how to relate the two. In fact, there is no canonical map. There are, however, a number of natural ways to make the connection. Two of these, discussed in [135] (see also [161]), are:

- We work with a coherent state representation $\hat{Z}|Z,\varphi\rangle = Z|Z,\varphi\rangle$ and $\hat{\varphi}|Z,\varphi\rangle = \varphi|Z,\varphi\rangle$ where, for once, we've used hats to denote the difference between the quantum operator $\hat{Z}$ and the classical matrix Z. We then diagonalize $Z = VDV^{-1}$ with $D = \text{diag}(z_1, \ldots, z_N)$ and express the resulting wavefunctions as $\psi(z_a) = \langle z_a|\Psi\rangle$. Essentially, we are using the eigenvalues of Z as coordinates on the phase space. (Non-diagonalisable matrices have zero measure.)

- Alternatively, we could decompose the complex operator matrix $\hat{Z} = \hat{X} + i\hat{Y}$ and subsequently work in a coherent state representation $\hat{X}|X\rangle = X|X\rangle$. This picture has the advantage that the matrices $\hat{X}$ and $\hat{Y}$ are conjugate, giving us the representation

  $$\hat{Z}^\dagger_{ab} = \frac{1}{\sqrt{2}}\left(X_{ab} - \frac{\partial}{\partial X_{ba}}\right) .$$

  Moreover, calculations in this approach are somewhat easier because the diagonalization $X = UXU^\dagger$ can be achieved by a unitary operator U. The resulting wavefunctions are written as $\psi(x_a) = \langle x_a|\Psi\rangle$. We then analytically continue $x_a \to z_a$ to provide holomorphic wavefunctions of the kind appropriate to describe the lowest Landau level.

Both of these approaches were described in [135]. The resulting wavefunctions differ in detail, but share their most important properties.

The first result of [135] is that the $k' = 1$ ground state, which we have chosen to label as $|\text{ground}\rangle_0 = |0\rangle$, is precisely the $\nu = 1$ Laughlin state describing a filled Landau level. That is:

$$|0\rangle = |\text{Laughlin}\rangle_1 \,.$$

For $k' > 1$, the map to the Laughlin wavefunction is not exact. Instead, the wavefunctions agree only at large separation

$$|\text{ground}\rangle_k \to |\text{Laughlin}\rangle_{k'} \quad \text{for} \quad |z_a - z_b| \gg 1/\pi\mu \,.$$

However, the matrix model states $|\text{ground}\rangle_k$ differ from the Laughlin states as the particle approach: the wavefunctions still vanish as $z_a \to z_b$, but not with the familiar zero-of-order m that is characteristic of the Laughlin wavefunction. Note that these differences only become visible at separations of order the magnetic length. (Indeed, one can obtain the so-called "X-representation" matrix model wavefunction from the Laughlin one by acting with exponentials of derivative operators $\ell_B(\partial/\partial z)$ on the polynomial part.)

As we described above, there is nothing privileged about the choice of coordinates used above – one may try various sets of coordinates and see if there is better short-distance agreement with the Laughlin wavefunction. However, as was found in [135, 161], there seems no obvious way to find an exact match to the Laughlin wavefunctions.

The connection to vortices sheds some light on this. Because vortices are extended objects, there is no "correct" way to specify their positions as they approach. Correspondingly, it is not obvious that their physics is captured by a wavefunction describing point particles. Instead, the important questions are those which are independent of the choice of coordinates. The fact that the long-distance correlations in the matrix model ground states (9.19) coincide with those of the Laughlin wavefunction suggests that these states describe the same universality class of quantum Hall fluids. In the rest of Part III, we show that this is indeed correct. We show that excitations of the matrix model describe chiral edge modes and quasiholes. In particular, the latter have charge $1/k'$ and fractional statistics, in agreement with the excitations of the Laughlin wavefunction.

9.4 Edge Modes

The classical excitations of the matrix model were described in [19]. There are edge excitations of the droplet and there quasihole excitations although, for finite N, there is no clear distinction between these. There are no quasiparticle excitations which, given

the spacetime picture in terms of vortices, is to be expected. We first study the edge modes and show that they form a chiral boson.

The linear perturbations of the solution (9.16), consistent with the constraint (9.12), were given in [19]. They are remarkably simple:

$$\delta_l Z = (Z_0^\dagger)^{l-1} \quad \text{and} \quad \delta_l \varphi = 0 \quad \text{with} \quad l = 1, \ldots, N\,. \tag{9.21}$$

These were interpreted in [19] as area-preserving deformations of the disc, restricted to the first N Fourier modes.

We now show that the dynamics is that of a chiral, relativistic boson. To do this, we write

$$Z(t) = Z_0 + \sum_{l=1}^{N} c_l(t) Z_0^{\dagger\, l-1}$$

with complex coefficients c_l. Plugging this ansatz into the action (9.14), we have the following expression for the effective dynamics of c_l,

$$S = \pi\mu \sum_{l,p=1}^{N} \int \mathrm{d}t\; i\,\mathrm{Tr}(Z_0^{l-1} Z_0^{\dagger\, p-1})\, c_l^\star \dot{c}_p + \Big[\mathrm{Tr}(\alpha\,[Z_0^{\dagger\, p-1}, Z_0^{l-1}]) - \omega\,\mathrm{Tr}(Z_0^{l-1} Z_0^{\dagger\, p-1})\Big]\, c_l^\star c_p$$

where we have dropped the constant contribution (9.17). We need to compute two traces, both involving Z_0 given in (9.16). The first is

$$\pi\mu\ \mathrm{Tr}\ Z_0^{l-1} Z_0^{\dagger\, p-1} \equiv \Theta_l \delta_{lp} \quad \text{with} \quad \Theta_l = \frac{k'^{l-1}}{l} N(N-1)\ldots(N-l+1)\,.$$

The second trace involves α and can be readily computed by invoking the relationship $\omega Z_0 = [\alpha, Z_0]$, to give $[\alpha, Z_0^{\dagger\, p}] = -p\omega Z_0^{\dagger\, p}$. The action for the perturbations can then be written in the simple form,

$$S = \sum_{l=1}^{N} \Theta_l \int \mathrm{d}t\ \left(i c_l^\star\, \dot{c}_l - \omega l c_l^\star c_l\right)\,. \tag{9.22}$$

This is the action for a real, chiral boson, defined on the edge of the Hall droplet. We parametrize the perimeter of the droplet by $\sigma \in [0, 2\pi R)$ with R given by (9.4). The continuum excitations then take the form

$$c(\sigma, t) = \frac{1}{\sqrt{2\pi}} \sum_{l=-\infty}^{\infty} e^{il\sigma/R} \sqrt{\frac{\Theta_l}{l}}\, c_l(t) \qquad \text{with} \quad c_{-l} = c_l^\star\,.$$

Then the action (9.22) becomes

$$S = -\int \mathrm{d}t d\sigma\ \partial_t c\, \partial_\sigma c + (\omega R)\partial_\sigma c\, \partial_\sigma c\,.$$

This is the form of the action for a chiral boson proposed in [132], now truncated to the lowest N Fourier modes. The action describes modes propagating in one direction around the disc with velocity $v = \omega R$. A previous derivation of a chiral boson edge theory from the matrix model was given in [163], albeit in a model with a different potential.

Note that as N increases, the radius of the disc (9.4) scales as $\sqrt{N}$, while the number of Fourier modes increases linearly with N. The density of modes therefore scales as $1/\sqrt{N}$, confirming the existence of a continuum (1+1)-dimensional limit as $N \to \infty$.

As mentioned above, this chiral boson is very natural: it is simply the natural way to parametrize the fluctuations of the incompressible disc formed by our quantum Hall droplet. In Part V, we will return to study the edge theories of the more general matrix model introduced in Part IV. However, even in that much richer theory, there is one sector which is simply this same chiral boson.

The Non-Commutative Description Revisited

The original motivation for the quantum Hall matrix model was to provide a finite N regularization of Susskind's non-commutative approach to quantum Hall fluids [20]. Taking the $N \to \infty$ limit of the matrix model, one can effectively drop the field φ and the constraint (9.12) becomes

$$[X^1, X^2] = i\frac{2\pi\mu}{k'} = i\frac{B}{k'}\,.$$

We interpret this as a non-commutative plane. Expanding the action around the state (9.16) gives rise to a Chern-Simons theory on this non-commutative plane, with fields multiplied using the Moyal product [20]. The perspective offered here shows that this non-commutative theory provides a hydrodynamic description of the dynamics of $N \to \infty$ BPS vortices.

There is no harmonic trap introduced in the non-commutative Chern-Simons description. Because it arises from the expansion around (9.16), all perturbative excitations of the theory are edge modes of an infinitely large disc, now consigned asymptotically to infinity. However, these perturbation excitations are not the end of the story. There are many other non-perturbative bulk excitations. These correspond to separating vortices or, as we will see in the next section, creating a hole in the fluid of vortices. The non-commutative Chern-Simons theory is capturing these modes.

However, we have already seen a different description of these modes from the perspective of the (2+1)-dimensional spacetime picture: they are the gapless, lowest Landau level of an interacting boson that we saw in Section 8.2. It appears that the Chern-Simons theory on the non-commutative plane is an alternative description of this lowest Landau level physics.

9.5 Quasiholes

Let us now return to a finite droplet of vortices. While the infinitesimal perturbations of the droplet describe edge modes, one can also consider finite deformations. Of course, if we make a large enough finite perturbation, then the droplet will eventually fragment into its component vortices. However, there are deformations for which the droplet retains its integrity, but with a hole carved out in the middle. These are the quasiholes of the quantum Hall effect.

There is a simple classical solution describing a quasihole placed at the centre of the vortex [19]. It arises by integrating the N^{th} Fourier mode,

$$Z = \sqrt{\frac{k'}{\pi\mu}} \begin{pmatrix} 0 & \sqrt{1+q} & & & \\ & 0 & \sqrt{2+q} & & \\ & & & \ddots & \\ & & & 0 & \sqrt{N-1+q} \\ \sqrt{q}e^{iN\omega t} & & & & 0 \end{pmatrix} . \tag{9.23}$$

This obeys the constraint (9.12) and equation of motion (9.15) with $\alpha = \omega\,\mathrm{diag}(N-1, N-2, \ldots, 2, 1, 0)$ and $\varphi = \varphi_0$.

This solution should be thought of as a deficit of magnetic field in the middle of the Hall droplet [19] (see also [167]). In other words, it is a quasihole. Using the maximum and minimum eigenvalues of $Z^\dagger Z$ as a proxy for the inner radius R_1 and the outer radius R_2 of this annulus, we find

$$R_1^2 \approx \frac{k'q}{\pi\mu} \quad \text{and} \quad R_2^2 \approx \frac{k'(N+q)}{\pi\mu}$$

which is consistent with the magnetic flux quantization (9.2) if f_{12} remains constant for $R_1 < r < R_2$. We can subject this interpretation to a further test. The angular momentum of the matrix model solution is given by

$$\mathcal{J} = \pi\mu \ \mathrm{Tr}\ Z^\dagger Z = \frac{k'N^2}{2} + k'Nq\,.$$

But we can also compute the angular momentum of an annular vortex by the same kind of calculation we used in (9.9). We find

$$\mathcal{J} \approx -\frac{\mu}{2}\int_{R_1}^{R_2} \mathrm{d}r\; 2\pi r^3 f_{12} = \frac{k'N^2}{2} + k'Nq$$

confirming the solution (9.23) as a classical quasihole.

There are, presumably, more complicated classical solutions, describing quasiholes displaced from the origin, rotating with frequency ω. Rather than searching for these classical solutions, we will instead describe their quantum counterparts.

Quantum Quasiholes

We claim that the quantum state describing m quasiholes, located at complex coordinates η_i, $i = 1, \ldots, m$, is

$$|\eta_1, \ldots, \eta_m\rangle_k \;\propto\; \prod_{i=1}^{m} \det(Z^\dagger - \eta_i^\dagger)\; |\text{ground}\rangle_k \tag{9.24}$$

where we have allowed for a normalization constant.

Let us first motivate this ansatz. Multiplying by $\det(Z^\dagger - \eta^\dagger)$ is equivalent to taking one of the baryonic operators in the ground state (9.19) and replacing each occurrence of $\varphi^\dagger Z^{\dagger p}$ by $\varphi^\dagger Z^{\dagger p}(Z-\eta)^\dagger$. Under the coherent state map of [135], where the eigenvalues of Z are used as coordinates, this gives

$$|\eta_1, \ldots, \eta_m\rangle_k \;\to\; \prod_a (z_a - \eta)|\text{Laughlin}\rangle_k$$

which is indeed the Laughlin wavefunction for quasiholes.

As we vary the positions η_i, the resulting states $|\eta_1, \ldots, \eta_m\rangle_k$ are not linearly independent. This reflects the fact that these holes are made from a finite number of underlying particles. Nonetheless, for $|\eta_i| < R$, with $R = \sqrt{k'N/\pi\mu}$ the size of the quantum Hall droplet (9.4), we expect the state to approximately describe m localized quasiholes. This interpretation breaks down as the quasiholes approach either each other or the edge of the droplet. Indeed, the states degenerate and become approximately the same for any value of $|\eta_i| \gg R$. We'll see the consequences of this below.

In the presence of a harmonic trap, the states (9.24) are not energy eigenstates unless $\eta_i = 0$. Nonetheless, it is simple to check that the time-dependent states

$$|e^{i\omega t}\eta_1, \ldots, e^{i\omega t}\eta_m\rangle_k$$

in which the quasiholes orbit the origin, solve the time-dependent Schrödinger equation. In what follows, we will compute the braiding of the time independent states (9.24).

In the quantum Hall effect, the quasiholes famously have fractional charge and fractional statistics. We now show this directly for the states (9.24). We follow the classic calculation of [133] in computing the Berry phase accumulated as quasiholes move in closed paths. However, there is a technical difference that is worth highlighting. In the usual Laughlin wavefunction, the overlap integrals are too complicated to perform directly. Instead, one resorts to the plasma analogy [9]. This requires an assumption that a classical 2d plasma exhibits a screening phase.

A second route to computing the braiding of quasiparticles is provided by the link to conformal field theories [16], where it is conjectured to be equivalent to the monodromy of conformal blocks. The primary focus has been on the richer subject of non-Abelian quantum Hall states. Different approaches include [168] and [169, 170], the latter once again relying on a plasma analogy. See also [171] for an alternative approach to braiding.

We will now show that the matrix model construction of the quasihole states (9.24) seems to avoid these issues and a direct attack on the problem bears fruit. We compute the Berry phase explicitly without need of a plasma analogy.

Fractional Charge

We start by computing the charge of the quasihole under the external gauge field. To do this, we consider a single excitation located at $\eta = re^{i\theta}$. We then adiabatically transport the quasihole in a circle by sending $\theta \to \theta + 2\pi$. If the quasihole has charge $q_{\rm QH}$ then we expect that the wavefunction will pick up an Aharonov-Bohm phase Θ proportional to the magnetic flux Φ enclosed in the orbit:

$$\Theta(r) = \Phi q_{\rm QH} = \pi r^2 B q_{\rm QH} = 2\pi^2 \mu r^2 \, q_{\rm QH} \tag{9.25}$$

where we've used the value of $B = 2\pi\mu$ computed in (9.3), with e the charge of a single vortex. There is a more direct expression for Θ, arising as the Berry phase associated to the adiabatic change of the wavefunction,

$$\Theta(r) = -i \int_0^{2\pi} {\rm d}\theta \; {}_k\langle \eta | \, \frac{\partial}{\partial \theta} | \eta \rangle_k \, . \tag{9.26}$$

Our task is to compute this phase. From this we extract $q_{\rm QH}$.

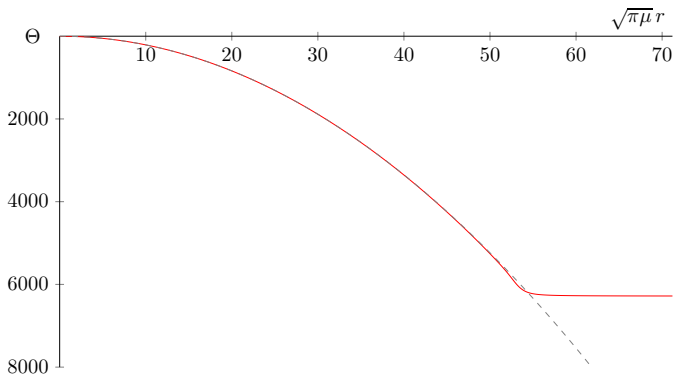


Figure 9.2: The Berry phase for a single quasihole in $N = 1000$ vortices with $k' = 3$. The phase Θ (solid, red) and the expected phase for a particle of charge $-1/k'$ in the field B (grey, dashed) are both plotted.

To do this, it will help to introduce some new notation. We define the states $|\Omega_l\rangle_k$, with $l = 0, \ldots, N-1$, via

$$|\Omega_l\rangle_k = \Big[\epsilon^{a_1\cdots a_N}\varphi^\dagger_{a_1}(\varphi^\dagger Z^\dagger)_{a_2}\cdots \quad (\varphi^\dagger Z^{\dagger\, l-1})_{a_l}(\varphi^\dagger Z^{\dagger\, l+1})_{a_{l+1}}\cdots(\varphi^\dagger Z^{\dagger\, N})_{a_N}\Big] \Big[\epsilon^{b_1\cdots b_N}\varphi^\dagger_{b_1}(\varphi^\dagger Z^\dagger)_{b_2}\cdots(\varphi^\dagger Z^{\dagger\, N-1})_{b_N}\Big]^{k-1}|0\rangle\,.$$

Each of these is an eigenstate of angular momentum, with $\mathcal{J} = \mathcal{J}_0 + \pi\mu(N-l)$. We can expand the quasihole state (9.24) in this basis as

$$|\eta\rangle_k \propto \sum_{l=0}^{N-1}(-\eta^\dagger)^l|\Omega_l\rangle_k\,.$$

Because the $|\Omega_l\rangle_{k'}$ have different angular momenta, they are orthogonal. We write their inner product as

$${}_k\langle\Omega_p|\Omega_l\rangle_k = \lambda(l;k')\,\delta_{lp}\,.$$

In terms of these inner products, the Berry phase (9.26) is simply written as

$$\Theta(r) = 2\pi i\,\frac{\sum_{l=0}^N il\lambda(l;k')\,r^{2l}}{\sum_{l=0}^N\lambda(l;k')\,r^{2l}}\,.$$

The computation of $\lambda(l;k')$ is not straightforward. (Indeed, this is the step in the usual calculation where one resorts to the plasma analogy.) We find the following result:

$$\lambda(l;k') = (\pi\mu)^{l-N}\binom{N}{l}\left[\prod_{a=0}^{N-l-1}(k'a+1)\right]\,{}_k\langle\text{ground}|\text{ground}\rangle_k\,. \tag{9.27}$$

We relegate the proof of this statement to Appendix C.

Rather remarkably, the resulting sum can be written in closed form. We find

$$\Theta(r) = -2\pi^2\mu r^2 \left(\frac{N}{(N-1)k'+1} \frac{{}_1F_1(1-N, 2-N-1/k', \pi\mu r^2/k')}{{}_1F_1(\ -N, 1-N-1/k', \pi\mu r^2/k')} \right) . \tag{9.28}$$

This is the ratio of confluent hypergeometric functions of the first kind.

The result (9.28) is plotted in Figure 9.2 for $N = 1000$ vortices and $k' = 3$. The plot shows clearly that, for $r < R$, the Berry phase Θ coincides with the expected Aharonov-Bohm phase (9.25) if the charge of the quasihole is taken to be

$$q_{\rm QH} = -\frac{1}{k'} .$$

This, of course, is the expected result [9, 128].

Our Berry phase computation also reveals finite size effects. The magnitude of the Berry phase reaches a maximum of $2\pi N$ at $r = R$, the edge of the droplet. Outside this disc, the Berry phase no longer increases and the picture in terms of quasiholes breaks down. One can also use the result above to determine the size of the edge effects; numerical plots reveal them to be small as long as $k' \ll N$.

There is another interpretation of the quasihole state (9.24): it is an excitation of the fundamental boson ϕ in the Hall phase (8.15). Now the Aharonov-Bohm phase arises because this particle has charge 1 under the statistical gauge field with magnetic field $f_{12} = -2\pi\mu/k'$. This is a pleasing, dual perspective. The vortices are solitons constructed from ϕ. But, equally, we see that we can reconstruct ϕ as a collective excitation of many vortices!

Fractional Statistics

Let us next consider the statistics of quasiholes as they are braided. To do this, we consider a state with two excitations, $|\eta_1, \eta_2\rangle_k$. It is simplest to place the first at the origin, $\eta_1 = 0$, and transport the second in a full circle. This is equivalent to exchanging the quasiholes twice and computes double the statistical phase. Of course, there is also a contribution from the Aharonov-Bohm phase $\Theta(r)$ described above and we must subtract this off. The resulting statistical phase $\Theta^{\rm stat}$ is then given by

$$2\Theta^{\rm stat}(r) = -i \int_0^{2\pi} {\rm d}\theta \ {}_k\langle 0, \eta| \frac{\partial}{\partial\theta} |0, \eta\rangle_k \ - \Theta(r)$$

where again $\eta = re^{i\theta}$.

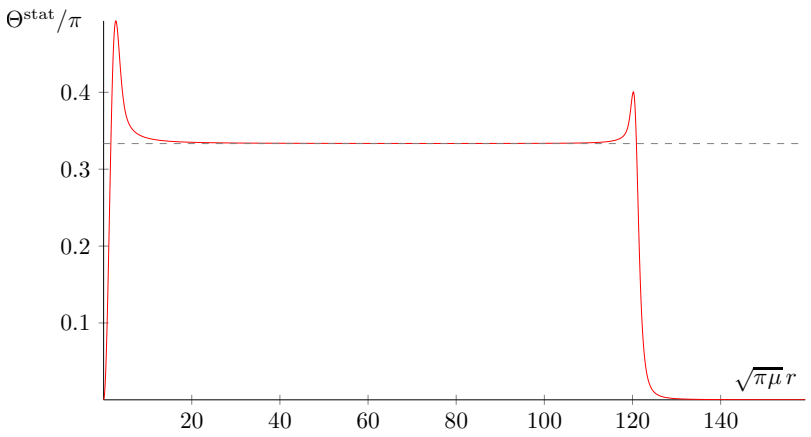


Figure 9.3: The statistical phase for a quasihole encircling a second quasihole at the origin for $N = 5000$ and $k' = 3$. The Berry phase Θ^{stat} (solid, red) is plotted, together with the expected phase for a particle of statistics π/k' (grey, dashed).

To compute the statistical phase, we need yet more inner products. We define the states

$$|\Omega_{0,l}\rangle_k = \left[\epsilon^{a_1\cdots a_N}(\varphi^\dagger Z^\dagger)_{a_1}(\varphi^\dagger Z^{\dagger\,2})_{a_2}\cdots \quad (\varphi^\dagger Z^{\dagger\,l})_{a_l}(\varphi^\dagger Z^{\dagger\,l+2})_{a_{l+1}}\cdots(\varphi^\dagger Z^{\dagger\,N+1})_{a_N}\right] \left[\epsilon^{b_1\cdots b_N}\varphi^\dagger_{b_1}(\varphi^\dagger Z^\dagger)_{b_2}\cdots(\varphi^\dagger Z^{\dagger\,N-1})_{b_N}\right]^{k-1}|0\rangle\,.$$

This is similar to $|\Omega_l\rangle_k$, defined previously, except now each factor of $Z^\dagger$ has been increased by 1. This is the effect of placing the extra quasihole at the origin. (For more general locations of the quasihole, we would need the obvious generalizations of these states $|\Omega_{l',l}\rangle_k$.) The states $|\Omega_{0,l}\rangle_k$ are again orthogonal. This time, we find the norm is given by

$$\frac{{}_k\langle\Omega_{0,l}|\Omega_{0,l}\rangle_k}{{}_{k'}\langle\text{ground}|\text{ground}\rangle_{k'}} = (\pi\mu)^{l-2N}\binom{N}{l}\left[\prod_{a=0}^{N-l-1}(k'a+1)\right] \times\left[\prod_{a=0}^{l-1}(k'a+1)\right]\left[\prod_{a=l}^{N-1}(k'a+2)\right]. \tag{9.29}$$

With these functions, it is straightforward to determine an expression for the statistical phase in terms of a sum over N states. Once again, this sum has a closed form, this time given using regularized hypergeometric functions by

$$2\Theta^{\text{stat}}(r) = \frac{2\pi^2\mu r^2}{k'}\left(N\frac{{}_2\tilde{F}_2(1+1/k',1-N;1+2/k',2-N-1/k';\,\pi\mu r^2/k')}{{}_2\tilde{F}_2(\quad 1/k',\quad -N;\quad 2/k',\,1-N-1/k';\,\pi\mu r^2/k')}\right) - \Theta(r)\,.$$

We plot this for $N = 5000$ and $k' = 3$ in Figure 9.3. All other plots with $k' \ll N$ have similar features. We see that there is clearly an intermediate, parametrically large

regime, in which the pair of particles are both far from the edge of the disc and far from each other, where their exchange statistics are given by

$$\Theta^{\text{stat}} = \frac{\pi}{k'} \,.$$

This is the expected result for a quasihole at filling fraction $\nu = 1/k'$.

10 Comments

Supersymmetry has long proven a powerful tool to understand physics at strong coupling in relativistic systems. It is clear that if this power could be transported to the non-relativistic realm, then supersymmetry could be employed to say something interesting about open problems in condensed matter physics.

In this spirit, there have been a number of recent papers in which mirror symmetry (which can be viewed as an exact particle-vortex duality in $d = 2+1$ interacting systems) has been explored in the presence of external sources. This has been used to study impurities [172, 173], non-Fermi liquids [174] and the physics of the lowest Landau level [175]. It would be interesting to follow the fate of mirror pairs (or Seiberg duals) under the non-relativistic limit.

Crucially, in our model, we have seen that it is possible for supersymmetry to provide a guide to understanding toy models even if we simply disregard their SUSY completion. The line of enquiry which supersymmetry guided us to in Part III is the one which we will continue to follow in Part IV, making very little further reference to supersymmetry. It is essentially relegated to a purely advisory role by the non-relativistic limit.

(Indeed, at least one other principle could guide one to the theory we used: recall that – without the chemical potential – it has conformal symmetry at this point. One perspective is that this is slightly less natural since we are ultimately interested in a non-conformal deformation of the theory. A more pragmatic reason is that it is typically easier to use supersymmetry to construct and analyse these theories.)

The Laughlin physics we have described so far, of course, is just the tip of the quantum Hall iceberg. A long-standing open problem has been how to generalize the quantum Hall matrix model of [19] to more general filling fractions such as the Jain hierarchy. (See [161] for an attempt.) But our perspective offers an approach. It is known that the most general Abelian quantum Hall state can be captured by the K-matrix approach [176], with an effective field theory given by several coupled Chern-Simons fields

$$L = \frac{1}{4\pi} K_{IJ} \epsilon^{\mu\nu\rho} a^I_\mu \partial_\nu a^J_\rho + \frac{1}{2\pi} A_\mu t_I \epsilon^{\mu\nu\rho} \partial_\nu a^I_\rho + \dots .$$

It is a simple matter to generalize this to a non-relativistic (and, if we wanted, supersymmetric) theory. However, the dynamics of vortices in these theories have not been

well studied. A matrix model for the vortex dynamics in these theories would presumably furnish a description of the most general Abelian quantum Hall states. (A matrix model for vortices in a class of theories with product gauge groups was proposed in [173, 177].) These are under investigation, but are not reported upon in this dissertation.

Another natural generalization – and the topic of this dissertation in Parts IV and V – is to look at vortices in non-Abelian $U(p)$ gauge theories. These were introduced in [155, 178]. The vortices now have an internal degree of freedom and the moduli space is given by

$$\pi\mu[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = k' \mathbf{1}_N$$

modulo $U(N)$ gauge transformations. Models like this have been previously discussed in the context of quantum Hall physics in [179, 180], but with rather different interpretations and approaches to those we have followed here. We would like to investigate which quantum Hall states these models describe.

Finally, we alluded in the D-brane derivation of the matrix model, to the famous ADHM construction [181]. This is an exact description of the moduli space of instantons in four dimensions, and the matrix model we have discussed is, loosely speaking, one half of the ADHM model. To be more precise, whereas we have a complex adjoint scalar Z and a complex fundamental ϕ, the fields of the ADHM model are doubled up: we have adjoint scalars Z, W and a fundamental and an antifundamental $\phi, \tilde{\phi}$. The pair (Z, W) transforms as a doublet under an $SU(2)_L$, and $(Z, W^\dagger)$ and $(\phi, \tilde{\phi}^\dagger)$ are $SU(2)_R$ doublets. Together, $SU(2)_L \times SU(2)_R/\mathbb{Z}_2 \cong SO(4)$ form the rotational symmetry group of four-dimensional Euclidean space.

This raises the interesting question of whether there is a (4+1)-dimensional version of the story we have been discussing. The answer seems to be yes: there is an appropriate non-relativistic theory in 5 dimensions which seems to have a moduli space of instanton solutions described by a non-relativistic matrix model based around the ADHM one. Rather elegantly, the notion of chirality in the conventional Hall effect is then replaced with the notion of chirality associated with the $SU(2)_L \times SU(2)_R$ decomposition of four dimensional rotations – just as the kinetic terms in the matrix model (9.13) naturally dictate a preferred direction of rotation under the $U(1)$ rotation group (for a given k), here, the kinetic terms break the $SU(2)_R$ symmetry, dictating that the left-handed symmetry is preferred. Similarly, just as the Chern-Simons term carries a choice of sign, in the 4+1 dimensional theory, the gauge theory kinetic terms requires a choice of self-dual or anti-self-dual configurations.

It would be interesting to use the approach presented here to further the study of such higher-dimensional Hall effects, a class of phenomena discussed in [182]. In particular, there is an emergent chiral 3+1 dimensional boundary theory to find, which would be very interesting.

PART IV

Non-Abelian Models

11 Introduction and Summary

Up until this point, we have been exploring in detail an old matrix model for the Laughlin states, introduced by Polychronakos [19], and inspired by earlier work [20]. However, the approach we have followed is far from limited to this case. Recall that the Chern-Simons theory studied in Part III was Abelian. This suggests a natural generalization: we should look at non-Abelian Chern-Simons theories.

Indeed, it turns out that non-Abelian gauge groups give rise to non-Abelian statistics for quasiholes in the quantum Hall phase. As we discuss below, pursuing this idea leads us directly to a wide variety of interesting electron states. They are classified in a natural and simple way by the Chern-Simons rank and level(s).

The new thing here is that the vortices which these non-Abelian theories sustain now carry spin [155, 178]. In Chapter 14, we will explain in analogy to our preceding work how these vortices give rise to a new, non-Abelian matrix model. However, we will postpone this until after Chapters 12 and 13 which contain respectively discussions of the matrix models themselves and the non-Abelian states they host.

A Class of Non-Abelian Quantum Hall States

Before we describe the role played by the matrix model, we first summarize some properties of the non-Abelian Hall states that will emerge.

The original Moore-Read state [16], and its extension to the series of Read-Rezayi states [86], describe spin polarized electrons. There are, however, a number of prominent non-Abelian Hall states in which the electrons carry an internal spin degree of freedom [183, 184, 185]. Typically, the quantum Hall ground states are singlets under the spin symmetry group. It is this kind of *non-Abelian spin-singlet state* which we shall investigate, although as we shall see there are interesting relationships between spin-singlet and spin-polarized states.

In the context of quantum Hall physics, the "spin" degrees of freedom can be more general than the elementary spin of the electron. For example, in bilayer systems the layer index plays a similar role to the spin degree of freedom and is sometimes referred to as a "pseudospin". In other systems, the electrons may carry more than two internal states. This occurs, for example, in graphene where one should include both spin and

valley degrees of freedom [186]. Here, we will consider systems in which each particle carries some number of internal states. This will include situations in which these states transform in a higher representation of $SU(2)$, but also situations in which the states transform under a general $SU(p)$ group. In all cases, we will refer to these internal states simply as the "spin" degrees of freedom of the particle.

When the symmetry group is $SU(2)$, one can construct non-Abelian spin-singlet states starting from the familiar Abelian (m, m, n) Halperin states [187]. It is well known that when particles carry spin $s = \frac{1}{2}$, only the Halperin states with $m = n + 1$ are spin-singlets [188]. Apparently less well-known is the statement that for particles carrying spin s, the (m, m, n) states, suitably interpreted, are spin-singlets when $m = n + 2s$. Moreover, the presence of the spin degrees of freedom changes the universality class of these states and, for $s > \frac{1}{2}$, they have non-Abelian topological order. In particular, when the particles have spin $s = 1$, it is possible to rewrite these states in Pfaffian form and they lie in the same universality class as the Moore-Read states.

When the symmetry group is $SU(p)$, the obvious $(m, \ldots, m, n \ldots, n)$ generalization of the Halperin states can again be used as the foundation to build non-Abelian states. When $m - n = 1$, these are spin-singlets if each particle transforms in the fundamental representation of $SU(p)$. More generally, when $m - n = k$ one can build spin-singlets if each particle transforms in the k^{th} symmetric representation of $SU(p)$.

The states that arise in this way are not novel. They were first introduced many years ago by Blok and Wen [183], albeit using the rather different construction of conformal blocks in an $SU(p)_k$ WZW model. The states have filling fraction

$$\nu = \frac{p}{k + np} \tag{11.1}$$

with p and k positive integers determined by the spin group and its representation, and n an arbitrary positive integer. For $p = 1$, these are simply the Laughlin states. For $p = k = 2$, these are spin-singlet generalizations of the Moore-Read states. For $p > 2$ and $k = 2$, these are spin-singlet generalizations of the Read-Rezayi states.

Chern-Simons Theories and Matrix Models

The effective description of the Blok-Wen states is a non-Abelian Chern-Simons theory. The gauge group and levels are given by

$$U(p)_{k,k+np} = \frac{U(1)_{(k+np)p} \times SU(p)_k}{\mathbb{Z}_p}\,. \tag{11.2}$$

The allowed level of the $U(1)$ factor is strongly constrained by the fact that this is a $U(p)$ rather than $U(1) \times SU(p)$ theory [189].

Viewed in a certain slant of light, the Blok-Wen states are the most natural non-Abelian quantum Hall states. Let us take a quick aside to explain this. The long-distance physics of all non-Abelian quantum Hall states is described by some variant of non-Abelian Chern-Simons theories. This means, of course, that Wilson lines in this theory carry some representation under the non-Abelian group which, for us, is $SU(p)$. The corresponding "colour" degrees of freedom are then interpreted as spin degrees of freedom of the underlying electron. This, in essence, is why non-Abelian quantum Hall states arise naturally from particles carrying internal spin.

In contrast, if one wants to describe the long-distance physics of spin-polarized non-Abelian Hall states, such as those of [16, 86], one must work somewhat harder. This involves the introduction of yet further quotients of the 3d Chern-Simons theory [189] to eliminate the spin degrees of freedom. This is the sense in which the Blok-Wen states are particularly natural[1].

Drawing on our experience from Part III, we are now in a position to guess how the matrix model arises. The electrons in the quantum Hall system correspond to vortices of the $U(p)$ Chern-Simons theory. The $U(N)$ matrix model is simply the description of the microscopic dynamics of N of these vortices. We only offer a construction of this matrix model for the choice $n = 1$ in (11.1) and (11.2); it seems likely that some generalization is possible, however. It is to be expected that quantizing these vortices results in the quantum Hall ground state. The matrix model provides the technology to do this explicitly.

The novelty in non-Abelian gauge theories is that the vortices are endowed with an internal orientation – spin degrees of freedom – as first explained in [155, 178]. We will show that this results in the non-Abelian quantum Hall states described above. (An earlier, somewhat orthogonal attempt to describe a quantum Hall fluid of non-Abelian vortices was made in [180].)

Plan of Attack

Part IV is written in a somewhat different order from the preceding introduction. In Chapter 12, we introduce the matrix model but do not explain its Chern-Simons origins. Instead, we will take the matrix model as the starting point and show that it

[1]Things look somewhat different when viewed from the boundary perspective. The same quotient that appears complicated in the 3d bulk can result in a very simple boundary theory, such as the Ising [16] or parafermion [86] conformal field theories.

describes particles with spin moving in the lowest Landau level. We will see that, upon quantization, the ground state lies in the same universality class as the non-Abelian quantum Hall states previously introduced by Blok and Wen [183].

In Chapter 13, we describe in some detail the Blok-Wen wavefunctions and their construction from spin generalizations of the Halperin-type states. We show, in particular, how they describe spin-singlet generalizations of the Moore-Read [16] and Read-Rezayi [86] states. One rather cute fact is that the Read-Rezayi states arise in this picture from $SU(p)_2$ Chern-Simons theory; this is related by level-rank duality to the more familiar $SU(2)_p$ coset constructions.

In Chapter 14, we return to the origin of the matrix model. We explain how it captures the dynamics of vortices in a Chern-Simons theory with gauge group (11.2). We then explore what the ideas of bosonization we discussed back in Chapter 6 have to say about these non-Abelian Chern-Simons theories. The upshot is that we will find an interesting fermionic dual of the vortex Chern-Simons theory, in which the vortices are replaced with baryons.

Finally, in Chapter 15, we complete the circle of ideas. We confirm that the matrix model wavefunctions, derived from the Chern-Simons theory, can be reconstructed as correlation functions in the boundary WZW model with algebra (11.2). In Part V, we will make the connection between the matrix model and the WZW model more direct: indeed, we will show how to construct the WZW currents in the matrix model, and show that the partition functions agree as $N \to \infty$ in the matrix model.

12 The Quantum Hall Matrix Model

The purpose of this chapter is to study a matrix model description of non-Abelian quantum Hall states. The model will describe N particles which we refer to as "electrons".

The matrix model is a $U(N)$ gauged quantum mechanics, with a gauge field which we denote as α. This gauge field is coupled to an $N \times N$ complex matrix Z, together with a set of N-dimensional vectors φ_i which are labelled by an index $i = 1, \ldots, p$. These transform under the gauge symmetry as

$$Z \to UZU^\dagger \quad \text{and} \quad \varphi_i \to U\varphi_i \quad \text{for} \quad U \in U(N)\,. \tag{12.1}$$

The dynamics is governed by the first-order action

$$S = \int \mathrm{d}t\, \frac{iB}{2}\, \mathrm{Tr}\, \left(Z^\dagger \mathcal{D}_t Z\right) + i\sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\, \mathrm{Tr}\, \alpha - \frac{\omega B}{2}\, \mathrm{Tr}\, Z^\dagger Z \tag{12.2}$$

with $\mathcal{D}_t Z = \partial_t Z - i[\alpha, Z]$ and $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - i\alpha\varphi_i$.

The action depends on three parameters: B, ω and k. We will see below that B is interpreted as the background magnetic field in which the electrons move, while ω is the strength of a harmonic trap which encourages the electrons to cluster close to the origin. Finally k, which appears in the combination $k' \equiv k + p$, is the coefficient of the quantum mechanical Chern-Simons term. Gauge invariance requires that k is an integer and we will further take it to be positive: $k \in \mathbb{N}$.

In addition to the $U(N)$ gauge symmetry, our model also enjoys an $SU(p)$ global symmetry, under which the φ_i rotate. When $p = 1$, (12.2) reduces to the action (9.14) studied in Part III. The model with general p was previously discussed in [179], albeit with a different interpretation from that offered here.

Getting a Feel for the Matrix Model

To gain some intuition for the physics underlying (12.2), let's first look at the example of a single particle. In this case $N = 1$ and so our matrix model is an Abelian $U(1)$ gauge theory, with dynamics

$$S_{N=1} = \int \mathrm{d}t\, \frac{iB}{2} Z^\dagger \dot{Z} + \sum_{i=1}^{p} i\varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p)\alpha - \frac{\omega B}{2} Z^\dagger Z\,.$$

In this case, the Z field decouples; the kinetic term, which is first order in time, describes the low-energy dynamics of an electron moving in a large external magnetic field B. When we come to the quantum theory, this will translate into the statement that the electron lies in the lowest Landau level. The term proportional to ω provides a harmonic trap for the electron.

Meanwhile, the φ_i variables describe the internal degrees of freedom of the electron. To see this, note that the equation of motion for α requires that $\sum_i |\varphi_i|^2 = k + p$ is constant. After dividing out by $U(1)$ gauge transformations, $\varphi_i \to e^{i\theta}\varphi_i$, we see that φ_i parametrize the space $\mathbb{CP}^{p-1}$. However, the action is first order in time derivatives, which means that $\mathbb{CP}^{p-1}$ should be viewed as the *phase space* of the system, as opposed to the configuration space. This is important. Because the phase space has finite volume, the quantization of φ_i will result in a finite-dimensional internal Hilbert space for the electron. In other words, the electron carries "spin".

As we emphasized above, this usage of the word "spin" is somewhat more general than its standard meaning in condensed matter physics (or high energy physics for that matter). Usually, one thinks of spin as referring to a representation of $SU(2)$; this corresponds to the choice $p = 2$ in our model. More generally, our internal degree of freedom transforms in some representation of $SU(p)$. The choice of representation is determined by the parameter k. (We will show below that the electrons sit in the k^{th} symmetric representation of $SU(p)$; in the case of $SU(2)$, this means that they carry spin $j = k/2$.)

We learn that the $U(1)$ matrix model describes a particle carrying spin, restricted to move in the lowest Landau level. The $U(N)$ matrix model simply describes N such particles. Roughly speaking, the N eigenvalues of the matrix Z correspond to the positions of the particles although, as we will see, there is some ambiguity in this when the particles are close. More precisely, we can again look at the equation of motion for the gauge field α. This results in the $u(N)$-valued constraint

$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N \,. \tag{12.3}$$

The phase space, $\mathcal{M}$, of the theory is now the space of solutions to (12.3), modulo the gauge action (12.1). This has real dimension $\dim \mathcal{M} = 2Np$. Our task is to quantize this phase space, with the harmonic potential $H = \frac{1}{2}\omega B \ \mathrm{Tr}\ Z^\dagger Z$ providing the Hamiltonian.

12.1 Quantization

In this section, we study the quantization of our matrix model (12.2). The canonical commutation relations inherited from the action (12.2) are

$$\frac{B}{2}[Z_{ab}, Z^\dagger_{cd}] = \delta_{ad}\delta_{bc} \quad \text{and} \quad [\varphi_{i\,a}, \varphi^\dagger_{j\,b}] = \delta_{ab}\delta_{ij} \tag{12.4}$$

with $a, b = 1, \ldots, N$ and $i, j = 1, \ldots, p$. We choose a reference state $|0\rangle$ obeying

$$Z_{ab}|0\rangle = \varphi_i|0\rangle = 0\,.$$

The Hilbert space is then constructed in the usual manner by acting on $|0\rangle$ with $Z^\dagger$ and $\varphi^\dagger_i$.

However, we still need to take into account the $U(N)$ gauge symmetry. This is implemented by requiring that all physical states obey the quantum version of Gauss's law (12.3). Normal ordering the terms in the matrix commutator, this reads

$$\frac{B}{2} : [Z, Z^\dagger] : + \sum_{i=1}^{p} \varphi_i\varphi_i^\dagger = (k+p)\mathbf{1}_N\,. \tag{12.5}$$

The traceless part of this equation is interpreted as the requirement that physical states are $SU(N)$ singlets. Meanwhile, the trace of this constraint requires all physical states to carry fixed charge under $U(1) \subset U(N)$. Here there is an ordering issue. Using the commutation relations (12.4), we find

$$\sum_{a=1}^{N}\sum_{i=1}^{p} \varphi_{i\,a}\varphi^\dagger_{i\,a} = (k+p)N \quad \Rightarrow \quad \sum_{a=1}^{N}\sum_{i=1}^{p} \varphi^\dagger_{i\,a}\varphi_{i\,a} = kN\,. \tag{12.6}$$

This tells us that all physical states carry charge kN under the $U(1)$. In other words, all states in the physical Hilbert space contain precisely kN copies of $\varphi^\dagger$ acting on $|0\rangle$.

The Spin of the Particle Revisited

We can now be more precise about the internal $SU(p)$ spin carried by each particle. Setting $N = 1$, the spin states of a single particle take the form

$$|\Omega_{i_1\ldots i_k}\rangle = \varphi^\dagger_{i_1}\ldots\varphi^\dagger_{i_k}|0\rangle\,.$$

Since each operator φ_i transforms in the fundamental of $SU(p)$, the spin states $|\Omega\rangle$ transform in the k^{th} symmetric representation. In particular, for $k = 1$ the electrons carry the fundamental representation of $SU(p)$.

Our main focus will be on quantum Hall states which are $SU(p)$ spin-singlets. Some simple group theory tells us that for this to happen we must have the number of electrons N divisible by p. Indeed, we will see below that the ground states simplify in this case.

12.2 The Ground States

We have already discussed, in Part III, the ground state of the matrix model with $p = 1$ (as originally constructed in [19]). We first review this example before explaining the straightforward generalization to $p > 1$.

The $p = 1$ Ground State

When $p = 1$, the electrons carry no internal spin. The constraint (12.6) tells us that all physical states have kN operators $\varphi^\dagger$ acting on $|0\rangle$. Further, the Hamiltonian arising from (12.2) is

$$H = \frac{\omega B}{2} \text{ Tr } Z^\dagger Z \tag{12.7}$$

which simply counts the number of $Z^\dagger$ operators acting on $|0\rangle$. The route to constructing the ground state is then straightforward: we need to act with kN copies of $\varphi^\dagger$, keeping the number of $Z^\dagger$ operators to a minimum. The subtleties arise from the requirement that the physical states are invariant under $SU(N)$ gauge transformations. Since we only have $\varphi^\dagger$ operators to play with, the only way to achieve this is to construct a baryon operator of the form

$$\epsilon^{a_1 \dots a_N} (Z^{l_1} \varphi)^\dagger_{a_1} \dots (Z^{l_N} \varphi)^\dagger_{a_N} .$$

However, because φ is bosonic, the antisymmetrization inherent in $\epsilon^{a_1 \dots a_N}$ causes this operator to vanish unless all the exponents l_a are distinct. Because we pay an energy cost (12.7) for each insertion of $Z^\dagger$, it follows that the lowest energy operator is given by

$$\epsilon^{a_1 \dots a_N} (Z^0 \varphi)^\dagger_{a_1} (Z\varphi)^\dagger_{a_2} \dots (Z^{N-1} \varphi)^\dagger_{a_N} .$$

The trace constraint then tells us that the ground state is given by

$$|\text{ground}\rangle_k = \left[\epsilon^{a_1 \dots a_N} (Z^0 \varphi)^\dagger_{a_1} (Z\varphi)^\dagger_{a_2} \dots (Z^{N-1} \varphi)^\dagger_{a_N}\right]^k |0\rangle .$$

The interplay between the gauge symmetry and the Hamiltonian has resulted in the construction of a state with interesting correlations between the positions of particles, encoded in the operator Z. This is what is increasingly apparent when one writes these states in the more familiar language of N-particle wavefunctions, revealing their close

relationship to the Laughlin wavefunctions. However, as we will now see, things can get even more interesting.

