Complex Space-time

by

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PREFACE

The dissertation is an account of work done in the Department of Applied Mathematics and Theoretical Physics, Cambridge, from October 1961 to October 1964, and then concurrently with employment at English Electric's Nelson Research Laboratories, Stafford (2 years), and at Hawker Siddeley Aviation, Kingston-on-Thames (1 year). I am grateful to the English Electric Company for sponsoring my research at Cambridge.

This dissertation has not been, and is not being, submitted for a degree or other qualification at any other university; and, except where otherwise stated, describes my own work.

In preparing a revised draft of the thesis I have derived much benefit from comments by Prof. C.W.Misner, and by Dr. D.W.Sciama. I should also like to thank Dr. R.J.Eden for being my supervisor.

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CONTENTS

CHAPTER 1

Unified Field Theories

§1.1 Preamble

Special relativity unified time with space in 1905. Subsequently, space-time was allowed to be curved, to represent gravitational forces. When after 1920 relativity ideas became widespread, the unlearned took up with an enthusiasm and awe matched only by their misconception this strange new doctrine of 'the fourth dimension'. The concept 'the world' had become at once more puzzling and yet richer, its structure more complex and therefore holding more boundless possibilities. To the professional, the mathematician, such 'mysticism' is in general distasteful, and tends to evaporate anyway on proper acquaintance with the theory. Nowadays it is 'obvious' that the events constituting the universe have as their setting a four-dimensional manifold.

Yet for all that, the one dimension of time is obstinately disparate from the three of space. The latter are accessible to an observer in a much 'freer' way than the former: we are presented with a succession of 'slices' of the four-dimensional world and cannot <u>directly</u> observe anything outside our present slice. Memory is no more than inference (in this respect on a par with prediction) from this-slice-now to the contents of other slices. Of course, we 'move' through a quasi-continuous sequence of such 'presents', but this is irrelevant to the (philosophically naive, admittedly) idea this discussion is intended to convey: the picture of an observer's experience-at-one-moment involving a three-dimensional continuum, and yet this experience being such that he can infer, but not directly test, the existence of another dimension.

So, one dimension is in some sense harder to 'navigate' than the other three. What if there were still others for which it was impossible: would their existence necessarily be without influence on the content of that part of the world which is directly accessible? Would the question of their reality be a question for metaphysics only, or even be perhaps 'unanswerable'? Only, surely, if one were prepared to treat the reality of future and past as a question equally outside the empirical realm. However, relativity theory makes this very reality its ontological cornerstone, assigning concepts like 'now' to logically secondary status. With such a precedent, it would appear that hypothetical 'extensions' of the phenomenal world are indeed a legitimate study for physics. Of course, one could still ask, of the supposed extra dimension(s) (i) why are they apparently unnavigable? and (ii) why does this particular 4-dimensional segment of the full (4+m)-dimensional manifold constitute 'our' world? Direct translation into the corresponding queries about 3-dimensional slices shows, however, that (i) and (ii) are strictly metaphysical, and their apparent unanswerability cannot therefore be cited (as antagonists of e.g. 5-dimensional theories may tend to) as constituting empirical evidence against the theory: any dismissal on these grounds would be purely *a priori*, and so can safely be ignored. Equally misplaced, however, would be attempts by the protagonists to answer such questions by assumptions about the geometry ("Toute théorie à plus de quatre dimensions fait toujours intervenir une 'condition cylindrique' ou une 'condition projective'. Celle-ci traduit le fait que les événements de l'univers quadridimensionnel ne peuvent dépendre de x^5." ([26] p.160)).

The present work stems from the following hypothesis: The space-time world of our experience is only the 'real part' of a complex world, so that, associated with each of the four directions in space-time, there is another, 'imaginary' direction, and there are events in the 'Überwelt' which are 'off the real axes', although as such not directly accessible to observation.

The motivation for making this hypothesis comes from what one could call the 'classical' unified field theory (UFT) problem. The latter came into being as an immediate consequence (both conceptual and temporal) of the theory of general relativity which, "having brought together the metric and gravitation, would have been completely satisfactory if the world had only gravitational fields and no electromagnetic fields. Now it is true that the latter can be included within the general theory of relativity by taking over and appropriately modifying Maxwell's equations of the electromagnetic field, but they do not then appear like the gravitational fields as structural properties of the space-time continuum, but as logically independent constructions. The two types of field are causally linked in this theory, but still not fused to an identity. It can, however, scarcely be imagined that empty space has conditions or states of two essentially different kinds, and it is natural to suspect that this only appears to be so because the structure of the physical continuum is not completely described by the Riemannian metric." [23] The author's idea, then, was that the presumably richer geometrical structure of a complex space-time might have room for electromagnetic (perhaps even scalar) fields. As initial grounds for thinking this to be a move in the right direction might be adduced the well-established method of generating appropriate electromagnetic interactions in quantum-mechanical field theory from phase ('gauge') transformations,[1] symbolically:

$$\psi \rightarrow e^{i\theta(x)}\psi \qquad (1.1)$$

which amount to position-dependent rotations in complex planes (the number of planes depending on the nature of ψ, as scalar, spinor, vector, etc.).

There have, of course, been many attempts at solving the classical UFT problem, and some are referred to in §1.3. To a good approximation (ignoring quantum-mechanical effects) the physical facts which all these theories are trying to re-express or re-derive are adequately characterized by the following set of equations, so that the production of some comparable set is a necessary, though not sufficient, criterion of success of a theory.

Einstein's field equations for gravitation are:

$$R_{\mu\nu} - \tfrac{1}{2} R g_{\mu\nu} = -\kappa T_{\mu\nu} \quad , \qquad (1.2)$$

the RHS representing 'matter' in some manner which must be specified in detail. Since they are covariant under the group of non-singular real coordinate transformations equations (1.2), regarded as partial differential equations for the $g_{\mu\nu}$, must be such as to involve four undetermined functions in any solution. This is ensured by the fact that only 6 of the 10 equations are independent, on account of the Bianchi identities:

$$R_\mu{}^\alpha{}_{;\alpha} - \tfrac{1}{2} \delta_\mu^\alpha R_{;\alpha} \equiv 0 . \qquad (1.3i)$$

(1.2) and (1.3i) are only mutually consistent if $T_{\mu\nu}$ satisfies conservation of energy-momentum: $\quad T_\mu{}^\alpha{}_{;\alpha} = 0 \qquad (1.3ii)$

The Maxwell-Lorentz field equations, in the absence of a material medium, are:[24]

$$F_{\mu\nu,\sigma} + F_{\nu\sigma,\mu} + F_{\sigma\mu,\nu} = 0 \qquad (1.4i)$$

$$F_\mu{}^{\cdot\alpha}{}_{;\alpha} = 4\pi J_\mu \qquad (1.4ii)$$

$$J^\mu{}_{;\mu} = 0 \qquad (1.4iii)$$

$$T^{(em)\nu}_{\mu\cdot} = \tfrac{1}{4\pi} (F_{\mu\alpha} F^{\nu\alpha} - \tfrac{1}{4} \delta_\mu^\nu F_{\alpha\beta} F^{\alpha\beta}) \qquad (1.4iv)$$

$$T^{(em)\alpha}_{\mu}{}_{;\alpha} = - F_\mu{}^{\cdot\alpha} J_\alpha \qquad (1.4v)$$

Of these, (i) & (ii) are Maxwell's equations for the propagation of the field (the RHS of (ii) must be a specified function of the non-electromagnetic fields present); (iii) (charge conservation) follows identically from (ii); while (v) is a consequence of (iv) (if (i) & (ii) are used) and, since it entails the well-substantiated Lorentz force on discrete charges, is the reason why the energy-momentum tensor has the particular form (iv) (cf. [25] p.131). (iv) also specifies, via (1.2), the gravitational effects produced by electromagnetic fields.

There is a neat derivation of all these equations from a single action principle of the form:

$$\delta \int \left[-\frac{1}{2\kappa} R - \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} + A_\mu J^\mu + L^{(m)} \right] \sqrt{-g}\ d^4 x = 0 \qquad (1.5)$$

where $L^{(m)}$ is the Lagrangian for any matter (non-gravitational and non-electromagnetic) fields, and where the $F_{\mu\nu}$ are not to be varied independently of the A_μ, but subject to the constraint $F_{\mu\nu} \equiv A_{\nu,\mu} - A_{\mu,\nu}$. (Strictly, the 3rd and 4th terms in the integrand are not always of precisely this form.)

As is the case for other UFT's, the present work can be viewed from two complimentary aspects: (i) as a purely mathematical study of a certain type of (in this case, complex) space, or (ii) as an investigation into how far the geometrical properties of the space mirror the behaviour of real physical fields. The mathematics of such a theory is either correct or incorrect; a decision on the second score, however, is a more delicate matter, and often amounts to no more (and no less!) than people getting bored with the theory, in the long run a sure indicator of its incorrectness. (Cf. W.Blake ('Proverbs of Hell'): "Truth can never be told so as to be understood, and not be believ'd.") The best estimate the author is able to make of the theory of complex space-time on physical grounds is, first, that (1.4i) finds a very natural place in it; second, that (1.4ii) and (surprisingly) (1.2) are more problematical; third, that there is not much sign of (1.4iv) or, therefore, of the Lorentz force (but see Chapter 5); overall, that the theory is <u>probably</u> false, considered as a classical field theory, and therefore also as a Cosmology (in the philosophical sense). It is firmly contended, though, that this could not have been forseen a priori, and that one of the prerogatives of theoretical physics is precisely to say "the world may be constructed like this" and then to spell out the consequences. Of course, not all worlds are worth constructing; and the current scientific consensus relegates classical UFT's to this class. Some possible grounds for this will now be considered.

The early years of the classical UFT problem coincided with a relatively primitive (by present standards) state of knowledge in fundamental physics, and it was then hoped that a complete solution of the problem of matter would result from a successful UFT, the latter being envisaged as a geometrical theory of a sort not too dissimilar from general relativity. For the Old Believers such a faith died hard, and, notwithstanding the advances in physics over the decades, the 1950's were in fact the heyday of UFT speculation. The history of the problem over this half-century is, however, a story of unremitting failure: Nature just does not seem interested any more in the game she played so willingly with Einstein in 1916. To explain this apparent change of heart, which is reflected, very properly, in physicists' own attitude, one might assert that what is wrong is that the classical UFT problem "ignores quantum theory". As is usually the case, however, such a generalized criticism would be at once both pertinent and wide of the mark. We shall distinguish three rather more precise remarks.

(i) Although the conceptual revolution entailed by quantum theory is profound and irreversible, quantum field theories arise directly out of their classical counterparts. This is true most obviously of quantum electrodynamics, but also applies to the (second-quantized) Klein-Gordon equation, it being relatively accidental that no-one had bothered to study the classical scalar wave equation with 'mass' term before quantum mechanics made it important. Although no longer in a position to claim final truth, any improved classical field theory would nevertheless react immediately, and presumably beneficially, on its quantized version (assuming that the latter could be constructed satisfactorily: with present techniques there would be big problems, as a UFT is necessarily more complex than general relativity).

(ii) The classical UFT problem is posed in the context of geometrizing electromagnetism. Subsequently, (at least) two more (weak and strong) interactions have become known which are quite as fundamental constituents of the universe as are the two known classically, and therefore potential candidates for 'geometrization' with equal a priori rights.

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Although underplayed compared with electromagnetism, certain types of scalar and vector 'meson' fields have in fact figured in the UFT literature, primarily in connection with multi-dimensional theories.[56] - [62] Whether or not these particular attempts are deemed successful (and it seems clear that they are at best rudimentary), the recognition of the non-uniqueness of electromagnetic amongst non-gravitational fields has obviously made the UFT problem more complex than was originally anticipated. Hopes for a satisfactory solution are thereby correspondingly diminished.

(iii) Classical physics operates exclusively with tensor fields. Since these are also the quantities which characterize the structure of the spaces studied by differential geometry, the formulation of the classical UFT problem in terms of some sort of 'geometrization' seemed eminently natural. Then quantum mechanics uncovered a new kind of field quantity: spinors; and, while tensors can be constructed from spinors, the converse is not true, so that a geometrical theory which was fully adequate to the facts would have to be framed ab initio in spinor form (cf. [77] p.521). Although, in the hands of their discoverer (Cartan) and of those who delighted in the older-fashioned kind of mathematics, spinors were sometimes presented in terms of conjugate points on ruled quadric surfaces and so on, they do not in fact appear to be very natural things in terms of which to define a line-element expression, and as this is what all UFT's to date have consid-

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§1.2 Finsler space theory

This section is disjoint from the rest of the dissertation. It describes an attempt (1962) at a 'geometrization' of the electromagnetic (and possibly scalar) fields in terms of a metric manifold which is not 'locally euclidean', in the sense that, with respect to an allowed coordinate system, the indicatrix of Carathéodory (locus of end-points of unit vectors) is not, as it is in Riemannian geometry, a quadric hypersurface centred on the origin. I argued that if an electromagnetic field, and correspondingly a non-vanishing vector potential ϕ_μ, is present, then this latter determines a preferred direction at any point of space-time; Riemannian geometry, on the other hand, is locally isotropic, which ties in with the difficulty of finding in its structure anything resembling electromagnetism. The moral seemed clear: generalize the Riemannian metric so as to destroy its local isotropy. Bearing in mind that one wishes to recover the Pythagorean line-element expression for vanishing electromagnetic field, the simplest choice seemed to be to keep the indicatrix a quadric surface but no longer centred on the origin i.e. to take as its equation:

$$g_{\mu\nu}(\xi^\mu + A^\mu)(\xi^\nu + A^\nu) = 1. \qquad (1.6)$$

The vector A^μ specifying the new position of the centre should be correlated with the physical electromagnetic potential ϕ^μ, while the symmetric tensor $g_{\mu\nu}$ might be expected to relate to the physical gravitational potentials (if $A^\mu = 0$ then it coincides with them). It is clear that, besides the anisotropy, 'distance' is no longer even a symmetric function, i.e. $d(P,P')$, the distance from a point P to a neighbouring point P' of the manifold, is not in general equal to $d(P',P)$; so that one is led to consider the notion of a directed path: a curve with an 'arrow' on it. There is a relevant parallel here with the Feynman graph formalism in particle physics, whereby particle-antiparticle conjugation (which, inter alia, changes the sign of charge) is represented by reversing the direction of the arrow on a (non-self-conjugate) particle line. But this is in a quantum-theoretical context. Classically, that the world-line of a charged, as opposed to uncharged, particle can be considered as being a directed line in the above sense is suggested by the form of the Lorentz force equation:

$$\frac{d^2x^\sigma}{ds^2} + \Gamma^\sigma_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = \frac{e}{m}F^\sigma_{\ \mu}\frac{dx^\mu}{ds} \qquad (1.7)$$

where the operation $ds \rightarrow -ds$, which could be interpreted as reversing the 'direction' of the world-line, is equivalent to changing the sign of the charge $\frac{e}{m} \rightarrow -\frac{e}{m}$.

Introducing in the usual way the concept of a 'line-element' via the requirement that, for an infinitessimal displacement dx^μ, $(\frac{dx^\mu}{ds})$ be a unit vector, (1.6) implies

$$ds = \frac{A_\mu dx^\mu \pm \sqrt{[(1-A_\alpha A^\alpha)g_{\mu\nu} + A_\mu A_\nu]\,dx^\mu dx^\nu}}{1 - A_\alpha A^\alpha}, \qquad (1.8)$$

where $A_\mu \equiv g_{\mu\nu}A^\nu$. The two-valuedness of the RHS of (1.8) is present also in the Riemannian case, but there it has no particular importance (one can always work with $|ds|$); here, however, changing branches of $\sqrt{\ }$ has the same effect on $|ds|$ as changing the sign of A_μ - the double-valuedness relating to the existence of two signs of charge. To make more explicit the structure of (1.8), it is convenient to define

$$\alpha_\mu \equiv \frac{A_\mu}{1-A_\alpha A^\alpha} \;; \qquad \gamma_{\mu\nu} \equiv \frac{g_{\mu\nu}}{1-A_\alpha A^\alpha} + \alpha_\mu \alpha_\nu \tag{1.9}$$

so that it becomes: $$ds = \alpha_\mu dx^\mu \pm \sqrt{\gamma_{\mu\nu}\, dx^\mu dx^\nu}. \tag{1.10}$$

Equation (1.10) is of the form

$$ds = \mathcal{L}(x^\alpha, dx^\alpha), \tag{1.11}$$

where $\mathcal{L}$ is positively homogeneous of the first degree in its second set of arguments, viz. $\mathcal{L}(x,\lambda y) = \lambda \mathcal{L}(x,y)$, for $\lambda > 0$. A metric space of this type is called a Finsler space, after the man who originated such geometries in his study of two-dimensional surfaces [P.Finsler, Dissertation, Göttingen, 1918]. It is interesting, though, as Weyl remarks ([4], p.138), that Riemann himself "many years ago pointed out that the metrical groundform might, with essentially equal right, be a homogeneous function of the fourth order in the differentials, or even a function built up in some other way, and that it need not even depend rationally on the differentials". Subsequently, contributions were made in 1925/6 by J.L.Synge,[5] J.H.Taylor,[6] and particularly L.Berwald,[7] but the first complete axiomatization of Finsler spaces was given by E.Cartan[8] in 1934. At least until the 1950's, Cartan's treatment became accepted as the standard one, and many papers appeared on the subject. Among these were several attempts[16, 17, 18, 20, 21, 22] to base 'unified' field theories on this geometry. Because of this, and because the work described in this section was in large measure an attempt to remedy what I came to regard as most unfortunate elements in this orthodox approach, the latter will now be briefly summarized (cf. [8], [9], [16]).

Cartan's starting-point is to introduce the idea of a space of 'line-elements' (éléments linéaires). A line-element (x, x') consists of its 'centre', the point (x^μ) $(\mu = 1, 2 .. n)$, together with a direction at that point, specified by the n homogeneous coordinates (x'^μ). Formally, the space is thus a (2n-1)-dimensional manifold. The coordinate transformation group considered is, however, of the (restricted) type:

$$\left.\begin{aligned} x^{\mu *} &= x^{\mu *}(x^\alpha) \\ x'^{\mu *} &= \frac{\partial x^{\mu *}}{\partial x^\alpha} x'^\alpha \end{aligned}\right\} \qquad (1.12)$$

A 'contravariant vector field' defined on this manifold is then a set of functions of (x, x') transforming under (1.12) as:

$$A^{\mu *}(x^*, x'^*) = \frac{\partial x^{\mu *}}{\partial x^\alpha} A^\alpha(x, x') \qquad (1.13)$$

and analogous formulae define other ranks of tensor. The 'basic function' (Grundfunction) from which all other quantities characterizing the geometrical structure of the manifold are to be derived is the scalar introduced in (1.11), namely $\mathcal{L}$. The argument of all tensor functions is henceforth understood to be (x, x'). Denote partial derivatives by a comma.

A 'metric tensor' is introduced by

$$g_{\mu\nu} \equiv \left(\tfrac{1}{2}\mathcal{L}^2\right)_{,\mu',\nu'} \qquad (1.14)$$

By homogeneity (Euler) this implies $\mathcal{L} = \sqrt{g_{\mu\nu} x'^\mu x'^\nu}$ (1.15)

It is assumed that (at least locally) $g \equiv \det \| g_{\mu\nu} \| \neq 0$ (1.16)

Therefore the contravariant inverse exists: $g^{\lambda\mu} g_{\mu\nu} = \delta^\lambda_\nu$ (1.17)

Write $l^\mu \equiv \dfrac{x'^\mu}{\mathcal{L}}$ (1.18)

Then $$l_\mu \equiv g_{\mu\nu} l^\nu = \mathcal{L}_{,\mu'} \tag{1.19}$$

Under 'parallel displacement' from (x,x') to (x+dx,x'+dx') a vector A^μ is assumed to change by

$$\delta A^\lambda = -\left(\Gamma^\lambda{}_{\cdot\mu\nu} A^\mu dx^\nu + C^\lambda{}_{\cdot\mu\nu} A^\mu dx'^\nu \right) \tag{1.20}$$

The requirement that lengths shall not change under parallel transport enables a corresponding formula for covariant vectors to be derived; and there is the usual extension to higher rank tensors. Certain other postulates, which will not be ennumerated here, are now made in order to determine the 'affine connection' components. One obtains (using the metric tensor to pull indices):

$$C_{\lambda\mu\nu} = \tfrac{1}{2} g_{\mu\nu,\lambda'} = (\tfrac{1}{4}\mathcal{L}^2)_{,\lambda',\mu',\nu'} \tag{1.21}$$

$$\Gamma_{\lambda\mu\nu} = \tfrac{1}{2}\left(g_{\lambda\mu,\nu} + g_{\lambda\nu,\mu} - g_{\mu\nu,\lambda} \right) + C_{\mu\nu\alpha} G^\alpha{}_{,\lambda'} - C_{\lambda\nu\alpha} G^\alpha{}_{,\mu'} \tag{1.22}$$

where $$2G_\mu \equiv (\tfrac{1}{2}\mathcal{L}^2)_{,\mu';\alpha}\, x'^\alpha - (\tfrac{1}{2}\mathcal{L}^2)_{,\mu} \tag{1.23}$$

By homogeneity, there is the identity $$C_{\lambda\mu\nu} l^\nu \equiv 0. \tag{1.24}$$

Observe that Γ is not symmetric in its last two indices. This is related to the fact that the RHS of (1.20) is not really in the most appropriate form. Consider the particular case in which the direction (x'+dx') is such that the result of parallel transport according to (1.20) of the vector l^μ(x,x') is just the vector $\dfrac{x'^\mu + dx'^\mu}{\mathcal{L}(x+dx,\, x'+dx')}$. One finds, noting (1.24), that this is characterized by

$$Dl^\lambda \equiv dl^\lambda + \Gamma^\lambda{}_{\cdot\mu\nu} l^\mu dx^\nu = dl^\lambda + \mathcal{L}^{-1} G^\lambda{}_{,\nu'} dx^\nu = 0 \tag{1.25}$$

(1.20) can now be re-written in terms of this new quantity Dl^λ, called by Cartan the 'absolute differential' of l^λ. One thereby obtains the following expression for the absolute differential of the vector A^μ:

$$D A^\lambda \equiv dA^\lambda - \delta A^\lambda = A^\lambda{}_{|\mu}\, dx^\mu + \mathcal{L}\, A^\lambda{}_{|\mu'}\, Dl^\mu \tag{1.26}$$

where $A^\lambda{}_{|\mu} \equiv A^\lambda{}_{,\mu} - A^\lambda{}_{,\alpha'}\, G^\alpha{}_{,\mu'} + \Gamma^{*\lambda}_{\cdot\beta\mu} A^\beta$ (1.27)

and $A^\lambda{}_{|\mu'} \equiv A^\lambda{}_{,\mu'} + C^\lambda_{\cdot\beta\mu} A^\beta$ (1.28)

In Berwald's notation, write $g_{\mu\nu(\sigma)} \equiv g_{\mu\nu,\sigma} - g_{\mu\nu,\alpha'}\, G^\alpha{}_{,\sigma'}$ (1.29)

Then the Γ^*'s <u>are</u> symmetric, and are given by

$$\Gamma^*_{\lambda\mu\nu} = \tfrac{1}{2}\left(g_{\lambda\mu(\nu)} + g_{\lambda\nu(\mu)} - g_{\mu\nu(\lambda)} \right). \tag{1.30}$$

There are the identities

$$\left.\begin{aligned} l_{\mu|\nu} &\equiv 0 \\ g_{\mu\nu|\sigma} &\equiv 0 \end{aligned}\right\} \tag{1.31}$$

the second being an analogue of the corresponding Riemannian result.

Curvature tensors are definable by parallel transfer of (e.g.) a vector round a closed circuit. Owing to the form of the RHS of (1.26) there will in fact be three distinct tensors:

$$\begin{aligned} R^\lambda_{\cdot\mu\nu\sigma} = {} & \left(\Gamma^{*\lambda}_{\cdot\mu\nu}\right)_{(\sigma)} - \left(\Gamma^{*\lambda}_{\cdot\mu\sigma}\right)_{(\nu)} + \Gamma^{*\lambda}_{\cdot\sigma\alpha}\Gamma^{*\alpha}_{\cdot\mu\nu} - \Gamma^{*\lambda}_{\cdot\nu\alpha}\Gamma^{*\alpha}_{\cdot\mu\sigma} \\ & + C^\lambda_{\cdot\mu\alpha}\left[\left(G^\alpha{}_{,\nu'}\right)_{(\sigma)} - \left(G^\alpha{}_{,\sigma'}\right)_{(\nu)} \right] \end{aligned} \tag{1.32}$$

$$P_{\lambda\mu\nu\sigma} = C_{\lambda\nu\sigma|\mu} - C_{\mu\nu\sigma|\lambda} + \left[C_{\mu\nu\alpha} C^\alpha_{\cdot\lambda\sigma|\beta} - C_{\lambda\nu\alpha} C^\alpha_{\cdot\mu\sigma|\beta} \right] x'^\beta \tag{1.33}$$

$$S^\lambda_{\cdot\mu\nu\sigma} = C^\lambda_{\cdot\sigma\alpha} C^\alpha_{\cdot\mu\nu} - C^\lambda_{\cdot\nu\alpha} C^\alpha_{\cdot\mu\sigma} \tag{1.34}$$

The first is the analogue of the Riemann tensor, and satisfies, in addition to anti-symmetry in its last two indices, the identities: $R_{\lambda\mu\nu\sigma} = -R_{\mu\lambda\nu\sigma}$ (cf. [11], p.107) (1.35)

$$R_{\lambda\{\mu\nu\sigma\}} = C_{\lambda\alpha\{\mu} R^\alpha_{\cdot|\beta|\nu\sigma\}}\, x'^\beta \tag{1.36}$$

$R^\lambda_{\cdot\mu\{\nu\sigma|\rho\}} = P^\lambda_{\cdot\mu\alpha\{\nu} R^\alpha_{\cdot|\beta|\sigma\rho\}}\, x'^\beta$ (cf. [11], p.111) (1.37)

where $\{\lambda\mu\nu\}$ signifies a sum over the three cyclic permutations.

Now, all this formalism bears a quite nice resemblance to that of a Riemannian space, but it is very difficult to give it any concrete, 'intuitive' meaning at all. This is particularly apparent if one attempts to 'do physics' in such a space. Physical theories represent measured quantities by scalars, vectors, etc. defined at each point of space-time. But in the line-element formalism just described a direction (x') has also to be specified before the corresponding quantity is defined. This is true even for the special case of what one might call a 'point'-vector field $v^{\mu}(x,x') = v^{\mu}(x)$ i.e. one independent of the directional arguments; for (i) its length is given by

$$\sqrt{g_{\mu\nu}(x,x')\,v^{\mu}v^{\nu}} \tag{1.38}$$

which is not independent of (x'), and (ii) its 'covariant derivatives' would also reintroduce x'-dependence, depending on the value assigned to the argument of the functions Γ^{*}, C in (1.27), (1.28). In this connection it is perhaps worth emphasising that the directional argument (x') of $\Gamma^{\lambda}_{\cdot\mu\nu}$ in (1.20) is quite uncorrelated with the spatial direction dx in which the parallel displacement is made.

The author was confirmed in his belief that an alternative approach was both necessary and possible by H.Busemann's book 'The geometry of geodesics' (1955). In the preface Busemann explains that the term 'Finsler space' does not appear in the title because it "means to many not only a type of space but also a definite approach: the space is considered as a set of line-elements to which euclidean metrics are attached. The main problems are connected with parallelism. In spite of the great success of Finsler's thesis, the later developement of this aspect lacks simple geometrical facts to the extent that their existence in non-Riemannian geometry has been doubted".[10] Using only topological methods (no differentiability assumptions or analytical tools) he succedes in showing that very many of the classic results of (global) Riemannian geometry carry over to Finsler spaces, and that therefore "there emerges the highly important problem of gaining a clear understanding of the true realm of Riemannian geometry, i.e. of recognizing the character of the theorems for which it is essential that the local unit spheres be ellipsoids rather than arbitrary convex surfaces with centre". The book is, however, "a geometric approach to qualitative problems in intrinsic differential geometry" (my italics), and to get a quantitative theory suitable for use in physics – in particular, a sharper characterization of local (rather than global) properties – it would seem that some sort of analytical approach is essential. Such a treatment will now be outlined.

We start from (1.11), which is considered as giving the length of the (infinitessimal) vector dx^μ qua element of a tangent vector space, $V_n(x)$ say, at the point x^μ. We now extend this statement to apply to all other vectors $v \in V_n$. This is just what is done in the case of a Riemannian manifold. Suppress the x-dependence, for the present – i.e. consider only one such V_n. We introduce, therefore, a scalar product which is to be such that

$$(v|v) = [\mathcal{L}(v)]^2. \qquad (1.39)$$

(Comparison with (1.38) illustrates the difference in approach.)

What can be said about the expression $(u|v)$ for $u \neq v$? Various possibilities present themselves, depending on how the 'non-linearity' entailed by (1.39) is incorporated. It turns out to be best to retain the concept of a dual space V_n' of linear mappings $v': V_n \to \mathbb{R}$ onto the reals, but to give up linearity of the function $G: V_n \to V_n'$ which associates co- with contra-variant components of the 'same' vector (in the Riemannian case G is of course a linear mapping, with matrix $g_{\mu\nu}$ with respect to appropriate bases). We therefore require:

$$(u|v) = u_\mu v^\mu = g_{\mu\nu}(u)\, u^\nu v^\mu \tag{1.40}$$

where $g_{\mu\nu}(u) = \dfrac{\partial^2}{\partial u^\mu \partial u^\nu}\left[\tfrac{1}{2}\mathcal{L}^2(u)\right]$ (1.41)

So that, in general $(u|v) \neq (v|u)$. (1.42)

This nonsymmetry of the scalar product reflects in the present formalism a remarkable theorem due to Blaschke (cf. [10] p.103) that if in a Minkowskian (i.e. flat Finsler) space of dimension greater than two perpendicularity between lines is symmetric, then the metric is euclidean.

By the assumed homogeneity of $\mathcal{L}$, $g_{\mu\nu}(u)$ is positively homogeneous of degree zero in the u^α. Make the requirement:

$$g(u) \equiv \det \| g_{\mu\nu}(u) \| \neq 0. \tag{1.43}$$

Then there is an inverse set of functions $g^{\mu\nu}(u)$ such that

$$u^\lambda = g^{\lambda\mu}(u)\, u_\mu . \tag{1.44}$$

Vector indices can therefore be raised and lowered as in Riemannian geometry, but it should be remarked that there is no clear-cut extension of the operation to higher rank tensors.

To construct a tensor calculus, we need an 'affine connection'

That the latter will involve a certain non-linearity, as did the scalar product, can be seen as follows. We require that the two definitions of geodesics as (i) minimum paths and (ii) autoparallel curves shall coincide. The first says:

$$\delta \int ds = \delta \int \mathcal{L}\left(x, \frac{dx}{ds}\right) ds = 0. \tag{1.45}$$

The Euler-Lagrange equations can be recast in the form:

$$\frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) + 2G^\mu\left(x, \frac{dx}{ds}\right) = 0, \tag{1.46}$$

where G is the same as in (1.23), but with directional argument $x' = \frac{dx}{ds}$. (1.46) is also equivalent to (cf. [11] p.52):

$$\frac{d}{ds}\left(\frac{dx^\mu}{ds}\right) + \left\{ {\mu \atop \alpha\beta} \right\}\left(\frac{dx}{ds}\right) \frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0 \tag{1.47}$$

where $\left\{ {\mu \atop \alpha\beta} \right\}\left(\frac{dx}{ds}\right)$ is formed from the $g_{\mu\nu}\left(\frac{dx}{ds}\right)$ by the Riemannian prescription for a Christoffel symbol. Now tie this in with definition (ii). As inspection of (1.47) shows, the change in a vector v^μ under parallel displacement from P (x) to P' (x + dx) cannot be linear in both v^μ and dx^μ. We require it to depend linearly on the components v^μ; it must therefore be allowed <u>to depend non-linearly on the displacement components</u> (dx^μ); we accordingly define:

$$\delta v^\lambda = - T^{*\lambda}{}_{\mu\nu}(dx)\, v^\mu dx^\nu \tag{1.48}$$

and thence the tensorial 'absolute differential':

$$Dv^\lambda \equiv dv^\lambda - \delta v^\lambda = dv^\lambda + T^{*\lambda}{}_{\mu\nu}(dx)\, v^\mu dx^\nu. \tag{1.49}$$

The T^* are assumed positively homogeneous of degree zero in dx^α. Similar expressions hold for other types of tensor, in the usual way. In particular:

$$D\,\delta^\mu_\nu \equiv 0. \tag{1.50}$$

An autoparallel curve satisfies:

$$0 = D\left(\frac{dx^\lambda}{ds}\right) = d\left(\frac{dx^\lambda}{ds}\right) + T^{*\lambda}{}_{\mu\nu}(ds)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} \tag{1.51}$$

i.e. $$\frac{d}{ds}\left(\frac{dx^\lambda}{ds}\right) + \Gamma^{*\lambda}_{\mu\nu}\left(\frac{dx}{ds}\right)\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0. \qquad (1.52)$$

Comparison of (1.47) with (1.52) restricts the Γ^*, but does not completely determine them in terms of the metric tensor. We can get a stronger restriction by requiring that the same geodesics also arise from parallel displacement of the covariant tangent vector $\frac{dx_\mu}{ds}$; we therefore require:

$$D\, g_{\mu\nu}\left(\frac{dx}{ds}\right) = 0 \qquad (1.53)$$

which, in turn, is equivalent to:

$$\left[\frac{\partial g_{\mu\nu}}{\partial x^\sigma} - \frac{\partial g_{\mu\nu}}{\partial \dot{x}^\alpha}\Gamma^{*\alpha}_{\cdot\beta\sigma}\dot{x}^\beta - g_{\alpha\nu}\Gamma^{*\alpha}_{\cdot\mu\sigma} - g_{\mu\alpha}\Gamma^{*\alpha}_{\cdot\nu\sigma}\right]dx^\sigma = 0 \qquad (1.54)$$

the directional arguments of the g's and Γ^*'s all being $\dot{x} \equiv \frac{dx}{ds}$. Setting (compare (1.46) and (1.52))

$$\Gamma^{*\lambda}_{\cdot\beta\sigma}\dot{x}^\beta = \frac{\partial G^\lambda}{\partial \dot{x}^\sigma} \qquad (1.55)$$

and requiring the square bracket itself in (1.54) to vanish, one can solve for the $\Gamma^*_{\mu\nu\sigma}(\dot{x})$, obtaining an expression <u>formally</u> identical to Cartan/Berwald's eqn.(1.30), with x' replaced by $\dot{x}$.

As is clear, this derivation has only provided <u>a</u> (quite natural) determination of the affine connection by the metric – other choices are possible. (There is a corresponding arbitrariness in Cartan's and Berwald's derivations.) It is not easy, however, to see the sort of additional criterion to appeal to in order to tighten up the deduction. So we shall work from the above particular solution. We also remark that although (1.49) and its counterparts define absolute differentials of tensors, <u>'covariant' derivatives are not defined</u>. This seems unavoidable in the context of the present treatment.

In spite of this difficulty in connection with differentiation, it is possible to construct a curvature tensor. In Riemannian geometry there are three common ways of doing this: (i) commutator of double covariant derivatives; (ii) parallel transport around infinitessimal circuit (holonomy group); (iii) geodesic deviation. In the present case, method (i) is ruled out; and (ii) even more so, since the result of such a parallel transport will, by (1.49), depend in a very complicated manner on the precise shape of the curve; (iii) is available, as will appear. This situation was noted, from his different point of view, by Busemann ([10] p.235): "In Riemannian spaces curvature has many different functions. It is not plausible that in Finsler spaces a single concept will suffice for all these functions; it is rather to be expected that different concepts, which happen to coincide in the Riemannian case, correspond to different functions.

"The great majority of the investigations on intrinsic geometry exploit, or can be modified so as to exploit, only one of the functions, and can therefore be extended to Finsler spaces." (his italics). His treatment is essentially in terms of definition (iii) which, via the Gauss-Bonnet theorem, can be formulated in terms of the angular excess of geodesic triangles.

The analytic formulation of (iii) we shall give is a precise parallel to the Riemannian case (cf.[12] p.90).

Definition (1.56): A vector v is orthogonal to a vector u if and only if $(v|u) = 0$.

(As already observed, this is not a symmetrical relation.)
Consider a two-dimensional surface $x^\mu = f^\mu(u,v)$ (1.57)
which is such that v = constant specifies a directed geodesic parametrized by its arc-length u, for each value of v; and such that there is a curve of the other system, $u = u_0$ say, to which all these geodesics are orthogonal. (Note: the curves u = constant are not directed curves - according to defn. (1.56) it is unnecessary that they should be.) Write

$$\frac{\partial f^\mu}{\partial u} \equiv p^\mu \quad ; \quad \frac{\partial f^\mu}{\partial v} \equiv q^\mu . \tag{1.58}$$

By assumption, then:

$$0 = (p|q)_{u=u_0} = g_{\mu\nu}(p)\, p^\nu q^\mu \Big|_{u=u_0} = \frac{\partial \mathcal{L}(p)}{\partial p^\mu} q^\mu \Big|_{u=u_0} \tag{1.59}$$

We first show that for any other value of u ($= u_1$, say) a similar equation to (1.59) holds; i.e. that the geodesics are orthogonal to all the curves u = constant. For, define

$$L(v) \equiv \int_{u_0}^{u_1} \mathcal{L}(x,p)\, du = u_1 - u_0 .$$

This is independent of v. Therefore

$$\begin{aligned} 0 = \frac{dL}{dv} &= \int_{u_0}^{u_1} \frac{\partial}{\partial v}\left[\mathcal{L}(x,p)\right] du = \int_{u_0}^{u_1} \left[\frac{\partial \mathcal{L}}{\partial x^\alpha} q^\alpha + \frac{\partial \mathcal{L}}{\partial p^\alpha}\frac{\partial^2 f^\alpha}{\partial u \partial v}\right] du \\ &= \frac{\partial \mathcal{L}}{\partial p^\alpha} q^\alpha \Big|_{u_0}^{u_1} - \int_{u_0}^{u_1}\left[\frac{d}{du}\left(\frac{\partial \mathcal{L}}{\partial p^\alpha}\right) - \frac{\partial \mathcal{L}}{\partial x^\alpha}\right] q^\alpha\, du \end{aligned} \tag{1.60}$$

Because the curves are geodesics, the integrand, and hence the integral, vanishes, as does, by (1.59), the contribution at the lower limit to the first term on the RHS. This proves the result. Now consider two neighbouring geodesics of the family, specified respectively by the parameters v, $v + dv$. Let P, P' be the points (u,v), $(u,v+dv)$ respectively. Then the vector $\overrightarrow{PP'}(u)$ has components $(q^\mu dv)$. By the result just established, the two geodesics will both be orthogonal to this vector, for all u. The distinction between a flat and curved space is that in the former the vector will be proportional to u. We therefore want to determine how $\vec{PP'}$, or equivalently q^μ, varies with u. Let $\frac{D}{Du}$ stand for the invariant derivative operator in the direction (p^μ). Then

$$\frac{Dq^\lambda}{Du} = \frac{\partial^2 f^\lambda}{\partial u \partial v} + \frac{\partial G^\lambda(p)}{\partial p^\alpha} q^\alpha ; \tag{1.61}$$

and the rate of 'geodesic deviation' is found to be

$$\begin{aligned} \frac{D^2 q^\lambda}{Du^2} &= \left[\left(\frac{\partial G^\lambda}{\partial p^\rho}\right)_{(\sigma)} - \left(\frac{\partial G^\lambda}{\partial p^\sigma}\right)_{(\rho)} \right] q^\rho p^\sigma \\ &= R^\lambda{}_{\cdot\nu\rho\sigma}(p)\, p^\nu q^\rho p^\sigma \end{aligned} \tag{1.62}$$

where in fact a term in $\frac{Dq^\mu}{Du}$ on the RHS has vanished, owing to (1.55) - a possible additional motivation for the latter. $R^\lambda{}_{\cdot\nu\rho\sigma}$ is formally identical to Cartan's expression (1.32), with x' = p. (1.62) has the same form as the Riemannian result, except for the direction-dependence of the $R^\lambda{}_{\cdot\nu\rho\sigma}$; so the difference is that a quadratic function of the direction of the geodesics (p) has become a more general function, homogeneous of degree two in the p^α. As in Riemannian geometry, therefore, (1.62) is a quite concrete result: one which could in principle be investigated by clocks and measuring rods, and so is a suitable ingredient for a geometrical theory of physics. (As already indicated, this problem of 'physical meaning' is a persistent one in Finsler-space theories, and published attempts [16, 18, 20, 21, 22] on these lines seem to gloss over it - a remark (in connection with e.g. derivation of 'field equations') like "Let a direction-field x'(x) be specified..." giving the game away at once.)

