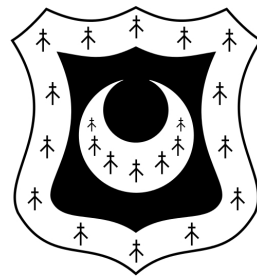


# Indices for Supersymmetric Quantum Field Theories in Four Dimensions



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## Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. No material in this thesis is part of any other thesis that I have submitted or will submit for a degree at this or any other university.

Mathieu Ehrhardt  
September 30, 2011

To my parents.

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# Abstract

In this thesis, we investigate four dimensional supersymmetric indices. The motivation for studying such objects lies in the physics of Seiberg's electric-magnetic duality in supersymmetric field theories. In the first chapter, we first define the index and underline its cohomological nature, before giving a first computation based on representation theory of free superconformal field theories. After listing all representations of the superconformal algebra based on shortening conditions, we compute the associated Verma module characters, from which we can extract the index in the appropriate limit. This approach only provides us with the free field theory limit for the index and does not account for the values of the  $R$ -charges away from free field theories. To circumvent this limitation, we then study a theory on  $\mathbb{R} \times S^3$  which allows for a computation of the superconformal index for multiplets with non-canonical  $R$ -charges. We expand the fields in harmonics and canonically quantise the theory to analyse the set of quantum states, identifying the ones that contribute to the index. To go beyond free field theory on  $\mathbb{R} \times S^3$ , we then use the localisation principle to compute the index exactly in an interacting theory, regardless of the value of the coupling constant. We then show that the index is independent of a particular geometric deformation of the underlying manifold, by squashing the sphere. In the final chapter, we show how the matching of the index can be used in the large  $N$  limit to identify the  $R$ -charges for all fields of the electric-magnetic theories of the canonical Seiberg duality. We then conclude by outlining potential further work.

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# Chapter 1

## Introduction

### 1.1 Dualities and Strong Coupling

Over the recent years, there has been a renewed interest in the study of higher dimensional conformal field theories. This is largely due in part to the discovery of dualities relating conformal field theories together [1, 2, 3, 4, 5] and also with gravitational theories in higher dimensions [6, 7, 8, 9]. The former is Seiberg's electro-magnetic duality, which relates two field theories at a common infrared conformal fixed points, while the second is the Maldacena's AdS/CFT correspondence for  $\mathcal{N} = 4$  supersymmetric theories in 4 dimensions and more recently has been extended to ABJM [7] and BLG [8, 9] theories in 3 dimensions. From the point of view of a physicist, these dualities are of paramount importance because they provide a potential handle for understanding strongly coupled systems.

Here, we will mainly be concerned with the calculation of indices in 4 dimensional supersymmetric field theories. The index is a generalisation of the Witten Index [10] to theories invariant under the superconformal group [11]. Indices are important in the study of dualities because they are computable at all values of the coupling constant, as they are invariant under continuous deformations of the theory, and are hence exactly computable at strong coupling, both perturbatively and non-perturbatively [12]. For this reason, they constitute a powerful tool in the study of dualities .

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### 1.2 Seiberg-duality

Seiberg duality [1] is an electro-magnetic duality giving two different and equivalent descriptions of the same physical reality in the infrared [13]. Concretely, the quarks and gluons in one theory correspond to solitonic objects on the other side of the duality. Importantly, the duality does not hold at all energy scales [14]. Hence, it can be seen as a weak low-energy realisation of the duality postulated by Montonen-Olive [15, 16, 17]. This duality is thought to be exactly realised for the full  $\mathcal{N} = 2, 4$  supersymmetric theories [18, 19, 20] in the form of  $S$ -duality<sup>1</sup>.

The canonical example of Seiberg duality relates an SQCD theory [21, 22] with the following anomaly free global symmetry,

$$H = SU(N_f) \times SU(N_f) \times U(1)_B \times U(1)_R \quad (1.1)$$

and  $SU(N_c)$  gauge group within the conformal window,

$$\frac{3}{2}N_c \leq N_f \leq 3N_c, \quad (1.2)$$

to a theory with  $SU(\tilde{N}_c)$  gauge group, and the same global symmetry, where,

$$\tilde{N}_c = N_f - N_c. \quad (1.3)$$

The electric  $SU(N_c)$  gauge theory has the content,

Field	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
$Q$	$f$	$f$	1	1	$\tilde{N}_c/N_f$
$\tilde{Q}$	$\bar{f}$	1	$\bar{f}$	-1	$\tilde{N}_c/N_f$
$\lambda$	adj.	1	1	0	1

Figure 1.1: Seiberg Electric Theory

while the  $SU(\tilde{N}_c)$  theory has the following content,

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<sup>1</sup>Although Montonen-Olive duality was postulated with a  $\mathbb{Z}_2$  transformation of the coupling constant,  $\mathcal{N} = 2, 4$  actually exhibit  $S$ -duality which involves an  $Sl(2, \mathbb{Z})$  transformation for the coupling constant complexified to include the theta angle  $\tau = \frac{1}{2\pi}\theta + \frac{1}{g^2}4\pi i$ .

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Field	$SU(\tilde{N}_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
$q$	$f$	$\bar{f}$	1	$N_c/\tilde{N}_c$	$N_c/N_f$
$\tilde{q}$	$\bar{f}$	1	$f$	$-N_c/\tilde{N}_c$	$N_c/N_f$
$\tilde{\lambda}$	adj.	1	1	0	1
$M$	1	$f$	$\bar{f}$	0	$2\tilde{N}_c/N_f$

Figure 1.2: Seiberg Magnetic Theory

One of the most non trivial tests of the duality is 't Hooft anomaly matching [23]. It was initially proposed as a consistency test for equivalent descriptions of strongly coupled QCD-like theories. It is based on the idea that one should be able to describe a theory both in terms of constituents and composite fields. Assuming an asymptotically free theory, with a global symmetry group  $H$ , one can compute the Adler-Bardeen-Jackiw [24] triangle anomaly for three  $H$  currents in the ultraviolet, with fundamental fermions – quarks – running in the loop. One can then weakly gauge the theory, with a coupling constant  $g \ll 1$ , and turning on an  $H$  background gauge field. Should the anomaly be non zero, one can add fermionic spectators coupling through  $H$  only to cancel the anomaly. One can then consider the effective theory below the strong interaction scale, and provided the symmetry  $H$  is not spontaneously broken and we keep the spectator fermions, the anomaly should still cancel with composite fermions running in the loop. If the symmetry is spontaneously broken, there are some other non-trivial results, but they have to do with the appearance of Goldstone bosons [25]<sup>1</sup>.

However non-trivial, this check is not sufficient in itself to have any degree of confidence that two theories are in fact dual [28]. Further non-trivial tests of the proposed duality are required.

- Supersymmetry is a powerful requirement in itself. Anomaly matching otherwise only constrains the fermionic spectrum of the theory [28], as only

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<sup>1</sup>In [26], one should note that although the  $U(1)_R$  symmetry is part of the anomaly free global symmetry group, a non-zero anomaly coefficient is matched in the electric and magnetic theories. There is no inconsistency, as the anomaly is calculated by turning on a non-physical background  $R$ -charge gauge field which allows for a non-zero anomaly result. The same holds for the Weyl anomaly [27], which only implies a breaking of conformal symmetry when a conformal theory is coupled with a background gravitational field.

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fermions run through the loop of the triangle anomaly.

- Preservation of any chiral symmetry is another requirement [28]. For Seiberg dual theories, this requirement is satisfied because both theories are conformal. In such a case massless composite fermions at low energy saturate the 't Hooft anomaly matching condition. However, in the case of a QCD-like confining theory in the infrared, Goldstone bosons associated with the breaking of chiral symmetry can essentially contribute to the 't Hooft anomaly matching constraint through their Wess-Zumino term. One could imagine a partial breaking of chiral symmetry, but QCD like theories do not break there vector-like theories [29], so either  $SU(N_f) \times SU(N_f)$  is unbroken or it goes to the diagonal or it breaks to the diagonal  $SU(N_f)_V$ .
- Finally, one should be able to consistently decouple any number of flavours by giving them a mass and integrating them out of the lagrangian. This should be consistently accounted for on both sides of the duality. This is the case for Seiberg duality [30, 26].

### 1.3 Index and Seiberg Duality

As was previously mentioned, the study of dualities is challenging because of the inevitable strong coupling hurdles, and 't Hooft anomaly matching has limitations. Hence one needs to define other quantities which, without being completely information free, do not depend on the full details of the interacting quantum theory, and are ultimately reliable regardless of the strength of the coupling. Indices provide us with such quantities, because of their topological properties, as they only depend on a set of states protected against perturbative corrections by shortening conditions for the supersymmetry algebra. Also non-perturbative corrections can in principle contribute to the index, however, topological considerations in four dimensions prevent them from contributing unlike in three dimensions, where monopoles contribute absolutely [12].

