

# Cosmological Models in Energy-Momentum-Squared Gravity

Charles V. R. Board\* and John D. Barrow†

*DAMTP, Centre for Mathematical Sciences, University of Cambridge,  
Wilberforce Rd., Cambridge CB3 0WA, U.K.*

## Abstract

We study the cosmological effects of adding terms of higher-order in the usual energy-momentum tensor to the matter lagrangian of general relativity. This is in contrast to most studies of higher-order gravity which focus on generalising the Einstein-Hilbert curvature contribution to the lagrangian. The resulting cosmological theories give rise to field equations of similar form to several particular theories with different fundamental bases, including bulk viscous cosmology, loop quantum gravity, K-essence, and brane-world cosmologies. We find a range of exact solutions for isotropic universes, discuss their behaviours with reference to the early and late-time evolution, accelerated expansion, and the occurrence or avoidance of singularities. We briefly discuss extensions to anisotropic cosmologies and delineate the situations where the higher-order matter terms will dominate over anisotropies on approach to cosmological singularities.

---

\* cvrb2@damtp.cam.ac.uk

† J.D.Barrow@damtp.cam.ac.uk

## I. INTRODUCTION

The twin challenges of naturally explaining two periods of accelerated expansion during the history of the universe engage the attentions of many contemporary cosmologists. The first period may have had a beginning and necessarily came to an end when the universe was young and hot: it is called a period of ‘inflation’ and it leaves observable traces in the cosmic microwave background radiation that are believed to have been detected. The second period of acceleration began only a few billion years ago and is observed in the Hubble flow traced by Type IA supernovae; it is not known if it will ever come to an end or is changing in any way. There are separate non-unique mathematical descriptions of each of these periods of acceleration but there is no single explanation of both of them both, nor any insight into whether or not they are related, or even random, occurrences. For these reasons, there is continuing interest in all the different ways in which expanding universes can undergo periods of accelerated expansion. In the case of late-time acceleration the simplest description of an effectively anti-gravitating stress, known as ‘dark energy’, is provided by introducing a cosmological constant ( $\Lambda$ ) into general relativity with a value arbitrarily chosen to match observations.

The best-fit theory of this sort is dubbed  $\Lambda$ CDM and in its simplest form is defined by six constants (which determine  $\Lambda$ ) that can be fixed by observation. One of those parameters is  $\Lambda$  and its required value is difficult to explain: it requires a theory that contributes an effective vacuum stress of magnitude  $\Lambda \sim (t_{pl}/t_0)^2 \sim 10^{-120}$  at a time of observation  $t_0 \sim 10^{17}s$ , where  $t_{pl} \sim 10^{-43}s$  is the Planck time [1]. Other descriptions that lead to slowly evolving scalar fields in place of a constant  $\Lambda$  have also been explored, together with a range of modified gravity theories that contribute anti-gravitating stresses. There are many such modifications and extensions of Einstein’s general relativity and they can be tuned to provide acceleration at early or late times. So far, almost all of these modifications to general relativity have focussed on generalising the gravitational lagrangian away from the linear function of the spacetime curvature,  $R$ , responsible for the Einstein tensor in Einstein’s equations. A much-studied family of theories of this sort are those deriving from a lagrangian of the form  $F(R)$ , where  $F$  is some analytic function. By contrast, in this paper we will explore some of the consequences of generalising the form of the matter lagrangian in a nonlinear way, to some analytic function of  $T_{\mu\nu}T^{\mu\nu}$ , where  $T_{\mu\nu}$  is the energy-momentum

tensor of the matter stresses. This is more radical than simply introducing new forms of fluid stress, like bulk viscosity or scalar fields, into the Einstein equations in order to drive acceleration in FLRW universes.

In II we discuss and motivate higher order contributions to gravity from matter terms. In III we derive the equations of motion for a generic  $F(R, T_{\mu\nu}T^{\mu\nu})$  modification of the action with bare cosmological constant, before specialising to the case  $F(R, T_{\mu\nu}T^{\mu\nu}) = R + \eta(T_{\mu\nu}T^{\mu\nu})^n$ . We then investigate several features of the isotropic cosmology in this theory IV and, finally, move to the anisotropic Bianchi Type I setting in V.

## II. BACKGROUND

### A. Field equations

Einstein's theory of general relativity (GR) with cosmological constant  $\Lambda$  can be derived from the variation of the action,

$$S = \frac{1}{2\kappa} \int \sqrt{-g}(R - 2\Lambda)d^4x + \int \sqrt{-g}L_m d^4x, \quad (1)$$

where  $\kappa = 8\pi G$  and  $L_m$  is the matter lagrangian, that we will take to describe a perfect fluid;  $R \equiv R_a^a$ , where  $R_b^a$  is the Ricci tensor, and  $g$  is the determinant of the metric itself. Here, and in all that follows, we use units in which  $c = 1$ .

An isotropic and homogeneous universe may be described by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric:

$$ds^2 = -dt^2 + a^2(t) \left( \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right), \quad (2)$$

where  $k$ , the curvature parameter, takes the values  $\{-1, 0, +1\}$  corresponding to open, flat and closed 3-spaces, respectively;  $t$  is the comoving proper time and  $a(t)$  is the expansion scale factor.

There are many proposals to modify or extend the  $\Lambda$ CDM cosmological picture. These fall broadly into two categories, depending on which side of the Einstein field equations is modified. We can modify the right-hand side of the Einstein equations by adding new forms of matter that will drive expansion either at early times, as in the theory of inflation, or at late times, such as in quintessence or K-essence scenarios [2]. Alternatively, we can modify

the left-hand side of the Einstein equations in order to modify the effect of gravity itself. There are several ways to do this, including  $F(R)$  theories [3] in which the Ricci scalar in (1) is replaced by some function  $f(R)$ , so called  $F(T)$  theories in which we modify the teleparallel equivalent of general relativity [7], or scalar-tensor theories in which a scalar field is coupled to the Ricci scalar.

## B. Higher-order Matter Contributions

The type of generalisation of general relativity we will explore in this paper looks to include higher-order contributions to the right-hand side of the Einstein equations, where the material stresses appear. This results in field equations that include new terms that enter at high densities and pressures, which may be anti-gravitational in their effects. Typically, they affect the cosmological model at high densities and may alter the conclusions regarding the appearance of spacetime singularities in the finite cosmological past. Conversely, we might expect their effects at late times and low cosmological densities to be very small. Even within general relativity, there is scope to include high-order matter contributions, as the Einstein equations have almost no content unless some prescription or constraint is given on the forms of matter stress. Thus, in the general-relativistic Friedmann models, we can introduce non-linear stresses defined by relations between pressure,  $p$ , and density,  $\rho$ , of the form  $\rho + p = \gamma\rho^n$ , [13], or  $f(\rho)$ , [14], with  $\gamma \geq 0$  and  $n$  constants, or include a bulk viscous stress into the equation of state of the standard form  $p = (\gamma - 1)\rho - 3H\varsigma(\rho)$ , where  $H$  is the Hubble expansion rate and  $\varsigma \geq 0$  is the bulk viscosity coefficient [15]. The so called Chaplygin and generalised Chaplygin gases are just special cases of these bulk viscous models, and choices of  $n$  or  $\varsigma \propto \rho^m$  introduce higher-order matter corrections. Similarly, the choice of self-interaction potential  $V(\phi)$  for a scalar field can also introduce higher-order matter effects into cosmology. Analogously, in scalar-tensor theories like Brans-Dicke (BD) which are defined by a constant BD coupling constant,  $\omega$ , generalisations are possible to the cases where  $\omega$  becomes a function of the BD scalar field. In all these extensions of the standard relativistic perfect fluid cosmology there will be several critical observational tests which will constrain them. In particular, in higher-order matter theories the inevitable deviations that can occur from the standard thermal history in the early radiation era will change the predicted abundances of helium-4 and deuterium and alter the detailed structure

