RADIAL BASIS FUNCTION METHODS FOR MULTIVARIABLE APPROXIMATION

by

IAN ROBERT HART JACKSON M.A.

of TRINITY HALL

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I. R. H. JACKSON

Summary

The problem of approximating functions of d variables ($d > 1$) has many diverse and useful applications. The idea of using radial basis function methods for such problems is motivated by the excellent results that they give in some practical problems, particularly that of multivariate interpolation to data given at a small number of irregularly positioned points.

In this dissertation it is first shown that, in some cases, radial basis functions do provide good spaces in which to look for approximations. Specifically, it is found that the best approximating functions from the linear space spanned by radial basis functions centred at arbitrary points in a bounded domain converge uniformly, on any slightly smaller domain, to any continuous function as the points become dense in the domain.

The main conclusion is a result about the rate of convergence of these approximations to suitably smooth functions when the points lie on a regular grid. Initially it is shown, both by elementary means and by using Fourier transforms of generalised functions, that, for some useful radial basis functions, the closure of the linear approximation space includes some low degree polynomials. Thus, it is shown how to deduce rates of convergence for best approximations in the case when the grid spacing decreases. Results are obtained for functions defined either on the whole space or on only a bounded domain. In particular, when the radial basis function is the identity and when d is odd, the rate of convergence is found to be $d + 1$. This generalises the well-known rate of convergence for linear interpolation to suitably smooth functions in one dimension.

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Declaration

In this dissertation all of the work is my own. The main results of Chapters 4 and 5, however, were discovered independently by Professor N. Dyn of Tel Aviv University. We are preparing a joint paper on these results, which has improved the presentation of this part of the dissertation.

Jesus Christ is the same yesterday and today and for ever

Hebrews 13:8

Preface

It is a pleasure to be able to acknowledge the many ways that I have been helped while spending three years doing research for this dissertation.

First and foremost my thanks go to my supervisor Professor M. J. D. Powell who, while being a great authority on numerical analysis, is concerned to share that expertise with others. He has been a source of many useful ideas for my research and he has given most generously of his time so that I have been able to make the best use of mine. I have learnt much from him, not only how to research but, most importantly for me, how to write technical papers. He has shown considerable patience over this and I am most grateful. The training I have received from him will certainly be of much use in the future years.

The Science and Engineering Research Council have provided the financial support for the three years which has enabled me to perform this research. Trinity Hall, my college, have also been generous with their facilities for graduates, particularly I would like to acknowledge their contributions enabling me to attend some conferences and the work of Dr. D. Thomas, the tutor for graduate students. Likewise the Department of Applied Mathematics and Theoretical Physics has provided a friendly place to work and has also been generous with sponsorship for conferences. However, it is to my colleagues in the Numerical Analysis Group that I must express particular thanks. They have together been a very stimulating and encouraging group but both Martin Buhmann, with whom I had many useful discussions on both radial basis functions and TEX, the latter of which he taught me how to use, and Arieh Iserles, who has managed almost single-handed to keep the group amused with his own brand of humour, deserve a special mention.

It has been interesting to be able to meet many visitors from overseas coming to the group and be able to benefit from the breadth of knowledge that they have brought with them. Particularly I think of C. K. Chui, N. Dyn, D. Levin and C. A. Micchelli.

I am most grateful to close friends for their support especially during the writing of the dissertation but above all to my parents who have supported me most faithfully during these three years and provided a relaxing and friendly home whenever I go there. It is out of appreciation for all that they have done that I dedicate this dissertation to them.

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CHAPTER 1 : INTRODUCTION

Section 1.1 : Multivariable Approximation

There has been much interest in recent years in approximating functions f by more simple functions s which are suitable to use for computational purposes. Much of the earlier work concerns one dimensional problems, that is approximating a function, usually over a finite region $[a, b]$ by some simpler function. This problem has been carefully analysed and answers are known to many of the important practical questions in that case (see e.g. Powell 1981). We restrict ourselves to the problem in more than one dimension where the current state of analysis is much less advanced. First we state the general problem which we wish to consider. We suppose that distinct **data points** $\{z_k \in \mathcal{R}^d : k = 1, 2, \ldots, n\}$ (with $d > 1$) are given, along with associated **function values** $\{f_k \in \mathcal{R} : k = 1, 2, \ldots, n\}$. We assume that each function value comes from some underlying function f so that

$$f(z_k) = f_k + \epsilon_k, \quad k = 1, 2, \ldots, n, \tag{1.1.1}$$

where $\{\epsilon_k \in \mathcal{R} : k = 1, 2, \ldots, n\}$ are small errors which may result from an inaccurate evaluation of the function. We wish to find a function s so that $s(z_k)$ is close to f_k. In the case of **multivariate interpolation** we satisfy the interpolation conditions

$$s(z_k) = f_k, \quad k = 1, 2, \ldots, n. \tag{1.1.2}$$

For approximation rather than interpolation we may require that one of the following inequalities hold:

$$\sum_{k=1}^{n} |s(z_k) - f_k| < \epsilon, \tag{1.1.3}$$

$$\sum_{k=1}^{n} (s(z_k) - f_k)^2 < \epsilon \tag{1.1.4}$$

or

$$\max\{|s(z_k) - f_k| : k = 1, 2, \ldots, n\} < \epsilon, \tag{1.1.5}$$

for some tolerance ϵ. More general formualtions than this may also be used, possibly including weights at the points z_k as well.

We begin by mentioning some types of applications that yield these problems.

Predictions.

It may be the case that the function f one wishes to analyse is very expensive to evaluate, either in real terms or in terms of the amount of computing time needed. This may cause a problem if, in the course of the analysis, we are going to require a knowledge of the function f at a large number of different data points. This can be alleviated if one can find a good approximation s to f over the whole region of interest, using only a relatively small number of function values of f. Then, instead of repeatedly evaluating f one can repeatedly evaluate the much more simple function s. Usually it will be the case that the function s will only be a good approximation to the function f inside the convex hull of the points at which f has been evaluated, although sometimes limited extrapolation may be possible. It may also be suitable to use the function s to estimate partial derivatives of f if these are required.

Optimization.

Another application, connected quite closely with predictions, is finding a local (or global) maximum (or minimum) of a function. In this case the location of the maximum is unknown. We may need to know the function f to very high precision near the maximum, but will not be greatly concerned about its behaviour away from this maximum, so long as it does not introduce a fictitious maximum.

Statistical Analysis.

It may be that, instead of requiring local information about the function f as in the previous two cases, we require some global information such as a mean, with respect to some weight function, or some correlation information. Here, in contrast to the previous case, an approximation which is uniformly good over the whole region of interest is required.

Storage.

Alternatively we may have a large number of evaluations of a function f from some experiment. We may need to store this data for some period before further analysis can be performed if, for instance, correlation is sought between this and similar data obtained at some later time. If the data has been oversampled in the experiment then it is advantageous to reduce the amount of data held in storage. If the data can be approximated by a function s which can recover the original data sufficiently accurately and takes much less storage, it is advantageous to use the function s instead.

However, when looking for suitable methods to solve a particular problem there may also be other constraints, apart from the application, which we need to take into consideration.

There is naturally a trade off between the accuracy of an approximation s and the time it takes to calculate that approximation. This may impose restrictions on either the time of solution or on the accuracy that can be obtained and usually some balance between the two must be achieved. It is also important to check that a method is well conditioned for the type of problem one wishes to solve and that there is sufficient storage to hold all the data for the problem at one time although, in the latter case, it may be possible to use a method that processes only a part of the data at any one time. If the function s is likely to be revised after the calculation, perhaps to achieve greater accuracy, then a method which allows for this will be preferred. It is also necessary to check on any parameters required by the method, as on occasions the function s can be affected critically by these.

Section 1.2 : Methods for Multivariable Approximation

In this section we shall just sketch a cross-section of the techniques most widely used for multivariable approximation problems. There are many such techniques and there have been several good review articles written recently including Schumaker (1976), Barnhill (1977), Franke (1982), Hayes (1987) and de Boor (1987). Mostly they are written from a practical viewpoint although that by de Boor (1987) contains several theoretical ideas. In many cases too the authors just consider the problem in two dimensions. We do not mention radial (or nodal) basis function methods in this section because they are considered in some detail in Section 1.3.

Tensor Product Methods.

These methods are especially good when the data are given on a rectangular grid. For example, in two dimensions suppose that we have data points

$$\{(x_i, y_k) : i = 1, 2, \ldots, p, \; k = 1, 2, \ldots, q\}, \tag{1.2.1}$$

and associated function values

$$\{f(x_i, y_k) : i = 1, 2, \ldots, p, \; k = 1, 2, \ldots, q\}, \tag{1.2.2}$$

We look for a solution which is a linear sum of products of univariate functions

$$s(x, y) = \sum_{i=1}^{p} \sum_{k=1}^{q} \alpha_{i,k} B_i(x) \tilde{B}_k(y), \tag{1.2.3}$$

where each B_i and $\tilde{B}_k$ is a univariate function. We can find the values of $\{\alpha_{i,k}\}$ by first solving q univariate problems on grid lines parallel to the first coordinate direction and then p univariate problems on grid lines parallel to the second coordinate direction. A good survey of approaches is given in (Light and Cheney 1986).

There have been attempts to try and extend this method, because of its obvious success, to the case when the data are irregularly distributed. This may involve setting up a regular grid and estimating function values at grid points from given function values nearby. Then the problem is solved in the usual way on the regular grid. A possible drawback of this approach is that the approximating function calculated on the regular grid will not in general interpolate the original function values, even if an interpolating method is used on the generated values on the regular grid. However, this difficulty can be overcome (e.g. Foley and Nielson 1980).

Blending Methods.

For ease we consider this method in two dimensions although the technique may be extended to higher dimensions. One forms a triangulation of the data points. In this and other methods where triangulations are needed it is often found that the Delanay triangulation is highly suitable (Lawson 1977). Then on a triangle with vertices $(x_i, y_i), (x_j, y_j), (x_k, y_k)$ we use the function

$$s(x,y) = w_i(x,y)Q_i(x,y) + w_j(x,y)Q_j(x,y) + w_k(x,y)Q_k(x,y), \tag{1.2.4}$$

where $Q_l(x,y)$ are nodal functions satisfying

$$Q_l(x_l, y_l) = f_l, \quad l = 1, 2, \ldots, n, \tag{1.2.5}$$

and $w_l(x,y)$ is a weight function, which in the case of interpolation would satisfy

$$w_l(x_m, y_m) = \begin{cases} 1 & \text{if } l = m; \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.6}$$

Examples of such methods are found in Franke (1982) who also considers the possibility of performing blending over rectangles.

Finite Element Methods.

This method again works, in two dimensions, on a triangulation of the data points. A finite element is chosen on each triangle and these are fitted together over the collection of triangles to give a good approximation and the required continuity. This method requires estimation of some partial derivatives at all the data points, and possibly at other places too, to enable the elements to be properly joined. The performance can depend greatly on the accuracy with which partial derivatives are calculated (Franke 1982). Also in this paper various good choices of finite elements may be found for the interpolation problem.

Multivariate B-splines.

Much work has been done recently on the question of finding multivariable functions which are analagous to the univariate B-splines in the case of irregularly spaced data. These functions have compact support, which should be as small as possible, and are region-wise polynomials with suitable continuity across the boundary of regions. Theoretically such techniques have proved very interesting but the computational effort to apply them in practice can cause difficulties (Grandine, 1986). A good introduction may be found in Höllig (1986a) and comprehensive reviews in Dahmen and Micchelli (1983), Höllig (1986b) and Chui (1987).

Repeated Surface Smoothing.

This method is a generalisation of the technique of repeated curve smoothing which is used in one dimension in many computer aided geometric design packages (Doo and Sabin 1978). Preliminary work has been done by Höllig (1986a) but the subject is still in very early stages although de Boor (1987) has high hopes of the utility of this approach for arbitrary distributions of data points.

Quasi-Interpolation.

Some of the previous schemes may be viewed together in the more general context of quasi-interpolation. In this case with each data point z_k we associate a function ψ_k which decays rapidly for large argument. We then form the approximation to f,

$$s(x) = \sum_{k=1}^{n} f_k \psi_k(x). \tag{1.2.7}$$

The naming arises because in the case of the multivariable interpolation if we find functions χ_k satisfying

$$\chi_k(z_j) = \begin{cases} 1 & \text{if } k = j; \\ 0 & \text{otherwise,} \end{cases} \tag{1.2.8}$$

then the function

$$s(x) = \sum_{k=1}^{n} f_k \chi_k(x), \tag{1.2.9}$$

interpolates the data. In many cases ψ_k has compact support (e.g. multivariable B-splines), but this condition is not necessary. We give much attention to this formulation later in the dissertation.

The problem of multivariable approximation is a large and difficult problem so all methods are bound to have shortcomings. In particular the methods just described become far more complicated as the dimension increases which is perhaps why many papers only consider calculations in two dimensions (e.g. Franke (1982) and Hayes (1987)). We now describe a method which practically overcomes this problem and also has the desirable properties of rotational and translational invariance.

Section 1.3 : Radial Basis Functions

This dissertation is concerned mainly with "radial basis function" type approximations, which were first suggested for the multivariable interpolation problem (1.1.2). The method of solution in this case is to look for an approximation from the linear space spanned by the n functions

$$x \mapsto \phi(\|x - z_k\|), \quad k = 1, 2, \ldots, n, \tag{1.3.1}$$

where the norm is Euclidean and where $\phi : \mathcal{R} \to \mathcal{R}$ is a continuous function known as a **radial basis function**, or sometimes as a nodal basis function. In this case the approximation is

$$s(x) = \sum_{k=1}^{n} \lambda_k \phi(\|x - z_k\|), \qquad x \in \mathcal{R}^d, \tag{1.3.2}$$

the $\{\lambda_k \ : \ k = 1, 2, \ldots, n\}$ being chosen to satisfy the interpolation conditions. Originally, Hardy (1971) suggested using either of the two radial basis functions $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$ or $(r^2 + c^2)^{-\frac{1}{2}}$, for some positive constant c.

Polynomials of low degree are not reproduced by a function (1.3.2), so it is sometimes useful to add to s a polynomial of total degree m which gives the form

$$s(x) = \sum_{k=1}^{n} \lambda_k \phi(\|x - z_k\|) + P_m(x), \qquad x \in \mathcal{R}^d. \tag{1.3.3}$$

It is now suitable to augment the interpolation conditions by the extra constraints that

$$\sum_{k=1}^{n} \lambda_k P(z_k) = 0, \tag{1.3.4}$$

for all polynomials P of total degree at most m, which regains a square system of linear equations in the parameters of s. This technique was first proposed in two dimensions for the basis function $\phi(r) = r^2 \log r$ and $m = 1$ by Duchon (1977). He actually arrived at this formulation by finding the solution of the interpolation problem which provides the minimum value of

$$\int_{\mathcal{R}^2} \left(\left| \frac{\partial^2 s(x)}{\partial x_1^2} \right|^2 + 2 \left| \frac{\partial^2 s(x)}{\partial x_1 \partial x_2} \right|^2 + \left| \frac{\partial^2 s(x)}{\partial x_2^2} \right|^2 \right) dx, \tag{1.3.5}$$

when the only restrictions on s are differentiability conditions.

We note that approximations of the form (1.3.2) and (1.3.3) may be used in cases where interpolation is not required. However, when performing interpolation the question naturally arises whether solutions to these two problems are well-defined. That is, for any distribution of the distinct points $\{z_k : k = 1, 2, \ldots, n\}$, do there exist functions s of the form (1.3.2) and (1.3.3) satisfying all the required constraints. In the first case the question is whether there exists $\{\lambda_k : k = 1, 2, \ldots, n\}$ such that

$$\sum_{k=1}^{n} \lambda_k \phi(\| z_j - z_k \|) = f_j, \quad j = 1, 2, \ldots, n. \tag{1.3.6}$$

There is a unique solution for all choices of $\{f_j : j = 1, 2, \ldots, n\}$ if and only if the n by n symmetric matrix

$$a_{k,j} = \phi(\| z_j - z_k \|) \tag{1.3.7}$$

is non-singular. In the second case we let D_m be the dimension of the space of polynomials of total degree at most m, and let

$$\{q_k : k = n+1, n+2, \ldots, n+D_m\} \tag{1.3.8}$$

be a basis for that space. Now the question is whether there exists $\{\lambda_k : k = 1, 2, \ldots, n + D_m\}$ such that

$$\sum_{k=1}^{n} \lambda_k \phi(\| z_j - z_k \|) + \sum_{k=n+1}^{n+D_m} \lambda_k q_k(z_j) = f_j, \quad j = 1, 2, \ldots, n, \tag{1.3.9}$$

and

$$\sum_{k=1}^{n} \lambda_k q_j(z_k) = 0, \quad j = n+1, n+2, \ldots, n+D_m. \tag{1.3.10}$$

We see that there is a unique solution for all choices of $\{f_j : j = 1, 2, \ldots, n\}$ if and only if the $n + D_m$ by $n + D_m$ symmetric matrix

$$a_{k,j} = \begin{cases} \phi(\| z_j - z_k \|) & \text{if } 1 \le k, j \le n; \\ q_k(z_j) & \text{if } 1 \le j \le n,\ n+1 \le k \le n+D_m; \\ q_j(z_k) & \text{if } 1 \le k \le n,\ n+1 \le j \le n+D_m; \\ 0 & \text{if } n+1 \le k, j \le n+D_m \end{cases} \tag{1.3.11}$$

is non-singular.

It is a measure of the lack of theoretical work done in the subject that a solution to the non-singularity problem for the multiquadric function was not found until 15 years after the publication of Hardy's (1971) paper. In a classic paper Micchelli (1986) proved among many other similar results that interpolation with the multiquadric basis function (1.3.2) is always possible and that it is also always possible for the method (1.3.3) for every value of m, so long as the only polynomial of degree at most m taking the value zero at all the data points is the zero polynomial. Such a set of data points we refer to as an m-**basic set**. The m-basic sets are linked to polynomial interpolation and hence to the Kirgin-Hakopyan conditions. This analysis is made more accessible in a subsequent paper by Powell (1987), although the results given there are less general than those of Micchelli.

Next we consider some of the functions ϕ that have been suggested, indicating in each case what is known about the non-singularity of the matrices (1.3.7) and (1.3.11) which arise in the interpolation problem, and what is known about their properties for interpolation and approximation, both theoretically and experimentally. Most of the interpolation results come from a very thorough survey paper by Franke (1982) in which he considers many methods in $\mathcal{R}^2$. They indicate that, for fairly small sets of data which are not too irregularly distributed, radial basis function methods provide the most accurate solutions to the interpolation problem.

(a) $\phi(r) = r$, Linear Radial Function.

The matrix (1.3.7) is non-singular for all choices of distinct data points and the matrix (1.3.11) is non-singular for all m, so long as the data points are an m-basic set. The latter case is included in the analysis of Duchon (1977) and Meinguet (1979) as with $m = (d-1)/2$ and d odd the solution to the $\phi(r) = r$ interpolation problem minimises an integral that is similar to (1.3.5) except that all partial derivatives of order $(d+1)/2$ (instead of 2) are used. In one dimension the method reduces to linear interpolation, but, perhaps surprisingly, in more than one dimension little appears to have been discovered experimentally about the performance of this function for approximation or interpolation.

(b) $\phi(r) = r^3$, Duchon Radial Cubics.

The matrix (1.3.11) is non-singular for all $m \geq 1$, so long as the data points are an m-basic set. This case is also considered in Duchon (1977) and Meinguet (1979) where it is shown that, with $m = (d+1)/2$ and d odd, the solution to the interpolation problem minimises an integral similar to (1.3.5) except that all partial derivatives of order $(d+3)/2$ are used. Franke (1982) found good results for interpolation in $\mathcal{R}^2$. Powell (1987) suggests the use of these functions for optimization calculations as partial derivatives are easy to calculate, but preliminary results were discouraging (Powell, private communication).

Similar remarks about non-singularity and variational problems apply when $\phi(r) = r^l$ for some odd integer $l \geq 5$, but little experimental work has been done. We also note that general approximation is inappropriate when $\phi(r) = r^l$ for some even integer l, because in this case $\phi(\|x-y\|)$ is a polynomial for all $y \in \mathcal{R}^d$. Hence the functions (1.3.2) and (1.3.3) are polynomials, so one cannot control well the flexibility of s by the value of n and the positions of the centres $\{z_k : k = 1, 2, \ldots, n\}$.

(c) $\phi(r) = r^2 \log r$, Duchon Thin Plate Splines.

These have already been mentioned in (1.3.5). The matrix (1.3.11) is non-singular for all $m \geq 1$, so long as the data points form an m-basic set. As in cases (a) and (b) a more general variational problem exists. If $m = (d+2)/2$ and d is even then the solution to the interpolation problem minimises an integral similar to (1.3.5) except that all partial derivatives of order $(d+2)/2$ are used. Very good results in two dimensions are reported by Franke (1982) and also by Dyn and Levin (1981 & 1983) and Dyn, Levin and Rippa (1986). Some interesting and promising work is presented by Rippa (1984) on smoothing the solution obtained from the interpolation problem.

Similar remarks about non-singularity and variational problems apply when $\phi(r) = r^l \log r$ for some positive even integer $l \geq 4$, but little experimental work has been done.

(d) $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$, $c > 0$, Hardy Multiquadrics.

We have already remarked that these were first suggested by Hardy (1971) and that the matrix (1.3.7) is always non-singular, as is the matrix (1.3.11) for all $m \geq 0$, so long as the data points form an m-basic set. Also the question of expressing the solution to the multiquadric interpolation problem as a minimum norm calculation in some reproducing kernel Hilbert space has been studied by, among others, Micchelli (1986) and Dyn (1987). Further, Franke (1982) found this to be the most accurate of all the methods he tried in practice for performing interpolation in two dimensions. His data were irregular but not too irregular so that an average "distance between neighbouring data points" could be defined. The method worked best when c had a value near to this distance, although the method was quite stable for a range of values of c. Similar results were also found experimentally by Carlson (1985) and Kansa (1986) and some theoretical justification for the case of data positioned on a regular grid is given by Buhmann (1988a).

(e) $\phi(r) = (r^2 + c^2)^{-\frac{1}{2}}$, $c > 0$, Inverse Multiquadrics.

These were also first suggested by Hardy (1971) and again the matrix (1.3.7) is always non-singular, as is the matrix (1.3.11) for all $m \geq 0$, so long as the data points form an m-basic set. Again some work has been done on expressing the solution of the inverse multiquadric interpolation problem as a minimum norm calculation in some reproducing kernel Hilbert space by, among others, Micchelli (1986) and Dyn (1987). Franke (1982) found these to be very good in practice, but not as good as the multiquadrics, for performing interpolation in two dimensions. He also found this method to be less stable with respect to variations in the parameter c. Inverse multiquadrics have also been analysed theoretically for data on a regular grid by Buhmann (1988a).

(f) $\phi(r) = \log(r^2 + c^2)$, $c > 0$, Shifted Logarithms.

These were introduced by Rippa (1984) and Dyn, Levin and Rippa (1986). The matrix (1.3.7) is always non-singular and so is the matrix (1.3.11) for all $m \geq 0$, so long as the data points form an m-basic set. This method was not examined by Franke (1982), so it hard to compare the results obtained by Rippa (1984) with other examples examined by Franke (1982). It is fair to say though that they produced good results especially under the conditions described in the multiquadric case (d).

Dyn, Levin and Rippa (1986) also consider $\phi(r) = (r^2 + c^2)^l \log(r^2 + c^2)$ for integers $l \geq 1$, although little experimental work has been done with these functions.

(g) $\phi(r) = e^{-cr^2}$, $c > 0$, Rotated Gaussians.

Franke (1982) remarks that these would have been his natural first choice of the function ϕ, but his results in this case were far worse than those obtained with, for instance, the multiquadric basis function. He also found the method to be very unstable with respect to the parameter c, and there seemed to be no readily accessible good choice for this parameter. Some work on a global optimization problem with this basis function is presented by Schagen (1984), and it may be possible to apply some of his techniques to other basis functions. Schagen points out that the matrix (1.3.7) is always non-singular for this method, and Micchelli (1986) has also shown that (1.3.11) is non-singular, for all $m \geq 0$, so long as the data points form an m-basic set.

One of the difficulties with applying interpolation with the basis functions (a)–(e) is that the matrices (1.3.7) and (1.3.11) tend to become very rapidly ill-conditioned for large n. They are in general full matrices and, if the function ϕ becomes large for large argument, the elements of the matrix tend to grow away from the diagonal. Some work on preconditioning the systems of equations (1.3.6) and (1.3.9) has been performed by Rippa (1984), Dyn, Levin and Rippa (1986) and Dyn (1987). Their techniques were developed initially for data distributed over a regular square grid but were later extended to the case of irregularly positioned points.

Cheney and Light (private communication) have begun to study basis functions of the type (1.3.1) for norms other than the Euclidean norm, but so far the results on theoretical orders of accuracy have not been encouraging.

Section 1.4 : Contents of Chapters 2–6

Chapters 2–5 contain some new results and some discussion on them is given in Chapter 6.

In Chapter 2 we consider the suitability of functions of the form (1.3.2) for performing approximation. Particularly we ask whether this space is dense in the space of all continuous functions over some bounded domain. We suppose we are given a set of points $\{z_k : k = 1, 2, \ldots\}$ which are dense in the bounded domain and a function f continuous on the closure of that domain. We ask if for any $\epsilon > 0$ there exists a function of the form

$$s(x) = \sum_{k=1}^{n} \lambda_k \phi(\|x - z_k\|) \tag{1.4.1}$$

which approximates f to accuracy ϵ. We find that this is possible in the case when we only require the approximation to f to be accurate at all points in the domain at least some fixed distance away from the boundary, provided that the radial basis function ϕ is homogeneous and there exists a function

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d, \tag{1.4.2}$$

with $\{\mu_j \in \mathcal{R} : j = 1, 2, \ldots, l\}$ and $\{x_j \in \mathcal{R}^d : j = 1, 2, \ldots, l\}$, which is absolutely integrable and has a non-zero integral.

We then consider the case of the identity basis function $\phi(r) = r$, which is homogeneous and consider trying to find a function ψ which is absolutely integrable and has non-zero integral. This is shown to be possible when d is odd but impossible when d is even.

Chapter 3 continues the analysis of the case $\phi(r) = r$. It considers quasi-interpolation on an infinite regular grid with integer spacing. In particular we address the case when f is a polynomial of low degree, and we find the unexpected result that functions

$$\psi(x) = \sum_{j=1}^{l} \mu_j \|x - x_j\|, \qquad x \in \mathcal{R}^d, \tag{1.4.3}$$

with $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$, exist when the dimension d is odd, which can reproduce all polynomials of degree d. Further, we show that it is not possible to reproduce all polynomials of degree $d+1$. The techniques involve no sophisticated analysis. We show first that the quasi-interpolant to f is a polynomial by proving that all sufficiently high order partial derivatives are zero and then we deduce that it is actually the correct polynomial.

Fourier transforms and generalised functions are used in Chapter 4, where we present some analysis that is more general than Chapter 3 for finding conditions on ϕ which allow certain polynomials to be reproduced by quasi-interpolation. These conditions allow us to deduce positive results for polynomial reproduction for cases (a)–(f) of Section 1.3. For instance it is found that when the dimension d is odd then functions ψ formed from the multiquadric basis function (example (d) in Section 1.3) can reproduce all polynomials of degree d. In case (g) of Section 1.3 it is found that no polynomial reproduction is possible.

In Chapter 5 we deduce results on the rate of convergence for quasi-interpolation schemes to suitably smooth functions as the mesh size of the infinite regular grid tends to zero, when the quasi-interpolation method reproduces low order polynomials. The error between the quasi-interpolant and the function f at x is the same as the error between the quasi-interpolant and P, the truncated Taylor series expansion of f, at x. If P is of sufficiently low degree so that it is reproduced by the quasi-interpolation method then the error is the value of the quasi-interpolant to $f - P$ at x. This is found to be small as near to x the function $f - P$ is small and away from x the function ψ decays quickly. Thus, it is not necessary that ψ has compact support only that it decays quickly. Results are obtained for rates of convergence both over the whole of $\mathcal{R}^d$ and over bounded domains, which are relevant to the radial basis functions considered in Chapter 4. Among other things we find that when $\phi(r) = r$ and d is odd we obtain an order of convergence $d + 1$ for some functions ψ (1.4.3). Thus, we find a rate of approximation which increases with increasing odd d, a remarkable and possibly very useful result.

Chapter 6 reviews the main results of the previous four chapters and comments on their practical implications. We also consider the work of Buhmann (1988b), which extends the ideas developed here for quasi-interpolation over an infinite regular grid to the case of interpolation over an infinite regular grid. Finally we comment on the main outstanding question in the subject; whether these results may extend to the case when we have scattered data rather than data on a regular grid.

Section 1.5 : Notation

We work in d dimensional real space $\mathcal{R}^d$. We denote our radial basis function by $\phi : \mathcal{R} \to \mathcal{R}$ so that in d dimensions we shall use the function $\phi(\|x\|)$, $x \in \mathcal{R}^d$, where the norm is the Euclidean (or 2-) norm. Wherever the norm symbol ($\|\cdot\|$) occurs without subscript it will imply this norm. In some places we employ the Chebyshev (or infinity) norm $\|\cdot\|_\infty$. The function ψ is usually a finite linear combination of translates of basis functions

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d, \tag{1.5.1}$$

where $\{\mu_j \in \mathcal{R} : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{R}^d : j = 1,2,\ldots,l\}$. Further, we shall often impose the restriction that each $x_j \in \mathcal{Z}^d$, where

$$\mathcal{Z}^d = \{y \in \mathcal{R}^d : \text{ each component of } y \text{ is an integer } \}. \tag{1.5.2}$$

We employ the standard (de Boor 1987) multi-index notation for multivariable functions which we now describe. We do not use any special notation to denote a vector, it should always be obvious from the context which quantities are scalars and which are vectors. We shall very rarely use components of vectors in this dissertation, but when they are needed the notation x_i denotes the i-th component of a vector x for $i = 1,2,\ldots,d$. In the multi-index notation we are concerned with the space

$$(\mathcal{Z}^+)^d = \{\alpha \in \mathcal{Z}^d : \alpha_i \geq 0,\ i = 1,2,\ldots,d\}, \tag{1.5.3}$$

and we shall often use α for a member of this space. Throughout the rest of the definitions we let $x, y \in \mathcal{R}^d$ and $\alpha, \alpha' \in (\mathcal{Z}^+)^d$. We say

$$\alpha \leq \alpha' \Leftrightarrow \alpha_i \leq \alpha_i',\ i = 1,2,\ldots,d, \tag{1.5.4}$$

$$\alpha < \alpha' \Leftrightarrow \alpha_i < \alpha_i',\ i = 1,2,\ldots,d, \tag{1.5.5}$$

and

$$\alpha \text{ is even } \Leftrightarrow \alpha_i \text{ is even},\ i = 1,2,\ldots,d. \tag{1.5.6}$$

We define

$$|\alpha| = \sum_{i=1}^{d} \alpha_i \tag{1.5.7}$$

and

$$x.y = \sum_{i=1}^{d} x_i y_i, \tag{1.5.8}$$

the usual inner product between x and y. We also define

$$x^\alpha = \prod_{i=1}^{d} x_i^{\alpha_i}, \tag{1.5.9}$$

$$\alpha! = \prod^{d} (\alpha_i)! \tag{1.5.10}$$

and, if $(c)_n$ (the Pochammer symbol or factorial function) is defined for $c \in \mathcal{R}$ by

$$(c)_0 = 1 \text{ and } (c)_n = (c+n-1)(c)_{n-1}, \quad n = 1, 2, \ldots, \tag{1.5.11}$$

we use

$$(c)_\alpha = \prod_{i=1}^{d} (c)_{\alpha_i}, \quad c \in \mathcal{R}. \tag{1.5.12}$$

Finally we introduce a notation for partial derivatives: The partial derivative of suitably smooth $f : \mathcal{R}^d \to \mathcal{R}$ of order α at the point x is the expression

$$\frac{\partial^{|\alpha|} f(x)}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \cdots \partial x_d^{\alpha_d}}. \tag{1.5.13}$$

We shall abbreviate this to either

$$\frac{\partial^\alpha f(x)}{\partial x^\alpha} \tag{1.5.14}$$

or

$$D^\alpha f(x), \tag{1.5.15}$$

where D is the vector of partial derivative operators

$$D = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_d} \right)^T. \tag{1.5.16}$$

We say that f has all partial derivatives of order at most m at the point y if

$$\left. \frac{\partial^\alpha f(x)}{\partial x^\alpha} \right|_{x=y} \tag{1.5.17}$$

exists for all $\alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| \le m$.

We also need, at various points in the dissertation, multivariable infinite sums of the form

$$\sum_{z \in \mathcal{Z}^d} \psi(z). \tag{1.5.18}$$

Unless specifically stated to the contrary we shall always check that the sum converges absolutely so that it has a well-defined value. The same remarks hold for multivariable infinite integrals of the form

$$\int_{\mathcal{R}^d} f(y)\, dy. \tag{1.5.19}$$

Unless specifically stated to the contrary we shall always check that the function f is absolutely integrable and so the integral converges absolutely to a well-defined value.

CHAPTER 2 : UNIFORM CONVERGENCE RESULTS

Section 2.1 : Local Uniform Convergence

In this section we shall establish a result showing local uniform convergence of sums of linear translates of radial basis functions to continuous functions over bounded open domains. Initially we explain the concept of local uniform convergence. We suppose that we are given a bounded open domain $\tilde{D} \subset \mathcal{R}^d$ and a function f defined on the closure of $\tilde{D}$, which we denote by $\mathrm{cl}(\tilde{D})$, and a sequence of functions $\{f_n : n = 0, 1, \ldots\}$ which are approximations to f.

We say that $\{f_n\}$ **converges** or **converges pointwise** to f over $\tilde{D}$ if, given $\epsilon > 0$ and $x \in \tilde{D}$, there exists n_0 such that, for all $n \geq n_0$,

$$|f(x) - f_n(x)| < \epsilon. \tag{2.1.1}$$

A stronger form of convergence is uniform convergence. In this case the same n_0 must be valid for all $x \in \tilde{D}$. Specifically, we say that $\{f_n\}$ **converges uniformly** to f over $\tilde{D}$ if, given $\epsilon > 0$, there exists n_0 such that, for all $n \geq n_0$,

$$\sup\{|f(x) - f_n(x)| : x \in \tilde{D}\} < \epsilon. \tag{2.1.2}$$

In some cases this condition is too restrictive as we may have good convergence away from the boundary of $\tilde{D}$, but convergence near the boundary may not be so fast. To allow for this case we say that $\{f_n\}$ **converges locally uniformly** to f over $\tilde{D}$ if, given any bounded open domain D with $\mathrm{cl}(D) \subset \tilde{D}$ and $\epsilon > 0$, there exists n_0 such that, for all $n \geq n_0$,

$$\sup\{|f(x) - f_n(x)| : x \in D\} < \epsilon. \tag{2.1.3}$$

It is local uniform convergence that is the appropriate notion for the result that we wish to prove.

We shall also be using a sequence of points within our domain $\tilde{D}$ and we shall require that the sequence of points leaves no holes in the domain. Making this notion precise, we suppose that for a given bounded open domain D^* and sequence of points $\{z_k \in D^* : k = 1, 2, \ldots\}$ we define τ_M, for $M = 1, 2, \ldots$, by

$$\tau_M = \sup\{\inf\{\|z - z_k\| : k = 1, 2, \ldots, M\} : z \in D^*\}. \tag{2.1.4}$$

The sequence $\{z_k \in D^* : k = 1, 2, \ldots\}$ **becomes dense** in D^* if $\tau_M \to 0$ as $M \to \infty$. We shall require our sequence of points to become dense in $\tilde{D}$.

Throughout this section it is assumed that there exist l, $\{\mu_j \in \mathcal{R} : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{R}^d : j = 1,2,\ldots,l\}$ such that the function

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d, \tag{2.1.5}$$

satisfies

$$\int_{\mathcal{R}^d} |\psi(x)|\, dx < \infty \tag{2.1.6}$$

and

$$\int_{\mathcal{R}^d} \psi(x)\, dx \neq 0. \tag{2.1.7}$$

The existence of such functions in special cases will be considered in Sections 2.2 and 2.3, and in Chapter 4.

Let $\widetilde{D}$ be a bounded open domain in $\mathcal{R}^d$ and let f be a continuous function

$$f : \mathrm{cl}(\widetilde{D}) \to \mathcal{R}. \tag{2.1.8}$$

We suppose that we are given any open bounded domain D with $\mathrm{cl}(D) \subset \widetilde{D}$ and any sequence of points $\{z_k \in \widetilde{D} : k = 1,2,\ldots\}$ which becomes dense in $\widetilde{D}$ (2.1.4). We seek an approximation to f of the form

$$g(x) = \sum_{k=1}^{N} \lambda_k \phi(\|x - z_k\|) \tag{2.1.9}$$

which approximates f uniformly over D to prescribed accuracy.

We find such a function g in three stages; first from the function ψ defined in (2.1.5), given certain assumptions about ϕ, a function γ is constructed which is an approximation to a delta-function, in the sense that f convolved with γ is close to f. Next a set of points is found such that a linear sum of the values of the integrand at these points is close to the convolution integral. Finally, it is shown from these results that, for any sequence of points which becomes dense in $\widetilde{D}$, there is a function of the form (2.1.9) which approximates f uniformly over D to any prescribed accuracy.

For the purpose of the proof it will be necessary to consider a third bounded open domain D', between D and $\widetilde{D}$, so that

$$\mathrm{cl}(D) \subset D' \text{ and } \mathrm{cl}(D') \subset \widetilde{D}. \tag{2.1.10}$$

It may be noted that if f were only defined on $\mathrm{cl}(D)$ then, by the Tietze extension theorem, f may be extended to a continuous function on $\mathrm{cl}(\widetilde{D})$ without increasing its maximum absolute value. So there is no loss of generality in assuming that f is defined on $\mathrm{cl}(\widetilde{D})$.

Lemma 2-1. *Suppose there exist l, $\{\mu_j \in \mathcal{R} : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{R}^d : j = 1,2,\ldots,l\}$ such that $\psi(x)$ as defined in (2.1.5) satisfies (2.1.6) and (2.1.7), bounded open domains D, D', $\widetilde{D}$ with $cl(D) \subset D'$, $cl(D') \subset \widetilde{D}$, a continuous function $f : cl(\widetilde{D}) \to \mathcal{R}$ and $\epsilon > 0$. Further, suppose that ϕ is homogeneous, so that there exist constants t, A such that $\phi(r) = Ar^t$, for all $r \in \mathcal{R}^+$. Then there exists a function*

$$\gamma(x) = \sum_{j=1}^{l} \nu_j \phi(\|x - x_j'\|), \qquad x \in \mathcal{R}^d, \tag{2.1.11}$$

such that, for all $x \in D$,

$$\left| f(x) - \int_{D'} \gamma(x-y) f(y)\, dy \right| < \epsilon. \tag{2.1.12}$$

Further, we may choose $\|x_j'\|$ so small that $y \in D' \Rightarrow y + x_j' \in \widetilde{D}$, $\quad j = 1,2,\ldots,l$.

