Geometry of Monopoles and Domain Walls

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DECLARATION

The research described in this thesis was performed in the Department of Applied Mathematics and Theoretical Physics between October 1995 and November 1998. The work contained in this dissertation is original, except where explicit reference to the results of others is given. Parts of this work, which are indicated in the text, were performed in collaboration and some of the results have appeared or will appear in published form:

- G. W. Gibbons, P. Rychenkova and R.Goto, *Hyperkähler quotient construction of BPS monopole moduli spaces*, Comm. Math. Phys. **186**: 581 (1997), hep-th/9608085.
- G. W. Gibbons and P. Rychenkova, *Threshold bound states of non-abelian monopoles*, DAMTP-1998-100.
- G. W. Gibbons and P. Rychenkova, *Cones, tri–Sasakian structures and superconformal invariance*, to appear in Phys. Lett. B, hep-th/9809158.
- G. W. Gibbons and P. Rychenkova, *Single-sided domain walls in M-theory*, submitted to J. Geom. Phys., hep-th/9811045.

This dissertation is not substantially the same as any I have submitted, or am submitting, for a degree or diploma or other qualification.

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SUMMARY

This dissertation explores interrelations between certain geometrical structures and string and M-theory. It is divided into three parts.

The first part is based on a study of BPS monopoles — highly topical objects both from a mathematical and physical point of view. The moduli spaces of BPS monopoles are hyperkähler manifolds which should possess certain features if the S-duality conjecture is to hold. Mathematical evidence is given that moduli spaces of various monopole configurations possess the requisite features. First, moduli spaces of fundamental monopoles, as well as other toric hyperkähler manifolds are obtained by hyperkähler quotient. The description in terms of the hyperkähler quotient of moduli spaces of monopole configurations with either fixed or massless monopoles is also found. The method of construction provides essential information on the global properties of the hyperkähler manifolds. This information is then used to exhibit quantum marginal bound states of fundamental monopoles by finding certain square-integrable harmonic forms on the Taubian–Calabi manifolds. Next, classical dynamics of distinct fundamental monopoles are studied. The physical picture and the mathematical proof are given for the non-existence of classical bound states of fundamental (possibly massless) monopoles. Simple scattering process of distinct monopoles is described, and the associated scaling solution to the geodesic equations on the moduli space is presented. The same technique is extended to the case of many $SU(2)$ monopoles, for which rigidly rotating solutions are also found.

The second part of the dissertation focuses on Kähler and hyperkähler manifolds with dilatation invariance. The study is motivated by the appearance of these manifolds as target spaces of certain superconformal theories with hypermultiplets. It is shown that the dilatation symmetry requires Kähler and hyperkähler manifolds to be cones over Sasakian and tri–Sasakian metrics respectively. Sasakian and tri–Sasakian geometries are described, and their significance in the developments relating to cone-branes and the AdS/CFT correspondence is discussed.

Finally, some single-sided BPS domain wall configurations in M-theory are investigated. These are smooth non-singular resolutions of Calabi–Yau orbifolds obtained by identifying the two sides of the wall under reflection. They may be thought of as domain walls at the end of the universe. Depending on their symmetries, the domain walls are classified according to the Bianchi scheme. The Kähler Ricci-flat examples are generalized to higher dimensions and, where possible, the orbifold singularities are resolved. Related domain wall type solutions with negative cosmological constant are also studied.

Но так и быть — рукой пристрастной
Прими собранье пестрых глав,
…
Небрежный плод моих забав,
Бессонниц, легких вдохновений,
Незрелых и увядших лет,
Ума холодных наблюдений
И сердца горестных замет.

А. С. Пушкин.

Моим Родителям

Contents

CHAPTER 1

Introduction

One of the striking features of supersymmetry is the pivotal role it has played in recent developments in many areas of theoretical physics. The presence of supersymmetry in a physical theory imposes constraints on the underlying geometrical structure. This dissertation explores a number of ways in which supersymmetry manifests itself through geometry in the context of string theory.

The work presented in this thesis is motivated by two conjectures which are made about the various interrelationships between string theory and supersymmetric gauge theory. One conjecture asserts that the full quantum type IIB string theory is invariant under $SL(2,\mathbb{Z})$ modular transformations. This non-perturbative symmetry is called S-duality, and the conjecture is referred to as the *S-duality conjecture*. The type IIB theory contains solitonic objects which are described by four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills (SYM) theory. The S-duality conjecture then generalizes the *Strong-weak coupling duality* conjecture of Montonen and Olive.

The second conjecture, due to Maldacena, is the *AdS/CFT correspondence*.[1] It asserts that, when compactified on $AdS_5 \times S^5$, the type IIB string theory is equivalent to the four-dimensional $\mathcal{N} = 4$ supersymmetric $SU(N)$ Yang–Mills theory in the limit of large N. $\mathcal{N} = 4$ SYM theory is a superconformal four-dimensional field theory (SCFT). It is possible, with certain care, to extend Maldacena's original conjecture to a correspondence between supergravity (and string theory) on an Einstein space M with negative cosmological constant and certain SCFT on the boundary of M.

Formulating, as well as testing the two conjectures relies on some remarkable features of four-

[1] AdS stands for Anti–de Sitter space-time and CFT stands for Conformal Field Theory.

dimensional maximally supersymmetric ($\mathcal{N} = 4$) Yang–Mills theory. The fact that $\mathcal{N} = 4$ SYM theory is finite and that the coupling constant receives no quantum corrections endows supersymmetric solitons of the gauge theory — BPS monopoles and instantons — with certain properties, which make them well suited for testing the above conjectures. In the case of S-duality, which is studied in the first part of the dissertation, supersymmetry imposes constraints on the geometry of the moduli space of BPS monopoles: the moduli space is required to be a hyperkähler manifold. Analyzing these hyperkähler geometries enables one to learn more about the spectrum of BPS monopoles and as a result to derive evidence in support of the S-duality conjecture.

Maldacena's conjecture relates $\mathcal{N} = 4$ SYM theory to ten-dimensional supergravity and it is with the supergravity side of the AdS/CFT correspondence that we concern ourselves. In the second part of the dissertation we study supergravity domain wall solutions and their space-time geometries. Depending on the amount of supersymmetry present in the problem, these manifolds admit additional geometrical structures besides the Riemannian metric. In particular, we are led to investigate Kähler cones over Sasakian metrics, hyperkähler cones over tri–Sasakian metrics, Calabi–Yau manifolds with Bianchi type symmetries, as well as related spaces with negative cosmological constant.

Let us now take a more detailed look at BPS monopoles and the role they play in S-duality. In the late 1970s it was conjectured [1, 2] that the $\mathcal{N} = 4$ supersymmetric $SU(2)$ gauge theory in four dimensions possesses an exact electro-magnetic duality under which the spectrum of electrically charged particles is mirrored by the spectrum of magnetically charged particles. This duality symmetry is realized by the $\mathbb{Z}_2$ group of transformations which map the weak coupling region of the theory to its strong coupling region, simultaneously interchanging states based on the elementary quanta with states based on simple soliton solutions. This symmetry is sometimes referred to as *Strong-weak coupling duality*.

Magnetically charged soliton solutions of SYM theory are supersymmetric embeddings of the classical Bogomol'nyi–Prasad–Sommerfield (BPS) monopole [3, 4]. BPS monopoles are minimal energy topological field configurations of Yang–Mills theory, and as such saturate the Bogomol'nyi energy bound [4]. The classical Bogomol'nyi bound remains true in the supersymmetric setting [5], but now the topological (magnetic) charge is interpreted as the central charge of the supersymmetry algebra. A state saturating the Bogomol'nyi bound must be annihilated by one half (or possibly some other fraction) of all supercharges present in the theory. Thus another way to characterize a BPS state is to say that it preserves a fraction of the total supersymmetry of the theory. It is a feature of $\mathcal{N} = 4$

SYM theory that it is finite, which in particular implies that the central charges do not receive quantum corrections, and hence the masses of the BPS states are not renormalized.

The Strong-weak coupling duality may be extended to $\mathcal{N} = 4$ SYM theories with arbitrary gauge groups [6]. In this case duality transformations relate the strongly coupled region of the gauge theory with gauge group G to the weakly coupled region of the gauge theory with gauge group G^* dual to G.[2] Note that for simply-laced gauge groups, the A, D and E series in the standard classification, G is self-dual (at least locally). Therefore, one finds that the duality symmetry relates the weakly and the strongly coupled regions of the same gauge theory.

Taking into account the effect of the theta angle [7], which is incorporated into the theory due to the presence of a non-trivial axion field, the $\mathbb{Z}_2$ group of the Strong-weak coupling duality is augmented to $SL(2, \mathbb{Z})$. As a consequence the spectrum of magnetic states includes BPS dyons — particles carrying both electric and magnetic charges, — which are obtained from monopole solutions by semiclassical quantization. In the current literature this duality is commonly referred to as S-duality.

As emphasized by Sen [8, 9], the S-duality conjecture makes non-trivial predictions about the spectrum of BPS dyons, and these predictions may be tested at weak coupling using semiclassical techniques. Sen reformulated the task in geometrical terms by showing that for gauge group $SU(2)$ spontaneously broken to $U(1)$ the existence of the predicted dyon states is equivalent to the existence of certain harmonic forms on the moduli space of classical BPS monopoles.

Predictions of S-duality for $\mathcal{N} = 4$ SYM theories with arbitrary gauge groups with both maximal and non-maximal symmetry breaking are more elaborate [10, 11, 12, 13, 14]. Predictions for the $SU(2)$ case are trivially embedded into the theories with higher rank gauge group. In addition, however, the matching of the electric and the magnetic spectra requires that there exist purely magnetic bound states of zero binding energy. Following the suggestion of Sen [9], these threshold bound states may also be re-interpreted as certain harmonic forms on the moduli space of relevant BPS monopole solutions.

In order to find the predicted harmonic forms with required properties it is necessary to possess the detailed knowledge of the geometry of moduli spaces of classical BPS monopoles. By counting the parameters of classical monopole solutions Weinberg showed [15, 16] that the dimension of the moduli spaces is a multiple of four. Moreover, these spaces were demonstrated to be hyperkähler manifolds [17, 18]. Therefore, the geometry we are led to study is the hyperkähler geometry.

[2] The roots of G^* are the co-roots of the roots of G.

Hyperkähler manifolds are objects of great interest in both differential geometry and theoretical physics. Hyperkähler spaces constitute a subset of complex manifolds which are simultaneously Kähler with respect to a two-sphere of complex structures. They have restricted holonomy and are thus an essential ingredient in the Berger's classification scheme of Riemannian manifolds according to their holonomy group [19, 20]. The holonomy group of a hyperkähler manifold is a subgroup of $Sp(k)$, which forces the real dimension of the manifold to be $4k$. The fact that the holonomy is contained in $Sp(k)$ also implies that a hyperkähler manifold is Ricci-flat.

There is not an exhaustive classification of hyperkähler manifolds, and every explicit example is a welcome addition to the lore. New examples of compact and non-compact hyperkähler spaces are often obtained as a result of resolving outstanding issues in physics. Since all hyperkähler manifolds are Ricci-flat they automatically solve the vacuum Einstein equations. This property is used, for instance, in constructing supersymmetric solutions to various supergravity theories. Many solutions of the eleven-dimensional supergravity were constructed in [21], where the authors considered direct products of the seven-dimensional Minkowski space with an asymptotically flat hyperkähler four-manifold; and of the three-dimensional Minkowski space with an asymptotically flat hyperkähler eight-manifold. Such solutions are BPS and are interpreted as a collection of intersecting p-branes, with a part of the hyperkähler factor playing the role of the space transverse to all branes.

Of central interest to this dissertation is the hyperkähler property of moduli spaces of BPS monopoles. In the mathematical language, these are moduli spaces of self-dual Yang–Mills connections on four-dimensional spaces admitting translational symmetry. To mention one of the major results, Atiyah and Hitchin constructed a hyperkähler four-manifold which is the moduli space of centred $SU(2)$ monopoles of topological charge two [17]. The authors made an ansatz for a rotationally invariant metric, and by using twistor methods and the self-duality equations they found the, essentially unique, solution.

In the physics community the problem of finding moduli spaces of BPS monopoles has been addressed in at least two different ways. The first relies on the *moduli space approximation* advocated by Manton [22] which states that the low-energy dynamics of BPS monopoles is equivalent to the geodesic motion on the moduli space. The exact metric on the moduli space can be computed directly by integrating over $\mathbb{R}^3$ the normalizable zero-modes of the classical monopole solution. A weak point of this method is that in most cases the family of monopole solutions is not known explicitly. In the second approach, also suggested by Manton [23], the metric is read off from the kinetic term of the

Lagrangian which describes interactions of well-separated monopoles. The main disadvantage of this method is that it yields information only about the asymptotic region of the moduli space, while the exact metric on the moduli space remains unknown.

We propose to remedy the situation in the following way. Note that hyperkähler manifolds are amenable to being studied from the point of view of symplectic geometry, and, in particular, the symplectic quotient construction can be adapted to work in the hyperkähler setting [24]. In this dissertation we use the hyperkähler quotient construction to obtain metrics on moduli spaces of BPS monopoles. Rather simple algebra leads to desired results. Unlike the latter of the two methods described above, the hyperkähler quotient produces the exact metric on the moduli space, and unlike the former method, it requires knowing only the symmetries of the monopole configuration and not the explicit form of the fields. Moreover, from the hyperkähler quotient construction one can easily deduce various global properties of the quotient manifold, such as its completeness and topological triviality. It is also straightforward to determine the isometry group of the metric and to see what subgroup of it preserves the hyperkähler structure.

Once the metric on the moduli space of a particular monopole configuration is known, one can explore both classical and quantum dynamics of the monopoles. The threshold bound states of monopoles in $\mathcal{N} = 4$ SYM theories with higher rank gauge groups predicted by S-duality are quantum in their nature. In other words, they are states in the Hilbert space of the $\mathcal{N} = 4$ quantum mechanics on the relevant moduli space [18, 25]. These quantum-mechanical bound states are in one-to-one correspondence with certain normalizable harmonic forms on the moduli space [26]. Thus the problem of checking S-duality amounts to the problem of finding such harmonic forms.

Turning to the classical dynamics of BPS monopoles we recall the moduli space approximation of Manton. According to the moduli space approximation [22], the low-energy dynamics of BPS monopoles is equivalent to the geodesic motion on the moduli space. Closed or bound geodesics on the moduli space would correspond to classical bound states of monopoles. While in the case of $SU(2)$ monopoles such classical bound states should exist, physical arguments prevent their appearance in the case of distinct fundamental monopoles. This result can be proved by a geometrical argument which exploits the hyperkähler quotient construction of the relevant moduli spaces. Although not solvable in general, the geodesic equations may be integrable for a particular ansatz. For example, one notices that geodesic equations on the moduli space of monopoles with no dyonic charges possess scaling symmetry. Thus making a scaling ansatz leads to an explicit solution. This solution describes either a simple scattering of distinct fundamental monopoles or that of well-separated $SU(2)$ monopoles.

Interestingly, there is an intimate connection between solutions of the Bogomol'nyi equation on $\mathbb{R}^3$ and solutions of the self-duality equations in one higher dimension. More precisely, periodic SYM instantons on $\mathbb{R}^4$ (in other words, instantons on $\mathbb{R}^3 \times S^1$) are equivalent to BPS monopoles on $\mathbb{R}^3$ for the same gauge group. Viewed from the string theory perspective, BPS $SU(N)$ k-monopoles are the intersections of N parallel Dirichlet 3-branes (D3-branes) with k Dirichlet strings (D-strings). Moving one of the D3-branes a finite distance away from the stack breaks the gauge group $SU(N)$ to $SU(N-1) \times U(1)$ and gives rise to a massive monopole.

SYM multi-instantons are realized as Dirichlet instantons in the presence of D3-branes of the type IIB theory. One of the ways to test Maldacena's conjecture [27, 28, 29] involves studying SYM multi-instantons for gauge group $SU(N)$. Remarkably, the moduli space of $SU(N)$ multi-instantons in the large N limit is $AdS_5 \times S^5$. On the supergravity side of the correspondence, $AdS_5 \times S^5$ represents the near-horizon geometry of N coincident D3-branes in the limit when N is large. Computations made on the string theory side [30] are in accord with those made on the gauge theory side [31], and hence the results support the suggested correspondence. Although the connection between SYM instantons and BPS monopoles is evident, it is yet unclear what the correct description of the latter is once the limit of large N is taken. In other words, it remains an open problem to find the "image" of BPS monopoles on the supergravity side of the AdS/CFT correspondence.

Let us investigate more closely the supergravity side of the AdS/CFT correspondence. In the original version of the conjecture the near-horizon geometry of N coincident D3-branes was considered in the limit of large N. The associated supergravity solution near the branes is $AdS_5 \times S^5$. The boundary of AdS_5 at infinity is the Minkowski four-space M_4. The CFT on M_4, which is conjectured to be equivalent to the supergravity theory in the bulk $AdS_5 \times S^5$, has maximal number of supersymmetries in four dimensions, $\mathcal{N} = 4$. Replacing the compact factor S^5 by some other Einstein five-manifold X_5 leads to reduced supersymmetry, and the corresponding four-dimensional SCFT differs from the large N limit of $\mathcal{N} = 4$ SYM theory. If the manifold X_5 preserves no supersymmetry, that is its holonomy group is unrestricted, then the construction of the field theory is difficult. The supergravity solution interpolating between the near-horizon $AdS_5 \times X_5$ geometry and the $M_4 \times C(X_5)$ geometry at the boundary ($C(X_5)$ is a metric cone over X_5) leads to the interpretation of the associated SCFT as the world-volume theory of D3-branes this time placed at a conical singularity of $M_4 \times C(X_5)$ [32].

It is possible to generalize the conjecture further and consider p-brane solutions in eleven dimensions interpolating between near-horizon geometries $AdS_{p+2} \times X$ and geometries $M_{p+1} \times C(X)$ at infinity [33]. Here X is a $(11-(p+2))$-dimensional Einstein manifold and $C(X)$ is a metric cone over X. One can consider, for example, M-theory 2-branes (M2-branes), $p = 2$, for which X is a seven-manifold. The dual three-dimensional SCFT would be the world-volume theory on the M2-branes placed at a singularity of $M_{p+1} \times C(X)$.

To identify or construct the SCFTs it is helpful to take manifolds X_5 and X with reduced holonomy so that they preserve some degree of supersymmetry. Thus one is led to consider Sasakian and tri–Sasakian manifolds, metric cones over which are Kähler and hyperkähler manifolds respectively. This is another instance where supersymmetry considerations determine the relevant geometrical structures.

There are reasons to believe (see e.g. [34]) that the AdS/CFT correspondence is a special case of a more widely applicable correspondence between supergravity theories[3] and quantum field theories in one lower dimension. Written in horospherical coordinates the d-dimensional AdS metric, which is invariant under $SO(d-2,2)$, is a special case of a domain wall metric. Hence the near-horizon geometry looks like a domain wall solution of a (possibly compactified) supergravity theory. It is then of interest to investigate domain wall solutions of ten- and eleven-dimensional supergravity theories and their compactifications. In particular, we can consider solutions of the form $\mathcal{M}_4 \times \mathbb{E}^{p-3,1}$, where $\mathcal{M}_4$ is a non-compact Riemannian manifold which is either Ricci-flat or has negative cosmological constant. Such a domain wall would be a solution of a $(p+1)$-dimensional supergravity theory. We are then looking for metrics on $\mathcal{M}_4$ which depend on only one coordinate transverse to the brane and are invariant under a transitively acting Lie group with three-dimensional orbits. Such groups have been classified by Bianchi [35]. Four of the Bianchi type groups, Bianchi types I, II, VI$_0$ and VII$_0$, yield domain wall solutions, some of which have already been found as supergravity domain walls.

Conventionally, orbifold domain wall metrics have a singularity at the location of the wall where a distributional energy source has been inserted. This treatment is unnatural for two reasons. First, there are no obvious sources in M-theory, and secondly, the induced metric on the domain wall itself is singular. Taking a different viewpoint one may try to resolve the singularity. It is indeed possible to do so, but in the process one needs to identify the two sides of the domain wall and regard the conventional solution as accurate only asymptotically. Near the wall this asymptotic metric becomes

[3] One needs to be careful to consider only those supergravity theories which are effective theories for some consistent quantum theories.

a non-singular gravitational instanton. In fact, for several of the Bianchi type solutions such four-dimensional gravitational instantons do indeed exist. These solutions may be thought of as domain walls at the end of the universe. Sufficiently far away from the wall the conventional picture of an orbifold domain wall pertains, while near the wall the space-time decompactifies.

Thesis Outline:

Chapter 2 provides the background material for Chapters 3 to 5 which deal with BPS monopoles, hyperkähler geometry and S-duality conjecture. Section 2.1.1 discusses BPS monopoles in gauge theories, explaining the physical framework and presenting various monopole solutions. S-duality predictions are discussed in Section 2.1.2, where we outline the predictions for higher rank gauge groups emphasizing the ones that we wish to check. In Section 2.1.3 we discuss various methods which have been used until now to construct hyperkähler metrics on monopole moduli spaces. The second half of this review chapter, Section 2.2, introduces the mathematical framework. We define a hyperkähler manifold in Section 2.2.1, and supply a number of examples in Section 2.2.2. Finally, our main mathematical tool for finding metrics on monopole moduli spaces — the hyperkähler quotient construction — is introduced in Section 2.2.3.

Chapter 3 is devoted to the application of the hyperkähler quotient construction. We first describe the features of the construction which are common to all forthcoming examples. In Section 3.2 we reproduce the following previously known hyperkähler manifolds: Euclidean Taub–NUT, cyclic ALE, cyclic ALF, Calabi and Taubian–Calabi. Then in Section 3.3 we construct moduli spaces of fundamental monopoles. Section 3.3.1 contains the hyperkähler quotient construction of the relative moduli space of distinct fundamental monopoles for arbitrary gauge group with maximal symmetry breaking — the so-called Lee–Weinberg–Yi manifold. One of the virtues of our method is that the construction may be easily modified to describe two degenerate limits of the Lee–Weinberg–Yi space. These are moduli spaces for theories containing non-abelian monopoles (Section 3.3.2) and moduli spaces for theories containing fixed (infinitely massive) monopoles (Section 3.3.3). Section 3.4 contains the construction of the Gibbons–Manton metric, which is the asymptotic metric on the moduli space of many $SU(2)$ monopoles. Finally, in Section 3.5 we discuss an alternative interpretation of monopole moduli spaces in terms of intersecting branes proposed by Hanany and Witten.

Chapter 4 contains results on normalizable harmonic forms whose existence is predicted by the S-duality conjecture. We discuss S-duality predictions only for theories with gauge group of rank

greater than one, i.e. gauge groups other than $SU(2)$. When the gauge group is broken to its maximal torus, there exists a unique purely magnetic threshold bound state of distinct fundamental monopoles. The construction of associated harmonic form is reviewed in Section 4.2. If the unbroken gauge group contains a non-abelian factor, one expects to find a unique threshold bound state of a number of massive and massless monopoles. In Section 4.3 a proposal is made for a candidate harmonic form, and the particular case of two massive fundamental monopoles is treated in detail.

In Chapter 5 some aspects of classical low-energy dynamics of fundamental monopoles are discussed. Section 5.1 contains the proof of the non-existence of closed or bound geodesics on the Lee–Weinberg–Yi and the Taubian–Calabi manifolds, which would correspond to classical bound states of distinct fundamental monopoles. In Section 5.2 we solve the geodesic equations on the Lee–Weinberg–Yi manifold with a homothety ansatz to find a solution that describes the simplest scattering process of distinct fundamental monopoles carrying no dyonic charges. Some comments pertinent to the case of well-separated $SU(2)$ monopoles with no dyonic charges are also made. In Section 5.2.3 we make another ansatz that describes a rigidly rotating configuration of monopoles in a plane and discuss the results.

In Chapter 6 we discuss rigid $\mathcal{N} = 2$ supersymmetric hypermultiplets in four dimensions. We demonstrate that the target space geometry has to be a metric cone over a tri–Sasakian manifold. We also discuss metric cones over Sasakian and Sasakian–Einstein manifolds and their applications in the context of cone-branes and the AdS/CFT correspondence. The Sasakian and tri–Sasakian geometries are reviewed in Section 6.1.

In Chapter 7 domain walls in M-theory are studied. We consider supergravity domain wall solutions invariant under transitive actions of various Bianchi type groups. Section 7.1 reviews briefly the Bianchi classification of Lie groups with three-dimensional orbits. Sections 7.2 and 7.5 describe four-dimensional Ricci-flat solutions, with particular emphasis on the Bianchi type II case which is BPS. In Section 7.3 we argue that it is desirable to resolve the singularity at the origin of the Bianchi type II four-manifold and propose such a resolution. The resulting manifold may be interpreted as a single-sided domain wall, or a "domain wall at the end of the universe". The solutions are described in a unifying fashion in terms of the Kähler potential which solves appropriate Monge–Ampère equations. In Sections 7.4 and 7.5.3 these equations are modified to include a negative cosmological constant and solved to find metrics with various Bianchi type symmetries. In Section 7.6 all the four-dimensional solutions are generalized, with the use of solutions to the Monge–Ampère equations, to higher-dimensional Calabi–Yau manifolds, as well as to related manifolds with negative cosmological constant.

Finally, Chapter 8 summarizes the results presented in the thesis.

CHAPTER 2

Monopoles and Hyperkähler Geometry: a Review

This review chapter is divided into two parts. In the first part, Section 2.1, we summarize a few important results concerning the conjectured S-duality symmetry of four-dimensional $\mathcal{N} = 4$ supersymmetric Yang–Mills theory. We concern ourselves with theories whose gauge groups have rank higher than one, although a few comments are made about the $SU(2)$ case. In Section 2.1.1 we first describe BPS monopole solutions of ordinary Yang–Mills–Higgs theory with gauge group $SU(2)$. These can be naturally embedded into $\mathcal{N} = 4$ $SU(2)$ SYM theory. We then move on to discuss soliton solutions of SYM theories with higher rank gauge group G, taking $G = SU(n)$ as an example. We explain what it means for the gauge group to be broken maximally or non-maximally. Then, in the case of maximal symmetry breaking, we describe how a number of soliton solutions of this theory may be obtained as embeddings of the basic $SU(2)$ monopole. In particular, we describe how fundamental monopole solutions are constructed. We briefly discuss monopoles in theories with non-abelian unbroken gauge group. Predictions of S-duality for both cases are reviewed in Section 2.1.2. This prompts us to look for metrics on the moduli spaces of monopoles. In Section 2.1.3 we discuss various methods of computing these metrics.

The second part of this chapter is devoted to hyperkähler geometry. In Section 2.2.1 we give a definition of a hyperkähler manifold, followed by a number of examples in Section 2.2.2. These include Euclidean Taub–NUT, asymptotically locally flat (ALF), asymptotically locally Euclidean (ALE) and Calabi metrics. The hyperkähler quotient construction is introduced in Section 2.2.3. There we also present two types of basic group actions from which all the group actions used in

Chapter 3 are constructed. These are real translations and $U(1)$ rotations. We find moment maps associated with these group actions.

2.1 S-duality and BPS Monopoles

2.1.1 Monopoles in Gauge Theories

Classical Euler–Lagrange equations of $SU(2)$ Yang–Mills–Higgs theory with a symmetry breaking potential admit static solitonic solutions [36, 37]. In the asymptotic region the scalar Higgs field is approximately constant and breaks the $SU(2)$ gauge symmetry to $U(1)$. In this region one can define a magnetic field in the usual sense. From far off the topological solitons look like point sources of the magnetic and the scalar Higgs fields, which is the reason for calling these solitons magnetic monopoles. The $SU(2)$ magnetic monopoles differ from the abelian Dirac monopole in that they have an extended core and are everywhere smooth field configurations.

If one regards the symmetry breaking as a boundary condition at infinity, a limit may be taken consistently in which the Higgs field potential vanishes. This is the Prasad–Sommerfield limit [3]. Topological soliton solutions persist. In fact, in the Prasad–Sommerfield limit there is a first order equation — the Bogomol'nyi equation [4] — describing static field configurations of the theory of minimal energy. Solutions of this equation automatically solve the Euler–Lagrange equations and are called Bogomol'nyi–Prasad–Sommerfield (BPS) monopoles. The explicit solution for a monopole of unit charge is given in [3]. This solution has four parameters, or moduli: three translational, associated with its position in $\mathbb{R}^3$, and one dyonic, associated with the $U(1)$ phase. Hence the parameter, or moduli space of a charge one BPS monopole is the flat $\mathbb{R}^3 \times U(1)$.

It is possible to define global magnetic charge of a BPS monopole which is topological in nature. This topological charge takes values in $\pi_2\left(\frac{SU(2)}{U(1)}\right) = \mathbb{Z}$ and hence is integral. BPS soliton solutions with topological (magnetic) charge k are called k-monopoles. The moduli space is the parameter space of all gauge-inequivalent solutions in one topological sector. Weinberg [15] showed that the dimension of this moduli space is $(4k-1)$, and gave a physical interpretation to these parameters. A k-monopole may be viewed as a superposition of k monopoles of unit charge with large inter-monopole separations [38]. Each 1-monopole is characterized by three positions and one $U(1)$ orientation. Since only the relative $U(1)$ orientation has physical significance, the total number of non-gauge zero-modes of a k-monopole is $3k + (k-1) = 4k - 1$. Adding one overall phase parameter, the overall $U(1)$ charge, renders the k-monopole moduli space $4k$-dimensional.

Yang–Mills–Higgs $SU(2)$ theory is embedded into the bosonic part of four-dimensional SYM theory with $\mathcal{N} = 4$ supersymmetry.[1] As before, the gauge and scalar Higgs fields take values in the Lie algebra $\mathfrak{su}(2)$ of $SU(2)$ and transform in the adjoint representation. BPS monopoles of YMH theory are also present in this supersymmetric extension. BPS monopoles are minimal energy configurations and as such saturate the Bogomol'nyi energy bound

$$M \geq |Z|,$$

where Z is, schematically, the central charge of the supersymmetry algebra [5]. States saturating the Bogomol'nyi bound are annihilated by one half of the supersymmetry charges. There are 16 supercharges in the $\mathcal{N} = 4$ supersymmetry algebra in four dimensions, and eight of them annihilate the BPS state. Hence BPS states fall into a $2^{\frac{8}{2}} = 16$-dimensional representation of the supersymmetry algebra, called the ultra-short representation.

Topological solitons are not unique to the $SU(2)$ gauge theory. Solutions of this kind exist in $\mathcal{N} = 4$ SYM theories with arbitrary gauge groups [16, 39]. In what follows we restrict our discussion to $SU(n)$ gauge theory, although all the concepts and remarks apply with some modification to arbitrary compact semi-simple gauge groups. The gauge and the Higgs fields take values in $\mathfrak{su}(n)$ and transform in the adjoint representation. Fixing the Higgs field to be constant at infinity $\phi_\infty = (t_1, \dots, t_n)$, $t_1 + \dots + t_n = 0$, spontaneously breaks the $SU(n)$ gauge symmetry to a subgroup $H \subset SU(n)$, which consists of elements of $SU(n)$ commuting with ϕ_∞.

Generically, all t_i's are distinct and H is the maximal torus $U(1)^{n-1}$ of $SU(n)$. We refer to this case as *maximally broken*. It might, however, happen that some of the t_i's coincide in which case the residual gauge symmetry is enhanced. For instance, when two t_i's are equal, one of the $(n-1)$ $U(1)$ factors is replaced by an $SU(2)$ factor. We refer to this case as *non-maximally broken*.

Let us treat the maximally broken case first. It is still possible to define a magnetic field in the asymptotic region, which now takes values in $\mathfrak{su}(n)$. Similarly, a set of magnetic charges may be globally defined which now take values in $\pi_2\left(\frac{SU(n)}{U(1)^{n-1}}\right) = \mathbb{Z}^{n-1}$. Thus a magnetic soliton solution is labelled by an $(n-1)$-vector $(n_1, \dots, n_{n-1})$ of topological charges of distinct types. A number of monopole solutions in this theory may be obtained by a principle embedding of the $SU(2)$ BPS monopole along a root of $\mathfrak{su}(n)$. There is a distinguished set of $(n-1)$ solutions called *fundamental*

[1] The supermultiplet of $\mathcal{N} = 4$ SYM theory contains six Higgs scalars, but we can choose to work in the unitary gauge, which breaks the $SO(6)$ $\mathcal{R}$-symmetry to $SO(5)$, and the theory has effectively one scalar field. We should also remark that the choice of unitary gauge naturally leads to the Prasad–Sommerfield limit, because the Higgs field potential $V(\phi^I) = tr[\phi^I, \phi^J]^2$, $I, J = 1, \dots, 6$, vanishes identically for one effective scalar field.

monopoles, that are obtained as embeddings of a unit charge $SU(2)$ monopole along a simple root of $su(n)$. The i-th fundamental monopole has charge $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the i-th place. The reason for calling these monopoles fundamental is twofold. First, being an embedding of a charge one $SU(2)$ solution, each fundamental monopole has four degrees of freedom: three translational and one phase. There is no room for "internal" structure. Secondly, a charge $(n_1, \dots, n_{n-1})$ monopole may be viewed as a superposition of n_1 monopoles of type $(1, 0, \dots, 0)$, n_2 monopoles of type $(0, 1, 0, \dots, 0)$, etc. This interpretation is supported by the BPS mass formula as well as by parameter counting. The moduli space of such a monopole has $4\sum_{i=1}^{n-1} n_i$ parameters, whose physical meaning is analogous to that of the parameters of an $SU(2)$ k-monopole.

Only monopoles obtained as $SU(2)$ embeddings along simple roots which are linked in the Dynkin diagram of $SU(n)$ interact. Note that there is no physical mechanism that mixes phases of fundamental monopoles of different types, i.e. of monopoles charged with respect to different $U(1)$ factors. From this fact we can infer that the moduli space of charge $(1, 1, \dots, 1)$ monopole possess a $U(1)^{n-1} = T^{n-1}$ symmetry. Moreover, for $n > 2$ there exists a spherically symmetric solution [16] of charge $(1, \dots, 1)$ with $4(n-1)$ normalizable zero-modes. This spherically symmetric solution may be interpreted as a superposition, at the same point, of the appropriate number of fundamental monopoles of different types. In the moduli space of the $(n_1, \dots, n_{n-1})$-monopole this solution corresponds to a fixed point of $SO(3)$ rotations.

Let us now turn to the case of non-maximal symmetry breaking. When $(n-r-1)$ t_i's coincide, the unbroken gauge symmetry is $H = K \times U(1)^r$, where K is a subgroup of $SU(n)$ of rank r. A number of topological charges can still be defined. They take values in $\pi_2\left(\frac{SU(n)}{K \times U(1)^r}\right) = \mathbb{Z}^r$, and we see that theories with non-abelian residual symmetry have fewer topological charges than theories whose unbroken gauge group is $H = U(1)^{n-1}$. A BPS monopole solution is still labelled by $(n-1)$ integers, r of which are topological charges and the remaining $(n-r-1)$ are non-abelian charges [16]. It does not seem possible to define a global non-abelian charge in the presence of a monopole [40]. There are also other conceptual problems that arise in connection with non-abelian charges, but it is beyond the scope of this brief review to dwell on them. We merely point out that monopole configurations may be found for which the non-abelian part of the long-range magnetic field vanishes and it is still possible to talk about a moduli space with a well defined metric. In the limit when the unbroken $H = U(1)^{n-1}$ gauge symmetry is enhanced to $H = K \times U(1)^r$, masses of fundamental monopoles associated with $(n-r-1)$ roots of the Cartan subalgebra of K tend to zero. These

fundamental monopoles become massless. An isolated massless soliton is not a well defined physical object since the size of its core becomes infinite. Even in the presence of a massive monopole the physical interpretation of massless solitons carrying a non-abelian charge is still not well understood.

A particular case of non-abelian symmetry breaking that we investigate in detail in the following chapters is $r = 2$. This theory admits two fundamental monopoles and $(n-3)$ massless monopoles, the latter forming a non-abelian cloud. The physics of this non-abelian cloud for $n = 4$ was studied in [41]. In analogy with monopole solutions obtained via an $SU(2)$ embedding along some root in the Lie algebra, SYM theory with unbroken gauge group $U(1) \times SU(n-2) \times U(1)$ admits a number of solutions, which are obtained by embedding the solution of the $SU(4)$ gauge theory broken to $U(1) \times SU(2) \times U(1)$ consisting of one massless and two massive monopoles [42]. This monopole solution has charge $(1, [1, \dots, 1], 1)$, where the $(n-3)$ integers inside square brackets denote non-abelian charges. In [12] an interpretation of this solution was put forward to the effect that it consists of two massive fundamental monopoles and a cloud of $(n-3)$ massless monopoles.

2.1.2 S-duality Predictions

As explained in the Introduction, S-duality is a conjectured exact symmetry of $\mathcal{N} = 4$ SYM theory which acts non-trivially on the gauge coupling $\tau = \theta/2\pi + ig^{-2}$, simultaneously acting on electric and magnetic quantum numbers of the states in the theory. The group of S-duality transformations for SYM theories with simply-laced gauge groups is $SL(2, \mathbb{Z})$. This duality symmetry is non-perturbative, in the sense that it is not expected to be valid order by order in the power series expansion in the gauge coupling g. Therefore, the only feasible way to test the S-duality conjecture is to study quantities, whose tree-level values are unchanged by quantum corrections [8].

States in the ultra-short multiplet of the $\mathcal{N} = 4$ supersymmetric theory, BPS states which break one half of the total supersymmetry, have the desirable property. Since they saturate the Bogomol'nyi energy bound, their masses are equal to the central charge of the supersymmetry algebra which is not renormalized. Hence if one assumes that $\mathcal{N} = 4$ SYM theory does not undergo any phase transitions, the dimension of a supersymmetry multiplet cannot change and the spectrum of BPS states remains unaltered for any value of the coupling. Electrically charged states saturating the Bogomol'nyi energy bound are gauge bosons, and magnetically charged BPS states are magnetic monopoles discussed in the previous section.

The $SL(2, \mathbb{Z})$ duality group acts on the complex gauge coupling $\tau = \theta/2\pi + ig^{-2}$ by fractional linear transformations:

$$\tau \to \frac{a\tau + b}{c\tau + d},$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The duality group has two generators: $\tau \to -1/\tau$ and $\tau \to \tau + 1$. If we set the theta angle to zero, $\theta = 0$, the first generates Strong-weak coupling duality of Montonen and Olive [1]. In this case the strongly coupled region of the theory is equivalent to the weakly coupled region with the roles of the elementary electric quanta (the gauge bosons) and the solitonic states (the magnetic monopoles) interchanged. In general, S-duality transformations map BPS states into BPS states.

Let us first consider $SU(2)$ $\mathcal{N} = 4$ SYM theory [9]. Given the existence of a gauge boson with electric and magnetic quantum numbers $\{1, 0\}$ at weak coupling, S-duality predicts the existence of BPS states with quantum numbers $\{a, -c\}$ at a transformed value of the coupling τ (a and c are necessarily relatively prime). Assuming that the spectrum of states varies continuously with coupling, i.e. the theory does not undergo a phase transition, we conclude that the states $\{a, -c\}$ must exist at all values of τ, in particular at weak coupling. States with charge quantum numbers $\{0, 1\}$ are unit charge $SU(2)$ monopoles of Section 2.1.1. States with charge quantum numbers $\{a, 1\}$, $a \neq 0$, are BPS dyons which may be obtained from the classical monopole solution by semiclassical quantization. In [9] Sen deduced that for S-duality symmetry to hold, there must exist bound states of c monopoles and/or dyons each carrying unit magnetic charge.