of the microwave background power spectrum. Also, as we studied for Brans-Dicke theory [33, 34], changes in the cold dark matter dominated era evolution can shift the time when matter and radiation densities are equal. This is the epoch when matter perturbations begin to grow and sensitively determines the peak of the matter power spectrum. At a later nonlinear stage of the evolution, higher-order gravity theories will effect the formation of galactic halos. This has been investigated for bulk viscous cosmologies by Li and Barrow [25]. These observational constraints will form the subject of a further paper and will not be discussed here.

If we depart from general relativity, then various simple quantum gravitational corrections are possible, and have been explored. The most well known are the Loop Quantum Gravity (LQG) [16] and brane-world [22] scenarios that contribute new quadratic terms to the Friedmann equation for isotropic cosmologies by replacing  $\rho$  by  $\rho(1 \pm O(\rho^2))$  in the Friedmann equation, where the  $-$  contribution is from LQG and the  $+$  in brane worlds. The impact on anisotropic cosmological models is more complicated and not straightforward to calculate [23, 24]. In particular, we find that simple forms of anisotropic stress are no longer equivalent to a  $p = \rho$  fluid as we are used to finding in general relativity. Our study will be of a type of higher-order matter corrections which modify the Friedmann equations in ways that include both of the aforementioned types of phenomenological modification to the form of the Friedmann equations, although the underlying physical theory does not incorporate the LQC or Braneworld models or reduce to them in a limiting case.

Standard  $F(R)$  theories of gravity [3] can be generalised to include a dependence of the form

$$S = \frac{1}{2\kappa} \int \sqrt{-g} F(R, L_m) d^4x. \quad (3)$$

This is in some sense an extremal extension of the Einstein-Hilbert action, as discussed in [9]. If the coupling between matter and gravity is non-minimal, then there will be an extra force exerted on matter, resulting in non-geodesic motion and a violation of the equivalence principle. This type of modification has been investigated in several contexts, particularly when the additional dependence on the matter lagrangian arises from  $F$  taking the form  $F(R, \mathcal{T})$  where  $\mathcal{T}$  is the trace of the energy-momentum tensor [10].

A theory, closely related to  $F(R, \mathcal{T})$  gravity, that allows the gravitational lagrangian to depend on a more complicated scalar formed from the energy-momentum tensor is provided

by  $F(R, \mathbf{T}^2)$ , where  $\mathbf{T}^2 \equiv T_{\mu\nu}T^{\mu\nu}$  is the scalar formed from the square of the energy-momentum tensor. This was first discussed in [11], and the special case with

$$F(R, \mathbf{T}^2) = R + \eta \mathbf{T}^2, \quad (4)$$

where  $\eta$  is a constant, was also discussed in [12], where the authors investigated the possibility of a bounce at early times when  $\eta < 0$  (although in that paper they used the opposite sign convention to us for  $\eta$ ), and also found an exact solution for charged black holes in the extended theory. In [31] a similar form, with additional cross terms between the Ricci and Energy-momentum tensors, was discussed as arising from quantum fluctuations of the metric tensor. Recently [32] investigated the late time acceleration of universes described by this model in the dust-only case, as well as using observations of the Hubble parameter to constrain the parameters of the theory.

We would expect the theory derived from (4) to provide different physics to the  $F(R, \mathcal{T})$  case. Indeed, one example of this is the case of a perfect fluid with equation of state  $p = -\frac{1}{3}\rho$ . The additional terms in  $F(R, \mathcal{T})$  will vanish as  $\mathcal{T} = 0$ , but in the  $F(R, \mathbf{T}^2)$  theory the extra terms in  $\mathbf{T}^2$  will not vanish and we will find new cosmological behaviour. In section III, we will investigate the cosmological solutions in a more general setting, where the  $\mathbf{T}^2$  term may be raised to an arbitrary power.

### III. FIELD EQUATIONS FOR $F(R, T_{\mu\nu}T^{\mu\nu})$ GRAVITY WITH COSMOLOGICAL CONSTANT

In [12] the Friedmann equations were derived in the case where  $F$  is given by (4), for a flat FLRW cosmology. A ‘bare’ cosmological constant was also included on the left-hand side of the field equations (rather than as an effective energy-momentum tensor for the vacuum). In [11], the field equations were derived without a cosmological constant and specialised to two particular models. We first derive the equations of motion with a cosmological constant for general  $F$ , before specialising to theories where the additional term takes the form  $(\mathbf{T}^2)^n$ , and determining the FLRW equations with general curvature. In GR, the cosmological constant can be considered to be, equivalently, either a ‘bare’ constant on the left-hand side of the Einstein equations, or part of the matter lagrangian. As discussed in [12], the two are no longer equivalent in this theory, due to the non-minimal nature of the curvature-matter

couplings. A similar inequivalence also occurs in other models that introduce non-linear matter terms, including Loop Quantum Cosmology. We will assume that the cosmological constant arises in its bare form as part of the gravitational action. This gives the modified action

$$S = \frac{1}{2\kappa} \int \sqrt{-g} (F(R, T^{\mu\nu} T_{\mu\nu}) - 2\Lambda) d^4x + \int L_m \sqrt{-g} d^4x, \quad (5)$$

where  $L_m$  is taken to be the same as the matter component contributed by  $T_{\mu\nu}$ . Since the gravitational lagrangian now depends on  $\mathbf{T}^2$ , we note that the new terms in the variation of the action will arise from the variation of this square, via  $\delta(T_{\mu\nu} T^{\mu\nu})$ . To calculate this, we define  $T_{\mu\nu}$  by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{\mu\nu}}. \quad (6)$$

We enforce the condition that  $L_m$  depends only on the metric components, and not on their derivatives, to find

$$T_{\mu\nu} = g_{\mu\nu} L_m - 2 \frac{\partial L_m}{\partial g^{\mu\nu}}. \quad (7)$$

Varying with respect to the inverse metric, we define

$$\theta_{\mu\nu} \equiv \frac{\delta(T_{\alpha\beta} T^{\alpha\beta})}{\delta g^{\mu\nu}} = -2L_m(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T) - TT_{\mu\nu} + 2T_\mu^\alpha T_{\nu\alpha} - 4T^{\alpha\beta} \frac{\partial^2 L_m}{\partial g^{\mu\nu} \partial g^{\alpha\beta}}, \quad (8)$$

where  $T$  is the trace of the energy-momentum tensor. Varying the action in this way, we find

$$\delta S = \frac{1}{2\kappa} \int \{F_R \delta R + F_{T^2} \delta(T_{\mu\nu} T^{\mu\nu}) - \frac{1}{2} g_{\mu\nu} F \delta g^{\mu\nu} + \Lambda + \frac{1}{\sqrt{-g}} \delta(\sqrt{-g} L_m)\} d^4x, \quad (9)$$

where subscripts denote differentiation with respect to  $R$  and  $\mathbf{T}^2$ , respectively.