The theory just outlined will now be applied to spaces flat and curved space is that in the former the vector will be proportional to u. We therefore want to determine how $\overrightarrow{PP'}$, or equivalently q^μ, varies with u. Let $\frac{D}{Du}$ stand for the invariant derivative operator in the direction (p^μ). Then

$$\frac{Dq^\lambda}{Du} = \frac{\partial^2 f^\lambda}{\partial u \partial v} + \frac{\partial G^\lambda(p)}{\partial p^\alpha} q^\alpha \quad ; \tag{1.61}$$

and the rate of 'geodesic deviation' is found to be

$$\begin{aligned} \frac{D^2 q^\lambda}{Du^2} &= \left[\left(\frac{\partial G^\lambda}{\partial p^\rho}\right)_{(\sigma)} - \left(\frac{\partial G^\lambda}{\partial p^\sigma}\right)_{(\rho)} \right] q^\rho p^\sigma \\ &= R^{\lambda}_{\cdot\nu\rho\sigma}(p)\, p^\nu q^\rho p^\sigma \end{aligned} \tag{1.62}$$

where in fact a term in $\frac{Dq^\mu}{Du}$ on the RHS has vanished, owing to (1.55) - a possible additional motivation for the latter. $R^{\lambda}_{\cdot\nu\rho\sigma}$ is formally identical to Cartan's expression (1.32), with x' = p. (1.62) has the same form as the Riemannian result, except for the direction-dependence of the $R^{\lambda}_{\cdot\nu\rho\sigma}$; so the difference is that a quadratic function of the direction of the geodesics (p) has become a more general function, homogeneous of degree two in the p^α. As in Riemannian geometry, therefore, (1.62) is a quite concrete result: one which could in principle be investigated by clocks and measuring rods, and so is a suitable ingredient for a geometrical theory of physics. (As already indicated, this problem of 'physical meaning' is a persistent one in Finsler-space theories, and published attempts [16, 18, 20, 21, 22] on these lines seem to gloss over it - a remark (in connection with e.g. derivation of 'field equations') like "Let a direction-field x'(x) be specified..." giving the game away at once.)

The theory just outlined will now be applied to spaces with basic function of the particular form (1.10), viz:

$$\mathcal{L}(x,v) = \alpha_\mu(x)\, v^\mu + \sqrt{\gamma_{\mu\nu}(x)\, v^\mu v^\nu} \tag{1.63}$$

The '+' sign has been chosen to make the metric single-valued; the opposite sign would lead to the geometry of a space (1.63) with the sign of α_μ everywhere reversed; so that (1.63), for various vector fields $\alpha_\mu(x)$, in fact covers all cases. (Compare this situation with that in regard to the initial definition of sign of charge in electromagnetic theory: if every charge in the universe were reversed in sign then this would make no observable difference, in classical electromagnetism at any rate.) Here are formulae for some of the geometrical objects in the space.

$$l^\mu(v) = \frac{v^\mu}{\mathcal{L}(v)} \tag{1.64}$$

$$l_\mu(v) = \frac{\partial \mathcal{L}(v)}{\partial v^\mu} = \frac{\gamma_{\mu\lambda} v^\lambda}{\sqrt{\gamma_{\alpha\beta} v^\alpha v^\beta}} + \alpha_\mu \tag{1.65}$$

Abbreviate the first term on RHS by $\varpi_\mu(v)$. From (1.41):

$$g_{\mu\nu}(v) = \gamma_{\mu\nu} + \alpha_\mu \alpha_\nu + \frac{(\gamma_{\mu\nu}\alpha_\lambda + \gamma_{\mu\lambda}\alpha_\nu + \gamma_{\nu\lambda}\alpha_\mu) v^\lambda}{(\gamma_{\alpha\beta} v^\alpha v^\beta)^{1/2}} - \frac{\gamma_{\mu\alpha} v^\alpha \gamma_{\nu\beta} v^\beta \alpha_\lambda v^\lambda}{(\gamma_{\alpha\beta} v^\alpha v^\beta)^{3/2}}$$

$$= \left[\frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta} v^\alpha v^\beta}}\right] (\gamma_{\mu\nu} - \varpi_\mu \varpi_\nu) + l_\mu l_\nu \tag{1.66}$$

$$\therefore\ 2C_{\mu\nu\rho} \equiv \frac{\partial g_{\mu\nu}}{\partial v^\rho} = \left[\frac{\mathcal{L}(v)}{\gamma_{\alpha\beta} v^\alpha v^\beta}\right] \Big[(\gamma_{\mu\nu} - \varpi_\mu \varpi_\nu)(\alpha_\rho - \alpha_\lambda l^\lambda l_\rho)\Big]_{\{\mu\nu\rho\}} \tag{1.67}$$

where { } represents sum over cyclic permutations. The remaining purely covariant quantity is (cf. (1.23)):

$$\begin{aligned} 2G_\mu = {} & \frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta} v^\alpha v^\beta}}\, \overset{(R)}{\Gamma}_{\mu\alpha\beta} v^\alpha v^\beta - \mathcal{L}(v)\, F_{\mu\alpha} v^\alpha \\ & + \alpha_\mu \left[\alpha_{\alpha,\beta} v^\alpha v^\beta + \frac{\frac{1}{2}\gamma_{\alpha\beta,\epsilon} v^\alpha v^\beta v^\epsilon}{\sqrt{\gamma_{\alpha\beta} v^\alpha v^\beta}} \right] \\ & + \varpi_\mu \left[\alpha_{\alpha,\beta} v^\alpha v^\beta - \frac{\frac{1}{2}(\gamma_{\alpha\beta,\epsilon} v^\alpha v^\beta v^\epsilon)(\alpha_\lambda v^\lambda)}{(\gamma_{\alpha\beta} v^\alpha v^\beta)} \right] \end{aligned} \tag{1.68}$$

where
$$F_{\mu\nu} \equiv \alpha_{\nu,\mu} - \alpha_{\mu,\nu} \tag{1.69}$$

and where the superscript $^{(R)}$ signifies: Riemannian Christoffel symbol formed from the $\gamma_{\mu\nu}$.

To proceed further, it is clearly necessary to evaluate the contravariant metric tensor, $g^{\mu\nu}(v)$. A direct inversion of the system (1.66) is not easy, so we do it indirectly. Let $\gamma^{\mu\nu}$ be the inverse matrix of $\gamma_{\mu\nu}$ (NB not its contravariant form). Then (1.65) yields the identity

$$\gamma^{\mu\nu}\left(\frac{v_\mu}{\mathcal{L}(v)} - \alpha_\mu\right)\left(\frac{v_\nu}{\mathcal{L}(v)} - \alpha_\nu\right) \equiv 1 . \tag{1.70}$$

Solving for $\mathcal{L}$ as a function of the covariant form of v :

$$\mathcal{L}(v) = \frac{-\alpha^\mu v_\mu + \sqrt{(1-\alpha_\lambda\alpha^\lambda)\gamma^{\mu\nu}v_\mu v_\nu + (\alpha^\lambda v_\lambda)^2}}{(1-\alpha_\lambda\alpha^\lambda)} \tag{1.71}$$

where $\quad \alpha^\lambda \equiv \gamma^{\lambda\mu}\alpha_\mu .$ (1.72)

In calculations one needs the useful identities:

$$\frac{\sqrt{(1-\alpha_\lambda\alpha^\lambda)\gamma^{\mu\nu}v_\mu v_\nu + (\alpha^\lambda v_\lambda)^2}}{\mathcal{L}(v)} = \frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}} \tag{1.73}$$

and, (cf. (1.65)): $\quad \ell^\mu(v) = \frac{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}{\mathcal{L}(v)}\left(\gamma^{\mu\lambda}\ell_\lambda(v) - \alpha^\mu\right).$ (1.74)

The equation $\quad g^{\mu\nu}(v) = \frac{\partial^2}{\partial v_\mu \partial v_\nu}\left[\frac{1}{2}\mathcal{L}^2(v)\right]$ (1.75)

now enables the $g^{\mu\nu}(v)$ to be computed:

$$g^{\mu\nu}(v) = \frac{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}{\mathcal{L}(v)}\left[\gamma^{\mu\nu} - (1-\alpha_\lambda\alpha^\lambda)\ell^\mu\ell^\nu - \alpha^\mu\ell^\nu - \alpha^\nu\ell^\mu\right] + \ell^\mu\ell^\nu . \tag{1.76}$$

One can verify that this is indeed the inverse of the set of functions (1.66). It is now possible to evaluate the determinant of $g_{\mu\nu}(v)$, as follows (again, the direct approach is not practicable). (1.67) and (1.76) give

$$\frac{\partial}{\partial v^\mu}\left[\log g(v)\right] = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial v^\mu} = 2g^{\alpha\beta}C_{\alpha\beta\mu} = (n+1)\frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}\left[\alpha_\mu - \alpha_\lambda\ell^\lambda\ell_\mu\right] \tag{1.77}$$

This differential equation can be integrated:

$$g(v) = f(x)\left[\frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}\right]^{n+1} \tag{1.78}$$

where f is independent of v , and can be found from:

$$\frac{\partial}{\partial x^\mu}\left[\log g(v)\right] = g^{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial x^\mu} = \left[\log\gamma\right]_{,\mu} + (n+1)\left[\log\frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}\right]_{,\mu} \tag{1.79}$$

where $\gamma \equiv \det\|\gamma_{\mu\nu}\|$. Therefore $\quad g(v) = \gamma\left[\frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta}v^\alpha v^\beta}}\right]^{n+1}$ (1.80) an unexpectedly simple result.

From (1.68) and (1.76) one finds, after a considerable amount of cancellation:

$$2\,G^{\lambda}(v) = \Gamma^{(R)\lambda}{}_{\mu\nu} V^{\mu} V^{\nu} - F^{\lambda}_{\cdot\mu} V^{\mu} \sqrt{\gamma_{\alpha\beta} V^{\mu} V^{\nu}} + \ell^{\lambda}(v)\left[\tfrac{1}{2} S_{\mu\nu} V^{\mu} V^{\nu} + \alpha_{\varepsilon} F^{\varepsilon}_{\cdot\mu} V^{\mu} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} \right] \tag{1.81}$$

where $F^{\lambda}_{\cdot\mu} \equiv \gamma^{\lambda\rho} F_{\rho\mu}$ and $S_{\mu\nu} \equiv \alpha_{\nu;\mu} + \alpha_{\mu;\nu}$

(semi-colon denoting Riemannian covariant derivative w.r.t. $\Gamma^{(R)}$).

Differentiating (1.81) gives:

$$\begin{aligned} 2\frac{\partial G^{\lambda}}{\partial v^{\kappa}} = {} & 2\,\Gamma^{(R)\lambda}_{\cdot\mu\kappa} V^{\mu} - F^{\lambda}_{\cdot\kappa} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} - F^{\lambda}_{\cdot\mu} V^{\mu} \varpi_{\kappa} \\ & + \ell^{\lambda}\left[S_{\mu\kappa} V^{\mu} + \alpha_{\varepsilon} F^{\varepsilon}_{\cdot\kappa} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} + \alpha_{\varepsilon} F^{\varepsilon}_{\cdot\mu} V^{\mu} \varpi_{\kappa}\right] \\ & + \mathcal{L}^{-1}(\delta^{\lambda}_{\kappa} - \ell^{\lambda}\ell_{\kappa})\left[\tfrac{1}{2} S_{\mu\nu} V^{\mu} V^{\nu} + \alpha_{\varepsilon} F^{\varepsilon}_{\cdot\mu} V^{\mu} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} \right] \end{aligned} \tag{1.82}$$

This enables one to form (cf. (1.29)):

$$\begin{aligned} {}'g_{\rho\sigma(\kappa)} \equiv {} & \frac{\partial g_{\rho\sigma}}{\partial x^{\kappa}} - 2\,C_{\rho\sigma\lambda} \frac{\partial G^{\lambda}(v)}{\partial v^{\kappa}} \\ = {} & \frac{\mathcal{L}(v)}{\sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}}} \gamma_{\rho\sigma,\kappa} + \ell_{\rho}\,\alpha_{\sigma,\kappa} + \ell_{\sigma}\,\alpha_{\rho,\kappa} \\ & + \left[\frac{\mathcal{L}(v)}{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}}\right] \left[\begin{aligned} & (\gamma_{\rho\sigma} - \varpi_{\rho}\varpi_{\sigma})(\alpha_{\lambda} - \alpha_{\varepsilon}\ell^{\varepsilon}\ell_{\lambda})\left(-\Gamma^{(R)\lambda}_{\cdot\kappa\mu} V^{\mu} + \tfrac{1}{2} F^{\lambda}_{\cdot\kappa} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}}\right) \\ & + (\alpha_{\rho} - \alpha_{\varepsilon}\ell^{\varepsilon}\ell_{\rho})\left(\Gamma^{(R)}_{\mu\sigma\kappa} V^{\mu} + \tfrac{1}{2} F_{\sigma\kappa} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} - \tfrac{1}{2} F_{\mu\kappa}\varpi_{\sigma} V^{\mu} - \tfrac{1}{2} F_{\mu\sigma}\varpi_{\kappa} V^{\mu}\right) \\ & + (\alpha_{\sigma} - \alpha_{\varepsilon}\ell^{\varepsilon}\ell_{\sigma})\left(\Gamma^{(R)}_{\mu\rho\kappa} V^{\mu} + \tfrac{1}{2} F_{\rho\kappa} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} - \tfrac{1}{2} F_{\mu\kappa}\varpi_{\rho} V^{\mu} - \tfrac{1}{2} F_{\mu\rho}\varpi_{\kappa} V^{\mu}\right) \end{aligned} \right] \\ & + \tfrac{1}{2}(\gamma_{\rho\sigma} - \varpi_{\rho}\varpi_{\sigma})\,\varpi_{\kappa} \frac{(\alpha_{\varepsilon} F^{\varepsilon}_{\cdot\mu} V^{\mu})}{\sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}}} \\ & - \mathcal{L}^{-1} C_{\rho\sigma\kappa}\left[\tfrac{1}{2} S_{\mu\nu} V^{\mu} V^{\nu} + \alpha_{\varepsilon} F^{\varepsilon}_{\cdot\mu} V^{\mu} \sqrt{\gamma_{\alpha\beta} V^{\alpha} V^{\beta}} \right] \end{aligned} \tag{1.83}$$

One can now immediately write down the expression for $\Gamma^{*}_{\lambda\mu\nu}(v)$ (cf. (1.30)), and thence calculate the $\Gamma^{*\lambda}_{\cdot\mu\nu}(v)$. The result for the latter is considerably more complicated even than (1.83) (which has been simplified as far as is possible), and the explicit evaluation of the curvature tensor from (1.32) seems out of the question. However, the computation becomes easy for what one might call the 'weak field, pure electromagnetic' case, viz. working to first order only in the α's, and setting $\gamma_{\mu\nu}$ = constant. Then

$$g_{\rho\sigma(\kappa)} \underset{(1)}{=} \varpi_\rho \alpha_{\sigma,\kappa} + \varpi_\sigma \alpha_{\rho,\kappa} \tag{1.84}$$

$$T^{*\lambda}_{\cdot\mu\nu} \underset{(1)}{=} \frac{\gamma_{\mu\alpha} V^\alpha F_{\nu}{}^{\cdot\lambda} + \gamma_{\nu\alpha} V^\alpha F_{\mu}{}^{\cdot\lambda} + S_{\mu\nu} V^\lambda}{2\sqrt{\gamma_{\alpha\beta} V^\alpha V^\beta}} \tag{1.85}$$

$$R^{\lambda}_{\cdot\mu\sigma\nu} \underset{(1)}{=} \frac{(F_{\sigma}{}^{\cdot\lambda}{}_{,\nu} - F_{\nu}{}^{\cdot\lambda}{}_{,\sigma})\gamma_{\mu\alpha} V^\alpha + F_{\mu}{}^{\cdot\lambda}{}_{,\nu}\gamma_{\sigma\alpha} V^\alpha - F_{\mu}{}^{\cdot\lambda}{}_{,\sigma}\gamma_{\nu\alpha} V^\alpha + F_{\nu\sigma,\mu} V^\lambda}{2\sqrt{\gamma_{\alpha\beta} V^\alpha V^\beta}} \tag{1.86}$$

which, it will be observed, depends on the α's only via the gauge-invariant combination $F_{\mu\nu}$. By contraction

$$R_{\mu\nu} \equiv R^{\lambda}_{\cdot\mu\lambda\nu} \underset{(1)}{=} \frac{(F_{\mu\alpha,\nu} + F_{\nu\alpha,\mu})V^\alpha - j_\mu \gamma_{\nu\alpha} V^\alpha - j_\nu \gamma_{\mu\alpha} V^\alpha}{2\sqrt{\gamma_{\alpha\beta} V^\alpha V^\beta}} \tag{1.87}$$

where $j_\mu \equiv F_{\mu}{}^{\cdot\alpha}{}_{,\alpha}$. In this approximation $R_{\mu\nu}$ is seen to be symmetric (in a Finsler space this is not a necessary result). A second contraction gives: $R \equiv R^{\cdot\nu}_{\nu} \underset{(1)}{=} 0.$ (1.88)

It is at this stage that the theory starts to peter out. For, the original aim was not just to compute a few geometrical quantities, but to show that the space manifests a behaviour resembling that of real charges and electric fields; and in this there has been conspicuously little success. There are two main difficulties. First, one wants some field equations, in order to restrict the space-time dependence of $\alpha_\mu(x)$ and $\gamma_{\mu\nu}(x)$ in some meaningful way. In euclidean or Riemannian manifolds a variational derivation of the equations has decided advantages (compatibility, Noether's theorems, etc.). In the present instance almost the only natural choice of Lagrangian density would be: $R(v)\sqrt{-g(v)}$, (1.89)

but its direction-dependence stands in the way of formulating a sensible action principle. On this score the theories of [16], [18], [22] must be adjudged as having little physical meaning.

The second, probably fatal, difficulty arises in relating α_μ to the physical 4-potential ϕ_μ. The former is dimensionless, so that if

$$\alpha_\mu = k\,\phi_\mu \tag{1.90}$$

then k must have dimensions $\left(\frac{\text{charge}}{\text{mass}}\right)$. Suppose $k = \frac{e}{m_0}$, where e is the electronic charge, and m_0 a mass of order that of the electron (or proton). Then the geodesics of the space are identical with the world-lines of particles with charge-to-mass ratio $\left(\frac{e}{m_0}\right)$ moving under the combined influence of the gravitational field derived from $\gamma_{\mu\nu}$ and a physical electromagnetic field

$$f_{\mu\nu} = \phi_{\nu,\mu} - \phi_{\mu,\nu} = \frac{m_0}{e}(\alpha_{\nu,\mu} - \alpha_{\mu,\nu}). \tag{1.91}$$

This fact is one of the motivations for choosing the form of metric (1.10) in the first place (cf. [13], [14], [15], [19], [21] which are the previous occasions on which this metric has been put forward, in no case with much elaboration). We shall gloss over the question of what <u>precise</u> value to take for m_0: since it does not attempt to consider strong interactions the theory is not likely to be correct in that much detail anyway. With such a choice of k, $\alpha_\mu \ll 1$ for all macroscopic fields; but at distances of order the classical radius of the electron ($= 2{\cdot}818 \times 10^{-13}$ cm) from a charge e the field is such that $\alpha_\mu \sim 1$, so that deviations from linearity would be expected (cf. e.g. the form of (1.71)) and therewith the possibility of a singular metric at finite distance from a 'point' charge.

Now, we have just assumed that the world-lines of charged particles should correspond to the geodesics of the space, whereas in fact for a non-linear theory the field equations themselves entail how the singularities of the field ('particles') must move. This is important in the present context for, although field equations are precisely what is lacking, it seems that even if they existed they would be unlikely to lead to the correct physics, as the following argument, rough though it is, intimates:

Although the curvature scalar has not been computed explicitly, it is clear from the form of (1.9), (1.66), (1.83), etc. that $\gamma_{\mu\nu}$ tends to figure in conjunction with products of two α's, and similarly when differentiated, so that (1.89), in addition to the Riemannian $R^{(R)}$ formed from the $\gamma_{\mu\nu}$, which must be an ingredient (consider the case $\alpha_\mu = 0$), will contain term(s) something like $F^{\mu\nu}F_{\mu\nu}$. Although prima facie just what is required, in order to get the correct contribution of the electromagnetic to the gravitational field, the Maxwell energy-momentum tensor, and thence the Lorentz force (cf. (1.4), (1.5)), nevertheless the presence of any such term is in fact disastrous, since by (1.91) it equals

$$\left(\frac{e}{m_0}\right)^2 f^{\mu\nu} f_{\mu\nu} \quad \simeq \quad 10^{39}\, \kappa\, f^{\mu\nu} f_{\mu\nu} \qquad (1.92)$$

where κ is Einstein's gravitational constant. This contribution is therefore 39 orders of magnitude too large. Stephenson & Kilmister's theory[19, 20, 21] is in fact rendered null by this observation which, by oversystematically calling all constants 1, they overlook.

No obvious progress seems possible. Of course, k could be scaled down by a factor $\sim 10^{20}$; but, among other disadvantages, that severs the connection with the Hamiltonian formulation of particle electrodynamics which was the basis of the original intuitive appeal of the theory.

After this work was essentially completed (1963), a former student of his drew to my attention the book[11] by H.Rund. In this monograph he presents, inter alia, his own criticism and revision of the orthodox Finsler space theory, made in the 1950's, from a standpoint and with results parallelling very closely those given here, though with much greater wealth of detail. In particular, he too treats the spaces as locally Minkowskian ([11] p.16) - rather than as locally euclidean, which, by concentrating attention on the so-called 'osculating Riemannian space', the 'line-element' approach manages to do. He also describes ([11] pp.111-9) the same construction as was given in (1.62) for the curvature tensor. The book is not concerned with unified field theory problems.

§1.3 Theories related to the present work

From the extensive literature on the UFT problem,[26] and also on (flat) spaces more general than the Minkowski and transformation/symmetry groups more general than the Lorentz, we restrict attention to ideas having some aspects in common with the theory of complex space-time. Actually, this still leaves a rather wide field: the theory has complex Hermitian metric tensor,[27][42] and complex symmetric affine connection;[44][47][54] it introduces extra dimensions;[50][54][55][56-62] and (the unimodular restriction of) its underlying flat-space transformation group, SU(4), is isomorphic with the real 6-dimensional orthogonal group and with the transformation group generated by (a real restriction of) the Dirac (Clifford) algebra in real Minkowski space-time,[91][96] and is related to certain 'internal' symmetry groups suggested in the context of elementary particle theory[84-92] (see §6.1).

Einstein's relativistic theory of gravitation is deducible from the following three data: (1) a 4-dimensional real manifold, defined w.r.t. the group of general (non-singular) coordinate transformations; (2) a real symmetric affine connection; (3) a real symmetric bilinear form (or 'metric tensor'), with signature $(\pm)2$. A natural way to try and construct a more comprehensive field theory is therefore to modify either (1), (2) or (3) (or any combination), and all classical UFT's have in fact proceeded in this manner.

The best-known example of a theory changing (1) is the

5-dimensional or 'projective relativity' theory, originated by J.Kaluza in 1921 and subsequently developed and recast in various forms by a number of physicists (see [25] pp.254-79 and [26] pp.156-241 for surveys of this work, which, apart from the extra-dimensionality, has no very close bearing on the present theory). Although at one stage favouring and contributing to such theories, Einstein himself came finally to concentrate his attention on a theory which kept (1), but gave up the symmetry requirements in (2) and (3). Since, in the original version,[27]-[29] the skew-symmetric contributions were taken to be pure-imaginary, and therefore the quantities themselves Hermitian, a brief account of the theory will be given here. (Schrödinger's 'purely affine' theory,[32] developed 1943 onwards, is often lumped together with Einstein's as the 'Einstein-Schrödinger theory', because it also presupposes a non-symmetric affine connection; however, it has no relevance to the present work.)

Introduce as 'metric tensor' the Hermitian matrix

$$g_{ik} = \overline{g_{ki}} \tag{1.93}$$

Assuming it non-singular, a contravariant metric tensor is definable by:

$$g^{im} g_{jm} = \delta^i_j \tag{1.94}$$

However, in view of their non-symmetry, one does not use these tensors for pulling indices ([32] p.109). Introduce a complex affine connection $\Gamma^i_{\cdot jk}$. The Ansatz

$$g_{ik,l} - g_{sk}\Gamma^s_{\cdot il} - g_{is}\Gamma^s_{\cdot lk} = 0 \tag{1.95}$$

goes into itself on complex conjugation (using (1.93)) if the Γ's are Hermitian:

$$\Gamma^i_{\cdot jk} = \overline{\Gamma^i_{\cdot kj}} \tag{1.96}$$

(1.95) are then just the right number of equations to determine the Γ's as functions of the g's. This is the reason for the choice of suffix order on the LHS.

The fact that the Γ's are no longer real means that there are really <u>two</u> affinities[29] in the space-time manifold, so that one has to distinguish two types of covariant derivative:

$$A^{\underset{+}{i}}{}_{;\ell} \equiv A^i{}_{,\ell} + \Gamma^i{}_{\cdot m\ell}\, A^m \qquad (1.97\text{i})$$

$$A^{\underset{-}{i}}{}_{;\ell} \equiv A^i{}_{,\ell} + \overline{\Gamma^i{}_{\cdot m\ell}}\, A^m = A^i{}_{,\ell} + \Gamma^i{}_{\cdot \ell m}\, A^m \qquad (1.97\text{ii})$$

and similarly for other tensor indices. (1.95) is the same as:

$$g_{ik,\ell} - g_{sk}\,\Gamma^s{}_{\cdot i\ell} - g_{is}\,\overline{\Gamma^s{}_{\cdot k\ell}} = 0 \qquad (1.98)$$

and has the quasi-Riemannian form: $$g_{\underset{+}{i}\underset{-}{k};\ell} = 0 \qquad (1.99)$$

Since $\delta_{\underset{+}{i}}{}^{\underset{-}{k}}{}_{;\ell} \neq 0$, the operations of covariant differentiation and of contraction no longer necessarily commute.

For any covariant (or contravariant) pair of indices write

$$\tfrac{1}{2}(A_{ik} + A_{ki}) \equiv A_{\underline{ik}} \;;\; \tfrac{1}{2}(A_{ik} - A_{ki}) \equiv A_{\underset{\vee}{ik}} \qquad (1.100)$$

(1.95) implies: $$(\sqrt{-g})_{,\ell} - \sqrt{-g}\,\Gamma^s{}_{\cdot\underline{\ell s}} = 0 \qquad (1.101)$$

so it is natural[28] to <u>define</u> the LHS of (1.101) to be the covariant derivative of the scalar density $\sqrt{-g}$. The formula for the covariant derivative of any tensor density is thereby fixed, via the product rule for differentiation.

Since there are two kinds of covariant derivative there will be four kinds of commutator of double differentiation:

$$\left.\begin{aligned} A_{\underset{+}{i};\underset{+}{\ell};m} - A_{\underset{+}{i};\underset{+}{m};\ell} &= -A_j R^j{}_{\cdot i\ell m} - 2\Gamma^s{}_{\cdot\underset{\vee}{\ell m}} A_{\underset{+}{i};s} \\ A_{\underset{+}{i};\underset{-}{\ell};m} - A_{\underset{+}{i};\underset{-}{m};\ell} &= -A_j R^j{}_{\cdot i\ell m} + 2\Gamma^s{}_{\cdot\underset{\vee}{\ell m}} A_{\underset{+}{i};s} \end{aligned}\right\} \qquad (1.102)$$

(plus two more equations which are obtainable by taking complex conjugates of both sides), where the curvature tensor is:

$$R^{j}_{\cdot ilm} = \Gamma^{j}_{\cdot il,m} - \Gamma^{j}_{\cdot im,l} - \Gamma^{j}_{\cdot sl}\Gamma^{s}_{\cdot im} + \Gamma^{j}_{\cdot sm}\Gamma^{s}_{\cdot il} \tag{1.103}$$

Define $$R_{hilm} \equiv g_{jh}\, R^{j}_{\cdot ilm} \tag{1.104}$$

Then the latter satisfies:[30]

$$R_{hilm} = - R_{himl} \tag{1.105i}$$

$$R_{hilm} = - \overline{R_{ihml}} \tag{1.105ii}$$

and also the Bianch-type identities:[30]

$$R_{\underset{-+-+}{iklm};n} + R_{\underset{-+++}{ikmn};l} + R_{\underset{-+--}{iknl};m} = 0 \tag{1.106}$$

Define $$R_{kl} \equiv R^{m}_{\cdot klm} = g^{mi}\, R_{iklm} \tag{1.107}$$

This is <u>not</u>, in general, Hermitian; one finds:[30]

$$\tfrac{1}{2}\left(R_{kl} - \overline{R_{lk}} \right) = \tfrac{1}{2}\left(\Gamma_{k,l} - \Gamma_{l,k} + \Gamma_{s}\, \Gamma^{s}_{\cdot \underset{\vee}{kl}} \right) \tag{1.108}$$

where $$\Gamma_{k} \equiv \Gamma^{s}_{\cdot \underset{\vee}{ks}} . \tag{1.109}$$

Note that $\Gamma^{i}_{\cdot \underset{\vee}{kl}}$ is a <u>tensor</u>, because of the usual transformation law for affinities, so that Γ_k is a vector, and also the RHS of (1.108) is, as required, a tensor. There is another contraction of the curvature tensor (identically zero in the Riemannian case):

$$R^{a}_{\cdot akl} = \Gamma^{a}_{\cdot ak,l} - \Gamma^{a}_{\cdot al,k}$$

$$= \Gamma_{l,k} - \Gamma_{k,l} \tag{1.110}$$

by virtue of (1.101).[28] Multiplying (1.106) by $(g^{mi}\, g^{kl})$ gives the doubly-contracted Bianchi identities:[30]

$$g^{kl}\left(R_{\underset{+-}{kl};n} - R_{\underset{++}{kn};l} - \overline{R_{\underset{--}{ln};k}} \right) = 0 \tag{1.111}$$

From the above, in particular (1.108), (1.110) and (1.111), it will have been apparent that a considerable simplification[28] [30] ensues if one postulates:

$$\Gamma_k = 0 \tag{1.112}$$

In conjunction with (1.95) this is equivalent to (cf. [32] p.110):

$$g^{\underset{\vee}{kl}}{}_{,l} = 0 \; . \tag{1.112'}$$

The most significant reason for postulating (1.112), however, comes from the search for field equations. Given suitable non-degeneracy, the equations (1.95) can be solved for the Γ's - although the explicit formulae are excessively complicated.[33][34] Consider this done. Then we want 16 field equations for the 16 unknowns g_{ik}, and by the general theory of such systems the equations must also satisfy 4 identities. In the Riemannian case (general relativity) the equations

$$R_{ik} = 0 \tag{1.113}$$

were appropriate. In the present case they amount to more than 16 equations, on account of the non-Hermiticity of R_{ik} (cf. (1.108)), and so are not permissible. If, on the other hand, one departs from (1.113), then one has to ensure anew the existence of identities, since the Bianchi identities will in general no longer fit the bill. A neat resolution of this dilemma was indicated by Einstein in [30], where he showed that if one postulated

$$\Gamma_k = 0 \tag{1.114i}$$

and

$$R_{\underline{ik}} = 0 \; , \tag{1.114ii}$$

then the Bianchi identities (1.111) reduced to:

$$g^{kl} \, R_{\{\underset{\vee}{kl},n\}} = 0 \; . \tag{1.115}$$

Therefore (1.111) constitute the required 4 identities not only for (1.113) but equally for the (less restrictive) set consisting of (1.114i), (1.114ii) and

$$R_{\{\underset{\vee}{kl},n\}} = 0 \; . \tag{1.114iii}$$

Discounting the Bianchi identities, the set (1.114) is prima facie $4+10+4 = 18$ equations for the 16 g_{ik}. Closer analysis shows, however, that there exist 2 further identities among the LHS's - there must exist such, since it is possible to derive the complete set (1.95) + (1.114) from a variational principle,[28-32] which ensures their compatibility. One of these additional identities is ([28] p.733):

$$(g^{\underset{\vee}{ik}} \Gamma_k)_{,i} = 0 \qquad (1.116)$$

The other one reduces the number of independent equations (1.114iii) from 4 to 3 according to Einstein & Straus ([28] p.735) but the justification for this assertion is obscure.

The above is a representative version of the theory - many variant formulations exist. Subsequent work has attempted to clarify the physical content, if any, of the formalism. The initial expectation was that the anti-symmetric part of g_{ik} should somehow relate to the electromagnetic field: (1.112') would then be like a Maxwell equation. However it is not even clear which of the two it should represent ((1.4i) or (1.4ii)), i.e. whether the physical F_{ik} is (proportional to) $g_{\underset{\vee}{ik}}$ or $g^{*}_{\underset{\vee}{ik}} = \frac{1}{2}\varepsilon_{iklm}\, g^{\underset{\vee}{lm}}$ - or even[34] a combination of the two. Still other identifications are possible: novel light was cast on the structure of the theory by Sciama's Vierbein reformulation,[41] [42] which tended to implicate $R^{a}{}_{aik}$ as the electromagnetic field (and correspondingly Γ_k as vector potential). However, none of these assignments mitigates what was discovered[36-39] to be the main shortcoming of the theory: to provide something like the Lorentz force. Of course, with extra terms added to the Lagrangian[39] a Maxwell-type energy-momentum tensor can be made to appear (cf. (1.5)); but the theory thereby ceases to be a UFT in the 'deductive' sense originally envisaged, and has less to recommend it than the conventional Maxwell-Einstein theory. All this applies equally to the real non-symmetric or the complex Hermitian versions of the theory. The former seems to stand very little chance of being a solution of the UFT problem for a further reason: it is probable, as argued by Sciama[40] and others, that non-symmetry of the g_{ik} is connected with something quite different: the presence of spin in the matter field.

Another UFT using complex tensors over a real manifold is Moffat's.[43][44] (Cf. also [47] - a more rudimentary theory.) Moffat assumes (using here a notation in which pure-imaginary quantities are explicitly displayed as such):

$$g_{\mu\nu} = \overset{(1)}{g}_{\mu\nu} + i\,\overset{(2)}{g}_{\mu\nu} \qquad (1.117)$$

with <u>both</u> tensors on the RHS real and <u>symmetric</u>. A symmetric complex affine connection $\Gamma^{\lambda}_{\cdot\mu\nu}$ is introduced w.r.t. which

$$g_{\mu\nu;\sigma} = 0. \qquad (1.118)$$

Therefore:

$$\Gamma^{\lambda}_{\cdot\mu\nu} \equiv \Gamma_{(1)}{}^{\lambda}_{\cdot\mu\nu} + i\,\Gamma_{(2)}{}^{\lambda}_{\cdot\mu\nu}$$
$$= \tfrac{1}{2} g^{\lambda\alpha}\left(g_{\alpha\mu,\nu} + g_{\alpha\nu,\mu} - g_{\mu\nu,\alpha}\right) \qquad (1.119)$$

Under the (real) transformation group being considered, $\Gamma_{(1)}{}^{\lambda}_{\cdot\mu\nu}$ transforms like a Riemannian connection, $\Gamma_{(2)}{}^{\lambda}_{\cdot\mu\nu}$ like a tensor. No explicit resolution of (1.119) into real and imaginary parts is obtained in the general case, though in the linear approximation (see below) and for particular solutions[46] it is possible.

The curvature tensor in formed in the standard way:

$$R^{\lambda}_{\cdot\mu\nu\sigma} \equiv R_{(1)}{}^{\lambda}_{\cdot\mu\nu\sigma} + i\, R_{(2)}{}^{\lambda}_{\cdot\mu\nu\sigma}$$

$$= \Gamma^{\lambda}_{\cdot\mu\nu,\sigma} - \Gamma^{\lambda}_{\cdot\mu\sigma,\nu} - \Gamma^{\lambda}_{\cdot\alpha\nu}\Gamma^{\alpha}_{\cdot\mu\sigma} + \Gamma^{\lambda}_{\cdot\alpha\sigma}\Gamma^{\alpha}_{\cdot\mu\nu} \quad (1.120)$$

There is the decomposition:

$$R_{(1)}{}^{\lambda}_{\cdot\mu\nu\sigma} = \left[\Gamma_{(1)}{}^{\lambda}_{\cdot\mu\nu,\sigma} - \Gamma_{(1)}{}^{\lambda}_{\cdot\alpha\nu}\Gamma_{(1)}{}^{\alpha}_{\cdot\mu\sigma} + \Gamma_{(2)}{}^{\lambda}_{\cdot\alpha\nu}\Gamma_{(2)}{}^{\alpha}_{\cdot\mu\sigma}\right] - \left[\nu \leftrightarrow \sigma\right]$$

$$R_{(2)}{}^{\lambda}_{\cdot\mu\nu\sigma} = \left[\Gamma_{(2)}{}^{\lambda}_{\cdot\mu\nu,\sigma} - \Gamma_{(1)}{}^{\lambda}_{\cdot\alpha\nu}\Gamma_{(2)}{}^{\alpha}_{\cdot\mu\sigma} - \Gamma_{(2)}{}^{\lambda}_{\cdot\alpha\nu}\Gamma_{(1)}{}^{\alpha}_{\cdot\mu\sigma}\right] - \left[\nu \leftrightarrow \sigma\right] \quad (1.121)$$

The usual Bianchi identities hold:

$$R^{\lambda}_{\cdot\mu\{\nu\sigma;\rho\}} = 0 \quad (1.122)$$

Define $R_{\mu\nu} \equiv R^{\alpha}_{\cdot\mu\nu\alpha}$; $R \equiv R^{\alpha}_{\cdot\alpha}$. (1.123)

Then there are the four complex identities

$$(R_{\mu}^{\alpha} - \tfrac{1}{2}\delta_{\mu}^{\alpha} R)_{;\alpha} = 0 . \quad (1.124)$$

Field equations are derived from a variational principle. Since the most natural choice, $R\sqrt{-g}$, is not suitable, being complex, he chooses, as Lagrangian density for the 'free' gravitational + electromagnetic fields, the real part of $R\sqrt{-g}$; if a matter term is also added to the Lagrangian, the field equations become:

$$R_{\mu\nu} - \tfrac{1}{2} R\, g_{\mu\nu} = -8\pi\,(G\, T^{(1)}_{\mu\nu} + i\, T^{(2)}_{\mu\nu}) \quad (1.125)$$

where $T^{(1)}_{\mu\nu}$ is postulated to be the usual matter energy-momentum tensor, and $T^{(2)}_{\mu\nu}$ "represents the charge-current distribution" ([44] p.478).

In an attempt to tie these field equations to physics, Moffat looks at the weak field approximation:

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(1)}_{\mu\nu} + i\, h^{(2)}_{\mu\nu} \quad (1.126)$$

where $\eta_{\mu\nu}$ are the Galilean values, and squares and cross-products of the h's are to be neglected. Write

$$h^{(2)'}_{\mu\nu} \equiv h^{(2)}_{\mu\nu} - \tfrac{1}{2}\eta_{\mu\nu}\, h^{(2)\alpha}_{\alpha} \tag{1.127}$$

and impose the 4 ('harmonic') conditions:

$$h^{(2)'\nu}_{\mu\ ,\nu} = 0 \tag{1.128}$$

The imaginary part of (1.125) then says:

$$\Box h^{(2)'}_{\mu\nu} = 16\pi T^{(2)}_{\mu\nu} \tag{1.129}$$

As for the RHS, he postulates that for a point-charge ε with 4-velocity $\frac{d\xi^\mu}{ds}$ it shall have the form:

$$T^{(2)\mu\nu} = \varepsilon\ \delta(x^1-\xi^1)\,\delta(x^2-\xi^2)\,\delta(x^3-\xi^3)\,\frac{d\xi^\mu}{ds}\frac{d\xi^\nu}{ds} \tag{1.130}$$

By considering the case of small spatial velocities, i.e:

$$\frac{d\xi^0}{ds} \sim 1\ ; \quad \frac{d\xi^k}{ds} \ll 1 \quad (k = 1,2,3) \tag{1.131}$$

Moffat identifies $h^{(2)'}_{0\mu}$ with the electromagnetic 4-potential and $-4\,T^{(2)}_{0\mu}$ with the current vector; four of (1.129) are then formally the same as the equations of Maxwell's theory in the Lorentz gauge:

$$\Box A_\mu = -4\pi j_\mu . \tag{1.132}$$

(However, an anti-symmetric tensor $F_{\mu\nu}$, which, after all, is the raison d'être of (1.132) in the Maxwell theory, does not appear naturally in the formalism, though of course it can be defined by $(h^{(2)'}_{0\nu,\mu} - h^{(2)'}_{0\mu,\nu})$.)

Returning now to the full (non-linear) field equations, set

$$R_{\mu\nu} = 0 \tag{1.133}$$

almost everywhere - i.e. with the exception of discrete singularities. Then Moffat has shown,[44][45] using the version of the EIH approximation method which is appropriate for slowly varying fields, and therefore for slow motion of the singularities, that a Lorentz force term does occur in the equations of motion. He claims ([44] p.487) to have also shown that the restriction to slow motion can be removed; if so, this would obviously be a very important achievement of the theory; but no published proof has appeared. It seems to the present author much more probable, in view of the 'quasi-Riemannian' structure of the whole theory, that this result is essentially restricted to low speeds; and that for high speeds, instead of remaining a linear function as in the Lorentz equation, the force would be seen to depend quadratically on the 4-velocity as in the geodesic equation of general relativity, for as is well known and readily verified the latter also reduces to an expression of precisely the Lorentz-force type to first order in the 3-velocity (for weak, slowly-varying fields). But this is only an 'intuitive' remark; the question could presumably be decided definitively by an appropriately refined EIH-type calculation. There is, however, a further difficulty in that, as pointed out by Kerr,[46] the LHS's of (1.125) and (1.133) satisfy the 4 complex and therefore 8 real identities (1.124), so that 4 more field equations are needed if the $g_{\mu\nu}$ are to be properly determined (i.e. up to only 4 arbitrary functions). It is not easy to see what these additional equations should be.