Ground States with $p \geq 2$

We now turn to the ground states when the electrons carry an internal spin. We anticipated above that the states will take a simpler form when N is divisible by p. And, indeed, this is the case.

N divisible by p

When N is divisible by p, there is a unique ground state. This is an $SU(p)$ singlet. To describe the construction of this state, we first group p creation operators $\varphi_i^\dagger$ together to form the $SU(p)$ baryon operator

$$\mathcal{B}(r)^\dagger_{a_1\cdots a_p} = \epsilon^{i_1\cdots i_p}(Z^r\varphi)^\dagger_{i_1\,a_1}\cdots(Z^r\varphi)^\dagger_{i_p\,a_p}\,.$$

This is a singlet under the $SU(p)$ global symmetry, but transforms in the p^{th} antisymmetric representation of the $U(N)$ gauge symmetry. To construct an $SU(N)$ singlet with the correct $U(1)$ charge required by (12.6), we make a "baryon of baryons". The ground state is then

$$|\text{ground}\rangle_k = \left[\epsilon^{a_1\cdots a_N}\mathcal{B}(0)^\dagger_{a_1\ldots a_p}\mathcal{B}(1)^\dagger_{a_{p+1}\cdots a_{2p}}\cdots\mathcal{B}(N/p-1)^\dagger_{a_{N-p+1}\cdots a_N}\right]^k|0\rangle\,. \tag{12.8}$$

This state has energy $E = \frac{\omega k N(N-p)}{2p}$. This time the requirements of the $U(N)$ gauge invariance have resulted in interesting correlations between both position and spin degrees of freedom of the electrons. We will devote the rest of this chapter and the next to describing the structure of these states.

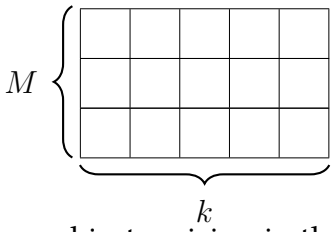
$N \equiv M \pmod p$

When N is not divisible by p, the ground state is no longer a singlet under the global $SU(p)$ symmetry. We write $N = Lp + M$ with $L, M \in \mathbb{Z}_{\geq 0}$. One can check that the ground states are

$$\begin{aligned}|\text{ground}\rangle_k \;=\; \prod_{l=1}^{k}\Big[&\epsilon^{a_1\cdots a_N}\mathcal{B}(0)^\dagger_{a_1\cdots a_p}\mathcal{B}(1)^\dagger_{a_{p+1}\cdots a_{2p}}\cdots\mathcal{B}(r-1)^\dagger_{a_{N-p-M+1}\cdots a_{N-M}}\\ &(Z^L\varphi_{i_{(l,1)}})^\dagger_{a_{N-M+1}}\cdots(Z^L\varphi_{i_{(l,q)}})^\dagger_{a_N}\Big]\,|0\rangle\end{aligned}$$

where $i_{(l,\alpha)}$, with $l = 1,\ldots,k$ and $\alpha = 1,\ldots,M$ are free indices labelling the degenerate ground states. These ground states transform in the k^{th}-fold symmetrization of the q^{th} antisymmetric representation of $SU(p)$. In terms of Young diagrams, this is the

following representation:



(12.9)

We have already seen these objects arising in the conformal theories studied in Part II. We'll see in Chapters 14 and 15 why these representations are special and might be expected to arise in quantum Hall states. In the meantime, we will primarily focus on the states (12.8) that arise when N is divisible by p.

12.3 The Wavefunctions

The description of the ground states given above is in terms of a coherent state representation for matrices. To make connections with the more traditional form of the wavefunctions, we need to find a map between the creation operators $Z^\dagger$ and the position space representation as discussed in Part III. We first briefly review the key points of the Abelian case, and then provide the generalization to the $SU(p)$ matrix model.

$p = 1$ and the Laughlin Wavefunctions

Recall that at the formal level, there was a clear similarity between the ground state for $p = 1$ theories,

$$|\text{ground}\rangle_k = \left[\epsilon^{a_1\ldots a_N}(Z^0\varphi)^\dagger_{a_1}(Z\varphi)^\dagger_{a_2}\ldots(Z^{N-1}\varphi)^\dagger_{a_N}\right]^k|0\rangle \tag{12.10}$$

and the Laughlin wavefunctions at filling fraction $\nu = 1/m$

$$\begin{aligned}\psi_m^{\text{Laughlin}}(z_a) &= \prod_{a<b}(z_a - z_b)^m e^{-B\sum|z_a|^2/4} \\ &= \left[\epsilon^{a_1\ldots a_N} z^0_{a_1} z^1_{a_2}\ldots z^{N-1}_{a_N}\right]^m e^{-B\sum|z_a|^2/4}.\end{aligned}$$

In Part III we described two strategies which essentially integrate out φ and the off-diagonal elements of Z to transform to a wavefunction representation. For $k = 0$, the wavefunctions coincided with the Slater determinant for a fully-filled Landau level

$$\langle z_a\,|\,\text{ground}\rangle_{k=0} = \prod_{a<b}(z_a - z_b)e^{-B\sum|z_a|^2/4}. \tag{12.11}$$

The exponential factor is the usual factor arising form the normalization of coherent states. The single factor of the Vandermonde determinant, which is not obvious in (12.10) when $k = 0$, is a Jacobian that arises in the transformation from matrix-valued objects to their eigenvalues.

Meanwhile, for $k \geq 1$, neither representation of the wavefunction coincided with the Laughlin state. Nonetheless, both had the property that

$$\langle z_a \,|\, \text{ground} \rangle_k \;\to\; \psi_{k+1}^{\text{Laughlin}}(z_a) \quad \text{as} \quad |z_a - z_b| \to \infty \,.$$

In other words, the wavefunctions that arise from the matrix model coincide with the Laughlin wavefunctions only at large distances. This ensures that the matrix model ground state has filling fraction

$$\nu = \frac{1}{k+1} \,.$$

However, the matrix model state differs from the Laughlin wavefunction as particles approach to within a magnetic length.

We reiterate that although the matrix model and Laughlin states differ in detail, this is not a matter of concern. There is nothing privileged about the Laughlin wavefunction: it is merely a representative of a universality class of states, characterized by their topological order – and moreover, it may be that there are other more mathematically convenient representatives. Indeed, in Part III, we explicitly saw two things: firstly, that the coherent state representation (12.10) reproduces the key aspects of Laughlin physics, and secondly, that for certain calculations it offers greater analytic control.

Wavefunctions for $p \geq 2$

For the case $p = 1$ described above, all physical states have the same dependence on $\varphi^\dagger$ excitations; they differ only in their $Z^\dagger$ excitations. This is the reason that no φ variables were needed when writing the wavefunctions. In contrast, when $p \geq 2$, different physical states can have a different structure of $\varphi_i^\dagger$ excitations. These capture the way the state transforms under the $SU(p)$ symmetry.

We repeat the procedure described above, moving from coherent state representation to wavefunction. For $k = 0$ the wavefunction knows nothing about the spin degrees of freedom. This means that the $k = 0$ wavefunction is again given by (12.11), describing a fully-filled Landau level with $\nu = 1$.

However, for $p \geq 2$ and $k > 1$, we have a new ingredient. Apart from the Vandermonde determinant (12.11), each time that a power of a particle coordinate z_a appears in the wavefunction, it is accompanied by a spin degree of freedom,

$$\sigma_a \;\in \{1, \ldots, p\}$$

where $a = 1, \ldots, N$ labels the particle. Concretely, a term in the wavefunction containing an operator $\varphi^\dagger_{ia}$ gives particle a a spin degree of freedom pointing in the i direction.

For example, when $k = 1$, the a^{th} particle has a single spin degree of freedom σ_a. This reflects the fact that, as we saw earlier, each particle transforms in the fundamental representation of $SU(p)$. More generally, the internal state of each particle is determined by k independent spin labels σ_a. As we will explain in some detail in Chapter 13, this is to be interpreted as specifying the k^{th} symmetric representation under $SU(p)$. (The symmetry simply arises from the fact that $\varphi^\dagger_{ia}\varphi^\dagger_{ja} = \varphi^\dagger_{ja}\varphi^\dagger_{ia}$.)

When N is divisible by p, the ground state wavefunction (12.8) is a an $SU(p)$ spin-singlet. The states have filling fraction

$$\nu = \frac{p}{k+p} \tag{12.12}$$

and have the property that

$$\langle z_a \,|\, \text{ground}\rangle_k \;\rightarrow\; \psi_{BW}(z_a) \quad \text{as} \quad |z_a - z_b| \rightarrow \infty$$

where $\psi_{BW}(z_a)$ are a class of non-Abelian wavefunctions constructed some time ago by Blok and Wen [183]. Like many non-Abelian quantum Hall states, the explicit description of the wavefunctions $\psi_{BW}(z_a)$ is straightforward, but somewhat fiddly. We devote the next chapter to a more detailed description of these quantum Hall states and their properties.

13 The Blok-Wen States

In this chapter, we describe the Blok-Wen wavefunctions in some detail. The original construction of [183] was in terms of conformal blocks of a $SU(p)$ WZW model and we will revisit this approach in Chapter 15. Here we provide an alternative, more down-to-earth construction of the states. We start with wavefunctions carrying spin under $SU(2)$, moving on to the more general $SU(p)$ case in Section 13.2.

13.1 Particles with $SU(2)$ Spin

The simplest examples of wavefunctions describing particles with spin are due to Halperin [187]. We take N particles, with N even, and split them into two groups of $N/2$ particles, with positions z_a and w_a where each index now runs over $a = 1, \ldots, N/2$. The (m, m, n) wavefunctions are

$$\psi(z, w) = \prod_{a<b}^{N/2}(z_a - z_b)^m \prod_{c<d}^{N/2}(w_c - w_d)^m \prod_{a,d}(z_a - w_d)^n \tag{13.1}$$

where, as throughout this chapter, we will omit the overall exponential factor common to all wavefunctions. Counting the angular momentum of particles shows that these states have filling fraction

$$\nu = \frac{2}{m+n}\,.$$

The (m, m, n) states (13.1) are really shorthand for wavefunctions with spin. As we review below, they should be dressed with explicit spin wavefunctions. This will result in the Blok-Wen states. These are actually a slightly more general class of states than those that emerge from the matrix model. We will see that the matrix model gives states with $n = 1$ and $m = k + 1$.

Usually one thinks of the Halperin states as describing spin-$\frac{1}{2}$ particles, with z_a and w_a labelling the positions of those which are spin-up and spin-down respectively. With this interpretation the $(n+1, n+1, n)$ states are spin-singlets. However, we will show that we can also view (13.1) as describing particles with spin $s > \frac{1}{2}$. This is perhaps surprising as these particles have $2s + 1$ spin states and it is not obvious how to decompose these into two groups. We will see that, with this interpretation, the (m, m, n) states are spin-singlets when $s = (m - n)/2$. Matching to the matrix model parameters, this means $s = k/2$.

Spin $\frac{1}{2}$

The standard interpretation of (13.1) is as a wavefunction for spin-$\frac{1}{2}$ particles. To dress the wavefunction with these spin states, it's useful to change notation slightly and label the positions of all N particles as z_a. Each particle carries a further internal spin degree of freedom σ_a which takes values $|\uparrow\rangle$ or $|\downarrow\rangle$. The (m, m, n) state for $m > n$ is then written as

$$\psi(z,\sigma) = \mathcal{A}\left[\prod_{a<b}^{N}(z_a - z_b)^n \prod_{a<b \text{ odd}}(z_a - z_b)^{m-n} \prod_{c<d \text{ even}}(z_c - z_d)^{m-n} \,|\uparrow\downarrow\uparrow\downarrow\ldots\uparrow\downarrow\rangle\right] \tag{13.2}$$

where $\mathcal{A}$ stands for antisymmetrization over all particles, exchanging both positions and spins. This wavefunction describes fermions for m odd and bosons for m even.

It is well known that only the states with $m - n = 1$ are spin-singlets [188]. In this case, the wavefunction factorizes as

$$\psi_{n+1,n+1,n}(z,\sigma) = \prod_{a<b}^{N}(z_a - z_b)^n \,\Phi(z,\sigma)\,.$$

This describes fermions for n even and bosons for n odd. Here the first factor takes the familiar Laughlin-Jastrow form, while the second factor is the Slater determinant of two fully filled Landau levels, one for the up spins and one for the down spins. The resulting wavefunction can be written as

$$\Phi(z,\sigma) \;=\; \mathcal{A}\Big[\prod_{a<b \text{ odd}}(z_a - z_b)\prod_{c<d \text{ even}}(z_c - z_d)\,|\uparrow_1\rangle|\downarrow_2\rangle|\uparrow_3\rangle\ldots|\downarrow_N\rangle\Big]$$

or, equivalently, as

$$\begin{aligned}\Phi(z,\sigma) \;&=\; \epsilon_{a_1\ldots a_N}(z_{a_1}z_{a_2})^0(z_{a_3}z_{a_4})^1\ldots(z_{a_{N-1}}z_{a_N})^{N/2-1} \\ &\qquad\times\Big[|\uparrow_{a_1}\rangle|\downarrow_{a_2}\rangle|\uparrow_{a_3}\rangle|\downarrow_{a_4}\rangle\ldots|\uparrow_{a_{N-1}}\rangle|\downarrow_{a_N}\rangle\Big]\,.\end{aligned} \tag{13.3}$$

In particular, this latter expression makes it clear that the spins are paired in singlet states of the form $|\uparrow_{a_1}\rangle|\downarrow_{a_2}\rangle - |\downarrow_{a_1}\rangle|\uparrow_{a_2}\rangle$.

Spin 1

So far, we have just reproduced the usual story. Suppose now that $m = n+2$. We claim that the following is a spin-singlet wavefunction for spin 1 particles,

$$\psi_{n+2,n+2,n}(z,\sigma) = \prod_{a<b}(z_a - z_b)^n\,\mathcal{P}\left[\Phi^2(z,\sigma)\right]\,. \tag{13.4}$$

This is a wavefunction for fermions when n is odd and bosons when n is even.

Our first task is to explain what this means. The factor Φ^2 includes two spin states for each particle. The tensor product of two spin 1/2 states gives $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. The operator $\mathcal{P}$ projects onto the symmetric $\mathbf{3}$. (In the present case, this operation is not required as it is implemented automatically by the form of Φ^2. However, we include it in our expression for clarity.) This means that we can interpret (13.4) as a quantum Hall state for spin 1 particles, with the map

$$|\uparrow\rangle|\uparrow\rangle = |1\rangle \quad , \quad |\downarrow\rangle|\downarrow\rangle = |-1\rangle \quad , \quad |\uparrow\rangle|\downarrow\rangle = |\downarrow\rangle|\uparrow\rangle = |0\rangle\,. \tag{13.5}$$

We further claim that (13.4) is a spin-singlet. We will first motivate this by looking at the kinds of terms that arise. We will subsequently provide a proof in the course of rewriting the wavefunction in a more familiar form.

Consider two particles, labelled 1 and 2, each of which carries spin $\frac{1}{2}$. The spin-singlet state is

$$|12\rangle_{\frac{1}{2}} = |\uparrow_1\rangle|\downarrow_2\rangle - |\downarrow_1\rangle|\uparrow_2\rangle$$

where the subscript $1/2$ is there to remind us that this is the singlet built from two spin 1/2 particles. The simplest terms that occur in (13.4) are of the form $|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}}$. Using the map (13.5), we have

$$|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}} \;\; = \;\; |1_1\rangle|-1_2\rangle + |-1_1\rangle|1_2\rangle - 2|0_1\rangle|0_2\rangle$$

which is indeed the singlet formed from two spin 1 states. To highlight this, we write the above equation as

$$|12\rangle_{\frac{1}{2}}|12\rangle_{\frac{1}{2}} = |12\rangle_1\,.$$

The next kind of term that arises in (13.4) involves four different particles. It is the cyclic term $|12\rangle_{\frac{1}{2}}|23\rangle_{\frac{1}{2}}|34\rangle_{\frac{1}{2}}|41\rangle_{\frac{1}{2}}$. We can similarly expand this in terms of spin 1 states and again find that only combinations of singlet states appear:

$$|12\rangle_{\frac{1}{2}}|23\rangle_{\frac{1}{2}}|34\rangle_{\frac{1}{2}}|41\rangle_{\frac{1}{2}} = |12\rangle_1|34\rangle_1 - |13\rangle_1|24\rangle_1 + |14\rangle_1|23\rangle_1\,.$$

The most general term in (13.4) has $2n$ particles. This too can be written as the linear combinations of n spin 1 singlet states. Rather than demonstrate this term by term, we will instead show that the wavefunction (13.4) has an alternative form written purely in terms of spin 1 singlets.

The Spin 1 Wavefunction as a Pfaffian

We will now show that the wavefunction (13.4) for spin 1 particles can be written as

$$\Phi^2(z,\sigma) = \mathrm{Pf}\left(\frac{|ab\rangle_1}{z_a - z_b}\right)\prod_{a<b}(z_a - z_b) \tag{13.6}$$

with $\mathrm{Pf}(M_{ab})$ the Pfaffian of the matrix M. This is a spin-singlet version of the Moore-Read state [16]. It is sensible because the spin 1 singlet $|ab\rangle_1$ is symmetric in the two spins, in contrast to $|ab\rangle_{\frac{1}{2}}$ which is antisymmetric.

It was noticed long ago [183, 190] that the $(3,3,1)$ state is closely related to the Pfaffian state. In [183] the particles were spin-1 but projected onto the $m = 0$ spin component; in [190] the particles were taken to be spin $1/2$ and the resulting state was not a spin-singlet. Our result (13.6) is clearly closely related to these earlier results, both of which are proven using the Cauchy identity. However, the proof of (13.6) requires more sophisticated machinery which appears not to have been available at the time of [183, 190].

The Proof:

The projective Hilbert space associated to the two spins is a Bloch sphere $\mathbb{CP}^1$. We parametrize this by the inhomogeneous coordinate ζ. Formally, we then set $|\downarrow_a\rangle = 1$ and $|\uparrow_a\rangle = \zeta_a$ and write Φ as the polynomial

$$\Phi(z,\zeta) = \frac{1}{2^{N/2}}\epsilon_{a_1\ldots a_N}\left[(z_{a_1}z_{a_2})^0\ldots(z_{a_{N-1}}z_{a_N})^{N/2-1}\right]\left[(\zeta_{a_1}-\zeta_{a_2})\ldots(\zeta_{a_{N-1}}-\zeta_{a_N})\right].$$

This has the advantage that the right-hand-side can be viewed as the determinant of a $N\times N$ matrix $\Delta[z;\zeta]$ with components given by

$$\Delta[z;\zeta]_{a,b} = \begin{cases} z_a^{b-1} & 1\le j\le \frac{N}{2} \\ \zeta_a z_a^{b-1} & \frac{N}{2}+1\le j\le 2N \end{cases} \tag{13.7}$$

To show the result (13.6), we then need to prove the polynomial identity

$$\det{}^2\Delta[z;\zeta] \stackrel{?}{=} \mathrm{Pf}\left(\frac{(\zeta_a-\zeta_b)^2}{z_a-z_b}\right)\prod_{a<b}(z_a-z_b)\,.$$

In fact, this identity is a special case of a more general result proven in [191]. Theorem 2.4 of this paper shows (among other things) that two matrices $\Delta[z;\zeta]$ and $\Delta[z;\eta]$, each defined by (13.7), obey the relation

$$\det\Delta[z;\zeta]\ \det\Delta[z;\eta] = \mathrm{Pf}\left(\frac{(\zeta_b-\zeta_a)(\eta_b-\eta_a)}{z_b-z_a}\right)\det(z_a^{b-1})\,.$$

Setting $\zeta_a = \eta_a$ yields the desired result. □

Higher Spin

The generalization to higher spins is now obvious. We construct the wavefunction

$$\psi_{n+2s,n+2s,n}(z,\sigma) = \prod_{a<b}(z_a - z_b)^n \, \mathcal{P}\left[\Phi^{2s}(z,\sigma)\right] \tag{13.8}$$

where $\mathcal{P}$ is there to remind us that the spin states for each particle are projected onto the fully symmetrized product. This means that this is a wavefunction for particles with spin s. Once again, the final state is a spin-singlet. This follows from some trivial group theory. The infinitesimal action of $SU(2)$ on the tensor product of $2s$ spin states is

$$T^\alpha = \sum_{a=1}^{N} t_a^\alpha \otimes \mathbf{1} \otimes \ldots \otimes \mathbf{1} + \text{symmetric}$$

where t_a^α is the operator in the fundamental representation acting on the a^{th} particle, and $\alpha = 1, 2, 3$ labels the three $su(2)$ generators. Because $\mathcal{P}$ projects onto an irrep, we have

$$T^\alpha \mathcal{P}\left[\Phi^2 s\right] = \mathcal{P}\left[\sum_a t_a^\alpha \Phi \otimes \Phi \otimes \ldots \otimes \Phi\right] + \text{symmetric}\,.$$

But each of these terms vanishes because Φ is itself a spin-singlet, which means that $\sum_a t_a^\alpha \Phi = 0$. This ensures that (13.8) is indeed a spin-singlet.

Although (13.8) provides an explicit description of the state, it would be pleasing to find a simple expression purely in terms of the singlets $|ab\rangle_s$, analogous to the Pfaffian (13.6) for $s = 1$. We have not been able to do this; it may simply not be possible due to the entanglement structure between higher numbers of spins.

While the Halperin states (13.1) describe Abelian quantum Hall states, our spin-singlet states (13.6) and (13.8) with spin $s \geq 1$ are all non-Abelian quantum Hall states. Indeed, it has long been known that dressing a quantum Hall state with spin degrees of freedom can change the universality class of the state. We will see in Chapter 15 that these states are associated to $SU(2)_{2s}$ WZW models.

13.2 Particles with $SU(p)$ Spin

We now generalize these ideas to particles that carry a "spin" under the group $SU(p)$. This means that each particle carries an internal Hilbert space which transforms under a particular representation of $SU(p)$.

The starting point is the p-component generalization of the Halperin states (13.1). We take N particles and split them into p groups, with positions $w_{i\,a}$, where $i = 1, \ldots, p$ and $a = 1, \ldots, N/p$. Then

$$\psi_{m,n}(z) = \left[\prod_{i=1}^{p}\prod_{a<b}^{N/p}(w_{i\,a} - w_{i\,b})^m\right]\left[\prod_{i<j}^{p}\prod_{c,d}^{N/p}(w_{i\,c} - w_{j\,d})^n\right] . \tag{13.9}$$

Multi-component states of this form were first discussed in [192]. More recently they have been studied in [193] for $p = 4$ to describe both spin and valley indices of electrons in graphene, and more generally in [194, 195]. The states (13.9) have filling fraction

$$\nu = \frac{p}{pn + (m - n)} . \tag{13.10}$$

It is natural to think of these wavefunctions as describing objects with p internal states. This corresponds to the situation where each particle sits in the fundamental representation, $\mathbf{p}$ of $SU(p)$. However, as we will see, there is also a generalization of our previous construction in which each particle has more internal states, corresponding to the symmetric representations of $SU(p)$.

Fundamental Representation

We start by describing the simplest situation where the particles sit in the fundamental representation, meaning that each carries an internal index, $\sigma_a \in \{1, 2, \ldots, p\}$. In this case, the wavefunctions (13.9) are spin-singlets when $m = n + 1$.

To see this, note that the smallest number of particles that can form a singlet state is p. To achieve this, the spin degrees of freedom are completely antisymmetrized into what we have been calling a "baryon",

$$B_{a_1\ldots a_p} = \epsilon^{\sigma_{a_1}\ldots\sigma_{a_p}}|\sigma_{a_1}\rangle \ldots |\sigma_{a_p}\rangle .$$

We can then form a spin-singlet state of type $(n + 1, n)$ by writing

$$\psi_{n+1,n}(z, \sigma) = \prod_{a<b}^{N}(z_a - z_b)^n \Phi_{(p)}(z, \sigma)$$

where, in analogy with (13.3), $\Phi_{(p)}$ describes p fully filled Landau levels, one for each type of spin,

$$\begin{aligned}\Phi_{(p)}(z, \sigma) \;&=\; \epsilon_{a_1\ldots a_N}(z_{a_1}\ldots z_{a_p})^0(z_{a_{p+1}}\ldots z_{a_{2p}})^1\ldots(z_{a_{N-p+1}}\ldots z_{a_N})^{N/p-1} \\ &\quad\times B_{a_1\ldots a_p}B_{a_{p+1}\ldots a_{2p}}\ldots B_{a_{N-p+1}\ldots a_N} .\end{aligned} \tag{13.11}$$

In the language of [86, 184], this state exhibits clustering at order p. This means that the factor $\Phi_{(p)}$ remains non-zero if the positions of up to p particles coincide. However, it vanishes if $p+1$ or more particles coincide. We will see in Chapter 15 that this wavefunction actually describes an Abelian quantum Hall state. To generate non-Abelian quantum Hall states, we need to look at higher representations of $SU(p)$.

Symmetric Representations

For $m > n+1$, we can still interpret (13.9) as a spin-singlet state, but now each particle must carry a spin in a higher representation. We define $k = m - n$ and write the wavefunction as

$$\psi_{n+k,n}(z,\sigma) = \prod_{a<b}^{N}(z_a - z_b)^n \, \mathcal{P}\left[\Phi_{(p)}^k(z,\sigma)\right] \tag{13.12}$$

where $\mathcal{P}$ projects onto the symmetrized product of spin states, meaning that each particle transforms in the k^{th} symmetric representation of $SU(p)$. These states are all spin-singlets, by the same argument that we gave in Section 13.1.

These states, still with $n = 1$, exactly reproduce the long-distance behaviour of the matrix model ground states described in the previous chapter, with the same values of k and p.

Relationship to Read-Rezayi States

When $k = 2$, our states describe particles transforming in the symmetric representation of $SU(p)$ with dimension $\frac{1}{2}p(p+1)$. They are p-clustered states with filling fraction

$$\nu = \frac{p}{pn+2}\,.$$

Both of these properties are shared by the Read-Rezayi states [86]. We will now show that our states are spin-singlet generalizations of the Read-Rezayi states.

A particularly simple form of the Read-Rezayi state was presented in [196],

$$\psi_n^{RR}(z) = \mathcal{S}\left[\prod_{i=1}^{p}\prod_{a<b}^{N/p}(w_{ia} - w_{ib})^2\right]\prod_{c<d}^{N}(z_c - z_d)^n \tag{13.13}$$

where $\mathcal{S}$ means that we symmetrize over all possible divisions of the particles into the p groups, while the z_a factor simply means that we include all particle positions rather than restricting to those in a specific group.

We will show that, after projection onto a particular spin state, the Blok-Wen wavefunction (13.12) coincides with the Read-Rezayi wavefunction (13.13), i.e.

$$\mathcal{P}\left[\Phi^k_{(p)}(z,\sigma)\right] \;\mapsto\; \mathcal{S}\left[\prod_{i=1}^{p}\prod_{a<b}^{N/p}(w_{i\,a}-w_{i\,b})^2\right] . \tag{13.14}$$

The Proof:

Our first task is to explain what $\mapsto$ means in the above expression. It's useful to first revisit the case of the Pfaffian (13.6). There, the spin-singlet wavefunction included the factor

$$|ab\rangle_1 = |1_a\rangle|{-1_b}\rangle + |{-1_a}\rangle|1_b\rangle - 2|0_a\rangle|0_b\rangle$$

and we can project onto a spin-polarized state simply by replacing $|ab\rangle_1 \mapsto |0_a\rangle|0_b\rangle$. This point was also made in [183].

Unfortunately, there is no analogous procedure for $SU(p)$ spins. There is, however, a generalization of the projection onto the highest spin states. The states $|1_a\rangle = |{\uparrow_a}\rangle|{\uparrow_a}\rangle$ and $|{-1_a}\rangle = |{\downarrow_a}\rangle|{\downarrow_a}\rangle$ have the property that both fundamental spins lie in the same direction. This is something which also makes sense for $SU(p)$ spins. We therefore define the projection $\mapsto$ in (13.13) as an operator which correlates the fundamental $SU(p)$ spins associated to each individual particle

$$\prod_{a=1}^{N}|\sigma_a\rangle|\sigma'_a\rangle \;\mapsto \prod_{a=1}^{N}\delta_{\sigma_a\sigma'_a} .$$

In particular, when we project the state $\mathcal{P}\left[\Phi^2_{(p)}(z,\sigma)\right]$, we correlate the two antisymmetrizations of spins inside the two $\Phi_{(p)}$ factors. The projection picks out the states in which these two spins associated to a given particle are the same. Spins associated to different particles can be different.

Having defined the projection, we turn to the structure of the state (13.12). It is helpful to think about collecting terms with some particular allocation of spin to each particle. For definiteness, let us consider the term where the particle at position $z_{(a-1)p+i} \equiv w_{i\,a}$ is given spin i. (Here $a = 1,\ldots,N/p$.) Now consider the polynomial in z which multiplies this spin state. It is the antisymmetrization over all ways of permuting particles at positions $w_{i\,a}$ and $w_{i\,a'}$ of

$$(w_{1\,1}\ldots w_{p\,1})^0(w_{1\,2}\ldots w_{p\,2})^1\cdots(w_{1\,N/p}\ldots w_{p\,N/p})^{N/p-1} .$$

But this is simply the product of p separate Laughlin-like factors $w^0_{a\,1} w^1_{a\,2} \cdots w^{N/p-1}_{a\,N/p}$, and hence the coefficient of the spin state is proportional to

$$\prod_{a=1}^{p} \prod_{a<b}^{N/p} (w_{ai} - w_{aj}) \tag{13.15}$$

which is essentially half of the Read-Rezayi state. To complete the argument, notice that whatever sign a particular spin allocation comes with, it comes with the same sign in both copies of $\Phi_{(p)}$. Hence overall, we obtain the square of this expression, *symmetrized* over all spin allocations. The projection does indeed result in the Read-Rezayi state (13.14). □

The Read-Rezayi states are associated to the parafermion CFT $SU(2)_k/U(1)_k$. Meanwhile, the states analysed here are associated to $SU(k)_2$. The two are related by level-rank duality. This means that our states include the non-Abelian anyons of the Read-Rezayi state; for example, $SU(3)_2$ includes the Fibonacci anyons. We will see how these emerge in Chapter 15 when we review the connection to conformal field theory. However, the Blok-Wen spin-singlet states arise from a CFT with no quotient, and hence contain additional anyonic degrees of freedom that are not part of the Read-Rezayi sequence of states.

14 Two Chern-Simons Theories

In Chapters 12 and 13, we've focussed only on the properties of the matrix model (12.2). In Section 14.1 of this chapter, we will generalize the work of Part III, and explain how this matrix model describes the dynamics of vortices in a (2+1)-dimensional Chern-Simons theory, but now one which is *non-Abelian*.

The exciting thing about our return to Chern-Simons theory is that the machinery of bosonization which we introduced in Section 6 becomes available. This will be the topic of Section 14.2, where we will introduce a fermionic description of quantum Hall physics.

14.1 The Bosonic Chern-Simons Theory

Our starting point is a Chern-Simons theory with gauge group

$$U(p)_{k,k'} = \frac{U(1)_{k'p} \times SU(p)_k}{\mathbb{Z}_p} \,. \tag{14.1}$$

Recall that the $\mathbb{Z}_p$ quotient places a strong restriction on the allowed values of k' which must obey

$$k' - k \in p\mathbb{Z} \,. \tag{14.2}$$

We denote the $U(1)$ gauge field as $\tilde{a}$ and the $SU(p)$ gauge field as a. (Note the slight switch of notation from Part III, where we denoted the $U(1)$ field by a as there was no $SU(p)$ part.) As before, both are to be thought of as emergent gauge fields in the condensed matter system. Their dynamics is governed by the Chern-Simons action

$$S_{CS} = -\int \mathrm{d}^3x \, \frac{k'p}{4\pi}\epsilon^{\mu\nu\rho}\tilde{a}_\mu\partial_\nu\tilde{a}_\rho + \frac{k}{4\pi}\,\mathrm{Tr}\;\epsilon^{\mu\nu\rho}(a_\mu\partial_\nu a_\rho - \frac{2i}{3}a_\mu a_\nu a_\rho) \,.$$

To this we couple non-relativistic matter. We consider N_f bosons ϕ_i, with $i = 1, \ldots, N_f$, each transforming in the $\mathbf{p}$ of $SU(p)$, with charge 1 under the $U(1)$. Their action is

$$S_{\text{matter}} = \int \mathrm{d}^3x \; i\phi_i^\dagger\mathcal{D}_0\phi_i - \frac{1}{2m}\mathcal{D}_n\phi_i^\dagger\mathcal{D}_n\phi_i - \frac{\pi}{mk'p}(\phi_i^\dagger\phi_i)^2 - \frac{\pi}{mk}(\phi_i^\dagger t^\alpha\phi_i)^2 \,.$$

Here the subscripts $\mu, \nu, \rho = 0, 1, 2$ are spacetime indices while $n = 1, 2$ is a spatial index only. The $SU(p)$ generators t^α are in the fundamental representation.

The coefficients of the ϕ^4 terms – which each describe the strength of a delta-function interaction between particles – are again "fine-tuned" to be proportional to the Chern-Simons levels $1/k'$ and $1/k$. This is the by now familiar conformal, supersymmetric RG endpoint.

The full action is then

$$S_{\text{3d}} = S_{\text{CS}} + S_{\text{matter}} - \int \mathrm{d}^3x \; \mu p \tilde{a}_0$$

where we've introduced a background charge μ. (We have not bothered to include the $\mu\mathcal{N}$ term which would be required by supersymmetry, since it does not affect the dynamics.) This causes the scalars to condense in the vacuum, breaking the gauge symmetry. This symmetry breaking is complete whenever $N_f \geq p$. In what follows we will take $N_f = p$. There is a unique ground state of the theory given by

$$\phi_i^a = \sqrt{\mu}\, \delta_i^a$$

with $a = 1, \ldots, p$ the gauge index and $i = 1, \ldots, p$ the flavour index. In this vacuum, the gauge and flavour symmetries are broken according to the pattern

$$U(1)_{\text{ gauge}} \times SU(p)_{\text{ gauge}} \times SU(p)_{\text{ flavour}} \;\longrightarrow\; SU(p)_{\text{ diag}} \,. \tag{14.3}$$

The low-energy physics of this broken phase is *not* that of a quantum Hall fluid. However, this can change in the presence of vortices: we need to once more create a Hall droplet.

Vortices

The symmetry breaking pattern (14.3) allows for the existence of vortex excitations in which the phase of ϕ winds. These have a rather nice property in this theory. The fine-tuning of the potential term described above means that vortices lie at the "Bogomolnyi point"; they satisfy first-order differential equations, rather than second order ones. The vortex equations are

$$\tilde{f}_{12} = \frac{2\pi}{k'p}\left(|\phi_i|^2 - p\mu\right) \quad , \quad f_{12}^\alpha = \frac{2\pi}{k}\phi_i^\dagger t^\alpha \phi_i \quad , \quad \mathcal{D}_z\phi_i = 0 \tag{14.4}$$

where the first and second of these equations are Gauss's law for the Abelian and non-Abelian gauge field respectively.

These equations coincide with the vortex equations that arise in certain non-Abelian relativistic theories [155, 178]. Their properties have been studied in some detail over the years (see, for example, [197, 198, 199]), especially in the case $k' = k$, which is con-

sistent with (14.2), where $U(1)$ and $SU(p)$ gauge fields naturally combine into a $U(p)$ gauge fields with the same level, meaning that Gauss's law reads

$$f_{12} = \frac{2\pi}{k}\left(\phi_i\phi_i^\dagger - \mu\right) .$$

The most striking fact about these equations is that, as with the Abelian case, they do not have a unique solution. Instead, for N vortices the most general solution has $2pN$ parameters [155]. (This is shown by index theorem techniques, generalizing previous results for Abelian vortices [146, 147].) These parameters can be thought of as labelling the positions and internal orientations of the N vortices. In particular, there are no forces between vortices. They can sit anywhere on the plane.

We can pick out a unique solution by adding an external harmonic trap. We again choose a trap which, when evaluated on vortices, is proportional to their angular momentum

$$V_{\text{trap}} = -\omega \int \mathrm{d}^2x \, \frac{\mu|z|^2}{2}\tilde{f}_{12} .$$

This, of course, changes the equations of motion. The previous vortex solutions now precess around the origin. There is a unique, stationary, lowest-energy state which occurs when all vortices coalesce at the origin to form a rotationally invariant configuration.

For a large number of vortices N the solution looks like a disc of radius

$$R \approx \sqrt{\frac{k'N}{\pi\mu p}} .$$

Inside this disc, the scalar fields vanish, $\phi \approx 0$ and Gauss's law is satisfied by the presence of a constant magnetic flux $\tilde{f}_{12} \approx -2\pi\mu/k'$. The end result is that we have manufactured a disc shaped region, inside of which lives an unbroken Chern-Simons theory with $U(p)$ gauge group (14.1). We view this as a region of quantum Hall fluid.

Quantizing the Vortices

We can start with a few simple observations. Our (2+1)-dimensional theory has a background charge density μ. As we saw in Part III, from the perspective of the vortices, this looks like an effective external magnetic field of strength

$$B = 2\pi\mu .$$

With this information, we can estimate the filling fraction of the quantum Hall fluid of vortices. In an area $A = \pi R^2$, the number of states in the lowest Landau level is $BA/2\pi = k'N/p$. Since we have filled this disc with N vortices, we expect a filling fraction

$$\nu = \frac{p}{k'} \,. \tag{14.5}$$

To understand the detailed properties of this Hall fluid, we must look in more detail at the microscopic dynamics of the vortices. As we mentioned above, in the absence of a trap, the most general solution to the vortex equations (14.4) has $2pN$ collective coordinates. We think of these as parametrizing the moduli space, a manifold we call $\mathcal{M}_{p,N}$. Each point on the moduli space corresponds to a different solution. Again, since the action is first-order rather than second-order in time derivatives, the moduli space $\mathcal{M}_{p,N}$ should be thought of as the *phase space* of vortices rather than the configuration space.

As with the Abelian theory discussed above, an expression for the first-order dynamics of these vortices appears in the literature [138]; but unfortunately we have the same issue as before: this result is somewhat abstract and, for a large number of closely packed vortices, not particularly useful.

Instead, we turn to a more versatile construction of the non-Abelian vortex moduli space $\mathcal{M}_{p,N}$ first derived in [155] using D-brane techniques. We introduce a complex $N \times N$ matrix Z and p complex N-vectors φ_i, $i = 1, \ldots, p$. Then the vortex moduli space is isomorphic to the space of solutions to

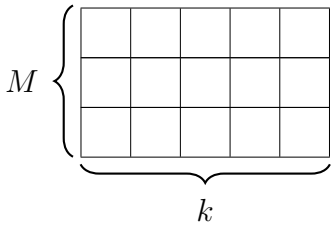
$$\frac{B}{2}[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = k' p \mathbf{1}_N$$

with solutions identified if they are related by $Z \to UZU^\dagger$ and $\varphi_i \to U\varphi_i$ where $U \in U(N)$. This, of course, is precisely the phase space of the matrix model (12.2), with the constraint above arising as Gauss's law (12.3). Moreover, the dynamics of the matrix model coincides with the dynamics expected on the vortex moduli space.

Again, the phase space of the matrix model and the vortex moduli space $\mathcal{M}_{p,N}$ are believed to coincide as complex manifolds, with the same Kähler class – but again, the symplectic form on the phase space inherited from the quotient construction does not coincide with that associated to vortex dynamics. This means that the matrix model should be used with some caution in extracting detailed properties of the vortices. However, our interest still lies in the universality class of the quantum Hall ground states and here the matrix model is expected to give the right answer. Indeed, we've seen that the ground state of the matrix model lies in the same universality class as the

Blok-Wen states. In the next chapter, we will confirm that these Blok-Wen states are indeed the ground state wavefunctions associated to the Chern-Simons theory (14.1).

It is simple to check that a single vortex transforms in the k^{th} symmetric representation of the $SU(p)_{\text{diag}}$ symmetry (14.10). This means that the vortex carries an internal "spin" degree of freedom; the wavefunction will depend on both the position z and the spin σ of each vortex. If we place N vortices in a harmonic trap, then the representation of the resulting ground state depends on the value of N modulo p. Writing $N \equiv M \pmod p$, the configuration of vortices transforms in the in the k^{th} symmetrization of the M^{th} antisymmetric representation. In terms of Young diagrams, this is



(14.6)

which agrees precisely with (12.9). In particular, when N is divisible by p the ground state is a singlet under $SU(p)$.

There is one final subtlety. Classically, the matrix model describes the dynamics of vortices when the $U(1)$ and $SU(p)$ levels are equal: $k' = k$. However, as we discussed back in Chapter 6, quantum effects lead to a shift of the level. In the 3d Chern-Simons theory, the $SU(p)$ level is renormalized at one-loop to $k \to k + p$. The matrix model captures the quantum dynamics when these shifted levels coincide. This requires

$$k' = k + p$$

which also satisfies the requirement (14.2). This is the value that we've used in (12.2) (and indeed throughout Chapter 12). In particular, we see that the filling fraction (14.5) becomes $\nu = p/(k+p)$ in agreement with the matrix model result (12.12).

(This also finally explains why in Part III, with $p = 1$, we bothered to give separate names to k and $k' = k + 1$.)

14.2 Bosonization in the Hall Regime

In Chapter 6, we discussed how certain beautiful dualities of relativistic Chern-Simons-matter theories manifested themselves in the non-relativistic spectrum of anyons. But the theories we are now investigating are of course simple deformations of the bosonic half of those dualities, so one might quite reasonably ask what happens to said duality.

There are two perspectives on our work here. One is that we are providing evidence for these bosonization dualities by studying each theory in the quantum Hall regime. Another is that we are looking for new, dual descriptions of the quantum Hall effect. Both seem interesting.

Let us begin by writing down the theories we shall be interested in:

Theory A: $U(p)_{k,\,k+np}$ coupled to N_f fundamental scalars.

Theory B: $U(k)_{-p+N_f}$ coupled to N_f fundamental fermions and, through a BF coupling, to $U(1)_n$.

They differ slightly from the pair in Chapter 6 in two respects:

- We are sticking with the Yang-Mills regularization in this section, so the apparent renormalization of the non-Abelian levels will happen organically rather than being imposed at the point of writing down the Lagrangian. (One way of thinking about this choice of regularization is to note that we are taking a non-relativistic limit of a matrix model description derived for a relativistic brane theory, which has a Yang-Mills term.)

- The shift in the Chern-Simons level of the fermionic theory now acts as $N_f/2 \to N_f$ rather than changing $N_f/2 \to 0$; this arises because of a different choice of which Dirac sea we integrate out in taking the non-relativistic limit. We will see a related difference of sign in the Lagrangian.

As we have seen, the dynamics of Theory A is particularly rich in a phase where the gauge symmetry is fully broken so that the theory admits topological vortex solutions. This only occurs when $N_f \geq p$. Here, we will focus on the specific case $N_f = p$, which is the minimal number of flavours to support such vortices, and also the case hosting quantum Hall physics. The beauty of the above conventions is the elegance of the resulting fermionic theory for this choice. The two dual theories are

Theory A: $U(p)_{k,\,k+np}$ coupled to $N_f = p$ fundamental scalars.

Theory B: $U(k)_0$ coupled to $N_f = p$ fundamental fermions and, through a BF coupling, to $U(1)_n$.

We note in passing that there are few concrete tests of the bosonization dualities with $N_f > 1$ and, indeed, it is thought to fail for N_f suitably large [127, 200]. Here we provide a fairly detailed test of the dualities with $N_f = p$.

Our interest of course lies in the quantum Hall regime of the two dual theories. We know that this occurs in the bosonic theory when we subject it to a chemical potential for its $U(1)$ factor; as we will see in some detail below, this is dual to an almost identical deformation of the fermionic theory. We will ultimately find that both theories describe the same quantum Hall states, but the way this arises in the two cases is rather different.

In Theory A, the emergence of quantum Hall physics involves the condensation of the scalar field and the dynamics of the resulting vortices as discussed throughout the rest of this dissertation. In contrast, in Theory B there is no scalar field to condense. This immediately poses the question: what is the dual of the condensed phase, and what excitations are dual to vortices? We will show that the fermions experience an effective background magnetic field, and the dual of the condensed phase is a fully filled Landau level; the vortices are dual to holes in this Landau level.

14.2.1 The Fermionic Chern-Simons Theory

Our task is to reproduce the properties of vortices described up to this point in terms of fermions. The theory consists of $N_f = p$ non-relativistic fermions ψ_i. These interact with a $U(k)_0$ gauge field; we denote the $SU(k)$ part as c and the $U(1) \subset U(k)$ part as $\tilde{c}$. As described in the introduction, this is subsequently coupled to a further $U(1)_n$ gauge field, b. The full action is

$$S = \int \mathrm{d}^3x \qquad \left[i\psi_i^\dagger \mathcal{D}_0 \psi_i - \frac{1}{2m} \vec{\mathcal{D}}\psi_i^\dagger \cdot \vec{\mathcal{D}}\psi_i - \psi_i^\dagger G \psi_i \right] + \frac{k}{2\pi} \epsilon^{\mu\nu\rho} \tilde{c}_\mu \partial_\nu b_\rho + \frac{n}{4\pi} \epsilon^{\mu\nu\rho} b_\mu \partial_\nu b_\rho - \frac{\mu k}{n} \tilde{c}_0 \,. \tag{14.7}$$

The third term in the action couples the fermions to the background magnetic field, $G = g_{12} + \tilde{g}_{12}\mathbf{1}_k$, where $g = \mathrm{d}c - i[c, c]$ and $\tilde{g} = \mathrm{d}\tilde{c}$ are the non-Abelian and Abelian field strengths respectively. This term arises from the non-relativistic limit of the Dirac equation, in the same way as the term discussed in Part III. (Note that it comes with the opposite sign; this simply depends on whether one integrates out particles or anti-particles in taking the non-relativistic limit, as can be seen for instance in [77].)