From this variation we obtain the field equations:

$$F_R R_{\mu\nu} - \frac{1}{2} F g_{\mu\nu} + \Lambda g_{\mu\nu} + (g_{\mu\nu} \nabla_\alpha \nabla^\alpha - \nabla_\mu \nabla_\nu) F_R = \kappa (T_{\mu\nu} - \frac{1}{\kappa} F_{\mathbf{T}^2} \theta_{\mu\nu}). \quad (10)$$

These reduce, as expected, to the field equations for  $F(R)$  gravity in the special case where  $F(R, \mathbf{T}^2) = F(R)$  [3] and to the Einstein equations with a cosmological constant when  $F(R, \mathbf{T}^2) = R$ .

We will assume that the matter component can be described by a perfect fluid,

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}, \quad (11)$$

where  $\rho$  is the energy density and  $p$  the pressure; hence

$$T_{\mu\nu}T^{\mu\nu} = \rho^2 + 3p^2. \quad (12)$$

Furthermore, we take the lagrangian  $L_m = p$ . This means that the final term in the definition of  $\theta_{\mu\nu}$  vanishes and allows us to calculate the form of  $\theta_{\mu\nu}$  independently of the function  $F$ . Substituting (11) into (8), we find

$$\theta_{\mu\nu} = -(\rho^2 + 4p\rho + 3p^2)u_\mu u_\nu. \quad (13)$$

We now proceed to specify a particular form for  $F(R, \mathbf{T}^2)$  which includes and generalises the models used in [11] and for energy-momentum-squared gravity in [12] (EMSG). This form is

$$F(R, T_{\mu\nu}T^{\mu\nu}) = R + \eta(T_{\mu\nu}T^{\mu\nu})^n, \quad (14)$$

where  $n$  need not be an integer. This corresponds to EMSG in the case  $n = 1$ , and to Models A and B of [11] when  $n = 1/2$  and  $n = 1/4$ , respectively; it reduces the field equations to

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa(T_{\mu\nu} + \frac{\eta}{\kappa}(T_{\alpha\beta}T^{\alpha\beta})^{n-1} \left[ \frac{1}{2}(T_{\alpha\beta}T^{\alpha\beta})g_{\mu\nu} - n\theta_{\mu\nu} \right]), \quad (15)$$

which we rewrite as

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}^{\text{eff}}, \quad (16)$$

where  $G_{\mu\nu}$  is the Einstein tensor, to show the relationship to general relativity. Continuing with the perfect fluid form of the energy-momentum tensor, this expands to give:

$$\begin{aligned} G_{\mu\nu} + \Lambda g_{\mu\nu} = & \kappa((\rho + p)u_\mu u_\nu + pg_{\mu\nu}) \\ & + \eta(\rho^2 + 3p^2)^{n-1} \left[ \frac{1}{2}(\rho^2 + 3p^2)g_{\mu\nu} + n(\rho + p)(\rho + 3p)u_\mu u_\nu \right]. \end{aligned} \quad (17)$$

#### IV. ISOTROPIC COSMOLOGY

If we assume a FLRW universe with curvature parameter  $k$ , we find the generalised Friedmann equation,

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \kappa \frac{\rho}{3} + \frac{\eta}{3}(\rho^2 + 3p^2)^{n-1} \left[ \left( n - \frac{1}{2} \right) (\rho^2 + 3p^2) + 4n\rho p \right], \quad (18)$$



and acceleration equation

$$\frac{\ddot{a}}{a} = -\kappa \frac{\rho + 3p}{6} + \frac{\Lambda}{3} - \frac{\eta}{3}(\rho^2 + 3p^2)^{n-1} \left[ \frac{n+1}{2}(\rho^2 + 3p^2) + 2n\rho p \right]. \quad (19)$$

If the matter field obeys a barotropic equation of state,  $p = w\rho$  with  $w$  constant, then the non-GR terms are all of the form  $\rho^{2n}$  multiplied by a constant. Thus, the generalised Friedmann equation becomes

$$\left( \frac{\dot{a}}{a} \right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \kappa \frac{\rho}{3} + \frac{\eta \rho^{2n}}{3} A(n, w), \quad (20)$$

where  $A$  is a constant depending on the choice of  $n$  and  $w$ , given by

$$A(n, w) \equiv (1 + 3w^2)^{n-1} \left[ \left( n - \frac{1}{2} \right) (1 + 3w^2) + 4nw \right], \quad (21)$$

and the acceleration equation becomes

$$\frac{\ddot{a}}{a} = -\kappa \frac{1 + 3w}{6} \rho + \frac{\Lambda}{3} - \frac{\eta \rho^{2n}}{3} B(n, w), \quad (22)$$

where  $B$  a constant given by

$$B(n, w) \equiv (1 + 3w^2)^{n-1} \left[ \frac{n+1}{2} (1 + 3w^2) + 2nw \right]. \quad (23)$$

Finally, we determine the generalised continuity equation, by differentiating the generalised Friedmann equation,

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \rho (1 + w) \left[ \frac{\kappa + \eta \rho^{2n-1} n (1 + 3w)}{\kappa + 2\eta \rho^{2n-1} A(n, w)} \right], \quad (24)$$

where we have written it in a form that makes clear the generalisation of the GR case.

We can see immediately that there is an interesting difference between the FLRW equations in GR and in EMSG. When  $\eta = 0$  there are solutions with finite  $a, \dot{a}$ , and  $\rho$  but infinite values of  $p$  and  $\ddot{a}$ . These are called sudden singularities [4–6] and can be constructed explicitly. In EMSG, where  $\eta \neq 0$ , the appearance of the pressure,  $p$ , explicitly in the Friedmann equation changes the structure of the equations and the same type of sudden singularity is no longer possible at this order in derivatives of  $a$ .

### A. Integrating the Continuity Equation

We now attempt to determine the cosmological behaviour of some cases where the modified continuity equation can be integrated exactly. We find four simply integrable cases: two of these are for fixed  $w$  independent of the value of  $n$ , the other two occur for specific values of  $w$  dependent on the choice of  $n$ , although we note that some of these integrable cases may coincide, depending on our choice of the exponent,  $n$ .

The first case that can be integrated is for the equation of state corresponding to dark energy,  $w = -1$ , where the entire right-hand side of (24) vanishes, and so  $\rho \equiv \rho_0$ , a constant. In this case we expect to find a solution to the modified Friedmann equation that is the same as the solution in GR except with altered constants, which results in a de Sitter solution where  $H \equiv \frac{\dot{a}}{a} = \text{constant}$ , and the universe expands exponentially.

Next, we consider the case  $w = -\frac{1}{3}$ , which corresponds to an effective perfect fluid representing a negative curvature, so the numerator in the modified continuity equation becomes simply  $\kappa$ , and we can integrate (24) since

$$\dot{\rho} \left( \frac{1}{\rho} + \frac{2\eta A(n, -\frac{1}{3})}{\kappa} \rho^{2n-2} \right) = -2 \frac{\dot{a}}{a}, \quad (25)$$

$$\frac{d}{dt} \left( \ln \rho - \frac{\eta \left(\frac{4}{3}\right)^n}{(2n-1)\kappa} \rho^{2n-1} \right) = \frac{d}{dt} (\ln a^{-2}), \quad (26)$$

$$\rho \exp \left( -\frac{\eta \left(\frac{4}{3}\right)^n}{(2n-1)\kappa} \rho^{2n-1} \right) = C a^{-2}, \quad (27)$$

with  $C > 0$  a constant of integration.