Since the duality symmetry is the symmetry of the full quantum theory, these bound states can not be detected at the classical level. Hence to find the predicted bound states we must quantize the collective zero-modes of the classical $SU(2)$ c-monopole solution, which requires knowing the metric on the moduli space of such a monopole.

Let us reformulate in geometrical terms the task of looking for quantum bound states of monopoles. Following [18] Blum showed [25] that the low-energy dynamics of BPS monopoles is equivalent to the $\mathcal{N} = 4$ supersymmetric quantum mechanics on the moduli space. This is a supersymmetric version of the moduli space approximation of Manton [22]. States in the Hilbert space of this quantum mechanics are in one-to-one correspondence with square-integrable real differential forms on the moduli space of the centred monopole configuration. Four anti-commuting supersymmetry operators may be identified with differential operators d, $\bar{\partial}_I, \bar{\partial}_J, \bar{\partial}_K$ acting on forms, where d is the exterior derivative and $\partial_I, \partial_J, \partial_K$ are the Dolbault operators associated to the three complex structures I, J, K ($d^\dagger$, $\partial_I^\dagger$, $\partial_J^\dagger$, $\partial_K^\dagger$ are their respective adjoints). The quantum-mechanical Hamiltonian becomes the

Hodge–de Rahm Laplacian

$$d\,d^\dagger + d^\dagger\,d = 2(\bar{\partial}_I\bar{\partial}_I^\dagger + \bar{\partial}_I^\dagger\bar{\partial}_I + \bar{\partial}_J\bar{\partial}_J^\dagger + \bar{\partial}_J^\dagger\bar{\partial}_J + \bar{\partial}_K\bar{\partial}_K^\dagger + \bar{\partial}_K^\dagger\bar{\partial}_K)\,.$$

A BPS state is an eigenstate of this Laplacian. For reasons explained in [9], we are particularly interested in eigenstates with zero eigenvalue, which are normalizable harmonic forms.

We conclude the discussion of the $SU(2)$ case by mentioning that the exact metric on the moduli space of charge c $SU(2)$ monopoles is known only for $c = 2$, it is the Atiyah–Hitchin manifold [17]. Sen found a unique normalizable harmonic two-form on the Atiyah–Hitchin space thus showing that the bound state predicted by S-duality exists.

Let us now discuss predictions of S-duality for theories with gauge group $SU(n)$. Already in the maximally broken case these predictions are more elaborate. Firstly, note that the $SU(2)$ S-duality predictions are "embedded" into the $n > 2$ case, i.e. associated to each charge $(0, 0, \ldots, k, \ldots, 0)$ fundamental monopole there is an infinite tower of dyonic bound states *à la* Sen. In addition, there are purely magnetic bound states of several fundamental monopoles as we shall demonstrate presently.

Electrically charged BPS states are gauge particles that aquire mass through the Higgs mechanism. For abelian unbroken gauge group $U(1)^{n-1}$ only $(n-1)$ photons with the charge vector corresponding to a Cartan generator stay massless, while the remaining $(dimSU(n) - (n-1))$ particles acquire mass. These are massive gauge bosons, and their number, $\frac{1}{2}(dimSU(n) - n + 1)$, is equal to the number of fundamental monopoles $(n-1)$ only for $n = 2$. In order that the electric and the magnetic spectra match there must appear $\frac{1}{2}(dimSU(n) - (n-1)) - (n-1)$ additional magnetic states.

Consider the simplest case $n = 3$, with gauge group $SU(3) \to U(1)^2$. There are three electrically charged vector mesons, whose charges in the two unbroken $U(1)$'s are $(1, 0)$, $(0, 1)$ and $(1, 1)$. From the BPS mass formula we know that the mass of the third gauge boson is equal to the sum of the masses of the first two. The S-duals of the first two objects are fundamental monopoles associated with the two simple roots of $\mathfrak{su}(3)$. Then the dual of the third one must be a charge $(1, 1)$ monopole, which we know from the previous section to be the bound state, with zero binding energy, of the two fundamental monopoles. Such a zero-energy bound state is called a *threshold bound state* because it is only marginally stable against decaying into its fundamental constituents. A very clear description of the simplest $SU(3)$ case can be found in [10].

For arbitrary values of n S-duality predicts the existence of a unique threshold bound state in the topological sector of charge $(1, 1, \ldots, 1)$. In geometrical terms this means that one expects to find a unique normalizable harmonic form on the moduli space of centred charge $(1, 1, \ldots, 1)$ monopole.

The predicted harmonic form was exhibited in [10, 43] for gauge group $SU(3) \to U(1)^2$, and in [44] for gauge group $SU(n) \to U(1)^{n-1}$.

When the gauge group is broken non-maximally to $H = K \times U(1)^r$, the electrically charged sector contains massless elementary excitations. If S-duality is to hold, one expects to find massless solitons charged under non-abelian gauge subgroup K. In addition, as in the maximally broken case, there are massive gauge bosons for every $U(1)$ factor in H. S-duality predictions for the massive sector are much the same as those in the maximally broken case. One should keep in mind, however, that some of the massive bosons and hence their solitonic duals will form non-trivial representations of K. A detailed discussion of S-duality for non-abelian gauge theories is given in [13, 14]. Here we shall focus on the theory with gauge group $SU(n) \to U(1) \times SU(n-2) \times U(1)$. Let us present a detailed counting for $n = 4$, showing that there must appear a threshold bound state of two massive fundamental monopoles and one massless one.

$SU(4) \to U(1) \times SU(2) \times U(1)$

All the fields transform in the adjoint representation of the gauge group. The elementary particle sector of $\mathcal{N} = 4$ SYM theory with gauge group $SU(4)$ has $15 = dim(Ad_{SU(4)})$ degrees of freedom. The unbroken gauge group $U(1) \times SU(2) \times U(1)$ has the following decomposition into irreducible representations:

$$\mathbf{15} = \underbrace{\mathbf{1} \oplus \mathbf{1} \oplus \mathbf{3}}_{\text{massless sector}} \oplus \underbrace{\mathbf{2} \oplus \bar{\mathbf{2}} \oplus \mathbf{2} \oplus \bar{\mathbf{2}} \oplus \mathbf{1} \oplus \bar{\mathbf{1}}}_{\text{massive sector}} . \qquad (2.1)$$

What are the states making up the components in this decomposition? Let us denote each elementary state by its electric charge vector with respect to the three components of the unbroken gauge group $U(1) \times SU(2) \times U(1)$. For clarity, we put square brackets around the non-abelian charge. With these conventions we can write massive electrically charged states in (2.1) as follows:

$$\left.\begin{array}{l}(1,[0],0)\\(1,[1],0)\end{array}\right\} \text{ doublet of } SU(2) \qquad (2.2)$$

$$\left.\begin{array}{l}(0,[0],1)\\(0,[1],1)\end{array}\right\} \text{ doublet of } SU(2)$$

$$(1,[1],1) \quad \text{singlet of } SU(2)$$

The first pair of states — a boson carrying one unit of abelian $U(1)$ charge and another boson carrying in addition a unit of non-abelian charge — transform in the fundamental representation $\mathbf{2}$ of $SU(2)$. Every boson has its antiparticle partner, hence there is a pair — anti-$(1,[0],0)$ and anti-$(1,[1],0)$ — transforming in the anti-fundamental representation $\bar{\mathbf{2}}$ of $SU(2)$. The same holds for the second pair of bosons in (2.2) which are charged with respect to the other $U(1)$ factor. The BPS mass formula does not distinguish between the two states in the doublet, since the non-abelian excitation is massless. However, the presence of a massless excitation is manifest in the symmetries of the moduli space.

The massless sector consists of three photons — one for each $U(1)$ factor and one for the $SU(2)$ factor, — a massless gauge boson $(0,[1],0)$ carrying a unit of non-abelian charge and its anti-particle. The non-abelian photon, the massless gauge boson and its anti-particle transform in the adjoint representation $\mathbf{3}$ of $SU(2)$, and the two other photons are singlets of $SU(2)$.

These massless and massive states account for all the available degrees of freedom in the system, and are in accord with decomposition (2.1). The state of most interest to us is the S-dual of the massive singlet $(1,[1],1)$. The BPS mass formula shows that the mass of this state is the sum of the masses of two states, one from each of the doublets in (2.2).

For $n > 4$ S-duality demands the existence of a threshold bound state of two fundamental and $(n-3)$ massless monopoles, which is the S-dual of the massive singlet $(1,[1,\ldots,1],1)$.

2.1.3 Metrics on Monopole Moduli Spaces

It is clear from the above discussion that in order to check predictions of S-duality one needs to know the metric on the moduli space of BPS monopoles and to search for certain harmonic forms on the moduli space. In this section we would like to summarize methods of constructing moduli space metrics which have been used until now.

The exact metric on the moduli space of monopole solutions may be constructed by direct computations in field theory as follows. Static BPS configurations of a given magnetic (topological) charge which are not related by local gauge transformations, are parametrized by m collective coordinates z_a, $a = 1,\ldots,m$. These collective coordinates are called moduli and are regarded as coordinates on the moduli space of these monopole configurations. In the moduli space approximation proposed by Manton [22] one assumes that small time-dependant fluctuations around a static BPS solution can be approximated by a configuration that is gauge equivalent to one of these BPS solutions. Physically, this assumption means that objects with small relative velocities and/or arbitrary small electric charges are, in some sense, almost BPS. It is a property of static BPS multi-monopole and multi-dyon solutions that there is no net force between the constituent objects — the electric and magnetic repulsion is balanced by the scalar attraction. Therefore, low-energy dynamics of the system is described by the kinetic term of the bosonic part of the Lagrangian only. It can be expressed (see e.g. [18]) in terms the background gauge zero-modes $\delta_a A_\mu$:

$$g_{ab} = \int d^3x \, \mathrm{tr}(\delta_a A_\mu \delta_b A_\mu) \, ,$$

where $A_\mu = (\Phi, A_i)$, $i = 1, 2, 3$, is the Higgs and the gauge fields. Thus low-energy dynamics of the fields is reduced to the dynamics of a point particle moving along geodesics on the n-dimensional moduli space. Although this method is quite straightforward, it has a definite drawback. In most cases the full family of solutions of a given magnetic charge is unknown and it is, therefore, impossible to calculate the complete set of background gauge zero-modes.

Another approach works for any monopole configuration but is valid only in the asymptotic region of the moduli space where monopoles are widely separated. It has been developed, and first applied by Manton [23] and consequently applied by Gibbons and Manton in [45]. It uses the fact that the long-range interactions between slowly moving monopoles are well understood, which allows one to construct the Lagrangian and deduce the metric g_{ab} from it. In [45] Gibbons and Manton applied this method to the case of $SU(2)$ monopoles of arbitrary charge. They found the asymptotic metric, which we refer to as the Gibbons–Manton metric. The Gibbons–Manton metric is not a good candidate for the exact metric on the moduli space since it has indefinite signature and develops singularities in the interior.

Lee, Weinberg and Yi [11] (see also [10, 46]) followed the same line of thought and constructed the asymptotic metric on the moduli space of distinct fundamental monopoles for any simple gauge group G broken to its maximal torus. They conjectured that this asymptotic metric, which we refer to as the Lee–Weinberg–Yi (LWY) metric, was, in fact, the exact metric on the moduli space. This statement was later proved to be true by Murray in [47] and by Chalmers in [48].

As we have already mentioned in Section 2.1.1, the number of normalizable zero-modes of a solution to the Bogomol'nyi equations is a multiple of four. Moreover, the $4n$-dimensional moduli space is hyperkähler (see Section 2.2). Atiyah and Hitchin [17] gave a proof of this fact based on mathematical properties of the Bogomol'nyi equations. An alternative proof due to Gauntlett [18] relies on his finding that quantum low-energy dynamics of monopoles is equivalent to the $\mathcal{N} = 4$

supersymmetric quantum mechanics on the moduli space. This quantum mechanical model is nothing else than the $\mathcal{N} = 4$ supersymmetric one-dimensional non-linear sigma-model, whose target space is the moduli space. Extended $\mathcal{N} = 4$ supersymmetry imposes constraints on the geometry of the target space: it requires the target space, and consequently the moduli space, to be a hyperkähler manifold.

It turns out that in some cases the isometries of the moduli space, deduced from the space-time and internal symmetries of the theory, together with the hyperkählerity requirement completely determine the metric on the moduli space. This approach was used by Atiyah and Hitchin [17] for the moduli space metric of charge 2 $SU(2)$ monopole and by Gauntlett and Lowe [10] and Lee, Weinberg and Yi [43] for the charge $(1,1)$ $SU(3)$ monopole.

Atiyah and Hitchin argued that the eight-dimensional moduli space of charge two $SU(2)$ monopole can be decomposed into a four-dimensional moduli space of centred monopoles, $\mathcal{M}^{rel}$, and the flat moduli space of the centre of mass:

$$\mathcal{M} = \mathbb{R}^3 \times \frac{S^1 \times \mathcal{M}^{rel}}{\mathbb{Z}^2} .$$

They showed further that $\mathcal{M}$ must be hyperkähler, and since $\mathbb{R}^3 \times S^1$ carries a flat hyperkähler structure, $\mathcal{M}^{rel}$ must also be hyperkähler. Besides, the isometry group of $\mathcal{M}^{rel}$ is just $SO(3)$. They looked for such a complete hyperkähler four-manifold by applying the twistor construction and solving the Monge–Ampère equations, and arrived at a metric expressed in terms of complete elliptic integrals now known as the Atiyah–Hitchin metric.

Now consider charge $(1,1)$ $SU(3)$ monopoles. The moduli space of charge $(1,1)$ monopoles was first studied by Connell [46] using Nahm data, and later by Gauntlett and Lowe [10] and Lee, Weinberg and Yi [43]. They showed that the moduli space can be isometrically decomposed as:

$$\mathcal{M} = \mathbb{R}^3 \times \frac{\mathbb{R} \times \mathcal{M}^{rel}}{\mathbb{Z}} .$$

The $\mathbb{R}^3$ factor corresponds to spatial translations of the centre of mass and the $\mathbb{R}$ factor to the overall $U(1)$ phase. All the information about the monopole interactions is contained in the metric on the relative, or centred, moduli space $\mathcal{M}^{rel}$. The exact form of the metric on $\mathcal{M}^{rel}$ may be deduced from the symmetry, hyperkählerity and the asymptotic behaviour of monopoles. The isometry group of $\mathcal{M}^{rel}$ for any monopole configuration must contain an $SO(3)$, or even an $SU(2)$, factor since the centre of mass motion has been taken out. From Section 2.1.1 we know that there exists a spherically symmetric monopole solution, which is the superposition of two fundamental monopoles at the origin of the relative moduli space. Hence the origin of $\mathcal{M}^{rel}$ is a fixed point of the $SO(3)$ action. There is no physical mechanism that mixes the $U(1)$ phases of distinct fundamental monopoles, hence the relative phase is conserved, implying that $\mathcal{M}^{rel}$ is axially symmetric. In other words, the isometry group of $\mathcal{M}^{rel}$ contains a $U(1)$ factor which preserves the hyperkähler structure. At large separations the two monopoles do not detect each other's presence, so the relative moduli space is asymptotically locally flat. The only hyperkähler metric in four dimensions which fits all the abovementioned requirements is the Euclidean Taub–NUT metric with positive mass parameter.[2]

Thus symmetry arguments are sufficient to uniquely determine $\mathcal{M}^{rel}$ of two monopoles for any gauge group. This is not the case for higher charge monopole solutions since there is no complete classification of hyperkähler manifolds.

2.2 Hyperkähler Geometry

2.2.1 Hyperkähler Manifolds

A $4n$-real-dimensional manifold $\mathcal{M}$ with a Riemannian metric g and three complex structures I, J, K is called hyperkähler if the following conditions hold:

- The three complex structures, which are endomorphisms on the tangent space $T\mathcal{M}$, obey the algebra of unit quaternions:

$$I^2 = J^2 = K^2 = -\mathbb{I}_{4n}, \quad IJ = K, \ etc.$$

- The Riemannian metric g is hermitian with respect to I, J and K:

$$g(X,Y) = g(IX, IY), \ etc.$$

  with $X, Y \in T\mathcal{M}$ vector fields on the tangent space of $\mathcal{M}$.

- The complex structures are covariantly constant with respect to the Levi–Civita connection ∇ of g:

$$\nabla I = \nabla J = \nabla K = 0.$$

Associated to each complex structure is a Kähler form — a nowhere vanishing real two-form — defined by:

$$\omega^I(X,Y) = g(X, IY), \quad \omega^J(X,Y) = g(X, JY), \quad \omega^K(X,Y) = g(X, KY).$$

[2] Hyperkähler four-manifolds with $SO(3)$ isometry have been classified in [17]; they are the flat $\mathbb{R}^4$, the self-dual Taub-NUT , the Atiyah–Hitchin and the Eguchi–Hanson metrics.

The three complex structures I, J, K are integrable if and only if the Kähler forms are parallel with respect to the Levi–Civita connection of g, or equivalently, if they are closed

$$d\omega^I = d\omega^J = d\omega^K = 0 \,.$$

It is sometimes convenient to regard a hyperkähler manifold as a complex-symplectic manifold. If we pick I to be the prefered complex structure, then ω^I is the Kähler form, and, in addition, $(\mathcal{M}, g, I)$ is endowed with a complex-symplectic form:

$$\omega^h = \omega^J + i\omega^K \,.$$

This complex-symplectic form is holomorphic with respect to I and is sometimes called the holomorphic Kähler form.

Every hyperkähler manifold is Ricci-flat. This follows directly from the fact that the holonomy group of a hyperkähler manifold is contained in $Sp(n)$, which in turn is a subgroup of $SU(2n)$. It is not difficult to prove that holonomy of a Riemannian manifold is contained in $SU(2n)$ if and only if the Ricci tensor vanishes identically [49]. Therefore a hyperkähler manifold is automatically a solution to vacuum Einstein's equations.[3] There is no complete classification of hyperkähler manifolds, making every explicit example of a hyperkähler metric a welcome contribution to the general lore. Most of the known hyperkähler spaces are non-compact and many are asymptotically flat. We give a few examples in the next section. Trivially, metric products of non-compact asymptotically flat hyperkähler manifolds are again manifolds of the same type. Explicit examples of compact hyperkähler manifolds are less plentiful. In four dimensions every hyperkähler manifold is a Calabi–Yau space, i.e. it is a Kähler Ricci-flat manifold, of which the $K3$ surface is an example. There are further examples of compact hyperkähler manifolds in dimensions eight and higher to be found in the mathematical literature, but these are not of immediate interest to the present work.

2.2.2 Examples

Flat space:

A trivial example of a hyperkähler manifold is the quaternionic flat space $\mathbb{R}^{4n} \equiv \mathbb{H}^n$ with its standard flat metric

$$ds^2 = \sum_a dq^a \, d\bar{q}^a \,, \qquad (2.3)$$

[3] For a concise discussion of hyperkähler manifolds and their properties see e.g. [50].

$a = 1, \ldots, n$, where we identify a point $(x_1^a, x_2^a, x_3^a, x_4^a) \in \mathbb{R}^{4n}$ with a quaternion $q^a \in \mathbb{H}^n$ as $q^a = x_1^a + i\,x_2^a + j\,x_3^a + k\,x_4^a$. A triplet of Kähler forms is

$$-\frac{1}{2}\sum_a dq^a \wedge d\bar{q}^a = i\omega^I + j\omega^J + k\omega^K,$$

where $\omega^I, \omega^J, \omega^K$ are the Kähler forms associated to the standard complex structures I, J, K on $\mathbb{R}^{4n}$:

$$\begin{aligned} \omega^I &= \sum_a dx_1^a \wedge dx_2^a - dx_3^a \wedge dx_4^a\,, \\ \omega^J &= \sum_a dx_1^a \wedge dx_3^a - dx_4^a \wedge dx_2^a\,, \\ \omega^K &= \sum_a dx_1^a \wedge dx_4^a - dx_2^a \wedge dx_3^a\,. \end{aligned} \tag{2.4}$$

For future convenience, let us rewrite this flat hyperkähler structure in complex coordinates $(z^a, w^a) \in \mathbb{C}^{2n}$. With $q^a = z^a + w^a j$ and $\bar{q}^a = \bar{z}^a - w^a j$, metric (2.3) and three Kähler forms (2.4) are:

$$\begin{aligned} g &= \sum_a dz^a d\bar{z}^a + dw^a d\bar{w}^a, \\ \omega^I &= \frac{1}{2}i\sum_a (dz^a \wedge d\bar{z}^a + dw^a \wedge d\bar{w}^a), \\ \omega^J &= \frac{1}{2}\sum_a (dz^a \wedge dw^a + d\bar{z}^a \wedge d\bar{w}^a), \\ \omega^K &= -\frac{1}{2}i\sum_a (dz^a \wedge dw^a - d\bar{z}^a \wedge d\bar{w}^a), \end{aligned} \tag{2.5}$$

and the complex-symplectic form $\omega^h = \omega^J + i\omega^K$ is

$$\omega^h = \sum_a dz^a \wedge dw^a.$$

Let us now give a few non-trivial examples of non-compact hyperkähler manifolds in four and higher dimensions.

Taub–NUT :

The Euclidean Taub–NUT metric on $\mathbb{R}^4$ is asymptotically locally flat, has isometry group $U(2) \approx U(1) \times SU(2)$ and may be written in the form:

$$ds^2 = V\,d\mathbf{r}^2 + V^{-1}(d\tau + \boldsymbol{\omega}(\mathbf{r}) \cdot d\mathbf{r})^2, \tag{2.6}$$

where $V = 1 + \frac{m}{r}$, $\mathbf{r}$ is the coordinate on flat $\mathbb{R}^3$, $r = |\mathbf{r}|$, and τ is a periodic coordinate with period 2π. The one-form $\omega = \boldsymbol{\omega}(\mathbf{r}) \cdot d\mathbf{r}$ is such that curl $\boldsymbol{\omega}(\mathbf{r}) =$ grad V, where curl and grad are standard operators on $\mathbb{R}^3$. This implies that V is a harmonic function on $\mathbb{R}^3$.

Constant m is the so-called mass parameter. When $m > 0$, the metric (2.6) is positive-definite and non-singular. The apparent singularity at the origin $r = 0$ is an artefact of spherical polar coordinates. Introducing a new radial coordinate $\rho = \sqrt{r}$ near $r = 0$, metric (2.6) becomes a flat metric written in terms of left-invariant $SU(2)$ one-forms. When $m < 0$, metric (2.6) changes signature from $(+ + + +)$ to $(- - - -)$ at $r = -m$ and is singular at that point.

The potential function V has to be a harmonic function on $\mathbb{R}^3$, and hence may include more than one poles. The following function with k centres, located at points $(\mathbf{x}_1, \ldots \mathbf{x}_k)$ in $\mathbb{R}^3$, also solves the self-duality equations:

$$V = 1 + \frac{1}{|\mathbf{r} - \mathbf{x}_1|} + \frac{1}{|\mathbf{r} - \mathbf{x}_2|} + \ldots + \frac{1}{|\mathbf{r} - \mathbf{x}_k|}. \tag{2.7}$$

The metric of the form (2.6) with V as in (2.7) is called a multi-centre Taub–NUT, or for short multi-Taub–NUT metric.

ALE:

Omitting the constant from the potential function V (2.7) of a two-centre Taub–NUT metric leads to a four-dimensional hyperkähler metric which is not asymptotically flat. This is the Eguchi–Hanson gravitational instanton [51]. It is asymptotically locally Euclidean (ALE) [52]. The Eguchi–Hanson metric is complete and is a minimal resolution of the singularity $\mathbb{C}^2/\mathbb{Z}_2$.

More generally, for any finite subgroup Γ of $SU(2)$ the singularity $\mathbb{C}^2/\Gamma$ has a minimal resolution. These non-singular manifolds were constructed by Kronheimer [53] using the hyperkähler quotient construction, who also showed that they were ALE. The finite subgroups Γ are in one-to-one correspondence with Dynkin diagrams for simple A_n, D_n and E_6, E_7, E_8 groups [54]. Subgroups $\Gamma = \mathbb{Z}_{n+1}$ correspond to the A_n series, and minimal resolutions of the A_n singularity are called cyclic ALE spaces. The Eguchi–Hanson manifold is then a minimal resolution of the A_1 singularity. Other cyclic ALE manifolds, which are also called multi-Eguchi–Hanson or multi-instanton, were first constructed by Gibbons and Hawking in [55] (see also [56]) by adding the appropriate number of simple poles to the potential function V. The multi-Eguchi–Hanson manifold with k centres located at points $(\mathbf{x}_1, \ldots \mathbf{x}_k)$ in $\mathbb{R}^3$ has metric (2.6) with

$$V = \frac{1}{|\mathbf{r} - \mathbf{x}_1|} + \frac{1}{|\mathbf{r} - \mathbf{x}_2|} + \ldots + \frac{1}{|\mathbf{r} - \mathbf{x}_k|}. \tag{2.8}$$

ALF:

Inspired by the relation between the Taub–NUT and the Eguchi–Hanson metrics we may wonder whether there are ALF counterparts to other ALE metrics of Kronheimer. In the cyclic case the answer is straightforward. The cyclic ALF metrics are the multi-Taub–NUT metrics, which were first written down by Hawking in [57].

The ALF boundary condition is compatible with the action of the binary dihedral group associated to the D_n series, and corresponding ALF metrics have been found. Cherkis and Kapustin [58] found the Kähler potential and the twistor space for these metrics. Kronheimer's quotient construction may be modified to prove their existence, although there is little hope of obtaining explicit expressions for the metrics in this way. The situation does not repeat itself in the E_n case. One can demonstrate [59] that the ALF boundary condition is not compatible with the action on $\mathbb{C}^2$ of the binary tetrahedral, octahedral and icosahedral groups associated with E_6, E_7 and E_8 groups respectively.

Calabi:

The Eguchi–Hanson metric is the standard hyperkähler structure on the cotangent bundle of complex projective line, $T^*\mathbb{C}P^1$, and as such can be viewed as the first member of another family of hyperkähler metrics. This is the family of Calabi metrics [60] on the total space of the cotangent bundle of complex projective space $T^*\mathbb{C}P^n$, whose holonomy is $Sp(n)$.

2.2.3 Hyperkähler Quotient

Definition of the hyperkähler quotient:

Hyperkähler manifolds may be usefully studied from the point of view of symplectic geometry. In symplectic geometry there is a construction called the symplectic quotient [61] which allows one to obtain new symplectic manifolds from known ones. It was first noticed by Hitchin, Karlhede, Linström and Roček [24] that this construction adapts naturally to hyperkähler geometry. Before we state their result, some preliminary explanations are in order.

Let (M, g, I, J, K) be a hyperkähler manifold and G a Lie group, with Lie algebra $\mathfrak{g}$, acting on $\mathcal{M}$. The action of a Lie group G, generated by vector $X \in T_p\mathcal{M}$, $p \in \mathcal{M}$, is said to be *isometric* if $\mathcal{L}_X g|_p = 0$, *holomorphic* if $\mathcal{L}_X I|_p = 0$, and *Hamiltonian* if $\mathcal{L}_X \omega^I|_p = 0$ at any point $p \in \mathcal{M}$, where $\mathcal{L}_X$ is the Lie derivative along X. By definition of the Lie derivative we have:

$$\mathcal{L}_X \omega^I = \iota(X) d\omega^I + d(\iota(X)\omega^I) = 0\,,$$

where $\iota(X)\omega^I$ denotes the interior product (contraction) of the Kähler form ω^I with the vector X. However, we know that the two-forms $\omega^I, \omega^J, \omega^K$ are closed, hence the one-forms $\iota(X)\omega^I$, *etc.* are

also closed. If the first de Rahm cohomology group $H^1(\mathcal{M}, \mathbb{R})$ vanishes there exist, at least locally, functions μ^I, μ^J, μ^K, such that

$$d\mu^I(X) = \iota(X)\omega^I, \; etc. \tag{2.9}$$

This defines the functions μ^I, μ^J and μ^K up to an additive constant. To be well defined globally, however, these functions have to be equivariant, i.e. commute with the actions of G on both $\mathcal{M}$ and $\mathfrak{g}^*$. For a general Lie group G there could be an obstruction to defining μ^I, μ^J and μ^K globally which lies in a cohomology group of $\mathfrak{g}$. Luckily the obstruction vanishes for semi-simple Lie groups or a torus. We can write the three moment maps more succinctly as a single map $\mu = (\mu^I, \mu^J, \mu^K)$

$$\mu : \mathcal{M} \to \mathfrak{g}^* \times Im\mathbb{H} = \mathfrak{g}^* \times \mathbb{R}^3\;.$$

The theorem of Hitchin *et al* [24] states that if (M, g, I, J, K) is a hyperkähler manifold and G is a Lie group, with Lie algebra $\mathfrak{g}$, which acts on $\mathcal{M}$ by isometries preserving the hyperkähler structure, there will be an associated moment map $\mu : \mathcal{M} \to \mathfrak{g}^* \otimes \mathbb{R}^3$. Consider level sets of the moment map $\mu = \zeta$, where ζ is an invariant element of $\mathfrak{g}^*$ under the co-adjoint action. Then

$$X_\zeta := \mu^{-1}(\zeta)/G$$

is also a hyperkähler manifold with a hyperkähler metric induced from g.

We shall not prove the theorem here, but point out that the proof emphasizes the role of the complex-symplectic form ω^h and relies on the analogous theorem for the symplectic (Kähler) quotient.

If G is compact and acts freely on $\mu^{-1}(\zeta)$, and $\mathcal{M}$ is complete, then X_ζ is also complete. The manifold $\mathcal{M}$ may admit another group K that acts tri-holomorphically, i.e. preserves all three complex structures of $\mathcal{M}$, and isometrically and whose action commutes with that of G. Such K will descend onto X_ζ as a group of tri-holomorphic isometries.

As an aside, everything said about Riemannian metrics carries over, at least locally, to metrics with signature $(4p,\; 4q)$. Unless stated otherwise, in the following we will be concerned with the positive-definite case.

Group actions and moment map calculations:

Let us consider two basic group actions which will arise time and again in the quotient constructions of Chapter 3. These are real translations and $U(1)$ rotations. Below we describe how these groups act on a flat hyperkähler four-space $\mathbb{H}$ and calculate the associated moment maps.

The flat hyperkähler structure (2.3) and (2.4) is invariant under real translations:

$$q \to q - t, \; t \in \mathbb{R} \,. \tag{2.10}$$

The moment map for this action is

$$\mu = \frac{1}{2}(q - \bar{q}) \,. \tag{2.11}$$

To see that this is the case, it is easiest to perform calculations in complex coordinates (quaternionic analysis has not yet been fully developed because of the anti-commutativity property of quaternions). In complex coordinates defined before (2.5) the $\mathbb{R}$ action (2.10) becomes

$$\begin{aligned} z^1 &\to z^1 - t \,, \\ z^2 &\to z^2 \,, \end{aligned}$$

and is generated by $X = \partial/\partial z^1 + \partial/\partial \bar{z}^1$. Given the standard hyperkähler structure (2.5), we can calculate the moment map from the definition (2.9):

$$d\mu^I = \frac{i}{2}(d\bar{z}^1 - dz^1) = \frac{i}{2}d(\bar{z}^1 - z^1)$$

$$\Rightarrow \mu^I = -\frac{i}{2}(z^1 - \bar{z}^1) \,,$$

and

$$d\mu^h = dz^2 \Rightarrow \mu^h = z^2 \,.$$

It can be easily checked that rewriting $\mu = i\mu^I + \mu^h j$ in terms of quaternion $q = z^1 + z^2 j$ gives precisely expression (2.11).

The flat hyperkähler structure is also invariant under right multiplications by unit quaternions p, such that $p\,\bar{p} = 1$:

$$q \to q\,p \,.$$

Unit quaternions are in one-to-one correspondence with elements of $SU(2)$, so this action is isomorphic to a right $SU(2)$ action. By contrast, left multiplications by unit quaternions

$$q \to p\,q$$

preserve the metric (2.3) but rotate the three Kähler forms. This may also be seen by noting that since the metric is flat we may identify the tangent space with $\mathbb{H}$. Then the complex structures I, J, K act on $\mathbb{H}$ by left multiplication by i, j, k. The action of I, J, K thus commutes with right actions.

A particular action of a $U(1)$ subgroup of $SU(2)$ that we shall be using is given by the following one parameter family of right multiplications:

$$q \to q\,e^{it}, \quad t \in (0, 2\pi] \,. \tag{2.12}$$

Its moment map is:

$$\mu = \frac{1}{2} q i \bar{q} \,. \tag{2.13}$$

This expression can again be deduced by performing the calculation in complex coordinates. In terms of (z^1, z^2) the $U(1)$ action (2.12) becomes

$$\begin{aligned} z^1 &\to e^{it}\, z^1 \,, \\ z^2 &\to e^{-it}\, z^2 \,. \end{aligned}$$

It is generated by vector $X = i(z^1\partial/\partial z^1 - \bar{z}^1\partial/\partial\bar{z}^1 - z^2\partial/\partial z^2 + \bar{z}^2\partial/\partial\bar{z}^2)$. As before, by definition of the moment map we find:

$$d\mu^I = -\frac{1}{2}(z^1 d\bar{z}^1 + \bar{z}^1 dz^1 - z^2 d\bar{z}^2 - \bar{z}^2 dz^2) = -\frac{1}{2}d(|z^1|^2 - |z^2|^2)$$

$$\Rightarrow \mu^I = -\frac{1}{2}(|z^1|^2 - |z^2|^2) \,,$$

and

$$d\mu^h = i(z^1 dz^2 + z^2 dz^1) = id(z^1 z^2) \Rightarrow \mu^h = i z^1\, z^2 \,.$$

Expression (2.13) follows straightforwardly from the above formulae by rewriting them in terms of q and $\bar{q}$.

Away from the origin, $q = 0$, the $U(1)$ action (2.12) is free. Using moment map (2.13) one can identify the orbit space of this action with $\mathbb{R}^3$, the origin corresponding to the fixed point set $q = 0$. Hence moment map (2.13) defines a Riemannian submersion $\mathbb{R}^4 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ whose fibres are circles S^1.

For later purposes it will prove useful to express the flat metric on $\mathbb{H}$ in coordinates adapted to this submersion. Any quaternion may be written as

$$q = a e^{i\psi/2} \,,$$

where ψ is real, $\psi \in (0, 4\pi]$, and a is a pure imaginary quaternion, $a = -\bar{a}$. Then the $U(1)$ action (2.12) is given by

$$\psi \to \psi + 2t \,.$$

The moment map (2.13) defines three cartesian coordinates $\mathbf{r}$ by

$$\mathbf{r} = qi\bar{q} = ai\bar{a} = -aia\,.$$

A short calculation reveals that flat metric (2.3) on $\mathbb{H}$ in coordinates $(\psi, \mathbf{r})$ becomes

$$ds^2 = \frac{1}{4}\Big(\frac{1}{r}d\mathbf{r}^2 + r(d\psi + \boldsymbol{\omega}\cdot d\mathbf{r})^2\Big), \tag{2.14}$$

with $r = |\mathbf{r}|$ and

$$\operatorname{curl}\boldsymbol{\omega} = \operatorname{grad}\left(\frac{1}{r}\right),$$

where the curl and grad operations are taken with respect to flat Euclidean metric on $\mathbb{R}^3$. Metric (2.14) is singular at $r = 0 \equiv q = 0$, but this is merely a coordinate artefact arising from the fact that the $U(1)$ action (2.12) has a fixed point at the origin. Away from that fixed point metric (2.14) is defined on the standard Dirac circle bundle over $\mathbb{R}^3 \setminus \{0\}$, and horizontal one-form $(d\tau + \boldsymbol{\omega}\cdot d\mathbf{r})$ defines the standard Dirac monopole connection.

CHAPTER 3

Hyperkähler Quotient Construction at Work

In this chapter we aim to construct, using the hyperkähler quotient, a number of metrics that play an important role in checking the S-duality hypothesis in $\mathcal{N} = 4$ SYM theory in four space-time dimensions. Our results concern mainly moduli spaces of monopoles in theories with higher rank gauge groups. The isometry group of the moduli space of distinct fundamental monopoles is larger than those of higher charge $SU(2)$ monopoles. As a consequence, moduli spaces of distinct fundamental monopoles are simpler to construct by the hyperkähler quotient.

Although the use of the hyperkähler quotient in this context is not in itself new, our treatment has the advantage that rather little machinery is necessary to obtain simple and tractable expressions for the metric. For example, Hitchin showed [62] that the ADHMN construction [63] of the moduli space of self-dual connections on $\mathbb{R}^3$ is equivalent to an infinite-dimensional hyperkähler quotient, where the moment map constraint is nothing else than the Bogomol'nyi equation. Another example is the construction of Dancer [64] which gives an eight-dimensional hyperkähler manifold which has an interpretation of the centred charge $(2, 1)$ monopole moduli space in theory with gauge group $SU(3) \to U(1) \times SU(2)$. The charge $(\cdot, 1)$ monopole in this configuration is fixed, that is it is infinitely massive. Further application of the hyperkähler quotient with a different group action to this eight-dimensional manifold leads to a one-parameter family of four-dimensional hyperkähler metrics. The metric for the zero value of the parameter corresponds to the double-cover of the Atiyah–Hitchin manifold.

The hyperkähler quotient construction affords a straightforward analysis of some global properties of monopole moduli spaces. It also enables us to make statements about isometries and geodesics on these manifolds, which are not otherwise apparent from the local forms of the metrics given in [10, 11, 45]. From Section 2.2.3 we know that the quotient manifold is complete if the group acts freely on the level sets of the moment map. If the quotient space has singularities, the quotient construction permits an easy analysis of possible resolutions. In some cases it is possible to find a set of global coordinates on the quotient space, which implies that the space is topologically trivial. We can also easily determine the group of tri-holomorphic isometries on the quotient manifold. This is done by finding transformations which leave invariant the moment map equations, as well as the metric on the original hyperkähler manifold.

The material in this chapter is derived primarily from [65] and is organized as follows. In Section 3.1 we describe some features of the construction which are common to all the forthcoming applications. Section 3.2 contains the hyperkähler quotient construction of a number of well known hyperkähler manifolds already mentioned in Section 2.2.2. These are Euclidean Taub–NUT, cyclic ALF and ALE, Calabi and Taubian–Calabi manifolds.

In Section 3.3 we use the hyperkähler quotient to re-derive the Lee–Weinberg–Yi manifold, which is the relative moduli space of distinct fundamental monopoles [43]. Our approach provides a natural setting for the investigation of certain degenerations of the Lee–Weinberg–Yi manifold. These degenerations lead to moduli spaces of fundamental monopoles for non-maximally broken gauge groups, as well as to moduli spaces of fixed, or infinitely massive monopoles.