The two theories described so far "are exposed to the objection that they are in disagreement with the principle that only irreducible quantities should be used in field theories... Therefore, I believe that cogent mathematical reasons, (for instance invariance postulates of a wider group of transformations) have to be given why a decomposition of the reducible quantities used in the theory (for instance R_{ik} , g_{ik} and Γ^{l}_{ik}) does not occur. This has not been done at all in the earlier literature. Einstein, however, was well aware of this objection, which he weighed carefully in his later work." ([3] p.226). This remark by Pauli expresses succintly a feeling one has about the desirability of constructing theories out of 'homogeneous' or in some sense 'unified' objects; but perhaps, like other group-theoretical arguments of an aprioristic nature, should not be wielded too indiscriminately (after all, $R_{i}^{\ k}$ is a reducible object under the full coordinate transformation group - R transforms as a scalar - but no-one objects to its playing a central role in gravitation theory). However, this is not to belittle Pauli's observation, but to emphasize that such points must be "weighed carefully".

One could also make the point that the 'metric tensor' is in all these theories singularly divorced from its origin in the notion of a scalar product - indeed, the refusal (in the Einstein theory) to use it for inter-converting co- and contravariant tensor components is rather like ending up with the grin and no Cheshire cat.

We turn now to a second group of theories related to the present work: those altering the datum (1) of general relativity (see p.32), by enlarging the transformation group and the dimensionality of the manifold. In his discussion of the UFT problem in [49] (pp.88-90) Einstein remarks: "Die gesuchte Struktur muss eine Verallgemeinerung des symmetrischen Tensors sein. Die Gruppe darf nicht enger sein als die der kontinuier-

-lichen Koordinaten-Transformationen. Wenn man nun eine reichere Struktur einführt, so wird die Gruppe die Gleichungen nicht mehr so stark determinieren wie im Falle des symmetrischen Tensors als Struktur. Deshalb wäre es am schönsten, wenn es gelänge, die Gruppe abermals zu erweitern in Analogie zu dem Schritte, der von der speziellen Relativität zur allgemeinen Relativität geführt hat. Im Besonderen habe ich versucht, die Gruppe der komplexen Koordinaten-Transformationen heranzuziehen. Alle derartigen Bemühungen waren erfolglos." He does not appear to have published these investigations.

The first reference to complex spaces in a physical context that I have found in the literature is by N.N.Ghosh,[48] in which he applies his rather peculiar matrix treatment of the dynamics of rigid bodies to the case where they are extended in and move in a complex space, with complex velocities, angular momenta, and so on.

The next reference is 'nearer home' as far as field theory is concerned - is in fact closest in spirit to the present work. It is a very short account by A.Crumeyrolle[55] of some aspects of his doctorate work (1961-3) on the geometry of a kind of manifold which is precisely analogous to a complex analytic manifold (see Chapter 2), but defined instead over the number field generated by $\{1, \varepsilon\}$, where $\varepsilon^2 = +1$. The following account of his results is based entirely on this summary article, as I have not obtained access to his dissertation. Introduce the coordinates, and their conjugates:

$$\left.\begin{aligned} z^{\mu} &= \tfrac{1}{2^{1/2}}(\, x^{\mu} + \varepsilon\, x^{\mu^*}) \\ z^{\mu^*} &= \tfrac{1}{2^{1/2}}(\, x^{\mu} - \varepsilon\, x^{\mu^*}) \end{aligned}\right\} \qquad (1.134)$$

Then V_{2n} is the manifold parametrized by the coordinates (z^{μ}), with transformations of the form:

$$z^{\mu'} = f^{\mu}(z^{\alpha}). \qquad (1.135)$$

He defines "la sous-variété diagonale", W_n , by:

$$z^{\mu} = z^{\mu^*} \qquad (1.136)$$

(cf. the 'real limit space' of Chapter 4). Let there be a non-symmetric affine connection in V_{2n} , with components $L^{i}_{\cdot jk}$ in the coordinate system $(x^{\alpha}, x^{\alpha^*})$, "le repère associé". (Cf. the $\Gamma^{\lambda\, \dagger c}_{\alpha\, \mu\nu}$ of Chapter 3.) He puts:

$$L^{i}_{\cdot jk} = L^{i^*}_{\cdot k^* j} \qquad (1.137)$$

where 'starring' a Latin index means add or subtract n , as appropriate. For the components when restricted to W_n write:

$$\left.\begin{aligned} L^{\alpha}_{\cdot \beta\gamma} &\equiv \mathcal{L}^{\alpha}_{\cdot \beta\gamma} \\ L^{\alpha^*}_{\cdot \beta\gamma} &\equiv \Lambda^{\alpha}_{\cdot \beta\gamma} \end{aligned}\right\} \qquad (1.138)$$

Then the former transforms as a connection, the latter as a tensor. V_{2n} has a metric tensor g_{ij} , which is symmetric but otherwise arbitrary. He now supposes that a (real) non-symmetric tensor $\mathcal{G}_{\alpha\beta}$ is given on the subspace W_n , and requires that when restricted to W_n g_{ij} shall have the components, still in repères associés:

$$\left.\begin{aligned} g_{\alpha\beta} &= 0 & \qquad g_{\alpha\beta^*} &= \mathcal{G}_{\alpha\beta} \\ g_{\alpha^*\beta} &= \mathcal{G}_{\beta\alpha} & \qquad g_{\alpha^*\beta^*} &= 0 \end{aligned}\right\} \qquad (1.139)$$

Requiring the covariant derivative of g_{ij} w.r.t. $L^{i}_{\cdot jk}$ to vanish "pour tout chemin de W_n " (p.2123), he obtains:

$$\left.\begin{aligned} \frac{\partial G_{\mu\nu}}{\partial x^\sigma} - G_{\alpha\nu}\mathcal{L}^{\alpha}_{\cdot\mu\sigma} - G_{\mu\alpha}\mathcal{L}^{\alpha}_{\cdot\sigma\nu} &= 0 \\ G_{\mu\alpha}\Lambda^{\alpha}_{\cdot\nu\sigma} + G_{\nu\alpha}\Lambda^{\alpha}_{\cdot\mu\sigma} &= 0 \\ G_{\alpha\mu}\Lambda^{\alpha}_{\cdot\sigma\nu} + G_{\alpha\nu}\Lambda^{\alpha}_{\cdot\sigma\mu} &= 0 \end{aligned}\right\} \qquad (1.140)$$

"c'est-à-dire le système d'Einstein-Schrödinger et des équations nouvelles susceptibles de décrire un champ inconnu" (p.2123). (There is no reference, however, to the curvature-tensor equations of the Einstein-Schrödinger theory.) He concludes by noting that in "repères adaptés", viz. (z^μ, z^{μ^*}), the components of the affine connection, $W^{i}_{\cdot jk}$ say, are:

$$\left.\begin{aligned} W^{\alpha}_{\cdot\beta\gamma} &= 2^{1/2}\,(L^{\alpha}_{\cdot\underline{\beta\gamma}} + \varepsilon\, L^{\alpha^*}_{\cdot\underline{\beta\gamma}}) \\ W^{\alpha}_{\cdot\beta^*\gamma^*} &= 2^{1/2}\,(L^{\alpha}_{\cdot\underset{\vee}{\beta\gamma}} + \varepsilon\, L^{\alpha^*}_{\cdot\underset{\vee}{\beta\gamma}}) \end{aligned}\right\} \qquad (1.141)$$

(using the Einstein-Straus notation for symmetric and anti-symmetric parts), so that the 'torsion vector' in V_{2n}, namely $W^{i}_{\cdot\underset{\vee}{ij}}$, vanishes; but that the Einstein-Schrödinger theory assignment, which would be $\mathcal{L}^{\alpha}_{\cdot\underset{\vee}{\alpha\beta}} = 0$, becomes in the present context $W^{\alpha}_{\cdot\alpha^*\beta^*} = 0$ (if $\Lambda^{\alpha}_{\cdot\beta\gamma} = 0$), a rather unnatural condition.

It remains to consider two contributions, which both appeared in J.Math.Phys. 7 early in 1966, i.e. almost at the time of, but slightly after, the present author's investigation (which was begun in Dec. 1965, and completed in all essentials by Jan. 1966). The first is a set of three papers by A.Das.[50-52] In (I), he looks at 'semi'-classical (i.e. un-second-quantized) field theory in flat complex space-time, coordinatized by z^{i+}, and $z^{i-} = \overline{z^{i+}}$, with:

$$ds^2 = \eta_{ij}\, dz^{i+} dz^{j-} \qquad (1.142)$$

He does not consider the full group leaving η_{ij} invariant, namely U(4), but only the 7-parameter subgroup $L_4^{\ddagger} \times U_1$, where $L_4^{\ddagger}$ represents the proper Lorentz group, U_1 the phase group:

$$z^{k\pm} \rightarrow e^{\pm i\theta} z^{k\pm} \tag{1.143}$$

"We shall physically interpret $|z^{k\pm}|$ as what we usually measure for positional coordinates, and $\arg z^{k\pm}$ are the electrical or internal coordinates." (p.46). He writes $2 z^{k\pm} = r^k e^{\pm i\theta_k}$. Consider the linear wave equation:

$$\left(\alpha^{k+} \frac{\partial}{\partial z^{k+}} + \alpha^{k-} \frac{\partial}{\partial z^{k-}} - im I \right) \psi = 0 \tag{1.144}$$

where

$$\left. \begin{aligned} \alpha^{k+}\alpha^{l+} + \alpha^{l+}\alpha^{k+} = \alpha^{k-}\alpha^{l-} + \alpha^{l-}\alpha^{k-} = 0 \\ \alpha^{k+}\alpha^{l-} + \alpha^{l-}\alpha^{k+} = \eta^{kl} I \end{aligned} \right\} \tag{1.145}$$

Das writes down what he claims is an irreducible representation of the α's but, as will be seen in Chapter 6, it is even reducible under U(4), so that under his restricted group $L_4^{\ddagger} \times U_1$ it is certainly not irreducible. (The present § 6.3 may well have been suggested by Das's work, and is an (in this respect) improved treatment of the 'spinor' equation in complex space-time.) He inserts an electromagnetic interaction into (1.144) via a prescription which resembles the usual one ($\partial_\mu \rightarrow \partial_\mu + i\varepsilon A_\mu$), though it is not entirely free from arbitrariness (he should, strictly, have complex quantities $A_{k\pm}$ and, by choosing a particular form of θ_k-dependence for ψ, arrives at equations containing "terms which reveal slight anisotropy in the physical space spanned by the four r^k's". Also given are expressions for energy-momentum tensors and conservation laws appropriate to the wave equation (1.144), and to the Klein-Gordon equation for a complex scalar field.

His paper (II) falls outside the scope of the present work.

In (III), he restricts attention to the subspace (with real dimension 5) defined by the constraints:

$$\arg \; z^{k+} = \theta \qquad (k = 1, .. 4) \,, \tag{1.146}$$

so that

$$z^{k\pm} = r^k e^{\pm i\theta} \tag{1.147}$$

On this subspace, the line-element (1.142) becomes:

$$ds^2 = \eta_{ik} (dr^i + i\, r^i d\theta)(dr^j - i\, r^j d\theta) \tag{1.148}$$

He now generalizes the transformation group, $L_4^{\ddagger} \times U_1$, of (I) to allow position-dependent phase transformations:

$$\left.\begin{aligned} r^{i'} &= a^i{}_j \; r^j \\ \theta' &= \theta + \lambda(r^i) \end{aligned}\right\} \qquad \begin{aligned} &(1.149\text{i}) \\ &(1.149\text{ii}) \end{aligned}$$

The line-element (1.148) is not form-invariant under this group, but becomes so if $(d\theta)$ is replaced by $(d\theta + A_k dr^k)$:

$$ds^2 = \eta_{ij}\, dr^i dr^j + r_\ell r^\ell (d\theta + A_k dr^k)^2, \tag{1.150}$$

with the new quantities $A_k(r^i)$ being required to transform as a covariant vector under (1.149i), and as

$$A'_k = A_k - \frac{\partial\lambda}{\partial r^k} \tag{1.151}$$

under (1.149ii). "For the equation of motion of a particle in complex space-time we shall postulate the geodesic principle" (p.62). (Actually, the motion would have to be in the 5-dimensional subspace just defined.) The Euler-Lagrange equation resulting from variation w.r.t. θ has the first integral:

$$m\, \eta_{ij} r^i r^j (\dot\theta + A_k \dot r^k) = \text{constant} \equiv q \tag{1.152}$$

The other four equations are:

$$\ddot r_i = \left(\frac{q}{m}\right) F_{ij}\, r^j + \left(\frac{q}{m}\right)^2 \frac{r_i}{(r_k r^k)^2} \tag{1.153}$$

where $F_{ij} \equiv \frac{\partial A_j}{\partial r^i} - \frac{\partial A_i}{\partial r^j}$. He interprets q as the charge of the particle, which according to (1.152) "corresponds to the sum of angular-momenta in complex planes" (p.62). (He does not remark that the RHS of (1.153) is singular everywhere on the light-cone through the coordinate origin.)

He next allows the $a^i_{\cdot j}$ in (1.149i) also to depend on the r^i. By defining covariant derivatives in a certain way he arrives at a contracted curvature tensor of the form:

$$P_{ik} \equiv P^m_{\cdot ikm} = R_{ik} + i\varepsilon F_{ik} \tag{1.154}$$

where R_{ik} is the Riemannian Ricci tensor formed from the $g_{ij}(r^k)$. He says (p.63): "The electro-gravitational field equations should be derived from any one or linear combinations of the square Lagrangians

$$\begin{aligned} L' &= P^{ik}\overline{P_{ik}} = R_{ik}R^{ik} + \varepsilon^2 F_{ik}F^{ik} \\ L'' &= P^{ijkl}\overline{P_{ijkl}} = R^{ijkl}R_{ijkl} + 4\varepsilon^2 F^{kl}F_{kl} \text{ ."} \end{aligned} \tag{1.155}$$

He points out that either of these lead to field equations which have as a particular case the Schwarzschild solution, but he shows that for $F_{ik} \not\equiv 0$ they do not contain the Nördstrom solution. (These Lagrangians will figure in Chapter 5.)

Das remarks, in conclusion, that it would be possible to consider the full transformation group:

$$z^{k'} = z^{k'}(z,\bar{z}) \quad ; \quad \text{conj.} \tag{1.156}$$

and that the corresponding metric tensor would have 36 components "and may consist of many other fields besides electro-gravitation" (p.63). (The theory we shall present stands in fact half way to this most general geometry and has a metric tensor with 16 (real) degrees of freedom.)

The remaining paper is by E.H.Brown,[54] and begins with a deductive 'proof' that space-time must be complex. Ignoring this, and putting aside the axiomatic and mathematical paraphernalia (which can be employed to make complex ideas simple, or the other way round; this paper seems to be an example of the latter), his theory runs as follows. Consider the line-element expression (using a notation which is essentially that explained in Chapter 2):

$$2\,ds^2 = g_{ij}(z^k)\,dz^i dz^j \qquad (1.157)$$

$$= (g_{\bar{\alpha}\beta}dz^{\bar{\alpha}}dz^{\beta} + g_{\alpha\bar{\beta}}dz^{\alpha}dz^{\bar{\beta}}) + (g_{\alpha\beta}dz^{\alpha}dz^{\beta} + g_{\bar{\alpha}\bar{\beta}}dz^{\bar{\alpha}}dz^{\bar{\beta}})$$

He points out that the two bracketed terms on the RHS are each separately real, if g_{ij} is self-adjoint (see Chapter 2 for this concept), and that "mathematical simplicity led Kähler to choose $g_{ij} = \{g_{\bar{\alpha}\beta}, g_{\alpha\bar{\beta}}\}$, with $g_{\alpha\beta} = g_{\bar{\alpha}\bar{\beta}} = 0$, as a metric tensor" (p.420). However, he then becomes guilty of "mathematical simplicity". Rightly saying that Kähler's assignment implies (after a few extra assumptions) the existence of a real scalar function Φ such that

$$g_{\alpha\bar{\beta}} = \Phi_{,\alpha,\bar{\beta}} \qquad (1.158)$$

he then 'deduces' that this implies $g_{\alpha\bar{\beta}} \equiv 0$, since he has previously 'shown' that all real scalars must be of the form $\Phi = \phi(z^{\alpha}) + \overline{\phi(z^{\alpha})}$. Although his proof is invalid, it is obviously permissible to choose to start from his line-element:

$$2\,ds^2 = g_{\alpha\beta}dz^{\alpha}dz^{\beta} + g_{\bar{\alpha}\bar{\beta}}dz^{\bar{\alpha}}dz^{\bar{\beta}} \qquad (1.159)$$

where the $g_{\alpha\beta}$ are assumed to be analytic functions of the z^{α}

alone (i.e. independent of the $z^{\bar{\alpha}}$), and vice versa for $g_{\bar{\alpha}\bar{\beta}}$. There is the block decomposition for the affine connection $\Gamma^{i}_{\cdot jk} = \{\Gamma^{\lambda}_{\cdot\mu\nu}(z^{\alpha}), \Gamma^{\bar{\lambda}}_{\cdot\bar{\mu}\bar{\nu}}(z^{\bar{\alpha}})\}$, and similarly for the curvature tensors. In this respect, and in the (complex) symmetry of the metric tensor, the theory has parallels with Moffat's. Brown's Axiom 5 says: "If the unit vector $w^{i}(z^{k}) = \frac{dz^{i}}{ds} = \{w^{\alpha}(z^{\alpha}), w^{\bar{\alpha}}(z^{\bar{\alpha}})\}$ is the complex four-velocity, the equations of a geodesic (the equations of motion of a particle) are then

$$\frac{dw^{i}}{ds} + \Gamma^{i}_{\cdot jk}\, w^{j} w^{k} = 0 \text{ ".} \tag{1.160}$$

Write

$$\left.\begin{aligned} w^{\mu} &\equiv u^{\mu} + i\, v^{\mu} \\ \Gamma^{\lambda}_{\cdot\mu\nu} &\equiv \Gamma_{(1)}{}^{\lambda}_{\cdot\mu\nu} + i\, \Gamma_{(2)}{}^{\lambda}_{\cdot\mu\nu} \end{aligned}\right\} \tag{1.161}$$

Then (1.160) splits into:

$$\frac{du^{\alpha}}{ds} + \Gamma_{(1)}{}^{\alpha}_{\cdot\beta\gamma}(u^{\beta}u^{\gamma} - v^{\beta}v^{\gamma}) - 2\Gamma_{(2)}{}^{\alpha}_{\cdot\beta\gamma}\, u^{\beta}v^{\gamma} = 0 \tag{1.162i}$$

$$\frac{dv^{\alpha}}{ds} + \Gamma_{(2)}{}^{\alpha}_{\cdot\beta\gamma}(u^{\beta}u^{\gamma} - v^{\beta}v^{\gamma}) + 2\Gamma_{(1)}{}^{\alpha}_{\cdot\beta\gamma}\, u^{\beta}v^{\gamma} = 0 \tag{1.162ii}$$

Now, w^{i} is a unit vector: $w_{i}w^{i} = 1 = u_{\alpha}u^{\alpha} - v_{\alpha}v^{\alpha}$. Neglect powers of v^{μ}. Then (1.162i) is like the Lorentz force equation and "suggests that $\frac{e}{mc^{3}}F^{\alpha}_{\cdot\beta}$ is a classical approximation to $2\Gamma_{(2)}{}^{\alpha}_{\cdot\beta\gamma}v^{\gamma}$ and that v^{α} (or, possibly, only its time-like component) is related to charge" (p.421). Again echoes of Moffat. (Of course, he has assumed in Axiom 5 that 'geodesic equation' and 'equation of motion of a particle' are synonymous, which begs the most difficult question of all in these UFT's.) I am not able to summarize with any confidence of having understood it the rest of his paper.

CHAPTER 2

Kähler Spaces. I

This chapter is an account of those aspects of the existing theory of what are known as 'Kähler' manifolds which are relevant to the theory presented here. The latter was developed independently of the Kähler space literature but, where there is overlap, is identical with it in content - though there are differences in method of derivation, notation and motivation. Parallels and divergences will be noted in the sequel, as occasion arises. The exposition is based primarily on [67], [68] and [69], which for ease of reference will in this chapter be called respectively S , YB , Y .

One first introduces the notion of an 'analytic manifold'. Consider a set of points parametrizable, in a neighbourhood, by continuous values of 2n real coordinates (x^i). Split the 2n indices into two groups of n, by writing

$$i \leftrightarrow \begin{cases} \mu & 1 \leq i \leq n \\ \bar{\mu} \equiv i-n & n+1 \leq i \leq 2n \end{cases} \tag{2.1}$$

Define $$z^\mu \equiv x^\mu + i\, x^{\bar{\mu}}. \tag{2.2}$$

Given a set of n independent (non-zero functional determinant) analytic (therefore infinitely differentiable) functions f^μ of the (z^μ), we can define an analytic coordinate transformation by:

$$\left.\begin{aligned} z^{\mu'} &= f^\mu(z^\alpha) \\ \det \left\| \frac{\partial z^{\mu'}}{\partial z^\nu} \right\| &\neq 0 \end{aligned}\right\} \tag{2.3}$$

Then the original set of points together with the group of all such analytic transformations is called (a neighbourhood of) an n-dimensional complex analytic manifold (YB p.118). Call it $\mathcal{C}_n$. (It will prove a convenient notation to denote, throughout this work, complex spaces by curly capital letters, real ones by ordinary capitals.)

Introduce a new set of complex coordinates by:

$$z^{\bar{\mu}} = \overline{z^{\mu}} \tag{2.4}$$

where the bar on the RHS signifies complex conjugate. Then, as (z^{μ}) ranges over the permissible coordinate systems for $\mathcal{C}_n$, $(z^{\bar{\mu}})$ will define a new complex analytic manifold, $\overline{\mathcal{C}}_n$ say, called the conjugate manifold of $\mathcal{C}_n$ (Y p.50).

Tensor analysis is constructed in the product manifold $\mathcal{C}_n \times \overline{\mathcal{C}}_n$ (Y pp.51-62). (Schouten calls this the "auxiliary X_{2n}" of the original X_n (S p.390).) Tensors are defined on this manifold, as objects transforming appropriately under the coordinate transformation group:

$$\left.\begin{aligned} z^{\mu'} &= f^{\mu}(z^{\alpha}) \\ z^{\bar{\mu}'} &= \bar{f}^{\mu}(z^{\bar{\alpha}}) \end{aligned}\right\} \tag{2.5}$$

where $\bar{f}^{\mu}$ is the complex conjugate function of f^{μ}, viz. the function of the n complex variables ξ^{α} which is such that $\bar{f}^{\mu}(\xi^{\alpha}) \equiv \overline{f^{\mu}(\bar{\xi}^{\alpha})}$.

A contravariant vector field on $\mathcal{C}_n \times \overline{\mathcal{C}}_n$ is a quantity $(v^{\mu}, v^{\bar{\mu}})$ transforming under (2.5) like:

$$\left.\begin{aligned} v^{\mu'} &= \frac{\partial z^{\mu'}}{\partial z^{\alpha}} v^{\alpha} \\ v^{\bar{\mu}'} &= \frac{\partial z^{\bar{\mu}'}}{\partial z^{\bar{\alpha}}} v^{\bar{\alpha}} \end{aligned}\right\} \tag{2.6}$$

Extension of the definition to covariant and higher-rank tensors is made in the usual way.

The vector with components $(\overline{v^{\bar{\mu}}}, \overline{v^{\mu}})$ is called the conjugate vector of $(v^{\mu}, v^{\bar{\mu}})$. A vector is self-conjugate (or self-adjoint, or real) if it is equal to its conjugate.

The quantity with components $(iv^{\mu}, -iv^{\bar{\mu}})$ is also a contravariant vector. (This fact is worthy of remark: it pin-points the distinguishing feature of complex tensor analysis relative to tensor analysis in a real manifold of twice the dimension.) Going back via (2.1) to the Latin-index notation, the above vector is derivable from $(v^{\mu}, v^{\bar{\mu}})$ by multiplication by the matrix (YB pp.154-5):

$$(h^{i}{}_{j}) \equiv \begin{pmatrix} i\,\delta^{\mu}_{\nu} & 0 \\ 0 & -i\,\delta^{\bar{\mu}}_{\bar{\nu}} \end{pmatrix}, \tag{2.7}$$

which satisfies $\quad h^{i}{}_{j}\, h^{j}{}_{k} = -\,\delta^{i}_{k}$. $\quad$ (2.8)

(Eqn.(2.8) expresses the defining property of what are known as 'almost complex spaces' (Y passim).)

Consider the submanifold of $C_n \times \overline{C}_n$ defined by

$$z^{\bar{\mu}} = \overline{z^{\mu}}. \tag{2.9}$$

When restricted to this subspace, a vector field $(v^{\mu}, v^{\bar{\mu}})$ is only a function of a single set of n complex coordinates, and is said to be a vector field over C_n; its transformation law (cf. (2.6)) becomes:

$$\left.\begin{aligned} v^{\mu'} &= \frac{\partial z^{\mu'}}{\partial z^{\alpha}}\, v^{\alpha} \\ v^{\bar{\mu}'} &= \overline{\left(\frac{\partial z^{\mu'}}{\partial z^{\alpha}}\right)}\, v^{\bar{\alpha}} \end{aligned}\right\} \tag{2.10}$$

Similarly for other tensors.

Introducing 2n new complex variables by (Y p.53):

$$\left.\begin{aligned} z^{\mu} &= \xi^{\mu} + i\,\xi^{\bar{\mu}} \\ z^{\bar{\mu}} &= \xi^{\mu} - i\,\xi^{\bar{\mu}} \end{aligned}\right\} \tag{2.11i}$$

with inverse:
$$\left.\begin{aligned} \xi^{\mu} &= \tfrac{1}{2}\left(z^{\mu} + z^{\bar\mu}\right) \\ \xi^{\bar\mu} &= \tfrac{1}{2i}\left(z^{\mu} - z^{\bar\mu}\right) \end{aligned}\right\} \qquad (2.11ii)$$

the restriction (2.9) then says: ξ^{μ} and $\xi^{\bar\mu}$ to be real. Yano calls $(\xi^{\mu}, \xi^{\bar\mu})$ a 'real coordinate system'. This is rather misleading, as (2.11) is not an allowable (i.e. type (2.5)) coordinate transformation (cf. S p.390). However, it is often necessary to make the transition (2.11), so, to avoid a logical hiatus, it is perhaps best to supplement Yano's exposition by explicitly <u>defining</u> the behaviour of (say) a contravariant vector under (2.11) by (cf. Y p.53):

$$\left.\begin{aligned} v^{\mu'} &= \tfrac{1}{2}\left(v^{\mu} + v^{\bar\mu}\right) \\ v^{\bar\mu'} &= \tfrac{1}{2i}\left(v^{\mu} - v^{\bar\mu}\right) \end{aligned}\right\} \qquad (2.12)$$

where $(v^{\mu'}, v^{\bar\mu'})$ are the components of the vector in the 'real' coordinate system. (2.12) can equally be written:

$$v^{i'} = T^{i'}{}_{j}\, v^{j} \qquad (2.13i)$$

where
$$\left(T^{i'}{}_{j}\right) \equiv \tfrac{1}{2}\begin{pmatrix} I_{(n)} & I_{(n)} \\ -i\,I_{(n)} & i\,I_{(n)} \end{pmatrix} \qquad (2.13ii)$$

($I_{(n)}$ being the unit matrix). (Cf. [64] p.464.)

Introduce a <u>metric tensor</u> g_{ij} satisfying the conditions:

$$g_{ji} = g_{ij} \qquad (2.14i)$$

$$g_{\mu\nu} = g_{\bar\mu\bar\nu} = 0 \qquad (2.14ii)$$

$$g_{\mu\bar\nu} = \overline{g_{\bar\mu\nu}} \qquad (2.14iii)$$

The second says that it is a so-called 'hybrid' quantity, the third that it is self-conjugate, the first and third together imply
$$g_{\mu\bar\nu} = \overline{g_{\nu\bar\mu}} \qquad (2.15)$$
i.e. that $(g_{\mu\bar\nu})$ is a Hermitian matrix. (2.14) characterize what Yano calls a 'Hermite space', Schouten an $\tilde{R}_n$. The

tensor is assumed non-singular, at least in some neighbourhood, so that it can be used to raise and lower indices. For example

$$h_{ij} \equiv g_{im} h^{m}_{\cdot j} \tag{2.16i}$$

has matrix (cf.(2.7)): $$(h_{ij}) = \begin{pmatrix} 0 & -ig_{\mu\bar\nu} \\ ig_{\bar\mu\nu} & 0 \end{pmatrix}. \tag{2.16ii}$$

Covariant derivatives in $C_n \times \bar{C}_n$ are constructed by means of Christoffel symbols formed from the g_{ij} according to the Riemannian prescription. Because of the restricted form (2.14) of the metric, one has:

$$\left.\begin{aligned} \Gamma^{\lambda}_{\cdot\mu\nu} &= \tfrac{1}{2} g^{\lambda\bar\alpha}\left(g_{\bar\alpha\mu,\nu} + g_{\bar\alpha\nu,\mu}\right) \\ \Gamma^{\lambda}_{\cdot\mu\bar\nu} &= \Gamma^{\lambda}_{\cdot\bar\nu\mu} = \tfrac{1}{2} g^{\lambda\bar\alpha}\left(g_{\mu\bar\alpha,\bar\nu} - g_{\mu\bar\nu,\bar\alpha}\right) \\ \Gamma^{\lambda}_{\cdot\bar\mu\bar\nu} &= 0 \end{aligned}\right\} \tag{2.17}$$

and three similar equations formed by replacing unbarred by barred indices and vice versa, an operation it is customary to abbreviate 'conj'.[64] The $\Gamma^{i}_{\cdot jk}$ transform under (2.5) like a Riemannian affine connection, viz:

$$\Gamma'^{i}_{\cdot jk} = \frac{\partial z^{i'}}{\partial z^{p}} \frac{\partial z^{q}}{\partial z^{j'}} \frac{\partial z^{r}}{\partial z^{k'}} \Gamma^{p}_{\cdot qr} + \frac{\partial z^{i'}}{\partial z^{p}} \frac{\partial^2 z^{p}}{\partial z^{j'}\partial z^{k'}} \tag{2.18}$$

The special form of (2.5) means that

$$\frac{\partial^2 z^{\mu}}{\partial z^{\alpha'}\partial z^{\bar\beta'}} = 0 \quad ; \quad \text{conj.} \tag{2.19}$$

so that the second term on the RHS of (2.18) vanishes for connection components of the form $\Gamma^{\lambda}_{\cdot\mu\bar\nu}$, $\Gamma^{\lambda}_{\cdot\bar\mu\nu}$; conj. The specification:

$$\Gamma^{\lambda}_{\cdot\mu\bar\nu} = 0 \quad ; \quad \text{conj.} \tag{2.20}$$

is therefore invariant under the group of allowed coordinate transformations. A Hermite space equipped with a connection satisfying (2.17) and (2.20) is a 'Kähler space', and will be denoted by $\mathcal{K}_n$. (In Schouten's terminology it is a $\tilde{V}_n$ (S p.397).) This geometry was first explicitly isolated in [66], although earlier work by Schouten and van Dantzig[63]-[65] had dealt with closely related (in certain features more general) geometries.

Denote covariant derivatives w.r.t. the $\Gamma^{i}_{\cdot jk}$ by a semi-colon. Then it is a simple matter to show (Y p.65) that the condition (2.20) is equivalent to:

$$h^{i}_{\cdot j\ ;k} = 0. \tag{2.21}$$

(2.17) implies that in a Kähler space:

$$g_{\mu\bar{\nu},\bar{\sigma}} = g_{\mu\bar{\sigma},\bar{\nu}} \quad ; \quad \text{conj.} \tag{2.22}$$

which in turn implies (S pp.397-8) the existence of a scalar function $\phi(z^{\alpha}, z^{\bar{\alpha}})$ such that

$$g_{\mu\bar{\nu}} = \phi_{,\mu,\bar{\nu}} . \tag{2.23}$$

Since in a K_n the only non-vanishing components of $\Gamma^{i}_{\cdot jk}$ are those of the form $\Gamma^{\lambda}_{\cdot\mu\nu}$, conj., the curvature tensor $R^{i}_{\cdot jkl}$ formed from the $\Gamma^{i}_{\cdot jk}$ by the Riemannian formula has as its only non-vanishing components:

$$\left.\begin{aligned} R^{\lambda}_{\cdot\mu\nu\bar{\sigma}} &= -R^{\lambda}_{\cdot\mu\bar{\sigma}\nu} && (2.24\text{i}) \\ &= (\Gamma^{\lambda}_{\cdot\mu\nu})_{,\bar{\sigma}} && (2.24\text{ii}) \end{aligned}\right\} \quad ; \quad \text{conj.}$$

It satisfies the relations, to be expected from its genesis:

$$\left.\begin{aligned} R_{\bar{\lambda}\mu\nu\bar{\sigma}} &= -R_{\mu\bar{\lambda}\nu\bar{\sigma}} && (2.25\text{i}) \\ R_{\bar{\lambda}\mu\nu\bar{\sigma}} &= R_{\nu\bar{\sigma}\bar{\lambda}\mu} && (2.25\text{ii}) \end{aligned}\right\} \quad ; \quad \text{conj.}$$

It is noteworthy that in a K_n the cyclic identities ($R_{i\{jkl\}} \equiv 0$) give no more information than the already-known (2.24i). Because (2.24) are the only remaining components, the Bianchi identities reduce to (S p.399):

$$R_{\bar{\lambda}\mu\nu\bar{\sigma};\bar{\rho}} - R_{\bar{\lambda}\mu\nu\bar{\rho};\bar{\sigma}} \equiv 0 \quad ; \quad \text{conj.} \tag{2.26}$$

although earlier work by Schouten and van Dantzig[63]-[65] had dealt with closely related (in certain features more general) geometries.

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It satisfies the relations, to be expected from its genesis:

$$\left.\begin{aligned} R_{\bar{\lambda}\mu\nu\bar{\sigma}} &= -R_{\mu\bar{\lambda}\nu\bar{\sigma}} && (2.25\text{i}) \\ R_{\bar{\lambda}\mu\nu\bar{\sigma}} &= R_{\nu\bar{\sigma}\bar{\lambda}\mu} && (2.25\text{ii}) \end{aligned}\right\} \quad ; \quad \text{conj.}$$

It is noteworthy that in a K_n the cyclic identities ($R_{i\{jk\ell\}} \equiv 0$) give no more information than the already-known (2.24i). Because (2.24) are the only remaining components, the <u>Bianchi identities</u> reduce to (S p.399):

$$R_{\bar{\lambda}\mu\nu\bar{\sigma};\bar{\rho}} - R_{\bar{\lambda}\mu\nu\bar{\rho};\bar{\sigma}} \equiv 0 \quad ; \quad \text{conj.} \tag{2.26}$$

This identity can also readily be derived directly from (2.24ii) – whence it appears as a kind of tensorial version of the 3-vector identity: curl grad $\equiv$ 0.

From (2.17), (2.24ii) and (2.23) one obtains (YB p.125):

$$R_{\lambda\bar{\mu}\nu\bar{\sigma}} = -g_{\lambda\bar{\mu},\nu,\bar{\sigma}} + g^{\bar{\alpha}\beta} g_{\lambda\bar{\alpha},\nu}\, g_{\bar{\mu}\beta,\bar{\sigma}} \qquad (2.27\text{i})$$

$$= -\phi_{,\lambda,\bar{\mu},\nu,\bar{\sigma}} + g^{\bar{\alpha}\beta} \phi_{\bar{\alpha},\lambda,\nu}\, \phi_{\beta,\bar{\mu},\bar{\sigma}} \qquad (2.27\text{ii})$$

The latter demonstrates that, in addition to (2.24i) and (2.25), $R_{\lambda\bar{\mu}\nu\bar{\sigma}}$ possesses the symmetry property:

$$R_{\lambda\bar{\mu}\nu\bar{\sigma}} = R_{\nu\bar{\mu}\lambda\bar{\sigma}} . \qquad (2.28)$$

The <u>Ricci tensor</u> $R_{ij} \equiv R^{m}{}_{imj}$ (opposite sign to Yano) has:

$$\left.\begin{aligned} R_{\mu\nu} &= 0 \\ R_{\mu\bar{\nu}} &= R^{\alpha}{}_{\mu\alpha\bar{\nu}} \end{aligned}\right\} ; \quad \text{conj.} \qquad (2.29)$$

Using (2.28) etc. one obtains:

$$R_{\mu\bar{\nu}} = -R^{\alpha}{}_{\mu\bar{\nu}\alpha} = -R^{\alpha}{}_{\alpha\bar{\nu}\mu} = R^{\alpha}{}_{\alpha\mu\bar{\nu}} = \Gamma^{\alpha}{}_{\alpha\mu,\nu}$$

$$= (\log\sqrt{g})_{,\mu,\bar{\nu}} \qquad (2.30)$$

where $g \equiv \det \| g_{ij} \| = \left[\det \| g_{\mu\bar{\nu}} \|\right]^2$, (2.31)

by (2.14ii) and (2.14iii).

Suppose the Kähler space is of <u>constant curvature</u>, in the sense that: $R_{ijkl} = K\,(g_{il}\, g_{jk} - g_{ik}\, g_{jl})$. (2.32)

There is only one independent non-trivial relation:

$$R_{\lambda\bar{\mu}\nu\bar{\sigma}} = K\, g_{\lambda\bar{\sigma}}\, g_{\bar{\mu}\nu}$$

The symmetry property (2.28) implies:

$$K\, g_{\lambda\bar{\sigma}}\, g_{\bar{\mu}\nu} = K\, g_{\nu\bar{\sigma}}\, g_{\bar{\mu}\lambda}$$

Multiply by $g^{\bar{\sigma}\lambda} g^{\nu\bar{\mu}}$ to obtain: $n^2 K = n\, K$ (2.33)

so that (Y p.69) <u>a $\mathcal{K}_n$ of constant curvature is flat</u> (if its dimension is greater than 1).

Finally, subspaces. Let (z^μ) be a coordinate system for a complex analytic manifold $\mathcal{C}_n$. Consider the equations:

$$z^\mu = z^\mu(u^\alpha) \qquad (\mu = 1,2 \,..\, n) \qquad (2.34)$$

where the u^α $(\alpha = 1,2 \,..\, m)$ are a set of $m < n$ complex parameters. (2.34) defines a proper subspace of $\mathcal{C}_n$, and (u^α) is a particular coordinate system for it. If one introduces also the set of all other coordinate systems derivable from (u^α) by analytic transformations (cf. (2.3)), then, by definition, one will have an m-dimensional analytic manifold, $\mathcal{C}_m$, which Yano (Y p.104) calls an analytic subspace of the $\mathcal{C}_n$. (This assumes that the set of functions in (2.34) is non-degenerate, in the sense that it specifies only (n-m) constraints; if the contrary, then one will have an analytic subspace $\mathcal{C}_r$, with $r < m$.)

Now suppose the original $\mathcal{C}_n$ is a $\mathcal{K}_n$. This means, in sum:
(1) There is a metric with g_{ij} satisfying (2.14)
(2) $g_{ij;k} = 0$ w.r.t. a symmetric connection $\Gamma^{i}_{\cdot jk}$
(3) h_{ij}, with components as in (2.16ii), satisfies $h_{ij;k} = 0$
Use the first few Roman and Greek letters for components in the subspace $\mathcal{C}_m$, e.g: $(a) \equiv (\alpha, \bar{\alpha})$ has range $(1,2 \,..\, 2m)$.
Define

$$\left.\begin{aligned} \hat{g}_{ab} &= \frac{\partial z^i}{\partial u^a}\frac{\partial z^j}{\partial u^b}\, g_{ij} \\ \hat{h}_{ab} &= \frac{\partial z^i}{\partial u^a}\frac{\partial z^j}{\partial u^b}\, h_{ij} \\ \hat{g}^{ab}\hat{g}_{bc} &= \delta^a_c \\ \hat{\Gamma}^a_{\cdot bc} &= \tfrac{1}{2}\hat{g}^{ae}\left(\hat{g}_{eb,c} + \hat{g}_{ec,b} - \hat{g}_{bc,e}\right) \end{aligned}\right\} \qquad (2.35)$$

and denote covariant derivatives in $\mathcal{C}_m$ w.r.t. $\hat{\Gamma}^a_{\cdot bc}$ by $\hat{;}$.
Then, because of the form of (2.34), one can verify (Y pp.104-6)

that with these definitions $\underline{\mathcal{C}_m \text{ is a } \mathcal{K}_m}$, in that

$$\left.\begin{aligned} (\hat{g}_{ab}) &= \begin{pmatrix} 0 & \hat{g}_{\alpha\bar{\beta}} \\ \hat{g}_{\bar{\alpha}\beta} & 0 \end{pmatrix}, \quad \text{with } \hat{g}_{\bar{\alpha}\beta} = \overline{\hat{g}_{\alpha\bar{\beta}}} \\ (\hat{h}_{ab}) &= \begin{pmatrix} 0 & -i\hat{g}_{\alpha\bar{\beta}} \\ i\hat{g}_{\bar{\alpha}\beta} & 0 \end{pmatrix} \\ \text{and} \quad \hat{h}_{ab;c} &= 0. \end{aligned}\right\} \quad (2.36)$$

CHAPTER 3

<u>Kähler Spaces. II</u>

§3.1 Complex numbers

The method of treating complex manifolds which is developed in this chapter and the next works entirely in terms of quantities which either are or are closely related to the real and imaginary parts of the complex numbers and tensors occuring, almost all the resulting formulae involving only real numbers. The motivation for this is primarily that it facilitates the study of the 'real limit space' (Chapter 4) which was found to be peculiarly intractable in the formalism the author employed in a first formulation of the theory (essentially the $(z^{\mu}, z^{\bar{\mu}})$ notation of Chapter 2). It also turns out that the nature of the geometry as that of a particular kind of 2n-dimensional real Riemannian space is particularly transparent in this formalism. There are, however, disadvantages. Some of the formulae, in particular those for the curvature tensors, are considerably neater in the complex-number formalism outlined in Chapter 2. Also, a rather odd 3-index symbol is introduced, for doing complex-number multiplication in terms of real quantities; this is the content of the present section.