We can also integrate the continuity equation when the correction factor in (24) is equal to 1, which occurs when

$$n(1 + 3w) = 2A(n, w). \quad (28)$$

The continuity equation then reduces to the standard GR form for these special values,  $w = w_*$ , and so we have

$$\rho = C a^{-3(1+w_*)}. \quad (29)$$

The final possibility that we consider is when

$$A(n, w) = n^2(1 + 3w), \quad (30)$$

in which case we can write (24) as

$$\frac{d}{dt}(\ln(\kappa\rho + n\eta\rho^{2n}(1 + 3w_*))) = \frac{d}{dt}(\ln a^{-3(1+w_*)}), \quad (31)$$

which integrates to

$$\kappa\rho + n\eta\rho^{2n}(1 + 3w_*) = Ca^{-3(1+w_*)}. \quad (32)$$

We note that, depending on the choice of exponent  $n$ , some of the second pair of solutions may exist for multiple choices of  $w$ , or may coincide with each other, or with the  $w = -1$ ,  $w = -\frac{1}{3}$  cases. Also, for some choices of  $n$ , there may be no solutions at all.

Finally, note that only one of these solutions allows easy integration of the modified Friedmann equation (20). This is the case when  $w = -1$  and so  $\rho = \rho_0$ . In this case the Friedmann-like equation becomes

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \alpha(\Lambda, n), \quad (33)$$

where  $\alpha$  is a constant given by

$$\alpha(\Lambda, n) \equiv \frac{\Lambda}{3} + \kappa\frac{\rho_0}{3} - \frac{\eta\rho_0^{2n}}{6}4^n. \quad (34)$$

The solution to the modified Friedmann equation is then given by

$$a(t) = \frac{1}{2\sqrt{\alpha}}(C\sqrt{\alpha} + \frac{k}{C\sqrt{\alpha}})\cosh(\sqrt{\alpha}t) \pm (C\sqrt{\alpha} - \frac{k}{C\sqrt{\alpha}})\sinh(\sqrt{\alpha}t) \quad (35)$$

where  $C$  is a new constant of integration. Equivalently, we can write this solution in terms of exponentials as

$$a(t) = \frac{1}{2\sqrt{\alpha}}\left(C\sqrt{\alpha}e^{\sqrt{\alpha}t} + \frac{k}{C\sqrt{\alpha}}e^{-\sqrt{\alpha}t}\right) \quad (36)$$

as well as its time reversal,  $t \rightarrow -t$ . Assuming  $\alpha > 0$ , we can see that this reduces to the expected de Sitter solution from general relativity in the case  $k = 0$ , as we would expect. If  $\alpha < 0$  then, writing instead  $\alpha \rightarrow -\alpha$ , there is a real solution only for negative curvature, where we must choose  $k = -C^2\alpha$ , giving the anti-de Sitter solution

$$a(t) = C\cos(\sqrt{\alpha}t). \quad (37)$$

It is important to note that because of the form of  $\alpha$ , unlike in the unmodified case, we do not necessarily require a negative cosmological constant to find this solution. We would

expect this anti-de Sitter analogue to appear whenever  $\eta > 0$ , for suitable choices of  $\rho_0$  and  $n$ .

This solution is very similar to the case of  $w = -1$  in GR, where we can rewrite the cosmological constant as a perfect fluid with this equation of state. This is possible in GR because the continuity equations for non-interacting multi-component fluids decouple, allowing us to treat them independently. Unfortunately, because of the additional non-linear terms arising in these  $F(R, \mathbf{T}^2)$  models (except in the special case  $n = 1/2$ ), we cannot decouple different fluids in this way and then subsequently superpose them in our Friedmann-like equations. This means that we cannot replace the curvature or cosmological constant terms with perfect fluids with  $w = -1/3$  and  $-1$  as in classical GR. However, for some choices of  $n$  and  $\eta$ , the correction terms can themselves provide an additional late-time or early inflationary repulsive force, removing the need for an explicit cosmological constant.

### B. Energy-momentum-squared gravity: the case $n = 1$

If we fix our choice of  $n$ , then we can say more about the behaviour of the specific solutions that arise. In what follows we consider primarily the case  $n = 1$  which was originally discussed in [12], under the name ‘energy-momentum squared gravity’. After specialising to  $n = 1$ , we can say more about the solutions to the continuity equation found in the previous section, and investigate the modified Friedmann equations. The form of the Friedmann equations, after setting  $n = 1$  in (20), (22) and (24), are:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2} = \frac{\Lambda}{3} + \kappa \frac{\rho}{3} + \frac{\eta \rho^2}{6} (3w^2 + 8w + 1) \quad (38)$$

$$\frac{\ddot{a}}{a} = \frac{\Lambda}{3} - \kappa \frac{1+3w}{6} \rho - \frac{\eta \rho^2}{3} (3w^2 + 2w + 1) \quad (39)$$

$$\dot{\rho} = -3 \frac{\dot{a}}{a} \rho (1+w) \frac{\kappa + \eta \rho (1+3w)}{\kappa + \eta \rho (3w^2 + 8w + 1)} \quad (40)$$

The new terms in the Friedmann equations are quadratic in the energy density, which we would expect to dominate in the very early universe as  $\rho \rightarrow \infty$ . Additionally, if we choose  $\eta < 0$ , then the modified Friedmann equations in this model are similar to the effective Friedmann equations arising in Loop Quantum Cosmology, [16], where

$$\left(\frac{\dot{a}}{a}\right) = \frac{\kappa}{3} \rho \left(1 - \frac{\rho}{\rho_{\text{crit}}}\right), \quad (41)$$

which may warrant further investigation. An analogous higher-order effect occurs in brane world cosmologies, where there is an effective equation of state with [18–21]

$$p^{eff} = \frac{1}{2\lambda}(\rho^2 + 2p\rho); \lambda > 0 \text{ constant.} \quad (42)$$

We briefly summarise the values of  $w$  for which the results of the previous section allow us to integrate the Friedmann equation and find the values of  $w$  that satisfy equations (28) and (30). If we set  $n = 1$  then (28) reduces to

$$3w^2 + 5w = 0, \quad (43)$$

which has solutions  $w = -\frac{5}{3}$  and  $w = 0$ . The  $w = 0$  solution describes ‘dust’ matter. The case  $w = -5/3$  corresponds to some form of phantom energy, which will result in a Big Rip singularity, [17], at finite future time.

Alternatively, solving (30) for  $n = 1$  gives

$$3w^2 + 2w - 1 = 0, \quad (44)$$

which has solutions  $w = -1$  and  $w = \frac{1}{3}$ . The first of these has already been found for all  $n$  as the first case above, whilst the second gives a solution corresponding to black body radiation. Hence, we have exact solutions to the continuity equation for the cases  $w = \{-\frac{5}{3}, -1, -\frac{1}{3}, 0, \frac{1}{3}\}$  which include the physically important cases of dust and radiation.