Proof. We assume that $\{\mu_j : j = 1,2,\ldots,l\}$ are scaled such that

$$\int_{\mathcal{R}^d} \psi(x)\, dx = 1 \tag{2.1.13}$$

and let

$$M_1 = \int_{\mathcal{R}^d} |\psi(x)|\, dx. \tag{2.1.14}$$

The function f is continuous on $\mathrm{cl}(\widetilde{D})$, so it is bounded there, and we let

$$M_2 = \sup\{|f(z)| : z \in \mathrm{cl}(\widetilde{D})\}. \tag{2.1.15}$$

Further, f is uniformly continuous on $\mathrm{cl}(\widetilde{D})$ so, given $\epsilon > 0$, there exists $\delta_1 > 0$ such that, for all $x, y \in \widetilde{D}$ with $\|x - y\| < \delta_1$,

$$|f(x) - f(y)| < \frac{\epsilon}{2M_1}, \tag{2.1.16}$$

where M_1 is defined in (2.1.14). Let

$$\delta_2 = \inf\{\|x - y\| : x \in \mathrm{cl}(D),\ y \in \mathcal{R}^d \backslash D'\} \tag{2.1.17}$$

and

$$\delta_3 = \inf\{\|y - z\| : y \in \mathrm{cl}(D'),\ z \in \mathcal{R}^d \backslash \widetilde{D}\}. \tag{2.1.18}$$

We note that δ_2 is strictly positive because $\mathrm{cl}(D)$ is compact, $\mathcal{R}^n \backslash D'$ is closed and the two sets are disjoint. Similarly δ_3 is strictly positive. The absolute integrability of $\psi(x)$ implies that given $\epsilon > 0$ there exists $R \in \mathcal{R}^+$ such that

$$\int_{\mathcal{R}^d \backslash S(0,R)} |\psi(x)|\, dx < \frac{\epsilon}{6M_2}, \tag{2.1.19}$$

where

$$S(y, R) = \{x \in \mathcal{R}^d \,:\, \|x - y\| \le R\}, \qquad y \in \mathcal{R}^d,\ R \in \mathcal{R}^+, \tag{2.1.20}$$

and M_2 is defined in (2.1.15). We pick $\delta > 0$ such that

$$R\delta < \min(\delta_1,\, \delta_2), \tag{2.1.21}$$

where R satisfies (2.1.19), and

$$\delta\|x_j\| < \delta_3, \qquad j = 1, 2, \ldots, l. \tag{2.1.22}$$

Now we define

$$\begin{aligned}\gamma(x) &= \frac{1}{\delta^d} \sum_{j=1}^{l} \mu_j \phi(\|\delta^{-1}x - x_j\|) \\ &= \frac{1}{\delta^{d+t}} \sum_{j=1}^{l} \mu_j \phi(\|x - \delta x_j\|), \qquad x \in \mathcal{R}^d,\end{aligned} \tag{2.1.23}$$

the second line using the condition on ϕ in the statement of the lemma. This will be the function γ described in (2.1.11) and hence the values of its parameters are $\nu_j = \delta^{-d-t}\mu_j,\ \ j = 1, 2, \ldots, l$ and $x'_j = \delta x_j,\ \ j = 1, 2, \ldots, l$. Defining also

$$S'(y) = S(y, \min(\delta_1,\, \delta_2)), \qquad y \in \mathcal{R}^d, \tag{2.1.24}$$

yields that

$$\int_{\mathcal{R}^d \setminus S'(0)} |\gamma(x)|\, dx = \frac{1}{\delta^d} \int_{\mathcal{R}^d \setminus S'(0)} |\psi(\delta^{-1}x)|\, dx < \frac{\epsilon}{6M_2}, \tag{2.1.25}$$

by (2.1.19) and (2.1.21). Further, (2.1.13) and (2.1.14) show that

$$\int_{\mathcal{R}^d} \gamma(x)\, dx = 1 \tag{2.1.26}$$

and

$$\int_{\mathcal{R}^d} |\gamma(x)|\, dx = M_1. \tag{2.1.27}$$

So, for $x \in D$,

$$\begin{aligned} E &= \left| f(x) - \int_{D'} f(y)\gamma(x - y)\, dy \right| \\ &= \left| \int_{\mathcal{R}^d} f(x)\gamma(x - y)\, dy - \int_{D'} f(y)\gamma(x - y)\, dy \right| \\ &\le \left| \int_{D'} (f(x) - f(y))\gamma(x - y)\, dy \right| + \left| \int_{\mathcal{R}^d \setminus D'} f(x)\gamma(x - y)\, dy \right|. \end{aligned} \tag{2.1.28}$$

Since $|f(x)| \le M_2$ by (2.1.15) and $S'(x) \subset D'$ by (2.1.17) and (2.1.24), it follows that

$$
\begin{aligned}
E &\le \left| \int_{D'} (f(x) - f(y))\gamma(x-y)\,dy \right| + M_2 \int_{\mathcal{R}^d \setminus S'(x)} |\gamma(x-y)|\,dy \\
&\le \left| \int_{S'(x)} (f(x) - f(y))\gamma(x-y)\,dy \right| + \left| \int_{D' \setminus S'(x)} (f(x) - f(y))\gamma(x-y)\,dy \right| \\
&\qquad + M_2 \int_{\mathcal{R}^d \setminus S'(0)} |\gamma(y)|\,dy \\
&< \frac{\epsilon}{2M_1} \int_{S'(x)} |\gamma(x-y)|\,dy + \left| \int_{D' \setminus S'(x)} (f(x) - f(y))\gamma(x-y)\,dy \right| + \frac{\epsilon}{6},
\end{aligned}
\tag{2.1.29}
$$

where the last line depends on (2.1.16), (2.1.24) and (2.1.25). Further, using (2.1.15), (2.1.25) and (2.1.27) we find

$$
\begin{aligned}
E &< \frac{\epsilon}{2M_1} \int_{\mathcal{R}^d} |\gamma(x-y)|\,dy + 2M_2 \int_{D' \setminus S'(x)} |\gamma(x-y)|\,dy + \frac{\epsilon}{6} \\
&< \frac{\epsilon}{2} + \frac{2\epsilon}{6} + \frac{\epsilon}{6} = \epsilon.
\end{aligned}
\tag{2.1.30}
$$

The proof is completed by noting that, from (2.1.18) and (2.1.22), $y \in D' \Rightarrow y + \delta x_j \in \tilde{D}, \quad j = 1, 2, \ldots, l.$ ■

Before proceeding we define a concept which will be needed in the remainder of the proof.

The family of functions $\{G_x : D' \to \mathcal{R}, \quad x \in D\}$ is **uniformly equicontinuous** on D' if, given $\eta > 0$, there exists $\delta > 0$ such that, for all $y_1, y_2 \in D'$, $x \in D$ with $\|y_1 - y_2\| < \delta$,

$$
|G_x(y_1) - G_x(y_2)| < \eta. \tag{2.1.31}
$$

We consider $\phi(\|x - y - z\|)$ as a function of y from D' to $\mathcal{R}$ where $x \in D$ and $z \in \mathcal{R}^d$ are such that $y \in D' \Rightarrow y + z \in \tilde{D}$, and will demonstrate that it is uniformly equicontinuous.

We let R be a positive constant such that $\tilde{D} \subset S(0, R)$. Hence $0 \le \|x - y - z\| \le 2R$ and ϕ is uniformly continuous on $[0, 2R]$. Therefore, given $\eta > 0$, there exists $\delta > 0$ such that, $a, b \in [0, 2R]$ and $|a - b| < \delta$ imply $|\phi(a) - \phi(b)| < \eta$. Therefore if $y_1, y_2 \in D'$ with $\|y_1 - y_2\| < \delta$ we have

$$
\big|\, \|x - y_1 - z\| - \|x - y_2 - z\| \,\big| < \delta \tag{2.1.32}
$$

which implies

$$
|\phi(\|x - y_1 - z\|) - \phi(\|x - y_2 - z\|)| < \eta. \tag{2.1.33}
$$

Hence, both $\phi(\|x - y\|)$ and $\gamma(x - y)$, as defined in (2.1.23), are uniformly equicontinuous on D' and so is the product $y \mapsto \gamma(x - y)f(y)$. Indeed, the above analysis shows that all three are also uniformly equicontinuous on $\tilde{D}$.

Lemma 2-2. *Given open bounded regions D', $\widetilde{D}$ with $cl(D') \subset \widetilde{D}$, a continuous function $f : cl(\widetilde{D}) \to \mathcal{R}$, a function*

$$\gamma(x) = \sum_{j=1}^{l} \nu_j \phi(\|x - x'_j\|), \qquad x \in \mathcal{R}^d, \tag{2.1.34}$$

where $y \in D' \Rightarrow y + x'_j \in \widetilde{D}$, $j = 1, 2, \ldots, l$, and $\epsilon > 0$, then there exist M, $\{\rho_i : i = 1, 2, \ldots, M\}$ and $\{y_i \in D' : i = 1, 2, \ldots, M\}$ such that

$$\left| \int_{D'} \gamma(x - y) f(y)\, dy - \sum_{i=1}^{M} \rho_i \sum_{j=1}^{l} \nu_j \phi(\|x - y_i - x'_j\|) \right| < \epsilon. \tag{2.1.35}$$

Proof. The function $y \mapsto \gamma(x - y) f(y)$ is uniformly equicontinuous on D' so, for any $\eta > 0$, we can choose $\delta > 0$ such that $y_1, y_2 \in D'$, $\|y_1 - y_2\| < \delta$ imply

$$\left| \gamma(x - y_1) f(y_1) - \gamma(x - y_2) f(y_2) \right| < \eta, \text{ for all } x \in D'. \tag{2.1.36}$$

Then pick any sequence of points $\{y_i : i = 1, 2, \ldots, M\} \subset D'$ for which $\tau_M < \delta$, where τ_M is defined as in (2.1.4), and let

$$A(y_i) = \{y \in D' : \|y - y_i\| \leq \|y - y_k\|, \quad k = 1, 2, \ldots, M\}. \tag{2.1.37}$$

Therefore (by (2.1.36)) $y \in A(y_i)$ implies

$$\left| \gamma(x - y) f(y) - \gamma(x - y_i) f(y_i) \right| < \eta. \tag{2.1.38}$$

Hence

$$\begin{aligned} &\left| \int_{D'} \gamma(x - y) f(y)\, dy - \sum_{i=1}^{M} |A(y_i)| \gamma(x - y_i) f(y_i) \right| \\ \leq &\sum_{i=1}^{M} \left| \int_{A(y_i)} \gamma(x - y) f(y)\, dy - |A(y_i)| \gamma(x - y_i) f(y_i) \right| \\ \leq &\eta \sum_{i=1}^{M} |A(y_i)| = \eta |D'|, \end{aligned} \tag{2.1.39}$$

where $|A(y_i)|$ and $|D'|$ are the Euclidean volumes of $A(y_i)$ and D' respectively. The proof is completed by letting $\rho_i = |A(y_i)| f(y_i)$, $\eta = \epsilon / |D'|$, and noting the form of $\gamma(x)$. ∎

The coefficients $\{\rho_i \nu_j\}$ and the points $\{y_i + x'_j\}$ of expression (2.1.35) become $\{\xi_i\}$ and $\{\bar{y}_i\}$ respectively in the following lemma and $\overline{M} = lM$.

Lemma 2-3. *Given bounded open regions* D, $\widetilde{D}$, $\{\xi_i : i = 1,2,\ldots,\overline{M}\}$, $\{\overline{y}_i \in \widetilde{D} : i = 1,2,\ldots,\overline{M}\}$, *a sequence of points* $\{z_k \in \widetilde{D} : k = 1,2,\ldots\}$ *becoming dense in* $\widetilde{D}$, *and* $\epsilon > 0$ *then there exist* N *and* $\{\lambda_k : k = 1,2,\ldots,N\}$ *such that, for all* $x \in D$,

$$\left| \sum_{i=1}^{\overline{M}} \xi_i \phi(\|x - \overline{y}_i\|) - \sum_{k=1}^{N} \lambda_k \phi(\|x - z_k\|) \right| < \epsilon. \tag{2.1.40}$$

Proof. Recalling that $\phi(\|x - y\|)$ is uniformly equicontinuous on $\widetilde{D}$, we choose $\delta > 0$ such that x_1, $x_2 \in \widetilde{D}$ and $\|x_1 - x_2\| < \delta$ imply

$$|\phi(\|x - x_1\|) - \phi(\|x - x_2\|)| < \epsilon \left(\sum_{i=1}^{\overline{M}} |\xi_i| \right)^{-1}, \qquad x \in D. \tag{2.1.41}$$

Now choose N such that for the sequence $\{z_k : k = 1,2,\ldots\}$ we have $\tau_N < \delta$, where τ_N is defined as in (2.1.4). Also for $i = 1,2,\ldots,\overline{M}$ let $z_i' \in \{z_k : k = 1,2,\ldots,N\}$ satisfy

$$\|z_i' - \overline{y}_i\| \le \|z_k - \overline{y}_i\|, \quad k = 1,2,\ldots,N. \tag{2.1.42}$$

Then $\|z_i' - \overline{y}_i\| < \delta, \quad i = 1,2,\ldots,\overline{M}$, and thus,

$$\begin{aligned} &\left| \sum_{i=1}^{\overline{M}} \xi_i \phi(\|x - \overline{y}_i\|) - \sum_{i=1}^{\overline{M}} \xi_i \phi(\|x - z_i'\|) \right| \\ &\le \sum_{i=1}^{\overline{M}} |\xi_i| \, |\phi(\|x - \overline{y}_i\|) - \phi(\|x - z_i'\|)| < \epsilon, \end{aligned} \tag{2.1.43}$$

the inequality in the last line being a consequence of (2.1.41). ■

In view of the three lemmas, the following key theorem has been proved:

Theorem 2-4. *Suppose that* ϕ *is homogeneous and we are given a function* $\psi(x)$ *as defined in* (2.1.5) *and satisfying* (2.1.6) *and* (2.1.7), *bounded open regions* D, $\widetilde{D}$ *with* $cl(D) \subset \widetilde{D}$, *a continuous function* $f : cl(\widetilde{D}) \to \mathcal{R}$, *a sequence of points* $\{z_k : k = 1,2,\ldots\}$ *which becomes dense in* $\widetilde{D}$, *and* $\epsilon > 0$ *then there exist* N *and* $\{\lambda_k : k = 1,2,\ldots,N\}$ *such that, for all* $x \in D$,

$$\left| f(x) - \sum_{k=1}^{N} \lambda_k \phi(\|x - z_k\|) \right| < \epsilon. \quad ■ \tag{2.1.44}$$

Before considering the construction of functions $\psi(x)$ defined by (2.1.5) and satisfying (2.1.6) and (2.1.7), it is worth looking more closely at an implication of the existence of $\psi(x)$. From such a function we form a radially symmetric function $\widetilde{\psi}(r)$ by

$$\widetilde{\psi}(r) = \frac{1}{|\partial S(0,r)|} \int_{\partial S(0,r)} \psi(x)\, dx, \qquad r \in \mathcal{R}^+, \tag{2.1.45}$$

where, corresponding to (2.1.20),

$$\partial S(0,r) = \{x \in \mathcal{R}^d : \|x\| = r\}, \qquad r \in \mathcal{R}^+, \tag{2.1.46}$$

and $|\partial S(0,r)|$ is the Euclidean volume of $\partial S(0,r)$. Hence with $r = \|y\|$

$$\int_{\mathcal{R}^d} |\tilde{\psi}(r)|\, dy \leq \int_{\mathcal{R}^d} |\psi(y)|\, dy < \infty \tag{2.1.47}$$

and

$$\int_{\mathcal{R}^d} \tilde{\psi}(r)\, dy = \int_{\mathcal{R}^d} \psi(y)\, dy \neq 0. \tag{2.1.48}$$

From (2.1.5),

$$\begin{aligned}\tilde{\psi}(r) &= \sum_{j=1}^{l} \mu_j \frac{1}{|\partial S(0,r)|} \int_{\partial S(0,r)} \phi(\|x - x_j\|)\, dx \\ &= \sum_{j=1}^{l} \mu_j \sigma(r, r_j), \qquad r \in \mathcal{R}^+,\end{aligned} \tag{2.1.49}$$

where $r_j = \|x_j\|$ and where

$$\sigma(r,s) = \frac{1}{|\partial S(0,r)|} \int_{\partial S(0,r)} \phi(\|x - y\|)\, dx, \qquad r, s \in \mathcal{R}^+, \tag{2.1.50}$$

y being any vector with $\|y\| = s$. Further, since it can be deduced by symmetry from (2.1.50) that $\sigma(r,s) = \sigma(s,r)$, we have

$$\tilde{\psi}(r) = \sum_{j=1}^{l} \mu_j \sigma(r_j, r), \qquad r \in \mathcal{R}^+. \tag{2.1.51}$$

We now show that the existence of a function (2.1.51) satisfying conditions (2.1.47) and (2.1.48) implies a uniform convergence result that is equivalent to Theorem 2-4. The homogeneity assumption in Lemma 2-1, that there exists some constant t such that $\phi(cr) = c^t \phi(r)$ for all $c, r \in \mathcal{R}^+$ becomes $\sigma(cr, cs) = c^t \sigma(r,s)$ for all $c, r, s \in \mathcal{R}^+$, and, guided by equation (2.1.23), we let $\gamma(\cdot)$ be the function

$$\begin{aligned}\gamma(x) &= \frac{1}{\delta^{d+t}} \sum_{j=1}^{l} \mu_j \sigma(\delta r_j, \|x\|) \\ &= \sum_{j=1}^{l} \nu_j \sigma(r'_j, \|x\|), \qquad x \in \mathcal{R}^d,\end{aligned} \tag{2.1.52}$$

where the value of δ comes from the method of proof of Lemma 2-1. Indeed, these changes imply that the analogues of Lemma 2-1 and its proof are true, the last line of the statement of the lemma being that $y \in D' \Rightarrow y + x'_j \in \widetilde{D}$ for all x'_j such that $\|x'_j\| = r'_j$ and any $j \in \{1, 2, \ldots, l\}$. Then Lemma 2-2 and its proof are also valid for the new function $\gamma(\cdot)$, after replacing $\phi(\|x - y_i - x'_j\|)$ by $\sigma(r'_j, \|x - y_i\|)$.

In view of equations (2.1.45) and (2.1.47), the existence of a suitable function $\widetilde{\psi}$ is a weaker assumption than that of ψ satisfying the conditions of Theorem 2-4. However, Theorem 2-4 can be deduced from the existence of $\widetilde{\psi}$, by applying the new form of Lemmas 2-1 and 2-2, and the old Lemma 2-3, after proving the following lemma.

Lemma 2-5. *Given a bounded open region D, any $y \in \mathcal{R}^d$, $r'_j \in \mathcal{R}^+$, and $\epsilon > 0$, there exist $\{\zeta_i : i = 1, 2, \ldots, \widetilde{M}\}$ and $\{w_i : \|w_i\| = r'_j,\ i = 1, 2, \ldots, \widetilde{M}\}$ such that, for all $x \in D$,*

$$\left| \sigma(r'_j, \|x - y\|) - \sum_{i=1}^{\widetilde{M}} \zeta_i \phi(\|x - y - w_i\|) \right| < \epsilon. \tag{2.1.53}$$

Proof. From (2.1.50),

$$\sigma(r'_j, \|x - y\|) = \frac{1}{|\partial S(0, r'_j)|} \int_{\partial S(0, r'_j)} \phi(\|x - y - w\|)\, dw. \tag{2.1.54}$$

It is easy to see that $\phi(\|x - y - w\|)$, now viewed as a function of w, is uniformly equicontinuous on $\partial S(0, r'_j)$. Now the argument proceeds exactly as in Lemma 2-2 to approximate the integral (here over $\partial S(0, r'_j)$) by a finite sum of the required form. ∎

The proof of the theorem corresponding to Theorem 2-4 can now be completed by using Lemma 2-3 exactly as it stands, so we have:

Theorem 2-6. *Suppose that ϕ is homogeneous and we are given $\widetilde{\psi}(r)$ as defined in (2.1.51) and satisfying (2.1.47) and (2.1.48), bounded open regions D, $\widetilde{D}$ with $cl(D) \subset \widetilde{D}$, a continuous function $f : cl(\widetilde{D}) \to \mathcal{R}$, a sequence of points $\{z_k : k = 1, 2, \ldots\}$ which becomes dense in $\widetilde{D}$, and $\epsilon > 0$, then there exist $\widetilde{N}$ and $\{\lambda_k : k = 1, 2, \ldots, \widetilde{N}\}$ such that, for all $x \in D$,*

$$\left| f(x) - \sum_{k=1}^{\widetilde{N}} \lambda_k \phi(\|x - z_k\|) \right| < \epsilon. \quad \blacksquare \tag{2.1.55}$$

Section 2.2 : Construction in Even Dimensions

In this and the subsequent section the following notation is used:

The **factorial function** is defined, for $a \in \mathcal{R}$, $n \in \mathcal{Z}^+$, by

$$(a)_0 = 1, \quad (a)_n = (a+n-1)(a)_{n-1} \text{ for } n \geq 1. \tag{2.2.1}$$

Thus we have $(a)_1 = a$, $(1)_n = n!$ and $(2)_n = (n+1)!$.

The **hypergeometric function** $F(a,b;c;z)$ is defined for $|z| < 1$ and $c \neq 0, -1, -2, \ldots$ by

$$F(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k. \tag{2.2.2}$$

This definition is also valid when $|z| = 1$ if $c - a - b > 0$ and $c \neq 0, -1, -2, \ldots$.

Two theorems that will be needed on hypergeometric functions are:

$$F(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \tag{2.2.3}$$

for $c - a - b > 0$ and $c \neq 0, -1, -2, \ldots$ (Abramowitz and Stegun 1970, 15.1.20), and

$$(1+z)^{-a} F\left(\tfrac{1}{2}a, \tfrac{1}{2}a + \tfrac{1}{2}; a - b + 1; \frac{4z}{(1+z)^2}\right) = F(a,b;a-b+1;z) \tag{2.2.4}$$

for $a - b + 1 \neq 0, -1, -2, \ldots$ and $|z| < 1$ (Abramowitz and Stegun 1970, 15.3.26).

Here the special case $\phi(r) = r$ is considered, so that in this and the subsequent section

$$\psi(x) = \sum_{j=1}^{l} \mu_j \|x - x_j\|, \quad x \in \mathcal{R}^d, \tag{2.2.5}$$

and

$$\tilde{\psi}(\|x\|) = \tilde{\psi}(s) = \sum_{j=1}^{l} \mu_j \sigma(r_j, s), \qquad x \in \mathcal{R}^d,\ s \in \mathcal{R}^+, \tag{2.2.6}$$

where, for x any vector with $\|x\| = s$,

$$\sigma(r_j, s) = \frac{1}{|\partial S(0, r_j)|} \int_{\partial S(0,r_j)} \|x - y\|\, dy, \qquad s \in \mathcal{R}^+. \tag{2.2.7}$$

The problem of finding a function $\psi(x)$ as defined in (2.2.5) and satisfying (2.1.6) and (2.1.7) is tackled. First the question of absolute integrability (2.1.6) is discussed. The function $\|x - x_j\|$ is infinitely differentiable except at $x = x_j$, where it is only continuous. A series expansion for $\psi(x)$ may be found for large $\|x\|$ by considering the identity

$$\begin{aligned}
\|x - x_j\| &= (\|x\|^2 - 2x.x_j + \|x_j\|^2)^{\frac{1}{2}} \\
&= \|x\| \left(1 - \frac{2x.x_j}{\|x\|^2} + \frac{\|x_j\|^2}{\|x\|^2}\right)^{\frac{1}{2}} \\
&= \|x\| - \frac{x.x_j}{\|x\|} + \frac{\|x\|^2\|x_j\|^2 - (x.x_j)^2}{2\|x\|^3} + \ldots,
\end{aligned} \tag{2.2.8}$$

the last line using

$$(1-\alpha)^{\frac{1}{2}} = \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k \alpha^k}{k!}, \qquad |\alpha| < 1. \tag{2.2.9}$$

We see from (2.2.5) and (2.2.8) that the condition for the $O(\|x\|)$ term to be zero in this expansion of $\psi(x)$ is

$$\sum_{j=1}^{l} \mu_j = 0. \tag{2.2.10}$$

Further, the condition for the $O(1)$ term to be zero is

$$-\|x\|^{-1} \sum_{j=1}^{l} \mu_j (x.x_j) = 0, \tag{2.2.11}$$

which must hold for all sufficiently large $\|x\|$ and hence for all x. Further, it can be deduced from the form of (2.2.8) that the $O(\|x\|^{-r})$ term has the form

$$\frac{1}{\|x\|^{2r+1}} \sum_{j=1}^{l} \mu_j Q_{r+1}(x, x_j), \qquad x \in \mathcal{R}^d, \tag{2.2.12}$$

where

$$Q_{r+1}(x,y) = \sum_{\{\alpha,\beta \in (\mathcal{Z}^+)^d : |\alpha| = |\beta| = r+1\}} A_{\alpha,\beta} x^\alpha y^\beta, \qquad x, y \in \mathcal{R}^d. \tag{2.2.13}$$

Here each $A_{\alpha,\beta} \in \mathcal{R}$ and we are using the multi-index notation defined at the end of the introduction. Therefore we view (2.2.12) as a polynomial in the components of x. From (2.2.13) each coefficient of this polynomial is of the form

$$\sum_{j=1}^{l} \mu_j \tilde{P}_{r+1}(x_j), \tag{2.2.14}$$

where $\tilde{P}_{r+1}$ is a homogeneous polynomial of degree $r+1$. Thus, the $O(\|x\|^{-r})$ term in the expansion of $\psi(x)$ for large argument vanishes if (2.2.14) is zero for every homogeneous polynomial $\tilde{P}_{r+1}$ of degree $r+1$.

Now conditions for absolute integrability can be given. The function ψ has a series expansion for large argument and hence it is necessary and sufficient that

$$\psi(x) = O(\|x\|^{-d-1}) \text{ as } \|x\| \to \infty. \tag{2.2.15}$$

By the remark after (2.2.14) it is sufficient that

$$\sum_{j=1}^{l} \mu_j P_{d+1}(x_j) = 0 \tag{2.2.16}$$

for every polynomial P_{d+1} of degree at most $d+1$. We also make the observation, which will prove useful in later analysis, that if $\psi(x) = O(\|x\|^{-r})$ as $\|x\| \to \infty$ then, because $\psi(x)$ has a series expansion for large $\|x\|$, for any $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = t$,

$$\frac{\partial^\alpha \psi(x)}{\partial x^\alpha} = O(\|x\|^{-r-t}) \text{ as } \|x\| \to \infty. \tag{2.2.17}$$

(The multi-index notation for partial derivatives is explained at the end of the introduction.) Equation (2.2.16) imposes only a finite number of conditions on the form of ψ and so there are certainly many such functions which are absolutely integrable; yet the following remarkable result will be proved.

Theorem 2-7. *Suppose that d is even and $\widetilde{\psi}$ as defined in* (2.2.6) *is absolutely integrable, then*

$$\int_{\mathcal{R}^d} \widetilde{\psi}(\|x\|)\, dx = 0.$$

Thus, remembering the construction used in (2.1.45), we deduce:

Corollary 2-8. *Suppose that d is even and ψ as defined in* (2.2.5) *is absolutely integrable, then*

$$\int_{\mathcal{R}^d} \psi(x)\, dx = 0.$$

Proof. To prove Theorem 2-7 we first discover the form of $\sigma(r,s)$. Converting (2.2.7) to spherical polar coordinates we find

$$\begin{aligned}\sigma(r,s) = {} & \frac{1}{|\partial S(0,r)|} \int_0^\pi \cdots \int_0^\pi \int_0^{2\pi} (s^2 + r^2 - 2rs\cos\theta)^{\frac{1}{2}} r^{d-1} \sin^{d-2}\theta \\ & \sin^{d-3}\phi_{d-3} \ldots \sin\phi_1\, d\phi_0\, d\phi_1 \ldots d\phi_{d-3}\, d\theta \\ = {} & K_d \int_0^\pi (s^2 + r^2 - 2rs\cos\theta)^{\frac{1}{2}} \sin^{d-2}\theta\, d\theta, \qquad r,s \in \mathcal{R}^+,\end{aligned} \tag{2.2.18}$$

where

$$K_d = 1 \Big/ \int_0^\pi \sin^{d-2}\theta\, d\theta. \tag{2.2.19}$$

Now, for $r \neq s$,

$$\begin{aligned}(s^2 + r^2 - 2rs\cos\theta)^{\frac{1}{2}} &= (r^2+s^2)^{\frac{1}{2}} \left(1 - \frac{2rs\cos\theta}{r^2+s^2}\right)^{\frac{1}{2}} \\ &= (r^2+s^2)^{\frac{1}{2}} \sum_{k=0}^\infty \frac{(-\frac{1}{2})_k}{k!} \alpha^k \cos^k\theta,\end{aligned} \tag{2.2.20}$$

where

$$\alpha = \frac{2rs}{(r^2+s^2)}. \tag{2.2.21}$$

Using

$$\int_0^{\pi} \cos^{m-1}\theta \sin^{d-2}\theta \, d\theta = \begin{cases} \dfrac{\Gamma(\frac{m}{2})\Gamma(\frac{d-1}{2})}{\Gamma(\frac{m+d-1}{2})} & \text{if } m \text{ is odd;} \\ 0 & \text{if } m \text{ is even,} \end{cases} \tag{2.2.22}$$

(e.g. Whittaker and Watson 1927, 12.42), we see that

$$\sigma(r,s) = K_d(r^2+s^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_{2k}\alpha^{2k}}{(2k)!} \frac{\Gamma(k+\frac{1}{2})\Gamma(\frac{d-1}{2})}{\Gamma(k+\frac{d}{2})}. \tag{2.2.23}$$

Since (2.2.19) and (2.2.22) imply

$$K_d = \frac{\Gamma(\frac{d}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{d-1}{2})}, \tag{2.2.24}$$

we have

$$\sigma(r,s) = (r^2+s^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_{2k}(\frac{1}{2})_k}{(2k)!(\frac{d}{2})_k}\alpha^{2k}. \tag{2.2.25}$$

Noting that $(-\frac{1}{2})_{2k} = (-\frac{1}{4})_k(\frac{1}{4})_k 2^{2k}$ and $(2k)! = (\frac{1}{2})_k k! \, 2^{2k}$ it follows that

$$\sigma(r,s) = (r^2+s^2)^{\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-\frac{1}{4})_k(\frac{1}{4})_k}{(\frac{d}{2})_k k!}\alpha^{2k}. \tag{2.2.26}$$

Now when $r > s$ we may use (2.2.2) and (2.2.21) to obtain

$$\sigma(r,s) = r\left(1+\frac{s^2}{r^2}\right)^{\frac{1}{2}} F\left(-\tfrac{1}{4}, \tfrac{1}{4}; \tfrac{d}{2}; \frac{4\frac{s^2}{r^2}}{(1+\frac{s^2}{r^2})^2}\right). \tag{2.2.27}$$

This is in a suitable form to apply (2.2.4) which yields

$$\sigma(r,s) = rF\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{s^2}{r^2}\right). \tag{2.2.28a}$$

Similarly, for $r < s$, the result is

$$\sigma(r,s) = sF\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{r^2}{s^2}\right). \tag{2.2.28b}$$

Further, by the continuity of $\sigma(r,s)$, both these results may be extended to include the case $r = s$.

Having established the form of $\sigma(r,s)$, and hence (by (2.2.6)) of $\widetilde{\psi}(s)$, it is necessary to calculate the integral of $\widetilde{\psi}(s)$. We consider

$$I = \int_{S(0,M)} \sigma(r, \|x\|) \, dx \tag{2.2.29}$$

where $M \gg r$. Denoting $\|x\|$ by s and using the radial symmetry of $\sigma(r,\|x\|)$ and (2.2.28), it follows that

$$I = L_d \int_0^r rs^{d-1} F\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{s^2}{r^2}\right) ds + L_d \int_r^M s^d F\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{r^2}{s^2}\right) ds, \tag{2.2.30}$$

where

$$L_d = |\partial S(0,1)|. \tag{2.2.31}$$

Denoting the first integral by I_0 and the second by I_1 we deduce from (2.2.2) that

$$I_0 = L_d r \int_0^r s^{d-1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2})_k k!} \frac{s^{2k}}{r^{2k}}\, ds \tag{2.2.32}$$

and, as this series converges uniformly over the range of integration,

$$\begin{aligned} I_0 &= L_d r \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2})_k k!} \frac{1}{r^{2k}} \int_0^r s^{2k+d-1}\, ds \\ &= L_d r^{d+1} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2})_k k!} \frac{1}{2(k+\frac{d}{2})} \\ &= \frac{L_d r^{d+1}}{d} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2}+1)_k k!}. \end{aligned} \tag{2.2.33}$$

Now expressions (2.2.2) and (2.2.3) may be used to yield

$$\begin{aligned} I_0 &= \frac{L_d r^{d+1}}{d} \frac{\Gamma(\frac{d}{2}+1)\Gamma(d+1)}{\Gamma(\frac{d+3}{2})\Gamma(d+\frac{1}{2})} \\ &= \tfrac{1}{2} L_d r^{d+1} \frac{\Gamma(\frac{d}{2})\Gamma(d+1)}{\Gamma(\frac{d+3}{2})\Gamma(d+\frac{1}{2})}. \end{aligned} \tag{2.2.34}$$

Similarly, the integrand of the term

$$I_1 = L_d \int_r^M s^d F\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{r^2}{s^2}\right) ds \tag{2.2.35}$$

has an expansion that converges uniformly over the range of integration, so, remembering that d is even,

$$\begin{aligned} I_1 &= L_d \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2})_k k!} r^{2k} \int_r^M s^{-2k+d}\, ds \\ &= L_d \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{1-d}{2})_k}{(\frac{d}{2})_k k!} r^{2k} \left(\frac{M^{-2k+d+1} - r^{-2k+d+1}}{-2(\frac{-1-d}{2}+k)} \right) \end{aligned} \tag{2.2.36}$$

$$= \frac{L_d}{1+d} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{-1-d}{2})_k}{(\frac{d}{2})_k k!} r^{2k}(M^{-2k+d+1} - r^{-2k+d+1}). \tag{2.2.37}$$

The value of this expression if $M^{-2k+d+1}$ is replaced by zero is

$$\tilde{I}_1 = -\frac{L_d r^{d+1}}{1+d} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{-1-d}{2})_k}{(\frac{d}{2})_k k!}. \tag{2.2.38}$$

Now, it follows from (2.2.2) and (2.2.3), that

$$\begin{aligned}\tilde{I}_1 &= -\frac{L_d r^{d+1}}{1+d}\frac{\Gamma(\frac{d}{2})\Gamma(d+1)}{\Gamma(\frac{d+1}{2})\Gamma(d+\frac{1}{2})} \\ &= -\tfrac{1}{2}L_d r^{d+1}\frac{\Gamma(\frac{d}{2})\Gamma(d+1)}{\Gamma(\frac{d+3}{2})\Gamma(d+\frac{1}{2})}.\end{aligned} \tag{2.2.39}$$

Equations (2.2.34), (2.2.37) and (2.2.39) imply that

$$I = I_0 + I_1 = \frac{L_d}{1+d}\sum_{k=0}^{\infty}\frac{(-\frac{1}{2})_k(\frac{-1-d}{2})_k}{(\frac{d}{2})_k k!} r^{2k} M^{-2k+d+1}. \tag{2.2.40}$$

Now, for the function $\widetilde{\psi}(\|x\|)$ of (2.2.6) to be absolutely integrable, it is required, by comparison with (2.2.15), that

$$\widetilde{\psi}(\|x\|) = O(\|x\|^{-d-1}) \text{ as } \|x\| \to \infty. \tag{2.2.41}$$

Hence, from (2.2.28*b*), this implies that

$$\sum_{j=1}^{l}\mu_j r_j^{2k} = 0, \quad k = 0, 1, \ldots, \tfrac{d}{2}. \tag{2.2.42}$$

Thus, there is no contribution from the first $(\frac{d}{2}+1)$ terms of the sum (2.2.40) to the integral

$$\int_{S(0,M)}\widetilde{\psi}(\|x\|)\,dx = \sum_{j=1}^{l}\mu_j\int_{S(0,M)}\sigma(r_j, \|x\|)\,dx. \tag{2.2.43}$$

It follows that

$$\int_{S(0,M)}\widetilde{\psi}(\|x\|)\,dx = O(M^{-1}), \tag{2.2.44}$$

and hence

$$\int_{\mathcal{R}^d}\widetilde{\psi}(\|x\|)\,dx = 0, \tag{2.2.45}$$

which completes the proof of Theorem 2-7. ■

Section 2.3 : Construction in Odd Dimensions

In this section the case $\phi(r) = r$ is again considered though now the dimension d is odd, and the problem of finding a function $\psi(x) : \mathcal{R}^d \to \mathcal{R}$ as defined in (2.2.5) and satisfying (2.1.6) and (2.1.7) is addressed. First, because of the similarity with Section 2.2 and because it will lead to the solution of the above problem, we prove:

Theorem 2-9. *Suppose that d is odd, $d \geq 3$, then there exist functions $\widetilde{\psi}(\|x\|)$ as defined in (2.2.6) which are absolutely integrable and satisfy*

$$\int_{\mathcal{R}^d} \widetilde{\psi}(\|x\|)\, dx \neq 0.$$

Proof. As in Section 2.2 the first step is to evaluate $\sigma(r,s)$ of (2.2.7). The analysis of Section 2.2 is still applicable because it is not dependent on d being even. Hence we obtain (2.2.28) that, for $r > s$,

$$\sigma(r,s) = rF\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{s^2}{r^2}\right), \tag{2.3.1a}$$

and, for $r < s$,

$$\sigma(r,s) = sF\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{r^2}{s^2}\right). \tag{2.3.1b}$$

Again, by the continuity of $\sigma(r,s)$, both these results may be extended to include the case $r = s$. In this case we find that $(\frac{1-d}{2})_k = 0$ for $k > \frac{d-1}{2}$, because $\frac{d-1}{2}$ is an integer. Hence, expressions (2.3.1a) and (2.3.1b) each only contain $\frac{d+1}{2}$ terms. In particular (2.3.1a) is a polynomial of degree $d-1$ in s. We recall that the corresponding equations (2.2.28) in Section 2.2 contain an infinite number of terms.

We continue to proceed as in Section 2.2 and calculate the integral of $\widetilde{\psi}(s)$. We consider

$$I = \int_{S(0,M)} \sigma(r, \|x\|)\, dx, \tag{2.3.2}$$

where $M \gg r$. Denoting $\|x\|$ by s, using the radial symmetry of $\sigma(r,\|x\|)$ and (2.3.1), it follows that (2.2.30) still holds, which gives the value

$$I = L_d r \int_0^r s^{d-1} F\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{s^2}{r^2}\right) ds + L_d \int_r^M s^d F\left(-\tfrac{1}{2}, \tfrac{1-d}{2}; \tfrac{d}{2}; \frac{r^2}{s^2}\right) ds, \tag{2.3.3}$$

where

$$L_d = |\partial S(0,1)|. \tag{2.3.4}$$

We again denote the first integral by I_0 and the second by I_1. The calculation of I_0 is exactly the same as in Section 2.2 and we obtain (2.2.34), which is the expression

$$I_0 = \tfrac{1}{2} L_d r^{d+1} \frac{\Gamma(\frac{d}{2})\Gamma(d+1)}{\Gamma(\frac{d+3}{2})\Gamma(d+\frac{1}{2})}. \tag{2.3.5}$$

To evaluate I_1 in a manner similar to Section 2.2 care must be taken to avoid dividing by zero in the equation corresponding to (2.2.36). However, this is no problem, if, recalling that we only have a finite sum, we change the top limit on the sum from ∞ to $\frac{d-1}{2}$. The argument then proceeds as in Section 2.2 to yield the equation corresponding to (2.2.37):

$$I_1 = \frac{L_d}{1+d} \sum_{k=0}^{\frac{d-1}{2}} \frac{(-\frac{1}{2})_k(\frac{-d-1}{2})_k}{(\frac{d}{2})_k k!} r^{2k}(M^{-2k+d+1} - r^{-2k+d+1}). \tag{2.3.6}$$

Thus the value of this expression when $M^{-2k+d+1}$ is replaced by zero is

$$\tilde{I}_1 = -\frac{L_d r^{d+1}}{d+1} \sum_{k=0}^{\frac{d-1}{2}} \frac{(-\frac{1}{2})_k(\frac{-d-1}{2})_k}{(\frac{d}{2})_k k!}. \tag{2.3.7}$$

Noting that $(\frac{-d-1}{2})_k = 0$ for $k > \frac{d+1}{2}$, we find

$$\begin{aligned} \tilde{I}_1 &= -\frac{L_d r^{d+1}}{1+d} \sum_{k=0}^{\infty} \frac{(-\frac{1}{2})_k(\frac{-d-1}{2})_k}{(\frac{d}{2})_k k!} + \frac{L_d r^{d+1}}{1+d} \frac{(-\frac{1}{2})_{\frac{d+1}{2}}(\frac{-d-1}{2})_{\frac{d+1}{2}}}{(\frac{d}{2})_{\frac{d+1}{2}}(\frac{d+1}{2})!} \\ &= -\frac{L_d r^{d+1}}{1+d} \frac{\Gamma(\frac{d}{2})\Gamma(d+1)}{\Gamma(\frac{d+1}{2})\Gamma(d+\frac{1}{2})} + L_d^* r^{d+1}, \end{aligned} \tag{2.3.8}$$

where the last line depends on (2.2.3), and where, after some simplification,

$$L_d^* = L_d \frac{(-1)^{\frac{d+1}{2}}(-\frac{1}{2})_{\frac{d+1}{2}}}{(1+d)(\frac{d}{2})_{\frac{d+1}{2}}}. \tag{2.3.9}$$

The presence of the $L_d^* r^{d+1}$ term in (2.3.8) is the main departure from the analysis of Section 2.2. Recalling (2.3.4), we find

$$L_d = |\partial S(0,1)| = \frac{-\pi^{\frac{d-1}{2}}}{(-\frac{1}{2})_{\frac{d+1}{2}}}, \tag{2.3.10}$$

where the final equality may be deduced by converting the integral to spherical polar coordinates and repeatedly using (2.2.22). Hence,

$$L_d^* = \frac{(-\pi)^{\frac{d-1}{2}}}{(1+d)(\frac{d}{2})_{\frac{d+1}{2}}}, \tag{2.3.11}$$

a non-zero constant only depending on d. It follows from (2.3.5), (2.3.6) and (2.3.8) that

$$I = I_0 + I_1 = L_d^* r^{d+1} + \frac{L_d}{1+d} \sum_{k=0}^{\frac{d-1}{2}} \frac{(-\frac{1}{2})_k(\frac{-d-1}{2})_k}{(\frac{d}{2})_k k!} r^{2k} M^{-2k+d+1}, \tag{2.3.12}$$

the other two terms, as in Section 2.2, exactly cancelling because $\Gamma\left(\frac{d+3}{2}\right) = \left(\frac{d+1}{2}\right)\Gamma\left(\frac{d+1}{2}\right)$. We recall that for absolute integrability we need (2.2.41). From (2.2.6) and (2.3.1*b*), this is the condition

$$\sum_{j=1}^{l} \mu_j r_j^{2k} = 0, \quad k = 0, 1, \ldots, \tfrac{d-1}{2}. \tag{2.3.13}$$

It then follows from the finiteness of the sum (2.3.1*b*) that $\tilde{\psi}(s)$ is identically zero for $s \geq \max\{r_j : j = 1, 2, \ldots, l\}$. So, when $\tilde{\psi}(\|x\|)$ is absolutely integrable, (2.3.12) implies

$$\int_{\mathcal{R}^d} \tilde{\psi}(\|x\|)\, dx = L_d^* \sum_{j=1}^{l} \mu_j r_j^{d+1}. \tag{2.3.14}$$

Now the Vandermonde matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1^2 & r_2^2 & \cdots & r_{\frac{d+3}{2}}^2 \\ r_1^4 & r_2^4 & \cdots & r_{\frac{d+3}{2}}^4 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{d+1} & r_2^{d+1} & \cdots & r_{\frac{d+3}{2}}^{d+1} \end{pmatrix} \tag{2.3.15}$$

is always non-singular for distinct $\{r_j : j = 1, 2, \ldots, \frac{d+3}{2}\}$. Hence, if $l = \frac{d+3}{2}$ and if the coefficients $\{\mu_j : j = 1, 2, \ldots, \frac{d+3}{2}\}$ satisfy (2.3.13) and are not all zero, then expression (2.3.14) is non-zero, as required. This completes the proof of Theorem 2-9. ■

It is interesting to ask whether, in this minimal case (in the sense of having the fewest possible number of rings) the function $\tilde{\psi}$ is always of one sign. To answer we recall from the remark following (2.3.13) that

$$\tilde{\psi}(s) = 0 \text{ for } s \geq \max\{r_j : j = 1, 2, \ldots, \tfrac{d+3}{2}\}. \tag{2.3.16}$$

Further, it can be deduced from (2.2.6) and (2.2.18) that, $\tilde{\psi}(\|x\|)$ is $d-1$ times continuously differentiable on $\mathcal{R}^d$. Consider $\tilde{\psi}(\|x\|)$ on a bi-infinite line through the origin and denote this function by $P : \mathcal{R} \to \mathcal{R}$. So P is a $d-1$ times continuously differentiable even function with d-th derivative discontinuities only at the $d+3$ points $\{\pm r_j : j = 1, 2, \ldots, \frac{d+3}{2}\}$. In the $\mathcal{R}^3$ case (when $d = 3$) (2.3.1) implies that the function $sP(s)$ is a piecewise cubic polynomial and hence a cubic spline, zero outside the range $[-\max\{r_j\}, \max\{r_j\}]$. Now the number of zeros may be estimated by the formula (Powell 1981, Theorem 19.1),

$$\text{no. of zeros} \;\leq\; \text{no. of intervals} \;-\; \text{degree of spline} \;-1,$$

which shows that $sP(s)$ has at most one zero in the range. Since there is a zero of the odd function $sP(s)$ at the origin, $P(s)$ takes values of only one sign and hence so does $\tilde{\psi}(s)$.