(Of course, the formulae arrived at in this chapter can be deduced from the work of Chapter 2 by making the appropriate changes of variable (cf. § 3.7). However, it was thought desirable to give a more unified presentation by developing the mathematical theory ab initio in terms of the formalism in which the 'physical' theory of complex space-time is actually expressed.)

Consider the complex number $A \equiv A_1 + i\,A_2$. It will be written (A_a), where, as throughout this work, small Latin indices always range over $(1, 2)$. Introduce a matrix:

$$(C^{ab}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{3.1}$$

This will be used to raise Latin suffices, so that (A^a) has for components the real numbers $(A_1, -A_2)$. The inverse, C_{ab}, has the same matrix, (3.1).

If two complex numbers, A and B, are multiplied together, the product is a third, C say, where:

$$C_1 + i\,C_2 = (A_1B_1 - A_2B_2) + i(A_1B_2 + A_2B_1) \tag{3.2}$$

To reproduce this fundamental property of complex arithmetic we introduce a quantity $p_{a}{}^{bc}$, and write:

$$C_a = p_{a}{}^{bc}\,A_b B_c \tag{3.3}$$

where summation over repeated indices at opposite levels is understood. Comparing (3.2) with (3.3), $p_{a}{}^{bc}$ must have:

$$\left.\begin{aligned} p_{1}{}^{11} &= p_{2}{}^{12} = p_{2}{}^{21} = -\,p_{1}{}^{22} = 1 \\ p_{1}{}^{12} &= p_{1}{}^{21} = p_{2}{}^{11} = p_{2}{}^{22} = 0 \end{aligned}\right\} \tag{3.4}$$

Symmetry in its last two indices corresponds to the commutativity of multiplication. We shall also need the form with the middle index in the covariant position:

$$\left.\begin{aligned} p_{11}{}^{1} &= p_{12}{}^{2} = p_{21}{}^{2} = -\,p_{22}{}^{1} = 1 \\ p_{11}{}^{2} &= p_{12}{}^{1} = p_{21}{}^{1} = p_{22}{}^{2} = 0 \end{aligned}\right\} \tag{3.5}$$

Two special cases of the general rule (3.3) are worth separate mention.

(i) Multiplication of any number z by the unit complex number $I = (1, 0)$ can be written:

$$(I \times z)_a = p_a{}^{bc} I_b z_c = \delta_a{}^c z_c \tag{3.6}$$

where the quantity introduced by the definition

$$\delta_a{}^c \equiv p_a{}^{bc} I_b \tag{3.7}$$

is, as the notation implies, the Kronecker delta symbol in the 2-dimensional 'Latin-index' space.

(ii) Multiplication by $i \equiv \sqrt{-1} = (0, 1)$ also has an alternative 2-index symbol representation:

$$(i \times z)_a = p_a{}^{bc} i_b z_c = e_a{}^c z_c \tag{3.8}$$

where the quantity introduced by $e_a{}^c \equiv p_a{}^{bc} i_b$ has matrix:

$$(e_a{}^b) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -(e^b{}_a) \tag{3.9}$$

§3.2 Complex vector spaces

This section deals with the question of equipping a (finite-dimensional) complex vector space with a Hermitian scalar product; it involves only elementary aspects of the theory of such 'unitary spaces'.[70][71]

Let $\mathcal{V}_n$ be an n-dimensional vector space over the field of complex numbers. Let $\{E_{(\mu)}\}$ be a set of n linearly independent vectors of $\mathcal{V}_n$, and therefore a basis for $\mathcal{V}_n$. (Greek indices always run from 1 to n, except where otherwise stated.) Consider the set of all linear mappings $f : \mathcal{V}_n \longrightarrow \mathbb{C}$ from $\mathcal{V}_n$ to the complex numbers. With the usual definitions of addition and scalar multiplication, this set of mappings is also an n-dimensional complex vector space, the 'conjugate'[70] or 'dual' space. Call it $\mathcal{V}_n^*$. There certainly exist n elements of $\mathcal{V}_n^*$, call them $\{F^*_{(\mu)}\}$, such that:

$$F^*_{(\mu)} \; E_{(\nu)} \;=\; \delta_\mu^\nu \tag{3.10}$$

They are linearly independent, and span $\mathcal{V}_n^*$. They will be taken as the canonical basis for $\mathcal{V}_n^*$, the basis 'complimentary to' $\{E_{(\mu)}\}$.

Introduce a scalar product into $\mathcal{V}_n$ as follows. With each element $v \in \mathcal{V}_n$ associate an element $v^* = G(v) \in \mathcal{V}_n^*$, where $G : \mathcal{V}_n \rightarrow \mathcal{V}_n^*$ is an anti-linear, or conjugate linear, mapping with inverse; then the quantity

$$(v|u) \;\equiv\; v^*u \;=\; G(v)\,u \tag{3.11}$$

is defined to be the scalar product of v with u. It will be required to satisfy in addition the Hermiticity condition:

$$(v|u) \;=\; \overline{(u|v)} \tag{3.12}$$

(An equally possible presentation would have been in terms of a linear mapping, $\hat{G}$ say, from V_n to the space of anti-linear mappings, $\hat{V}_n^{*}$ say. It is relatively immaterial at what stage the presence of the operation of complex conjugation in (3.12) is allowed for (cf. [70] p.102).)

We now want a representation of the mapping G. Let v be the vector:
$$v = \sum_{\mu=1}^{n} (v_1^{\mu} + i\, v_2^{\mu})\, E_{(\mu)} \tag{3.13}$$
It will be said to have the components v_a^{μ} w.r.t. this basis. Write its image under G as:
$$v^{*} = G(v) = \sum_{\mu} (v_{\mu}^{1} - i\, v_{\mu}^{2})\, F^{*}_{(\mu)} \tag{3.14}$$
Then the anti-linearity of G is found to entail the existence of a matrix relation of the form:
$$v_{\mu}^{a} = g_{\mu\nu}^{ab}\, v_{b}^{\nu} \tag{3.15}$$
with summation over repeated indices of both kinds. The $(2n \times 2n)$ real matrix $(g_{\mu\nu}^{ab})$ will be called, more particularly in the context of the next and following sections, the <u>metric tensor</u>. Its properties follow from those of the scalar product. By the preceding equations, we have:
$$\begin{aligned}(v|u) &= v^{*}u = \Big[\sum_{\mu}(v_{\mu}^{1} - i\, v_{\mu}^{2})\, F^{*}_{(\mu)}\Big]\Big[\sum_{\nu}(u_1^{\nu} + i\, u_2^{\nu})\, E_{(\nu)}\Big] \\ &= (v_{\mu}^{1} u_1^{\mu} + v_{\mu}^{2} u_2^{\mu}) + i(v_{\mu}^{1} u_2^{\mu} - v_{\mu}^{2} u_1^{\mu})\end{aligned} \tag{3.16}$$
Using (3.5), this can be written:
$$(v|u)_a = p_{ab}^{\ \ c}\, v_{\mu}^{b}\, u_c^{\mu} \tag{3.17}$$
(Compare the formula $(v|u) = v_{\mu} u^{\mu}$ for the case of a real scalar product in a real vector space.) Combining (3.15) and (3.17), the Hermiticity (3.12) implies:
$$\left.\begin{aligned} p_{1b}^{\ \ c}\,(g_{\mu\nu}^{bd}\, v_d^{\nu})\, u_c^{\mu} &= p_{1b}^{\ \ c}\,(g_{\mu\nu}^{bd}\, u_d^{\nu})\, v_c^{\mu} \\ p_{2b}^{\ \ c}\,(g_{\mu\nu}^{bd}\, v_d^{\nu})\, u_c^{\mu} &= -\, p_{2b}^{\ \ c}\,(g_{\mu\nu}^{bd}\, u_d^{\nu})\, v_c^{\mu}\end{aligned}\right\} \tag{3.18}$$

But these must be identities in v and u, so that:

$$\left.\begin{aligned} p_{1b\cdot}^{\ \ c}\, g_{\mu\nu}^{bd} &= p_{1b\cdot}^{\ \ d}\, g_{\nu\mu}^{bc} \\ p_{2b\cdot}^{\ \ c}\, g_{\mu\nu}^{bd} &= -\, p_{2b\cdot}^{\ \ d}\, g_{\nu\mu}^{bc} \end{aligned}\right\} \tag{3.19}$$

Defining: $\quad g_{\mu\nu}^{11} \equiv \gamma_{\mu\nu} \quad$ and $\quad g_{\mu\nu}^{12} \equiv \omega_{\mu\nu}\ ,$ (3.20i)

the content of the 8 equations (3.19) is:

$$\left.\begin{aligned} g_{\mu\nu}^{21} &= \omega_{\nu\mu} = -\,\omega_{\mu\nu} \\ g_{\mu\nu}^{22} &= \gamma_{\mu\nu} = \gamma_{\nu\mu} \end{aligned}\right\} \tag{3.20ii}$$

In other words, $(g_{\mu\nu}^{ab})$ is <u>symmetric</u>:

$$g_{\mu\nu}^{ab} = g_{\nu\mu}^{ba} \tag{3.21}$$

and can be written in the partitioned form:

$$(g_{\mu\nu}^{ab}) = \begin{pmatrix} \gamma_{\mu\nu} & \omega_{\mu\nu} \\ -\,\omega_{\mu\nu} & \gamma_{\mu\nu} \end{pmatrix} \tag{3.22}$$

We shall call any $(2n \times 2n)$ real matrix, $H_{\mu\nu}^{ab}$ say, with these properties 'Hermitian'. Correspondingly, an 'anti-Hermitian' matrix $A_{\mu\nu}^{ab}$ will have

$$\left.\begin{aligned} A_{\mu\nu}^{11} &= A_{\mu\nu}^{22} = -\,A_{\nu\mu}^{11} \\ A_{\mu\nu}^{12} &= -\,A_{\mu\nu}^{21} = -\,A_{\nu\mu}^{21} \end{aligned}\right\} \tag{3.23}$$

(and so will be anti-symmetric). From any Hermitian matrix we can form a new matrix by the prescription:

$$(\, i \times H\,)_{\mu\nu}^{ab} \equiv e^{a}_{\cdot c}\, H_{\mu\nu}^{cb} = \begin{pmatrix} H_{(21)} & H_{(22)} \\ -\,H_{(11)} & -\,H_{(12)} \end{pmatrix} \tag{3.24}$$

which is readily verified to be anti-Hermitian, corresponding to an analogous result in the context of a more conventional use of these terms.

Using (3.15), (3.17) and (3.20), the scalar product can be expressed in the form of one complex equation:

$$\begin{aligned} (v|u) &\equiv (v|u)_1 + i\,(v|u)_2 \\ &= (\,\gamma_{\mu\nu} + i\,\omega_{\mu\nu})(v_1^{\nu} - i\,v_2^{\nu})(u_1^{\mu} + i\,u_2^{\mu}) \end{aligned} \tag{3.25}$$

By definition, G is non-singular. Therefore the inverse matrix exists:
$$g^{ab}_{\mu\nu}\; g^{\nu\sigma}_{bc} = \delta^\sigma_\mu\, \delta^a_c \qquad (3.26)$$
One can solve (3.26) in terms of partitioned matrices (cf. (3.22)), obtaining:
$$(g^{\mu\nu}_{ab}) = \begin{pmatrix} \gamma^{\mu\nu} & -\omega^{\mu\nu} \\ \omega^{\mu\nu} & \gamma^{\mu\nu} \end{pmatrix} \qquad (3.27)$$
where $\gamma^{\mu\nu} = (\tau^{-1})^{\mu\nu}$ and $\omega^{\mu\nu} = (\gamma^{-1}\omega\tau^{-1})^{\mu\nu}$, with $\tau \equiv \gamma + \omega\gamma^{-1}\omega$.

Define $$G_{\mu\nu} \equiv \gamma_{\mu\nu} + i\,\omega_{\mu\nu}\,. \qquad (3.28)$$
There is the following simple relation between the determinant of this (n × n) complex (Hermitian) matrix and that of the (2n × 2n) real symmetric matrix (3.22):
$$\det \| g^{ab}_{\mu\nu} \| = \left[\det \| G_{\mu\nu} \| \right]^2. \qquad (3.29)$$
This result is readily established by taking determinants of both sides of the matrix equation:
$$\begin{pmatrix} I & I \\ iI & -iI \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{2} G_{\mu\nu} \\ \frac{1}{2}\overline{G_{\mu\nu}} & 0 \end{pmatrix} \begin{pmatrix} I & iI \\ I & -iI \end{pmatrix} = \begin{pmatrix} \gamma_{\mu\nu} & \omega_{\mu\nu} \\ -\omega_{\mu\nu} & \gamma_{\mu\nu} \end{pmatrix} \qquad (3.30)$$
By Hermiticity, $\det \| G_{\mu\nu} \|$ is of course real. There are the following formulae for it, in 2, 3, and 4 dimensions:

<u>2-D</u>: $\det \| G_{\mu\nu} \| = \gamma - \omega$

<u>3-D</u>: $\det \| G_{\mu\nu} \| = \gamma \left[1 - (\gamma^{-1})^{\mu\alpha} (\gamma^{-1})^{\nu\beta}\, \omega_{\alpha\beta}\, \omega_{\mu\nu} \right]$ (3.31)

<u>4-D</u>: $\det \| G_{\mu\nu} \| = \gamma \left[1 - (\gamma^{-1})^{\mu\alpha} (\gamma^{-1})^{\nu\beta}\, \omega_{\alpha\beta}\, \omega_{\mu\nu} \right] + \omega$

where $\gamma \equiv \det \| \gamma_{\mu\nu} \|$ and $\omega \equiv \det \| \omega_{\mu\nu} \|$. These results on the determinant of the metric tensor have practical utility since, as will appear, a certain curvature tensor which plays a central role is completely specified if $\det \| g^{ab}_{\mu\nu} \|$ is known.

§3.3 Transformations

We shall first look at what happens when one chooses new bases in the $\mathcal{V}_n$ of §3.2. This will then be related to coordinate transformations in a certain type of manifold.

Suppose, then, that a new set of base vectors for $\mathcal{V}_n$, $\{E'_{(\mu)}\}$, is related linearly to the old, in the sense that there is a non-singular complex matrix A such that:

$$\left.\begin{aligned} E_{(\mu)} &= \sum_\nu A_{\nu\mu} E'_{(\nu)} \\ \text{with inverse:}\quad E'_{(\mu)} &= \sum_\nu A^{-1}_{\nu\mu} E_{(\nu)} \end{aligned}\right\} \tag{3.32}$$

This change of basis induces a change in the conjugate basis $\{F^*_{(\mu)}\}$ of $\mathcal{V}^*_n$ to some new set of vectors, $\{F^{*\prime}_{(\mu)}\}$ say, which must satisfy (cf. (3.10)):

$$F^{*\prime}_{(\mu)} \; E'_{(\nu)} = \delta^\nu_\mu \tag{3.33}$$

From this one can deduce that:

$$\left.\begin{aligned} F^*_{(\mu)} &= \sum_\nu A^{-1}_{\mu\nu} F^{*\prime}_{(\nu)} \\ F^{*\prime}_{(\mu)} &= \sum_\nu A_{\mu\nu} F^*_{(\nu)} \end{aligned}\right\} \tag{3.34}$$

These changes of basis cause the components of a (fixed) vector v to change, in the following manner:

$$\left.\begin{aligned} v^{\mu\prime}_1 + i\, v^{\mu\prime}_2 &= \sum_\alpha A_{\mu\alpha} (v^\alpha_1 + i\, v^\alpha_2) \\ v^{1\prime}_\mu - i\, v^{2\prime}_\mu &= \sum_\alpha A^{-1}_{\alpha\mu} (v^1_\alpha - i\, v^2_\alpha) \end{aligned}\right\} \tag{3.35}$$

where dashes represent components w.r.t. the new bases. Let us determine the new matrix of the mapping G. We need to split (3.35) into real and imaginary parts. Suppose

$$A_{\mu\nu} \equiv A^{(1)}_{\mu\nu} + i\, A^{(2)}_{\mu\nu} \tag{3.36}$$

and write:

$$(\mathcal{A}^{ab}_{\mu\nu}) \equiv \begin{pmatrix} A^{(1)}_{\mu\nu} & -A^{(2)}_{\mu\nu} \\ A^{(2)}_{\mu\nu} & A^{(1)}_{\mu\nu} \end{pmatrix} \tag{3.37}$$

Then equations (3.35) are:

$$\left.\begin{aligned} v^{\mu\,\prime}_{a} &= \sum_{\alpha,s} \mathcal{Q}^{a\,s}_{\mu\,\alpha}\, v^{\alpha}_{s} \\ v^{a\,\prime}_{\mu} &= \sum_{\alpha,s} \mathcal{Q}^{-1\,s\,a}_{\alpha\,\mu}\, v^{s}_{\alpha} \end{aligned}\right\} \tag{3.38}$$

Therefore (3.15) holds also for the dashed quantities, with:

$$g^{a\,b\,\prime}_{\mu\,\nu} = \sum_{\alpha,s}\sum_{\beta,t} \mathcal{Q}^{-1\,s\,a}_{\alpha\,\mu}\,\mathcal{Q}^{-1\,t\,b}_{\beta\,\nu}\, g^{s\,t}_{\alpha\,\beta} \tag{3.39}$$

It is readily verified that this is still a Hermitian matrix, and that its $\gamma'_{\mu\nu}$, $\omega'_{\mu\nu}$ are determined by the equation:

$$\gamma'_{\mu\nu} + i\,\omega'_{\mu\nu} = \sum_{\alpha}\sum_{\beta} A^{-1}_{\alpha\mu}\,(\gamma_{\alpha\beta} + i\,\omega_{\alpha\beta})\,\overline{A^{-1}}_{\beta\nu}\,. \tag{3.40}$$

We can now construct a theory of complex metric manifolds. Let R_{2n} be a real 2n-dimensional manifold, parametrized by coordinates (z^{μ}_{a}), with (at present) the group of general non-singular coordinate transformations:

$$z^{\mu\,\prime}_{a} = z^{\mu\,\prime}_{a}(z^{\alpha}_{s}) \tag{3.41}$$

Consider a point P with coordinates z^{μ}_{a}, and a neighbouring point P', $z^{\mu}_{a} + dz^{\mu}_{a}$, where the dz^{μ}_{a} are infinitessimal. As P' varies (always in the neighbourhood of P), the quantities dz^{μ}_{a} span a 2n-dimensional real vector space, the tangent vector space at P. We now require that it shall be an n-dimensional complex vector space, $\mathcal{V}_n(P)$, in the sense that the quantities $(dz^{\mu}_{1} + i\,dz^{\mu}_{2})$ shall be the components of a vector of $\mathcal{V}_n(P)$.

Consider the effect of the coordinate transformation (3.41). For fixed P, P', the relation between the components of $\overrightarrow{PP'}$ in the two coordinate systems is:

$$dz^{\mu\,\prime}_{a} = \frac{\partial z^{\mu\,\prime}_{a}}{\partial z^{\alpha}_{s}}\, dz^{\alpha}_{s} \tag{3.42}$$

Compare this with (3.38). It is clear that the change in components which (3.42) signifies is more general than that of (3.38), since the transformation matrix in the former does not in general have the partitioned structure (3.37), and so is not equivalent to a change of basis in $\mathcal{V}_n^c$ (P) (in fact, it 'mixes up' the vectors of $\mathcal{V}_n$ and those of the complex conjugate vector space $\overline{\mathcal{V}_n}$). We shall accordingly restrict the allowable transformations (3.41) so as to ensure that the matrix in (3.42) does have the structure (3.37), i.e. we require:

$$\left.\begin{aligned} \frac{\partial z_1^{\mu'}}{\partial z_1^{\nu}} &= \frac{\partial z_2^{\mu'}}{\partial z_2^{\nu}} \\ \frac{\partial z_1^{\mu'}}{\partial z_2^{\nu}} &= -\frac{\partial z_2^{\mu'}}{\partial z_1^{\nu}} \end{aligned}\right\} \qquad (3.43)$$

These Cauchy-Riemann equations, however, just say that $(z_1^{\mu'} + i z_2^{\mu'})$ is to be an analytic function of the $(z_1^{\alpha} + i z_2^{\alpha})$ only (i.e. independent of $(z_1^{\alpha} - i z_2^{\alpha})$). Under these conditions, (3.41) will be called an analytic coordinate transformation, and the manifold R_{2n} with its transformation group thus restricted will be called a complex analytic manifold, denoted $\mathcal{C}_n$. If the tangent $\mathcal{V}_n^c$'s are furnished, as in §3.2, with Hermitian scalar products, the $\mathcal{C}_n$ could be called, following Yano, a Hermite space, $\mathcal{H}_n$. We shall henceforth restrict attention exclusively to such spaces. (It would be possible, as suggested by A.Das (cf. §1.3), to try and construct a field theory in terms of manifolds supporting transformation groups not restricted to analyticity (cf. also [67] p.389, note 2); the resulting theory has, however, little to recommend it.)

We can now say that the analytic coordinate transformation induces the change of basis in $\mathcal{V}_n^c$(P) given by:

$$\mathfrak{a}^{ab}_{\mu\nu} = \frac{\partial z^{\mu'}_a}{\partial z^\nu_b} \qquad (3.44)$$

or: $$A_{\mu\nu} = \frac{\partial z^{\mu'}_1}{\partial z^\nu_1} + i\,\frac{\partial z^{\mu'}_2}{\partial z^\nu_1} \qquad (3.45)$$

Using the relations: $$\frac{\partial z^{\mu'}_a}{\partial z^\nu_b}\,\frac{\partial z^\nu_b}{\partial z^{\sigma'}_c} = \delta^\mu_\sigma\,\delta^c_a \qquad (3.46)$$

together with the Cauchy-Riemann (C-R) equations, one finds:

$$A^{-1}_{\mu\nu} = \frac{\partial z^\mu_1}{\partial z^{\nu'}_1} + i\,\frac{\partial z^\mu_2}{\partial z^{\nu'}_1} \qquad (3.47)$$

These equations together with (3.39) or (3.40) determine the metric tensor field in the new coordinate system.

§3.4 Affine connection

A tensor calculus will now be set up, in a way which corresponds as closely as possible with the Riemannian theory (cf. e.g. [25]). Again we start from 'first principles'.

Let $v^{\mu}_{a}(z^{\alpha}_{s})$ be C^1 functions of the z^{α}_{s} in some neighbourhood, and transform (cf. (3.38) + (3.44)) as:

$$v^{\mu'}_{a} = \frac{\partial z^{\mu'}_{a}}{\partial z^{\nu}_{b}} v^{\nu}_{b} \tag{3.48}$$

under the analytic coordinate transformation (3.41). Such a quantity will be called a <u>contravariant vector field</u> on $\mathcal{H}_n$. (N.B. it is not necessarily an <u>analytic</u> function of the $(z^{\alpha}_{1} + iz^{\alpha}_{2})$ only.) Consider two neighbouring points P, P', of $\mathcal{H}_n$, at which the field has the values v^{μ}_{a}, $v^{\mu}_{a} + dv^{\mu}_{a}$ respectively, with:

$$dv^{\mu}_{a} = v^{\mu}_{a},{}^{s}_{\alpha}\, dz^{\alpha}_{s} \tag{3.49}$$

to first order in the coordinate differentials. dv^{μ}_{a} is not a tensor, since the transformation matrix in (3.48) in general differs from P to P'; in fact, differentiation of (3.48) gives:

$$(v^{\mu}_{a},{}^{c}_{\sigma})' = \frac{\partial z^{\mu'}_{a}}{\partial z^{\alpha}_{s}} \frac{\partial z^{\beta}_{t}}{\partial z^{\sigma'}_{c}} \left(v^{\alpha}_{s},{}^{t}_{\beta}\right) + \frac{\partial^2 z^{\mu'}_{a}}{\partial z^{\alpha}_{s}\partial z^{\beta}_{t}} \frac{\partial z^{\beta}_{t}}{\partial z^{\sigma'}_{c}} v^{\alpha}_{s} \tag{3.50}$$

So we define a vector $v^{\mu}_{a} + \delta v^{\mu}_{a}$ at P' which is said to result from the <u>parallel displacement</u> of the vector v^{μ}_{a} from P to P', with δv^{μ}_{a} bilinear in the vector and displacement:

$$\delta v^{\mu}_{a} = -\Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma} v^{\nu}_{b}\, dz^{\sigma}_{c}\,. \tag{3.51}$$

$(v^{\mu}_{a} + dv^{\mu}_{a}) - (v^{\mu}_{a} + \delta v^{\mu}_{a})$ <u>is</u> a vector (at P'), and the limit P' $\longrightarrow$ P yields the <u>covariant derivative</u>:

$$v^{\mu}_{a};{}^{c}_{\sigma} = v^{\mu}_{a},{}^{c}_{\sigma} + \Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma} v^{\nu}_{b} \tag{3.52}$$

The LHS will be a tensor only if, from (3.50), the Γ's transform as:

$$\left(\Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma}\right)' = \frac{\partial z^{\mu\,\prime}_{a}}{\partial z^{\lambda}_{m}} \frac{\partial z^{\alpha}_{s}}{\partial z^{\nu\,\prime}_{b}} \frac{\partial z^{\beta}_{t}}{\partial z^{\sigma\,\prime}_{c}} \Gamma^{\lambda\, s\, t}_{m\, \alpha\, \beta} + \frac{\partial z^{\mu\,\prime}_{a}}{\partial z^{\alpha}_{s}} \frac{\partial^2 z^{\alpha}_{s}}{\partial z^{\nu\,\prime}_{b}\, \partial z^{\sigma\,\prime}_{c}} \qquad (3.53)$$

The covariant derivative of a covariant vector field can be defined by means of the following requirement: scalar products, in the form of (3.17), are unchanged under parallel displacement of the vectors. If the result of parallelly transferring v^a_μ from P to P' is $v^a_\mu + \delta v^a_\mu$, this requirement translates into:

$$p_{ab}^{\ \ c}\,(\,v^b_\mu\, \delta u^\mu_c + u^\mu_c\, \delta v^b_\mu\,) = 0 \qquad (3.54)$$

Taking $a = 1$ leads to:

$$\delta v^a_\mu = +\,\Gamma^{\lambda\, a\, c}_{b\, \mu\, \sigma}\, v^b_\lambda\, dz^\sigma_c \qquad (3.55)$$

so that the corresponding covariant derivative is:

$$v^{a}_{\mu\, ;}{}^{c}_{\sigma} = v^{a}_{\mu\, ,}{}^{c}_{\sigma} - \Gamma^{\lambda\, a\, c}_{b\, \mu\, \sigma}\, v^b_\lambda \; . \qquad (3.56)$$

Taking $a = 2$ leads to a restriction on the Γ's:

$$\left.\begin{aligned} \Gamma^{\mu\, 1\, c}_{1\, \nu\, \sigma} &= \Gamma^{\mu\, 2\, c}_{2\, \nu\, \sigma} \\ \Gamma^{\mu\, 2\, c}_{1\, \nu\, \sigma} &= -\Gamma^{\mu\, 1\, c}_{2\, \nu\, \sigma} \end{aligned}\right\} \qquad (3.57)$$

The affine connection can be related to the metric tensor via the following requirement: $(v^a_\mu + \delta v^a_\mu)$ is the covariant counterpart (in the sense of (3.15)) of $(v^\mu_a + \delta v^\mu_a)$ w.r.t. the metric tensor at P'. This translates into:

$$(v^a_\mu + \delta v^a_\mu) = (g^{a\, b}_{\mu\, \nu} + g^{a\, b}_{\mu\, \nu ,}{}^{c}_{\sigma}\, dz^\sigma_c)(v^\nu_b + \delta v^\nu_b) \qquad (3.58)$$

which in turn entails: $\qquad g^{a\, b}_{\mu\, \nu\, ;}{}^{c}_{\sigma} = 0 \; . \qquad (3.59)$

The metric tensor is symmetric (see (3.21)), so that (3.59) only determines the Γ's uniquely if they also are symmetric:

$$\Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma} = \Gamma^{\mu\, c\, b}_{a\, \sigma\, \nu} \qquad (3.60)$$

The solution is then:

$$\Gamma^{\lambda\, b\, c}_{e\, \nu\, \sigma} = \tfrac{1}{2}\, g^{\lambda\, \mu}_{e\, a}\, (g^{a\, b}_{\mu\, \nu ,}{}^{c}_{\sigma} + g^{a\, c}_{\mu\, \sigma ,}{}^{b}_{\nu} - g^{b\, c}_{\nu\, \sigma ,}{}^{a}_{\mu}) \, . \qquad (3.61)$$

An $\mathcal{H}_n$ satisfying (3.57), (3.59) and (3.60) is a <u>Kähler space</u>, $\mathcal{K}_n$. From now on we shall restrict attention to such spaces, mainly on the grounds of simplicity, and because there seems to be no physical motivation for considering more general possibilities. (3.21), (3.59) and (3.60) imply that the R_{2n} is now inter alia a (real) <u>Riemannian</u> space.

Because $g^{ab}_{\mu\nu}$ has the particular structure (3.22), it might be thought that (3.57) are consequences of (3.61); but this is not so: they entail certain restrictions on the <u>derivatives</u> of the metric tensor, as will now appear. When combined with (3.60), (3.57) imply that there are <u>only two</u> distinct Γ's, $\Gamma_{(a)\cdot\nu\sigma}^{\lambda}$ say, where:

$$\left.\begin{aligned} \Gamma_{(1)\cdot\nu\sigma}^{\lambda} &\equiv \Gamma^{\lambda 1 1}_{1\nu\sigma} = \Gamma^{\lambda 2 1}_{2\nu\sigma} = \Gamma^{\lambda 1 2}_{2\nu\sigma} = -\Gamma^{\lambda 2 2}_{1\nu\sigma} \\ \Gamma_{(2)\cdot\nu\sigma}^{\lambda} &\equiv \Gamma^{\lambda 2 2}_{2\nu\sigma} = \Gamma^{\lambda 1 2}_{1\nu\sigma} = \Gamma^{\lambda 2 1}_{1\nu\sigma} = -\Gamma^{\lambda 1 1}_{2\nu\sigma} \end{aligned}\right\} \qquad (3.62)$$

A precisely similar set of relations - call them (3.62') - exist also for the completely-covariant quantities

$$\Gamma^{abc}_{\mu\nu\sigma} \equiv g^{ae}_{\mu\lambda}\Gamma^{\lambda bc}_{e\nu\sigma} = \tfrac{1}{2}\left(g^{ab}_{\mu\nu,}{}^{c}_{\sigma} + g^{ac}_{\mu\sigma,}{}^{b}_{\nu} - g^{bc}_{\nu\sigma,}{}^{a}_{\mu}\right), \qquad (3.63)$$

with $\Gamma^{(a)}_{\mu\nu\sigma}$ related to the $\Gamma_{(a)\cdot\nu\sigma}^{\lambda}$ by:

$$\left.\begin{aligned} \Gamma^{(1)}_{\mu\nu\sigma} &= \gamma_{\mu\lambda}\Gamma_{(1)\cdot\nu\sigma}^{\lambda} - \omega_{\mu\lambda}\Gamma_{(2)\cdot\nu\sigma}^{\lambda} \\ \Gamma^{(2)}_{\mu\nu\sigma} &= \gamma_{\mu\lambda}\Gamma_{(2)\cdot\nu\sigma}^{\lambda} + \omega_{\mu\lambda}\Gamma_{(1)\cdot\nu\sigma}^{\lambda} \end{aligned}\right\} \qquad (3.64\text{i})$$

with the inverse:

$$\left.\begin{aligned} \Gamma_{(1)\cdot\nu\sigma}^{\lambda} &= \gamma^{\lambda\mu}\Gamma^{(1)}_{\mu\nu\sigma} + \omega^{\lambda\mu}\Gamma^{(2)}_{\mu\nu\sigma} \\ \Gamma_{(2)\cdot\nu\sigma}^{\lambda} &= \gamma^{\lambda\mu}\Gamma^{(2)}_{\mu\nu\sigma} - \omega^{\lambda\mu}\Gamma^{(1)}_{\mu\nu\sigma} \end{aligned}\right\} \qquad (3.64\text{ii})$$

Inserting (3.63) into (3.62') entails:

$$\left.\begin{aligned} \gamma_{\mu\sigma,}{}^{1}_{\nu} - \gamma_{\nu\sigma,}{}^{1}_{\mu} &= -\omega_{\mu\sigma,}{}^{2}_{\nu} + \omega_{\nu\sigma,}{}^{2}_{\mu} \\ \gamma_{\mu\sigma,}{}^{2}_{\nu} - \gamma_{\nu\sigma,}{}^{2}_{\mu} &= \omega_{\mu\sigma,}{}^{1}_{\nu} - \omega_{\nu\sigma,}{}^{1}_{\mu} \end{aligned}\right\} \qquad (3.65)$$

An $\mathcal{H}_n$ satisfying (3.57), (3.59) and (3.60) is a <u>Kähler space</u>, $\mathcal{K}_n$. From now on we shall restrict attention to such spaces, mainly on the grounds of simplicity, and because there seems to be no physical motivation for considering more general possibilities. (3.21), (3.59) and (3.60) imply that the R_{2n} is now inter alia a (real) <u>Riemannian</u> space.

Because $g^{ab}_{\mu\nu}$ has the particular structure (3.22), it might be thought that (3.57) are consequences of (3.61); but this is not so: they entail certain restrictions on the <u>derivatives</u> of the metric tensor, as will now appear. When combined with (3.60), (3.57) imply that there are <u>only two</u> distinct Γ's, $\Gamma_{(a)\cdot\nu\sigma}^{\;\;\lambda}$ say, where:

$$\left.\begin{aligned} \Gamma_{(1)\cdot\nu\sigma}^{\;\;\lambda} &\equiv \Gamma^{\lambda 1 1}_{1\nu\sigma} = \Gamma^{\lambda 2 1}_{2\nu\sigma} = \Gamma^{\lambda 1 2}_{2\nu\sigma} = -\Gamma^{\lambda 2 2}_{1\nu\sigma} \\ \Gamma_{(2)\cdot\nu\sigma}^{\;\;\lambda} &\equiv \Gamma^{\lambda 2 2}_{2\nu\sigma} = \Gamma^{\lambda 1 2}_{1\nu\sigma} = \Gamma^{\lambda 2 1}_{1\nu\sigma} = -\Gamma^{\lambda 1 1}_{2\nu\sigma} \end{aligned}\right\} \qquad (3.62)$$

A precisely similar set of relations - call them (3.62') - exist also for the completely-covariant quantities

$$\Gamma^{abc}_{\mu\nu\sigma} \equiv g^{ae}_{\mu\lambda}\Gamma^{\lambda bc}_{e\nu\sigma} = \tfrac{1}{2}\left(g^{ab\,,c}_{\mu\nu,\sigma} + g^{ac\,,b}_{\mu\sigma,\nu} - g^{bc\,,a}_{\nu\sigma,\mu}\right), \qquad (3.63)$$

with $\Gamma^{(a)}_{\mu\nu\sigma}$ related to the $\Gamma_{(a)\cdot\nu\sigma}^{\;\;\lambda}$ by:

$$\left.\begin{aligned} \Gamma^{(1)}_{\mu\nu\sigma} &= \gamma_{\mu\lambda}\Gamma_{(1)\cdot\nu\sigma}^{\;\;\lambda} - \omega_{\mu\lambda}\Gamma_{(2)\cdot\nu\sigma}^{\;\;\lambda} \\ \Gamma^{(2)}_{\mu\nu\sigma} &= \gamma_{\mu\lambda}\Gamma_{(2)\cdot\nu\sigma}^{\;\;\lambda} + \omega_{\mu\lambda}\Gamma_{(1)\cdot\nu\sigma}^{\;\;\lambda} \end{aligned}\right\} \qquad (3.64\text{i})$$

with the inverse:

$$\left.\begin{aligned} \Gamma_{(1)\cdot\nu\sigma}^{\;\;\lambda} &= \gamma^{\lambda\mu}\Gamma^{(1)}_{\mu\nu\sigma} + \omega^{\lambda\mu}\Gamma^{(2)}_{\mu\nu\sigma} \\ \Gamma_{(2)\cdot\nu\sigma}^{\;\;\lambda} &= \gamma^{\lambda\mu}\Gamma^{(2)}_{\mu\nu\sigma} - \omega^{\lambda\mu}\Gamma^{(1)}_{\mu\nu\sigma} \end{aligned}\right\} \qquad (3.64\text{ii})$$

Inserting (3.63) into (3.62') entails:

$$\left.\begin{aligned} \gamma_{\mu\sigma,\overset{1}{\nu}} - \gamma_{\nu\sigma,\overset{1}{\mu}} &= -\omega_{\mu\sigma,\overset{2}{\nu}} + \omega_{\nu\sigma,\overset{2}{\mu}} \\ \gamma_{\mu\sigma,\overset{2}{\nu}} - \gamma_{\nu\sigma,\overset{2}{\mu}} &= \omega_{\mu\sigma,\overset{1}{\nu}} - \omega_{\nu\sigma,\overset{1}{\mu}} \end{aligned}\right\} \qquad (3.65)$$

By cyclically permuting indices and adding, one obtains:

$$\omega_{\mu\nu,\overset{a}{\sigma}} + \omega_{\nu\sigma,\overset{a}{\mu}} + \omega_{\sigma\mu,\overset{a}{\nu}} = 0 \tag{3.66}$$

(cf. Maxwell's equation (1.4i)). (3.65) + (3.66) imply:

$$\left.\begin{aligned} \omega_{\mu\nu,\overset{1}{\sigma}} &= \gamma_{\mu\sigma,\overset{2}{\nu}} - \gamma_{\nu\sigma,\overset{2}{\mu}} \\ \omega_{\mu\nu,\overset{2}{\sigma}} &= -\gamma_{\mu\sigma,\overset{1}{\nu}} + \gamma_{\nu\sigma,\overset{1}{\mu}} \end{aligned}\right\} \tag{3.67}$$

so that the derivatives of $\omega_{\mu\nu}$ are in fact completely determined by those of $\gamma_{\mu\nu}$. In terms of the latter, one has:

$$\Gamma^{(a)}_{\mu\nu\sigma} = \tfrac{1}{2}\left(\gamma_{\mu\nu,\overset{a}{\sigma}} + \gamma_{\mu\sigma,\overset{a}{\nu}} - \gamma_{\nu\sigma,\overset{a}{\mu}}\right). \tag{3.68}$$

One may note that in a $\mathcal{K}_4$, for example, there are (2×40) distinct components of the connection and that the (10×8) different derivatives of the $\gamma_{\mu\nu}$, alone, are sufficient to produce just this multiplicity. However, there is in fact a <u>single</u> function, $\Omega(z^\mu_a)$ say, which determines not only the derivatives of both $\omega_{\mu\nu}$ and $\gamma_{\mu\nu}$ but also these quantities themselves. The existence of this 'basic function' is not so immediately evident in the present formalism as it was in that of Chapter 2 (the author only tumbled to it in the course of explicit calculations of the 2-dimensional case), but can be demonstrated as follows. Multiply the second equation of (3.65) by i and subtract from the first, giving:

$$G_{\mu\sigma,\overset{1}{\nu}} - i\,G_{\mu\sigma,\overset{2}{\nu}} = G_{\nu\sigma,\overset{1}{\mu}} - i\,G_{\nu\sigma,\overset{2}{\mu}} \tag{3.69}$$

Write

$$z^\mu \equiv z^\mu_1 + iz^\mu_2 \; ; \quad \overline{z^\mu} = z^\mu_1 - iz^\mu_2 \tag{3.70}$$

Then

$$2\frac{\partial}{\partial z^\mu} = \frac{\partial}{\partial z^\mu_1} - i\frac{\partial}{\partial z^\mu_2} \; ; \quad 2\frac{\partial}{\partial \overline{z^\mu}} = \frac{\partial}{\partial z^\mu_1} + i\frac{\partial}{\partial z^\mu_2} \tag{3.71}$$

so that (3.69) can be written:

$$\frac{\partial G_{\mu\sigma}}{\partial z^\nu} = \frac{\partial G_{\nu\sigma}}{\partial z^\mu} \tag{3.72}$$

This, together with the complex conjugate equation, leads, by steps as given in [67] pp.397-8, to the existence of a real function $\Omega(z, \bar{z})$ such that:

$$G_{\mu\nu} = 4 \frac{\partial^2 \Omega}{\partial z^\mu \partial \bar{z}^\nu} \tag{3.73}$$

Clearly, Ω is uniquely determined up to addition of the real part of an arbitrary analytic function, i.e. $\Omega \rightarrow \Omega^*$, where

$$\Omega^* = \Omega + \tfrac{1}{2}\left[f(z^\alpha) + \overline{f(z^\alpha)}\right]. \tag{3.74}$$

Using (3.71), (3.73) implies:

$$\left.\begin{aligned} \gamma_{\mu\nu} &= \Omega_{,\overset{1}{\mu},\overset{1}{\nu}} + \Omega_{,\overset{2}{\mu},\overset{2}{\nu}} \\ \omega_{\mu\nu} &= \Omega_{,\overset{1}{\mu},\overset{2}{\nu}} - \Omega_{,\overset{2}{\mu},\overset{1}{\nu}} \end{aligned}\right\} \tag{3.75}$$

One can show, by comparing this equation with the corresponding one in the dashed coordinate system, that under (3.41) Ω remains unchanged in value (at a given point of $\mathcal{K}_n$), i.e. transforms as a <u>scalar</u>, within the latitude allowed by (3.74).