Our most elaborate example is presented in Section 3.4. There we construct a new class of metrics, which include, as a special case, a positive mass parameter version of the Gibbons–Manton metric [45].

Some of the moduli space metrics constructed in Sections 3.3 and 3.4 have figured in the studies of three-dimensional SYM theories [66, 67]. Three-dimensional $\mathcal{N}=4$ SYM theories appear as world-volume theories of certain configurations of intersecting branes in the ten-dimensional type IIB string theory. In Section 3.5 we present this alternative interpretation of monopole moduli spaces, in particular of the massless and the infinitely massive degenerations of the Lee–Weinberg–Yi manifold. In the appropriate limit, the near-horizon geometry of the intersecting brane configuration is likely to be of interest for the AdS/CFT correspondence.

3.1 General Setup of the Construction

In all examples of hyperkähler manifolds in this chapter we start with a $4(m+d)$-dimensional Euclidean space $\mathcal{M}=\mathbb{R}^{4(m+d)}=\mathbb{H}^{m+d}$ with its standard flat hyperkähler structure (2.3) and (2.4). As group G, acting on $\mathcal{M}$, we take a d-dimensional subgroup of the Euclidean group of motions, $E(4m+4d)$, on $\mathcal{M}$. The group G is generated by real translations and left $U(1)$ rotations described in Section 2.2.3. The quotient manifold X_ζ is then $4m$-dimensional. In all our examples there is a tri-holomorphic action of the torus group $T^m=U(1)^m$ on $\mathcal{M}=\mathbb{H}^{m+d}$, which commutes with the action of G and hence descends onto the quotient X_ζ as the group of tri-holomorphic isometries.

The general local form of the metric admitting a tri-holomorphic torus action was written down by Lindström and Roček [68] and Pedersen and Poon [69]:

$$ds^2=\frac{1}{4}G_{ab}d\mathbf{r}_a\cdot d\mathbf{r}_b+\frac{1}{4}G^{ab}(d\tau_a+\boldsymbol{\omega}_{ac}\cdot d\mathbf{r}_c)(d\tau_b+\boldsymbol{\omega}_{bd}\cdot d\mathbf{r}_d)\,, \tag{3.1}$$

where $a,b=1\dots m$, G^{ab} is the inverse of G_{ab}, and the T^m action is generated by Killing vector fields $\partial/\partial\tau_a$. Unless stated otherwise, we assume Einstein summation convention. In order that the metric (3.1) is hyperkähler, the matrix G_{ab} and the one-forms $\boldsymbol{\omega}_{ab}$ must satisfy the following linear partial differential equations:

$$\begin{aligned}\frac{\partial}{\partial x_a^i}\,G_{bc}&=\frac{\partial}{\partial x_b^i}\,G_{ac}\,,\\ \frac{\partial}{\partial x_a^i}\,\omega_{bc}^j&-\frac{\partial}{\partial x_b^j}\,\omega_{ac}^i=\epsilon^{ijk}\frac{\partial}{\partial x_a^k}\,G_{bc}\,,\end{aligned}$$

where $i,j,k=1,2,3$, x_a^i are components of $\mathbf{r}_a$ and ω_{ab}^i are components of $\boldsymbol{\omega}_{ab}$. Clearly, given matrix G_{ab}, the second equation determines one-forms $\boldsymbol{\omega}_{ab}$ up to a gauge equivalence. Thus to identify a metric of the form (3.1) one needs only to calculate G_{ab}. We shall use this fact later to relate metrics obtained by the quotient construction to previously known forms.

If $r_a=|\mathbf{r}_a|$ and $G_{ab}=\delta_{ab}/r_a$, (3.1) becomes a metric product on $\mathbb{H}^m$ of m flat factors (2.14). The torus action is tri-holomorphic and the associated moment map in the present case is given (up to a scalar multiple) by $\mu=\mathbf{r}_a$, which may be used to parameterise the space of orbits of the T^m action.

3.2 Known Spaces

In this section we obtain some well known and widely used manifolds by the hyperkähler quotient. Not all constructions are new, some have already been known to mathematicians. For example, the

Taub–NUT metric is often used as an example of the symplectic quotient [61], while the cyclic ALE spaces are a special case of Kronheimer's hyperkähler quotient construction [53]. Our notation, however, has the advantage of being rather simple and explicit.

3.2.1 Euclidean Taub–NUT Metric

The prototype case of constructions we are interested in is that of the Euclidean Taub–NUT space (2.6). We will describe this in great detail in order to omit explicit computations in the forthcoming examples. Let us start with space

$$\mathcal{M} = \mathbb{H} \times \mathbb{H} \tag{3.2}$$

parametrized by quaternionic coordinates (q, w). Let $G = \mathbb{R}$, $t \in \mathbb{R}$, act by rotations (2.12) on the first $\mathbb{H}$ factor and by translations (2.10) on the second $\mathbb{H}$ factor:

$$(q, w) \to (qe^{it}, w - \lambda t), \quad \lambda \in \mathbb{R}. \tag{3.3}$$

From Section 2.2.3 the moment map for this action is:

$$\begin{aligned} \mu &= \frac{1}{2} q\, i\, \bar{q} + \frac{\lambda}{2}(w - \bar{w}) \\ &= \frac{1}{2}\mathbf{r} + \lambda\, \mathbf{y}\,, \end{aligned} \tag{3.4}$$

where $\mathbf{r} = q\, i\, \bar{q}$ and $w = (y + \mathbf{y}), y \in \mathbb{R}$. The flat metric on $\mathcal{M}$ is

$$ds^2 = \frac{1}{4}\left(\frac{1}{r} d\mathbf{r}^2 + r(d\psi + \boldsymbol{\omega} \cdot d\mathbf{r})^2\right) + dy^2 + d\mathbf{y}^2. \tag{3.5}$$

Action (3.3) corresponds to $(\psi, y) \to (\psi + 2t, y - \lambda\, t)$, which leaves $\tau = \psi + \frac{2y}{\lambda}$ invariant. Without loss of generality we set

$$\zeta = 0\,. \tag{3.6}$$

On the five-dimensional intersection of the three level sets $\mu^{-1}(0)$ one has $\mathbf{y} = -\mathbf{r}/2\lambda$, and hence the induced metric on $\mu^{-1}(0)$ is:

$$ds^2 = \frac{1}{4}\left(\frac{1}{r} d\mathbf{r}^2 + r(d\tau + \frac{2}{\lambda} dy + \boldsymbol{\omega} \cdot d\mathbf{r})^2\right) + dy^2 + \frac{1}{4\lambda^2} d\mathbf{r}^2\,. \tag{3.7}$$

The metric on the quotient $X_0 = \mu^{-1}(0)/\mathbb{R}$ is obtained by projecting (3.7) orthogonally to the Killing vector field $\partial/\partial y$. Completing the square in (3.7) gives

$$\begin{aligned} ds^2 &= \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right) d\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1} \left(d\tau + \boldsymbol{\omega} \cdot d\mathbf{r}\right)^2 \\ &+ \left(\frac{r}{\lambda^2} + 1\right)\left(dy + \frac{r\lambda}{2} \frac{(d\tau + \boldsymbol{\omega} \cdot d\mathbf{r})}{(\frac{r}{\lambda^2} + 1)}\right)^2 . \end{aligned}$$

Finally, the metric on the quotient $X_0 = \mu^{-1}(0)/\mathbb{R}$ is

$$ds^2 = \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right) d\mathbf{r}^2 + \frac{1}{4}\left(\frac{1}{r} + \frac{1}{\lambda^2}\right)^{-1} (d\tau + \boldsymbol{\omega} \cdot d\mathbf{r})^2\,. \tag{3.8}$$

This is the standard form of the Taub–NUT metric with positive mass parameter λ^2. It appears to be singular at $r = 0$. However it is easily seen that $\mathbb{R}$ acts freely on $\mu^{-1}(0)$, and hence the singularity at $r = 0$ is merely a coordinate singularity which may be removed provided that the $U(1)$ coordinate τ is identified with an appropriate period. To obtain global coordinates on the quotient space note that on the zero-set $\mu^{-1}(0)$, $\mathbf{y} = -q\, i\, \bar{q}/2$ and the $\mathbb{R}$ action shifts y, so we may set $y = 0$. Thus q serves as a global coordinate and we see that the Taub–NUT metric is topologically equivalent to $\mathbb{R}^4$. Also, if $\lambda \to \infty$ the Taub–NUT metric (3.8) degenerates to a flat metric (2.14) on $\mathbb{R}^4$.

The Taub–NUT metric with negative mass parameter may be obtained in an analogous way, however instead of starting with the positive-definite metric on $\mathcal{M} = \mathbb{H} \times \mathbb{H}$ we start with the flat metric of signature $(4, 4)$:

$$ds^2 = dq\, d\bar{q} - dw\, d\bar{w}.$$

Following all the same steps yields the same metric (3.8) with λ^2 replaced by $-\lambda^2$.

The $\mathbb{R}$ action (3.3) commutes with the $U(1)$ action:

$$(q, w) \to (qe^{i\alpha}, w)$$

which descends to the Taub–NUT metric (3.8) as the tri-holomorphic action $\tau \to \tau + 2\alpha$. In addition, the following action of unit quaternions

$$(q, w) \to (p\, q, p\, w\, \bar{p}),$$

$p\bar{p} = 1$, commutes with both of the previous actions and leaves $\mu^{-1}(0)$ invariant. After dividing out by a discrete factor to ensure that the action of the isometry group is effective, we deduce that the full isometry group of the Taub–NUT metric is $U(2) = (SU(2) \times U(1))/\mathbb{Z}^2$.

3.2.2 Cyclic ALF Metrics

Cyclic ALF metrics are the multi-Taub–NUT metrics of [57] which we have already mentioned in Section 2.2.2. Take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}$$

with coordinates (q_a, w), $a = 1, \ldots, m$, and $G = \mathbb{R}^m$ with action

$$\begin{aligned} q_a &\to q_a e^{it_a}\,, \quad \text{(no sum)} \\ w &\to w - \sum t_a\,. \end{aligned}$$

The moment map of this action is:

$$\mu_a = \frac{1}{2}\mathbf{r}_a + \mathbf{y}\,, \tag{3.9}$$

where $\mathbf{r}_a = q_a\, i\, \bar{q}_a$ and $\mathbf{y} = (w - \bar{w})/2$. Then $\mu^{-1}(\zeta)$ is given by

$$\frac{1}{2}\mathbf{r}_a = \zeta_a - \mathbf{y}\,.$$

Making the following redefinitions:

$$\mathbf{y} = \frac{1}{2}\mathbf{r}\,, \quad \zeta_a = \frac{1}{2}\mathbf{x}_a\,,$$

the level set of the moment map is given by

$$\mathbf{r}_a = \mathbf{x}_a - \mathbf{r}\,.$$

The metric on X_ζ takes the multi-centre form

$$ds^2 = \frac{1}{4}V d\mathbf{r}^2 + \frac{1}{4}V^{-1}(d\tau + \boldsymbol{\omega}\cdot d\mathbf{r})^2, \tag{3.10}$$

with

$$V = 1 + \sum \frac{1}{|\mathbf{r} - \mathbf{x}_a|}$$

and

$$\operatorname{curl}\boldsymbol{\omega} = \operatorname{grad} V\,.$$

Because

$$\mathbf{r}_a - \mathbf{r}_b = \mathbf{x}_a - \mathbf{x}_b\,,$$

we require $\zeta_a \neq \zeta_b$, $\forall a$ and b, in order that the $\mathbb{R}^m$ action be free. Constants $\mathbf{x}_a$ specify relative positions of the centres. Note that all the coefficients multiplying the inverse distances in the expression for V are the same, they are all set equal to one. In fact they can be equal to any other constant as long as they remain the same. Only then potential orbifold singularities at points $\mathbf{r} = \mathbf{x}_a$ can be resolved by identifying τ periodically. The isometry group of multi-Taub–NUT metrics, $m > 1$, is just $U(1)$ unless all the centres lie on a straight line in which case there is an extra $U(1)$ symmetry. When $m = 1$ we recover the Taub–NUT metric (3.8).

3.2.3 Cyclic ALE Metrics

Cyclic ALE metrics are the multi-instanton, or multi-Eguchi–Hanson metrics that we have encountered in Section 2.2.2. This time take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}$$

with coordinates (q_a, q), $a = 1, \ldots, m$, and $G = T^m = (t_1, \ldots, t_m)$ with action

$$\begin{aligned} q_a &\to q_a e^{it_a}\,, \quad \text{(no sum over } a)\,, \\ q &\to q e^{-i\sum t_a}\,. \end{aligned}$$

The moment maps for this action are

$$\mu_a = \frac{1}{2}(q_a\, i\, \bar{q}_a - q\, i\, \bar{q})\,.$$

If $\mathbf{r}_a = q_a i\, \bar{q}_a$ and $\mathbf{r} = q\, i\, \bar{q}$, then $\mu^{-1}(\zeta)$ is given by

$$\frac{1}{2}\mathbf{r}_a = \zeta_a + \frac{1}{2}\mathbf{r}\,.$$

Making the following redefinition:

$$\zeta_a = \frac{1}{2}\mathbf{x}_a\,,$$

the level sets of the moment maps are given by

$$\mathbf{r}_a = \mathbf{x}_a + \mathbf{r}\,,$$

and the metric on X_ζ takes the multi-centre form (3.10) with

$$V = \frac{1}{r} + \sum \frac{1}{|\mathbf{r} + \mathbf{x}_a|}\,.$$

As in the ALF case, we must require $\zeta_a \neq \zeta_b$ to avoid orbifold singularities at the points where two or more centres coincide. The comments made about the isometry group and orbifold singularities of cyclic ALF metrics for $m > 1$ hold equally well for cyclic ALE metrics. Note that the case $m = 1$ coincides with the Eguchi–Hanson metric, whose isometry group is $U(2)$, on $T^*(\mathbb{CP}^1)$ which is the first of the Calabi series of metrics that we construct next.

3.2.4 Calabi Metrics

The construction of the Calabi metric on $T^*(\mathbb{CP}^m)$ is perhaps the oldest of the hyperkähler quotient constructions [60]. We choose

$$\mathcal{M} = \mathbb{H}^{m+1}$$

with coordinates q_a ,$a = 1, \dots, m+1$, and $G = U(1)$ with action

$$q_a \to q_a\, e^{it}, \quad t \in (0, 2\pi]\,, \tag{3.11}$$

and moment map

$$\mu = \frac{1}{2}\sum q_a\, i\, \bar{q}_a = \frac{1}{2}\sum \mathbf{r}_a\,,$$

where $\mathbf{r}_a = q_a\, i\, \bar{q}_a$. The level set of the moment map $\mu^{-1}(\zeta)$ is given by

$$\mu = \frac{1}{2}\sum \mathbf{r}_a = \boldsymbol{\zeta}\,,$$

where the three-vector $\boldsymbol{\zeta}$ must be non-vanishing if the action (3.11) is to be free. Let us make the following redefinition to make the formulae tidier:

$$\boldsymbol{\zeta} = \frac{1}{2}\mathbf{x}\,.$$

Then the potential function G_{ij} in (3.1) is:

$$\begin{aligned} G_{ii} &= \frac{1}{|\mathbf{x} - \sum \mathbf{r}_i|} + \frac{1}{r_i}\,, \\ G_{ij} &= \frac{1}{|\mathbf{x} - \sum \mathbf{r}_i|}\,, \quad i \neq j\,, \end{aligned} \tag{3.12}$$

$i, j = 1, \dots, m$. Action (3.11) commutes with the tri-holomorphic action of $SU(m+1)$ given by

$$q_a \to q_a\, U_{ac}\,,$$

where U_{ac} is a $(m+1) \times (m+1)$ quaternion valued matrix with no j or k components satisfying

$$U_{ac}\bar{U}_{ab} = \delta_{cb}\,,$$

$$\det U = 1.$$

Left multiplications by unit quaternions

$$q_a \to p\, q_a$$

induce the rotation of $\mathbf{r}_a$'s. Choosing p such that this is an $SO(2)$ rotation about the $\boldsymbol{\zeta}$ direction, this left action will leave $\mu^{-1}(\zeta)$ invariant. Such an $SO(2)$ action will preserve a single complex structure.

Thus the Calabi metric is invariant under an effective action of $U(m+1)/\mathbb{Z}_{m+1}$, of which $SU(m+1)/\mathbb{Z}_{m+1}$ acts tri-holomorphically. With respect to a privileged complex structure we have a holomorphic effective action of $U(m+1)/\mathbb{Z}_{m+1}$. The principal orbits are of the form $U(m+1)/U(m-1) \times U(1)$. There is a degenerate orbit of the form $U(m+1)/U(m) \times U(1) \approx \mathbb{CP}^m$ corresponding to the zero section of $T^*(\mathbb{CP}^m)$. If $\boldsymbol{\zeta} = 0$ the metric becomes incomplete — it has an orbifold singularity at $q = 0$.

A recent theorem of Swann and Dancer [70] shows that the Calabi metric is the unique complete cohomogeneity one[1] hyperkähler metric of dimension greater than four.

3.2.5 Taubian–Calabi Metrics

The passage between ALE and ALF boundary conditions in the cyclic case is accomplished by adding a constant factor to the potential function V (see Section 2.2.2). By analogy, we could entertain the idea that the asymptotic form of the Calabi metric (3.12) may be altered by adding a constant term to the potential function G_{ij} (now a matrix). In fact, this change can be implemented by altering the group action to include a real translation on one of the $\mathbb{H}$ factors in $\mathcal{M}$ of Section 3.2.4. As a result we obtain a family of Taubian–Calabi metrics on $\mathbb{R}^{4m}$. The name Taubian–Calabi is due to [71]. Take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}$$

parametrized by quaternion coordinates (q_a, w), $a = 1, \dots, m$, and $G = \mathbb{R}$ with action

$$\begin{aligned} q_a &\to q_a\, e^{it}\,, \\ w &\to w - t\,, t \in \mathbb{R}\,. \end{aligned} \tag{3.13}$$

The moment map is

$$\mu = \frac{1}{2}\sum q_a\, i\bar{q}_a + \frac{(w - \bar{w})}{2}\,.$$

Without loss of generality we can choose the zero-set of the moment map $\mu^{-1}(0)$ which is given by

$$\frac{1}{2}\sum \mathbf{r}_a + \mathbf{y} = 0\,, \tag{3.14}$$

where, as previously, $\mathbf{r}_a = q_a\, i\bar{q}_a$ and $\mathbf{y} = 1/2(w - \bar{w})$. There is a T^m action on the $\mathbb{H}^m$ factor which commutes with G, therefore the metric on the quotient X_0 is of the general form (3.1) with metric

[1] A manifold is cohomogeneity one if the generic, or principle, orbit of the isometry group has real codimension one.

components G_{ab} given by:

$$G_{aa} = 1 + \frac{1}{r_a}, \quad G_{ab} = 1, \quad a \neq b\,. \tag{3.15}$$

The quotient space X_0 has a tri-holomorphic right action of $U(m)/\mathbb{Z}_m$ and a left action of $SU(2)$ by unit quaternions. Hence the total isometry group of the Taubian–Calabi metric is, up to a discrete factor, $U(m) \times SU(2)$. The principle orbits of $U(m)$ are $U(m)/U(m-2)$ which are $(4m-4)$-dimensional. The left action of $SU(2)$ rotates the q_a's, and therefore $\mathbf{r}_a$'s, but leaves invariant the phases of q_a's, and hence it increases the dimension of the principle orbit by two. We conclude that the principle orbits of the Taubian–Calabi metric are of codimension two. A straightforward generalization of the argument given for the Taub–NUT metric implies that the q_a's may serve as global coordinates, and we get a complete metric on $\mathbb{R}^{4m}$. Setting $m = 1$ gives the Taub–NUT metric.

In addition to continuous symmetries the Taubian–Calabi metrics admit many discrete symmetries. There are m reflections $R_a : q_a \to -q_a$ and S_m permutation group on m letters, both acting tri-holomorphically. Their fixed point sets are totally geodesic and hyperkähler. In this way we see that the $4m$-dimensional Taubian–Calabi manifold contains, as a totally geodesic submanifold, the $4n$-dimensional Taubian–Calabi manifold for $n < m$.

3.3 Moduli Spaces of Fundamental Monopoles

We are now ready to construct hyperkähler metrics on moduli spaces of fundamental monopoles. We shall first construct the Lee–Weinberg–Yi (LWY) metric on the relative moduli space of charge $(1, 1, \ldots, 1)$ monopoles when the gauge group $SU(m+2)$ is broken to its maximal torus $U(1)^{m+1}$. This metric was first presented in [11], where its form was deduced from an asymptotic (Liénard–Wiechert) analysis of interactions between fundamental monopoles.

From the physics of monopoles and their interactions we gather the following facts about the relative moduli space $\mathcal{M}^{rel}$ of charge $(1, 1, \ldots, 1)$ monopoles. The origin of $\mathcal{M}^{rel}$ is fixed by the rotation group $SO(3)$ due to the existence of a spherically symmetric solution. Moreover, $SO(3)$ rotates the three complex structures and hence does not act tri-holomorphically. We also know that there is no process that changes electric charges of individual monopoles, thus the isometry group of $\mathcal{M}^{rel}$ contains a T^m factor, which acts preserving the hyperkähler structure. Knowing these facts we

can find a set-up for the hyperkähler quotient construction that yields the Lee–Weinberg–Yi metric. This is done in Section 3.3.1

From the quotient construction we shall see that the Lee–Weinberg–Yi metric is determined by m linearly independent vectors in $\mathbb{R}^m$ whose matrix of inner products gives the reduced mass matrix of distinct fundamental monopoles. It is conceivable that some of these m vectors vanish or become infinitely large. Nevertheless the hyperkähler quotient construction still goes through. The new hyperkähler spaces correspond in some sense to degenerations of the Lee–Weinberg–Yi manifold. We interpret these degenerations as limits in which some of the monopole masses vanish or become infinite. A special case of the moduli space metric in the massless limit has already been considered in [12]. In Section 3.3.2 we identify this metric as the Taubian–Calabi metric of Section 3.2.5 and obtain its generalizations. In Section 3.3.3 we interpret of the second degeneration as the limit in which some of the monopoles are fixed, or infinitely massive. This interpretation is supported by the brane picture of Section 3.5, as well as by the Nahm data analysis performed in [72].

3.3.1 Lee–Weinberg–Yi Metric

The metric we are about to construct is the metric on the relative moduli space of $(m+1)$ distinct fundamental monopoles in $\mathcal{N} = 4$ SYM theory with gauge group $SU(m+2)$ broken to $U(1)^{m+1}$. The Lee–Weinberg–Yi metric on $\mathbb{R}^{4m}$ is perhaps the simplest generalization of the Taub–NUT metric to higher dimensions. When $m = 1$ we recover Taub–NUT metric (3.8) which is the exact metric on the relative moduli space of distinct fundamental $SU(3)$ monopoles [10, 43, 46]. Take

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^m$$

with coordinates (q_a, w_a), $a = 1, \ldots, m$, and $G = \mathbb{R}^m = (t_1, \ldots, t_m)$ with action

$$q_a \to q_a\, e^{it_a} \quad (\text{no sum over } a)\,, \qquad w_a \to w_a - \lambda_a{}^b t_b\,. \tag{3.16}$$

The action of G commutes with the tri-holomorphic action of $K = T^m = U(1)^m = (\alpha_1, \ldots, \alpha_m)$ given by

$$q_a \to q_a\, e^{i\alpha_a} \quad (\text{no sum over } a)\,, \qquad w_a \to w_a\,. \tag{3.17}$$

Moment maps of the $\mathbb{R}^m$ action are

$$\mu_a = \frac{1}{2} q_a\, i\, \bar{q}_a + \frac{1}{2}\lambda_a^{\ b}(w_b - \bar{w}_b)\,, \tag{3.18}$$

where the $m \times m$ real matrix $\lambda_a^{\ b}$ is taken to be non-singular. The Lee–Weinberg–Yi metric is then the induced metric on $\mu^{-1}(0)/\mathbb{R}^m$. The zero-sets of the moment maps are given by

$$\mu_a = \frac{1}{2}\mathbf{r}_a + \lambda_a^{\ b}\mathbf{y}_b = 0\,, \tag{3.19}$$

where $\mathbf{r}_a = q_a\, i\, \bar{q}_a$ and $\mathbf{y}_a = 1/2(w_a - \bar{w}_a)$. A short calculation shows that the metric on the quotient is

$$\begin{aligned} ds^2 &= \frac{1}{4}G_{ab} d\mathbf{r}_a \cdot d\mathbf{r}_b + \frac{1}{4}G^{ab}(d\tau_a + \boldsymbol{\omega}(\mathbf{r}_a)\cdot d\mathbf{r}_a)(d\tau_b + \boldsymbol{\omega}(\mathbf{r}_b)\cdot d\mathbf{r}_b)\,, \\ G_{ab} &= \frac{\delta_{ab}}{r_a} + \mu_{ab}\,, \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} \mu_{ab} &= (\nu^t\,\nu)_{ab}\,, \\ \nu &\equiv \lambda^{-1}\,, \end{aligned} \tag{3.21}$$

and

$$\mathrm{curl}_c\, \boldsymbol{\omega}(\mathbf{r}_a) = \mathrm{grad}_c \left(\frac{1}{r_a}\right)\,.$$

The tri-holomorphic T^m action is generated by m vectors $\partial/\partial\tau_a$. Condition (3.19) is invariant under the following action of unit quaternions:

$$\begin{aligned} q_a &\to p\, q_a\,, \\ w_a &\to p\, w_a \bar{p}\,. \end{aligned}$$

Hence the metric on $\mu^{-1}(0)/\mathbb{R}^m$ is preserved by this $SU(2)$ action, as well as by the tri-holomorphic T^m action (3.17). As remarked above, to specify a $4m$-dimensional hyperkähler metric with a tri-holomorphic T^m action it suffices to know the $m \times m$ matrix $G_{ab}(\mathbf{r}_a)$. The remaining parts of the metric may then be deduced by direct computation. In the present case G_{ab} depends on the matrix μ_{ab}. Physically, μ_{ab} is (up to an overall constant factor) the reduced mass matrix of fundamental monopoles. Geometrically, μ_{ab} is related to the matrix of inner products of the m linearly independent translation vectors defining the $\mathbb{R}^m$ action (3.16). These translation vectors $v^{(b)} \in \mathbb{R}^m$, $b = 1, \ldots, m$, have components:

$$(v^{(b)})_a = \lambda_a^{\ b}\,.$$

Thus

$$g(v^{(a)}, v^{(b)}) = (\lambda^t\,\lambda)^{ab} = \mu^{ab}\,.$$

If one thinks of the vectors $v^{(b)}$ as defining a lattice Λ in $\mathbb{R}^m$ with metric $g(v^{(a)}, v^{(b)})$, then μ_{ab} are the components of the metric on the reciprocal lattice Λ^*.

As long as the matrix $\lambda_a^{\ b}$ is invertible we may use equations (3.19) to eliminate $\mathbf{y}_b$'s in favour of $\mathbf{r}_a$'s on $\mu^{-1}(0)$. It then follows, as it did for the Taub–NUT metric, that m quaternions q_a serve as global coordinates on the Lee–Weinberg–Yi manifold which makes it homeomorphic to $\mathbb{R}^{4m}$. The LWY metric is complete since the $\mathbb{R}^m$ action on $\mu^{-1}(0)$ is free and is regular at infinity.

If the matrix $\lambda_a^{\ b}$ becomes singular or diverges, i.e. if the translation vectors $v^{(a)}$ cease to be linearly independent or become infinite respectively, we are led to various degenerate cases. In these cases the reduced mass matrix μ_{ab} drops in rank or diverges. Physically, this is associated with enhanced gauge symmetries due to the appearance of massless monopoles or due to some monopoles acquiring infinite mass respectively.

So far we have considered monopoles of a specific gauge group $SU(m+2)$. However, our construction may be easily generalized to work for any semi-simple group of rank $(m+1)$. In the notation of [11], λ_a's (not to be confused with the translation matrix λ_a^{b}) are essentially inner products between the simple roots of the Dynkin diagram of the gauge group. One replaces the flat metric on $\mathcal{M}$ by a related flat metric

$$\lambda_a dq_a d\bar{q}_a + dw_a d\bar{w}_a\,,$$

the Kähler forms by

$$-\frac{1}{2}\lambda_a dq_a \wedge d\bar{q}_a - \frac{1}{2}dw_a \wedge d\bar{w}_a\,,$$

and the action (3.16) by

$$\begin{aligned} q_a &\to q_a e^{it_a\lambda_a} \quad (\text{no sum over } a)\,, \\ w_a &\to w_a - \lambda_a^{\ b} t_b\,. \end{aligned}$$

The form of the moment map (3.18) remains unchanged, and the resulting metric is precisely of the form given in [11].

3.3.2 Moduli Spaces of Massless Monopoles

It is interesting to ask how the LWY metric would change if some of the monopoles became massless. Massless monopole appear in gauge theories where the unbroken gauge group contains a non-abelian factor. This question was addressed by Lee, Weinberg and Yi in [12]. They argued that it makes sense to talk about moduli spaces only for monopole configurations with no net non-abelian charge, in which case no non-normalizable zero-modes appear. They also suggested that the metric on the moduli space of such configurations is the massless limit of the metric on the moduli space of distinct fundamental monopoles (the Lee–Weinberg–Yi metric). This argument allowed the authors of [12] to obtain the metric in the case when the gauge group is

$$SU(m+2) \to SU(m) \times U(1)^2 ,$$

but did not yield an explicit answer in the more general case

$$SU(m+2) \to SU(m+2-k) \times U(1)^k , \tag{3.22}$$

$k = 2, \dots, m+1$. For simplicity we consider only unitary gauge groups, although, with minor modifications, the results are also valid for symplectic groups of rank $(m+1)$. Note that the situation becomes more complicated when one considers non-simply-laced gauge groups, for which monopoles corresponding to composite roots do not account for all the missing states in the magnetic spectrum [73].

Using the hyperkähler quotient method we construct the metrics in the case of the symmetry breaking (3.22). We analyze in detail the metric on the moduli space of monopoles for the case $m = 2, k = 2$. To illustrate how the hyperkähler quotient construction of Section 3.3.1 has to be modified we begin by considering the case $m = 1$.

Degeneration of Taub–NUT to flat metric:

The exact metric on the relative moduli space of distinct fundamental $SU(3)$ monopoles is the Taub–NUT metric with positive mass parameter [10, 43, 46]. If one of the monopoles becomes massless, the Taub–NUT metric degenerates to a flat metric on $\mathbb{R}^4$.

In the notation of Section 3.2.1 this is equivalent to taking $\lambda \to \infty$. Define $\nu = \lambda^{-1}$, so that $\nu \to 0$ when $\lambda \to \infty$. In order that the action (3.3) be well defined we must introduce a new parameter $\tilde{t}$ and a new quaternionic coordinate $\tilde{w}$ in the following way:

$$\nu \tilde{t} = t, \quad \tilde{w} = \nu w . \tag{3.23}$$

The action (3.3) then becomes:

$$\begin{aligned} q &\to q\, e^{i\nu\tilde{t}} , \\ \tilde{w} &\to \tilde{w} - \nu\tilde{t} . \end{aligned} \tag{3.24}$$

In the limit $\nu \to 0$ the $U(1)$ action (3.24) is trivial and the metric on the quotient space X_0 is just the flat metric (2.14).

$\mathbf{SU(m+2) \to SU(m) \times U(1)^2}$:

Physically this situation corresponds to having two massive and $(m-1)$ massless monopoles. We now show that in this case the Lee–Weinberg–Yi metric degenerates to the Taubian–Calabi metric of Section 3.2.5.

When $(m-1)$ monopoles become massless, the reduced mass matrix μ_{ab} drops in rank to rank $=$ 1. By (3.21) this is equivalent to ν_{ab} having rank one and the translation vectors $v^{(a)}$ not being linearly independent. By analogy with (3.23) we define new group parameters $\tilde{t}_a$ and new quaternionic coordinates $\tilde{w}_a$ to be:

$$\nu_a{}^b \tilde{t}_b = t_a, \quad \tilde{w}_a = \nu_a{}^b w_b .$$

Then the action (3.16) becomes:

$$\begin{aligned} q_a &\to q_a\, e^{i\nu_a{}^b\tilde{t}_b} , \\ \tilde{w}_a &\to \tilde{w}_a - \nu_a{}^b\tilde{t}_b . \end{aligned} \tag{3.25}$$

When the rank of ν_{ab} is one there is only one independent coordinate $\tilde{w}_a$ and the $\mathbb{R}^m$ action (3.25) reduces to the $\mathbb{R}$ action (3.13). Then all the elements of μ_{ab} are equal and the Lee–Weinberg–Yi metric (3.20) on $\mathbb{R}^{4m}$ degenerates to the Taubian–Calabi metric (3.15) on $\mathbb{R}^{4m}$. In taking the quotient we ignore the contributions to the flat metric on $\mathcal{M}$ from the $(m-1)$ coordinates $\tilde{w}_a$ which transform trivially under the redefined action (3.25).

From Section 3.2.5 we know that the part of the full isometry group of the Taubian–Calabi metric which acts tri-holomorphically and effectively is $U(m)/\mathbb{Z}_m$. This result agrees, up to a discrete factor, with that of [12]. Giving a physical interpretation to the degrees of freedom of metric (3.15) is not straightforward. It does not make sense to consider a soliton of zero energy on its own. Such an object is not localized in space and has no definite size. However in the presence of a massive monopole the behaviour of massless solitons changes. They form a massless cloud that carries a net non-abelian charge and is characterized by one *size* parameter $R = \sum r_a$. The quantity R is clearly invariant under the full isometry group $U(m) \times SU(2)$ since the metric on $\mathcal{M}$, and consequently $\sum q_a\bar{q}_a$, is preserved by both the $U(m)$ action and the $SU(2)$ action, and $q_a\bar{q}_a = |q_a i \bar{q}_a| = r_a$. Hence R is an invariant of the isometry group.

Let us focus on the simplest non-trivial case $m = 2$.

$\mathbf{SU(4) \to SU(2) \times U(1)^2}$:

The Taubian–Calabi metric (3.15) on $\mathbb{R}^8$ is:

$$\begin{aligned} 4\,ds^2 \;=\;& \Big(1+\frac{1}{r_1}\Big)d\mathbf{r}_1^2 + 2d\mathbf{r}_1\cdot d\mathbf{r}_2 + \Big(1+\frac{1}{r_2}\Big)d\mathbf{r}_2^2 \\ &+\frac{1}{1+r_1+r_2}\Big[r_1(1+r_2)(d\tau_1+\boldsymbol{\omega}(\mathbf{r}_1)\cdot d\mathbf{r}_1)^2 \\ &-2r_1r_2(d\tau_1+\boldsymbol{\omega}(\mathbf{r}_1)\cdot d\mathbf{r}_1)(d\tau_2+\boldsymbol{\omega}(\mathbf{r}_2)\cdot d\mathbf{r}_2)+r_2(1+r_1)(d\tau_2+\boldsymbol{\omega}(\mathbf{r}_2)\cdot d\mathbf{r}_2)^2\Big]\,, \end{aligned} \tag{3.26}$$

where

$$\mathrm{curl}\boldsymbol{\omega}(\mathbf{r}_i) = \mathrm{grad}\left(\frac{1}{r_i}\right),\quad i=1,2\,.$$

The tri-holomorphic action of $U(2)$ preserves $(\mathbf{r}_1+\mathbf{r}_2)$ and has four-dimensional orbits. The left $SU(2)$ action preserves the length of any vector and the inner products between three-vectors but rotates the vectors around. Thus it rotates $(\mathbf{r}_1+\mathbf{r}_2)$ keeping $|\mathbf{r}_1+\mathbf{r}_2|$ unchanged. This makes the principle orbits six-dimensional.

Metric (3.26) describes the moduli space of distinct centred fundamental $SU(4)$ monopoles of charge $(1,1,1)$ in the limit when one of the monopoles becomes massless. We denote such a state by a charge vector $(1,[1],1)$, where square brackets are placed around the charge of the fundamental monopole mass vanishes. There are eight parameters on the moduli space: $\mathbf{r}_1,\tau_1,\mathbf{r}_2,\tau_2$. Four of them correspond to positions of the two massive monopoles relative to the centre of mass coordinates and the relative $U(1)$ phase. Thus there are four parameters left to describe the massless monopole. Orbits of the $SU(2)$ action are ellipsoids. An ellipsoid with the two massive monopoles situated at the foci is the massless cloud. The physics of this cloud was recently investigated in [41]. The fields of the two massive monopoles are not altered when the $(0,[1],0)$ monopole moves around the ellipsoid.

To gain some insight into the behaviour of the metric (3.26) we consider it in two interesting limits. Note that the vector $\mathbf{r}_1$ points from the $(1,0,0)$ monopole to the $(0,[1],0)$ monopole, and $\mathbf{r}_2$ points from the $(0,[1],0)$ monopole to the $(0,0,1)$ monopole. Consider the first limiting case: $\mathbf{r}_1=\mathbf{r}_2$. The massless $SU(2)$ monopole is situated midway between the two massive ones. The metric (3.26) and the hyperkähler structure are invariant under the interchange of the two quaternionic coordinates q_1 and q_2. The fixed point set of this isometry $q_1 = q_2$ (or, equivalently, $\mathbf{r}_1=\mathbf{r}_2$ and $\tau_1=\tau_2$) is a totally geodesic submanifold. From (3.26) we see that the metric on this totally geodesic submanifold is isomorphic to the Taub–NUT metric. This is a very interesting situation: the two massive $SU(4)$ monopoles $(1,0,0)$ and $(0,0,1)$ behave like two distinct fundamental $SU(3)$ monopoles $(1,0)$ and $(0,1)$ and do not notice the presence of the massless monopole! The $SU(2)$ ellipsoid degenerates to an open interval between the two fundamental monopoles.

The second limit is when the two massive monopoles are situated on top of each other: $\mathbf{r}_1=-\mathbf{r}_2$ and $\tau_1=-\tau_2$. These equations define a totally geodesic submanifold, the metric on which is flat. We could have anticipated this result on the following grounds. As the two massive monopoles coincide we effectively have one massive and one massless monopole which is analogous to the $SO(5)\approx Sp(2)\to Sp(1)\times U(1)$ case studied in [12], and the metric on the relative moduli space of such a configuration was shown to be flat. In this case the ellipsoid becomes a sphere with the massive monopoles at its centre.