On extending the method to $\mathcal{R}^5$, $P(s)$ may be defined as before, except that this time $sP(s)$ is a piecewise rational function and $s^3P(s)$ has degree 7, so it is not a quintic spline. However, using (2.3.1) for the form of $\tilde{P}(s) = sP(s)$, it is seen that, if $0 < r_1 < r_2 < r_3 < r_4$, we have fourth derivatives of the form

$$\tilde{P}^{(4)}(s) = \begin{cases} A_1 s & \text{if} \quad |s| \le r_1; \\ A_2 s \pm B_2 s^{-6} & \text{if } r_1 \le |s| \le r_2; \\ A_3 s \pm B_3 s^{-6} & \text{if } r_2 \le |s| \le r_3; \\ A_4 s \pm B_4 s^{-6} & \text{if } r_3 \le |s| \le r_4, \end{cases} \tag{2.3.17}$$

where $\{A_j : j = 1,2,3,4\}$ and $\{B_j : j = 2,3,4\}$ are constants and the positive signs are taken for s positive and negative signs for s negative. Now, as $\tilde{P}(s) = 0$ for $|s| \ge r_4$, it follows that $\tilde{P}^{(4)}(s)$ has no zeros in the two intervals $r_3 \le |s| < r_4$, and, by the expression above, at most one in each of the other five intervals. From this estimate and the finite support of $\tilde{P}$ it can be deduced (by repeatedly using the mean value theorem) that $\tilde{P}(s)$ has at most one zero in the range $(-r_4, r_4)$. However, there is a zero at the origin, and hence $\tilde{\psi}(s)$ takes values of only one sign.

Unfortunately this argument does not generalise to higher dimensional spaces, but the result may still hold.

We now return to the question of finding a function $\psi(x)$ as defined in (2.2.5), and satisfying (2.1.6) and (2.1.7). So far we have only considered the radially symmetric case and, although we have shown that the conditions of Theorem 2-6 can hold and so we do obtain a local uniform convergence result, the question of the existence of a suitable $\psi(x)$ is still open. First we consider the conditions on $\psi(x)$ which will ensure absolute integrability. Recalling the discussion in Section 2.2, in particular (2.2.16), it was found to be sufficient that

$$\sum_{j=1}^{l} \mu_j P_{d+1}(x_j) = 0 \tag{2.3.18}$$

where P_{d+1} is any polynomial of degree at most $d+1$. However, this implies

$$\sum_{j=1}^{l} \mu_j \|x_j\|^{2k} = 0, \quad k = 0, 1, \ldots, \tfrac{d+1}{2}, \tag{2.3.19}$$

and then equations (2.2.5), (2.2.7), (2.3.2) and (2.3.12) yield that, for M sufficiently large,

$$\int_{S(0,M)} \psi(x)\,dx = \sum_{j=1}^{l} \mu_j \int_{S(0,M)} \|x - x_j\|\,dx = \sum_{j=1}^{l} \mu_j \int_{S(0,M)} \sigma(\|x_j\|, \|x\|)\,dx = 0. \tag{2.3.20}$$

This observation gives the following theorem:

Theorem 2-10. *Suppose* $\{\mu_j : j = 1, 2, \ldots, l\}$ *and* $\{x_j \in \mathcal{R}^d : j = 1, 2, \ldots, l\}$ *are chosen to satisfy condition* (2.3.19). *Then, for* $\psi(x)$ *defined in* (2.2.5), *we have*

$$\int_{\mathcal{R}^d} \psi(x)\, dx = 0. \quad \blacksquare$$

Although Theorem 2-9 assumes spherical symmetry, Theorems 2-9 and 2-10 suggest that the conditions (2.2.16) are not necessary for the absolute integrability of the function (2.2.5). Indeed in the spherically symmetric case, the only conditions for absolute integrability are (2.3.13), many of the conditions analogous to (2.2.16) being redundant because of the symmetry. We consider the derivation of these conditions from a series expansion for $\tilde{\psi}(\|x\|)$ derived from (2.2.6), (2.2.7) and (2.2.8) which gives

$$\tilde{\psi}(\|x\|) = \sum_{j=1}^{l} \frac{\mu_j}{|\partial S(0, r_j)|} \int_{\partial S(0,r_j)} \|x\| \left(1 - \frac{2x.x_j}{\|x\|^2} + \frac{\|x_j\|^2}{\|x\|^2}\right)^{\frac{1}{2}} dx_j. \tag{2.3.21}$$

For large $\|x\|$ this may be expanded into an asymptotic series in $\|x\|$ and integrated term by term. From (2.2.12), the term in $O(\|x\|^{-d})$ is

$$P^*_{d+1}(x) = \sum_{j=1}^{l} \frac{\mu_j}{|\partial S(0, r_j)|} \int_{\partial S(0,r_j)} Q_{d+1}(x, x_j)\, dx_j, \tag{2.3.22}$$

and P^*_{d+1} must be identically zero due to condition (2.2.15). However, substituting the definition (2.2.13) into (2.3.22), noting that $|\beta| = d + 1$ in this case, and defining

$$K_\beta = \frac{1}{|\partial S(0, 1)|} \int_{\partial S(0,1)} y^\beta\, dy, \tag{2.3.23}$$

we find

$$P^*_{d+1}(x) = \left(\sum_{j=1}^{l} \mu_j r_j^{d+1}\right) \sum_{\{\alpha,\beta \in (\mathcal{Z}^+)^d : |\alpha| = |\beta| = d+1\}} A_{\alpha,\beta} K_\beta x^\alpha, \tag{2.3.24}$$

and the term in brackets is non-zero in Theorem 2-9 and (2.3.14). Therefore, because P^*_{d+1} is identically zero, we must have

$$\sum_{\{\beta \in (\mathcal{Z}^+)^d : |\beta| = d+1\}} A_{\alpha,\beta} K_\beta = 0, \tag{2.3.25}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = d+1$. Viewing $A_{\alpha,\beta}$ as a square matrix, and noting that K_β is positive for some values of β ($|\beta| = d + 1$ allows every non-zero component of β to be even), equation (2.3.25) states that $A_{\alpha,\beta}$ is a singular matrix and K_β is an element of the null space.

This information will be used to find parameters that annihilate the $O(\|x\|^{-d})$ term in the expansion of $\psi(x)$. Indeed, (2.2.12) and (2.2.13) show that the condition for this term to be zero is that, for all $\alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| = d+1$,

$$\sum_{\{\beta \in (\mathcal{Z}^+)^d : |\beta| = d+1\}} A_{\alpha,\beta} \left(\sum_{j=1}^{l} \mu_j x_j^\beta \right) = 0. \tag{2.3.26}$$

Therefore it is sufficient to satisfy

$$\sum_{j=1}^{l} \mu_j x_j^\beta = C K_\beta, \tag{2.3.27}$$

for some constant C and all $\beta \in (\mathcal{Z}^+)^d \ : \ |\beta| = d+1$, which is possible for large l because different multiples of powers of components of x are linearly independent functions from $\mathcal{R}^d$ to $\mathcal{R}$. Further, if we also satisfy the condition

$$\sum_{j=1}^{l} \mu_j P_d(x_j) = 0 \tag{2.3.28}$$

for every polynomial P_d of degree at most d, then $\psi(x) = O(\|x\|^{-d-1})$ for large $\|x\|$, which gives the required absolute integrability. In this case (2.1.48) and (2.3.14) imply

$$\int_{\mathcal{R}^d} \psi(x)\, dx = L_d^* \sum_{j=1}^{l} \mu_j \|x_j\|^{d+1}. \tag{2.3.29}$$

We regard $\|x_j\|^{d+1}$ as a polynomial in the components of x_j, in order to apply equation (2.3.27), where K_β has the value (2.3.23). Thus we obtain

$$\int_{\mathcal{R}^d} \psi(x)\, dx = \frac{L_d^* C}{|\partial S(0,1)|} \int_{\partial S(0,1)} \|y\|^{d+1}\, dy = L_d^* C, \tag{2.3.30}$$

which is non-zero for $C \neq 0$. In particular we have $\int_{\mathcal{R}^d} \psi(x)\, dx = 1$ if, from (2.3.11),

$$C = \frac{1}{L_d^*} = \frac{(1+d)(\frac{d}{2})_{\frac{d+1}{2}}}{(-\pi)^{\frac{d-1}{2}}}. \tag{2.3.31}$$

In this case, from (2.3.10), (2.3.23) and (2.3.27), our other conditions are that for $\beta \in (\mathcal{Z}^+)^d$: $|\beta| = d+1$,

$$\begin{aligned} \sum_{j=1}^{l} \mu_j x_j^\beta &= \frac{(1+d)(\frac{d}{2})_{\frac{d+1}{2}}(-\frac{1}{2})_{\frac{d+1}{2}}}{(-\pi)^{\frac{d-1}{2}}\left(-(\pi)^{\frac{d-1}{2}}\right)} \int_{\partial S(0,1)} y^\beta\, dy \\ &= \frac{(-1)^{\frac{d+1}{2}}(1+d)(-\frac{1}{2})_{d+1}}{\pi^{d-1}} \int_{\partial S(0,1)} y^\beta\, dy. \end{aligned} \tag{2.3.32}$$

This analysis proves the following result:

Theorem 2-11. *When d is odd, there exist functions $\psi(x)$ of the form* (2.2.5) *which satisfy* (2.1.6) *and* (2.1.7). ∎

The question arises whether these functions can ever be of compact support. The answer is negative, for, consider $\psi(x)$ on any line not passing through any point x_j. We denote this function from $\mathcal{R}$ to $\mathcal{R}$ by g and note that it has the form

$$g(x) = \sum_{j=1}^{l} \mu_j (c_j^2 + (x - r_j)^2)^{\frac{1}{2}}, \qquad x \in \mathcal{R}, \tag{2.3.33}$$

where $c_j > 0$, $j = 1, 2, \ldots, l$. g can be extended to a complex analytic function $\tilde{g}$ on $\{z \in \mathcal{C} : |\mathrm{Im}(z)| < \min\{c_j : j = 1, 2, \ldots, l\}\}$ with

$$\tilde{g}(z) = \sum_{j=1}^{l} \mu_j (c_j^2 + (z - r_j)^2)^{\frac{1}{2}}. \tag{2.3.34}$$

Indeed in this range $\mathrm{Re}(c_j^2 + (z - r_j)^2) > 0$, so we can define $(re^{i\theta})^{\frac{1}{2}} = r^{\frac{1}{2}} e^{i\theta/2}$ for $-\pi/2 < \theta < \pi/2$.

If ψ had compact support, then certainly $\tilde{g}(z) = 0$ for $\mathrm{Im}(z) = 0$ and $\mathrm{Re}(z)$ sufficiently large and hence, by the principle of isolated zeros for analytic functions, $\tilde{g}(z) = 0$ for all z with $\mathrm{Im}(z) = 0$. Hence $\psi(x) = 0$ for all $x \notin \{x_j : j = 1, 2, \ldots, l\}$. So, by continuity, $\int_{\mathcal{R}^d} \psi(x)\, dx = 0$, a contradiction.

The final problem we address in this section is that of constructing an absolutely integrable function $\psi(x)$ in three dimensions satisfying $\int_{\mathcal{R}^3} \psi(x)\, dx = 1$. We let $x_j = (a_j, b_j, c_j)^T$, $j = 1, 2, \ldots, l$, and then it is sufficient, from (2.3.28), that

$$\sum_{j=1}^{l} \mu_j P_3(a_j, b_j, c_j) = 0 \tag{2.3.35}$$

for any polynomial P_3 of degree at most 3, and, from (2.3.23) and (2.3.27), we may take all the fourth order moments to be zero except for

$$\sum_{j=1}^{l} \mu_j a_j^4 = \sum_{j=1}^{l} \mu_j b_j^4 = \sum_{j=1}^{l} \mu_j c_j^4 = -3/\pi,$$

$$\sum_{j=1}^{l} \mu_j a_j^2 b_j^2 = \sum_{j=1}^{l} \mu_j a_j^2 c_j^2 = \sum_{j=1}^{l} \mu_j b_j^2 c_j^2 = -1/\pi. \tag{2.3.36}$$

These conditions are derived from (2.3.32) and the fact that $y = (y_1, y_2, y_3)^T$ satisfies

$$\int_{\partial S(0,1)} y_1^4\, dy = 4\pi/5, \qquad \int_{\partial S(0,1)} y_1^2 y_2^2\, dy = 4\pi/15, \tag{2.3.37}$$

and, by symmetry, all other fourth degree moments, not calculable by permuting suffices, are zero.

Thus there is a function $\psi(x)$ with $l = 15$:

$$\begin{array}{ll}
\mu_j & x_j \\
-8/\pi & (0,0,0); \\
-1/\pi & (0,0,\pm1),\ (0,\pm1,0),\ (\pm1,0,0); \\
4/\pi & (\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}),\ (-\tfrac{1}{2},-\tfrac{1}{2},-\tfrac{1}{2}),\ (-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}),\ (\tfrac{1}{2},-\tfrac{1}{2},\tfrac{1}{2}); \\
-1/2\pi & (1,1,-1),\ (-1,-1,-1),\ (-1,1,1),\ (1,-1,1).
\end{array} \tag{2.3.38}$$

Also a function with $l = 21$:

$$\begin{array}{ll}
\mu_j & x_j \\
-17/4\pi & (0,0,0); \\
1/\pi & (0,0,\pm1),\ (0,\pm1,0),\ (\pm1,0,0); \\
-1/8\pi & (0,0,\pm2),\ (0,\pm2,0),\ (\pm2,0,0); \\
-1/8\pi & (\pm1,\pm1,\pm1).
\end{array} \tag{2.3.39}$$

Also a function with $l = 35$:

$$\begin{array}{ll}
\mu_j & x_j \\
-49/8\pi & (0,0,0); \\
71/48\pi & (0,0,\pm1),\ (0,\pm1,0),\ (\pm1,0,0); \\
-25/96\pi & (0,0,\pm2),\ (0,\pm2,0),\ (\pm2,0,0); \\
1/48\pi & (0,0,\pm3),\ (0,\pm3,0),\ (\pm3,0,0); \\
-1/6\pi & (\pm1,\pm1,\pm1); \\
1/384\pi & (\pm2,\pm2,\pm2).
\end{array} \tag{2.3.40}$$

It may easily be checked that these functions satisfy all the required conditions. We also note that (2.3.39) satisfies

$$\sum_{j=1}^{l} \mu_j \tilde{P}_t(x_j) = 0, \tag{2.3.41}$$

for any homogeneous polynomial $\tilde{P}_t$ of degree t, for $t = 5$. The coefficients (2.3.40) satisfy the equations for $t = 5, 6, 7$, properties which will prove useful in subsequent chapters.

CHAPTER 3 : POLYNOMIAL REPRODUCTION PART I

Section 3.1 : Polynomial Preservation

One of the main conclusions of this dissertation is the accuracy of radial basis function approximations to suitably smooth functions. Anticipating, for now, the results of Chapter 5 where we show that the rate of convergence depends on the maximum degree of polynomial that can be reproduced, we set out to find this maximum possible degree. We must first define what we mean by polynomial preservation and reproduction.

We shall be working on the infinite regular grid formed by all the points in d-dimensional space all of whose components are integers, which we shall denote by $\mathcal{Z}^d$.

We let Π_m be the space of polynomials from $\mathcal{R}^d$ to $\mathcal{R}$ of total degree at most m:

$$\Pi_m = \left\{ \sum_{\{\beta\in(\mathcal{Z}^+)^d : |\beta|\le m\}} A_\beta x^\beta, \qquad x \in \mathcal{R}^d \right\}, \tag{3.1.1}$$

where each $A_\beta \in \mathcal{R}$ and we are using the multi-index notation defined at the end of the introduction. We define the **degree** of such a polynomial to be the maximum value of $|\beta|$ that satisfies $A_\beta \neq 0$.

In the case where $P \in \Pi_m$ and the function ψ is chosen so that the sum is absolutely convergent, we define

$$s(x) = \sum_{z\in\mathcal{Z}^d} P(z)\psi(x-z), \qquad x \in \mathcal{R}^d. \tag{3.1.2}$$

In this case we say that ψ **preserves** polynomials of degree m if $P \in \Pi_m \Rightarrow s \in \Pi_m$, the degree of s being at most the degree of P, and we say that ψ **reproduces** polynomials of degree m if $P \in \Pi_m \Rightarrow s \equiv P$.

We shall concentrate, as in most of Chapter 2, on the case $\phi(r) = r$ and so we shall take

$$\psi(x) = \sum_{j=1}^{l} \mu_j \|x - x_j\|, \qquad x \in \mathcal{R}^d. \tag{3.1.3}$$

We impose the condition

$$x_j \in \mathcal{Z}^d, \quad j = 1, 2, \ldots, l, \tag{3.1.4}$$

which is fundamental to our analysis of polynomial preservation and reproduction. This is not a great restriction for we see that examples (2.3.39) and (2.3.40) both satisfy this condition and we can easily arrange that (2.3.38) does so too by multiplying each point x_j by 2 and dividing each value μ_j by 16. We shall prove some polynomial reproduction properties in this chapter by direct methods (i.e. they do not employ Fourier transforms or generalised functions). It will be seen that some of the techniques can easily be extended to radial basis functions other than $\phi(r) = r$. We shall not concentrate on attempting to give the most general formulation of these results as Chapter 4 presents a more general and powerful technique that does depend on Fourier transforms and generalised functions.

First we return to the question of finding a function ψ (3.1.3) which decays sufficiently fast so that the sum (3.1.2) is absolutely convergent. A sufficient condition is that

$$\psi(x) = O(\|x\|^{-d-1-m}) \text{ as } \|x\| \to \infty. \tag{3.1.5}$$

We shall also add the condition

$$\int_{\mathcal{R}^d} \psi(x)\, dx = 1, \tag{3.1.6}$$

which will prove essential in order to reproduce a constant. Our analysis of Sections 2.2 and 2.3 shows that we must take d odd. We recall from (2.2.14), (2.3.28) and (2.3.32) that conditions (3.1.5) and (3.1.6) are satisfied if

$$\sum_{j=1}^{l} \mu_j \tilde{P}_t(x_j) = 0, \qquad t = 0, 1, \ldots, d, d+2, d+3, \ldots, d+1+m, \tag{3.1.7}$$

$\tilde{P}_t$ being any homogeneous polynomial of degree t, and if, for all $\beta \in (\mathcal{Z}^+)^d : |\beta| = d+1$, we have

$$\sum_{j=1}^{l} \mu_j x_j^\beta = \frac{(-1)^{\frac{d+1}{2}}(1+d)(-\frac{1}{2})_{d+1}}{\pi^{d-1}} \int_{\partial S(0,1)} y^\beta\, dy. \tag{3.1.8}$$

In this section it is proved that any ψ of the form (3.1.3), which satisfies conditions (3.1.4), (3.1.7) and (3.1.8), preserves polynomials of degree m, so long as $m \le d$. In Section 3.2 we go further and prove that such a ψ reproduces polynomials of degree m, so long as $m \le d$. Polynomial preservation is a consequence of the following two lemmas:

Lemma 3-1. *Let d be odd, $P \in \Pi_m$ and ψ (3.1.3) satisfy conditions (3.1.4), (3.1.7) and (3.1.8). In this case the sum (3.1.2) for s is absolutely convergent and further there is a polynomial of degree at most the degree of P which interpolates $\{s(z) : z \in \mathcal{Z}^d\}$.*

Proof. We have already remarked that a function ψ satisfying (3.1.7) and (3.1.8) also satisfies (3.1.5). Therefore the right hand side of (3.1.2) is absolutely convergent as also are all sums

$$\sum_{y \in \mathcal{Z}^d} y^\alpha \psi(y), \qquad \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m. \tag{3.1.9}$$

For $x \in \mathcal{Z}^d$, we make the change of variables $y = x - z$ in (3.1.2). Thus we obtain

$$\begin{aligned} s(x) &= \sum_{y \in \mathcal{Z}^d} P(x-y)\psi(y) \\ &= \sum_{y \in \mathcal{Z}^d} \sum_{\{\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m\}} \frac{1}{\alpha!} \frac{\partial^\alpha P(x)}{\partial x^\alpha} (-y)^\alpha \psi(y) \\ &= \sum_{\{\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m\}} \frac{1}{\alpha!} \frac{\partial^\alpha P(x)}{\partial x^\alpha} \sum_{y \in \mathcal{Z}^d} (-y)^\alpha \psi(y), \qquad x \in \mathcal{Z}^d, \end{aligned} \tag{3.1.10}$$

the middle line expanding $P \in \Pi_m$ about x and the change of order of summation in the last line being justified because all sums are absolutely convergent. This final expression is just a polynomial in x of degree at most that of P, which completes the proof. ∎

Lemma 3-2. *Let d be odd, $P \in \Pi_m$ with $m \le d$, and ψ (3.1.3) satisfy conditions (3.1.4), (3.1.7) and (3.1.8). In this case the function defined by (3.1.2), namely $\{s(x) : x \in \mathcal{R}^d\}$, is a polynomial of degree at most that of P.*

Proof. Suppose we are given $\epsilon > 0$. Take any $x \in \mathcal{R}^d$ and $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = d + m + 1$. Recalling the definition of ψ (3.1.3), we let

$$q = \max\{\|x_j\|_\infty : j = 1, 2, \ldots, l\}. \tag{3.1.11}$$

Now, for

$$R \ge \max(1/\epsilon, 2(\|x\| + q)), \tag{3.1.12}$$

we define

$$\Delta_R = \{y \in \mathcal{R}^d : \|y\|_\infty \le R\}, \tag{3.1.13}$$

$$s_R(y) = \sum_{z \in \mathcal{Z}^d \cap \Delta_R} P(z)\psi(y-z), \qquad y \in \mathcal{R}^d, \tag{3.1.14}$$

and

$$\bar{s}_R(y) = \sum_{z \in \mathcal{Z}^d \setminus \Delta_R} P(z)\psi(y-z), \qquad y \in \mathcal{R}^d. \tag{3.1.15}$$

We shall show that both $\frac{\partial^\alpha s_R(x)}{\partial x^\alpha}$ and $\frac{\partial^\alpha \bar{s}_R(x)}{\partial x^\alpha}$ can be made arbitrarily small, for suitable choice of R. It will follow that $\frac{\partial^\alpha s(x)}{\partial x^\alpha}$ is zero, for all such α, and hence s is a polynomial. The bound on the degree of s is obtained from Lemma 3-1.

Because $\|y - z\|$ is infinitely differentiable as a function of y away from $y = z$, $\bar{s}_R(y)$ is infinitely differentiable for $\|y\|_\infty \le R - q$ and in particular at $y = x$ by (3.1.12). Moreover, because ψ satisfies (3.1.5) (see the proof of Lemma 3-1), it follows from (2.2.17) and our choice of α that

$$\left| \frac{\partial^\alpha \psi(y)}{\partial y^\alpha} \right| \le A\|y\|^{-2d-2m-2}, \qquad \|y\|_\infty > q. \tag{3.1.16}$$

Combining these two observations we find

$$\begin{aligned}\left|\frac{\partial^\alpha \bar{s}_R(x)}{\partial x^\alpha}\right| &\le \sum_{z\in\mathcal{R}^d\setminus\Delta_R} |P(z)| \left|\frac{\partial^\alpha \psi(x-z)}{\partial x^\alpha}\right| \\ &\le \sum_{z\in\mathcal{R}^d\setminus\Delta_R} A'\|z\|^m\, A\|x-z\|^{-2d-2m-2} \\ &\le A'A\,2^{2d+2m+2} \sum_{z\in\mathcal{R}^d\setminus\Delta_R} \|z\|^{-2d-m-2}, \end{aligned} \qquad (3.1.17)$$

where the last line uses $\|x-z\| \ge \|z\| - \|x\| \ge \frac{1}{2}\|z\|$ which follows from (3.1.12). Hence we can find $\tilde{R}$ such that, for all $R \ge \tilde{R}$,

$$\left|\frac{\partial^\alpha \bar{s}_R(x)}{\partial x^\alpha}\right| < \epsilon. \qquad (3.1.18)$$

We add this condition as a further requirement on R and so we choose

$$R = \max(1/\epsilon, 2(\|x\|+q), \tilde{R}). \qquad (3.1.19)$$

Using (3.1.3) and (3.1.4) we may rearrange the finite sum (3.1.14) for $s_R(y)$ to the form

$$s_R(y) = \sum_{z'\in\mathcal{Z}^d} \mu(z')\|y-z'\|, \qquad y\in\mathcal{R}^d, \qquad (3.1.20)$$

where

$$\mu(z') = \sum_{\{x_j\,:\,\|z'-x_j\|_\infty\le R,\; j\in[1,l]\}} \mu_j P(z'-x_j), \qquad z'\in\mathcal{Z}^d. \qquad (3.1.21)$$

Remembering (3.1.11), this expression yields that

$$\mu(z') = 0 \text{ for } \|z'\|_\infty > R+q, \qquad (3.1.22)$$

for the sum is empty, and

$$\mu(z') = \sum_{j=1}^{l} \mu_j P(z'-x_j) \text{ for } \|z'\|_\infty \le R-q, \qquad (3.1.23)$$

because all possible terms are included. Hence, from (3.1.7) and $P\in\Pi_m$ with $m\le d$, we find that

$$\mu(z') = 0 \text{ for } \|z'\|_\infty \le R-q. \qquad (3.1.24)$$

This is the point where we have used both the restrictions (3.1.4) and $m\le d$. Equations (3.1.19) and (3.1.21) also show that there exists a constant A_1 independent of z' and R such that

$$|\mu(z')| \le A_1 R^m, \qquad R-q < \|z'\|_\infty \le R+q. \qquad (3.1.25)$$

Hence, because equations (3.1.19) and (3.1.24) imply that s_R is infinitely differentiable at x, we have

$$\left|\frac{\partial^\alpha s_R(x)}{\partial x^\alpha}\right| = \left|\sum_{\{z' \in \mathcal{Z}^d : R-q<\|z'\|_\infty \le R+q\}} \mu(z') \frac{\partial^\alpha \|x-z'\|}{\partial x^\alpha}\right|$$
$$\le \sum_{\{z' \in \mathcal{Z}^d : R-q<\|z'\|_\infty \le R+q\}} A_1 R^m A_2 \|x-z'\|^{-d-m}, \qquad (3.1.26)$$

for some constant A_2 independent of z' and R, because $|\alpha| = d+m+1$. So, noting that (3.1.19) implies $\|x-z'\| \ge \|z'\| - \|x\| \ge R - q - \|x\| \ge \frac{1}{2}R$, it follows that

$$\left|\frac{\partial^\alpha s_R(x)}{\partial x^\alpha}\right| \le A_1 A_2 2^{d+m} \sum_{\{z' \in \mathcal{Z}^d : R-q<\|z'\|_\infty \le R+q\}} R^{-d} \qquad (3.1.27)$$
$$\le A_3 R^{-1}, \qquad (3.1.28)$$

for some constant A_3 independent of R, because the number of points in the sum in (3.1.27) is bounded by a constant multiple of R^{d-1}.

Recalling that both s_R and $\bar{s}_R$ are infinitely differentiable at x we find that so is s. Hence we may combine (3.1.18), $R \ge 1/\epsilon$ which follows from (3.1.19), and (3.1.28) to yield

$$\left|\frac{\partial^\alpha s(x)}{\partial x^\alpha}\right| < (1 + A_3)\epsilon. \qquad (3.1.29)$$

However, because ϵ is arbitrary, the above expression is exactly zero. This is true for all $x \in \mathcal{R}^d$ and all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = d+m+1$, and hence $s \in \Pi_{d+m}$. Combining this observation with Lemma 3-1 shows that in fact s is a polynomial of degree at most that of P and in particular $s \in \Pi_m$ which completes the proof of the lemma. ■

Lemma 3-2 shows that such a function ψ preserves polynomials of degree m, so long as $m \le d$. We apply the result to the functions ψ for $d = 3$ given at the end of Chapter 2. If we rescale the function (2.3.38) as suggested in the remark after (3.1.4) then it satisfies (3.1.4), (3.1.7) (with $m = 0$) and (3.1.8) and hence it preserves constant functions. Similarly the function (2.3.39), satisfying (3.1.4), (3.1.7) (with $m = 1$) and (3.1.8) preserves all linear functions, while (2.3.40) preserves all cubics, the maximum degree allowed by Lemma 3-2.

Section 3.2 : Polynomial Reproduction

In this section we shall progress from the polynomial preservation result of Section 3.1 to deduce a polynomial reproduction result. We work under the same conditions as in Section 3.1, so we have a function ψ (3.1.3) satisfying (3.1.4), (3.1.7) and (3.1.8) and we define $s(x)$ as in (3.1.2). We wish to prove that $P \in \Pi_m \Rightarrow s \equiv P$. We deduce from Lemma 3-2 that the expression (3.1.10) for $s(x)$ is valid for all $x \in \mathcal{R}^d$. Hence if, for $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq m$, we define

$$C_\alpha = \sum_{y \in \mathcal{Z}^d} (-y)^\alpha \psi(y), \tag{3.2.1}$$

it is sufficient for $s \equiv P$ that $C_0 = 1$ and $C_\alpha = 0$ for $\alpha \in (\mathcal{Z}^+)^d : 0 < |\alpha| \leq m$. This condition is also necessary if the result is to be true for all $P \in \Pi_m$. We estimate the values C_α for $\alpha \in (\mathcal{Z}^+)^d$: $|\alpha| \leq m$ by a sequence of results:

Lemma 3-3. *Let ψ (3.1.3) satisfy (3.1.4), (3.1.7) and (3.1.8). Then C_α, as defined in (3.2.1), has the value*

$$C_\alpha = \int_{\mathcal{R}^d} (-w)^\alpha \psi(w)\, dw. \tag{3.2.2}$$

Proof. Let $w \in \mathcal{R}^d$ and $x \in \mathcal{Z}^d$. From (3.1.2), we find

$$\begin{aligned} s(x+w) &= \sum_{z \in \mathcal{Z}^d} P(z)\psi(x+w-z) \\ &= \sum_{y \in \mathcal{Z}^d} P(x-y)\psi(y+w) \\ &= \sum_{\{\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq m\}} \frac{1}{\alpha!} \frac{\partial^\alpha P(x+w)}{\partial x^\alpha} \sum_{y \in \mathcal{Z}^d} (-(y+w))^\alpha \psi(y+w), \end{aligned} \tag{3.2.3}$$

the middle line using the change of variables $y = x - z$, and the final one the expansion of $P \in \Pi_m$ about $x + w$. Comparing this equation with (3.1.10) and recalling from Lemma 3-2 that s is a polynomial throughout $\mathcal{R}^d$, we see that

$$C_\alpha = \sum_{y \in \mathcal{Z}^d} (-(y+w))^\alpha \psi(y+w), \tag{3.2.4}$$

for any $w \in \mathcal{R}^d$. Hence, we find that

$$\begin{aligned} C_\alpha &= \int_{\{w \in \mathcal{R}^d : \|w\|_\infty \leq \frac{1}{2}\}} C_\alpha\, dw \\ &= \int_{\{w \in \mathcal{R}^d : \|w\|_\infty \leq \frac{1}{2}\}} \sum_{y \in \mathcal{Z}^d} (-(y+w))^\alpha \psi(y+w)\, dw \\ &= \sum_{y \in \mathcal{Z}^d} \int_{\{w \in \mathcal{R}^d : \|w\|_\infty \leq \frac{1}{2}\}} (-(y+w))^\alpha \psi(y+w)\, dw, \end{aligned} \tag{3.2.5}$$

the change in order of summation and integration being permitted because of the fast decay of ψ (3.1.5) which has been deduced already from (3.1.7) and (3.1.8). Now performing the change of variables $w' = y + w$, we obtain

$$\begin{aligned} C_\alpha &= \sum_{y \in \mathcal{Z}^d} \int_{\{w' \in \mathcal{R}^d : \|w'-y\|_\infty \le \frac{1}{2}\}} (-w')^\alpha \psi(w')\, dw' \\ &= \int_{\mathcal{R}^d} (-w')^\alpha \psi(w')\, dw', \end{aligned} \tag{3.2.6}$$

as required. ■

To evaluate C_α it is sufficient to work with the integral expression given by Lemma 3-3. To do this we establish some further lemmas:

Lemma 3-4. *Let d be odd, let P be any homogeneous polynomial of degree t from $\mathcal{R}^d$ to $\mathcal{R}$ and let $\tilde{w}$ be any* given *point of $\mathcal{R}^d$. In this case the expression*

$$I = I(M) = \int_{\{w \in \mathcal{R}^d : \|w\| \le M\}} \|w - \tilde{w}\|\, P(w)\, dw, \tag{3.2.7}$$

for $M > \|\tilde{w}\|$, is a polynomial in M of degree at most $d + t + 1$.

Proof. By rotating axes, which only transforms P into another homogeneous polynomial of the same degree, we can assume, without loss of generality, that $\tilde{w}$ is on the first coordinate axis, i.e. $\tilde{w} = (\eta, 0, \ldots, 0)^T$, where $\eta = \|\tilde{w}\|$. We also assume, by the linearity of the integral, that P has the form

$$P(w) = w_1^{t_1} \cdots w_d^{t_d}, \qquad w \in \mathcal{R}^d. \tag{3.2.8}$$

By symmetry we have $I \equiv 0$ if any of the integers $\{t_i : i = 2, 3, \ldots, d\}$ are odd. Therefore we may restrict ourselves to the case when these integers are all even, and then $t - t_1 = t_2 + \cdots + t_d$ is even too.

We can change from Cartesian to spherical polar coordinates through the usual transformation

$$\begin{aligned} w_1 &= r\cos\theta_1 \\ w_2 &= r\sin\theta_1 \cos\theta_2 \\ &\vdots \\ w_{d-1} &= r\sin\theta_1 \cdots \sin\theta_{d-2}\cos\theta_{d-1} \\ w_d &= r\sin\theta_1 \cdots \sin\theta_{d-2}\sin\theta_{d-1}. \end{aligned} \tag{3.2.9}$$

In this frame

$$\|w - \tilde{w}\| = (r^2 + \eta^2 - 2r\eta\cos\theta_1)^{\frac{1}{2}}, \tag{3.2.10}$$

where $r = \|w\|$. We also note that on substituting (3.2.8) into (3.2.7) the integrations over $\theta_{d-1}, \theta_{d-2}, \ldots, \theta_2$ can be evaluated and only multiply the value of I by some constant, A say. Hence

$$I = A \int_{r=0}^{M} \int_{\theta_1=0}^{\pi} (r^2 + \eta^2 - 2r\eta\cos\theta_1)^{\frac{1}{2}} (\cos\theta_1)^{t_1} (\sin\theta_1)^{t+d-t_1-2}\, r^{t+d-1}\, d\theta_1\, dr. \tag{3.2.11}$$

This yields the required result in the case $\eta = 0$. Otherwise we make use of the elementary identity

$$r^{t+d-3}(\cos\theta_1)^{t_1}(\sin\theta_1)^{t+d-t_1-3} = (r\cos\theta_1)^{t_1}(r^2 - r^2\cos^2\theta_1)^{\frac{1}{2}(t+d-t_1-3)}, \tag{3.2.12}$$

remembering that $(t + d - t_1 - 3)$ is even. Here we make the substitution

$$r\cos\theta_1 = [(r^2 + \eta^2) - (r^2 + \eta^2 - 2r\eta\cos\theta_1)]/2\eta, \tag{3.2.13}$$

for $\eta \neq 0$, which gives a relation of the form

$$r^{t+d-3}(\cos\theta_1)^{t_1}(\sin\theta_1)^{t+d-t_1-3} = \sum_{k=0}^{t+d-3} q_k(r)(r^2 + \eta^2 - 2r\eta\cos\theta_1)^k, \tag{3.2.14}$$

where each $q_k(r)$ is a polynomial in r. It follows that

$$\begin{aligned} I &= A \int_{r=0}^{M} \int_{\theta_1=0}^{\pi} r^2 \sin\theta_1 \sum_{k=0}^{t+d-3} q_k(r)(r^2 + \eta^2 - 2r\eta\cos\theta_1)^{k+\frac{1}{2}}\, d\theta_1\, dr \\ &= A \int_{r=0}^{M} r \sum_{k=0}^{t+d-3} \frac{q_k(r)}{\eta(2k+3)} \left[(r^2 + \eta^2 - 2r\eta\cos\theta_1)^{k+\frac{3}{2}}\right]_{\theta_1=0}^{\pi} dr \\ &= A \int_{r=0}^{M} r \sum_{k=0}^{t+d-3} \frac{q_k(r)}{\eta(2k+3)} \left[(r+\eta)^{2k+3} - |r-\eta|^{2k+3}\right] dr. \end{aligned} \tag{3.2.15}$$

Thus I is a polynomial in M, for $M > \|\tilde{w}\|$. We also note from (3.2.7), that for some constants A and A_1,

$$\begin{aligned} |I| &\leq \int_{\{w \in \mathcal{R}^d : \|w\| \leq M\}} \|w - \tilde{w}\|\, A\|w\|^t\, dw \\ &\leq A \int_{\{w \in \mathcal{R}^d : \|w\| \leq M\}} (2M)M^t\, dw \\ &\leq A_1 M^{d+t+1}. \end{aligned} \tag{3.2.16}$$

Hence I is a polynomial in M of degree at most $d + t + 1$, for $M > \|\tilde{w}\|$, as required. ∎

We remark that $(1 - 2xz + z^2)^{-\frac{1}{2}}$ is the generating function for the Legendre polynomials and that it is also possible to prove the result using this fact.

Lemma 3-5. *Let d be odd, let P be any homogeneous polynomial of degree t from $\mathcal{R}^d$ to $\mathcal{R}$ and let $\tilde{w}$* be any given *point of $\mathcal{R}^d$. In this case expression (3.2.7), for $M > \|\tilde{w}\|$, has the form*

$$I = I(M, \tilde{w}) = \int_{\{w \in \mathcal{R}^d : \|w\| \leq M\}} \|w - \tilde{w}\| P(w)\, dw = \sum_{k=0}^{d+t+1} \tilde{P}_k(\tilde{w}) M^{d+t+1-k}, \tag{3.2.17}$$

where each $\tilde{P}_k$ is a homogeneous polynomial of degree k.