The superconformal index was initially defined [11] as a generalisation of the Witten Index [10] to theories invariant under the full superconformal group. The Witten index is an integer which counts the number of supersymmetric vacua,

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and hence allows one to study supersymmetry breaking. The Witten Index is defined as  $\text{Tr}(-1)^F$ , with  $F$  acting on fermionic states  $|f\rangle$  and bosonic states  $|b\rangle$  as,

$$(-1)^F|b\rangle = 0, \quad (-1)^F|f\rangle = -|f\rangle, \quad (1.4)$$

The superconformal index on the other hand counts states in the cohomology of a particular supercharge  $\mathcal{Q}$  in the superconformal algebra.

$$\text{Cohom}(\mathcal{Q}) = \{|\phi\rangle \mid \mathcal{Q}|\phi\rangle = 0, |\phi\rangle \neq \mathcal{Q}|\psi\rangle\} \quad (1.5)$$

The index itself does not depend on the choice of supercharge  $\mathcal{Q}$ . While most papers take  $\mathcal{Q}^2 = 0$ , we will take the supercharge to square a twisted hamiltonian, as in [31]. As there is an infinite set of states in the cohomology of  $\mathcal{Q}$ , one should use an appropriately regulated version of the Witten Index. Denoting  $\{C_i\}$  the set of generators of the symmetry algebra which commute with  $\mathcal{Q}$ , the superconformal index is defined as,

$$I(\mu_i) = \text{Tr}(-1)^F e^{\mu_i C_i}. \quad (1.6)$$

Note that the set of generators  $\{C_i\}$  are part of the superconformal algebra but also include the relevant flavour generators of the theory under consideration. The gauge generators play a role in the computation of the index, however, they are only important in the projection of the index formula onto gauge singlets. Importantly, the theory does not in fact have to be invariant under the full superconformal group as was realised in [32, 33], and the superconformal index should really be called a supersymmetric index. In fact, such quantity had been proposed long ago [34] for  $\mathcal{N} = 1$  field theories on  $\mathbb{R} \times S^3$  although recently re-discovered. For superconformal theories, both definitions are equivalent, through radial quantisation. This definition of the index as supersymmetric as opposed to superconformal is important for calculational purposes, as the index can then be computed for theories with arbitrary  $R$ -charges [33], and these are less constrained than in a superconformal setting. This might also have important theoretical implications, as candidates for dualities outside the conformal window have been proposed [35].

A crucial realisation by Dolan and Osborn [36] was the fact that the matching



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of the electric and the magnetic index corresponds to a mathematical identity by Spiridonov [37, 38] generalising the Nasrallah-Rahman theorem. The matching of indices for more general dualities is a general identity by Rains [39]. Consequently, despite the specificity of the state dependence of the index, its matching is a highly non-trivial mathematical property of elliptic hypergeometric integrals. The latter integrals arise when projecting onto gauge singlet the multiplarticle index, or Plethystic [40, 41] of the single particle index.

A number of papers followed. In [42, 43] a large number of Seiberg dual pairs were written down based on elliptic-hypergeometric identities. Duality was also shown to hold for situations not requiring such identities, for instance for  $\mathcal{N} = 1$  superconformal field theories with AdS duals [44]. Finally, one should note that the usefulness of these identities is not confined to Seiberg dual theories. In the context of  $\mathcal{N} = 2$  superconformal field theories defined by Gaiotto [45] as compactification of six dimensional  $(0, 2)$  theories, elliptic hypergeometric identities appear in the calculation of a four dimensional indices for theories associated to an  $n$ -punctured Riemann surface [46, 47]. The index is also relevant in the study of AGT type of relations [48, 49] between two dimensional Liouville theory and four dimensional  $\mathcal{N} = 2$  gauge theory on the four sphere [50].

### 1.4 $R$ -charges and $a$ , $Z$ -maximisation

In all the previous considerations, the  $R$ -charges of the fields in the theory play paramount role, both in the matching of the index, but also in the anomaly matching procedure. In the case of  $SU(N_c)/SU(\tilde{N}_c)$  duality, the charges are fixed by physical consistency requirements.

- the superconformal  $U(1)_R$  has to commute with all non abelian flavour symmetries.
- it also has to commute with any charge conjugation operator. For instance, in SQCD, baryons and antibaryons should have the same  $R$ -charge, and so should left-handed and right-handed-quarks.
- Finally, the  $R$ -charge should be anomaly free, in the sense explained before.

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In the case of SQCD, this actually determines the  $R$ -charge unambiguously and leads to the  $R$ -charge assignment in figure (1.1), (1.2). However, in some case these physical requirements are insufficient to determine the  $R$ -charges unambiguously, as in the case of SQCD with an extra adjoint and no superpotential. In this case, the  $a$ -maximisation [51] procedure allows one to determine the appropriate  $R$ -charges. This construction was echoed more recently in the discovery of the  $Z$ -maximisation procedure for 3-dimensional quantum field theories [52].

### 1.5 Motivation & Outline

Despite some impressive results regarding indices and dualities, there are still some unresolved issues surrounding these objects. A well known feature of the index is its topological nature. It is invariant under continuous deformations of the theory which modify the chosen supercharge  $\mathcal{Q}$  as well as the modified hamiltonian  $\mathcal{H}$ ,

$$\mathcal{Q}^2 = \mathcal{H}, \tag{1.7}$$

but preserve the set of generators  $\{C_i\}$  which enter the definition of the index (1.6). Then, one might be tempted to see the transition from free field theory to interacting theory and strong coupling as a continuous modification of the value of the coupling constant away from zero. Consequently, one might expect the index formulas for a free superconformal theory to hold in the context of an interacting theory. However this is not the case, for 2 reasons.

- There is no physical continuous deformation of  $\mathcal{N} = 1$  that takes one from strong coupling to weak coupling while preserving conformality, allowing  $\mathcal{Q}$  to be defined as part of the superconformal algebra [36].
- The  $R$ -charge dependence for the chiral field is not captured by the previous calculation. Even if we had a renormalisation path taking us to strong coupling, it would be unclear how the renormalisation of the  $R$ -charge would modify the picture above, as the numerical values of the free field  $R$ -charges is implicitly entering the index formula at free field, and cannot hence be extracted. In other words, while an index formula can be relatively easily justified for free field theories, using representation theory [36], whereby the

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$R$  charges take the value  $\frac{2}{3}$ , in an interacting theory the  $R$ -charges can be renormalised away from free field values and this justification fails.

In fact, it will become clear later that the  $R$ -charge dependence cannot be captured with this approach because of the drastic restrictions imposed by conformal invariance. By focusing on theories invariant under the  $SU(2, 1)_L \times SU(2)_R$  subalgebra of the superconformal algebra which includes  $\mathcal{Q}$ ,  $\{C_i\}$ , one can compute an index with an arbitrary  $R$ -charge dependence. For superconformal theories, one recovers the appropriate index formula, and the  $SU(2, 1)_L \times SU(2)_R$  subalgebra can then be interpreted as the minimally  $\mathcal{N} = 1$  supersymmetric completion of the isometry group of  $\mathbb{R} \times S^3$ , which is the relevant space for radial quantisation of a four dimensional conformal field theory.

This observation allows for an exact computation of the index with arbitrary  $R$ -charges, using localisation of the compactified theory on  $S^1 \times S^3$ . This procedure leads to exact formulas for expectation values of  $\mathcal{Q}$  closed operators, and is not limited to index computation. Such approach is valid even for theories not amenable to a standard perturbative treatment, where superconformal symmetry does not extend continuously to the free field theory, on spaces appropriate for radial quantisation, here  $\mathbb{R} \times S^3$ . This is different in flavour to other scenarios where localisation techniques have been applied to computing indices, for example [12], where superconformal symmetry holds for all values of the coupling. Such localisation actions are at least applicable to superconformal field theories, when the subgroup symmetry composed with translations, extends to superconformal symmetry on  $\mathbb{R}^4$ . Such SCFTs, of course, include those even at IR fixed points, not amenable to a perturbative analysis. So, ultimately, we are computing indices for a larger class of supersymmetric theories on  $\mathbb{R} \times S^3$ , for which localisation applies, and which include IR superconformal fixed points of relevance for Seiberg duality. Localisation provides an alternative proof of the conjecture by Romelsberger [32, 33] of the general form of the index in four dimensions, and a rigorous derivation of its exact-value in interacting theories using localisation arguments.

In the first chapter, we define the superconformal algebra and review all possible representations, focusing on the relevant shortening conditions. This then allows us to compute the single particle index for free superconformal theories, using limits of various characters to obtain the results. In the second chapter, we

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focus on theories living on  $\mathbb{R} \times S^3$  invariant under the  $SU(2, 1)_L \times SU(2)_R$  subalgebra. After defining the relevant differential operators and Killing spinors, we define the chiral and vector representations used to construct supersymmetric and covariant actions on  $\mathbb{R} \times S^3$ . We can then repeat the analysis of the first chapter, after expanding the field in spherical harmonics and canonically quantising the theory. In the third chapter, we compute the index as a path integral. We review the localisation framework and the appropriate twisting procedure which allows a consistent computation of the index after compactifying the time direction to  $S^1$ . This then allows for a consistent localisation principle to be formulated in order to justify the exactness of the index formula,

$$\begin{aligned}
 I(t, x, h) &= \int d\mu(g) \text{Pexp}(i(t, x, g, h)), \tag{1.8} \\
 i(t, x, g, h) &= \sum_i \frac{t^{r_i} \chi_{R_{G,i}}(g) \chi_{R_{H,i}}(h) - t^{2-r_i} \chi_{\bar{R}_{G,i}}(g) \chi_{\bar{R}_{H,i}}(h)}{(1-tx)(1-tx^{-1})} \\
 &\quad + \frac{1-t(x+x^{-1})}{(1-tx)(1-tx^{-1})} \chi_{\text{Adj.}}(g), \tag{1.9}
 \end{aligned}$$

with  $i$  labelling the chiral multiplets with  $R$ -charges  $r_i$ , transforming in gauge group  $G$  representations  $R_{G,i}$ , flavour group  $H$  representations  $R_{H,i}$ , with gauge group and flavour group elements  $g \in G$ ,  $h \in H$ . The terms proportional to  $t^{r_i}$  are due to each chiral field, those proportional to  $t^{2-r_i}$  are due to conjugate anti-chiral fields and the adjoint representation contribution is due to vector field multiplet. Having shown explicitly that the free field theory Lagrangian is  $\mathcal{Q}$ -exact, we can reduce the path integral to an exact one loop determinant calculation around saddle points of the free field action, as in [53]. The existence of zero modes living on non-trivial instanton backgrounds on  $S^1 \times S^3$  means that there are no non-perturbative contributions to the index in four dimensions, unlike in three dimensions [12].