Note that the duality maps the chemical potential μp of Theory A into a chemical potential $\mu k/n$ of Theory B. This map can be explicitly checked (at least in the Abelian case) using the techniques of [120, 121]; for non-Abelian gauge groups considered here, the map between chemical potentials includes a rescaling by the rank of the gauge group. As an alternative, one can change the term in (14.7) for a chemical potential for b; in this case it takes the simpler form $-\mu k\tilde{c}_0/n \to +\mu b_0$. The physics which follows is identical. This allows a clearer extension to $n = 0$.

Our task is to reproduce the quantum Hall physics found in the bosonic theory. The essence of the problem becomes immediately apparent if we look at the constraints enforced by Gauss's law. Because the $SU(k)$ Chern-Simons level is vanishing, the dynamics of the non-Abelian field c is solely governed by the Yang-Mills regulator whose coupling is taken to be large; thus, this gauge theory is confined and only $SU(k)$ singlets are allowed. In contrast, the Abelian Gauss's law arising from $\tilde{c}$ and b read

$$\begin{aligned} \psi_i^\dagger \psi_i - \frac{\mu k}{n} + \frac{k}{2\pi} \mathrm{d}b &= 0\,, \\ \frac{k}{2\pi} \mathrm{d}\tilde{c} + \frac{n}{2\pi} \mathrm{d}b &= 0\,. \end{aligned} \tag{14.8}$$

Now we see the difficulty. There is only one obvious, translationally invariant solution, given by $\mathrm{d}b = -(k/n)\mathrm{d}\tilde{c}$ and

$$\underline{\textbf{Phase 1':}} \qquad \mathrm{d}\tilde{c} = \tilde{g}_{12} = -\frac{2\pi\mu}{k} \quad , \quad \langle \psi_i^\dagger \psi_i \rangle = 0\,.$$

This provides the dual to Phase 1 of the bosonic theory. However, life is more difficult if we want to write down the dual of Phase 2 in the bosonic theory because we cannot simply condense the fermions to saturate the background charge. How, then, to construct Phase 2?

To do this, we work self-consistently. Suppose the state has a constant, background Abelian field with strength $\tilde{g}_{12}$. The fermionic excitations then form Landau levels. However, crucially, the presence of the $\psi_i^\dagger \tilde{g}_{12} \psi_i$ term in the action (14.7) means that the lowest Landau level costs zero energy.[1] This means that there is a second, translationally invariant ground state in which the lowest Landau level is fully filled. The density of states in a Landau level is $|\tilde{g}_{12}|/2\pi$ and, including both flavour and colour degrees of freedom, there are kp different fermions which we can excite. Hence the fully filled lowest Landau level has $\langle \psi_i^\dagger \psi_i \rangle = kp|\tilde{g}_{12}|/2\pi$. The self-consistent solution to (14.8) is then

$$\underline{\textbf{Phase 2':}} \qquad \tilde{g}_{12} = -\frac{2\pi\mu}{k'} \quad , \quad \langle \psi_i^\dagger \psi_i \rangle = \frac{\mu k p}{k'}$$

where $k' = k + np$. We claim that this phase is dual to Phase 2 of Theory A.[2]

[1] This is a familiar fact for relativistic fermions, and the direct coupling to the field strength arises because (14.7) is the non-relativistic limit of a relativistic theory, as we discussed in Part II. The different sign of this direct coupling in this chapter is responsible for the different physics relative to the fermionic Landau levels of Part III.

[2] One could also consider such self-consistent solutions for bosons. In this language, the condensed Phase 2 for bosons corresponds to filling the lowest Landau level an infinite number of times, a luxury not available for fermions. Filling a finite number of times would appear to correspond to a fractionally filled Landau level for the fermions; it would be interesting to explore this connection further.

14.2.2 Holes as Vortices

Our next task is to understand the excitations above Phase $2'$. These are the dual to the vortices in Theory A. Since all physical states must be $SU(k)$ singlets, the lowest energy excitations are baryonic holes in the lowest Landau level. In the absence of a trap, these cost zero energy and are created by operators

$$H_{i_1\ldots i_k}(\vec{x}) = \epsilon^{m_1\ldots m_k}\psi_{i_1 m_1}(\vec{x})\cdots\psi_{i_k m_k}(\vec{x}) \tag{14.9}$$

where the colour indices range from $m = 1,\ldots,k$ and the flavour indices from $i = 1,\ldots,p$. Gauss's law (14.8) ensures that each hole is accompanied by a flux $\mathrm{d}b = -2\pi$ and $\tilde{g}_{12} = 2\pi n/k$. We will now show that these holes share the same properties as the vortices in Theory A.

Theory B has an $SU(p)$ flavour symmetry. In Phase 2′, this should be identified with the $SU(p)_{\text{ diag}}$ symmetry which emerges in Theory A:

$$U(p)_{\text{ gauge}} \times SU(p)_{\text{ flavour}} \longrightarrow SU(p)_{\text{ diag}}\,. \tag{14.10}$$

Since the fermionic operators in (14.9) are anti-commuting, the hole operators $H_{i_1\ldots i_k}$ must transform in the k^{th} symmetric representation of $SU(p)$. This coincides with the transformation of a single vortex in Theory A.

What happens as we introduce more and more baryonic holes? Clearly, we start to construct a region that takes us back to Phase $1'$. Just as it was useful to understand Phase 1 of the bosonic theory through the lens of the vortices, here we would like to understand Phase $1'$ through the lens of the holes. The first step is to notice that the holes feel as if they are moving in a background magnetic field. This is because they carry flux $\tilde{g}_{12} = 2\pi n/k$ and, by the same kind of duality argument we used in the bosonic theories, the $(\mu k/n)\tilde{c}_0$ term in the action mimics a magnetic field for any magnetic excitation. The strength of this effective magnetic field is $B = 2\pi\mu$.

Meanwhile, the maximum density of holes is $\rho_h = \langle\psi_i^\dagger\psi_i\rangle/k = \mu p/k'$, because each hole consists of k ψ excitations. This means that the holes can be packed at filling fraction

$$\nu = \frac{\rho_h}{B/2\pi} = \frac{p}{k'}\,.$$

This coincides with the filling fraction of vortices that we saw in Theory A, for instance in equation (14.5).

The mapping of quantum numbers and density already provides good evidence that non-Abelian vortices map to holes in the lowest Landau level. The BPS nature of the vortices is associated to vanishing energy of states in the lowest Landau level.

Our next task is to construct wavefunctions for these states. Since the holes created by ψ_{ia} experience a background magnetic field, wavefunctions for a single-hole are just the familiar lowest Landau level states. In symmetric gauge, the quantum fields can be expanded in angular momentum modes as

$$\psi_{im}(z,\bar{z}) = \sum_{q=0}^{\infty} z^q\, e^{-B|z|^2/4}\, \chi^q_{im} \tag{14.11}$$

where χ^q_{im} is the creation operator for a fermion, labelled by i and m, in the q^{th} angular momentum state of the lowest Landau level.

We now look at states with p holes. This is trickier as we should take into account the interaction between holes. We will proceed by neglecting this. Partial justification comes from the fact that the $SU(k)$ gauge interactions are strongest and we have already taken these into account in forming the baryonic holes. Nonetheless, one may expect some residual short range interactions which we do not have control over. The fact that ultimately the ground state is gapped (and the agreement with the dual description) suggests that this is valid.

To provide an energetic distinction between different hole excitations, we introduce a harmonic trap. As in Theory A, it is simplest to take the trap to be proportional to the angular momentum q of the holes, with the convention that Phase 2′ has vanishing energy. For each spatial wavefunction, we have pk fermionic states ψ_{im}. Each hole is constructed from k of these states. This means that the first p holes sit in the lowest, $q = 0$, state; the next p holes sit in the $q = 1$ state, and so on.

What representation of $SU(p)$ does the resulting ground state sit in? To see this, note that we can equally well write the single hole creation operator (14.9) as

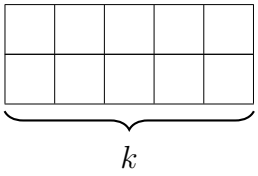
$$H_{i_1,\ldots,i_k} = \mathrm{Sym}_i[\psi_{i_1,1}\ldots\psi_{i_k,k}]$$

where the symmetrization is over all flavour indices. Now consider the product over two, spatially coincident holes,

$$H_{i_1,\ldots,i_k} H_{j_1,\ldots,j_k} = \mathrm{Sym}_{i,j}[\psi_{i_1,1}\ldots\psi_{i_k,k}\psi_{j_1,1}\ldots\psi_{j_k,k}]$$

where we symmetrize independently over i indices and over j indices. Clearly this state is antisymmetric under exchange of each pair, such as (i_1, j_1). The upshot is that this

state transforms in the k^{th} symmetrization of the antisymmetric representation or, in terms of Young diagrams,



and so forth. Thus we see that the ground state of $N \equiv M \pmod p$ holes transforms in the same representation (14.6) as the ground state of vortices.

Before writing down the many-hole wavefunction, there is one final thing we should remember. The holes are composite fermions/bosons; they have charge k and flux $2\pi n/k$. This means that when one hole circles another, it picks up a $2\pi n$ phase. To reflect this, we should include the factor $\prod(z_a - z_b)^n$ in the wavefunction.

We've now described all the ingredients which go into constructing the wavefunction for N holes. The only remaining difficulty is notational. For simplicity, we take N divisible by p. Each hole, $a = 1, \ldots, N$, has an associated $SU(p)$ spin $\mathcal{H}_a$ which lies in the k^{th} symmetric representation of $SU(p)$

$$\mathcal{H}_a(\vec{x}) = (H_a)_{i_1 \ldots i_k}(\vec{x})\, |\sigma_{i_1}\rangle \ldots |\sigma_{i_k}\rangle$$

where, as for the vortices, $|\sigma\rangle \in \{1, \ldots, p\}$. The wavefunction is then given by the overlap

$$\Psi(z, \sigma) = \prod_{a<b}^{N} (z_a - z_b)^n \, \langle \text{LLL}|\, \mathcal{H}^\dagger_{a_1}(z_1, \bar{z}_1) \ldots \mathcal{H}^\dagger_{a_N}(z_N, \bar{z}_N)\, |N\rangle$$

where $\langle \text{LLL}|$ is the ground state for Phase 2′, while $|N\rangle$ is the state with the N holes removed in successive lowest angular momentum modes. To construct the explicit wavefunction now involves only Wick contractions of the creation operators χ^q_{im} which appear in (14.11). Despite its simplicity, this step is a little fiddly. It is easiest to focus on a specific colour index, say $m = 1$. One can check that the resulting terms in the wavefunction are precisely those that appear in $\Phi(z, \sigma)$ defined in (13.11). Repeating this for each $m = 1, \ldots, k$, we find the Blok-Wen wavefunction (13.12), where the symmetrization naturally occurs for the reasons described above.

14.2.3 Level Rank Duality

Comparing the construction of the wavefunction for holes and vortices, we see that there is an interesting interplay the roles played by $SU(k)$ and $SU(p)$ on the two sides

of the duality. This is the essence of level-rank duality. In this section, we review some representation theory which highlights this connection.

In building the hole wavefunctions, we find that each state in the lowest Landau level comes in pk varieties, each associated to a fermionic annihilation operator $\psi_{i,m}$ with $i = 1, \ldots, p$ and $m = 1, \ldots, k$. These states naturally carry a representation of $u(pk)_1$. This then has a decomposition into

$$u(1)_{pk} \times su(k)_p \times su(p)_k \subset u(pk)_1 \,. \tag{14.12}$$

The first factor, $u(1)_{pk}$ simply counts the number of excited fermions. The second and third factors correspond to our gauge and flavour groups respectively. (The levels arise because there is a truncation on the dimension of each representation, which follows simply from the fact that we have a finite number of Grassmann operators to play with.) Gauge invariance means that we want to restrict to $SU(k)$ singlets. The question we would like to ask is: which $SU(p)$ representations then emerge?

The general decomposition (14.12) has been well studied, not least because of the important role it plays in level-rank duality. We label representations under the left-hand side using triplets $(q, R, \tilde{R})$, where q is the number of excited fermions and R and $\tilde{R}$ denote the Young diagrams for the representations of $su(k)_p$ and $su(p)_k$ respectively. Suppose that the representation R appears on the left-hand side: then it is accompanied by $\tilde{R} = R^T$, or its orbit under outer automorphisms.

Recall from Chapter 6 that the *outer automorphism group* of $SU(p)_k$ is $\mathbb{Z}_p$, generated by the basic outer automorphism operator σ which maps $\tilde{R} \mapsto \sigma(\tilde{R})$, whose Young diagram one constructs by adding rows of length k atop the diagram and then reducing it.

The upshot of this is that the only representations of $u(1)_{pk} \times su(k)_p \times su(p)_k$ that can appear are $\big(|R| + Mk \pmod{kp}, R, \sigma^M(R^T)\big)$, with $M = 0, 1, \ldots, p-1$. Here R^T denotes the transpose of the Young diagram R, and $|R|$ is the number of boxes it contains.

For us, the above construction is particularly simple because we are interested in the singlet representation R. These have $|R| = 0$, and R^T is the singlet representation of $su(p)$. Under the action of outer automorphisms, the singlet representation is mapped into representations which contain N complete rows of k boxes, with $u(1)_{pk}$ charge Nk. This means that the operators H^N, with $N < p$, transform in the representation (14.6) which we saw for vortices in Theory A.

The discussion above was restricted to $N < p$ baryonic holes. Each spatially distinct state in the lowest Landau level has pk fermionic states. This means that if we remove p

baryons then we empty one spatial bucket, leaving the state a singlet once more. Then we must begin again from the next bucket, and the process repeats. So for N baryonic holes, the representation is again given by (14.6), where $N \equiv M \pmod p$. This again matches the representation theory of the vortices.

14.2.4 Discussion

As we are seeing, there are a bewildering number of descriptions of quantum Hall states. Many of these are related by dualities, including level-rank dualities such as those we have been discussing. Let us try to place our results within this wider context.

The original effective field theory for the Laughlin state is due to Zhang, Hansson and Kivelson [10]. It consists of an Abelian Chern-Simons theory with non-integer level set by the filling fraction. The Chern-Simons field is coupled to non-relativistic scalars which, through the process of flux attachment, become the electrons of the system. An alternative description was offered by Lopez and Fradkin [13], which again consists of an Abelian Chern-Simons field at non-integer level, this time coupled to fermions. The equivalence of these two descriptions for the long distance physics can be viewed as a simple example of 3d bosonization, albeit restricted to the non-relativistic regime of quantum mechanics.

The fact that the Chern-Simons level in [10, 13] is fractional means that these theories miss aspects of the physics related to topological order. This was rectified in the work of Wen and Zee [176], who presented an effective description of quantum Hall states in terms of Abelian Chern-Simons theories with integer-valued levels. These are related to the earlier papers through a kind of particle-vortex duality. In particular, the vortices now play the role of the electrons in the system. The gauge fields are coupled to scalars whose excitations describe the quasiholes with anyonic statistics.

This seems to be the first time that a fermionic version of the Wen-Zee class of theories has been constructed. This is what the bosonization duality achieves. For example, we have demonstrated that the Laughlin state at filling fraction $\nu = 1/(k+1)$ is described by a $U(k)_{0,-k} \cong [U(1)_{-k^2} \times SU(k)_0]/\mathbb{Z}_k$ Chern-Simons theory coupled to just a single species of fermion. This viewpoint appears to be closely related to the partonic construction of [21, 22].

The bosonic "Theory A" that we have described should be viewed in the same spirit as the Wen-Zee theories, with the obvious exception that it is a non-Abelian gauge theory. It is a $U(p)_{k,k'}$ Chern-Simons theory whose vortices are to be thought of as the "electrons", now endowed with internal spin degrees of freedom. The resulting quantum Hall states were previously introduced by Blok and Wen. The bosonization duality

now tells us that the duals of these non-Abelian states can be constructed by considering $SU(k)$ singlets, coupled to further Abelian gauge fields. This is reminiscent of the partonic description of these states previously presented in in [201, 183].

15 The View from Conformal Field Theory

In the previous chapter, we used vortices to construct a disc-like region of space in which the low-energy dynamics is described by an unbroken $U(p)$ Chern-Simons theory. The microscopic dynamics of these vortices are described by the matrix model (12.2) whose ground states lie in the same universality class as the Blok-Wen wavefunctions. In this chapter, we close the circle and describe these states from the perspective of the boundary.

Our vortex construction has presented us with a Chern-Simons theory on a manifold with boundary, where the boundary is now the edge of the large vortex. On general grounds, we expect this boundary to support a chiral $U(p)_{k,k+p}$ WZW model [83, 202]. This should manifest itself in two ways.

First, the excitations of the matrix model should coincide with the excitations of a (suitably discretized) WZW model. We will return to this in Part V. Secondly, the ground state wavefunction – which, as we have seen, is of the Blok-Wen type – should arise as the correlation function in the conformal field theory [16]. This is sometimes known as the *bulk-boundary correspondence*.[1] This, of course, was how Blok and Wen originally derived their wavefunctions [183]. Here we review this construction, including the effect of the Abelian factor in the gauge group.

Let's first review some simple properties of the WZW models. The irreducible representations of the $SU(p)$ Kac-Moody algebra at level k are labelled by the corresponding representation of the $SU(p)$ Lie algebra. The latter are well known to be described by Young tableaux with up to $p-1$ rows. The representations of $SU(p)_k$ are those Young tableaux which have no more than k boxes in the first row.

Each irreducible representation of the Kac-Moody algebra gives rise to a primary operator in the corresponding WZW model. We call these operators $\mathcal{O}_R$ where R denotes the representation. The usual candidates for quantum Hall wavefunctions are the correlation functions of strings of chiral operators

$$\langle \mathcal{O}_R(z_1) \dots \mathcal{O}_R(z_N) \rangle \tag{15.1}$$

[1] It can be thought of as a baby version of de Sitter holography. A review of the bulk-boundary correspondence applied to quantum Hall physics can be found in the lecture notes [203].

where $\mathcal{O}_R$ is the "electron operator" in the CFT or, more generally, the operator associated to the particle which forms the quantum Hall state. (More precisely, these should be thought of as conformal blocks of the non-chiral WZW theory.)

There is, however, a problem in identifying (15.1) as a wavefunction: for most choices of $\mathcal{O}_R$, there is no unique answer due to monodromies in the correlation function as z_a are varied. Instead, the number of conformal blocks is the number of singlets that arises when the many copies of R are fused together. Typically this number will increase exponentially with N. Of course, this growth of conformal blocks is precisely what's needed to describe non-Abelian quasiholes in a quantum Hall state, but this should only occur for correlation functions in which quasihole operators are inserted. For a sensible quantum Hall interpretation, we want to have a unique ground state, and this means that (15.1) should yield a unique answer when only electron operators are inserted.

There is, fortunately, a choice of R for which (15.1) has a unique answer. We take N to be a multiple of p and choose the representation R which is maximally symmetric. In terms of Young diagrams, it is a single row of k boxes as in the following diagram:

$$R = \underbrace{\square\square\square\square\square}_{k} \tag{15.2}$$

For $SU(2)_k$, this corresponds to the spin $s = k/2$ representation; for $SU(p)_k$ it is the k^{th} symmetric representation. This, of course, is precisely the representation carried by the particles described by our wavefunctions (13.12).

To see that there is indeed a unique singlet when we take (15.2), we need to look at the fusion rules once more. For $SU(2)_k$, it is straightforward to show that the fusion of two spin $s = k/2$ representations leaves only the singlet $s = 0$. Written in terms of the dimension $\mathbf{d} = 2s + 1$ of the representation, this reads

$$(\mathbf{k+1}) \star (\mathbf{k+1}) = \mathbf{1}\,.$$

For $SU(3)_k$, one finds that the k^{th} symmetric representation, which we denote Sym_k, has fusion rules

$$\mathrm{Sym}_k \star \mathrm{Sym}_k = \overline{\mathrm{Sym}}_k$$

while

$$\mathrm{Sym}_k \star \overline{\mathrm{Sym}}_k = \mathbf{1}$$

which tells us that three copies of Sym_k can fuse only to the singlet. More generally, for $SU(p)_k$ we define the representations

$$Y_i = \quad i \left\{ \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline & & & & \\ \hline & & & & \\ \hline \end{array} \right. \underbrace{\qquad\qquad\qquad\qquad\qquad}_{k}$$

which consist of k columns (the maximum number allowed by the level) and i rows. These are precisely the representations that we saw in Section 12.2 when discussing the ground states of the matrix model. In this notation, our maximally symmetric representation is $Y_1 = \mathrm{Sym}_k$, while $Y_{N-1} = \overline{\mathrm{Sym}}_k$. The Y_i have the nice property that they fuse only among themselves [204, 205] according to

$$Y_i \star Y_j = Y_{i+j \pmod p} \,.$$

This is enough to ensure that

$$\overbrace{\mathrm{Sym}_k \star \mathrm{Sym}_k \star \ldots \star \mathrm{Sym}_k}^{p} = \mathbf{1}$$

which tells us that (15.1) has a unique answer when R is taken to be the maximally symmetric representation. Now our job is to compute it.

15.1 The Wavefunction as a Correlation Function

The standard tool to compute correlation functions in WZW models is a constraint linking their spatial dependence and group transformation properties known as the Knizhnik-Zamolodchikov (KZ) equation [206]. Usually this is employed to compute 4-point functions but since we expect a unique solution to (15.1), we can hope to use it in the present case to compute higher-point functions.

The KZ equation reads

$$\left(\frac{\partial}{\partial z_a} - \frac{1}{k+p} \sum_{b \neq a}^{N} \frac{T_a^\alpha \otimes T_b^\alpha}{z_a - z_b} \right) \langle \mathcal{O}_R(z_1) \ldots \mathcal{O}_R(z_N) \rangle = 0 \tag{15.3}$$

where T^α is the Hermitian generator for the k-th symmetric representation. These obey the $SU(p)$ algebra $[T^\alpha, T^\beta] = i f^{\alpha\beta\gamma} T^\gamma$, with the normalization $f^{\alpha\gamma\delta} f^{\beta\gamma\delta} = 2p\, \delta^{\alpha\beta}$, where p appears in its role as the dual Coxeter number of $SU(p)$. (As promised, the structure of the Hamiltonian (5.29) governing the quantum mechanics of superconformal anyons is seen to be echoed here.)

Solving the Knizhnik-Zamolodchikov Equation

We will now show that the KZ equation (15.3) is solved by

$$\langle \mathcal{O}_R(z_1)\dots\mathcal{O}_R(z_N)\rangle = \prod_{a<b}^{N}(z_a-z_b)^{-k/p}\,\mathcal{P}\left[\Phi^k_{(p)}(z,\sigma)\right] \tag{15.4}$$

with $\Phi_{(p)}$ is defined in (13.11).

It will be useful to first rewrite our ansatz in a slightly more concrete form. As discussed in the run up to (13.15), it's simple to check that, up to an unimportant normalization,

$$\mathcal{P}\left[\Phi^k_{(p)}(z,\sigma)\right] = \Bigg[\mathcal{A}\quad \prod_{1\le a<b\le N/p}(z_a-z_b)|\sigma_1\rangle\dots|\sigma_1\rangle\ \otimes \prod_{N/p<a<b\le 2N/p}(z_a-z_b)|\sigma_2\rangle\dots|\sigma_2\rangle$$
$$\otimes\dots\dots\otimes \prod_{(p-1)N/p<a<b\le N}(z_a-z_b)\ \ |\sigma_p\rangle\dots|\sigma_p\rangle\Bigg]^k. \tag{15.5}$$

Here we have placed the first N/p particles in the same spin state, the next N/p in a different spin state and so on. The $\mathcal{A}$ symbol means that we then antisymmetrize over all particles.

The generators T^α in (15.3) can be viewed as acting symmetrically on what were originally k distinct fundamental factors,

$$T^\alpha \;=\; \overbrace{t^\alpha\otimes\mathbf{1}\otimes\cdots\otimes\mathbf{1}}^{k} + \text{symmetric permutations}$$

with t^α the generator in the fundamental representation. The normalization in the KZ equation ensures that we have $t^\alpha_{ij}t^\alpha_{kl} = \delta_{il}\delta_{jk} - \frac{1}{p}\delta_{ij}\delta_{kl}$, where the group indices $i,j,k,l = 1,\dots,p$. This means that if the operator $T^\alpha_a\otimes T^\alpha_b$ acts on a state where particles a and b have the same spin, this tensor product returns the same spin state multiplied by a factor $(1-1/p)$. By contrast, if the two particles have different spins, it returns a superposition of the same state with a factor $-1/p$, and a state with the particles swapped with no factor.

We'll start by considering the action of the non-derivative part of the KZ operator on $\mathcal{P}[\Phi^k_{(p)}]$. Expanding out (15.5) will result in a slew of terms, each of them containing k factors. Let's look at one of these terms – call it X. We're going to figure out the coefficient in front of X after the action of the $T\otimes T$ term in the KZ operator. There are two contributions. One arises when $T\otimes T$ acts on X itself. The other arises from $T\otimes T$ hitting other terms in the expansion of $\mathcal{P}[\Phi^k_{(p)}]$ so that they are mapped into X. We deal with these in turn.

Let's first look at the action of $T \otimes T$ on X. Suppose that particle a has the same spin as particle b in d_b of the k factors. Then acting with $T \otimes T$ will, among other things, return X with a coefficient

$$-\frac{1}{k+p}\sum_{b\neq a}^{N}\frac{kd_b(1-1/p)-k(k-d_b)/p}{z_a-z_b}=-\frac{1}{k+p}\sum_{b\neq a}^{N}\frac{kd_b-k^2/p}{z_a-z_b}\,. \tag{15.6}$$

This is our first result.

Now let's see how we can get terms proportional to X by the action of $T \otimes T$ on some other term Y. This can occur only if Y differs from X by swapping the spins of i and j in just one factor. Suppose that, within Y, particle i has spin a and particle j has spin $b \neq a$. Then the $T \otimes T$ term in the KZ operator will map Y to X with coefficient

$$+\frac{1}{k+p}\frac{1}{z_a-z_b}\prod_{(d\neq a|i)}\frac{z_b-z_d}{z_a-z_d}\prod_{(d\neq b|j)}\frac{z_a-z_d}{z_b-z_d}$$

where the notation $(d \neq a|i)$ means that we take the product over all particles $d \neq a$ which carry spin i. Something nice now happens when this expression is summed over all particles j which carry spin b; the terms combine to give

$$+\frac{1}{k+p}\left[-\sum_{(d\neq a|i)}\frac{1}{z_a-z_d}+\sum_{(d|i)}\frac{1}{z_a-z_d}\right]. \tag{15.7}$$

This is our second result.

The total coefficient multiplying the term X after the action of $T \otimes T$ is then given by the sum of (15.6) and (15.7). It is

$$\sum_{b\neq a}\frac{k/p-d_b}{z_a-z_b}\,. \tag{15.8}$$

The key point is that this coefficient is precisely cancelled by the derivative term in the KZ equation, since $(z_a - z_b)$ appears with the power $-k/p + d_b$ in the correlation function (15.4). Note that the actual coefficient of a given term X typically includes, in addition to (15.8), a symmetry factor to account for the fact that X may appear many times in the original expansion of $\mathcal{P}[\Phi^k_{(p)}]$. This is not relevant for our final result; the same symmetry factor appears every time X arises. This concludes our proof that the correlation function (15.4) indeed solves the KZ equation (15.3).

The Full Wavefunction

The solution to the KZ equation ensures that our wavefunction (13.12) can be written as the product

$$\psi_{n+k,n}(z,\sigma) = \prod_{a<b}^{N}(z_a - z_b)^{n+k/p}\,\langle \mathcal{O}_R(z_1)\ldots\mathcal{O}_R(z_N)\rangle\, e^{-\sum_a |z_a|^2/4\ell_B^2} \tag{15.9}$$

where, for once, we've restored the exponential factor common to all lowest Landau level wavefunctions. The first factor is, of course, a Laughlin wavefunction and can be expressed as a correlation function for a free compact boson. The fractional exponent is unusual, but has been seen before in constructing Halperin wavefunctions from a CFT [16, 207] where the same factorization into "spin" degrees of freedom, captured by $\langle \mathcal{O}_R(z_1)\ldots\mathcal{O}_R(z_N)\rangle$, and "charge" degrees of freedom captured by the Laughlin wavefunction also occurs.

The full wavefunction can be written as a correlation function in the WZW model with algebra

$$U(p)_{k,k+np} = \frac{U(1)_{(k+np)p}\times SU(p)_k}{\mathbb{Z}_p}\,. \tag{15.10}$$

Note that our matrix model describes the Blok-Wen states with $n = 1$. Happily, in that case, the WZW model (15.10) indeed arises as the description of the boundary dynamics of the Chern-Simons theory with gauge group (14.1).

The $U(1)$ part is described by a compact chiral boson ϕ and the correlation function

$$\langle \prod_{a=1}^{N} e^{i\sqrt{(k+np)/p}\phi(z_i)}\, e^{-\int \mathrm{d}^2z'\sqrt{(k+np)/p}\,\phi(z')/2\pi\ell_B^2}\rangle = \prod_{a<b}^{N}(z_a - z_b)^{n+k/p}\, e^{-\sum_a |z_a|^2/4\ell_B^2}$$

gives the Laughlin part of the wavefunction (15.9) in the usual manner [16].

Quasiholes as Non-Abelian Anyons

The presence of the $SU(p)_k$ factor ensures that our quantum Hall states have non-Abelian anyons for $k > 1$. These quasiholes are associated to primary operators in the WZW conformal field theory and their properties are expected to be determined by the fusion rules and braiding inherited from the CFT. Although this story is well known (see, for example, [130]), we pause here to point out a few of the more prominent examples.

An Example: Ising Anyons

Ising anyons are well known to appear in the Moore-Read state which is associated to

the $SU(2)_2$ WZW model [208]. The primary operators carry spin $s = 0, 1/2$ and 1. We denote these representations using their dimension $\mathbf{d} = 2s + 1$. As we saw above, we identify the electron with the spin 1, or $\mathbf{d} = \mathbf{3}$ representation. The $\mathbf{d} = \mathbf{2}$ primary is then identified with the quasihole, with the corresponding fusion rules given by

$$\mathbf{2} \star \mathbf{2} = \mathbf{1} \oplus \mathbf{3} \quad , \quad \mathbf{2} \star \mathbf{3} = \mathbf{2} \quad , \quad \mathbf{3} \star \mathbf{3} = \mathbf{1} \, .$$

These are the fusion rules for Ising anyons.

Another Example: Fibonacci Anyons

Fibonacci anyons are known to arise as the quasiholes in the $\mathbb{Z}_3$ parafermionic Read-Rezayi state. This is governed by the coset model $SU(2)_3/U(1)$. As we have seen above, these states arise in our construction as the $SU(3)_2$ WZW model. This is related to the parafermion CFT through level-rank duality and a quotient. The anyon is associated to the primary operator which transforms in the adjoint representation $\mathbf{8}$ of $SU(3)$. It is simple to compute the fusion rules in $SU(3)_2$ to find

$$\mathbf{8} \star \mathbf{8} = \mathbf{1} \oplus \mathbf{8} \, .$$

This is indeed the fusion rule for Fibonacci anyons. A nice review of these objects can be found in [209].

It remains an open problem to identify these anyonic states directly within the matrix model. For the Laughlin states, we have already seen that the matrix model provides a construction of quasihole states which are analytically more tractable than the traditional approach. It seems plausible that the matrix model may also prove useful in understanding the properties of non-Abelian anyons. This is certainly something to return to in the future.

PART V

Edge Theories

16 Introduction

In this part of the dissertation, our goal is to describe in much more detail how our simple quantum mechanical matrix model is related to the chiral WZW conformal field theory in $d = 1 + 1$ dimensions.

Recall that the matrix model consists of a $U(N)$ gauge field α coupled to a complex adjoint scalar Z and p fundamental scalars φ_i, $i = 1, \ldots, p$. We will choose units such that $B = 2$ for simplicity, so the first order action is

$$S = \int \mathrm{d}t \left[i \ \mathrm{Tr} \left(Z^\dagger \mathcal{D}_t Z \right) + i \sum_{i=1}^{p} \varphi_i^\dagger \mathcal{D}_t \varphi_i - (k+p) \, \mathrm{Tr} \, \alpha - \omega \ \mathrm{Tr} \ Z^\dagger Z \right] . \tag{16.1}$$

The covariant derivatives are $\mathcal{D}_t Z = \partial_t Z - i[\alpha, Z]$ and $\mathcal{D}_t \varphi_i = \partial_t \varphi_i - i\alpha\varphi_i$ and Tr denotes the trace over $U(N)$ gauge indices. Here and in the following k is a positive integer.

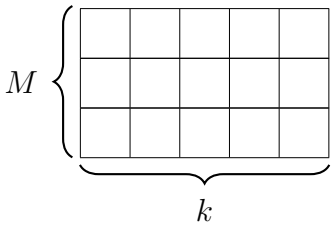
In addition to the $U(N)$ gauge symmetry, the quantum mechanics has an $SU(p)$ global symmetry. We will show that, in the large N limit, this matrix model captures the physics of the $SU(p)_k$ WZW conformal field theory. Specifically, we demonstrate the following two results:

- The left-moving $\widehat{su(p)}$ affine Lie algebra at level k can be constructed from the quantum mechanical operators Z and φ_i.
- The partition function of the matrix model can be computed exactly, for all N, as a function of both temperature and chemical potentials for the $SU(p)$ global symmetry. The result (18.19) is an expansion in *Schur polynomials* and *Kostka polynomials* (both of which will be defined below). In the large N limit, the partition function is proportional to the partition function of the chiral $SU(p)_k$ WZW model.

This second property requires some elaboration as the matrix model partition function depends in a rather delicate way on how we take the large N limit. To recover the chiral WZW partition function – also known as the vacuum character – one should set N divisible by p and subsequently take the large N limit.

One can also ask what happens if we take the large N limit when $N \equiv M \pmod{p}$. In this case, we show that the quantum mechanics partition function is equal to the character of the WZW model associated to a primary in a representation which is perhaps best described as the "k-fold symmetrization of the M^{th} antisymmetric representation

of $SU(p)$". In terms of Young diagrams, this representation is the (by now extremely familiar) rectangular diagram depicted here:



(16.2)

Relationship to Chern-Simons Theory

The connection between the matrix model (16.1) and the WZW model is of course not coincidental; as we have discussed, both are related to Chern-Simons theories. Before we derive the results above, let us first recall why they are not unexpected.

We have already established that the matrix model (16.1) describes the dynamics of vortices in a (2+1)-dimensional Chern-Simons theory, coupled to non-relativistic matter. This Chern-Simons theory has gauge group and levels

$$U(p)_{k,k'} = \frac{U(1)_{k'p} \times SU(p)_k}{\mathbb{Z}_p} \tag{16.3}$$

where $k' = k + p$.

The vortices sit in a harmonic trap which forces them to cluster around the origin, where they form a droplet of size $\sim \sqrt{N}$. Outside this region, the gauge group $U(p)$ is broken; inside it is unbroken.

The upshot is that the solitonic vortex provides a way to engineer Chern-Simons theory on a manifold with boundary, where the role of the boundary is played by the edge of the vortex. It is well known that the gapless excitations of the Chern-Simons theory are chiral edge modes, described by a WZW model with algebra $U(p)_{k,k'}$ [83, 202]. The advantage of the present set-up is that we can identify the microscopic origin of these edge modes as the excitations of the vortices. These excitations are captured by the matrix model (16.1).

The vortex perspective also provides a way to understand the delicate manner in which we should take the large N limit. One of the other facts explored in Part IV was that the vortices have a unique, $SU(p)$ singlet, ground state only when N is divisible by p. As we described above, with this restriction in place, the large N limit of the partition function coincides with the partition function of the WZW model.

In contrast, when $N \equiv M \pmod p$, the ground state of the vortices is not unique; rather, it transforms in the representation (16.2). This explains why taking the large N limit keeping $N \equiv M \pmod p$ results in the character of the Kac-Moody algebra associated to this representation.

Relationship to the Quantum Hall Effect

Of course, our original interest in the matrix model (16.1) was through its connection to the quantum Hall effect.

As we have already briefly outlined, there is a deep connection between the bulk properties of quantum Hall states and the $d = 1 + 1$ conformal field theory which describes the dynamics of the edge modes. This connection was first highlighted in [16] where it was shown that the bulk wavefunction can be reconstructed as a CFT correlation function. This relationship was subsequently used to derive several interesting non-Abelian quantum Hall states [16, 86, 184].

However, one can also go the other way. Starting from a quantum Hall wavefunction, one can enumerate its full set of excitations. These can then be matched to the excitations of the boundary conformal field theory. This was first done by Wen for Abelian quantum Hall states [15, 210] and later extended to a number of paired, non-Abelian quantum Hall states in [143].

The connection between the matrix model (16.1) and the WZW model highlighted here falls naturally into this larger quantum Hall narrative. Indeed, we have already seen that the Blok-Wen states – which are the ground states of our matrix model – can be reconstructed from correlation functions in the $U(p)_{k,k'}$ WZW model. The results of this part can be thought of as a derivation of the converse story: the excitations of the matrix model coincide with those of the boundary CFT.

The excitations arising from the $p = 1$ matrix model were previously shown to coincide with those of a chiral boson. For $p \geq 2$, the story is much richer as the partition function now depends on both temperature and chemical potentials for the $SU(p)$ flavour symmetry. Nonetheless, our results show that the excitations above the quantum Hall state do coincide with those of the boundary conformal field theory.

The Plan of Action

This part contains two main results. In Chapter 17 we construct the Kac-Moody current algebra from the quantum mechanics. In Chapter 18 we compute the partition function of the matrix model and explain how to take the large N limit.

The computation of the partition function involves a number of results from the theory of symmetric functions. In the interests of keeping this work self-contained, in Section 18.1 there is a review of the properties of Schur, Hall-Littlewood and Kostka polynomials, which are the lead characters in our story. Appendix E contains further details about Kostka polynomials. Other appendices describe our conventions for affine Lie algebras and the details of the current algebra computation.

17 The Current Algebra

Our aim in this chapter is to explain how the $N \to \infty$ limit of the matrix model (16.1) is related to the $d = 1 + 1$ WZW conformal field theory. The smoking gun for the emergence of a WZW model is, of course, a current algebra. In this chapter we will show how to construct such an algebra from the matrix model degrees of freedom Z and φ_i.

The key point is that the $U(N)$ gauge symmetry ensures that Z and φ are not independent. In particular, Gauss's law of the matrix model (16.1) constrains the degrees of freedom to obey

$$[Z, Z^\dagger] + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N \,. \tag{17.1}$$

We'll see that the current algebra arises, in part, due to these constraints.

Both the classical and quantum matrix models exhibit the Kac-Moody algebra. The difference between the two appears only to be a shift of the level. We will prove that the classical matrix model has an $\widehat{su(p)}$ algebra at level $k+p$. In the quantum theory we find level k. However, the extra complications in the quantum theory mean that the proof of the existence of the algebra relies on two conjectured identities which we present below.

This shift of the level can already be seen in the quantum version of the constraint equation (17.1). In the quantum theory, the individual matrix and vector entries Z_{ab} and $\varphi_{i\,a}$ become operators, obeying the canonical commutation relations

$$[Z_{ab}, Z^\dagger_{cd}] = \delta_{ad}\delta_{bc} \quad \text{and} \quad [\varphi_{i\,a}, \varphi^\dagger_{j\,b}] = \delta_{ab}\delta_{ij} \,. \tag{17.2}$$

We choose a reference state $|0\rangle$ obeying $Z_{ab}|0\rangle = \varphi_{i\,a}|0\rangle = 0$ and construct a Hilbert space by acting with $Z^\dagger_{ab}$ and $\varphi^\dagger_{i\,a}$. The quantum version of Gauss's law (17.1) is interpreted as the requirement that physical states are $SU(N)$ singlets; this can be written in normal ordered form as

$$:[Z, Z^\dagger]: + \sum_{i=1}^{p} \varphi_i \varphi_i^\dagger = (k+p)\mathbf{1}_N \,. \tag{17.3}$$

Here the : : determines the order in which operators appear – with Z moved to the right – but not the way that $U(N)$ group indices are contracted; this is determined by the matrix commutator $[\,,\,]$. Meanwhile, the level determines the charge under $U(1) \subset U(N)$ that physical states must carry. Taking the trace of Gauss's law, and using the commutation relations (17.2), gives

$$\sum_{a=1}^{N}\sum_{i=1}^{p} \varphi_{i\,a}\varphi^{\dagger}_{i\,a} = (k+p)N \quad \Rightarrow \quad \sum_{a=1}^{N}\sum_{i=1}^{p} \varphi^{\dagger}_{i\,a}\varphi_{i\,a} = kN\,. \tag{17.4}$$

We will see below that a similar normal ordering issue shifts the level of the Kac-Moody algebra.

17.1 The Currents

It is straightforward to construct generators of the *positive graded* current algebra in the matrix model. The problem factorizes into $U(1)$ and $SU(p)$ parts. The $U(1)$ currents are simply

$$\tilde{\mathcal{J}}^m = \mathrm{Tr}\ Z^m$$

while the $SU(p)$ adjoint-valued currents are

$$\tilde{\mathcal{J}}^m_{ij} = i\left(\varphi^{\dagger}_i Z^m \varphi_j - \frac{1}{p}\delta_{ij}\,\varphi^{\dagger}_k Z^m \varphi_k\right)\,.$$

Here $i, j, k = 1, \ldots, p$ are flavour indices, while $m \geq 0$ denotes the grading.

It is simple to show that the commutators (17.2) imply that these currents give a representation of half of the Kac-Moody algebra,

$$[\tilde{\mathcal{J}}^m_{ij}, \tilde{\mathcal{J}}^n_{kl}] \;=\; i\left(\delta_{il}\tilde{\mathcal{J}}^{m+n}_{kj} - \delta_{kj}\tilde{\mathcal{J}}^{m+n}_{il}\right) \tag{17.5}$$

while $[\tilde{\mathcal{J}}^m, \tilde{\mathcal{J}}^n] = [\tilde{\mathcal{J}}^m, \mathcal{J}^n_{ij}] = 0$. This holds for any N. This same expression holds in both the quantum theory and the classical theory where, in the latter, the commutation relations (17.2) should be replaced by classical Poisson brackets.

While the result (17.5) is heartening, our interest really lies in the full Kac-Moody algebra and, in particular, the central extension term. Here we will see the difference between classical and quantum theories.

The central charge of the $U(1)$ current is harder to pin down due to a possible rescaling. For this reason, we focus on the $SU(p)$ currents. Here too there is a normalization issue, but one that will turn out to be uniquely fixed. To this end, we rescale the

positive-graded currents

$$\mathcal{J}^m_{ij} = \left(\frac{(k+p)N}{p}\right)^{-m/2} \tilde{\mathcal{J}}^m_{ij} \qquad m \geq 0\,.$$

Note that these still obey the algebra (17.5) since the overall scaling is a power of m. We will see that only these rescaled currents will give rise to the full Kac-Moody algebra. (One can compare this to the normalization of the Abelian Fourier modes back in Section 9.4.) We then define the negative graded currents as

$$\mathcal{J}^m_{ij} = \mathcal{J}^{|m|\,\dagger}_{ji} \qquad\qquad m < 0$$

and similarly for $\tilde{\mathcal{J}}$. These too obey the graded Lie algebra (17.5) if we restrict to $m, n < 0$.

Of course, the central term only arises when we consider mixed commutators of the form $[\mathcal{J}^m_{ij}, \mathcal{J}^{-n}_{kl}]$ with $m, n > 0$. These are trickier to compute because now the constraint (17.1) comes into play. However, things simplify somewhat in the $N \to \infty$ limit. We will show that the currents obey the Kac-Moody algebra

$$[\mathcal{J}^m_{ij}, \mathcal{J}^n_{kl}] \;\sim\; i(\delta_{il}\mathcal{J}^{m+n}_{kj} - \delta_{kj}\mathcal{J}^{m+n}_{il}) + km\;\delta_{m+n,0}\left(\delta_{jk}\delta_{il} - \frac{1}{p}\delta_{ij}\delta_{kl}\right). \tag{17.6}$$

Here $\sim$ means up to $1/N$ corrections. Moreover, the operators in this equation should act on states that are constructed from the vacuum $|0\rangle$ by acting with fewer than $\mathcal{O}(N)$ creation operators.

The rest of this chapter is devoted to the derivation of (17.6). (We also show this structure arises perturbatively, in a sense made clear in the appendix, in the Poisson brackets of the *classical* theory. In that setting we obtain an algebra at the unshifted level $k + p$.)

17.2 Deriving the Kac-Moody Algebra

The novelty in deriving (17.6) arises from the commutator $[Z, Z^\dagger]$ terms between currents travelling in opposite directions. We take $m, n > 0$ and look at

$$\begin{aligned}
[\tilde{\mathcal{J}}^m_{ij}, \tilde{\mathcal{J}}^{-n}_{kl}] &= [\varphi^\dagger_i Z^m \varphi_j, \varphi^\dagger_k Z^{\dagger n}\varphi_l] \\
&= \delta_{jk}\varphi^\dagger_i Z^m Z^{\dagger n}\varphi_l - \delta_{il}\varphi^\dagger_k Z^{\dagger n} Z^m \varphi_j + \varphi^\dagger_{ia}\varphi^\dagger_{kb}[Z^m_{ac}, Z^{\dagger n}_{bd}]\varphi_{jc}\varphi_{ld} \\
&= \delta_{jk}\varphi^\dagger_i [Z^m, Z^{\dagger n}]\varphi_l + \delta_{jk}\varphi^\dagger_i Z^{\dagger n} Z^m \varphi_l \\
&\quad -\delta_{il}\varphi^\dagger_k Z^{\dagger n} Z^m \varphi_j + \varphi^\dagger_{ia}\varphi^\dagger_{kb}[Z^m_{ac}, Z^{\dagger n}_{bd}]\varphi_{jc}\varphi_{ld}
\end{aligned} \tag{17.7}$$

where all $U(N)$ group indices are contracted in the obvious manner, apart from in the final term, where we've written them explicitly.