It is interesting to note that a necessary and sufficient condition for (3.57) + (3.60) to hold is: the Γ's can be made to vanish at any one point of C_n by an analytic coordinate transformation. This is closely parallel to the corresponding Riemannian result, and throws extra light on the nature of the Kählerian requirement (3.57). We shall demonstrate only the sufficiency - proof of the converse is almost as straightforward. Suppose, then, that in the new (dashed) coordinate system the $\left(\Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma}\right)'$ all vanish at the point with coordinates $(z^{\mu'}_a)$. Then in the original system the Γ's must have had the values, at this point:

$$\Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma} = \frac{\partial z^\mu_a}{\partial z^{\alpha'}_s} \frac{\partial^2 z^{\alpha'}_s}{\partial z^\nu_b \, \partial z^\sigma_c} \tag{3.76}$$

(use (3.53), with dashed and undashed indices interchanged).

The RHS is obviously symmetric in $({}^{b}_{\nu}, {}^{c}_{\sigma})$ (condition (3.60)), while the C-R equations (3.43) ensure that it also satisfies (3.57).

We note, finally, that the theory of geodesics goes through precisely as for a Riemannian space. By (3.12), the scalar product of any vector with itself is real:

$$(v|v) = \overline{(v|v)} \tag{3.77}$$

Putting $a = 1$ in (3.17), this real number is just:

$$(v|v)_1 = v^a_\mu v^\mu_a \tag{3.78}$$

Take v to be the infinitessimal displacement vector dz, and call the corresponding real number ds^2. Then there is the <u>line-element</u> formula:

$$\left.\begin{aligned} ds^2 &= dz^a_\mu dz^\mu_a = g^{ab}_{\mu\nu}\, dz^\mu_a dz^\nu_b \\ &= \gamma_{\mu\nu}(dz^\mu_1 dz^\nu_1 + dz^\mu_2 dz^\nu_2) + \omega_{\mu\nu}(dz^\mu_1 dz^\nu_2 - dz^\mu_2 dz^\nu_1) \\ &= G_{\mu\nu}\, dz^\mu d\overline{z^\nu}\,. \end{aligned}\right\} \tag{3.79}$$

Either by auto-parallel displacement of the unit vector $\frac{dz^\mu_a}{ds}$, or from the variational principle:

$$\delta \int ds = 0 \,, \tag{3.80}$$

one arrives at the geodesic equation:

$$\frac{d^2 z^\mu_a}{ds^2} + \Gamma^{\mu\, b\, c}_{a\, \nu\, \sigma} \frac{dz^\nu_b}{ds} \frac{dz^\sigma_c}{ds} = 0. \tag{3.81}$$

§3.5 Curvature

The following few equations are completely standard, and will be presented without comment; they hold for $\mathcal{K}_n$ qua real 2n-dimensional Riemannian manifold.

$$v^{\mu\ \ s\ \ t}_{|a;\alpha;\beta} - v^{\mu\ \ t\ \ s}_{|a;\beta;\alpha} = v^{\nu}_{b}\, R^{\mu b s t}_{|a\nu\alpha\beta}\ , \tag{3.82}$$

where
$$R^{\mu b s t}_{|a\nu\alpha\beta} = \left[\Gamma^{\mu b s\ \ t}_{|a\nu\alpha,\beta} - \Gamma^{\mu u s}_{|a\rho\alpha}\Gamma^{\rho b t}_{|u\nu\beta}\right] - \left[{}^{s}_{\alpha} \leftrightarrow {}^{t}_{\beta}\right]. \tag{3.83}$$

$$\begin{aligned} R^{abst}_{\mu\nu\alpha\beta} &\equiv g^{ac}_{\mu\lambda}\, R^{\lambda b s t}_{c\nu\alpha\beta} \\ &= \left[\Gamma^{abs\ \ t}_{\mu\nu\alpha,\beta} - g^{\rho\sigma}_{|uv}\, \Gamma^{uat}_{\rho\mu\beta}\Gamma^{vbs}_{\sigma\nu\alpha}\right] - \left[{}^{s}_{\alpha} \leftrightarrow {}^{t}_{\beta}\right]. \end{aligned} \tag{3.84}$$

$$R^{abst}_{\mu\nu\alpha\beta} = - R^{bast}_{\nu\mu\alpha\beta} = R^{bats}_{\nu\mu\beta\alpha} = R^{stab}_{\alpha\beta\mu\nu} \tag{3.85}$$

$$R^{a\{bst\}}_{\mu\{\nu\alpha\beta\}} = 0 \tag{3.86}$$

$$R^{ab\{st\ \ u\}}_{\mu\nu\{\alpha\beta;\gamma\}} = 0 \tag{3.87}$$

Now utilize the special form (3.62) of the Γ's. To do this, it is simplest to work in a geodesic coordinate system (cf. [67] p.156) for the point of $\mathcal{K}_n$ under consideration (we have already seen (p.75) that this is possible). Then (3.84) becomes:

$$R^{abst}_{\mu\nu\alpha\beta} = \Gamma^{abs\ \ t}_{\mu\nu\alpha,\beta} - \Gamma^{abt\ \ s}_{\mu\nu\beta,\alpha} \tag{3.88}$$

Inserting (3.62'), one finds just three distinct types of tensor component:

$$\left.\begin{aligned} R^{(1)}_{\mu\nu\alpha\beta} &\equiv R^{1111}_{\mu\nu\alpha\beta} \\ R^{(2)}_{\mu\nu\alpha\beta} &\equiv R^{2111}_{\mu\nu\alpha\beta} \\ R^{(3)}_{\mu\nu\alpha\beta} &\equiv R^{1212}_{\mu\nu\alpha\beta} \end{aligned}\right\} \tag{3.89}$$

all others being expressible in terms of one of these, e.g. $R^{2212}_{\mu\nu\alpha\beta} = - R^{(2)}_{\beta\alpha\mu\nu}$.

Inserting into (3.88) the values of the $\Gamma^{(a)}_{\mu\nu\sigma}$ from (3.68), and using (3.67), one can obtain the following formulae:

$$\left.\begin{aligned}
\overset{(1)}{R}_{\mu\nu\alpha\beta} &= \tfrac{1}{2}\left(\omega_{\mu\nu,\overset{1}{\alpha},\overset{2}{\beta}} - \omega_{\mu\nu,\overset{2}{\alpha},\overset{1}{\beta}}\right) = \tfrac{1}{2}\left(\omega_{\alpha\beta,\overset{1}{\mu},\overset{2}{\nu}} - \omega_{\alpha\beta,\overset{2}{\mu},\overset{1}{\nu}}\right) \\
\overset{(2)}{R}_{\mu\nu\alpha\beta} &= \tfrac{1}{2}\left(\gamma_{\mu\nu,\overset{1}{\alpha},\overset{2}{\beta}} - \gamma_{\mu\nu,\overset{2}{\alpha},\overset{1}{\beta}}\right) = \tfrac{1}{2}\left(\omega_{\alpha\beta,\overset{1}{\mu},\overset{1}{\nu}} + \omega_{\alpha\beta,\overset{2}{\mu},\overset{2}{\nu}}\right) \\
\overset{(3)}{R}_{\mu\nu\alpha\beta} &= \tfrac{1}{2}\left(\gamma_{\mu\nu,\overset{1}{\alpha},\overset{1}{\beta}} + \gamma_{\mu\nu,\overset{2}{\alpha},\overset{2}{\beta}}\right) = \tfrac{1}{2}\left(\gamma_{\alpha\beta,\overset{1}{\mu},\overset{1}{\nu}} + \gamma_{\alpha\beta,\overset{2}{\mu},\overset{2}{\nu}}\right)
\end{aligned}\right\} \quad (3.90)$$

These entail the following symmetry properties (valid in <u>all</u> coordinate systems):

$$\left.\begin{aligned}
\overset{(1)}{R}_{\mu\nu\alpha\beta} &= -\overset{(1)}{R}_{\mu\nu\beta\alpha} = -\overset{(1)}{R}_{\nu\mu\alpha\beta} = \overset{(1)}{R}_{\alpha\beta\mu\nu} \\
\overset{(2)}{R}_{\mu\nu\alpha\beta} &= -\overset{(2)}{R}_{\mu\nu\beta\alpha} = +\overset{(2)}{R}_{\nu\mu\alpha\beta} \\
\overset{(3)}{R}_{\mu\nu\alpha\beta} &= +\overset{(3)}{R}_{\mu\nu\beta\alpha} = +\overset{(3)}{R}_{\nu\mu\alpha\beta} = \overset{(3)}{R}_{\alpha\beta\mu\nu}
\end{aligned}\right\} \quad (3.91)$$

Also, the content of the 'cyclic' identity (3.86) is:

$$\left.\begin{aligned}
\overset{(1)}{R}_{\mu\{\nu\alpha\beta\}} &= 0 \\
\overset{(2)}{R}_{\mu\{\nu\alpha\beta\}} &= 0 \\
\overset{(1)}{R}_{\mu\nu\alpha\beta} &= \overset{(3)}{R}_{\mu\alpha\nu\beta} - \overset{(3)}{R}_{\mu\beta\nu\alpha}
\end{aligned}\right\} \quad (3.92)$$

(which could also have been deduced from (3.90)).

How many linearly independent components of the tensor $R^{abst}_{\mu\nu\alpha\beta}$ are there? The third of (3.92) shows that $\overset{(1)}{R}_{\mu\nu\alpha\beta}$, which has all the symmetry properties of a Riemannian curvature tensor, can be eliminated from the count. $\overset{(3)}{R}_{\mu\nu\alpha\beta}$ is then not restricted by (3.92), and so has

$$N_3 \equiv \tfrac{1}{2}\,\frac{n(n+1)}{2}\left[\frac{n(n+1)}{2}+1\right] \quad (3.93)$$

different components. The symmetry conditions (3.91) give

$$N_2 \equiv \frac{n(n+1)}{2}\cdot\frac{n(n-1)}{2} \quad (3.94)$$

as the number of different components of $\overset{(2)}{R}_{\mu\nu\alpha\beta}$; there remains only the second of (3.92) to take into account. If $\alpha=\beta$ or $\beta=\nu$ or $\nu=\alpha$, it gives only the (known) anti-symmetry of $R^{(2)}$ in its last two indices. So ν, α, β must all be different, which totals $\frac{1}{6}n(n-1)(n-2)$ possibilities, and for each of these μ can take on any of its n values.

How many of these equations are linearly independent? If $n \leqslant 2$ there is no problem because there are then no equations of this type; so take $n \geqslant 3$. If $\mu = \nu$ the equation is:

$$R^{(2)}_{\mu\mu\alpha\beta} + R^{(2)}_{\mu\alpha\beta\mu} + R^{(2)}_{\mu\beta\mu\alpha} = 0 \qquad (3.95)$$

(no summation), and none of the three components on the LHS appears in any other of the equations, so the particular one (3.95) is certainly independent of all the others. The same is true if $\mu = \alpha$ or if $\mu = \beta$. Equations of this type total $\frac{1}{2}n(n-1)(n-2)$. The remaining equations, of which there are $N \equiv \frac{1}{6}n(n-1)(n-2)(n-3)$, will have all four indices different. Consider any particular one, together with the three others formed by cyclically permuting all four indices. Then it is readily verified that these four equations contain between them six of the components of $R^{(2)}$ each repeated twice ($4 \times 3 = 6 \times 2$), and further that none of these components figures in any of the other $(N-4)$ equations. In other words, the set of N equations splits up into $\frac{N}{4}$ disjoint subsets. But in each subset only 3 of the 4 equations are linearly independent, since the 1st minus the 2nd plus the 3rd minus the 4th gives identically zero. So, finally, the second of (3.92) amounts to $\frac{1}{2}n(n-1)(n-2) + \frac{3N}{4} = \frac{1}{8}n(n-1)(n-2)(n+1)$ <u>linearly independent</u> relations among the components of $R^{(2)}$. Subtracting this number from $N_{(3)} + N_{(2)}$, one obtains

$$\left[\tfrac{1}{2}n(n+1)\right]^2 \qquad (3.96)$$

as the number of independent components of the curvature tensor $R^{abst}_{\mu\nu\alpha\beta}$. (In a K_4, for example, there are 100.) To the author's knowledge this simple result (3.96) is not in the literature.

Just as for $R^{a\,b\,s\,t}_{\mu\nu\alpha\beta}$, there are only three distinct kinds of component of $R^{\lambda\,b\,s\,t}_{e\,\nu\alpha\beta}$, namely:

$$\left.\begin{aligned} R_{(1)\cdot\nu\alpha\beta}^{\;\;\lambda} &\equiv R^{\lambda\,1\,1\,1}_{1\,\nu\,\alpha\,\beta} = \gamma^{\lambda\mu}\, \overset{(1)}{R}_{\mu\nu\alpha\beta} - \omega^{\lambda\mu}\, \overset{(2)}{R}_{\mu\nu\alpha\beta} \\ R_{(2)\cdot\nu\alpha\beta}^{\;\;\lambda} &\equiv R^{\lambda\,1\,1\,1}_{2\,\nu\,\alpha\,\beta} = \gamma^{\lambda\mu}\, \overset{(2)}{R}_{\mu\nu\alpha\beta} + \omega^{\lambda\mu}\, \overset{(1)}{R}_{\mu\nu\alpha\beta} \\ R_{(3)\cdot\nu\alpha\beta}^{\;\;\lambda} &\equiv R^{\lambda\,2\,1\,2}_{1\,\nu\,\alpha\,\beta} = \gamma^{\lambda\mu}\, \overset{(3)}{R}_{\mu\nu\alpha\beta} + \omega^{\lambda\mu}\, \overset{(2)}{R}_{\alpha\beta\mu\nu} \end{aligned}\right\} \tag{3.97}$$

The remainder of this section will be concerned with contractions of the curvature tensor. Use the p-symbol to contract (cf. (3.17)) $R^{\mu\,b\,s\,t}_{a\,\nu\,\alpha\,\beta}$ over its first and third index-pairs, giving the two tensors:

$$B^{(r)\,b\,t}_{\;\;\;\nu\beta} \equiv p_{r\cdot s}^{\;a}\, R^{\mu\,b\,s\,t}_{a\,\nu\,\mu\,\beta} \,. \tag{3.98}$$

Contraction, instead, over the first and <u>second</u> indices does not give identically zero, as in the Riemannian case, but nevertheless yie]ds no new tensors, since

$$\begin{aligned} p_{r\cdot b}^{\;a}\, R^{\mu\,b\,s\,t}_{a\,\mu\,\alpha\,\beta} &= p_{r\cdot b}^{\;a}\, (R^{\mu\,s\,b\,t}_{a\,\alpha\,\mu\,\beta} - R^{\mu\,t\,b\,s}_{a\,\beta\,\mu\,\alpha}) \\ &= B^{(r)\,s\,t}_{\;\;\;\alpha\beta} - B^{(r)\,t\,s}_{\;\;\;\beta\alpha} \,. \end{aligned} \tag{3.99}$$

(3.98) says:

$$\left.\begin{aligned} B^{(1)\,b\,t}_{\;\;\;\nu\beta} &= R^{\mu\,b\,1\,t}_{1\,\nu\,\mu\,\beta} + R^{\mu\,b\,2\,t}_{2\,\nu\,\mu\,\beta} = R^{\mu\,b\,a\,t}_{a\,\nu\,\mu\,\beta} \\ B^{(2)\,b\,t}_{\;\;\;\nu\beta} &= R^{\mu\,b\,1\,t}_{2\,\nu\,\mu\,\beta} - R^{\mu\,b\,2\,t}_{1\,\nu\,\mu\,\beta} \end{aligned}\right\} \tag{3.100}$$

By utilizing the relations among the components of the curvature tensor which have just been derived, one finds that the quantities in (3.100) all derive from one symmetric and one anti-symmetric matrix, in the following manner:

$$\left.\begin{aligned} R_{(S)\mu\nu} &\equiv B^{(1)\,1\,1}_{\;\;\;\mu\nu} = B^{(1)\,2\,2}_{\;\;\;\mu\nu} = -\,B^{(2)\,1\,2}_{\;\;\;\mu\nu} = +\,B^{(2)\,2\,1}_{\;\;\;\mu\nu} \\ R_{(A)\mu\nu} &\equiv B^{(2)\,1\,1}_{\;\;\;\mu\nu} = B^{(2)\,2\,2}_{\;\;\;\mu\nu} = +\,B^{(1)\,1\,2}_{\;\;\;\mu\nu} = -\,B^{(1)\,2\,1}_{\;\;\;\mu\nu} \end{aligned}\right\} \tag{3.101}$$

author's knowledge this simple result (3.96) is not in the literature.

Just as for $R^{abst}_{\mu\nu\alpha\beta}$, there are only three distinct kinds of component of $R^{\lambda bst}_{e\nu\alpha\beta}$, namely:

$$\left.\begin{array}{lllll} R_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} & \equiv & R^{\lambda 1 1 1}_{1\nu\alpha\beta} & = \gamma^{\lambda\mu} R^{(1)}_{\mu\nu\alpha\beta} & - \omega^{\lambda\mu} R^{(2)}_{\mu\nu\alpha\beta} \\ R_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} & \equiv & R^{\lambda 1 1 1}_{2\nu\alpha\beta} & = \gamma^{\lambda\mu} R^{(2)}_{\mu\nu\alpha\beta} & + \omega^{\lambda\mu} R^{(1)}_{\mu\nu\alpha\beta} \\ R_{(3)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} & \equiv & R^{\lambda 2 1 2}_{1\nu\alpha\beta} & = \gamma^{\lambda\mu} R^{(3)}_{\mu\nu\alpha\beta} & + \omega^{\lambda\mu} R^{(2)}_{\alpha\beta\mu\nu} \end{array}\right\} \quad (3.97)$$

The remainder of this section will be concerned with contractions of the curvature tensor. Use the p-symbol to contract (cf. (3.17)) $R^{\mu bst}_{|a\nu\alpha\beta}$ over its first and third index-pairs, giving the two tensors:

$$B^{(r)bt}{}_{\nu\beta} \equiv p_{r}{}^{a}{}_{\cdot s}\, R^{\mu bst}_{|a\nu\mu\beta} . \quad (3.98)$$

Contraction, instead, over the first and <u>second</u> indices does not give identically zero, as in the Riemannian case, but nevertheless yields no new tensors, since

$$\begin{aligned} p_{r}{}^{a}{}_{\cdot b}\, R^{\mu bst}_{|a\mu\alpha\beta} &= p_{r}{}^{a}{}_{\cdot b}\, (R^{\mu sbt}_{|a\alpha\mu\beta} - R^{\mu tbs}_{|a\beta\mu\alpha}) \\ &= B^{(r)st}{}_{\alpha\beta} - B^{(r)ts}{}_{\beta\alpha} . \end{aligned} \quad (3.99)$$

(3.98) says:

$$\left.\begin{array}{l} B^{(1)bt}{}_{\nu\beta} = R^{\mu b1t}_{|1\nu\mu\beta} + R^{\mu b2t}_{|2\nu\mu\beta} = R^{\mu bat}_{|a\nu\mu\beta} \\ B^{(2)bt}{}_{\nu\beta} = R^{\mu b1t}_{|2\nu\mu\beta} - R^{\mu b2t}_{|1\nu\mu\beta} \end{array}\right\} \quad (3.100)$$

By utilizing the relations among the components of the curvature tensor which have just been derived, one finds that the quantities in (3.100) all derive from one symmetric and one anti-symmetric matrix, in the following manner:

$$\left.\begin{array}{l} R_{(S)\mu\nu} \equiv B^{(1)11}{}_{\mu\nu} = B^{(1)22}{}_{\mu\nu} = -B^{(2)12}{}_{\mu\nu} = +B^{(2)21}{}_{\mu\nu} \\ R_{(A)\mu\nu} \equiv B^{(2)11}{}_{\mu\nu} = B^{(2)22}{}_{\mu\nu} = +B^{(1)12}{}_{\mu\nu} = -B^{(1)21}{}_{\mu\nu} \end{array}\right\} \quad (3.101)$$

Comparison with (3.20) to (3.24) shows that $B^{(1)ab}_{\mu\nu}$ is Hermitian, $B^{(2)ab}_{\mu\nu}$ is anti-Hermitian, and that they are related by (cf. (3.24)):
$$(i \times B^{(1)})^{ab}_{\mu\nu} = - B^{(2)ab}_{\mu\nu} . \qquad (3.102)$$
So there is essentially only one contracted curvature tensor; and it is completely specified by the complex Hermitian matrix $(R_{(S)\mu\nu} + i R_{(A)\mu\nu})$ - just as $(\gamma_{\mu\nu} + i\omega_{\mu\nu})$ specifies the metric tensor.

The shortest route to an explicit formula for the contracted curvature tensor is via (3.99) which, with $r = 2$, gives:
$$B^{(2)st}_{\alpha\beta} = \tfrac{1}{2} (R^{\mu 1 st}_{\cdot 2\mu\alpha\beta} - R^{\mu 2 st}_{\cdot 1\mu\alpha\beta}) \qquad (3.103)$$
$$\left.\begin{aligned} \therefore\quad R_{(S)\mu\nu} &= B^{(2)21}_{\mu\nu} = R_{(3)\cdot\alpha\mu\nu}^{\alpha} = \Gamma^{\alpha 2 1\ 2}_{\cdot 1\alpha\mu,\nu} - \Gamma^{\alpha 2 2\ 1}_{\cdot 1\alpha\nu,\mu} \\ R_{(A)\mu\nu} &= B^{(2)11}_{\mu\nu} = R_{(2)\cdot\alpha\mu\nu}^{\alpha} = \Gamma^{\alpha 1 1\ 1}_{\cdot 2\alpha\mu,\nu} - \Gamma^{\alpha 1 1\ 1}_{\cdot 2\alpha\nu,\mu} \end{aligned}\right\} \qquad (3.104)$$
But (3.64ii) and (3.67) lead to:
$$\Gamma^{\lambda}_{(c)\cdot\lambda\sigma} = \tfrac{1}{2} (\gamma^{\lambda\mu}\gamma_{\lambda\mu,\overset{c}{\sigma}} + \omega^{\lambda\mu}\omega_{\mu\lambda,\overset{c}{\sigma}}) = \tfrac{1}{4} g^{\lambda\mu}_{ab} g^{ab}_{\lambda\mu,\overset{c}{\sigma}} = (\tfrac{1}{4}\log g)_{,\overset{c}{\sigma}} \qquad (3.105)$$
Write
$$\Psi \equiv \tfrac{1}{4} \log g = \tfrac{1}{2} \log (\det \| G_{\mu\nu} \|) \qquad (3.106)$$
Then, combining (3.104) and (3.105):
$$\left.\begin{aligned} R_{(S)\mu\nu} &= \Psi_{,\overset{1}{\mu},\overset{1}{\nu}} + \Psi_{,\overset{2}{\mu},\overset{2}{\nu}} \\ R_{(A)\mu\nu} &= \Psi_{,\overset{1}{\mu},\overset{2}{\nu}} - \Psi_{,\overset{2}{\mu},\overset{1}{\nu}} \end{aligned}\right\} \qquad (3.107)$$
The behaviour of Ψ under the analytic coordinate transformation (3.41) is worth noting. (3.40) and (3.47) imply that, in terms of the complex coordinate notation introduced in (3.70), the transformation law for $G_{\mu\nu}$ can be written:
$$G'_{\mu\nu} = \frac{\partial z^{\alpha}}{\partial z^{\mu'}} \overline{\left(\frac{\partial z^{\beta}}{\partial z^{\nu'}}\right)} G_{\alpha\beta} \qquad (3.108)$$
$$\therefore\quad \Psi' = \Psi + \tfrac{1}{2} \left[\log (\det \| \tfrac{\partial z^{\lambda}}{\partial z^{\mu'}} \|) + \log (\det \overline{\| \tfrac{\partial z^{\lambda}}{\partial z^{\mu'}} \|}) \right] \qquad (3.109)$$
If the first index-pair of $B^{(r)ab}_{\mu\nu}$ is raised, the resulting mixed tensors satisfy relations precisely similar to (3.101) -

call them (3.101') - with

$$\left.\begin{aligned} R_{(S)}{}^{\lambda}{}_{\cdot\nu} &= \gamma^{\lambda\mu} R_{(S)\mu\nu} + \omega^{\lambda\mu} R_{(A)\mu\nu} \\ R_{(A)}{}^{\lambda}{}_{\cdot\nu} &= \gamma^{\lambda\mu} R_{(A)\mu\nu} - \omega^{\lambda\mu} R_{(S)\mu\nu} \end{aligned}\right\} \qquad (3.110)$$

Defining the possible contractions of $B^{(r)\lambda b}{}_{a\nu}$ by:

$$B^{(q,r)} \equiv p_{q}{}^{a}{}_{\cdot b}\, B^{(r)\lambda b}{}_{a\lambda} \qquad (3.111)$$

one finds:

$$\left.\begin{aligned} B^{(1,1)} &= B^{(2,2)} = 2\, R_{(S)}{}^{\lambda}{}_{\cdot\lambda} \\ B^{(1,2)} &= B^{(2,1)} = 0 \end{aligned}\right\} \qquad (3.112)$$

so that there is essentially <u>one</u>, real, curvature scalar.

Various contractions of the Bianchi identities (3.87) can be made. In contrast to the Riemannian case, it is possible to get identities involving only the contracted curvature tensor by a <u>single</u> contraction (namely, over the first and second index-pairs); when written out, however, these have precisely the structure (3.66) + (3.67), and so are equivalent to a statement of the existence of <u>a</u> Ψ such that (3.107) hold; they therefore tell us nothing new. The doubly-contracted identities are a fortiori already contained in (3.107); they take the quasi-Riemannian form:

$$\left(B^{(1)\alpha b}{}_{s\nu} - \tfrac{1}{2}\,\delta^{\alpha}_{\nu}\,\delta^{b}_{s}\, B^{(1)\lambda a}{}_{a\lambda} \right)_{;\,\alpha}^{\;\;s} = 0, \qquad (3.113)$$

which can also be written:

$$\left.\begin{aligned} R_{(S)}{}^{\alpha}{}_{\cdot\nu;\alpha}^{\;1} - R_{(S)}{}^{\lambda}{}_{\cdot\lambda;\nu}^{\;1} - R_{(A)}{}^{\alpha}{}_{\cdot\nu;\alpha}^{\;2} &= 0 \\ R_{(S)}{}^{\alpha}{}_{\cdot\nu;\alpha}^{\;2} - R_{(S)}{}^{\lambda}{}_{\cdot\lambda;\nu}^{\;2} + R_{(A)}{}^{\alpha}{}_{\cdot\nu;\alpha}^{\;1} &= 0 \end{aligned}\right\} \qquad (3.113')$$

§3.6 Subspaces

This section is concerned with placing in a rather more general context the theory of analytic subspaces to which Yano's discussion[69] (cf. Chapter 2) is confined. The purpose is to emphasize and make explicit (with a proof which is the author's) the fact that choosing the imbedding functions to be analytic does not give the whole class of Kähler manifolds which are contained in the original $\mathcal{K}_n$. A theorem due to Calabi[72] which complements and is more profound than this essentially straightforward point is given at the end of the section. (It should be added that in [68] p.176 Bochner gives examples of manifolds not imbeddable complex-analytically in a flat manifold of any finite dimension; and that, from its authors' different standpoint, a related remark occurs in [71] p.534.)

We work primarily in terms of complex coordinates, defined as in (3.70). Also, only local imbedding is considered (global questions are of course much harder). Let a $\mathcal{K}_n$ be given, with coordinate system (z^{μ}_{a}). Introduce the constraints:

$$z^{\mu} = f^{\mu}(\xi^{\alpha}, \overline{\xi^{\alpha}}) \qquad (3.114)$$

where the f^{μ} are n functions of the $m<n$ complex variables ξ^{α} and of their complex conjugates $\overline{\xi^{\alpha}}$. (As in Chapter 2, let α, β .. range over 1 to m, while μ, ν .. range over 1 to n.) (3.114) defines a certain subspace of the $\mathcal{K}_n$, of real dimension 2m provided that the system of constraints has its maximum rank (cf. [67] p.75); call it S_{2m}. Under

what conditions is S_{2m} a $\mathcal{K}_m$? The question is not as it stands well posed, since the method of equipping the S_{2m} with a metric has not been specified. Let the $\mathcal{K}_n$ have basic function $\Omega(z^\mu, \overline{z^\mu})$, so that its metric is determined by the complex equation (3.73). Write

$$\Omega(f^\mu(\xi^\alpha, \overline{\xi^\alpha}), \overline{f^\mu(\xi^\alpha, \overline{\xi^\alpha})}) \equiv \hat{\Omega}(\xi^\alpha, \overline{\xi^\alpha}), \tag{3.115}$$

and consider the following possibility.

Definition I: The metric tensor for S_{2m} is

$$(\hat{g}^{st}_{\alpha\beta}) = \begin{pmatrix} \hat{\gamma}_{\alpha\beta} & \hat{\omega}_{\alpha\beta} \\ -\hat{\omega}_{\alpha\beta} & \hat{\gamma}_{\alpha\beta} \end{pmatrix} \tag{3.116}$$

where $$\hat{\gamma}_{\alpha\beta} + i\,\hat{\omega}_{\alpha\beta} \equiv \hat{G}_{\alpha\beta} = 4\frac{\partial^2 \hat{\Omega}}{\partial \xi^\alpha \partial \overline{\xi^\beta}}. \tag{3.117}$$

It is clear that, with this definition, S_{2m} is a $\mathcal{K}_m$. However, this is not in fact a very useful way of defining the metric, since if P, P' are two neighbouring points of $\mathcal{K}_n$ both lying in S_{2m} then the distance $|\overrightarrow{PP'}|$ is, by (3.79):

$$\begin{aligned} ds &= \left[g^{ab}_{\mu\nu}\, dz^\mu_a\, dz^\nu_b \right]^{1/2} = \left[G_{\mu\nu}\, dz^\mu \overline{dz^\nu} \right]^{1/2} \\ &= \left[4\frac{\partial^2 \Omega}{\partial z^\mu \partial \overline{z^\nu}} \left(\frac{\partial f^\mu}{\partial \xi^\alpha} d\xi^\alpha + \frac{\partial f^\mu}{\partial \overline{\xi^\alpha}} d\overline{\xi^\alpha} \right) \left(\frac{\partial \overline{f^\nu}}{\partial \overline{\xi^\beta}} d\overline{\xi^\beta} + \frac{\partial \overline{f^\nu}}{\partial \xi^\beta} d\xi^\beta \right) \right]^{1/2} \end{aligned} \tag{3.118}$$

qua displacement in $\mathcal{K}_n$, and:

$$\begin{aligned} \hat{ds} &= \left[\hat{g}^{st}_{\alpha\beta}\, d\xi^\alpha_s\, d\xi^\beta_t \right]^{1/2} = \left[\hat{G}_{\alpha\beta}\, d\xi^\alpha\, d\overline{\xi^\beta} \right]^{1/2} \\ &= \left[4\frac{\partial^2 \hat{\Omega}}{\partial \xi^\alpha \partial \overline{\xi^\beta}} d\xi^\alpha\, d\overline{\xi^\beta} \right]^{1/2} \end{aligned} \tag{3.119}$$

qua displacement in S_{2m}, and <u>in general $ds \neq \hat{ds}$</u>. We therefore abandon this definition, and adopt

Definition II: The metric tensor in S_{2m} is:

$$\hat{g}^{st}_{\alpha\beta} = \sum_{\mu,a} \sum_{\nu,b} \frac{\partial z^\mu_a}{\partial \xi^\alpha_s} \frac{\partial z^\nu_b}{\partial \xi^\beta_t} g^{ab}_{\mu\nu}. \tag{3.120}$$

This ensures (cf. (3.118)) that the distance $|\overrightarrow{PP'}|$ <u>is</u> the same in the two spaces: the manifold S_{2m} is said to be imbedded <u>isometrically</u> in $\mathcal{K}_n$. S_{2m} is however no longer necessarily a $\mathcal{K}_m$. If the f^μ are analytic functions of the ξ^α only, then, using the C-R equations, (3.120) can be translated into:

$$(\hat{g}^{st}_{\alpha\beta}) = \begin{pmatrix} \hat{\gamma}_{\alpha\beta} & \hat{\omega}_{\alpha\beta} \\ -\hat{\omega}_{\alpha\beta} & \hat{\gamma}_{\alpha\beta} \end{pmatrix} \qquad (3.121i)$$

where
$$\hat{\gamma}_{\alpha\beta} + i\,\hat{\omega}_{\alpha\beta} = \frac{\partial z^\mu}{\partial \xi^\alpha}\frac{\partial \bar{z}^\nu}{\partial \bar{\xi}^\beta} G_{\mu\nu} = 4\,\frac{\partial^2 \hat{\Omega}}{\partial \xi^\alpha \partial \bar{\xi}^\beta}\,, \qquad (3.121ii)$$

so that S_{2m} is a $\mathcal{K}_m$. This is the case of <u>analytic subspaces</u>, mentioned in Chapter 2, and first treated in [65] pp.335-8. However, one can readily show that for S_{2m} to be a $\mathcal{K}_m$ it is not <u>necessary</u> that the f^μ be analytic. <u>Proof.</u> Consider any $\mathcal{K}_m$, with coordinate system (ξ^α_s), and metric tensor $(\hat{g}^{st}_{\alpha\beta})$. The latter is a Riemannian metric in a real 2m-dimensional space, R_{2m} say. This R_{2m} can be imbedded isometrically in a real euclidean space, E_N say, where $N \leqslant \frac{2m(2m+1)}{2}$ (cf. [67] p.268). Let the metric tensor of the latter have as eigenvalues (+1) r times, (-1) (N-r) times. This E_N can in turn be imbedded in an $E_{N'}$ which is such that it has eigenvalues (+1) r' times, (-1) (N'-r') times, where r', (N'-r') are the smallest <u>even</u> integers not less than r, (N-r) respectively. $E_{N'}$ is a flat $\mathcal{K}_{\frac{N'}{2}}$, with basic function

$$\Omega = \frac{1}{4}\sum_{A=1}^{N'/2} \varepsilon_A \left[(dz^A_1)^2 + (dz^A_2)^2\right] \qquad (3.122i)$$

where
$$\varepsilon_A = \begin{cases} +1 & 1 \leqslant A \leqslant \frac{1}{2}r' \\ -1 & \frac{1}{2}r' < A \leqslant \frac{1}{2}N' \end{cases} \qquad (3.122ii)$$

We have, therefore, a $\mathcal{K}_m$ which is imbedded isometrically in the sense of Definition II in a $\mathcal{K}_{\frac{N'}{2}}$. Suppose the contrary of what is to be proved. Then there exists a set of $\frac{1}{2}N'$ analytic functions of the ξ^α, $F^A(\xi^\alpha)$ say, such that $\mathcal{K}_m$ is the subspace of $\mathcal{K}_{\frac{N'}{2}}$ determined by:

$$z^A = F^A(\xi^\alpha) \qquad (A = 1, 2 \,..\, \tfrac{1}{2}N') \tag{3.123}$$

Therefore the metric in $\mathcal{K}_m$ is derivable (cf. (3.121ii)) from the following basic function:

$$\hat{\Omega}(\xi^\alpha, \overline{\xi^\alpha}) = \Omega(F^A, \overline{F^A}) = \tfrac{1}{4}\sum_{A=1}^{N'/2} \varepsilon_A \left|F^A(\xi^\alpha)\right|^2 \tag{3.124}$$

But the initial $\mathcal{K}_m$ was arbitrary, so it can be chosen so as to have a basic function which is not expressible as the sum of squares of moduli of $\frac{1}{2}N'$ (which is $\leqslant m^2 + \frac{1}{2}m + 1$) analytic functions of the ξ^α, thereby giving a contradiction.

Reverting to analytic imbedding, a beautiful result of L.Calabi[72] should be mentioned. Define $\mathcal{E}_\infty$ to be the infinite-dimensional euclidean Kähler space with metric derived from a basic function of form (3.122i) but with the summation over $(-\infty, +\infty)$, and with $\varepsilon_A = \begin{cases} +1 & 1 \leqslant A < \infty \\ -1 & -\infty < A \leqslant -1 \end{cases}$

Points of $\mathcal{E}_\infty$ are those with finite norm in the sense of

$$\sum_{-\infty}^{\infty} |z^A|^2 < \infty$$

(so the $\mathcal{E}_\infty$ is a Hilbert space). He proves that any $\mathcal{K}_n$ can be imbedded isometrically and analytically in $\mathcal{E}_\infty$.
(There is also a converse result.)

§3.7 Formulae relating formalisms I and II

Write the change of variables (2.11) as:

$$\left.\begin{aligned} z^{\mu} &= z^{\mu}_{1} + i\, z^{\mu}_{2} \\ z^{\bar\mu} &= z^{\mu}_{1} - i\, z^{\mu}_{2} \end{aligned}\right\} \qquad (3.125)$$

and apply it to the quantities of Chapter 2 to convert them, using essentially the matrix $(T^{i'}{}_{j})$ of (2.13), to their components in Yano's 'real coordinate system', i.e. into the formalism of the present chapter. In the following formulae 'conj' means interchange barred and unbarred indices on the LHS and i and $-i$ on the RHS.

$$v^{\mu} = v^{\mu}_{1} + i\, v^{\mu}_{2} \quad ; \quad \text{conj.} \qquad (3.126)$$

$$v_{\mu} = \tfrac{1}{2}\, v^{1}_{\mu} + \tfrac{1}{2i}\, v^{2}_{\mu} \quad ; \quad \text{conj.} \qquad (3.127)$$

$$g_{\mu\bar\nu} = \tfrac{1}{2}\left(\gamma_{\mu\nu} + i\, \omega_{\mu\nu} \right) \quad ; \quad \text{conj.} \qquad (3.128)$$

$$g^{\mu\bar\nu} = 2\left(\gamma^{\mu\nu} + i\, \omega^{\mu\nu} \right) \quad ; \quad \text{conj.} \qquad (3.129)$$

$$\Gamma_{\bar\mu\nu\sigma} = \tfrac{1}{2}\left(\Gamma^{(1)}_{\mu\nu\sigma} - i\, \Gamma^{(2)}_{\mu\nu\sigma} \right) \quad ; \quad \text{conj.} \qquad (3.130)$$

$$\Gamma^{\lambda}{}_{\cdot\nu\sigma} = \Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\sigma} - i\, \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\sigma} \quad ; \quad \text{conj.} \qquad (3.131)$$

$$R_{\mu\bar\nu\alpha\bar\beta} = \tfrac{1}{4}\left(\overset{(1)}{R}_{\mu\nu\alpha\beta} - \overset{(3)}{R}_{\mu\nu\alpha\beta} - i\, \overset{(2)}{R}_{\mu\nu\alpha\beta} - i\, \overset{(2)}{R}_{\alpha\beta\mu\nu} \right) \; ; \quad \text{conj.} \qquad (3.132)$$

There are also the relations inverse to (3.132):

$$\left.\begin{aligned} R_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2}\left(R^{\lambda}{}_{\cdot\nu\alpha\bar\beta} + R^{\lambda}{}_{\cdot\nu\bar\alpha\beta} + R^{\bar\lambda}{}_{\cdot\bar\nu\bar\alpha\beta} + R^{\bar\lambda}{}_{\cdot\bar\nu\alpha\bar\beta} \right) \\ R_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2i}\left(R^{\lambda}{}_{\cdot\nu\alpha\bar\beta} + R^{\lambda}{}_{\cdot\nu\bar\alpha\beta} - R^{\bar\lambda}{}_{\cdot\bar\nu\bar\alpha\beta} - R^{\bar\lambda}{}_{\cdot\bar\nu\alpha\bar\beta} \right) \\ R_{(3)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2}\left(R^{\lambda}{}_{\cdot\nu\alpha\bar\beta} - R^{\lambda}{}_{\cdot\nu\bar\alpha\beta} + R^{\bar\lambda}{}_{\cdot\bar\nu\bar\alpha\beta} - R^{\bar\lambda}{}_{\cdot\bar\nu\alpha\bar\beta} \right) \end{aligned}\right\} \qquad (3.133)$$

By combining this last equation with (2.24ii) + (3.131) one can obtain the following expressions, alternative to (but also deducible from) those which result from (3.83):

$$\left.\begin{aligned} R_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2}\left(\Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{1}{\beta}} - \Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{1}{\alpha}} + \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{2}{\beta}} - \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{2}{\alpha}} \right) \\ R_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2}\left(\Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{2}{\beta}} - \Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{2}{\alpha}} - \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{1}{\beta}} + \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{1}{\alpha}} \right) \\ R_{(3)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{2}\left(\Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{1}{\beta}} + \Gamma_{(1)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{1}{\alpha}} + \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\alpha,\overset{2}{\beta}} + \Gamma_{(2)}{}^{\lambda}{}_{\cdot\nu\beta,\overset{2}{\alpha}} \right) \end{aligned}\right\} \qquad (3.134)$$

CHAPTER 4

The Real Limit Space M_n

§4.1 Definition

Let $\mathcal{K}_n$ be any Kähler space, and (z^μ_a) a coordinate system for it. Consider the manifold, M_n say, of real dimension n, defined by the constraints:

$$z^\mu_2 = 0 \qquad (4.1)$$

Let $(z^{\mu'}_a)$ be a new coordinate system for $\mathcal{K}_n$, derived from the old by the analytic transformation:

$$z^{\mu'}_1 + i z^{\mu'}_2 = f^\mu(z^\alpha_1 + i z^\alpha_2) \qquad (4.2)$$

Then the constraints:

$$z^{\mu'}_2 = 0 \qquad (4.1')$$

do not in general give the same subspace, M_n. The latter can therefore only be studied meaningfully when the coordinate transformation group in $\mathcal{K}_n$ is suitably restricted, as will now be shown.