# Chapter 2

## Index and Representation Theory

### 2.1 The Superconformal Algebra

#### 2.1.1 Constructing the Conformal Algebra

We first review here the construction of the conformal and superconformal algebra in 4 dimensions, following the discussion in [54]. By definition, the conformal group in  $d$  dimensions is defined as the set of transformations which leave the metric invariant up to a positive factor,

$$\begin{aligned}x &\rightarrow x', \\ ds^2 &\rightarrow ds'^2 = \Omega(x)ds^2,\end{aligned}\tag{2.1}$$

with the line element defined as,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu\tag{2.2}$$

Consider an infinitesimal coordinate change,

$$x^\mu \rightarrow x^\mu + \epsilon^\mu.\tag{2.3}$$

The transformation of the line element reads,

$$ds^2 \rightarrow ds^2 + (\partial_\mu\epsilon_\nu + \partial_\nu\epsilon_\mu)dx^\mu dx^\nu.\tag{2.4}$$

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For equations (2.1) and (2.4) to be compatible, one has to impose,

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial^\rho \epsilon_\rho) \eta_{\mu\nu} \quad (2.5)$$

Also, one gets the following expression for  $\Omega$  in (2.1),

$$\Omega(x) = 1 + \frac{2}{d} \partial \epsilon \quad (2.6)$$

It follows that,

$$(\eta_{\mu\nu} \square + (d-2) \partial_\mu \partial_\nu) \partial \epsilon = 0 \quad (2.7)$$

Consequently,  $\epsilon$  corresponds to the following transformations,

1. Translations:  $\epsilon^\mu = b^\mu$
2. Rotations:  $\epsilon^\mu = \omega^\mu{}_\nu x^\nu$
3. Scalings:  $\epsilon^\mu = \lambda x^\mu$
4. Special conformal transformations:  $\epsilon^\mu = b^\mu x^2 - 2(b \cdot x) x^\mu$

This then leads up to the associated set of Killing vectors, which generate the associated transformations,

1. Translations:  $P_\mu = -i \partial_\mu$
2. Rotations:  $M_{\mu\nu} = -i(x_\mu \partial_\nu - x_\nu \partial_\mu)$
3. Scalings:  $H = x^\mu \partial_\mu$
4. Inversions:  $K_\mu = i(2x_\mu x^\nu \partial_\nu - x^2 \partial_\mu)$

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One should note that the special conformal transformations correspond to an inversion  $x^\mu \rightarrow x^\mu/x^2$  plus a translation. These lead to the conformal algebra,

$$[H, P_\mu] = P_\mu \quad (2.8)$$

$$[H, K_\mu] = -K_\mu \quad (2.9)$$

$$[K_\mu, P_\nu] = 2(\eta_{\mu\nu}H - iM_{\mu\nu}) \quad (2.10)$$

$$[M_{\mu\nu}, K_\rho] = i(\eta_{\mu\rho}P_\nu - \eta_{\nu\rho}P_\mu) \quad (2.11)$$

$$[M_{\mu\nu}, P_\rho] = i(\eta_{\mu\rho}K_\nu - \eta_{\nu\rho}K_\mu) \quad (2.12)$$

$$[M_{\mu\nu}, M_{\rho\sigma}] = i(\eta_{\mu\rho}M_{\nu\sigma} + \eta_{\nu\sigma}M_{\mu\rho} - \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\nu\rho}M_{\mu\sigma}) \quad (2.13)$$

This then leads [55] to the following action of the algebra generators acting on a quasi-primary field  $\mathcal{O}_I(x)$ ,

$$[P_\mu, \mathcal{O}_I] = i\partial_\mu \mathcal{O}_I(x), \quad (2.14)$$

$$[H, \mathcal{O}_I] = i(x \cdot \partial + \Delta)\mathcal{O}_I(x), \quad (2.15)$$

$$[K_\mu, \mathcal{O}_I] = i(x^2\partial_\mu - 2x_\mu x \cdot \partial + 2\Delta x_\mu)\mathcal{O}_I(x) + 2\mathcal{O}_J(x)(s_{\mu\nu})_I^J x^\nu, \quad (2.16)$$

$$[M_{\mu\nu}, \mathcal{O}_I] = \mathcal{L}_{\mu\nu}\mathcal{O}_I(x) + (s_{\mu\nu})_I^J \mathcal{O}_J(x), \quad (2.17)$$

with  $\Delta$  the scaling dimension of  $\mathcal{O}$  and  $s_{\mu\nu}$  the appropriate spin matrices. The differential rotation generators  $\mathcal{L}_{\mu\nu}$  are given by the following,

$$\mathcal{L}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu). \quad (2.18)$$

These, as well as the spin matrices satisfy the following commutation relations,

$$[\mathcal{L}_{\mu\nu}, \mathcal{L}_{\rho\sigma}] = i(\eta_{\mu\sigma}\mathcal{L}_{\nu\rho} + \eta_{\nu\rho}\mathcal{L}_{\mu\sigma} - \eta_{\mu\rho}\mathcal{L}_{\nu\sigma} - \eta_{\nu\sigma}\mathcal{L}_{\mu\rho}), \quad (2.19)$$

$$[s_{\mu\nu}, s_{\rho\sigma}] = i(\eta_{\mu\sigma}s_{\nu\rho} + \eta_{\nu\rho}s_{\mu\sigma} - \eta_{\mu\rho}s_{\nu\sigma} - \eta_{\nu\sigma}s_{\mu\rho}), \quad (2.20)$$

### 2.1.2 Radial Quantisation

Conformal field theories can be studied in the framework of radial quantisation. Within this framework, the euclidean path integral with operator insertions can be regarded as wave functions, over the unit ball surrounding the origin. Dual

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wave functions can be obtained through the action of inversions. Under an inversion, the rotations are taken to rotations, the dilation operator is taken to minus itself, the translation generators are taken to the special conformal transformation generators. This leads to the following hermiticity conditions,

$$M_{\mu\nu}^+ = M_{\mu\nu}, \quad P_\mu = K_\mu^+. \quad (2.21)$$

Overall, radial quantisation is equivalent to the study of the field theory on  $\mathbb{R} \times S^3$ . The time translation generator corresponds to the dilation operator while the generators  $M_{\mu\nu}$  correspond to the isometry generators of the spatial  $S^3$ . In this context, the generators  $P_\mu$  and  $K_\mu$  act as ladder operators which take quantum states to different energy levels.

### 2.1.3 The Superconformal Algebra

When extending this symmetry group to include  $\mathcal{N} = 1$  supersymmetry, one has to introduce one set of special supersymmetry generators  $S_\alpha, \bar{S}_{\dot{\alpha}}$  in addition to the familiar supersymmetry generators  $Q_\alpha, \bar{Q}_{\dot{\alpha}}$  of  $\mathcal{N} = 1$  supersymmetry. One then obtains the  $SU(2, 2|1)$  superconformal algebra and following [36] notations, the bosonic commutation relations are given by,

$$[\mathcal{M}_A^B, \mathcal{M}_C^D] = \delta_C^B \mathcal{M}_A^D - \delta_A^D \mathcal{M}_C^B, \quad (2.22)$$

while the supercharge commutation relations are given by,

$$\{Q_A, \bar{Q}^B\} = 4 \mathcal{M}_A^B + 3 \delta_A^B R, \quad (2.23)$$

$$\{Q_A, Q_B\} = \{\bar{Q}^A, \bar{Q}^B\} = 0, \quad (2.24)$$

the mixed commutation relations are given by,

$$[\mathcal{M}_A^B, Q_C] = \delta_C^B Q_A - \frac{1}{4} \delta_A^B Q_C, \quad (2.25)$$

$$[\mathcal{M}_A^B, \bar{Q}^C] = -\delta_A^C \bar{Q}^B + \frac{1}{4} \delta_A^B \bar{Q}^C, \quad (2.26)$$



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and finally, the external  $R$ -charge,

$$[R, \mathcal{Q}_A] = -\mathcal{Q}_A, \quad [R, \bar{\mathcal{Q}}^B] = \bar{\mathcal{Q}}^B, \quad (2.27)$$

with the following definitions, for the bosonic operators,

$$\mathcal{M}_A{}^B = \begin{pmatrix} M_\alpha{}^\beta + \frac{1}{2}\delta_\alpha{}^\beta H & \frac{1}{2}P_{\alpha\dot{\beta}} \\ \frac{1}{2}K^{\dot{\alpha}\beta} & \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} - \frac{1}{2}\delta^{\dot{\alpha}}{}_{\dot{\beta}} H \end{pmatrix}, \quad \delta_A{}^B = \begin{pmatrix} \delta_\alpha{}^\beta & 0 \\ 0 & \delta^{\dot{\alpha}}{}_{\dot{\beta}} \end{pmatrix}, \quad (2.28)$$

and the supercharges,

$$\mathcal{Q}_A = \begin{pmatrix} Q_\alpha \\ \bar{S}^{\dot{\alpha}} \end{pmatrix}, \quad \bar{\mathcal{Q}}^B = (S^\beta, \bar{Q}_{\dot{\beta}}), \quad (2.29)$$

The rotation generators will be denoted by,

$$M_\alpha{}^\beta = -J_m(\sigma_m)_\alpha{}^\beta, \quad \bar{M}^{\dot{\alpha}}{}_{\dot{\beta}} = -\bar{J}_m(\sigma_m)^{\dot{\alpha}}{}_{\dot{\beta}} \quad (2.30)$$

where  $J_m, \bar{J}_m$  denote the  $SU(2)_L$  and  $SU(2)_R$  generators of the  $SU(2, 2|1)$  generators (2.26). These satisfy the standard  $SU(2)$  commutation relations,

$$[J_m, J_n] = i \varepsilon_{mnp} J_p, \quad [\bar{J}_m, \bar{J}_n] = i \varepsilon_{mnp} \bar{J}_p. \quad [J_m, \bar{J}_n] = 0, \quad (2.31)$$

with  $m, n, p = 1, 2, 3$  the spatial indices, and  $\varepsilon_{mnp}$  the completely antisymmetric Levi-Civita symbol such that  $\varepsilon_{123} = 1$ . The hermiticity properties of the generators can be chosen in two different ways. When studying a theory on flat space, one will take the following hermiticity conditions,

$$Q_\alpha^\dagger = \bar{Q}_{\dot{\alpha}}, \quad S^{\alpha\dagger} = \bar{S}^{\dot{\alpha}}, \quad (2.32)$$

while the hermiticity properties which will be chosen in a radially quantised setting will be given by,

$$Q_\alpha^+ = S^\alpha, \quad \bar{Q}_{\dot{\alpha}}^+ = -\bar{S}^{\dot{\alpha}} \quad (2.33)$$