Our first goal is to simplify the two commutators in this expression. We deal with them in turn. For the first, we write

$$[Z^m, Z^{\dagger n}] = \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} Z^r Z^{\dagger s}[Z, Z^\dagger] Z^{\dagger n-1-s} Z^{m-1-r} \,. \tag{17.8}$$

We're going to replace the factor $[Z, Z^\dagger]$ appearing here with some combination of $\varphi_i\varphi_i^\dagger$ using Gauss's law (17.3). However, Gauss's law is not an identity between operators; instead it holds only when evaluated on physical states $|\text{phys}\rangle$. If we don't include the normal ordering in (17.3), then the constraint is written as

$$\Big([Z, Z^\dagger] + \varphi_i\varphi_i^\dagger\Big)\,|\text{phys}\rangle = (k+p+N)|\text{phys}\rangle \,. \tag{17.9}$$

To this end, we consider the operator $\varphi_i^\dagger[Z^m, Z^{\dagger n}]\varphi_l$ acting on a physical state. Then, after some manipulation, we can use (17.9) to write

$$\begin{aligned}
\varphi_i^\dagger[Z^m, Z^{\dagger n}]\varphi_l|\text{phys}\rangle &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} \varphi_i^\dagger Z^r Z^{\dagger s}[Z, Z^\dagger] Z^{\dagger n-1-s} Z^{m-1-r}\varphi_l|\text{phys}\rangle \\
&= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1} \varphi_i^\dagger Z^r Z^{\dagger s}(k+p-\varphi_{i'}\varphi_{i'}^\dagger) Z^{\dagger n-1-s} Z^{m-1-r}\varphi_l|\text{phys}\rangle \\
&= -\sum_{r=0}^{m-1}\sum_{s=0}^{n-1} \varphi_i^\dagger Z^r Z^{\dagger s}\varphi_{i'}\varphi_{i'}^\dagger Z^{\dagger n-1-s} Z^{m-1-r}\varphi_l|\text{phys}\rangle \qquad (17.10) \\
&\qquad\qquad +(k+p)n\sum_{r=0}^{m-1} \varphi_i^\dagger Z^r Z^{\dagger n-1} Z^{m-1-r}\varphi_l|\text{phys}\rangle \,.
\end{aligned}$$

This term above proportional to $(k+p)n$ is key: it will become the central term in the algebra. We'll come back to this shortly. Meanwhile, the first term combines nicely with the second commutator in (17.7). Using the expansion (17.8), it can be written as

$$\begin{aligned}
\varphi_{ia}^\dagger\varphi_{kb}^\dagger[Z_{ac}^m, Z_{bd}^{\dagger n}]\varphi_{jc}\varphi_{ld} &= \varphi_{ia}^\dagger\varphi_{kb}^\dagger\left(\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(Z^rZ^{\dagger s})_{ad}(Z^{\dagger n-1-s}Z^{m-1-r})_{bc}\right)\varphi_{jc}\varphi_{ld} \\
&= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(\varphi_i^\dagger Z^r Z^{\dagger s}\varphi_l)(\varphi_k^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_j) + \delta_{kl}\Big(\cdots\Big)
\end{aligned}$$

where the term proportional to δ_{kl} arises from commuting $\varphi_{kb}^\dagger$ past φ_{ld}. It can be neglected simply because we are ultimately interested in the kl-traceless part of this expression.

The four-φ term above is very close to that appearing in (17.10); it differs only in its $U(p)$ indices and overall sign. Adding the two together gives

$$\begin{aligned}\Theta &= \sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\Big[(\varphi_i^\dagger Z^r Z^{\dagger s}\varphi_l)(\varphi_k^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_j)\\ &\qquad -\delta_{jk}\,(\varphi_i^\dagger Z^r Z^{\dagger s}\varphi_{i'})(\varphi_{i'}^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_l)\Big]\,.\end{aligned}$$

We can manipulate the index structure to exploit this similarity: we separate the two double sums into their trace and traceless parts with respect to δ_{il}, δ_{jk}, $\delta_{ii'}$ and $\delta_{i'l}$. Doing this we find that the products of traces cancel between the two pairs, as do half of the trace-traceless terms, leaving only traceless-traceless terms which we neglect on the grounds that they are subleading in the large N limit. We're left with

$$\begin{aligned}\Theta &\sim \frac{1}{p}\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}\Big[\delta_{il}(\varphi_{i'}^\dagger Z^r Z^{\dagger s}\varphi_{i'})(\varphi_k^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_j)\\ &\qquad -\delta_{jk}(\varphi_{i'}^\dagger Z^r Z^{\dagger s}\varphi_{i'})(\varphi_i^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_l)\Big]\,.\end{aligned}$$

The first term above and the third term in (17.7) are both proportional to δ_{il}; similarly, the second term above and the second term in (17.7) are both proportional to δ_{jk}. In each case, the two terms combine together in the large N limit. This follows from the following identity:

Identity 1: For $m \geq n$,

$$\varphi_i^\dagger Z^{\dagger n}Z^m\varphi_l - \frac{1}{p}\sum_{r=0}^{m-1}\sum_{s=0}^{n-1}(\varphi_{i'}^\dagger Z^r Z^{\dagger s}\varphi_{i'})(\varphi_i^\dagger Z^{\dagger n-1-s}Z^{m-1-r}\varphi_l) \sim \left(\frac{(k+p)N}{p}\right)^n \varphi_i^\dagger Z^{m-n}\varphi_l$$

where $\sim$ again means up to $1/N$ corrections. Further, we are neglecting a trace, proportional to δ_{il} on both sides. A similar expression holds when $n > m$.

The proof of this identity in the classical theory is already somewhat involved, so we relegate it to Appendix D. The additional commutators (17.2) make it much more challenging in the quantum case, and so no proof is included, though it has been checked for small n and m. In what follows, we will make the natural assumption that this identity generalizes directly to the quantum case.

It remains only to discuss the second term in (17.10); this is our central term. We again decompose it into the trace and traceless components with respect to the i, l indices. At large N, the traceless component is subleading; we have

$$(k+p)n\delta_{jk}\sum_{r=0}^{m-1}\varphi_i^\dagger Z^r Z^{\dagger n-1}Z^{r-1-r}\varphi_l \sim \frac{(k+p)n}{p}\delta_{jk}\delta_{il}\sum_{r=0}^{m-1}\varphi_{i'}^\dagger Z^r Z^{\dagger n-1}Z^{m-1-r}\varphi_{i'}\,.$$

To proceed, we need a second large N identity. This time the identity takes a different form in the classical and quantum theories. Evaluated on classical matrices, the identity reads

Identity 2 (Classical Version): For $m \geq n$,

$$\sum_{r=0}^{m-1} \varphi_{i'}^\dagger Z^r Z^{\dagger n-1} Z^{m-1-r} \varphi_{i'} \sim p \left(\frac{(k+p)N}{p}\right)^n \delta_{mn} \,.$$

We present a proof of this identity in Appendix D.

Meanwhile, in the quantum theory there is an extra term which arises due to the shift $k+p \to k$ seen in (17.4). The corresponding large N identity now reads as follows:

Identity 2 (Quantum Version): For $m \geq n$,

$$\sum_{r=0}^{m-1} \varphi_{i'}^\dagger Z^r Z^{\dagger n-1} Z^{m-1-r} \varphi_{i'} \sim p \left(\frac{(k+p)N}{p}\right)^n \left(1 - \frac{p}{k+p}\right) \delta_{mn} \,.$$

We will not offer a general proof of this result in the quantum theory. Nonetheless, as before the existence of the new factor can be checked straightforwardly in a number of simple examples.

Putting all of this together, we arrive at our final result. In the large N limit, up to terms proportional to δ_{ij} and δ_{kl}, we have

$$[\varphi_i^\dagger Z^m \varphi_j, \varphi_k^\dagger Z^{\dagger n} \varphi_l] \;\sim\; \left(\frac{(k+p)N}{p}\right)^n \left[\delta_{jk} \varphi_i^\dagger Z^{m-n} \varphi_l - \delta_{il} \varphi_k^\dagger Z^{m-n} \varphi_j + kn\, \delta_{mn}\, \delta_{jk} \delta_{il}\right] .$$

Written in terms of currents, this is equivalent to the Kac-Moody algebra (17.6).

18 The Partition Function

In this chapter, we recapitulate from [5] the computation of the partition function of the matrix model. In the limit of large particle number, $N \to \infty$, we will show that this partition function is proportional to a character of the chiral $\hat{A}_{p-1}$ current algebra at level k (17.6).

There is a well-established machinery for solving matrix models in the $N \to \infty$ limit; the usual route is through the path integral which, at large N, can typically be evaluated by finding an appropriate saddle point for the Wilson lines arising from the gauge field α. Here we will do better and compute the partition function exactly for all values of N. The resulting formula for the partition function, given in equation (18.19), can then be analysed directly in the large N limit. However, the nature of this limit is subtle; in particular, it depends on the value of N modulo p, and does not seem to have a direct interpretation in terms of a saddle point of the original matrix integral.[1]

In fact, our formula for the partition function of the matrix model can be related [211, 212, 213] to the partition function of a certain integrable lattice model in two dimensions which gives rise to conformal field theory with affine Lie algebra symmetry in the continuum limit. This limit has been studied in detail in [211], and the results therein lead to a closed formula for the large N limit of the matrix model partition function as an affine character.

Our partition function will depend on both the (inverse) temperature β and the chemical potentials μ_i for the $U(1)$ Cartan elements of the $SU(p)$ global symmetry. Including these chemical potentials, the Hamiltonian for the matrix model (16.1) is

$$H = \omega \ \mathrm{Tr}\, Z^\dagger Z - \sum_{i=1}^{p} \mu_i \varphi_i^\dagger \varphi_i \,. \tag{18.1}$$

[1] If one tries to take the standard large N approach, the hurdle is to find a correct way to implement the level constraint (17.4). One can show that integrating out the fundamental matter φ_i results in a Wilson line for the $SU(N)$ gauge field α which sits in the kN^{th} symmetric representation [179]. Because this representation scales with N, it shifts the saddle point in a complicated manner, which seems to make the process much harder. It would be interesting to investigate how to evaluate the partition function using large N techniques.

The Hamiltonian is trivial: it counts the number of $Z^\dagger$ and $\varphi_i^\dagger$ excitations, weighting them by ω and μ_i respectively. Evaluated on any physical state, the Hamiltonian gives

$$H|\text{phys}\rangle = \left(\omega\Delta - \sum_{i=1}^{p} \mu_i J_i\right)|\text{phys}\rangle$$

where the quantum numbers Δ and J_i are integers labelling each state.

Our interest is in the partition function

$$\mathcal{Z}(q, x_i) = \text{Tr}_{\mathcal{H}}\ e^{-\beta H} = \text{Tr}_{\mathcal{H}}\ q^{\Delta} \prod_{i=1}^{p} x_i^{J_i}$$

where Tr is the trace over all states in the physical Hilbert space $\mathcal{H}$, and we define $q = e^{-\beta\omega}$ and $x_i = e^{\beta\mu_i}$.

All the complexity in the problem lies not in the Hamiltonian, but instead in the non-trivial structure of the physical Hilbert space originating in the constraints imposed by the $U(N)$ gauge symmetry. Our strategy is to first enumerate all gauge non-invariant states and only later project onto the gauge invariant subset. With this in mind, we introduce further fugacities for each Cartan element of the gauge symmetry, $U(1)^N \subset U(N)$. We call these fugacities ω_a with $a = 1, \ldots, N$.

If we ignore the restrictions of gauge invariance, then the Hilbert space is simple to define: it consists of any number of $Z_{ab}^\dagger$ or $\varphi_{a\,i}^\dagger$ operators acting on $|0\rangle$. Let's deal with each species of operator in turn. The Z operators lie in the adjoint representation of $U(N)$ and are singlets under $SU(p)$. They carry quantum numbers of $\omega_a^{+1}\omega_b^{-1}$ (for some $a \neq b$) and $\Delta = 1$. Taking the trace over states of the form $Z^{\dagger\,r}|0\rangle$ for all possible r gives the contribution to the partition function of the form

$$\mathcal{Z}_Z = \prod_{a,b=1}^{N} \frac{1}{1 - q\omega_a/\omega_b}\,. \tag{18.2}$$

Meanwhile, the φ operators transform in the fundamental of both $U(N)$ and $SU(p)$. This means that they come with a factor $\omega_a^{+1}x_i^{+1}$ for some a and i. They have $\Delta = 0$. Taking the trace over states of the form $\varphi^{\dagger\,r}|0\rangle$ gives the contribution to the partition function

$$\mathcal{Z}_\varphi = \prod_{a=1}^{N}\prod_{i=1}^{p} \frac{1}{1 - \omega_a x_i}\,. \tag{18.3}$$

We now impose the requirements of gauge invariance. The physical states making up $\mathcal{H}$ must be $SU(N)$ singlets. Further, the level constraint (17.4) requires that they carry

charge k under the $U(1)$ centre of $U(N)$ but are singlets under $SU(N) \subset U(N)$. This can be imposed by contour integration, giving us the expression

$$\mathcal{Z}(q, x_i) = \frac{1}{N!} \left(\prod_{a=1}^{N} \frac{1}{2\pi i} \oint \frac{d\omega_a}{\omega_a^{k+1}} \right) \prod_{b \neq c} \left(1 - \frac{\omega_b}{\omega_c} \right) \mathcal{Z}_Z \, \mathcal{Z}_\varphi \tag{18.4}$$

where the contour of integration is the unit circle in the complex plane for each integration variable. Here the product factor arises from the Haar measure on the group manifold of $U(N)$. The factor of ω^{k+1} in the denominator ensures that the only contributions we pick up in the contour integral are those with correct overall charge.

Our strategy for evaluating the partition function will be to expand the integrand of (18.4) in a suitable basis of polynomials. The integration variables w_a and the fugacities x_i are invariant under permutations corresponding to the Weyl groups of $U(N)$ and $SU(p)$ respectively. This means that the partition function can be expanded in terms of symmetric polynomials. Before proceeding we pause to review some elementary facts about these functions.

18.1 A Digression on Symmetric Functions

In this section we review some standard facts about symmetric functions. For further details and proofs of the statements reviewed below see [214]. As symmetric functions are labelled by partitions we will begin by reviewing basic features of the latter.

A *partition* λ is a non-increasing sequence of non-negative integers,

$$\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_{\ell(\lambda)} > \lambda_{\ell(\lambda)+1} = 0 \, .$$

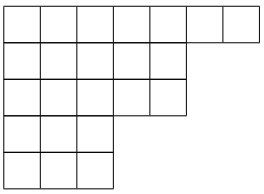
The number $\ell(\lambda)$ of non-zero elements in the sequence is called the *length* of the partition. The sum of all the elements, $|\lambda| = \sum_{i \geq 1} \lambda_i$, is called the *weight* of the partition. We will write $\mathcal{P}$ for the set of all partitions.

The *multiplicity* $m_j(\lambda)$ of the positive integer j is the number of times that j appears in the partition λ; i.e.

$$m_j(\lambda) = |\{i \geq 1 : \lambda_i = j\}| \, .$$

We can specify a partition either by listing its non-zero parts, $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ or by specifying multiplicities. For example the partition $(7, 5, 5, 3, 3)$ can alternatively be written as $(7^1, 5^2, 3^2)$ where the exponent of each entry indicates its multiplicity. We will use this notation extensively below.

A partition λ can be represented graphically by a *Young diagram* $\mathbb{Y}(\lambda)$. This is an array of boxes where the i^{th} row contains λ_i boxes. Each row is aligned so that the left-most boxes sit under each other. For example, the Young diagram for the partition $(7^1, 5^2, 3^2)$ looks like this:



Concretely the set $\mathbb{Y}(\lambda)$ contains boxes $x = (r, s)$ labelled by their coordinate r and s specifying the row and column respectively of the diagram relative to the top left hand corner of the diagram. The Young diagram $\mathbb{Y}(\lambda)$ therefore contains boxes $x = (r, s)$ with $r = 1, \ldots, \ell(\lambda)$ and, for each value of r, $s = 1, \ldots, \lambda_r$.

The *transpose* λ^T of the partition λ is obtained by interchanging the rows and columns of the Young diagram $\mathbb{Y}(\lambda)$. Explicitly the non-zero parts of λ^T are

$$\lambda_i^T = |\{j \geq 1 : \lambda_j \geq i\}|$$

for $i = 1, \ldots, \ell(\lambda^T) = \lambda_1$. For example, $(7^1, 5^2, 3^2)^T = (5^3, 3^2, 1^2)$. Finally, we also define the function $n : \mathcal{P} \to \mathbb{Z}_{\geq 0}$ by

$$n[\lambda] = \sum_{i \geq 1} (i-1)\lambda_i \,.$$

We now turn to symmetric functions. Let $X = \{x_1, \ldots, x_n\}$ denote a set of n variables. A symmetric function $f(X) = f(x_1, x_2, \ldots, x_n)$ is any polynomial of the x_i invariant under the action of the permutation group S_n acting on the variables X, so

$$f\left(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)}\right) = f(x_1, x_2, \ldots, x_n) \qquad \forall \sigma \in S_n \,.$$

We will frequently use the shorthand notation $\sigma\{g(X)\} = g(x_{\sigma(1)}, x_{\sigma(2)}, \ldots, x_{\sigma(n)})$ for the action of a permutation $\sigma \in S_n$ on an arbitrary function $g(X)$.

The set of all symmetric polynomials forms a vector space. Though it is infinite-dimensional, it is naturally written as a direct sum of vector subspaces of finite dimension corresponding to symmetric polynomials of fixed degree. The basis vectors are labelled by partitions $\mu \in \mathcal{P}$ with at most n parts: $\ell(\mu) \leq n$. In particular, one possible choice of basis vectors are monomial symmetric functions, given by

$$m_\mu(X) = \sum_{\sigma \in S_n / S_n^\mu} \sigma\left\{x_1^{\mu_1} x_2^{\mu_2} \ldots x_n^{\mu_n}\right\} \,. \tag{18.5}$$

Here S_n^μ denotes the stabilizer of the monomial $X^\mu = x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n}$ in S_n and thus the sum is taken over distinct permutations $\sigma\{X^\mu\} = x_{\sigma(1)}^{\mu_1} x_{\sigma(2)}^{\mu_2} \dots x_{\sigma(n)}^{\mu_n}$ of the monomial X^μ. The degree of the monomial symmetric function $m_\mu(X)$ corresponds to the weight $|\mu|$ of the permutation μ. One can easily define an inner product on the space of symmetric functions with respect to which the monomial symmetric functions form an orthonormal basis:

$$\langle m_\lambda, m_\mu \rangle \equiv \frac{1}{n!} \left(\prod_{i=1}^{n} \frac{1}{2\pi i} \oint_C \frac{dx_i}{x_i} \right) m_\lambda(X)\, m_\mu\left(X^{-1}\right) = \delta_{\lambda,\mu}\,.$$

Here the contour of integration is the unit circle C in the complex x_i-plane for $i = 1, 2, \dots, n$ and X^{-1} denotes the n variables $\{x_1^{-1}, x_2^{-1}, \dots, x_n^{-1}\}$.

Another possible set of basis vectors for the space of symmetric functions of n variables is provided by the *Schur functions*. For each partition $\mu \in \mathcal{P}$, we define the Schur function

$$s_\mu(X) = \sum_{\sigma \in S_n} \sigma \left\{ x_1^{\mu_1} x_2^{\mu_2} \dots x_n^{\mu_n} \prod_{i>j} \frac{1}{\left(1 - \frac{x_i}{x_j}\right)} \right\}.$$

Although not immediately apparent from this definition, the Schur function $s_\mu(X)$, like the monomial symmetric function $m_\mu(X)$, is a polynomial in the variables X of degree $|\mu|$. The significance of the Schur functions for our problem lies in their close relation to the representation theory of the Lie algebra $u(n)$. The finite-dimensional, irreducible representations of $u(n)$ are inherited from those of its complexification $gl(n, \mathbb{C})$. Each such representation is labelled by a partition λ of length $\ell(\lambda) \leq n$. Equivalently, the representation is labelled by the Young diagram $\mathbb{Y}(\lambda)$. As discussed in more detail below, the Schur function $s_\lambda(X)$, evaluated on the n variables $X = \{x_1, \dots, x_n\}$ is essentially the character of the corresponding representation R_λ. This correspondence is a consequence of the famous Schur-Weyl duality between the representation theory of $u(n)$ and that of the permutation group.

Like the monomial symmetric functions discussed above, the Schur functions provide a basis for the vector space of symmetric functions. Indeed one can construct a "matrix" K giving the explicit linear transformation between these two bases by writing

$$s_\lambda(X) = \sum_\mu K_{\lambda,\mu} m_\mu(X)\,. \tag{18.6}$$

Here $K_{\lambda,\mu}$ is zero unless $|\lambda| = |\mu|$. The non-vanishing entries of $K_{\lambda,\mu}$ are all positive integers, known as *Kostka numbers*. Thinking of s_λ as the character of R_λ, each monomial in the Schur polynomial corresponds to a weight of the representation and the

corresponding coefficient is simply the multiplicity of this weight. More precisely, each monomial symmetric function m_μ appearing on the RHS of (18.6) corresponds to a family of $gl(n,\mathbb{C})$ weights related by the action of the Weyl group. The Kostka number $K_{\lambda,\mu}$ is precisely the common multiplicity of these weights in R_λ.

The Kostka numbers also have a second interpretation in the representation theory of $u(n)$ which will be important below. Let $\mathbf{n}^j$ denote the j^{th} symmetric power of the fundamental representation. Then $K_{\lambda,\mu}$ is the multiplicity of the irreducible representation R_λ in the decomposition of the tensor product

$$\mathcal{T}(\mu) \;=\; \mathbf{n}^{\mu_1}\otimes\mathbf{n}^{\mu_2}\otimes\ldots\otimes\mathbf{n}^{\mu_{\ell(\mu)}}\,. \tag{18.7}$$

The Schur functions form a complete basis for the symmetric functions in n variables. They are orthonormal with respect to a modified inner product $\langle\,,\rangle_S$ defined by

$$\langle s_\lambda, s_\mu\rangle_S \equiv \frac{1}{n!}\left(\prod_{i=1}^n \frac{1}{2\pi i}\oint_C \frac{dx_i}{x_i}\right)\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)\, s_\lambda\left(X\right)\, s_\mu\left(X^{-1}\right) = \delta_{\lambda,\mu}\,. \tag{18.8}$$

In the group theoretic context described above, this relation is just the familiar orthogonality of $U(n)$ characters with respect to integration over the group manifold with the Haar measure. The completeness of the Schur functions as a basis is expressed by the *Cauchy identity*. For any two sets of variables $X=\{x_1,\ldots,x_n\}$ and $Y=\{y_1,\ldots,y_m\}$ we have

$$\prod_{i=1}^n\prod_{j=1}^m \frac{1}{1-x_iy_j} \;=\; \sum_\lambda s_\lambda(X)s_\lambda(Y)\,. \tag{18.9}$$

The sum on the right-hand side can be taken over all partitions λ as the product of Schur functions in the summand will vanish identically for $\ell(\lambda)>\min\{n,m\}$.

As stated above, our goal will be to evaluate the matrix model partition function by expanding the integrand of (18.4) in terms of symmetric functions. Because of the presence in this integrand of the Haar measure factor, together with the adjoint partition function $\mathcal{Z}_Z$, it will be convenient to introduce yet another inner product $\langle\,,\rangle_P$ on the space of symmetric functions depending on an arbitrary complex parameter q. For any two symmetric functions $f(X)$ and $g(X)$ we define

$$\langle f,g\rangle_P \;\equiv\; \frac{1}{n!}\left(\prod_{i=1}^n \frac{1}{2\pi i}\oint_C \frac{dx_i}{x_i}\right)\,\frac{\prod_{i\neq j}\left(1-\frac{x_i}{x_j}\right)}{\prod_{i\neq j}\left(1-q\frac{x_i}{x_j}\right)}\, f\left(X\right)\, g\left(X^{-1}\right)\,. \tag{18.10}$$

Note that our new inner product reduces to $\langle\,,\rangle_S$ in the special case $q=0$. Can we find a new set of basis functions, generalizing the Schur functions, which are orthogonal with

respect to new measure with $q \neq 0$? In fact the *Hall-Littlewood polynomials* have exactly this property. Moreover, many of the properties of the Schur polynomials discussed above are generalized in a nice way. For each partition $\lambda \in \mathcal{P}$ we define

$$P_\lambda(X;q) \;=\; \frac{1}{\mathcal{N}_\lambda} \sum_{\sigma \in S_n} \sigma \left\{ x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \prod_{i>j} \frac{\left(1 - q\frac{x_i}{x_j}\right)}{\left(1 - \frac{x_i}{x_j}\right)} \right\} . \tag{18.11}$$

The normalization factor is given by

$$\mathcal{N}_\lambda = \frac{\varphi_{n-\ell(\lambda)} \prod_{j\geq 1} \varphi_{m_j(\lambda)}}{(1-q)^n}$$

where

$$\varphi_m = \prod_{j=1}^{m} \left(1 - q^j\right)$$

and $m_j(\lambda)$ denotes the multiplicity of the positive integer j in the partition λ as defined above. As before, $P_\lambda(X;q)$, is a homogeneous polynomial in the variables X of degree $|\lambda|$. It is useful to rewrite the definition (18.11) as

$$P_\lambda(X;q) \;=\; \sum_{\sigma \in S_n/S_n^\lambda} \sigma \left\{ x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n} \prod_{\lambda_i<\lambda_j} \frac{\left(1 - q\frac{x_i}{x_j}\right)}{\left(1 - \frac{x_i}{x_j}\right)} \right\} \tag{18.12}$$

where the sum is over distinct permutations of the monomial $X^\lambda = x_1^{\lambda_1} x_2^{\lambda_2} \dots x_n^{\lambda_n}$.

As already mentioned, we have the orthogonality property

$$\langle P_\lambda, P_\mu \rangle_P = \frac{1}{\mathcal{N}_\lambda}\, \delta_{\lambda,\mu}\,. \tag{18.13}$$

An even more striking fact is that, with the given normalization, each term in $P_\lambda(X;q)$ is itself a polynomial in the parameter q with integer coefficients. One might instead choose to normalize these functions to achieve orthonormality with respect to the inner product $\langle\,,\rangle_P$; however then the basis functions would no longer be polynomial in q.

The completeness of the resulting basis is expressed in a generalization of the Cauchy identity. As before we consider two sets of variables, called $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_m\}$. We now have

$$\prod_{i=1}^{n} \prod_{j=1}^{m} \frac{1 - q x_i y_j}{1 - x_i y_j} \;=\; \sum_\lambda b_\lambda(q) P_\lambda(X;q) P_\lambda(Y;q) \tag{18.14}$$

where $b_\lambda(q) = \prod_{j\geq 1} \varphi_{m_j(\lambda)}(q)$. From the definition (18.11), the Hall-Littlewood polynomial $P_\lambda(X;q)$ reduces to the Schur function $s_\lambda(X)$ for $q = 0$. On setting $q = 1$ in the definition, we also find $P_\lambda(X,1) = m_\lambda(X)$ where m_μ is the monomial symmetric function defined in (18.5) above. Again we can find a "matrix" describing the change of basis from Schur to Hall-Littlewood. Relation (18.6) is now generalized to

$$s_\lambda(X) = \sum_\mu K_{\lambda,\mu}(q) P_\mu(X;q) \ . \tag{18.15}$$

For each choice of λ, $\mu \in \mathcal{P}$, the matrix elements $K_{\lambda,\mu}(q)$ are polynomials in the parameter q. They are known as *Kostka polynomials* (see e.g. Chapter III.6 of [214]) and they will play a central role in our evaluation of the partition function. An explicit combinatoric formula for the Kostka polynomials due to Kirillov and Reshetikhin is given in Appendix E. Here we will list some of their main features[2]:

- They are polynomials in q of degree $n[\mu] - n[\lambda]$ with leading coefficient equal to unity.
- All non-zero coefficients are positive integers.
- $K_{\lambda,\mu}(q) = 0$ unless $|\lambda| = |\mu|$.
- They reduce to the Kostka numbers for $q = 1$: $K_{\lambda,\mu}(1) = K_{\lambda,\mu}$ for all partitions λ and μ.
- $K_{\lambda,\mu}(0) = \delta_{\mu,\nu}$.

These properties ensure that the Kostka polynomials can be regarded as a graded generalization of the Kostka numbers. As the Kostka numbers $K_{\lambda,\mu}$ count the number of occurrences of the representation R_λ in the tensor product $\mathcal{T}(\mu)$ defined in (18.7), the corresponding Kostka polynomial $K_{\lambda,\mu}(q)$ receives a contribution q^Δ for some $\Delta \in \mathbb{Z}_{\geq 0}$, for each such occurrence. Hence as we vary the partition λ, the Kostka polynomial assigns an integer-valued "energy" $\Delta(\lambda,\mu)$ to each irreducible component of the tensor product $\mathcal{T}(\mu)$. It is useful to think of the representation space of $\mathcal{T}(\mu)$ as the Hilbert space of a spin chain with $\ell(\mu)$ sites with a $u(n)$ spin in the representation $\mathfrak{n}^{\mu_i}$ at the i^{th} site. Remarkably, the energy Δ precisely corresponds to one of the Hamiltonians of the Heisenberg spin chain with these spins. In fact Δ is essentially the lattice momentum for a spin chain with periodic boundary conditions. The Bethe ansatz solution of this system provides an efficient combinatoric description of the corresponding Kostka polynomials and leads directly to the explicit formulae given in Appendix E.

[2]The last two listed properties follow easily from the definition (18.15) and the third follows from the invertibility of the change of basis proven in [214] (see Eqn (2.6) in Chapter III of this reference). The first two properties are highly non-trivial and were first proven in [215].

To evaluate the partition function we will need one more class of symmetric functions known as[3] *Modified Hall-Littlewood polynomials* [216, 217] $Q'_\mu(X;q)$. For our purposes, this polynomial is defined by the formula

$$Q'_\mu(X;q) = \sum_\lambda K_{\lambda,\mu}(q) s_\lambda(X) \ . \tag{18.16}$$

Importantly, this definition yields a non-zero answer for partitions μ of any length.[4] Thus, unlike the other symmetric functions defined above, $Q'_\mu(X;q)$ does not vanish identically[5] for partitions with $\ell(\mu) > n$.

As discussed above, each Schur function s_λ is the character of a $u(n)$ representation R_λ. Meanwhile, the Kostka polynomial $K_{\lambda,\mu}(q)$ is non-zero only for irreducible representations R_λ occurring in the tensor product $\mathcal{T}(\mu)$ defined in (18.7). Further, for each λ, $K_{\lambda,\mu}(q)$ receives a contribution q^Δ for each occurrence of the irrep R_λ in $\mathcal{T}(\mu)$ where Δ is the appropriate spin chain Hamiltonian. Putting these facts together we learn that $Q'(X;q)$ has a natural interpretation as the partition function of a spin chain defined on the tensor product space $\mathcal{T}(\mu)$.

Using the properties of the Schur functions and Kostka polynomials, we see that $Q'_\mu(X;q)$ is a homogeneous polynomial in the variables $X = \{x_1, \ldots, x_n\}$ of degree $|\mu|$. Moreover the coefficients are themselves polynomials in q with positive integer coefficients. The polynomial Q'_μ has the following key property: it is adjoint to the ordinary Hall-Littlewood polynomials P_μ with respect to the inner product $\langle\, , \rangle_S$ for Schur functions [217]. For any two sets of variables $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$ we have

$$\begin{aligned} \prod_{i=1}^n \prod_{j=1}^m \frac{1}{1 - x_i y_j} &= \sum_\lambda s_\lambda(X) s_\lambda(Y) \\ &= \sum_{\lambda,\rho} K_{\lambda,\rho}(q)\, P_\rho(Y;q)\, s_\lambda(X) \\ &= \sum_\rho Q'_\rho(X;q) P_\rho(Y;q) \end{aligned} \tag{18.17}$$

using (18.15) and (18.16). The final sum on the RHS can be taken over all partitions ρ but the summand will vanish unless $\ell(\rho) \leq m$.

[3] These are also sometimes referred to as *Milne polynomials* in the mathematical literature.

[4] This follows because, although the RHS of (18.16) vanishes identically for partitions λ of length greater than n, there is no such constraint for the partition μ appearing in the Kostka polynomial $K_{\lambda,\mu}(t)$. Indeed we evaluate several examples of this type in the following using the combinatorial algorithm of Appendix E.

[5] Note that in some references the definition of the modified Hall-Littlewood polynomial in n variables is nevertheless restricted to the case $\ell(\mu) \leq n$.

18.2 Back to the Partition Function

We are now ready to compute the partition function $\mathcal{Z}$ defined in (18.4). The partition function is symmetric in the $u(p)$ fugacities $X = \{x_1, x_2, \ldots, x_p\}$ so we can expand it in terms of Schur functions. As each Schur function corresponds to the character of a finite-dimensional irreducible representation of $U(p)$, the resulting expansion determines the multiplets of the $SU(p) \subset U(p)$ global symmetry present in the matrix model spectrum. The integrand of the partition function is also a symmetric function of the $U(N)$ fugacities $\Omega = \{\omega_1 \ldots, \omega_N\}$ and we may thus expand it in terms of a suitable set of basis functions.

To proceed to the answer by the shortest path, we will use the Cauchy identity in the form (18.17) to expand the factor of the integrand corresponding to the fundamental-valued fields as

$$\mathcal{Z}_\varphi = \prod_{a=1}^{N} \prod_{i=1}^{p} \frac{1}{1-\omega_a x_i} = \sum_\lambda Q'_\lambda(X;q) P_\lambda(\Omega;q)\,.$$

In contrast, the corresponding factor $\mathcal{Z}_Z$ for the adjoint-valued field will be left unexpanded as part of the integration measure. For the next step, we use the definition of the Hall polynomials in its second form (18.12) to write

$$\frac{1}{\prod_{a=1}^{N} \omega_a^k} = P_{(k^N)}\left(\Omega^{-1};q\right)$$

where, as above, (k^N) denotes the partition with N non-zero parts each equal to k. The resulting integral over the variables Ω can then be written, using (18.13), as an inner product

$$\begin{aligned}
\mathcal{Z} &= \sum_\lambda Q'_\lambda(X;q) \;\times\; \frac{1}{N!}\left(\prod_{a=1}^{N} \frac{1}{2\pi i}\oint_C \frac{d\omega_a}{\omega_a}\right) \frac{\prod_{a\neq b}\left(1-\frac{\omega_a}{\omega_b}\right)}{\prod_{a,b}\left(1-q\frac{\omega_a}{\omega_b}\right)} P_\lambda\left(\Omega;q\right)\; P_{(k^N)}\left(\Omega^{-1};q\right) \\
&= \sum_\lambda Q'_\lambda(X;q) \;\times\; \frac{1}{(1-q)^N}\langle P_\lambda, P_{(k^N)}\rangle_P \qquad (18.18) \\
&= \frac{1}{\varphi_N(q)}\; Q'_{(k^N)}(X;q)\,.
\end{aligned}$$

Thus our final result for the partition function of the matrix model is

$$\mathcal{Z} \;=\; \prod_{j=1}^{N} \frac{1}{(1-q^j)} \;\sum_\lambda K_{\lambda,(k^N)}(q)\, s_\lambda(X) \qquad (18.19)$$

where the sum on the right-hand side runs over all partitions $\lambda \in \mathcal{P}$. However, as explained above the summand vanishes unless we have $|\lambda| = |(k^N)| = kN$ and $\ell(\lambda) \leq p$.

Ground State Energy

Our result for the partition function (18.19) holds for all positive integral values of the level, k, the rank p of the global symmetry and the particle number N. In the remainder of this section we will extract the ground state energy $E_0(k, p, N)$, which simply corresponds to the leading power of q appearing in the expansion of the partition function for $|q| \ll 1$, and compare it with our expectations based on the analysis of the ground state from Part IV.

We begin with the Abelian case $p = 1$. In this case there is only one partition $\lambda = (kN)$ which satisfies the conditions $|\lambda| = kN$, and $\ell(\lambda) \leq 1$. In this special case, the formulae for the Kostka polynomial given in Appendix E simplify, giving[6]

$$K_{(kN),(k^N)}(q) = q^{n\left[(k^N)\right]}$$

where, as above, $n[\mu] = \sum_{i\geq 1}(i-1)\mu_i$. Thus the partition function for $p = 1$ reads

$$\mathcal{Z}_{p=1} = x^{kN} q^{\frac{k}{2}N(N-1)} \prod_{j=1}^{N} \frac{1}{(1-q^j)} \tag{18.20}$$

where $x = x_1$ acts as a fugacity for the $U(1)$ charge which is fixed by the D-term constraint. The ground state energy,

$$E_0(k, 1, N) \;=\; \frac{k}{2}N(N-1)\,,$$

agrees with the identification of the ground state given in Part IV. The remaining factor in (18.20) is a plethystic exponential accounting for excitations corresponding to all possible products of the N independent single-trace operators $\mathrm{Tr}(Z^l)$ with $l = 1, 2, \ldots, N$. This partition function for the $p = 1$ matrix model was previously computed in [163].

The partition function for general $p \geq 1$ is somewhat richer; it also depends on the fugacities x_i for the $SU(p)$ Cartan elements. To understand the form of this partition function, we start by recalling that the Kostka polynomial $K_{\lambda,(k^N)}(q)$ specializes for $q = 1$ to the Kostka number $K_{\lambda,(k^N)}$. This in turn coincides with the multiplicity of the irreducible representation of $u(p)$ specified by the partition λ in the tensor product

$$\mathcal{T}_N = \mathbf{p}^k \otimes \mathbf{p}^k \otimes \ldots \otimes \mathbf{p}^k$$

of N copies of the k^{th} symmetric power of the fundamental representation $\mathbf{p}$. The corresponding Kostka polynomial is a gradation of the Kostka number where each power of q appears with a non-negative integer coefficient. As the Kostka polynomial $K_{\lambda,(k^N)}(q)$

[6]Here we see explicitly that $K_{\lambda,\mu}(t)$ can be non-zero when $\ell(\mu) > \ell(\lambda)$ as mentioned above.

appears in (18.19) multiplied by the corresponding Schur function $s_\lambda(X)$, we deduce that the full spectrum of the matrix model transforms in the reducible $u(p)$ representation $\mathcal{T}_N$. More precisely, the overall prefactor of $\prod_{j=1}^N (1-q^j)^{-1}$ means we actually have an infinite number of copies of this representation. The additional information contained in the partition function is the energy of each irreducible component in the tensor product.

To go further we will need to use the combinatoric description of the Kostka polynomials given in Appendix E. We will start with the easiest case $k = 1$ where an explicit formula is available. Here we have

$$K_{\lambda,(1^N)}(q) \;=\; q^{n\left[\lambda^T\right]}\,\frac{\prod_{j=1}^N (1-q^j)}{H(q)}$$

where $H(q)$ is the hook-length polynomial given by

$$H(q) \;=\; \prod_{x\in\mathbb{Y}(\lambda)} \left(1-q^{h(x)}\right) .$$

Here the product is over the boxes $x = (r, s)$ of the Young diagram $\mathbb{Y}(\lambda)$ corresponding to the partition λ and $h(x) = \lambda_r + \lambda_s^T - r - s + 1 > 0$ is the length of the hook passing through box x.

As we described above, λ^T denotes the *transpose* of the partition λ, obtained by interchanging the rows and columns of the Young diagram $\mathbb{Y}(\lambda)$. Explicitly the non-zero parts of λ^T are

$$\lambda_i^T = |\{j \geq 1 : \lambda_j \geq i\}|$$

for $i = 1, \ldots, \ell(\lambda^T) = \lambda_1$. To find the ground state energy of the model we must therefore minimize the quantity

$$n\left[\lambda^T\right] = \sum_{i\geq 1}(i-1)\lambda_i^T$$

as we vary λ over partitions with $|\lambda| = N$ and $\ell(\lambda) \leq p$. These restrictions correspond to demanding that $|\lambda^T| = |\lambda| = N$ and that $\lambda_i^T \leq p$ for all $i \geq 1$. Writing $N = Lp + M$ for non-negative integers L and $M < p$, the minimum occurs for the partition $\lambda_0^T = (p^L, M)$ corresponding to

$$\lambda_0 = \left((L+1)^M, L^{p-M}\right) .$$

Thus the leading term in the partition sum for $|q| \ll 1$ is

$$\mathcal{Z} \simeq q^{E_0(N)}\, s_{((L+1)^M, L^{p-M})}\,(x_1, \ldots, x_p)$$

with vacuum energy

$$E_0(1, p, N) \;\; = \;\; n\left[(p^L, M)\right] = \frac{1}{2}L(L-1)p + LM$$

for $N = Lp + M$. The Schur polynomial corresponds to the representation of $U(p) \simeq U(1) \times SU(p)$ with $U(1)$ charge N, which coincides with the M^{th} antisymmetric power of the fundamental. Again, this yields complete agreement with the properties of the ground state discussed in Part IV.

Although the formulae for the Kostka polynomials are more complicated, the generalization of this analysis to $k > 1$ is straightforward. As we discuss in Appendix E, the minimum energy is obtained for the partition

$$\lambda_0 = \left((kL + k)^M, (kL)^{p-M}\right)$$

and takes the value

$$E_0(k, p, N) \;\; = \;\; \frac{k}{2}L(L-1)p + kLM$$

for $N = Lp + M$. The resulting ground state has $U(1)$ charge kN and transforms in an irreducible representation of $SU(p)$ corresponding to a k-fold symmetrization of the M^{th} antisymmetric power of the fundamental representation. This is the representation (16.2) that we mentioned in the introduction; it is in agreement with the results of Part IV.

18.3 The Continuum Limit

In this section we will investigate the $N \to \infty$ limit of the partition function (18.19). As we have just seen, the ground state energy and its quantum numbers under the global $U(p)$ symmetry depend sensitively on the value of N modulo p. This means that in order to get a sensible limit, we must hold this value fixed as $N \to \infty$. Setting

$$N = Lp + M$$

for non-negative integers $M < p$ and L, we therefore take the limit $L \to \infty$ with M and p held fixed.

It is also convenient to factorize the partition function as

$$\mathcal{Z} \;=\; q^{E_0} w^{kN/p} \hat{\mathcal{Z}}$$

where

$$E_0 \;=\; E_0(k,p,N) = \frac{k}{2} L(L-1)p + kLM$$

is the ground state energy and $w = x_1 x_2 \ldots x_p$ is the fugacity for the $u(1)$ centre of $u(p)$. The reduced partition function

$$\hat{\mathcal{Z}} \;=\; \prod_{j=1}^{N} \frac{1}{(1-q^j)}\; q^{-E_0} \sum_{\lambda} K_{\lambda,(k^N)}(q)\, w^{-kN/p} s_\lambda(X) \tag{18.21}$$

thus encodes the energies and $u(1)$ charges of the states in the spectrum relative to those of the ground state. As we will see, it is $\hat{\mathcal{Z}}$ rather than the original partition function which has a non-singular $N \to \infty$ limit. The main result of this section is (18.30) which says (roughly) that

$$\lim_{N\to\infty} \hat{\mathcal{Z}} = \prod_{j=1}^{\infty} \frac{1}{(1-q^j)} \;\; \chi_{R_{k,M}}(q;X)\ . \tag{18.22}$$

Here the "roughly" refers to a slight notational subtlety regarding the difference between the $U(p)$ fugacities labelled by X and the $SU(p)$ fugacities; this will be explained below. The key part of the result is that $\chi_{R_{k,M}}$ denotes the character of the $\hat{A}_p$ affine Lie algebra at level k associated to the representation $R_{k,M}$ of $SU(p)$. When $M = 0$, so N is divisible by p, this is the vacuum character which coincides with the partition function of the WZW model. Meanwhile, for $M \neq 0$, $R_{k,M}$ is the k-fold symmetrization of the M^{th} antisymmetric representation, namely (16.2). The remainder of this section is devoted to the derivation of (18.22).

As discussed above, each Schur function $s_\lambda(X)$ appearing in the sum on the RHS of (18.21) is the character of the irreducible representation of $u(p)$ corresponding to the partition λ. In the following it will be convenient to decompose the global symmetry as $u(p) \simeq u(1) \oplus su(p)$. Recall that the finite-dimensional, irreducible representations of $su(p)$ are in one-to-one correspondence with Young diagrams having at most $p-1$ rows or, equivalently with partitions $\tilde{\lambda}$ having $\ell(\tilde{\lambda}) < p$. In contrast, representations of $u(p)$ correspond to diagrams with at most p rows or to partitions λ with $\ell(\lambda) \leq p$. Given an irreducible representation of $u(p)$, we obtain a unique irreducible representation of $su(p)$ by removing all columns of height p from the corresponding Young diagram. Similarly, for any partition λ with $\ell(\lambda) \leq p$ we may find a unique partition $\tilde{\lambda}$ with $\ell(\tilde{\lambda}) < p$ such that $\lambda_i = \tilde{\lambda}_i + Q$ with $i = 1, \ldots, p$ for some non-negative integer Q. In the

following we will abbreviate this relation as

$$\lambda = \tilde{\lambda} + (Q^p)\,. \tag{18.23}$$

The Kostka polynomial $K_{\lambda,(k^N)}(q)$ is only non-zero if $|\lambda| = |(k^N)| = kLp + kM$. Given any partition $\tilde{\lambda}$ with $\ell(\tilde{\lambda}) < p$ and $|\tilde{\lambda}| \leq kN$, we may find a unique partition $\lambda = \tilde{\lambda} + (Q^p)$ obeying this constraint if and only if $|\tilde{\lambda}| - kM$ is divisible by p, in which case we set

$$Q = Q(\tilde{\lambda}) = kL - \frac{1}{p}\Big(|\tilde{\lambda}| - kM\Big)\,. \tag{18.24}$$

As the Schur function $s_\lambda(X)$ is a homogeneous polynomial of degree $|\lambda|$ in the variables $X = \{x_1, x_2, \ldots, x_p\}$, we may write

$$s_\lambda(X) = w^{|\lambda|/p} s_{\tilde{\lambda}}(\tilde{X})$$

where

$$\tilde{X} = X/w^{1/p} = \{x_1 w^{-1/p}, \ldots, x_p w^{-1/p}\}\,. \tag{18.25}$$

In particular note that, by construction, $\tilde{x}_1\tilde{x}_2\ldots\tilde{x}_p = 1$, which implies $s_\lambda(\tilde{X}) = s_{\tilde{\lambda}}(\tilde{X})$. Using the above results, we can trade the sum over all partitions λ appearing in (18.21) for a sum over $\tilde{\lambda}$ of length $\ell(\tilde{\lambda}) < p$. This gives

$$\hat{\mathcal{Z}} \;=\; \prod_{j=1}^{N} \frac{1}{(1-q^j)}\; q^{-E_0} \sum_{\tilde{\lambda},\ \ell(\tilde{\lambda})<p} K_{\tilde{\lambda}+(Q(\tilde{\lambda})^p),(k^N)}(q)\, s_{\tilde{\lambda}}(\tilde{X}) \tag{18.26}$$

where $Q(\tilde{\lambda})$ is a non-negative integer given by (18.24) when $|\tilde{\lambda}| \equiv kM \pmod p$ and $|\tilde{\lambda}| \leq kN$, and is set to zero otherwise.