$\mathbf{SU(m+2) \to SU(m+2-k) \times U(1)^k}$:

We can now construct metrics on moduli spaces for all the intermediate cases of symmetry breaking. These are generalizations of Taubian–Calabi metrics (3.15) which can be obtained by the hyperkähler quotient starting with

$$\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^{k-1}\,.$$

In this case $(m+1-k)$ monopoles become massless, $k=2,\dots,m$, and hence the rank of μ_{ab} (and of ν_{ab}) is equal to $(k-1)$. The group action is (3.25), where only $(k-1)$ coordinates $\tilde{w}_i$, $i=1,\dots,k-1$, are independent. The potential function G_{ab} is the same as in (3.20) with μ_{ab} of the form:

$$\mu_{ab} = \left(\begin{array}{c|c} \mu'_{ab} & \cdots \\ \hline \vdots & \ddots \end{array}\right)$$

where μ'_{ab} is the $(k-1)\times(k-1)$ reduced mass matrix of k distinct fundamental monopoles. It is not difficult to see that (up to a discrete factor) the isometry group for this manifold is:

$$SU(2)\times U(m-(k-2))\times U(1)^{k-2}\,.$$

3.3.3 Moduli Spaces of Fixed Monopoles

In this section we consider another interesting degeneration of the Lee–Weinberg–Yi metric which occurs when the translation matrix λ_{ab} itself, and not its inverse ν_{ab}, drops in rank. If λ_{ab} has rank one, the $\mathbb{R}^m$ action (3.16)

$$\begin{aligned} q_a &\to q_a\, e^{it_a} \quad (\text{no sum over } a)\,, \\ w_a &\to w_a - \lambda_a{}^b t_b\,, \end{aligned}$$

reduces to the $\mathbb{R}^m$ action (3.9), and there is effectively one w_a coordinate. Proceeding with the hyperkähler quotient construction of Section 3.2.2 we obtain the cyclic ALF manifold with m centres. This is the relative moduli space of $(m+1)$ distinct fundamental monopoles, where the positions of all but one monopoles are fixed, i.e. all except one monopoles are infinitely heavy. The free monopole described by the one w_a coordinate moves in the background of m infinitely massive monopoles.

If the rank of λ_{ab} is $1 < k < m$, the metric on the relative moduli space of k free monopoles moving in the background of $(m+1-k)$ fixed monopoles is a $4k$-dimensional generalization of cyclic ALF spaces.

3.4 Asymptotic Metric for Many $SU(2)$ Monopoles

A more complicated example which also generalizes the Taub–NUT space is the asymptotic metric on the moduli space of charge m $SU(2)$ monopoles [45]. We begin by constructing the higher dimensional analogue of the Taub–NUT metric with positive mass parameter (3.8), and then construct the higher dimensional analogue of the Taub–NUT metric with negative mass parameter. It is this latter case that describes the behaviour of many well-separated $SU(2)$ monopoles. The hyperkähler quotient construction of the positive mass parameter metric has also been performed independently by R. Goto [74]. We take

$$\mathcal{M} = \mathbb{H}^{\frac{1}{2}m(m-1)} \times \mathbb{H}^m \tag{3.27}$$

with coordinates (q_{ab}, w_a) , $a = 1, \ldots, m$, $a < b$, and the group $G = \mathbb{R}^{\frac{1}{2}m(m-1)} = \{(t_{ab})\}$ with action

$$\begin{aligned} q_{ab} &\to q_{ab}e^{it_{ab}} , \\ w_a &\to w_a - \sum_c t_{ac} , \end{aligned} \tag{3.28}$$

where $t_{ac} = -t_{ca}$ for $c < a$. The $\frac{1}{2}m(m-1)$ moment maps are

$$\mu_{ab} = \frac{1}{2}\mathbf{r}_{ab} + (\mathbf{y}_a - \mathbf{y}_b) , \tag{3.29}$$

where $\mathbf{r}_{ab} = q_{ab}\, i\, \bar{q}_{ab}$ and $\mathbf{y}_a = (w_a - \bar{w}_a)/2$. The level set $\mu^{-1}(\zeta)$ of the moment maps is given by

$$\frac{1}{2}\mathbf{r}_{ab} = -(\mathbf{y}_a - \mathbf{y}_b) + \zeta_{ab} . \tag{3.30}$$

Using (3.30) one can eliminate the $\mathbf{r}_{ab}$'s in favour of the $\mathbf{r}_a$'s on $\mu^{-1}(\zeta)$. Projecting orthogonally to the orbits of action (3.28) we can eliminate the $\frac{1}{2}m(m-1)$ phases of the quaternions q_{ab}'s and use the m quaternions w_a as local coordinates on X_ζ. Let us make the following redefinitions:

$$\mathbf{y}_a = \frac{1}{2}\mathbf{r}_a , \; \zeta_{ab} = \frac{1}{2}\mathbf{x}_{ab} .$$

In this notation the metric on the quotient X_ζ is of the form (3.1) with potential functions given by:

$$\begin{aligned} G_{aa} &= 1 + \sum_{b \neq a} \frac{1}{|\mathbf{r}_a - \mathbf{r}_b - \mathbf{x}_{ab}|} \quad \text{(no sum over } a) , \\ G_{ab} &= -\frac{1}{|\mathbf{r}_a - \mathbf{r}_b - \mathbf{x}_{ab}|} . \end{aligned} \tag{3.31}$$

As pointed out in [45] the case of positive mass parameter appears to be relevant to the motion of $a = 1$ black holes.

The metric constructed by Gibbons and Manton in [45] is the negative mass parameter version of (3.31). To arrive at the Gibbons–Manton metric one starts with the following flat metric on $\mathcal{M}$ (3.27):

$$ds^2 = dq_{ab}d\bar{q}_{ab} - dw_a d\bar{w}_a$$

and chooses $\zeta_{ab} = 0$. Physically, coordinates w_a correspond to positions and internal phases of m unit charge well-separated $SU(2)$ monopoles. Just as in the case of the Lee–Weinberg–Yi manifold more complicated metrics may be constructed by introducing weights.

The quotient construction is invariant under m real translations of the w_a's:

$$\begin{aligned} q_{ab} &\to q_{ab} , \\ w_a &\to w_a + t_a , \end{aligned}$$

and it appears that the isometry group of (3.31) contains an $\mathbb{R}^m$ factor. However, the phases of the monopoles encoded into the real parts of w_a's have to be periodically identified, which replaces $\mathbb{R}^m$ by T^m.

The global behaviour of these metrics is quite complicated, despite the simplicity of the construction. Let us point out that metrics (3.31) are non-singular if and only if the ζ_{ab}'s satisfy the following conditions:

$$\sum_{a,b \in T} \zeta_{ab} \neq 0 .$$

Here T is a cycle of arbitrary length in a simplex, whose vertices are associated to the quaternions w_a and edges to the quaternions q_{ab}. Thus the problem of calculating the topology of the quotient space has been restated in graph-theoretical terms. Goto [74] found that the Euler number of the quotient space is $\chi(X_\zeta) = m^{m-2}$.

As in the case of Taubian–Calabi metrics, metric (3.31) with $\zeta_{ab} \neq 0$ admits various discrete symmetries, e.g. reflections and permutation groups, as tri-holomorphic isometries. It follows that the $4m$-dimensional metric contains totally geodesic copies of the first non-trivial case $m = 2$. Since the Gibbons–Manton metric differs from the exact metric on the moduli space of many $SU(2)$ monopoles by exponentially small terms, the last statement is presumably related to the observation of Bielawski that the exact $SU(2)$ moduli space of m monopoles always admits a totally geodesic copy of the Atiyah–Hitchin manifold [75].

3.5 Brane Interpretation of Monopole Moduli Spaces

It is instructive to view $SL(2,\mathbb{Z})$ duality of four-dimensional $\mathcal{N} = 4$ SYM theory as the field-theoretic counterpart of the more fundamental $SL(2,\mathbb{Z})$ duality, or S-duality, of the ten-dimensional type IIB string theory. Precise formulation of this correspondence can be achieved in the context of Dirichlet branes [76]. BPS monopoles of four-dimensional SYM theories are viewed as solitons of the world-volume theories on the brane. The relevant configuration of intersecting solitonic and Dirichlet branes was proposed by Hanany and Witten [67]. The motivation for [67] came from the desire to learn more about certain phenomena displayed by three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories [77]. Findings of [77] prompted Hanany and Witten to investigate the relation between moduli spaces of BPS monopoles and the Coulomb branch of the three-dimensional effective world-volume theories on the brane. Here we present intersecting brane configurations considered in [67] in order to view moduli spaces of fundamental monopoles of Section 3.3 in a different light. The two limits in which the Lee–Weinberg–Yi metric degenerates, studied in Sections 3.3.2 and 3.3.3, can be interpreted in terms of relative separations of parallel branes in the ten-dimensional Minkowski space-time M_{10}.

An intriguing property of three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories is that, in certain cases, the Coulomb branch of vacua is isomorphic, as a hyperkähler manifold, to the moduli space of BPS monopoles of four-dimensional $\mathcal{N} = 4$ supersymmetric gauge theory. Seiberg and Witten [77] found that the Coulomb branch of the three-dimensional $SU(2)$ gauge theory with no matter multiplets is isomorphic to the moduli space of charge two $SU(2)$ monopole — the Atiyah–Hitchin manifold. This result was generalized to three-dimensional gauge theories with arbitrary unitary gauge groups in [78], where it was demonstrated that the Coulomb branch of the $(2+1)$

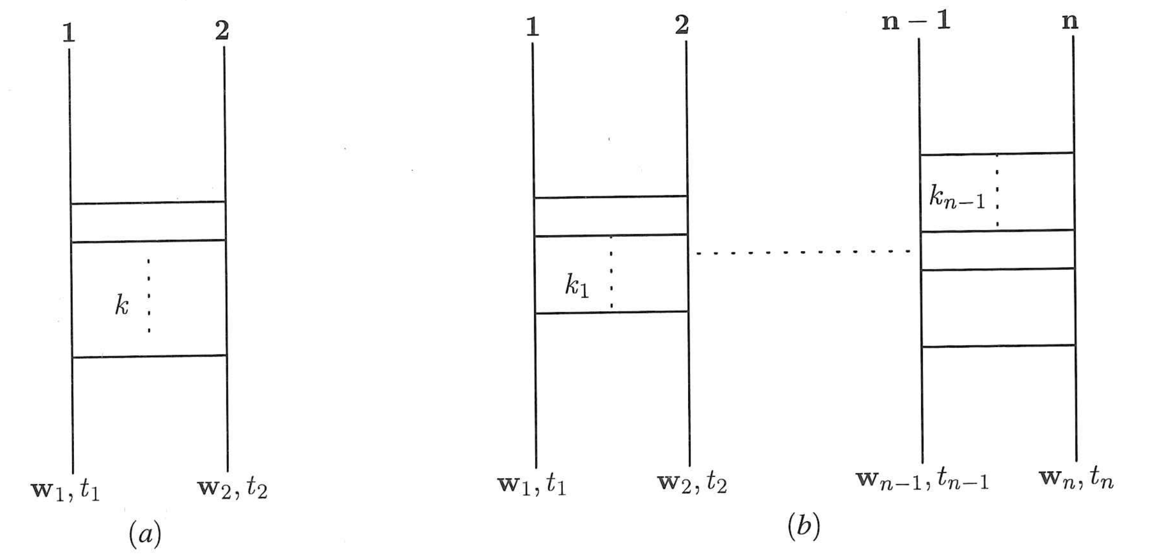

Figure 3.1: Brane configuration: (a) two parallel NS5-branes with k D3-branes stretched between them; (b) n parallel NS5-branes with D3-branes stretched between them.

$SU(k)$ theory with no matter is precisely the moduli space of charge k $SU(2)$ monopole in $(3+1)$ dimensions.

Various interpretations of this duality have been offered, most of which make a connection with string, M- or F-theories. Here we shall discuss the interpretation due to Hanany and Witten [67] which most naturally accommodates our results on moduli spaces of fundamental monopoles.

Consider the following supersymmetric brane configuration in the type IIB string theory on M_{10} with coordinates $(x^0 = t, x^1, \ldots, x^9)$ (fig. 3.1a). We have two parallel solitonic (NS) 5-branes, whose $(5+1)$-dimensional world volumes are parametrized by $(x^0, x^1, x^2, x^3, x^4, x^5)$. Positions of the two 5-branes, which are separated in the x^6 direction, in the four-space transverse to the branes are $t_1 = (x^6)_1$, $\mathbf{w}_1 = (x^7, x^8, x^9)_1$ of the first brane, and $t_2 = (x^6)_2$, $\mathbf{w}_2 = \mathbf{w}_1$ of the second brane respectively. In addition, there are k parallel D3-branes suspended between the two 5-branes, whose $(3+1)$-dimensional world volumes are parametrized by (x^0, x^1, x^2, x^6). The position of the i-th D3-brane in the transverse six-space is $\mathbf{y}_i = (x^3, x^4, x^5)_i$, $\mathbf{w}_1 = \mathbf{w}_2$, where $i = 1, \ldots, k$. Hence NS5-branes and D3-branes share three world-volume directions (x^0, x^1, x^2) and can, in some sense, be regarded as intersecting.

It is well known that the effective world-volume theory of k parallel D3-branes is the three-dimensional $\mathcal{N} = 4$ $U(k)$ gauge theory. Non-zero relative separation of all k branes breaks the gauge group $U(k)$ to $U(1)^k$. Thus to an observer on a D3-brane the low-energy theory is a $U(k)$ electric gauge theory with no matter. The Coulomb branch of the moduli space of vacua is parametrized by $3k$ transverse positions, $\mathbf{y}_i$, of the branes and k scalars, b_i, dual to the vector fields living on the brane.[2] Hence the Coulomb branch is a $4k$-dimensional manifold. Moreover, since the gauge theory has extended $\mathcal{N} = 4$ supersymmetry, this $4k$-dimensional manifold is hyperkähler.

How does an observer on a 5-brane see this situation? To simplify the terminology, we suppress the two dimensions (x^1, x^2) that are common to all branes. Thus an observer on a 5-brane sees the four-dimensional $\mathcal{N} = 4$ $U(2)$ gauge theory, with $U(2)$ broken to $U(1)^2$ by the relative separation of the 5-branes. The centre of $U(2)$ plays no role in what follows, and hence the 5-branes are seen to carry an $SU(2)$ gauge theory broken to $U(1)$. The expectation value of the Higgs field controlling the gauge symmetry breaking is nothing else than the separation between the 5-branes in the x^6 direction, $x^6 = t_2 - t_1$. Since the 5-branes have two infinite directions more than the D3-branes, the 5-brane theory can be treated classically. From the point of view of the 5-brane theory, the ends of a D3-brane look like magnetic monopoles. Thus the 5-brane observer sees a charge k $SU(2)$ monopole.

The intersecting brane configuration that we have just described breaks one half of the total supersymmetry, which guarantees that the $SU(2)$ k-monopole is BPS. From Section 2.1.3 we have learned that the classical moduli space of an $SU(2)$ k-monopole is a hyperkähler manifold labelled by $4k$ real parameters. These parameters are the $3k$ positions of the D3-branes, $\mathbf{y}_i$, plus k scalars, b_i. But these are precisely the moduli parametrizing the Coulomb branch of the three-dimensional $\mathcal{N} = 4$ $U(k)$ gauge theory.

We conclude that the Coulomb branch of the three-dimensional $\mathcal{N} = 4$ $U(k)$ gauge theory with no matter is isometric, as a hyperkähler manifold, to the classical moduli space of $SU(2)$ k-monopoles of four-dimensional $\mathcal{N} = 4$ SYM theory. Recall that the asymptotic metric on this moduli space is the Gibbons–Manton metric constructed in Section 3.4. For $k = 2$ the asymptotic metric is the Taub–NUT metric with negative mass parameter, which differs from the exact metric on the moduli space, the Atiyah–Hitchin manifold, by exponentially small terms. These exponentially small corrections arise as instanton corrections to the Coulomb branch of the quantum $(2+1)$-dimensional gauge theory. This interpretation of instanton corrections in quantum gauge theory was exploited by Fraser and Tong [79] in an attempt to compute the exact metric on the moduli space of many $SU(2)$ monopoles.

The brane configuration in fig. 3.1a can be generalized to accommodate moduli spaces of fundamental monopoles in four-dimensional theories with arbitrary gauge groups which we discussed

[2] In three dimensions, the Hodge dual of a vector is a scalar.

in Section 3.3. The new BPS configuration of intersecting branes (fig. 3.1 b) consists of n parallel NS5-branes with the values of the x^6 coordinate $x^6 = t_1, \dots, t_6$. The i-th brane is connected to the $(i+1)$-th brane by k_i D3-branes, $i = 1, \dots, n-1$. As before, this configuration can be viewed in two different ways. An observer on a D3-brane sees three-dimensional $\mathcal{N} = 4$ gauge theory with gauge group $U(k_1) \times U(k_2) \times \dots \times U(k_{n-1})$. Massless hypermultiplets of the theory transform in the $(\mathbf{k}_1, \bar{\mathbf{k}}_2) \oplus (\mathbf{k}_2, \bar{\mathbf{k}}_3) \oplus \dots \oplus (\mathbf{k}_{n-2}, \bar{\mathbf{k}}_{n-1})$ representation. An observer on a 5-brane sees a classical charge $(k_1, k_2, \dots, k_{n-1})$ monopole in four-dimensional $\mathcal{N} = 4$ $SU(n)$ gauge theory. Depending on the positions of the 5-branes in the x^6 direction, the $SU(n)$ gauge symmetry is broken maximally or non-maximally. Generalizing the analysis carried out for two 5-branes, we conclude that the Coulomb branch of the three-dimensional $\mathcal{N} = 4$ $U(k_1) \times U(k_2) \times \dots \times U(k_{n-1})$ gauge theory is isometric, as a hyperkähler manifold, to the classical moduli space of charge $(k_1, k_2, \dots, k_{n-1})$ monopole in four-dimensional $\mathcal{N} = 4$ $SU(n)$ gauge theory.

When all $k_i = 1$, the gauge group of the D3-brane theory is the abelian $U(1)^{n-1}$. This theory contains $(n-2)$ neutral hypermultiplets, and we can factor out a surplus $U(1)$ to arrive at a $U(1)^{n-2}$ gauge theory with $(n-2)$ charged hypermultiplets. On the other hand, this is the charge $(1, 1, \dots, 1)$ monopole of four-dimensional SYM theory with maximally broken $SU(n)$ gauge group. The relative moduli space of this monopole is the Lee–Weinberg–Yi metric of Section 3.3. In the simplest case, $n = 3$, it is the Taub–NUT metric with positive mass parameter. Hence we expect the Coulomb branch of the three-dimensional $U(1)$ gauge theory with one charged hypermultiplet to be the smooth Taub–NUT manifold. This is precisely what was found in [77] using different techniques. Note also that the three-dimensional $\mathcal{N} = 4$ theory with abelian gauge group does not contain instantons, and hence the metric on the Coulomb branch is given *exactly* by the one-loop formula with no exponentially small corrections. The metric must also be invariant under shifts of the scalars parametrizing the Coulomb branch. In the monopole language this means that the asymptotic metric on the moduli space of monopoles in the case of maximal gauge symmetry breaking is exact, and that it possesses a tri-holomorphic torus action. This is precisely what we found in Section 3.3

Recall that in Section 3.3 we discussed two limits in which the Lee–Weinberg–Yi metric can degenerate. The first limit arises when the unbroken gauge symmetry is enhanced to non-abelian symmetry. This happens when some of the expectation values $(t_1, \dots, t_n)$ of the Higgs field at infinity are equal. In the language of 5-branes, the Higgs field expectation values are positions of the 5-branes in the x^6 direction. Thus when two or more 5-branes coincide, the unbroken gauge symmetry of four-dimensional $\mathcal{N} = 4$ SYM theory is enhanced to contain a non-abelian factor. Moreover, masses of fundamental monopoles are proportional, and their sizes are inverse proportional, to the Higgs field expectation values, and hence to the separations between the 5-branes. Therefore, when any two 5-branes coincide, corresponding monopoles become massless. For example, if all but two of the n 5-branes are coincident, the unbroken gauge symmetry is $SU(n-2) \times U(1)^2$, and $(n-3)$ fundamental monopoles become massless. The monopole moduli space in this case is the Taubian–Calabi manifold of Section 3.2.5.

Performing a mirror transformation to convert the NS5-branes to D5-branes, followed by T-duality transformations to reduce the D3-branes to solitonic strings, we obtain a new intersecting brane configuration of the type IIB theory. Such a system of strings ending on D3-branes was considered by Diaconescu [80]. He showed that the end points of strings on 3-branes correspond to BPS monopoles of four-dimensional $\mathcal{N} = 4$ gauge theory. Now take a limit of the number n of D3-branes going to infinity, and have all but one D3-brane coincide. This configuration can be analyzed in the context of the AdS/CFT correspondence [27]. The effective CFT on the boundary is $\mathcal{N} = 4$ supersymmetric and contains at least on massive BPS monopole. It is conjectured to be equivalent to the supergravity theory on the near-horizon geometry of the intersecting brane configuration.

The second limit in which the Lee–Weinberg–Yi metric degenerates corresponds to keeping one or more fundamental monopoles fixed. In the brane picture, if one of the NS5-branes is moved out to infinity, the corresponding fundamental monopoles shrink to zero size and become singular points of infinite mass. Monopole configurations containing infinitely massive, or fixed, monopoles were also investigated by Houghton using Nahm data [72].

CHAPTER 4

Threshold Bound States of Monopoles

In this chapter we shall present evidence in support of the S-duality conjecture in the context of $\mathcal{N} = 4$ supersymmetric gauge theories with higher rank gauge groups. We have already demonstrated in Section 2.1.2 the connection between predicted purely magnetic bound states of zero binding energy and normalizable harmonic forms on the relevant moduli spaces. Here we present the predicted harmonic forms.

In Chapter 3 we have obtained metrics on moduli spaces of fundamental monopoles in theories with both maximal and non-maximal gauge symmetry breaking. When the gauge group $SU(n+2)$ is broken to its maximal torus $U(1)^{n+1}$, the relative moduli space of $(n+1)$ distinct fundamental monopoles is the Lee–Weinberg–Yi metric on $\mathbb{R}^{4n}$. The charge $(1, 1, \dots, 1)$ monopole is interpreted as a threshold bound state of $(n+1)$ distinct fundamental monopoles. This bound state is quantum-mechanical in nature and should appear in the spectrum of the Hamiltonian of the supersymmetric quantum mechanics on the moduli space. Being a zero-energy bound state, it corresponds to a unique normalizable harmonic form on the Lee–Weinberg–Yi manifold.

In the simplest case $SU(3) \to U(1)^2$, the Lee–Weinberg–Yi manifold is the self-dual Taub–NUT (TN) metric. A (anti)self-dual two-form on the TN manifold may be constructed as a linear combination of three two-forms forming a basis of (anti)self-dual two-forms. The coefficient functions are required to satisfy an ODE which can be solved explicitly. Only one of the solutions yields a non-degenerate square-integrable $U(1)$-invariant harmonic two-form predicted by S-duality [10, 43]. This argument, however, can not be generalized to $n > 1$. The candidate harmonic $2n$-form for $n > 1$ was constructed by Gibbons in [44]. His form has the correct transformation properties under the isometry group — it is T^n-invariant — and is square-integrable, but the question of self-duality and uniqueness still remains open. We present the construction of Gibbons in Section 4.2 and offer a few speculative arguments of how one can go about proving the uniqueness property.

When the gauge group $SU(n+2)$ is broken non-maximally to $SU(n+2-r) \times U(1)^r$, there are r topological charges and corresponding to them are r massive fundamental monopoles. This time, however, the long-range magnetic fields of these massive solitons transform non-trivially under the unbroken $SU(n+2-r)$, and fundamental monopoles aquire a non-abelian magnetic charge. The study of monopoles with non-abelian long-range magnetic fields shows that it is not possible to apply global non-abelian gauge transformations to these solutions to produce a dyonic object, as can be done with solitons charged under an abelian magnetic field. It turns out that one cannot define a global non-abelian charge in the presence of a monopole [40]. In addition to these non-abelian massive monopoles, massless solitonic objects appear in a theory with non-abelian gauge symmetry. These are, presumably, dual to the massless gauge bosons appearing in the electric spectrum of the theory. Taking a massless limit of the classical monopole solution leads to the size of the monopole core becoming infinitely spread out, with the fields tending to their vacuum expectation values. The strange behaviour of massless solitons is checked by the presence of a massive monopole. It is, in fact, possible to find combinations of massive and massless monopoles whose long-range non-abelian fields, and, consequently, non-abelian charges, cancel. Massless monopoles in such a configuration combine to form a sort of cloud carrying non-abelian charge [12]. Properties of this non-abelian cloud for $n = 2$, $r = 2$ were studied in [41].

The case we treat in detail in Section 4.3 is when the gauge group $SU(n+2)$ is broken to $U(1) \times SU(n) \times U(1)$. The material in this section is derived from [81]. The monopole configuration with vanishing non-abelian charges consists of two massive fundamental monopoles and $(n-1)$ massless ones. As we found in Section 3.3.2, the relative moduli space of this solution is the Taubian–Calabi metric on $\mathbb{R}^{4n}$. S-duality predicts (see Section 2.1.2) the existence of a unique threshold bound state of two massive and $(n-1)$ massless monopoles which transforms as a singlet under the unbroken non-abelian gauge group $SU(n)$. Such a state, in turn, corresponds to a unique square-integrable $SU(n)$-invariant harmonic $2n$-form on the Taubian–Calabi $4n$-dimensional manifold. In Section 4.3 we present a candidate harmonic form, whose construction was left as an outstanding problem in [12]. The proposed harmonic $2n$-form possesses correct transformation properties, and for $n = 2$ can be

explicitly checked to be square-integrable. As in the maximally broken case, we have no rigorous proof of either its uniqueness or self-duality. We have also not been able to demonstrate its square-integrability for $n > 2$. Nevertheless, the fact that the candidate form converges in the $n = 2$ case and has correct symmetry property is a good incentive to believe that our conjecture is correct.

The problem of existence and uniqueness of the predicted harmonic forms can also be addressed in the following way. Square-integrable harmonic forms are elements of L^2 cohomology classes on the moduli space. It is desirable to be able to calculate the dimensions of the L^2 cohomology groups of the relevant manifolds, and to prove a vanishing type theorem. However, the theory of square-integrable cohomology on non-compact spaces (see e.g. [82]) is still in its infancy and does not offer definite answers to these questions.

As an aside, let us mention that S-duality predictions for $SU(2)$ $\mathcal{N} = 4$ SYM theory were formulated by Sen [9], who exhibited the required harmonic two-form on the Atiyah–Hitchin manifold. Since only the asymptotic form of the metric on the moduli space of higher charge $SU(2)$ monopoles is known, the required harmonic forms cannot be found by explicit computations. Segal and Selby [83] proved the existence, although not the uniqueness, of the middle-dimensional harmonic form on the moduli space of many $SU(2)$ monopoles. However the techniques used in [83] are not applicable to moduli spaces of distinct fundamental monopoles. For the rest of the chapter we consider gauge theories with gauge groups of rank higher than one.

4.1 Threshold Bound States and Harmonic Forms

Both for theories with maximal and non-maximal gauge symmetry breaking S-duality predicts the existence of a unique threshold bound state of monopoles of certain topological charge. This state should be present in the spectrum of the Hamiltonian of the $\mathcal{N} = 4$ supersymmetric quantum mechanics on the relevant moduli space. States in the Hilbert space of this quantum mechanics are in one-to-one correspondence with real square-integrable differential forms on the moduli space [18, 25]. The quantum-mechanical Hamiltonian is the usual Hodge–de Rahm Laplacian acting on forms:

$$d\,d^\dagger + d^\dagger\,d\,,$$

where d and $d^\dagger$ are the exterior derivative operator and its adjoint. On even-dimensional manifolds $d^\dagger = -*d*$. All BPS states are eigenstates of the Hodge–de Rahm Laplacian. In particular, the BPS bound state of zero binding energy predicted by S-duality is an eigenstate of the Laplacian with zero eigenvalue. Differential forms in the kernel of the Hodge–de Rahm Laplacian are harmonic. Moreover, the threshold bound state is a singlet under the adjoint action of the unbroken gauge group. And since the unbroken gauge symmetry manifests itself as tri-holomorphic isometry of the moduli space, the square-integrable harmonic form should be invariant under the action of the group of tri-holomorphic isometries of the moduli space metric.

The threshold bound state is unique, which requires the harmonic form to be unique as well. Recall that there is a natural isomorphism between r-forms and $(4n-r)$-forms on a $4n$-dimensional differential manifold realized by the Hodge star operator $*$, the so-called Hodge duality. Therefore, to avoid getting duplicates via Hodge duality, the desired harmonic form should be a self-dual (or anti-self-dual) middle-dimensional differential form (a $2n$-form). This is a necessary but not sufficient condition for the $2n$-form to be unique.

In brief, we are looking for non-degenerate square-integrable middle-dimensional (anti)self-dual harmonic forms on the relevant moduli space, which are invariant under the action of the tri-holomorphic isometry group of the moduli space metric.

4.2 L^2 Harmonic Form on the Lee–Weinberg–Yi Manifold

In this section we consider $\mathcal{N}=4$ SYM theories with gauge group $SU(n+2)$ broken to $U(1)^{n+1}$. The relative moduli space of $(n+1)$ distinct fundamental monopoles is the Lee–Weinberg–Yi metric on $\mathbb{R}^{4n}$ constructed in Section 3.3. The Lee–Weinberg–Yi manifold is topologically trivial, and hence harmonic forms that we find do not come from the non-trivial topology.

4.2.1 Harmonic two-form on the Taub–NUT Manifold

Consider the simplest case $n=1$, the gauge group is $SU(3)\to U(1)^2$. The predicted bound state is the charge $(1,1)$ monopole that manifests itself as a non-degenerate square-integrable $U(1)$-invariant (anti)self-dual harmonic two-form on the Taub–NUT space.

The Taub–NUT manifold is well understood, and the required two-form has already been constructed in a different context in [84]. Let us demonstrate how this can be done. The Taub–NUT metric (3.8) written in terms of three left-invariant $SU(2)$ one-forms is:

$$ds^2 = V\,dr^2 + V\,r^2(\sigma_1^2+\sigma_2^2) + V^{-1}\,\sigma_3^2\,,$$

where $V=1+\frac{1}{r}$ and $d\sigma_i=\frac{1}{2}\epsilon_{ijk}\sigma_j\wedge\sigma_k$, $i,j,k=1,2,3$. An orthonormal basis for the above metric is given by:

$$e^0 = V^{1/2}\,dr\,,\quad e^1 = rV^{1/2}\sigma_1\,,\quad e^2 = rV^{1/2}\sigma_2\,,\quad e^3 = V^{-1/2}\sigma_3\,.$$

In terms of these one-forms, the three Kähler forms, that form a basis of self-dual two-forms, are:

$$\omega_i^{(+)} = 2e^0\wedge e^i + \epsilon_{ijk}\,e^j\wedge e^k\,. \tag{4.1}$$

And the three two-forms forming a basis of anti-self-dual two-forms are:

$$\omega_i^{(-)} = 2e^0\wedge e^i - \epsilon_{ijk}\,e^j\wedge e^k\,. \tag{4.2}$$

Then the most general (anti)self-dual form can be written as:

$$sefl-dual = \sum_i f_i^{(+)}(r)\,\omega_i^{(+)}\,,\quad anti-self-dual = \sum_i f_i^{(-)}(r)\,\omega_i^{(-)}\,. \tag{4.3}$$

The two-form we are looking for must be harmonic. Being (anti)self-dual, the two-forms (4.3) are harmonic if they are closed. The requirement for the two-forms (4.3) to be closed yields the following first order differential equation for $f^{(\pm)}(r)$:

$$\frac{df^{(\pm)}}{dr} + V\,f^{(\pm)} = 0\,.$$

It follows immediately that the only non-singular normalizable harmonic two-form is:

$$\Omega = \frac{dV}{dr}dr\wedge\sigma_3 + V^{-1}\sigma_1\wedge\sigma_2 = d(V\,\sigma_3)\,, \tag{4.4}$$

which is self-dual. From the second equality we see that it is invariant under $U(1)$ transformations generated by the vector field $\partial/\partial\tau$ dual to σ_3.

The two-from (4.4) can also be constructed in a different way [44]. Take the Killing field $K=\partial/\partial\tau$ that generates the tri-holomorphic $U(1)$ isometry of the Taub–NUT manifold. The one-form dual to $\partial/\partial\tau$ with respect to metric (3.8) has components $A_\alpha = g_{\alpha\beta}K^\beta$ and satisfies the Killing equation $A_{(\alpha;\beta)}=0$, where $(;)$ denotes the covariant derivative with respect to the Levi–Civita connection of (3.8). Now consider the two-form $F=dA$ with components $F_{\alpha\beta}=A_{[\alpha;\beta]}$. It is harmonic, i.e. closed and co-closed, which follows from the Ricci identity valid for any Killing vector K^μ

$$2\,K^\alpha{}_{;[\beta\gamma]} = R^\alpha{}_{\delta\beta\gamma}\,K^\delta.$$

Contract this expression on α and β and recall that every hyperkähler manifold is Ricci-flat, i.e. $R_{\alpha\beta}=0$, to get:

$$\nabla * F_{\alpha\beta} = 0\,.$$

Acting on F, ∇ is nothing else than the Hodge–de Rahm Laplacian $(d*d*+*d*d)$ acting on forms, and hence we deduce that

$$dF = 0\,, \quad d*F = 0\,.$$

Thus by virtue of the construction, the two-form $F = dA$ is harmonic and is preserved by the tri-holomorphic Killing vector K. Moreover, it is square-integrable:

$$\|F\|^2 = \int |F|^2\, d\,Vol < \infty\,,$$

which follows from the analysis of the behaviour of $|F|$ at infinity, i.e. for large r. Clearly, the dependence of F on the compact coordinates is not important for the estimate, since integration over those yields some bounded constant value. By construction, $|F| \sim 1/r^2$ for large r, and the unit volume on the Taub–NUT manifold $d\,Vol \sim r^2 dr \cdot d(compact\ part)$. Then $\|F\|^2 \sim \frac{1}{r}$ for $r \to \infty$, which is sufficiently fast to render F square-integrable.

The required middle-dimensional form on the Lee–Weinberg–Yi manifold can be obtained by generalizing this construction. This was done by Gibbons in [44].

4.2.2 Harmonic $2n$-form on the Lee–Weinberg–Yi Manifold

As discussed in Section 3.3.1, the group of tri-holomorphic isometries of the Lee–Weinberg–Yi manifold (3.20) is T^n. Hence the harmonic $2n$-form on the Lee–Weinberg–Yi space should be T^n-invariant. Torus T^n is generated by n vectors $K^a = \partial/\partial\tau^a$, $a = 1, \ldots, n$ (NB a is a label and not a coordinate index). Denote by A^a n Killing one-forms dual to K^a. By analogy with the Taub–NUT example, we construct n two-forms

$$F^a = dA^a\,,$$

every one of which is harmonic by the same argument as before. Now, the norm of the two-form $|F^a| \sim \frac{1}{r_a}$ at infinity. Asymptotically the Lee–Weinberg–Yi manifold differs from the metric product of n Taub–NUT spaces parametrized by $(\mathbf{r}_a, \tau_a)$ each, by a constant matrix μ_{ab} (the reduced mass matrix). Hence the volume of the Lee–Weinberg–Yi metric has the same asymptotic dependence on radial coordinates r_a as does a n-fold product of Taub–NUT metrics:

$$\prod_{a=1}^{a=n} r_a^3\,.$$

A harmonic $2n$-form may be constructed by taking n-fold wedge products of the two-forms F^a. But the only combination which gives a square-integrable and T^n-invariant $2n$-form is clearly:

$$\Omega = F^1 \wedge F^2 \wedge \ldots \wedge F^n\,.$$

This is the required non-degenerate normalizable harmonic $2n$-form on the Lee–Weinberg–Yi manifold. Note that although Ω is square-integrable, the $(2n-1)$-form B, such that $\Omega = d\,B$, is not. Thus Ω is a non-trivial element of the middle L^2 cohomology class on the Lee–Weinberg–Yi manifold.

Self-duality of Ω does not follow directly from this construction, and should be checked explicitly. Direct computations are rather cumbersome and have not been performed. Even if the candidate form is self-dual the issue of uniqueness still remains. Ideally, if one could prove the uniqueness of Ω, one would not have to check the self-duality property explicitly.

Uniqueness of the harmonic form:

To the best of my knowledge no rigorous proof of uniqueness is known. Below is a somewhat sketchy argument of how one could go about proving the uniqueness property. Recall from the hyperkähler quotient construction of the Lee–Weinberg–Yi metric in Section 3.3.1 that the family of these metrics is, in some sense, parametrized by the reduced mass matrix μ_{ab}. One can picture the space of μ as a cone: the open cone corresponds to a generic non-singular μ, the axis corresponds to a diagonal μ reducing the LWY metric on $\mathbb{R}^{4n}$ to a direct product of n Taub–NUT metrics (TN^n), and the boundary of the cone is where μ drops in rank. As far as the square-integrability of harmonic forms goes, we are most interested in how the volume of the manifold grows with r_a at infinity. Since the difference between the LWY metric and a product of Taub–NUT metrics TN^n is the constant matrix μ, it is reasonable to suppose that the two spaces have the same L^2 cohomology.

The last relation can be formulated in more precise terms. Using the hyperkähler quotient, one constructs a quasi-isometry between LWY and TN^n, i.e. a diffeomorphism f

$$f : \mathcal{M}_{LWY} \to TN^n\,,$$

such that if g is the metric on $\mathcal{M}_{LWY}$ and h is the metric on TN^n,

$$c\,g \le f^* h \le c^{-1} g, \quad c = const,$$

where f^* is the push-forward map acting on tensors. The inverse f^{-1} exists and from the above equation we have

$$c^{-1}\,h \le (f^{-1})^* g \le c\,h.$$

Invoking a general mathematical theorem we deduce the equivalence of the L^2 cohomology on the two spaces. The next step is to use the uniqueness of the harmonic two-form (4.4) on the Taub–NUT manifold and to extend this result to a direct product of Taub–NUT manifolds. More work is needed to make this argument into a rigorous proof.

4.3 L^2 Harmonic Form on the Taubian–Calabi Manifold

Consider $\mathcal{N} = 4$ SYM theory with gauge group $SU(n+2)$ broken to $U(1) \times SU(n) \times U(1)$. The relative moduli space of the charge $(1, [1, \dots, 1], 1)$ monopole is the Taubian–Calabi (TC) metric on $\mathbb{R}^{4n}$ constructed in Section 3.2.5.

$$ds^2 = G_{ab} d\mathbf{r}_a \cdot d\mathbf{r}_b + G^{-1}_{ab}(d\tau_a + \boldsymbol{\omega}_{(a)} \cdot d\mathbf{r}_a)(d\tau_b + \boldsymbol{\omega}_{(b)} \cdot d\mathbf{r}_b) \,, \tag{4.5}$$

where $a, b = 1, \dots, n$ and G_{ab} is given by:

$$\begin{aligned} G_{aa} &= 1 + \frac{1}{r_a} \,, \\ G_{ab} &= 1, \quad a \neq b \,. \end{aligned} \tag{4.6}$$

The subgroup of the isometry group of (4.5) that preserves its hyperkähler structure is $U(n)$, which is considerably larger than the T^n tri-holomorphic isometry of the Lee–Weinberg–Yi manifold.

S-duality predicts the existence of a unique threshold bound state of two massive and $(n-1)$ massless fundamental monopoles, which corresponds to a unique $SU(n)$-invariant square-integrable harmonic $2n$-form on the Taubian–Calabi $4n$-manifold. The Taubian–Calabi manifold is topologically trivial, and hence square-integrable harmonic forms come from the geometry rather than the topology of the manifold. In order to construct the predicted harmonic form on the Taubian–Calabi manifold we exploit ideas presented in the previous section.