Proof. Making the change of variables $y = cw$, for some positive constant c, we find

$$\int_{\{w \in \mathcal{R}^d : \|w\| \leq M\}} \|w - \tilde{w}\|\, P(w)\, dw = \int_{\{y \in \mathcal{R}^d : \|y\| \leq cM\}} \|c^{-1}y - \tilde{w}\|\, P(c^{-1}y) c^{-d}\, dy$$
$$= c^{-d-t-1} \int_{\{y \in \mathcal{R}^d : \|y\| \leq cM\}} \|y - c\tilde{w}\|\, P(y)\, dy. \tag{3.2.18}$$

This shows that $I(M, \tilde{w}) = c^{-d-t-1} I(cM, c\tilde{w})$, for all $c > 0$, and hence, in view of Lemma 3-4,

$$I(M, \tilde{w}) = \sum_{k=0}^{d+t+1} f_k(\tilde{w}) M^{d+t+1-k}, \tag{3.2.19}$$

where each f_k, $k = 0, 1, \ldots, d + t + 1$ is a homogeneous function of degree k. To show that f_k is in fact a homogeneous polynomial of degree k we shall need two observations: Firstly we may rewrite $I(M, \tilde{w})$ as

$$I(M, \tilde{w}) = \int_{\{y \in \mathcal{R}^d : \|y + \tilde{w}\| \leq M\}} \|y\|\, P(y + \tilde{w})\, dy. \tag{3.2.20}$$

This shows that, for $\|\tilde{w}\| < M$, I is the integral of a smooth function of $\tilde{w}$ over a region which is smoothly varying with respect to $\tilde{w}$. Hence, for $\|\tilde{w}\| < M$, I as a function of $\tilde{w}$ has all partial derivatives of all orders. Secondly we see that any first order partial derivative of a suitably smooth homogeneous function of positive integer degree k is a homogeneous function of degree $k - 1$ for, differentiating $f_k(c\tilde{w}) = c^k f_k(\tilde{w})$, we obtain

$$c \frac{\partial f_k(c\tilde{w})}{\partial \tilde{w}_l} = c^k \frac{\partial f_k(\tilde{w})}{\partial \tilde{w}_l}, \qquad l = 1, 2, \ldots, d. \tag{3.2.21}$$

Hence a partial derivative of order at least $k+1$ applied to a suitably smooth homogeneous function of degree k gives either a function with a singularity at the origin or the zero function.

Hence applying any partial derivative of order $d + t + 2$ to (3.2.19) we find that the left hand side has no singularity at the origin in view of the first observation and so the right hand side must be identically zero in view of the second observation. Because this is true for all partial derivatives of this order then each f_k must be a homogeneous polynomial of degree k, as required. ■

Theorem 3-6. *Let C_α be defined as in* (3.2.1). *In this case*

$$C_\alpha = \begin{cases} 1 & \text{if } \alpha = 0; \\ 0 & \text{if } \alpha \in (\mathcal{Z}^+)^d \ : \ 0 < |\alpha| \le m. \end{cases} \tag{3.2.22}$$

Proof. Let $\alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| \le m$. We note that $(-w)^\alpha$ is a homogeneous polynomial of degree $|\alpha|$ and so Lemma 3-3, (3.1.3) and Lemma 3-5 yield that

$$\begin{aligned} C_\alpha &= \int_{\mathcal{R}^d} (-w)^\alpha \psi(w)\, dw \\ &= \lim_{M\to\infty} \left(\int_{\{w\in\mathcal{R}^d : \|w\|\le M\}} (-w)^\alpha \psi(w)\, dw \right) \\ &= \lim_{M\to\infty} \left(\sum_{j=1}^{l} \mu_j \int_{\{w\in\mathcal{R}^d : \|w\|\le M\}} (-w)^\alpha \|w - x_j\|\, dw \right) \\ &= \lim_{M\to\infty} \left(\sum_{j=1}^{l} \mu_j \sum_{k=0}^{d+|\alpha|+1} \tilde{P}_k(x_j) M^{d+|\alpha|+1-k} \right), \end{aligned} \tag{3.2.23}$$

where each $\tilde{P}_k$ is a homogeneous polynomial of degree k, depending on α. The limit must exist because C_α (3.2.1), for $\alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| \le m$, has a convergent sum, by virtue of the fast decay of ψ (3.1.5). Hence only the M^0 term of expression (3.2.23) can be non-zero, so we have

$$C_\alpha = \sum_{j=1}^{l} \mu_j \tilde{P}_{d+|\alpha|+1}(x_j), \tag{3.2.24}$$

for some homogeneous polynomial $\tilde{P}_{d+|\alpha|+1}$ of degree $d + |\alpha| + 1$, depending on α. Now (3.1.7), along with $|\alpha| \le m$, yield that $C_\alpha = 0$ for $\alpha \ne 0$. When $\alpha = 0$, Lemma 3-3 shows that

$$C_0 = \int_{\mathcal{R}^d} \psi(w)\, dw, \tag{3.2.25}$$

which we have already chosen to be 1 (3.1.6). Therefore the theorem is true. ■

We recall from our remarks at the beginning of the section that Theorem 3-6 is sufficient to enable us to deduce polynomial reproduction from the polynomial preservation results of Section 3.1. Therefore, combining the results of the two sections we have proved:

Theorem 3-7. *Let ψ* (3.1.3) *satisfy* (3.1.4), (3.1.7) *and* (3.1.8). *In this case ψ reproduces all polynomials of degree m, so long as $m \le d$.* ■

It follows that the conclusions we drew about polynomial preservation for the functions defined at the end of Chapter 2 also hold for polynomial reproduction. In particular we deduce that the function (2.3.40) reproduces all cubic polynomials, the maximum degree allowed by Theorem 3-7.

The question arises whether it may ever be possible to reproduce all polynomials of degree $d+1$ for ψ of the form (3.1.3). We now relax the condition that the points $\{x_j : j = 1, 2, \ldots, l\}$ are to be lattice points, which strengthens our conclusion because the answer to the question is negative. We can find functions ψ of the form (3.1.3) which give a convergent sum in the right hand side of expression (3.1.2) for all polynomials $P \in \Pi_{d+1}$; a sufficient condition is (3.1.5) with $m = d+1$ which will hold for any such ψ satisfying (3.1.7) with $m = d+1$ and (3.1.8).

We let ψ be any function of the form (3.1.3) which reproduces all polynomials of degree d. We shall find a polynomial of degree $d+1$ which is not reproduced by this function.

Equation (2.2.12) shows that $\psi(x)$ has a series expansion for large $\|x\|$. Hence, as polynomials of degree d are reproduced, ψ must certainly decay to zero at least as fast as $\|x\|^{-d-1}$. Hence, because ψ reproduces constants, we have

$$\begin{aligned}
1 = \int_{\{y \in \mathcal{R}^d : \|y\|_\infty \le \frac{1}{2}\}} 1\, dy &= \int_{\{y \in \mathcal{R}^d : \|y\|_\infty \le \frac{1}{2}\}} \sum_{z \in \mathcal{Z}^d} \psi(y-z)\, dy \\
&= \sum_{z \in \mathcal{Z}^d} \int_{\{y \in \mathcal{R}^d : \|y\|_\infty \le \frac{1}{2}\}} \psi(y-z)\, dy \\
&= \sum_{z \in \mathcal{Z}^d} \int_{\{y \in \mathcal{R}^d : \|y'+z\|_\infty \le \frac{1}{2}\}} \psi(y')\, dy' \\
&= \int_{\mathcal{R}^d} \psi(y')\, dy', \qquad (3.2.26)
\end{aligned}$$

the change in order of summation and integration being justified by the speed of decay of ψ. Thus, as the speed of decay also yields that ψ is absolutely integrable, equation (2.3.29) shows that there exists $\beta \in (\mathcal{Z}^+)^d : |\beta| = d+1$ for which

$$\sum_{j=1}^{l} \mu_j x_j^\beta \neq 0. \qquad (3.2.27)$$

The answer to our question is negative because we shall find that the function

$$T(x) = \sum_{z \in \mathcal{Z}^d} z^\beta \psi(x-z), \qquad x \in \mathcal{R}^d, \qquad (3.2.28)$$

is not differentiable at some point in

$$C = \{y \in \mathcal{R}^d : 0 \le y_i < 1, \quad i = 1, 2, \ldots, d\}. \qquad (3.2.29)$$

Therefore ψ fails to reproduce the polynomial $x^\beta \in \Pi_{d+1}$.

Our argument depends on the conditions that arise because ψ reproduces all polynomials of degree at most d. We let $P \in \Pi_d$, we define s and q as in (3.1.2) and (3.1.11) and we choose

$R = q + 1$. We also define Δ_R, s_R and $\bar{s}_R$ as in (3.1.13)–(3.1.15) respectively. By assumption, $s \equiv P$ and so is differentiable throughout C (3.2.29). Further, by our choice of R, $\bar{s}_R$ is also differentiable throughout C and so s_R must also be differentiable there.

For each x_j, $j = 1, 2, \ldots, l$ we let z_j be the point on the integer lattice so that $x_j - z_j \in C$. We define

$$z'_j = x_j - z_j, \qquad j = 1, 2, \ldots, l. \tag{3.2.30}$$

In the case where $\{x_j : j = 1, 2, \ldots, l\}$ lie on the integer lattice we would have each $z'_j = 0$, but this is no longer necessarily the case. Using (3.1.3) we may rearrange the finite sum (3.1.14) for $s_R(x)$ to the form

$$s_R(x) = \sum_{\{y \in \mathcal{R}^d \,:\, y - x_j \in \mathcal{Z}^d,\ \|y - x_j\|_\infty \le R \text{ for some } j \in [1,l]\}} \mu(y)\|x - y\|, \tag{3.2.31}$$

where, in particular,

$$\mu(z'_k) = \sum_{\{j \in [1,l] \,:\, z'_j = z'_k\}} \mu_j P(z'_j - x_j), \quad k = 1, 2, \ldots, l. \tag{3.2.32}$$

However, s_R is differentiable throughout C and so each $\mu(z'_k)$ must be zero, which, on multiplying (3.3.32) by $(z'_k)^\alpha$ for any $\alpha \in (\mathcal{Z}^+)^d$ and summing over all such distinct points, gives

$$\sum_{j=1}^{l} \mu_j P(z'_j - x_j)(z'_j)^\alpha = 0, \tag{3.2.33}$$

for all $P \in \Pi_d$ and $\alpha \in (\mathcal{Z}^+)^d$. This is equivalent to the condition that

$$\sum_{j=1}^{l} \mu_j P(-x_j)(z'_j)^\alpha = 0, \tag{3.2.34}$$

for all $P \in \Pi_d$ and $\alpha \in (\mathcal{Z}^+)^d$.

Now we return to the polynomial x^β. From the function T (3.2.28) we define (similar to (3.1.14) and (3.1.15))

$$T_R(x) = \sum_{z \in \mathcal{Z}^d \cap \Delta_R} z^\beta \psi(x - z), \qquad x \in \mathcal{R}^d, \tag{3.2.35}$$

and

$$\overline{T}_R(x) = \sum_{z \in \mathcal{Z}^d \setminus \Delta_R} z^\beta \psi(x - z), \qquad x \in \mathcal{R}^d. \tag{3.2.36}$$

We may rearrange T_R in the same way as (3.2.31) and obtain

$$T_R(x) = \sum_{\{y \in \mathcal{R}^d \,:\, y - x_j \in \mathcal{Z}^d,\ \|y - x_j\|_\infty \le R \text{ for some } j \in [1,l]\}} \lambda(y)\|x - y\|, \tag{3.2.37}$$

where, similar to (3.2.32),

$$\lambda(z'_k) = \sum_{\{j\in[1,l]\,:\,z'_j=z'_k\}} \mu_j(z'_j - x_j)^\beta, \quad k = 1, 2, \ldots, l. \tag{3.2.38}$$

Adding together the equations for all distinct points z'_k gives

$$\sum_{j=1}^{l} \mu_j(z'_j - x_j)^\beta = \sum_{j=1}^{l} \mu_j \sum_{\{\alpha\in(\mathcal{Z}+)^d\,:\,\alpha\leq\beta\}} \frac{1}{\alpha!} \left.\frac{\partial^\alpha x^\beta}{\partial x^\alpha}\right|_{x=-x_j} (z'_j)^\alpha. \tag{3.2.39}$$

Now, recalling that $\frac{\partial^\alpha x^\beta}{\partial x^\alpha} \in \Pi_d$ for all $\alpha > 0$, we may use (3.2.34) to deduce that the sum of coefficients has the value

$$\sum_{j=1}^{l} \mu_j(-x_j)^\beta. \tag{3.2.40}$$

This is non-zero, by (3.2.27), and hence there must be at least one non-zero $\lambda(z'_k)$ (3.2.38). We recall that $z'_k \in C$ (3.2.30) and so the function T_R is not differentiable throughout C.

However, by the choice of R, the function $\overline{T}_R$ is differentiable there and so we have the desired conclusion that T is not differentiable throughout C, which we have already remarked is suffficient to give a negative answer to our question.

CHAPTER 4 : POLYNOMIAL REPRODUCTION PART II

Section 4.1 : Introduction to Generalised Functions

In a few places in this chapter we shall be using the techniques of generalised functions, as they provide very easy solutions to some questions which we shall encounter. To make the dissertation as self contained as possible we provide a brief introduction to generalised functions and a summary of the main results that we shall be using later in the chapter.

We shall be following the approach of Jones (1982) who provides a very readable account of the subject, yet is not afraid to go into details when the results require it. This book contains all but one of the results we shall be using, the remaining one coming from Gel'fand and Shilov (1964), and is a useful source of information on generalised functions.

Generalised functions enable a rigorous meaning to be given to a class of functions containing members which cannot be treated by the standard analytical techniques. Perhaps the most well known of these is the Dirac delta function. They also enable the conventional Fourier transform to be extended to this wider class of functions. It is the extension of the Fourier transform which we will find particularly useful.

We begin by a series of definitions which will lead to one for generalised functions:

A function $\gamma : \mathcal{R}^d \to \mathcal{R}$ is said to be **good** if it possesses all partial derivatives of all orders everywhere in $\mathcal{R}^d$ and

$$\|x\|^k \frac{\partial^\alpha \gamma(x)}{\partial x^\alpha} \to 0 \text{ as } \|x\| \to \infty, \tag{4.1.1}$$

for all $k \in \mathcal{Z}^+$ and all $\alpha \in (\mathcal{Z}^+)^d$. For example the function $e^{-\|x\|^2}$ is a good function, but $e^{-\|x\|}$ is not because it does not possess all partial derivatives at the origin.

A sequence $\{\gamma_m : m = 1, 2, \ldots\}$ of good functions is said to be **regular** if, for every good γ,

$$\lim_{m\to\infty} \int_{\mathcal{R}^d} \gamma_m(x)\gamma(x)\, dx \tag{4.1.2}$$

exists and is finite.

Two regular sequences which give rise to the same limit for every good γ are said to be **equivalent**. An equivalence class of regular sequences is said to be a **generalised function**.

If $\{\gamma_m\}$ is one regular sequence in the equivalence class then we say that the generalised function is defined by $\{\gamma_m\}$. Further, if

$$\lim_{m\to\infty} \int_{\mathcal{R}^d} \gamma_m(x)\gamma(x)\, dx = \int_{\mathcal{R}^d} g(x)\gamma(x)\, dx, \tag{4.1.3}$$

for every good γ, we say that g is a generalised function defined by $\{\gamma_m\}$.

We say that two generalised functions g_1 and g_2 are equal if

$$\int_{\mathcal{R}^d} g_1(x)\gamma(x)\,dx = \int_{\mathcal{R}^d} g_2(x)\gamma(x)\,dx, \tag{4.1.4}$$

for every good γ.

First we note that many well known functions are also generalised functions. We let K_1 be the function space

$$K_1 = \left\{ f : \mathcal{R}^d \to \mathcal{R} : \int_{\mathcal{R}^d} \frac{|f(x)|}{(1+\|x\|^2)^N}\,dx < \infty \text{ for some } N \in \mathcal{Z}^+ \right\}. \tag{4.1.5}$$

Jones (1982) deduces

Theorem 4-1. *If $f \in K_1$ then f is a generalised function. In particular all continuous functions of at most polynomial growth are generalised functions.*

We now present some simple results of these definitions, without proof, and introduce some further preliminary definitions which will enable us to state some further important theorems.

Suppose that g_1 and g_2 are generalised functions and that $\{\gamma_{1m}\}$ and $\{\gamma_{2m}\}$ are any members of the equivalence classes of regular sequences for g_1 and g_2 respectively. In this case the sequence $\{\gamma_{1m} + \gamma_{2m}\}$ is regular and all such sequences form an equivalence class for the generalised function denoted by $g_1 + g_2$.

Similarly suppose that g is a generalised function and that $\{\gamma_m\}$ is any member of the equivalence class of regular sequences for g. In this case the sequence $\{\gamma_m(ax - y)\}$ is regular for any $y \in \mathcal{R}^d$ and $a \in \mathcal{R}$ and all such sequences form an equivalence class for the generalised function denoted by $g(ax - y)$. We say that g is **homogeneous** of degree t if $g(ax) = a^t g(x)$ for all $x \in \mathcal{R}^d$ and $a \in \mathcal{R}$.

With the same hypotheses as in the preceding paragraph we find that, for every good γ, applying integration by parts yields

$$\int_{\mathcal{R}^d} \frac{\partial \gamma_m(x)}{\partial x_t}\gamma(x)\,dx = -\int_{\mathcal{R}^d} \gamma_m \frac{\partial \gamma(x)}{\partial x_t}\,dx, \qquad m \in \mathcal{Z}^+, \quad t \in [1,d]. \tag{4.1.6}$$

The function $\frac{\partial \gamma(x)}{\partial x_t}$ is still good and so the sequence $\left\{\frac{\partial \gamma_m(x)}{\partial x_t}\right\}$ is regular, for any integer $t \in [1,d]$, and all such sequences form an equivalence class for the generalised function $\frac{\partial g(x)}{\partial x_t}$ defined by

$$\int_{\mathcal{R}^d} \frac{\partial g(x)}{\partial x_t}\gamma(x) = -\int_{\mathcal{R}^d} g(x)\frac{\partial \gamma(x)}{\partial x_t}\,dx, \tag{4.1.7}$$

for any good γ. We call $\frac{\partial g(x)}{\partial x_i}$ a **generalised partial derivative** of g. The definition can be justified, as it can be proved that if both f and its conventional partial derivative $\frac{\partial f(x)}{dx_i}$ are in K_1 then its generalised partial derivative is equal to the conventional partial derivative. As a consequence, it can be deduced that all partial derivatives exist for each generalised function. For instance in $\mathcal{R}$ the function $x^{-3/2}$ is a generalised function as, although not absolutely integrable in a neighbourhood of 0, it is the generalised partial derivative of the function $-2x^{-1/2}$ which is in K_1.

We say that a function $\sigma : \mathcal{R}^d \to \mathcal{R}$ is **fairly good** if σ possesses all partial derivatives of all orders everywhere in $\mathcal{R}^d$ and there exists some $N \in \mathcal{Z}^+$ such that, for all $\alpha \in (\mathcal{Z}^+)^d$,

$$\|x\|^{-N} \frac{\partial^\alpha \sigma(x)}{\partial x^\alpha} \to 0 \text{ as } \|x\| \to \infty. \tag{4.1.8}$$

In particular any good function, any polynomial and the function $\sigma(x) = e^{i\lambda . x}$, for any $\lambda \in \mathcal{R}^d$, are all fairly good. It is clear too that if σ is fairly good and γ is good then the product $\sigma\gamma$ is good.

Suppose now that σ is fairly good, g is a generalised function and that $\{\gamma_m\}$ is any member of the equivalence class of regular sequences for g. In this case the sequence $\{\sigma\gamma_m\}$ is regular and all such sequences form an equivalence class for the generalised function denoted by σg.

The **Fourier transform** $\hat{f}$ for an absolutely integrable function f is defined by

$$\hat{f}(\lambda) = \int_{\mathcal{R}^d} f(x) e^{-i\lambda . x}\, dx, \qquad \lambda \in \mathcal{R}^d. \tag{4.1.9}$$

Jones (1982) proves the following 4 major theorems which are useful for our applications.

Theorem 4-2. *If $\gamma(x)$ is a good function then so is its Fourier transform $\hat{\gamma}(\lambda)$, defined as in* (4.1.9).

Theorem 4-3. *Suppose that g is a generalised function defined by $\{\gamma_m\}$ and that for each $m \in \mathcal{Z}^+$ $\hat{\gamma}_m$ is the Fourier transform of γ_m. Then, from Theorem 4-2, each $\hat{\gamma}_m$ is good and indeed $\{\hat{\gamma}_m\}$ is a regular sequence. Further all such regular sequences give rise to the same generalised function which is denoted by $\hat{g}$.*

We call $\hat{g}$ the **generalised Fourier transform** of g.

Theorem 4-4. *Let the generalised Fourier transforms of g, g_1 and g_2 be $\hat{g}$, $\hat{g}_1$ and $\hat{g}_2$ respectively. Then we have the following table of generalised Fourier transforms:*

(a)	$g_1(x) + g_2(x)$	$\hat{g}_1(\lambda) + \hat{g}_2(\lambda)$;
(b)	$g(x - y)$	$e^{-i\lambda . y}\hat{g}(\lambda)$;
(c)	$g(cx)$	$c^{-d}\hat{g}(c^{-1}\lambda), \quad 0 < c \in \mathcal{R}$;

(d) $x_t g(x)$ $\qquad i\frac{\partial \hat{g}(\lambda)}{\partial \lambda_t}$, $\qquad t \in \mathcal{Z},\ 1 \le t \le d;$

(e) $\hat{g}(\lambda)$ $\qquad (2\pi)^d g(-x);$

(f) $e^{-i\lambda.y}\hat{g}(\lambda)$ $\qquad (2\pi)^d g(-x+y);$

(g) $\hat{g}(c\lambda)$ $\qquad (2\pi)^d c^{-d} g(-c^{-1}x),$ $\qquad 0 < c \in \mathcal{R};$

(h) $\frac{\partial \hat{g}(\lambda)}{\partial \lambda_t}$, $\qquad -(2\pi)^d i x_t g(-x),$ $\qquad t \in \mathcal{Z},\ 1 \le t \le d.$

We remark that these properties are all well known for the conventional Fourier transform applied to suitably smooth and quickly decaying functions. In that case all (except (e)) are simple consequences of (4.1.9) and previous parts of the theorem. In the case of f absolutely integrable, (e) gives us the usual inversion formula for conventional Fourier transforms:

$$f(x) = \frac{1}{(2\pi)^d}\int_{\mathcal{R}^d} \hat{f}(\lambda)e^{ix.\lambda}\,d\lambda, \qquad x \in \mathcal{R}^d. \tag{4.1.10}$$

Theorem 4-5. *Let f be an absolutely integrable function with conventional Fourier transform $\hat{f}$. We notice that $f \in K_1$ (4.1.5) and so is a generalised function (Theorem 4-1). Let its generalised Fourier transform be F (Theorem 4-3). In this case $\hat{f} \equiv F$, in the sense of generalised functions.*

This theorem justifies the definition of the generalised Fourier transform given after Theorem 4-3, and from now on we shall often drop the describing words "conventional" and "generalised" from Fourier transforms and just refer to them all as Fourier transforms.

We deduce from these theorems two simple corollaries which we will apply later:

Corollary 4-6. *Suppose that $\phi^* : \mathcal{R}^d \to \mathcal{R}$ is continuous and of at most polynomial growth, and let $\{\mu_j : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{R}^d : j = 1,2,\ldots,l\}$ be chosen so that*

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi^*(x - x_j), \qquad x \in \mathcal{R}^d, \tag{4.1.11}$$

is absolutely integrable. In this case ψ has a continuous, conventional Fourier transform

$$\hat{\psi}(\lambda) = \sum_{j=1}^{l} \mu_j e^{-i\lambda.x_j}\hat{\phi}^*(\lambda), \qquad \lambda \in \mathcal{R}^d, \tag{4.1.12}$$

where $\hat{\phi}^*$ is the generalised Fourier transform of ϕ^*.

Corollary 4-7. *Suppose that $\hat{\psi}$ as defined in (4.1.12) is absolutely integrable and that $\hat{\phi}^*$ is the generalised Fourier transform of some generalised function ϕ^*. Then $\psi(x)$ as defined in (4.1.11) is the conventional inverse transform of the function $\hat{\psi}$ and so is continuous.*

We note that when we are working with a function ϕ which is a radial basis function we may replace ϕ^* in Corollaries 4-6 and 4-7 by the function $\phi(\|\cdot\|)$.

Section 4.2 : A Form of ϕ to Reproduce Polynomials

In this section we study an alternative method for reproducing polynomials. We suppose that we are given a continuous function $\phi : \mathcal{R}^+ \to \mathcal{R}$ and also a function ψ of the standard form

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d. \tag{4.2.1}$$

First, for the sake of clarity, we review the definition of polynomial reproduction given in Chapter 3. As usual Π_m is the space of polynomials from $\mathcal{R}^d$ to $\mathcal{R}$ of total degree at most m:

$$\Pi_m = \left\{ \sum_{\{\beta \in (\mathcal{Z}+)^d : |\beta| \leq m\}} A_\beta x^\beta, \qquad x \in \mathcal{R}^d \right\}, \tag{4.2.2}$$

where $A_\beta \in \mathcal{R}$ and where we are using the multi-index notation defined at the end of the introduction. We define the **degree** of such a polynomial to be the maximum value of $|\beta|$ that satisfies $A_\beta \neq 0$.

If $P \in \Pi_m$ and the function ψ is chosen so that the following sum is absolutely convergent, we define

$$s(x) = \sum_{z \in \mathcal{Z}^d} P(z)\psi(x - z), \qquad x \in \mathcal{R}^d. \tag{4.2.3}$$

In this case we say that ψ **preserves** polynomials of degree m if $P \in \Pi_m \Rightarrow s \in \Pi_m$, the degree of s being at most the degree of P, and we say that ψ **reproduces** polynomials of degree m if $P \in \Pi_m \Rightarrow s \equiv P$.

We shall find conditions on ϕ which ensure that there exist l, $\{\mu_j : j = 1, 2, \ldots, l\}$ and $\{x_j : j = 1, 2, \ldots, l\}$ so that the function ψ (4.2.1) reproduces polynomials of degree m. First we find conditions on the function ψ (4.2.1) which allow reproduction of polynomials of degree m, and then we translate them into conditions on ϕ.

The approach that we initially follow dates back to a classic paper by Schoenberg (1946) in which the author considers the quasi-interpolation formula

$$s(x) = \sum_{z \in \mathcal{Z}} f(z)\psi(x - z), \qquad x \in \mathcal{R}; \tag{4.2.4}$$

where $f, \psi : \mathcal{R} \to \mathcal{R}$ and ψ is continuous. He provides sufficient conditions on ψ for such formulae to preserve and reproduce polynomials of degree m. His definition of polynomial preservation is slightly different to that used here and the following theorem is in a form that allows for this. Schoenberg proves (originally Theorem 2 on page 64):

Theorem 4-8. *If there exist positive constants A and B such that*

$$|\psi(x)| \leq Ae^{-Bx}, \qquad x \in \mathcal{R}, \tag{4.2.5}$$

and if $\hat{\psi}(\lambda)$ is the Fourier transform of ψ (4.1.9), then

(a) ψ in (4.2.4) preserves polynomials of degree m if $\hat{\psi}(\lambda)$ has zeros of order $m+1$ for all non-zero integral multiples of 2π,

(b) ψ in (4.2.4) reproduces polynomials of degree m if, in addition to the conditions (a) being satisfied, $\hat{\psi}(\lambda) - 1$ has a zero of order $m+1$ at 0.

A generalization of this theorem to higher dimensions is given in several papers, including Fix and Strang (1969), Dahmen and Micchelli (1984) and de Boor and Jia (1985). It is not sufficient for our purposes, however, because it depends on exponential decay in ψ, whereas the functions we consider only have algebraic decay. Therefore we shall prove the theorem in multiple dimensions under weaker assumptions that are suitable for our work. The crucial point in the proof is the conditions under which the Poisson summation formula holds so we consider them first.

Theorem 4-9 : Poisson Summation Formula. *Let $h : \mathcal{R}^d \to \mathcal{R}$ be an absolutely integrable function and let*

$$C = \{y \in \mathcal{R}^d \ : 0 \leq y_i < 1, \ i = 1, 2, \ldots, d\}. \tag{4.2.6}$$

Then the sum

$$\sum_{n \in \mathcal{Z}^d} h(y+n), \qquad y \in C, \tag{4.2.7}$$

(which may not converge absolutely) converges in norm to a function g in $L_1(C)$, which is the space of absolutely integrable functions on C. Further, the function g has the Fourier series expansion

$$g(y) = \sum_{l \in \mathcal{Z}^d} \hat{h}(2\pi l) e^{2\pi i l . y}, \qquad y \in C. \tag{4.2.8}$$

Proof. See Stein and Weiss (1971, Chapter 7, Theorem 2.4). ■

This theorem has a highly useful corollary which may also be called the Poisson summation formula.

Corollary 4-10 : Poisson Summation Formula. *Let $h : \mathcal{R}^d \to \mathcal{R}$ be continuous and absolutely integrable, let*

$$\sum_{n \in \mathcal{Z}^d} h(y+n), \qquad y \in \mathcal{R}^d, \tag{4.2.9}$$

converge absolutely to a continuous function and let

$$\sum_{l \in \mathcal{Z}^d} |\hat{h}(2\pi l)| < \infty. \tag{4.2.10}$$

These conditions imply the equation

$$\sum_{n \in \mathcal{Z}^d} h(y+n) = \sum_{l \in \mathcal{Z}^d} \hat{h}(2\pi l) e^{2\pi i l.y}, \qquad y \in \mathcal{R}^d, \tag{4.2.11}$$

and that both these sums are absolutely convergent.

Proof. By Theorem 4-9, equation (4.2.8) gives the Fourier series of the continuous function (4.2.9). By condition (4.2.10) the sum on the right hand side of (4.2.11) is absolutely convergent. Therefore the function on the left hand side is equal to its Fourier series (Stein and Weiss 1971, Chapter 7, Corollary 1.8), which is equation (4.2.11), and the left hand side is absolutely convergent by assumption. ■

We can now embark on proving sufficient conditions on ψ to ensure polynomial preservation and reproduction:

Theorem 4-11. *Let $\psi : \mathcal{R}^d \to \mathcal{R}$ be a continuous function such that, for all $\alpha \in (\mathcal{Z}^+)^d \, : \, |\alpha| \le m$,*

$$\int_{\mathcal{R}^d} |y^\alpha \psi(y)| \, dy < \infty, \tag{4.2.12}$$

and

$$\sum_{n \in \mathcal{Z}^d} (y+n)^\alpha \psi(y+n), \qquad y \in \mathcal{R}^d, \tag{4.2.13}$$

is continuous. Suppose further that

$$\left. \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} = 0, \tag{4.2.14}$$

for all $\beta \in (\mathcal{Z}^+)^d \backslash \{0\}$ and all $\alpha \in (\mathcal{Z}^+)^d \, : \, |\alpha| \le m$ (where the multi-index notation for partial derivatives is defined in the introduction). Then ψ in formula (4.2.3) preserves polynomials of degree m. If, in addition,

$$\left. \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 0} = \begin{cases} 1 & \text{if } \alpha = 0; \\ 0 & \text{if } \alpha \in (\mathcal{Z}^+)^d, \quad 0 < |\alpha| \le m, \end{cases} \tag{4.2.15}$$

then formula (4.2.3) reproduces polynomials of degree m.

We remark that if $\psi(x) = O(\|x\|^{-d-m-\epsilon})$ as $\|x\| \to \infty$ for any $\epsilon > 0$ then condition (4.2.12) is satisfied and also the sum in (4.2.13) converges uniformly and hence is continuous.

Proof. We deduce from (4.2.12) that

$$\hat{\psi}(\lambda) = \int_{\mathcal{R}^d} \psi(y)e^{-i\lambda.y}\, dy, \qquad \lambda \in \mathcal{R}^d, \tag{4.2.16}$$

is m times continuously differentiable, as we may differentiate under the integral sign when both the integrand and its derivative are absolutely integrable. Therefore conditions (4.2.14) and (4.2.15) do indeed make sense.

We take $P \in \Pi_m$, x any fixed point in $\mathcal{R}^d$ and define the continuous function ρ_x by

$$\rho_x(y) = P(y)\psi(x-y), \qquad y \in \mathcal{R}^d. \tag{4.2.17}$$

We shall prove the theorem by applying the Poisson summation formula (Corollary 4-10) to the function ρ_x. Condition (4.2.13) implies that

$$\sum_{n\in\mathcal{Z}^d} \rho_x(y+n), \qquad y \in \mathcal{R}^d, \tag{4.2.18}$$

is continuous, which is condition (4.2.9), and condition (4.2.12) implies that ρ_x is absolutely integrable. Next we evaluate, for any $\beta \in \mathcal{Z}^d$,

$$\begin{aligned}\hat{\rho}_x(2\pi\beta) &= \int_{\mathcal{R}^d} P(z)\psi(x-z)e^{-2\pi i\beta.z}\, dz \\ &= e^{-2\pi i\beta.x}\int_{\mathcal{R}^d} P(x-y)\psi(y)e^{2\pi i\beta.y}\, dy \\ &= e^{-2\pi i\beta.x}\int_{\mathcal{R}^d} P(x-y)\psi(y)e^{-i\lambda.y}\, dy\Big|_{\lambda=-2\pi\beta}.\end{aligned} \tag{4.2.19}$$

Writing

$$\hat{D} = \left(\frac{\partial}{\partial\lambda_1}, \frac{\partial}{\partial\lambda_2}, \ldots, \frac{\partial}{\partial\lambda_d}\right)^T \tag{4.2.20}$$

and recalling that $P(x-y)$ is a polynomial of degree at most m in y, we may use Theorem 4-4 (d) to express (4.2.19) as

$$\hat{\rho}_x(2\pi\beta) = e^{-2\pi i\beta.x}\left[P\left(x - i\hat{D}\right)\hat{\psi}\right]\Big|_{\lambda=-2\pi\beta}. \tag{4.2.21}$$

Therefore, using (4.2.14), we find

$$\hat{\rho}_x(2\pi\beta) = \begin{cases} 0 & \text{if } \beta \in \mathcal{Z}^d\backslash\{0\}; \\ \left[P(x-i\hat{D})\hat{\psi}\right]\Big|_{\lambda=0} & \text{if } \beta = 0. \end{cases} \tag{4.2.22}$$

We note that, because $\hat{\psi}$ is m times differentiable, this expression when $\beta = 0$ is well defined. Hence (4.2.10) holds if we replace $\hat{h}$ by $\hat{\rho}_x$, and we have already found that the other conditions of Corollary 4-10 are satisfied. Thus we may apply the Poisson summation formula with $y = 0$ to the function ρ_x yielding, for all $x \in \mathcal{R}^d$, that expression (4.2.3) has the value

$$\begin{aligned} s(x) &= \sum_{n \in \mathcal{Z}^d} P(n)\psi(x-n) \\ &= \sum_{n \in \mathcal{Z}^d} \rho_x(n) \\ &= \sum_{\beta \in \mathcal{Z}^d} \hat{\rho}_x(2\pi\beta) \\ &= \left[P(x - i\hat{D})\hat{\psi}\right]\Big|_{\lambda=0}, \end{aligned} \tag{4.2.23}$$

the last line using (4.2.22). The final line is a polynomial of degree at most the degree of P, completing the proof of the first part of the theorem.

We notice too that (4.2.23) and (4.2.15) give, for all $x \in \mathcal{R}^d$,

$$\begin{aligned} s(x) &= \left[P(x - i\hat{D})\hat{\psi}\right]\Big|_{\lambda=0} \\ &= \left[P(x)\hat{\psi}\right]\Big|_{\lambda=0} \\ &= P(x), \end{aligned} \tag{4.2.24}$$

which yields the second part of the theorem. ■

Theorem 4-11 gives sufficient conditions for an interpolation formula of the type (4.2.3) to preserve and reproduce polynomials of degree m. In order to discover if these conditions are also necessary, we prove the following theorem:

Theorem 4-12. *Suppose that $\psi : \mathcal{R}^d \to \mathcal{R}$ satisfies both*

$$\int_{\mathcal{R}^d} |y^\alpha \psi(y)|\, dy < \infty, \tag{4.2.25}$$

and

$$\sum_{\beta \in \mathcal{Z}^d} \left| \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} < \infty, \tag{4.2.26}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m$. Further, suppose that the formula (4.2.3) preserves polynomials of degree m. In this case condition (4.2.14) of Theorem 4-11 holds:

$$\left. \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} = 0, \tag{4.2.27}$$

for all $\beta \in \mathcal{Z}^d \backslash \{0\}$ and $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m$. Further, if the formula (4.2.3) reproduces polynomials of degree m then condition (4.2.15) of Theorem 4-11 also holds:

$$\left. \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 0} = \begin{cases} 1 & \text{if } \alpha = 0; \\ 0 & \text{if } \alpha \in (\mathcal{Z}^+)^d \backslash \{0\}, \end{cases} \tag{4.2.28}$$

for all $\alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \le m$.

Proof. As at the beginning of the proof of Theorem 4-11 we note that (4.2.25) implies that $\hat{\psi}$ has all partial derivatives of order at most m existing and so condition (4.2.26) makes sense. We let $P \in \Pi_m$, $x \in \mathcal{R}^d$ and define, as in the proof of Theorem 4-11 (4.2.17),

$$\rho_x(y) = P(y)\psi(x-y), \qquad y \in \mathcal{R}^d. \tag{4.2.29}$$

We wish to prove the result by applying Corollary 4-10 to the function ρ_x, so we must check that it satisfies all the conditions.

Condition (4.2.25) shows that ρ_x is absolutely integrable. The fact that ψ preserves polynomials of degree m implies that the function of y,

$$\sum_{n \in \mathcal{Z}^d} \rho_x(y+n), \qquad y \in \mathcal{R}^d, \tag{4.2.30}$$

is a polynomial of degree at most that of P and hence is continuous, as required. We also recall from (4.2.21) that

$$\hat{\rho}_x(2\pi\beta) = e^{-2\pi i\beta . x} \left[P(x - i\hat{D})\hat{\psi}\right]\Big|_{\lambda=-2\pi\beta} \tag{4.2.31}$$

and so, by assumption (4.2.26),

$$\sum_{\beta \in \mathcal{Z}^d} |\hat{\rho}_x(2\pi\beta)| < \infty, \tag{4.2.32}$$

which is condition (4.2.10). Hence we may apply the Poisson summation formula (Corollary 4-10) to the function ρ_x.

We continue the proof by induction on the degree of the polynomial. We consider first the case of polynomial preservation and start with $m = 0$. In this case we take P of degree 0, say $P \equiv 1$, and expression (4.2.3) is some constant K_0 by hypothesis. Hence, using (4.2.29) with $P \equiv 1$, applying the Poisson summation formula (Corollary 4-10) to ρ_x and using (4.2.31) with $P \equiv 1$, we obtain

$$\begin{aligned} K_0 &= \sum_{n \in \mathcal{Z}^d} \psi(x-n) \\ &= \sum_{n \in \mathcal{Z}^d} \rho_x(n) \\ &= \sum_{l \in \mathcal{Z}^d} \hat{\rho}_x(2\pi l) \\ &= \sum_{l \in \mathcal{Z}^d} e^{-2\pi i l . x}\hat{\psi}(-2\pi l). \end{aligned} \tag{4.2.33}$$

Now, if we multiply (4.2.33) by $e^{2\pi i\beta . x}$ for $\beta \in \mathcal{Z}^d$ and integrate over

$$C_0 = \{z \in \mathcal{R}^d : \|z\|_\infty \le \tfrac{1}{2}\}, \tag{4.2.34}$$

which is the unit cube centred at 0, we find

$$\hat{\psi}(-2\pi\beta) = \begin{cases} K_0 & \text{if } \beta = 0; \\ 0 & \text{if } \beta \in \mathcal{Z}^d\backslash\{0\}, \end{cases} \tag{4.2.35}$$

the required result for the case $m = 0$.