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### 2.2 $SU(2, 1)_r$ Algebra and the Index

Just like for the Witten index, the superconformal index definition rests upon the choice of a particular supercharge  $\mathcal{Q}$  and the associated hamiltonian  $\mathcal{H}$  defined as <sup>1</sup>,

$$\mathcal{Q}^2 = \mathcal{H}, \quad (2.34)$$

with this definition, the fermionic supercharge satisfies the following reality condition,

$$\mathcal{Q} = \mathcal{Q}^+, \quad (-1)^F \mathcal{Q} = -\mathcal{Q}(-1)^F. \quad (2.35)$$

The difference between the Witten index and the superconformal index is due to the different definitions for the hermitian conjugation and hence for the hamiltonian. For the Witten index, the appropriate hermitian conjugation operation is the usual dagger operation which takes  $Q_\alpha$  to  $\bar{Q}_{\dot{\alpha}}$  and leads to the square of the chosen charge  $\mathcal{Q}$  to be the actual hamiltonian  $P_0$  of the theory. For the superconformal index on the other hand, the hermitean conjugation <sup>+</sup> takes  $Q_\alpha$  to  $S^\alpha$ . and leads to the definition of a twisted hamiltonian. Here, we take the following convention,

$$\mathcal{Q} = \frac{1}{\sqrt{2}}(Q_1 + S^1), \quad (2.36)$$

which, following equation 1.7, leads to the expression for the modified hamiltonian  $\mathcal{H}$ ,

$$\mathcal{H} = H + \frac{3}{2}R - 2J_3. \quad (2.37)$$

The latter definition is the one prescribed by radial quantisation of superconformal field theory [26], and is associated with a scalar product determined by two point correlation functions [56]. The Witten index [10] can be defined in a regularised manner as  $\text{Tr}(-1)^F e^{-\beta\mathcal{H}}$ . For the superconformal index, there are more states contributing and we need to identify the elements  $\{C_i\}$  of the superconformal algebra which commute with the supercharge  $\mathcal{Q}$  and can be used to fully regularise the definition of the index.

The bosonic subalgebra of the  $SU(2, 2|1)$  superconformal algebra that com-

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<sup>1</sup>An alternative definition often found in the literature is  $\{\mathcal{Q}, \mathcal{Q}^+\} = 2\mathcal{H}$ . The two definitions are equivalent and are related by the following transformation  $\mathcal{Q} \rightarrow \mathcal{Q} + \mathcal{Q}^+$ . With the alternative definition,  $\mathcal{Q}^2 = 0$ .

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mutates with the charge  $\mathcal{Q}$  is an  $SU(2, 1)_r$  algebra:

$$[\mathcal{M}_A^B, \mathcal{M}_C^D] = \delta_C^B \mathcal{M}_A^D - \delta_A^D \mathcal{M}_C^B \quad (2.38)$$

and one can use its Cartans  $\{C_i\}$  to weigh the counting. Given our chosen supercharge (2.36), the appropriate  $SU(2, 1)_r$  algebra is given by,

$$\mathcal{M}_A^B = \begin{pmatrix} \bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \frac{1}{2} \delta_{\dot{\beta}}^{\dot{\alpha}} \mathcal{R} & -\mathcal{P}^{\dot{\alpha}} \\ \bar{\mathcal{P}}_{\dot{\beta}} & \mathcal{R} \end{pmatrix}, \quad \delta_A^B = \begin{pmatrix} \delta_{\dot{\beta}}^{\dot{\alpha}} & 0 \\ 0 & 1 \end{pmatrix}, \quad (2.39)$$

with:

$$\bar{\mathcal{P}}_{\dot{\beta}} = \frac{1}{2} P_{2\dot{\beta}}, \quad \mathcal{P}^{\dot{\alpha}} = -\frac{1}{2} \bar{K}^{\dot{\alpha}2}. \quad (2.40)$$

The Cartans of the algebra are,

$$\{C_i\} = \{\mathcal{R}, \bar{J}_3\}, \quad (2.41)$$

with the following definition,

$$\mathcal{R} = \frac{2}{3}(H + J_3), \quad (2.42)$$

and the index then reads:

$$I(t, x) = \text{Tr}(-1)^F t^{\mathcal{R}} x^{2\bar{J}_3}. \quad (2.43)$$

The index only depends on the cohomology of  $\mathcal{Q}$ , the set of states  $\mathcal{Q}$ -closed without being  $\mathcal{Q}$ -exact. To justify this, assume a state such that,  $\mathcal{Q}|\varphi\rangle \neq 0$ . Then, as  $\mathcal{Q}$  commutes with the generators  $C_i$ , the states  $|\varphi\rangle$  and  $\mathcal{Q}|\varphi\rangle$  have the same  $C_i$  eigenvalues, but opposite spin statistics. Hence the  $(-1)^F$  factor implies the cancellation of both contributions. Hence the index only depends on the  $\mathcal{Q}$ -cohomology, which is the kernel of  $\mathcal{H}$  [57]. Assuming a state  $\chi$  annihilated by  $\mathcal{Q}$ , then by definition of  $\mathcal{H}$ , then  $\chi$  is in  $\text{Ker } \mathcal{H}$ . Conversely, if  $\mathcal{H}\chi = 0$  then,

$$\langle \chi | \mathcal{H} | \chi \rangle = 0 = \langle \chi | \mathcal{Q}^+ \mathcal{Q} | \chi \rangle = \|\mathcal{Q}\chi\|^2 = 0, \quad (2.44)$$

where we've used the reality condition (2.35) on the charge  $\mathcal{Q}$ . If  $\chi$  is  $\mathcal{Q}$ -exact,

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specifically  $\chi = \mathcal{Q}\xi$  then  $\xi$  is a zero eigenstate of  $\mathcal{H}$ . This in turn means that  $\chi = 0$  by equation (2.44). Consequently,  $\chi$  is part of the  $\mathcal{Q}$ -cohomology. Hence, the index only receives contributions from states in  $\text{Ker } \mathcal{H}$ .

One can then refine the definition of the index (2.43) to highlight this exclusive  $\text{Ker } \mathcal{H}$  dependence. Rewrite  $\mathcal{R}$  in the following fashion,

$$\mathcal{R} = -R + 2J_3 + \frac{2}{3}\mathcal{H} \quad (2.45)$$

and more generally

$$\mathcal{R}_c = -R + 2J_3 - c\mathcal{H} \quad (2.46)$$

Consequently, the index (2.43) does not depend on the parameter  $c$  which appears in the definition of  $\mathcal{R}_c$  equation (2.46). Hence, denoting  $t^c = y$ , one can write:

$$I(t, x) = \text{Tr} (-1)^F y^{\mathcal{H}} t^{-R+2J_3} x^{2\bar{J}_3}. \quad (2.47)$$

The result should be independent of  $y$  for consistency. There are other equivalent convention choices, but the result for the index is independent of such choice. The table below summarises the alternative possible choices. Finally, for theories

$SU(2, 1)_r: \mathcal{M}_A^B$	$\mathcal{Q}$	$2\mathcal{P} \ 2\bar{\mathcal{P}}$	$\frac{3}{2}\mathcal{R}, J$	$\mathcal{H}$
$\begin{pmatrix} M_\alpha^\beta + \frac{1}{2}\delta_\alpha^\beta \mathcal{R} & \mathcal{P}_\alpha \\ -\bar{\mathcal{P}}^\beta & -\mathcal{R} \end{pmatrix}$	$\begin{matrix} \bar{Q}_1 - \bar{S}^1 \\ \bar{Q}_2 - \bar{S}^2 \end{matrix}$	$\begin{matrix} P_{\alpha 2} - \bar{K}^{2\beta} \\ P_{\alpha 1} - \bar{K}^{1\beta} \end{matrix}$	$\begin{matrix} H - \bar{J}_3, J_3 \\ H + \bar{J}_3, J_3 \end{matrix}$	$\begin{matrix} H + 2\bar{J}_3 - \frac{3}{2}R \\ H - 2\bar{J}_3 - \frac{3}{2}R \end{matrix}$
$\begin{pmatrix} \bar{M}_{\dot{\beta}}^{\dot{\alpha}} - \frac{1}{2}\delta_{\dot{\beta}}^{\dot{\alpha}} \mathcal{R} & -\mathcal{P}^{\dot{\alpha}} \\ \bar{\mathcal{P}}_{\dot{\beta}} & \mathcal{R} \end{pmatrix}$	$\begin{matrix} Q_1 + S^1 \\ Q_2 + S^2 \end{matrix}$	$\begin{matrix} -\bar{K}^{\dot{\alpha}2} P_{2\dot{\beta}} \\ -\bar{K}^{\dot{\alpha}1} P_{1\dot{\beta}} \end{matrix}$	$\begin{matrix} H + J_3, \bar{J}_3 \\ H - J_3, \bar{J}_3 \end{matrix}$	$\begin{matrix} H - 2J_3 + \frac{3}{2}R \\ H + 2J_3 + \frac{3}{2}R \end{matrix}$

Figure 2.1: All  $SU(2, 1)_r$  Subalgebras of the Superconformal Algebra

invariant under a global symmetry group  $H$  such as flavour symmetry and/or gauge group  $G$ , the definition of the index then includes extra factors,

$$I(t, x, h) = \int_G d\mu(g) \mathcal{I}(t, x, g, h). \quad (2.48)$$

$$\mathcal{I}(t, x, g, h) = \text{Tr} (-1)^F y^{\mathcal{H}} t^{-R+2J_3} x^{2\bar{J}_3} g h. \quad (2.49)$$

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with  $g, h$  elements of the gauge group  $G$  and global symmetry group  $H$ . Note that  $g, h$  are finite elements of the groups, as opposed to generators of the respective Lie algebras. Also, the gauge and flavour groups have different roles. The flavour symmetry element is a part of the definition of the index as these are global symmetries which commute with supercharges, whereas the gauge group integration measure appears as one has to project the final answer on gauge singlet.