It will also be useful to make the relation between Schur functions and the characters of the simple Lie algebra $su(p)$ and its complexification $A_{p-1} = sl(p,\mathbb{C})$ more explicit (see Appendix F for the Lie algebra conventions we are using here). The finite-dimensional irreducible representations of A_{p-1} are of course labelled by dominant integral weights $\Lambda \in \mathcal{L}_W^+$. Each such Λ has an expansion in terms of the fundamental weights $\{\Lambda_{(1)}, \ldots, \Lambda_{(p-1)}\}$. We write

$$\Lambda = \sum_{j=1}^{p-1} \psi_j \Lambda_{(j)}$$

with coefficients $\psi_j \in \mathbb{Z}_{\geq 0}$ known as Dynkin labels. Let R_Λ denote the corresponding A_{p-1} representation with representation space $\mathcal{V}_\Lambda$. The *character* of R_Λ is a function

of variables $Z = \{z_1, \ldots, z_{p-1}\}$ which encodes the weights of this representation or, equivalently, the eigenvalues of the matrices $R_\Lambda(h^i)$ representing the Cartan subalgebra generators h^i with $i = 1, \ldots, p-1$ in the Chevalley basis. Explicitly we define

$$\chi_\Lambda(Z) \;=\; \mathrm{Tr}_{\mathcal{V}_\Lambda}\left[\prod_{j=1}^{p-1} z_j^{R_\Lambda(h^j)}\right].$$

As mentioned above irreducible representations of $su(p)$ can also be labelled by partitions $\tilde{\lambda}$ with $\ell(\tilde{\lambda}) < p$. For each dominant integral weight $\Lambda \in \mathcal{L}_W^+$, the corresponding partition $\tilde{\lambda}(\Lambda)$ has parts $\tilde{\lambda}_i = \sum_{j=i}^{p-1} \psi_j$. The character of R_Λ can then be related to the Schur function of the partition $\tilde{\lambda}(\Lambda)$:

$$\chi_\Lambda(Z) = s_{\tilde{\lambda}(\Lambda)}(\tilde{X})$$

where the variables $\tilde{X} = \{\tilde{x}_1, \ldots, \tilde{x}_p\}$, obeying $\tilde{x}_1\tilde{x}_2\ldots\tilde{x}_p = 1$, are related to $Z = \{z_1, \ldots, z_{p-1}\}$ by

$$\begin{aligned} z_1 &= \tilde{x}_1 \\ z_2 &= \tilde{x}_1\tilde{x}_2 \\ &\vdots \\ z_{p-1} &= \tilde{x}_1\tilde{x}_2\ldots\tilde{x}_{p-1}\,. \end{aligned} \tag{18.27}$$

Irreducible representations R_Λ of A_{p-1} are further classified by the *congruence class* of the corresponding highest weight Λ, given in terms of the Dynkin labels by the value $\mathbb{P}(\Lambda)$ of $\psi_1 + 2\psi_2 + \ldots (p-1)\psi_{p-1}$ modulo p. Equivalently $\mathbb{P}(\Lambda)$ is equal modulo p to the weight $|\tilde{\lambda}|$ of the partition $\tilde{\lambda}$ corresponding to Λ or to the number of boxes in the corresponding Young diagram. We denote by $\mathcal{L}_W^+(\mathbb{P})$ the subset of the positive weight lattice $\mathcal{L}_W^+$ corresponding to positive weights in $\mathbb{P}^{\text{th}}$ congruence class.

The above results mean that we can rewrite the sum over partitions $\tilde{\lambda}$ appearing in the reduced partition function (18.26) as a sum over dominant integral weights Λ of A_{p-1} in the congruence class $\mathbb{P}(\Lambda) = kM$ mod p. Our final rewriting of the reduced partition function is

$$\hat{\mathcal{Z}} \;=\; \prod_{j=1}^{N} \frac{1}{(1-q^j)} \sum_{\Lambda \in \mathcal{L}_W^+(kM)} K_\Lambda(q)\chi_\Lambda(Z)\,. \tag{18.28}$$

Here

$$K_\Lambda(q) \;=\; q^{-E_0(k,p,N)} K_{\lambda(\Lambda),(k^N)}(q)$$

and

$$\lambda(\Lambda) = \tilde{\lambda}(\Lambda) + (Q(\Lambda)^p)$$

where

$$Q(\Lambda) = \max\left\{0, kL - \frac{1}{p}\left(|\tilde{\lambda}(\Lambda)| - kM\right)\right\} .$$

We will now switch gears and consider something seemingly quite unrelated to the above discussion; the representation theory of the affine Lie algebra $\hat{A}_{p-1}$. (Again, see Appendix F for conventions.) We work in a Chevalley basis with generators $\{h^i, e^i, f^i\}$ where the index i now runs from zero to $p-1$. A complete basis also includes the *derivation* or grading operator L_0 associated with the imaginary root.

The weights of any representation of $\hat{A}_{p-1}$ lie in the affine weight lattice, whose basis vectors are the fundamental weights $\hat{\Lambda}_{(j)}$ with $j = 0, 1, \ldots, p-1$. The *integrable* representations of $\hat{A}_{p-1}$ are labelled by a highest weight

$$\hat{\Lambda} = \sum \hat{\psi}_j \hat{\Lambda}_{(j)}$$

whose p Dynkin labels $\{\hat{\psi}_j\}$ are non-negative integers. Each integrable representation has a definite *level* which is a non-negative integer given by the sum of the Dynkin indices,

$$k = \hat{\psi}_0 + \hat{\psi}_1 + \ldots + \hat{\psi}_{p-1} .$$

The resulting representations $R_{\hat{\Lambda}}$ are the affine analogues of the finite-dimensional irreducible representations R_Λ of the simple Lie algebra A_{p-1} discussed above. We denote the corresponding representation space $\mathcal{V}_{\hat{\Lambda}}$. The character $\chi_{\hat{\Lambda}}(q; Z)$ of the representation $R_{\hat{\Lambda}}$ is a function of the variables q and $Z = \{z_1, \ldots, z_{p-1}\}$ which encodes the weights of the representation or, equivalently, the eigenvalues of the representatives of the Cartan generators h^i, for $i = 1, \ldots p-1$ of the global subalgebra $A_{p-1} \subset \hat{A}_{p-1}$ together with those of the derivation L_0 acting in $\mathcal{V}_{\hat{\Lambda}}$. Explicitly we define

$$\chi_{\hat{\Lambda}}(q; Z) \;\; = \;\; \mathrm{Tr}_{\mathcal{V}_{\hat{\Lambda}}}\left[q^{-R_{\hat{\Lambda}}(L_0)} \prod_{j=1}^{p-1} z_j^{R_{\hat{\Lambda}}(h^j)}\right] .$$

Any representation of $\hat{A}_{p-1}$ must also provide a representation of the global subalgebra A_{p-1}. Thus the affine character must have an expansion in terms of A_{p-1} characters of

the form

$$\chi_{\hat{\Lambda}}(q;Z) \;\; = \;\; \sum_{\Lambda\in\mathcal{L}_W^+} b_{\hat{\Lambda}}^{\Lambda}(q)\,\chi_{\Lambda}(Z)\,.$$

The coefficients $b_{\hat{\Lambda}}^{\Lambda}(q)$ are polynomials in q with non-negative integral coefficients. They are known as the *branching functions* for the embedding of A_{p-1} in $\hat{A}_{p-1}$. Another way to characterize them is to pick out only those vectors in the representation space $\mathcal{V}_{\hat{\Lambda}}$ which are highest weight with respect to the global generators. Thus we define, for each dominant integral weight Λ of A_{p-1} with $\Lambda = \sum_{i=1}^{p-1}\psi^i\Lambda_{(i)}$, the following subspace:

$$\mathcal{V}_{\hat{\Lambda}}^{\Lambda} \;\; = \;\; \{|\hat{\lambda}\rangle \in \mathcal{V}_{\hat{\Lambda}} : h^i|\hat{\psi}\rangle = \psi^i|\hat{\psi}\rangle, e^i|\hat{\psi}\rangle = 0 \;\; i=1,\ldots,p-1\}\,.$$

Then we have

$$b_{\hat{\Lambda}}^{\Lambda}(q) = \mathrm{Tr}_{\mathcal{V}_{\hat{\Lambda}}^{\Lambda}}\left[q^{-R_{\hat{\Lambda}}(L_0)}\right]\,.$$

Remarkably a relation between the large $N = Lp + M$ limit (with fixed M and p) of the object $K_\Lambda(q)$ defined in (18.28) above and a particular affine branching function of $\hat{A}_{p-1}$ is obtained in [211], proving an earlier conjecture of [218]. In particular, we must consider the integrable representation $R_{\hat{\Lambda}}$ with $\hat{\Lambda} = k\Lambda_{(M)}$. The primary states in the representation (i.e. those with the lowest L_0 eigenvalue) transform in the A_{p-1} representation with $\Lambda_0 = k\Lambda_{(M)}$; this corresponds to the k-fold symmetrization of the M^{th} antisymmetric power of the fundamental representation. This is indeed the expected representation (16.2) for the ground state of the model. This representation has congruence class $\mathbb{P}(\Lambda_0) \equiv kM \pmod p$ and the remaining dominant integral weights Λ for which $b_{\hat{\Lambda}}^{\Lambda}(q)$ is non-zero necessarily lie in the same congruence class. Corollary 4.8 of [211] states that, for all $\Lambda \in \mathcal{L}_W^+(kM)$, we have

$$\lim_{N\to\infty} \;\; K_\Lambda(q) \;\; = \;\; b_{k\Lambda_{(M)}}^{\Lambda}(q)\,. \tag{18.29}$$

This result has its origin [212] in the relation between the Kostka polynomials and the partition function of an integrable A_{p-1} spin chain to which we alluded above. Under favourable conditions, the relevant spin chain is believed to go over to the $SU(p)$ Wess-Zumino-Witten model in the continuum limit [219, 220]. Kostka polynomials also appear [213] in the partition function of the so-called *RSOS models*, which yield coset conformal field theories with affine Lie algebra symmetry in the continuum limit.

Incorporating the above limit in the reduced partition function as given in (18.28) we reach our final result

$$\lim_{N\to\infty} \hat{\mathcal{Z}} = \prod_{j=1}^{\infty}\frac{1}{(1-q^j)} \;\; \chi_{k\Lambda_{(M)}}(q;Z) \tag{18.30}$$

where the variables $Z = \{z_1, \ldots, z_{p-1}\}$ are related to the $su(p)$ fugacities of the matrix model by equations (18.25) and (18.27). The prefactor encoding the excitation spectrum of the $u(1)$ sector of the model precisely corresponds to the partition function of a chiral boson.

PART VI

Quantum Supergravity

19 Introduction and Summary

The purpose of this part of the dissertation is to study four-dimensional $\mathcal{N} = 1$ supergravity compactified on a spatial circle. We will see that this background is quantum mechanically unstable: the circle dynamically expands and the ground state is Minkowski space with all three spatial dimensions non-compact. To see this, we need to explore the non-perturbative physics of gravity.

Quantum mechanical instabilities of Kaluza-Klein compactifications have a long history. In the absence of supersymmetry, a Casimir force is generated perturbatively with a competition between bosonic fields, which cause the circle to contract, and fermionic fields which cause the circle to expand [41, 42]. More scary instabilities lurk at the non-perturbative level, with space teetering on the brink of tunnelling into a bubble of nothing [221].

The existence of supersymmetry removes both instabilities described above.[1] But another remains. As we show in some detail, a Casimir force is now generated by gravitational instantons. This results in a superpotential which schematically takes the form

$$\mathcal{W} \sim \exp\left(-\frac{\pi R^2}{4G_N} - i\sigma\right) \tag{19.1}$$

where R is the radius of the spatial circle and σ is dual to the Kaluza-Klein photon, $\mathrm{d}\sigma \sim {}^\star F$. The existence of the superpotential (19.1) was first proposed in [222] on the basis of fermion zero mode counting. It is also closely related to the superpotentials arising from D6-brane instantons wrapping G2-holonomy manifolds described in [223]. Our goal is to develop the full quantum supergravity computation which results in (19.1).

One motivation for performing the instanton calculation in some detail is that $\mathcal{N} = 1$ supergravity offers a testing ground in which some of the old ideas of Euclidean quantum gravity can be explored, but where many of the accompanying difficulties do not arise. It thus provides an opportunity for precision Euclidean quantum gravity. Indeed, as we will see, it is possible to compute the numerical prefactor in (19.1). In doing these calculations, we will meet a number of issues that seem surprising and well worth highlighting.

[1] As such, unlike in the quantum Hall parts of this dissertation, the dynamical role of supersymmetry is non-trivial. Nonetheless, the careful reader will observe that the general points we will make (about extra scales, instanton selection rules, and so on) do not seem to depend on this supersymmetry.

The Scale of Gravitational Instantons

The natural energy scale associated to any quantum gravity effect is usually thought to lie far in the ultra-violet: it might be the Planck scale, the string scale or something still more exotic. However, in situations where gravitational instantons play a role, this is not the only scale of importance. The partition function for quantum gravity comes equipped with a hidden infra-red scale, $\Lambda_{\rm grav}$. This arises through dimensional transmutation from the logarithmic running of the coefficient $\alpha(\mu)$ of the Gauss-Bonnet term,

$$\Lambda_{\rm grav} = \mu \exp\left(-\frac{\alpha(\mu)}{2\alpha_1}\right) .$$

Here α_1 is an appropriate beta-function for the Gauss-Bonnet term. Of course, gravity is not a renormalizable theory and so, in some sense, includes an infinite number of extra scales associated to the higher-derivative operators. These are all ultra-violet scales, naturally of order of the Planck mass or other UV cut-off. In contrast, the scale $\Lambda_{\rm grav}$ is distinguished by the fact that, like its Yang-Mills counterpart $\Lambda_{\rm QCD}$, it can be naturally exponentially suppressed relative to the Planck scale.

The Gauss-Bonnet term is topological and the scale $\Lambda_{\rm grav}$ plays no role in perturbative physics around flat space. However, it becomes important when summing over gravitational instantons with non-trivial topology. Moreover, in supersymmetric theories, $\Lambda_{\rm grav}$ is naturally complex, with the phase supplied by the gravitational theta angle. The complexified $\Lambda_{\rm grav}$ lives in a chiral multiplet and, indeed, we will see that it provides (part of) the pre-factor for the superpotential (19.1). A discussion of this new scale can be found in Section 21.1 and 21.5.

Summing over Topologies

One conceptual issue that arises in our computation is the question of what topologies we should include in the path integral. We are interested in physics on $\mathcal{M} \cong \mathbb{R}^{1,2} \times S^1$. In Euclidean space, this manifold has boundary $\partial\mathcal{M} \cong S^2 \times S^1$. However, the gravitational instantons that we meet have boundaries with different topologies. They are the multi-Taub-NUT spaces, whose boundary is isomorphic to the Lens space L_k in which the S^1 is non-trivially fibered over the S^2 with winding k. In Section 22.1, it is argued that we should, nonetheless, include these in the path integral. The superpotential (19.1) arises from the simplest Taub-NUT space in which the S^1 winds once around S^2.

There are further gravitational instantons whose boundary has the topology of S^1 fibered over $\mathbb{RP}^2 \cong S^2/\mathbb{Z}_2$. The Atiyah-Hitchin manifold falls in this class and has the right number of zero modes to contribute to the superpotential. However, the principles put forward in Section 22.1 will lead us to conclude that this class of solutions should

be discarded.

One-Loop Determinants

The final issue to highlight is of a more technical nature. In any instanton calculation, one should compute the one-loop determinants around the background of the classical solution. In supersymmetric theories, there is a pairing between the bosonic and fermionic non-zero modes and, correspondingly, a naive expectation that these determinants should cancel. However, for non-compact spaces such as Taub-NUT, the spectrum of operators is continuous and although the range of bosonic and fermionic eigenvalues coincides, their densities need not. As we shall see, the resulting determinants in Taub-NUT indeed do not cancel but, nonetheless, are computable. They are closely related to the boundary terms that appear in index theory. These determinants are computed in Section 22.2.

The one-loop determinants contribute to the pre-factor of (19.1). Ignoring numerical factors, the superpotential is more precisely given by

$$\mathcal{W} \sim \Lambda_{\text{grav}}^{41/24}\, R^{-7/24} \exp\left(-\frac{\pi R^2}{4G_N} - i\sigma\right) .$$

The presence of a power of R in the pre-factor appears to be in tension with the expected holomorphic nature of the superpotential. We will, however, find that there is a one-loop correction to the complex structure relating R and σ and that the superpotential above is indeed holomorphic as expected. This discussion can be found in Section 21.4.

The Plan

We begin in Chapter 20 by describing a few simple classical aspects of $\mathcal{N} = 1$ supergravity and its Kaluza-Klein compactification to three dimensions. Chapter 21 is devoted to perturbative aspects. It begins with a summary of the most important results, including the one-loop divergences that give rise to the new scale Λ_{grav}, as well as the finite renormalization of the kinetic terms. The remainder of Chapter 21 describes these calculations in more detail. Chapter 22 covers the instanton computation. Again it starts with a summary, focussing in particular on the gravitational instantons of interest and a discussion of the kind of asymptotic boundaries that we should admit. The majority of Chapter 22 is concerned with the computation of the one-loop determinants around the background of Taub-NUT.

Reading Chapter 20, Section 21.1 and Section 22.1, together with the ultimate punchline, should give the reader the conceptual aspects of the work whilst avoiding the meat of the calculations.

20 Classical Aspects

We work with $\mathcal{N} = 1$ supergravity in $d = 3 + 1$ dimensions, focussing on the minimal theory containing only a graviton and gravitino. The bulk four-dimensional action is given by

$$S = \frac{M_{\rm pl}^2}{2} \int \mathrm{d}^4x \, \sqrt{-g} \left(\mathcal{R}_{(4)} + \bar{\psi}_\mu \gamma^{\mu\nu\rho} \mathcal{D}_\nu \psi_\rho \right) . \tag{20.1}$$

We use the notation of the (reduced) Planck mass $M_{\rm pl}^2 = 1/8\pi G_N$ instead of the Newton constant G_N. Here $\mathcal{R}_{(4)}$ is the 4d Ricci scalar, with the subscript to distinguish it from its 3d counterpart that we will introduce shortly. There is also the standard Gibbons-Hawking boundary term which we have not written explicitly.

The action is to be thought of as a functional of the Majorana gravitino ψ_μ and the vierbein e_μ^a where $\mu, \nu = 0, 1, 2, 3$ are spacetime indices and $a, b = 0, 1, 2, 3$ are tangent space indices. Here we follow the standard notation of suppressing the spinor indices on the gravitino, whose covariant derivative is given by

$$\mathcal{D}_\nu \psi_\rho = \partial_\nu \psi_\rho + \frac{1}{4} \hat{\omega}_{ab\nu} \gamma^{ab} \psi_\rho \, .$$

In this formalism, the spin connection $\hat{\omega}_{ab\mu}$ that appears in the covariant derivative differs from the purely geometric spin connection by the addition of a gravitino torsion term: $\hat{\omega}_{ab\mu} = \omega_{ab\mu}(e) + H_{ab\mu}$ with

$$H_{ab\mu} = -\frac{1}{4} e_a^\nu e_b^\rho \left(\bar{\psi}_\mu \gamma_\rho \psi_\nu - \bar{\psi}_\nu \gamma_\mu \psi_\rho - \bar{\psi}_\rho \gamma_\nu \psi_\mu \right) .$$

The action is, of course, invariant under diffeomorphisms and local supersymmetry transformations. The latter act as $\delta e_\mu^a = \frac{1}{2} \bar{\epsilon} \gamma^a \psi_\mu$ and $\delta \psi_\mu = \mathcal{D}_\mu \epsilon$.

The classical theory also enjoys a $U(1)_R$ symmetry which acts by axial rotations on ψ. As we describe in more detail in Chapters 21 and 22, this $U(1)_R$ symmetry is anomalous in the quantum theory. (Although, as we will see, it mixes with a $U(1)_J$ bosonic symmetry that will be described shortly and a combination of the two survives.)

20.1 Reduction on a Circle

More specifically, our interest lies in the dynamics of $\mathcal{N} = 1$ supergravity when compactified on a manifold $\mathcal{M} \cong \mathbb{R}^{1,2} \times S^1$. We denote the physical radius of the circle as R.

We choose the spin structure such that the fermions are periodic around the compact direction and supersymmetry is preserved.

At distances larger than the compactification scale R, the dynamics is effectively three-dimensional. The metric degrees of freedom are parametrized by the familiar Kaluza-Klein ansatz,

$$ds^2_{(4)} = \frac{L^2}{R^2} ds^2_{(3)} + \frac{R^2}{L^2} \left(dz + A_i dx^i\right)^2 \tag{20.2}$$

where $z \in [0, 2\pi L)$ is the periodic coordinate. Here R, A_i and the 3d metric $g^{(3)}_{ij}$ are dynamical degrees of freedom, while L is a fixed, fiducial scale. It is natural to pick coordinates such that $R(x) \to L$ asymptotically and we will eventually do so but, for now, we leave L arbitrary.

Evaluated on this background, the Einstein-Hilbert action becomes

$$\begin{aligned} S_{\text{eff}} &= \frac{M_{\text{pl}}^2}{2} \int \mathrm{d}^4x \sqrt{-g}\, \mathcal{R}_{(4)} \\ &= \frac{M_3}{2} \int \mathrm{d}^3x \sqrt{-g_{(3)}} \left[\mathcal{R}_{(3)} - 2 \left(\frac{\partial R}{R}\right)^2 - \frac{1}{4} \frac{R^4}{L^4} F_{ij} F^{ij} \right] \end{aligned}$$

with $M_3 = 2\pi L M_{\text{pl}}^2$ the 3d Planck scale and $F_{ij} = \partial_i A_j - \partial_j A_i$ the graviphoton field strength.

In three dimensions, it is often useful to dualize the gauge field in favour of a periodic scalar σ. This is particularly true if we are interested in instanton physics [50]. The dual photon can be viewed as Lagrange multiplier which imposes the Bianchi identity,

$$\mathcal{L}_\sigma = \frac{\sigma}{4\pi L} \epsilon^{ijk} D_i F_{jk} \,. \tag{20.3}$$

With the magnetic charge quantized in integral units, σ has periodicity 2π. Integrating out the field strength, we can write the low-energy effective action in dual form,

$$S_{\text{eff}} = \int \mathrm{d}^3x \sqrt{-g_{(3)}} \left[\frac{M_3}{2} \mathcal{R}_{(3)} - M_3 \left(\frac{\partial R}{R}\right)^2 - \frac{1}{M_3} \frac{L^2}{R^4} \left(\frac{\partial \sigma}{2\pi}\right)^2 \right] . \tag{20.4}$$

This action enjoys a new $U(1)_J$ symmetry which acts by shifting the dual photon: $\sigma \to \sigma + c$. All other fields are left invariant under this symmetry. The symmetry is preserved in perturbation theory but, as we will see in Chapter 22, is broken by instanton effects.

Our goal is to determine the quantum corrections to the effective action (20.4). We describe perturbative corrections in Chapter 21 and instanton corrections in Chapter

22.

Fermions

This bosonic effective action has a fermionic counterpart which is dictated by supersymmetry. Let us work for now with a Majorana basis of 4d gamma matrices,

$$\gamma^i = \begin{pmatrix} 0 & \gamma^i_{3d} \\ \gamma^i_{3d} & 0 \end{pmatrix} \quad i = 0, 1, 2 \quad , \quad \gamma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{20.5}$$

with $\gamma^i_{3d} = (i\sigma^2, \sigma^3, \sigma^1)$. Upon dimensional reduction, the 4d Majorana gravitino ψ_μ decomposes into a 3d spin-3/2 Dirac fermion λ_i and 3d spin-1/2 Dirac fermion χ. To perform this reduction, it's simplest to work with the frame index, so that $\psi_a = e^\mu_a \psi_\mu$. Further, to make life easy for ourselves, we restrict to the flat background $\mathbb{R}^{1,2} \times S^1$ with metric (20.2) and make the spinor ansatz

$$\psi_i = \begin{pmatrix} \mathrm{Re}\,\lambda_i + (\gamma_{3d})_i\,\mathrm{Im}\,\chi \\ \mathrm{Im}\,\lambda_i + (\gamma_{3d})_i\,\mathrm{Re}\,\chi \end{pmatrix} \quad \text{and} \quad \psi_z = \begin{pmatrix} \mathrm{Re}\,\chi \\ \mathrm{Im}\,\chi \end{pmatrix} . \tag{20.6}$$

The gravitino kinetic term in (20.1) then becomes

$$\begin{aligned} S_{\text{fermions}} &= \int \mathrm{d}^4x\,\sqrt{-g}\,\frac{M_{\rm pl}^2}{2}\,\bar\psi_\mu \gamma^{\mu\nu\rho}\partial_\nu\psi_\rho \\ &= \int \mathrm{d}^3x\,\sqrt{-g_{(3)}}\,\frac{M_3 L}{R}\left(\frac{1}{2}\bar\lambda_i\epsilon^{ijk}\partial_i\lambda_k - \bar\chi\not{\partial}\chi\right) . \end{aligned} \tag{20.7}$$

After dividing out by local supersymmetry transformations, the spin-3/2 fermion λ_i carries no propagating degrees of freedom. (This is the supersymmetric analogue of the statement that the 3d metric carries no propagating degrees of freedom.) In contrast, the spin-1/2 fermion χ carries two propagating degrees of freedom; these are the supersymmetric partners of R and σ. We will postpone a more detailed discussion of how supersymmetry relates R, σ and χ to Sections 21.4 and 22.4.

20.2 Topological Terms

In addition to the Einstein-Hilbert action, there are two topological terms that will play a role in our story. Both are higher derivative terms, with dimensionless coefficients. They are the Gauss-Bonnet term

$$S_\alpha = \frac{\alpha}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\;{}^\star\mathcal{R}^\star_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma} \tag{20.8}$$

which integrates to the Euler characteristic of the manifold, and the Pontryagin class

$$S_\theta = \frac{\theta}{16\pi^2}\int \mathrm{d}^4x\sqrt{-g}\,{}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}\,. \tag{20.9}$$

If we care only about perturbative physics on $\mathbb{R}^3 \times S^1$, then we can neglect these terms. However, when we start to sum over manifolds of different topology, they become important.

Usually, when working in an effective field theory, we keep all relevant and marginal terms in the action, neglecting only the irrelevant operators on the grounds that they are suppressed by some high mass scale. In the present case, there are two further four-derivative terms which come with dimensionless coefficients: $\mathcal{R}^2$ and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$. However, both can be absorbed into the Einstein-Hilbert term through a redefinition of the metric [224]. For this reason, we need only consider S_α and S_θ above.

In supergravity, the Gauss-Bonnet (20.8) and Pontryagin (20.9) terms can be written as an F-term [225, 226] (using the so-called "chiral projection operator"). This, in turn, means that the two coupling constants α and θ combine into the complex coupling

$$\tau_{\text{grav}} = \alpha + 2i\theta \tag{20.10}$$

which naturally lives in a chiral multiplet. We will see later that τ_{grav} appears in the instanton generated superpotential.

21 Perturbative Aspects

In this chapter we describe the results of quantum fluctuations of the graviton and gravitino around the background $\mathbb{R}^{1,2} \times S^1$. There are two kinds of effects: those from divergences that arise already in four dimensions; and finite corrections to the low-energy effective action which are suppressed by the dimensionless combination $1/M_{\rm pl}^2 R^2$.

21.1 Summary

We open this chapter by summarizing the main results. The remainder of the chapter contains details of the computations.

Finite Corrections

Finite corrections to the effective action occur when the theory is compactified on $\mathbb{R}^{1,2} \times S^1$ and arise due to loops wrapping the spatial circle. The results depend on R, the radius of the circle and so are non-local from the four-dimensional perspective. For this reason, they are not sensitive to the ultra-violet details of the theory and can therefore be reliably calculated.

These finite corrections were first computed in the Kaluza-Klein context in [41, 42], where they manifested themselves as a Casimir force, causing the Kaluza-Klein circle to either shrink or expand. (The analogous calculation was performed earlier in the thermal context [227].) The effective 3d potential is given by[1]

$$V_{\rm eff} = -\frac{N_B - N_F}{720\pi}\frac{L^3}{R^6}\,. \tag{21.1}$$

Here N_B is the number of massless bosonic degrees of freedom; these make the Kaluza-Klein circle contract. N_F the number of massless fermionic degrees of freedom; these make the circle expand. Of course, in supersymmetric theories $N_B = N_F$ and Kaluza-Klein compactifications are perturbatively stable. The presence of fermions with periodic boundary conditions means that the bubble-of-nothing instability is absent in this theory [221], but other gravitational instantons, discussed in Chapter 22, will contribute.

[1] The standard Casimir potential in four dimensions scales as $1/R^3$. The $1/R^6$ scaling seen here arises after a Weyl transformation to the 3d Einstein frame.

Although the perturbative potential vanishes, there are still finite one-loop effects of interest. These renormalize the kinetic terms in the effective action (20.4). Much of this chapter is devoted to computing these effects; we will show that the low-energy effective action becomes

$$\begin{aligned}\mathcal{L}_{\text{eff}} &= \frac{1}{2}\left(M_3 + \frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)} - \left(M_3 - \frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2 \\ &\qquad - \left(M_3 + \frac{11}{24\pi}\frac{L}{R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2 . \end{aligned} \tag{21.2}$$

This is the one-loop effective action. One should certainly expect that there will be further corrections, both from higher-loops and from non-perturbative effects. Nonetheless, this will suffice for our purposes. The most important fact that we will need is the observation that the renormalization of the R and σ kinetic terms come with different coefficients. This will prove important later when we reconcile this with supersymmetry: it results in a one-loop shift in the complex structure and means R and σ sit together in a chiral multiplet with lowest component

$$\mathcal{S} = 2\pi^2 M_{\text{pl}}^2 R^2 + \frac{7}{48}\log(M_{\text{pl}}^2 R^2) + i\sigma\,.$$

The log term above arises from the one-loop correction. This will be described in Section 21.4.

Anomalies and One-Loop Divergences

It is well known that the S-matrix is one-loop finite in pure Einstein gravity [224] and two-loop finite in pure $\mathcal{N} = 1$ supergravity [228, 229]. Nonetheless, these theories do suffer from divergences at one-loop which, while not appearing in the S-matrix, can affect the physics. As we review, these divergences are related to anomalies.

For our purposes, the most important one-loop divergence is associated to the Gauss-Bonnet term (20.8). This, of course, is a total derivative in four-dimensions but will be important when we come to discuss gravitational instanton physics. The coefficient α is dimensionless and runs logarithmically at one-loop [224] as

$$\alpha(\mu) = \alpha_0 - \alpha_1 \log\left(\frac{M_{\text{UV}}^2}{\mu^2}\right) \tag{21.3}$$

where α_0 is the coupling at the UV cut-off which we denote as M_{UV}. In general, for a theory with N_s free massless spin-s fields, the beta-function is given by [230, 231, 232, 233]

$$\alpha_1 = \frac{1}{48\cdot 15}\left(848N_2 - 233N_{3/2} - 52N_1 + 7N_{1/2} + 4N_0\right)\,.$$

The computation leading to this result is closely related to the trace anomaly for massless fields in fixed, curved spacetime. Indeed, for spins $s \leq 1$, the coefficients above are the same as $c - a$ of the trace anomaly. The running coupling $\alpha(\mu)$ results in an RG-invariant scale,

$$\Lambda_{\text{grav}} = \mu \exp\left(-\frac{\alpha(\mu)}{2\alpha_1}\right) . \tag{21.4}$$

For the pure supergravity theory that is our focus here, we have $N_0 = N_{1/2} = N_1 = 0$ while $N_{3/2} = N_2 = 1$ which gives $\alpha_1 = 41/48$.

In the original discussions of Euclidean quantum gravity, the suggestion seems to have been that Λ_{grav} (or sometimes μ) should be identified with the Planck scale. (See, for example, [40].) In contrast, here we view Λ_{grav} as a new scale which emerges in quantum gravity through dimensional transmutation; it dictates the length at which topological fluctuations are unsuppressed by the Gauss-Bonnet term. Like its counterpart Λ_{QCD} in Yang-Mills theory, Λ_{grav} can naturally be exponentially smaller than the Planck scale. As we will see shortly, like its Yang-Mills counterpart, it provides the scale at which instanton effects become important.

In the previous chapter, we saw that α sits in a background chiral multiplet with the gravitational theta-term θ. These combine into the complex coupling $\tau_{\text{grav}} = \alpha + 2i\theta$. This means that the scale $\Lambda_{\text{grav}} = \mu e^{-\tau/2\alpha_1}$ is also naturally complex in supergravity and sits in a chiral multiplet.

There is one further one-loop divergence that will play a role in our story. This is responsible for the axial anomaly for the $U(1)_R$ symmetry with current $J^\mu_5 = i\bar{\psi}_\nu \gamma^{\nu\mu\rho}\gamma^5\psi_\rho$. In general, the anomaly is given by [234, 235, 231, 232]

$$\nabla_\mu J^\mu_5 = \frac{1}{24 \cdot 16\pi^2}\left(21N_{3/2} - N_{1/2}\right) {}^\star\mathcal{R}_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma} . \tag{21.5}$$

For us, $N_{1/2} = 0$ and $N_{3/2} = 1$. As usual, the anomaly can be compensated by shifts on the gravitational theta angle which means that we should view Λ_{grav} as carrying $U(1)_R$ charge.

21.2 One-Loop Determinants

In this section, we find the determinants arising from one-loop fluctuations of the graviton, the gravitino and their ghosts. This material is standard fare but, since we will need this for a number of subsequent calculations, we take the time to go through it in some detail.

The Graviton and its Ghost

Throughout this part of the dissertation, we use the background field method. We work in Euclidean space and write the metric as a background $g_{\mu\nu}$, which is taken to obey the Einstein equations, with a fluctuation $h_{\mu\nu}$,

$$g_{\mu\nu} \to g_{\mu\nu} + h_{\mu\nu}\,.$$

From now on, all covariant derivatives and curvatures are to be thought of with respect to the background. It is useful to further decompose the fluctuations into the trace $h = g^{\mu\nu}h_{\mu\nu}$ and traceless parts $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{4}g_{\mu\nu}h$.

We expand the Einstein-Hilbert action to quadratic order in $h_{\mu\nu}$ following, for example, [236]. The residual gauge freedom $h_{\mu\nu} \to h_{\mu\nu} + \nabla_\mu\xi_\nu + \nabla_\nu\xi_\mu$ is fixed by imposing the condition

$$\nabla^\mu\left(h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h\right) = 0\,.$$

The resulting Faddeev-Popov determinants are exponentiated in the usual fashion via the introduction of ghosts which, in this context, are anti-commuting complex vectors.

The Einstein-Hilbert action is, famously, unbounded below. In the present context, this shows up in the negative-definite operator ∇^2 for the trace fluctuations h. We follow the prescription of [236] and rotate the contour to integrate over imaginary conformal factors so that we work with the positive definite operator

$$\Delta_0 = -\nabla^2\,. \tag{21.6}$$

For the ghosts and traceless fluctuations, no such rotation is necessary. The operators for these other fields are most conveniently written using tangent space indices. This means, for example, that we write the metric fluctuation as $h_{ab} = e^\mu_a e^\nu_b h_{\mu\nu} = e^\mu_{(a}\,\delta e_{b)\mu}$. (Note that the asymmetric components of e^a_μ are non-propagating.) The fluctuation operator for the symmetric, traceless spin-2 field $\bar{h}_{ab}$ and is given by

$$(\Delta_2)_{ab;cd} = -\frac{1}{4}\eta_{ac}\eta_{bd}\nabla^2 + \left(\frac{1}{4}\eta_{ac}\eta_{bd}\mathcal{R} - \frac{1}{2}\eta_{ac}\mathcal{R}_{bd} - \frac{1}{2}\mathcal{R}_{acbd}\right)\,. \tag{21.7}$$

Meanwhile the fluctuation operator for the spin-1 ghosts takes the form

$$(\Delta_1)_{a;b} = -\eta_{ab}\nabla^2 - \mathcal{R}_{ab}\,. \tag{21.8}$$

Note that in each of (21.6), (21.7) and (21.8), the subscript on Δ_s labels the spin of the field and therefore determines the appropriate Laplacian ∇^2. Integrating out the gravi-

ton and its ghost at one-loop then results in the determinant factor

$$\Gamma_B = \frac{\det \Delta_1}{\det^{1/2} \Delta_2 \, \det^{1/2} \Delta_0} \,. \tag{21.9}$$

The Gravitino and its Ghost

The quantization of the spin-3/2 fermion was described in [237, 238]. (See also [230].) We again need to fix the redundancy of local supersymmetry transformations. The standard choice is $\gamma^\mu \psi_\mu = 0$. After gauge fixing, the kinetic term for the gravitino reads

$$\mathcal{L}_{\text{gravitino}} = \frac{i}{2} \bar{\psi}_a \left(\gamma^b \not{D} \gamma^a \right) \psi_b \,.$$

In computing the one-loop determinants, it is somewhat simpler to work with the squares of Dirac operators. For the spin-3/2 gravitino, this is given by

$$\left(\Delta_{3/2}\right)_{a;b} = \left(\gamma^c \not{D} \gamma_a\right) \left(\gamma_b \not{D} \gamma_c\right) = -\eta_{ab} \nabla^2 - \frac{1}{2} \mathcal{R}_{cdab} \gamma^{[c} \gamma^{d]} + \mathcal{R}_{ab} \tag{21.10}$$

where we have left the Dirac spinor indices implicit in this expression. Meanwhile, the gravitino is accompanied by three commuting, spin-1/2 Majorana ghosts. These come with the simple Dirac operator $i \not{D}$ which, after squaring, becomes

$$\Delta_{1/2} = (i \not{D})^2 = -\nabla^2 + \frac{1}{4} \mathcal{R} \,. \tag{21.11}$$

Integrating out the gravitino and its ghosts then gives rise to the one-loop determinants

$$\Gamma_F = \frac{\det^{1/4} \Delta_{3/2}}{\det^{3/4} \Delta_{1/2}} \,. \tag{21.12}$$

The One-Loop Effective Action

Each of the one-loop fluctuation operators introduced above takes the form

$$\Delta_s = -\nabla^2 - E_s$$

where, for each spin $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, the operator includes a spin-dependent term E_s, linear in the curvature $\mathcal{R}_{abcd}$ and is given, respectively, in (21.6), (21.11), (21.8), (21.10) and (21.7). Of course, the Laplacian ∇^2 also hides a spin structure since acting on the spin s field,

$$\nabla_\mu = \partial_\mu + \frac{1}{2} \omega_{ab\mu} t^{ab}_{(s)}$$

where $t^{ab}_{(s)}$ are the spin-s Lorentz generators (or, in Euclidean space, rotation generators).

The one-loop determinants from gravitons (21.9) and gravitinos (21.12) can be exponentiated to give the one-loop contribution to the effective action. This can be written as

$$S_{\text{1-loop}} = -\sum_{s=0}^{2} \zeta_s \, \log\det \Delta_s \tag{21.13}$$

where the coefficients ζ_s are the exponents of the various operators, given by

$$\zeta_s = \left(-\frac{1}{2}, -\frac{3}{4}, +1, +\frac{1}{4}, -\frac{1}{2}\right) \qquad s = 0,\ \frac{1}{2},\ 1,\ \frac{3}{2},\ 2\,.$$

The number of off-shell degrees of freedom of a spin-s field are $d_s = (1, 4, 4, 16, 9)$. (Recall that the spin-2 operator acts on the traceless part of symmetric tensors which is why $d_2 = 9$.) Note that $\vec{\zeta}\cdot\vec{d} = 0$. This, of course, is the manifestation of supersymmetry in the guise of an equal number of bosonic and fermionic off-shell degrees of freedom.

In the rest of this chapter, we will compute various terms in the expansion of (21.13). We will also return to compute the ratio of determinants $\Gamma_B\Gamma_F$ in Section 22.2 in a self-dual background where, as we show, considerable simplifications occur.

21.3 Two-Derivative Effective Action

Let us begin by computing the finite corrections to the low-energy effective that we previewed in (21.2). Specifically, we will evaluate the one-loop effective action (21.13) in a gradient expansion around the flat background $\mathbb{R}^3 \times S^1$, keeping only terms with two derivatives or fewer. As we will see, supersymmetry means that many of the contributions vanish.

We take the flat metric to be given by (20.2) with $A_i = 0$ and R constant. We denote this metric as $\hat{g}_{\mu\nu}$ and the associated Laplacian as $\hat{\nabla}^2$. Each of the terms in the low-energy effective action can then be expanded as

$$\begin{aligned}
\log\det\Delta_s &= \text{Tr}\log[-\hat{\nabla}^2] + \text{Tr}\log[1 - \hat{\nabla}^{-2}(\Delta_s + \hat{\nabla}^2)] \\
&\approx \text{Tr}\log[-\hat{\nabla}^2] + \text{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2) \\
&\qquad -\frac{1}{2}\,\text{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2)(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2) + \ldots
\end{aligned} \tag{21.14}$$

where higher order terms do not contribute to the two-derivative effective action. The leading term above, involving only the flat Laplacian $\hat{\nabla}^2$, gives the perturbative contribution to the Casimir energy advertized previously in (21.1). For us, supersymmetry ensures it vanishes after summing over all spins. (This is due to the relation $\vec{\zeta}\cdot\vec{d} = 0$.)

Subsequent terms in the expansion also enjoy cancellations. To see this, let us first look at the second term, $\mathrm{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2)$. Expanding the Laplacian, the general fluctuation operator can be written as

$$\Delta_s = -g^{\mu\nu}\partial_\mu\partial_\nu - \frac{1}{2}g^{\mu\nu}\{\partial_\mu, \omega_{ab\nu}t^{ab}_{(s)}\} - \frac{1}{4}\omega_{ab\mu}\omega_{cd}{}^{\mu}t^{ab}_{(s)}t^{cd}_{(s)} + g^{\mu\rho}\Gamma^{\nu}_{\mu\rho}\nabla_\nu - E_s\,. \tag{21.15}$$

The sum over different spins $s = 0, \ldots, 2$ will mean that any term which doesn't have an explicit spin dependence will vanish. That immediately kills the ∂^2 term and the term with the Christoffel symbol. The term linear in $t^{ab}_{(s)}$ vanishes as soon as the trace over spin indices is taken. This leaves us with

$$\sum_{s=0}^{2}\zeta_s\ \mathrm{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2) = \sum_{s=0}^{2}\zeta_s\ \mathrm{Tr}\,(-\hat{\nabla}^{-2})\left[-\frac{1}{4}\omega_{ab\mu}\omega_{ab}{}^{\mu}t^{ab}_{(s)}t^{cd}_{(s)} - E_s\right].$$

Here the trace Tr should be taken over both spin and momentum quantum numbers. We deal with the spin trace first. We have

$$\mathrm{tr}_{\,\mathrm{spin}}[t^{ab}_{(s)}t^{cd}_{(s)}] = a_s(-\delta^{ac}\delta^{bd} + \delta^{bc}\delta^{ad}) \tag{21.16}$$

where the coefficients a_s are related to the Casimirs of the representation of the Lorentz group[2] and are given by

$$a_s = (0, 1, 2, 12, 12)\,.$$

Meanwhile, the trace over spin indices of E_s is proportional to the Ricci scalar of the background:

$$\mathrm{tr}_{\,\mathrm{spin}}\ E_s = -b_s\mathcal{R} \quad \text{with} \quad b_s = (0, 1, -1, 4, 6) \tag{21.17}$$

This allows us to express the contribution to the one-loop effective action in terms of traces over momentum states only.

$$\sum_{s=0}^{2}\zeta_s\ \mathrm{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2) = \frac{1}{2}(\vec{a}\cdot\vec{\zeta})\ \mathrm{Tr}\,[-\hat{\nabla}^{-2}\omega_{abc}\omega^{abc}] + (\vec{b}\cdot\vec{\zeta})\ \mathrm{Tr}\,[-\hat{\nabla}^{-2}\mathcal{R}]\,. \tag{21.18}$$

We will come back and perform these momentum integrals shortly. But first, we also need to include the contributions from the third term in (21.14):

$$X_3 = -\frac{1}{2}\sum_{s=0}^{2}\zeta_s\ \mathrm{Tr}\,(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2)(-\hat{\nabla}^{-2})(\Delta_s + \hat{\nabla}^2)\,.$$

[2]The irreducible representation (j_1, j_2) has dimension $d = (2j_1 + 1)(2j_2 + 1)$ and the appropriate group theory gives $a = d/3[(j_1(j_1 + 1) + j_2(j_2 + 1)]$. (See, for example, [231, 232].)