Suppose now that the result (4.2.27) is true for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le r-1$ where r is any integer in $[1, m-1]$. We let

$$\left.\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\right|_{\lambda=0} = (-i)^{|\alpha|} K_\alpha, \tag{4.2.36}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le m$, and we note that K_0 ties in with our previous definition. Without loss of generality we may consider $P(x) = x^{\alpha'}$ for any $\alpha' \in (\mathcal{Z}^+)^d : |\alpha'| = r$. Therefore, in view of (4.2.3), Corollary 4-10 and (4.2.31), we find that, for all $x \in \mathcal{R}^d$,

$$\begin{aligned} s(x) &= \sum_{n\in\mathcal{Z}^d} P(n)\psi(x-n) \\ &= \sum_{n\in\mathcal{Z}^d} \rho_x(n) \\ &= \sum_{\beta\in\mathcal{Z}^d} \hat{\rho}_x(2\pi\beta) \\ &= \sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta . x} \left[P(x - i\hat{D})\hat{\psi}\right]\Big|_{\lambda=-2\pi\beta} \qquad (4.2.37) \\ &= \sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta . x} \sum_{\{\alpha\in(\mathcal{Z}^+)^d : \alpha\le\alpha'\}} \left.\frac{\partial^\alpha P(x)}{\partial x^\alpha} i^{|\alpha|} \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial\lambda^\alpha}\right|_{\lambda=-2\pi\beta} \\ &= \sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta . x} \alpha'! \, i^r \left.\frac{\partial^{\alpha'}\hat{\psi}(\lambda)}{\partial\lambda^{\alpha'}}\right|_{\lambda=-2\pi\beta} + \sum_{\{\alpha\in(\mathcal{Z}^+)^d : \alpha<\alpha'\}} \frac{\partial^\alpha P(x)}{\partial x^\alpha} K_\alpha, \qquad (4.2.38) \end{aligned}$$

the last line depending on (4.2.27) (which by hypothesis is true for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \le r-1$) and (4.2.36). Recalling that $s \in \Pi_r$, as (4.2.3) is polynomial preserving, and noting that the final sum is a polynomial of degree at most r, it follows that the first sum is a polynomial of degree at most r. However, the first sum is periodic in x with period 1 and so must be equal to some constant A, which is the identity

$$\sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta . x} \alpha'! \, i^r \left.\frac{\partial^{\alpha'}\hat{\psi}(\lambda)}{\partial\lambda^{\alpha'}}\right|_{\lambda=-2\pi\beta} = A, \qquad x \in \mathcal{R}^d. \tag{4.2.39}$$

Multiplying by $e^{2\pi i\beta'.x}$ and integrating with respect to x over C_0 (see (4.2.34)), we obtain

$$\alpha'!\,i^r \left.\frac{\partial^{\alpha'}\hat{\psi}(\lambda)}{\partial\lambda^{\alpha'}}\right|_{\lambda=-2\pi\beta'} = 0, \qquad \beta' \in \mathcal{Z}^d\backslash\{0\}, \tag{4.2.40}$$

which gives condition (4.2.27) for any $\alpha' \in (\mathcal{Z}^+)^d : |\alpha'| = r$, as required.

In the case where the formula (4.2.3) reproduces polynomials the argument is somewhat simpler. The case $r = 0$ is as above, taking $K_0 = 1$. Now assume that the result (4.2.28) is true for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq r-1$ and then consider $P(x) = x^{\alpha'}$, for any $\alpha' \in (\mathcal{Z}^+)^d : |\alpha'| = r$. From (4.2.3) and (4.2.38), we obtain that, for any $x \in \mathcal{R}^d$,

$$\begin{aligned} P(x) &= \sum_{n\in\mathcal{Z}^d} P(n)\psi(x-n) \\ &= \sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta.x}\alpha'!\,i^r \left.\frac{\partial^{\alpha'}\hat{\psi}(\lambda)}{\partial\lambda^{\alpha'}}\right|_{\lambda=-2\pi\beta} + \sum_{\alpha<\alpha'} \frac{\partial^\alpha P(x)}{\partial x^\alpha} K_\alpha, \\ &= \sum_{\beta\in\mathcal{Z}^d} e^{-2\pi i\beta.x}\alpha'!\,i^r \left.\frac{\partial^{\alpha'}\hat{\psi}(\lambda)}{\partial\lambda^{\alpha'}}\right|_{\lambda=-2\pi\beta} + P(x), \end{aligned} \tag{4.2.41}$$

the last line depending on the fact that (4.2.28) and (4.2.36) give $K_\alpha = 0$ for all $\alpha \in (\mathcal{Z}^+)^d : 0 < |\alpha| \leq r-1$. Integrating this sum with respect to x over C_0 we obtain the second line of (4.2.28) for any $\alpha' \in (\mathcal{Z}^+)^d : |\alpha'| = r \geq 1$, which completes the proof. ■

We are moving towards the main theorem of the section, that is conditions on ϕ which will ensure that certain polynomials can be reproduced. However, there are still a few more preliminaries to set up:

Lemma 4-13. *If $\hat{\psi} : \mathcal{R}^d \to \mathcal{R}$ is the Fourier transform of a function ψ which satisfies*

$$\int_{\mathcal{R}^d} \left|\frac{\partial^\alpha\hat{\psi}(\lambda)}{\partial\lambda^\alpha}\right| d\lambda < \infty, \tag{4.2.42}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = q$ then $\psi(x) = o(\|x\|^{-q})$ as $\|x\| \to \infty$.

Proof. Stein and Weiss (1971) include the Riemann-Lebesgue Lemma (Theorem 1.2) which states that, if $f : \mathcal{R}^d \to \mathcal{R}$ is absolutely integrable, then $\hat{f}(\lambda) \to 0$ as $\|\lambda\| \to \infty$. A comparison with

$$\int_{\mathcal{R}^d} \frac{\partial^\alpha\hat{\psi}(\lambda)}{\partial\lambda^\alpha} e^{-i\lambda.x}\, d\lambda = (2\pi)^d(-i)^{|\alpha|}x^\alpha\psi(-x), \tag{4.2.43}$$

which is obtained from Theorem 4-4 (h), yields the result. ■

We shall also need a technical definition in the main theorem: we say that h is a **b-regularly differentiable function**, for $b \in \mathcal{R}$, if h has all partial derivatives existing away from 0 and

$$\frac{\partial^\alpha h(\lambda)}{\partial\lambda^\alpha} = O(\|\lambda\|^{b-|\alpha|}) \text{ as } \|\lambda\| \to 0, \tag{4.2.44}$$

for all $\alpha \in (\mathcal{Z}^+)^d$. We draw together some simple consequences of this definition into a lemma:

Lemma 4-14. *Suppose that* $\{\tilde{P}_s, \tilde{q}_s \,:\, s = 0,1,\ldots\}$ *are homogeneous polynomials of degree* s. *Then* $\tilde{P}_t(\cdot)(\tilde{q}_r(\cdot))^{-1}$ *is a* $(t-r)$*-regularly differentiable function. Further, assuming both sums converge in a neighbourhood of 0,* $\left(\sum_{s=t}^{\infty} \tilde{P}_s(\cdot)\right)\left(\sum_{s=r}^{\infty} \tilde{q}_s(\cdot)\right)^{-1}$ *is also* $(t-r)$*-regularly differentiable. Further, if* g *is* t*-regularly differentiable and* h *is* r*-regularly differentiable, then* gh *is* $(t+r)$*-regularly differentiable and* $g+h$ *is* $\min(t,r)$*-regularly differentiable. Finally the function* $\log(\|\cdot\|)$ *is* $(-\delta)$*-regularly differentiable for any* $\delta > 0$.

In each case the statement may be verified without difficulty and we omit the proof in order to concentrate on more important results.

Now we may state the main theorem in which we look for conditions on $\phi^* : \mathcal{R}^d \to \mathcal{R}$ so that a finite linear combination of integer translates

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi^*(x - x_j), \qquad x \in \mathcal{R}^d, \tag{4.2.45}$$

$\{x_j \in \mathcal{Z}^d \,:\, j = 1,2,\ldots,l\}$, when used in formula (4.2.3), reproduces polynomials of some degree.

Theorem 4-15. *Let* $\phi^* : \mathcal{R}^d \to \mathcal{R}$ *be a function with generalised Fourier transform* $\hat{\phi}^*$ *having all partial derivatives on* $\mathcal{R}^d \backslash \{0\}$ *and satisfying*

$$\int_{\{\lambda \in \mathcal{R}^d : \|\lambda\| \geq 1\}} \left| \frac{\partial^\alpha \hat{\phi}^*(\lambda)}{\partial \lambda^\alpha} \right| d\lambda < \infty, \tag{4.2.46}$$

for all $\alpha \in (\mathcal{Z}^+)^d$. *Suppose that near* $\lambda = 0$ *we have an expansion for* $\hat{\phi}^*(\lambda)$ *in functions of* λ *of increasing degree:*

$$\hat{\phi}^*(\lambda) = \frac{1}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda)} + h(\lambda), \tag{4.2.47}$$

where $r \geq 1$, *where each* $\tilde{P}_s$ *is a homogeneous polynomial of degree* s *and we assume* $\tilde{P}_r \not\equiv 0$, *and where the remainder term* h *is a* $(t-2r+\epsilon')$*-regularly differentiable function for some* $0 < \epsilon' < 1$. *Suppose too that* ϕ^* *is such that any function* ψ *of the form (4.2.45) satisfying*

$$\psi(x) = o(\|x\|^{-d-k}) \text{ as } \|x\| \to \infty, \tag{4.2.48}$$

for any $k = 0,1,\ldots$ *also satisfies both*

$$\int_{\mathcal{R}^d} |y^\alpha \psi(y)| dy < \infty, \qquad \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \leq k, \tag{4.2.49}$$

and the condition that the function

$$\sum_{n \in \mathcal{Z}^d} (y+n)^\alpha \psi(y+n), \qquad y \in \mathcal{R}^d, \tag{4.2.50}$$

is continuous, for all $\alpha \in (\mathcal{Z}^+)^d \, : \, |\alpha| \le k$. *In this case there exist* l, $\{\mu_j \, : \, j = 1,2,\ldots,l\}$ *and* $\{x_j \in \mathcal{Z}^d \, : \, j = 1,2,\ldots,l\}$ *such that* ψ *of the form (4.2.45), when used in formula (4.2.3), reproduces all polynomials of degree at most* $m = \min(r-1, t-r)$.

Conversely, suppose that among all possible expansions of the form (4.2.47) for ϕ^* *near* $\lambda = 0$ *we let* t *take on the maximum possible value, if that is finite, or* $2r-1$ *if that is infinite, recalling that we are allowed to take* $\tilde{P}_s \equiv 0$ *in (4.2.47) so long as* $s > r$. *Suppose also that*

$$\sum_{\beta \in \mathcal{Z}^d \setminus \{0\}} \left| \frac{\partial^\alpha \hat{\phi}^*(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} < \infty, \tag{4.2.51}$$

for all $\alpha \in (\mathcal{Z}^+)^d$, *and that*

$$\hat{\phi}^*(2\pi\beta) \neq 0 \tag{4.2.52}$$

for some $\beta \in \mathcal{Z}^d \setminus \{0\}$. *Further, let* ψ *be any function of the form (4.2.45) (where we still require* $\{x_j \in \mathcal{Z}^d \, : \, j = 1,2,\ldots,l\}$*) that reproduces polynomials of degree* k *and that satisfies*

$$\int_{\mathcal{R}^d} |y^\alpha \psi(y)| \, dy < \infty, \tag{4.2.53}$$

for all $\alpha \in (\mathcal{Z}^+)^d \, : \, |\alpha| \le k$. *Then* $m = \min(r-1, t-r)$ *is the maximum possible value of* k.

Before proving Theorem 4-15, a few remarks are in order to explain some of the technical conditions in the statement of the theorem. In some cases our conditions are not the weakest possible and in some may not be minimal. We have aimed to provide a set of conditions that can be applied to practical calculations, which are unlikely to include functions with abstruse properties. We first remark that Theorem 4-15 can be used when ϕ is a radial basis function, in which case we replace ϕ^* in the above statement by the function $\phi(\|\cdot\|)$.

Conditions (4.2.47), (4.2.48)–(4.2.50) and (4.2.53) can sometimes be combined into one condition, for the smoothness of the function ϕ^* near ∞ is related to the smoothness of the function $\hat{\phi}^*$ near 0, although a useful formulation is not easily accessible. However the present conditions are easy to apply and a combined one may be less amenable.

We remark that conditions (4.2.48)–(4.2.50) and (4.2.53) all restrict ψ rather than ϕ^*. In many cases they can be verified easily and to try to translate them into constraints on ϕ^* tends to be less convenient. On a similar note we see that, although neither condition (4.2.46) nor (4.2.51) implies the other, the possibility that one holds and not the other is unlikely to occur in practice.

The crucial condition for the validity of the theorem is (4.2.47), because in several useful cases when it holds it can be shown that the other conditions hold too. For example see Corollaries 4-18 and 4-19.

is continuous, for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq k$. *In this case there exist* l, $\{\mu_j : j = 1, 2, \ldots, l\}$ *and* $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$ *such that* ψ *of the form (4.2.45), when used in formula (4.2.3), reproduces all polynomials of degree at most* $m = \min(r-1, t-r)$.

Conversely, suppose that among all possible expansions of the form (4.2.47) for ϕ^* *near* $\lambda = 0$ *we let* t *take on the maximum possible value, if that is finite, or* $2r-1$ *if that is infinite, recalling that we are allowed to take* $\tilde{P}_s \equiv 0$ *in (4.2.47) so long as* $s > r$. *Suppose also that*

$$\sum_{\beta \in \mathcal{Z}^d \setminus \{0\}} \left| \frac{\partial^\alpha \hat{\phi}^*(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} < \infty, \tag{4.2.51}$$

for all $\alpha \in (\mathcal{Z}^+)^d$, *and that*

$$\hat{\phi}^*(2\pi\beta) \neq 0 \tag{4.2.52}$$

for some $\beta \in \mathcal{Z}^d \setminus \{0\}$. *Further, let* ψ *be any function of the form (4.2.45) (where we still require* $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$*) that reproduces polynomials of degree* k *and that satisfies*

$$\int_{\mathcal{R}^d} |y^\alpha \psi(y)|\, dy < \infty, \tag{4.2.53}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq k$. *Then* $m = \min(r-1, t-r)$ *is the maximum possible value of* k.

Before proving Theorem 4-15, a few remarks are in order to explain some of the technical conditions in the statement of the theorem. In some cases our conditions are not the weakest possible and in some may not be minimal. We have aimed to provide a set of conditions that can be applied to practical calculations, which are unlikely to include functions with abstruse properties. We first remark that Theorem 4-15 can be used when ϕ is a radial basis function, in which case we replace ϕ^* in the above statement by the function $\phi(\|\cdot\|)$.

Conditions (4.2.47), (4.2.48)–(4.2.50) and (4.2.53) can sometimes be combined into one condition, for the smoothness of the function ϕ^* near ∞ is related to the smoothness of the function $\hat{\phi}^*$ near 0, although a useful formulation is not easily accessible. However the present conditions are easy to apply and a combined one may be less amenable.

We remark that conditions (4.2.48)–(4.2.50) and (4.2.53) all restrict ψ rather than ϕ^*. In many cases they can be verified easily and to try to translate them into constraints on ϕ^* tends to be less convenient. On a similar note we see that, although neither condition (4.2.46) nor (4.2.51) implies the other, the possibility that one holds and not the other is unlikely to occur in practice.

The crucial condition for the validity of the theorem is (4.2.47), because in several useful cases when it holds it can be shown that the other conditions hold too. For example see Corollaries 4-18 and 4-19.

Proof. First we deduce from the statement of the theorem that conditions (4.2.12)–(4.2.15) can be satisfied for a function ψ (4.2.45), with $m = \min(r-1, t-r)$. Then we deduce from Theorem 4-11 that such a function reproduces all polynomials of degree m. Guided by Corollary 4-6, which shows us the form of $\hat{\psi}$, we expand $e^{-i\lambda.y}$ near $\lambda = 0$ and find

$$e^{-i\lambda.y} = \sum_{s=0}^{\infty} \frac{(-i\lambda.y)^s}{s!}. \tag{4.2.54}$$

Hence, denoting the term in Corollary 4-6 by $g(\lambda)$, we expand it as

$$\begin{aligned} g(\lambda) &= \sum_{j=1}^{l} \mu_j e^{-i\lambda.x_j} \\ &= \sum_{j=1}^{l} \mu_j \sum_{s=0}^{\infty} \frac{(-i\lambda.x_j)^s}{s!} \\ &= \sum_{s=0}^{\infty} \tilde{q}_s(\lambda), \end{aligned} \tag{4.2.55}$$

where each $\tilde{q}_s(\lambda)$ is a homogeneous polynomial of degree s in λ. For example we see that

$$\tilde{q}_0(\lambda) = \sum_{j=1}^{l} \mu_j, \tag{4.2.56}$$

$$\tilde{q}_1(\lambda) = (-i) \sum_{j=1}^{l} \mu_j \lambda.x_j \tag{4.2.57}$$

and further polynomials can be calculated similarly. In particular, if $\alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = s$, the coefficient of λ^α in $\tilde{q}_s(\lambda)$ is

$$(-i)^s \sum_{j=1}^{l} \frac{\mu_j x_j^\alpha}{s!} \frac{(|\alpha|)!}{\alpha!} = \frac{(-i)^{|\alpha|}}{\alpha!} \sum_{j=1}^{l} \mu_j x_j^\alpha. \tag{4.2.58}$$

We recall that products of powers of components from $\mathcal{R}^d$ to $\mathcal{R}$ are linearly independent functions. Therefore we can choose l, $\{\mu_j \,:\, j = 1, 2, \ldots, l\}$ and $\{x_j \in \mathcal{Z}^d \,:\, j = 1, 2, \ldots, l\}$ such that

$$\tilde{q}_s(\lambda) \equiv 0 \text{ for } s < r \tag{4.2.59}$$

and

$$\tilde{q}_s(\lambda) = \tilde{P}_s(\lambda) \text{ for } r \le s \le t', \tag{4.2.60}$$

where

$$t' = \min(2r-1, t) \tag{4.2.61}$$

and $\tilde{P}_s$ is defined as in (4.2.47). In this case, taking

$$\hat{\psi}(\lambda) = \sum_{j=1}^{l} \mu_j e^{-i\lambda.x_j} \hat{\phi}^*(\lambda), \qquad \lambda \in \mathcal{R}^d, \tag{4.2.62}$$

in Corollary 4-6 and using (4.2.45) and (4.2.55), we find that, near $\lambda = 0$,

$$\begin{aligned}
\hat{\psi}(\lambda) &= \sum_{s=0}^{\infty} \tilde{q}_s(\lambda)\hat{\phi}^*(\lambda) \\
&= \left(\sum_{s=r}^{t'} \tilde{P}_s(\lambda) + \sum_{s=t'+1}^{\infty} \tilde{q}_s(\lambda) \right) \hat{\phi}^*(\lambda) \\
&= \frac{\sum_{s=r}^{t'} \tilde{P}_s(\lambda) + \sum_{s=t'+1}^{\infty} \tilde{q}_s(\lambda)}{\sum_{s=r}^{t} \tilde{P}_s(\lambda)} + h(\lambda) \sum_{s=r}^{\infty} \tilde{q}_s(\lambda) \\
&= 1 + \frac{\sum_{s=t'+1}^{t} (\tilde{q}_s(\lambda) - \tilde{P}_s(\lambda)) + \sum_{s=t+1}^{\infty} \tilde{q}_s(\lambda)}{\sum_{s=r}^{t} \tilde{P}_s(\lambda)} + h(\lambda) \sum_{s=r}^{\infty} \tilde{q}_s(\lambda),
\end{aligned} \tag{4.2.63}$$

where the leftmost summation in the formula is interpreted as zero if $t' = t$. Hence, because h is $(t - 2r + \epsilon')$-regularly differentiable, it follows from Lemma 4-13 that

$$\hat{\psi}(\lambda) = 1 + \tilde{h}(\lambda), \tag{4.2.64}$$

where $\tilde{h}$ is a $\min(t' + 1 - r, t - r + \epsilon')$-regularly differentiable function. Therefore, remembering $t' \le t$ (4.2.61), (4.2.15) is satisfied for $m = t' - r = \min(r - 1, t - r)$.

We also note that the function $g(\lambda)$ defined in (4.2.55) satisfies $g(\lambda + 2\pi n) = g(\lambda)$ for all $n \in \mathcal{Z}^d$ because we have chosen $x_j \in \mathcal{Z}^d$, $j = 1, 2, \ldots, l$. Hence, near $2\pi\beta$ for $\beta \in \mathcal{Z}^d \backslash \{0\}$, say at $\lambda = 2\pi\beta + \delta$, we find, from (4.2.59) and (4.2.62),

$$\hat{\psi}(2\pi\beta + \delta) = \sum_{s=r}^{\infty} \tilde{q}_s(\delta)\hat{\phi}^*(2\pi\beta + \delta). \tag{4.2.65}$$

Because we have assumed that all partial derivatives of $\hat{\phi}^*$ exist at $2\pi\beta$, it follows that $\hat{\psi}$ satisfies (4.2.14) for $m \le r - 1$, and for all $\beta \in \mathcal{Z}^d \backslash \{0\}$.

It remains to show that ψ is continuous and satisfies (4.2.12) and (4.2.13) for $m = \min(r - 1, t - r)$. We are going to apply Lemma 4-13 with $q = m + d = \min(r - 1, t - r) + d$. Therefore, for $\alpha \in (\mathcal{Z}^+)^d$, $\lambda \neq 0$, we see that (4.2.62) gives

$$\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} = \sum_{\{\alpha' \in (\mathcal{Z}^+)^d : \alpha' \le \alpha\}} \binom{\alpha}{\alpha'} \frac{\partial^{(\alpha - \alpha')} g(\lambda)}{\partial \lambda^{(\alpha - \alpha')}} \frac{\partial^\alpha \hat{\phi}^*(\lambda)}{\partial \lambda^\alpha}, \tag{4.2.66}$$

where g is defined in (4.2.55). Noting that all terms involving g are just trigonometric polynomials and hence bounded over $\mathcal{R}^d$, we deduce from (4.2.46) and (4.2.66) that, for all $\alpha \in (\mathcal{Z}^+)^d$,

$$\int_{\{\lambda \in \mathcal{R}^d \,:\, \|\lambda\| \geq 1\}} \left| \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right| d\lambda < \infty. \tag{4.2.67}$$

Hence, to satisfy the conditions of Lemma 4-13 with $q = \min(r-1, t-r) + d$, we only need to check that they are satisfied on $\{\lambda \in \mathcal{R}^d \,:\, \|\lambda\| \leq 1\}$. On this bounded region the only singularity of any partial derivative of $\hat{\psi}(\lambda)$ occurs at $\lambda = 0$, so we only need absolute integrability in a neighbourhood of 0.

Using (4.2.64) and the regular differentiability of $\tilde{h}$, we find that, for all $\alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \geq 1$,

$$\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} = O(\|\lambda\|^{t'-r+\epsilon'-|\alpha|}) \text{ as } \|\lambda\| \to 0, \tag{4.2.68}$$

recalling we have chosen $\epsilon' < 1$. Hence the conditions of Lemma 4-13 are satisfied for all $\alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = t' - r + d$, so we have the relation

$$\psi(x) = o(\|x\|^{-d-(t'-r)}) \text{ as } \|x\| \to \infty. \tag{4.2.69}$$

Equation (4.2.69) gives us (4.2.48) with $k = t' - r = m$ and hence our assumptions imply (4.2.49) and (4.2.50) which are just conditions (4.2.12) and (4.2.13), as required. We also note, from Corollary 4-7, that ψ is continuous. Therefore the first part of the theorem is a consequence of Theorem 4-11.

It remains only to show the converse, namely, given the conditions (4.2.51), (4.2.52) and (4.2.53) and that t takes its maximum value, $m = \min(r-1, t-r)$ is the maximum integer such that all polynomials of order m are reproduced.

We shall obtain a contradiction by supposing that there exists a function ψ of the form (4.2.45) which reproduces all polynomials of degree $\min(r-1, t-r) + 1$. We employ two stages; first we shall show that conditions (4.2.25) and (4.2.26) are satisfied for $m = \min(r-1, t-r) + 1$. Hence, by Theorem 4-12, conditions (4.2.27) and (4.2.28) must also hold with this value of m. Secondly we shall show that this function ψ is unable to satisfy both (4.2.27) and (4.2.28) for this value of m which is the required contradiction.

Let ψ be a function of the form (4.2.45) which reproduces all polynomials of degree at most $m = \min(r-1, t-r) + 1$. We see that (4.2.53) is identical to condition (4.2.25). Further (4.2.51) and (4.2.66) imply that

$$\sum_{\beta \in \mathcal{Z}^d \setminus \{0\}} \left| \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right|_{\lambda = 2\pi\beta} < \infty. \tag{4.2.70}$$

Because the $\beta = 0$ term is missing from this expression, we deduce from the remark just after (4.2.16) that all partial derivatives of $\hat{\psi}$ of order at most m exist and are continuous. Hence (4.2.70) implies (4.2.26). It follows from Theorem 4-12 that (4.2.27) and (4.2.28) hold with $m = \min(r-1, t-r)+1$.

To obtain the contradiction we now demonstrate that ψ cannot satisfy both (4.2.27) and (4.2.28) with this value of m. We take $\beta \in \mathcal{Z}^d\backslash\{0\}$ for which $\hat{\phi}^*(2\pi\beta) \neq 0$ (4.2.52), and we recall $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$. Because we already have found $\hat{\psi}(0) = 1$ (4.2.28) we require that, in (4.2.62), $\tilde{q}_r \equiv \tilde{P}_r \not\equiv 0$, where $\tilde{q}_r$ and $\tilde{P}_r$ are defined in (4.2.55) and (4.2.47) respectively. Therefore (4.2.65) shows that some partial derivative of $\hat{\psi}$ of order r is non-zero at $2\pi\beta$. Hence equation (4.2.27) fails for $m > r-1$. So, to satisfy both (4.2.27) and (4.2.28) for $m = \min(r-1, t-r)+1$, the maximal value of t in (4.2.47) must be less than $2r-1$, and so in (4.2.61) $t' = t$.

Next we try to satisfy (4.2.28) for $m = \min(r-1, t-r)+1 = t-r+1$. In this case the homogeneous function of degree $t-r+1$ in (4.2.63) would be identically zero, which gives the equation

$$\frac{\tilde{q}_{t+1}(\lambda)}{\tilde{P}_r(\lambda)} + h(\lambda)\tilde{q}_r(\lambda) = o(\|\lambda\|^{t-r+1}). \tag{4.2.71}$$

Hence, remembering $\tilde{P}_r \equiv \tilde{q}_r$, we would have

$$h(\lambda) = -\frac{\tilde{q}_{t+1}(\lambda)}{(\tilde{P}_r(\lambda))^2} + o(\|\lambda\|^{t-2r+1}). \tag{4.2.72}$$

Thus (4.2.47) would imply that, near $\lambda = 0$,

$$\begin{aligned}
\hat{\phi}^*(\lambda) &= \frac{1}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda)} - \frac{\tilde{q}_{t+1}(\lambda)}{(\tilde{P}_r(\lambda))^2} + o(\|\lambda\|^{t-2r+1}) \\
&= \frac{1}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda)} - \frac{\tilde{q}_{t+1}(\lambda)}{(\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda))^2} + o(\|\lambda\|^{t-2r+1}) \\
&= \left(\frac{1}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda)}\right)\left(1 + \frac{\tilde{q}_{t+1}(\lambda)}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda)}\right)^{-1} + o(\|\lambda\|^{t-2r+1}) \\
&= \frac{1}{\tilde{P}_r(\lambda) + \tilde{P}_{r+1}(\lambda) + \cdots + \tilde{P}_t(\lambda) + \tilde{q}_{t+1}(\lambda)} + o(\|\lambda\|^{t-2r+1}),
\end{aligned} \tag{4.2.73}$$

which contradicts the maximality of t.

This analysis shows that Theorem 4-12 fails if $m = \min(r-1, t-r)+1$. Hence our assumption of the existence of a function ψ reproducing polynomials of degree $m = \min(r-1, t-r)+1$ was wrong, completing the proof. ■

We now consider what happens if $\hat{\phi}^*$ does not have the form (4.2.47) near $\lambda = 0$. In many cases it is found that no polynomial reproduction is possible.

Theorem 4-16. *Let $\phi^* : \mathcal{R}^d \to \mathcal{R}$ have Fourier transform $\hat{\phi}^*$ which is not of the form* (4.2.47) *near $\lambda = 0$ and let $\hat{\phi}^*$ satisfy* (4.2.51). *If there is an absolutely integrable function ψ of the form* (4.2.45) *(with $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$) which can reproduce a constant, then, near $\lambda = 0$,*

$$\hat{\phi}^*(\lambda) = A + o(1), \tag{4.2.74}$$

for some $A \in \mathcal{R}$, and $\hat{\phi}^*(2\pi\beta) = 0$ for all $\beta \in \mathcal{Z}^d \backslash \{0\}$.

Proof. The method of proof is similar to the second half of the proof of Theorem 4-15 where we obtain a contradiction by using Theorem 4-12. We suppose that ψ is an absolutely integrable function (4.2.45) which reproduces constants. The absolute integrability implies (4.2.25) with $m = 0$. Further, (4.2.51) shows that

$$\sum_{\beta \in \mathcal{Z}^d \backslash \{0\}} |\hat{\psi}(2\pi\beta)| < \infty. \tag{4.2.75}$$

Combining this with the fact that $\hat{\psi}(0)$ is finite, because ψ is absolutely integrable, yields (4.2.26) with $m = 0$. Hence, in this case (4.2.27) and (4.2.28) hold with $m = 0$, namely

$$\hat{\psi}(2\pi\beta) = \begin{cases} 1 & \text{if } \beta = 0; \\ 0 & \text{for all } \beta \in \mathcal{Z}^d \backslash \{0\}. \end{cases} \tag{4.2.76}$$

We recall that, near $\lambda = 0$, $\hat{\psi}$ is of the form (4.2.62) and we also have the expansion (4.2.55)

$$\sum_{j=1}^{l} \mu_j e^{-i\lambda . x_j} = \sum_{s=0}^{\infty} \tilde{q}_s(\lambda). \tag{4.2.77}$$

Hence, it is impossible to satisfy $\hat{\psi}(0) = 1$ if, near $\lambda = 0$, $\hat{\phi}^*$ is neither of the form (4.2.47) nor (4.2.74). It remains to consider the case when, near $\lambda = 0$, $\hat{\phi}^*$ is of the form (4.2.74). Equation (4.2.62) shows that

$$\hat{\psi}(0) = \sum_{j=1}^{l} \mu_j \hat{\phi}^*(0) \tag{4.2.78}$$

and so we must have

$$\sum_{j=1}^{l} \mu_j \neq 0. \tag{4.2.79}$$

Recalling that (4.2.52) is satisfied in this case, we take $\beta \in \mathcal{Z}^d \backslash \{0\}$ with $\hat{\phi}^*(2\pi\beta) \neq 0$. In this case we find (from (4.2.62)) that

$$\begin{aligned} \hat{\psi}(2\pi\beta) &= \sum_{j=1}^{l} \mu_j e^{-i2\pi\beta . x_j} \hat{\phi}^*(2\pi\beta) \\ &= \sum_{j=1}^{l} \mu_j \hat{\phi}^*(2\pi\beta) \neq 0, \end{aligned} \tag{4.2.80}$$

the last line using $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$ and (4.2.79). This provides the required contradiction to (4.2.76), which completes the proof. ■

We now indicate some cases in which the conditions of Theorem 4-15 to ensure polynomial reproduction can be simplified. These will be useful when we come to analyse some examples in Section 4.3.

Lemma 4-17. *Let $\psi : \mathcal{R}^d \to \mathcal{R}$ be continuous and let*

$$\psi(x) = \tau(x) + o(\|x\|^{-l}) \text{ as } \|x\| \to \infty, \tag{4.2.81}$$

where τ is a homogeneous function of degree $-l$ which is not identically zero on $\mathcal{R}^d\backslash\{0\}$. In this case, if for some positive integer k, ψ satisfies (4.2.48) it also satisfies both (4.2.49) and (4.2.50). Further, if, for some positive integer k, the function ψ reproduces polynomials of degree k then $k < l - d$ and ψ satisfies (4.2.53).

Proof. Suppose that ψ satisfies (4.2.48). In this case $k < l - d$ and we suppose $k = l - d - \delta$ with $\delta > 0$. It follows that (4.2.49) holds and also that the series (4.2.50) converges uniformly and so the resulting function is continuous, as required.

The proof of the last statement of the theorem is not quite so simple. Let ψ reproduce polynomials of degree k. Because τ is not identically zero and homogeneous we may pick z with $\|z\| = 1$ for which $\tau(z) = a \neq 0$. Without loss of generality we assume that $a > 0$. Because ψ is continuous it can be deduced that τ, being homogeneous, is continuous on $\mathcal{R}^d\backslash\{0\}$. Hence, we can pick $\eta > 0$ so that

$$\|y - z\| \leq \eta \Rightarrow \tau(y) > a/2. \tag{4.2.82}$$

We let

$$S = \{y : \|y\| = 1, \ \|y - z\| < \eta\} \tag{4.2.83}$$

and we pick M so that

$$|\psi(y)| \geq \tfrac{1}{2}|\tau(y)|, \qquad \|y\| \geq M, \quad y/\|y\| \in S. \tag{4.2.84}$$

As ψ reproduces polynomials of degree k, all sums of the form

$$\sum_{z \in \mathcal{Z}^d} z^\alpha \psi(z), \tag{4.2.85}$$

for $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq k$, must be absolutely convergent. Therefore, noting that

$$\sum_{i=1}^{d} |z_i|^k \geq b\|z\|^k, \qquad z \in \mathcal{R}^d \tag{4.2.86}$$

and some $b > 0$, it follows that

$$\sum_{z \in \mathcal{Z}^d} \|z\|^k \psi(z) \tag{4.2.87}$$

is also absolutely convergent. However, using (4.2.84) and the degree $-l$ homogeneity of τ,

$$\begin{aligned}
\sum_{z \in \mathcal{Z}^d} \|z\|^k |\psi(z)| &\geq \sum_{\{z \in \mathcal{Z}^d : \|z\| \geq M,\, z/\|z\| \in S\}} \|z\|^k |\psi(z)| \\
&\geq \tfrac{1}{2} \sum_{\{z \in \mathcal{Z}^d : \|z\| \geq M,\, z/\|z\| \in S\}} \|z\|^k |\tau(z)| \\
&= \tfrac{1}{2} \sum_{\{z \in \mathcal{Z}^d : \|z\| \geq M,\, z/\|z\| \in S\}} \|z\|^k \|z\|^{-l} |\tau(z/\|z\|)| \\
&\geq \tfrac{a}{4} \sum_{\{z \in \mathcal{Z}^d : \|z\| \geq M,\, z/\|z\| \in S\}} \|z\|^{k-l},
\end{aligned} \tag{4.2.88}$$

the last line using (4.2.82) and (4.2.83). Further, using (4.2.83) again, we find that this sum is at least some constant multiple of

$$\sum_{\{z \in \mathcal{Z}^d : \|z\| \geq M\}} \|z\|^{k-l}. \tag{4.2.89}$$

This sum converges only if $k < l - d$. However, in this case all integrals of the form

$$\int_{\mathcal{R}^d} y^\alpha \psi(y)\, dy, \tag{4.2.90}$$

for $\alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq k$, are absolutely convergent. Therefore condition (4.2.53) is satisfied, which completes the proof. ■

In some cases Lemma 4-17 enables us to give a simpler statement of Theorem 4-15 and also to deduce more about functions ψ of the form (4.2.45). One such formulation is

Corollary 4-18. *Suppose that $\phi^* : \mathcal{R}^d \to \mathcal{R}$ is a function with a generalised Fourier transform $\hat{\phi}^*$. Suppose also that $\hat{\phi}^*$ has all partial derivatives of all orders on $\mathcal{R}^d \backslash \{0\}$, satisfies (4.2.46) and that there exist homogeneous polynomials of degree s $\{\tilde{P}_s : s = r, r+1, \ldots\}$ (with $\tilde{P}_r \not\equiv 0$) so that, near $\lambda = 0$,*

$$\hat{\phi}^*(\lambda) = \frac{1}{\sum_{s=r}^{\infty} \tilde{P}_s(\lambda)}. \tag{4.2.91}$$

In this case any ψ of the form (4.2.45) which tends to zero for large argument satisfies either

$$\psi(x) = o(\|x\|^{-k}) \text{ as } \|x\| \to \infty, \tag{4.2.92}$$

for all $k \in \mathcal{Z}^+$, or

$$\psi(x) = \tau(x) + o(\|x\|^{-l}) \text{ as } \|x\| \to \infty, \tag{4.2.93}$$

where τ is a homogeneous function of degree $-l$ which is not identically zero on $\mathcal{R}^d\backslash\{0\}$, l being a positive integer.

Further, there exist functions ψ of the form (4.2.45) which reproduce all polynomials of degree $r-1$. All such functions satisfy

$$\hat{\psi}(\lambda) = 1 + \bar{h}(\lambda), \tag{4.2.94}$$

near $\lambda = 0$, for some r-regularly differentiable function $\bar{h}$, and

$$|\psi(x)| \leq A/(1 + \|x\|^{d+r}), \qquad x \in \mathcal{R}^d. \tag{4.2.95}$$

Moreover, there exist such functions ψ satisfying (4.2.93) for each integer $l \geq d + r$. Further, if ϕ^* also satisfies conditions (4.2.51)–(4.2.52), then $r-1$ is the maximum degree for which all polynomials can be reproduced by a function of the form (4.2.45).

Proof. Any ψ of the form (4.2.45) which tends to zero for large argument has Fourier transform $\hat{\psi}$ of the form (4.2.62). Near $\lambda = 0$, we may form the expansion (4.2.55):

$$g(\lambda) = \sum_{j=1}^{l} \mu_j e^{-i\lambda . x_j} = \sum_{s=0}^{\infty} \tilde{q}_s(\lambda). \tag{4.2.96}$$

Thus, near $\lambda = 0$,

$$\hat{\psi}(\lambda) = \frac{\sum_{s=0}^{\infty} \tilde{q}_s(\lambda)}{\sum_{s=r}^{\infty} \tilde{P}_s(\lambda)}. \tag{4.2.97}$$

We may expand the right hand side as a power series in λ, near $\lambda = 0$:

$$\hat{\psi}(\lambda) = \sum_{s=-r}^{\infty} \tilde{f}_s(\lambda), \tag{4.2.98}$$

where each $\tilde{f}_s$ is a homogeneous rational function of degree s. There are two possibilities: firstly the expression (4.2.98) may be a power series in λ. In this case $\hat{\psi}$ has all partial derivatives of all orders at 0. Therefore, because $\hat{\phi}^*$ also satisfies (4.2.46), we find, using (4.2.66) and the boundedness of g,

$$\int_{\mathcal{R}^d} \left| \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} \right| d\lambda < \infty, \tag{4.2.99}$$

for all $\alpha \in (\mathcal{Z}^+)^d$, and hence Lemma 4-13 gives us (4.2.92). Otherwise we denote the leading order term in the expansion (4.2.98), which is not a polynomial, by σ and suppose that σ is homogeneous of degree $\tilde{l}$. We note that $\hat{\psi}$ must be absolutely integrable in a neighbourhood of the origin to ensure that ψ tends to zero for large argument and hence we must have $\tilde{l} \geq -d$. In this case, near $\lambda = 0$,

$$\hat{\psi}(\lambda) = P(\lambda) + \sigma(\lambda) + \tilde{h}(\lambda), \tag{4.2.100}$$

where P is a polynomial and $\tilde{h}$ is an $(\tilde{l}+1)$-regularly differentiable function. We want to deduce (4.2.93) when (4.2.100) holds. Intuitively we argue that the highest order singularity of $\hat{\psi}$ at the origin comes from the term σ and hence the leading order term in an expansion of ψ for large $\|x\|$ will come from the Fourier transform of σ, but it is neccesary to proceed more carefully to make this deduction watertight.

Suppose that equation (4.2.100) is valid in $\|\lambda\| \leq \delta$ and let ρ be a smooth function satisfying $0 \leq \rho(\lambda) \leq 1$ and

$$\rho(\lambda) = \begin{cases} 1 & \text{if } \|\lambda\| \leq \frac{1}{2}\delta; \\ 0 & \text{if } \|\lambda\| \geq \delta. \end{cases} \tag{4.2.101}$$

Because $\hat{\phi}^*$ has all partial derivatives of all orders on $\mathcal{R}^d \backslash \{0\}$ and satisfies (4.2.46) we may use (4.2.66) to deduce that all partial derivatives of order $d+\tilde{l}$ of the function

$$\hat{\psi}(\lambda) - \rho(\lambda)\sigma(\lambda), \qquad \lambda \in \mathcal{R}^d, \tag{4.2.102}$$

are absolutely integrable over $\{\lambda \in \mathcal{R}^d : \|\lambda\| \geq \delta/2\}$. Further, (4.2.100) shows that they are all absolutely integrable over $\{\lambda \in \mathcal{R}^d : \|\lambda\| \leq \delta/2\}$ and so they are absolutely integrable over the whole of $\mathcal{R}^d$. Now, Lemma 4-13 yields that the inverse Fourier transform of (4.2.102) decays as $o(\|x\|^{-d-\tilde{l}})$ as $\|x\| \to \infty$. Hence, to show that the inverse Fourier transform of $\hat{\psi}$ satisfies (4.2.93) for some some function τ, homogeneous of degree $-l = -d - \tilde{l}$, it is sufficient to show that these conditions are obtained by the inverse Fourier transform of

$$\rho(\lambda)\sigma(\lambda) = (\rho(\lambda) - 1)\sigma(\lambda) + \sigma(\lambda), \qquad \lambda \in \mathcal{R}^d. \tag{4.2.103}$$

The two functions on the right hand side of (4.2.103) both have generalised inverse Fourier transforms. We note that all partial derivatives of $\rho(\lambda) - 1$ of order at least 1 are smooth functions supported on $\{\lambda \in \mathcal{R}^d : \delta/2 \leq \|\lambda\| \leq \delta\}$. Hence all partial derivatives of order $d+\tilde{l}+1$ of $(\rho(\lambda)-1)\sigma(\lambda)$ are absolutely integrable over $\mathcal{R}^d$ and so Lemma 4-13 yields that the inverse transform of this function decays as $o(\|x\|^{-d-\tilde{l}-1})$ as $\|x\| \to \infty$. Further, the function σ is homogeneous of integer degree $\tilde{l} > -d$ and hence its inverse Fourier transform τ is homogeneous of degree $-d-\tilde{l} \leq -1$, which can be deduced from Theorem 4-4 (g). We note further that its inverse Fourier transform is not identically zero on $\mathcal{R}^d \backslash \{0\}$ as we have chosen σ not to be a polynomial. This completes the verification of (4.2.93).