The equation of state  $p = 0$  corresponds to pressureless dust or non-relativistic cold dark matter, and as shown above, we recover the same dependence of energy density on the scale factor as in the GR case,

$$\rho = Ca^{-3}. \quad (45)$$

If we combine this with the modified acceleration and Friedmann equations for  $w = 0$  we find

$$a\ddot{a} + 2\dot{a}^2 + k = \frac{\Lambda}{2}a^2 + \frac{\kappa}{4C}a^{-1}. \quad (46)$$

If we consider only flat space ( $k = 0$ ) then we find

$$a(t) = (4\Lambda)^{-\frac{1}{3}}((C^2 + D + 1)\cosh\left(\sqrt{\frac{3\Lambda}{2}}t\right) + (C^2 + D - 1)\sinh\left(\sqrt{\frac{3\Lambda}{2}}t\right) - 2C)^{\frac{1}{3}}, \quad (47)$$

where  $D$  is a constant of integration, and we have eliminated a further constant by a covariant translation of the time coordinate. We can then find  $\rho$  explicitly, using (45). We

can see, however, that this form of the solution does not capture the case  $\Lambda = 0$ . In this case, instead we find the solution

$$a(t) = \left(\frac{3}{8C}\right)^{\frac{1}{3}} (C^2 t^2 - 16D)^{\frac{1}{3}}, \quad (48)$$

which gives the GR dust behaviour of  $a \sim t^{\frac{2}{3}}$  at large  $t$ .

In the case of  $w = -\frac{1}{3}$ , we can write

$$\rho \exp\left(-\frac{4\eta}{3\kappa}\rho\right) = C a^{-2}. \quad (49)$$

After differentiation and multiplication by  $a^2$ , we can write

$$\frac{\dot{a}}{a} = \frac{\dot{\rho}}{\rho} \left(1 - \frac{4\eta}{3\kappa}\rho\right) \quad (50)$$

and so in the case  $k = 0$  we can write the Friedmann equation in terms of  $\rho$  without any exponentials, as

$$\left(\frac{\dot{\rho}}{\rho}\right)^2 \left(1 - \frac{4\eta}{3\kappa}\rho\right)^2 \frac{1}{4C^2} = \frac{\Lambda}{3} + \frac{\kappa}{3}\rho - \frac{2\eta}{9}\rho^2. \quad (51)$$

Finally, in the case of  $w = \frac{1}{3}$ , which corresponds to radiation, [12] gave a solution in the case of flat space,  $a(t) \propto \sqrt{\cosh(\alpha t)}$  where  $\alpha \equiv \sqrt{\frac{4\Lambda}{3}}$ . We see that in this case we can write the continuity equation as

$$\kappa\rho + 2\eta\rho^2 = C a^{-4}, \quad (52)$$

and that in the Friedmann and acceleration equations, the density terms are of equal magnitude but opposite sign. We can then sum the two to find our equation for  $a(t)$

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a} + \frac{k}{a^2} = \frac{2\Lambda}{3}, \quad (53)$$

which we solve by use of the substitution  $y = a^2$  to find

$$a^2(t) = \frac{1}{4\Lambda} ((1 + 9k^2 - 12\Lambda D) \cosh(\alpha t) + (1 - 9k^2 + 12\Lambda D) \sinh(\alpha t) + 6k) \quad (54)$$

for all  $\Lambda$ ,  $k$  non-zero, with  $\alpha \equiv \sqrt{\frac{4\Lambda}{3}}$ , as above. In the  $\Lambda = 0$  subcase, we find the solutions

$$a^2(t) = \begin{cases} Dt - kt^2 & k \neq 0 \\ Dt & k = 0 \end{cases}. \quad (55)$$

### C. De Sitter-like solutions

De Sitter solutions arise in EMSG theory. They have constant density and Hubble parameter, which includes the case  $w = -1$ . In  $\Lambda$ CDM we expect this to arise in two situations. The first is when we have  $\rho \equiv 0$ , that is an empty universe whose expansion is controlled solely by  $\Lambda$ , and the second is the similar dark-energy equation of state  $w = -1$  for which the perfect fluid behaves as a cosmological constant. In EMSG we find there is an extra family of de Sitter solutions. We describe them first for general  $n$ , then specialise to EMSG.

Since we are searching for solutions with  $H \equiv H_0$  and  $\rho \equiv \rho_0$ , from (20) we must have  $k = 0$ , and the Friedmann equation then reduces to an algebraic one for  $H^2$  in terms of  $\rho_0$ . Similarly, since  $\dot{H} = 0$ , (22) reduces to another relation for  $H^2$ . Equating the two to remove  $H^2$  and simplifying, we find that  $\rho_0$  must satisfy

$$\rho_0(1+w)(\kappa + n\eta(1+3w^2)^{n-1}(1+3w)\rho_0^{2n-1}) = 0. \quad (56)$$

There are the two standard solutions,  $w = -1$  and  $\rho_0 = 0$ , but the additional factor gives us another family of solutions, with

$$\rho_0^{2n-1} = -\frac{\kappa}{n\eta(1+3w^2)^{n-1}(1+3w)}. \quad (57)$$

In the case of EMSG, when we choose  $n = 1$ , this condition reduces to

$$\rho_0 = -\frac{\kappa}{\eta(1+3w)} \quad (58)$$

which gives us a constant density, exponentially expanding solution for every equation of state,  $w$ , excluding  $w = -\frac{1}{3}$ , for an appropriate sign of  $\eta$ . The existence of this extra de Sitter solution is reminiscent of its appearance in GR cosmologies with bulk viscosity [15]

This unusual situation suggests that we investigate the stability of these  $n = 1$  solutions. We consider a linear perturbation about the constant density solution by writing

$$\rho = \rho_0(1 + \delta), \quad (59)$$

$$H = H_0(1 + \epsilon), \quad (60)$$

The perturbed continuity equation is then given by

$$\rho_0 \dot{\delta} = -3(1+w)H_0(1+\epsilon)\rho_0(1+\delta) \frac{\kappa + \eta\rho_0(1+\delta)(1+3w)}{\kappa + \eta\rho_0(1+\delta)(3w^2 + 8w + 1)}, \quad (61)$$

If we use the expression for  $\rho_0$  given in (58), we can reduce this to

$$\dot{\delta} = -3(1+w)(1+3w)H_0(1+\epsilon)(1+\delta)\delta \frac{1}{(3w^2+5w)(1+\frac{3w^2+8w+1}{3w^2+5w}\delta)}. \quad (62)$$