4.3.1 Candidate Harmonic $2n$-form

The tri-holomorphic isometry group of (4.5) $U(n)$ is locally isomorphic to a direct product $U(n) \approx SU(n) \times U(1)$. Generators of the maximal torus $T^n \subset U(n)$ are the vector fields $\partial/\partial\tau_a$. Then vector K

$$K = \sum_{a=1}^{n} \frac{\partial}{\partial \tau_a}$$

generates the $U(1)$ subgroup of $U(n)$ in the above factorization of $U(n)$. The Killing vector K commutes with all the generators of the $SU(n)$ subgroup. It is this fact that prompts us to use K for constructing a $SU(n)$-invariant harmonic form on TC^{4n}. Physically, K generates the relative $U(1)$ charge of the two massive fundamental monopoles.

Killing one-form A dual to K with respect to metric (4.5) is

$$A = \sum_a G^{-1}_{ab}(d\tau_b + \boldsymbol{\omega}_{(b)} \cdot d\mathbf{r}_b) \,. \tag{4.7}$$

Given G_{ab} (4.6), we compute:

$$\sum_a G^{-1}_{ab} = \frac{r_b}{1 + \sum r_c} \,. \tag{4.8}$$

Let $F = dA$. By the same logic as before, the two-form F so constructed is both closed and co-closed:

$$dF = 0 \,, \quad d * F = 0 \,.$$

Take an n-fold wedge product of the two-forms:

$$\Omega = \underbrace{F \wedge \ldots \wedge F}_{n \text{ times}} \,. \tag{4.9}$$

By virtue of the construction Ω is harmonic and $SU(n)$-invariant. We *conjecture* that:

Ω is the unique square-integrable harmonic $2n$-form on the Taubian–Calabi $4n$-manifold.

To show that the square norm of Ω is finite:

$$||\Omega||^2 = \int_{\mathcal{M}} |\Omega|^2 \, d\,Vol < \infty \,, \tag{4.10}$$

we need to evaluate the asymptotic behaviour of the unit volume $d\,Vol$ of (4.5) as well as $|\Omega|$.

4.3.2 Estimates of the Unit Volume

The $4n$ coordinates on TC^{4n} split into n compact T^n-fibre coordinates τ_a and n sets of spherical polar coordinates $\mathbf{r}_a = (r_a, \theta_a, \phi_a)$ on n copies of flat $\mathbb{R}^3$.

The unit volume on $\mathcal{M}$ is:

$$d\,Vol = (det g_{ij})^{1/2} \prod_a dr_a \, d\theta_a \, d\phi_a \, d\tau_a \,,$$

where the wedge product of forms is assumed. The determinant of the Taubian–Calabi metric g_{ij}, $i, j = 1, \dots, 4n$, is:

$$(det g_{ij})^{1/2} = det G_{bc} \cdot \prod_a r_a^2 \sin\theta_a \,,$$

where the product factor comes from n flat $\mathbb{R}^3$ components of the metric expressed in spherical polar coordinates. It is a simple matter to evaluate the determinant of the matrix G_{ab} in (4.6):

$$det\, G_{ab} = \frac{1+\sum_c r_c}{r_1\, r_2\, \ldots\, r_n} .$$

Then the unit volume $d\,Vol$ is:

$$d\,Vol = (1+\sum_c r_c) \prod_a r_a\, \sin\theta_a\, dr_a\, d\theta_a\, d\phi_a\, d\tau_a . \tag{4.11}$$

Since the ranges of τ_a, θ_a and ϕ_a are compact, performing the integration in (4.10) over these variables yields some bounded constant value. Thus we are concerned only with the dependence of $d\,Vol$ on the radial coordinates r_a. There are n radial coordinates r_a, and the notion of infinity is somewhat ambiguous. To proceed we shall introduce an effective radial coordinate t in place of these n original coordinates, and by infinity we shall mean the region of TC^{4n} where t is large. The idea is to replace n degrees of freedom with infinite ranges by one degree of freedom with an infinite range and $(n-1)$ compact degrees of freedom.

For clarity consider the $n=2$ case. Set the radial three-vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ to be:

$$\begin{aligned} \mathbf{r}_1 &= \hat{\mathbf{r}}_1\, t\, \sin\alpha , \\ \mathbf{r}_2 &= \hat{\mathbf{r}}_2\, t\, \cos\alpha , \end{aligned}$$

where $t \in (0,\infty)$, $\alpha \in [0, 2\pi]$, and $\hat{\mathbf{r}}_1, \hat{\mathbf{r}}_2$ are unit vectors. Then the sizes of $\mathbf{r}_1$ and $\mathbf{r}_2$ are:

$$r_1 = t\, \sin\alpha\, ,\; r_2 = t\, \cos\alpha\, ,$$

and hence

$$dr_1 \wedge dr_2 = t\, dt \wedge d\alpha .$$

With this re-parametrization, the unit volume (4.11) for $n=2$ becomes:

$$d\,Vol = (1+t(\sin\alpha+\cos\alpha))\, t^2\, \sin\alpha\, \cos\alpha\, t\, dt\, d\alpha ,$$

which, in the limit of large t, has the following dependence on the effective radial coordinate t:

$$d\,Vol \sim t^4\, dt . \tag{4.12}$$

It is obvious how this re-parametrization generalizes to $n>2$. Hence the unit volume of the Taubian–Calabi metric on $\mathbb{R}^{4n}$ in the limit of large t grows as:

$$d\,Vol \sim t^{2n}\, dt .$$

This provides the lower bound on how quickly $|\Omega|$ should decay at infinity. Unfortunately, estimating $|\Omega|$ is not as straightforward as it was in the Lee–Weinberg–Yi case. Let us write down components of Ω.

4.3.3 Components of Ω

It is convenient to write components of the $2n$-form Ω in an orthonormal basis. Let us pick an orthonormal basis of frames for (4.5). The frame fields and their inverses are labelled by a pair of indices $(0a)$ or (ra), with $a = 1, \ldots, n$ and $r = 1, 2, 3$. The orthonormal basis of one-forms is:

$$\begin{aligned} e^{(0a)} &= (G^{-1/2})_{ab}\, (d\tau_b + \boldsymbol{\omega}_{(b)} \cdot d\mathbf{r}_b) , \\ e^{(ra)} &= (G^{1/2})_{ab} \tilde{e}^{(rb)} . \end{aligned} \tag{4.13}$$

$G^{1/2}$ is a symmetric matrix such that $G_{ab} = G^{1/2}_{ac}\, G^{1/2}_{cb}$ and $G^{-1/2}$ is its inverse. In spherical polar coordinates the one-forms $\tilde{e}^{(ra)}$ are:

$$\begin{aligned} \tilde{e}^{(1a)} &= dr_a , \\ \tilde{e}^{(2a)} &= r_a\, d\theta_a , \\ \tilde{e}^{(3a)} &= r_a\, \sin\theta_a\, d\phi_a . \end{aligned} \tag{4.14}$$

In terms of these frames a flat metric on the a-th $\mathbb{R}^3$ factor is $d\mathbf{r}_a^2 = \sum_r (\tilde{e}^{(ra)})^2$. The inverses of (4.13) are:

$$\begin{aligned} E_{(0a)} &= (G^{1/2})_{ab}\, \frac{\partial}{\partial\tau_b}, \\ E_{(ra)} &= (G^{-1/2})_{ab}\, (\tilde{E}_{(rb)} - \omega_{r(b)} \frac{\partial}{\partial\tau_b}) . \end{aligned} \tag{4.15}$$

The vectors $\tilde{E}_{(rb)}$ are dual to $\tilde{e}^{(ra)}$ (4.14). Here $\omega_{r(b)}$ stands for the r-th component of the connection one-forms $\omega_b = \boldsymbol{\omega}_{(b)} \cdot d\mathbf{r}_b$ in spherical polar coordinates. In terms of frames (4.13) the Taubian–Calabi metric (4.5) is simply

$$ds^2 = \sum_a (e^{(0a)})^2 + \sum_{a,r} (e^{(ra)})^2 .$$

Given (4.7) and (4.8), the two-form $F = d\,A$ is

$$F = \frac{r_a}{1+\sum r_c} d(d\tau_a + \omega_a) + d\left(\frac{r_a}{1+\sum r_c}\right) \wedge (d\tau_a + \omega_a) , \tag{4.16}$$

which, written in terms of frames (4.13), becomes:

$$F = \frac{\partial}{\partial r_b}\left(\frac{r_a}{1+\sum r_c}\right) G^{1/2}_{ad} G^{-1/2}_{bf} e^{(1f)} \wedge e^{(0d)} - \frac{1}{r_a(1+\sum r_c)} G^{-1/2}_{ad} G^{-1/2}_{af} e^{(2d)} \wedge e^{(3f)} , \tag{4.17}$$

where one has to remember to sum over all the repeated indices, i.e. over a, b, d and f.

Introducing a collective index $A = ((0a), (ra))$, a component of Ω looks like

$$\Omega_{A_1 B_1 \ldots A_n B_n} = \sum_{\mathbf{P}} F_{A_1 B_1} \ldots F_{A_n B_n}, \tag{4.18}$$

where $\mathbf{P}$ denotes all possible permutations of distinct indices A_a, B_b. The general expression is rather cumbersome and we shall not give it here.

As it turns out, the question of convergence is a very delicate one, and it is not sufficient to make rough estimates of $|\Omega|$. In fact, it appears to be essential that the square of the norm of Ω, $|\Omega|^2$, is computed exactly. Certain cancellations of undesirable terms — terms that do not decay sufficiently fast at infinity — occur, leading to square-integrability of Ω. We have succeeded in evaluating $|\Omega|^2$ explicitly by computing directly components of Ω only in the case $n = 2$.

4.3.4 Square-integrability of Ω on $\mathbf{TC}^8$

Already in this case the algebra is rather involved but still tractable. To evaluate $|\Omega|^2$ we used the algebraic package Maple to keep track of the combinatorics. The exact expression for $|\Omega|^2$ offers little insight, and here we shall only present the result.

Substituting the effective radial coordinate t in place of r_1 and r_2 and taking the limit $t \to \infty$ we find:

$$|\Omega|^2 \sim \frac{1}{t^6}\,.$$

This is a narrow escape! Putting together what we know about the asymptotic behaviour of the harmonic four-form and the unit volume of the Taubian–Calabi metric on $\mathbb{R}^8$ (4.12), we see that the square norm (4.10) of the candidate harmonic form Ω converges:

$$\|\Omega\|^2 \sim \int \frac{1}{t^6}\, t^4\, dt = \frac{1}{t} < \infty\,.$$

We have not yet been able to extend this argument to $2n$-forms for $n > 2$.

CHAPTER 5

Classical Bound States of Fundamental Monopoles

In this chapter we study some aspects of classical dynamics of fundamental monopoles. It was demonstrated by Manton [22] that the classical dynamics of slowly moving monopoles or dyons is equivalent to the geodesic flow on the relevant moduli space. In brief, the argument runs as follows. Since there is no net force between static BPS solitons — the repulsive magnetic force is cancelled by the attractive scalar force — there exist multi-monopole configurations saturating the Bogomol'nyi energy bound. In the case of $SU(2)$ monopoles, a multi-monopole solution can be interpreted as a superposition of many $SU(2)$ monopoles of unit charge, at least when the constituents are well-separated. In the case of $SU(n)$ monopoles, there exists a multi-monopole solution that can be everywhere interpreted as a superposition of a number of distinct fundamental monopoles. When the fields are allowed to vary with time, they will evolve along a path in the relevant moduli space, provided that the initial data correspond to slow motion tangent to the moduli space. These time-dependent solutions are almost BPS. Since the potential energy of BPS solitons is constant, it is only the kinetic term of the underlying fields that induces a metric on the moduli space. One concludes that the relevant path on the moduli space along which the fields evolve is a geodesic. This is the moduli space approximation. Thus if we wish to study classical dynamics of slowly moving monopoles, we need to solve geodesic equations on the relative moduli space.

In Section 5.1 we demonstrate the non-existence of closed or bound[1] geodesics on the Lee–Weinberg–Yi and the Taubian–Calabi manifolds. We explain why it is not surprising that no classical

[1] By bound we mean confined to a compact set at all times.

bound states of monopoles exist in the associated gauge theories.

In Section 5.2 we study the classical scattering of distinct fundamental monopoles. Although we do not expect the motion to be integrable in general, we are able to solve the geodesic equations in a special case. The scaling solution which we present in Section 5.2.1 describes the simplest scattering of fundamental monopoles with zero dyonic charges: individual monopoles approach one another from infinity, pass through the origin and move out to infinity, their velocity at infinity approaching a constant value.

When all the conserved (dyonic) charges vanish, equations of motion for a system of many well-separated $SU(2)$ monopoles are, up to an overall sign, identical to the equations of motion for a system of fundamental monopoles. Hence, one can again make the scaling ansatz and integrate the equations of motion. We find that the qualitative behaviour of individual $SU(2)$ monopoles is different from the behaviour of distinct fundamental monopoles.

The analysis in Section 5.2.2 shows that scaling solutions for distinct fundamental monopoles carrying constant dyonic charges do not exist.

Another type of simple monopole motion is rigid rotations. Making the appropriate ansatz in Section 5.2.3, we find that such motions are not allowed for distinct fundamental monopoles (which is consistent with conclusions of Section 5.1). Interestingly, for many well-separated $SU(2)$ monopoles with zero dyonic charges the ansatz yields a solution, i.e. there exist closed geodesics on the Gibbons–Manton manifold representing rigidly rotating $SU(2)$ monopoles. However, solving the equations explicitly yields circular geodesics in the unphysical region of the Gibbons–Manton metric. Physically, there are no closed orbits for pure monopoles since the net force is repulsive. This is in accord with the analysis of the two-monopole system performed in [85]. From physical considerations, as well as from reference [85] we know that closed orbits of well-separated $SU(2)$ dyons do exist. In this case, however, the dyons cannot all lie in the same plane due to the presence of electrical charges. To demonstrate this point we substitute the rotational ansatz into the equations of motion on the reduced Gibbons–Manton manifold with non-zero dyonic charges and show that although a solution exists it lies in the unphysical region of the Gibbons–Manton space.

5.1 Non-existence of Bound Geodesics

The low-energy dynamics of two well-separated $SU(2)$ monopoles was studied by Gibbons and Manton in [85], where they found bound orbits for two well-separated $SU(2)$ dyons. The existence of

bound orbits is not surprising, since the dyons are oppositely charged with respect to one unbroken $U(1)$, which gives rise to attractive Coulomb forces. The situation is different for two or more distinct fundamental $SU(n+2)$ monopoles. Distinct fundamental monopoles carry dyonic charges with respect to different $U(1)$ factors in the unbroken gauge group $U(1)^{n+1}$, which does not give rise to attractive forces. As seen from the asymptotic analysis of monopole interactions performed by Lee, Weinberg and Yi [43], the forces between these dyons are repulsive. By the moduli space approximation, classical bound states of monopoles or dyons correspond to closed or bound geodesics on the relevant moduli space. Hence we do not expect to find any closed or even bound geodesics on the Lee–Weinberg–Yi manifold, which is the relative moduli space of the charge $(1, 1, \ldots, 1)$ monopoles.

Let us now give an argument for the non-existence of bound geodesics on the Lee–Weinberg–Yi manifold. The same argument is valid for geodesics on the Taubian–Calabi manifold, although a physical interpretation of the low-energy dynamics of massless monopoles is less clear.

If the topology of the moduli space is complicated, one may invoke a general result of Benci and Giannoni [86] for open manifolds to establish the existence of closed geodesics. However, if the manifold is topologically trivial, such arguments give no information.

For topologically trivial manifolds, such as the Lee–Weinberg–Yi (3.20) and the Taubian–Calabi (3.15) spaces, one may instead use the following criterion. If there exists an everywhere distance-increasing vector field $\mathbf{V}$, then there are no closed or bound geodesics on this manifold. The distance-increasing condition means that the Lie derivative of the metric along $\mathbf{V}$ satisfies:

$$\mathcal{L}_{\mathbf{V}} g(\mathbf{X}, \mathbf{Y}) > 0 \,, \tag{5.1}$$

for all $4n$-vectors $\mathbf{X}, \mathbf{Y}$, or equivalently

$$\mathbf{V}_{(a;b)} \mathbf{X}^a \mathbf{Y}^b > 0 \,.$$

Along a geodesic with a tangent vector $\mathbf{L}$ one therefore has:

$$\frac{d}{dt} g(\mathbf{V}, \mathbf{L}) = \mathcal{L}_{\mathbf{V}} g(\mathbf{L}, \mathbf{L}) > 0 \,. \tag{5.2}$$

Now if this is a bound or closed geodesic one may average over a time period T. The left-hand side of (5.2) tends to zero as $T \to \infty$, while the right-hand side is some positive constant. This is a contradiction.

The existence of a distance-increasing vector field on the Lee–Weinberg–Yi (3.20) and the Taubian–Calabi (3.15) manifolds can be easily demonstrated if we make use of the hyperkähler quotient construction. Both the Lee–Weinberg–Yi and the Taubian–Calabi manifolds were obtained by choosing the zero-set as the level set the moment map. Why this detail is important becomes clear presently. The vector field $\mathbf{V}$ is induced on the quotient space X_0 from the following $\mathbb{R}^+ = \{\alpha\}$ action on $\mathcal{M} = \mathbb{H}^m \times \mathbb{H}^p$:

$$\begin{aligned} q_a i\bar{q}_a &\rightarrow \alpha^{1/2} q_a i\bar{q}_a \,, \\ \psi_a &\rightarrow \psi_a \,, \\ \mathrm{Re} w_i &\rightarrow \mathrm{Re} w_i \,, \\ \mathrm{Im} w_i &\rightarrow \alpha\, \mathrm{Im} w_i \,, \quad \alpha > 0 \,, \end{aligned} \tag{5.3}$$

where $q_a = r_a e^{-i\psi_a/2}$, $a = 1, \ldots, m$ and $i = 1, \ldots, p$. This $\mathbb{R}^+$ action leaves invariant the level set $\mu^{-1}(0)$ and commutes with the $\mathbb{R}^m$, T^m and $SU(2)$ actions. It descends to give a well defined $\mathbb{R}^+$ action on $X_0 = \mu^{-1}(0)/\mathbb{R}^m$ which stabilises the origin $q_a = 0$, and corresponds on the Lee–Weinberg–Yi metric to the spherically symmetric monopole. Action (5.3) is clearly distance-increasing on $\mathcal{M}$, so its restriction to $\mu^{-1}(0)/\mathbb{R}^m$ is also distance-increasing. Note that the argument just given is a more geometric version of the generalized Virial Theorem discussed earlier in [44].

5.2 Scattering of Fundamental Monopoles

The exact metric on the moduli space of n distinct fundamental $SU(n+1)$ monopoles is the Lee–Weinberg–Yi metric on $\mathbb{R}^{4n}$. For the sake of capturing the full physical picture, we begin with the moduli space metric with the centre of mass coordinates included and reinstate factors of g (the gauge coupling) and π into the solution. The exact metric is (see e.g. [43]):

$$ds^2 = G_{ij} d\mathbf{x}_i \cdot d\mathbf{x}_j + G^{-1}_{ij}(d\tau_i + \mathbf{W}_{ik} \cdot d\mathbf{x}_k)(d\tau_j + \mathbf{W}_{jl} \cdot d\mathbf{x}_l) \,, \tag{5.4}$$

where the elements of the matrix G_{ij} are

$$\begin{aligned} G_{ii} &= m_i + \frac{g^2}{4\pi} \sum_{j \neq i} \frac{\lambda_{ij}}{r_{ij}} \,, \\ G_{ij} &= -\frac{g^2}{4\pi} \frac{\lambda_{ij}}{r_{ij}} \,, \quad i \neq j \,. \end{aligned} \tag{5.5}$$

G^{-1} is the inverse of G and

$$\begin{aligned} W_{ii} &= \sum_{j \neq i} \lambda_{ij} \mathbf{w}_{ij} \,, \\ W_{ij} &= -\lambda_{ij} \mathbf{w}_{ij} \,, \quad i \neq j \,, \end{aligned} \tag{5.6}$$

where $i, j = 1, \ldots, n$, $\mathbf{x}_i$ is the position of the i-th monopole in $\mathbb{R}^3$, $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$ is the separation vector between the i-th and the j-th monopoles, and $r_{ij} = |\mathbf{r}_{ij}|$. $\mathbf{w}_{ij}$ is the value at $\mathbf{x}_i$ of the Dirac potential due to the j-th monopole, which satisfies

$$\nabla_j \times \mathbf{w}_{ij}(\mathbf{r}_{ij}) = -\frac{\mathbf{r}_{ij}}{r^3_{ij}} \,. \tag{5.7}$$

Positive constants λ_{ij} depend on the gauge group. In fact, for gauge group $SU(n+1)$ $\lambda_{ij} = 1$ for all pairs (i, j), and in the following we shall restrict ourselves to this case.

We can study dynamics of the fundamental monopoles by finding geodesics on the Lee–Weinberg–Yi manifold which describe monopole interactions. The Lee–Weinberg–Yi manifold (5.4) can be viewed as the total space of the T^n principle bundle over a conformally flat $\mathbb{R}^{3n}$ base. We refer to the base manifold $\mathbb{R}^{3n}$ as the reduced moduli space $\mathcal{M}_{red}$. Recall that the moduli space metric is invariant under the T^n action. Hence from the physical point of view it is simplest to project the geodesics onto $\mathcal{M}_{red}$ and encode the motion in the toric fibres into conserved quantities associated with the T^n isometry.

In other words, n of the $4n$ variables in the Lagrangian obtained from the Lee–Weinberg–Yi metric:

$$\mathcal{L} = G_{ij} \frac{d\mathbf{x}_i}{dt} \cdot \frac{d\mathbf{x}_j}{dt} + G^{-1}_{ij} \Big(\frac{d\tau_i}{dt} + \mathbf{W}_{ik} \cdot \frac{d\mathbf{x}_k}{dt} \Big) \Big(\frac{d\tau_j}{dt} + \mathbf{W}_{jl} \cdot \frac{d\mathbf{x}_l}{dt} \Big) \tag{5.8}$$

are not dynamical. Instead one can identify n conserved charges:

$$Q^i = G^{-1}_{ij} \Big(\frac{d\tau_j}{dt} + \mathbf{W}_{jk} \cdot \frac{d\mathbf{x}_k}{dt} \Big) \,,$$

and eliminate τ_i's from (5.8) in favour of the conserved charges Q^i to obtain the effective Lagrangian on $\mathcal{M}_{red}$:

$$\mathcal{L}_{eff} = G_{ij} \frac{d\mathbf{x}_i}{dt} \cdot \frac{d\mathbf{x}_j}{dt} - G_{ij} Q^i Q^j + Q^i \mathbf{W}_{ik} \cdot \frac{d\mathbf{x}_k}{dt} . \tag{5.9}$$

Two cases shall be considered separately: (i) all Q^i vanish and (ii) non-zero Q^i.

Scattering of monopoles:

Let us first discuss scattering of monopoles. When all dyonic charges vanish the effective Lagrangian and the equations of motion on the reduced moduli space possess scaling invariance. This property prompts one to look for scaling solutions (sometimes also called similarity or homothety solutions). To illustrate what is meant by a scaling solution consider the Lagrangian for a particle moving in some potential $V(\mathbf{x})$:

$$\mathcal{L} = \frac{\dot{\mathbf{x}}^2}{2} - V(\mathbf{x}) ,$$

with equations of motion

$$\ddot{\mathbf{x}} = -\frac{d}{d\mathbf{x}} V(\mathbf{x}) \equiv -V'(\mathbf{x}) .$$

If the potential $V(\mathbf{x})$ is such that

$$V'(a\,\mathbf{x}) = a^p\, V'(\mathbf{x})$$

for some function of time $a(t)$, and p a rational number, then one can make the following ansatz:

$$\mathbf{x}(t) = a(t)\,\mathbf{y} .$$

All the time-dependence is contained in the scalar function $a(t)$, and the vector $\mathbf{y}$ is independent of time. The equations of motion then become:

$$\ddot{a}\,\mathbf{y} = -\,a^p\, V'(\mathbf{y}) .$$

Since the original Lagrangian possesses scaling invariance, the equations of motion are equivalent to the following system:

$$\begin{aligned} \frac{\ddot{a}}{a^p} &= C , \\ \mathbf{y} &= -\frac{1}{C} V'(\mathbf{y}) , \end{aligned}$$

where C is some constant. The first ODE yields the time-dependent scaling factor, while solutions of the second equation, if they exist, are called *central configurations*. Solving the latter is equivalent to finding critical points $\partial W/\partial \mathbf{y} = 0$ of the function W

$$W \equiv \frac{C}{2}\mathbf{y}^2 + V(\mathbf{y}) .$$

Physical systems of this type, i.e. where the potential scales, are encountered, for example, in celestial mechanics. The evolution equation of the scaling factor is akin to the Friedmann equation in an FRW cosmological model. The problem of looking for central configuration in this case is the n-body problem of gravitating bodies. The classification of all central configuration of such a system is an outstanding problem. Another example of a system possessing scaling invariance is the Newtonian matrix cosmology discussed in [87], where the authors find some scaling solutions to the equations of motion.

When all conserved charges Q^i of distinct fundamental monopoles vanish, one intuitively expects to find a solution which would describe the simplest scattering process: individual monopoles arranged in some symmetric configuration, moving with constant speeds at infinity, come closer together, losing their individual identity, pass through the origin, forming the spherically symmetric charge $(1,1,\ldots,1)$ monopole, move apart, regaining their individual character, and return to free particle motion at infinity (possibly in a different symmetric configuration). This motion, expressed in terms of particle dynamics on the reduced space, can be described by a scaling solution.

In the following we shall also discuss similarity geodesics on the asymptotic moduli space of many $SU(2)$ monopoles (the Gibbons–Manton manifold). We shall demonstrate that such solutions are permitted.

In the case of non-vanishing dyonic charges Q^i, equations of motion on the reduced moduli space of fundamental monopoles do not possess such a scaling symmetry, and hence one does not expect the scaling ansatz to lead to a solution.

Rigidly rotating monopole configurations:

Another type of monopole motions which can be described by a relatively simple ansatz is rigid rotations. As in the scaling ansatz, the time-dependence of the positions of monopoles is under control. One can imagine a system of monopoles in a plane rotating rigidly about some axis perpendicular to it, in other words so that the distances between individual monopoles are fixed. Corresponding to this motion is a closed geodesic on the moduli space. To find such a solution on the reduced moduli space one should make the following ansatz:

$$\dot{\mathbf{x}}_i = \boldsymbol{\omega} \times \mathbf{x}_i , \tag{5.10}$$

where $\boldsymbol{\omega}$ is the constant angular velocity common to all monopoles in the system. It follows from (5.10) that:

$$\dot{\mathbf{r}}_{ij} = \boldsymbol{\omega} \times \mathbf{r}_{ij} ,$$

where $\mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j$. Hence

$$\dot{r}_{ij} = \frac{d}{dt}(\mathbf{r}_{ij}\cdot\mathbf{r}_{ij})^{\frac{1}{2}} = \frac{1}{r_{ij}}\dot{\mathbf{r}}_{ij}\cdot\mathbf{r}_{ij} = 0 ,$$

which is the statement that the individual particles do not move with respect to one another. The assumption that the monopoles are contained in a plane and rotate about an axis perpendicular to it implies that $\omega \cdot \mathbf{r}_{ij} = 0$ and hence

$$\ddot{\mathbf{r}}_{ij} = -|\omega|^2 \mathbf{r}_{ij} \, .$$

Substituting this ansatz into the Euler–Lagrange equations we find that the constant angular velocity factors out and its specific value is unimportant. One is left with a set of equations containing no time-dependant terms analogous to equations describing central configurations in the scaling ansatz.

Since rigid rotations are represented by closed geodesics on the moduli space, they correspond to classical bound states of monopoles. As we have demonstrated in Section 5.1 for distinct fundamental monopoles classical bound states do not exist. Thus when in Section 5.2.3 we use the ansatz (5.10) to solve the geodesic equations on the Lee–Weinberg–Yi manifold (with vanishing dyonic charges), we find that the equations do not admit a solution. This is only consistent with what we already knew. On the contrary, in the case of many well-separated $SU(2)$ monopoles with no dyonic charges the rotational ansatz (5.10) leads to equations which admit solutions. These solutions, however, are in the unphysical region of the Gibbons–Manton space. This is to be expected since the net force between two $SU(2)$ monopoles is repulsive. Making the same ansatz for well-separated $SU(2)$ dyons again leads to an unphysical solution since the dyons cannot all lie in the same plane as our ansatz presupposes.

5.2.1 Scattering of Uncharged Fundamental Monopoles

Setting $Q^i = 0$ and using (5.5) in (5.9) gives:

$$\mathcal{L} = \frac{1}{2}\sum_{i=1}^{n}\Big(m_i + \frac{g^2}{4\pi}\sum_{j\neq i}\frac{1}{r_{ij}}\Big)\dot{\mathbf{x}}_i^2 - \frac{1}{2}\frac{g^2}{4\pi}\sum_{i,j\neq i}\frac{1}{r_{ij}}\dot{\mathbf{x}}_i\cdot\dot{\mathbf{x}}_j \, . \tag{5.11}$$

The Euler–Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \mathbf{x}_k} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{x}}_k} = 0$$

for the above Lagrangian are:

$$m_k \ddot{\mathbf{x}}_k = \frac{g^2}{4\pi}\sum_{i\neq k}\Big(\frac{1}{2}\frac{\dot{r}_{ik}^2 \mathbf{r}_{ik}}{r_{ik}^3} + \frac{\ddot{\mathbf{r}}_{ik}}{r_{ik}} - \frac{\dot{r}_{ik}\dot{\mathbf{r}}_{ik}}{r_{ik}^2}\Big) \, , \tag{5.12}$$

where $\dot{\mathbf{x}}_k = \frac{d}{dt}\mathbf{x}_k$. Now let us make the following ansatz:

$$\mathbf{x}_i = a(t)\, \mathbf{a}_i \, . \tag{5.13}$$

Then $\mathbf{r}_{ij} = a(t)\, \mathbf{a}_{ij}$ and $r_{ij} = a\, a_{ij}$. Equations of motion (5.12) become

$$m_k \mathbf{a}_k = \frac{g^2}{4\pi}\sum_{i\neq k}\frac{\mathbf{a}_{ik}}{a_{ik}}\frac{1}{\ddot{a}}\Big(\frac{\ddot{a}}{a} - \frac{1}{2}\Big(\frac{\dot{a}}{a}\Big)^2\Big) \, . \tag{5.14}$$

All t-dependence is now collected on the right-hand side of the equation. The common practice in solving this type of problems in celestial mechanics is to require that the t-dependence amounts to a constant. We shall do the same and separate (5.14) into the t-dependent part and the part describing dynamics in the transverse space:

$$\frac{1}{\ddot{a}}\Big(\frac{\ddot{a}}{a} - \frac{1}{2}\Big(\frac{\dot{a}}{a}\Big)^2\Big) = C \, , \tag{5.15}$$

$$m_k \mathbf{a}_k = C\frac{g^2}{4\pi}\sum_{i\neq k}\frac{\mathbf{a}_{ik}}{a_{ik}} \, , \tag{5.16}$$

where C is a real constant that may be positive or negative. Note that $C = 0$ is not allowed for a physical solution of (5.14). Moreover, there is a trivial scaling symmetry of (5.14) which allows us to rescale C. Hence it is sufficient to consider two values of the constant $C = \pm 1$.

Let us look for solutions of (5.15). First of all, $a(t) = const$ trivially solves the equation and describes a static solution, which is not of much interest to us. Another obvious guess is a power-law solution $a(t) = t^p$. However, equation (5.15) is inhomogeneous and clearly does not admit a solution of this type, unless $C = 0$, which is not physically meaningful. Nevertheless, it is possible to solve (5.15).

Letting $a = x$ and $\dot{a} = y$, we can rewrite (5.15) as:

$$\begin{aligned} \dot{x} &= y \, , \\ \dot{y} &= \frac{y^2}{2x(1 - Cx)} \, , \end{aligned} \tag{5.17}$$

which implies

$$\frac{dy}{dx} = \frac{y}{2x(1 - Cx)} \, . \tag{5.18}$$

If $C = 1$, (5.18) can be integrated to give

$$y = v_\infty \sqrt{\frac{x}{x-1}} \, , \tag{5.19}$$

where v_∞ is a constant of integration.

If $C = -1$, (5.18) can be integrated to give

$$y = v_\infty \sqrt{\frac{x}{x+1}} \, . \tag{5.20}$$

We can visualize these solutions by sketching them on a phase diagram (fig. 5.1). For large values

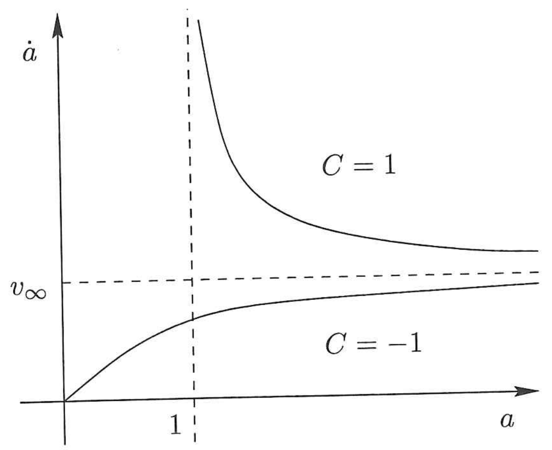

Figure 5.1: Phase-plane diagram of the two solutions.

of a, the velocity of the particles approaches a constant value v_∞. If we follow the curve for $C = 1$ towards small a, we find that at $a = 1$ the velocity $\dot{a}$ blows up. This is not a physically realistic behaviour for distinct fundamental monopoles.

The solution for $C = -1$ shows that the monopoles reach the origin $a = 0$. This is the type of behaviour we expect for distinct fundamental monopoles. On the moduli space, the Lee–Weinberg–Yi manifold, the origin is just a coordinate singularity. If the Lee-Weinberg-Yi metric (5.4) were expressed in Cartesian coordinates, we would expect the proper time along a geodesic through the origin to be proportional to the radial distance. Coordinates on the Lee–Weinberg–Yi manifold (5.4) are, in some sense, a square root of the spherical polar coordinates. To be more precise, the point $r = 0$ on the Taub-NUT manifold (3.8) is merely a coordinate singularity (see Section 2.2.2), and in the neighbourhood of the origin the metric looks flat if we replace r by ρ^2. The geodesics described by the $C = -1$ curve pass through the origin and do so in finite proper time t which we estimate from (5.17) and (5.18) to be

$$t \sim \sqrt{x}\,.$$

This is precisely the expected dependence, since the origin is a regular point of the manifold.

We should now check for which value of C, if any, equation (5.16) has solutions. In other words, whether it is possible to find configurations of fundamental monopoles compatible with this equation (cf. central configurations in the simple example described above). Finding central configurations of

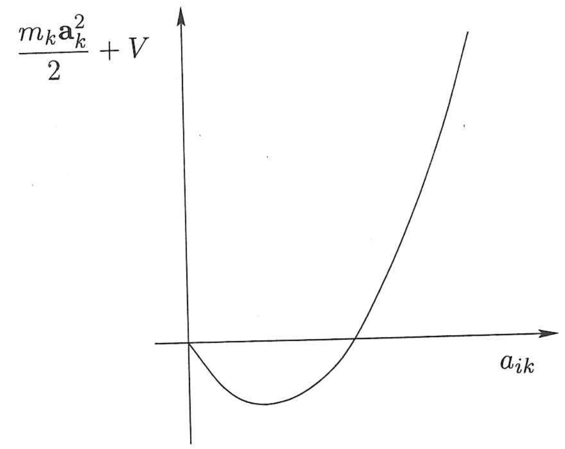

Figure 5.2: Schematic dependence of W on a_{ik} .

(5.16) is equivalent to finding critical points $\frac{\partial}{\partial \mathbf{a}_k} W = 0$ of the function $W \equiv \frac{m_k \mathbf{a}_k^2}{2} + V$:

$$\frac{\partial}{\partial \mathbf{a}_k}\Big(\frac{m_k \mathbf{a}_k^2}{2} + V\Big) = 0\,. \tag{5.21}$$

Comparing this with (5.16) we find V to be

$$V = C\,\frac{g^2}{4\pi}\sum_{i\neq k} a_{ik}\,.$$

When $C = 1$ the potential V is positive and the total energy is always positive, hence W has no critical points. Intuitively, critical points of W should exist for $C = -1$. In this case in the neighbourhood of $\mathbf{a}_i = 0$ V decreases from zero sufficiently fast, and the schematic dependence of W on a_{ik} is sketched on fig. 5.2. Finding and classifying all central configurations that solve (5.16) is a challenging problem, in the same way as it is challenging to find all central configurations of a gravitating n-body problem. However, a few symmetric configurations are easy to spot. For two distinct fundamental monopoles $a_{12} = const$ is clearly a solution. For three monopoles an equilateral triangle is a solution. We can see that any sufficiently symmetric planar (or possibly non-planar) arrangement of n particles about the origin would correspond to a critical point of W.

Note that equations (5.17) are invariant under the change of sign of t: $d/dt \to -d/dt$ and hence $y \to -y$. This means that the problem is symmetric under time-reversal. Positive t corresponds to monopoles moving away from the origin, while negative t corresponds to monopoles moving towards the origin.

We shall now discuss the case of many well-separated $SU(2)$ monopoles. Compare equations of motion (5.12) on the Lee–Weinberg–Yi manifold with equations of motion on the Gibbons–Manton manifold in [45] for $Q^i = 0$. Rewritten in the notation adopted here the equations of motion are:

$$m_k \ddot{\mathbf{x}}_k = -\frac{g^2}{4\pi} \sum_{i \neq k} \left(\frac{1}{2} \frac{\dot{r}_{ik}^2 \mathbf{r}_{ik}}{r_{ik}^3} + \frac{\ddot{\mathbf{r}}_{ik}}{r_{ik}} - \frac{\dot{r}_{ik} \dot{\mathbf{r}}_{ik}}{r_{ik}^2} \right). \tag{5.22}$$

The only difference between equations (5.22) and (5.12) is the overall change of sign of the right-hand side, hence these equations are also integrable for the ansatz (5.13). However, the constant C in (5.15) and (5.16) and in the two sets of solutions (5.19) and (5.20) must be replaced by another constant $\tilde{C} = -C$. It is the first solution (5.19) with $C = 1$, and hence $\tilde{C} = -1$, that is now physically applicable. Equation (5.16) becomes:

$$m_k \mathbf{a}_k = \tilde{C} \frac{g^2}{4\pi} \sum_{i \neq k} \frac{\mathbf{a}_{ik}}{a_{ik}},$$

which admits solutions for $\tilde{C} = -1$. Monopole configurations solving this equation are the same monopole configurations as those solving equation (5.16) for $C = -1$.