The proof of polynomial reproduction is now quite straightforward. Near $\lambda = 0$, we may rearrange expression (4.2.91) for $\hat{\phi}^*$ to the form

$$\hat{\phi}^*(\lambda) = \frac{1}{\sum_{s=r}^{2r-1} \tilde{P}_s(\lambda)} + h(\lambda), \tag{4.2.104}$$

where

$$h(\lambda) = \frac{-\sum_{s=2r}^{\infty} \tilde{P}_s(\lambda)}{\left(\sum_{s=r}^{\infty} \tilde{P}_s(\lambda)\right)\left(\sum_{s=r}^{2r-1} \tilde{P}_s(\lambda)\right)}. \tag{4.2.105}$$

From Lemma 4-14 we find that h is a 0-regularly differentiable function. Equation (4.2.104) corresponds to (4.2.47) in Theorem 4-15 with $t = 2r - 1$ and $\epsilon' = 1$. Therefore, as we have already assumed (4.2.46) and Lemma 4-17 gives us (4.2.48)–(4.2.50), we may use Theorem 4-15 to deduce that there exists a function ψ of the form (4.2.45) which reproduces all polynomials of degree $r-1$. Further, as (4.2.51) and (4.2.52) are assumed in the last part of the theorem and as Lemma 4-17 gives us (4.2.53), $r - 1$ is the maximum degree for which all polynomials can be reproduced by a function ψ of the form (4.2.45).

To complete the proof it only remains to show that any ψ of the form (4.2.45) reproducing polynomials of degree $r - 1$ satisfies (4.2.94) and (4.2.95) and that every $l \geq d + r$ is attainable in (4.2.93). Corollary 4-7 shows that any such ψ is continuous and, because it must tend to zero for large argument, it either satisfies (4.2.92) or (4.2.93). In the latter case Lemma 4-17 shows $l \geq d + r$ and so either case yields (4.2.95). This also implies that ψ satisfies (4.2.25) and (4.2.26) with $m = r - 1$ and so by Theorem 4-12 it satisfies (4.2.28) with $m = r - 1$. Combining this observation with the form of $\hat{\psi}$ near $\lambda = 0$ (4.2.97) yields (4.2.94). It remains to show that any $l \geq d + r$ is attainable in (4.2.93).

We let k be any non-negative integer and in (4.2.97) we choose

$$q_s \equiv 0, \quad s = 0, 1, \ldots, r-1, \tag{4.2.106}$$

$$\tilde{q}_s \equiv \tilde{P}_s, \quad s = r, r+1, \ldots, 2r+k-1 \tag{4.2.107}$$

and $\tilde{q}_{2r+k}$ so that

$$\frac{\tilde{q}_{2r+k} - \tilde{P}_{2r+k}}{\tilde{P}_r} \tag{4.2.108}$$

is not a polynomial. Because this choice includes the conditions (4.2.59) and (4.2.60), such a function reproduces all polynomials of degree $r-1$ by Theorem 4-15. Further, near $\lambda = 0$, we have the expansion

$$\begin{aligned}
\hat{\psi}(\lambda) &= \frac{\sum_{s=0}^{\infty} \tilde{q}_s(\lambda)}{\sum_{s=0}^{\infty} \tilde{P}_s(\lambda)} \\
&= \frac{\sum_{s=r}^{2r+k-1} \tilde{P}_s(\lambda) + \tilde{q}_{2r+k}(\lambda)}{\sum_{s=r}^{2r+k-1} \tilde{P}_s(\lambda) + \tilde{P}_{2r+k}(\lambda)} + h_1(\lambda) \\
&= 1 + \frac{\tilde{q}_{2r+k}(\lambda) - \tilde{P}_{2r+k}(\lambda)}{\sum_{s=r}^{2r+k-1} \tilde{P}_s(\lambda)} + h_2(\lambda)
\end{aligned}$$

$$= 1 + \frac{\tilde{q}_{2r+k}(\lambda) - \tilde{P}_{2r+k}(\lambda)}{\tilde{P}_r(\lambda)} + h_3(\lambda). \qquad (4.2.109)$$

where h_1, h_2 and h_3 are $(r+k)$-regularly differentiable functions. We see that the leading term which is not a polynomial is homogeneous of degree $r+k$. Hence, recalling that, when σ in (4.2.100) was homogeneous of degree $\tilde{l}$, ψ satisfied (4.2.93) with $l = -d - \tilde{l}$ (the paragraph after 4.2.103)), we find that ψ satisfies (4.2.93) with $l = -d - r - k$, as required. ■

Further, the following easy corollary shows that, if $\hat{\phi}^*$ retains its behaviour near 0 over all of $\mathcal{R}^d$, then the conditions for polynomial reproduction are even simpler.

Corollary 4-19. *Let $\phi^* : \mathcal{R}^d \to \mathcal{R}$ be a function with generalised Fourier transform $\hat{\phi}^*$. Suppose that there exist integers $\{d+1 \leq r_i \leq t_i \,:\, i = 1,2,\ldots,n\}$ and homogeneous polynomials $\tilde{P}_{i,s} \,:\, i = 1,2,\ldots,n, \quad s = r_i, r_i+1,\ldots,t_i$ of degree s (with each $\tilde{P}_{i,r_i} \not\equiv 0$) such that*

$$\hat{\phi}^*(\lambda) = \sum_{i=1}^{n} \frac{1}{\sum_{s=r_i}^{t_i} \tilde{P}_{i,s}(\lambda)}, \qquad \lambda \in \mathcal{R}^d. \qquad (4.2.110)$$

Suppose also that $\hat{\phi}^*$ has no singularities on $\mathcal{R}^d \backslash \{0\}$. In this case any function ψ of the form (4.2.45) which tends to zero for large argument either satisfies (4.2.92) or (4.2.93). There exist such functions ψ which reproduce all polynomials of degree $\min\{r_i \,:\, i = 1,2,\ldots,n\} - 1$ and this is the maximum degree for which all polynomials can be reproduced by such a function ψ. Further, all such functions satisfy (4.2.94) and (4.2.95) and there exist such functions which satisfy (4.2.93) for any integer $l \geq d + \min\{r_i \,:\, i = 1,2,\ldots,n\}$.

Proof. The function $\hat{\phi}^*$ is of the form (4.2.91) for small $\|\lambda\|$. It has no singularities on $\mathcal{R}^d \backslash \{0\}$ and hence has all partial derivatives there. It satisfies (4.2.46) and (4.2.51) because we have chosen each $r_i \geq d+1$. The fact that we only have a finite sum implies that (4.2.52) is satisfied and so the result follows from Corollary 4-18. ■

Section 4.3 : Examples of Suitable ϕ

In this section we apply the analysis that was developed in Section 4.2 to show that four of the following families of functions ϕ, examples of which occur frequently in the existing literature (see Section 1.3), do reproduce polynomials.

(a) r^b, $\quad b > 0$;

(b) $(r^2 + c^2)^{\frac{b}{2}}$, $\quad b \in \mathcal{R},\ c > 0$;

(c) $r^b \log r$, $\quad b > 0$;

(d) $(r^2 + c^2)^{\frac{b}{2}} \log(r^2 + c^2)^{\frac{1}{2}}$, $\quad b \in \mathcal{R},\ c > 0$;

(e) e^{-cr^b}, $\quad b > 1,\ c > 0$.

Although (a) and (c) are special cases of (b) and (d) respectively, we consider them separately because the analysis is quite different. Here we summarise the main results that will be obtained before we embark on the details. When we work in $\mathcal{R}^d$ we find:

Theorem 4-20. *Let m be the maximum integer so that all polynomials of degree m are reproduced by a function of the form* (4.2.45)

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d, \tag{4.3.1}$$

with l finite and $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$. For different functions ϕ the values of m are as follows:

$$\begin{array}{rl}
\phi(r) & m \\
r^b & \begin{cases} b+d-1 & \text{if } b \text{ and } d \text{ are both positive odd integers;} \\ \text{None} & \text{otherwise;} \end{cases} \\
(r^2+c^2)^{\frac{b}{2}} & \begin{cases} b+d-1 & \text{if } b+d \text{ is a positive even integer} \\ & \text{and } b \text{ is not a non-negative even integer;} \\ \text{None} & \text{otherwise;} \end{cases} \\
r^b \log r & \begin{cases} b+d-1 & \text{if } b \text{ and } d \text{ are both positive even integers;} \\ \text{None} & \text{otherwise;} \end{cases} \\
(r^2+c^2)^{\frac{b}{2}} \log(r^2+c^2)^{\frac{1}{2}} & \begin{cases} b+d-1 & \text{if } b+d \text{ is a positive even integer} \\ & \text{and } b \text{ is a non-negative even integer;} \\ \text{None} & \text{otherwise;} \end{cases} \\
e^{-cr^b} & \text{None, for any } b > 1.
\end{array}$$

Here *None* indicates that even $m = 0$ is not admissible for any absolutely integrable ψ. In each case we shall establish the result by evaluating the Fourier transform $\hat{\phi}$ of ϕ and then checking to see whether the conditions of Theorems 4-15 or 4-16 or Corollary 4-19 are satisfied. The details in the individual cases are as follows:

Analysis for the case $\phi(r) = r^b$, $b > 0$.

The restriction on b ensures the continuity of ϕ at the origin. If b is a positive even integer then any function ψ of the form (4.3.1) is a polynomial of degree b. Hence there can be no absolutely integrable ψ which is not identically zero. Hence, there is no non-trivial polynomial reproduction by quasi-interpolation. Therefore we now restrict b so that it is not a positive even integer. In this case the required Fourier transform (Jones 1982, Theorem 7.31) is the function

$$\hat{\phi}(\lambda) = \frac{\Gamma\left(\frac{b+d}{2}\right) 2^{b+d}\pi^{\frac{d}{2}}}{\Gamma\left(-\frac{b}{2}\right) \|\lambda\|^{b+d}}, \qquad \lambda \in \mathcal{R}^d. \tag{4.3.2}$$

When b and d are not both positive odd integers then $\hat{\phi}(\lambda)$ is not of the form (4.2.47) near $\lambda = 0$. We see that (4.2.51) is satisfied, because $b > 0$, and hence Theorem 4-16 shows that it is not possible to reproduce constant functions with an absolutely integrable function ψ.

It remains to consider the case when b and d are both positive odd integers and we use Corollary 4-19. We take $n = 1$, $r_1 = t_1 = b + d$ and

$$\tilde{P}_{1,b+d}(\lambda) = \frac{\Gamma\left(-\frac{b}{2}\right) \|\lambda\|^{b+d}}{\Gamma\left(\frac{b+d}{2}\right) 2^{b+d}\pi^{\frac{d}{2}}}, \qquad \lambda \in \mathcal{R}^d. \tag{4.3.3}$$

Hence Corollary 4-19 shows that there is a function ψ of the form (4.3.1) which reproduces all polynomials of degree $b + d - 1$, and that it is not possible to find one which reproduces all polynomials of degree $b + d$.

For future reference we also note that Corollary 4-19 yields that (4.2.95) holds for any function ψ of the form (4.3.1) which reproduces polynomials of degree $b + d - 1$ and hence it satisfies

$$|\psi(x)| \leq A/(1 + \|x\|^{b+2d}), \qquad x \in \mathcal{R}^d, \tag{4.3.4}$$

for some constant A. Further, it also yields that there exist such functions satisfying (4.2.93) for any integer $l \geq 2d + b$, and the expansion for large $\|x\|$ consists of homogeneous terms of integer degree. Finally, we note that (4.2.94) is also satisfied, near $\lambda = 0$, and so we have

$$\hat{\psi}(\lambda) = 1 + \bar{h}(\lambda), \tag{4.3.5}$$

where $\bar{h}$ is a $(b + d)$-regularly differentiable function.

Now we present sufficient conditions for reproducing all polynomials of degree at most $m \leq b + d - 1$. Near $\lambda = 0$, we have an expression of the form (4.2.47) for $\hat{\phi}$, in which

$$\tilde{P}_s(\lambda) = \begin{cases} 0 & \text{if } 0 \leq s \leq b + d - 1; \\ \dfrac{\Gamma\left(-\frac{b}{2}\right) \|\lambda\|^{b+d}}{\left(\frac{b+d}{2} - 1\right)! \, 2^{b+d}\pi^{\frac{d}{2}}} & \text{if } s = b + d; \\ 0 & \text{if } b + d + 1 \leq s \leq b + d + m. \end{cases} \tag{4.3.6}$$

We also see that in this case the remainder term $h(\lambda)$ is identically zero. Hence equations (4.2.59) and (4.2.60) give sufficient conditions to reproduce all polynomials of degree $m \leq b+d-1$, namely

$$\tilde{q}_s(\lambda) = \begin{cases} 0 & \text{if } 0 \leq s \leq b+d-1; \\ \dfrac{\Gamma\left(-\frac{b}{2}\right) \|\lambda\|^{b+d}}{\left(\frac{b+d}{2}-1\right)! \, 2^{b+d} \pi^{\frac{d}{2}}} & \text{if } s = b+d; \\ 0 & \text{if } b+d+1 \leq s \leq b+d+m. \end{cases} \tag{4.3.7}$$

Equation (4.2.58) shows that $\tilde{q}_s(\lambda) = A\|\lambda\|^s$ for s even if and only if $\{\mu_j : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{Z}^d : j = 1,2,\ldots,l\}$ satisfy

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} (-1)^{\frac{s}{2}} A \left(\frac{s}{2}\right)! \dfrac{\alpha!}{\left(\frac{\alpha}{2}\right)!} & \text{if } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = s, \end{cases} \tag{4.3.8}$$

where we recall that α is even if each component of α is even. Hence (4.3.7) yields

$$\begin{aligned} \sum_{j=1}^{l} \mu_j x_j^\alpha &= \begin{cases} (-1)^{\frac{b+d}{2}} \dfrac{\Gamma\left(-\frac{b}{2}\right)\left(\frac{b+d}{2}\right)!}{\left(\frac{b+d}{2}-1\right)! \, 2^{b+d}\pi^{\frac{d}{2}}} \dfrac{\alpha!}{\left(\frac{\alpha}{2}\right)!} & \text{if } |\alpha| = b+d \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq b+d+m, \end{cases} \\ &= \begin{cases} \dfrac{(-1)^{\frac{d-1}{2}}(b+d)\,\alpha!}{2^{b+d+1}\pi^{\frac{d-1}{2}} \left(\frac{1}{2}\right)_{\frac{b+1}{2}} \left(\frac{\alpha}{2}\right)!} & \text{if } |\alpha| = b+d \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq b+d+m, \end{cases} \end{aligned} \tag{4.3.9}$$

where we have used the relation

$$\Gamma\left(-\tfrac{b}{2}\right) = \frac{\Gamma\left(\frac{1}{2}\right)}{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{b}{2}\right)} = \frac{(-1)^{\frac{b+1}{2}} \pi^{\frac{1}{2}}}{\left(\frac{1}{2}\right)_{\frac{b+1}{2}}}, \tag{4.3.10}$$

and where $\left(\frac{1}{2}\right)_{\frac{b+1}{2}}$ is defined in (2.2.1). Putting $b = 1$ in the formula (4.3.9), we obtain from (2.3.32)

$$\begin{aligned} \int_{\partial S(0,1)} y^\alpha \, dy &= \begin{cases} \dfrac{(-1)^{\frac{d+1}{2}} \pi^{d-1}}{(1+d)\left(-\frac{1}{2}\right)_{1+d}} \dfrac{(-1)^{\frac{d-1}{2}}(1+d)\,\alpha!}{2^{1+d}\pi^{\frac{d-1}{2}} \left(\frac{\alpha}{2}\right)!} & \text{if } |\alpha| = 1+d \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 1+d, \end{cases} \\ &= \begin{cases} \dfrac{\pi^{\frac{d-1}{2}}}{\left(\frac{1}{2}\right)_d 2^d} \dfrac{\alpha!}{\left(\frac{\alpha}{2}\right)!} & \text{if } |\alpha| = 1+d \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 1+d, \end{cases} \end{aligned} \tag{4.3.11}$$

an interesting result in its own right, which corroborates (2.3.37) when $d = 3$.

Finally, applying (4.3.9) to the examples at the end of Chapter 2, we note that (2.3.38), when scaled as after (3.1.4), can reproduce all constants, (2.3.39) can reproduce all linear polynomials and (2.3.40) can reproduce all cubic polynomials, as we would expect from the analysis in Chapter 3.

Analysis for the case $\phi(r) = (r^2 + c^2)^{\frac{b}{2}}$, $\qquad b \in \mathcal{R},\ c > 0$.

We remark initially that Buhmann (1988b) has presented the analysis when b takes on the values 1 and -1. As in the previous case we note that, if b is a non-negative even integer, then any function ψ of the form (4.3.1) is a polynomial of degree b. Hence there can be no absolutely integrable ψ which is not identically zero, and so there is no non-trivial polynomial reproduction by quasi-interpolation. Therefore we now restrict b so that it is not a non-negative even integer. The required Fourier transform (Gel'fand and Shilov 1964 Chapter 3; 2.8, (5)) is the function

$$\hat{\phi}(\lambda) = \frac{2^{\frac{b+d}{2}+1}\pi^{\frac{d}{2}}c^{\frac{b+d}{2}}}{\Gamma(-\frac{b}{2})}\ \frac{K_{\frac{b+d}{2}}(c\|\lambda\|)}{\|\lambda\|^{\frac{b+d}{2}}}, \qquad \lambda \in \mathcal{R}^d, \tag{4.3.12}$$

where K is a modified Bessel (or Macdonald's) function (Abramowitz and Stegun 1970, 9.6). We first aim to satisfy (4.2.47) near $\lambda = 0$. Thus, we examine the asymptotic behaviour of $K_{\frac{b+d}{2}}(z)$ for small positive z. We find (Abramowitz and Stegun 1970, 9.6.6, 9.6.7 and 9.6.9) that

$$K_{\frac{b+d}{2}}(z) \sim \begin{cases} \frac{1}{2}\Gamma\left(\left|\frac{b+d}{2}\right|\right)\left(\frac{1}{2}z\right)^{-\left|\frac{b+d}{2}\right|} & \text{if } \frac{b+d}{2} \neq 0; \\ -\log z & \text{if } \frac{b+d}{2} = 0. \end{cases} \tag{4.3.13}$$

Comparing (4.3.12) and (4.3.13), we find that $\phi(\lambda)$ cannot be of the form (4.2.47) near $\lambda = 0$ when $b+d$ is not an even integer. Therefore, by Theorem 4-16, it is not possible to reproduce constants in this case with an absolutely integrable ψ if condition (4.2.51) is satisfied. This condition is obtained because, for large, positive x and any ν, we have the relation (Abramowitz and Stegun 1970, 9.7.2)

$$K_\nu(x) \sim \sqrt{\frac{\pi}{2x}}e^{-x}. \tag{4.3.14}$$

which is sufficient.

It remains to consider the case when $b+d$ is a positive even integer and b is not a non-negative even integer. We find (Abramowitz and Stegun 1970, 9.6.11) that

$$\hat{\phi}(\lambda) = \frac{2^{b+d}\pi^{\frac{d}{2}}\|\lambda\|^{-b-d}}{\Gamma(-\frac{b}{2})}\sum_{k=0}^{\frac{b+d}{2}-1}\frac{(\frac{b+d}{2}-k-1)!}{k!}(-\tfrac{1}{4}c^2\|\lambda\|^2)^k \ + \ \bar{h}(\lambda), \tag{4.3.15}$$

where the leading term of $\bar{h}(\lambda)$ is a non-zero multiple of $\log\|\lambda\|$ and the remainder is a 0-regularly differentiable function. From Lemma 4-14 we find that $\bar{h}(\lambda)$ is a $(-\delta)$-regularly differentiable function for any $\delta > 0$, but we have to express (4.3.15) in the form (4.2.47).

Denoting "$\frac{1}{4}c^2\|\lambda\|^2$" and "$\frac{b+d}{2}$" by "$x$" and "$n$" respectively, it is sufficient to rearrange the finite sum

$$\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}(-x)^k \tag{4.3.16}$$

to the form

$$\frac{1}{\sum_{l=0}^{n-1} A_l x^l} + \tilde{h}(x), \tag{4.3.17}$$

where each $A_l \in \mathcal{R}$ and $\tilde{h}(x)$ is a (univariate) n-regularly differentiable function. Indeed, in this case we would have

$$\begin{aligned}\hat{\phi}(\lambda) &= \frac{2^{b+d}\pi^{\frac{d}{2}}\|\lambda\|^{-b-d}}{\Gamma(-\frac{b}{2})} \frac{1}{\sum_{l=0}^{\frac{b+d}{2}-1} A_l(\frac{1}{4}c^2\|\lambda\|^2)^l} + h(\lambda) \\ &= \frac{1}{\sum_{l=0}^{\frac{b+d}{2}-1}(A_l\Gamma(-\frac{b}{2})\pi^{-\frac{d}{2}}2^{-b-d-2l}c^{2l})\|\lambda\|^{b+d+2l}} + h(\lambda),\end{aligned} \tag{4.3.18}$$

where

$$h(\lambda) = \bar{h}(\lambda) + \frac{2^{b+d}\pi^{\frac{d}{2}}}{\Gamma(-\frac{b}{2})} \frac{\tilde{h}(\frac{1}{4}c^2\|\lambda\|^2)}{\|\lambda\|^{b+d}} \tag{4.3.19}$$

is a $(-\delta)$-regularly differentiable function for any $\delta > 0$. Expression (4.3.18) is of the form (4.2.47) with $r = b + d$ and $t = 2b + 2d - 1$ (taking $\tilde{P}_t \equiv 0$), h being a $(t - 2r + \epsilon')$-regularly differentiable function for any $0 < \epsilon' < 1$.

In order to express the series (4.3.16) in the form (4.3.17), we recall the definition of the factorial function (2.2.1), which gives

$$\begin{aligned}\sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!}(-x)^k &= (n-1)!\sum_{k=0}^{n-1} \frac{1}{k!(1-n)_k} x^k \\ &= (n-1)! \frac{1}{\left(1 + \sum_{k=1}^{n-1} \frac{1}{k!(1-n)_k} x^k\right)^{-1}} \\ &= (n-1)! \frac{1}{\sum_{m=0}^{\infty}(-1)^m \left(\sum_{k=1}^{n-1} \frac{1}{k!(1-n)_k} x^k\right)^m}.\end{aligned} \tag{4.3.20}$$

From this expression we can pick out the coefficients of the terms x^l, $l = 0, 1, \ldots, n-1$ and find that $A_0 = 1/(n-1)! = 1/(\frac{b+d}{2} - 1)!$ and for $1 \leq l \leq n-1$,

$$\begin{aligned}A_l &= \frac{1}{(n-1)!} \sum_{w=1}^{l} (-1)^w \sum_{\{\alpha \in (\mathcal{Z}^+ \setminus \{0\})^w : |\alpha| = l\}} \frac{1}{\alpha!(1-n)_\alpha} \\ &= \frac{1}{(\frac{b+d}{2} - 1)!} \sum_{w=1}^{l} (-1)^w \sum_{\{\alpha \in (\mathcal{Z}^+ \setminus \{0\})^w : |\alpha| = l\}} \frac{1}{\alpha!(1 - \frac{b+d}{2})_\alpha},\end{aligned} \tag{4.3.21}$$

where we are using the multi-index notation defined at the end of the introduction. For example, the suitable values of α when $l = 4$ and $w = 2$ are $(1, 3)$, $(2, 2)$ and $(3, 1)$.

We now check that all remaining conditions of Theorem 4-15 are satisfied to deduce that it is possible to reproduce all polynomials of degree $b+d-1$ with a function ψ of the form (4.3.1) and that no higher degree is possible.

We have already deduced (4.2.51) from (4.3.14) and (4.3.12) and we note that (4.2.46) holds for the same reason. Also (4.2.52) holds for all $\beta \in \mathcal{Z}^d \backslash \{0\}$, because $K_{\frac{b+d}{2}}(x)$ is positive for all positive x (Abramowitz and Stegun 1970, remark after 9.6.1).

We also note that, for fixed $y \in \mathcal{R}^d$,

$$(\|x-y\|^2+c^2)^{\frac{b}{2}} = \|x\|^b \left(1+\frac{-2x.y+\|y\|^2+c^2}{\|x\|^2}\right)^{\frac{b}{2}}. \tag{4.3.22}$$

This may be expanded as a power series for sufficiently large $\|x\|$ and hence we have an expansion

$$(\|x-y\|^2+c^2)^{\frac{b}{2}} = \sum_{k=0}^{\infty} f_{b-k}(x), \tag{4.3.23}$$

for all sufficiently large $\|x\|$, where each f_{b-k} is homogeneous of degree $b-k$. For example

$$f_b(x) = \|x\|^b \text{ and } f_{b-1}(x) = -b\|x\|^{b-2}x.y. \tag{4.3.24}$$

Thus, for large $\|x\|$, the leading term in the expansion of any function ψ of the form (4.2.45) is a homogeneous term of some integer degree. By Lemma 4-17 this is sufficient to satisfy (4.2.48)–(4.2.50) and (4.2.53).

Hence we have deduced (4.2.46), (4.2.47) with $r = b+d$, $t = 2b+2d-1$ and $\epsilon' = 1$ (in (4.3.18) and (4.3.19)) and (4.2.48)–(4.2.53) and so Theorem 4-15 implies that it is possible to reproduce all polynomials of degree $b+d-1$ with a function ψ of the form (4.3.1) and that no higher degree is possible.

For future reference we note that any such ψ reproducing polynomials of degree $b+d-1$ is continuous and, by the remark after (4.3.24), its expansion for large argument consists of homogeneous terms of integer degree. Now Lemma 4-17 yields that

$$|\psi(x)| \le A/(1+\|x\|^{b+2d}), \qquad x \in \mathcal{R}^d, \tag{4.3.25}$$

for some constant A. We also consider the behaviour of its Fourier transform $\hat{\psi}$ near $\lambda = 0$. Because such a function ψ satisfies (4.2.25) and (4.2.26) for $m = b+d-1$ it also satisfies (4.2.28) for this value of m. We also note, from the remark after (4.3.12), that the leading order term in (4.3.19) is a non-zero multiple of $\log\|\lambda\|$ and the remainder a 0-regularly differentiable function. From these two observations and (4.3.18) it can be deduced that, near $\lambda = 0$,

$$\hat{\psi}(\lambda) = 1 + B\|\lambda\|^{b+d}\log\|\lambda\| + h_1(\lambda), \tag{4.3.26}$$

for some non-zero constant B, where h_1 is a $(b+d)$-regularly differentiable function. This expansion can be used to yield, by similar analysis to that used in the proof of Corollary 4-18, that the leading order term in the expansion for any such ψ cannot decay faster than $O(\|x\|^{b+2d})$ as $\|x\| \to \infty$. However, we shall deduce this as an easy corollary of the convergence order deduced for this basis function in Section 5.3.

We go further and present sufficient conditions for the reproduction of all polynomials of degree $m \le b+d-1$. Recalling that (4.3.18) is of the form (4.2.47), we find that sufficient conditions, given by equations (4.2.59) and (4.2.60) in the proof of Theorem 4-15, are

$$\tilde{q}_s(\lambda) = \begin{cases} \dfrac{A_{\frac{s-b-d}{2}}\Gamma(-\frac{b}{2})c^{s-b-d}\|\lambda\|^s}{\pi^{\frac{d}{2}}2^s} & \text{if } s \text{ is even and } b+d \le s \le b+d+m; \\ 0 & \text{if } 0 \le s \le b+d-1 \text{ or if } s \text{ is odd and } s \le b+d+m. \end{cases} \tag{4.3.27}$$

It is more useful to have conditions on $\{\mu_j : j = 1, 2, \ldots, l\}$ and $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$ and we use (4.3.8) to deduce these from (4.3.27), as each non-zero $\tilde{q}_s$ is a multiple of $\|\lambda\|^s$. This gives the equivalent conditions

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{(-1)^{\frac{|\alpha|}{2}} A_{\frac{|\alpha|-b-d}{2}}\Gamma(-\frac{b}{2})c^{|\alpha|-b-d}(\frac{|\alpha|}{2})!}{\pi^{\frac{d}{2}}2^{|\alpha|}} \dfrac{\alpha!}{(\frac{\alpha}{2})!} & \text{if } b+d \le |\alpha| \le b+d+m \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le b+d+m. \end{cases} \tag{4.3.28}$$

Recalling that $A_0 = 1/(\frac{b+d}{2}-1)!$, we note that the condition when $|\alpha| = b+d$ is exactly the same as the condition in the case when $\phi(r) = r^b$ (4.3.9), which is what is expected as $c \to 0$.

For example we specialise to the case $b = 1$ and $d = 3$. In this case to reproduce constants expression (4.3.28) gives the equations

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{-\alpha!}{4\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 4 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le 4, \end{cases} \tag{4.3.29}$$

where we recall that $\Gamma(-\frac{1}{2}) = -2\pi^{\frac{1}{2}}$. To reproduce all linear polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 5, \tag{4.3.30}$$

to reproduce all quadratic polynomials we also require

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{3c^2\alpha!}{16\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 6 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 6, \end{cases} \tag{4.3.31}$$

and to reproduce all cubic polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 7. \tag{4.3.32}$$

We notice that when $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$, the weights (2.3.38) are suitable for reproducing constant functions in 3 dimensions for all c. Also (2.3.39) reproduces all linear polynomials, as does (2.3.40). The choice (2.3.40), however, does not reproduce quadratic polynomials when $c > 0$, because condition (4.3.31) fails. Any function that is to reproduce cubic polynomials for $c > 0$ must have weights which depend on c.

It can be verified that the following function with $l = 59$ is sufficient:

$$\begin{array}{rl}
\mu_j & x_j \\
(-49 - 81c^2)/8\pi & (0,0,0); \\
(142 + 369c^2)/96\pi & (\pm1,0,0),(0,\pm1,0),(0,0,\pm1); \\
(-25 - 54c^2)/96\pi & (\pm2,0,0),(0,\pm2,0),(0,0,\pm2); \\
(2 + 3c^2)/96\pi & (\pm3,0,0),(0,\pm3,0),(0,0,\pm3); \\
(-4 - 27c^2)/24\pi & (\pm1,\pm1,\pm1); \\
(1 + 72c^2)/384\pi & (\pm2,\pm2,\pm2); \\
3c^2/32\pi & (\pm2,\pm1,0),(\pm2,0,\pm1),(\pm1,\pm2,0), \\
 & (0,\pm2,\pm1),(\pm1,0,\pm2),(0,\pm1,\pm2).
\end{array} \tag{4.3.33}$$

We also consider the case $b = -1$ and $d = 3$. In this case the maximum degree of polynomials that can be reproduced is 1. To reproduce all constants expression (4.3.28) implies

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{-\alpha!}{4\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 2 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le 2, \end{cases} \tag{4.3.34}$$

and to reproduce all linear polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 3. \tag{4.3.35}$$

In this case it is easy to find basis functions which reproduce all linear polynomials (the maximum degree possible), for example:

$$\begin{array}{rl}
\mu_j & x_j \\
3/2\pi & (0,0,0); \\
-1/4\pi & (\pm1,0,0),(0,\pm1,0),(0,0,\pm1).
\end{array} \tag{4.3.36}$$

Finally we consider the case when $b = -2$ and $d = 4$. Again the maximum degree of polynomials that can be reproduced is 1. In this case to reproduce all constants we require

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{-\alpha!}{4\pi^2(\frac{\alpha}{2})!} & \text{if } |\alpha| = 2 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| \leq 2, \end{cases} \tag{4.3.37}$$

and to reproduce all linear polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d \ : \ |\alpha| = 3. \tag{4.3.38}$$

Thus, for example, the following basis function reproduces all linear polynomials (the maximum degree possible):

$$\begin{array}{cl} \mu_j & x_j \\ 2/\pi^2 & (0,0,0,0); \\ -1/4\pi^2 & (\pm 1,0,0,0),(0,\pm 1,0,0),(0,0,\pm 1,0),(0,0,0,\pm 1). \end{array} \tag{4.3.39}$$

Analysis for the case $\phi(r) = r^b \log r, \quad b > 0$.

In this case the restriction $b > 0$ is necessary for the continuity of ϕ at 0. The required Fourier transform (Jones 1982, Theorem 7.34) is the function

$$\hat{\phi}(\lambda) = \begin{cases} \dfrac{(-1)^{\frac{b}{2}+1}\Gamma(\frac{b+d}{2})(\frac{b}{2})!\,2^{b+d-1}\pi^{\frac{d}{2}}}{\|\lambda\|^{b+d}} & \text{if } b \text{ is a positive even integer;} \\ \dfrac{\Gamma(\frac{b+d}{2})2^{b+d}\pi^{\frac{d}{2}}}{\Gamma(-\frac{b}{2})\|\lambda\|^{b+d}}(\frac{1}{2}\mathsf{K}(\frac{b+d}{2}-1)+\frac{1}{2}\mathsf{K}(-\frac{b}{2}-1)-\log(\frac{\|\lambda\|}{2})) & \text{otherwise,} \end{cases} \tag{4.3.40}$$

where

$$\mathsf{K}(z) = \frac{1}{\Gamma(z+1)}\frac{d\Gamma(z+1)}{dz}, \qquad z \neq -1,-2,\ldots. \tag{4.3.41}$$

When b and d are not both positive even integers then $\hat{\phi}$ cannot be of the form (4.2.47) near $\lambda = 0$. Because $b > 0$, (4.2.51) holds and so Theorem 4-16 shows that it is not possible to reproduce all constants with an absolutely integrable function ψ.

Therefore we restrict b and d to be positive even integers. As in the case $\phi(r) = r^b$, Corollary 4-19 shows that all polynomials of degree $b + d - 1$ can be reproduced by a function of the form (4.3.1) and that this is the maximum degree possible.

For future reference we note that all the remarks we made about the function $\phi(r) = r^b$ in the paragraph including (4.3.4) and (4.3.5) also hold in this case.

Now, after allowing for the difference in the constants of expressions (4.3.2) and (4.3.40), the technique used to deduce conditions on $\{\mu_j : j = 1,2,\ldots,l\}$ and $\{x_j : j = 1,2,\ldots,l\}$ to enable polynomials of degree m to be reproduced, so long as $m \le b + d - 1$, is exactly as in the case $\phi(r) = r^b$ and yields:

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{(-1)^{\frac{d}{2}-1}(b+d)}{(\frac{b}{2})!\,2^{b+d}\pi^{\frac{d}{2}}} \dfrac{\alpha!}{(\frac{\alpha}{2})!} & \text{if } |\alpha| = b+d \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le b+d+m. \end{cases} \tag{4.3.42}$$

We consider the case when $b = d = 2$. The maximum degree of polynomials that can be reproduced is 3. In this case to reproduce all polynomials of degree m, $0 \le m \le 3$, conditions (4.3.42) require

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{\alpha!}{4\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 4 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le 4+m. \end{cases} \tag{4.3.43}$$

Therefore the following basis function, with $l = 21$, reproduces all cubic polynomials:

μ_j	x_j
$175/48\pi$	$(0,0)$;
$-71/48\pi$	$(\pm1,0),(0,\pm1)$;
$25/96\pi$	$(\pm2,0),(0,\pm2)$;
$-1/48\pi$	$(\pm3,0),(0,\pm3)$;
$1/3\pi$	$(\pm1,\pm1)$;
$-1/192\pi$	$(\pm2,\pm2)$.

(4.3.44)

To show the increase in complexity for reproduction of higher order polynomials we consider the case $b = 4$ and $d = 2$. Now the maximum degree of polynomials that can be reproduced is 5, and to reproduce all polynomials of degree m, $0 \le m \le 5$, the conditions (4.3.42) are the equations

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{3\alpha!}{32\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 6 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le 6+m. \end{cases} \tag{4.3.45}$$

With some endeavour it may be checked that the following basis function, with $l = 37$, reproduces all quintic polynomials:

$$
\begin{array}{rl}
\mu_j & x_j \\
-737132/245760\pi & (0,0); \\
342432/245760\pi & (\pm1,0),(0,\pm1); \\
-91104/245760\pi & (\pm2,0),(0,\pm2); \\
21904/245760\pi & (\pm3,0),(0,\pm3); \\
-3082/245760\pi & (\pm4,0),(0,\pm4); \\
208/245760\pi & (\pm5,0),(0,\pm5); \\
-93696/245760\pi & (\pm1,\pm1); \\
8112/245760\pi & (\pm2,\pm2); \\
-512/245760\pi & (\pm3,\pm3); \\
21/245760\pi & (\pm4,\pm4).
\end{array} \tag{4.3.46}
$$

Analysis for the case $\phi(r) = (r^2+c^2)^{\frac{b}{2}} \log(r^2+c^2)^{\frac{1}{2}}, \quad b \in \mathcal{R}$.

Neither Jones (1982) nor Gel'fand and Shilov (1964) give a form for the generalised Fourier transform of th function ϕ, but it is not difficult to calculate it using, for instance, the technique of proof found in Jones (1982, Theorem 7-34). We find

$$
\hat{\phi}(\lambda) = \begin{cases} (-1)^{\frac{b}{2}+1} 2^{\frac{b+d}{2}} (\frac{b}{2})! \, \pi^{\frac{d}{2}} c^{\frac{b+d}{2}} \dfrac{K_{\frac{b+d}{2}}(c\|\lambda\|)}{\|\lambda\|^{\frac{b+d}{2}}} & \text{if } b \text{ is a non-negative even integer;} \\ \left[\left(\frac{1}{2}\mathsf{K}(-\frac{b}{2}-1) - \frac{1}{2}\log(\frac{\|\lambda\|}{2c})\right) K_{\frac{b+d}{2}}(c\|\lambda\|) + \dfrac{d}{db}\left(K_{\frac{b+d}{2}}(c\|\lambda\|)\right)\right] \dfrac{2^{\frac{b+d}{2}+1}\pi^{\frac{d}{2}} c^{\frac{b+d}{2}}}{\Gamma(-\frac{b}{2})\|\lambda\|^{\frac{b+d}{2}}} & \text{otherwise,} \end{cases} \tag{4.3.47}
$$

where $\mathsf{K}(z)$ is defined in (4.3.41). There is no simple expression for $\frac{d}{db} K_{\frac{b+d}{2}}(c\|\lambda\|)$, but it may be deduced from Abramowitz and Stegun (1970, 9.6.42–46) that it is not possible for the second line of (4.3.47) ever to be of the form (4.2.47) near $\lambda = 0$. The analysis which was used in case (b) (4.3.13) shows that the first line cannot be of that form if $b + d$ is not an even integer. In both these cases (4.2.51) holds by (4.3.14) and so Theorem 4-16 shows that in these cases it is not possible to reproduce constants with an absolutely integrable function ψ.

Therefore we restrict $b+d$ to be a positive even integer and b to be a non-negative even integer. To deduce polynomial reproduction in this case we use Theorem 4-15 as for case (b). The analysis given there implies that (4.2.47) holds with $r = b+d$, $t = 2b+2d-1$ and $0 < \epsilon' < 1$ as do (4.2.46), (4.2.51) and (4.2.52). To check the other conditions we proceed as in case (b) and look for a series expansion for $\phi(x-y)$. For fixed $y \in \mathcal{R}^d$, we find

$$\begin{aligned}(\|x-y\|^2+c^2)^{\frac{b}{2}}\log((\|x-y\|^2+c^2)^{\frac{1}{2}}) \\ = (\|x\|^2 - 2x.y + \|y\|^2 + c^2)^{\frac{b}{2}} \tfrac{1}{2}\left(\log\|x\|^2 + \log\left(1 + \frac{-2x.y+\|y\|^2+c^2}{\|x\|^2}\right)\right).\end{aligned} \tag{4.3.48}$$

The first term on the right hand side is just a polynomial in x, as b is a non-negative even integer. The final term may be expanded as a power series for sufficiently large $\|x\|$. Hence, we have an expansion, valid for sufficiently large $\|x\|$,

$$(\|x-y\|^2+c^2)^{\frac{b}{2}}\log((\|x-y\|^2+c^2)^{\frac{1}{2}}) = (\|x\|^2-2x.y+\|y\|^2+c^2)^{\frac{b}{2}}\log\|x\| + \sum_{k=0}^{\infty} f_{b-k}(x), \tag{4.3.49}$$

where each f_{b-k} is homogeneous of degree $b-k$. Thus, any ψ of the form (4.3.1) which decays as $\|x\| \to \infty$ has an expansion whose leading order term is homogeneous of some negative integer degree. In this case Lemma 4-17 yields that conditions (4.2.48)–(4.2.50) and (4.2.53) all hold. Hence, all conditions of Theorem 4-15 are satisfied and so functions of the form (4.3.1) can reproduce all polynomials of degree $b+d-1$ and this is the maximum possible degree.