One should note that a generalised version of the index has been proposed in [58]. In this approach, the flavour symmetry is gauged. This approach can be useful to generalise a duality which includes a global symmetry whose mapping through the duality is known. One can then generate new dualities by gauging this global symmetry. In that case, the integral (2.48) is generalised to include the measure over the group  $H$ .

### 2.3 First Computation: Free Flat Space Theory

The first step in computing the index is the single particle index, defined as,

$$i(t, x, g, h) = \text{Tr}_{\text{s.p.}} (-1)^F y^{\mathcal{H}} t^{-R+2J_3} x^{2\bar{J}_3} gh. \quad (2.50)$$

with  $\text{Tr}_{\text{s.p.}}$  the trace restricted to single particle states. One can then compute the contributions from all multiparticle states, which is given by the Plethystic exponential,

$$\text{Pexp}[i(t, x, g, h)] = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} i(t^n, x^n, g^n, h^n)\right). \quad (2.51)$$

One then needs to project this quantity onto gauge singlet to get the index,

$$I(t, x, h) = \int_G d\mu(g) \text{Pexp} i(t, x, g, h), \quad (2.52)$$

with  $d\mu(g)$  the invariant measure for the gauge group  $G$ .

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### 2.3.1 Superconformal Algebra Short Representations

Given a superconformal primary field  $\mathcal{O}$ , through the state-operator correspondence [59], one can define the associated state by acting on the vacuum with the operator at the origin,

$$\mathcal{O}(0)|0\rangle = |\mathcal{O}\rangle \quad (2.53)$$

Assuming  $\mathcal{O}$  to be a superconformal primary operator, then  $\mathcal{O}$  is annihilated by  $S, \bar{S}, K$  generators of the superconformal algebra, and defines a the lowest state of a multiplet. Also define,

$$H|\mathcal{O}\rangle = \Delta|\mathcal{O}\rangle, \quad R|\mathcal{O}\rangle = r|\mathcal{O}\rangle. \quad (2.54)$$

One can then define the operator  $\mathcal{O}$  to be a chiral field,

$$\mathcal{O} = \phi, \quad \bar{Q}_{\dot{\alpha}}|\phi\rangle = 0, \quad (2.55)$$

From the trace part of the  $\{\bar{Q}, \bar{S}\}|\phi\rangle = 0$  commutation, one gets the usual relation between scaling dimension  $\Delta$  and the  $R$ -charge  $r$ ,

$$\Delta = \frac{3}{2}r, \quad (2.56)$$

Also, from the traceless part of the same anticommutation relation  $\{\bar{Q}, \bar{S}\}$ , it is clear that  $\phi$  falls into a  $(j, 0)$  representation of  $SU(2)_l \times SU(2)_r$ . Similarly an antichiral field lies in  $(0, j)$  representation of  $SU(2)_l \times SU(2)_r$ , and,

$$Q_{\alpha}\bar{\phi} = 0, \quad \Delta = -\frac{3}{2}r. \quad (2.57)$$

### 2.3.2 The Chiral Multiplet

Let us first focus on the chiral multiplet, which corresponds to the the  $j = 0$  representation of  $SU(2)_r$  case from the above discussion. The transformations are given by,

$$[Q_{\alpha}, \phi] = \psi_{\alpha}, \quad \{Q_{\alpha}, \psi_{\beta}\} = \epsilon_{\alpha\beta}F, \quad [Q_{\alpha}, F] = 0. \quad (2.58)$$

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with  $\epsilon_{12} = -1$ . For consistency with the superconformal algebra,

$$\{S^\beta, \psi_\alpha\} = 6r \delta_\alpha^\beta \phi, \quad [S^\alpha, F] = -2(3r-2)\psi^\alpha, \quad (2.59)$$

$$\{\bar{Q}_{\dot{\alpha}}, \psi_\alpha\} = 2i \partial_{\alpha\dot{\alpha}} \phi, \quad [\bar{Q}_{\dot{\alpha}}, F] = -2i \partial_{\alpha\dot{\alpha}} \psi^\alpha. \quad (2.60)$$

The corresponding antichiral multiplet has the following transformations,

$$[\bar{Q}_{\dot{\alpha}}, \bar{\phi}] = \bar{\psi}_{\dot{\alpha}}, \quad \{\bar{Q}_{\dot{\alpha}}, \bar{\psi}_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} \bar{F}, \quad [\bar{Q}_{\dot{\alpha}}, \bar{F}] = 0, \quad (2.61)$$

and,

$$\{\bar{S}^{\dot{\beta}}, \bar{\psi}_{\dot{\alpha}}\} = 6r \delta_{\dot{\alpha}}^{\dot{\beta}} \bar{\phi}, \quad [\bar{S}^{\dot{\alpha}}, \bar{F}] = -2(3r-2)\bar{\psi}^{\dot{\alpha}}, \quad (2.62)$$

$$\{Q_\alpha, \bar{\psi}_{\dot{\alpha}}\} = -2i \partial_{\alpha\dot{\alpha}} \bar{\phi}, \quad [Q_\alpha, \bar{F}] = 2i \partial_{\alpha\dot{\alpha}} \bar{\psi}^{\dot{\alpha}}. \quad (2.63)$$

Assuming a free field theory,

$$F = 0, \quad \Rightarrow \quad \partial_{\alpha\dot{\alpha}} \psi^\alpha = 0, \quad \partial^2 \phi = 0, \quad (2.64)$$

and most importantly,

$$r = \frac{2}{3}. \quad (2.65)$$

and similarly for an antichiral field,

$$\bar{F} = 0, \quad \Rightarrow \quad r = -\frac{2}{3} \quad (2.66)$$

All the information given so far is valid for the lowest lying state in the multiplet. The rest of the multiplet is generated through the repeated use of  $P$  translation generators on the lowest lying state we've just described. The states which contribute to the index are those obtained through the action of the  $P$  generators part of the  $SU(2,1)_r$  algebra described in equations (2.39) and (2.40). Hence the bosonic states contributing to the index for the scalar chiral field are given by,

$$P_{21}^m P_{22}^n |\bar{\phi}\rangle, \quad \forall m, n = 0, 1, 2 \dots \quad (2.67)$$

The fermionic states contributing to the index are descendents of  $\psi_2$ . In a free

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theory  $F$  vanishes, hence  $\psi_2$  is annihilated by  $Q_1$ . Consequently, the fermionic states contributing to the index are given by,

$$P_{21}^m P_{22}^n |\psi_2\rangle, \quad \forall m, n = 0, 1, 2 \dots \quad (2.68)$$

The single particle index for the chiral/antichiral multiplet is then given by the following,

$$i_\phi(t, x) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} (tx)^m (tx^{-1})^n (t^{\frac{2}{3}} - t^{\frac{4}{3}}) = \frac{t^{\frac{2}{3}} - t^{\frac{4}{3}}}{(1-tx)(1-tx^{-1})} \quad (2.69)$$

The infinite sums above arise from the contributions of all the descendent states, while the numerator terms in the final expression arise from the antichiral and chiral contribution respectively.

### 2.3.3 The Vector Multiplet Index

As was noted previously, the traceless part of the  $\{\bar{Q}, \bar{S}\}$  anticommutation relations implies that a chiral field falls into a  $(j, 0)$  representation of  $SU(2)_l \times SU(2)_r$ . If  $j = \frac{1}{2}$ , we are considering the usual supersymmetric vector multiplet. The supersymmetry transformations are then given by the following,

$$\{Q_\alpha, \lambda_\beta\} = f_{\alpha\beta} + \epsilon_{\alpha\beta} iD, \quad [Q_\alpha, f_{\beta\gamma}] = \epsilon_{\alpha\beta} \mu_\gamma + \epsilon_{\alpha\gamma} \mu_\beta, \quad (2.70)$$

$$[Q_\alpha, D] = i\mu_\alpha, \quad \{Q_\alpha, \mu_\beta\} = 0, \quad (2.71)$$

$$[S^\gamma, f_{\alpha\beta}] = 2(3r+1)\delta_\alpha^\gamma \lambda_\beta, \quad [S^\beta, D] = 3(r-1)i\lambda^\beta, \quad (2.72)$$

$$\{S^\beta, \mu_\alpha\} = -3(r-1)\epsilon^{\beta\gamma} f_{\alpha\gamma} - (3r+1)i\delta_\alpha^\beta D, \quad (2.73)$$

with  $f_{\alpha\beta} = f_{\beta\alpha}$ , and

$$[\bar{Q}_{\dot{\alpha}}, f_{\alpha\beta}] = 2i\partial_{\alpha\dot{\alpha}}\lambda_\beta, \quad [\bar{Q}_{\dot{\alpha}}, D] = -\partial_{\alpha\dot{\alpha}}\lambda^\alpha, \quad (2.74)$$

$$\{\bar{Q}_{\dot{\alpha}}, \mu_\alpha\} = i\epsilon^{\beta\gamma}\partial_{\gamma\dot{\alpha}}f_{\alpha\beta} + \partial_{\alpha\dot{\alpha}}D. \quad (2.75)$$



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The minimal spinor in four dimensions is Majorana, or two Weyl spinors  $\lambda$  and  $\bar{\lambda}$  with the reality condition

$$\bar{\lambda}_\alpha = \lambda_\alpha^+ \quad (2.76)$$

which in turn leads to the following reality condition on  $D$ ,

$$D = D^+ = \bar{D}. \quad (2.77)$$

The  $[Q, D]$  commutation relation then leads to,

$$\mu_\alpha = -i \partial_{\alpha\dot{\alpha}} \bar{\lambda}^{\dot{\alpha}}, \quad (2.78)$$

Consistency with  $\{\bar{Q}, \mu\}$  and  $\{S, \mu\}$  then imposes the abelian Bianchi identity,

$$\epsilon^{\beta\gamma} \partial_{\gamma\dot{\alpha}} f_{\alpha\beta} + \bar{\epsilon}^{\dot{\gamma}\dot{\beta}} \partial_{\alpha\dot{\gamma}} \bar{f}_{\dot{\alpha}\dot{\beta}} = 0, \quad r = 1. \quad (2.79)$$

where the usual field strength can be retrieved in the following fashion,

$$F_{\alpha\dot{\alpha},\beta\dot{\beta}} = \epsilon_{\alpha\beta} \bar{f}_{\dot{\alpha}\dot{\beta}} + \bar{\epsilon}_{\dot{\alpha}\dot{\beta}} f_{\alpha\beta} \quad (2.80)$$

and the the value of the  $R$ -charge  $r = 1$  for the gaugino  $\lambda$  is imposed by consistency of the supersymmetry algebra. This in turn implies a zero value for the  $R$ -charge of the field strength  $f_{\alpha\beta}$ ,  $\bar{f}_{\dot{\alpha}\dot{\beta}}$  and  $D$ , and hence no anomalous dimensions are possible.