The Gibbons–Manton metric (3.31) is a good approximation to the exact metric on the moduli space of higher charge $SU(2)$ monopoles. It differs from the still unknown exact metric by exponentially small terms. In the asymptotic region of the moduli space described by the Gibbons–Manton metric, a higher charge $SU(2)$ monopole may be regarded as a superposition of many well-separated charge one $SU(2)$ monopoles. This approximation breaks down when the inter-monopole distances are comparable with the size of their cores. If we follow the geodesic (5.19) into the interior, we see that it is not regular at $a = 1$. This behaviour of the solution is perfectly natural, since the approximation is no longer valid for short inter-monopole distances, i.e. in the neighbourhood of $a = 1$. Similarity solutions like (5.19) exist only in the asymptotic region of the exact moduli space, and we do not expect the precise form of the solution to remain valid in the interior. The k charge one $SU(2)$ monopoles sent in from infinity in a highly symmetrical arrangement, lose their identity and presumably form an $SU(2)$ k-monopole with appropriate symmetry. A number of such symmetric $SU(2)$ k-monopole solutions were found in [88, 89, 90] by solving the Nahm data numerically.

To sum up, we have found a set of geodesics on the Lee–Weinberg–Yi manifold describing a simple scattering of many distinct fundamental monopoles carrying no dyonic charges. These geodesics are nothing other than similarity solutions to the equations of motion of the effective Lagrangian on the reduced moduli space. Individual fundamental monopoles behave like free point particles at infinity.

Sent in from infinity in a symmetric configuration, they approach one another forming a spherically symmetric charge $(1, 1 \ldots 1)$ monopole at the origin. Reversing the direction of time, we can watch this spherically symmetric monopole split up into fundamental constituents which move away in a symmetric configuration and return to free particle motion at infinity. A qualitative argument suggests that if the initial arrangement of the fundamental monopoles is not symmetrical, the system would have a non-zero angular momentum, monopoles would not reach the origin and the spherically symmetric monopole would not be formed. A quantitative description of this process should be possible.

We have also noticed that the equations of motion on the reduced Gibbons–Manton manifold are, up to a sign, identical to the equations of motion on the reduced Lee–Weinberg–Yi manifold. The sign difference accounts for the difference in the qualitative behaviour between many well-separated $SU(2)$ monopoles and distinct fundamental monopoles. The Gibbons–Manton metric changes signature and develops singularities when two or more charge one $SU(2)$ monopoles come close together, thus it is a valid approximation to the exact metric only in the asymptotic regions of the moduli space. We therefore expect the similarity solution to break down in the region near the multi-monopole core. This is precisely the effect we have observed.

5.2.2 Scattering of Charged Fundamental Monopoles

Next we would like to see whether the similarity ansatz (5.13) solves the equations of motion of $\mathcal{L}_{eff}$ (5.9) for distinct fundamental monopoles with non-vanishing charges Q^i. We go through the same steps as we have done in the previous case.

For $Q^i \neq 0$ and using (5.5) and (5.6) the effective Lagrangian (5.9) becomes:

$$\begin{aligned} \mathcal{L}_{eff} &= \frac{1}{2} \sum_{i=1}^{n} \Big(m_i + \frac{g^2}{4\pi} \sum_{j \neq i} \frac{\lambda_{ij}}{r_{ij}} \Big) \Big(\dot{\mathbf{x}}_i^2 - \frac{(Q^i)^2}{g^2} \Big) \\ &- \frac{1}{2} \frac{g^2}{4\pi} \sum_{i,j \neq i} \frac{\lambda_{ij}}{r_{ij}} \Big(\dot{\mathbf{x}}_i \cdot \dot{\mathbf{x}}_j - \frac{Q^i Q^j}{g^2} \Big) - \frac{g}{4\pi} \sum_{i,j \neq i} Q^i \lambda_{ij} \left(\mathbf{w}_{ij} \cdot \mathbf{r}_{ij} \right). \end{aligned} \tag{5.23}$$

The last term in the Lagrangian (5.23) vanishes due to (5.7). The Euler–Lagrange equations of (5.23) are:

$$m_k \ddot{\mathbf{x}}_k = \frac{g^2}{4\pi} \sum_{i \neq k} \lambda_{ik} \Big(-\frac{1}{2} \frac{\mathbf{r}_{ik}}{r_{ik}^3} \frac{(Q^i - Q^k)^2}{g^2} + \frac{1}{2} \frac{\dot{r}_{ik}^2 \mathbf{r}_{ik}}{r_{ik}^3} + \frac{\ddot{\mathbf{r}}_{ik}}{r_{ik}} - \frac{\dot{r}_{ik} \dot{\mathbf{r}}_{ik}}{r_{ik}^2} \Big). \tag{5.24}$$

The only difference between these and the equations of motion in the uncharged case (5.12) is the presence of the first term in parenthesis on the right-hand side. The presence of this term breaks scaling symmetry of the problem. Therefore, it is not surprising that making the same ansatz, $\mathbf{x}_i = a(t)\,\mathbf{a}_i$, we find that the equations of motion

$$m_k \mathbf{a}_k = \frac{g^2}{4\pi} \sum_{i \neq k} \lambda_{ik} \frac{\mathbf{a}_{ik}}{a_{ik}} \frac{1}{\ddot{a}} \left(-\frac{1}{2} \frac{(Q^i - Q^k)^2}{g^2} \frac{1}{a^2\, a_{ik}^2} + \frac{\ddot{a}}{a} - \frac{1}{2} \left(\frac{\dot{a}}{a} \right)^2 \right) \tag{5.25}$$

do not separate into a t-independent part and the evolution equation for the scaling factor $a(t)$. If we require that the two terms involving $a(t)$ are constant independently:

$$\begin{aligned} \ddot{a}\, a^2 &= C_1 \,, \\ \frac{1}{\ddot{a}} \left(\frac{\ddot{a}}{a} - \frac{1}{2} \left(\frac{\dot{a}}{a} \right)^2 \right) &= C_2 \,, \end{aligned} \tag{5.26}$$

we arrive at an over-constrained system, admitting no solutions that satisfy both equations simultaneously. This suggests that similarity solutions to (5.25) do not exist, which is consistent with the result of Section 5.1.

5.2.3 Rigidly Rotating Monopole Configurations

Let us now make the ansatz (5.10)

$$\dot{\mathbf{x}}_i = \boldsymbol{\omega} \times \mathbf{x}_i \,.$$

Consider first a system of distinct fundamental monopoles with all dyonic charges vanishing, $Q^i = 0$. Substitute the above ansatz into the equations of motion (5.12). The first and third terms in parenthesis on the right-hand side of the equation vanish since $\dot{r}_{ij} = 0$ and, eliminating the constant $|\omega|^2$, we have:

$$m_k \mathbf{x}_k = \frac{g^2}{4\pi} \sum_{i \neq k} \frac{\mathbf{r}_{ik}}{r_{ik}} \,. \tag{5.27}$$

This equation contains no t-dependent terms and it is, up to a constant, identical to equation (5.16) with $C = 1$. Thus solving (5.27) is equivalent to finding central configurations of equations (5.16). Recall from the discussion in Section 5.2.1 that central configurations of (5.16) exist only for $C = -1$. We conclude that there are no solutions of (5.27). This conclusion is consistent with the result of Section 5.1, where we have proved that no closed geodesics exist on the Lee–Weinberg–Yi manifold.

Consider now a system of many well-separated $SU(2)$ monopoles. In this case equations of motion on the reduced moduli space are (5.22), and substituting the rotational ansatz we have:

$$m_k \mathbf{x}_k = -\frac{g^2}{4\pi} \sum_{i \neq k} \frac{\mathbf{r}_{ik}}{r_{ik}} \,. \tag{5.28}$$

This equation is precisely equation (5.16) with $C = -1$, which does admit central configurations. At this stage one has to proceed with caution. Recall that the system consists of $SU(2)$ monopoles with no electrical charges, hence the only force between them is the repulsive magnetic force. Consequently, one does not expect to find any classical bound states or, equivalently, any closed geodesics.

This apparent contradiction is easily resolved. Consider a system of two monopoles. One can solve equations (5.28) explicitely to find $r \equiv r_{12}$:

$$r = \frac{g^2}{4\pi} \frac{1}{\mu} \,,$$

where $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass of two monopoles. This is the value of r where the metric on the relative moduli space of two well-separated $SU(2)$ monopoles — the Taub–NUT metric with negative mass parameter — develops a singularity and changes signature. Hence in this region the asymptotic moduli space metric is no longer a good approximation to the exact moduli space (the Atiyah–Hitchin metric). Therefore the solution we found is unphysical. The conclusion is valid for more than two monopoles.

Let us apply the rotational ansatz to monopole configurations with non-vanishing dyonic charges. Based on the results of Section 5.1 we do not expect to find solutions in the case of distinct fundamental monopoles, let us therefore turn to well-separated $SU(2)$ dyons. Given ansatz (5.10) the equations of motions (see [45]) reduce to:

$$m_k\, \mathbf{x}_k = -\frac{g^2}{4\pi} \sum_{i \neq k} \left(1 + \frac{(Q^i - Q^k)^2}{|\omega|^2 g^2} \frac{1}{r_{ik}^2} \right) \frac{\mathbf{r}_{ik}}{r_{ik}} \,. \tag{5.29}$$

Consider a system of two dyons and solve equations (5.29) to find:

$$\mu = \frac{g^2}{4\pi} \frac{1}{r} \left(1 + \frac{1}{r^2} \frac{q^2}{g^2 \omega^2} \right) \,, \tag{5.30}$$

where $q \equiv Q^1 - Q^2$ and, as before, $r = r_{12}$ and μ is the reduced mass. The authors of [85] studied orbits of two well-separated dyons and concluded that closed orbits exist but the motion of dyons is not planar. In fact all dyon orbits are conic sections. With a non-vanishing electric charge, it is not possible for the dyons to move in the same plane. Rather the planes to which the motion of individual dyons is confined are parallel. Hence it is not surprising that the solution of (5.30) is unphysical. To reproduce the circular orbits of [85] we need to modify the ansatz (5.10) to allow $\boldsymbol{\omega} \cdot \mathbf{r}_{ij} \neq 0$, while $\boldsymbol{\omega} \cdot \mathbf{x}_i = 0$. In this instance the resulting equations are more complicated than (5.29).

CHAPTER 6

Cones, tri–Sasakian Structures and Superconformal Invariance

In this chapter, based primarily on [91], we explore two geometrical structures, namely Sasakian and tri–Sasakian structures defined naturally on odd-dimensional manifolds, as well as their relationship to Kähler and hyperkähler geometries. Developments in superconformal field theories and other areas of string theory have indicated that Sasakian and tri–Sasakian geometries arise as the underlying mathematical structures.

There has been great interest in rigid conformally invariant supersymmetric field theories. In particular de Wit, Kleijn and Vandoren [92] have studied $\mathcal{N}=2$ models containing hypermultiplets taking values in a hyperkähler manifold $(\mathcal{M}, g_{\mu\nu}, I_a{}^{\mu}{}_{\nu})$, where $\mu, \nu = 1, \ldots, 4k = dim\mathcal{M}$ and $a = 1, 2, 3$. They find the following necessary condition that the target manifold admits an infinitesimal dilatation invariance: $(\mathcal{M}, g_{\mu\nu})$ admits a vector field Ψ^{μ} such that

$$\boxed{\Psi^{\mu}{}_{;\nu} = \delta^{\mu}_{\nu}} \tag{6.1}$$

In the following sections we point out that condition (6.1) implies (regardless of any hyperkähler condition) that $\mathcal{M}$ is a cone, $C(B)$, over a base manifold B, i.e. in coordinates $x^{\mu} = (r, x^i)$, $i = 1, \ldots, dim\mathcal{M} - 1$, the metric $g_{\mu\nu}$ on the cone $C(B)$ is

$$g_{\mu\nu}\, dx^{\mu}\, dx^{\nu} = dr^2 + r^2\, h_{ij}\, dx^i\, dx^j\ , \tag{6.2}$$

where $h_{ij}(x^k)$ is the metric on the base B which depends only on x^k. Moreover, in these coordinates

$$\Psi = r\frac{\partial}{\partial r}\ ,$$

and the dilatation acts on $C(B)$ as

$$(r, x^i) \rightarrow (\lambda r, x^i), \quad \lambda \in \mathbb{R}^+ .$$

The differential operator $\Psi = r\partial/\partial r$ is sometimes called the Eulerian vector field.

In the case that $(\mathcal{M}, g_{\mu\nu})$ is a Ricci-flat Kähler manifold, condition (6.1) implies that the vector field Ψ^μ is holomorphic, that the base manifold B carries a Sasakian structure and hence the metric $g_{\mu\nu}$ admits a holomorphic Killing vector field

$$K^\mu = I^\mu_\nu \Psi^\nu ,$$

where I^μ_ν is the complex structure. Presumably this case arises in $\mathcal{N} = 1$ rigid superconformally invariant theories [32]. If $(\mathcal{M}, g_{\mu\nu})$ is Ricci-flat then B must be Sasakian–Einstein. We do not know whether rigid $\mathcal{N} = 1$ superconformal invariance implies that the metric should be Ricci-flat.

In the case that $(\mathcal{M}, g_{\mu\nu})$ is hyperkähler, the base manifold B admits a tri–Sasakian structure and the metric $g_{\mu\nu}$ also admits an $SU(2)$ action by isometries which permutes the complex structures I_1, I_2 and I_3. In this case the metric is necessarily Ricci-flat and the base manifold is necessarily Einstein.

This chapter is organized as follows. In Section 6.1 we shall introduce Sasakian and tri–Sasakian geometries, not yet well known to physicists, emphasizing their connection with Kähler and hyperkähler geometries. In Section 6.2 we study equation (6.1) in an arbitrary metric $g_{\mu\nu}$ and show that it leads to equation (6.2). In Section 6.3 we assume $(\mathcal{M}, g)$ is Kähler. In Section 6.4 we assume $(\mathcal{M}, g)$ is hyperkähler. In Section 6.5 we discuss a general homothety which does not satisfy (6.1). Section 6.6 contains examples, and in Section 6.7 we discuss applications of the results. We find it remarkable how simply our main results follow from equation (6.1) and, although there are a number of discussions of cone geometries in the pure mathematics literature, the simple and direct treatment presented here is likely to be especially appealing to field theorists.

6.1 Sasakian and tri–Sasakian Geometries

Historically, the notion of Sasakian geometry arose from the study of contact geometry [93], and it is usually defined via contact structure by adding to it a Riemannian metric with some additional conditions (see e.g. [94]).

A $(2n + 1)$ real-dimensional manifold S with a Riemannian metric h and a triple (ξ, η, Φ) of tensor fields — where ξ is a Killing vector, η is a one-form and Φ is a $(1, 1)$ tensor fields on S — is called Sasakian if the following conditions hold:

- The Killing vector field ξ (called the *characteristic vector field*) and the one-form η (called the *characteristic one-form*), which is dual to ξ, i.e. $\eta(Y) = g(\xi, Y)$ for all vectors $Y \in TS$, satisfy

  $$\eta(\xi) = 1 ,$$

  in other words the Killing vector ξ has unit length;

- The Riemannian metric h is compatible with the Sasakian structure (ξ, η, Φ)

  $$h(\Phi\, X, \Phi\, Y) = h(X, Y) - \eta(X)\eta(Y) ,$$

  where $X, Y \in TS$ are any two vector fields on S;

- The $(1, 1)$ tensor field Φ which is an endomorphism on TS defined by $\Phi(X) = \nabla_X\, \xi$ (∇ is the Levi–Civita connection of h) satisfies

  $$\Phi \circ \Phi(Y) = -Y + \eta(Y)\, \xi$$

  and

  $$(\nabla_X\, \Phi)(Y) = h(\xi, Y)\, X - h(X, Y)\, \xi .$$

There are other properties which may be derived from, or are equivalent to those, but we shall omit them here and refer the reader to the book by Blair [94] and a more recent publication by Boyer, Galicki and Mann [95] and references therein. The above definition parallels the standard definition of a Kähler manifold. In fact the two geometries are intimately related. It is this relation that can be used to provide a more economical definition of a Sasakian manifold offering greater geometrical insight:

A Riemannian manifold (S, h) of real dimension k is Sasakian if the holonomy group of the metric cone $C(S) = \mathbb{R}_+ \times S$ on S with metric $g = dr^2 + r^2\, h$ reduces to a subgroup of $U(\frac{k+1}{2})$. Then $k = 2n + 1$, $n \geq 1$ and the manifold $(C(S), g)$ is Kähler.

This definition is consistent with the conclusions reached in the forthcoming sections.

A Sasakian manifold does not automatically satisfy the Einstein equations. Sasakian–Einstein manifolds form a subclass of all Sasakian metrics for which the holonomy group of the metric cone

$C(S)$ reduces to a subgroup of the unitary group $SU(\frac{k+1}{2})$. This implies that the Kähler manifold $C(S)$ with the cone metric g is Ricci-flat.

Some typical examples of Sasakian–Einstein manifolds are the Euclidean space $\mathbb{R}^{2n+1}$, the sphere S^{2n+1} and the real projective space $\mathbb{RP}^{2n+1}$ with their canonical metrics.

Recall how hypercomplex geometry arises from the notion of complex geometry (see e.g. [19]), when instead of one complex structure a manifold carries three complex structures satisfying certain commutation relations. In much the same way the notion of Sasakian geometry may be refined to manifolds carrying Sasakian 3-structures, otherwise known as tri–Sasakian manifolds.

A $(4n+3)$ real-dimensional $(n \geq 1)$ manifold S with a Riemannian metric h and three Sasakian structures (ξ^a, η^a, Φ^a), $a = 1, 2, 3$, is tri–Sasakian if the three Killing vector fields ξ^a are such that

$$h(\xi^a, \xi^b) = \delta_{ab} ,$$

and

$$[\xi^a, \xi^b] = 2\epsilon_{abc}\, \xi^c ,$$

and moreover the tensor fields satisfy the following conditions:

$$\begin{aligned} \eta^a(\xi^b) &= \delta^{ab} , \\ \Phi^a\, \xi^b &= -\epsilon^{abc}\, \xi^c , \\ \Phi^a \circ \Phi^b - \xi^a \otimes \eta^b &= -\epsilon^{abc}\Phi^c - \delta^{ab}\, \mathbb{I} . \end{aligned}$$

As before a more insightful definition of a tri–Sasakian manifold may be given in terms of the requirement of holonomy reduction:

A Riemannian manifold (S, h) of real dimension k is tri–Sasakian if the holonomy group of the metric cone $C(S) = \mathbb{R}_+ \times S$ on S with metric $g = dr^2 + r^2\, h$ reduces to a subgroup of $Sp(\frac{k+1}{2})$. Then $k = 4n+3$, $n \geq 1$ and the manifold $(C(S), g)$ is hyperkähler.

Since a hyperkähler manifold is necessarily Ricci-flat, every tri–Sasakian manifold is Einstein. The simplest examples of tri–Sasakian manifolds are the sphere S^{4n+3} and the real projective space $\mathbb{RP}^{4n+3}$ with their canonical metrics. Every three-dimensional Sasakian–Einstein manifold must also carry a tri–Sasakian structure (cf. every four-dimensional Ricci-flat Kähler manifold is hyperkähler). For an in depth discussion of tri–Sasakian manifolds and orbifolds see [95, 96] and references therein.

6.2 Cones and Dilatations

A manifold $(\mathcal{M}, g_{\mu\nu})$ regardless of signature of $g_{\mu\nu}$ admits a conformal Killing vector field Ψ if and only if

$$\mathcal{L}_\Psi g_{\mu\nu} = \phi g_{\mu\nu} = \Psi_{\mu;\nu} + \Psi_{\nu;\mu} \tag{6.3}$$

for some smooth function ϕ. If ϕ is constant, Ψ^μ is said to generate a *homothety*. If Ψ^μ is hypersurface orthogonal, i.e.

$$\Psi_\mu = \partial_\mu f \iff \Psi_{\mu;\nu} = \Psi_{\nu;\mu} \tag{6.4}$$

for some function f, we say that $(\mathcal{M}, g)$ admits an infinitesimal *dilatation*. Since equations (6.3) and (6.4) are equivalent to equation (6.1) this is the situation we are interested in. It follows that

$$\nabla_\mu \nabla_\nu f = g_{\mu\nu}. \tag{6.5}$$

Moreover, defining

$$V = g^{\mu\nu}\partial_\mu f\, \partial_\nu f = g_{\mu\nu}\Psi^\mu\Psi^\nu ,$$

we have

$$\partial_\mu V = 2\partial_\mu f . \tag{6.6}$$

We choose the arbitrary constant of integration such that

$$V = 2f .$$

Now we pick f as one of the coordinates and find the metric $g_{\mu\nu}$ on $\mathcal{M}$ to be

$$ds^2 = \frac{df^2}{2f} + g_{ij}(x^k, f)\, dx^i\, dx^j . \tag{6.7}$$

There is no cross term $df\, dx^i$ in the metric because Ψ^μ is orthogonal to the surface $f = const.$

Finally, we write out the equation

$$\mathcal{L}_\Psi g_{\mu\nu} = g_{\mu\nu,\lambda}\Psi^\lambda + g_{\mu\lambda}\Psi^\lambda{}_{,\nu} + g_{\nu\lambda}\Psi^\lambda{}_{,\mu} = 2g_{\mu\nu} . \tag{6.8}$$

Using the fact that

$$\Psi^\mu = g^{\mu\nu}\partial_\nu f = 2f\delta^\mu_0 ,$$

and substituting (6.7) into (6.8), we obtain for the (i, j) component

$$f\,\frac{\partial g_{ij}}{\partial f} = g_{ij}\,. \tag{6.9}$$

If we define

$$r^2 = 2f\,,$$

the solution of (6.9) may be written as

$$g_{ij} = r^2 h_{ij}(x^k)\,,$$

and the basic result (6.2) follows.

In fact we need not assume that the metric $g_{\mu\nu}$ is Riemannian, but if Ψ^μ were time-like we would need to adjust the signs in (6.2). Note that since $V \neq 0$ we cannot have Ψ^μ light-like. From (6.5) we find that

$$\nabla_\lambda \nabla_\mu \nabla_\nu f = 0\,.$$

But $(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu)\,\Psi^\alpha = R^\alpha{}_{\beta\mu\nu}\,\Psi^\beta$ gives

$$R^\alpha{}_{\beta\mu\nu}\,\Psi^\beta = 0\,,$$

which contracted on α and μ gives

$$\Psi^\beta R_{\beta\nu} = 0\,. \tag{6.10}$$

Obviously (6.10) is incompatible with $g_{\mu\nu}$ being an Einstein metric with non-vanishing scalar curvature. However, it is not incompatible with $g_{\mu\nu}$ being Ricci-flat, and indeed this will be true if the metric h_{ij} on the base is Einstein such that

$$R_{ij} = (n-1)\,h_{ij}\,,$$

where $n = dim\mathcal{M}$. Such a metric $g_{\mu\nu}$ is called a Ricci-flat cone.

Note that the assumption (6.4) that the homothety Ψ^μ is hypersurface orthogonal played an essential role. For example, Chave, Tod and Valent [97] have exhibited a Ricci-flat (hyperkähler) four-metric admitting a homothety which is *not* hypersurface orthogonal.

We conclude this section by remarking that cones have arisen in supergravity theories under the guise of "generalized dimensional reduction". For example, Lavrinenko *et al* [98] have used the scaling invariance of supergravity theories to construct solutions which are eleven-dimensional cones over ten-dimensional base manifolds.

6.3 Kählerian Cones and Sasakian Structures

Now we suppose that $\mathcal{M} = C(B)$ is a Kähler manifold with a (covariantly constant) complex structure I. We have

$$\begin{aligned} \mathcal{L}_\Psi I^\mu{}_\nu &= I^\mu{}_{\nu;\sigma}\Psi^\sigma - I^\sigma{}_\nu \Psi^\nu{}_{;\sigma} + I^\mu{}_\sigma \Psi^\sigma{}_{;\nu} \\ &= -I^\mu{}_\nu + I^\mu{}_\nu = 0\,. \end{aligned} \tag{6.11}$$

Hence necessarily Ψ is a holomorphic vector field. The reader is cautioned however that since

$$\mathcal{L}_\Psi \omega = d(\iota_\Psi \omega) + \iota_\Psi(d\omega) = 2\omega\,,$$

where ω is the Kähler form with components $\omega_{\mu\nu} = g_{\mu\sigma} I^\sigma{}_\nu$, we have

$$d(\iota_\Psi \omega) = 2\omega\,,$$

and hence Ψ is not Hamiltonian and there is no conventional moment map.

The vector field

$$K^\mu = I^\mu{}_\nu \Psi^\nu \tag{6.12}$$

satisfies

$$K_{\mu;\nu} = \omega_{\mu\nu}\,.$$

Thus K is a Killing field and it is easily seen to be holomorphic and to commute with Ψ. In addition, K is a Hamiltonian vector field whose moment map is f, and hence the level sets of the moment map coincide with the base manifold B. K^μ is tangent to the base manifold B and is therefore a Killing field of the metric h_{ij}. From (6.12) we have

$$K_\mu K^\mu = \Psi_\nu \Psi^\nu = V\,.$$

Thus the length of the vector K^μ is constant along the base manifold B. Choosing $V = 1$ as our base manifold we have the following structure on B:

- a one-form $\eta_i = K_i$
- a vector field $\xi^i = K^i$
- an endomorphism $\Phi^i{}_j \equiv I^i_j$ of the tangent bundle TB of B

- a metric h_{ij} .

It is straightforward to check that $(B,\ h_{ij},\ \xi^i,\ \eta_i,\ \Phi^i{}_j)$ satisfies the conditions to be a Sasakian manifold defined in Section 6.1.

Note that we have not assumed that M is Ricci-flat. If we did so, then (B, h_{ij}) would necessarily be a Sasakian–Einstein manifold.

6.4 Hyperkählerian Cones and tri–Sasakian Structures

Now we suppose $\mathcal{M} = C(B)$ is hyperkähler and hence necessarily Ricci-flat. The base metric must therefore be Einstein. The vector field Ψ is tri-holomorphic, i.e. it preserves the three complex structures I_a and their algebra. There are three Killing vector fields K_a tangent to B and commuting with Ψ:

$$K_a^\mu = I_a{}^\mu{}_\nu \Psi^\nu \,.$$

However now

$$\mathcal{L}_{K_a} I_b = -\,2\,\epsilon_{abc}\, I_c$$

and

$$[K_a, K_b] = -\,2\,\epsilon_{abc}\, K_c \,.$$

Thus we have a non-triholomorphic $SU(2)$ action on $\mathcal{M}$ which descends to the base manifold B. We now have the following structure on B:

- three one-forms $\eta_i^a = (K_a)_i$
- three vector fields $(\xi^a)^i = (K_a)^i$
- three endomorphisms $(\Phi^a)^i_{\ j} \equiv (I_a)^i_{\ j}$ of the tangent bundle TB of B
- a metric h_{ij} .

It can be easily verified that the tensor fields satisfy all conditions in the definition of a tri–Sasakian structure given in Section 6.1, and hence the base manifold B is tri–Sasakian. Each generator K_a of the $SU(2)$ action is holomorphic with respect to its own complex structure I_a, and f is the associated moment map. The emergence of an extra $SU(2)$ isometry group was noticed in [92]. For more information and references to the mathematical literature on tri–Sasakian structures the reader is directed to [33, 95, 96].

6.5 Hypersurface Non-orthogonality

As we have emphasized above the assumption (6.4) that the homothetic Killing field Ψ is hypersurface orthogonal is essential for our result. Suppose that Ψ is a homothety which is not hypersurface orthogonal. Defining

$$F_{\mu\nu} = \partial_\mu \Psi_\nu - \partial_\nu \Psi_\nu \,,$$

one finds that

$$\mathcal{L}_\Psi\, I^\mu_{\ \nu} = I^\mu_{\ \sigma} F^\sigma_{\ \nu} - F^\mu_{\ \sigma} I^\sigma_{\ \nu} \,.$$

Thus, in general, Ψ need not be holomorphic with respect to any complex structure. Moreover, defining K as in (6.12) we have

$$\mathcal{L}_K g_{\mu\nu} = \omega_{\mu\sigma} F^\sigma_{\ \nu} + \omega_{\nu\sigma} F^\sigma_{\ \mu} \,.$$

Therefore we do not necessarily have an extra isometry. One might wonder whether, assuming $\mathcal{M}$ is hyperkähler, any non-trivial homothety could exist. In their paper Chave *et al* [97] gave a family of four-dimensional hyperkähler metrics with a tri-holomorphic homothety which is not hypersurface orthogonal. The metric is of the form

$$e^{2t}\left\{\frac{1}{W}(dt + A)^2 + W\tilde{h}_{ij} dx^i dx^j\right\} ,$$

where the metric on the base gives a $(4, 0)$ sigma model and W and A satisfy monopole-like equations.

6.6 Symmetry Enhancement and Examples

There is no shortage of examples of tri–Sasakian manifolds (see [33] and references therein). However, unless we take B to be a sphere S^{4k-1} with its standard tri–Sasakian structure, the manifold $\mathcal{M}$ will be singular at the vertex $r = 0$. In some cases the singularity may be removed to give a non-singular hyperkähler manifold which no longer admits an exact dilatation symmetry but continues to do so approximately at infinity.

The obvious examples are the ALE cones for which the base B is S^3/Γ, where Γ is a finite subgroup $\Gamma \subset SU(2) \subset SO(4)$. They may be thought of as the quotient of $\mathbb{R}^4$ by Γ with an orbifold fixed point at the origin. As is well known [53] this may be blown up to give a non-singular manifold. It is instructive to consider the multi-centre case (see e.g. [55]). This may be constructed as the hyperkähler quotient [65]

$$\mathbb{H}^{n+1} //\ (U(1))^n \,.$$

The level sets of the moment maps are

$$\mu_\alpha = q_\alpha \, i \, \bar{q}_\alpha + q \, i \, \bar{q} = \zeta_\alpha \,, \tag{6.13}$$

where $\alpha = 1, \ldots, m$, and the quaternions (q_α, q) parametrize $\mathbb{H}^{m+1}$. The quantities $(\zeta_\alpha - \zeta_\beta)$ correspond to the relative separation of the centres. Now let $\zeta_\alpha \to 0$ for all α. We get the orbifold limit in which the sizes of all two-cycles shrink to zero. In the same limit the level sets (6.13) become invariant under the dilatation of $\mathbb{H}^{m+1}$ given by

$$(q_\alpha, q) \longrightarrow (\lambda q_\alpha, \lambda q) \,,$$

which descends to the quotient orbifold. Thus the appearance of the dilatation symmetry is associated with the shrinking of two-cycles. Note that a general ALE metric has no $SU(2)$ isometry, tri-holomorphic or not. As we approach the orbifold limit the isometry group is enhanced to include $\mathbb{R}_+ \times SU(2)$ where $\mathbb{R}_+$ corresponds to dilatations. Note also that although there are many hyperkähler manifolds with non-triholomorphic $SU(2)$ actions they are not all cones. Neither are they necessarily asymptotically conical. For example, all BPS monopole moduli spaces presented in Chapter 3 admit such an $SU(2)$ or $SO(3)$ action, which arises from rotations in physical space, but they do not admit dilatations because of the scale set by the monopole mass.

An interesting question for further study is whether one can construct non-locally flat dilatation invariant hyperkähler manifolds using the hyperkähler quotient construction on a flat space.

6.7 Discussion

Cones over Sasakian and tri–Sasakian manifolds have made an appearance in M-theory [32, 33]. One considers p-brane solutions of the form

$$H^{-\alpha}(-dt^2 + d\mathbf{x}_p^2) + H^{\frac{2}{\beta}} g_C \,,$$

with $H = 1 + (\alpha/r)^\beta$ and g_C the metric on a Ricci-flat cone with base B. These interpolate between $\mathbb{E}^{p,1} \times C(B)$ at infinity and $AdS_{p+2} \times B$ near the throat. This supergravity solution corresponds to a large number, k, of Dirichlet p-branes.

The general belief is that the $U(1)$ factor of the world-volume $U(k)$ gauge theory is associated with the centre of mass motion. The $(9-p)$ scalars give the transverse coordinates of the branes. The amount of supersymmetry of the world-volume theory is expected to agree with the amount of supersymmetry of the supergravity background.

If $p = 3$ it is tempting to make a connection with the four-dimensional rigid $\mathcal{N} = 2$ conformally invariant theories considered in [92]. However, although cones appear both in the construction of the bulk space-time and as the target space of the world-volume theory the cones are, in general, not the same. The base B of the cone used to construct the bulk space-time is five-dimensional and Einstein–Sasakian. The base of the cone of the target space of a putative $\mathcal{N} = 2$ world-volume theory must be $(4n-1)$-dimensional and tri–Sasakian. Moreover, the amounts of supersymmetry of the supergravity solution and the world-volume theory do not agree.

We have a better bet with $\mathcal{N} = 1$ superconformal theories based on six-dimensional Calabi–Yau cones. The idea would be that the six centre of mass coordinates of the D3-branes in the ten-dimensional type IIB string theory should assemble into three complex Higgs fields of the world-volume theory. This appears to coincide with the example considered in [32]: one takes $B = (SU(2) \times SU(2))/U(1)$ with its Sasakian–Einstein structure.

For the M2-brane the cone of the supergravity solution is seven-dimensional and this could be taken to be tri–Sasakian. One might then contemplate identifying a hypermultiplet of the $(2+1)$-dimensional world-volume theory with the coordinates transverse to the M2-brane. However, this looks rather artificial and suggests that one should look elsewhere for the geometrical origin of the hypermultiplets. By analogy with our discussion for the D3-brane it would seem to be more fruitful to follow [32] and consider three-dimensional $\mathcal{N} = 2$ world-volume theories[1] associated to an eight-dimensional Calabi–Yau cone. The case analyzed in [32] is $B = SO(5)/SO(3)$ with its standard Sasakian–Einstein structure.

[1] The counting $\mathcal{N} = 2$ is from the three-dimensional point of view.

CHAPTER 7

Single-sided Domain Walls

It is now widely recognized that topological defects with p spatial dimensions (p-branes) invariant under half the maximum number of supersymmetries — BPS configurations — play a central role in non-perturbative string theory and M-theory. If p is less than $(n-3)$, where n is the space-time dimension, then such objects can be studied in at least two limits. One is the *light* approximation in which the gravitational field that the objects generate is ignored. Treated classically, the world-volume theory of such objects is described by a Dirac–Born–Infeld type action. The other is the *heavy* approximation, in which the gravitational field generated by the objects is taken into account and one looks for solutions of the supergravity equations of motion.

If $p < n-3$ heavy branes give rise to asymptotically flat metrics in directions transverse to the brane, and from a distance they behave more or less like light branes moving in a flat background. However, if $p = n-3$ (vortices) or $p = n-2$ (domain walls), the metrics they generate are not asymptotically flat. For vortices — for example, the 7-brane of the ten-dimensional type IIB theory [99] — the metric has an angular deficit. In the case of domain walls, their effect on space-time can be even more drastic. For example, conventional domain walls, even in the thin-wall approximation, bring about the compactification of space [100]. This happens as follows. Either side of the domain wall is isometric to the interior of a time-like hyperboloid in Minkowski spacetime $\mathbb{E}^{n-1,1}$. To get the entire spacetime one glues two such domain walls back to back. The induced metric is continuous across the domain wall but the second fundamental form has a discontinuity which gives the distributional stress tensor. Another feature of conventional domain walls, which is more or less obvious from the description just given, is that one does not expect to have more than one in a static configuration.

Domain walls of an unconventional (orbifold) type play an important role in Hořava and Witten's approach to the $E_8 \times E_8$ heterotic string theory in M-theory [101, 102]. They also have a drastic global effect on the structure of space-time. Of course, in addition to their gravitational fields one must take into account the effects of anomalies and the four-form field strength.

In this chapter, which is based on [103], we study the global structure of some other space-times containing BPS domain walls that have arisen in M-theory. A striking feature of M-theory is the extent to which configurations in eleven dimensions are non-singular even though they may appear to be singular in lower dimensions. We shall therefore be particularly interested in everywhere non-singular configurations.

The organization of the chapter is as follows. Section 7.1 contains the background material. It describes briefly the Bianchi classification scheme, since the domain wall solutions of M-theory that we intend to study are invariant under group actions of various Bianchi types. Since most geometries that arise are Kähler we introduce the Monge–Ampère equation, which allows us to give a more uniform description of the metrics. We then study various solutions of M-theory which have symmetries of a domain wall. In Section 7.2 vacuum Bianchi type I and II solutions are presented. The most important example is BPS and is based on the Bianchi type II group, otherwise known as the Heisenberg group. Usually this is regarded as an orbifold solution with a singularity. In Section 7.3 we argue that it is desirable to resolve the singularity and show how this may be achieved to give a complete non-singular solution representing a single-sided domain wall. Bianchi type VI_0 and VII_0 domain walls are presented in Section 7.5, and the singularity of the latter solution is argued to be resolvable. A description of the four-metrics in terms of the Kähler potential which solves the appropriate Monge–Ampère equation is given.

These four-dimensional Ricci-flat examples are generalized to higher dimensional Calabi–Yau metrics in Section 7.6 (except in the Bianchi type I case for which the metric exists in odd as well as in even dimensions).

Related Bianchi type solutions, some of which are BPS, with negative cosmological constant are presented in Sections 7.4 and 7.5.3, and their higher-dimensional generalizations are considered in Section 7.6. We believe that these are relevant for the AdS/CFT correspondence and other future applications of eleven-dimensional supergravity.

Before closing the discussion of single-sided domain walls, in Section 7.7 we comment on solutions invariant under other Bianchi type groups, namely the Bianchi type III.

7.1 Bianchi Domain Walls

We shall consider p-brane solutions of the form:

$$M_4 \times \mathbb{E}^{p-3,1},$$

where M_4 is a non-compact Riemannian four-manifold which is either Ricci-flat or has negative cosmological constant. If $p = 3$ we would be considering domain walls in five space-time dimensions.

We are looking for metrics on M_4 which depend only on one coordinate t, transverse to the domain wall. The metric should be homogeneous in the directions parallel to the wall. Mathematically, this means that we are looking for cohomogeneity one, or hypersurface homogeneous, metrics invariant under the action of Lie group G which acts transitively on three-dimensional orbits. In the cases we are interested in G may be taken to be three-dimensional and the possible groups have been classified by Bianchi (see e.g. [35]). The problem is very similar to that encountered in studying homogeneous Lorentzian cosmologies and we shall freely use standard results from that subject [104]. The Bianchi types relevant for this chapter are type I, II, VI_0 and VII_0 . Domain walls of types I and II are discussed in Section 7.2 while the treatment of the more "exotic" solutions is relegated to Section 7.5.

In the following we shall find that all Ricci-flat solutions are singular and describe how the singularity of the type II solution may be resolved. The resolution of this singularity gives a complete Ricci-flat Kähler manifold that we shall call the BKTY metric.[1]

In the light of the AdS/CFT correspondence [27, 28, 29] and its generalization [34] it is instructive to investigate domain walls in the Anti–de Sitter background. Sections 7.4 and 7.5.3 are therefore devoted to the study of four-manifolds of the abovementioned Bianchi types with negative cosmological constant.

7.1.1 Bianchi Models

Let us now be more specific about the four-manifolds M_4 in question. Spaces of interest are homogeneous manifolds with the following ansatz for the metric:

$$ds^2 = dt^2 + a^2(t)\ (\sigma^1)^2 + b^2(t)\ (\sigma^2)^2 + c^2(t)\ (\sigma^3)^2\ . \tag{7.1}$$

[1]The name BKTY is derived from the initials of the authors of [105, 106, 107] who constructed this space as a certain degeneration of the $K3$ surface.