<u>Theorem</u> (4.3): A necessary and sufficient condition for M_n to be an invariant subspace under the transformation (4.2) is:

$$f^\mu(z^\alpha) = \bar{f}^\mu(z^\alpha) \qquad (4.4)$$

(For the definition of the complex conjugate of a <u>function</u> see the remarks following (2.5).) Functions satisfying (4.4) will be called <u>real</u> analytic functions, and correspondingly (4.2) a <u>real</u> analytic coordinate transformation.

<u>Proof</u>: To prove necessity, suppose that M_n <u>is</u> invariant. Then

$$z^\mu_2 = 0 \implies z^{\mu'}_2 = 0$$

$$\therefore \; f^\mu(z^\alpha_1) = \overline{f^\mu(z^\alpha_1)} = \bar{f}^\mu(z^\alpha_1)$$

The two analytic functions f^μ and $\bar{f}^\mu$ therefore coincide when all their arguments are real. By the fundamental theorem on uniqueness of analytic continuation of functions of several complex variables (cf. [74] p.34), this entails that f^μ and $\bar{f}^\mu$ coincide everywhere in their domain of analyticity. This argument is reversible, which therefore proves sufficiency also.

For the remainder of the chapter, only real analytic coordinate transformations in $\mathcal{K}_n$ will be considered, so that M_n is well defined.

If $Q^{\alpha}_{s}\cdots{}^{\beta}_{t}\,(z^\mu_a)$ is any (tensorial or otherwise) field defined over some domain of $\mathcal{K}_n$, then the value of the quantity Q at any point of M_n will be distinguished by enclosing it in angular brackets, $\langle Q^{\alpha}_{s}\cdots{}^{\beta}_{t}\rangle$, and will be called a 'real limit' value of Q. An equation which obviously holds identically is:

$$\langle z^\mu_2\rangle \equiv 0 \tag{4.5}$$

Define:

$$x^\mu \equiv \langle z^\mu\rangle = \langle z^\mu_1\rangle \tag{4.6}$$

The coordinate system (z^μ_a) on $\mathcal{K}_n$ will be said to 'induce' the coordinate system (x^μ) on M_n. If $(x^{\mu'})$ is the coordinate system induced by $(z^{\mu'}_a)$, then:

$$x^{\mu'} = \langle z^{\mu'}\rangle = \langle f^\mu(z^\alpha)\rangle = f^\mu(z^\alpha_1) = f^\mu(x^\alpha) \tag{4.7}$$

which is a real coordinate transformation in M_n 'induced' by the real analytic transformation (4.2) in $\mathcal{K}_n$. M_n, a real n-dimensional manifold with this transformation group, will be called the <u>real limit space</u>. (A preliminary study of this space appeared early in the literature, in [65] pp.344-6. The connection with Crumeyrolle's 'sous-variété diagonale' has already been noted in §1.3.)

At any point $P = (x^\mu)$ of M_n there is a real n-dimensional tangent vector space, $V_n(P)$ say, spanned by any n linearly independent infinitessimal displacements $\overrightarrow{PP'} = (dx^\mu)$, $P' \in M_n$. There is clearly a (1-1) correspondance between the vectors $\in V_n(P)$ and those vectors $\in \dot{\mathcal{V}}_n(P)$ whose components v_λ^μ satisfy $v_2^\mu = 0$. Using this correspondance, one can define in the following way an 'induced' <u>Hermitian scalar product</u> on M_n. Let dx^μ, dy^μ be any two infinitessimal displacements in M_n, at P. Then, distinguishing this scalar product in M_n by using square brackets, (3.25) implies:

$$[dy|dx] = \langle \gamma_{\mu\nu} + i\,\omega_{\mu\nu} \rangle\, dx^\mu\, dy^\nu . \qquad (4.8)$$

If the two displacements coincide, we get the <u>line-element</u> expression for M_n :

$$ds^2 = \langle \gamma_{\mu\nu} \rangle\, dx^\mu\, dx^\nu . \qquad (4.9)$$

(4.9) shows that <u>lengths</u> are determined in M_n, as in a Riemannian space, by a real symmetric matrix. The anti-symmetric $\langle \omega_{\mu\nu} \rangle$, however, does not only enter into the geometry of M_n via its influence on curvature (see below); (4.8) shows that even when undifferentiated it will affect <u>angular</u> measure in the space.

The transformation properties of real limit values of quantities under (4.7) will now be exhibited. For any quantity Q, differentiation w.r.t. z_1^μ clearly commutes with the bracket operation:

$$\frac{\partial}{\partial z_1^\mu} \langle Q_s^\alpha \cdots {}_t^\beta \rangle = \left\langle \frac{\partial}{\partial z_1^\mu} Q_s^\alpha \cdots {}_t^\beta \right\rangle \qquad (4.10)$$

(4.5), (4.6) and the C-R equations therefore imply:

$$\left.\begin{aligned} \left\langle \frac{\partial z_1^{\mu'}}{\partial z_1^{\nu}} \right\rangle = \left\langle \frac{\partial z_2^{\mu'}}{\partial z_2^{\nu}} \right\rangle &= \frac{\partial x^{\mu'}}{\partial x^{\nu}} \\ \left\langle \frac{\partial z_2^{\mu'}}{\partial z_1^{\nu}} \right\rangle = \left\langle \frac{\partial z_1^{\mu'}}{\partial z_2^{\nu}} \right\rangle &= 0 \end{aligned}\right\} \tag{4.11}$$

A real limit value, $\langle Q^{\mu} \rangle$ say, transforming under (4.7) as:

$$\langle Q^{\mu'} \rangle = \frac{\partial x^{\mu'}}{\partial x^{\alpha}} \langle Q^{\alpha} \rangle \tag{4.12}$$

will be called a real contravariant vector field in M_n, with analogous definitions for other tensors. The metric tensor in $\mathcal{K}_n$ transforms under (4.2) as (cf. (3.39) + (3.44)):

$$g_{\mu\nu}^{a b'} = \frac{\partial z_s^{\alpha}}{\partial z_a^{\mu'}} \frac{\partial z_t^{\beta}}{\partial z_b^{\nu'}} g_{\alpha\beta}^{s t} \tag{4.13}$$

Taking real limits of both sides one finds, using (4.11), that $\langle \gamma_{\mu\nu} \rangle$ and $\langle \omega_{\mu\nu} \rangle$ each transform separately under (4.7) as real second-rank tensors. In a similar way one finds that $\langle R_{(S)\mu\nu} \rangle$, $\langle R_{(A)\mu\nu} \rangle$, and $\langle R_{(i)\cdot\nu\alpha\beta}^{\lambda} \rangle$ (i = 1, 2, 3), are all real tensors of the appropriate ranks (2 and 4). Differentiation of (4.11) and use of the C-R equations gives:

$$\left\langle \frac{\partial^2 z_2^{\alpha'}}{\partial z_2^{\mu} \partial z_2^{\nu}} \right\rangle = 0 \tag{4.14}$$

From this, and the transformation law (3.53) for the $T_{a\nu\sigma}^{\lambda b c}$, one finds that $\langle T_{(1)\cdot\nu\sigma}^{\lambda} \rangle$ transforms like a real Riemannian affine connection, and $\langle T_{(2)\cdot\nu\sigma}^{\lambda} \rangle$ as a real third-rank <u>tensor</u>; from (3.64i) the transformation behaviour of the quantities $\langle T_{\mu\nu\sigma}^{(a)} \rangle$ can also be found.

(It is perhaps worth remarking that there is no inconsistency between the tensorial character of $\langle T_{(2)\cdot\nu\sigma}^{\lambda} \rangle$ under (4.7) and the fact that all the $T_{a\nu\sigma}^{\lambda b c}$ can be 'transformed away' at any one point of $\mathcal{K}_n$; the resolution lies in the realization that to accomplish the latter it will in general be necessary to make a <u>non</u>-real analytic coordinate transformation.)

§4.2 The structure of M_n

In this section the geometrical structure of M_n will be approached from a rather different angle, thereby complementing and extending the results just obtained. (The method of analysis and the conclusions drawn do not appear to have antecedents in the literature.)

It has been assumed throughout the preceding work, usually implicitly, that the basic function $\Omega(z^\mu_a)$ is 'sufficiently' differentiable - up to fifth order, occasionally. In some neighbourhood of the real limit space there therefore exists the following Taylor-type expansion:

$$\Omega = a^{(0)} + a^{(1)}_\mu z^\mu_2 + \frac{1}{2!} a^{(2)}_{\mu\nu} z^\mu_2 z^\nu_2 + \frac{1}{3!} a^{(3)}_{\mu\nu\sigma} z^\mu_2 z^\nu_2 z^\sigma_2 \ \ldots \tag{4.15}$$

where the $a^{(\tau)}$ are functions of the $\langle z^\alpha_1 \rangle \equiv x^\alpha$, given by:

$$a^{(\tau)}_{\mu\nu\ldots\theta} = \left\langle \frac{\partial^{(\tau)}\Omega}{\partial z^\mu_2 \partial z^\nu_2 \ldots \partial z^\theta_2} \right\rangle \tag{4.16}$$

and so are totally symmetric in their indices. (3.75) implies:

$$\left.\begin{aligned} \langle \gamma_{\mu\nu} \rangle &= a^{(0)}_{,\mu,\nu} + a^{(2)}_{\mu\nu} \\ \langle \omega_{\mu\nu} \rangle &= a^{(1)}_{\nu,\mu} - a^{(1)}_{\mu,\nu} \end{aligned}\right\} \tag{4.17}$$

Now, Ω is only determinate up to the addition of the real part of an arbitrary analytic function, so that a completely equivalent basic function is (cf. (3.74)):

$$\begin{aligned} \Omega^* &= \Omega + \tfrac{1}{2}\left[f(z^\alpha) + \overline{f(z^\alpha)} \right] \\ &= \Omega + f_1(z^\alpha_1, z^\alpha_2), \end{aligned} \tag{4.18}$$

if $$f(z^\alpha) \equiv f_1(z^\alpha_1, z^\alpha_2) + i\, f_2(z^\alpha_1, z^\alpha_2). \tag{4.19}$$

By expanding f_1 in a Taylor series at $z^\alpha_2 = 0$, and using the C-R equations, it is readily established that the expansion of Ω^* (cf. (4.15)) is such that:

$$\left.\begin{aligned} a^{(0)*} &= a^{(0)} + \langle f_1 \rangle \\ a^{(1)*}_{\mu} &= a^{(1)}_{\mu} - \langle f_2 \rangle_{,\mu}^{\;\;1} \\ a^{(2)*}_{\mu\nu} &= a^{(2)}_{\mu\nu} - \langle f_1 \rangle_{,\mu,\nu}^{\;\;1\;1} \end{aligned}\right\} \qquad (4.20)$$

and so on. (It is readily verified that 'starring' the RHS of (4.17) leads to just the same metric, as it must.) Now, $\langle f_1 \rangle$ and $\langle f_2 \rangle$ are two arbitrary and independent functions of x^μ. Choosing $\langle f_1 \rangle = - a^{(0)}$ ensures that $\langle \Omega^* \rangle = 0$; such a choice of basic function will be called 'canonical'. The possibility of this choice immediately dampens any hopes that M_n might contain a scalar field suitable for 'geometrizing' a physical meson field; this is rather ironic, in view of the fact that the whole geometry is derivable from a scalar, Ω. The freedom implied by the second of (4.20) is a precise counterpart of the gauge invariance of electromagnetic theory.

We now examine more closely the nature of the coefficients up to the fourth order in the power series (4.15); so, rename these first few and (choosing a canonical basic function) write:

$$\Omega = A_\mu z_2^\mu + \tfrac{1}{2!} g_{\mu\nu} z_2^\mu z_1^\nu + \tfrac{1}{3!} S_{\mu\nu\sigma} z_2^\mu z_1^\nu z_2^\sigma + \tfrac{1}{4!} H_{\mu\nu\sigma\kappa} z_2^\mu z_1^\nu z_2^\sigma z_2^\kappa \ldots \qquad (4.21)$$

The relation of the first two coefficients to the metric tensor of M_n is (cf. (4.17)):

$$\left.\begin{aligned} \langle \gamma_{\mu\nu} \rangle &= g_{\mu\nu} \\ \langle \omega_{\mu\nu} \rangle &= A_{\nu,\mu} - A_{\mu,\nu} \end{aligned}\right\} \qquad (4.22)$$

It is a reasonable presupposition that the third and fourth coefficients in (4.21) will play a part in determining the affine connection and curvature tensors of M_n; it is the purpose of the remainder of the present section to elucidate this remark.

Under the (real) analytic coordinate transformation (4.2), inducing the real transformation (4.7) in M_n, one finds, from the defining equation (4.16), that the coefficients in (4.21) transform as follows:

$$\left.\begin{aligned}
A'_{\mu} &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}} A_{\alpha} \\
g'_{\mu\nu} &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} g_{\alpha\beta} \\
S'_{\mu\nu\sigma} &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} \frac{\partial x^{\varepsilon}}{\partial x^{\sigma'}} S_{\alpha\beta\varepsilon} - \frac{\partial x^{\alpha}}{\partial x^{\{\mu'}} \frac{\partial^2 x^{\beta}}{\partial x^{\nu'} \partial x^{\sigma'\}}} A_{\alpha,\beta} - \frac{\partial^3 x^{\alpha}}{\partial x^{\mu'} \partial x^{\nu'} \partial x^{\sigma'}} A_{\alpha} \\
H'_{\mu\nu\sigma\kappa} &= \frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\nu'}} \frac{\partial x^{\varepsilon}}{\partial x^{\sigma'}} \frac{\partial x^{\theta}}{\partial x^{\kappa'}} H_{\alpha\beta\varepsilon\theta} - \frac{\partial x^{\alpha}}{\partial x^{\{\mu'}} \frac{\partial^3 x^{\beta}}{\partial x^{\nu'} \partial x^{\sigma'} \partial x^{\kappa'\}}} g_{\alpha\beta} \\
&\quad - \left(\frac{\partial x^{\alpha}}{\partial x^{\mu'}} \frac{\partial x^{\beta}}{\partial x^{\{\nu'}} \frac{\partial^2 x^{\varepsilon}}{\partial x^{\sigma'} \partial x^{\kappa'\}}} + \frac{\partial x^{\alpha}}{\partial x^{\{\kappa'}} \frac{\partial x^{\beta}}{\partial x^{\sigma'}} \frac{\partial^2 x^{\varepsilon}}{\partial x^{\nu'\}} \partial x^{\mu'}} \right) g_{\alpha\beta,\varepsilon}
\end{aligned}\right\} \tag{4.23}$$

where the curly bracket notation, as introduced previously, indicates summation over the terms obtained by cyclic permutation of the indices. (4.23) only hold for canonical basic functions. The last two equations show the non-tensorial nature of $S_{\mu\nu\sigma}$ and $H_{\mu\nu\sigma\kappa}$. It is, however, possible to associate with each of them a closely related totally symmetric <u>tensor</u>; and these tensors, $T_{\mu\nu\sigma}$ and $K_{\mu\nu\sigma\kappa}$ say, can then be used in conjunction with A_{μ} and $g_{\mu\nu}$ and their derivatives w.r.t. the x^{μ} to provide a covariant description of the geometry of M_n <u>up to the level of the curvature tensors</u> - one would need in addition an infinite number of other tensors, of fifth and higher ranks, corresponding to the infinite power series (4.21), to characterize the complete geometry in terms of quantities accessible on M_n alone (cf. the remarks in §1.1).

In the rest of this section the $\langle\ \rangle$ signs will be omitted, since all quantities will be real limit values; correspondingly $\frac{\partial}{\partial z_1^\mu}$ (written $,\dot{\mu}$) and $\frac{\partial}{\partial x^\mu}$ (written $,\mu$ – a notation anticipated in (4.22) & (4.23)) are interchangeable; and 'tensor' will always mean tensor under M_n's real transformation group (4.7).

Introduce the following quantities formed from $g_{\mu\nu} = \langle \gamma_{\mu\nu} \rangle$ as if it was a Riemannian metric tensor.

The inverse matrix: $$g^{\lambda\mu}\, g_{\mu\nu} = \delta^\lambda_\nu \tag{4.24}$$

$$\Gamma^{(R)\lambda}_{\cdot\nu\sigma} = \tfrac{1}{2} g^{\lambda\mu} (g_{\mu\nu,\sigma} + g_{\mu\sigma,\nu} - g_{\nu\sigma,\mu}) \tag{4.25}$$

$$R^{(R)\lambda}{}_{\cdot\nu\alpha\beta} = \left[\Gamma^{(R)\lambda}_{\cdot\nu\alpha,\beta} - \Gamma^{(R)\lambda}_{\cdot\theta\alpha}\,\Gamma^{(R)\theta}_{\cdot\nu\beta}\right] - \left[\,,\alpha \leftrightarrow \beta\right] \tag{4.26}$$

We shall now find $T_{\mu\nu\sigma}$, in subsection (i), and $K_{\mu\nu\sigma\kappa}$ in (ii), the procedures in the two cases being closely parallel.

(i) By (3.75) etc. it is clear that $S_{\mu\nu\sigma}$ will only figure in (real limit) expressions which involve first derivatives of $\gamma_{\mu\nu}$ w.r.t. the z_2^α. $\Gamma^{(2)}_{\mu\nu\sigma}$ is one such:

$$\Gamma^{(2)}_{\mu\nu\sigma} = \tfrac{1}{2} (A_{\nu,\dot{\mu},\dot{\sigma}} + A_{\sigma,\dot{\mu},\dot{\nu}} - A_{\mu,\dot{\nu},\dot{\sigma}} + S_{\mu\nu\sigma}) \tag{4.27}$$

but it is not a tensor. From the transformation properties of $\Gamma_{(1)}{}^{\lambda}_{\cdot\nu\sigma}$ (cf. § 4.1) and of the 'Riemannian' object $\Gamma^{(R)\lambda}_{\cdot\nu\sigma}$ it follows that $(\Gamma_{(1)}{}^{\lambda}_{\cdot\nu\sigma} - \Gamma^{(R)\lambda}_{\cdot\nu\sigma})$ is a tensor. Therefore so is $\omega_{\mu\lambda}(\Gamma_{(1)}{}^{\lambda}_{\cdot\nu\sigma} - \Gamma^{(R)\lambda}_{\cdot\nu\sigma})$. Therefore, by (3.64i), we can introduce the following <u>tensor</u>:

$$J_{\mu\nu\sigma} \equiv \Gamma^{(2)}_{\mu\nu\sigma} - \omega_{\mu\lambda}\,\Gamma^{(R)\lambda}_{\cdot\nu\sigma} \tag{4.28}$$

J is symmetric in its last two indices, so we can define a <u>totally symmetric</u> tensor by:

$$T_{\mu\nu\sigma} \equiv J_{\{\mu\nu\sigma\}} \tag{4.29}$$

Combining (4.27) - (4.29) gives:

$$S_{\mu\nu\sigma} = \tfrac{2}{3} T_{\mu\nu\sigma} - \tfrac{1}{3} A_{\{\mu,\nu,\sigma\}} - \tfrac{2}{3} \omega_{\lambda\{\mu} \Gamma^{(R)\lambda}_{\cdot\,\nu\sigma\}} \tag{4.30}$$

Combining (4.27), (4.28) and (4.30) gives the (surprisingly simple) result:

$$J_{\mu\nu\sigma} = \tfrac{1}{3} T_{\mu\nu\sigma} + \tfrac{1}{3} (\omega_{\mu\nu|\sigma} + \omega_{\mu\sigma|\nu}) \tag{4.31}$$

where a vertical stroke signifies covariant differentiation w.r.t. the affine connection $\Gamma^{(R)\lambda}_{\cdot\,\nu\sigma}$.

In spaces for which $A_\mu \equiv 0$, $S_{\mu\nu\sigma}$ does transform as a tensor (cf. (4.23)). (4.30) shows that this tensor is just $\tfrac{2}{3} T_{\mu\nu\sigma}$, which is justification for considering T as the tensorial counterpart of S.

A remark on the form of (4.31). Its 'gauge-invariance' (the fact that the A_μ's only appear via the combination $\omega_{\mu\nu}$) is noteworthy, and is connected with the existence of just the 'right' number of derivatives of $\omega_{\mu\nu}$ w.r.t. the x^μ, in the following sense. The tensor on the LHS of (4.31) has $\tfrac{1}{2} n^2(n+1)$ (= 40 in M_4) linearly independent components, because $\Gamma^{(2)}_{\mu\nu\sigma}$ has. Any totally symmetric third-rank tensor, and therefore T, has only $\tfrac{1}{6} n(n+1)(n+2)$ (= 20 in M_4), so that some quantity other than T and with at least $\tfrac{1}{3} n(n^2-1)$ different components must also enter into the RHS. Now, the number of derivatives of the $\omega_{\mu\nu}$ w.r.t. x^μ is $\tfrac{1}{2} n^2(n-1)$, but these are connected by the 'Maxwell'-type identities:

$$\omega_{\{\mu\nu,\sigma\}} \equiv 0 \tag{4.32}$$

of which there are $\tfrac{1}{6} n(n-1)(n-2)$; so that the number of <u>linearly independent</u> derivatives of the $\omega_{\mu\nu}$ is $\tfrac{1}{3} n(n^2-1)$ (= 20 in M_4).

(ii) It is clear that $H_{\mu\nu\sigma\kappa}$ will only figure in expressions which involve second derivatives of $\gamma_{\mu\nu}$ w.r.t. the z^{α}_{2}. $T^{(2)}_{\mu\nu\sigma,\overset{2}{\kappa}}$ is one such:

$$T^{(2)}_{\mu\nu\sigma,\overset{2}{\kappa}} = \tfrac{1}{2}\left(\gamma_{\sigma\kappa,\overset{1}{\mu},\overset{1}{\nu}} + \gamma_{\nu\kappa,\overset{1}{\mu},\overset{1}{\sigma}} - \gamma_{\mu\kappa,\overset{1}{\nu},\overset{1}{\sigma}} + H_{\mu\nu\sigma\kappa}\right) \tag{4.33}$$

but it is not a tensor. Neither $R^{(1)}_{\mu\nu\sigma\kappa}$ nor $R^{(2)}_{\mu\nu\sigma\kappa}$ involve this quantity (cf. (3.90)), but from (3.84) we find:

$$\begin{aligned} R^{(3)}_{\mu\nu\sigma\kappa} = {} & T^{(2)}_{\mu\nu\sigma,\overset{2}{\kappa}} + T^{(1)}_{\mu\nu\kappa,\overset{1}{\sigma}} - T^{(1)}_{\alpha\mu\kappa}\, T_{(1)\cdot\nu\sigma}^{\alpha} \\ & - T^{(2)}_{\alpha\mu\kappa}\, T_{(2)\cdot\nu\sigma}^{\alpha} - T^{(1)}_{\alpha\mu\sigma}\, T_{(1)\cdot\nu\kappa}^{\alpha} - T^{(2)}_{\alpha\mu\sigma}\, T_{(2)\cdot\nu\kappa}^{\alpha} \end{aligned} \tag{4.34}$$

$R^{(3)}$ is symmetric in its last two (and in its first two) indices, so we can define a <u>totally symmetric</u> tensor by:

$$K_{\mu\nu\sigma\kappa} = R^{(3)}_{\mu\{\nu\sigma\kappa\}} \tag{4.35}$$

Combining (4.33) - (4.35) gives:

$$\begin{aligned} H_{\mu\nu\sigma\kappa} = {} & \tfrac{2}{3} K_{\mu\nu\sigma\kappa} - \tfrac{1}{3}\gamma_{\{\nu\sigma,\kappa\},\mu} \\ & + \tfrac{4}{3}\left(T^{(1)}_{\alpha\mu\{\kappa}\, T_{(1)\cdot\nu\sigma\}}^{\alpha} + T^{(2)}_{\alpha\mu\{\kappa}\, T_{(2)\cdot\nu\sigma\}}^{\alpha}\right) \end{aligned} \tag{4.36}$$

Combining (4.33), (4.34) and (4.36) (or alternatively, judicious use of (3.91) and (3.92)) gives:

$$R^{(3)}_{\mu\nu\sigma\kappa} = \tfrac{1}{3} K_{\mu\nu\sigma\kappa} + \tfrac{1}{3}\left(R^{(1)}_{\mu\sigma\nu\kappa} + R^{(1)}_{\mu\kappa\nu\sigma}\right) \tag{4.37}$$

In spaces for which $g_{\mu\nu} \equiv 0$, $H_{\mu\nu\sigma\kappa}$ does transform as a tensor (cf. (4.23)). (4.36) entails that this tensor is just $\tfrac{2}{3}K_{\mu\nu\sigma\kappa}$, which is justification for considering K as the tensorial counterpart of H.

A remark on the form of (4.37). The tensor on the LHS has $\tfrac{1}{8}n(n+1)(n^2+n+2)$ (= 55 in M_4) linearly independent components (cf. (3.93)). Any totally symmetric fourth-rank tensor, and therefore K, has only $\tfrac{1}{24}n(n+1)(n+2)(n+3)$ (= 35 in M_4), so that some quantity other than K and with at least $\frac{1}{12}n^2(n^2-1)$ different components must also enter into the RHS. But this is precisely the number of linearly independent components of $\overset{(1)}{R}_{\mu\nu\sigma\kappa}$ since (cf. §3.5) it has all the symmetry properties of a Riemannian curvature tensor and the latter is well known[25] to have this number of components (= 20 in M_4).

The section concludes with formulae from which all geometrical quantities up to the curvature tensors can be found in terms of the set:

Tensor	Number of components In M_n	In M_4
A_μ	n	4
$g_{\mu\nu}$	$\frac{1}{2}n(n+1)$	10
$T_{\mu\nu\sigma}$	$\frac{1}{6}n(n+1)(n+2)$	20
$K_{\mu\nu\sigma\kappa}$	$\frac{1}{24}n(n+1)(n+2)(n+3)$	35

(4.38)

(4.31) gives $J_{\mu\nu\sigma}$ in terms of these (it is independent of $K_{\mu\nu\sigma\kappa}$). From (4.28), and by use of the identities:

$$\left.\begin{aligned} \gamma^{\mu\alpha}\gamma_{\alpha\nu} + \omega^{\mu\alpha}\omega_{\alpha\nu} &= \delta^\mu_\nu \\ \omega^{\mu\alpha}\gamma_{\alpha\nu} - \gamma^{\mu\alpha}\omega_{\alpha\nu} &= 0 \end{aligned}\right\} \qquad (4.39)$$

(which are a transcription of (3.26)), one finds:

$$\left.\begin{aligned} \overset{(1)}{T}_{\mu\nu\sigma} &= \overset{(R)}{T}_{\mu\nu\sigma} \\ \overset{(2)}{T}_{\mu\nu\sigma} &= \omega_{\mu\lambda}\, T^{(R)\lambda}{}_{\cdot\nu\sigma} + J_{\mu\nu\sigma} \\ T_{(1)}{}^{\lambda}{}_{\cdot\nu\sigma} &= T^{(R)\lambda}{}_{\cdot\nu\sigma} + \omega^{\lambda\mu} J_{\mu\nu\sigma} \\ T_{(2)}{}^{\lambda}{}_{\cdot\nu\sigma} &= \gamma^{\lambda\mu} J_{\mu\nu\sigma} \end{aligned}\right\} \qquad (4.40)$$

(The fourth equation is in agreement with a result established in §4.1: $T_{(2)}{}^{\lambda}{}_{\cdot\nu\sigma}$ is a tensor.) Contraction of the third equation over (λ, ν) gives (cf. (3.105)):

$$\Gamma_{(1)\cdot\alpha\sigma}^{\alpha} = \frac{1}{2}\frac{\partial}{\partial x^{\sigma}}\left(\log\det\|\gamma_{\mu\nu} + i\,\omega_{\mu\nu}\|\right) \tag{4.41}$$

Contraction of the fourth equation gives:

$$\left.\begin{aligned} \Gamma_{(2)\cdot\alpha\sigma}^{\alpha} &= \tfrac{1}{3}T_{\sigma} - \tfrac{1}{3}j_{\sigma} \\ \text{where}\quad T_{\sigma} &\equiv \gamma^{\alpha\beta}T_{\sigma\alpha\beta} \\ \text{and}\quad j_{\sigma} &\equiv \gamma^{\alpha\beta}\omega_{\sigma\alpha|\beta} \end{aligned}\right\} \tag{4.42}$$

The following formulae for the first two curvature tensors are valid throughout $\mathcal{K}_n$ (not just in the real limit), and come from (3.83):

$$\left.\begin{aligned} R_{(1)\cdot\nu\alpha\beta}^{\lambda} &= \left[\Gamma_{(1)\cdot\nu\alpha,\beta}^{\lambda} - \Gamma_{(1)\cdot\rho\alpha}^{\lambda}\Gamma_{(1)\cdot\nu\beta}^{\rho} + \Gamma_{(2)\cdot\rho\alpha}^{\lambda}\Gamma_{(2)\cdot\nu\beta}^{\rho}\right] - \left[\alpha\leftrightarrow\beta\right] \\ R_{(2)\cdot\nu\alpha\beta}^{\lambda} &= \left[-\Gamma_{(2)\cdot\nu\alpha,\beta}^{\lambda} + \Gamma_{(1)\cdot\rho\alpha}^{\lambda}\Gamma_{(2)\cdot\nu\beta}^{\rho} + \Gamma_{(2)\cdot\rho\alpha}^{\lambda}\Gamma_{(1)\cdot\nu\beta}^{\rho}\right] - \left[\alpha\leftrightarrow\beta\right] \end{aligned}\right\} \tag{4.43}$$

Inserting (4.40), these two tensors are therefore obtainable in terms of A_{μ}, $g_{\mu\nu}$, $T_{\mu\nu\sigma}$, and their derivatives w.r.t. x^{α}. The third curvature tensor depends in addition on $K_{\mu\nu\sigma\kappa}$, and from (3.97) and (4.37) is:

$$R_{(3)\cdot\nu\alpha\beta}^{\lambda} = \tfrac{1}{3}\gamma^{\lambda\mu}K_{\mu\nu\alpha\beta} + \tfrac{1}{3}\gamma^{\lambda\mu}\left(R_{\mu\alpha\nu\beta}^{(1)} + R_{\mu\beta\nu\alpha}^{(1)}\right) + \omega^{\lambda\mu}R_{\alpha\beta\mu\nu}^{(2)} \tag{4.44}$$

Finally, the contracted curvature tensors. From (3.104) and (4.42):

$$R_{(A)\mu\nu} = \tfrac{1}{3}\left(T_{\nu,\mu} - T_{\mu,\nu}\right) - \tfrac{1}{3}\left(j_{\nu,\mu} - j_{\mu,\nu}\right) \tag{4.45}$$

while from (3.104) and (4.44), with the help of (3.92):

$$\begin{aligned} R_{(S)\mu\nu} &= \tfrac{1}{3}K_{\mu\nu} + \tfrac{2}{3}\left(R_{(1)\cdot\mu\alpha\nu}^{\alpha} - \omega^{\alpha\beta}R_{\mu\nu\alpha\beta}^{(2)}\right), \\ \text{where}\quad K_{\mu\nu} &\equiv \gamma^{\alpha\beta}K_{\mu\nu\alpha\beta}. \end{aligned} \tag{4.46}$$

It will be convenient to divide all these geometrical quantities into two classes: 'self-conjugate' (not the same as Yano's use of the term, cf. Chapter 2) and 'anti-self-conjugate'. The distinction arises as follows. Consider the change:

$$\Omega \equiv \Omega(z_1^\mu, z_2^\mu) \longrightarrow \Omega^* \equiv \Omega(z_1^\mu, -z_2^\mu) \qquad (4.47)$$

Under (4.47) a self-conjugate quantity goes into itself, whereas an anti-self-conjugate one changes sign. All the equations in this section (with the exception of the series expansion of Ω itself) will be observed to consist of sums of homogeneous terms (i.e. all self-conjugate or all anti-self-conjugate). The classification is:

Self-conjugate: $g_{\mu\nu}$, $\gamma_{\mu\nu}$, $H_{\mu\nu\sigma\kappa}$, $K_{\mu\nu\sigma\kappa}$, $\Gamma^{(R)\lambda}{}_{.\nu\sigma}$, $R^{(R)\lambda}{}_{.\nu\alpha\beta}$, $\Gamma_{(1)}{}^{\lambda}{}_{.\nu\sigma}$, $\det \| \gamma_{\mu\nu} + i\,\omega_{\mu\nu} \|$, $R_{(1)}{}^{\lambda}{}_{.\nu\alpha\beta}$, $R_{(3)}{}^{\lambda}{}_{.\nu\alpha\beta}$, $R_{(S)\mu\nu}$.

Anti-self-conjugate: A_μ , $\omega_{\mu\nu}$, $S_{\mu\nu\sigma}$, $T_{\mu\nu\sigma}$, $J_{\mu\nu\sigma}$, $\Gamma_{(2)}{}^{\lambda}{}_{.\nu\sigma}$, j_μ , $R_{(2)}{}^{\lambda}{}_{.\nu\alpha\beta}$, $R_{(A)\mu\nu}$.

It is natural to look on the latter collection as the 'electromagnetic' quantities, the former as containing the 'gravitational' (and perhaps some other) field.

§4.3 Analytic continuation

The preceding section has provided a description of M_n. In this section, and the following chapter, the problem of the relation of M_n to its 'parent' $\mathcal{K}_n$ will be investigated. From the standpoint of physical theory, the eventual aim is to discover the nature of the 'Überwelt' ($\mathcal{K}_4$) with, as data, only the known fields on space-time (M_4) (cf. §1.1). The problem is not entirely dissimilar from that faced by the cosmologist who, from data on a very 'thin' null shell, must try to reconstruct the whole universe in space and time (cf. [83] p.330). In the present case there is no precedent, and few hints as to how one should proceed; so the remainder of this work really consists only of suggestions and experiments - no solutions are claimed.

Let X_n be any given real n-dimensional Riemannian manifold, with metric tensor $g^{(R)}_{\mu\nu}$, affine connection $\Gamma^{(R)\lambda}_{\cdot\nu\sigma}$, and curvature tensor $R^{(R)\lambda}_{\cdot\nu\alpha\beta}$. Then it is possible to construct a $\mathcal{K}_n$ whose real limit space M_n has the following properties:

(i) All anti-self-conjugate quantities vanish

(ii) $\langle \gamma_{\mu\nu} \rangle = g^{(R)}_{\mu\nu}$; $\langle \gamma^{\mu\nu} \rangle = g^{(R)\mu\nu}$

(iii) $\langle \Gamma_{(1)\cdot\nu\sigma}^{\lambda} \rangle = \Gamma^{(R)\lambda}_{\cdot\nu\sigma}$

(iv) $\langle R_{(1)\cdot\nu\alpha\beta}^{\lambda} \rangle = R^{(R)\lambda}_{\cdot\nu\alpha\beta}$

This real n-dimensional manifold M_n is therefore very similar geometrically to the original X_n, but it should be noted that it supports also the fourth-rank tensor $R_{(3)\cdot\nu\alpha\beta}^{\lambda}$ (and therefore $R_{(s)\mu\nu}$), which has no counterpart in X_n. The

$\mathcal{K}_n$ will be called an analytic continuation of X_n, the name deriving from the method of construction, which will now be given.

Let (x^μ) be a coordinate system for X_n. The latter can be imbedded in a euclidean E_m $(m \leqslant \frac{1}{2}n(n+1))$ (cf. [67] p.268). Let the line-element expression for E_m be:

$$\left.\begin{aligned} ds^2 &= \sum_{k=1}^{m} \varepsilon_k (du^k)^2 \\ \varepsilon_k &= \pm 1 \end{aligned}\right\} \tag{4.48}$$

and let the imbedding be:

$$u^k = f^k(x^\mu) \tag{4.49}$$

These two equations give, as line-element for the subspace X_n:

$$ds^2 = \sum_{k=1}^{m} \varepsilon_k \frac{\partial f^k}{\partial x^\mu} \frac{\partial f^k}{\partial x^\nu} dx^\mu dx^\nu \tag{4.50}$$

so that the (real) functions f^k are solutions of:

$$\sum_k \varepsilon_k \frac{\partial f^k}{\partial x^\mu} \frac{\partial f^k}{\partial x^\nu} = g^{(R)}_{\mu\nu} \tag{4.51}$$

For each k, let $f^k(z^\mu)$ be the analytic function of n complex variables which is the analytic continuation[74] of the function $f^k(x^\mu)$ of n real variables (and restrict attention to some domain of analyticity in the neighbourhood of the real axes). We now show that the $\mathcal{K}_n$ with basic function:

$$\Omega(z^\mu_a) = \frac{1}{4} \sum_k \varepsilon_k \left| f^k(z^\mu) \right|^2 \tag{4.52}$$

satisfies conditions (i) - (iv), and so is an analytic continuation of X_n. (Being so coordinate-dependent this construction is unlikely to be unique, but the extent of its non-uniqueness has not been established.) Split the functions f^k into their real and imaginary parts:

$$f^k(z^\mu) \equiv f^k_1(z^\mu_1, z^\mu_2) + i\, f^k_2(z^\mu_1, z^\mu_2) \tag{4.53}$$

From (3.75) and (4.52) the metric in $\mathcal{K}_n$ is (using the C-R equations):

$$\left.\begin{aligned} \gamma_{\mu\nu} &= \sum_k \varepsilon_k (f^k_{1,\mu} f^k_{1,\nu} + f^k_{2,\mu} f^k_{2,\nu}) \\ \omega_{\mu\nu} &= \sum_k \varepsilon_k (f^k_{2,\mu} f^k_{1,\nu} - f^k_{1,\mu} f^k_{2,\nu}) \end{aligned}\right\} \tag{4.54}$$

But the f^k are real functions (cf. (4.4)), so:

$$\left.\begin{aligned} \langle f^k_2 \rangle &= 0 \\ \text{and therefore} \quad \langle f^k_1 \rangle &= \langle f^k \rangle = f^k(z^\mu_1) \equiv f^k(x^\mu) \end{aligned}\right\} \tag{4.55}$$

(As previously in the chapter, z^μ_1 and x^μ are used interchangeably.) So the real limit of (4.54) is, using (4.51):

$$\left.\begin{aligned} \langle \gamma_{\mu\nu} \rangle &= g^{(R)}_{\mu\nu} \\ \langle \omega_{\mu\nu} \rangle &= 0 \end{aligned}\right\} \tag{4.56}$$

The vanishing of $\langle\omega_{\mu\nu}\rangle$ entails that $\langle\gamma^{\mu\nu}\rangle$ is just the inverse of the matrix $\langle\gamma_{\mu\nu}\rangle$ (cf. (4.39)); so (ii) holds. By differentiating the first of (4.54) and using the C-R equations:

$$\left.\begin{aligned} \Gamma^{(1)}_{\mu\nu\sigma} &= \sum_k \varepsilon_k (f^k_{1,\mu} f^k_{1,\nu,\sigma} + f^k_{2,\mu} f^k_{2,\nu,\sigma}) \\ \Gamma^{(2)}_{\mu\nu\sigma} &= \sum_k \varepsilon_k (f^k_{2,\mu} f^k_{1,\nu,\sigma} - f^k_{1,\mu} f^k_{2,\nu,\sigma}) \end{aligned}\right\} \tag{4.57}$$

Taking real limits, the second equation shows that $\langle\Gamma^{(2)}_{\mu\nu\sigma}\rangle$, and therefore $\langle J_{\mu\nu\sigma}\rangle$, vanishes; (i), (iii) and (iv) are now immediately verifiable, using (4.40) and (4.43).

No <u>particular</u> value (e.g. zero) is to be expected for the third curvature tensor. One finds, from (4.34):

$$\begin{aligned} \langle R^{(3)}_{\mu\nu\sigma\kappa} \rangle = &\sum_k \varepsilon_k (\langle f^k_1 \rangle_{,\mu,\kappa} \langle f^k_1 \rangle_{,\nu,\sigma} + \langle f^k_1 \rangle_{,\mu,\sigma} \langle f^k_1 \rangle_{,\nu,\kappa}) \\ &- \Gamma^{(R)}_{\alpha\mu\kappa} \Gamma^{(R)\alpha}_{\cdot\,\nu\sigma} - \Gamma^{(R)}_{\alpha\mu\sigma} \Gamma^{(R)\alpha}_{\cdot\,\nu\kappa} \end{aligned} \tag{4.58}$$

whence $\langle K_{\mu\nu\sigma\kappa}\rangle$ can be found. There are also the relations, from (4.44) and (4.46) respectively:

$$\langle R_{(3)}{}^{\lambda}_{\cdot\,\nu\alpha\beta} \rangle = \tfrac{1}{3} g^{(R)\lambda\mu} \langle K_{\mu\nu\alpha\beta} \rangle + \tfrac{1}{3}(R^{(R)\lambda}{}_{\alpha\nu\beta} + R^{(R)\lambda}{}_{\beta\nu\alpha}) \tag{4.59}$$

$$\langle R_{(S)\mu\nu} \rangle = \tfrac{1}{3} \langle K_{\mu\nu} \rangle + \tfrac{2}{3} R^{(R)}_{\mu\nu} \tag{4.60}$$

The prescription just given enables a 'complex environment' to be constructed for any of the space-times of interest in general relativity (for example), but the resulting M_n's suffer from the disadvantage, so far as electromagnetism is concerned, that all the anti-self-conjugate quantities are zero. Their vanishing is, in fact, a direct consequence of the first of equations (4.55). In order to construct spaces which, in the real limit, have non-vanishing 'electromagnetic' fields, one could therefore give up the requirement that the f^k be real analytic functions, by (for example) allowing some of the constants entering into the expressions for $f^k(x^\mu)$ to become complex. This could be called 'generalized analytic continuation', and will be exemplified in subsections (ii) and (iii) below. But first, in (i), we present a simple instance of analytic continuation proper, yielding our first concrete example of a non-trivial (non-flat) Kähler space.