Also, for future reference we note that the remarks made in case (b) (the paragraph containing (4.3.25) and (4.3.26)) also hold in this case.

Conditions on $\{\mu_j : j = 1,2,\ldots,l\}$ and $\{x_j \in \mathcal{Z}^d : j = 1,2,\ldots,l\}$ similar to (4.3.28) can be evaluated by the same analysis used in case (b) and they yield that to reproduce polynomials of degree $m \le b+d-1$ we require

$$\sum_{j=1}^{l} \mu_j x_j^{\alpha} = \begin{cases} \dfrac{(-1)^{\frac{b+|\alpha|}{2}+1} A_{\frac{|\alpha|-b-d}{2}}\, c^{|\alpha|-b-d}(\frac{|\alpha|}{2})!}{2^{|\alpha|-1}\pi^{\frac{d}{2}}(\frac{b}{2})!}\dfrac{\alpha!}{(\frac{\alpha}{2})!} & \text{if } b+d \le |\alpha| \le b+d+m \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d : |\alpha| \le b+d+m, \end{cases} \tag{4.3.50}$$

where A_l has the value (4.3.21). We notice that the conditions when $|\alpha| = b+d$ are exactly the same as those when $\phi(r) = r^b \log r$ (4.3.42), which is expected as $c \to 0$.

We consider the case $b = d = 2$. In this case the maximum degree of polynomials that can be reproduced is 3. The conditions to reproduce constants are:

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{\alpha!}{4\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 4 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \le 4. \end{cases} \tag{4.3.51}$$

To reproduce all linear polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = 5, \tag{4.3.52}$$

to reproduce all quadratic polynomials we also require

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{-3c^2\alpha!}{16\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 6 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = 6, \end{cases} \tag{4.3.53}$$

and to reproduce all cubic polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = 7. \tag{4.3.54}$$

Therefore, when $c > 0$, the weights (4.3.44) reproduce only linear polynomials. Any function ψ that is to reproduce cubic polynomials must have weights which depend on c.

For example it can be verified that the following function with $l = 21$ is sufficient:

$$\begin{array}{rl} \mu_j & x_j \\ 175 + 85c^2/48\pi & (0,0); \\ -71 - 117c^2/48\pi & (\pm1,0),(0,\pm1); \\ 25 + 45c^2/96\pi & (\pm2,0),(0,\pm2); \\ -1 - 3c^2/48\pi & (\pm3,0),(0,\pm3); \\ 4 + 9c^2/12\pi & (\pm1,\pm1); \\ -1 - 9c^2/192\pi & (\pm2,\pm2). \end{array} \tag{4.3.55}$$

Our final example is the case $b = 0$ and $d = 2$. Now the maximum degree of polynomial that can be reproduced is 1. In this case to reproduce constants we require

$$\sum_{j=1}^{l} \mu_j x_j^\alpha = \begin{cases} \dfrac{\alpha!}{2\pi(\frac{\alpha}{2})!} & \text{if } |\alpha| = 2 \text{ and } \alpha \text{ is even;} \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \le 2, \end{cases} \tag{4.3.56}$$

and to reproduce all linear polynomials we require in addition

$$\sum_{j=1}^{l} \mu_j x_j^{\alpha} = 0 \text{ for all } \alpha \in (\mathcal{Z}^+)^d : |\alpha| = 3. \tag{4.3.57}$$

Therefore the following function, with $l = 5$, reproduces all linear polynomials:

$$\begin{array}{ll} \mu_j & x_j \\ -2/\pi & (0,0); \\ 1/2\pi & (\pm 1, 0), (0, \pm 1). \end{array} \tag{4.3.58}$$

Analysis for the case $\phi(r) = e^{-cr^b}$, $\quad b > 1,\ c > 0$.

In this case we use Theorem 4-16 to deduce that it is not possible to reproduce constants and so we check conditions (4.2.51) and (4.2.52). The function ϕ is not smooth only near $x = 0$ and there the leading order singularity is homogeneous of degree b (if b is not an even integer). Because we have $b > 1$ all partial derivatives of ϕ of order at most $d + 1$ are absolutely integrable over $\mathcal{R}^d$. Hence Lemma 4-13 shows that

$$\hat{\phi}(\lambda) = o(\|\lambda\|^{-d-1}) \text{ as } \|\lambda\| \to \infty, \tag{4.3.59}$$

and so condition (4.2.51) is satisfied. Further

$$\int_{\mathcal{R}^d} e^{-c\|x\|^b} e^{-2\pi i \beta . x}\, dx = \int_{\mathcal{R}^d} e^{-c\|x\|^b} \cos(2\pi\beta_1 x_1) \cos(2\pi\beta_2 x_2) \cdots \cos(2\pi\beta_d x_d)\, dx. \tag{4.3.60}$$

It can now be deduced, although we omit the details, that, because the function $e^{-c\|x\|^b}$ is strictly monotically decreasing away from 0, (4.2.52) is satisfied for all $\beta \in \mathcal{Z}^d \backslash \{0\}$. These two conditions are sufficient for us to apply Theorem 4-16 and find that it is not possible to reproduce constants with an absolutely integrable ψ of the form (4.3.1).

CHAPTER 5 : RATE OF CONVERGENCE

Section 5.1 : Convergence over $\mathcal{R}^d$.

In this section we shall be deducing results about global rates of convergence from polynomial reproduction properties worked out in Chapter 4. Suppose that $\psi : \mathcal{R}^d \to \mathcal{R}$ is a function that reproduces polynomials of degree m and $f : \mathcal{R}^d \to \mathcal{R}$ is a suitably smooth function which does not grow too fast. We may form the **quasi-interpolant** to f:

$$a_h(x) = \sum_{z \in (h\mathcal{Z})^d} f(z)\psi(h^{-1}(x-z)), \qquad x \in \mathcal{R}^d, \tag{5.1.1}$$

where

$$(h\mathcal{Z})^d = \{y \in \mathcal{R}^d : (h^{-1}y_1, h^{-1}y_2, \ldots, h^{-1}y_d)^T \in \mathcal{Z}^d\}. \tag{5.1.2}$$

We study the rate of decay of the error $\|a_h - f\|_\infty$ as $h \to 0$.

In this section and Section 5.2 we work with a general continuous function ψ which reproduces polynomials and then apply these results in Section 5.3 to some of the cases considered in Section 4.3 where ψ is a linear sum of translates of radial basis functions

$$\psi(x) = \sum_{j=1}^{l} \mu_j \phi(\|x - x_j\|), \qquad x \in \mathcal{R}^d, \tag{5.1.3}$$

and where $\{x_j \in \mathcal{Z}^d : j = 1, 2, \ldots, l\}$. In many practical cases to reproduce polynomials of degree m with absolutely convergent sums we require

$$\psi(x) = o(\|x\|^{-d-m}) \text{ as } \|x\| \to \infty, \tag{5.1.4}$$

and we shall restrict our attention to the case when

$$|\psi(x)| \le A/(1 + \|x\|^{d+m+k}), \qquad x \in \mathcal{R}^d, \tag{5.1.5}$$

for some constants A and $k > 0$. We call such a function which reproduces polynomials of degree m an **m,k-basis function**, for $k > 0$. As we consider many examples of $m,1$-basis functions (e.g. Examples (a)–(d) in Section 4.3) we abbreviate "$m,1$-basis function" to "**m-basis function**". We see that an m,k-basis function satisfies

$$|\psi(x)| \le A, \qquad x \in \mathcal{R}^d, \tag{5.1.6}$$

and

$$|\psi(x)| \le A\|x\|^{-d-m-k}, \qquad x \in \mathcal{R}^d. \tag{5.1.7}$$

The results obtained, although of a theoretical rather than a practical nature, will allow us to deduce corresponding results for rates of convergence over a finite domain. This will be demonstrated in Section 5.2. Initially we prove a straightforward theorem showing the general technique of analysis. A sharpened version of Theorem 5-1 is given later (Theorem 5-6).

Theorem 5-1. *Suppose that ψ is a m,k-basis function and that $f \in C^{m+1}(\mathcal{R}^d)$ with all partial derivatives of orders m and $m+1$ being bounded over $\mathcal{R}^d$. Then the error of the quasi-interpolant* (5.1.1) *satisfies the bound*

$$\|a_h - f\|_\infty \le \begin{cases} A_0 h^{m+k} & \text{if } 0 < k < 1; \\ A_0^* h^{m+1}|\log h| & \text{if } k = 1; \\ A_0^+ h^{m+1} & \text{if } k > 1, \end{cases} \tag{5.1.8}$$

for sufficiently small h, where A_0, A_0^* and A_0^+ are independent of h.

Proof. Fix a point $x \in \mathcal{R}^d$ and let $p \in \Pi_m$ be the Taylor series approximation for f about x taking all terms up to and including degree m. We note that

$$p(x) = f(x). \tag{5.1.9}$$

Also, because all partial derivatives of f of order $m+1$ are bounded over $\mathcal{R}^d$, by a standard Taylor series argument,

$$|p(z) - f(z)| \le A_{m+1}\|z - x\|^{m+1}, \qquad z \in \mathcal{R}^d, \tag{5.1.10}$$

for some constant A_{m+1} independent of x. Similarly, because all partial derivatives of f of order m are bounded over $\mathcal{R}^d$ then,

$$|p(z) - f(z)| \le A_m\|z - x\|^m, \qquad z \in \mathcal{R}^d, \tag{5.1.11}$$

where again A_m is independent of x.

Now

$$\begin{aligned} |f(x) - a_h(x)| &= \left| f(x) - \sum_{z \in (h\mathcal{Z})^d} f(z)\psi(h^{-1}(x-z)) \right| \\ &= \left| p(x) - \sum_{z \in (h\mathcal{Z})^d} f(z)\psi(h^{-1}(x-z)) \right|, \end{aligned} \tag{5.1.12}$$

using (5.1.9). We may now use the fact that polynomials of degree m are reproduced to give

$$|f(x) - a_h(x)| = \left| \sum_{z \in (h\mathcal{Z})^d} (p(z) - f(z))\psi(h^{-1}(x-z)) \right|. \tag{5.1.13}$$

This sum is now in a suitable form for estimation as $|p(z) - f(z)|$ is small for z near x and $\psi(h^{-1}(x-z))$ decays rapidly away from x. In fact we split up the sum into three parts:

$$S_1 = \{z \in (h\mathcal{Z})^d \ : \ \|z - x\| \le ch\} \tag{5.1.14}$$

$$S_2 = \{z \in (h\mathcal{Z})^d \ : \ ch < \|z - x\| \le \delta\} \tag{5.1.15}$$

$$S_3 = \{z \in (h\mathcal{Z})^d \ : \ \delta < \|z - x\|\} \tag{5.1.16}$$

where c is any constant satisfying

$$c \ge 2d^{\frac{1}{2}}, \tag{5.1.17}$$

where δ is any fixed positive real and where h is assumed sufficiently small so that $S_2 \ne \emptyset$. We see that

$$|f(x) - a_h(x)| \le |T_1| + |T_2| + |T_3| \tag{5.1.18}$$

where

$$T_i = \sum_{z \in S_i} (p(z) - f(z))\psi(h^{-1}(x-z)), \qquad i = 1,2,3. \tag{5.1.19}$$

To bound T_1 we use (5.1.6), (5.1.10) and (5.1.14) to yield

$$\begin{aligned} |T_1| &\le A_{m+1} A \sum_{z \in S_1} \|z - x\|^{m+1} \\ &\le A_{m+1} A (ch)^{m+1} \sum_{z \in S_1} 1 \\ &\le A_1 h^{m+1}, \end{aligned} \tag{5.1.20}$$

where A_1 is a constant independent of x and h.

To bound T_2 we deduce from (5.1.7) and (5.1.10) that

$$|T_2| \le A_{m+1} A h^{d+m+k} \sum_{z \in S_2} \|z - x\|^{-d+1-k}. \tag{5.1.21}$$

To estimate this final sum we aim, in the manner of the one dimensional integral test, to bound it by an integral which is easy to evaluate. We take, for each point $z \in S_2$, a cube

$$G_z = \{y \ : \ \|y - z\|_\infty \le h/2\}. \tag{5.1.22}$$

We have the inequality

$$\begin{aligned} \|z - x\|^{-d+1-k} &\le \|z - x\|^{-d+1-k} \frac{\int_{G_z} \|y - x\|^{-d+1-k}\, dy}{\mathrm{vol}(G_z) \inf\{\|y - x\|^{-d+1-k} \ : \ y \in G_z\}} \\ &= \frac{\sup\{\|y - x\|^{d-1+k} \ : \ y \in G_z\}}{\mathrm{vol}(G_z)\|z - x\|^{d-1+k}} \int_{G_z} \|y - x\|^{-d+1-k}\, dy \\ &\le \frac{(c + \sqrt{d}/2)^{d-1+k}}{h^d c^{d-1+k}} \int_{G_z} \|y - x\|^{-d+1-k}\, dy, \end{aligned} \tag{5.1.23}$$

the last line following from (5.1.15). Now summing (5.1.23) over all points $z \in S_2$, using the bound (5.1.17) on c and the fact that the integrand is positive, we obtain for sufficiently small h

$$
\begin{aligned}
|T_2| &\le \hat{A}_2 h^{m+k} \int_{\{y\,:\,3ch/4 \le \|y-x\| \le \delta + \sqrt{d}h/2\}} \|y-x\|^{-d+1-k}\, dy \\
&\le \breve{A}_2 h^{m+k} \int_{3ch/4}^{3\delta/2} s^{-k}\, ds \\
&= \begin{cases} \breve{A}_2 h^{m+k}((3\delta/2)^{1-k} - (3ch/4)^{1-k})/(1-k) & \text{if } 0 < k < 1; \\ \breve{A}_2 h^{m+1}(\log(3\delta/2) - \log(3ch/4)) & \text{if } k = 1; \\ \breve{A}_2 h^{m+k}(-(3\delta/2)^{1-k} + (3ch/4)^{1-k})/(k-1) & \text{if } k > 1 \end{cases} \qquad (5.1.24) \\
&\le \begin{cases} A_2 h^{m+k} & \text{if } 0 < k < 1; \\ A_2^* h^{m+1} |\log h| & \text{if } k = 1; \\ A_2^+ h^{m+1} & \text{if } k > 1, \end{cases} \qquad (5.1.25)
\end{aligned}
$$

for constants $\hat{A}_2$, $\breve{A}_2$, A_2, A_2^* and A_2^+ all independent of x and h.

Finally to bound T_3 we can use (5.1.7) and (5.1.11) to yield

$$
|T_3| \le A A_m h^{d+m+k} \sum_{z \in S_3} \|z - x\|^{-d-k}. \qquad (5.1.26)
$$

This sum may be analysed by similar methods to those used for T_2 and it follows that

$$
\begin{aligned}
|T_3| &\le \breve{A}_3 h^{m+k} \int_{\delta/2}^{\infty} s^{-1-k}\, ds \qquad (5.1.27) \\
&= A_3 h^{m+k}, \qquad (5.1.28)
\end{aligned}
$$

where $\breve{A}_3$ and A_3 are constants independent of x and h.

This estimate combined with (5.1.18), (5.1.20) and (5.1.25) yields (5.1.8) which completes the proof of the theorem. ■

We note the following elementary converse to this theorem, which is well known (e.g. Fix and Strang (1969, Theorem 2)).

Lemma 5-2. *Suppose that*

$$
\|a_h - f\|_\infty = o(h^m), \qquad (5.1.29)
$$

as $h \to 0$, for all $f \in C^m(\mathcal{R}^d)$ with all partial derivatives of orders m and $m+1$ bounded over $\mathcal{R}^d$. In this case ψ reproduces all polynomials of degree m.

Proof. Let P be a homogeneous polynomial of degree $m' \le m$. By assumption

$$
\left| \sum_{z \in (h\mathcal{Z})^d} P(z)\psi(h^{-1}(x-z)) - P(x) \right| = o(h^m), \qquad (5.1.30)
$$

as $h \to 0$. We pick $y \in \mathcal{R}^d$ and take $x = hy$: then

$$\left|\sum_{z\in(h\mathcal{Z})^d} P(z)\psi(h^{-1}(x-z)) - P(x)\right| = o(h^m)$$

$$\Rightarrow \left|\sum_{z'\in\mathcal{Z}^d} h^{m'}P(z')\psi(y-z') - h^{m'}P(y)\right| = o(h^m)$$

$$\Rightarrow \left|\sum_{z'\in\mathcal{Z}^d} P(z')\psi(y-z') - P(y)\right| = o(h^{m-m'}). \tag{5.1.31}$$

The left hand side of (5.1.31) is now independent of h and so must be zero which means that P is reproduced by ψ. Hence all polynomials of degree at most m are reproduced by ψ. ∎

However Theorem 5-1, especially (5.1.25), leaves open the question whether it may be possible to remove the $|\log h|$ term in the case $k = 1$. This is especially important to us as we have already remarked that the examples we considered in Section 4.3 can be $m, 1$-basis functions. Initially we note the following simple corollary from the proof of Theorem 5-1.

Corollary 5-3. *If ψ is an m-basis function, f satisfies the conditions of Theorem 5-1 and T_2 is defined as in* (5.1.19) *then, for all $x \in \mathcal{R}^d$,*

$$|(f(x) - a_h(x)) - T_2| \le A_4 h^{m+1}, \tag{5.1.32}$$

for some constant A_4 independent of x and h.

Proof. From (5.1.13) and (5.1.19) we see that

$$|(f(x) - a_h(x)) - T_2| \le |T_1| + |T_3|. \tag{5.1.33}$$

Now (5.1.20) and (5.1.28) yield the corollary. ∎

This makes it clear that to answer the question we must investigate the estimation of T_2 more carefully; to this end we prove a lemma.

Lemma 5-4. *Suppose that ψ is a m-basis function and that there is a constant $\tilde{A}$ such that*

$$\left|\frac{\partial\psi(y)}{\partial y_i}\right| \le \tilde{A}||y||^{-d-2-m}, \quad i = 1, 2, \ldots, d, \tag{5.1.34}$$

for all sufficiently large $||y||$. Suppose also that $f \in C^{m+2}(\mathcal{R}^d)$ with all partial derivatives of order m, $m+1$ and $m+2$ bounded over $\mathcal{R}^d$. Then, with T_2 defined as in (5.1.19), *we have*

$$\left|T_2 + h^{m+1}\int_{\{y\,:\,||y||\le\delta h^{-1}\}} \tilde{p}_{m+1}(-y)\psi(y)\,dy\right| \le \dot{A}_0 h^{m+1}, \tag{5.1.35}$$

where

$$\tilde{p}_{m+1}(y) = \sum_{\{\alpha\in(\mathcal{Z}^+)^d\,:\,|\alpha|=m+1\}} \frac{1}{\alpha!}\frac{\partial^\alpha f(x)}{\partial x^\alpha}\, y^\alpha, \qquad y\in\mathcal{R}^d, \tag{5.1.36}$$

is a homogeneous polynomial of degree $m+1$, where δ is any positive constant and where $\dot{A}_0$ is independent of x and h.

Proof. Before embarking we note that (5.1.34) is satisfied for m-basis functions ψ of the form (5.1.3) if the expansion of ϕ for large argument consists of homogeneous terms and if $\phi(\|x\|)$ is continuously differentiable for sufficiently large $\|x\|$. These conditions are satisfied by examples (a)–(d) considered in Section 4.3. We also note that if (5.1.34) is satisfied for all $\|y\|\ge\tilde{c}$ we can replace the condition on c in (5.1.17) by

$$c = \max(2d^{\frac{1}{2}},\tilde{c}). \tag{5.1.37}$$

In this case the proof of Theorem 5-1 and Corollary 5-3 are still valid. From now on we assume that this choice of c has been made.

From (5.1.19)

$$\left|T_2 + \sum_{z\in S_2}\tilde{p}_{m+1}(z-x)\psi(h^{-1}(x-z))\right| = \left|\sum_{z\in S_2}(p(z)+\tilde{p}_{m+1}(z-x)-f(z))\psi(h^{-1}(x-z))\right| \tag{5.1.38}$$

and, because all partial derivatives of f of order $m+2$ are bounded over $\mathcal{R}^d$,

$$|p(z)+\tilde{p}_{m+1}(z-x)-f(z)| \le A_{m+2}\|z-x\|^{m+2}, \qquad z\in\mathcal{R}^d, \tag{5.1.39}$$

where A_{m+2} is a constant independent of x. Hence, using (5.1.7) with $k=1$, the expression (5.1.38) is at most

$$A_{m+2}Ah^{d+m+1}\sum_{z\in S_2}\|z-x\|^{-d+1}. \tag{5.1.40}$$

The method of analysis that is used to obtain (5.1.25) from (5.1.21) shows that (5.1.40) is at most $\dot{A}h^{m+1}$ for some constant $\dot{A}$ independent of x and h. Therefore, to prove the lemma, it is sufficient to show that

$$\left|\sum_{z\in S_2}\tilde{p}_{m+1}(z-x)\psi(h^{-1}(x-z)) - h^{m+1}\int_{\{y\,:\,\|y\|\le\delta h^{-1}\}}\tilde{p}_{m+1}(-y)\psi(y)\,dy\right| \le \dot{A}_1h^{m+1}, \tag{5.1.41}$$

for some constant $\dot{A}_1$ independent of x and h. Recalling that $\tilde{p}_{m+1}$ is homogeneous of degree $m+1$, this is equivalent to showing that

$$\left|\sum_{z\in\tilde{S}_2}\tilde{p}_{m+1}(z-h^{-1}x)\psi(h^{-1}x-z) - \int_{\{y\,:\,\|y\|\le\delta h^{-1}\}}\tilde{p}_{m+1}(-y)\psi(y)\,dy\right| \le \dot{A}_1, \tag{5.1.42}$$

where

$$\tilde{S}_2 = \{z \in \mathcal{Z}^d \ : c < \|z - h^{-1}x\| \leq \delta h^{-1}\}. \tag{5.1.43}$$

For each $z \in \tilde{S}_2$ we let

$$\tilde{G}_z = \{y \ : \ \|y - z\|_\infty \leq \tfrac{1}{2}\} \tag{5.1.44}$$

and because $\tilde{G}_z$ has unit volume we find

$$\begin{aligned}
&\left|\tilde{p}_{m+1}(z - h^{-1}x)\psi(h^{-1}x - z) - \int_{\tilde{G}_z} \tilde{p}_{m+1}(y - h^{-1}x)\psi(h^{-1}x - y)\, dy\right| \\
&\qquad \leq \tfrac{d}{2} \max\left[\sup\left\{\left|\frac{\partial}{\partial y_i}[\tilde{p}_{m+1}(y - h^{-1}x)\psi(h^{-1}x - y)]\right| \ : \ y \in \tilde{G}_z\right\} \ : \ i = 1, 2, \ldots, d\right],
\end{aligned} \tag{5.1.45}$$

by the mean value theorem. To estimate the term on the right we recall from (5.1.36) and our assumptions on f that

$$|\tilde{p}_{m+1}(y)| \leq \tilde{A}_{m+1}\|y\|^{m+1}, \qquad y \in \mathcal{R}^d, \tag{5.1.46}$$

and also that

$$\left|\frac{\partial \tilde{p}_{m+1}(y)}{\partial y_i}\right| \leq \tilde{A}_m \|y\|^m, \qquad y \in \mathcal{R}^d, \qquad i = 1, 2, \ldots, d, \tag{5.1.47}$$

where $\tilde{A}_m$ and $\tilde{A}_{m+1}$ are constants independent of x. It follows from (5.1.7), (5.1.34) and the remark in the second paragraph of the proof that, for $z \in \tilde{S}_2$, the right hand side of (5.1.45) is at most

$$\begin{aligned}
&\tfrac{d}{2} \sup\{(\tilde{A}_m A + \tilde{A}_{m+1}\tilde{A})\|y - h^{-1}x\|^{-d-1} \ : \ y \in \tilde{G}_z\} \\
&\qquad \leq \tfrac{d}{2}(\tilde{A}_m A + \tilde{A}_{m+1}\tilde{A})\|z - h^{-1}x\|^{-d-1} \frac{c^{d+1}}{(c - \sqrt{d}/2)^{d+1}},
\end{aligned} \tag{5.1.48}$$

the last inequality being a consequence of the definitions (5.1.43) and (5.1.44). Replacing the right hand side of (5.1.45) with this expression and summing over all $z \in \tilde{S}_2$ we obtain

$$\begin{aligned}
&\left|\sum_{z \in \tilde{S}_2} \left(\tilde{p}_{m+1}(z - h^{-1}x)\psi(h^{-1}x - z) - \int_{\tilde{G}_z} \tilde{p}_{m+1}(y - h^{-1}x)\psi(h^{-1}x - y)\, dy\right)\right| \\
&\qquad \leq \dot{A}_2 \sum_{z \in \tilde{S}_2} \|z - h^{-1}x\|^{-d-1},
\end{aligned} \tag{5.1.49}$$

where $\dot{A}_2$ is independent of x and h. The sum on the right is bounded independently of x and h and so it follows from (5.1.42) that it remains only to show that

$$\left|\int_{\cup \tilde{G}_z : z \in \tilde{S}_2} \tilde{p}_{m+1}(y - h^{-1}x)\psi(h^{-1}x - y)\, dy - \int_{\{y : \|y\| \leq \delta h^{-1}\}} \tilde{p}_{m+1}(-y)\psi(y)\, dy\right| \tag{5.1.50}$$

is also bounded independently of x and h. From the definitions of $\tilde{G}_z$ (5.1.44) and $\tilde{S}_2$ (5.1.43) we see that this difference is at most

$$\int_{\{y\,:\,\|y\|\leq c+\sqrt{d}/2\}} |\tilde{p}_{m+1}(-y)\psi(y)|\,dy + \int_{\{y\,:\,\delta h^{-1}-\sqrt{d}/2\leq\|y\|\leq\delta h^{-1}+\sqrt{d}/2\}} |\tilde{p}_{m+1}(-y)\psi(y)|\,dy. \quad (5.1.51)$$

However it follows from (5.1.5) and (5.1.46) that both these integrals are bounded independently of x and h for sufficiently small h, thus completing the proof of the lemma. ■

Corollary 5-3 and Lemma 5-4 show that, because the coefficients $\frac{\partial^\alpha f(x)}{\partial x^\alpha}$ in (5.1.36) are bounded independently of x by assumption, the question as to the possible redundancy of the $|\log h|$ term of Theorem 5-1 can be reduced to the study of the integral

$$I_0 = \int_{\{y\,:\,\|y\|\leq\delta h^{-1}\}} y^\alpha \psi(y)\,dy \quad (5.1.52)$$

with $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = m+1$. In particular, if I_0 is uniformly bounded, we can dispense with the $|\log h|$ term. In the following lemma we give an expression for this integral in an important case which will be shown to hold when ψ comes from the examples considered in Section 4.3.

Lemma 5-5. *Let ψ be a m-basis function, let I_0 be the integral* (5.1.52) *for any $\alpha \in (\mathcal{Z}^+)^d$: $|\alpha| = m+1$, and let $\hat{\psi}$ denote the Fourier transform of ψ. If, for some complex number K and real constants $\bar{A}_0$, $\bar{A}_1$ and $\epsilon > 0$, we have*

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial\lambda^\alpha} - K\log\|\lambda\|\right| \leq \bar{A}_0, \qquad 0 < \|\lambda\| \leq \epsilon, \quad (5.1.53)$$

and

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial\lambda^\alpha}\right| \leq \bar{A}_1, \qquad \|\lambda\| \geq \epsilon, \quad (5.1.54)$$

then

$$|I_0 - i^{m+1}K\log h| \quad (5.1.55)$$

is bounded independently of h, where $i = \sqrt{-1}$. (Note that we allow $K = 0$.)

Proof. The function $v^* : \mathcal{R} \to \mathcal{R}$ defined by

$$v^*(s) = \begin{cases} 0 & \text{if } s \leq 0 \\ \exp(-1/s^2) & \text{if } s > 0 \end{cases} \quad (5.1.56)$$

is smooth. Hence $\tilde{v} : \mathcal{R} \to \mathcal{R}$ defined by

$$\tilde{v}(s) = \frac{v^*(\delta - s)}{v^*(\delta - s) + v^*(s - \delta/2)} \quad (5.1.57)$$

is smooth, positive and satisfies

$$\tilde{v}(s) = \begin{cases} 1 & \text{if } s \le \delta/2 \\ 0 & \text{if } s \ge \delta, \end{cases} \tag{5.1.58}$$

where δ occurs in the definition (5.1.52). We define $v : \mathcal{R}^d \to \mathcal{R}$ by

$$v(y) = \tilde{v}(\|y\|) \tag{5.1.59}$$

and finally $u_h : \mathcal{R}^d \to \mathcal{R}$ by

$$u_h(y) = v(hy). \tag{5.1.60}$$

Thus u_h is smooth, positive and satisfies

$$u_h(y) = \begin{cases} 1 & \text{if } \|y\| \le \delta h^{-1}/2 \\ 0 & \text{if } \|y\| \ge \delta h^{-1}. \end{cases} \tag{5.1.61}$$

It follows that

$$\begin{aligned} &\left| \int_{\{y\,:\,\|y\| \le \delta h^{-1}\}} y^\alpha \psi(y)\,dy - \int_{\mathcal{R}^d} u_h(y) y^\alpha \psi(y)\,dy \right| \\ &\qquad \le \int_{\{y\,:\,\delta h^{-1}/2 \le \|y\| \le \delta h^{-1}\}} |y^\alpha||\psi(y)|\,dy \\ &\qquad \le A \int_{\{y\,:\,\delta h^{-1}/2 \le \|y\| \le \delta h^{-1}\}} \|y\|^{m+1}\|y\|^{-d-1-m}\,dy, \end{aligned} \tag{5.1.62}$$

the last line depending on (5.1.7). This integral is a multiple of $\int_{\delta h^{-1}/2}^{\delta h^{-1}} s^{-1}\,ds$ which is independent of h. Hence it is sufficient to prove the lemma when we replace I_0 by

$$I_1 = \int_{\mathcal{R}^d} u_h(y) y^\alpha \psi(y)\,dy. \tag{5.1.63}$$

Both $u_h(y)$ and $y^\alpha \psi(y)$ are in $L_2(\mathcal{R}^d)$ and hence we may use the Parseval-Plancheral Theorem (Friedlander 1982, Corollary 9.2.1) to deduce that, because the Fourier transform $\hat{u}_h$ is real,

$$I_1 = \frac{i^{m+1}}{(2\pi)^d} \int_{\mathcal{R}^d} \hat{u}_h(\lambda) \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\,d\lambda, \tag{5.1.64}$$

this expression for the Fourier transform of $y^\alpha \psi(y)$ coming from Theorem 4-4 (d). Now the assumptions (5.1.53) and (5.1.54) give

$$\begin{aligned} &\left| \int_{\mathcal{R}^d} \hat{u}_h(\lambda) \frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\,d\lambda - \int_{\mathcal{R}^d} \hat{u}_h(\lambda) K \log\|\lambda\|\,d\lambda \right| \\ &\qquad \le \int_{\{\lambda\,:\,\|\lambda\| \le \epsilon\}} \bar{A}_0 |\hat{u}_h(\lambda)|\,d\lambda + \int_{\{\lambda\,:\,\|\lambda\| \ge \epsilon\}} (\bar{A}_1 + |K|\,|\log\|\lambda\|\,|)\,|\hat{u}_h(\lambda)|\,d\lambda \\ &\qquad \le \bar{A}_0 \int_{\mathcal{R}^d} |\hat{v}(\omega)|\,d\omega + \int_{\{\omega\,:\,\|\omega\| \ge \epsilon h^{-1}\}} (\bar{A}_1 + |K|\,|\log h| + |K|\,|\log\|\omega\|\,|)\,|\hat{v}(\omega)|\,d\omega, \end{aligned} \tag{5.1.65}$$

the final line following from the substitution $\omega = h^{-1}\lambda$ and the observation that

$$\hat{u}_h(\lambda) = h^{-d}\hat{v}(h^{-1}\lambda). \tag{5.1.66}$$

Both integrals are bounded independently of h, for h sufficiently small, because we note that v is good (4.1.1) and hence so is $\hat{v}$ (Theorem 4-2): in particular $\hat{v}$ is continuous and decays faster than any polynomial. Hence it is sufficient to prove the lemma when we replace I_0 by

$$I_2 = \frac{i^{m+1}}{(2\pi)^d}\int_{\mathcal{R}^d} \hat{u}_h(\lambda) K \log\|\lambda\|\, d\lambda. \tag{5.1.67}$$

Performing the change of variables $\omega = h^{-1}\lambda$, we have

$$\begin{aligned} I_2 &= \frac{i^{m+1}K}{(2\pi)^d}\int_{\mathcal{R}^d} \hat{v}(\omega)(\log h + \log\|\omega\|)\, d\omega \\ &= \frac{i^{m+1}K}{(2\pi)^d}\log h\,(2\pi)^d v(0) + \bar{A}, \end{aligned} \tag{5.1.68}$$

for some constant $\bar{A}$ independent of h, the final line following from the formula for inverse Fourier transforms and $\hat{v}$ being good. Noting finally that $v(0) = 1$, the proof of the lemma is complete. ∎

We can now provide a slightly sharper result than Theorem 5-1, in the case when ψ is a suitable m-basis function.

Theorem 5-6. *Let ψ be an m-basis function for which*

$$\left|\frac{\partial\psi(y)}{\partial y_i}\right| \le \tilde{A}\|y\|^{-d-m-2}, \quad i = 1, 2, \ldots, l \tag{5.1.69}$$

and all sufficiently large $\|y\|$ and

$$\left|\frac{\partial^\alpha\hat{\psi}(\lambda)}{\partial\lambda^\alpha}\right| \le \bar{A}_0, \qquad \lambda \in \mathcal{R}^d\backslash\{0\} \tag{5.1.70}$$

and $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = m+1$. In this case for any $f \in C^{m+2}(\mathcal{R}^d)$ with all partial derivatives of order m, $m+1$ and $m+2$ bounded over $\mathcal{R}^d$ there exists a constant A^* (independent of h) such that, for all sufficiently small h,

$$\|a_h - f\|_\infty \le A^* h^{m+1}. \tag{5.1.71}$$

However, if ψ is an m-basis function satisfying (5.1.69) and, for some $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = m+1$, there are a non-zero complex constant K and $\epsilon > 0$ such that

$$\left|\frac{\partial^\alpha\hat{\psi}(\lambda)}{\partial\lambda^\alpha} - K\log\|\lambda\|\right| \le \bar{A}_1, \qquad 0 < \|\lambda\| \le \epsilon, \tag{5.1.72}$$

and

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\right| \leq \bar{A}_2, \qquad \|\lambda\| \geq \epsilon, \tag{5.1.73}$$

then there exist functions f of the above form such that

$$\|a_h - f\|_\infty \geq a^* h^{m+1} |\log h| \tag{5.1.74}$$

for sufficiently small h and some constant a^*.

Proof. The first part follows directly from Corollary 5-3, Lemma 5-4 and Lemma 5-5 with $K = 0$. For the second part we take some particular $\alpha' \in (\mathcal{Z}^+)^d \,:\, |\alpha'| = m+1$ satisfying (5.1.72) and (5.1.73), and we choose any $f \in C^{m+2}(\mathcal{R}^d)$ with all partial derivatives of order m, $m+1$ and $m+2$ bounded over $\mathcal{R}^d$ such that there exists some $x \in \mathcal{R}^d$ for which

$$\frac{\partial^\alpha f(x)}{\partial x^\alpha} = \begin{cases} (-1)^{m+1}\alpha! & \text{if } \alpha = \alpha'; \\ 0 & \text{for all other } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| = m+1. \end{cases} \tag{5.1.75}$$

In this case equation (5.1.35) in the statement of Lemma 5-4 becomes

$$\left| T_2 + h^{m+1} \int_{\{y \,:\, \|y\| \leq \delta h^{-1}\}} y^{\alpha'} \psi(y)\, dy \right| \leq \dot{A}_0 h^{m+1}. \tag{5.1.76}$$

Now the result follows from Corollary 5-3 and Lemmas 5-4 and 5-5. ■

Section 5.2 : Convergence over a Bounded Domain.

We now come to the more practical question of determining the rate of convergence for quasi-interpolation over a bounded region, again on a regular grid. We take Ω to be an open, bounded region of $\mathcal{R}^d$ and suppose that we have a function f, suitably smooth, on cl(Ω), the closure of Ω. We now define the **quasi-interpolant** to f on Ω by

$$\tilde{a}_h(x) = \sum_{z \in (h\mathcal{Z})^d \cap \Omega} f(z)\psi(h^{-1}(x-z)). \tag{5.2.1}$$

We cannot expect convergence on the whole of Ω but we look for convergence on a domain smaller by size δ:

$$\Omega_\delta = \{y \in \Omega : \|y - z\| \leq \delta \Rightarrow z \in \Omega\}. \tag{5.2.2}$$

In some cases we may be able to take $\delta \to 0$ as $h \to 0$ but the mesh size of the discretization excludes $\delta = o(h)$. We assume

$$ch \leq \delta < M, \tag{5.2.3}$$

where c occurs in (5.1.17) and

$$M = \sup\{\inf\{\|y - z\| : z \notin \Omega\} : y \in \Omega\}, \tag{5.2.4}$$

the second inequality implying that Ω_δ is not empty. As in Section 5.1 we restrict our attention to the case where ψ is an m,k-basis function for some $k > 0$.

In some cases it is possible to establish a rate of convergence by first extending the function f to a function f_E over the whole of $\mathcal{R}^d$. If f_E satisfies the conditions of Theorem 5-1 or 5-6 then a rate of convergence can be deduced from an estimate of the error between $\tilde{a}_h$ and the quasi-interpolant (5.1.1) to f_E. Suitable f_E can be provided in many cases by the Whitney Extension Theorem (Hörmander 1983, 2.3.6), although it cannot be applied for all domains Ω. We instead give a slightly longer but more general proof which involves an extra estimation of the error when performing quasi-interpolation to a polynomial.

Lemma 5-7. *Let Ω, Ω_δ be defined as above, ψ a m,k-basis function, $q \in \Pi_m$ and $x \in \Omega_\delta$. Then, for sufficiently small h,*

$$\left| q(x) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} q(z)\psi(h^{-1}(x-z)) \right| \leq A_q h^{m+k} \delta^{-m-k}, \tag{5.2.5}$$

where A_q is a constant independent of x, δ and h, depending on

$$\sup\left\{ \left| \frac{\partial^\alpha q(y)}{\partial y^\alpha} \right| : y \in \Omega,\ \alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq m \right\}. \tag{5.2.6}$$

Proof. We see that, because q is reproduced over $\mathcal{R}^d$ and because $x \in \Omega_\delta$,

$$\left| q(x) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} q(z)\psi(h^{-1}(x-z)) \right| = \left| \sum_{z \in (h\mathcal{Z})^d \setminus \Omega} q(z)\psi(h^{-1}(x-z)) \right|$$
$$\leq \sum_{\{z \in (h\mathcal{Z})^d \,:\, \delta < \|z-x\|\}} |q(z)|\,|\psi(h^{-1}(x-z))|. \tag{5.2.7}$$

Now the identity

$$q(z) = \sum_{\{\alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \leq m\}} \frac{1}{\alpha!} \left. \frac{\partial^\alpha q(y)}{\partial y^\alpha} \right|_{y=x} (z-x)^\alpha \tag{5.2.8}$$

implies

$$|q(z)| \leq \frac{\dot{A}_q}{\delta^m} \|z-x\|^m, \qquad \|z-x\| \geq \delta, \tag{5.2.9}$$

where $\dot{A}_q$ depends on (5.2.6). Using inequalities (5.1.7) and (5.2.9) we see that the right hand side of (5.2.7) is at most

$$\sum_{\{z \in (h\mathcal{Z})^d \,:\, \delta < \|z-x\|\}} \frac{\dot{A}_q}{\delta^m} A\, h^{d+m+k} \|z-x\|^{-d-k} \leq \tilde{A}_q h^{m+k} \delta^{-m} \int_{\delta/2}^{\infty} s^{-1-k}\, ds, \tag{5.2.10}$$

the inequality following from the method of analysis used to obtain (5.1.25) from (5.1.21) in the proof of Theorem 5-1 and (5.2.3). Evaluation of this integral completes the proof of the lemma. ■

The other preliminary is to define the space of suitable functions:

$$C^k(\mathrm{cl}(\Omega)) = \left\{ f \in C^k(\Omega) \,:\, \frac{\partial^\alpha f(y)}{\partial y^\alpha} \text{ extends continuously to } \mathrm{cl}(\Omega) \text{ for all } \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \leq k \right\}. \tag{5.2.11}$$

Now we can prove two theorems, analogous to Theorems 5-1 and 5-6, establishing the rates of convergence over Ω.