The fermionic states contributing to the index are then given by,

$$P_{21}^m P_{22}^n |\bar{\lambda}_\alpha\rangle, \quad \forall m, n = 0, 1, 2 \dots \quad (2.81)$$

but given the constraints,

$$\partial_{22} \bar{\lambda}_1 = \partial_{21} \bar{\lambda}_2 \quad (2.82)$$

the set of non-redundant fermionic states contributing to the index are given by,

$$\{P_{21}^m P_{22}^n |\bar{\lambda}_1\rangle, P_{22}^n |\bar{\lambda}_2\rangle\}, \quad (2.83)$$

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while the contributing bosonic states are,

$$P_{21}^m P_{22}^n |f_{22}\rangle. \quad (2.84)$$

Consequently, the index for the vector multiplet is given by,

$$i_\lambda(t, x) = -\frac{tx}{(1-tx)(1-tx^{-1})} - \frac{tx^{-1}}{1-tx^{-1}} + \frac{t^2}{(1-tx)(1-tx^{-1})} \quad (2.85)$$

or equivalently,

$$i_\lambda(t, x) = \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})}, \quad \chi_2(x) = x + x^{-1} \quad (2.86)$$

For non trivial gauge  $G$  and flavour symmetry  $F$ , denote  $g, h$  respective elements of  $G, H$ . Denoting the corresponding representations as  $R_g, R_f$ , the definition of the single particle index expression is modified in the following fashion,

$$i_\lambda(t, x, g) = \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})} \chi_{\text{adj.}}(g), \quad (2.87)$$

$$i_\phi(t, x, g, h) = \frac{t^{\frac{2}{3}} \chi_{R_g}(g) \chi_{R_f}(h) - t^{\frac{4}{3}} \chi_{R_g}(g) \chi_{\bar{R}_f}(h)}{(1-tx)(1-tx^{-1})}, \quad (2.88)$$

with gauge and flavour symmetry group characters defined as the trace of the group element  $U$  in the representation  $R_g$ ,

$$\chi_{R_g}(U) = \text{Tr}_{R_g} U. \quad (2.89)$$

### 2.4 Indices and Characters

In this section we emphasize the close relationship between indices and conformal and superconformal characters. This allows one to understand the state dependence of the index. We first list the various possible representations of the superconformal algebra and the various possible shortening conditions. We differentiate between long, semi short and short representations of the algebra, the latter being otherwise known as chiral representations.

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### 2.4.1 Primary States and Verma Modules

In this section we list all possible representations of the superconformal algebra. We will denote the representations as  $V_{\Delta,r,j,\bar{j}}^{f,\bar{f}}$ . The subscripts denote the quantum numbers of the primary state which acts as a seed of the representation,

$$(H, R, J_3, \bar{J}_3)|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = (\Delta, R, J_3, \bar{J}_3)|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}}, \quad (2.90)$$

and the primary state is both a lowest lying state with respect to the set of  $H$ -eigenvalues of the states in the representation as well as a highest weight state wrt the  $SU(2)_l \times SU(2)_r$  part of the algebra,

$$(K^{\dot{\alpha}}, S^\alpha, \bar{S}^{\dot{\alpha}}, J_+, \bar{J}_+)|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0. \quad (2.91)$$

The Verma module  $\mathcal{V}_{\Delta,r,j,\bar{j}}^{f,\bar{f}}$  associated with each primary state can be obtained by acting with the positive dimension generators of the superconformal algebra as well as the  $J_-, \bar{J}_-$  generators,

$$\mathcal{V}_{\Delta,r,j,\bar{j}}^{f,\bar{f}} = \{P_{\alpha\dot{\alpha}}^{N_{\alpha\dot{\alpha}}} Q_\alpha^{N_\alpha} \bar{Q}_{\dot{\alpha}}^{\bar{N}_{\dot{\alpha}}} J_-^N \bar{J}_-^{\bar{N}} |\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} |N, \bar{N}, N_{\alpha\dot{\alpha}} \in \mathbb{N}, N_\alpha, N_{\dot{\alpha}} = 0, 1\} \quad (2.92)$$

while the actual representation is obtained by the usual unitarity constraints on  $SU(2)$  and hence restrict  $N, \bar{N}$  to  $0 \dots 2j$  and  $0 \dots 2\bar{j}$  respectively. Finally the  $f, \bar{f}$  superscripts in the definition of the representation subscripts denote the fraction of the  $Q$  and  $\bar{Q}$  respectively annihilate the primary states.

When those BPS conditions are imposed, the Verma modules are truncated. Consider the left-moving  $Q$  supercharges. They fall into a  $\frac{1}{2}$  representation of  $SU(2)_l$ , with  $Q_2$  the highest weight state. Hence, when acting on an  $SU(2)_l$  highest weight state  $|\text{hw}_L\rangle$ , then  $Q_2|\text{hw}_L\rangle$  also is an  $SU(2)_l$  highest weight state. However this is not the case for  $Q_1$ . A most convenient way of visualising a representation  $V$  or a Verma module  $\mathcal{V}$  will be to list the highest weight states of  $SU(2)_l \times SU(2)_r$  obtained by acting with  $Q$  on the primary state. Given the properties of  $Q_\alpha$ , those states generated by  $Q_2$  and a modified version of  $Q_1$  denoted

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$\tilde{Q}_1$  such that,

$$[J_+, \tilde{Q}_1]|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0, \quad (2.93)$$

and the simplest choice for such a supercharge is given by,

$$\tilde{Q}_1 = Q_1 + \frac{1}{2j} Q_2 J_-, \quad j \neq 0, \quad (2.94)$$

$$\tilde{Q}_2 = \bar{Q}_2 - \frac{1}{2\bar{j}} \bar{Q}_1 \bar{J}_-, \quad \bar{j} \neq 0. \quad (2.95)$$

where we have also given the result of the similar analysis for  $SU(2)_r$ <sup>1</sup>. in the other sections. Hence we are now equipped with a set of supercharges such that,

$$[J_+, \tilde{Q}_\alpha] = [\bar{J}_+, \tilde{Q}_{\dot{\alpha}}] = 0, \quad (2.96)$$

when acting on a primary state, and

$$[J_3, \tilde{Q}_1] = -\frac{1}{2}\tilde{Q}_1, \quad [\bar{J}_3, \tilde{Q}_2] = -\frac{1}{2}\tilde{Q}_2, \quad (2.97)$$

We have extended the definition to  $\tilde{Q}_2 = Q_2$  as well as  $\tilde{Q}_1$ . One can then define the following semi-short representations,

$$f = \frac{1}{2}, \quad \tilde{Q}_1|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0, \quad \Delta = \Delta_{r,j}, \quad \text{when } j \neq 0, \quad (2.98)$$

$$\bar{f} = \frac{1}{2}, \quad \tilde{Q}_2|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0, \quad \Delta = \Delta_{-r,\bar{j}}, \quad \text{when } \bar{j} \neq 0, \quad (2.99)$$

When  $j$  or  $\bar{j} = 0$ , the natural shortening condition is given by,

$$f = \frac{1}{2}, \quad Q^2|\Delta, r, 0, \bar{j}\rangle^{\text{h.w.}} = 0, \quad \Delta = \Delta_{r,0}, \quad (2.100)$$

$$\bar{f} = \frac{1}{2}, \quad \bar{Q}^2|\Delta, r, j, 0\rangle^{\text{h.w.}} = 0, \quad \Delta = \Delta_{-r,0}. \quad (2.101)$$

The shortening conditions impose the above values for the scaling dimensions of the primary state, and implicitly the rest of the multiplet, with,

$$\Delta_{r,j} = -\frac{3}{2}r + 2 + 2j. \quad (2.102)$$

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<sup>1</sup>In this character based approach, the supercharges above will play a role similar to the supercharge  $Q$  defined in equation (2.36). One can regard this as a convenient change of basis.