(i) 2-dimensional surface of sphere. Parametrize the surface, X_2, of a sphere of radius a in three dimensions by the polar angles (θ, ϕ). An imbedding of X_2 in E_3 is:

$$\left.\begin{aligned} x &= a \sin\theta \cos\phi \\ y &= a \sin\theta \sin\phi \\ z &= a \cos\theta \end{aligned}\right\} \tag{4.61}$$

Therefore its analytic continuation, $\mathcal{K}_2$, has basic function:

$$\begin{aligned} \Omega &= \tfrac{1}{4} a^2 (|\sin\theta \cos\phi|^2 + |\sin\theta \sin\phi|^2 + |\cos\theta|^2) \\ &= \tfrac{1}{4} a^2 (\cosh 2\theta_2 \cosh^2\phi_2 - \cos 2\theta_1 \sinh^2\phi_2) \end{aligned} \tag{4.62}$$

Treating θ as the first coordinate, ϕ the second, the metric in $\mathcal{K}_2$ is:

$$\left.\begin{aligned} \gamma_{11} &= a^2(\cosh 2\theta_2 \cosh^2\phi_2 + \cos 2\theta_1 \sinh^2\phi_2) \\ \gamma_{12} &= \gamma_{21} = \tfrac{1}{2}a^2 \sinh 2\theta_2 \sinh 2\phi_2 \\ \gamma_{22} &= a^2(\sinh^2\theta_2 + \sin^2\theta_1)\cosh 2\phi_2 \\ \omega_{12} &= -\omega_{21} = \tfrac{1}{2}a^2 \sin 2\theta_1 \sinh 2\phi_2 \end{aligned}\right\} \qquad (4.63)$$

By inspection, the real limit values are verified to be those of X_2. The $\Gamma^{\lambda bc}_{\alpha\nu\sigma}$, and thence $R^{\lambda bst}_{\alpha\nu\alpha\beta}$, can be computed from (4.63). The results are rather complicated; but in the real limit $R_{(1)}{}^{\lambda}{}_{\nu\alpha\beta}$ reduces, as it must, to the curvature tensor of X_2, while:

$$\left.\begin{aligned} \langle R_{(s)11}\rangle &= 2 & \langle R_{(s)22}\rangle &= 2\sin^2\theta_1 \\ \langle R_{(s)12}\rangle &= 0 & \langle R_{(s)}\rangle &= \frac{4}{a^2} \end{aligned}\right\} \qquad (4.64)$$

These may be compared with the Ricci tensor of X_2:

$$\left.\begin{aligned} R^{(R)}_{11} &= -1 & R^{(R)}_{22} &= -\sin^2\theta \\ R^{(R)}_{12} &= 0 & R^{(R)} &= -\frac{2}{a^2} \end{aligned}\right\} \qquad (4.65)$$

So $\langle K_{\mu\nu}\rangle$ does not vanish (cf. (4.60)).

Although X_2 is a space of constant curvature, $\mathcal{K}_2$ is not (the curvature scalar is a function of θ_1, θ_2 and ϕ_2); but in fact it could not be a space of constant curvature (see Chapter 2 for a proof).

(ii) de Sitter universe. Consider the spatially flat Robertson-Walker line-element expression ([76] p.102):

$$ds^2 = S^2(t)\left[\sum_{i=1}^{3}(dx^i)^2\right] - dt^2 \qquad (4.66)$$

These universes are imbeddable in the E_5 with line-element:

$$ds^2 = \sum_{i=1}^{3}(dx'^i)^2 + d\xi^2 - d\eta^2 \qquad (4.67)$$

by means of the constraints:

$$\left.\begin{aligned} x'^{i} &= x^{i} S(t) \\ \xi + \eta &= S(t) \\ \xi - \eta &= F(t) - \left[\sum_i (x^i)^2\right] S(t) \end{aligned}\right\} \qquad (4.68)$$

provided that $\dfrac{dF}{dt} = -1\Big/\dfrac{dS}{dt}$. (4.69)

(This imbedding is obtainable as a simple generalization of the work on pp.346-7 of [75].) The Riemannian X_4 with line-element (4.66) therefore has as analytic continuation the $\mathcal{K}_4$ with basic function:

$$\Omega = \tfrac{1}{8}(S\bar{F} + \bar{S}F) + \tfrac{1}{2} S\bar{S}\left[\sum_{i=1}^{3}(x_2^i)^2\right] \qquad (4.70)$$

where $S \equiv S(t_1 + i\, t_2)$, and similarly F.

We now specialize to the de Sitter metric, viz:

$$\left.\begin{aligned} S(t) &= e^{t/R} \\ F(t) &= R^2 e^{-t/R} \end{aligned}\right\} \qquad (4.71)$$

and (cf. (4.69)):

where R is a real constant. The corresponding $\mathcal{K}_4$ has, as non-vanishing components of the metric:

$$\left.\begin{aligned} \gamma_{ii} &= e^{2t_1/R} \quad \text{(no summation)} \\ \gamma_{44} &= -\cos\frac{2t_2}{R} + \frac{2}{R^2} e^{2t_1/R}\left[\sum_{i=1}^{3}(x_2^i)^2\right] \\ \omega_{i4} &= -\frac{2x_2^i}{R} e^{2t_1/R} \end{aligned}\right\} \qquad (4.72)$$

The real limit values are as expected. The real limit of the 'anomalous' object $R_{(s)\mu\nu}$ is as follows:

$$\langle R_{(s)ii}\rangle = \frac{2}{R^2} e^{2t_1/R} \qquad \langle R_{(s)44}\rangle = -\frac{2}{R^2} \qquad (4.73)$$

which may be compared with the (Riemannian) Ricci tensor:

$$\overset{(R)}{R}_{ii} = \frac{3}{R^2} e^{2t/R} \qquad \overset{(R)}{R}_{44} = -\frac{3}{R^2} \qquad (4.74)$$

Therefore (cf. (4.60) for this case $\langle K_{\mu\nu}\rangle = 0$.

If we make a generalized analytic continuation by taking the constant R in (4.71) to be complex then, although $\langle \omega_{\mu\nu} \rangle$ is found to no longer vanish, at the same time $\langle \gamma_{\mu\nu} \rangle$ ceases to be of the de Sitter form: a sinusoidal oscillation appears in the t_1-dependence of the real limit metric. However, the de Sitter metric can also be put, by a coordinate transformation, in the time-independent form:

$$\left.\begin{aligned} g_{ij} &= \delta_{ij} + \frac{x^i x^j}{R^2 - r^2} \\ g_{44} &= -1 + \frac{r^2}{R^2} \\ g_{i4} &= 0 \end{aligned}\right\} \tag{4.75}$$

and in this form will be amenable to the method given in (iii).

(iii) <u>Static, spherically symmetric space-times.</u> Consider the E_6:

$$ds^2 = \sum_{i=1}^{3} (dx^i)^2 + d\xi^2 - d\eta^2 \pm d\zeta^2 \tag{4.76}$$

with the constraints:

$$\left.\begin{aligned} \xi &= E(r)\cosh(at + b) \\ \eta &= E(r)\sinh(at + b) \\ \zeta &= F(r) \end{aligned}\right\} \tag{4.77}$$

where $r^2 \equiv \sum_{i=1}^{3} (x^i)^2$. There results the X_4 with metric

$$\left.\begin{aligned} g_{ij}^{(R)} &= \delta_{ij} + \left[\left(\frac{dE}{dr}\right)^2 \pm \left(\frac{dF}{dr}\right)^2\right] \frac{x^i x^j}{r^2} \\ g_{i4}^{(R)} &= 0 \\ g_{44}^{(R)} &= -E^2 a^2 \end{aligned}\right\} \tag{4.78}$$

which is equivalent to the general static spherically symmetric space-time (cf. [25] p.200). As analytic continuation there is the K_4 with basic function:

$$\Omega = \tfrac{1}{4}\left[\sum_i |x^i|^2 + |E(r)|^2 \cos(2at_2) \pm |F(r)|^2\right] \tag{4.79}$$

A generalized analytic continuation can be obtained by making the constants a and b in (4.77) complex. However, if $a_2 \neq 0$, a t_1-dependence appears in the real limit metric, so we keep a real; then (4.79) becomes generalized to:

$$\Omega = \tfrac{1}{4}\left[\sum_i |x^i|^2 + |E(r)|^2 \cos(2at_2 + 2b_2) \pm |F(r)|^2 \right] \quad (4.80)$$

(independent of b_1). If the functions E and F are chosen such that:

$$\langle \gamma_{\mu\nu} \rangle = g^{(R)}_{\mu\nu} \quad (4.81)$$

where the RHS is any particular static spherically symmetric metric in these coordinates, then (4.80) implies that at the same time there appears the following spherically symmetric 'electrostatic' field:

$$\left. \begin{aligned} \langle \omega_{ij} \rangle &= 0 \qquad (i,j = 1,2,3), \\ \langle \omega_{i4} \rangle &= \left(\frac{\tan 2b_2}{2a}\right)\left(\frac{d}{dr}\langle \gamma_{44} \rangle\right)\frac{x^i}{r} \; . \end{aligned} \right\} \quad (4.82)$$

In the case of the <u>Schwarzschild metric</u> (i.e. $\langle \gamma_{44} \rangle = -1 + \frac{2m}{r}$) the second of (4.82) becomes:

$$\langle \omega_{i4} \rangle = -\left(\frac{m \tan 2b_2}{a}\right)\frac{x^i}{r^3} \quad (4.83)$$

which suggests that this <u>displacement of the whole Schwarzschild space-time in the imaginary t-direction</u> through a distance ($\frac{b_2}{a}$) has caused the particle at the origin to acquire a 'charge' of amount $-\left(\frac{m \cdot \tan 2b_2}{a}\right)$.

In the case of the <u>de Sitter metric</u> (cf. (4.75):

$$\left. \begin{aligned} E(r) &= (R^2 - r^2)^{1/2} \\ F(r) &= 0 \\ a &= \frac{1}{R} \end{aligned} \right\} \quad (4.84)$$

Substitution in (4.80) yields the following real limit metric:

$$\left. \begin{aligned} \langle \gamma_{\mu\nu} \rangle &= \text{as in (4.75)} \\ \langle \omega_{ij} \rangle &= 0 \\ \langle \omega_{i4} \rangle &= \left(\frac{\tan 2b_2}{R}\right) x^i \end{aligned} \right\} \quad (4.85)$$

which suggests an interpretation in terms of a uniform charge density $\rho = \frac{3}{4\pi} \frac{\tan 2b_2}{R}$, creating a radial electrostatic field proportional to distance from the origin. One finds for the real limit of the contracted curvature tensor the following values:

$$
\left.\begin{aligned}
\langle R_{(s)ij} \rangle &= \frac{2}{\varpi}\left(\delta_{ij} + \frac{x^i x^j}{R^2 - r^2} \right) - \frac{2R^2 \sin^2 2b_2}{\varpi^2} \frac{x^i x^j}{R^2 - r^2} \\
\langle R_{(s)i4} \rangle &= 0 \\
\langle R_{(s)44} \rangle &= - \frac{2\cos^2 2b_2}{\varpi}\left(1 - \frac{r^2}{R^2} \right) \\
\langle R_{(A)ij} \rangle &= 0 \\
\langle R_{(A)i4} \rangle &= \frac{\sin 4b_2}{\varpi^2} \frac{r^2}{R} x^i
\end{aligned}\right\} \qquad (4.86)
$$

where $\varpi \equiv R^2 \cos^2 2b_2 + r^2 \sin^2 2b_2$.

CHAPTER 5

Field Equations for the Metric

The field equations in (say) the Einstein UFT fall naturally into two groups: the first connecting the metric with the affine connection, the second being a restriction of some kind on the curvature tensor(s). In the present context, the first group is straightforward; the second is very problematic and no definite conclusions are reached.

The first group of field equations is (cf. (3.57) etc.):

$$\left.\begin{aligned} g^{ab\ c}_{\mu\nu;\sigma} &= 0 \\ \Gamma^{\lambda 1 c}_{1\nu\sigma} = \Gamma^{\lambda 2 c}_{2\nu\sigma} &= \Gamma^{\lambda c 2}_{2\sigma\nu} \ , \\ \Gamma^{\lambda 2 c}_{1\nu\sigma} = -\Gamma^{\lambda 1 c}_{2\nu\sigma} &= -\Gamma^{\lambda c 1}_{2\sigma\nu} \ . \end{aligned}\right\} \tag{5.1}$$

For $c = 1$, the first of (5.1) implies, using the second and third, two equations which can be combined into the single complex equation:

$$G_{\mu\nu,\sigma} - G_{\alpha\nu}\left(\Gamma_{(1)}{}^{\alpha}{}_{\cdot\mu\sigma} - i\,\Gamma_{(2)}{}^{\alpha}{}_{\cdot\mu\sigma}\right) - G_{\mu\alpha}\left(\Gamma_{(1)}{}^{\alpha}{}_{\cdot\nu\sigma} + i\,\Gamma_{(2)}{}^{\alpha}{}_{\cdot\nu\sigma}\right) = 0 \tag{5.2}$$

At any point P of M_n (in particular) this equation relates the derivatives w.r.t. the x^σ of the Hermitian matrix $G_{\mu\nu}$ to the affine connection components $\Gamma_{(a)}{}^{\lambda}{}_{\cdot\nu\sigma}$ at P.

Compare (the real limit of) (5.2) with the corresponding equation of Einstein's UFT (cf. (1.98)). They are similar in form except that Einstein's Γ's are Hermitian, the present ones complex symmetric.

Compare (5.2) with the corresponding equation of Moffat's UFT (cf. (1.118)). They are similar in form except that

(a) his $g_{\mu\nu}$ are complex symmetric, the present $G_{\mu\nu}$ Hermitian, and (b) his (in contrast to Einstein's and the present theory's) covariant derivatives are formed solely with the $\Gamma^{\lambda}_{\mu\nu}$ of (1.119), the complex conjugate quantities playing no part.

Compare (5.2) with the corresponding equation of Crumeyrolle's theory (the first of (1.140)). Given the difference in the underlying number field the two equations are formally the same. However, his choice of metric tensor on W_4 is <u>not</u> analogous to the present theory's, since (1.139) is stated in 'repères associés', and in this coordinate system our equivalent of (1.139) is (cf. (3.20)):

$$\left.\begin{array}{llll} g_{\alpha\beta} = \gamma_{\alpha\beta} & & g_{\alpha\beta^*} = \omega_{\alpha\beta} \\ g_{\alpha^*\beta} = \omega_{\beta\alpha} & & g_{\alpha^*\beta^*} = \gamma_{\alpha\beta} \end{array}\right\} \tag{5.3}$$

Only if (1.139) was to be read as holding in 'repères adaptés' would his g_{ij} on W_4 be (the real limit of) a Kähler-type metric.

A result which follows from (5.1) is (cf. (3.66)):

$$\omega_{\{\mu\nu,\dot{\sigma}\}} = 0 \tag{5.4}$$

Like (5.2) this holds throughout $\mathcal{K}_n$, and therefore also on M_n. (5.4) does not seem to have counterparts in the theories just mentioned (see, in this connection, [28] p.737).

If the last two equations of (4.40) are inserted into (the real limit of) (5.2), then the real and imaginary parts of the latter just reduce to the known results:

$$\left.\begin{array}{r} \gamma_{\mu\nu|\sigma} = 0 \\ \omega_{\mu\nu|\sigma} + J_{\nu\mu\sigma} - J_{\mu\nu\sigma} = 0 \end{array}\right\} \tag{5.5}$$

We now derive an identity which implies the existence of a conserved ('current') vector field in M_n. The contravariant form of the first of (5.1) is:

$$g^{\mu\nu}_{ab};^{c}_{\sigma} = 0 \tag{5.6}$$

The equation with $a = 2$, $b = c = 1$, and contracted over (ν,σ) is:

$$\omega^{\mu\nu}{}_{,\nu} + \omega^{\mu\alpha}\Gamma_{(1)}{}^{\nu}{}_{\cdot\alpha\nu} + \gamma^{\mu\alpha}\Gamma_{(2)}{}^{\nu}{}_{\cdot\alpha\nu} - \gamma^{\alpha\nu}\Gamma_{(2)}{}^{\mu}{}_{\cdot\alpha\nu} = 0 \tag{5.7}$$

Using (3.105), this can be written:

$$\frac{1}{\sqrt{-G}}\left(\sqrt{-G}\,\omega^{\mu\nu}\right)_{,\nu} = -\gamma^{\mu\alpha}\Gamma_{(2)}{}^{\nu}{}_{\cdot\alpha\nu} + \gamma^{\alpha\nu}\Gamma_{(2)}{}^{\mu}{}_{\cdot\alpha\nu} \tag{5.7'}$$

Take real limits of both sides, and use (4.40), (4.31) & (5.4):

$$\begin{aligned}\frac{1}{\langle\sqrt{-G}\rangle}\langle\sqrt{-G}\,\omega^{\mu\nu}\rangle_{,\nu} &= \langle(-\gamma^{\mu\alpha}\gamma^{\nu\beta} + \gamma^{\alpha\nu}\gamma^{\mu\beta})\,J_{\beta\alpha\nu}\rangle \\ &= \langle\gamma^{\mu\alpha}\gamma^{\nu\beta}(J_{\beta\alpha\nu} - J_{\alpha\beta\nu})\rangle \\ &= \langle\gamma^{\mu\alpha}\gamma^{\nu\beta}\omega_{\alpha\beta|\nu}\rangle = \langle\gamma^{\mu\alpha}j_{\alpha}\rangle\end{aligned} \tag{5.8}$$

Define $\quad \langle j^{\mu}\rangle \equiv \langle\gamma^{\mu\alpha}j_{\alpha}\rangle \qquad (5.9)$

Then the anti-symmetry of $\omega^{\mu\nu}$ implies that

$$\langle\sqrt{-G}\,j^{\mu}\rangle_{,\mu} = 0 \tag{5.10}$$

So it would be quite natural to identify $\langle\sqrt{-G}\;j^{\mu}\rangle$ as being proportional to the physical electric current vector density on M_4, and correspondingly $\int_D j^0\sqrt{-G}\;d^3x^i$ as the total charge contained in the region D of M_4.

We turn now to the second group of field equations, those involving the curvature tensors. Concerning the status of field equations of this kind, the philosophy adopted here is the following. A possible formulation of the relativistic theory of gravitation (cf. the EIH approach) is to start from a general Riemannian manifold, X_4 say, and then require that

$$R_{\mu\nu} = 0 \tag{5.11}$$

<u>almost everywhere</u>, i.e. with the exception of isolated singularities, which are to be identified with material particles. This is a considerable specialization of the X_4. Mutatis mutandis, we adopt the same viewpoint here, which is a less ambitious one than requiring the field equations to hold everywhere, with globally non-singular solutions (cf. e.g. [28] p. 737). The remainder of the chapter is concerned with investigating various analogues of (5.11), the goal being to find a set of equations for the various geometrical quantities on $\mathcal{K}_4$ and M_4 which is complete, compatible, and at the same time has solutions which may be expected to exhibit the correct 'physical' behaviour (general relativity, as a limiting case, is a useful guide here). (This is of course the goal of all classical UFT's.)

The strongest condition on the metric of $\mathcal{K}_4$ would be:

$$R^{abst}_{\mu\nu\alpha\beta} = 0 \tag{5.12}$$

Such a $\mathcal{K}_4$ is euclidean, so there is a coordinate system in which its basic function is:

$$\Omega = \frac{1}{4}\sum_{\alpha=1}^{4} \varepsilon_\alpha \left|z^\alpha\right|^2 \tag{5.13}$$

where $\varepsilon_\alpha = \pm 1$. In an arbitrary (allowable) coordinate system its basic function is of the form:

$$\Omega = \frac{1}{4}\sum_{\alpha=1}^{4} \varepsilon_\alpha \left|f^\alpha(z^\beta)\right|^2 \tag{5.14}$$

With the notation of (4.53), the metric tensor which follows from (5.14) is (cf. (4.54)):

$$\left.\begin{aligned} \gamma_{\mu\nu} &= \sum_\alpha \varepsilon_\alpha (f^\alpha_{1,\mu}{}^{\prime}\, f^\alpha_{1,\nu}{}^{\prime} + f^\alpha_{2,\mu}{}^{\prime}\, f^\alpha_{2,\nu}{}^{\prime}) \\ \omega_{\mu\nu} &= \sum_\alpha \varepsilon_\alpha (f^\alpha_{2,\mu}{}^{\prime}\, f^\alpha_{1,\nu}{}^{\prime} - f^\alpha_{1,\mu}{}^{\prime}\, f^\alpha_{2,\nu}{}^{\prime}) \end{aligned}\right\} \tag{5.15}$$

Now consider M_4. Suppose the functional determinant of the $f_1^{\alpha}(x^{\beta})$ is non-zero, so that

$$x^{\alpha'} = f_1^{\alpha}(x^{\beta}) \tag{5.16}$$

represents an allowable coordinate transformation on M_4. Treating the $f_2^{\alpha}(x^{\beta})$ just as four scalar functions of the x^{β} (or of the $x^{\beta'}$), write:

$$\phi_{\alpha} \equiv \eta_{\alpha\beta} f_2^{\beta} \tag{5.17}$$

where $\eta_{\alpha\beta} \equiv \varepsilon_{\alpha} \delta_{\alpha\beta}$ (no summation). Then <u>in the dashed coordinate system</u>:

$$\left.\begin{aligned} \langle \gamma'_{\mu\nu} \rangle &= \eta_{\mu\nu} + \eta^{\alpha\beta} \phi_{\alpha,\mu'} \phi_{\beta,\nu'} \\ \langle \omega'_{\mu\nu} \rangle &= \phi_{\nu,\mu'} - \phi_{\mu,\nu'} \\ \langle T'_{(1)\mu\nu\sigma} \rangle &= \eta^{\alpha\beta} \phi_{\alpha,\mu'} \phi_{\beta,\nu';\sigma'} \\ \langle T'_{(2)\mu\nu\sigma} \rangle &= - \phi_{\mu,\nu';\sigma'} \end{aligned}\right\}. \tag{5.18}$$

(These are not tensor equations; they only give the <u>values</u> of these quantities in the particular coordinate system $(x^{\mu'})$ for M_4.) As for curvature tensors, (5.12) of course implies:

$$\left.\begin{aligned} \langle R_{(i)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} \rangle &= 0 \qquad (i = 1,2,3) \\ \text{and therefore} \quad \langle K_{\mu\nu\sigma\kappa} \rangle &= 0 \end{aligned}\right\} \tag{5.19}$$

On the other hand, considered as a Riemannian space with metric tensor $\langle \gamma_{\mu\nu} \rangle$ (cf. (4.9)), M_4 is not flat, i.e. $R^{(R)\lambda}{}_{\cdot\nu\alpha\beta}$, which can be computed from the first of (5.18) as a function of the ϕ_{α}, is non-vanishing. $\langle T_{\mu\nu\sigma} \rangle$, which can be found from (5.18), is also non-zero.

Having characterized the (rather restricted) class of M_4's which are compatible with the field equation (5.12) for $\mathcal{K}_4$, we next consider a less restrictive condition than the latter, namely, one derivable from a variational principle of the form:

$$\delta \int_{\mathcal{K}_4} L \sqrt{g} \quad d^8 z^{\mu}_{a} = 0 \qquad (5.20)$$

where $\quad g \equiv \det \| g^{ab}_{\mu\nu} \| = \left[\det \| G_{\mu\nu} \| \right]^2 \equiv G^2$, say $\qquad (5.21)$

and L is some function of the $g^{ab}_{\mu\nu}$ and their derivatives w.r.t. the z^{μ}_{a}. Although other possibilities have been considered, we discuss here only the consequences resulting from the simplest choice for L (and the one most closely analogous to the general relativity Lagrangian for the free gravitational field), namely the curvature scalar of $\mathcal{K}_4$:

$$L = B^{(1,1)} = B^{(1)\mu\ a}_{\ \ a\mu} = 2\, R_{(S)}{}^{\mu}{}_{.\mu} = 2(\gamma^{\mu\nu} R_{(S)\mu\nu} - \omega^{\mu\nu} R_{(A)\mu\nu}) \qquad (5.22)$$

The Euler-Lagrange equations are:

$$R_{(S)\mu\nu} = 0 \qquad (5.23i)$$

$$R_{(A)\mu\nu} = 0 \qquad (5.23ii)$$

Although prima facie 16 different equations these can in fact, using (3.106) + (3.107), immediately be integrated twice to give the single equation:

$$G = e^{\frac{1}{2}\left[f(z^{\mu}) + \overline{f(z^{\mu})} \right]} \qquad (5.24)$$

where f is an arbitrary analytic function. Since, by (3.75), $G_{\mu\nu}$, and therefore G, is a function of Ω and its deriv- atives, (5.24) is a non-linear differential equation for Ω. It might be thought that the only solutions are the flat-space ones, (5.12). That this is not so will be demonstrated by explicitly constructing two non-flat spaces satisfying (5.23), the first a $\mathcal{K}_2$, the second a $\mathcal{K}_4$.

Consider the $\mathcal{K}_2$ defined by the basic function:

$$\Omega = x_1 \log \left[\frac{x_1}{\cos(\alpha t_1 + \beta t_2)} \right] \qquad (5.25)$$

Its metric is (with $z^0 \equiv t$, $z^1 \equiv x$):

$$\left.\begin{aligned} \gamma_{00} &= (\alpha^2+\beta^2)\, x_1 \sec^2(\alpha t_1 + \beta t_2) \\ \gamma_{01} &= \gamma_{10} = \alpha \tan(\alpha t_1 + \beta t_2) \\ \gamma_{11} &= 1/x_1 \\ \omega_{01} &= -\omega_{10} = -\beta \tan(\alpha t_1 + \beta t_2) . \end{aligned}\right\} \qquad (5.26)$$

Therefore $G = (\alpha^2+\beta^2) = \text{constant}$ (5.27)

so that (5.24) is clearly satisfied. By direct calculation it is found that all the curvature tensor components $R_{(i)}{}^{\lambda}{}_{\cdot\nu\alpha\beta}$ are non-zero (with the exception of $R_{(3)}{}^{1}{}_{\cdot 011}$), so it is certainly not a flat Kähler space. If $\beta = 0$, all anti-self-conjugate quantities are found to vanish. This could be forseen from the form of (5.25): Ω is then independent of the z^{μ}_{2}, so that the power series (4.15) consists only of the first term.

As the second example, consider the following $\mathcal{K}_4$, which could be called a 'spatially isotropic' complex space-time:

$$\left.\begin{aligned} \Omega &= \tfrac{1}{4}|t|^2 - \tfrac{1}{4} f(\sigma) \\ \text{where} \quad \sigma &\equiv \sum_{i=1}^{3} |x^i|^2 \end{aligned}\right\} \qquad (5.28)$$

and f is some real function. The metric is (with $z^0 \equiv t$):

$$\left.\begin{aligned} G_{00} &= 1 \\ G_{0i} &= 0 \\ G_{ij} &= -f'\delta_{ij} - f'' \overline{x^i} x^j \end{aligned}\right\} \qquad (5.29)$$

where $f' \equiv \frac{df}{d\sigma}$. This matrix has determinant:

$$-G = (f')^3 + (f')^2 f'' \sigma \qquad (5.30)$$

(5.24) will certainly be satisfied if we make

$$(f')^3 + (f')^2 f'' \sigma = \text{constant} = 1, \text{ say} \qquad (5.31)$$

This equation integrates to:

$$f' = \left[1 + \left(\frac{\sigma_0}{\sigma}\right)^3\right]^{1/3}$$
$$\therefore \quad f'' = -\frac{\sigma_0^{\,3}}{\sigma^4}\left[1 + \left(\frac{\sigma_0}{\sigma}\right)^3\right]^{-2/3} \tag{5.32}$$

Using for the moment the notation of Chapter 2, one finds that the curvature tensor components $R^{\lambda}{}_{\nu\sigma\kappa}$ vanish unless all the indices are space-like, in which case:

$$R^{i}{}_{jk\ell} = \left(\frac{f''}{f'}\right)\left[\delta_{\ell k}\delta_{ij} + \delta_{\ell j}\delta_{ik} - \frac{4}{\sigma}\overline{x^i}(x^k\delta_{\ell j} + x^j\delta_{\ell k}) + \frac{4}{\sigma^2}\overline{x^\ell}\,\overline{x^i}\,x^j x^k\right]$$
$$+ \left(\frac{f''}{f'}\right)'\left[\overline{x^\ell}(x^k\delta_{ij} + x^j\delta_{ik}) - \frac{4}{\sigma}\overline{x^\ell}\,\overline{x^i}\,x^j x^k\right] \tag{5.33}$$

Having shown that the equations (5.23) possess non-trivial solutions in $\mathcal{K}_n$, we now look at what they imply about M_n. For the rest of the chapter all quantities will be real limit values, so we henceforth omit the $\langle\ \rangle$ signs. Using (4.46) and (4.45), (5.23) imply:

$$R_{(1)}{}^{\alpha}{}_{\mu\alpha\nu} - \omega^{\alpha\beta}R^{(2)}_{\mu\nu\alpha\beta} = -\tfrac{1}{2}K_{\mu\nu} \tag{5.34i}$$
$$j_{\nu,\mu} - j_{\mu,\nu} = T_{\nu,\mu} - T_{\mu,\nu} \tag{5.34ii}$$

Consider the latter first. Writing out its LHS, one obtains the following propagation equation for $\omega_{\mu\nu}$:

$$\gamma^{\alpha\beta}\omega_{\mu\nu|\alpha|\beta} + \omega_{\mu\rho}(\gamma^{\alpha\beta}R^{(R)\rho}{}_{\cdot\beta\nu\alpha}) - \omega_{\nu\rho}(\gamma^{\alpha\beta}R^{(R)\rho}{}_{\cdot\beta\mu\alpha})$$
$$+ \omega^{\alpha\beta}R^{(R)}_{\mu\nu\beta\alpha} - \gamma^{\alpha\beta}{}_{|\mu}\,\omega_{\nu\alpha|\beta} + \gamma^{\alpha\beta}{}_{|\nu}\,\omega_{\mu\alpha|\beta} = T_{\mu,\nu} - T_{\nu,\mu} \tag{5.35}$$

This may be compared with the propagation equation for the physical electromagnetic field $F_{\mu\nu}$ in general-relativistic Maxwell-Lorentz electrodynamics (cf. [81] p.176):

$$g^{(R)\alpha\beta}F_{\mu\nu|\alpha|\beta} + F_{\mu\rho}(g^{(R)\alpha\beta}R^{(R)\rho}{}_{\cdot\beta\nu\alpha}) - F_{\nu\rho}(g^{(R)\alpha\beta}R^{(R)\rho}{}_{\cdot\beta\mu\alpha})$$
$$+ F^{\alpha\beta}R^{(R)}_{\mu\nu\beta\alpha} = 4\pi(J_{\mu,\nu} - J_{\nu,\mu}) \tag{5.36}$$

where J_μ is the physical current vector (cf. (1.4)). The close similarity between the last two equations suggests that T_μ should be correlated with J_μ, and so <u>should vanish in</u>

<u>the absence of charged matter</u>.

$R^{(R)}_{\mu\nu}$ (5.34ii) are clearly equivalent to:

$$j_\mu = T_\mu + \Sigma_{,\mu} \tag{5.37}$$

where Σ is an arbitrary scalar function. (There is a certain similarity here with Schrödinger's theory (cf. [82] p.21).)

Turn now to (5.34i). Consider first the case of a 'self-conjugate' or 'non-electromagnetic' space-time, defined as an M_4 on which all anti-self-conjugate quantities vanish. Then:

$$\left.\begin{aligned}
\gamma^{\lambda\mu} &= g^{(R)\lambda\mu} \\
T_{(1)}{}^{\lambda}{}_{\cdot\nu\sigma} &= T^{(R)\lambda}{}_{\cdot\nu\sigma} \\
R_{(1)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= R^{(R)\lambda}{}_{\cdot\nu\alpha\beta} \\
R_{(3)}{}^{\lambda}{}_{\cdot\nu\alpha\beta} &= \tfrac{1}{3}\, g^{(R)\lambda\mu} K_{\mu\nu\alpha\beta} + \tfrac{1}{3}\,(R^{(R)\lambda}{}_{\cdot\alpha\nu\beta} \dot{+} R^{(R)\lambda}{}_{:\beta\nu\alpha}) \\
R_{(s)\mu\nu} &= \tfrac{1}{3} K_{\mu\nu} + \tfrac{2}{3} R^{(R)}_{\mu\nu}
\end{aligned}\right\} \tag{5.38}$$

So (5.34i) reduces to:

$$R^{(R)}_{\mu\nu} = -\tfrac{1}{2} K_{\mu\nu} \tag{5.39}$$

Comparison with the corresponding general relativity equation suggests that $K_{\mu\nu}$ should be correlated with the physical energy-momentum tensor $T_{\mu\nu}$ (cf. (1.2)), and so <u>should vanish in the absence of matter</u>. We shall assume this correlation even when anti-self-conjugate fields are not absent.

With these identifications in mind, we look at the case of a 'source-free' space-time, defined as one satisfying:

$$T_\mu = 0 \tag{5.40i}$$

$$K_{\mu\nu} = 0 \tag{5.40ii}$$

almost everywhere. Now, the 10 equations (5.34i) (with $K_{\mu\nu} = 0$) involve not only the $\gamma_{\mu\nu}$ but also the 20 components

$T_{\mu\nu\sigma}$, as may be seen by writing it more explicitly as:

$$R^{(R)}_{\mu\nu} + \tfrac{1}{2}\omega^{\alpha\beta}(\omega_{\mu\lambda}\, R^{(R)\lambda}{}_{\nu\alpha\beta} + \omega_{\nu\lambda}\, R^{(R)\lambda}{}_{\mu\alpha\beta}) - \tfrac{1}{4}\omega^{\alpha\beta}(\omega_{\alpha\beta|\mu|\nu} + \omega_{\alpha\beta|\nu|\mu})$$
$$+ \omega^{\alpha\beta}\, T_{\mu\nu\alpha|\beta} + (\gamma^{\rho\sigma}\gamma^{\alpha\beta} + 3\,\omega^{\rho\sigma}\omega^{\alpha\beta})(J_{\rho\mu\nu}\, J_{\sigma\alpha\beta} - J_{\rho\mu\alpha}\, J_{\sigma\nu\beta}) = 0 \tag{5.41}$$

in which $T_{\mu\nu\sigma}$ enters also via the $J_{\mu\nu\sigma}$ (see (4.31)). So it is necessary to supply field equations for the $T_{\mu\nu\sigma}$ also; and the most natural choice is to replace (5.40i) by the stronger requirement:

$$T_{\mu\nu\sigma} = 0 \tag{5.42}$$

The souce-free M_4 would now be characterized by the following set of equations:

$$j_{\nu,\mu} - j_{\mu,\nu} = 0 \tag{5.43i}$$
$$R^{(R)}_{\mu\nu} + E_{\mu\nu} = 0 \tag{5.43ii}$$
$$T_{\mu\nu\sigma} = 0 \tag{5.43iii}$$

where the second equation is an abbreviation for (5.41) (with $T_{\mu\nu\sigma} = 0$). There are the same number of equations as unknowns, which is 4 too many: being covariant under the real transformation group, the LHS's should, for the usual reasons, satisfy four identities; but no such identities exist. (The situation is similar to that faced by Einstein and Straus when, in an early version of their theory, they tried to justify $R_{ik} = 0$ as field equation rather than the weaker (1.114ii) + (1.114iii).) The nature of the difficulty can be pin-pointed by noting that the first term in (5.43ii) satisfies the Bianchi identities so that, for consistency, so must $E_{\mu\nu}$; but explicit calculation shows that it does not in general, the condition that it should amounting to 4 equations in the $\omega_{\mu\nu}$, which, when (5.43i) is also taken into account, one has no reason to expect can be satisfied other than by:

$$\omega_{\mu\nu} \equiv 0 \ ;$$

so one is left with a solution manifold which is merely that of the empty-space field equations (5.11) of general relativity.

No satisfactory way of weakening, or of making consistent, the set (5.43) has been found in the context of the present approach. We therefore go on to consider field equations for M_4 derived by means of a variational principle <u>on M_4</u> ; this has the advantage of guaranteeing the existence of the required four identities among the equations, but the very considerable (in the author's opinion) conceptual disadvantage of leaving the 'parent' $\mathcal{K}_4$ out of the picture.

Consider the variational principle:

$$\delta \int_{M_4} L \sqrt{-G}\, d^4x \equiv \int_{M_4} \left(T^{\mu\nu} \delta \gamma_{\mu\nu} - \Omega^{\mu\nu} \delta \omega_{\mu\nu} \right) \sqrt{-G}\, d^4x = 0 , \qquad (5.44)$$

L being a function of the metric on M_4 and of its derivatives w.r.t. the x^μ, and the Euler-Lagrange equations being:

$$T^{\mu\nu} = 0 \qquad (5.45\text{i})$$

$$\Omega^{\mu\nu} = 0 \qquad (5.45\text{ii})$$

Using the standard technique (see e.g. [3] §23) for deriving conservation laws by making infinitessimal transformations of the group(s) under which the Lagrangian is invariant, in this case real coordinate transformations in M_4 , one can show that the LHS's of (5.45) are connected by the following four identities:

$$\gamma_{\mu\beta} T^{\beta\alpha}{}_{|\alpha} + \omega_{\mu\beta} \Omega^{\beta\alpha}{}_{|\alpha} - \tfrac{1}{2}\left(\gamma_{\mu\beta} T^{\beta\epsilon} + \omega_{\mu\beta} \Omega^{\beta\epsilon} \right) \omega^{\rho\sigma} \omega_{\rho\sigma|\epsilon} = 0 \qquad (5.46)$$

where (4.40) and (5.4) have been used. In subsections (i) - (iii) we shall investigate three possible Lagrangians, restricting attention throughout to the source-free case (see above), i.e:

$$T_{\mu\nu\sigma} = K_{\mu\nu} = 0 \qquad (5.47)$$

so that both $R_{(S)\mu\nu}$ and $R_{(A)\mu\nu}$ are functions only of the <u>metric</u> and its derivatives w.r.t. x^μ.

(i) At first sight the most natural choice for L is:

$$L = R_{(s)}{}^{\mu}{}_{\cdot\mu} = \tfrac{2}{3}\,\gamma^{\mu\nu}\left(R_{(1)}{}^{\alpha}{}_{\cdot\mu\alpha\nu} - \omega^{\alpha\beta} R^{(2)}_{\mu\nu\alpha\beta}\right) + \tfrac{1}{3}\,\omega^{\mu\nu}\left(j_{\mu,\nu} - j_{\nu,\mu}\right) \qquad (5.48)$$

The corresponding tensors $\Gamma^{\mu\nu}$, $\Omega^{\mu\nu}$, have been computed, but are excessively complicated. In the static, spherically symmetric case, however, the field equations become relatively simple. We may take, as non-vanishing components of the metric:

$$\left.\begin{aligned} \gamma_{00} &= e^{\nu} & \gamma_{11} &= -e^{\lambda} \\ \gamma_{22} &= -r^2 & \gamma_{33} &= -r^2\sin^2\theta \\ \omega_{10} &= -\,\omega_{01} = \frac{F\,e^{\frac{1}{2}(\lambda+\nu)}}{(1-F^2)^{1/2}} \end{aligned}\right\} \qquad (5.49)$$

whence

$$\sqrt{-G} = \frac{r^2 \sin\theta\; e^{\frac{1}{2}(\lambda+\nu)}}{(1-F^2)^{1/2}} \qquad (5.50)$$

$$R_{(s)}{}^{\mu}{}_{\cdot\mu} = \frac{4}{3r^2} - \frac{2}{3}e^{-\lambda}\left\{\begin{aligned} &\left[\nu'' + \tfrac{1}{2}\nu'(\nu'-\lambda')\right]\left[1+2F^2\right]\left[1-F^2\right] + \tfrac{2}{r}(\nu'-\lambda') \\ &+ \tfrac{1}{r^2}\left[2 - \tfrac{40}{9}F^2\right] + 2FF'' + (\nu'-\lambda')FF' \\ &- \left[\tfrac{16}{9} - \tfrac{4}{1-F^2}\right]\frac{FF'}{r} - \left[\tfrac{4}{9} - \tfrac{2}{3(1-F^2)}\right](F')^2 \end{aligned}\right\} \qquad (5.51)$$

Inserting these expressions in the variational principle, one can obtain in a straightforward manner the three Euler-Lagrange equations for the three unknown functions of r. A class of solutions with $F \not\equiv 0$ should exist, but has not been explicitly found (it might, of course, be empty).

In the non-electromagnetic case (cf. (5.38)) the field equations derived from (5.48) are just:

$$R^{(R)}_{\mu\nu} = 0$$

i.e. Einstein's empty-space field equations, while the identities (5.46) go over into the Bianchi identities.