Bianchi type	n_1	n_2	n_3	Group of motions
I	0	0	0	$\mathbb{R}^3$
II	0	0	1	Nil
VI_0	1	-1	0	$E(1,1)$
VII_0	1	1	0	$E(2)$

Table 7.1: Several Bianchi type groups

Here t is the imaginary time and the metric coefficients are functions of t only. The one-forms $\{\sigma^k\}$, $k = 1, 2, 3$, are left-invariant one-forms of the three-dimensional group G of isometric motions and as such satisfy:

$$d\sigma^k = -\frac{1}{2} n_k \, \epsilon_{ijk} \, \sigma^i \wedge \sigma^j \,, \quad \text{no sum over } k \,,$$

where constants $\{n_k\}$ are the structure constants of G. The four-manifolds may be classified according to their group of isometric motions. This is the Bianchi classification in which each type corresponds to a particular set of values of the structure constants $\{n_k\}$. In the following manifolds of four Bianchi types will arise, whose properties are summarized in Table 7.1. Note that all four groups of isometric motions are solvable, in fact they all have one non-trivial commutator. The Einstein equations for metric (7.1) reduce to the following set of second-order ODEs:

$$-R^0_{\ 0} = \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} \,, \tag{7.2}$$

$$R^1_{\ 1} = \frac{(\dot{a}bc)^{\cdot}}{abc} + \frac{1}{2}\frac{1}{a^2b^2c^2}\left[n_1^2a^4 - (n_2b^2 - n_3c^2)^2\right] \,, \tag{7.3}$$

$$R^2_{\ 2} = \frac{(a\dot{b}c)^{\cdot}}{abc} + \frac{1}{2}\frac{1}{a^2b^2c^2}\left[n_2^2b^4 - (n_1a^2 - n_3c^2)^2\right] \,, \tag{7.4}$$

$$R^3_{\ 3} = \frac{(ab\dot{c})^{\cdot}}{abc} + \frac{1}{2}\frac{1}{a^2b^2c^2}\left[n_3^2c^4 - (n_1a^2 - n_2b^2)^2\right] \,, \tag{7.5}$$

where $\dot{a} = da/dt$, *etc.* If the metric on M_4 is Ricci-flat, i.e. $R^a_{\ b} = 0$, equations (7.2)-(7.5) are integrable in most cases. The resulting manifolds are singular.[2]

For non-Ricci-flat manifolds, in particular for manifolds with $R^a_{\ b} = \Lambda\delta^a_{\ b}$, $\Lambda < 0$, the Einstein equations are not in general integrable. However, a number of solutions with extra symmetries exist. For example, in Section 7.4 we discuss the Bergmann metric — a Bianchi type II solution with negative cosmological constant. Unlike the Ricci-flat Bianchi type II solution, the Bergmann metric is complete.

[2] Many self-dual four-dimensional vacuum solutions of various Bianchi types have been found in [108].

7.1.2 Monge–Ampère Equation

From the theory of Kähler manifolds it is known that the Kähler metric may be obtained from a real-valued function of complex holomorphic coordinates $z, \bar{z} = \{z^a, \bar{z}^a\}$ called the Kähler potential:

$$g_{a\bar{b}} = \partial_a \, \partial_{\bar{b}} \, K(z, \bar{z}) \,.$$

Here $\partial_a = \partial/\partial z^a$ and $\partial_{\bar{b}} = \partial/\partial \bar{z}^b$. A Kähler manifold is Kähler–Einstein if the Kähler metric $g_{a\bar{b}}$ satisfies the Einstein equations:

$$\mathcal{R}_{a\bar{b}} = \Lambda \, g_{a\bar{b}} \,.$$

These are equivalent to the requirement that the Kähler potential satisfy the so-called Monge–Ampère equation obtained as follows. The Ricci tensor is given by:

$$\mathcal{R}_{a\bar{b}} = -\,\partial_a \, \partial_{\bar{b}} \log \det g(z, \bar{z})) \,,$$

and hence the Kähler–Einstein condition reduces to

$$\det(\partial_a \, \partial_{\bar{b}} \, K) = e^{-\Lambda K} \,. \tag{7.6}$$

In general this is a complex partial differential equation solving which is not straightforward. If, however, the manifold possesses certain amount of symmetry, the Monge–Ampère equation may reduce to an ODE. In the following sections we shall deduce Monge–Ampère equations and their solutions for most of the Kähler manifolds that we study.

7.2 Vacuum Solutions of Bianchi type I and II

We shall begin by assuming that the domain walls are invariant under three translations, i.e. that they are Bianchi type I, or Kasner, but we shall find that to be supersymmetric objects they should instead be invariant under the nilpotent Bianchi type II group Nil often called the Heisenberg group.

7.2.1 Kasner walls

One's first idea might be to choose the metric on M_4 to depend only on one "transverse" coordinate, call it t, and to be independent of the other three coordinates (x^1, x^2, x^3) say. Thus the metric admits an isometric action of $\mathbb{R}^3$ or, if we identify, T^3 and falls into the vacuum Bianchi I, or Kasner class of solutions [109]:

$$ds^2 = dt^2 + t^{2\alpha_1}(dx^1)^2 + t^{2\alpha_2}(dx^2)^2 + t^{2\alpha_3}(dx^3)^2, \tag{7.7}$$

where constant $\alpha_1, \alpha_2, \alpha_3$ satisfy:

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 = \alpha_1^2 + \alpha_2^2 + \alpha_3^2\,. \tag{7.8}$$

There are two problems with this metric. The first problem is that if metric (7.7) is not flat, it is singular at the domain wall $t = 0$; and the second problem is that it is not BPS.

Consider, for example the rotationally symmetric case

$$(\alpha_1, \alpha_2, \alpha_3) = (\frac{4}{3}, \frac{4}{3}, \frac{2}{3})\,.$$

Metric (7.7) is then singular at $t = 0$, but complete as $t \to \infty$. Thus, in accordance with our general remarks made in the beginning of the chapter, it is not asymptotically flat in the usual sense although the curvature falls off as t^{-2}.

7.2.2 BPS Walls: Bianchi type II

To be BPS the manifold M_4 must admit at least one, and hence at least two, covariantly constant spinors. If the solution admits at least one tri-holomorphic Killing vector it may be cast in the form:

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V d\mathbf{x}^2, \tag{7.9}$$

where $\mathbf{x} = (x^1, x^2, x^3)$ with

$$\text{curl}\,\omega = \text{grad}\,V.$$

One may either regard the ignorable coordinate τ as lying in the world-volume of the p-brane or as a Kaluza–Klein coordinate. Obviously, one may entertain both interpretations simultaneously, in which case one is considering the double-dimensional reduction of a brane in a lower dimensional space [110]. For the time being we will not tie ourselves down on this point. In order to get a domain wall solution we want some sort of invariance under two further translations and we are naturally led to choose for the harmonic function V

$$V = z,$$

where we now interpret the coordinate z as a transverse coordinate. With $x^1 \equiv x$, $x^2 \equiv y$, $x^3 \equiv z$ metric (7.9) becomes:

$$ds^2 = z dz^2 + z(dx^2 + dy^2) + z^{-1}(d\tau - x dy)^2. \tag{7.10}$$

The transverse proper distance is given by

$$t = \frac{2}{3} z^{\frac{3}{2}}.$$

The metric is complete as $z \to +\infty$, the curvature again falls off as t^{-2} but it clearly has a singularity at $z = 0$, at which the signature changes from $(++++)$ to $(----)$. We shall return to this point shortly.

The Monge–Ampère equation and the Kähler potential for this metric will be given in Section 7.4.3, where suitable complex coordinates are introduced.

7.2.3 Geometrical Considerations and the Heisenberg Group

Evidently metric (7.10) is not invariant under translations in the y coordinate; nevertheless it admits a three-dimensional group of isometries. The metric may be written in the general form (7.1) so that the group of isometric motions is manifest:

$$ds^2 = dt^2 + \left(\frac{3t}{2}\right)^{-\frac{2}{3}} (\sigma^3)^2 + \left(\frac{3t}{2}\right)^{\frac{2}{3}} ((\sigma^1)^2 + (\sigma^2)^2)\,, \tag{7.11}$$

where $\{\sigma^k\}$ are left-invariant one forms on the Nil, or the Heisenberg group. From this point on we shall refer to the metric (7.10) (or (7.11)) as the *Heisenberg* metric.

The Heisenberg group may be defined as the nilpotent group $Nil = \{g\}$ of 3×3 real-valued upper triangular matrices:

$$g = \begin{pmatrix} 1 & x & \tau \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The Lie algebra of Nil has as a basis[3]:

$$\mathbf{e}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{e}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\mathbf{e}_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

[3] We adhere to the conventions that if a group G with Lie algebra $[\mathbf{e}_a, \mathbf{e}_b] = C_a{}^c{}_b \mathbf{e}_c$ acts on the left on a manifold M then the Killing vector fields $\mathbf{R}_a$ have Lie brackets $[\mathbf{R}_a, \mathbf{R}_b] = -C_a{}^c{}_b \mathbf{R}_c$, while the left-invariant one-forms $g^{-1}dg = \mathbf{e}_a\,\sigma^a$ satisfy $d\sigma^c = -\frac{1}{2} C_a{}^c{}_b \sigma^a \wedge \sigma^b$.

and the only non-vanishing commutator is

$$\left[\mathbf{e}_1, \mathbf{e}_2\right] = \mathbf{e}_3 \, .$$

The basis elements $\{\mathbf{e}_k\}$ correspond to three right-invariant Killing vector fields

$$\begin{aligned} \mathbf{R}_1 &= \frac{\partial}{\partial x} + y\frac{\partial}{\partial \tau}, \\ \mathbf{R}_2 &= \frac{\partial}{\partial y}, \\ \mathbf{R}_3 &= \frac{\partial}{\partial \tau}, \end{aligned}$$

for which the only non-vanishing commutator is

$$\left[\mathbf{R}_1, \mathbf{R}_2\right] = -\mathbf{R}_3 \, ; \tag{7.12}$$

and three left-invariant one-forms

$$\begin{aligned} \sigma^1 &= dx \, , \\ \sigma^2 &= dy \, , \\ \sigma^3 &= d\tau - x dy \, , \end{aligned} \tag{7.13}$$

whence

$$d\sigma^3 = -\sigma^1 \wedge \sigma^2 \, .$$

In addition, this metric admits a rotational Killing vector of the form

$$\mathbf{m} = -x\frac{\partial}{\partial y} + y\frac{\partial}{\partial x} - \frac{x^2 - y^2}{2}\frac{\partial}{\partial \tau}. \tag{7.14}$$

This Killing field $\mathbf{m}$ induces a rotation of σ^1 into σ^2 but leaves σ^3 invariant.

The four-dimensional Heisenberg manifold (7.10) is Ricci-flat Kähler and hence carries a hyper-kähler structure. To exhibit the three Kähler forms let us introduce the following orthonormal basis of one-forms $\{e^0, e^k\}$:

$$\begin{aligned} e^0 &= z^{-\frac{1}{2}} (d\tau - x dy) \, , \\ e^1 &= z^{\frac{1}{2}} \, dx \, , \\ e^2 &= z^{\frac{1}{2}} \, dy \, , \\ e^3 &= z^{\frac{1}{2}} \, dz \, . \end{aligned} \tag{7.15}$$

In terms of these frames the three self-dual two-forms which are the Kähler forms are:

$$\Omega_x = e^0 \wedge e^1 + e^2 \wedge e^3 \, , \quad \text{and cyclic permutations} \, , \tag{7.16}$$

and for the Heisenberg metric (7.10) these become

$$\begin{aligned} \Omega_x &= (d\tau - x dy) \wedge dx + z \, dy \wedge dz \, , \\ \Omega_y &= (d\tau - x dy) \wedge dy + z \, dz \wedge dx \, , \\ \Omega_z &= (d\tau - x dy) \wedge dz + z \, dx \wedge dy \, . \end{aligned} \tag{7.17}$$

It is easily seen that the self-dual two-forms Ω_x, Ω_y and Ω_z — the Kähler forms — are closed and hence harmonic. They are clearly invariant under the action of the Heisenberg group. However only Ω_z is invariant under the circle action generated by the rotational Killing field $\mathbf{m}$ (7.14).

7.2.4 Circle Bundles and Volume Growth

If one wishes to identify the coordinates x and y to obtain a two-dimensional torus one is forced to make appropriate identifications of the coordinate τ. The result is a circle bundle over a two-torus. Such bundles M_k are indexed by an integer k which is essentially the Chern class. They are often referred to as *Nilmanifolds*.

If the periods of (x, y, τ) are (L_x, L_y, L_τ) then one must have:

$$k = \frac{L_x L_y}{L_\tau} \in \mathbb{Z}. \tag{7.18}$$

If that is true then

$$\exp(L_x \mathbf{e}_1) \exp(L_y \mathbf{e}_2) \exp(-L_x \mathbf{e}_1) \exp(-L_y \mathbf{e}_2) = \exp(L_x L_y \mathbf{e}_3) \, ,$$

and hence $\exp(L_x \mathbf{e}_1)$, $\exp(L_y \mathbf{e}_2)$ and $\exp(L_\tau \mathbf{e}_3)$ will close on a discrete group $\mathcal{N}_k$. One then has:

$$M_k = Nil/\mathcal{N}_k \, ,$$

which clearly admits a global right action of $U(1) = \exp(z\mathbf{e}_3)$.

The curvature of the connection pulled back to the base T^2 is

$$F = d\,\sigma^3 = dy \wedge dx \, ,$$

and the Dirac quantization condition is

$$\frac{1}{L_\tau} \int_{T^2} F = k \in \mathbb{Z} \, . \tag{7.19}$$

We shall see in Section 7.3.6 that the relevant value of the integer k in our case is $k = 3$. Formula (7.19) takes on a more conventional appearance if one chooses $L_\tau = 2\pi$. Alternatively, one could think of $e = 2\pi/L_\tau$ as an electric charge. It has been known for some time that a Kaluza–Klein reduction on the Heisenberg group gives rise to a uniform magnetic field [111]. Interestingly, since the present solution is BPS, it should be stable against production of monopole–anti-monopole pairs. This is in contrast to other examples of magnetic fields in Kaluza–Klein theories, for example in vacua studied in [112, 113] such monopole–anti-monopole pairs are produced.

The curves of constant (x, y, τ) are geodesics orthogonal to the group orbits and the coordinate t is the radial distance. If the orbits are compact we may estimate how the four-volume of a geodesic ball increases with t by calculating the four-volume of the metric (7.10) between $t = t_1$ say and t. This is easily seen to grow with t as $t^{\frac{4}{3}}$. We shall use this fact in Section 7.3.5 to compare with the work of Bando, Kobayashi, Tian and Yau [105, 106, 107] on an exact metric on the complement of a smooth cubic curve in $\mathbb{CP}^2$.

7.3 Resolution of the Singularity

In this section we describe the physical motivation for resolving the singularity of the Heisenberg manifold (7.10) and analyze the underlying mathematical structure of the proposed resolution.

7.3.1 8-branes, 6-branes and T-duality

The metric (7.10) has been reached previously by a different route. The massive type IIA ten-dimensional theory of Romans [114] admits BPS solutions corresponding to Dirichlet 8-branes whose properties have been discussed by Polchinski and Witten [115] and Bergshoeff *et al* [99]. The solutions are based on a harmonic function of the coordinate transverse to the 8-branes, which has discontinuities at the location of the 8-branes. The relation of Romans' theory to eleven-dimensional supergravity theory is unclear.[4] However, under a double T-duality with respect to two coordinates lying in the 8-brane, x and y say, it may be reduced to a 6-brane solution of the IIA theory compactified to eight dimensions. Under T-duality, the coordinates x and y become transverse coordinates and, strictly speaking, because the solution is independent of the coordinates x and y, one has a superposition of 6-branes. A 6-brane solution of the ten-dimensional type IIA theory may be lifted to eleven dimensions to give a BPS 7-brane wrapped around the eleventh dimension. In other words the

[4] See however [116].

eleven-dimensional 7-brane is a Ricci-flat metric of the form:

$$\mathbb{E}^{6,1} \times M_4 \,, \qquad (7.20)$$

where M_4 is a multi-Taub–NUT metric of the form (7.9):

$$ds^2 = V^{-1}(d\tau + \omega_i dx^i)^2 + V d\mathbf{x}^2,$$

with

$$\text{curl}\,\omega = \text{grad}\,V \,.$$

The coordinate τ is the eleventh direction. Coordinates $\mathbf{x}$ are transverse to the 6-brane. A single 6-brane corresponds to the Taub–NUT metric (see Section 3.2.1) with positive mass which has

$$V = 1 + \frac{1}{r} \,.$$

In order to get a superposition of 6-branes which is independent of x and y (at least up to gauge transformations) one should choose

$$V = z \,,$$

and this is indeed what Bergshoeff *et al* [99] find.

7.3.2 Sources

As it stands, metric (7.10) is singular at $z = 0$. In fact this singularity resembles the singularity in the self-dual Taub–NUT metric with negative mass parameter for which $V = 1 - 1/r$ in (7.9) (see Section 3.2.1). On the three-surface $r = 1$ the metric changes signature from $(++++)$ to $(----)$. The Taub–NUT metric with negative mass parameter is asymptotic to a complete topologically non-singular self-dual Riemannian manifold — the Atiyah–Hitchin manifold [17]. The presence of the singularity at $r = 1$ is a clear indication of the fact that the Taub–NUT approximation is broken already at values of r greater than one. It is natural to suppose that something similar may be happening in the case of the Heisenberg metric (7.10). Indeed, in the next section we shall make a concrete proposal for the exact metric.

However, Bergshoeff *et al* [99] and others writing on supergravity domain walls [117] do something else. They replace z by $|z|$ which results in a configuration symmetric under the reflection $z \to -z$. The justification for this procedure is that one has inserted a distributional source at $z = 0$ representing the domain wall, and the regions $z > 0$ and $z < 0$ correspond to the two sides of the domain wall. Geometrically this resembles but is not equivalent to the procedure of Israel [118] used in classical general relativity, who describes a shell of matter by gluing together two smooth space-times $M^{\pm}$ across a hypersurface Σ. The Israel matching conditions are that the two metrics $g^{\pm}_{ij}$ induced on Σ from $M^{\pm}$ agree. One then evaluates the distributional stress tensor from the discontinuity in the second fundamental forms $(K^{+}_{ij} - K^{-}_{ij})$ across Σ.

From the point of view of M-theory there are two objections to doing this in the present case:

- There are no obvious sources in M-theory,
- The induced metric on the hypersurface Σ given by $z = 0$ is singular.

7.3.3 Orbifold Walls

An alternative attitude to the singularity of (7.10) at $z = 0$ would be to identify the region $z > 0$ with the region $z < 0$. The singularity would then be viewed as a consequence of the fact that the reflection $z \to -z$ has a fixed point set. Thus one has something analogous to the two orbifold domain walls at the ends of an interval in Hořava and Witten's compactification of the eleven-dimensional M-theory on $S^1 \times \mathbb{Z}_2$ to give the $E_8 \times E_8$ heterotic theory in ten dimensions [101, 102]. In the formulation of [119] one considers the following eleven-dimensional metric on $\mathbb{E}^{3,1} \times S^1/\mathbb{Z}^2 \times X^6$:

$$ds^2 = \frac{1}{H}\, g^4_{\mu\nu} + H^2\, dy^2 + H\, g^6_{AB}\,,$$

where $g^4_{\mu\nu}$ is the four-metric on the flat Minkowski space-time $\mathbb{E}^{3,1}$, y is the coordinate on the interval $S^1/\mathbb{Z}^2$ ranging from $-\pi\rho$ to $\pi\rho$, and g^6_{AB} is the metric on the compact Calabi–Yau space X^6. Function H is a harmonic function linear in y and invariant under the reflection $y \to -y$. In addition, there is a non-vanishing four-form field strength in the eleven-dimensional theory. In the effective five-dimensional theory obtained by generalized Kaluza–Klein dimensional reduction on the Calabi–Yau space X^6, this solution can be viewed as a pair of 3-brane domain walls on the orbifold fixed planes $y = 0$ and $y = \pi\rho$. The 3-branes are in fact the M-theory 5-branes with two world-volume dimensions "wrapped" on a two-cycle in X^6.

In our case the solution is defined on $\mathbb{E}^{6,1} \times \mathbb{R}_+ \times Nil$, where $\mathbb{R}_+$ is parametrized by $z > 0$, and the three-manifold Nil parametrized by (x, y, τ) is the group manifold of the Heisenberg group. We may think of this as a 9-brane solution of eleven-dimensional supergravity, where the world-volume of the 9-brane is taken to be $Nil \times \mathbb{E}^{6,1}$. Replacing Nil by $M_k = Nil/\mathcal{N}_k$ defined in Section 7.2.4 amounts to "wrapping" the 9-brane on the S^1 bundle over T^2. Just as in the Hořava–Witten case we do not have the full $SO(9,1)$ Lorentz invariance, rather it is broken to $SO(6,1) \times Nil$.

7.3.4 Scherk–Schwarz Reduction to Seven Dimensions

In the light of the comments above, particularly the absence of the Lorentz invariance, perhaps the most attractive interpretation of the solution (7.10) is that adopted by Lavrinenko *et al* [98]. One regards it as a solution of the so-called "massive" eight-dimensional theory which is obtained by reducing eleven-dimensional supergravity *à la* Scherk and Schwarz [120]. In other words, one restricts the eleven-dimensional theory to solutions invariant under the action of the three-dimensional Heisenberg group. The resulting theory has a potential for the scalar fields arising from the reduction, and, as a consequence, there is no solution with the eight-dimensional Poincaré invariance. Lavrinenko *et al* [98] therefore propose using the BPS solution (7.10). In their interpretation z is, as with us, the transverse coordinate (i.e. the eighth coordinate) and (x, y, τ) are the ninth, tenth and eleventh coordinates in no particular order. Since the size of the x and y directions goes to infinity as $z \to \infty$, and the size of the τ direction goes to zero, the Scherk–Schwarz reduction is not really a compactification even if one identifies the coordinates so as to obtain an S^1 bundle over T^2. It is, however, certainly a consistent truncation of the theory.

7.3.5 Blowing up the Singularity

If the configuration (7.10) really does come from M-theory we still face the problem of the source. There are two possibilities:

- *Either* to follow Bergshoeff *et al* [99] and Lavrinenko *et al* [98] and take the view that the domain wall has two sides,
- *Or* to adopt the orbifold interpretation and identify the regions of positive and negative z.

Both approaches give rise to a singularity. The question arises as to whether one can somehow smooth out the singularity? We are going to argue that the answer is *no* if we adopt the first course and *yes* if we adopt the second. Assuming that only gravity with no extra form-fields is present, we thus seek a non-singular Ricci-flat BPS metric which is asymptotic to the Heisenberg metric (7.10).

To see that the first approach is ruled out we note that if the singularity could be resolved, then keeping coordinates (x, y, τ) would give a complete Ricci-flat metric on $\mathbb{R} \times \Sigma$, where Σ is a closed complete three-manifold. In particular, the manifold would have two "ends", i.e. two infinite regions. However, if this were true we could construct a "line" between the two ends, that is a geodesic which minimizes the length between any two points lying on it. But by the Cheeger–Gromoll theorem this is impossible (see e.g. [50]). Thus we are forced to adopt the second course of action which is investigated in detail in the following section.

Before doing so it is perhaps worthwhile pointing out the analogy of the situation in question with the case of the blow up of $\mathbb{E}^4/\mathbb{Z}_2$. One might have wondered if it is possible to glue together two copies of $\mathbb{E}^4$ to get a Ricci-flat wormhole-like structure with topology $\mathbb{R} \times \mathbb{RP}^3$. Again, by the Cheeger–Gromoll theorem this cannot happen. In fact, we know that the correct blow up of $\mathbb{E}^4/\mathbb{Z}_2$ is the Eguchi–Hanson manifold on the cotangent bundle of $\mathbb{CP}^1$ [51] and that this manifold has only one infinite region.

Another completely analogous situation is the Taub–NUT approximation to an orientifold plane. This is obtained by taking the metric (7.9) with $V = 1 - 1/r$ and making a further identification [121]. The metric is incomplete because of the singularity at $r = 1$. This singularity cannot be resolved by joining together two copies of the Taub–NUT metric across $r = 1$ because this would also produce a manifold with two ends. The correct way to blow up the singularity of the Taub–NUT metric is to pass to the Atiyah–Hitchin manifold.

7.3.6 Complement of a Cubic in $\mathbb{CP}^2$: the BKTY Metric

We now turn to the problem of finding, or more properly speaking identifying, the exact metric of which the Heisenberg metric (7.10) is an asymptotic approximation. This task is greatly facilitated by the extremely helpful review of Kobayashi [59] on degenerations of the metric on $K3$ and in what follows we shall rely heavily on that reference. The general self-dual four-metric on $K3$ has (including an overall scale) a 58 parameter moduli space. As we move to the boundary of the moduli space in certain directions the four-metric may decompactify, while remaining complete and non-singular. Among the degenerations discussed by Kobayashi there is one he refers to as type II (not to be confused with Bianchi type II). It may be constructed by considering the complement $M_4 = \mathbb{CP}^2 \setminus C$ of a smooth cubic curve C in the complex plane $\mathbb{CP}^2$. This has a Kähler metric: the Fubini–Study metric which is incomplete because the cubic has been removed. However, using general existence theorems for solutions of the Monge–Ampère equation Yau, Tian, Bando and Kobayashi [105, 106, 107] have shown that there exists a complete non-singular Ricci-flat Kähler (and hence self-dual) metric on M_4. Clearly the metric must blow up on the cubic C which corresponds to infinity.

Consider now the neighbourhood of the cubic C. The curve itself is topologically a two-torus T^2.

A normal neighbourhood consists of a disc bundle over T^2. The centre of the disc corresponds to infinity in M_4. The radial direction corresponds to a geodesic in the self-dual metric. A surface of constant radius is a circle bundle over the torus. This is the three-dimensional Nilmanifold.

Kobayashi tells us that, as we approach infinity, the Nilmanifold collapses in such a way that the metric spheres are an S^1 bundle over T^2, the size of the S^1 falls off as $t^{-1/3}$ (t is the radial distance) and the size of each cycle in T^2 grows as $t^{1/3}$. The volume of a metric ball grows as $t^{4/3}$. This is exactly the behaviour of the Heisenberg metric (7.11) . It is therefore very plausible that the metrics constructed by Yau, Tian, Bando and Kobayashi do indeed asymptote to the metric (7.11). In what follows we shall assume that this is true.

The topology of M_4 is non-trivial[5]: it is not simply connected and has

$$H_1(M_4) = \mathbb{Z}_3\,, \quad H_2(M_4) = \mathbb{Z} \oplus \mathbb{Z}\,.$$

Hence if arguments like those in [122] apply, the manifold should admit at least two normalizable anti-self-dual two-forms. Using the analysis of [122] one deduces that there should be a $2 \times 3 = 6$ dimensional family of transverse traceless zero modes of the Lichnerowicz operator. Adding the trivial overall scaling we expect to find a seven-dimensional family of metrics.

7.3.7 Gravitational Action

Complete Ricci-flat (vacuum) Einstein manifolds are gravitational instantons. Let us evaluate their gravitational action. If $\mathcal{M}$ is a non-compact manifold or a compact manifold with boundary $\partial\mathcal{M}$ the gravitational action is:

$$-\frac{1}{16\pi}\int_{\mathcal{M}} R - \frac{1}{8\pi}\int_{\partial\mathcal{M}} Tr\mathcal{K}\,, \tag{7.21}$$

where R is the Ricci tensor and $\mathcal{K}$ is the second fundamental form on $\mathcal{M}$. The first term is the contribution from the bulk which vanishes for Ricci-flat manifolds; the second term is the contribution from the boundary (possibly boundary at infinity). Traditionally only four-manifolds were regarded as gravitational instantons, however expression (7.21) is valid for complete Ricci-flat Einstein manifolds in any dimension.

Let us estimate the contribution from the boundary. Let $\mathbf{n}$ be a vector normal to the boundary $\partial\mathcal{M}$, then the second term in (7.21) is:

$$\frac{1}{8\pi}\int_{\partial\mathcal{M}} Tr\mathcal{K} = \frac{1}{8\pi}\frac{\partial}{\partial\mathbf{n}}\left(\mathrm{Vol}\,\partial\mathcal{M}\right).$$

[5] We thank Ryushi Goto for this computation.

By Vol $\partial\mathcal{M}$ we mean the unit volume of the boundary. For four-dimensional Ricci-flat manifolds, if t is the radial distance, the boundary term contribution to the action is finite if Vol $\partial\mathcal{M}$ grows no faster than linear in t. This implies that the volume growth of a large metric sphere at infinity should be no faster than t^2. Similarly, for higher-dimensional instantons the "critical" volume growth for which the boundary contribution to the action is finite (but not necessarily vanishing) is t^2.

The BKTY manifold possesses a non-compact complete Ricci-flat Kähler metric with the Heisenberg end and can thus be viewed as a gravitational instanton. Since it is Ricci-flat, its gravitational action receives no contribution from the first term in (7.21). At infinity the BKTY metric looks like the Heisenberg metric (7.11). The boundary of (7.11) at large values of t looks like an S^1 bundle over T^2, where the two-torus is parametrized by (x, y), and τ is the fibre coordinate. Hence the second term in (7.21) is:

$$\frac{1}{8\pi}\int_{\partial\mathcal{M}} Tr\mathcal{K} = \frac{1}{8\pi}\frac{\partial}{\partial\mathbf{n}}(\mathrm{Vol}\,\partial\mathcal{M}) = \frac{1}{8\pi}\frac{\partial}{\partial t}\left[\left(\frac{3}{2}t\right)^{1/3}\mathcal{V}\right].$$

$\mathcal{V} = L_x L_y L_\tau$, where L_x, L_y and L_τ are the periods of x, y and τ as described in Section 7.2.4, and we get

$$\left(\frac{3}{2}t\right)^{-\frac{2}{3}}\frac{L_x L_y L_\tau}{16\pi}.$$

Note that the unit volume of $\partial\mathcal{M}$ grows as $t^{1/3}$, which is slower than the critical estimate t, hence it is not surprising that the gravitational action of the BKTY instanton is finite and tends to zero as $t \to +\infty$.

7.4 Bianchi types I and II with Negative Cosmological Constant

In this section we look for p-brane solutions of the form $M_4 \times \mathbb{E}^{p-3,1}$, where now M_4 is not a Ricci-flat manifold but rather a four-manifold with negative cosmological constant. Such solutions may be interpreted as branes in the Anti–de Sitter background and are likely to be of interest in connection with the AdS/CFT correspondence [27, 28, 29, 34].

In this section we consider four-manifolds of Bianchi types I and II. The relevant metrics must be solutions of the Einstein equations (7.2)-(7.5) with $R^a_{\ b} = \Lambda\delta^a_{\ b}$, $\Lambda < 0$, and appropriate values of the structure constants $\{n_k\}$ given in Table 7.1.

7.4.1 Bianchi type I

In this case $n_k = 0$ and the Einstein equations are integrable. While there is no polynomial solutions like the Kasner metric (7.7), the solution is obtained by replacing t^{α_k} in (7.7) with

$$\left(\frac{\sinh(\sqrt{-3\Lambda}\,t)}{\tanh\left(\frac{\sqrt{-3\Lambda}}{2}\,t\right)}\right)^{\frac{1}{3}}\left(\tanh\left(\frac{\sqrt{-3\Lambda}}{2}\,t\right)\right)^{\alpha_k},$$

where the powers α_k again satisfy relations (7.8). Setting $\alpha_1 = 1$, we get a complete non-singular (in contrast with the singular Kasner metric) instanton if the coordinate x is suitably identified. This Kasner–Anti–de Sitter metric could be used to construct domain walls. Note, however, that like its vacuum counterpart (7.7) this metric is not BPS.

7.4.2 Bianchi type II: the Bergmann Metric

Substituting the relevant structure constants into the Einstein equations (7.2)-(7.5) we find $b(t) = c_0\,a(t) + c_1$. However, we are only interested in self-dual metrics and for these the constants c_0, c_1 are $c_0 = 1, c_1 = 0$. Hence we necessarily have $b(t) = a(t)$, and the Einstein equations reduce to:

$$-\Lambda = 2\frac{\ddot{a}}{a} + \frac{\ddot{c}}{c}, \qquad (7.22)$$

$$-\Lambda = \frac{(\dot{a}ac)^{\cdot}}{a^2c} - \frac{c^2}{2a^4}, \qquad (7.23)$$

$$-\Lambda = \frac{(\dot{c}a^2)^{\cdot}}{a^2c} + \frac{c^2}{2a^4}. \qquad (7.24)$$

It is not straightforward to solve equations (7.22)-(7.24), in fact it is not clear whether they are integrable in general. However, there exists a special solution for which

$$a(t) = A\,e^{\alpha t}, \quad c(t) = B\,e^{\gamma t},$$

where α, γ, A, B are constants. Substituting this ansatz into (7.22)-(7.24) we find:

$$\alpha^2 = -\frac{\Lambda}{6} = \frac{B^2}{4A^4}, \quad \gamma = 2\alpha.$$

Since only the ratio A/B plays a role we are free to choose $A = 1$, which gives $B = \sqrt{-\frac{2\Lambda}{3}}$. Clearly, the solution is invariant under time-reversal $t \to -t$; and hence it suffices to consider negative parameters $\alpha < 0$. The Bianchi type II four-metric thus becomes:

$$ds^2 = dt^2 + e^{-2t\sqrt{-\frac{\Lambda}{6}}}\left((\sigma^1)^2 + (\sigma^2)^2\right) + \sqrt{-\frac{2\Lambda}{3}}\,e^{-4t\sqrt{-\frac{\Lambda}{6}}}(\sigma^3)^2, \qquad (7.25)$$

where $\{\sigma^k\}$ are the left-invariant one-forms (7.13). A convenient choice of the cosmological constant is $\Lambda = -2/3$, for which (7.25) becomes:

$$ds^2 = dt^2 + e^{-t}\left((\sigma^1)^2 + (\sigma^2)^2\right) + e^{-2t}(\sigma^3)^2 . \tag{7.26}$$

In metric (7.26) one recognizes the Bergmann metric — a non-compact Kähler symmetric space $SU(2,1)/U(2)$. Note that the Bergmann metric is complete (unlike the vacuum Bianchi type II metric (7.10)). It is also BPS.

Another form of the Bergmann metric where the $U(2)$ group action is manifest is:

$$ds^2 = \frac{dR^2}{(1+\frac{\Lambda}{6}R^2)^2} + \frac{R^2}{4(1+\frac{\Lambda}{6}R^2)^2}(\sigma^3)^2 + \frac{R^2}{4(1+\frac{\Lambda}{6}R^2)}\left((\sigma^1)^2+(\sigma^2)^2\right) . \tag{7.27}$$

Expressed in this form the metric is a limiting case of the general $U(2)$-invariant Taub–NUT–AdS family of metrics [123] when $N \to \infty$ (N is the NUT charge). Such a family can be written in the form (see equation (2.6) in reference [123]):

$$\frac{U(r)}{f(r)}dr^2 + 4n^2\frac{f(r)}{U(r)}(\sigma^3)^2 + r^2 U(r)\left((\sigma^1)^2+(\sigma^2)^2\right) , \tag{7.28}$$

where $f(r) = 1 + \frac{\Lambda}{3}r^2(1+\frac{4N}{r})$ and $U(r) = 1 + \frac{2N}{r}$. Now, to take the limit of large NUT charge rescale the radial coordinate $r = \rho/N$ in the above formulae and take $N \to \infty$. Metric (7.28) becomes:

$$ds^2 = \frac{4\,d\rho^2}{2\rho\,(1+\frac{4\Lambda}{3}\rho)} + 2\rho\left(1+\frac{4\Lambda}{3}\rho\right)(\sigma^3)^2 + 2\rho\left((\sigma^1)^2+(\sigma^2)^2\right) , \tag{7.29}$$

and taking $2\rho = \frac{R^2}{4(1+\frac{\Lambda}{6}R^2)}$ we get back to expression (7.27).

7.4.3 Horospheres

To elucidate the geometrical structure of the Bergmann manifold and the role played by the Heisenberg group we shall now describe the way the Bergmann manifold arises as the set of horospheres of an odd-dimensional Anti–de Sitter space. We shall also obtain the Monge–Ampère equation for a Bianchi type II four-manifold. A solution to the equation with a negative cosmological constant gives the Kähler potential for the Bergmann metric. We make use of the defined complex coordinates and solve the Ricci-flat Monge–Ampère equation to obtain the Kähler potential for the Heisenberg metric (7.11).

Suppose G/H is a non-compact Riemannian symmetric space. The Iwasawa theorem (see e.g. [124]) tells us that every element $g \in G$ may be uniquely expressed as

$$g = h\,a\,n\,,$$

where $h \in H \subset G, a \in A \subset G, n \in N \subset G$; A is abelian and N is nilpotent subgroups of G. This means that we may also think of G/H as a solvable group $G_{solv} = A \ltimes N$ with a left-invariant metric, where $\ltimes$ denotes a semi-direct product. The orbits of N in G/H are called *horospheres*. The set of horospheres is labelled by elements of A. They are permuted by elements of H. The simplest example would be an n-dimensional real hyperbolic space $G/H = \mathcal{H}^n$, $A = \mathbb{R}_+$ and $N = \mathbb{R}^{n-1}$. If we think of $\mathcal{H}^n$ as a quadric in $\mathbb{E}^{n,1}$ (($n+1$)-dimensional Minkowski space-time) then the horospheres are the intersections of the quadric with a family of parallel null hypersurfaces related by boosts. There is a similar description of an n-dimensional Anti–de Sitter space AdS_n regarded as a quadric in $\mathbb{E}^{n-1,2}$. This description of the hyperbolic and the Anti–de Sitter spaces will be useful in Section 7.7.

The case of a complex hyperbolic n-space $\mathcal{H}^n_{\mathbb{C}}$ is slightly more complicated. Thinking of $\mathbb{E}^{2n,2}$ as $\mathbb{C}^{n,1}$, the $(2n+1)$-dimensional Anti–de Sitter space, AdS_{2n+1}, is given by the quadric

$$|z^0|^2 - \sum_{a=1}^{n}|z^a|^2 = 1\,.$$

Then the complex hyperbolic n-space $\mathcal{H}^n_{\mathbb{C}}$ is obtained by identifying z^a with $e^{i\theta}z^a$, $a = 1,\ldots,n$. Thus $(z^0,\ldots,z^n)$ are homogeneous coordinates on $\mathcal{H}^n_{\mathbb{C}}$. The nilpotent group N turns out to be the $(2n-1)$-dimensional Heisenberg group (see Section 7.6). Let us see how this works in detail. It is helpful to recall that the inhomogeneous coordinates ζ^a are defined in the usual way as $\zeta^a = z^a/z^0$ and make manifest the action of $U(n)$ on $\mathcal{H}^n_{\mathbb{C}}$. Our aim is to find a set of coordinates which make manifest the action of the Heisenberg group N.