Theorem 5-8. *Let Ω, Ω_δ be defined as at the beginning of the section, let ψ be a m,k-basis function and let ϵ be any positive constant. Then, for any $f \in C^{m+1}(cl(\Omega))$, $x \in \Omega_\delta$, sufficiently small h, and δ satisfying* (5.2.3) *and*

$$\delta \geq \begin{cases} \epsilon & \text{if } 0 < k < 1; \\ \epsilon |\log h|^{-1/(1+m)} & \text{if } k = 1; \\ \epsilon h^{(k-1)/(m+k)} & \text{if } k > 1, \end{cases} \tag{5.2.12}$$

the inequality

$$|f(x) - \tilde{a}_h(x)| \leq \begin{cases} \tilde{A} h^{m+k} & \text{if } 0 < k < 1; \\ \tilde{A}^* h^{m+1} |\log h| & \text{if } k = 0; \\ \tilde{A}^+ h^{m+1} & \text{if } k > 0 \end{cases} \tag{5.2.13}$$

holds, where $\tilde{A}$ and $\tilde{A}^*$ are constants independent of x and h.

Proof. As in the proof of Theorem 5-1 we let $p \in \Pi_m$ be the truncated Taylor series for f about the point x containing all terms up to and including degree m. Then we have the bound

$$\begin{aligned} |f(x) - \tilde{a}_h(x)| &= \left| p(x) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} f(z)\psi(h^{-1}(x-z)) \right| \\ &\leq \left| \sum_{z \in (h\mathcal{Z})^d \cap \Omega} (p(z) - f(z))\psi(h^{-1}(x-z)) \right| + A_f h^{m+k} \delta^{-m-k}, \end{aligned} \tag{5.2.14}$$

the final line depending on Lemma 5-7. Here we use A_f instead of A_p because p is the Taylor series for f and hence the constant depends only on $\sup\left\{ \left| \frac{\partial^\alpha f(y)}{\partial y^\alpha} \right| \,:\, y \in \Omega,\ \alpha \in (\mathcal{Z}^+)^d \,:\, |\alpha| \leq m \right\}$ which, because $f \in C^{m+1}(\mathrm{cl}(\Omega))$, is bounded and independent of x, δ and h. The sum in the final line is a subset of that analysed in Theorem 5-1 and we may estimate it by similar techniques. We define, using (5.1.14)–(5.1.16),

$$\hat{S}_i = S_i \cap \Omega, \qquad i = 1, 2, 3, \tag{5.2.15}$$

and

$$\hat{T}_i = \sum_{z \in \hat{S}_i} (p(z) - f(z))\psi(h^{-1}(x-z)), \qquad i = 1, 2, 3. \tag{5.2.16}$$

Because $x \in \Omega_\delta$, (5.2.3) implies $\hat{T}_1 = T_1$ and $\hat{T}_2 = T_2$, and we see that

$$\left| \sum_{z \in (h\mathcal{Z})^d \cap \Omega} (p(z) - f(z))\psi(h^{-1}(x-z)) \right| \leq |\hat{T}_1| + |\hat{T}_2| + |\hat{T}_3|. \tag{5.2.17}$$

The estimates of $\hat{T}_1$ and $\hat{T}_2$ are exactly as in the proof of Theorem 5-1, because, when $\|z - x\| \leq \delta$, the line joining x to z is completely within Ω. Hence (5.1.20) and (5.1.25) yield, for some constants A_1, A_2, A_2^* and A_2^+ independent of x, δ and h, and all sufficiently small h,

$$|\hat{T}_1| \leq A_1 h^{m+1} \tag{5.2.18}$$

and

$$|\hat{T}_2| \leq \begin{cases} A_2 h^{m+k} & \text{if } 0 < k < 1; \\ A_2^* h^{m+1} |\log h| & \text{if } k = 1; \\ A_2^+ h^{m+1} & \text{if } k > 1. \end{cases} \tag{5.2.19}$$

The estimation of $\hat{T}_3$, however, is not the same as previously because, when $z \in \hat{S}_3$, the line joining x to z may not lie completely within Ω and hence (5.1.11) may not hold. Therefore we use (5.2.9) to give the bound

$$|p(z) - f(z)| \leq \frac{\tilde{A}_f}{\delta^m} \|z - x\|^m + \sup\{|f(y)| \,:\, y \in \Omega\}, \qquad z \in \hat{S}_3, \tag{5.2.20}$$

where, by the remark after (5.2.14), $\tilde{A}_f$ depends on $\sup\left\{\left|\frac{\partial^\alpha f(y)}{\partial y^\alpha}\right| : y \in \Omega,\ \alpha \in (\mathcal{Z}^+)^d : |\alpha| \leq m\right\}$. Hence, we have the bound, similar to (5.1.11),

$$|p(z) - f(z)| \leq \hat{A}_m \delta^{-m} \|z - x\|^m, \qquad z \in \hat{S}_3, \tag{5.2.21}$$

where $\hat{A}_m = \tilde{A}_f + \sup\{|f(y)| : y \in \Omega\}$ is independent of x, δ and h. This estimate, combined with the analysis that obtains (5.1.27) from (5.1.19) yields

$$|\hat{T}_3| \leq A_3 h^{m+k} \delta^{-m-k}, \tag{5.2.22}$$

for all sufficiently small h, where A_3 is independent of x, δ and h. Now (5.2.14), (5.2.17), (5.2.18), (5.2.19) and (5.2.22), combined with the bound (5.2.12) on δ, imply (5.2.13) which completes the proof of the theorem. ■

Theorem 5-9. *Let Ω, Ω_δ be defined as at the beginning of the section, where we now take δ to be any fixed number. Let ψ be an m-basis function such that*

$$\left|\frac{\partial \psi(y)}{\partial y_i}\right| \leq \tilde{A} \|y\|^{-d-m-2}, \quad i = 1, 2, \ldots, d \tag{5.2.23}$$

for all sufficiently large $\|y\|$ and

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\right| \leq \bar{A}_0, \qquad \lambda \in \mathcal{R}^d \backslash \{0\}, \tag{5.2.24}$$

for all $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = m+1$. In this case for any $f \in C^{m+2}(cl(\Omega))$ there exists a constant A^+, independent of h, such that, for all sufficiently small h and $x \in \Omega_\delta$,

$$\left| f(x) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} f(z) \psi(h^{-1}(x - z)) \right| \leq A^+ h^{m+1}. \tag{5.2.25}$$

However, if ψ is an m-basis function satisfying (5.2.23) and if for some $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = m+1$ there are a complex non-zero constant K and $\epsilon > 0$ such that

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha} - K \log \|\lambda\|\right| \leq \bar{A}_1, \qquad 0 < \|\lambda\| \leq \epsilon, \tag{5.2.26}$$

and

$$\left|\frac{\partial^\alpha \hat{\psi}(\lambda)}{\partial \lambda^\alpha}\right| \leq \bar{A}_2, \qquad \|\lambda\| \geq \epsilon, \tag{5.2.27}$$

then there exist functions f of the above form such that, for some $x \in \Omega_\delta$,

$$\left| f(x) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} f(z)\psi(h^{-1}(x-z)) \right| \geq a^+ h^{m+1} |\log h|, \tag{5.2.28}$$

for sufficiently small h and some constant a^+.

Proof. Defining p as in the proof of Theorem 5-1, we see that, as in (5.2.14),

$$\left| (f(x) - \tilde{a}_h(x)) - \sum_{z \in (h\mathcal{Z})^d \cap \Omega} (p(z) - f(z))\psi(h^{-1}(x-z)) \right| \leq A_f h^{1+m} \delta^{-1-m}. \tag{5.2.29}$$

Hence, because δ is fixed and the whole of the region S_2 (5.1.14) is within our quasi-interpolating sum, we find that, as in Corollary 5-3,

$$|(f(x) - \tilde{a}_h(x)) - T_2| \leq \tilde{A}_4 h^{m+1}, \tag{5.2.30}$$

for sufficiently small h, where T_2 is defined in (5.1.19) and $\tilde{A}_4$ is independent of x and h. The result now follows from the method of proof of Theorem 5-6. ■

Section 5.3 : Examples of Suitable ϕ

In this section we consider examples (a)–(d) of Section 4.3, to find out the rates of convergence that are achieved.

We begin by defining the actual rate of convergence for a function ψ and a suitable space of functions C. We say that a function ψ has **actual rate of convergence** τ over C (where τ is a function defined for all sufficiently small positive h) if, for any $f \in C$, there is a constant A such that the quasi-interpolant a_h (5.1.1) satisfies

$$\|a_h - f\|_\infty \leq A\tau(h), \tag{5.3.1}$$

for all sufficiently small h, but there exists $f \in C$ for which

$$a\tau(h) \leq \|a_h - f\|_\infty \leq A\tau(h), \tag{5.3.2}$$

for all sufficiently small h, where a is another positive constant. The function space

$$C_{m,r} = \{f \in C^{m+r}(\mathcal{R}^d) : \text{ all partial derivatives of } f \text{ of orders } m, m+1, \ldots, m+r \text{ are bounded over } \mathcal{R}^d\}, \tag{5.3.3}$$

is suitable for our analysis.

Theorem 5-10. *Any function ψ of the form (5.1.3) which reproduces all polynomials of degree $b+d-1$ gives actual rates of convergence that are stated in the following table. Further, it is impossible to obtain a better rate with any function ψ of the form (5.1.3).*

	$\phi(r)$	Actual Rate	Conditions
(a)	r^b	h^{b+d}	$f \in C_{b+d-1,2}$ and b and d both positive odd integers;
(b)	$(r^2+c^2)^{\frac{b}{2}}$	$h^{b+d}\lvert\log h\rvert$	$f \in C_{b+d-1,1}$, $b+d$ a positive even integer and b not a non-negative even integer;
(c)	$r^b \log r$	h^{b+d}	$f \in C_{b+d-1,2}$ and b and d both positive even integers;
(d)	$(r^2+c^2)^{\frac{b}{2}}\log(r^2+c^2)^{\frac{1}{2}}$	$h^{b+d}\lvert\log h\rvert$	$f \in C_{b+d-1,1}$, $b+d$ a positive even integer and b a non-negative even integer.

Proof. In each of the cases (a)–(d) Theorem 4-20 shows that there exist functions ψ of the form (5.1.3) which can reproduce all polynomials of degree $b+d-1$. We have already remarked in Section 4.3 that such functions satisfy (4.3.4) and (4.3.25), which is condition (5.1.5) for a $(b+d-1)$-basis function. Hence, for all cases, Theorem 5-1 yields that, for any $f \in C_{b+d-1,1}$, we have the bound

$$\|a_h - f\|_\infty \le A h^{b+d}|\log h|, \tag{5.3.4}$$

for some constant A independent of h and all sufficiently small h. Further, Lemma 5-2 states that we can obtain this rate of convergence only when ψ of the form (5.1.3) reproduces all polynomials of degree $b+d-1$. This proves the last assertion of the theorem, so henceforth we shall restrict attention to functions ψ which are $(b+d-1)$-basis functions.

To proceed further we wish to apply Theorem 5-6, so we must check whether its conditions are satisfied. We have remarked in Section 4.3 (just after (4.3.4) for case (a) and at corresponding points for the the other examples) that, in each case, the expansion of any ψ for large argument consists of homogeneous terms of integer degrees at most $-2d-b$. It follows from the first paragraph of the proof of Lemma 5-4 that in each case (5.1.69) is satisfied. Further, (4.3.5), which holds for cases (a) and (c), shows that condition (5.1.70) is satisfied in these cases for $m = b+d-1$ and sufficiently small $\|\lambda\|$. This condition also holds for large $\|\lambda\|$ which may be deduced from the form of the Fourier transform $\hat{\phi}$ ((4.3.2) and (4.3.40)) and (4.2.62). Hence, in cases (a) and (c), Theorem

5-6 yields that, for any $f \in C_{b+d-1,2}$, we have the bound

$$\|a_h - f\|_\infty \leq Ah^{b+d}, \tag{5.3.5}$$

for some constant A and all sufficiently small h.

Equations (5.3.4) and (5.3.5) show that the rates of convergence described in the theorem are attained for any ψ of the form (5.1.3) reproducing all polynomials of degree $b+d-1$, and it remains to show that they are "actual". We first consider the cases (b) and (d) and, in order to use the other half of Theorem 5-6, we check its conditions. We have already remarked that (5.1.69) is satisfied for these examples. Equation (4.3.26) also holds in these cases, which shows that every $\alpha \in (\mathcal{Z}^+)^d : |\alpha| = b+d$ with α even satisfies (5.1.72) for some non-zero K. Further, equation (5.1.73) may be deduced in these cases from (4.3.14), the form of the Fourier transforms ((4.3.12) and (4.3.47)) and (4.2.62). Hence Theorem 5-6 yields the existence of functions $f \in C_{b+d-1,2}$ satisfying

$$\|a_h - f\|_\infty \geq ah^{b+d}|\log h|, \tag{5.3.6}$$

for some constant a and all sufficiently small h. The fact that $C_{b+d-1,2} \subset C_{b+d-1,1}$ completes the proof of the actual rate of convergence for cases (b) and (d).

To show that the rate h^{b+d} is actual for cases (a) and (c) we aim to get a contradiction. Therefore we suppose that a function ψ of the form (5.1.3) exists which reproduces all polynomials of degree $b+d-1$ and which gives a rate of convergence

$$\|a_h - f\|_\infty = o(h^{b+d}) \text{ as } h \to 0, \tag{5.3.7}$$

for any function $f \in C_{b+d-1,2}$.

First we suppose that ψ is a $b+d-1, k$-basis function for some $k > 1$ (we have already remarked in Section 4.3 (after (4.3.4)) that k must be an integer). We aim to proceed in a similar manner to Lemma 5-2 and show that all polynomials of degree $b+d$ are reproduced by ψ which would contradict Theorem 4-20. However, we must argue carefully because a homogeneous polynomial of degree $b+d$ is not in the space $C_{b+d-1,2}$. (This method of analysis has also been used recently and independently by Buhmann and Chui (private communication).)

We pick some $x \in \mathcal{R}^d$, we let f be any homogeneous polynomial of degree $b+d$ and we let ρ be any smooth function which satisfies $0 \leq \rho(y) \leq 1$ and

$$\rho(y) = \begin{cases} 1 & \text{if } \|y - x\| \leq \delta/2; \\ 0 & \text{if } \|y - x\| \geq \delta, \end{cases} \tag{5.3.8}$$

for some $\delta > 0$. We have $\rho f \in C_{b+d-1,2}$ and hence, by assumption (5.3.7),

$$\left| \sum_{z \in (h\mathcal{Z})^d} \rho(z) f(z) \psi(h^{-1}(x-z)) - \rho(x) f(x) \right| = o(h^{b+d}). \tag{5.3.9}$$

It follows from $\rho(x) = 1$ that

$$\begin{aligned} &\left| \sum_{z \in (h\mathcal{Z})^d} f(z) \psi(h^{-1}(x-z)) - f(x) \right| \\ &\qquad \leq \left| \sum_{z \in (h\mathcal{Z})^d} f(z) \psi(h^{-1}(x-z)) - \sum_{z \in (h\mathcal{Z})^d} \rho(z) f(z) \psi(h^{-1}(x-z)) \right| + o(h^{b+d}). \end{aligned} \tag{5.3.10}$$

Further, by (5.3.8), the term inside the modulus signs on the right hand side of (5.3.10) can be bounded from above by

$$\sum_{\{z \in (h\mathcal{Z})^d : \|z-x\| \geq \delta/2\}} |f(z)|\, |\psi(h^{-1}(x-z))|. \tag{5.3.11}$$

Recalling that ψ is a $b+d-1, k$-basis function for some $k > 1$, this sum may be estimated by the techniques already described in this chapter and is found to be bounded above by a constant multiple of h^{b+d+1} for sufficiently small h, and uniformly bounded for x within a compact set. Hence (5.3.10) yields

$$\left| \sum_{z \in (h\mathcal{Z})^d} f(z) \psi(h^{-1}(x-z)) - f(x) \right| = o(h^{b+d}), \tag{5.3.12}$$

which is true for all $x \in \mathcal{R}^d$ and uniformly for all x in some compact set.

Now take some fixed $y \in \mathcal{R}^d$ and $x = hy$. We note that, as we are only interested in sufficiently small h, all the choices x are within a compact set. Hence, recalling that f is homogeneous of degree $b+d$, we find that, as $h \to 0$,

$$\begin{aligned} &\left| \sum_{z \in (h\mathcal{Z})^d} f(z) \psi(h^{-1}(x-z)) - f(x) \right| = o(h^{b+d}) \\ \Rightarrow &\left| \sum_{z' \in \mathcal{Z}^d} h^{b+d} f(z') \psi(y-z') - h^{b+d} f(y) \right| = o(h^{b+d}) \\ \Rightarrow &\left| \sum_{z' \in \mathcal{Z}^d} f(z') \psi(y-z') - f(y) \right| = o(1). \end{aligned} \tag{5.3.13}$$

However, the left hand side is independent of h and hence the polynomial f is reproduced. Thus all homogeneous polynomials of degree $b+d$ are reproduced by ψ. Hence, in view of the fact that

ψ reproduces all polynomials of degree $b+d-1$, we deduce that all polynomials of degree $b+d$ are reproduced, which contradicts Theorem 4-20.

When ψ decays no faster than $\|y\|^{-2d-b}$ we must argue slightly differently because, for instance, a sum of the form (5.3.11) would not be convergent. Our method of analysis is similar to that in the second half of the proof of Lemma 4-17. The leading order term in the expansion of ψ for large argument is continuous and homogeneous of degree $-2d-b$. We denote this term by g. Let z be a point with $\|z\| = 1$ for which $g(z) = a \neq 0$, and without loss of generality we suppose that $a > 0$. Now, because g is continuous, there exists some $\epsilon > 0$ such that $\|y - z\| < \epsilon \Rightarrow g(y) \geq a/2$. We let

$$S_1 = \{y \in \mathcal{R}^d \, : \, \|y\| = 1, \; \|y - z\| < \epsilon\} \tag{5.3.14}$$

and

$$S_2 = \{y \in \mathcal{R}^d \, : \, \|y\| = 1, \; \|y - z\| < \epsilon/2\}. \tag{5.3.15}$$

Hence, because g is homogeneous, we find

$$y/\|y\| \in S_1 \Rightarrow g(y) \geq (a/2)\|y\|^{-2d-b}. \tag{5.3.16}$$

We choose M so that, for $y/\|y\| \in S_1$ and $\|y\| \geq M$, we have

$$\psi(y) \geq \tfrac{1}{2} g(y) \geq 0. \tag{5.3.17}$$

We pick $x \in \mathcal{R}^d$, we let $\tilde{S}_1$ be part of a sphere centred at x:

$$\tilde{S}_1 = \{y \in \mathcal{R}^d \, : \, -(y-x)/\|y-x\| \in S_1, \; \delta/4 \leq \|y-x\| \leq \delta\}, \tag{5.3.18}$$

for some $\delta > 0$, and we let $\tilde{S}_2$ be the strictly interior subset of $\tilde{S}_1$:

$$\tilde{S}_2 = \{y \in \mathcal{R}^d \, : -(y-x)/\|y-x\| \in S_2, \; \delta/2 \leq \|y-x\| \leq 3\delta/4\}. \tag{5.3.19}$$

We note that $y \in \tilde{S}_1$ and $h \leq \delta/(4M)$ imply $h^{-1}(x-y)/\|h^{-1}(x-y)\| \in S_1$ and $\|h^{-1}(x-y)\| \geq M$: hence, by (5.3.17),

$$\psi(h^{-1}(x-y)) \geq \tfrac{1}{2} g(h^{-1}(x-y)) \geq 0. \tag{5.3.20}$$

We let ρ be a smooth function satisfying $0 \leq \rho(y) \leq 1$ and

$$\rho(y) = \begin{cases} 0 & \text{if } y \notin \tilde{S}_1; \\ 1 & \text{if } y \in \tilde{S}_2. \end{cases} \tag{5.3.21}$$

Now $\rho \in C_{b+d-1,2}$ and we consider performing quasi-interpolation to this function at the point x. In this case, for $h \leq \delta/(4M)$,

$$|a_h(x) - \rho(x)| = |a_h(x)| = \left| \sum_{z \in (h\mathcal{Z})^d} \rho(z)\psi(h^{-1}(x-z)) \right|$$
$$\geq \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)=1\}} \psi(h^{-1}(x-z)),$$

because ρ is non-negative and (5.3.20) shows that $\psi(h^{-1}(x-z))$ is non-negative at all points where ρ is non-zero. Hence, using the first inequality in (5.3.17) and then (5.3.16), we deduce

$$|a_h(x) - \rho(x)| \geq \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)=1\}} \tfrac{1}{2} g(h^{-1}(x-z)), \tag{5.3.22}$$
$$\geq (a/4)\, h^{b+2d} \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)=1\}} \|z - x\|^{-2d-b}. \tag{5.3.23}$$

Hence, for sufficiently small h,

$$|a_h(x) - \rho(x)| \geq \tilde{a} h^{b+d} \int_{\delta/2}^{3\delta/4} s^{-d-b-1}\, ds$$
$$\geq a^* h^{b+d}, \tag{5.3.24}$$

where $\tilde{a}$ and a^* are positive constants. This completes the proof that h^{b+d} is the actual rate of convergence in cases (a) and (c) and hence also completes the proof of the theorem. ■

We remark that in cases (b) and (d) any basis function reproducing polynomials of degree $b+d-1$ can decay no faster than $\|y\|^{-2d-b}$, for otherwise we could obtain a better rate of convergence from Theorem 5-1 by choosing $k > 1$.

We also remark that, if in cases (a) and (c) the function ψ is a $b+d-1, k$-basis function for some $k > 1$, then Theorem 5-1 implies that it is unneccesary to impose the restriction $f \in C_{b+d-1,2}$ in order to remove the $|\log h|$ term. It would be sufficient that $f \in C_{b+d-1,1}$. However, as we shall now demonstrate, if ψ is just a $(b+d-1)$-basis function, then there exist functions in $C_{b+d-1,1}$ for which we do not obtain the rate of convergence h^{b+d}.

The proof is similar to the final part of Theorem 5-10 but we have to argue slightly more carefully. We take ψ to be an $(b+d-1)$-basis function which decays no faster than $\|y\|^{-2d-b}$ for large argument. We define g, z, a, ϵ, S_1, S_2 and M as in the proof of Theorem 5-10 (the paragraph containing (5.3.14)) and we note that equations (5.3.16) and (5.3.17) hold. Let $\tilde{\rho}$ be a smooth function which satisfies $0 \leq \tilde{\rho}(y) \leq 1$ and

$$\tilde{\rho}(y) = \begin{cases} 1 & \text{if } \|y\| \leq \delta/2; \\ 0 & \text{if } \|y\| \geq \delta, \end{cases} \tag{5.3.25}$$

for some $0 < \delta < 1$. We let $\sigma : \{y \in \mathcal{R}^d : \|y\| = 1\} \to \mathcal{R}$ be smooth and satisfy $0 \le \sigma(y) \le 1$ and

$$\sigma(y) = \begin{cases} 1 & \text{if } -y \in S_2; \\ 0 & \text{if } -y \notin S_1. \end{cases} \tag{5.3.26}$$

We let τ be the function

$$\tau(y) = \begin{cases} \sigma(y/\|y\|) & \text{if } y \neq 0; \\ 0 & \text{if } y = 0. \end{cases} \tag{5.3.27}$$

Finally we let $f : \mathcal{R}^d \to \mathcal{R}$ be the function

$$f(y) = \begin{cases} 0 & \text{if } y = 0 \text{ or } \|y\| = 1; \\ \dfrac{\|y\|^{b+d}}{|\log(\|y\|)|}\tilde{\rho}(y)\tau(y) & \text{otherwise.} \end{cases} \tag{5.3.28}$$

We must check that $f \in C_{b+d-1,1}$. The only problem is checking whether all partial derivatives of f of order at most $b+d$ are continuous at 0. This condition is a consequence of the exponent of $\|y\|$, the smoothness of $\tilde{\rho}$, the homogeneity of τ and the unboundedness of $|\log(\|y\|)|$. We consider the error between f and its quasi-interpolant at the origin. For sufficiently small h we have

$$\begin{aligned} |a_h(0) - f(0)| = |a_h(0)| &= \left| \sum_{z \in (h\mathcal{Z})^d} f(z)\psi(-h^{-1}z) \right| \\ &\ge \left| \sum_{\{z \in (h\mathcal{Z})^d : \|z\| \ge Mh\}} \frac{\|z\|^{b+d}}{|\log(\|z\|)|}\tilde{\rho}(z)\tau(z)\psi(-h^{-1}z) \right| \\ &\quad - \sum_{\{z \in (h\mathcal{Z})^d : \|z\| < Mh\}} \frac{\|z\|^{b+d}}{|\log(\|z\|)|}|\psi(-h^{-1}z)|, \end{aligned} \tag{5.3.29}$$

where in the second sum we have used $0 \le \tilde{\rho}(z) \le 1$ and $0 \le \tau(z) \le 1$. The second sum is bounded above by a constant multiple of h^{b+d} so long as $h < 1/(2M)$, say. We also note that, when $\|z\| \ge Mh$ and $\tilde{\rho}(z)\tau(z) > 0$, then as in (5.3.20) we have $\psi(-h^{-1}z) \ge 0$. Hence, we find

$$\begin{aligned} |a_h(0) - f(0)| &\ge \left| \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)\tau(z)=1,\ \|z\| \ge Mh\}} \frac{\|z\|^{b+d}}{|\log(\|z\|)|}\psi(-h^{-1}z) \right| - Ah^{b+d} \\ &\ge \left| \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)\tau(z)=1,\ \|z\| \ge Mh\}} \frac{\|z\|^{b+d}}{|\log(\|z\|)|}\tfrac{1}{2}g(-h^{-1}z) \right| - Ah^{b+d}, \end{aligned} \tag{5.3.30}$$

the final line using (5.3.17), (5.3.25) and (5.3.27). Therefore (5.3.16) yields

$$|a_h(0) - f(0)| \ge (a/4)h^{2d+b} \sum_{\{z \in (h\mathcal{Z})^d : \rho(z)\tau(z)=1,\ \|z\| \ge Mh\}} \frac{\|z\|^{b+d}}{|\log(\|z\|)|}\|z\|^{-2d-b} - Ah^{b+d}. \tag{5.3.31}$$

To complete the estimation we may deduce from (5.3.25) and (5.3.31) that, for some constants $\tilde{a}, a^* > 0$ and all sufficiently small h,

$$|a_h(0) - f(0)| \geq \tilde{a} h^{b+d} \int_{Mh}^{\delta/2} \frac{1}{s|\log s|} \, ds - A h^{b+d}$$
$$\geq a^* h^{b+d} |\log(|\log h|)|. \tag{5.3.32}$$

Hence, when f is the function (5.3.28), we do not obtain the rate of convergence h^{b+d}.

Theorem 5-10 has a simple corollary that is a consequence of Theorems 5-8 and 5-9:

Corollary 5-11. *Let Ω be an open bounded region and Ω_δ a region smaller by size δ, for some fixed δ. If we change the definition* (5.3.3) *to*

$$C_{m,r} = C^{m+r}(cl(\Omega)), \tag{5.3.33}$$

where $C^{m+r}(cl(\Omega))$ is defined in (5.2.11), *then, for any function ψ of the form* (5.1.3) *which reproduces all polynomials of degree $b + d - 1$, the table in the statement of Theorem 5-10 gives the actual rates of convergence over Ω_δ.*

Proof. The only part that does not follow immediately from the proofs of Theorems 5-8, 5-9 and 5-10 is the fact that the rate given is actual in cases (a) and (c). However, our definitions of ρ ((5.3.8) and (5.3.21)) both have $\rho(y) = 0$ for $\|y - x\| \geq \delta$. Hence, when $x \in \Omega_\delta$, performing quasi-interpolation to the two functions (in (5.3.9) and (5.3.22)) over the whole of $\mathcal{R}^d$ is the same as quasi-interpolation over Ω. Therefore, the arguments we have used in the proof of Theorem 5-10 remain valid. ∎

Our final remark is for the benefit of those readers who are more at home with the techniques of Chapter 3 than with the Fourier transforms of Chapters 4 and 5. Fourier transforms do not occur in Chapter 5 until the estimation of the integral (5.1.52). In the case of $\phi(r) = r$, analysed in Chapter 3, the parameter α in (5.1.52) satisfies $|\alpha| = d + 1$ (and d is odd) and

$$I_0 = \int_{\{y : \|y\| \leq \delta h^{-1}\}} y^\alpha \psi(y) \, dy$$
$$= \sum_{j=1}^{l} \mu_j \int_{\{y : \|y\| \leq \delta h^{-1}\}} y^\alpha \|y - x_j\| \, dy. \tag{5.3.34}$$

Noting that y^α is a homogeneous polynomial of degree $d + 1$, we may apply Lemma 3-4 and find that

$$\int_{\{y : \|y\| \leq \delta h^{-1}\}} y^\alpha \|y - x_j\| \, dy \tag{5.3.35}$$

is a polynomial in h^{-1} for $h^{-1}\delta \geq \|x_j\|$. Hence I_0 (5.3.34) is a polynomial in h^{-1} for $h^{-1}\delta \geq \max\{\|x_j\| : j = 1, 2, \ldots, l\}$. However, recalling that any $\psi(x)$ which reproduces polynomials of degree d decays at least as fast as $\|x\|^{-2d-1}$, for large $\|x\|$, and also using $|\alpha| = d + 1$, the integral I_0 cannot grow faster than $|\log h|$ as $h \to 0$. It follows from these two observations that the integral remains finite as $h \to 0$, as required in the remark after (5.1.52). Thus, using the work in Chapter 3, we can complete a proof of the rate of convergence h^{d+1} when $\phi(r) = r$, without invoking Fourier transforms or generalised functions.

CHAPTER 6 : DISCUSSION

Section 6.1 : Discussion of Results

For practical purposes the most important result that we have derived in this dissertation is Corollary 5-11 in which we have proved rates of convergence for performing quasi-interpolation to sufficiently smooth functions over a bounded domain on a regular grid. Although results about rates of convergence over $\mathcal{R}^d$ as given in Theorem 5-10 are interesting theoretically, they are not themselves of practical use. Results on polynomial reproduction, Theorem 4-20, are also very interesting theoretically but for practical purposes they are only a means to an end. In this section will shall mainly be discussing the results in Corollary 5-11.

The first point that we note is that although, for example, the function $\phi(r) = r$ gives a better actual rate of convergence than $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$ in Corollary 5-11 the difference is not very large. For practical problems where one is just interested in ensuring that the error is less than some fixed tolerance it is possible that the difference in the constants in front of the rates of convergence may have a larger effect than the logarithm term.

For a fixed basis function the rate of convergence increases with the dimension. This is in marked contrast to many methods where one needs to work harder and harder to obtain just the same rates of convergence in higher dimensions. We consider the implications of this by considering performing quasi-interpolation with the basis function $\phi(r) = r$ in the two cases when $d = 3$ and $d = 5$. Suppose that in each case we have functions

$$\psi_d(x) = \sum_{j=1}^{l_d} \mu_{j,d} \|x - x_{j,d}\|, \qquad x \in \mathcal{R}^d, \tag{6.1.1}$$

which reproduce polynomials of degree d, $d = 3, 5$. We perform quasi-interpolation over the cubes

$$B_d = \{y \in \mathcal{R}^d \ : \ \|y\|_\infty < 1\}, \quad d = 3, 5, \tag{6.1.2}$$

to two functions f_d with

$$f_d \in C^{d+2}(\mathrm{cl}(B_d)), \quad d = 3, 5. \tag{6.1.3}$$

In each case we require the error between f_d and its quasi-interpolant to be less than some tolerance ϵ on

$$(B_d)_{0.1} = \{y \in \mathcal{R}^d \ : \ \|y\|_\infty < 0.9\}, \quad d = 3, 5. \tag{6.1.4}$$

Corollary 5-11 implies that if $a_{h_d,d}$ is the quasi-interpolant to f_d on a regular grid of spacing h_d then

$$|a_{h_d,d}(x) - f_d(x)| \le A_d h_d^{d+1}, \quad d = 3, 5 \tag{6.1.5}$$

and all $x \in (B_d)_{0.1}$. Thus, to satisfy the tolerance it would be sufficient to take

$$h_d = (\epsilon/A_d)^{1/(1+d)}, \quad d = 3, 5. \tag{6.1.6}$$

Hence the number of data points we would need to take is

$$N_d = (2/h_d)^d = (2A_d/\epsilon)^{d/(1+d)}, \quad d = 3, 5. \tag{6.1.7}$$

The ratio of the numbers is

$$\frac{N_5}{N_3} = \frac{(2A_5)^{5/6}}{(2A_3)^{3/4}}\epsilon^{(3/4)-(5/6)} = \frac{(2A_5)^{5/6}}{(2A_3)^{3/4}}\epsilon^{-1/12}, \tag{6.1.8}$$

which grows very slowly as ϵ decreases. Indeed, the ratio does not even double when ϵ decreases by a factor of 4000.

However, this does not quite tell the whole story. It is probable that the constant A_5 in (6.1.8) is much larger than A_3, the estimates in Chapter 5 indicate as much. The function ψ_5 will also be much more complicated than ψ_3. Similar examples of this kind of increase in complexity have already been noted in Section 4.3. There is the further problem that the function $\psi_5(x)$ which is a linear combination of increasing functions must decay at least as fast as $\|x\|^{-11}$. This can lead to large rounding errors in calculating the function ψ_5 for large argument. However, Powell (private communication) has found that for some choices of ψ it is possible to perform some of the cancellation analytically and so help to alleviate this difficulty.

For a further example we suppose that we are performing quasi-interpolation in d dimensions and wish to use a function ψ coming from one of the basis functions $\phi(r) = (r^2 + c^2)^{\frac{b}{2}}$. We suppose that d is odd and $b + d$ is a positive even integer in order that a suitable ψ exists. Increasing the value of b results in increasing rates of convergence so one may favour a large value of b. However, a large value of b implies a much more complicated function ψ and it is also probable, as in the case of splines in one dimension, that a larger value of b will give far worse localization properties. It is an interesting question as to which are good values of b for different values of d for practical problems. The best value may depend on the accuracy with which one wishes to solve the problem. No experiments have been done on this, although we mention the related results of Franke (1982); when performing interpolation in two dimensions, he found that multiquadrics (with $b = 1$) give better results than inverse multiquadrics (with $b = -1$).

Suppose that we perform interpolation over some bounded region Ω on a regular grid of spacing h. We let I_h be the interpolation operator. The interpolation operators $\{I_h\}$ are uniformly bounded over Ω_δ if $\|I_h\| \le M$ for all h and some constant M, where the norm is taken over Ω_δ. In this case, as we shall demonstrate by a standard argument, we can deduce that convergence of interpolants to a suitably smooth function f is of at least the same rate as that given in Corollary 5-11. For, with $x \in \Omega_\delta$,

$$\begin{aligned} |(I_h(f))(x) - f(x)| &\le |(I_h(f))(x) - a_h(x)| + |a_h(x) - f(x)| \\ &\le |I_h(x) - (I_h(a_h))(x)| + |a_h(x) - f(x)| \\ &\le (\|I_h\| + 1)|a_h(x) - f(x)|. \end{aligned} \tag{6.1.9}$$

However, deducing that the interpolation operators are uniformly bounded over Ω_δ is not an easy question. It may also be the case that interpolants converge over the whole of Ω and not just over Ω_δ, although the convergence near the boundary may be of a slower rate. Results of this nature have been obtained in some cases by Duchon (1977) and Arcangéli and Rabut (1986) by analysis which depends heavily on the variational principle (1.3.5) and its generalisations. However, we now consider what has turned out to be a more tractable extension of the work presented in this dissertation.

Section 6.2 : Extension to Interpolation over $\mathcal{R}^d$

We have worked through this dissertation with the function ψ (5.3.1) being a finite linear combination of basis functions. It is a reasonable question to ask whether the same analysis can ever hold if we took instead a function ψ of the form

$$\psi(x) = \sum_{l \in \mathcal{Z}^d} \mu_l \phi(\|x - l\|), \qquad x \in \mathcal{R}^d, \tag{6.2.1}$$

where $\{\mu_l : l \in \mathcal{Z}^d\}$ must be chosen so that the sum converges absolutely for all $x \in \mathcal{R}^d$. Buhmann (1988b) has considered this approach although he arrived at it by starting with the function χ with Fourier transform

$$\hat{\chi}(\lambda) = \frac{\hat{\phi}(\lambda)}{\sum_{\beta \in \mathcal{Z}^d} \hat{\phi}(2\pi\beta + \lambda)}, \qquad \lambda \in \mathcal{R}^d, \tag{6.2.2}$$

for some radial basis function ϕ. In the paper he considers the cases $\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$ and $\phi(r) = (r^2 + c^2)^{-\frac{1}{2}}$ in odd dimensions although he remarks that the technique is more general and he is working on a more comprehensive paper. He is able to show that

$$\chi(x) = \sum_{l \in \mathcal{Z}^d} c_l \phi(\|x - l\|), \qquad x \in \mathcal{R}^d, \tag{6.2.3}$$

for some constants $\{c_l : l \in \mathcal{Z}^d\}$, that the sum is absolutely convergent, that the function χ decays for large argument, and that

$$\chi(n) = \begin{cases} 1 & \text{if } n = 0; \\ 0 & \text{if } n \in \mathcal{Z}^d \backslash \{0\}. \end{cases} \tag{6.2.4}$$

This is a highly important result for it shows that χ is a cardinal function for performing interpolation on an infinite regular grid. Further, using the techniques of Chapter 4 he is able to show that his functions χ reproduce polynomials of the same degree as that found in Theorem 4-20. Using the techniques of Chapter 5 he is able to obtain slightly better convergence results in that he is able to get rid of the logarithm terms in Theorem 5-10 with this function χ. Thus he has proved convergence orders for interpolation over an infinite regular grid. He also remarks, without proof, that these functions χ are not restricted to either odd or even dimensions as in the case of functions ψ of the form (5.3.1). We remark that the formulation of the Fourier transform (6.2.2) of the cardinal function was first derived in one dimension by Schoenberg (1946) and has also been considered for a more restrictive class of basis functions by Maydych and Nelson (1988). These results are of significant theoretical interest but unfortunately, because there is no obvious analogue of (6.2.2), they do not provide an obvious method for deducing the convergence order in the case of interpolation over a bounded region on a regular grid.

Section 6.3 : Scattered Data

The practical value of approximating functions by performing quasi-interpolation over a regular grid is unclear as good techniques, such as tensor products of B-splines already exist. However, preliminary experiments (Powell, private communication) in evaluating some of the cardinal functions (6.2.3) in 2 and 3 dimensions have proved very encouraging. These functions were originally suggested for the problem of scattered data and it is expected that this will become their main application. In this case we must define what we mean by a rate of convergence. We suppose that we have a bounded domain Ω and an infinite sequence of points $\{z_k \in \Omega : k = 1, 2, \ldots\}$ which become dense in Ω.

We define h_N (c.f Section 2.1, especially (2.1.4)) by

$$h_N = \sup\{\inf\{\|y - z_k\| : k = 1, 2, \ldots, N\} : y \in \Omega\}. \tag{6.3.1}$$

The methods would be particularly useful for scattered data if their rates of convergence with respect to h are the same as the ones we have deduced for a regular grid. We must recall that this time when performing quasi-interpolation it is not sufficient just to find one function ψ, it is necessary to find one function corresponding to each data point (1.2.7). The only result known in this direction is the case $\phi(r) = r$ in one dimension, which is the same as linear interpolation. Here the rate of convergence h^2 is still attained for scattered data.

It is clear that the technique of proof used in Chapter 4 to prove polynomial reproduction which is heavily dependent on Fourier transforms and hence on a regular grid will not be able to work in this case. It is possible that an order of convergence result for scattered data will depend for its proof on some result like Lemma 5-7 which shows that polynomials are "almost reproduced" over Ω_δ. There is more of a chance that techniques similar to those in Chapter 3 could be developed to obtain a proof of such a result. Some of the ideas developed by Dyn, Levin and Rippa (1986) in the context of preconditioning the interpolation matrices for scattered data could also be useful in attacking such a problem. The author would like to offer a conjecture. We say that scattered data $\{z_k \in \Omega : k = 1, 2, \ldots\}$ are quasi-regular if there exist $A, M > 0$ such that for all values of h defined by (6.3.1) the number of points in any sphere of diameter Mh contained within Ω is bounded above by A. If the scattered data $\{z_k\}$ are quasi-regular then rates of convergence $h^{b+d-1}|\log h|$ will be obtained for quasi-interpolation schemes over this data in the four cases (a)–(d) of Corollary 5-11.

However, be this true or not, the theoretical results that have been derived in this dissertation far exceed all expectations from when this research began. They have given a stimulus to the method of radial basis function approximation and I look forward to the future developments which I hope will continue to astound.

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