Since (5.45) are required to hold almost everywhere (i.e. with the exception of isolated singularities) they should, in so far as they are non-linear, predict, for any particular choice of Lagrangian L, definite equations of motion for these singularities. With (5.48) as Lagrangian, however, there is sufficient analogy with Einstein's UFT for one to be fairly sure in advance that no Lorentz force will appear.[36]-[38] This is clearly connected with the fact that in the present Lagrangian the curvature scalar formed from the $\gamma_{\mu\nu}$, $R^{(R)}$, is adjoined to quantities like $\gamma^{\mu\nu}\omega^{\alpha\beta}\omega_{\alpha\beta|\mu|\nu}$, and similarly in the case of Einstein's Lagrangian; whereas the Lagrangian for the physical electromagnetic field is of the form $F^{\mu\nu}F_{\mu\nu}$ (cf. (1.5)), and this form does lead, by an EIH type calculation, to the correct (Lorentz) force on electric monopole singularities. In (ii) and (iii) we discuss moves which can be made to meet this situation.

(ii) Choose $\quad L = R_{(s)}{}^{\mu}{}_{\cdot\mu} - 2\lambda$ (5.52)

Let $T^{\mu\nu}$, $\Omega^{\mu\nu}$ represent the Hamiltonian derivatives of $(R_{(s)}{}^{\mu}{}_{\cdot\mu}\sqrt{-G})$ w.r.t. $\gamma_{\mu\nu}$, $\omega_{\mu\nu}$ respectively. By the usual formula for differentiation of determinants, one finds:

$$\delta\sqrt{-G} = \tfrac{1}{2}\sqrt{-G}\,(\gamma^{\mu\nu}\delta\gamma_{\mu\nu} - \omega^{\mu\nu}\delta\omega_{\mu\nu}) \tag{5.53}$$

The field equations resulting from (5.52) are therefore:

$$T^{\mu\nu} = \lambda \gamma^{\mu\nu} \qquad (5.54i)$$

$$\Omega^{\mu\nu} = \lambda \omega^{\mu\nu} \qquad (5.54ii)$$

Suppose the $\omega_{\mu\nu}$ are connected with the physical $F_{\mu\nu}$ by:

$$\omega_{\mu\nu} = \eta F_{\mu\nu} \qquad (5.55)$$

where, since the LHS is dimensionless, the constant η has dimensions $\left(\frac{\text{charge} \times \text{length}}{\text{mass}}\right)$. We want:

$$\Omega^{\mu\nu} \omega_{\mu\nu} \sim \kappa F^{\mu\nu} F_{\mu\nu} \qquad (5.56)$$

(κ = Einstein's gravitational constant), because then (5.46) will be something like the Lorentz force equation. Combining the last three equations gives:

$$\lambda \eta^2 \sim \kappa \qquad (5.57)$$

In this relation, both λ and η are unknown. The following 'plausibility argument' gives an <u>upper bound on η</u>. The classical radius of the electron is (taking c = 1):

$$r_o = \frac{e^2}{m_e} \doteqdot 2.8 \times 10^{-13} \text{ cm} \qquad (5.58)$$

All field strengths with which classical electromagnetism deals are therefore less than

$$F^{(max)} = \frac{e}{r_o^2} \qquad (5.59)$$

and for all such fields the linear Maxwell theory is a good description; if the present <u>non</u>-linear theory is to provide an equally good account, these 'observable' fields must be such that (in coordinates for which $\gamma_{\mu\nu} \sim 1$):

$$\omega_{\mu\nu} \ll 1 \qquad (5.60)$$

Therefore

$$\eta < \frac{r_o^2}{e} \qquad (5.61)$$

This implies, with (5.57), the following <u>lower bound on λ</u> :

$$\lambda > \frac{\kappa e^2}{r_0^4} \sim \frac{10^{-40}}{r_0^2} \tag{5.62}$$

(5.62) means that it is not possible to take, as the form of (5.54i) might at first suggest, λ to be of cosmological dimensions ($\sim 10^{-55}$ cm^{-2}). The next most natural choice, which is consistent with (5.62), is:

$$\lambda = \frac{1}{r_0^2} \tag{5.63}$$

This leads to $\omega_{\mu\nu} \sim 10^{-20}$ at the surface of an electron, so that linearity for observable fields would be well satisfied. Problems arise in connection with (5.54i), however: the gravitational field will now propagate (in the linear approximation) according to an analogue of the Klein-Gordon equation rather than the wave equation, which seems definitely unacceptable (although it should be mentioned that Lanczos' theory,[78]-[80] employing a Riemannian manifold, is based on a proportionality such as (5.54i) with essentially the value (5.63) for λ).

(iii) Choose

$$\begin{aligned} L &= B^{(1)\mu\nu}{}_{ab}\, B^{(1)ab}{}_{\mu\nu} \\ &= 2\,(R_{(S)}{}^{\mu\nu} R_{(S)\mu\nu} + R_{(A)}{}^{\mu\nu} R_{(A)\mu\nu}) \end{aligned} \tag{5.64}$$

where (cf. (3.101) & (3.110)):

$$\left.\begin{aligned} R_{(S)}{}^{\lambda\kappa} &\equiv B^{(1)\lambda\kappa}{}_{11} = \gamma^{\kappa\nu} R_{(S)}{}^{\lambda}{}_{\cdot\nu} - \omega^{\kappa\nu} R_{(A)}{}^{\lambda}{}_{\cdot\nu} \\ R_{(A)}{}^{\lambda\kappa} &\equiv B^{(1)\lambda\kappa}{}_{12} = \gamma^{\kappa\nu} R_{(A)}{}^{\lambda}{}_{\cdot\nu} + \omega^{\kappa\nu} R_{(S)}{}^{\lambda}{}_{\cdot\nu} \end{aligned}\right\} \tag{5.65}$$

The second term on the RHS of (5.64) resembles the Maxwell Lagrangian for the free electromagnetic field, and mainly on this account a Lagrangian of type similar to (5.64) for the combined gravitational and electromagnetic fields has been put forward on a number of occasions and in a variety of contexts, e.g. [52] p.63, [77] p.532, [79], [80], [81] p.230. In contrast to case (i) above, however, there is now a departure from general relativity even for a non-electromagnetic M_4, for, by (5.38), the action principle then reduces to:

$$\delta \int_{M_4} R^{(R)\mu\nu} \, R^{(R)}_{\mu\nu} \sqrt{-g^{(R)}} \; d^4x \quad = \quad 0 \tag{5.66}$$

where $g^{(R)} \equiv \det \| g^{(R)}_{\mu\nu} \|$. This latter Lagrangian is gauge-invariant in Weyl's sense, and for that reason figured in his theory (see also [81] p.141). The following field equations result from (5.66) (the equations given in [98] are slightly erroneous):

$$(R^{\mu}{}_{\alpha} R^{\nu\alpha} - \tfrac{1}{4} g^{\mu\nu} R^{\alpha\beta} R_{\alpha\beta}) + \tfrac{1}{2} (R^{\mu\sigma|\nu} + R^{\nu\sigma|\mu} - R^{\mu\nu|\sigma})_{|\sigma} = 0 \tag{5.67}$$

(where the superscript ${}^{(R)}$ has been dropped, for simplicity). Contraction over (μ, ν), and use of the Bianchi identities, implies that the curvature scalar satisfies the wave equation:

$$\Box R \equiv g^{\alpha\beta} R_{|\alpha|\beta} = 0 \tag{5.68}$$

All solutions of (5.11) simultaneously satisfy (5.67), but not of course vice versa. Whether (5.64) is a permissible choice of Lagrangian depends, therefore, inter alia, on whether it can be shown that (5.67) does not lead to physically unacceptable conclusions. This seems to be an open question (cf. [77] p.533).

CHAPTER 6

Fields in $\mathcal{E}_4$

§6.1 Introduction

The previous chapters have treated curved $\mathcal{K}_n$'s. We shall now consider only flat ones:

$$\left.\begin{aligned} \Omega &= \tfrac{1}{4}\sum_{\alpha=1}^{4} \varepsilon_\alpha \, |z^\alpha|^2 \\ \varepsilon_\alpha &= \pm 1 \end{aligned}\right\} \qquad (6.1)$$

The metric which corresponds to this basic function and coordinate system is:

$$\left.\begin{aligned} \gamma_{\mu\nu} &= \eta_{\mu\nu} \equiv \varepsilon_\mu \delta_{\mu\nu} \quad \text{(no summation)} \\ \omega_{\mu\nu} &= 0 \end{aligned}\right\} . \qquad (6.2)$$

At the same time the coordinate transformations will be restricted to the affine group:

$$z^{\mu'} = \sum_\alpha A_{\mu\alpha} z^\alpha \qquad (6.3)$$

i.e. only position-independent, homogeneous transformations are allowed, and they are further required to leave the metric (6.2) invariant, so that (cf. (3.40)):

$$\sum_\alpha \sum_\beta A^{-1}_{\alpha\mu} \, \eta_{\alpha\beta} \, \overline{A^{-1}_{\beta\nu}} = \eta_{\mu\nu} \qquad (6.4)$$

We shall use the rather ugly term 'quasi-unitary' for these transformations, and write the group as U(4) (it is unfortunate that group-theoretical nomenclature seems to be adapted primarily to groups deriving from positive definite forms, so that in the contrary case there are almost as many notations as authors). Under these conditions the $\mathcal{K}_4$ is a (quasi-) unitary space, and will be written $\mathcal{E}_4$.

The motivation for this specialization is the following.

As discussed in §1.1, one would not expect a 'complete' physics to be constructible purely out of the sort of geometrical tensors considered so far. So one is led to examine the behaviour of 'extraneous' (in the sense of non-geometrical) fields on $\mathcal{K}_4$. This is a potentially very large area of investigation and will be no more than touched on here: it leads almost at once outside the scope of the present work and into quantum theory. In this chapter we treat only the question of the existence of Dirac spinors in $\mathcal{E}_4$ (see §6.3). §6.2 assembles oneor two results relevant to the purpose. The present section concludes with some remarks on the subject of 'internal' symmetry groups.

Suppose (cf. § 1.1) that, as observers confined to M_4, our experience is only a partial view, a slice, of the 'real' physics, that of $\mathcal{K}_4$. The latter will (presumably) involve interactions which are covariant under U(4). The only coordinate transformations permissible on M_4 are (cf. Chapter 4) real transformations of the x^μ, which in the present context means: the Lorentz group, L_4. The latter is a 6 real parameter proper subgroup of U(4) (we consider homogeneous transformations only). The covariance of an interaction under the wider group U(4) might therefore be interpreted by an observer on M_4 in terms of (covariance under L_4, together with) some 'internal' symmetry property.

Turning now to elementary particle physics, we find the following situation. The structure of the (inhomogeneous)

Lorentz group gives rise to a classification of particles (strictly: of irreducible representations) according to their mass and spin. The classificatory possibilities latent in the group are thereby exhausted. However, the observed particle interaction symmetries either demand or at least invite the introduction of other quantum numbers: charge, baryonic charge, hypercharge, etc., which are not connected with space-time coordinate transformations. Thus there arises the concept of <u>isospace</u>. Originally, the latter was taken to be three-dimensional, and provided a 'geometrical' characterization of charge multiplets (in terms of isotopic spin). Subsequently, a real four-dimensional space was suggested[85]-[87] in the attempt to incorporate strangeness ([88], and [89] Chapter V, review this work). More recently, attention has been primarily focussed on SU(3), SU(6), and related groups. However, the group called here U(4) has also been put forward.[90]-[92] In general the internal transformation group is unconnected with space-time coordinate transformations, so that the overall symmetry of (say) the Lagrangian is just the direct product of the two distinct types of group; attempts to modify this state of affairs meet with great difficulties.

Is there any connection between the situations, one hypothetical, one actual, portrayed in the two preceding paragraphs? It would seem not entirely inconceivable that a case could be made for trying to relate the 4 'unphysical' degrees of freedom, z^{μ}_{2} , of a complex space-time to the coordinates of (for example) a four-dimensional isospace; but without a much clearer picture of the supposed nature of fields in $\mathcal{E}_4$ it is difficult to see how one might progress towards a less indefinite answer to the question.

§6.2 U(4)

$U(4)$[90][95] is, like $O(4)$ or $L(4)$, a <u>semi</u>-simple group. It contains the 'phase' transformations:

$$z^{\mu\prime} = e^{i\theta} z^{\mu} \qquad (\theta \text{ real}) \tag{6.5}$$

as an invariant subgroup, so that there is the direct product decomposition:

$$U(4) = SU(4) \times U(1) \tag{6.6}$$

Representations of $U(4)$ can therefore be classified according to their behaviour under $U(1)$ and $SU(4)$ separately. The latter is a 15 real parameter simple group, and will now be considered in more detail.

The complex Lie algebras ([93] §53) of the $SU(4)$'s associated with the various possible signatures (choices of ε_μ) in (6.1) are identical, differences only showing up in the real algebras. These distinguishing features will be ignored here, since questions of compactness, etc. are not the present concern, and none of the results of this or the following section are sensitive to choice of signature. The Lie algebra of $SU(4)$ is, in Cartan's classification, A_3 ; it is of rank 3 and has the following Schouten (or Dynkin) diagram:

$$\circ\!\!-\!\!-\!\!-\!\!\circ\!\!-\!\!-\!\!-\!\!\circ \tag{6.7}$$

Since this is also the diagram for D_3 (the 6-dimensional orthogonal group, $O(6)$), there is the isomorphism:

$$SU(4) \cong O(6) \tag{6.8}$$

The next few results are obtainable by straightforward application of the standard techniques, as collected in [94] in particular. Calling the three simple roots of the Lie algebra α, β, γ, its 12 roots are:

$$\pm\{\alpha, \beta, \gamma, \alpha+\beta, \beta+\gamma, \alpha+\beta+\gamma\} \tag{6.9}$$

From this, the canonical structure constants of the algebra can readily be computed. The dimension of the irreducible representation with diagram:

$$\overset{\lambda_\alpha}{\circ}\!-\!\!\!-\!\overset{\lambda_\beta}{\circ}\!-\!\!\!-\!\overset{\lambda_\gamma}{\circ}$$

(λ_α, λ_β, λ_γ non-negative integers) is found to be:

$$N = \tfrac{1}{12}\mu_\alpha \mu_\beta \mu_\gamma (\mu_\alpha + \mu_\beta)(\mu_\beta + \mu_\gamma)(\mu_\alpha + \mu_\beta + \mu_\gamma) \tag{6.10}$$

where $\mu_\alpha \equiv \lambda_\alpha + 1$, and similarly μ_β, μ_γ. The following table gives all the irreducible representations up to power 3.

Representation	Greatest weight	Dimension N	Weight diagram
$\circ-\circ-\circ$	0	1	[1]
$\overset{1}{\circ}-\circ-\circ$	$\frac{3}{4}\alpha + \frac{1}{2}\beta + \frac{1}{4}\gamma$	4	[1,1,1,1]
$\circ-\overset{1}{\circ}-\circ$	$\frac{1}{2}\alpha + \beta + \frac{1}{2}\gamma$	6	[1,1,2,1,1]
$\overset{2}{\circ}-\circ-\circ$	$\frac{3}{2}\alpha + \beta + \frac{1}{2}\gamma$	10	[1,1,2,2,2,1,1]
$\overset{1}{\circ}-\circ-\overset{1}{\circ}$	$\alpha + \beta + \gamma$	15	[1,2,3,3,3,2,1]
$\overset{1}{\circ}-\overset{1}{\circ}-\circ$	$\frac{5}{4}\alpha + \frac{3}{2}\beta + \frac{3}{4}\gamma$	20	[1,2,3,4,4,3,2,1]
$\circ-\overset{2}{\circ}-\circ$	$\alpha + 2\beta + \gamma$	20	[1,1,3,3,4,3,3,1,1]
$\overset{3}{\circ}-\circ-\circ$	$\frac{9}{4}\alpha + \frac{3}{2}\beta + \frac{3}{4}\gamma$	20	[1,1,2,3,3,3,3,2,1,1]
$\overset{2}{\circ}-\circ-\overset{1}{\circ}$	$\frac{7}{4}\alpha + \frac{3}{2}\beta + \frac{5}{4}\gamma$	36	[1,2,4,5,6,6,5,4,2,1]
$\overset{2}{\circ}-\overset{1}{\circ}-\circ$	$2\alpha + 2\beta + \gamma$	45	[1,2,4,5,7,7,7,5,4,2,1]
$\circ-\overset{3}{\circ}-\circ$	$\frac{3}{2}\alpha + 3\beta + \frac{3}{2}\gamma$	50	[1,1,3,4,6,6,8,6,6,4,3,1,1]
$\overset{1}{\circ}-\overset{2}{\circ}-\circ$	$\frac{7}{4}\alpha + \frac{5}{2}\beta + \frac{5}{4}\gamma$	60	[1,2,4,6,8,9,9,8,6,4,2,1]
$\overset{1}{\circ}-\overset{1}{\circ}-\overset{1}{\circ}$	$\frac{3}{2}\alpha + 2\beta + \frac{3}{2}\gamma$	64	[1,3,5,8,10,10,10,8,5,3,1]

where, in the last column, (1,1,2,1,1) means and so on.

The second row corresponds to the vector representation of SU(4), the spinor one of $O_+(6)$.[97]

The third row corresponds to skew-symmetric tensor, and vector, respectively, for the two groups.

The fifth row corresponds to a zero-trace second rank Hermitian tensor, and a skew-symmetric second rank tensor, respectively. Use of the above table alone enables most products of low-order representations to be reduced unambiguously.

Whereas $O(n)$ is not simply connected, so that its basic representations are the two-valued (spinor) ones, SU(4) does not possess two-valued representations ([99] pp.268-70).

We conclude by establishing an isomorphism which lies at the basis of the work in §6.3, namely:

$$U(4) \cong O_+(8) \cap Sp(8) \tag{6.11}$$

where $_+$ denotes the proper (positive determinant) subgroup, and $Sp(8)$ is the <u>real</u> symplectic group in 8 dimensions.

In terms of the decompositions (3.36) and (3.37), (6.3) and (6.4) become:

$$\begin{pmatrix} z_1^{\mu'} \\ z_2^{\mu'} \end{pmatrix} = \begin{pmatrix} A^{(1)}_{\mu\alpha} & -A^{(2)}_{\mu\alpha} \\ A^{(2)}_{\mu\alpha} & A^{(1)}_{\mu\alpha} \end{pmatrix} \begin{pmatrix} z_1^{\alpha} \\ z_2^{\alpha} \end{pmatrix} \tag{6.12}$$

and

$$\begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} = \begin{pmatrix} A^{(1)T} & A^{(2)T} \\ -A^{(2)T} & A^{(1)T} \end{pmatrix} \begin{pmatrix} \eta & 0 \\ 0 & \eta \end{pmatrix} \begin{pmatrix} A^{(1)} & -A^{(2)} \\ A^{(2)} & A^{(1)} \end{pmatrix} \tag{6.13}$$

where ^{T} denotes the transposed matrix. (6.13) just says:

$$\left.\begin{array}{l} A^{(1)T}\eta\, A^{(1)} + A^{(2)T}\eta\, A^{(2)} = \eta \\ A^{(1)T}\eta\, A^{(2)} - A^{(2)T}\eta\, A^{(1)} = 0 \end{array}\right\} \tag{6.14}$$

which are the real and imaginary parts of (6.4). Consider the general affine transformation of the 8-dimensional real space coordinatized by the z^{μ}_{a} , i.e:

$$\begin{pmatrix} z^{\mu'}_{1} \\ z^{\mu'}_{2} \end{pmatrix} = \begin{pmatrix} T^{(11)}_{\mu\alpha} & T^{(12)}_{\mu\alpha} \\ T^{(21)}_{\mu\alpha} & T^{(22)}_{\mu\alpha} \end{pmatrix} \begin{pmatrix} z^{\alpha}_{1} \\ z^{\alpha}_{2} \end{pmatrix} \qquad (6.15)$$

The condition that this be a <u>real symplectic</u> transformation, in the sense of leaving invariant the anti-symmetric bilinear form with matrix: $\begin{pmatrix} 0 & I_{(4)} \\ -I_{(4)} & 0 \end{pmatrix}$

and that it should simultaneously be an <u>orthogonal</u> transformation, in the sense of leaving invariant the symmetric bilinear form with matrix the LHS of (6.13), is found to entail that $(T^{(ab)}_{\mu\alpha})$ must have a partitioned structure as in (6.12) and must satisfy equation (6.13). If, in addition, it is noted that the determinant of a transformation matrix of type (6.12) is inherently positive, being a perfect square, the proof of (6.11) is complete. (Of course, the result is not specially dependent on the value $n = 4$, but holds for all positive n.)

§6.3 Linear wave equation

Spinors, in the sense of two-valued representations of the coordinate transformation group, do not exist in $\mathcal{E}_4$. Nevertheless, it is still possible to linearize the d'Alembertian operator, by a procedure precisely parallelling Dirac's original one. Throughout this section we shall suppose the metric to be positive definite: the transition to a space-time of Minkowski signature is straightforward, and the presentation is merely made more cumbersome by having continually to differentiate between time and space values of the indices.

The wave operator in $\mathcal{E}_4$ is:

$$\Box \equiv 4\sum_{\mu=1}^{4} \frac{\partial^2}{\partial z^\mu \partial \bar{z}^\mu} = \sum_{\mu=1}^{4}\sum_{a=1}^{2}\left(\frac{\partial}{\partial z^\mu_a}\right)^2 = \sum_{i=1}^{8}\left(\frac{\partial}{\partial z^i}\right)^2 \tag{6.16}$$

where in the last step an identification like (2.1) is involved. Following the procedure given in Brauer & Weyl's classic paper[102] (cf. also [99] pp.270-4), we introduce 8 quantities p^i which enable $\Box$ to be written as a perfect square:

$$\Box = \left(p^i \frac{\partial}{\partial z^i} \right)^2 \tag{6.17}$$

(introducing the summation convention). Equating the last two equations gives:

$$p^i p^j + p^j p^i = 2\delta^{ij} \tag{6.18}$$

so that the set of 'units' $\prod_{i=1}^{8}(p^i)^{e_i}$ with $e_i = 0$ or 1 spans a 256-dimensional Clifford algebra. (Gamba[105] has also treated spinors in 8 dimensions, and the work of Das[50] has already been mentioned.) A standard matrix representation of (6.18) is as follows (it will be distinguished by a hat):

$$\left.\begin{aligned}
\hat{P}^1 &= 1'' \times 1 \times 1 \times 1 & \hat{P}^5 &= 1''' \times 1 \times 1 \times 1 \\
\hat{P}^2 &= 1 \times 1'' \times 1 \times 1 & \hat{P}^6 &= 1' \times 1''' \times 1 \times 1 \\
\hat{P}^3 &= 1' \times 1' \times 1'' \times 1 & \hat{P}^7 &= 1' \times 1' \times 1''' \times 1 \\
\hat{P}^4 &= 1' \times 1' \times 1 \times 1'' & \hat{P}^8 &= 1' \times 1' \times 1' \times 1'''
\end{aligned}\right\} \quad (6.19)$$

where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $1' = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $1'' = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $1''' = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ (6.20)

and $\times$ represents outer or Kronecker multiplication. The first four matrices are real, the second four pure imaginary. Any other (16×16) representation of the algebra is obtainable from this one, by a non-singular 'spin-frame' transformation S:

$$P^i = S^{-1} \hat{P}^i S \quad . \qquad (6.21)$$

and S is uniquely determined up to a (complex) scalar factor – this follows from Schur's lemma, and can be verified directly: the only matrix which commutes with all the $\hat{P}$'s is (a multiple of) the unit matrix $I_{(16)}$. For reasons which will appear shortly, we use this freedom of choice of basis in the spin frame to convert (6.19) into a different representation, P^i say, by choosing as S in (6.21) a 'quasi'-permutation matrix, namely one that permutes the rows and columns of $\hat{P}^i$ according to the following scheme:

$$\downarrow\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 \\ 1 & 12 & 11 & -6 & 10 & 7 & -8 & 13 & 9 & 4 & 3 & -14 & 2 & 15 & -16 & 5 \end{pmatrix} \qquad (6.22)$$

where the notation means that e.g. under the row interchange operation the 6th row of P^i is $-$ (the 4th row of $\hat{P}^i$), and so on. The matrix S is unitary. The resulting matrices P^i all have (unlike the $\hat{P}^i$) the following partitioned form:

$$(P^i) = \begin{pmatrix} O_{(8)} & \pi^i \\ \pi^{i\dagger} & O_{(8)} \end{pmatrix} \tag{6.23}$$

where $\dagger$ denotes Hermitian conjugate. Explicitly:

$$\pi^1 = I_{(8)} \qquad \pi^5 = \left(\begin{array}{cccc|cccc} i &&&&&&& \\ & -i &&&&&& \\ && -i &&&&& \\ &&& -i &&&& \\ \hline &&&& -i &&& \\ &&&&& i && \\ &&&&&& i & \\ &&&&&&& i \end{array}\right)$$

$$\pi^2 = \left(\begin{array}{cccc|cccc} 0&1&0&0 &&&& \\ -1&0&0&0 &&&& \\ &&&& 0&0&0&1 \\ &&&& 0&0&-1&0 \\ \hline &&&& 0&1&0&0 \\ &&&& -1&0&0&0 \\ 0&0&0&1 &&&& \\ 0&0&-1&0 &&&& \end{array}\right) \qquad \pi^6 = \left(\begin{array}{cccc|cccc} 0&i&0&0 &&&& \\ i&0&0&0 &&&& \\ &&&& 0&0&0&i \\ &&&& 0&0&-i&0 \\ \hline &&&& 0&-i&0&0 \\ &&&& -i&0&0&0 \\ 0&0&0&-i &&&& \\ 0&0&i&0 &&&& \end{array}\right) \tag{6.24}$$

$$\pi^3 = \left(\begin{array}{cccc|cccc} 0&0&1&0 &&&& \\ &&&& 0&0&0&-1 \\ -1&0&0&0 &&&& \\ &&&& 0&1&0&0 \\ \hline &&&& 0&0&1&0 \\ 0&0&0&-1 &&&& \\ &&&& -1&0&0&0 \\ 0&1&0&0 &&&& \end{array}\right) \qquad \pi^7 = \left(\begin{array}{cccc|cccc} 0&0&i&0 &&&& \\ &&&& 0&0&0&-i \\ i&0&0&0 &&&& \\ &&&& 0&i&0&0 \\ \hline &&&& 0&0&-i&0 \\ 0&0&0&i &&&& \\ &&&& -i&0&0&0 \\ 0&-i&0&0 &&&& \end{array}\right)$$

$$\pi^4 = \left(\begin{array}{cccc|cccc} 0&0&0&1 &&&& \\ &&&& 0&0&1&0 \\ &&&& 0&-1&0&0 \\ -1&0&0&0 &&&& \\ \hline &&&& 0&0&0&1 \\ 0&0&1&0 &&&& \\ 0&-1&0&0 &&&& \\ &&&& -1&0&0&0 \end{array}\right) \qquad \pi^8 = \left(\begin{array}{cccc|cccc} 0&0&0&i &&&& \\ &&&& 0&0&i&0 \\ &&&& 0&-i&0&0 \\ i&0&0&0 &&&& \\ \hline &&&& 0&0&0&-i \\ 0&0&-i&0 &&&& \\ 0&i&0&0 &&&& \\ &&&& -i&0&0&0 \end{array}\right)$$

(where blank spaces are occupied by zeros).

These are all unitary matrices; that they must be, follows by squaring both sides of (6.23). Also, π^1 is Hermitian, the other 7 anti-Hermitian. (These particular properties are <u>not</u> independent of the signature of $\mathcal{E}_4$.) The equivalent of the "γ^5" of Dirac 4-spinor theory is:

$$\prod_{i=1}^{8} P^i = \begin{pmatrix} I_{(8)} & O \\ O & -I_{(8)} \end{pmatrix} \tag{6.25}$$

Consider the 'Klein-Gordon' wave equation in $\mathcal{E}_4$:

$$(\Box - \kappa^2)\Psi = 0 \qquad (6.26)$$

where κ is some (real or pure imaginary) constant. Using (6.17), this can be linearized to give the 'Dirac' wave equation:

$$(P^i \frac{\partial}{\partial z^i} - \kappa)\Psi = 0 \qquad (6.27)$$

where Ψ is a (16×1) matrix. We now look at the way this equation behaves under coordinate transformations.

Let O_{ik} be the matrix of an orthogonal transformation of the z^i:

$$z^{i'} = \sum_k O_{ik} z^k \qquad (6.28)$$

with
$$\delta_{ij} = \sum_k \sum_\ell O^{-1}_{ki} \delta_{k\ell} O^{-1}_{\ell j} \qquad (6.29)$$

Let $P^{i'}$ be the set of matrices which result from treating the index i as a (contravariant) vector index:

$$P^{i'} = \sum_k O_{ik} P^k \qquad (6.30)$$

The $P^{i'}$ will also satisfy the anti-commutation relations (6.18), and because the Clifford algebra possesses only <u>inner</u> automorphisms there must therefore exist a matrix, $S(O)$ say, such that:

$$P^i = [S(O)]^{-1} P^{i'} S(O) \qquad (6.31)$$

Combining the last two equations:

$$P^i = \sum_k [S(O)]^{-1} O_{ik} P^k S(O) \qquad (6.32)$$

which says that under the coordinate and spin transformations <u>combined</u> the P's, and therefore the Dirac equation (6.27), are invariant. O may be said to <u>induce</u> this spin transformation $S(O)$. The relation between O and S is, because of the presence of an arbitrary multiplying factor on S, a 'projective' homomorphism; by suitable normalization (cf. [99]

p.273) the factor can be reduced to ± 1, giving a two-valued homomorphism. The behaviour of Ψ under $S(O)$ is:

$$\Psi' = [S(O)]^{-1} \Psi \tag{6.33}$$

So it transforms according to an irreducible but two-valued representation of $O(8)$. However, only some of the orthogonal transformations (6.28) correspond to coordinate transformations in $\mathcal{E}_4$ (see § 6.2); so it is prima facie unlikely that Ψ will transform irreducibly under $U(4)$. It is the main purpose of the rest of the section to discover how it does transform. This will be done in two stages, corresponding to the two terms on the RHS of (6.11): we first show how Ψ transforms under $O_+(8)$ - this is common knowledge, from quantum theory - and then find what effect the simplectic condition has.

Since only the proper orthogonal group has to be considered we may restrict attention to the neighbourhood of the identity. If

$$O = I_{(8)} + \varepsilon\omega \tag{6.34}$$

then the orthogonality condition (6.29) says:

$$\omega^T + \omega = 0 \tag{6.35}$$

There are 28 linearly independent anti-symmetric (8×8) matrices, and they are spanned by the set $M^{(rs)} \equiv - M^{(sr)}$ $(r < s)$ where:

$$M^{(rs)}_{ij} = \begin{cases} +1 & i = r \quad j = s \\ -1 & i = s \quad j = r \\ 0 & \text{otherwise} \end{cases} \tag{6.36}$$

Consider the infinitessimal transformation corresponding to one of these generators:

$$z^{i'} = \sum_k (I + \varepsilon M^{(rs)})_{ik} \; z^k \tag{6.37}$$

In this case an explicit solution of the equation (6.32) for S(0) is obtainable, in the usual way, and substitution in (6.33) gives as the induced transformation of Ψ :

$$\Psi' = (I + \tfrac{1}{2} \varepsilon P^r P^s) \Psi \tag{6.38}$$

We now consider infinitessimal unitary transformations in $\mathcal{E}_4$, and for this purpose revert temporarily to the complex coordinates z^μ. If, in (6.3):

$$A = I_{(4)} + \varepsilon a \tag{6.39}$$

then the unitarity condition (6.4) says (since now $\eta_{\mu\nu} = \delta_{\mu\nu}$):

$$a^\dagger + a = 0 \tag{6.40}$$

There are 16 linearly independent anti-Hermitian (4 × 4) matrices, and they are spanned by the following set:

$$F = \begin{pmatrix} -i & & & \\ & -i & & \\ & & -i & \\ & & & -i \end{pmatrix}$$

$$K_1 = \begin{pmatrix} i & & & \\ & 0 & & \\ & & 0 & \\ & & & -i \end{pmatrix} \qquad K_2 = \begin{pmatrix} 0 & & & \\ & i & & \\ & & 0 & \\ & & & -i \end{pmatrix} \qquad K_3 = \begin{pmatrix} 0 & & & \\ & 0 & & \\ & & i & \\ & & & -i \end{pmatrix}$$

$$\underset{(\mu<\nu)}{R^{(\mu\nu)}_{\alpha\beta}} = \begin{cases} +1 & \alpha = \mu \quad \beta = \nu \\ -1 & \alpha = \nu \quad \beta = \mu \\ 0 & \text{otherwise} \end{cases} \tag{6.41}$$

$$\underset{(\mu<\nu)}{I^{(\mu\nu)}_{\alpha\beta}} = \begin{cases} -i & \alpha = \mu \quad \beta = \nu \\ -i & \alpha = \nu \quad \beta = \mu \\ 0 & \text{otherwise} \end{cases}$$

Write the corresponding real 8-dimensional matrices with curly capital letters (cf. (3.37)); then, in terms of the matrices $M^{(rs)}$ introduced above:

$$
\begin{aligned}
\mathcal{F} &= M^{(15)} + M^{(26)} + M^{(37)} + M^{(48)} \\
\mathcal{K}_1 &= -M^{(15)} + M^{(48)} \\
\mathcal{K}_2 &= -M^{(26)} + M^{(48)} \\
\mathcal{K}_3 &= -M^{(37)} + M^{(48)} \\
\mathcal{R}^{(\mu\nu)} &= M^{(\mu\nu)} + M^{(\mu+4,\nu+4)} \\
\mathcal{J}^{(\mu\nu)} &= M^{(\mu,\nu+4)} + M^{(\nu,\mu+4)}
\end{aligned}
\tag{6.42}
$$

Combining (6.42) with (6.38) enables the transformation properties of Ψ under U(4) to be found. By explicitly computing the transformation matrices, from (6.24), the following two rather remarkable facts emerge.

(i) If the 16 components of Ψ are relabelled according to the scheme:

$$
\begin{aligned}
\Psi^1 \equiv s \qquad & \Psi^5 \equiv t \\
(\Psi^2, \Psi^3, \Psi^4, \Psi^6, \Psi^7, \Psi^8) &\equiv (w_{12}, w_{13}, w_{14}, w_{43}, w_{24}, w_{32}) \\
(\Psi^9, \Psi^{10}, \Psi^{11}, \Psi^{12}) &\equiv (u_1, u_2, u_3, u_4) \\
(\Psi^{13}, \Psi^{14}, \Psi^{15}, \Psi^{16}) &\equiv (v^1, v^2, v^3, v^4)
\end{aligned}
\tag{6.43}
$$

and if the 6 components on the second line are duplicated by:

$$w_{\mu\nu} \equiv -w_{\nu\mu} \quad (\mu \neq \nu) \tag{6.44}$$

then under the 15 generators of SU(4), i.e. the set (6.41) but excluding F, Ψ is found to transform in the manner implied by the index labelling in (6.43). So, instead of transforming irreducibly under SU(4), it transforms according to a representation which is in its explicitly reduced form, with diagonal 'block' structure signified by the decomposition:

$$16 = 1 + 4 + 6 + 4 + 1 \tag{6.45}$$

This (by hindsight) was the reason for making the change (6.22).

(ii) Consider now the behaviour of Ψ under the phase subgroup U(1), with generator F. Again by inspection of the matrices, it turns out that the components of Ψ transform as tensor densities. (This fact is of course irrelevant for the unimodular transformations SU(4) (cf. [99] p.264), which is why it is not in evidence in (i) above.)

Definition: A quantity transforming under (6.3) as a tensor but with inclusion of a factor

$$\left(\det \| A_{\mu\nu} \|\right)^{-w} \left(\det \| \overline{A_{\mu\nu}} \|\right)^{-w'}$$

is called a tensor density of weight w and anti-weight w' (cf. [67] p.12).

Take the determinant of (6.39), with $a = F$:

$$\det \| I_{(4)} + \varepsilon F \| = 1 - 4 i \varepsilon \tag{6.46}$$

This means that a density with the above weights will acquire the additional factor

$$1 + 4 i (w - w') \varepsilon$$

in its transformation law for F. The values of $(w - w')$ for the components of Ψ are found to be as follows:

$$\begin{array}{cc} s & -\frac{1}{2} \\ t & +\frac{1}{2} \\ u_\mu & -\frac{1}{2} \\ v^\mu & +\frac{1}{2} \\ w_{\mu\nu} & -\frac{1}{2} \end{array} \tag{6.47}$$

If, now, the wave equation (6.27) is written out explicitly using the matrices P^i as given by (6.23) + (6.24), and in terms of the symbols introduced in (6.43), the 16 equations are:

$$
\begin{aligned}
u^{\bar{\mu}}{}_{,\bar{\mu}} \;-\; \tfrac{1}{2}\kappa s &= 0 \\
s_{,\mu} \;-\; w_{\mu}{}^{\bar{\nu}}{}_{,\bar{\nu}} \;-\; \tfrac{1}{2}\kappa u_{\mu} &= 0 \\
(u_{\nu,\mu} - u_{\mu,\nu}) - \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma}(v^{\sigma,\rho} - v^{\rho,\sigma}) - \tfrac{1}{2}\kappa w_{\mu\nu} &= 0 \\
t_{,\bar{\mu}} + w^{*}{}_{\bar{\mu}}{}^{\nu}{}_{,\nu} - \tfrac{1}{2}\kappa v_{\bar{\mu}} &= 0 \\
v^{\mu}{}_{,\mu} - \tfrac{1}{2}\kappa t &= 0
\end{aligned}
\tag{6.48}
$$

where (cf. (3.71)) $\quad {}_{,\mu} \equiv \frac{\partial}{\partial z^{\mu}}$

and $\quad w^{*\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\sigma\rho}\, w_{\sigma\rho}$ (6.49)

and where the metric tensor $(\delta_{\mu\bar{\nu}})$ has been used to raise and lower indices (the components $u^{\bar{\mu}}$ being in fact, with the present signature, numerically indistinguishable from u_{μ}). $\varepsilon_{\mu\nu\sigma\rho}$ $(\varepsilon^{\mu\nu\sigma\rho})$ have weights $-1\,(+1)$ respectively, and anti-weights zero, so one can readily check that the LHS's in (6.48) do transform homogeneously under U(4).

It is worth emphasizing that (6.48) is inter alia a perfectly ordinary Dirac equation in E_8 , only the notation being unusual.

In conclusion, we consider briefly what happens to (6.48) if the restriction to flatness of the $\mathcal{K}_n$, and/or the restriction of (2.3) to the affine group (6.3), both imposed at the start of the chapter, are lifted. Since (6.48) is in the complex-coordinate notation, we use the formalism of Chapter 2 for convenience.

Although covariant derivatives have been defined for tensors in a Kähler space, tensor densities have not figured at all in the preceding chapters (nor in the literature, so far as the author is aware). We therefore <u>define</u> the covariant derivative of a scalar density, Σ say, with weights w and w', to be of the form:

$$\Sigma_{;\mu} \equiv \Sigma_{,\mu} + w\Delta_{\mu}\Sigma + w'\Delta'_{\mu}\Sigma \,, \tag{6.50}$$

formulae for higher rank tensor densities following by invoking the product rule for differentiation, in the usual way. A relation between the 'connection' quantities Δ_{μ}, Δ'_{μ}, and the metric of K_n is obtainable by comparing the equations:

$$(\det \| g_{\mu\bar{\nu}} \|)_{;\sigma} = g^{\mu\bar{\nu}} g_{\mu\bar{\nu};\sigma} \doteq 0 \tag{6.51}$$

$$(\det \| g_{\mu\bar{\nu}} \|)_{;\sigma} = (\det \| g_{\mu\bar{\nu}} \|)_{,\sigma} - (\Delta_{\sigma} + \Delta'_{\sigma})(\det \| g_{\mu\bar{\nu}} \|) \tag{6.52}$$

the second one holding because the determinant of the metric tensor has weight and anti-weight -1. One deduces that

$$\Delta_{\mu} + \Delta'_{\mu} = \Gamma^{\alpha}_{\cdot\alpha\mu} \tag{6.53}$$

but that $(\Delta_{\mu} - \Delta'_{\mu})$ is undetermined by the metric tensor. If we call this arbitrary quantity Ω_{μ}, (6.50) becomes:

$$\Sigma_{;\mu} = \Sigma_{,\mu} + \left[\tfrac{1}{2}(w + w')\Gamma^{\alpha}_{\cdot\alpha\mu} + \tfrac{1}{2}(w - w')\Omega_{\mu}\right]\Sigma \tag{6.54}$$

Similarly, by consideration of the complex conjugate equation, one finds:

$$\Sigma_{;\bar{\mu}} = \Sigma_{,\bar{\mu}} + \left[\tfrac{1}{2}(w + w')\Gamma^{\bar{\alpha}}_{\cdot\bar{\alpha}\bar{\mu}} - \tfrac{1}{2}(w - w')\overline{\Omega_{\mu}}\right]\Sigma \tag{6.55}$$

All that needs to be done to make (6.48) generally covariant is to replace commas by semi-colons. Because of the quantities' density character (see (6.47)), the Ω_{μ} will then enter the set of equations precisely as do the physical electromagnetic 4-potentials A_{μ} in the Dirac equation for a charged spinor field. (There is a connection here with Sciama's 'unitary Vierbein' formulation of electromagnetic theory (cf. [42] p.428).) The fact that the Ω_{μ} are unrelated to the metric casts doubts, in the author's view, on whether the metric tensor is the 'seat' of the electromagnetic field after all, and therefore on presuppositions lying at the basis of the theory of complex space-time as presented here.

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