Once again, any term linear in $t^{ab}_{(s)}$ upon taking the trace over spin indices, while any term without a spin structure vanishes after summing over different spins due to supersymmetry. After the dust settles, we find that just two terms are relevant,

$$\begin{aligned} X_3 &= -\sum_{s=0}^{2} \mathrm{Tr}\left[\frac{1}{2}(-\hat{\nabla}^{-2})t^{ab}_{(s)}\omega_{ab}{}^{\mu}\partial_\mu(-\hat{\nabla}^{-2})t^{cd}_{(s)}\omega_{cd}{}^{\nu}\partial_\nu + (-\hat{\nabla}^{-2})E_s(-\hat{\nabla}^{-2})\,(g-\hat{g})^{\mu\nu}\,\partial_\mu\partial_\nu\right] \\ &\approx (\vec{a}\cdot\vec{\zeta})\ \mathrm{Tr}\left[(-\hat{\nabla}^{-2})^2\omega_{ab}{}^{\mu}\omega^{ab\nu}\partial_\mu\partial_\nu\right] + (\vec{b}\cdot\vec{\zeta})\ \mathrm{Tr}\left[(-\hat{\nabla}^{-2})^2\,\mathcal{R}\,\Delta g^{\mu\nu}\partial_\mu\partial_\nu\right] \end{aligned} \tag{21.19}$$

with $\Delta g^{\mu\nu} = g^{\mu\nu} - \hat{g}^{\mu\nu}$. To reach the second line, we move derivatives past some of the fields; this is allowed since it only makes a difference to higher derivative terms in the effective action.

The remaining traces in (21.18) and (21.19) are over momentum. Since we are working on $\mathbb{R}^3 \times S^1$, this involves both an integral and a discrete sum[3] for the momentum $k_4 = n/L$, with $n \in \mathbb{Z}$, for modes on S^1:

$$\mathrm{Tr} \longrightarrow \frac{1}{2\pi L}\sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3}\,.$$

With this, the expressions (21.18) and (21.19) for the one-loop contribution to the two-derivative effective action combine to become

$$\begin{aligned} S_{\text{1-loop}} = -\frac{1}{2\pi L}\sum_n \int \frac{\mathrm{d}^3k}{(2\pi)^3} \quad & \left\{(\vec{a}\cdot\vec{\zeta})\left[\frac{\omega_{abc}\omega^{abc}}{2\hat{g}^{\mu\nu}k_\mu k_\nu} - \frac{\omega_{ab}{}^{\mu}\omega^{ab\nu}k_\mu k_\nu}{(\hat{g}^{\mu\nu}k_\mu k_\nu)^2}\right]\right. \\ & \left. + (\vec{b}\cdot\vec{\zeta})\left[\frac{1}{\hat{g}^{\mu\nu}k_\mu k_\nu} - \frac{\Delta g^{\mu\nu}k_\mu k_\nu}{(\hat{g}^{\mu\nu}k_\mu k_\nu)^2}\right]\mathcal{R}\right\}. \end{aligned} \tag{21.20}$$

These integrals suffer both quadratic and logarithmic divergences which we need to tame. Our method of choice is Pauli-Villars regularization.

Pauli-Villars Regularization

Pauli-Villars offers perhaps the most straightforward method of regularization. We start by providing all of our original fields with a small mass m. This will act as an infra-red cut-off and ultimately we send $m \to 0$. (In practice, this means that we need only replace $k^2 \to k^2 + m^2$ in the denominator of integrals.)

The UV divergences are tamed by introducing very heavy ghost particles with mass M_{UV}. We will ultimately take $M_{\text{UV}} \to \infty$. Introducing one such field is enough to remove logarithmic divergences, but we also have a quadratic divergence to deal with. This requires the introduction of two further fields; a physical field with mass-squared

[3]Strictly speaking, to compute the Wilsonian effective action we should drop the $n = 0$ zero-mode in the sum. These terms can be interpreted as counterterms for the 3d theory.

$\gamma M_{\rm UV}^2$ and a ghost with mass-squared $(\gamma-1)M_{\rm UV}^2+m^2$ where γ is an arbitrary parameter on which no physical quantity should depend. The upshot is that the integrands in (21.20) are replaced by their regulated form such as

$$\frac{1}{\hat{g}^{\mu\nu}k_\mu k_\nu} \to \left[\frac{1}{\hat{g}^{\mu\nu}k_\mu k_\nu + m^2}\right]_{\rm PV}$$

where we introduce the notation

$$\left[f(m^2)\right]_{\rm PV} = f(m^2) - f(M_{\rm UV}^2) + f(\gamma M_{\rm UV}^2) - f((\gamma-1)M_{\rm UV}^2 + m^2)\,. \tag{21.21}$$

Our goal is to now evaluate the integrals (21.20) using this regularization procedure.

Extracting the Divergent Piece

Before we proceed, it will help to better understand the origin of the divergent pieces and, more importantly, the finite pieces. Because the divergences arise from the UV, it should come as no surprise to learn that they are the same regardless of whether we work on $\mathbb{R}^4$ or $\mathbb{R}^3\times S^1$. In contrast, the finite terms that we seek are proportional to $1/R^2$ and are only present when we are on the circle. For this reason, it is useful to write

$$\frac{1}{2\pi L}\sum_n \frac{{\rm d}^3k}{(2\pi)^3} = \int \frac{{\rm d}^4k}{(2\pi)^4} + \left[\frac{1}{2\pi L}\sum_{k_4=n/L}\int\frac{{\rm d}^3k}{(2\pi)^3} - \int\frac{{\rm d}^4k}{(2\pi)^4}\right].$$

All divergences are contained in the first term. Meanwhile, we will see that the second term, which captures the difference between physics on the circle and in the plane, contains only finite pieces.

As it stands, the integrands in (21.20) are not quite rotationally invariant, even when integrated over $\mathbb{R}^4$. This is because the background flat metric gives

$$\hat{g}^{\mu\nu}k_\mu k_\nu = (R^2/L^2)\mathbf{k}^2 + (L^2/R^2)k_4^2\,.$$

To proceed, we rescale the 3-momentum $\mathbf{k}\to(R^2/L^2)\mathbf{k}$. Then, the integrand in (21.20) becomes isotropic. On grounds of rotational invariance, the divergent piece of the one-loop effective action, arising from integrating over $\int {\rm d}^4k$, is then given by

$$\begin{aligned} S_{\text{divergent}} \;=\; -\frac{L^4}{R^2}\int\frac{{\rm d}^4k}{(2\pi)^4}\,\Bigg\{&(\vec{a}\cdot\vec{\zeta})\left[\frac{1}{2(k^2+m^2)} - \frac{k^2}{4(k^2+m^2)^2}\right]_{\rm PV}\omega_{abc}\omega^{abc} \\ &+ (\vec{b}\cdot\vec{\zeta})\left[\frac{1}{k^2+m^2} - \frac{\hat{g}_{\mu\nu}\Delta g^{\mu\nu}k^2}{4(k^2+m^2)^2}\right]_{\rm PV}\mathcal{R}\Bigg\} \end{aligned} \tag{21.22}$$

where the factor of L^4/R^4 arises from the aforementioned rescaling of the momentum and is identified as $\sqrt{\hat{g}}$.

The regulated integrals in the above expression are easily computed. They are given by

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{1}{k^2+\mu^2}\right]_{\text{PV}} = -\frac{1}{16\pi^2}\left[m^2\log m^2\right]_{\text{PV}}$$

and

$$\int \frac{\mathrm{d}^4k}{(2\pi)^4}\left[\frac{k^2}{(k^2+\mu^2)^2}\right]_{\text{PV}} = -\frac{1}{8\pi^2}\left[m^2\log m^2\right]_{\text{PV}} .$$

We see that the ω^2 terms in (21.22) cancel. (This is perhaps rather surprising; if you consider the unregulated integrands with $m^2 = 0$ then the two terms appear to differ by a factor of 2. But, of course, such unregulated integrals are ill-defined. The same cancelling factor of 2 can also be seen in dimensional regularization as discussed, for example, in [239].)

The term proportional to $\mathcal{R}$ in (21.22) does not vanish. Instead, it gives

$$\begin{aligned} S_{\text{divergent}} &= \frac{\vec{b}\cdot\vec{\zeta}}{16\pi^2}\left[m^2\log m^2\right]_{\text{PV}}\left(1-\frac{1}{2}\hat{g}_{\mu\nu}\Delta g^{\mu\nu}\right)\sqrt{\hat{g}}\mathcal{R} \\ &= -\frac{15}{64\pi^2}\left[m^2\log m^2\right]_{\text{PV}}\sqrt{g}\mathcal{R} \end{aligned} \tag{21.23}$$

where the Δg term acts simply to change the fiducial metric $\sqrt{\hat{g}}$ into the background metric $\sqrt{g}$ (to the order at which we are working). This term is divergent but can be absorbed through a renormalization of Newton's constant. As we will see later, it agrees with the divergence computed using heat kernel methods.

Extracting the Finite Pieces

As described above, the finite terms in the effective action (21.20) arise from the difference between physics on the circle and physics on the plane.

$$S_{\text{finite}} = -\left[\frac{1}{2\pi L}\sum_n\int\frac{\mathrm{d}^3k}{(2\pi)^3} - \int\frac{\mathrm{d}^4k}{(2\pi)^4}\right] \quad \left\{(\vec{a}\cdot\vec{\zeta})\left[\frac{\omega_{abc}\omega^{abc}}{2\hat{g}^{\mu\nu}k_\mu k_\nu} - \frac{\omega_{ab}{}^{\mu}\omega^{ab\nu}k_\mu k_\nu}{(\hat{g}^{\mu\nu}k_\mu k_\nu)^2}\right]\right.$$
$$\left. + (\vec{b}\cdot\vec{\zeta})\left[\frac{1}{\hat{g}^{\mu\nu}k_\mu k_\nu} - \frac{\Delta g^{\mu\nu}k_\mu k_\nu}{(\hat{g}^{\mu\nu}k_\mu k_\nu)^2}\right]\mathcal{R}\right\} .$$

We again rescale the 3-momentum $\mathbf{k} \to (R^2/L^2)\mathbf{k}$. Isotropy and parity ensure that the terms with $k_\mu k_\nu$ in the numerator are once again diagonal, but we now have to treat the $\mathbb{R}^3$ and S^1 components separately. The relevant integrals are

$$\frac{1}{2\pi L}\left(\sum_n - \int \mathrm{d}n\right)\frac{L^4}{R^4}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{(n/L)^2+\mathbf{k}^2+m^2}\right]_{\text{PV}} \longrightarrow \frac{1}{48\pi^2}\frac{L^2}{R^4}$$

and

$$\frac{1}{2\pi L}\left(\sum_n - \int \mathrm{d}n\right)\frac{L^4}{R^4}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{\mathbf{k}^2}{\left((n/L)^2+\mathbf{k}^2+m^2\right)^2}\right]_{\mathrm{PV}} \longrightarrow \frac{1}{32\pi^2}\frac{L^2}{R^4}$$

and

$$\frac{1}{2\pi L}\left(\sum_n - \int \mathrm{d}n\right)\frac{L^4}{R^4}\int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\left[\frac{(n/L)^2}{\left((n/L)^2+\mathbf{k}^2+m^2\right)^2}\right]_{\mathrm{PV}} \longrightarrow -\frac{1}{96\pi^2}\frac{L^2}{R^4}$$

where $\longrightarrow$ means that we have dropped terms which vanish as we remove the regulators, $m^2 \to 0$ and $M^2 \to \infty$. This leaves behind only finite contributions as promised. The final result is

$$S_{\text{finite}} = -\int \mathrm{d}^4x\sqrt{g}\;\frac{1}{48\pi^2}\frac{1}{R^2}\left\{(\vec{a}\cdot\vec{\zeta})\,\omega_{ab4}\omega^{ab4} + (\vec{b}\cdot\vec{\zeta})\,\mathcal{R}\right\}$$

where, as in the divergent case, the role of the Δg terms is to ensure that the R that appears here is now the dynamical field rather than the fixed, asymptotic value of $\hat{g}$. Substituting the three-dimensional expressions for ω and $\mathcal{R}$ we have

$$\begin{aligned} S_{\text{finite}} = -\int \mathrm{d}^3x\sqrt{g_{(3)}} \quad & \frac{1}{24\pi}\frac{L}{R^2}\left\{(\vec{a}\cdot\vec{\zeta})\left[2\left(\frac{\partial R}{R}\right)^2 + \frac{1}{4}\frac{R^4}{L^4}F^2\right]\right. \\ & \left. +\ (\vec{b}\cdot\vec{\zeta})\left[\mathcal{R}_{(3)} + 2\left(\frac{\partial R}{R}\right)^2 + \frac{1}{4}\frac{R^4}{L^4}F^2 - 2\nabla^2\log R\right]\right\}. \end{aligned}$$

We now integrate the last term by parts, discarding the total derivative, leaving us with

$$S_{\text{finite}} = -\int \mathrm{d}^3x\sqrt{g_{(3)}}\;\frac{1}{24\pi}\frac{L}{R^2}\left\{(\vec{b}\cdot\vec{\zeta})\,\mathcal{R}_{(3)} + 2(\vec{a}-\vec{b})\cdot\vec{\zeta}\left(\frac{\partial R}{R}\right)^2 + \frac{1}{4}(\vec{a}+\vec{b})\cdot\vec{\zeta}\,\frac{R^4}{L^4}F^2\right\}.$$

Note that the finite renormalizations to the scalar R and field strength F are different. This will prove important shortly since it can be interpreted as a one-loop correction to the complex structure. Putting this together with the tree-level contributions, we find that the one-loop effective action in Euclidean space is given by

$$\begin{aligned} S_{\text{eff}} \;=\; \int \mathrm{d}^3x\sqrt{g_{(3)}} \Bigg\{ & \frac{1}{2}\left(M_3 + \frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)} + \left(M_3 - \frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2 \\ & + \frac{1}{2}\left(M_3 + \frac{11}{24\pi}\frac{L}{R^2}\right)\frac{1}{4}\frac{R^4}{L^4}F^2\Bigg\}. \end{aligned}$$

It remains only to rotate back to Lorentzian signature and to subsequently dualize the photon in favour of the periodic scalar σ. The result is the effective action

$$\begin{aligned}\mathcal{S}_{\text{eff}} &= \int \mathrm{d}^3x\sqrt{-g_{(3)}}\left\{\frac{1}{2}\left(M_3+\frac{5}{16\pi}\frac{L}{R^2}\right)\mathcal{R}_{(3)}-\left(M_3-\frac{1}{6\pi}\frac{L}{R^2}\right)\left(\frac{\partial R}{R}\right)^2\right.\\ &\qquad\left.-\left(M_3+\frac{11}{24\pi}\frac{L}{R^2}\right)^{-1}\frac{L^2}{R^4}\left(\frac{\partial\sigma}{2\pi}\right)^2\right\}\end{aligned} \tag{21.24}$$

as previously advertized in (21.2).

21.4 Supersymmetry and the Complex Structure

We now describe how the low-energy effective action is consistent with supersymmetry. After dimensional reduction, the propagating bosonic fields R and σ lie in a chiral multiplet [222]. (The most general form of the 3d supergravity action with chiral multiplets was presented in [240].) The lowest component of the chiral multiplet is given by

$$\mathcal{S} = 2\pi^2 M_{\text{pl}}^2 R^2 + i\sigma \tag{21.25}$$

and the classical action (20.4) for this complex scalar takes the form

$$S = -M_3\int \mathrm{d}^3x\,\sqrt{-g_{(3)}}\,\frac{1}{(\mathcal{S}+\mathcal{S}^\dagger)^2}\partial\mathcal{S}\partial\mathcal{S}^\dagger \tag{21.26}$$

which is derived from the classical Kähler potential

$$K = -\log(\mathcal{S}+\mathcal{S}^\dagger)\,. \tag{21.27}$$

The presence of the Planck mass M_{pl} in the complex structure (21.25) means that this chiral multiplet does not survive the rigid limit in which gravity is decoupled. (The distinction between rigid and gravitational theories was stressed, in particular, in [241].) This, in turn, means that we cannot use the fact that R sits in a chiral multiplet to restrict the way it appears in superpotentials when rigid supersymmetric gauge theories are compactified on a circle as in [242, 243][4].

One-Loop Corrected Complex Structure

As we have just seen, the kinetic terms are corrected at one-loop. This in principle affects both the complex structure and Kähler potential. For our present purposes, we are only concerned with the shift to the complex structure.

[4]Thanks to N. Seiberg for discussions on these issues.

The renormalization of the complex structure can be seen from the fact that the $(\partial R)^2$ and $(\partial\sigma)^2$ terms pick up different $1/R^2$ corrections in (21.24). (Strictly speaking, we should first perform a conformal transformation so that we are working in the Einstein frame, but this only affects the complex structure at order $1/R^4$ and so can be neglected at one-loop order.) It is simple to check that the one-loop corrected complex structure is given by

$$\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + \frac{7}{48}\log(M_{\rm pl}^2 R^2) + i\sigma\,. \tag{21.28}$$

(Tracing the origin of this shift, we see that it depends on the $\vec{a}$ coefficients defined in (21.16), but is independent of the $\vec{b}$ coefficients defined in (21.17).) We will have use for this later when we compute the instanton-generated superpotential.

21.5 Divergences and the Heat Kernel

The gradient expansion employed in Section 21.3 is the simplest approach for computing the effective action at the two derivative level. However, it becomes increasingly cumbersome as we look to higher derivatives. In particular, as described at the beginning of Chapter 21, we are interested in computing the logarithmic running of the coefficient of the Gauss-Bonnet term. For this, we turn to the heat kernel method. The results of this section are not new but, for completeness, we describe the essence of the computation. Further details can be found in the original paper [231, 232]. A clear review of heat kernel methods can be found in [244].

The heat kernel approach starts by writing the one-loop effective action (21.13) as

$$S_{\text{1-loop}} = -\sum_{s=0}^{2}\zeta_s\,\log\det\Delta_s = \sum_{s=0}^{2}\zeta_s\int\frac{dt}{t}\;\text{Tr}\,\left[e^{-t(\Delta_s+m^2)}\right]_{\rm PV}$$

which is true up to an (infinite) constant which we can safely ignore. Ultra-violet divergences show up in the $t\to 0^+$ limit of the integral. The standard expansion gives[5]

$$\text{Tr}\left[e^{-t\Delta_s}\right] \sim t^{-2}B_0 + t^{-1}B_2 + B_4 + \mathcal{O}(t)$$

where the Schwinger-DeWitt coefficients B_k are geometric quantities, constructed from the data in the operator $\Delta_s = -\nabla^2 - E_s$, with $\nabla_\mu = \partial_\mu + \frac{1}{2}\omega_{ab\mu}t^{ab}_{(s)}$. The leading divergence is simply the cosmological constant term,

$$B_0(\Delta_s) = \frac{1}{16\pi^2}\int \mathrm{d}^4x\sqrt{g}\;\text{tr}\;1\,.$$

[5]On manifolds with boundary, further terms may arise in the heat kernel approach. These can give rise, for example, to renormalization of the coefficient of the Gibbons-Hawking term. Here we focus only on bulk divergences.

This vanishes when we sum over the spins $s = 0, 1/2, 1, 3/2, 2$ by virtue of supersymmetry, which guarantees $\vec{\zeta} \cdot \vec{d} = 0$ as we saw above. The quadratic divergences are contained in the B_2 coefficient which is given by

$$B_2(\Delta_s) = \frac{1}{16\pi^2} \int \mathrm{d}^4x \sqrt{g}\ \mathrm{tr}\left(E_s + \frac{1}{6}\mathcal{R}\right) .$$

Here, the $\mathcal{R}/6$ term contains no spin dependence and once again cancels due to supersymmetry. The trace of E_s is given in (21.17), leaving us with

$$\sum_s \zeta_s\, B_2(\Delta_s) = -\frac{\vec{b}\cdot\vec{\zeta}}{16\pi^2} \int \mathrm{d}^4x \sqrt{g}\, \mathcal{R}\,.$$

This is the renormalization of Newton's constant. One can easily check that it agrees with the quadratic divergence (21.23) that we computed using the gradient expansion previously.

For our purposes, the most important quantities are the logarithmic divergences contained in B_4. This is given by

$$\begin{aligned} B_4(\Delta_s) \;=\; & \frac{1}{16\pi^2} \int \mathrm{d}^4x \sqrt{g}\ \mathrm{tr}\left(\frac{1}{6}\nabla^2 E_s + \frac{1}{6}\mathcal{R}E_s + \frac{1}{2}E_s^2 + \frac{1}{72}\mathcal{R}^2 - \frac{1}{180}\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}\right. \\ & \left. + \frac{1}{180}\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \frac{1}{48}t^{ab}_{(s)}\mathcal{R}_{ab\mu\nu}t^{cd}_{(s)}\mathcal{R}_{cd}{}^{\mu\nu}\right). \end{aligned} \tag{21.29}$$

The story is, by now, familiar. Any terms without spin dependence vanish due to supersymmetry. The $\nabla^2 E_s$ term survives, but results in divergences for $\nabla^2\mathcal{R}$ which is a total derivative and vanishes on the backgrounds we are interested in. For this reason, we ignore this term. Meanwhile, the $\mathcal{R}E_s$ term results in a logarithmic divergence to $\mathcal{R}^2$. Both $\mathcal{R}^2$ terms and $\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu}$ terms can be absorbed into the Einstein-Hilbert action through a field redefinition [224]. Indeed, this is the heart of the statement that the S-matrix of pure Einstein-Hilbert gravity is one-loop finite.

The upshot of this is that the only terms that we care about are those that give rise to logarithmic divergences for $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$. This receives contributions from E_s^2 and the last, $t\mathcal{R}t\mathcal{R}$ term. In particular,

$$\mathrm{tr}(E_s^2) = c_s \mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \ldots \qquad \text{with} \qquad c_s = (0, 0, 0, 2, 3)\ .$$

Putting this together with (21.16), we have

$$\sum_{s=0}^{2} \zeta_s\, B_4(\Delta_s) = \frac{1}{32\pi^2} \int \mathrm{d}^4x\ \sqrt{g}\left(\vec{c} - \frac{1}{12}\vec{a}\right)\cdot\vec{\zeta}\ (\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \ldots)\ .$$

The same field redefinitions of the metric that we described above allow us to massage the . . . terms above so that they become the Gauss-Bonnet term, with the integral given by the Euler character

$$\chi = \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\ \left(\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} - 4\mathcal{R}_{\mu\nu}\mathcal{R}^{\mu\nu} + \mathcal{R}^2\right) .$$

The one-loop effective action therefore contains the logarithmically divergent term

$$S_{\text{one-loop}} = -\frac{41}{48}\log(\mu^2/m^2)\,\chi \tag{21.30}$$

where, in the Pauli-Villars scheme (21.21), $\mu^2 = \frac{\gamma-1}{\gamma}M_{\text{UV}}^2$. This is the origin of the running of the Gauss-Bonnet coefficient described in (21.3).

We note that the interpretation of this "running" as a scale-dependent coupling constant comes with a caveat. In gauge theories, the running coupling $g^2(\mu)$ tells us how the strength of local interactions varies with the energy scale of the process. But, in the gravitational context, there is no local process associated to the Gauss-Bonnet term. Instead, it knows only about the global properties of the space. The real physics in this running coupling is the emergence of the infra-red scale Λ_{grav} defined in (21.4) which tells us characteristic scale at which manifolds with different topologies contribute to the path integral.

22 Non-Perturbative Aspects

In this chapter we describe the instanton corrections to the low-energy effective action. We will show that they generate a superpotential term for the chiral multiplet $\mathcal{S}$. The techniques of gravitational instanton computations were pioneered in the late 1970s [44, 53, 245] and much of this chapter is devoted to reviewing and extending this machinery. We must start, however, with a brief introduction to gravitational instantons and the role they play in $\mathcal{N} = 1$ supergravity.

22.1 Gravitational Instantons

Gravitational instantons are saddle points of the four-dimensional path integral. In supersymmetric theories, we can restrict attention to (anti)-self-dual solutions to the Einstein equations satisfying

$$\mathcal{R}_{\mu\nu\rho\sigma} = \pm^{\star}\mathcal{R}_{\mu\nu\rho\sigma}\,. \tag{22.1}$$

Such backgrounds preserve half of the supersymmetry. This means that supersymmetry transformations generate only two fermionic Goldstino zero modes, which is the right number to contribute towards a superpotential in $\mathcal{N} = 1$ theories [246]. The self-duality requirement (22.1) is a necessary, but not sufficient, condition for instantons to contribute to the superpotential; there may also be further fermionic zero modes which do not arise from broken supersymmetry which we describe below.

For theories on $\mathbb{R}^3 \times S^1$, the gravitational instantons are Kaluza-Klein monopoles [51, 52] which in the present context are perhaps best referred to as "Kaluza-Klein instantons". From the low-energy 3d perspective, these solutions look like Dirac monopoles and the calculation can be thought of as a gravitational completion of Polyakov's famous computation [50]. The contribution of these "Kaluza-Klein instantons" has been discussed previously in the non-supersymmetric context in [247] and, more recently, in [54].

The simplest class of gravitational instantons are the multi-Taub-NUT metrics [44],

$$\mathrm{d}s^2 = U(\mathbf{x})\mathrm{d}\mathbf{x}\cdot\mathrm{d}\mathbf{x} + U(\mathbf{x})^{-1}\left(\mathrm{d}z + \mathbf{A}\cdot\mathrm{d}\mathbf{x}\right)^2 \tag{22.2}$$

with

$$U(\mathbf{x}) = 1 + \frac{L}{2}\sum_{a=1}^{k}\frac{1}{|\mathbf{x}-\mathbf{X}_a|} \quad \text{and} \quad \nabla\times\mathbf{A} = \pm\nabla U\,.$$

The metric is smooth when $z \in [0, 2\pi L)$ and the $\mathbf{X}_a$ are distinct. For $\nabla\times\mathbf{A} = \pm\nabla U$, the Riemann tensor obeys $\mathcal{R}_{\mu\nu\rho\sigma} = \mp^\star\mathcal{R}_{\mu\nu\rho\sigma}$.

The Taub-NUT metric takes the same form as our Kaluza-Klein ansatz (20.2) with $U = L^2/R^2$. However, because $U \to 1$ asymptotically, it means that we have made a coordinate choice in which the fiducial length L is taken to be the physical asymptotic length of the circle: $R(\mathbf{x}) \to L$.

One might wonder about the relevance of Taub-NUT spaces to the Euclidean path integral. Ultimately, we are of course interested in physics on $\mathbb{R}^{1,2}\times S^1$ and, after a Wick rotation, the boundary of space is $S^2\times S^1$. Yet for $k \neq 0$, the boundary of the manifold is the S^1 is fibered non-trivially over the S^2. For example, with $k = 1$, the boundary is topologically S^3. The question at hand is whether we should sum over these different boundary conditions in the path integral.

A similar question arises in gauge theories in flat space, where the issue is whether one should sum over topologically non-trivial bundles at infinity. Here the answer is certainly yes: a trivial gauge bundle can be smoothly deformed into an instanton-anti-instanton pair which are subsequently moved far apart. Such configurations certainly contribute to the path integral but locality and cluster decomposition then requires us to also sum over individual instanton bundles. (See, for example, [241] for a recent discussion of this topic.) However, these same arguments also hold in the present case: we can equally well locally nucleate a NUT-anti-NUT pair which can then be moved far apart. This suggests that should sum over all asymptotic windings. (There is, admittedly, one loophole which is the lack of local observables in a theory of gravity but this does not seem to be a serious objection to the argument.)

Another way to motivate including non-trivial S^1 bundles is to consider a parallel to a more familiar story with gauge theory instantons. There, one imposes 'initial' and 'final' conditions in Euclidean time and boundary conditions at spatial infinity which require local decay everywhere, but allow for non-trivial global behaviour of the solution. For us, where the distinction between initial and boundary conditions is blurred, the obvious analogy is to consider 'initial' and 'final' surfaces which are asymptotically flat hemispheres of S^2 with a (necessarily trivial) S^1 bundle, and require them to be glued in a locally smooth, flat manner. The non-trivial global behaviour now arises due to the possibility of this gluing creating a non-trivial bundle of S^1 over the S^2.

We conclude that, despite the different boundary conditions, we should be summing over Taub-NUT configurations to determine the low-energy physics on $\mathbb{R}^{1,2} \times S^1$. We would reach the same conclusion by considering the low-energy world where we would expect to sum over different Dirac monopole configurations provided they have a suitable microscopic completion [50]. (We would also reach the same conclusion in the very high-energy world of string theory, where these Taub-NUT instantons can be viewed as D6-brane instantons wrapping manifolds of G2-holonomy [223].)

The multi-Taub-NUT solution (22.2) enjoys $3k$ bosonic zero modes, parametrized by the centres $\mathbf{X}_a$, and $2k$ spin-3/2 fermionic zero modes [53][1]. Although this result is well known, we will provide a slightly different derivation of the index theorem for the fermionic zero modes in Section 22.3 en route to calculating the one-loop determinants. For now, we merely note that only the $k = 1$ Taub-NUT solution, with two fermionic zero modes, can contribute to the superpotential [222].

The Action

The Einstein-Hilbert action evaluated on the Taub-NUT space with charge $k = 1$ is, after subtracting appropriate counterterms, given by [245, 249],

$$S_{\rm TN} = 2\pi^2 M_{\rm pl}^2 R^2$$

where R here is interpreted as the asymptotic radius of the circle. (In the coordinates (22.2), we could just as well have written $S_{\rm TN} = 2\pi^2 M_{\rm pl}^2 L^2$.) However, there are a number of further contributions to the action. The first comes from the dual 3d photon which, as first observed by Polyakov, acts as a chemical potential for the topological instanton charge [50]. This follows from the coupling (20.3): the 3d field strength arising from the metric (22.2) has charge $\int_{S^2} F = 2\pi L$, which ensures that the single Taub-NUT instanton also comes with a factor of

$$\mathcal{S} = 2\pi^2 M_{\rm pl}^2 R^2 + i\sigma \,.$$

This coincides with the classical complex structure (21.25). Of course, this had to be the case since the superpotential will come with the factor $\mathcal{W} \sim e^{-\mathcal{S}}$. Turning this observation on its head, it could be viewed as a particularly simple derivation of the action of Taub-NUT, a subject which has previously enjoyed some controversy before the definitive analysis of [249].

[1]For Yang-Mills instantons, the number of zero modes can be simply determined by integrating the anomaly. In the present case there is a mismatch between the integrated anomaly (21.5) and the number of zero modes due to the presence of boundary terms. These are known as eta-invariants [248] and will also play a role when we come to discuss the one-loop determinants around the background of the gravitational instanton.

Further contributions come from the total derivative terms: these are the Gauss-Bonnet term (20.8) and the Pontryagin term (20.9). For Taub-NUT, the integral of the Gauss-Bonnet term gives the Euler character,

$$\chi = \frac{1}{32\pi^2}\int \mathrm{d}^4x\sqrt{g}\,{}^\star\mathcal{R}^\star_{\mu\nu\rho\sigma}\,\mathcal{R}^{\mu\nu\rho\sigma} = 1\,.$$

(There is no boundary contribution.) This means that the Taub-NUT instanton will contribute to the superpotential in the form

$$\mathcal{W} \sim e^{-\mathcal{S}}e^{-\tau_{\rm grav}}\,. \tag{22.3}$$

This is the promised superpotential (19.1). Here $\tau_{\rm grav}$ is given by (20.10) and, like $\mathcal{S}$, is naturally complex and lives is the lowest component of a chiral multiplet. This superpotential drives the moduli $\mathcal{S}$ to large values, decompactifying the Kaluza-Klein circle.

The Computation

The rest of this chapter is devoted to understanding more fully the computations involved in deriving (22.3). The key extra ingredient is the evaluation of the one-loop determinants around the background of Taub-NUT. We will find that, despite the existence of supersymmetry, these determinants do not cancel. Instead, after removing the zero-modes, the determinants are computed to be (up to a numerical constant)

$$\text{dets} \sim \mu^{41/24}R^{-7/24}$$

where μ is the UV cut-off. This provides the prefactor to the superpotential (22.3) which becomes

$$\mathcal{W} \sim \mu^{41/24}R^{-7/24}e^{-\mathcal{S}}e^{-\tau_{\rm grav}}\,.$$

Now we can see how all the pieces fit together. As we explained in Chapter 21, the Gauss-Bonnet coupling $\tau_{\rm grav}$ runs at one-loop and so depends on μ. This combines with the $\mu^{41/24}$ factor that arises from the determinant and whose exponent agrees with the beta-function for $\tau_{\rm grav}$. Together they give the RG-invariant scale $\Lambda_{\rm grav}$ defined in (21.4). Meanwhile, the factor of $R^{-7/24}$ coming from the determinants can be exponentiated and has the right coefficient to shift the chiral multiplet $\mathcal{S}$ to its one-loop corrected value given in (21.28). The net result is that the superpotential takes the simple form

$$\mathcal{W} \sim \Lambda_{\rm grav}^{41/24}\,e^{-\mathcal{S}}$$

where the $\sim$ is hiding a numerical coefficient and factors of $M_{\rm pl}$ which ensure that the dimensions work out.

However, before diving into the detail of these computations, let us make a few general comments on these instanton effects.

Relation to Three-Dimensional Gauge Theories

There is a close analogy between our gravitational computation and the quantum dynamics of $\mathcal{N} = 1$ $SU(2)$ Yang-Mills theory compactified on $\mathbb{R}^{1,2} \times S^1$. In both cases, the low-energy physics comprises a $U(1)$ gauge field and a neutral scalar, with the only difference classically lying in the form of the Kähler potential (21.27).

In the case of Yang-Mills theory, there are two contributions to the low-energy effective action. The first, considered long ago in [250], arises from monopoles in the three-dimensional effective gauge theory and results in a run-away potential on the Coulomb branch, parametrized by the chiral multiplet Φ. The second contribution is inherently four-dimensional in origin; it arises from monopoles twisted around the spatial S^1, sometimes known as calarons. This second contribution carries the quantum numbers of a four-dimensional instanton, $e^{2\pi i \tau_{\rm YM}}$ with $\tau_{\rm YM} = 2\pi/\theta_{\rm YM} + 4\pi i/g^2_{\rm YM}$. The net result is the superpotential [242, 243],

$$\mathcal{W}_{\rm YM} \sim e^{-\Phi} + e^{+\Phi} e^{2\pi i \tau_{\rm YM}} \, .$$

The gravitational instanton contribution (22.3) is analogous to the second term above[2]. Both are associated to physics in four dimensions that does not strictly have a counterpart in three dimensions. Both also drive the moduli to the region where the heavy states – whether W-bosons or Kaluza-Klein modes – become light. In the Yang-Mills case, this is the strongly coupled region and the W-bosons do not ultimately become massless; in the gravitational case, this is the weakly coupled region and the Kaluza-Klein modes do become massless.

Of course, in the Yang-Mills case the first term stabilizes the Coulomb branch scalar and the theory on S^1 has two, isolated vacua. There seems to be no analogue of the first term in the gravitational context. The reason is simply that the strict three-dimensional theory is $U(1)$ and not $SU(2)$ and the former has no microscopic monopoles of its own.

Other Topologies and Moduli Fixing

The Taub-NUT metrics (22.2) are not the only self-dual gravitational instantons which asymptote to a space with one compact direction. For our purposes, the other relevant instanton is the Atiyah-Hitchin manifold $\mathcal{M}_{AH}$. This admits a smooth hyperKähler metric with isometry group $SO(3)$, as opposed to the $SO(3) \times U(1)$ isometry of Taub-

[2]Thanks to N. Seiberg for discussions on this issue.

NUT [251]. This means that the Kaluza-Klein modes around the asymptotic S^1 are excited in this solution.

The Atiyah-Hitchin manifold has 3 bosonic zero modes and 2 fermionic zero modes, the right number to contribute to the superpotential[3]. Let us first proceed naively and ask what would happen if we *were* to admit Atiyah-Hitchin as a contribution to the path integral. While Taub-NUT has winding, or magnetic charge, 1, Atiyah-Hitchin has winding number -4. (See, for example, [242].) By supersymmetry, this means that its complexified action should be $S_{AH} = -4\mathcal{S}$. The minus sign is important here. It is related to the fact that, viewed as a soliton, Atiyah-Hitchin has negative mass. As explained in [54], it follows from the breaking of the $U(1)$ isometry, and the fact that spatial kinetic terms act like a negative mass in gravity. (It is also related to the fact that M-theory compactified on Atiyah-Hitchin reduces to type IIA string theory in the presence of an orientifold $O6$-plane [253, 254] and orientifolds have negative tension.) Including contributions from both Taub-NUT and Atiyah-Hitchin would give rise to the superpotential

$$\mathcal{W} \sim e^{-\mathcal{S}} e^{-\tau_{\text{grav}}} + e^{+4\mathcal{S}} e^{-\tau_{\text{grav}}} \,.$$

The theory appears to now have a ground state with the radius R fixed at some value (albeit at the Planck scale where the analysis is not trustworthy). The presence of the Atiyah-Hitchin manifold here is reminiscent of the role orientifolds play in more complicated models of moduli stabilization [255].

Nonetheless, there is reason to doubt that we should include $\mathcal{M}_{AH}$ as a saddle in the path integral. This is because the asymptotic structure of $\mathcal{M}_{AH}$ is given by a S^1 bundle over $\mathbb{RP}^2 \cong S^2/\mathbb{Z}_2$ rather than a bundle over S^2. It is not clear whether such an asymptotic change of topology should be allowed in the sum over geometries.

Of course, we have just argued that we should be summing over different asymptotic S^1 bundles and we could try to repeat the nucleation argument that we made above for NUTs. Now the object that lies at the centre of Atiyah-Hitchin is a "bolt", a 2-cycle with topology $\mathbb{RP}^2$ and size $\sim R$. The non-local nature makes it less clear whether bolts and anti-bolts can be smoothly nucleated from the vacuum. Furthermore, the "gluing" argument that we presented above suggests that we should not include Atiyah-Hitchin in the path integral.

While we do not yet know the complete rules for performing the path integral over manifolds we different topology, these arguments suggest that it is at the very least

[3]The double cover of Atiyah-Hitchin also admits a smooth hyperKähler metric, but this space has 6 bosonic zero modes and 4 fermionic zero modes so cannot contribute to the superpotential. (The 6 bosonic zero modes consist of 3 translations and 3 deformations described in [252].)

consistent (and probably required) to only include S^1 bundles over S^2 in the path integral. This means that we do not sum over discrete quotients of the asymptotic S^2 and ignore the contribution from Atiyah-Hitchin. The same conclusion was reached in [54]. Henceforth, we proceed under this assumption.

It is pleasing to note that our justifications for these choices are well motivated physically, and that they seem to chime together well. Along with the fact that the rest of this computation works out nicely, this seems to suggest that these are the right sort of criteria to apply.

22.2 Determinants Again

In Section 21.2 we computed the ratio of determinants arising from one-loop fluctuations around a general background. They are

$$\Gamma = \frac{\det \Delta_1 \, \det^{1/4} \Delta_{3/2}}{\det^{1/2} \Delta_2 \, \det^{1/2} \Delta_0 \, \det^{3/4} \Delta_{1/2}} \tag{22.4}$$

where Δ_s is the Laplacian-type operator acting on a field of spin s. The definitions of each of them can be found in Section 21.2. The purpose of this section is to compute this ratio of determinants explicitly in the Taub-NUT background. We will find that, despite the existence of supersymmetry, the bosonic and fermionic determinants do not cancel. Nonetheless, there is sufficient simplification that the ratio can be evaluated exactly.

Determinants in an Anti-Self-Dual Background

We start by finding a simplified expression for the ratio of determinants in an anti-self-dual background obeying $\mathcal{R}_{\mu\nu\rho\sigma} = -{}^\star\mathcal{R}_{\mu\nu\rho\sigma}$. The key observation is that the self-dual part of the spin connection is flat. This means that it is possible to choose coordinates such that

$$\omega_{ab}{}^{\mu} = -\frac{1}{2}\epsilon_{abcd}\,\omega^{cd\mu} \tag{22.5}$$

and moreover, the coordinates in which we have written the Taub-NUT metric (22.2) have this property.

To see the implications of this, it is useful to change from the Majorana basis of gamma matrices introduced in (20.5) to a chiral basis. In Euclidean space, these are given by

$$\gamma^a = \begin{pmatrix} 0 & \sigma^a \\ \bar{\sigma}^a & 0 \end{pmatrix} \qquad a = 1,2,3,4 \quad \text{and} \quad \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

with $\sigma^a = (1, \vec{\sigma})$ and $\bar{\sigma}^a = (1, -\vec{\sigma})$. In such a basis, Dirac spinors decompose in the familiar left-handed (undotted) and right-handed (dotted) chiral spinors,

$$\psi = \begin{pmatrix} \chi_\alpha \\ \bar{\lambda}^{\dot{\alpha}} \end{pmatrix} . \tag{22.6}$$

In what follows, we work with this chiral decomposition, using indices $\alpha, \dot{\alpha} = 1, 2$. (This contrasts with our earlier work with 4d Majorana spinors.) The utility of this is that the spin connection acting on right-handed spinors is $\frac{1}{2}\omega_{ab\mu}\bar{\sigma}^{ab}_{\dot{\alpha}\dot{\beta}}$ where $\bar{\sigma}^{ab} = \frac{1}{2}\bar{\sigma}^{[a}\sigma^{b]}$ is self-dual and so, in the coordinates in which (22.5) holds, vanishes when contracted with the spin connection. Meanwhile, the spin connection acting on left-handed spinors involves $\omega_{ab\mu}\sigma^{ab}_{\alpha\beta}$ and does not vanish since $\sigma^{ab} = \frac{1}{2}\sigma^{[a}\bar{\sigma}^{b]}$ is anti-self-dual. This means that the chiral Dirac operator acting on left-handed fermions – which we call $\bar{\sigma}^\mu\nabla^+_\mu$ – includes a spin connection, but the chiral operator acting on right-handed fermions – which we call $\sigma^\mu\nabla^-_\mu$ – does not. (Of course, both of these covariant derivatives do contain Levi-Civita connections when acting on objects which also carry vector indices.)

We will show that each of the operators Δ_s has a natural decomposition into operators that act on left-handed or right-handed spinors. This is simplest to see for the spin $s = 1/2$ operator, where we have

$$\Delta_{1/2} = (i\not{D})^2 = \begin{pmatrix} -\sigma^\mu\nabla^-_\mu\bar{\sigma}^\nu\nabla^+_\nu & 0 \\ 0 & -\bar{\sigma}^\mu\nabla^+_\mu\sigma^\nu\nabla^-_\nu \end{pmatrix} \equiv \begin{pmatrix} \Delta_{1/2\,+} & 0 \\ 0 & \Delta_{1/2\,-} \end{pmatrix} .$$

Moreover, the self-duality of the spin connection means that the operator on right-handed fermions simplifies yet further. It is given by

$$\Delta_{1/2\,-} = \Delta_0\,\mathbf{1}_2 \quad \Rightarrow \quad \det\Delta_{1/2\,-} = (\det\Delta_0)^2 .$$

To perform a similar decomposition for higher spin operators, we need to work a little harder. We start with Δ_1 defined in (21.8). A self-dual background has $\mathcal{R}_{\mu\nu} = 0$, so the operator involves only the Laplacian acting on vectors. To decompose this in terms of spinors, we use the fact that the background admits two, orthogonal, covariantly constant (and, in fact, actually constant) right-handed spinors, $\xi^{\dot{\alpha}}_{(i)}$. These obey the simple equation

$$\nabla^-_\mu\xi_{(i)} = 0 \quad i = 1, 2$$

where, in the coordinates in which (22.5) holds, $\nabla^-_\mu\xi_{(i)} \equiv \partial_\mu\xi_{(i)}$.