Let us first introduce complex null coordinates u and v:

$$u = \frac{1}{\sqrt{2}}(z^0 + z^n)\,,$$
$$v = \frac{1}{\sqrt{2}}(z^0 - z^n)\,.$$

Define z and w^i, $i = 1,\ldots,n-1$, to be

$$z = \frac{u}{v}\,,\quad w^i = \frac{z^i}{v}\,.$$

In terms of inhomogeneous coordinates (ζ^i, ζ^n) these are

$$\begin{aligned} z &= \frac{2}{1-\zeta^n} - 1 \,, \\ w^i &= \frac{\sqrt{2}\,\zeta^i}{1-\zeta^n} \,. \end{aligned} \tag{7.30}$$

A complex hyperbolic n-space is topologically the interior of a unit ball in $\mathbb{C}^n$ and the map $(\zeta^i, \zeta^n) \to (w^i, z)$ provides a bi-holomorphism from the interior of the unit ball in $\mathbb{C}^n$ into the interior of the paraboloid

$$z + \bar{z} > \sum_i |w^i|^2 \,.$$

As we have mentioned above $\mathcal{H}^n_{\mathbb{C}}$ is obtained from AdS_{2n+1} as the base of the Hopf fibration, with the time-like Hopf fibre parametrized by θ such that

$$\begin{aligned} u &= \frac{z^0 + z^n}{\sqrt{2}} = \frac{z\, e^{i\theta}}{(z+\bar{z} - \sum_i |w^i|^2)^{\frac{1}{2}}} \,, \\ v &= \frac{z^0 - z^n}{\sqrt{2}} = \frac{e^{i\theta}}{(z+\bar{z} - \sum_i |w^i|^2)^{\frac{1}{2}}} \,, \\ z^i &= \frac{w^i\, e^{i\theta}}{(z+\bar{z} - \sum_i |w^i|^2)^{\frac{1}{2}}} \,. \end{aligned} \tag{7.31}$$

The real quantity $(z + \bar{z} - \sum_i |w^i|^2)$ is invariant under the action of the Heisenberg group N parametrized by (a^i, b):

$$\begin{aligned} w^i &\to w^i + a^i \,, \\ z &\to z + i\,b + \sum_i \frac{1}{2}|a^i|^2 + \bar{a}^i\, w^i \,. \end{aligned} \tag{7.32}$$

Considered as a subgroup of $SU(n,1) \subset SO(2n,2)$ N acts on $\mathcal{H}^n_{\mathbb{C}}$ as

$$\begin{aligned} u &\to u + (i\,b + \sum_i \frac{1}{2}|a^i|^2)\, v + \bar{a}^i\, z^i \,, \\ v &\to v \,, \\ z^i &\to z^i + a^i \,. \end{aligned}$$

Finally, the abelian group $A = \mathbb{R}_+$ parametrized by λ acts as $(z, w^i) \to (\lambda^2\, z, \lambda\, w^i)$ or

$$\begin{aligned} u &\to \lambda\, u \,, \\ v &\to \lambda^{-1}\, v \,, \\ z^i &\to z^i \,. \end{aligned}$$

Having identified all group actions, let us now formulate the problem in terms of the Kähler potential. The Kähler potential for the metric on the horospheres may be obtained from the Kähler potential on the AdS_{2n+1} manifold which in terms of inhomogeneous coordinates ζ^a is given by

$$K(\zeta^a, \bar{\zeta}^a) = -\log(\sum_{i=1}^{n-1} |\zeta^i|^2 + |\zeta^n|^2 - 1) \,.$$

From (7.30)

$$\sum_i |\zeta^i|^2 + |\zeta^n|^2 - 1 = 2\,\frac{|w^i|^2 - (z+\bar{z})}{(z+1)(\bar{z}+1)} \,,$$

and hence the Kähler potential on the horospheres becomes (up to a Kähler gauge transformation)

$$K = -\log(z + \bar{z} - \sum_i |w^i|^2) \,. \tag{7.33}$$

Let us derive the Monge–Ampère equation to which Kähler potential (7.33) is a solution. Since the resulting metric contains the higher-dimensional extension of the Heisenberg group (see Section 7.6) as a group of isometries we must assume that the Kähler potential depends only on the real quantity $f = z + \bar{z} - \sum_i |w^i|^2$. Then the Monge–Ampère equation (7.6) becomes an ordinary differential equation:

$$(K')^{n-1}\, K'' = (-1)^{n-1}\, e^{-\Lambda K} \,, \tag{7.34}$$

where $K' = dK/df$.

Let us make the connection with the form of the four-dimensional Bergmann metric (7.26). In this case $n = 2$, and there are two complex coordinates (z, w). We can pass from this parametrization to the parametrization of (7.26) in terms of (t, x, y, τ) as follows:

$$\begin{aligned} z - \bar{z} &= i\left(\tau - \frac{x\,y}{2}\right) \,, \\ z + \bar{z} &- w\bar{w} = f \,, \\ w &= \frac{1}{2}(x + iy) \,, \\ t &= e^f \,. \end{aligned} \tag{7.35}$$

For $n = 2$ the Monge–Ampère equation (7.34) becomes

$$K'\, K'' = -\,e^{-\Lambda K} \,. \tag{7.36}$$

In terms of the Kähler potential $K(f)$ the compatible Kähler metric is

$$ds^2 = -K'\, dw \wedge d\bar{w} + K''\, (dz - \bar{w}\, dw)(d\bar{z} - w\, d\bar{w}) \,. \tag{7.37}$$

The Kähler potential (7.33) for $n = 2$

$$K = -\log f = -\log\log t \tag{7.38}$$

is clearly a solution of equation (7.36) for $\Lambda = -3$. The Kähler metric $g_{a\bar{b}} = \partial_a \partial_{\bar{b}} K$ — the Bergmann metric in complex coordinates — obtained from the Kähler potential (7.38) is:

$$ds^2 = \frac{1}{f}\, dw \wedge d\bar{w} + \frac{1}{f^2}\,(dz - \bar{w}\, dw)(d\bar{z} - w\, d\bar{w}) \,.$$

Rewriting this metric using definitions (7.35) we get the Bergmann metric in the standard form (7.26).

Let us solve the Monge–Ampère equation (7.36) in the Ricci-flat case $\Lambda = 0$. Here we present the four-dimensional case $n = 2$ leaving the treatment of the higher-dimensional example to Section 7.6. Solution of (7.36) should yield the Kähler potential for the Heisenberg metric (7.11). Integrating the equation

$$K'\, K'' = -1 \,,$$

we get

$$K' = \sqrt{c - 2f}\,, \quad K'' = -\frac{1}{\sqrt{c-2f}}\,,$$

where c is the integration constant which, without loss of generality, we may set to zero. Substituting these expressions into (7.37) and ignoring an overall constant factor we obtain the Heisenberg metric (7.11). Note that the preferred complex structure, with respect to which the Kähler potential is defined, is the one whose associated Kähler form is $U(1)$-invariant. It is the two-form Ω_z (7.17) presented in Section 7.2.3.

7.5 Exotic Asymptotics: Bianchi types VII$_0$ and VI$_0$

In this section we propose to investigate p-brane solutions whose asymptotics are more unusual than the ones considered in Section 7.2. We turn to Bianchi types VII$_0$ and VI$_0$ whose groups of isometric motions are $E(2)$ and $E(1,1)$ respectively (see Table 7.1).[6]

We do not discuss the most general manifolds of the above types but rather focus on self-dual metrics, which are BPS. Vacuum four-metrics of this kind are Ricci-flat Kähler metrics. The Einstein

[6] We consider type VII$_0$ spaces before type VI$_0$ spaces because the type VII$_0$ metric is, in some sense, simpler since its isometry group is $E(2)$.

equations (7.2)-(7.5) in the vacuum self-dual case reduce to a set of first-order ODEs:

$$\frac{2}{a}\,a' = -n_1\, a^2 + n_2\, b^2 + n_3\, c^2 \,, \tag{7.39}$$

$$\frac{2}{b}\,b' = -n_2\, b^2 + n_3\, c^2 + n_1\, a^2 \,, \tag{7.40}$$

$$\frac{2}{c}\,c' = -n_3\, c^2 + n_1\, a^2 + n_2\, b^2 \,. \tag{7.41}$$

For convenience we have introduced another radial coordinate η in place of t such that $dt = abc\, d\eta$ and $(\;)'$ denotes differentiation with respect to η.

In Sections 7.5.1 and 7.5.2 we solve equations (7.39)-(7.41) to obtain self-dual vacuum Bianchi type VII$_0$ and VI$_0$ metrics and discuss their properties. In Section 7.5.3 we discuss Bianchi type VII$_0$ and VI$_0$ manifolds with negative cosmological constant.

7.5.1 Vacuum Solutions of Bianchi type VII$_0$ and Solvmanifolds

The group of isometries of a self-dual Bianchi type VII$_0$ metric is $E(2)$, whose structure constants are $n_1 = n_2 = 1$ and $n_3 = 0$ and a set of left-invariant one-forms is:

$$\sigma^1 = \cos\tau\, dx + \sin\tau\, dy \,, \tag{7.42}$$

$$\sigma^2 = -\sin\tau\, dx + \cos\tau\, dy \,,$$

$$\sigma^3 = d\tau \,.$$

Self-duality equations (7.39)-(7.41) become:

$$\frac{2}{a}\,a' = -a^2 + b^2 \,,$$

$$\frac{2}{b}\,b' = -b^2 + a^2 \,,$$

$$\frac{2}{c}\,c' = a^2 + b^2 \,.$$

These are easily solved to yield the metric:

$$ds^2 = \frac{\lambda^2}{2}\,\sinh 2\eta\,\left(d\eta^2 + (\sigma^3)^2\right) + \coth\eta\,(\sigma^1)^2 + \tanh\eta\,(\sigma^2)^2 \,, \tag{7.43}$$

where λ is the integration constant. Let us estimate the volume of a large metric ball as was done in Section 7.2 for the Heisenberg manifold. Introducing, as before, an effective radial coordinate t to return to the ansatz (7.1), we find that the metric volume grows as t^2 for large t.

Interestingly, this is the predicted volume growth of another type of a degeneration of the $K3$ surface in Kobayashi's review [59]. In fact, Kobayashi proved the existence and the completeness of a gravitational instanton whose three-dimensional hypersurfaces $t = const$ represent a collapse of *Solvmanifolds*.[7] The non-compact complete metric on a degeneration of the K3 surface is expected to have quadratic volume growth of large metric spheres and have as an asymptotic metric the standard flat metric on $\mathbb{C}^* \times \mathbb{C}^*$. It is not known explicitly.

The present situation parallels the one we have already encountered with the Heisenberg metric. The Heisenberg metric (7.10) is singular at the origin, but the singularity can be resolved by passing to another self-dual metric, the BKTY gravitational instanton, whose asymptotic form is exponentially close to the Heisenberg metric. The singularity at the origin of the Bianchi type VII_0 metric (7.43) may be resolved by passing to a non-singular manifold, whose existence and completeness is guaranteed by the general theorem of Kobayashi [59].

As we have pointed out, the large metric spheres have quadratic volume growth for large t. According to the estimates in Section 7.3.7, this is the critical volume growth for which the boundary term contribution to the gravitational action is constant and finite.

Alternatively, metric (7.43) may be obtained by solving the Monge–Ampère equation (7.6) for a Ricci-flat Kähler metric with appropriate symmetries. If we assume that the Kähler potential K is independent of the imaginary parts of $z^1 = u^1 + i\,v^1$ and $z^2 = u^2 + i\,v^2$, we obtain a metric with two commuting holomorphic isometries. If one further assumes that K depends only on the combination $\sqrt{(u^1)^2 + (u^2)^2}$ one gains an extra $SO(2)$ isometric action. From the first glance the resulting metric appears to be invariant under the direct product $SO(2) \times \mathbb{R}^2$ but, in fact, it turns out that the group of isometries is the semi-direct product $SO(2) \ltimes \mathbb{R}^2 \equiv E(2)$. Thus one obtains a Bianchi type VII_0 metric.

A short explicit calculation reveals that in polar coordinates (r, τ)

$$u^1 = r\cos\tau\,, \quad u^2 = r\sin\tau\,,$$

the Kähler potential depends only on r and the metric becomes:

$$ds^2 = K''\,dr^2 + r\,K'\,(\sigma^3)^2 + \frac{K'}{r}\left((\sigma^1)^2 + (\sigma^2)^2\right) + \left(K'' - \frac{K'}{r}\right)(\sigma^1)^2\,, \tag{7.44}$$

where $K' = dK/dr$ and $\{\sigma^k\}$ are the left-invariant $E(2)$ one-forms (7.42). Then the Monge–Ampère equation reduces to an ODE first written down by Calabi [125]:

$$\frac{K''\,K'}{r} = e^{-\Lambda K}\,, \tag{7.45}$$

[7] Usually the Bianchi type VI_0 group $E(1,1)$ is referred to as *Solv* or Solvable group. It is clear, however, that it is the type VII_0 manifold that has the volume growth predicted by Kobayashi. Its associated isometry group $E(2)$ is also solvable.

with Λ the cosmological constant. When $\Lambda = 0$, the vacuum case, Calabi found that

$$K'(r) = \sqrt{r^2 - a^2}\,, \tag{7.46}$$

where a is the integration constant. Metric (7.44) with (7.46) is precisely the metric (7.43). Incidentally, this metric is the helicoid metric found by Aliev *et al* [126] who obtained it using the connection between the real Monge–Ampère equation and minimal surfaces. It is singular at $\eta = 0$.

7.5.2 Vacuum Solutions of Bianchi type VI_0

The group of motions preserving Bianchi type VI_0 metrics is $E(1,1)$. From Table 7.1 the structure constants are $n_1 = 1$, $n_2 = -1$ and $n_3 = 0$, and hence the left-invariant one-forms are:

$$\begin{aligned} \sigma^1 &= \cosh\tau\,dx + \sinh\tau\,dy\,, \\ \sigma^2 &= \sinh\tau\,dx + \cosh\tau\,dy\,, \\ \sigma^3 &= d\tau\,. \end{aligned} \tag{7.47}$$

The self-duality equations (7.39)-(7.41) become:

$$\begin{aligned} \frac{2}{a}a' &= -(a^2+b^2)\,, \\ \frac{2}{b}b' &= a^2+b^2\,, \\ \frac{2}{c}c' &= a^2-b^2\,. \end{aligned}$$

These can be easily solved to give the following Ricci-flat Kähler metric:

$$ds^2 = \frac{\lambda^2}{2}\sin 2\eta\left(d\eta^2 + (\sigma^3)^2\right) + \cot\eta\,(\sigma^1)^2 + \tan\eta\,(\sigma^2)^2\,, \tag{7.48}$$

where λ is the integration constant. This metric is incomplete at the origin $\eta = 0$.

Such self-dual metrics of Bianchi type VI_0 were also displayed by Aliev *et al* [126] and were referred to as catenoid metrics.

One can find a description of the metric (7.48) in terms of the Kähler potential, as was done in the previous section for the type VII_0 metric (7.43). Again assuming that the Kähler potential K is independent of the imaginary parts of $z^1 = u^1 + i\,v^1$ and $z^2 = u^2 + i\,v^2$, we obtain a metric with two commuting holomorphic isometries. Now assume that K depends only on the combination $\sqrt{(u^1)^2 - (u^2)^2}$ thus gaining an $SO(1,1)$ isometric action. The resulting metric has as its group of isometries the semi-direct product $SO(1,1) \ltimes \mathbb{R}^2 \equiv E(1,1)$.

Defining new coordinates (r, τ) as

$$u^1 = r\cosh\tau\,, \quad u^2 = r\sinh\tau\,,$$

we find that the Kähler potential depends only on r and the metric becomes:

$$ds^2 = K''\,dr^2 - r\,K'\,(\sigma^3)^2 + \frac{K'}{r}\left((\sigma^1)^2 + (\sigma^2)^2\right) + \left(K'' - \frac{K'}{r}\right)(\sigma^1)^2\,, \qquad (7.49)$$

where $\{\sigma^k\}$ are the left-invariant $E(1,1)$ one-forms (7.47). The Monge–Ampère equation in this case differs from Calabi's equation (7.45) by a sign:

$$\frac{K''\,K'}{r} = -e^{-\Lambda K}\,. \qquad (7.50)$$

In the vacuum case, $\Lambda = 0$, the Monge–Ampère equation

$$\frac{K''\,K'}{r} = -1$$

is solved by

$$K'(r) = \sqrt{a^2 - r^2}\,, \qquad (7.51)$$

where a is the integration constant. This is the metric (7.48).

7.5.3 Bianchi type VII$_0$ and VI$_0$ with Negative Cosmological constant

If the cosmological constant Λ is negative, Calabi [125] proved that there exists a solution of (7.45) which gives a complete non-singular metric on $\mathbb{R}^4$. Unlike the analogous metric of the Bianchi type II (the Bergmann metric of Section 7.4.2), but like the Kasner–Anti–de Sitter metric of Section 7.4.1, this metric is not homogeneous.

Analogously, Calabi's argument concerning the existence of a solution of equation (7.45) with negative cosmological constant is applicable to the Bianchi type VI$_0$ case. It may thus be argued that solutions of (7.50) exist, although the completeness of the metrics has to be demonstrated separately.

7.6 Higher-dimensional Examples of Domain Walls

In this section we would like to give examples of domain walls of the form $M \times \mathbb{E}^{p-3,1}$ in eleven dimensions, where manifold M remains hypersurface homogeneous but has dimension higher than four. Firstly, we describe Calabi–Yau manifolds which are the higher-dimensional generalizations of

the BKTY instanton of Section 7.3.6. We find their asymptotic metrics by solving the vacuum Einstein equations in $2n$ dimensions. We then give particular examples of such asymptotic metrics which arise as extensions of the vacuum Bianchi type II, or Heisenberg metric (7.11) to higher dimensions. In addition, we present three series of higher-dimensional metrics originating from four-metrics of other Bianchi types: type I (7.7), type VII$_0$ (7.43) and type VI$_0$ (7.48) metrics. We do so by generalizing the relevant Monge–Ampère equations and arguing the existence of solutions, which provide the Kähler potentials for the metrics in question.

By the extensions of Bianchi type metrics from four to arbitrary number of dimensions we mean the following. The Bianchi type I isometry group $\mathbb{R}^3$ extended to $(n+1)$ dimensions is simply $\mathbb{R}^n$. The Bianchi type II three-dimensional group of isometries given by the left-invariant one-forms (7.13) may be easily generalized to a $(2n+1)$-dimensional group parametrized by (x_i, y_i, τ), where $i = 1, \ldots, n$, with the following left-invariant one-forms:

$$\begin{aligned} \sigma_i^1 &= dx_i\,, \\ \sigma_i^2 &= dy_i\,, \\ \sigma^3 &= d\tau - \sum_{i=1}^{n} x_i\,dy_i\,, \end{aligned} \qquad (7.52)$$

satisfying

$$d\sigma^3 = -\sum_i \sigma_i^1 \wedge \sigma_i^2\,.$$

The Bianchi type VI$_0$ and VII$_0$ groups are three-dimensional groups $E(1,1)$ and $E(2)$ respectively. In higher dimensions these become $E(n-1,1)$ and $E(n)$ respectively.

7.6.1 Higher-dimensional BKTY Metrics and Their Asymptotic Forms

In this section we shall again use reference [59] in which Kobayashi proves the existence theorem for complete Ricci-flat Kähler metrics on $X - D$ with $c_1(X) = [D]$, where X is a Fano manifold[8] and D is a complex codimension one hypersurface in X. Here $[D]$ is a Poincaré dual of D. From Yau's solution to Calabi's conjecture one can infer that D carries a Ricci-flat Kähler metric. Although the gravitational instanton is not known explicitly, Kobayashi provides some detailed information on the asymptotic form of the metric. It has the following properties.

Let $t(p)$ measure the distance from some fixed point in $X - D$ to a point $p \in X - D$. Then far away from the chosen fixed point, i.e. for large t, the metric spheres have a structure of an S^1 bundle over

[8] X is Fano if its first Chern class is positive, $c_1 > 0$.

D. The size of the fibre, with respect to the induced metric on the metric spheres, decays as $t^{-\frac{n-1}{n+1}}$, while the radius of the $(n-1)$ complex-dimensional base grows as $t^{\frac{1}{n+1}}$. Given this information one can estimate the volume growth of a large metric ball to be:

$$Vol \sim \int t^{2(n-1)\cdot\frac{1}{n+1}} \cdot t^{-\frac{n-1}{n+1}} dt \sim t^{\frac{2n}{n+1}} . \tag{7.53}$$

In this section we shall use the above information to make an ansatz for the asymptotic form of the metric and to show that it is an exact solution of the vacuum Einstein equations. We find that although the solution is Ricci-flat and Kähler it is singular. In fact, it bears the same relation to the gravitational instantons of Kobayashi as does the Heisenberg metric to the BKTY gravitational instanton: outside a compact set the complete metric differs from this asymptotic form by exponentially small terms.

It can be easily seen that the Heisenberg manifold (7.11) is a special case, $n = 2$, of this set-up. As we have already described in Section 7.3.6, Kobayashi points out that for $n = 2$ the BKTY gravitational instanton arises as a certain degeneration of the $K3$ surface. The metric spheres at large t represent a collapse of a Nilmanifold to a flat T^2, and the volume of a metric ball grows as $t^{4/3}$. The Ricci-flat metric on D is necessarily flat only in this case.

Consider the following ansatz compatible with the above remarks:

$$ds^2 = dt^2 + a^2(t) g_{ab} dx^a dx^b + c^2(t)(d\tau - 2A)^2 . \tag{7.54}$$

Here g_{ab} is the complete Ricci-flat Kähler metric on D, $a, b = 1, \cdots, 2(n-1)$; τ is the periodic coordinate on the canonical bundle over D and A is a one-form that depends only on x^a such that its exterior derivative is proportional to the Kähler form on D, $dA = -\sigma J$, σ is a constant.

The Einstein equations for (7.54) reduce to a system of second order ODEs for the functions $a(t)$ and $c(t)$:

$$0 = \frac{\ddot a}{a} + \frac{\dot a \dot c}{ac} + (2n-3)\left(\frac{\dot a}{a}\right)^2 + 2\sigma^2 \frac{c^2}{a^4} , \tag{7.55}$$

$$0 = \frac{\ddot c}{c} + 2(n-1)\frac{\ddot a}{a} , \tag{7.56}$$

$$0 = \frac{\ddot c}{c} + 2(n-1)\frac{\dot a \dot c}{ac} - 2(n-1)\sigma^2 \frac{c^2}{a^4} . \tag{7.57}$$

Let us look for solutions with polynomial dependence on the radial coordinate t of the form:

$$c(t) = \mu t^{\lambda_1} , a(t) = \nu t^{\lambda_2} , \tag{7.58}$$

where $\mu, \nu, \lambda_1, \lambda_2$ are constants. Substituting (7.58) into equations (7.55)-(7.57) we find:

$$0 = \lambda_2(\lambda_2 - 1) + \lambda_1\lambda_2 + (2n-3)\lambda_2^2 + 2\sigma^2 \frac{\mu^2}{\nu^4} t^{2(1+\lambda_1-2\lambda_2)} , \tag{7.59}$$

$$0 = \lambda_1(\lambda_1 - 1) + 2(n-1)\lambda_2(\lambda_2 - 1) , \tag{7.60}$$

$$0 = \lambda_1(\lambda_1 - 1) + 2(n-1)\lambda_1\lambda_2 - 2(n-1)\sigma^2 \frac{\mu^2}{\nu^4} t^{2(1+\lambda_1-2\lambda_2)} . \tag{7.61}$$

According to the ansatz (7.58) equations (7.59)-(7.61) must reduce to algebraic equations for the constants $\mu, \nu, \lambda_1, \lambda_2$. Hence we find that λ_1 and λ_2 must satisfy:

$$\lambda_1 = 2\lambda_2 - 1 . \tag{7.62}$$

Substituting (7.62) into (7.60) we obtain a quadratic equation for λ_2

$$(n+1)\lambda_2^2 - (n+2)\lambda_2 + 1 = 0 , \tag{7.63}$$

which is solved by:

$$\lambda_2^{(1)} = \frac{1}{n+1} , \quad \lambda_2^{(2)} = 1 . \tag{7.64}$$

From (7.62) we have:

$$\lambda_1^{(1)} = -\frac{n-1}{n+1} , \quad \lambda_1^{(2)} = 1 . \tag{7.65}$$

We discard the pair $(\lambda_1^{(2)}, \lambda_2^{(2)}) = (1, 1)$ since it satisfies both (7.59) and (7.61) only for $n = 0$. We are thus left with the other pair of solutions $(\lambda_1, \lambda_2) \equiv (\lambda_1^{(1)}, \lambda_2^{(1)}) = (-\frac{n-1}{n+1}, \frac{1}{n+1})$. Let us now find the constants μ, ν and σ. From (7.61), or equivalently from (7.59), we have

$$\sigma^2 \frac{\mu^2}{\nu^4} = \frac{1}{(n+1)^2} . \tag{7.66}$$

The values of μ and ν for $n = 2$ may be read off from the Heisenberg metric (7.11): $\mu = (3/2)^{-1/3}$ and $\nu = (3/2)^{1/3}$. With these values (7.66) gives

$$\sigma = \frac{1}{2} . \tag{7.67}$$

Since the parametrization of the one-form A should not depend on the dimension of M, we are compelled to choose constants μ and ν to satisfy (7.66) with $\sigma = 1/2$ and consistent with their values for $n = 2$. An appropriate choice is:

$$\mu = \left(\frac{n+1}{2n}\right)^{-\frac{n-1}{n+1}} , \quad \nu = \left(\frac{n+1}{2n}\right)^{\frac{1}{n+1}} . \tag{7.68}$$

Absorbing constant $-\sigma = -1/2$ into the definition of the one-form A we can now write down the asymptotic metric for the $2n$-dimensional BKTY gravitational instanton:

$$ds^2 = dt^2 + \left(\frac{n+1}{2n}t\right)^{-2\frac{n-1}{n+1}} (d\tau + A)^2 + \left(\frac{n+1}{2n}t\right)^{\frac{2}{n+1}} g_{ab}dx^a dx^b \,, \tag{7.69}$$

where now dA is precisely the Kähler form on D.

Metric (7.69) has indeed the volume growth predicted by Kobayashi:

$$Vol \sim \int t^{-\frac{n-1}{n+1}} \cdot t^{2(n-1)\cdot\frac{1}{n+1}} dt \sim t^{\frac{2n}{n+1}} \,.$$

7.6.2 Bianchi type I

Metrics of Kasner type (7.7) exist in arbitrary number of dimensions; the metric becomes:

$$ds^2 = dt^2 + t^{2\alpha_1}(dx^1)^2 + t^{2\alpha_2}(dx^2)^2 + \ldots + t^{2\alpha_n}(dx^n)^2,$$

where

$$\sum_{k=1}^{n} \alpha_k = 1 = \sum_{k=1}^{n} \alpha_k^2 \,. \tag{7.70}$$

Just as the Kasner metric (7.7), these metrics are not BPS.

7.6.3 Bianchi type II

A particular case of the asymptotic BKTY $2n$-dimensional metric (7.69) is one whose isometry group is the higher-dimensional Bianchi type II group. In this case the arbitrary Calabi–Yau $(2n-2)$-dimensional metric $g_{ab}dx^a\, dx^b$ is flat:

$$g_{ab}dx^a\, dx^b = \sum_{i=1}^{n-1} (\sigma_i^1)^2 + (\sigma_i^2)^2 \,.$$

and the term involving the connection on the canonical bundle over $g_{ab}dx^a\, dx^b$ is simply $(\sigma^3)^2$; where $\{\sigma_i^1, \sigma_i^2, \sigma^3\}$ are the one-forms (7.52).

To find the Kähler potential generating the metric (7.69) for the special case of Bianchi type II isometry group we solve the Monge–Ampère equation (7.34) with $\Lambda = 0$:

$$(K')^{n-1} K'' = (-1)^{n-1} \,. \tag{7.71}$$

It is sufficient to know $K'(f)$, where $f = z + \bar{z} - \sum_i |w^i|^2$, since the higher-dimensional Bianchi type II metric is expressed in terms of K' and K'' as follows:

$$ds^2 = -K' \sum_i dw^i \wedge d\bar{w}^i + K'' \,(dz - \sum_i \bar{w}^i\, dw^i)(d\bar{z} - w^i\, d\bar{w}^i) \,,$$

which after coordinate redefinitions (7.35), with (w, x, y) replaced by (w^i, x^i, y^i) becomes

$$ds^2 = -K' \sum_i ((dx^i)^2 + (dy^i)^2) + K''\, df^2 + K''\,(d\tau - \sum_i x^i\, dy^i)^2 \,. \tag{7.72}$$

Integrating equation (7.71) we find:

$$K' = (-1)^{\frac{n-1}{n}} (n\, f)^{\frac{1}{n}} \,,$$

and hence

$$K'' = (-1)^{\frac{n-1}{n}} (n\, f)^{-\frac{n-1}{n}} \,.$$

To compare with the metric we have obtained by solving the self-duality equations let us define a new radial coordinate t such that

$$f = \left(\frac{n+1}{2n}\, t\right)^{\frac{2n}{n+1}} \,.$$

Written in terms of t the metric (7.72) is the same as (7.69) up to an overall constant factor.

7.6.4 Bianchi type VII$_0$ and VI$_0$

The analysis based on solving the Monge–Ampère equations (7.45) and (7.50) may be extended to arbitrary number of dimensions. In fact in the case of Bianchi type VII$_0$ it was done by Calabi in [125]. If M is parametrized by n complex coordinates z^a, $a = 1, \ldots, n$, and one assumes, as was done in Section 7.5.1, that the Kähler potential K is independent of the imaginary parts of $z^a = u^a + i\, v^a$ and is solely a function of $r \equiv \sqrt{(u^1)^2 + \ldots + (u^n)^2}$, the resulting metric will have $E(n)$ isometry group. The Monge–Ampère equation reduces to

$$\left(\frac{K'}{r}\right)^{n-1} K'' = 1 \,.$$

It is solved by

$$K(r) = \int_0^r (c + r^{\frac{1}{n}})\, dr \,, \quad c = const \,.$$

The manifold is a higher-dimensional vacuum Bianchi type VII$_0$ metric whose isometry group is $E(n)$. It is incomplete. Arguments analogous to the ones just given extend to the Bianchi type VI$_0$ metrics.

7.7 Other Bianchi Types: Bianchi type III

We have not attempted here to survey all known cohomogeneity one Einstein metrics. Even in four dimensions this would be a formidable task. Some pertinent references in that case are [104, 108]. However we would like to comment on the Bianchi type III situation since it may well prove relevant for various applications of Anti–de Sitter space-time.

The most general diagonal Lorentzian Bianchi type III local solution is given in [127]. A simple analytic continuation of the metric in [127] gives a Riemannian metric with negative scalar curvature $\Lambda < 0$:

$$ds^2 = \frac{3}{\Lambda}\left(\frac{d\tau^2}{\sinh^2\tau} + \frac{d\Omega_{-1}^2}{\sinh^2\tau} + \frac{d\alpha^2}{\tanh^2\tau}\right), \qquad (7.73)$$

which is presumably the most general local solution with this signature. Setting

$$t = \log\tanh\frac{\tau}{2}$$

gives

$$ds^2 = \frac{3}{\Lambda}\left(dt^2 + \sinh^2 t\, d\Omega_{-1}^2 + \cosh^2 t\, d\alpha^2\right).$$

In (7.73) $d\Omega_{-1}^2$ is the standard metric on the hyperbolic two-space $\mathcal{H}^2$. The isometry group of the manifold is therefore $SO(2,1)\times SO(2)$. The group $SO(2,1)$ has a two-dimensional subgroup $\tilde{G}_2$ which acts transitively on $\mathcal{H}^2$ and combined with $SO(2)$ we get a three-dimensional Lie group with three-dimensional orbits whose Lie algebra corresponds to Bianchi type III. Explicitly we consider $\mathcal{H}^2$ in horospheric coordinates (x,y):

$$\frac{dx^2+dy^2}{y^2},$$

and $\tilde{G}_2 = \mathbb{R}\ltimes\mathbb{R}$ is generated by $\partial/\partial x$ and $x\,\partial/\partial x + y\,\partial/\partial y$.

In fact the metric (7.73) is that of hyperbolic four-space $\mathcal{H}^4$ (cf. Section 7.4.3). This may be seen by isometrically embedding (7.73) into $\mathbb{E}^{4,1}$ as:

$$(X^0)^2-(X^1)^2-(X^2)^2-(X^3)^2-(X^4)^2 = 1,$$

where

$$\begin{aligned}
X^3 &= \cosh t\,\cos\alpha,\\
X^4 &= \cosh t\,\sin\alpha,\\
X^0+X^1 &= \frac{1}{y}\sinh t,\\
X^0-X^1 &= \left(y+\frac{x^2}{y}\right)\sinh t,\\
X^2 &= \frac{x}{y}\sinh t.
\end{aligned}$$

It is obvious that the Bianchi type III solution (7.73) may be extended to $(n+2)$ dimensions by replacing the metric on $\mathcal{H}^2$ by that on $\mathcal{H}^n$. The group $\tilde{G}_2$ is replaced by the group $\tilde{G}_n = \mathbb{R}\ltimes\mathbb{R}^{n-1}$, generated by $\partial/\partial x^i$ and $x^i\,\partial/\partial x^i + y\,\partial/\partial y$, $i = 1,\dots,n-1$. Then the generalization of the Bianchi type III group is $(\mathbb{R}\ltimes\mathbb{R}^{n-1})\times SO(2)$.

CHAPTER 8

Conclusion

Let us recapitulate the results presented in this thesis. The work in the first part of this dissertation (Chapters 2 to 5) was motivated by the S-duality conjecture. It states that $\mathcal{N} = 4$ supersymmetric Yang–Mills theory is invariant under the $SL(2, \mathbb{Z})$ group of symmetry transformations which act non-trivially on both the coupling as well as on the states in the theory. We investigated the consequences of this conjecture in the context of SYM theories with higher rank gauge groups.

In Chapter 2 we reformulated predictions of the S-duality hypothesis in geometrical terms: the problem of studying the spectrum of BPS solitons of the SYM theory was reduced to the problem of investigating the spectrum of normalizable eigenstates of the Hodge–de Rahm Laplacian on the moduli space of these solitons. More specifically, if S-duality is a true symmetry of the theory, one expects to find certain normalizable harmonic forms on the relevant moduli spaces.

The first step towards exhibiting these harmonic forms was taken in Chapter 3, where exact metrics on monopole moduli spaces were constructed. It is well known that moduli spaces of solutions to the Bogomol'nyi equations are hyperkähler manifolds, and it is this property that was exploited to obtain the moduli space metrics. It is a common lore of hyperkähler geometry that hyperkähler manifolds are amenable to being quotiented. We advocated the use of the hyperkähler quotient construction in obtaining the desired hyperkähler metrics. This approach allowed us not only to obtain the metrics explicitly, but also to gain insight into some global properties of the manifolds. Since all the spaces we proposed to study were toric hyperkähler manifolds, computations involved in implementing the quotient construction simplified dramatically. To exemplify the hyperkähler quotient construction we first obtained a number of well known hyperkähler manifolds. We then turned to moduli spaces of fundamental monopoles and re-derived the Lee–Weinberg–Yi metric. One of the advantages of our approach is that we were able to deduce immediately that the Lee–Weinberg–Yi metric was complete and topologically trivial — a result which had previously required a lengthy analysis. The hyperkähler quotient construction led naturally to the discussion of two degenerations of the Lee–Weinberg–Yi metric. One corresponded to the metric on the moduli space of monopoles in theories with non-maximally broken gauge symmetry. We identified a subset of these metrics to be the Taubian–Calabi manifolds and analyzed their global properties. The other degeneration corresponded to the moduli space metric of a configuration of fundamental monopoles, some of which were fixed, or infinitely massive. We also constructed a family of new hyperkähler metrics, a special case of which was the metric on the moduli space of many well-separated $SU(2)$ monopoles. We concluded the chapter by presenting an alternative interpretation for the monopole moduli spaces in terms of world-volume theories of certain intersecting D-brane configurations in the type IIB string theory.

In Chapter 4 we exhibited harmonic forms on the moduli spaces of fundamental monopoles, which are associated with threshold bound states predicted by S-duality. The new result in this chapter was the candidate harmonic form on the Taubian–Calabi manifold. Although we could not offer a rigorous proof of the uniqueness of the candidate harmonic form, the fact that it possesses correct symmetry properties and is normalizable, at least in the eight-dimensional case, strongly suggests that the candidate form is indeed the desired harmonic form. This is the first quantitative evidence supporting the S-duality conjecture for theories with non-abelian unbroken gauge groups.

In Chapter 5 we investigated classical dynamics of fundamental monopoles. The study of classical dynamics of slowly moving BPS monopoles can be conveniently relegated to the study of the geodesic flow on the monopole moduli space. We first proved that there were no classical bound states of fundamental monopoles by proving the non-existence of closed or bound geodesics on the Lee–Weinberg–Yi and the Taubian–Calabi manifolds. Our argument relied on the hyperkähler quotient construction of these metrics. We then studied classical scattering of distinct fundamental monopoles. Although the equations of motion are not integrable in general, we found scaling solutions (otherwise known as similarity or homothety solutions). These scaling solutions describe the simplest scattering of monopoles carrying no dyonic charges. We also found that such similarity solutions exist on the moduli space of well-separated $SU(2)$ monopoles.

The work in the second part of the dissertation (Chapters 6 and 7) derives its motivation from the

AdS/CFT correspondence principle. Most of the results in Chapter 6 are deduced from one condition: the target space of hypermultiplets in the rigid $\mathcal{N} = 2$ SCFT is required to be a hyperkähler manifold invariant under dilatations. We demonstrated that the manifold admitting a dilatation (regardless of any hyperkähler condition) must be a metric cone. In fact, the base manifolds should be either Sasakian or tri–Sasakian, and cones over them are Kähler and hyperkähler respectively. We offered a few speculative remarks regarding possible applications of these novel geometries to cone-branes and the AdS/CFT correspondence.

Finally, Chapter 7 was devoted to domain wall solutions in M-theory. We studied domain walls which are products of a four-manifold with flat Minkowski space of an appropriate dimension. We restricted ourselves to considering four-metrics of cohomogeneity one with isometry groups of Bianchi type. We first looked for Ricci-flat solutions. Our main example was the Bianchi type II metric which we called the Heisenberg metric. It is BPS and has a singularity at the location of the domain wall. In the conventional interpretation of the solution one inserts a distributional source at the singularity. However, there are no obvious sources in M-theory, and hence it is reasonable to seek a resolution of the singularity. We argued that there is a natural way in which the singularity may be resolved: we identified the two sides of the domain wall and replaced the Heisenberg manifold by a smooth gravitational instanton whose asymptotic form is exponentially close to the Heisenberg metric. In this way we obtained a "domain wall at the end of the universe". A similar procedure was applied to the Bianchi type VII_0 four-metric. The solutions were generalized to higher dimensions, and a smooth Calabi–Yau resolution was found in the Bianchi type II case. Motivated by the AdS/CFT correspondence, we also found cohomogeneity one metrics with Bianchi type symmetries with negative cosmological constant. In addition, we gave a description of all Kähler spaces which arose in this chapter in terms of the Kähler potential.

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