From the perturbation of the modified Friedmann equation we find that  $\epsilon \sim \delta$  which means that after expanding to first order in  $\delta$ , we have

$$\dot{\delta} = -3H_0\delta \frac{(1+w)(1+3w)}{(3w+5)w}, \quad (63)$$

so small perturbations evolve as

$$\delta \propto \exp\left(-3H_0 \frac{(1+w)(1+3w)}{(3w+5)w}\right) \quad (64)$$

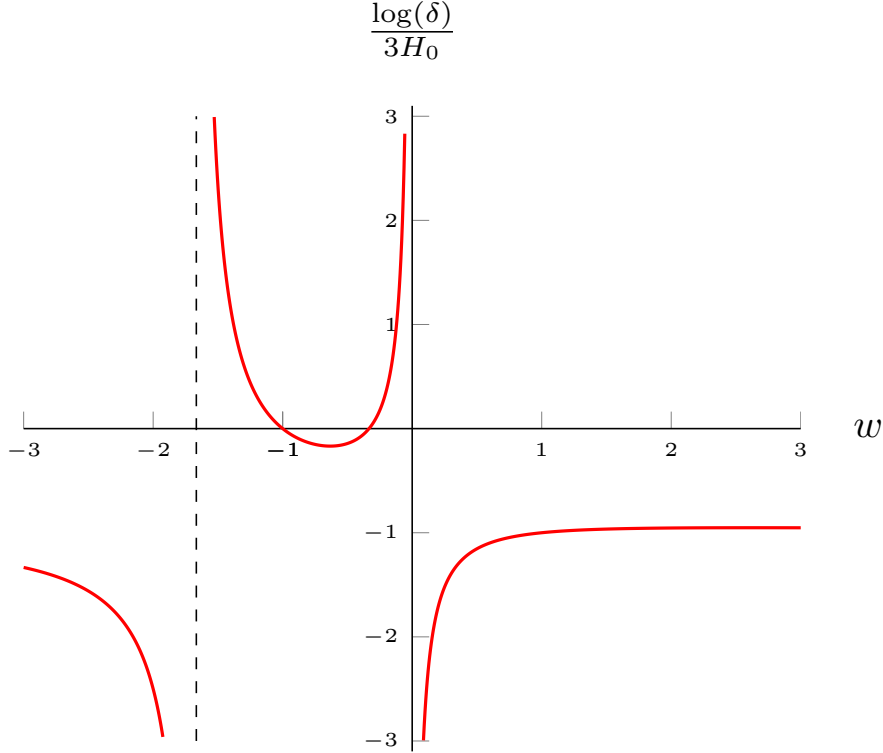


FIG. 1. The plot shows the value of the exponent, and hence the stability of the solutions, in (64) (where we have divided by  $3H_0$ ). The asymptotes are found at  $w = -\frac{5}{3}$  and  $w = 0$ , whilst the zeroes are at  $w = -1$  and  $w = -\frac{1}{3}$ . The solutions will be stable for values of  $w$  where the graph is negative, and unstable otherwise.

The exponent in (64) is plotted in Figure 1, where we can see that these de Sitter-like solutions are indeed stable for a wide range of  $w$  values. This gives us an exponentially



expanding universe for (almost) any equation of state as long as we set the density to the correct constant value. In particular, these solutions will be stable for  $w < -\frac{5}{3}$ ,  $-1 < w < -\frac{1}{3}$  and  $w > 0$ , and unstable for  $-\frac{5}{3} < w < -1$  and  $-\frac{1}{3} < w < 0$ . It is also the case that, depending on the sign of the parameter  $\eta$ , some of these solutions will be unphysical, as they require negative energy density. For  $\eta < 0$ , there will be no physical solutions for  $w < -\frac{1}{3}$ , whilst for  $\eta > 0$  there will be no solutions for  $w > -\frac{1}{3}$ .

#### D. Early times: the bounce and high-density limits

Examining the modified Friedmann equation (38) in the case  $k \geq 0$ , we can see that as the left-hand side of the equation is a sum of positive terms, we must have

$$\Lambda + \kappa\rho + \eta\rho^2 A(1, w) \geq 0, \quad (65)$$

which can be split into two cases, for  $\eta A(1, w) < 0$  and  $\eta A(1, w) > 0$ , respectively. The first case occurs for

$$\eta < 0 \quad \text{and} \quad \{w < \alpha_- \text{ or } w > \alpha_+\}, \quad (66)$$

$$\eta > 0 \quad \text{and} \quad \{\alpha_- < w < \alpha_+\}, \quad (67)$$

where

$$\alpha_{\pm} = -\frac{-4 \pm \sqrt{13}}{3} \quad (68)$$

are the roots of

$$A(1, w) \equiv 3w^2 + 8w + 1 = 0. \quad (69)$$

In this case we have a maximum possible density given by

$$\rho_{max} = \frac{\kappa}{2A(1, w)\eta} \left( -1 + \sqrt{1 - \frac{4\eta\Lambda A(1, w)}{\kappa^2}} \right), \quad (70)$$

indicating that a bounce occurs in this case, avoiding an initial singularity. In the second case, where  $\eta A(1, w) > 0$ , there is no bounce and no maximum energy density.

We now consider the solutions when  $k = 0$  in the high-density limit, where we assume the correction terms dominate over the  $\rho$  and  $\Lambda$  terms. We consider the case of general  $n$ ,

and find an analytic solution. The Friedmann and acceleration equations reduce to

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{\eta}{3}\rho^{2n}A(n, w), \quad (71)$$

$$\frac{\ddot{a}}{a} = -\frac{\eta}{3}\rho^{2n}B(n, w). \quad (72)$$

From these, we can eliminate  $\rho$  to find

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{A(n, w)}{B(n, w)}\frac{\ddot{a}}{a} = 0. \quad (73)$$

which has the solution

$$a(t) = D[(A + B)t - C]^{\frac{A}{A+B}} \quad (74)$$

where  $C$  and  $D$  are new constants of integration. We can then solve for the density:

$$\rho(t) = \left(\frac{3A}{\eta}\right)^{\frac{1}{2n}} ((A + B)t - C)^{-\frac{1}{n}}. \quad (75)$$

This solution is real (and thus not unphysical) only if  $A(n, w)/\eta$  is positive. In the case of EMSG, this condition reduces to the requirement that  $\eta$  and  $3w^2 + 8w + 1$  must have the same sign. So, the two regions where this solution exists are,

$$\eta > 0 \quad \text{and} \quad \{w < \alpha_- \text{ or } w > \alpha_+\}, \quad (76)$$

$$\eta < 0 \quad \text{and} \quad \{\alpha_- < w < \alpha_+\}, \quad (77)$$

where

$$\alpha_{\pm} = -\frac{-4 \pm \sqrt{13}}{3} \quad (78)$$

are the roots of  $3w^2 + 8w + 1$ . These are complementary to the conditions for the bounce to occur, as previously discussed. This is as we would expect, with the high-density approximation failing at a maximum density, as in the case of a bounce.

## V. ANISOTROPIC COSMOLOGY

There are several ways of introducing anisotropy into our cosmological models. We will consider the simplest generalisation of FLRW, in which we have a flat, spatially homogeneous universe, with anisotropic scale factors. This is the Bianchi type  $I$  universe, with metric given by [28]

$$ds^2 = -dt^2 + a^2(t)dx^2 + b^2(t)dy^2 + c^2(t)dz^2, \quad (79)$$

where  $a(t)$ ,  $b(t)$  and  $c(t)$  are the expansion scale factors in the  $x, y$  and  $z$  directions, respectively.