The constant spinors $\xi_{(i)}$ allow us to decompose any (complexified) field so that the dynamical degrees of freedom live in irreducible representations of $su(2)_L \subset so(4)$. We

first demonstrate this with the vector field A^a which we write it in the usual bi-spinor form as $A^{\alpha\dot\alpha} = A_a(\sigma^a)^{\alpha\dot\alpha}$. The existence of the pair of constant spinors $\xi^{\dot\alpha}$ allows us to write a general complex vector in this background as

$$A^{\alpha\dot\alpha} = \sum_{i=1}^{2} a^{\alpha}_{(i)}\xi^{\dot\alpha}_{(i)}$$

where the dynamical degrees of freedom are now the two left-handed spinors $a_{(i)}$. When sandwiched between two such vectors, Δ_1 reads

$$\tilde{A}^{\dagger\, a}(\Delta_1)_{a;b}A^b = \sum_{i=1}^{2} \xi^{\dagger}_{(i)}\xi_{(i)}\,(\tilde{a}^{\dagger}_{(i)}\nabla^2 a_{(i)})$$

where we are exploiting the fact that the $\xi_{(i)}$ with $i = 1, 2$ are orthogonal to eliminate the cross-terms. The upshot of this argument is that in a self-dual background, we can write

$$\det \Delta_1 = (\det \Delta_{1/2\,+})^2\,.$$

We can now move on to discuss $\Delta_{3/2}$ defined in (21.10). This involves a new element since the Riemann tensor now appears. We make use of the fact that, after replacing the spatial indices with bi-spinors, an anti-self-dual Riemann tensor can be written as

$$\mathcal{R}_{\alpha\dot\alpha\,\beta\dot\beta\,\gamma\dot\gamma\,\delta\dot\delta} = \mathcal{C}_{\alpha\beta\gamma\delta}\,\epsilon_{\dot\alpha\dot\beta}\,\epsilon_{\dot\gamma\dot\delta}$$

where $\mathcal{C}_{\alpha\beta\gamma\delta}$ is the totally symmetric, anti-self-dual Weyl tensor. As in the spin-1/2 case, the $\Delta_{3/2}$ operator naturally decomposes into left and right-moving parts,

$$\det \Delta_{3/2} = \det \Delta_{3/2\,-}\ \det \Delta_{3/2\,+}\,.$$

To get more of a handle on these determinants, we again decompose a spin-3/2 fermion in terms of the covariantly constant spinors $\xi_{(i)}$. We have to treat the left and right-moving pieces somewhat differently. A general, complex right-handed spinor can be decomposed as

$$\psi^{\alpha\dot\alpha\dot\beta} \equiv (\sigma^\mu)^{\alpha\dot\alpha}\psi^{\dot\beta}_{\mu} = f^{\alpha}_{(1)}\xi^{\dot\alpha}_{(1)}\xi^{\dot\beta}_{(1)} + f^{\alpha}_{(2)}\xi^{\dot\alpha}_{(1)}\xi^{\dot\beta}_{(2)} + f^{\alpha}_{(3)}\xi^{\dot\alpha}_{(2)}\xi^{\dot\beta}_{(1)} + f^{\alpha}_{(4)}\xi^{\dot\alpha}_{(2)}\xi^{\dot\beta}_{(2)}\,.$$

The 8 dynamical degrees of freedom are now contained in four, left-moving spinors $f_{(i)}$, $i = 1, 2, 3, 4$. Perhaps unsurprisingly, the Riemann tensor does not act on this part of ψ. The same kind of argument that we used for Δ_1 shows that $\Delta_{3/2\,-}$ does not mix the different $f_{(i)}$, and we find

$$\det \Delta_{3/2\,-} = (\det \Delta_{1/2\,+})^4\,.$$

The decomposition of the left-handed spin-3/2 field involves a new ingredient. We write

$$\psi^{\alpha\dot\alpha\beta} \equiv (\sigma^\mu)^{\alpha\dot\alpha}\psi_\mu^\beta = \sum_{i=1}^{2} F_{(i)}^{\alpha\beta}\xi_{(i)}^{\dot\alpha} + \phi_{(i)}\epsilon^{\alpha\beta}\xi_{(i)}^{\dot\alpha}\,. \tag{22.7}$$

Now the dynamical degrees of freedom are contained in two scalars $\phi_{(i)}$ and two, symmetric tensors $F_{(i)}^{\alpha\beta}$. The Riemann tensor does not affect the scalar fields $\phi_{(i)}$; these merely contribute a factor of $(\det\Delta_0)^2$ to $\det\Delta_{3/2+}$. However, the Riemann tensor does affect the operators acting on the symmetric tensors $F_{(i)}$. To see how, we look at the contraction

$$\begin{aligned}
(\bar\sigma^a)^{\dot\alpha\alpha}\left(\Delta_{3/2+}\right)_{a\gamma;b}{}^{\delta}(\sigma^b)_{\beta\dot\beta} &= \bar\sigma^{a\,\dot\alpha\alpha}\left(-\eta_{ab}\delta_\gamma^\delta\nabla^2 - \frac{1}{2}\mathcal{R}_{cdab}\sigma^c_{\gamma\dot\gamma}\bar\sigma^{d\,\dot\gamma\delta}\right)\sigma^b_{\beta\dot\beta} \\
&= -2\delta_{\dot\beta}^{\dot\alpha}\,\delta_\beta^\alpha\,\delta_\gamma^\delta\nabla^2 - \frac{1}{2}\mathcal{C}_\gamma{}^{\delta\alpha}{}_\beta\,\delta_{\dot\gamma}^{\dot\gamma}\,\delta_{\dot\beta}^{\dot\alpha} \\
&\equiv 2\delta_{\dot\beta}^{\dot\alpha}\,\Delta_C{}^{\alpha}{}_{\gamma;\,\beta}{}^{\delta}
\end{aligned}$$

where we define a new operator Δ_C which acts on anti-self-dual 2-forms (which transform in the $(1,0)$ representation of $SO(4)$ rotations) and involves the Weyl tensor:

$$(\Delta_C)^{\alpha\beta}{}_{\gamma\delta} = -\delta_\gamma^\alpha\,\delta_\delta^\beta\nabla^2 - \frac{1}{2}C^{\alpha\beta}{}_{\gamma\delta}\,.$$

Our expression for the left-moving spin-3/2 determinant is then

$$\det\Delta_{3/2+} = (\det\Delta_C)^2\,(\det\Delta_0)^2\,.$$

The same operator Δ_C also shows up in the determinant of Δ_2. The traceless part of the metric is decomposed as

$$\bar h^{\alpha\dot\alpha\,\beta\dot\beta} = H_{(1)}^{\alpha\beta}\xi_{(1)}^{\dot\alpha}\xi_{(1)}^{\dot\beta} + H_{(2)}^{\alpha\beta}\xi_{(1)}^{(\dot\alpha}\xi_{(2)}^{\dot\beta)} + H_{(3)}^{\alpha\beta}\xi_{(2)}^{\dot\alpha}\xi_{(2)}^{\dot\beta} \tag{22.8}$$

where the nine dynamical degrees of freedom are now contained in three symmetric tensors, $H_{(i)}^{\alpha\beta}$, with $i=1,2,3$. The Laplacian operator Δ_2 is defined in (21.7) and also contains a Riemann tensor term. To understand its action on the $H_{(i)}$, we again look at the contraction

$$\begin{aligned}
\bar\sigma^{a\,\dot\alpha\alpha}\bar\sigma^{b\,\dot\beta\beta}(\Delta_2)_{ab;cd}\sigma^{(c}_{\gamma\dot\gamma}\sigma^{d)}_{\delta\dot\delta} &= \bar\sigma^{a\,\dot\alpha\alpha}\bar\sigma^{b\,\dot\beta\beta}\left(-\frac{1}{4}\eta_{ac}\eta_{bd}\nabla^2 - \frac{1}{2}\mathcal{R}_{acbd}\right)\sigma^{(c}_{\gamma\dot\gamma}\sigma^{d)}_{\delta\dot\delta} \\
&= \frac{1}{2}\left[-\delta_\gamma^\alpha\,\delta_\delta^\beta\,\delta_{\dot\gamma}^{\dot\alpha}\,\delta_{\dot\delta}^{\dot\beta}\,\nabla^2 - \frac{1}{2}\mathcal{R}^{\alpha\dot\alpha}{}_{\gamma\dot\gamma}{}^{\beta\dot\beta}{}_{\delta\dot\delta} + (\gamma\dot\gamma\leftrightarrow\delta\dot\delta)\right] \\
&= \frac{1}{2}\delta_{\dot\gamma}^{\dot\alpha}\,\delta_{\dot\delta}^{\dot\beta}\,(\Delta_C)^{\alpha\beta}{}_{\gamma\delta}
\end{aligned}$$

where, at each step, one should understood these operators to be acting on suitably

symmetrized objects. This means that we have

$$\det \Delta_2 = (\det \Delta_C)^3 = \frac{(\det \Delta_{3/2\,+})^{3/2}}{(\det \Delta_0)^3}\,.$$

Putting all this together, we find that the ratio of determinants (22.4) in an anti-self-dual background can be written as

$$\Gamma = \left(\frac{\det \Delta_{3/2\,+}}{\det \Delta_{3/2\,-}}\right)^{-1/2} \left(\frac{\det \Delta_{1/2\,+}}{\det \Delta_{1/2\,-}}\right)^{+1/4}\,. \tag{22.9}$$

The determinants take the form of ratios of chiral Dirac operators. This is characteristic of instanton computations in supersymmetric theories. Indeed, since the spectrum of non-vanishing eigenvalues of $\Delta_{s\,+}$ (with $s = 1/2, 3/2$) is identical to the spectrum of $\Delta_{s\,-}$ one might naively think that these determinants cancel. (This was the conclusion reached in [53] based on an explicit bijection between the bosonic and fermionic eigenfunctions in of the operators in (22.4).) However, this is too quick. The spectra of both $\Delta_{s\,+}$ and $\Delta_{s\,-}$ contain a continuum of scattering states, and while the range of eigenvalues of the two operators coincide, their densities are not necessarily the same. Below we will compute Γ in a multi-Taub-NUT background and show that it is non-trivial.

The non-cancellation of determinants around self-dual backgrounds has precedent. It occurs in three-dimensional supersymmetric gauge theories where the instantons are 't Hooft-Polyakov monopoles [256, 257]. (The spectral asymmetry of the Dirac operators had been appreciated earlier in the renormalization of monopole states in four-dimensional gauge theories [258].) The non-cancellation of determinants also arises in supersymmetric quantum mechanics where the instantons are kinks [259]. (Again, the first appearance of this can be traced to the mass renormalization of kinks in two dimensional theories [260]; a detailed review of these effects can be found in [261].)

Finally, we mention that closely related results have been seen recently in the computation of the elliptic genus in non-compact sigma-models, where the non-cancellation of a continuum of scattering states results in a holomorphic anomaly [262, 263]. This effect also occurs for Taub-NUT sigma-models [264]. It would be interesting to see if there is any deeper relationship between these two effects.

Evaluating the Determinants

We now turn to the task of evaluating the determinants explicitly. This is possible because there is a close relationship between the ratio of determinants in (22.9) and the (regularized) index for the appropriate Dirac operator [256]. To see this we first define

the regularized ratio

$$D(m^2) = \frac{\det \Delta_+ + m^2}{\det \Delta_- + m^2}\,. \tag{22.10}$$

This expression could apply to either $s = 1/2$ or $s = 3/2$ operators. Here m^2 plays the role of an infra-red regulator; its presence will allow us to easily extract the zero modes from the determinants later. Now consider

$$\mathcal{I}(m^2) = \frac{\partial \log D}{\partial \log m^2} = \mathrm{Tr}\left[\frac{m^2}{\Delta_+ + m^2} - \frac{m^2}{\Delta_- + m^2}\right].$$

This is the regularized index of the Dirac operator. The index itself is given by

$$\mathcal{I} = \lim_{m^2 \to 0} \mathcal{I}(m^2)$$

and counts $n_+ - n_-$ where $n_\pm$ is the number of zero modes of $\Delta_\pm$.

In what follows, we want to treat both $s = 1/2$ and $s = 3/2$ operators at once. We can do this at the expense of introducing some new notation. We return to the original 4-component spinor notation, with the Dirac operator written as $\hat{\gamma}\cdot\nabla$. For the spin-1/2 field, we simply choose $\hat{\gamma}^\mu = \gamma^\mu$. But, for the spin-3/2 field, the Dirac operator in (20.1) means we should pick $(\hat{\gamma}^\mu)_{\rho\sigma} = -\frac{1}{2}\gamma_\sigma\gamma^\mu\gamma_\rho$, where the additional indices are contracted with the spacetime indices of ψ_μ.

For both cases, we have $\{\hat{\gamma}^a, \hat{\gamma}^b\} = 2\delta^{ab}$, and $\hat{\gamma}^5 = \hat{\gamma}^1\hat{\gamma}^2\hat{\gamma}^3\hat{\gamma}^4 = \gamma^5$ so that $\{\hat{\gamma}^5, \hat{\gamma}^a\} = 0$. We should also bear in mind that the Lorentz generators t^{ab} are different for the two spins.

With this new notation, we can write the regularized index as

$$\mathcal{I}(m^2) = \mathrm{Tr}\left[\hat{\gamma}^5 \frac{m^2}{-(\hat{\gamma}\cdot\nabla)^2 + m^2}\right].$$

We now split this expression for $\mathcal{I}(m^2)$ into two terms. One of these will be somewhat subtle and we should be careful in proceeding. Wary of this, we will work with a form of zeta-function regularization. This means first introducing a new parameter z and replacing the expression in square brackets above with

$$\hat{\gamma}^5 \frac{m^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^{1+z}} = \hat{\gamma}^5 \frac{1}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^{z}} + \hat{\gamma}^5 \frac{(\hat{\gamma}\cdot\nabla)^2}{(-(\hat{\gamma}\cdot\nabla)^2 + m^2)^{1+z}}\,. \tag{22.11}$$

We will ultimately set $z = 0$. The first term above naively looks like it reduces to $\hat{\gamma}^5$ when we set $z = 0$. But this is too hasty: it ignores the presence of the anomaly. To see this, we use the same heat kernel techniques that we employed in Section 21.5. Taking

the trace, the first term above reads

$$\mathrm{Tr}\left[\hat{\gamma}^5 \frac{1}{(-(\hat{\gamma}\cdot\nabla)^2+m^2)^z}\right] = \mathrm{Tr}\left[\hat{\gamma}^5 \frac{1}{\Gamma(z)}\int_0^\infty \frac{\mathrm{d}t}{t^{1-z}} e^{-\left(-(\hat{\gamma}\cdot\nabla)^2+m^2\right)t}\right] .$$

This is the same kind of integral that we saw in Section 21.5. Up to terms which vanish as $z \to 0$, the result is very almost the expression B_4 given in (21.29); the only difference is the presence of $\hat{\gamma}^5$ in the spinor trace. This kills most of the terms and changes $\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$ expression in (21.29) into ${}^*\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma}$. The end result is

$$\lim_{z\to 0}\mathrm{Tr}\left[\hat{\gamma}^5 \frac{1}{(-(\hat{\gamma}\cdot\nabla)^2+m^2)^z}\right] = \frac{\alpha_s}{24\cdot 16\pi^2}\int \mathrm{d}^4x\sqrt{g}\,{}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} .$$

This is the promised contribution from the axial anomaly. The coefficient α_s depends on the spin of the operator and is given by[4]

$$\alpha_{1/2} = 1 \quad \text{and} \quad \alpha_{3/2} = -20 .$$

We now turn to the second term in (22.11). This term is less delicate and we can happily set $z = 0$ from the beginning without repercussion. (We will, however, still implicitly use zeta-function regularization later when we come to evaluate it.) This term is, in fact, a total derivative, and the full regularized index takes the form

$$\mathcal{I}(m^2) = \frac{\alpha_s}{24\cdot 16\pi^2}\int \mathrm{d}^4x\sqrt{g}\,{}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} + \int \mathrm{d}S_\mu\,\sqrt{g_{\text{bdy}}}\,J^\mu \tag{22.12}$$

where $\sqrt{g_{\text{bdy}}}$ is the square-root of the induced metric on the boundary and the current J^μ is defined by

$$J^\mu = \lim_{y\to x}\ \frac{1}{2}\,\mathrm{tr}\,\langle y|\,\hat{\gamma}^5\hat{\gamma}^\mu \frac{\hat{\gamma}\cdot\nabla}{(-(\hat{\gamma}\cdot\nabla)^2+m^2)}\,|x\rangle\ . \tag{22.13}$$

The two contributions in (22.12) are typical for index theorems on manifolds with boundary. (See, for example, [265], for a discussion of index theorems for gravitational instantons. A similar structure is also seen in index theorems for Yang-Mills-Dirac operators on $\mathbb{R}^3\times S^1$ [266].)

So far our discussion has been for a general anti-self-dual metric. At this point we restrict to the multi-Taub NUT spaces of interest, with metric given in (22.2). They have Pontryagin class

$$\frac{1}{16\pi^2}\int \mathrm{d}^4x\sqrt{g}\,{}^\star\mathcal{R}_{\mu\nu\rho\sigma}\mathcal{R}^{\mu\nu\rho\sigma} = -2k . \tag{22.14}$$

[4]In the expression for the axial anomaly (21.5), the spin-3/2 and spin-1/2 contributions differ by a factor of -21. This is because, in computing the physical anomaly, the factor of -21 includes the contribution from three spin-1/2 ghosts. These have different chiral charges and change the $\alpha_{3/2} = -20$ that arises in the present computation into the -21 that appears in (21.5).

To compute the boundary in (22.12), we use some standard machinery [267]. The current is a local response to a nearby (as $x \to y$) excitation and its flux through the boundary can be computed using only the asymptotic form of the metric (22.2). Since the volume scales as r^2, anything decaying as $1/r^3$ or faster in the current vanishes. Accordingly, if we expand the covariant derivatives as $\nabla_\mu = \partial_\mu + \frac{1}{2}t^{ab}\omega_{ab\mu}$, we have

$$J^\mu = \frac{1}{2}\,\mathrm{tr}\,\langle x|\,\hat\gamma^5\hat\gamma^\mu\hat\gamma^\nu \quad \left(\partial_\nu + \frac{1}{2}\omega_{ab\nu}t^{ab}\right)\left[\frac{1}{(-\Delta_0+m^2)}\right.$$
$$\left. +\frac{1}{(-\Delta_0+m^2)}\omega_{ab}{}^{\rho}t^{ab}\partial_\rho\frac{1}{(-\Delta_0+m^2)} + \ldots\right]|x\rangle\ .$$

The leading terms vanish using $\mathrm{tr}\,\hat\gamma^5\hat\gamma^a\hat\gamma^b = 0$. Keeping only terms which survive asymptotically, we find

$$J^\mu \longrightarrow \frac{1}{2}\,\mathrm{tr}\left[\hat\gamma^5\hat\gamma^\mu\hat\gamma^\nu t^{ab}\right]\omega_{ab}{}^{\rho}\,\langle x|\left[\frac{1}{2}\frac{g_{\nu\rho}}{(-\partial^2+m^2)} + \frac{\partial_\nu\partial_\rho}{(-\partial^2+m^2)^2}\right]|x\rangle\ .$$

The overall coefficient is determined by the trace of gamma matrices. It differs for spin-1/2 and spin-3/2:

$$\frac{1}{2}\,\mathrm{tr}\left[\hat\gamma^5\hat\gamma^\mu\hat\gamma^\nu t^{ab}\right] = \beta_s\epsilon^{\mu\nu ab} \quad \text{with} \quad \beta_{1/2}=1 \quad \text{and} \quad \beta_{3/2}=4\,.$$

Using the self-duality of the spin connection (22.5), we can then write

$$\begin{aligned} J^\mu &\longrightarrow -\beta_s\,\omega^{\mu\nu\rho}\,\langle x|\left[\frac{g_{\nu\rho}}{(-\partial^2+m^2)} + \frac{2\partial_\nu\partial_\rho}{(-\partial^2+m^2)^2}\right]|x\rangle \\ &= -\beta_s\,\omega^{\mu\nu\rho}g^{-1/2}\frac{1}{2\pi L}\sum_n\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{g_{\nu\rho}}{(k^2+m^2)} - \frac{2k_\nu k_\rho}{(k^2+m^2)^2}\right] \end{aligned}$$

where we have introduced a Fourier basis to integrate over the 4d momenta $k^\mu = (\mathbf{k}, n/L)$. Our interest is in the outward flux, J^i where we will take $i = 1,2,3$ to be a tangent space index for simplicity. Asymptotically, the metric is locally flat and we have $k^2 = \mathbf{k}^2 + n^2/L^2$. Using the explicit form of the spin connection, one finds that only the $\nu,\rho = 4$ components contribute, and the relevant current is given by

$$\begin{aligned} J^i &\longrightarrow -\frac{\beta_s}{2}(\partial_i\log U)\,\frac{1}{2\pi L}\sum_n\int\frac{\mathrm{d}^3k}{(2\pi)^3}\left[\frac{1}{(\mathbf{k}^2+n^2/L^2+m^2)} - \frac{2}{(\mathbf{k}^2+n^2/L^2+m^2)^2}\frac{n^2}{L^2}\right] \\ &= +\frac{\beta_s}{2}\partial_i\left(1+\frac{Lk}{2|\mathbf{x}|}\right)\frac{1}{8\pi^2 L}\sum_n\left[\left(\frac{n^2}{L^2}+m^2\right)^{1/2} + \left(\frac{n^2}{L^2}+m^2\right)^{-1/2}\frac{n^2}{L^2}\right] \\ &= -\frac{\beta_s k}{32\pi^2 L}\,\frac{x^i}{|\mathbf{x}|^3}\sum_n\left[\left(n^2+m^2L^2\right)^{1/2} + \left(n^2+m^2L^2\right)^{-1/2}n^2\right] \end{aligned}$$

where we have taken the liberty of regularizing the linearly divergent term that appears in going from the first to the second line. Finally, we need the fact that the asymptotic flux is given by

$$\int \mathrm{d}S_i \sqrt{g_{\text{bdy}}}\, \frac{x^i}{|\mathbf{x}|^3} = 8\pi^2 L\,.$$

Putting this together with the Taub-NUT Pontryagin class (22.14), the regularized index (22.12) can be written as

$$\mathcal{I}(m^2) = -\frac{\alpha_s k}{12} - \frac{\beta_s k}{4} \sum_{n\in\mathbb{Z}} \left[\left(n^2 + m^2 L^2\right)^{1/2} + \left(n^2 + m^2 L^2\right)^{-1/2} n^2 \right] . \tag{22.15}$$

The Index

Let us pause to compute the index of the Dirac operator in the multi-Taub-NUT backgrounds. As we saw previously, the index is given by $\mathcal{I}(m^2 = 0)$. In this limit, the sum above reduces to $4\zeta(-1) = -1/3$. (The sum is over both positive and negative integers which gives a factor of 2.) Combined with the contribution from the Pontryagin class, we find

$$\mathcal{I} = -\frac{k}{12} + \frac{k}{12} = 0 \quad \text{for spin-1/2}$$

and

$$\mathcal{I} = +\frac{20k}{12} + \frac{4k}{12} = 2k \quad \text{for spin-3/2}\,.$$

This agrees with the results of [265]. This also confirms a statement that we made earlier: if we are interested in contributions to the superpotential, only the single Taub-NUT, with $k = 1$, will play a role. Nonetheless, for completeness we will compute the determinants around an arbitrary multi-Taub-NUT background.

It is instructive to return to the decomposition of spin-3/2 fermions in a self-dual background (22.7). We see that the degrees of freedom include two anti-self-dual two-forms, $F^{\alpha\beta}$, transforming in the $(1,0)$ representation of $SO(4)$. These are the objects that carry the zero modes. The same objects appear in the decomposition of the metric (22.8) which contains three anti-self-dual two-forms $H^{\alpha\beta}$. This is the reason why the metric (22.2) has $3k$ bosonic zero modes. These are identified with the positions $\mathbf{X}_a$ of the NUTs.

Back to the Determinants

We now return to the task of computing the determinants. The sums in our expression (22.15) for $\mathcal{I}(m^2)$ are divergent. Although we have used zeta-function regularization

in the derivation of the first term in (22.15), at this stage it is important that we return to Pauli-Villars regularization so that we can correctly match the finite terms with our one-loop counterterm (21.30). We have

$$\begin{aligned}\log D(m^2) - \log D_0 &= \int_0^1 \frac{d\lambda}{\lambda}\left[I(\lambda m^2)\right]_{\rm PV} \\ &= -\frac{\beta_s k}{4}\sum_n \left[2\sqrt{n^2+m^2L^2} - 2|n| - 4|n|\log\left(\frac{1}{2}+\frac{1}{2}\sqrt{1+\frac{m^2L^2}{n^2}}\right)\right]_{\rm PV}\end{aligned}$$

where $\log D_0 = \lim_{\lambda\to 0}\left[\log D(\lambda m^2)\right]_{\rm PV}$ is the logarithmic ratio of determinants in the limit in which all four fields in the Pauli-Villars regulator become massless. The equality on the second line follows after noting that any m^2-independent piece in $\mathcal{I}(m^2)$ vanishes in the Pauli-Villars regulator.

The sum above is now finite for each Pauli-Villars field individually. In the limit $m^2 \to 0$, the sum vanishes which means that it receives no contributions from the original field. But it still receives contributions from the three additional terms in the regularization (21.21). Each of these has a large mass given by $M_{\rm UV}$ (or $\gamma M_{\rm UV}$ or $(\gamma-1)M_{\rm UV}+m^2$) and we are interested in the asymptotic form of the sum in the limit $M_{\rm UV}\to\infty$. We find that

$$\int_0^1 \frac{d\lambda}{\lambda}\left[I(\lambda m^2)\right]_{\rm PV} \longrightarrow \frac{\beta_s}{12}\left[\log(\mu^2R^2)+C'\right]$$

where $\mu^2 = (\gamma-1)M_{\rm UV}^2/\gamma$ is the appropriate Pauli-Villars scale. The same quantity appeared in the one-loop counterterm (21.30). The constant is given by $C' = -\log 4 + 1 - 24\zeta'(-1)$.

The quantity $\log D_0 = \lim_{\lambda\to 0}\left[\log D(\lambda m^2)\right]_{\rm PV}$ is dominated by the zero modes. As we saw above, there are $\mathcal{I}=0$ zero-modes for spin-1/2 operators and $\mathcal{I}=2k$ zero modes for spin-3/2. We have

$$D_0 = \frac{(\lambda m^2)^{\mathcal{I}}(\lambda\gamma M_{\rm UV}^2)^{\mathcal{I}}}{(\lambda M_{\rm UV}^2)^{\mathcal{I}}(\lambda(\gamma-1)M_{\rm UV}^2+\lambda m^2)^{\mathcal{I}}} \longrightarrow \left(\frac{m^2}{\mu^2}\right)^{\mathcal{I}}.$$

We now have everything that we need to compute the one-loop determinants (22.9) about the k-centred Taub-NUT background. In the limit $m^2\to 0$, the determinants take the form

$$\Gamma = \left(\frac{\det\Delta_{3/2\,+}}{\det\Delta_{3/2\,-}}\right)^{-1/2}\left(\frac{\det\Delta_{1/2\,+}}{\det\Delta_{1/2\,-}}\right)^{+1/4} = m^{-2k}\,\Gamma'$$

which reflects the fact that $2k$ zero modes are carried by $\Delta_{3/2\,+}$. The truncated determinants Γ' are given by

$$\Gamma' = (\mu^2)^{41k/48} \left(\frac{R^2}{A}\right)^{-7k/48} \tag{22.16}$$

where the constant numerical factor is

$$A = 4e^{24\zeta'(-1)-1}\,.$$

We have seen the numbers that appear in (22.16) before! The fraction $41/48$ appeared as the beta-function for the running Gauss-Bonnet coupling (21.30). This is not a coincidence. The fraction $7/48$ appeared in the one-loop shifted complex structure (21.28). This is not a coincidence either.

22.3 Zero Modes and Jacobians

In any instanton computation, one should isolate the zero modes and replace their contribution to the path integral with a normal integration over the associated collective coordinates. In doing so, we pick up a Jacobian factor for our troubles. For gravitational instantons, this procedure was described in [45].

Bosonic Zero Modes

We restrict our attention to the Taub-NUT metric (22.2) with $k = 1$. This metric has three collective coordinates which are identified with the position $\mathbf{X}$ of the nut. The three corresponding zero modes arise from translations and suitably gauge-fixed versions of them can be conveniently constructed by taking the Lie derivative of the metric along one of the three vector fields $\partial/\partial x^i$, $i = 1, 2, 3$,

$$h^{(i)}_{\mu\nu} = \mathcal{L}_i g_{\mu\nu} = 2\nabla_\mu \nabla_\nu x^{(i)}\,.$$

These zero modes are pure gauge. However, they arise from large gauge transformations which do not die off sufficiently fast at infinity and so should be thought of as physical. To see that they satisfy the transverse trace-free gauge condition, we use the facts that in our background we have

$$\nabla^2 x^{(i)} = g^{\mu\nu}\Gamma^i_{\mu\nu} = 0$$

and also that we can commute certain derivatives through each other since $\mathcal{R}_{\mu\nu} = 0$. Taking these together, one finds

$$\nabla^\mu \left(\nabla_\mu \nabla_\nu x^{(i)}\right) = \nabla_\nu \nabla^2 x^{(i)} = 0\,, \qquad g^{\mu\nu}\left(\nabla_\mu \nabla_\nu x^{(i)}\right) = \nabla^2 x^{(i)} = 0\,.$$

To compute the Jacobian, we need an inner product between the modes. This is inherited from the action and is given by,

$$\begin{aligned}\frac{M_{\rm pl}^2}{2}\int \mathrm{d}^4x\sqrt{g}\,\frac{1}{2}h^{(i)}_{\mu\nu}h^{(j)\,\mu\nu} &= M_{\rm pl}^2\int \mathrm{d}^4x\sqrt{g}\,\left(\nabla_\mu\nabla_\nu x^{(i)}\right)\left(\nabla^\mu\nabla^\nu x^{(j)}\right)\\ &= M_{\rm pl}^2\int \mathrm{d}S^\mu\sqrt{g_{\rm bdy}}\,\left(\nabla_\mu\nabla_\nu x^{(i)}\right)\left(\nabla^\nu x^{(j)}\right)\\ &= 2\pi M_{\rm pl}^2 L\int \mathrm{d}^2x\; e^\mu_k\,\frac{x^k}{r}r^2\left(-\Gamma^i_{\mu\nu}\right)\left(g^{\nu j}\right)\\ &= \pi M_{\rm pl}^2 L\int \mathrm{d}^2x\,\frac{x^k}{r}r^2\left(-\delta^{ij}\partial_k-\delta^i_k\partial^j+\delta^j_k\partial^i\right)U\\ &= 2\pi^2 M_{\rm pl}^2L^2\,\delta^{ij}\end{aligned}$$

which we recognize as the Taub-NUT action, $S_{\rm TN}=2\pi^2M_{\rm pl}^2L^2$. The upshot is that the integral over the three bosonic collective coordinates comes with the measure

$$\int \mathrm{d}\mu_B=\int\frac{\mathrm{d}^3X}{(2\pi)^{3/2}}\,S_{\rm TN}^{3/2}\,. \tag{22.17}$$

Fermionic Zero Modes

As we saw above, the gravitino has two zero modes in the $k=1$ Taub-NUT background. These are Goldstino modes, arising from broken supersymmetry but, like their bosonic counterparts, are physical as they arise from large gauge transformations of the form $\psi_\mu=\nabla_\mu\epsilon$. The ϵ parameter satisfies the gauge fixing condition

$$\gamma^\mu\psi_\mu=\not{D}\epsilon=0\,.$$

The gravitino introduced in the original action (20.1) is a Majorana fermion. However, there is no Majorana condition in Euclidean space and, for this reason, it is simplest to work with a two component Weyl spinor formalism where

$$\psi_\mu=\begin{pmatrix}\psi_{\mu\alpha}\\ \bar\psi_\mu{}^{\dot\alpha}\end{pmatrix}\,.$$

The zero mode for this two-component spinor is then $\psi_{\mu\alpha}=\nabla_\mu\epsilon_\alpha$, $\alpha=1,2$, and the zero mode equation reduces to

$$\bar\sigma^\mu\nabla_\mu\epsilon=-i\sigma^i\,\frac{\partial_i\left(U^{1/2}\epsilon\right)}{U}=0$$

which has normalizable solutions of the form

$$\epsilon=\frac{1}{U^{1/2}}\xi$$

for any constant spinor ξ_α. (These are not to be confused with the right-handed spinors $\xi_{\dot\alpha}$ introduced at the beginning of Section 22.2 which are associated to the unbroken supersymmetry. In contrast, the left-handed spinors ξ_α are associated to the broken supersymmetry.)

The fermionic zero modes are accompanied by the measure

$$\int \mathrm{d}\mu_F = \int \mathrm{d}^2\xi\ \mathcal{J}_F^{-1}\,.$$

The fermionic Jacobian, $\mathcal{J}_F$, is given by the overlap of zero modes,

$$\begin{aligned}\mathcal{J}_F &= \frac{M_{\rm pl}^2}{2}\int \mathrm{d}^4x\mathrm{d}^2\xi\ \sqrt{g}\,(\nabla^\mu\epsilon)^\alpha\,(\nabla_\mu\epsilon)_\alpha \\ &= \frac{M_{\rm pl}^2}{2}\int \mathrm{d}S^\mu\mathrm{d}^2\xi\ \sqrt{g_{\rm bdy}}\,\epsilon^\alpha\,(\nabla_\mu\epsilon)_\alpha \\ &= \pi M_{\rm pl}^2 L\int \mathrm{d}^2x\ x^i\,r\,(\partial_i U^{-1/2}) = \frac{1}{2}S_{\rm TN}\end{aligned}$$

where, in the last line, we use the normalization $\int \mathrm{d}^2\xi\ \xi^2 = 1$.

Putting this together with the bosonic measure (22.17), we find that the integration over all collective coordinates is accompanied by the Jacobian factor

$$\int \mathrm{d}\mu_B\mathrm{d}\mu_F = \int \frac{\mathrm{d}^3X}{(2\pi)^{3/2}}\int \mathrm{d}^2\xi\ 2S_{\rm TN}^{1/2}\,. \tag{22.18}$$

22.4 Computing the Superpotential

We now have all the ingredients necessary to compute the instanton-generated superpotential. We start by computing the two-point function of the 3d spin-1/2 fermion χ which arises under dimensional reduction (20.6) from ψ_4. As we have just seen, in the background of Taub-NUT we can turn on a fermionic zero mode. For χ, this is given by

$$\chi_\alpha = \frac{1}{2}\omega_{ab4}\left(\sigma^{ab}\xi\right)_\alpha = \frac{\partial_i U}{U^{3/2}}\ \left(\sigma^{i4}\xi\right)_\alpha\,.$$

Far from the NUT itself, the zero mode becomes

$$\chi_\alpha \to \pi L S_F(x-X)_\alpha^{\ \beta}\xi_\beta$$

where $S_F(x) = \gamma^i_{\rm 3d}x_i/4\pi x^3$ is the flat-space propagator. This form will suffice for our instanton computation. Using our results for the action (22.3), the one-loop determinants

(22.16) and the measure (22.18), we have the two-point function

$$\langle \chi_\alpha(x)\chi_\beta(y)\rangle = \int \frac{\mathrm{d}^3X}{(2\pi)^{3/2}} \int \mathrm{d}^2\xi \quad 2S_{\text{TN}}^{1/2}\, \mu^{41/24} \left(\frac{R^2}{A}\right)^{-7/48} e^{-2\pi^2 M_{\text{pl}}^2 R^2 + i\sigma} e^{-\tau^\star_{\text{grav}}} \times \pi^2 L^2 S_F(x-X)_\alpha^{\ \gamma}\xi_\gamma\, S_F(y-X)_\beta^{\ \delta}\xi_\delta \,.$$

Let's firstly explain why the various fractions that appear in the determinants are not coincidental. The power of the Pauli-Villars scale $\mu^{41/24}$ combines with the $e^{-\tau^\star_{\text{grav}}}$ factor to give rise to the RG-invariant scale that we introduced in (21.4),

$$(\Lambda^\star_{\text{grav}})^{41/24} = \mu^{41/24} e^{-\alpha(\mu)+2i\theta} \,.$$

As we explained in Chapter 20, the complexified Λ_{grav} sits in a chiral multiplet and so can appear in the superpotential. Meanwhile, the power of $(R^2)^{-7/48}$ combines with the instanton action to give $e^{-\mathcal{S}^\star}$ where $\mathcal{S}$ is the one-loop corrected complex structure introduced in (21.28),

$$\mathcal{S} = 2\pi^2 M_{\text{pl}}^2 R^2 + \frac{7}{48}\log(M_{\text{pl}}^2 R^2) + i\sigma \,.$$

Once again, $\mathcal{S}$ is the lowest component of a chiral multiplet and so can naturally appear in a superpotential. (There are further powers of R buried in the factor $S_{\text{TN}}^{1/2}$ in the two-point function but, as we will now see, these do not appear in the superpotential.)

Continuing with the computation, we have

$$\langle \chi_\alpha(x)\chi_\beta(y)\rangle = \frac{A^{7/48}}{2(2\pi)^{3/2}} \left(\frac{\Lambda^\star_{\text{grav}}}{M_{\text{pl}}}\right)^{41/24} S_{\text{TN}}^{3/2}\, e^{-\mathcal{S}^\star} \int \mathrm{d}^3X\; S_F(x-X)_{\alpha\gamma} S_F(y-X)_{\beta\delta}\epsilon^{\gamma\delta} \,.$$

We want to write down a low-energy effective action for χ which captures this two-point vertex. This can be simply done if the kinetic term (20.7) around a flat background is supplemented by the interaction term

$$S_\chi = \int \mathrm{d}^3x\, \sqrt{-g_{(3)}}\; M_3 \left[\bar\chi \not{\partial}\chi + \frac{M_3 A^{7/48}}{4(2\pi)^{3/2}} \left(\frac{\Lambda_{\text{grav}}}{M_{\text{pl}}}\right)^{41/24} S_{\text{TN}}^{3/2}\, e^{-\mathcal{S}}\, \chi\chi + \text{h.c.}\right] \tag{22.19}$$

where we are now working in the choice of coordinates of (22.2) such that $R(x) \to L$ asymptotically. We would like to determine the supersymmetric completion of this interaction term.

Supersymmetric Effective Action

The spin-1/2 fermion χ is related to the superpartner of our complex scalar $\mathcal{S}$ defined classically by (21.25). However, there is an important normalization that must be de-

termined. We denote by Ψ the spin-1/2 Dirac fermion that sits in the chiral multiplet with $\mathcal{S}$. By supersymmetry, the kinetic term for Ψ must agree with that of $\mathcal{S}$ in (21.26), namely

$$S_\Psi = M_3 \int \mathrm{d}^3x \, \sqrt{-g_{(3)}} \, \frac{1}{(\mathcal{S}+\mathcal{S}^\dagger)^2} \bar{\Psi} \not{D} \Psi \, .$$

Restricting to a flat background, and comparing to (20.7), we learn that the correctly normalized superpartner of $\mathcal{S}$ is given by

$$\Psi = 2\pi M_3 R \chi \, .$$

The instanton-generated $\Psi\Psi$ vertex in the low-energy effective action arises from a superpotential. The general form involves a number of terms. (See, for example, [268] for the general form in four-dimensions, or [240] for the three-dimensional effective action.) However, to the order that we're working, only the leading term contributes and the fermionic part of the action should take the form

$$S_\Psi = \int \mathrm{d}^3x \sqrt{g_{(3)}} \, M_3 \left[(\partial\bar{\partial} K) \, \bar{\Psi} \not{D} \Psi + \frac{1}{2} (e^{K/2} \partial\partial \mathcal{W}) \, \Psi\Psi + \text{h.c.} \right] \tag{22.20}$$

where for the purposes of this calculation it suffices to use the classical Kähler potential $K = -\log(\mathcal{S}+\mathcal{S}^\dagger)$ defined in (21.27). Comparing the two expressions (22.19) and (22.20), we find that the superpotential is given by

$$\mathcal{W} = C M_3 \left(\frac{\Lambda_{\text{grav}}}{M_{\text{pl}}} \right)^{41/24} e^{-\mathcal{S}}$$

with the overall constant

$$C = \frac{\left(4e^{24\zeta'(-1)-1}\right)^{7/48}}{2(4\pi)^{3/2}} \, .$$

Note that the superpotential is not invariant under the $U(1)_J$ symmetry which shifts the dual photon. Further, the Yukawa vertex in (22.20) explicitly breaks the $U(1)_R$ symmetry under which the gravitino is charged; this is a manifestation of the axial anomaly (21.5). However, a combination of the $U(1)_J$ and $U(1)_R$ symmetries survives.

The Potential

The supersymmetric completion of the Yukawa term is a potential for the chiral multiplet. In three-dimensional supergravity, this is given by (see, for example, [240, 222])

$$V = M_3 \, e^K \left((\partial\bar{\partial} K)^{-1} \, |D\mathcal{W}|^2 - 4|\mathcal{W}|^2 \right)$$

with $DW = \partial W + (\partial K)W$. This potential includes some critical points at $\mathcal{S} \sim \mathcal{O}(1)$. They are not to be trusted as they lie outside the semi-classical regime of large $\mathcal{S}$ where we performed our calculation. Instead, at large $\mathcal{S}$, the potential is dominated by the $|W'|^2$ term and takes the runaway form

$$V \sim M_3^3 (R\Lambda_{\text{grav}})^{41/24} \exp\left(-4\pi^2 M_{\text{pl}}^2 R^2\right) .$$

We learn that the Kaluza-Klein compactification of $\mathcal{N} = 1$ supergravity on $\mathbb{R}^3 \times S^1$ is not a ground state of the theory. This instanton-generated potential causes the circle to decompactify to large radius R.

PART VII

Conclusion

23 Summary

Throughout this dissertation, we have been interested in pursuing precision in the complicated world of non-perturbative quantum dynamics. Both in condensed matter or quantum gravity, the difficulty of the problems we must deal with means physics has often proceeded heuristically. Yet by cleverly constructing toy models (Chern-Simons-matter theories on the one hand, minimal supergravity models on the other) and making use of the sophisticated mathematical machinery of theoretical physics, we can make real inroads into these theories.

One of the key guiding principles is to exploit supersymmetry to tell us what theories to look at. In the non-relativistic context of anyon physics and the quantum Hall effect, this simply meant choosing parameters cleverly. This allowed us to exploit our knowledge of superconformal algebras on the one hand, and supersymmetric solitons in the form of vortices on the other. In the very much relativistic context of quantum gravity, supersymmetry really meant adding in some fermions to our problem. This exposed the effects we were interested in, and rendered even non-perturbative calculations tractable – again, knowledge of supersymmetric solitons proved to be very useful.

Importantly, in neither case do we end up very far from the physics we were looking for, despite the move to more tractable models. It is very clear from all of our concrete results in non-relativistic theories that we are working at the very least in the same universality class as one would normally elect to. Over in the world of quantum gravity, the existence and role of Λ_{grav} is independent of the presence of supersymmetry; equally supersymmetry did not have much to say about the choice of which manifolds to include in the path integral.

In both settings, the most obvious thing gained the ability to do an impressive amount of direct calculation: analytically computing anomalous dimensions, quasihole properties, exactly evaluating partition functions, calculating effective potentials, and so on. Perhaps more significant, though, are the consequences of being able to work through the details: we learn about the way bosonization works; we see how to phrase the relation between microscopic Hall wavefunctions and Wess-Zumino-Witten current algebras; we discover new equivalent models of the QHE; we can test how sensible our ideas about quantum gravity actually are, identify and isolate new quantities of interest, and make sure we have the right dependence on all parameters of our theory; and so on.

So at a high level, one take-home message of this work is that it is still worth taking the time to identify good, *solvable* models exhibiting effects we are interested in – and solving them. You never know what you will find along the way. This approach is of course one lying at the heart of theoretical physics. But crucially, by importing new advances in other areas (like supersymmetric solitons) into the problems we face, the number of models which are solvable can be extended. It is easy to miss the consequences of an advance in one field for another, but every time we miss such a synergy we simply make life harder for ourselves.

Having looked at the general messages of the dissertation, let us review the more specific ideas presented in the various parts of the dissertation, and assess their significance.

Anyons and Quantum Hall Physics

Here is an executive summary of some of the computational achievements in Parts II-V:

- The ideas of supersymmetry, brought to non-relativistic field theory, have surprising power. This holds even if we throw away the supersymmetric partners in the field theory.
- We can compute the chiral part of the spectrum of non-relativistic superconformal field theories, and see how bosonization is implemented in such a setting.
- We can model the quantum Hall effect using matrix models – even in a wide class of non-Abelian states.
- These matrix models afford better control over these states than previous approaches. We are able to analyse them in remarkable detail:
  - In the Abelian models at least we may calculate quasihole charges and statistics analytically.
  - We can directly link microscopic electron wavefunctions to Chern-Simons theory by quantizing Chern-Simons vortices.
  - We can see how in detail a truncation of Wess-Zumino-Witten theory emerges from the quantum mechanics of many electrons.
- We can also exploit bosonization dualities to generate new theories of quantum Hall states. In particular, we can construct a fermionic parton-like field theory. This gives a different perspective on the structure of the wavefunction.

As we will outline in Chapter 24, this suggests several natural avenues to pursue. But it is worth taking a moment to reflect on what has been achieved already and how it might help with such future work.

Exact calculations in strongly coupled systems are few and far between. It is pleasing that emergent properties of a subtle many-body system (the excitations of a vortex droplet) can be calculated from first principles. Of course, as we will outline below the dream is to use this to model and explain more complicated phenomena, either by investigating the models we have in more detail, or by deforming them in interesting ways. If we can do this in a clever way, perhaps we will begin untangling some of the issues described below by direct calculations.

Complementary to this, broadening the collection of models we have for these states makes it easier to see how to deform the idealized quantum Hall states we typically study. It also offers potential insight into the formation of these states: we see a different emergent gauge symmetry, for instance, in the dual fermion version of the Chern-Simons theory.

Progress in systematizing the complicated picture of theories and probing the depth and nature of their interrelations is itself very helpful. Knowing for instance how to construct currents for large numbers of particles which can be directly modelled with Wess-Zumino-Witten theories gives a clear picture of how to use results about the latter to understand the microscopic state.

In general, we have attempted to enrich the toolbox of theoreticians studying anyonic and quantum Hall physics, using ideas borrowed from high-energy theory. Hopefully this will inform future efforts to explore these remarkable phenomena.

Quantum Gravity

Meanwhile, in Part VI, we explored the world of effective quantum gravity. We have already highlighted the main points, but let us reiterate them briefly to see where we stand:

- Generically[1], there is a logarithmic running of the Gauss-Bonnet term in the Wilsonian action of quantum gravity theories. This causes dimensional transmutation, and leads to a new scale Λ_{grav} associated with the contributions of topologically distinct spacetimes. It can naturally be parametrically separated from ultra-violet scales like the Planck scale.

- There are settings where this scale dictates the physics, such as the quantum instability of $\mathbb{R}^{1,2} \times S^1$, which we saw arises from instantons.

[1]Meaning without very specific matter content, as with lots of supersymmetry – more than $\mathcal{N} = 1$ – at low energies.

- There are several reasonable physical principles which guide our choice of what instantons to consider in this computation, and they seem to agree: nucleation of instanton-anti-instanton pairs; imposition of 'initial' and 'final topologies' at infinity; analysis of the 3d effective theory; and string theory models of the ultraviolet completion.

- It is possible to evaluate subtle non-cancelling one-loop determinants in self-dual backgrounds using the power of supersymmetry. This relies on the very neat organization of the fields in a twistor-like manner into their spinor irreps.

It is surprisingly satisfying to see that all the various moving parts of this calculation – one-loop divergences around flat space, corrections to complex structures, selection of instantons, determinant calculations in instanton backgrounds, and so on – fit together to give a consistent picture. This seems a very good sanity check on the computation. But more importantly, it is a sanity check on the ideas we are applying to the theory.

It seems clear in light of this computation that we can ask a straightforward question of an effective gravity theory and get a reasonable answer, which is heartening. It also seems clear that we must appreciate the role of Λ_{grav} in such computations, which is interesting. And it seems believable that we have obtained a good rationale for identifying which instantons contribute to the path integral here, which is exciting.

Given how few and far between detailed computations are in this field, it is comforting to find that they can be done – and intriguing that we can learn new things by doing them.

24 Outlook

As mentioned above, the general strategy advocated in this dissertation – solving special cases or toy models – is one familiar to all physicists. But again it is worth emphasizing that none of the progress in the Hall effect would have been possible without the insights about matrix models which are credited to string theory; and it was a set of computations in flat space gauge theories which we have adapted to the context of non-perturbative quantum gravity.

The outlook for this kind of approach to physics is good. There is a lot of conversation between condensed matter theorists, string theorists, numerical relativists, higher-spin gravity theorists, mathematical physicists, geometers and so on, particularly in light of ideas like the AdS/CFT correspondence which have driven inter-disciplinary collaborations. It is good that we exploit this not simply to work on projects in the intersection of these areas, but to import techniques wholesale from one area into another. The progress in bosonization dualities is also very much of this nature, with a basic idea from condensed matter driving progress in various other areas like large N gauge theories studied by string theorists.