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This expression can be derived from the shortening condition and imposing the zero-norm condition on  $\tilde{Q}_1|\text{h.w.}\rangle$  and the corresponding state for  $\bar{f} = 1/2$  while making use of the following relation,

$$\frac{1}{2}\{\tilde{Q}_1, \tilde{Q}_1^+\} = H + \frac{3}{2}R - 2J_3 - 2, \quad (2.103)$$

$$\frac{1}{2}\{\tilde{\bar{Q}}_2, \tilde{\bar{Q}}_2^+\} = H - \frac{3}{2}R - 2\bar{J}_3 - 2, \quad (2.104)$$

One could wonder why the shortening conditions (2.98), (2.99) do not involve  $\tilde{Q}_2 = Q_2$ . The reason is, assuming a representation whereby,

$$Q_2|\text{h.w.}\rangle = 0 \Rightarrow \Delta + \frac{3}{2}r + 2j = 0, \quad (2.105)$$

where we have used the fact that,

$$\{Q_2, Q_2^+\} = 2(H + \frac{3}{2}R + 2J_3) = 2\mathcal{H}_2. \quad (2.106)$$

One then notices that,

$$\{\mathcal{H}_2, \tilde{Q}_1\} = -2\tilde{Q}_1, \quad (2.107)$$

hence the state  $\tilde{Q}_1|\text{h.w.}\rangle$  such that its  $\mathcal{H}_2$  eigenvalue is negative, eventhough  $\mathcal{H}_2$  is positive semi-definite given its definition (2.106). For physical application, we only consider unitary representations, hence the definition of the semi short representation (2.98), (2.99).

Short representations, or chiral multiplets are defined as,

$$f = 1, \quad Q_\alpha|\Delta, r, 0, \bar{j}\rangle^{\text{h.w.}} = 0, \quad \Delta = -\frac{3}{2}r, \quad (2.108)$$

$$\bar{f} = 1, \quad \bar{Q}_{\dot{\alpha}}|\Delta, r, j, 0\rangle^{\text{h.w.}} = 0, \quad \Delta = \frac{3}{2}r, \quad (2.109)$$

with the scaling dimension obtained from the traceless of the zero norm condition corresponding to the shortening condition, while the  $j = 0, \bar{j} = 0$  conditions come from its traceless part.

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### 2.4.2 Short, Semi-Short, and Long Representations

We can now give a diagrammatic representation of the long multiplet in a similar fashion as in [55].

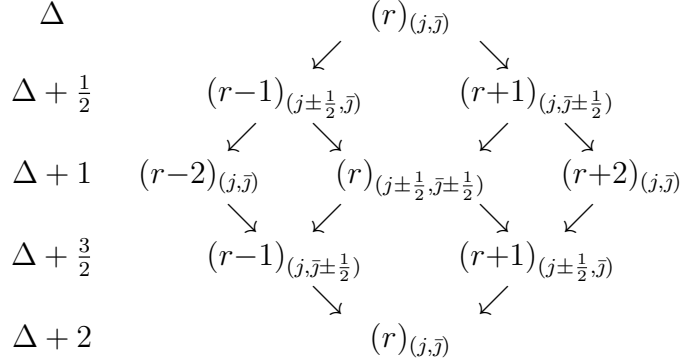


Figure 2.2: Long Representation  $V_{\Delta, r, j, \bar{j}}^{0,0}$

In figure (2.2), we have represented the states obtained by acting on the primary state with the various  $\tilde{Q}$  and  $\tilde{\bar{Q}}$  supercharges, and are  $SU(2)_l \times SU(2)_r$  highest weight states. One can hence generate the entire  $SU(2)$  representations by acting with lowering operators on these states. The translation generators  $P$  also act as ladder operators in the sense that they increase or decrease the values of  $(j, \bar{j})$ . One can then use a Clebsch-Gordan decomposition, and

$$P_{21}|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = C|\Delta, r, j + \frac{1}{2}, \bar{j} + \frac{1}{2}\rangle + \dots, \quad (2.110)$$

$$P_{12}|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = C|\Delta, r, j - \frac{1}{2}, \bar{j} - \frac{1}{2}\rangle + \dots, \quad (2.111)$$

$$P_{11}|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = C|\Delta, r, j - \frac{1}{2}, \bar{j} + \frac{1}{2}\rangle + \dots, \quad (2.112)$$

$$P_{22}|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = C|\Delta, r, j + \frac{1}{2}, \bar{j} - \frac{1}{2}\rangle + \dots, \quad (2.113)$$

where the constant  $C$  is the appropriate Clebsch-Gordan coefficient and the state written is a highest weight state for  $SU(2)_r \times SU(2)_l$  but is not a highest-weight state in the sense that it is not primary. The dots denote extra-states coming from the Clebsch-Gordan decomposition which are not highest-weight states in any sense of the word. Hence the  $P$  generators allow to generate  $SU(2)$  representations with all possible values of  $(j, \bar{j})$  half-integers from zero to infinity.

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One can then give the corresponding diagram for semi short representations as shown in figure (2.3). while the other semi-short representation's diagram is

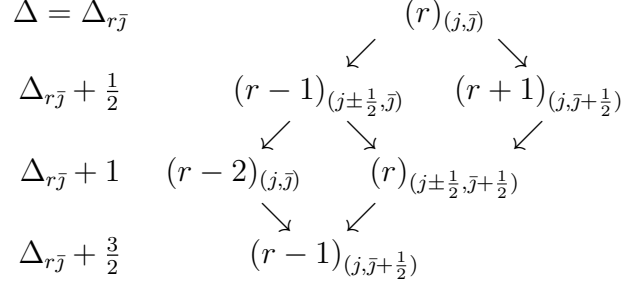


Figure 2.3: Semi-Short Representation  $V_{\Delta_{r\bar{j}},r,j,\bar{j}}^{0,\frac{1}{2}}$

shown in figure (2.4).

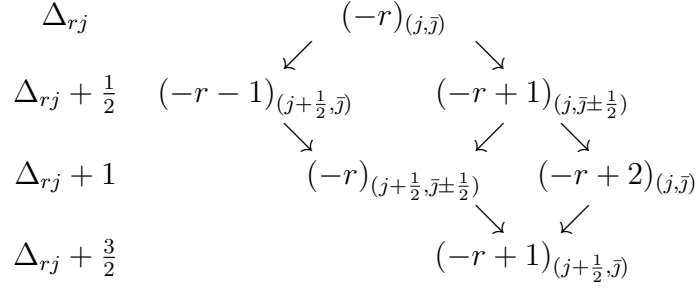


Figure 2.4: Semi-Short Representation  $V_{\Delta_{rj},-r,j,\bar{j}}^{\frac{1}{2},0}$

The  $f = 0, \bar{f} = 1$  and  $f = 1, \bar{f} = 0$  representations are given in figure (2.5).

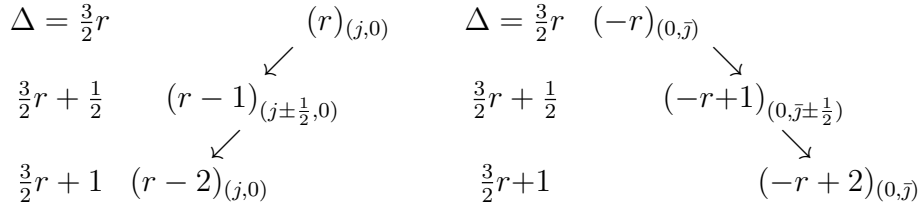


Figure 2.5: Short Representations  $V_{\frac{3}{2}r,r,j,0}^{0,1}$  left, and  $V_{\frac{3}{2}r,-r,0,\bar{j}}^{1,0}$  right

Let us now consider representations which satisfy shortening conditions for both supercharge sectors. In that case, one needs to be careful to not overcount or undercount states due to the  $\{Q, \bar{Q}\} = P$  commutation relation.

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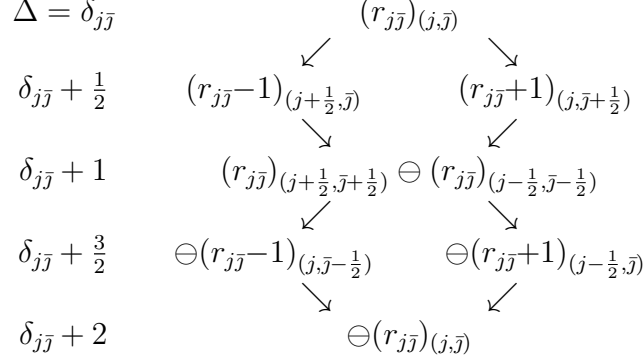


Figure 2.6: Semi-Short Representation  $V_{\delta_{j\bar{j}}, r_{j\bar{j}}, j, \bar{j}}^{\frac{1}{2}, \frac{1}{2}}$

The values of the scaling dimension of the primary state as well as the  $R$ -charge are obtained by imposing  $\Delta = \Delta_{r,j} = \Delta_{-r,\bar{j}}$  as defined in equation (2.102) and solving for  $(\Delta, r)$ ,

$$\delta_{j\bar{j}} = 2 + j + \bar{j}, \quad r_{j\bar{j}} = \frac{2}{3}(j - \bar{j}) \quad (2.114)$$

Also, one has to impose the following conservation condition,

$$\{\tilde{Q}_1, \tilde{Q}_2\}|\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0 \quad (2.115)$$

with,

$$\{\tilde{Q}_1, \tilde{Q}_2\} = 2P_{12} - \frac{1}{2j\bar{j}}J_-P_{21}\bar{J}_- + \frac{1}{j}J_-P_{22} - \frac{1}{\bar{j}}P_{11}\bar{J}_- \quad (2.116)$$

Consequently, one has to subtract the set of states descending from  $(\delta_{j\bar{j}}, r_{j\bar{j}})_{(j-\frac{1}{2}, \bar{j}-\frac{1}{2})}$  as well as its  $Q_2$  and  $\bar{Q}_1$  descendants as indicated in figure (2.6).