Assuming that the energy-momentum tensor takes the form of a perfect fluid with principal pressures,  $p_1$ ,  $p_2$  and  $p_3$ , so  $L_m = \frac{1}{3}(p_1 + p_2 + p_3)$ , we can derive the field equations for Bianchi I universes in our higher-order matter theories:

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}\dot{a}}{ca} = \kappa\rho + \frac{\eta}{6}(\rho^2 + \sum_{i=1}^3 p_i^2)^{n-1} \left[ (6n-3)\rho^2 + 8n\rho \sum_{i=1}^3 p_i + 2n(\sum_{i=1}^3 p_i)^2 - 3 \sum_{i=1}^3 p_i^2 \right], \quad (80)$$

$$\frac{\dot{b}\dot{c}}{bc} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -\kappa p_1 + \frac{\eta}{6}(\rho^2 + \sum_{i=1}^3 p_i^2)^{n-1} \left[ 2n(\rho + p_1 - p_2 - p_3)(2p_1 - p_2 - p_3) - 3 \sum_{i=1}^3 p_i^2 \right], \quad (81)$$

$$\frac{\dot{c}\dot{a}}{ca} + \frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} = -\kappa p_2 + \frac{\eta}{6}(\rho^2 + \sum_{i=1}^3 p_i^2)^{n-1} \left[ 2n(\rho + p_2 - p_3 - p_1)(2p_2 - p_3 - p_1) - 3 \sum_{i=1}^3 p_i^2 \right], \quad (82)$$

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} = -\kappa p_3 + \frac{\eta}{6}(\rho^2 + \sum_{i=1}^3 p_i^2)^{n-1} \left[ 2n(\rho + p_3 - p_1 - p_2)(2p_3 - p_1 - p_2) - 3 \sum_{i=1}^3 p_i^2 \right]. \quad (83)$$

In the case of an isotropic pressure fluid ( $p_1 = p_2 = p_3 = p$ ):

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\dot{b}\dot{c}}{bc} + \frac{\dot{c}\dot{a}}{ca} = \kappa\rho + \frac{\eta}{2}(\rho^2 + 3p^2)^{n-1}((2n-1)\rho^2 + 8n\rho p + (6n-3)p^2), \quad (84)$$

$$\frac{\dot{b}\dot{c}}{bc} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = -\kappa p - \frac{\eta}{2}(\rho^2 + 3p^2)^{n-1}3p^2, \quad (85)$$

$$\frac{\dot{c}\dot{a}}{ca} + \frac{\ddot{c}}{c} + \frac{\ddot{a}}{a} = -\kappa p - \frac{\eta}{2}(\rho^2 + 3p^2)^{n-1}3p^2, \quad (86)$$

$$\frac{\dot{a}\dot{b}}{ab} + \frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} = -\kappa p - \frac{\eta}{2}(\rho^2 + 3p^2)^{n-1}3p^2. \quad (87)$$

The first of these is the generalised Friedmann equation.

Qualitatively, we expect that the higher-order density and pressure terms will dominate at early times to modify or remove (depending on the sign of  $\eta$ ) the initial singularity when  $n > 1/2$ , but will have negligible effects at late times, when the dynamics will approach the flat isotropic FLRW model. At early times, we know that in GR the singularity will be anisotropic and dominated by shear anisotropy whenever  $-\rho/3 < p < \rho$ . In order to determine the dominant effects as  $t \rightarrow 0$  we will simplify to the case of isotropic perfect fluid pressures ( $p_1 = p_2 = p_3 = w\rho$ ). Now, we determine the dependence of the highest-order matter terms on the scale factors,  $a, b$  and  $c$  from the generalisation of the conservation equation (24) with an anisotropic metric (79). For the case with general  $n$ , this is

$$\dot{\rho} = - \left( \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \right) \rho(1+w) \left[ \frac{\kappa + \eta\rho^{2n-1}n(1+3w)}{\kappa + 2\eta\rho^{2n-1}A(n,w)} \right], \quad (88)$$

and so the behaviour of the density is just

$$\rho \propto (abc)^{-\Gamma}, \quad (89)$$

where

$$\Gamma(n, w) = (1+w) \left[ \frac{\kappa + \eta\rho^{2n-1}n(1+3w)}{\kappa + 2\eta\rho^{2n-1}A(n,w)} \right]. \quad (90)$$

The higher-order density terms will dominate the evolution at early times when  $n > 1/2$  and we see that, in these cases,  $\Gamma$  is independent of  $\rho$  and  $\eta$  as  $\rho \rightarrow \infty$ , since in this limit,

$$\Gamma(n, w) \rightarrow \frac{n(1+3w)(1+w)}{2A(n, w)}. \quad (91)$$

In the cosmology obtained by setting  $n = 1$  in (80)-(83) we will have domination by the nonlinear matter terms, which will drive the expansion towards isotropy as  $t \rightarrow 0$  if  $\rho^2$  diverges faster than  $(abc)^{-2}$  as  $abc \rightarrow 0$ . Thus, the condition for an isotropic initial singularity in  $n = 1$  theories is that  $\Gamma(1, w) > 2$ , or

$$\frac{(1+3w)(1+w)}{2A(1, w)} > 2 \quad (92)$$

When this condition holds as  $t \rightarrow 0$ , the dynamics will approach the flat FLRW metric with

$$a(t) \propto b(t) \propto c(t) \propto t^{2/\Gamma(1, w)}. \quad (93)$$

When  $\Gamma(1, w) < 2$ , the dynamics will approach the vacuum Kasner metric with

$$(a, b, c) = (t^{q_1}, t^{q_2}, t^{q_3}), \quad (94)$$

$$\sum_{i=1}^3 q_i = \sum_{i=1}^3 q_i^2 = 1. \quad (95)$$

This condition simplifies to four cases:

$w > 0$	anisotropic singularity
$\alpha_+ < w < 0$	isotropic singularity
$\alpha_- < w < \alpha_+$	anisotropic singularity
$w < \alpha_-$	isotropic singularity

Here, the constants  $\alpha_+$  and  $\alpha_-$  take the values determined earlier in (78)).

In general, for arbitrary  $n$ , the higher-order correction terms on the right-hand side of the field equations (17) are proportional to  $\eta\rho^{2n}$  when  $p = w\rho$ , and so the condition for an isotropic singularity as  $t \rightarrow 0$  becomes

$$\Gamma(n, w) > 2n, \quad (96)$$

and the dynamics approach

$$a(t) \propto b(t) \propto c(t) \propto t^{2/\Gamma(n, w)}. \quad (97)$$

The case for general  $n$  and  $w$  is problematic to simplify succinctly due to the exponential dependence on  $n$ . However, we can consider specific physically relevant equations of state individually.

For dark energy ( $w = -1$ ) and curvature ( $w = -\frac{1}{3}$ ) ‘fluids’, we find that  $\Gamma(n) = 0$ , for all  $n$ , and so the condition for an isotropic initial singularity will depend only on whether  $n$  itself is positive or negative.

For  $w = 0$ , dust, we find

$$\Gamma(n, 0) = \frac{n}{2n-1}, \quad (98)$$

for  $n \neq \frac{1}{2}$ . which leads to isotropy only when  $\frac{1}{2} < n < \frac{3}{4}$ .

For radiation,  $w = \frac{1}{3}$ , an isotropic singularity will occur if

$$\frac{n}{(\frac{4}{3})^{n-1}(2n - \frac{1}{2})} > 2n, \quad (99)$$

whilst for  $w = 1$  we find that the condition for isotropy is

$$\frac{n}{4^{n-1}(2n - \frac{1}{2})} > 2n. \quad (100)$$

In both the latter cases, we require  $n \neq \frac{1}{4}$ .