But let us focus on offering up some suggestions for future work in light of the specific results we have obtained in anyonic physics and in quantum gravity:

Anyons and the Quantum Hall Effect

Indeed, there are a great many things which one might hope to do, particularly using our new approaches to the quantum Hall effect. Here are just a few ideas and suggestions:

- There are still loose ends in the ongoing research reported at the end of Part II; it is clear that bosonization is essentially working as it should, but there are a few details to straighten out. It would be interesting to see if there are unanswered questions about the relativistic dualities which can be tested in the playground of the non-relativistic section.

- We could try and understand more of the phenomenology of the quantum Hall effect from our microscopic models. For example:

  - Magnetoroton modes are surprisingly universal and even quantitatively robust, and so may be within grasp of our models; reproducing the distinctive Bessel function-like dispersion relation would be exciting [269].
  - It would be nice to understand the role of different effective trapping potentials in the energetics of our models; analysis of hard confinement, for example, has shown that this can make qualitative differences to the energetics of quasiholes [270].
  - It may be possible to deform our theory slightly away from the supersymmetric point and understand what other effects electron interactions can have on our theory. This might provide some additional insight into these states.
  - These models could be amenable to adding impurities. If we can do this and retain our control, we could get a handle on some interesting aspects of more realistic Hall physics. (In some supersymmetric theories, impurities can be well understood, as in [173].)

- It would also be interesting to assess the entanglement structure of our Hall states, and possibly compute things like entanglement entropies. Calculating these quantities has proved very productive elsewhere in theoretical physics, and it would be very useful to have a highly non-trivial model in which we could, say, compute them exactly.

- Another project, mentioned at the end of Part III and already under way, is the analysis of product gauge groups. Understanding K-matrix theories and hopefully ultimately the hierarchy in a new way would be fascinating. String theory literature again offers a suggestion of where to look: quiver gauge theories. The author has already made some progress using these models as inspiration.

- The other idea mentioned at the end of Part III, of exploring (4+1)-dimensional quantum Hall states, certainly also seems to be interesting. The author is also pursuing this line of interest.

- Along we the way, we pointed various possible projects such as those on bosonization mentioned in Section 14.2.4, and earlier suggestions about mirror symmetry in Chapter 10. On top of this, it would be interesting to understand to what extent the bosonization dualities we have discussed underlie more general non-Abelian dualities in $d = 2 + 1$ dimensions. For example, are they related to other approaches such as [80, 271]? Can they be used as building blocks to derive non-Abelian particle-vortex dualities, or their supersymmetric counterparts constructed in [272, 273]?

- A more ambitious goal would be to try and find ways to extend these models to cover the transitions between Hall plateaus. This is a real challenge, since we do

not have any real understanding of what guiding principles are available in constructing effective models, and from experimental data this physics seems less universal than that in the middle of the plateau. Yet perhaps the K-matrix approach, with its connections to hierarchy physics, will unlock some of the secrets of these poorly understood states.

- Finally, one last hard problem: giving models of the microscopic mechanisms by which certain Hall states are preferred. Again, messy, small-scale physics is clearly important here – but our model is remarkably powerful in connecting smaller scales to larger ones, so perhaps it is possible to deform the model to simulate this and unpick the remarkable structure of Figure 1, with which this dissertation began.

Quantum Gravity

Over in the quantum gravity picture, there are also now some definite questions to ask. Two of the most obvious first:

- Where does Λ_{grav} crop up?
- What is the value of Λ_{grav} in our universe?

It would be especially intriguing if $\Lambda_{\text{grav}} \ll M_{\text{pl}}$, so that there was an additional scale present in our universe at observable energy scales which we had simply missed. The reason for us missing it, of course, is that it contributes only to subtle non-perturbative gravitational effects, and we still don't know exactly what they are.

(At the risk of indulging ourselves, one might point out that we made reference to the rather speculative work of authors like Coleman and Carlip in the introduction on issues such as the cosmological constant problem. It seems certain that our work should at least have *something* to say about ideas such as summation over wormholes and bubbling universes. It is also tempting to speculate about the possibility that having a new low energy scale Λ_{grav} might help explain certain other mysterious low-energy scales which we have observed: the cosmological constant and the Higgs mass, for example.)

If we want to search for this scale in our universe, of course, one issue which should be addressed is exactly what is robust when we remove the comfort blanket of supersymmetry. Certainly the basic argument that it appears in summations over distinct topologies is clear: the logarithmic running is present in generic gravitational theories, and it always contributes a factor dependent upon the Euler characteristic of the manifold. But is this swamped by other terms without the protection of supersymmetry? There is no particular reason to expect so, but we should certainly check.

It would be nice to take our principles for identifying instantons and formulate them in a very mathematical way (perhaps in the language of gluing manifolds together), and then to try and classify the contributions to key universes of direct interest to us. This is made more difficult, but also more interesting, by how poorly understood the classification of 4-manifolds is.

It might also prove interesting to pick some simple variations of the problem we have tackled (perhaps compactifying more dimensions, for instance) and see if and how the story changes. What aspects of the instability are generic, for example, and which depend on dimension and so forth?

Additionally, one might ask whether our scale has any role to play in holography, where non-perturbative effects such as black hole formation in the bulk theory are well-known to be significant. If so, what is the role of the scale in the boundary theory?

Finally, one last idea which might prove interesting is trying to understand our new scale and our rules about instantons in the context of candidate ultra-violet completions of quantum gravity. For instance, what sets the scale of Λ_{grav} in string theory? If we can identify observable quantities dependent on Λ_{grav}, can this be a test of potential quantum gravity models?

Perhaps having a good example to glance back at will eventually shed some more light on this subtle but fundamental area of physics.

PART VIII

Appendices

A Non-Relativistic Limits

Lagrangian such as (4.18) and (8.1) can be derived by starting from a relativistic Chern-Simons-matter theory, with $\mathcal{N} = 2$ supersymmetry, and taking a limit in which anti-particles decouple [30]. A number of other non-relativistic theories with different gauge groups, and more supersymmetry, have been constructed in this manner [77, 78, 79, 67].

In this appendix, we review this non-relativistic limit. We construct a more general theory than that of [30], with gauge group $U(N_c)$ and N_f matter multiplets transforming in the fundamental representation of the gauge group. (For simplicity, we do not give different levels to the $U(1)$ and $SU(N_c)$ parts, but it is hopefully clear that it generalizes.) We also show how the chemical potential term μ can arise in this limit.

We restrict our attention to the bosonic fields and, only at the end, describe the generalization to the fermions. The bosonic Lagrangian for the $\mathcal{N} = 2$ supersymmetric $U(N_c)$ Yang-Mills Chern-Simons theory is

$$\begin{aligned} S_{\text{rel}} \;=\; -\int \mathrm{d}^3x\, \frac{1}{4e^2}\,\mathrm{Tr}\,(f_{\mu\nu}f^{\mu\nu}) + \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\,(a_\mu\partial_\nu a_\rho - \frac{2i}{3}a_\mu a_\nu a_\rho) + \frac{1}{2e^2}\,\mathrm{Tr}\,(\mathcal{D}_\mu\sigma)^2 \\ + \sum_{i=1}^{N_f} |\mathcal{D}_\mu\phi_i|^2 + \phi_i^\dagger\sigma^2\phi_i + \frac{e^2}{2}\,\mathrm{Tr}\,\Big(\sum_i \phi_i\phi_i^\dagger - \frac{k\sigma}{2\pi} - v^2\Big)^2 . \end{aligned} \tag{A.1}$$

Here σ is the real, adjoint scalar which accompanies a_μ in the vector multiplet, while ϕ_i are fundamental scalars that live in chiral multiplets. We have included a Fayet-Iliopoulos term v^2, but not real masses for the ϕ_i. This can be done and results in different inertial masses in the non-relativistic limit.

Before proceeding, it's useful to perform some simple dimensional analysis. We work with $\hbar = 1$. This, of course relates energy to inverse time scales. However, as we are ultimately interested in non-relativistic physics, we retain the speed of light c. This means that we have two dimensionful quantities, length L and time T.

The factors of c in (A.1) are currently hidden in the notation. The measure is

$$\mathrm{d}^3x = c\,\mathrm{d}t\,\mathrm{d}^2x \tag{A.2}$$

while the derivatives are

$$|\mathcal{D}_\mu\phi|^2 = -\frac{1}{c^2}|\mathcal{D}_t\phi|^2 + |\mathcal{D}_p\phi|^2$$

with $p = 1, 2$ indexing spatial directions. Similarly, $a_0 = a_t/c$. The action is dimensionless. The other fields have dimensions $[a_t] = T^{-1}$ and $[a_\alpha] = [\sigma] = L^{-1}$ and $[\phi] = L^{-1/2}$. The parameters have dimension $[k] = 0$ and $[e^2] = [v^2] = L^{-1}$.

We first take the infra-red limit, $e^2 \to \infty$, to remove the Yang-Mills term. This also imposes the D-term as a constraint:

$$\frac{k\sigma}{2\pi} = \sum_i \phi_i \phi_i^\dagger - v^2 \,. \tag{A.3}$$

Using this to integrate out the adjoint scalar σ, the scalar potential terms in (A.1) become

$$V = \left(\frac{2\pi}{k}\right)^2 \sum_i \text{Tr } \phi_i \phi_i^\dagger \Big(\sum \phi_j \phi_j^\dagger - v^2 \Big)^2 \,. \tag{A.4}$$

This kind of sextic potential is standard in supersymmetric Chern-Simons theories. The next step is to take the non-relativistic limit by discarding anti-particle excitations. To this end, we make the ansatz

$$\phi_i(x,t) = \frac{1}{\sqrt{2mc}} \tilde{\phi}_i(x,t) e^{-imc^2 t} \,. \tag{A.5}$$

Here m is the mass of ϕ, which we read off from the quadratic term in the potential (A.4). It takes the value

$$m = \frac{2\pi v^2}{kc} \,.$$

The key point of the non-relativistic limit is that $\tilde{\phi}$ varies much more slowly that the frequencies mc^2 set by the mass gap. In particular, this means that the ansatz (A.5) prohibits anti-particle excitations which scale as $e^{+imc^2 t}$. Plugging the ansatz (A.5) into the kinetic terms gives, after an integration by parts,

$$\frac{1}{c^2}|\mathcal{D}_t \phi|^2 = \frac{1}{2mc}\left(\frac{1}{c^2}|\mathcal{D}_t \tilde{\phi}|^2 + 2im\tilde{\phi}^\dagger \mathcal{D}_t \tilde{\phi} + m^2 c^2 |\tilde{\phi}|^2 \right) \,.$$

The overall factor of $1/c$ is cancelled by the factor of c in the measure (A.2). The third term, $m^2c^2|\tilde{\phi}|^2$ is designed to cancel the quadratic term in the potential. We now take the non-relativistic limit $c \to \infty$. In doing so, we're left only with the term linear in time derivatives. We can repeat this for all other terms in the action. In particular, taking a similar scaling of the potential (A.4) leaves us only with the quartic coupling

$$V = -\frac{\pi}{kmc} \sum_{ij} (\tilde{\phi}_j \tilde{\phi}_i)(\tilde{\phi}_i^\dagger \tilde{\phi}_j) \,.$$

The same scaling can be applied to the fermions in the original $\mathcal{N} = 2$ theory. The end result is a $U(N_c)$ Chern-Simons theory, coupled to N_f fundamental matter multiplets. To describe it, we revert to the notation $\tilde{\phi} \to \phi$. The final non-relativistic action is

$$\begin{aligned} S = \int \mathrm{d}t\, \mathrm{d}^2x \qquad & \sum_{i=1}^{N_f} i\phi_i^\dagger \mathcal{D}_t \phi_i + i\psi_i^\dagger \mathcal{D}_t \psi_i - \frac{k}{4\pi}\,\mathrm{Tr}\,\epsilon^{\mu\nu\rho}(a_\mu \partial_\nu a_\rho - \frac{2i}{3} a_\mu a_\nu a_\rho) \\ & -\frac{1}{2m}\sum_{i=1}^{N_f}\Big(\mathcal{D}_p\phi_i^\dagger \mathcal{D}_p\phi_i + \mathcal{D}_p\psi_i^\dagger \mathcal{D}_p\psi_i + \psi_i^\dagger f_{12}\psi_i\Big) \\ & -\frac{\pi}{mk}\sum_{i,j}\Big[(\phi_j^\dagger\phi_i)(\phi_i^\dagger\phi_j) - (\phi_j^\dagger\psi_i)(\psi_i^\dagger\phi_j) + 2(\phi_i^\dagger\phi_j)(\psi_j^\dagger\psi_i)\Big] \end{aligned} \tag{A.6}$$

which agrees with (4.18). For $U(1)$ with $N_f = 1$, this is also the action (8.1) when the chemical potential $\mu = 0$. (We have used the notation ∂_t rather than ∂_0 in this appendix.)

As highlighted in Part II, the action (A.6) is invariant under superconformal transformations [30, 66].

Adding a Chemical Potential

The action (8.1) also includes a chemical potential μ which plays a crucial role in our quantum Hall story. It is straightforward to add an analogous to term to the relativistic Lagrangian (A.1). It is

$$\mathcal{L}_\mu = \mu\,\mathrm{Tr}\,(a_0 - \sigma)\,. \tag{A.7}$$

Obviously this breaks $d = 2 + 1$ Lorentz invariance. It preserves two of the four supercharges. Indeed, such terms are well known in the context of quantum mechanics models with $\mathcal{N} = (0, 2)$ supersymmetry and were first introduced in [274]. In taking the infra-red limit, the σ term in (A.7) gets replaced by $\sum \phi_i\phi_i^\dagger$ through the constraint (A.3). The resulting interaction terms of the non-relativistic theory are

$$V = \frac{\pi}{mk}\sum_{i,j}\Big[(\phi_j^\dagger\phi_i)(\phi_i^\dagger\phi_j) - \mu\phi_i^\dagger\phi_i - (\phi_j^\dagger\psi_i)(\psi_i^\dagger\phi_j) + 2(\phi_i^\dagger\phi_j)(\psi_j^\dagger\psi_i)\Big]\,.$$

Despite the fact that the relativistic theory with the deformation (A.7) preserves only one complex supercharge, both supercharges (8.8) and (8.9) are recovered after taking the non-relativistic limit. However, as we have seen, only Q_2 remains a symmetry of the spectrum.

B The Geometry of the Vortex Moduli Space

In this appendix, we review a few basic facts about the geometry of the vortex moduli space. Suppose that we have at our disposal the most general solution to the vortex equation with winding N,

$$\phi(x; X) \quad \text{and} \quad a_z(x; X)\,.$$

We define $2N$ *zero modes* $(\delta_a\phi, \delta_a a_z)$ to be the infinitesimal deformations which take us from one solution to another:

$$\delta_a\phi = \frac{\partial\phi}{\partial X^a} + i\alpha_a\phi \quad \text{and} \quad \delta_a a_z = \frac{\partial a_z}{\partial X^a} + \partial_z\alpha_a\,. \tag{B.1}$$

Here $\alpha_a(x; X)$ is an accompanying gauge transformation. By construction, these zero modes solve the linearized versions of the vortex equations (9.1) for any choice of $\alpha(x, X)$. This ambiguity is fixed by further requiring that the zero modes obey the background gauge condition,

$$\partial_z\,\delta_a a_z + \partial_{\bar z}\,\delta_a a_{\bar z} = \frac{2\pi}{k'}\left(i\phi\delta_a\phi^\dagger - i\phi^\dagger\delta_a\phi\right)\,. \tag{B.2}$$

The metric on the vortex moduli space $\mathcal{M}_N$ is constructed by taking the overlap of the zero modes

$$g_{ab} = \int \mathrm{d}^2x\,\frac{k'}{\pi}\left(\delta_a a_{\bar z}\,\delta_b a_z + \delta_a a_z\,\delta_b a_{\bar z}\right) + \left(\delta_a\phi^\dagger\,\delta_b\phi + \delta_a\phi\,\delta_b\phi^\dagger\right)\,. \tag{B.3}$$

In relativistic theories, this metric plays an important role: the low-energy dynamics of the vortices is described by a sigma-model on $\mathcal{M}_N$ with metric g_{ab}. The metric is known to be free of singularities. It is also Kähler, inheriting its complex structure from the natural action of complex conjugation on the fields. The associated Kähler form is

$$\Omega_{ab} = i\int \mathrm{d}^2x\,\frac{k'}{\pi}\left(\delta_a a_{\bar z}\,\delta_b a_z - \delta_a a_z\,\delta_b a_{\bar z}\right) + \left(\delta_a\phi^\dagger\,\delta_b\phi - \delta_a\phi\,\delta_b\phi^\dagger\right)\,. \tag{B.4}$$

We now show that this Kähler form governs the first order dynamics of vortices in our model. We will prove that the effective action for vortices is given by

$$S_{\text{vortex}} = \int \mathrm{d}t\;\mathcal{F}_a(X)\dot X^a \quad \text{with} \quad \mathrm{d}\mathcal{F} = \Omega\,.$$

This result was previously derived in [138, 151, 152].

We work in the usual spirit of the moduli space: we promote the collective coordinates of the static solutions to be time dependent: $X^a(t)$. We then substitute this time-dependent ansatz into the kinetic terms of the action (8.1). This results in an effective vortex action,

$$S = \int \mathrm{d}^3x \frac{ik'}{2\pi}\left(a_{\bar{z}}\dot{a}_z - a_z\dot{a}_{\bar{z}}\right) + \frac{i}{2}\left(\phi^\dagger\dot{\phi} - \dot{\phi}^\dagger\phi\right) \equiv \int \mathrm{d}t\, \mathcal{F}_a(X)\dot{X}^a \tag{B.5}$$

with

$$\mathcal{F}_a(X) = \frac{i}{2}\int \mathrm{d}^2x\, \frac{k'}{\pi}\left(a_{\bar{z}}\frac{\partial a_z}{\partial X^a} - \frac{\partial a_{\bar{z}}}{\partial X^a}a_z\right) + \left(\phi^\dagger\frac{\partial\phi}{\partial X^a} - \frac{\partial\phi^\dagger}{\partial X^a}\phi\right).$$

Note that the kinetic terms in (B.5) contain time derivatives rather than covariant time derivatives. This is because the a_0 terms in (8.1) multiply Gauss's law and so necessarily vanish. Correspondingly, the expression for $\mathcal{F}_a$ above contains partial derivatives of fields which differ from the zero modes defined in (B.1) as they are missing the contribution from the gauge transformation $\alpha_a(x;X)$.

The 2-form $\tilde{\Omega} = \mathrm{d}\mathcal{F}$ is

$$\tilde{\Omega}_{ab} = \frac{\partial\mathcal{F}_a}{\partial X^b} - \frac{\partial\mathcal{F}_b}{\partial X^a} = i\int \mathrm{d}^2x\, \frac{k'}{\pi}\left(\frac{\partial a_{\bar{z}}}{\partial X^a}\frac{\partial a_z}{\partial X^b} - \frac{\partial a_z}{\partial X^a}\frac{\partial a_{\bar{z}}}{\partial X^b}\right) + \left(\frac{\partial\phi^\dagger}{\partial X^a}\frac{\partial\phi}{\partial X^b} - \frac{\partial\phi}{\partial X^a}\frac{\partial\phi^\dagger}{\partial X^b}\right).$$

Our goal is to show that $\tilde{\Omega}_{ab} = \Omega_{ab}$, the Kähler form defined in (B.4). The expressions look similar. They differ because the expression for Ω_{ab} includes extra contributions from the gauge fixing terms. We now show that these terms vanish.

The proof is very similar to that given recently in [275] in the context of first order motion on the instanton moduli space. We take the difference

$$\begin{aligned}\Omega_{ab} - \tilde{\Omega}_{ab} &= i\int \mathrm{d}^2x\, \frac{k'}{\pi}\left(\frac{\partial a_{\bar{z}}}{\partial X^a}\partial_z\alpha_b - \frac{\partial a_z}{\partial X^a}\partial_{\bar{z}}\alpha_b\right) + \left(i\frac{\partial\phi^\dagger}{\partial A^a}\alpha_b\phi + i\phi^\dagger\alpha_a\frac{\partial\phi}{\partial X^a}\right) - (a\leftrightarrow b)\\ &= -\int \mathrm{d}^2x\, \alpha_b\frac{\partial}{\partial X^a}\left(-\frac{k'}{2\pi}f_{12} + \phi^\dagger\phi\right) - (a\leftrightarrow b)\end{aligned}$$

where we have integrated by parts to get to the second line. But the term in brackets vanishes, courtesy of Gauss's law (9.1). We learn that $\mathrm{d}\mathcal{F} = \Omega$, the Kähler form, as advertized. Note that the proof above did not need us to use the background gauge fixing condition (B.2). While the metric (B.3) is sensitive to the background gauge condition, the Kähler form (B.4) is not.

C Overlap of Matrix Model States

Our derivation of the fractional charge and statistics of quasiholes relied on expressions for the norms of matrix model states given in (9.27) and (9.29). These results have been derived previously, most notably in the context of the Calogero-Sutherland-Moser model. Because these results are stated in a slightly different language, we use this appendix to explain the connection.

The quantum Hall matrix model is well known to be equivalent to the bosonic integrable Calogero-Sutherland-Moser model [19, 276]. This describes identical particles in one spatial dimension, placed in a harmonic trap and interacting via a specific inverse-square potential. To see the connection we begin, following [136], by working with a coherent state representation of all matrix model states. Firstly, expand $Z = X + iY$ into Hermitian and anti-Hermitian parts, and let the overcomplete states $|X, \phi\rangle$ be defined by

$$\hat{X}\,|X,\phi\rangle = X\,|X,\phi\rangle\,, \qquad \hat{\varphi}\,|X,\phi\rangle = \phi\,|X,\phi\rangle$$

together with the normalization

$$\int e^{-\bar{\phi}\phi}\mathrm{d}\phi\mathrm{d}\bar{\phi}\prod_{a,b}\mathrm{d}X_{ab}\,|X,\phi\rangle\,\langle X,\phi| \equiv 1$$

where we have added hats to emphasize which symbols denote the quantum operators. With respect to these states, we can write all states in terms of a wavefunction by taking inner products with $\langle X, \phi|$. This in turn gives us a way to compute the inner products of matrix model states by computing integrals over X, ϕ. In what follows, we work with the convention $\pi\mu = 1$.

On these wavefunctions, $Z^\dagger$ has the representation

$$Z^\dagger_{ab} \equiv \frac{1}{\sqrt{2}}\left(X_{ab} - \frac{\partial}{\partial X_{ba}}\right)$$

analogous to the raising operator of the more familiar Hermite polynomials. Hence, up to an overall normalization, the states we are interested in all have wavefunctions of the form

$$\Phi_f(X,\phi) = f(Z^\dagger)\left[\epsilon^{a_1\cdots a_N}\bar{\phi}_{a_1}(\bar{\phi}X)_{a_2}\cdots(\bar{\phi}X^{N-1})_{a_N}\right]^k e^{-\frac{1}{2}\operatorname{Tr}X^2}$$

where f is some homogeneous, gauge-invariant polynomial. Specifically, we have the following correspondence:

$$\begin{aligned} |\text{ground}\rangle_k : &\quad f(B) = 1 \\ |\Omega_l\rangle_k : &\quad f(B) = B^{a_1}_{[a_1} B^{a_2}_{a_2} \cdots B^{a_{N-l}}_{a_{N-l}]} \\ |\Omega_{0,l}\rangle_k : &\quad f(B) = \det B \cdot B^{a_1}_{[a_1} B^{a_2}_{a_2} \cdots B^{a_{N-l}}_{a_{N-l}]} \end{aligned}$$

At given N, k', we will denote the state with a given choice of f simply by $|f\rangle$.

One can evaluate the action of f on the state to obtain instead

$$\Phi_f(X,\phi) = \tilde{f}(X) \left[\epsilon^{a_1\cdots a_N} \bar{\phi}_{a_1} (\bar{\phi}X)_{a_2} \ldots (\bar{\phi}X^{N-1})_{a_N}\right]^k e^{-\frac{1}{2}\operatorname{Tr} X^2}$$

where at leading order $\tilde{f}(B) \sim 2^{(\deg f)/2} f(B)$.

Now the relationship to the states of the Calogero model is seen by performing a change of variables: diagonalize X via $X = UDU^{-1}$, where $D_{ab} = x_a \delta_{ab}$. Defining the Vandermonde determinant

$$\Delta = \epsilon^{a_1\cdots a_N} x^0_{a_1} \cdots x^{N-1}_{a_N} = \prod_{a<b} (x_a - x_b)$$

one sees that the wavefunction becomes

$$\Phi_f(X,\phi) \;\; = \tilde{f}(D) \cdot \Delta^k \, e^{-\frac{1}{2}x^2} \cdot \textstyle\prod_a (\bar{\phi}U)^k_a$$

Note that $\tilde{f}(D) \equiv \tilde{f}(x)$ is simply a polynomial in x, whose leading behaviour we can determine from f. Also, we can see that U, ϕ have decoupled from x.

Hence, taking account of the Jacobian Δ^2 for our change of variables, at a given N, k' all inner products satisfy

$$\langle f|g\rangle = c_{N,k} \int \mathrm{d}^N x \, e^{-x^2} \Delta^{2k'} \tilde{f}(x) \tilde{g}(x) \tag{C.1}$$

where $c_{N,k}$ is a calculable constant which we do not need for our computation.

As is shown in detail in [136], the key observation now is that the action of the matrix model Hamiltonian H on our wavefunctions is given by $H \equiv \Delta^{-1} H_{\text{Cal}} \Delta$, where H_{Cal} is the Hamiltonian of the Calogero model at statistical parameter k'. But the eigenstates of the Calogero model are known; they correspond precisely to the Hi-Jack polynomials, the multi-variable generalizations of the Hermite polynomials which are orthogonal with respect to the measure in (C.1). These are labelled by partitions λ.

One may readily check that in fact $\tilde{f}, \tilde{g}$ in equation (C.1) must be multiples of the generalized Hermite polynomials discussed in Section 3 of [277]. But now we can refer to Proposition 3.7 of that paper which is readily unpacked to give the ratios between the norms of general states. Concretely, their $H_\lambda(x)$ have leading term

$$H_\lambda(x) \sim 2^{|\lambda|} \frac{(x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N} + \text{distinct permutations})}{\text{number of distinct permutations}}$$

and norms

$$\frac{\int H_\lambda^2(x)\, \mathrm{d}\mu(x)}{\int \mathrm{d}\mu(x)} = 2^{|\lambda|} \prod_{(c,d)\in\lambda} \frac{(k'\, l_\lambda(c,d) + (a_\lambda(c,d)+1))(k'(l_\lambda(c,d)+1) + a_\lambda(c,d))}{k'(N-(c-1)) + (d-1)}\,.$$

Here, $|\lambda| = \sum_a \lambda_a$ is the number of cells in the corresponding Young diagram, and $a_\lambda(c,d)$ and $l_\lambda(c,d)$ are respectively the arm and leg length of the cell with coordinates (c,d) in that diagram.

All that remains is to work out what choice of λ and normalization correspond to the examples of f given above for the matrix model states. It is easily found that

$$\begin{aligned}
|\text{ground}\rangle_k : &\quad \tilde{f} = H_{(0,0,\ldots,0)} \\
|\Omega_l\rangle_k : &\quad \tilde{f} = 2^{(N-l)/2} \tbinom{N}{l} H_{(1,1,\ldots,1,0,0,\ldots,0)} \\
|\Omega_{0,l}\rangle_k : &\quad \tilde{f} = 2^{(2N-l)/2} \tbinom{N}{l} H_{(2,2,\ldots,2,1,1,\ldots,1)}
\end{aligned}$$

where there are $N-l$ instances of 1 (resp. 2) in the second (resp. third) partition and then (9.27) and (9.29) both follow on evaluating the above product.

It is hopefully clear how this generalizes to arbitrary states in the matrix model, especially if one realizes the close relationship between the partition λ and the original definition of the matrix model states $|\Omega_l\rangle_k$ and $|\Omega_{0,l}\rangle_k$.

D Proofs of Two Classical Identities

In this appendix we prove the two classical identities that we used to exhibit the existence of a Kac-Moody algebra. We work in units with $B = 2$. Assuming $m \geq n$, they are

Identity 1:

$$\varphi_i^\dagger Z^{\dagger n} Z^m \varphi_l - \frac{1}{p} \sum_{r=0}^{m-1} \sum_{s=0}^{n-1} (\varphi_{i'}^\dagger Z^r Z^{\dagger s} \varphi_{i'})(\varphi_i^\dagger Z^{\dagger n-1-s} Z^{m-1-r} \varphi_l) - \delta_{il}(\cdots) \\ \sim \left(\frac{kN}{p}\right)^n \varphi_i^\dagger Z^{m-n} \varphi_l - \delta_{il}(\cdots) \tag{D.1}$$

Identity 2 (Classical Version):

$$\sum_{r=0}^{m-1} \varphi_{i'}^\dagger Z^r Z^{\dagger n-1} Z^{m-1-r} \varphi_{i'} \sim p \left(\frac{(k+p)N}{p}\right)^n \delta_{mn} \tag{D.2}$$

where $\sim$ means up to $1/N$ corrections and, in the first identity, we subtract off the il trace on both sides.

The phrase "up to order $1/N$ corrections" implicitly includes a restriction on the kind of classical solutions on which we should evaluate these expressions. Roughly speaking, the solutions shouldn't deviate by $\mathcal{O}(N)$ from the ground state. We start by describing in more detail what this means.

For the $p = 1$ matrix model, the ground state was given in Part III (and in [19])

$$Z = Z_0(N) \equiv \sqrt{k} \begin{pmatrix} 0 & 1 & & & \\ & 0 & \sqrt{2} & & \\ & & & \ddots & \\ & & & 0 & \sqrt{N-1} \\ & & & & 0 \end{pmatrix} \quad \text{and} \quad \varphi = \varphi_0(N) \equiv \sqrt{k} \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \sqrt{N} \end{pmatrix}$$

together with $\alpha = (\omega^2/B)\,\mathrm{diag}(N-1, N-2, \ldots, 2, 1, 0)$.

It is simple to embed these solutions in the more general matrix model. The number of ground states now depends on the value of N modulo p. It is simplest when N is

divisible by p. In this case there is a unique ground state which takes the block diagonal form

$$Z = Z_0(N/p) \otimes \mathbf{1}_p \quad , \quad \varphi = \varphi_0(N/p) \otimes \mathbf{1}_p \tag{D.3}$$

where we've written the φ_i (with $i = 1, \ldots, p$) as an $N \times p$ matrix, denoted by φ.

If N is not divisible by p then there are multiple classical ground states, transforming in the representation (16.2), as we established earlier. For example, if $N \equiv 1 \pmod p$ then each of the blocks has $Z_0((N-1)/p)$, except for one which has $Z_0((N+p-1)/N)$. There are p such choices; these ground states transform in the $\mathbf{p}$ of the $SU(p)$ global symmetry. Similarly, if $N \equiv q \pmod p$ then there are $\binom{p}{q}$ ground states, transforming in the q^{th} antisymmetric representation of $SU(p)$.

In what follows, we will assume that N is divisible by p. Now we can make our statement about $\mathcal{O}(1/N)$ corrections more precise. We should treat $\varphi \sim O(N^{1/2})$ and $Z \sim O(N^{1/2})$, since in the ground state the largest components of either scale like the square root of N, and even when contracting indices there is only one non-zero entry per row or column. (This is important to check because there are $O(N)$ components, which could upset our counting.) We will evaluate the identities on states which differ from the ground state by $\mathcal{O}(1)$ when measured naturally by the norm squared of δZ and $\delta\phi$. It is important that these states still satisfy Gauss's law (17.1).

These restrictions make it fairly straightforward to prove the classical version of Identity 2. Consider a linear expansion of the left-hand side around the ground state in powers of $N^{1/2}$; we obtain the zeroth order term plus something we can bound by $\epsilon N^{n-1/2}$. If we decide to neglect terms of this order, we can simply substitute the expression for the ground state into the left hand side. It is trivial to check that $Z^\dagger \varphi_i = 0$, and hence the only contribution is from $\varphi_i^\dagger Z^{\dagger n-1} Z^{m-1} \varphi_i$.

Next, observe that

$$Z^{\dagger m} Z^m \varphi_i = (k+p)^m \frac{(N/p-1)!}{(N/p-1-m)!} \varphi_i \sim \left(\frac{(k+p)N}{p}\right)^m \varphi_i \,.$$

Upon using $Z^\dagger \varphi_i = 0$ once more, the δ_{nm} factor in (D.2) follows. The final ingredient is to observe $\varphi_i^\dagger \varphi_i = kN$, completing the proof of (D.2).

Identity 1 is a little harder to prove. Let us start by rewriting it slightly:

$$\varphi_i^\dagger Z^{\dagger n} Z^{n+m} \varphi_l - \frac{1}{p} \sum_{r=0}^{n+m-1} \sum_{s=0}^{n-1} (\varphi_{i'}^\dagger Z^r Z^{\dagger s} \varphi_{i'})(\varphi_i^\dagger Z^{\dagger n-1-s} Z^{n+m-1-r} \varphi_l) \sim' \left(\frac{(k+p)N}{p}\right)^n \varphi_i^\dagger Z^m \varphi_l$$

where for brevity the prime $'$ denotes (asymptotic) equality of the il-traceless parts. We will proceed by firstly showing that only one term in the double sum contributes at leading order, namely that obtained at $r = s = 0$, reducing the problem to proving

$$\varphi_i^\dagger Z^{\dagger n} Z^{n+m} \varphi_l - \frac{kN}{p} (\varphi_i^\dagger Z^{\dagger n-1} Z^{n+m-1} \varphi_l) \sim' \left(\frac{(k+p)N}{p} \right)^n \varphi_i^\dagger Z^m \varphi_l \,.$$

Then we will inductively demonstrate that

$$\varphi_i^\dagger Z^{\dagger n} Z^{n+m} \varphi_l \sim' (n+1) \left(\frac{(k+p)N}{p} \right)^n \varphi_i^\dagger Z^m \varphi_l \tag{D.4}$$

from which the original identity follows immediately.

So to begin, let us estimate the size of the terms we wish to keep. The traceless part of the right-hand side vanishes in the ground state, so we must sacrifice at least one term for something of order ϵ; this is then generically non-vanishing. Therefore, the right-hand side is of order $\mathcal{O}(\epsilon N^{n+(m+1)/2})$.

We can now consider a single term of the double sum at general (r, s). The traceless part of the second bracket, $(\varphi_i^\dagger Z^{\dagger n-1-s} Z^{n+m-1-r} \varphi_l)$, vanishes in the ground state, and hence is at most order $\mathcal{O}(\epsilon N^{n+(m-r-s-1)/2})$. Thus the first bracket must be at least of order $\mathcal{O}(N^{1+(r+s)/2})$. But this N-scaling is only possible if all terms in the first bracket come from the ground state, when this term vanishes by the observations above – except for $r = s = 0$.

This leaves us only with deriving (D.4). We will induct on n to establish this; note that the case $n = 0$ is trivial. Write

$$\varphi_i^\dagger Z^{\dagger n} Z^{n+m} \varphi_l =' \varphi_i^\dagger Z^m Z^{\dagger n} Z^n \varphi_l - \varphi_i^\dagger [Z^m, Z^{\dagger n}] Z^n \varphi_l$$

The first term is simple to handle. Since at leading order $\varphi_i^\dagger Z^m = 0$, we can safely make the approximation $Z^{\dagger n} Z^n \varphi_i \sim ((k+p)N/p)^n \varphi_i$.

The second term can be expanded into a double sum, and simplified slightly using the asymptotic version of Gauss's law, $[Z, Z^\dagger] \sim -\varphi\varphi^\dagger$. Then almost all terms can be shown to be subleading, using the ideas above, except for the one where we have the $\varphi\varphi^\dagger$ appearing at the far left. Hence

$$\begin{aligned} \varphi_i^\dagger Z^{\dagger n} Z^{n+m} \varphi_l &\sim' \left(\frac{(k+p)N}{p} \right)^n \varphi_i^\dagger Z^m \varphi_l + \varphi_i^\dagger \varphi_j \varphi_j^\dagger Z^{\dagger n-1} Z^{m-1} Z^n \varphi_l \\ &\sim' \left(\frac{(k+p)N}{p} \right)^n \varphi_i^\dagger Z^m \varphi_l + \left(\frac{(k+p)N}{p} \right) \varphi_i^\dagger Z^{\dagger n-1} Z^{n-1+m} \varphi_l \end{aligned}$$

where we have also used the trick of separating the ij and jl traces out, discarding more irrelevant terms. Finally, applying the inductive hypothesis to the second term, we establish (D.4), and hence identity 1.

E Kostka Polynomials

In this appendix we give an explicit description of the Kostka polynomial $K_{\lambda,\mu}(q)$ due to Kirillov and Reshetikhin [278].

Given λ, $\mu \in \mathcal{P}$, we define a sequence of partitions $\nu^{(K)}$ with $K = 0, 1, 2, \ldots, \ell(\lambda) - 1$ with $\nu^{(0)} = \mu$ and

$$|\nu^{(K)}| = \sum_{j \geq K+1} \lambda_j \,. \tag{E.1}$$

For each such sequence we define the *vacancy numbers*

$$\mathbb{P}_n^{(K)} \;=\; \sum_{j \geq 1} \Big[\min\{n, \nu_j^{(K+1)}\} - 2\min\{n, \nu_j^{(K)}\} + \min\{n, \nu_j^{(K-1)}\} \Big]$$

for all positive integers n and $K = 0, 1, 2, \ldots, \ell(\lambda) - 1$ with the understanding that $\nu^{(\ell(\lambda))} \equiv 0$. An *admissible configuration*, $\{\nu\}$ is any such sequence of partitions with non-negative vacancy numbers, i.e.

$$\mathbb{P}_n^{(K)} \geq 0$$

for all values of n and K. The *charge* $c(\{\nu\})$ of an admissible configuration is defined as

$$c\left(\{\nu\}\right) \;=\; n[\mu] \,+\, \sum_{K=1}^{\ell(\lambda)-1} \left(\mathbb{M}\left[\nu^{(K)}, \nu^{(K)}\right] \,-\, \mathbb{M}\left[\nu^{(K)}, \nu^{(K-1)}\right] \right)$$

where, for any two partitions ρ, $\kappa \in \mathcal{P}$, we define the function $\mathbb{M} : \mathcal{P} \times \mathcal{P} \to \mathbb{Z}_{\geq 0}$ by

$$\mathbb{M}\left[\rho, \kappa\right] \;=\; \sum_{i,j \geq 1} \min\{\rho_i, \kappa_j\} \,.$$

Finally the Kostka polynomial can be defined as a sum over all admissible configurations; explicitly,

$$K_{\lambda,\mu}(q) \;=\; \sum_{\{\nu\}} q^{c[\{\nu\}]} \prod_{K=1}^{\ell(\lambda)-1} \prod_{n \geq 1} \begin{bmatrix} \mathbb{P}_n^{(K)} + m_n\left(\nu^{(K)}\right) \\ m_n\left(\nu^{(K)}\right) \end{bmatrix}_q$$

where we define the q-binomial coefficient

$$\left[\begin{array}{c} m \\ n \end{array} \right]_q = \frac{\varphi_m(q)}{\varphi_n(q)\varphi_{m-n}(q)}$$

and, as in the text, we have $\varphi_n(q) = \prod_{j=1}^{n}(1-q^j)$.

Ground State Energy

As in the text we set $\mu = (k^N)$ and look for the ground state by searching for the partition λ, satisfying $|\lambda| = kN$ and $\ell(\lambda) \leq p$, such that $K_{\lambda,(k)^N}(q)$ yields the leading term in the q expansion for $|q| \ll 1$. The general formula given above can be simplified [278] in the case $k = 1$ where $\mu = (1^N)$. We find

$$K_{\lambda,(1^N)}(q) = q^{n\left[\lambda^T\right]} \frac{\prod_{j=1}^{N}(1-q^j)}{H(q)}$$

where $H(q)$ is the hook-length polynomial given by

$$H(q) = \prod_{x \in \mathbb{Y}(\lambda)} \left(1 - q^{h(x)}\right) .$$

Here the product is over the boxes $x = (r, s)$ of the Young diagram $\mathbb{Y}(\lambda)$ corresponding to the partition λ and $h(x) = \lambda_r + \lambda_s^T - r - s + 1 > 0$ is the length of the hook passing through box x.

As explained in the text, the minimum for the case $k = 1$ is attained for the partition $\lambda_0 = ((L+1)^M, L^{p-M})$ where $N = Lp+M$ for non-negative integers L and $M < p$ which gives a ground state energy

$$E_0(1, p, N) = \frac{1}{2}L(L-1)p + LM .$$

It is instructive to reproduce this result from the general formula given above in the case $\mu = (1^N)$ and $\lambda = \lambda_0$. According to the recipe we must find a sequence of partitions $\nu^{(K)}$ for $K = 0, 1, 2, \ldots, \ell(\lambda_0) - 1 = p - 1$ with $\nu^{(0)} = \mu = (1^N)$ and

$$|\nu^{(K)}| = \sum_{j \geq K+1} (\lambda_0)_j = \begin{cases} (M-K)(L+1) + L(P-M) & 0 \leq K \leq M \\ L(p-K) & M+1 \leq K \leq p-1 \end{cases}$$

with non-negative occupation numbers which minimizes the charge $c(\{\nu\})$. It not hard to see that this is achieved by maximizing the number of parts in each partition $\nu^{(K)}$.

Thus we set

$$\nu^{(K)} = \begin{cases} \left(1^{((M-K)(L+1)+L(P-M))}\right) & 0 \leq K \leq M \\ \left(1^{(L(p-K))}\right) & M+1 \leq K \leq p-1 \end{cases} .$$

One may then check that the corresponding occupation numbers are non-negative and that

$$c\left(\{\nu\}\right) \;=\; E_0(1,p,N) = \frac{1}{2}L(L-1)p + LM\,.$$

The above configuration has a straightforward generalization to $k \geq 1$. As in the text we set

$$\lambda_0 = ((kL+k)^M, (kL)^{p-M})$$

and one simply scales each partition $\nu^{(K)}$ in the configuration by a factor of k setting

$$\nu^{(K)} = \begin{cases} \left(k^{((M-K)(L+1)+L(P-M))}\right) & 0 \leq K \leq M \\ \left(k^{(L(p-K))}\right) & M+1 \leq K \leq p-1 \end{cases} .$$

The vacancy numbers are remain non-negative and the charge of the configuration scales linearly with k. Thus the new ground state energy is

$$c\left(\{\nu\}\right) \;=\; E_0(k,p,N) = \frac{k}{2}L(L-1)p + kLM$$

which is the result stated in the main text.

F Affine Lie Algebra Conventions

Here we give our conventions for the simple Lie algebra $A_{p-1} = sl(p, \mathbb{C})$ and its Affine counterpart $\hat{A}_{p-1}$.

For A_{p-1}, we work in a Chevalley basis with generators $\{h^i, e^i, f^i; i = 1, \ldots p-1\}$ with brackets

$$[h^i, h^j] = 0 \quad , \quad [h^i, e^j] = A_{ji} e^j \quad , \quad [h^i, f^j] = -A_{ji} f^j \quad , \quad [e^i, f^j] = \delta_{ij} h^i$$

where A_{ij} is the A_{p-1} Cartan matrix. Weights of each irreducible representation lie in the weight lattice

$$\mathcal{L}_W \quad = \quad \mathrm{Span}_{\mathbb{Z}}\{\Lambda_{(i)}, i = 1, \ldots, p-1\}$$

whose basis vectors are the fundamental weights $\Lambda_{(i)}$. Finite-dimensional, irreducible representations R_Λ are labelled by a highest weight Λ, lying in the positive weight lattice

$$\mathcal{L}_W^+ \quad = \quad \mathrm{Span}_{\mathbb{Z}_{\geq 0}}\{\Lambda_{(i)}, i = 1, \ldots, p-1\}\,.$$

We denote the corresponding representation space $\mathcal{V}_\Lambda$. For each weight of R_Λ there is an element

$$\Psi = \sum_{i=1}^{p-1} \psi^i \Lambda_{(i)} \tag{F.1}$$

of $\mathcal{L}_W$ with Dynkin labels $\psi^i \in \mathbb{Z}$. We then have a basis vector $|\Psi\rangle$ of $\mathcal{V}_\Lambda$ which is a simultaneous eigenvector of the Cartan generators satisfying

$$R_\Lambda\left(h^i\right)|\Psi\rangle \quad = \quad \psi^i|\Psi\rangle$$

for $i = 1, \ldots, p-1$.

For the affine Lie algebra $\hat{A}_{p-1}$ we have Chevalley generators $\{h^i, e^i, f^i; i = 0, \ldots p-1\}$ with brackets

$$[h^i, h^j] = 0 \quad , \quad [h^i, e^j] = \hat{A}_{ji} e^j \quad , \quad [h^i, f^j] = -\hat{A}_{ji} e^j \quad , \quad [e^i, f^j] = \delta_{ij} h^i$$

where $\hat{A}_{ij}$ is the affine Cartan matrix. The basis elements with $i > 0$ generate an A_{p-1} subalgebra. Weights of an integrable representation have an expansion

$$\hat{\Psi} \;=\; \sum_{i=0}^{p-1} \hat{\psi}^i \hat{\Lambda}_{(i)} + n\delta \tag{F.2}$$

for integers $\hat{\psi}_i$ and n where δ is the imaginary root. The fundamental weights of $\hat{A}_{p-1}$ can be written as

$$\hat{\Lambda}_{(i)} \;=\; \hat{\Lambda}_{(0)} + \Lambda_{(i)}$$

for $i > 0$, where $\Lambda_{(i)}$ are fundamental weights of the global A_{p-1} subalgebra.

The integrable representations $R_{\hat{\Lambda}}$ of $\hat{A}_{p-1}$ are characterized by a highest weight $\hat{\Lambda}$ with non-negative Dynkin labels, and have the representation space $\mathcal{V}_{\hat{\Lambda}}$. Each weight of $R_{\hat{\Lambda}}$ has an expansion of the form (F.2). The corresponding basis vector $|\hat{\Psi}\rangle$ of $\mathcal{V}_{\hat{\Lambda}}$ is a simultaneous eigenvector of the Cartan generators, with

$$R_{\hat{\Lambda}}\left(h^i\right)|\hat{\Psi}\rangle \;=\; \hat{\psi}^i|\hat{\Psi}\rangle$$

for $i = 1, \ldots, p-1$, and the derivation or grading operator, with

$$-R_{\hat{\Lambda}}\left(L_0\right)|\hat{\Psi}\rangle \;=\; n|\hat{\Psi}\rangle\,.$$

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