A similar effect will occur in more general anisotropic universes, like those of Bianchi type  $VII_h$  or  $IX$ , which are the most general containing open and closed FLRW models, respectively. In type  $IX$ , the higher-order matter terms will prevent the occurrence of chaotic behaviour with  $w < 1$  fluids on approach to an initial or final singularity in a  $\mathbf{T}^{2n}$  theory when  $n > 1$ . Thus we see that in these theories the general cosmological behaviour on approach to an initial and (in type  $IX$  universes) final singularity is expected to be isotropic in the wide range cases we have determined, when  $\Gamma(n, w) > 2n$ . This simplifying effect of adding higher-order effects can also be found in the study of other modifications to GR, for example those produced by the addition of quadratic  $R_{ab}R^{ab}$  terms to the gravitational lagrangian, [29, 30]. These also render isotropic singularities stable for normal matter (unlike in GR). If  $T_{ab}$  is not perfect fluid but has anisotropic terms (for example, because of a magnetic field or free streaming gravitons [27]) they will add higher-order anisotropic stresses.

## VI. CONCLUSIONS

We have considered a class of theories which generalise general relativity by adding higher-order terms of the form  $(T^{\mu\nu}T_{\mu\nu})^n$  to the matter lagrangian, in contrast to theories which add higher-order curvature terms to the Einstein-Hilbert lagrangians, as in  $f(R)$  gravity theories. The family of theories which lead phenomenologically to higher-order matter contributions to the classical gravitation field equations of the sort studied here includes loop quantum gravity, and bulk viscous fluids, K-essence, or brane-world cosmologies in GR. This generalisation of the matter stresses is expected to create changes in the evolution of simple cosmological models at times when the density or pressure is high but to recover the predictions of general relativistic cosmology at late times in ever-expanding universes where the density is small. However, we find that there is a richer structure of behaviour if we generalise GR by adding arbitrary powers of the scalar square of the energy-momentum tensor to the action. In particular, we find a range of exact solutions for isotropic universes, discuss their behaviours with reference to the early and late-time evolution, accelerated expansion,

and the occurrence or avoidance of singularities. Finally, we discuss extensions to the simplest anisotropic cosmologies and delineated the situations where the higher-order matter terms will dominate over the anisotropic stresses on approach to cosmological singularities. This leads to a situation where the general cosmological solutions of the field equations for our higher-order matter theories are seen to contain isotropically expanding universes, in complete contrast to the situation in general relativistic cosmologies. In future work we will discuss the observational consequences of higher-order stresses for astrophysics.

*Acknowledgements* The authors are supported by the Science and Technology Funding Council (STFC) of the UK and would like to thank T. Harko and O. Akarsu for helpful comments.

- 
- [1] J. D. Barrow and D. J. Shaw, Phys. Rev. Lett. **106**, 101302 (2011)
  - [2] E. J. Copeland, M. Sami, and S. Tsujikawa, Int. J. Mod. Phys. D **15**, 1753, (2006).
  - [3] A. D. Felice and S. Tsujikawa, Living Rev. Rel. **13**, 3 (2010).
  - [4] J.D. Barrow, G. Galloway and F.J. Tipler, Mon. Not. Roy. astron. Soc. **223**, 835 (1986)
  - [5] J.D. Barrow, Class. Quantum Gravity **21**, L79 (2004)
  - [6] J.D. Barrow, Class. Quantum Grav. **21**, 5619 (2004)
  - [7] A. Paliathanasis, J. D. Barrow, and P. G. L. Leach, Phys. Rev. D **94**, 023525 (2016)
  - [8] T. Clifton, P. G. Ferreira, A. Padilla, and C. Skordis, Phys. Reports., **513**, 1 (2012).
  - [9] T. Harko and F. S. N. Lobo, Eur. Phys. J. C **70**, 373 (2010)
  - [10] T. Harko, F. S. N. Lobo, S. Nojiri, and S. D. Odintsov, Phys. Rev. D **84**, 024020 (2011)
  - [11] N. Katirci and M. Kavuk, Eur. Phys. J. Plus **129**, 163, (2014)
  - [12] M. Roshan and F. Shojai, Phys. Rev. D **94**, 044002 (2016)
  - [13] J.D. Barrow, Phys. Lett. B **235**, 40 (1990)
  - [14] S. Nojiri and S. D. Odintsov, Phys. Rev. D **72**, 023003 (2005)
  - [15] J.D. Barrow, Nucl. Phys. B **310**, 743 (1988)
  - [16] M. Bojowald, Living Rev. Rel. **8**, 11 (2005), A. Ashtekar, T. Pawłowski and P. Singh, Phys. Rev. D **73**, 124038 (2006), A. Ashtekar, T. Pawłowski and P. Singh, Phys. Rev. Lett. **96**, 141301 (2006), A. Ashtekar, T. Pawłowski and P. Singh, Phys. Rev. D **74**, 084003 (2006), A. Ashtekar, T. Pawłowski, P. Singh and K. Vandersloot, Phys.

- Rev. D **75**, 024035 (2007), K. Vandersloot, Phys. Rev. D **75**, 023523 (2007), T. Cailleteau, A. Cardoso, K. Vandersloot and D. Wands, Phys. Rev. Lett. **101**, 251302 (2008), A. Ashtekar and P. Singh, Class. Quant. Grav. **28**, 213001 (2011)
- [17] R.R. Caldwell, M. Kamionkowski, and N.N. Weinberg, Phys. Rev. Lett. **91**, 071301 (2003)
- [18] J.D. Barrow and S. Hervik, Class. Quant. Grav. **19**, 155 (2002)
- [19] J.D. Barrow and R. Maartens, Phys. Lett B **532**, 153 (2002)
- [20] A.A. Coley, arXiv:gr-qc/0312073, P. Dunsby, N. Goheer, M. Bruni, and A.A. Coley, Phys. Rev. D **69**, 101303 (2004)
- [21] A.A. Coley, Y. He, W. C. Lim, Class. Quant. Grav. **21**, 1311 (2004)
- [22] P. Brax and C. van de Bruck, Class. Quant. Grav. **20**, R201 (2003)
- [23] P. Singh and E. Wilson-Ewing, Class. Quant. Grav. **31**, 035010 (2014)
- [24] B. Gupta and P. Singh, Phys. Rev. D **85**, 044011 (2012)
- [25] B. Li and J.D. Barrow, Phys. Rev. D **79**, 103521 (2009)
- [26] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and Other Topological Defects*, (Cambridge UP, Cambridge, 1994)
- [27] J.D. Barrow, Phys. Rev. D **55**, 7451 (1997)
- [28] L. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th rev. edn., (Pergamon, Oxford, 1974)
- [29] J.D. Barrow and J. Middleton, Phys. Rev. D **75**, 123515 (2007)
- [30] J.D. Barrow and J. Middleton, Phys. Rev. D **77**, 103523 (2008)
- [31] X. Liu, T. Harko and S. D. Liang, Eur. Phys. J. C **76** (2016) no.8, 420
- [32] O. Akarsu, N. Katirci and S. Kumar, arXiv:1709.02367 [gr-qc].
- [33] A. R. Liddle, A. Mazumdar and J.D. Barrow, Phys. Rev. D **58**, 027302 (1998)
- [34] X. Chen and M. Kamionkowski, Phys. Rev. D **60**, 104036 (1999)