Finally the  $f = \frac{1}{2} \bar{f} = 1$  and  $f = 1 \bar{f} = \frac{1}{2}$  are given in figure (2.7). In the chiral case, the scaling dimension of the primary field is obtained by imposing both constraints (2.102) and (2.99) and solving for  $(\Delta, r)$ ,

$$\Delta_j = j + 1, \quad r_j = \frac{2}{3}(j + 1). \quad (2.117)$$

In order to account for the conservation conditions,

$$\{\tilde{Q}_1, \bar{Q}_{\dot{\alpha}}\}|\Delta, r_j, j, 0\rangle^{\text{h.w.}} = 0, \quad \{\tilde{Q}_2, Q_{\alpha}\}|\Delta, r_j, 0, \bar{j}\rangle^{\text{h.w.}} = 0, \quad (2.118)$$



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one has to subtract the  $(r_j)_{(j-\frac{1}{2},\frac{1}{2})}$  and its  $Q_2$  descendent. However, too many states are subtracted by imposing such a constraint as the translation  $P$  generators are commuting. This then implies that,

$$\left[ \{\tilde{Q}_1, \bar{Q}_{\dot{\alpha}}\}, \{\tilde{Q}_1, \bar{Q}_{\dot{\beta}}\} \right] = 0, \quad \left[ \{\tilde{Q}_2, Q_{\alpha}\}, \{\tilde{Q}_2, Q_{\beta}\} \right] = 0, \quad (2.119)$$

consequently we need to add back the states  $(r_j)_{(j-1,0)}$  and scaling dimension  $\Delta_j + \frac{5}{2}$  to the multiplet and its  $Q_2$  descendent in the  $f = \frac{1}{2}$ ,  $\bar{f} = 1$  case as indicated in figure (2.7).

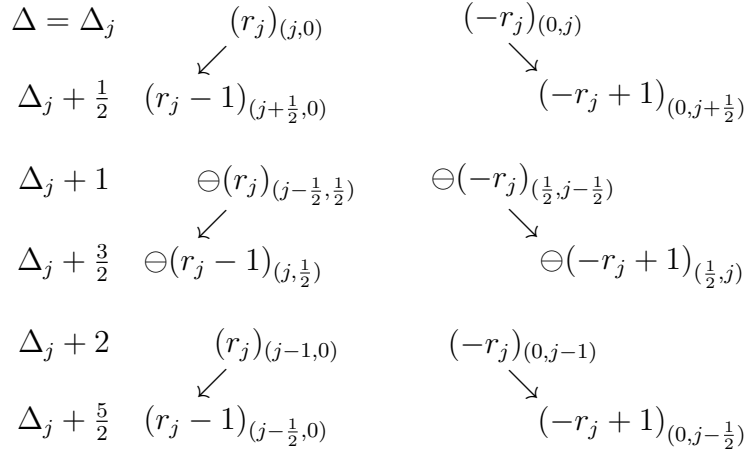


Figure 2.7: Short Representations  $V_{\Delta_j, r_j, j, 0}^{\frac{1}{2}, 1}$  left, and  $V_{\Delta_j, -r_j, 0, j}^{\frac{1}{2}, \frac{1}{2}}$  right

### 2.4.3 Multiplet Decompositions at Unitarity Thresholds

As seen previously, the possible shortening conditions on the multiplets lead to restrictions on the scaling dimensions and  $R$ -charges of states in the multiplet, and the shorter the multiplet, the more constrained its quantum numbers. This leads to long multiplet decomposition at unitarity thresholds, i.e. when a long multiplet saturates the shortening bounds, it can be decomposed into shorter

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multiplets,

$$V_{\Delta_{r\bar{j}}, r, j, \bar{j}}^{0,0} = V_{\Delta_{r\bar{j}}, r, j, \bar{j}}^{0, \frac{1}{2}} \oplus V_{\Delta_{r+1, \bar{j}-\frac{1}{2}}, r+1, j, \bar{j}-\frac{1}{2}}^{0, \frac{1}{2}}, \quad (2.120)$$

$$V_{\Delta_{r\bar{j}}, -r, j, \bar{j}}^{0,0} = V_{\Delta_{r\bar{j}}, -r, j, \bar{j}}^{\frac{1}{2}, 0} \oplus V_{\Delta_{r+1, \bar{j}-\frac{1}{2}}, -r-1, j, \bar{j}-\frac{1}{2}}^{\frac{1}{2}, 0}, \quad (2.121)$$

$$V_{\delta_{j\bar{j}}, r_{j\bar{j}}, j, \bar{j}}^{\frac{1}{2}, \frac{1}{2}} = V_{\delta_{j\bar{j}}, r_{j\bar{j}}, j, \bar{j}}^{0, \frac{1}{2}} \ominus V_{\delta_{j\bar{j}+\frac{1}{2}}, r_{j\bar{j}-1}, j, \bar{j}-\frac{1}{2}}^{\frac{1}{2}, 0} \quad (2.122)$$

$$= V_{\delta_{j\bar{j}}, r_{j\bar{j}}, j, \bar{j}}^{\frac{1}{2}, 0} \ominus V_{\delta_{j\bar{j}+\frac{1}{2}}, r_{j\bar{j}+1}, j, \bar{j}-\frac{1}{2}}^{0, \frac{1}{2}}, \quad (2.123)$$

### 2.4.4 Character Calculations

One can then compute the corresponding characters for all the previous representations, where the characters are by definition given by,

$$\chi^{f, \bar{f}}(s, u, x, \bar{x}) = \text{Tr}_{V_{f, \bar{f}}} (s^{2H} u^R x^{2J_3} \bar{x}^{2\bar{J}_3}), \quad (2.124)$$

One can compute those quantities by looking at the previous diagrams for the various representations given the following prescription,

$$(\Delta, (r)_{j, \bar{j}}) \leftrightarrow s^{2\Delta} u^r \chi_j(x) \chi_{\bar{j}}(\bar{x}) P(s, x, \bar{x}) \quad (2.125)$$

where the spin- $j$   $SU(2)$  character is defined as,

$$\chi_j(x) = \text{Tr}_j x^{2J_3} = \sum_{m=-j}^j x^{2m} = \frac{x^{2j+1} - x^{-2j-1}}{x - x^{-1}} \quad (2.126)$$

while the action of the translation generators  $P_{\alpha\dot{\alpha}}$  gives another factor,

$$P(s, x, \bar{x}) = \prod_{\epsilon, \eta = \pm 1} \frac{1}{1 - s^2 x^\epsilon \bar{x}^\eta}, \quad (2.127)$$

Based on this analysis, one can then give the following characters,

$$\chi_{\Delta, r, j, \bar{j}}^{0,0}(s, u, x, \bar{x}) = s^{2\Delta} u^r \chi_j(x) \chi_{\bar{j}}(\bar{x}) P(s, x, \bar{x}) Q(s, u, x) \bar{Q}(s, u, \bar{x}) \quad (2.128)$$

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with the supercharge factors defined as,

$$Q(s, u, x) = \prod_{\epsilon=\pm 1} (1 + su^{-1}x^\epsilon), \quad (2.129)$$

$$\bar{Q}(s, u, \bar{x}) = \prod_{\epsilon=\pm 1} (1 + su\bar{x}^\epsilon), \quad (2.130)$$

The semi-short representations' characters are given by,

$$\begin{aligned} \chi_{r,j,\bar{j}}^{0,\frac{1}{2}}(s, u, x, \bar{x}) &= s^{3r+4\bar{j}+4} u^r (\chi_{\bar{j}}(\bar{x}) + su\chi_{\bar{j}+\frac{1}{2}}(\bar{x}))\chi_j(x)P(s, x, \bar{x})Q(s, u, x), \\ \chi_{r,j,\bar{j}}^{\frac{1}{2},0}(s, u, x, \bar{x}) &= s^{-3r+4j+4} u^r (\chi_j(x) + su^{-1}\chi_{j+\frac{1}{2}}(x))\chi_{\bar{j}}(\bar{x})P(s, x, \bar{x})\bar{Q}(s, u, \bar{x}), \end{aligned}$$

and,

$$\chi_{r,j}^{0,1}(s, u, x, \bar{x}) = s^{3r}u^r \chi_j(x)P(s, x, \bar{x})Q(s, u, x), \quad (2.131)$$

$$\chi_{r,j}^{1,0}(s, u, x, \bar{x}) = s^{3r}u^{-r} \chi_{\bar{j}}(\bar{x})P(s, x, \bar{x})\bar{Q}(s, u, \bar{x}). \quad (2.132)$$

Further semi-short characters are given by,

$$\begin{aligned} \chi_{j,\bar{j}}^{\frac{1}{2},\frac{1}{2}}(s, u, x, \bar{x}) &= s^{4+2j+2\bar{j}}u^{\frac{2}{3}(j-\bar{j})}P(s, x, \bar{x})(D_{j,\bar{j}}(s, x, \bar{x}) + D_{j+\frac{1}{2},\bar{j}+\frac{1}{2}}(s, x, \bar{x}) \\ &\quad u D_{j,\bar{j}+\frac{1}{2}}(s, x, \bar{x}) + u^{-1} D_{j+\frac{1}{2},\bar{j}}(s, x, \bar{x})), \end{aligned} \quad (2.133)$$

where  $D_{j,\bar{j}}(s, x, \bar{x})$  is defined as,

$$D_{j,\bar{j}}(s, x, \bar{x}) = \chi_j(x)\chi_{\bar{j}}(\bar{x}) - s^2\chi_{j-\frac{1}{2}}(x)\chi_{\bar{j}-\frac{1}{2}}(\bar{x}), \quad (2.134)$$

where the minus sign comes from the subtracted state arising from the conservation condition (2.115). Finally the short representations' characters are given by,

$$\begin{aligned} \chi_j^{\frac{1}{2},1}(s, u, x, \bar{x}) &= s^{2j+2}u^{\frac{2}{3}(j+1)}P(s, x, \bar{x})(C_j(s, x, \bar{x}) + su^{-1}C_{j-\frac{1}{2}}(s, x, \bar{x})), \\ \chi_{\bar{j}}^{1,\frac{1}{2}}(s, u, x, \bar{x}) &= s^{2\bar{j}+2}u^{-\frac{2}{3}(\bar{j}+1)}P(s, x, \bar{x})(C_{\bar{j}}(s, \bar{x}, x) + suC_{\bar{j}-\frac{1}{2}}(s, \bar{x}, x)), \end{aligned} \quad (2.135)$$

with,

$$C_j(s, x, \bar{x}) = \chi_j(x) - s^2\chi_{j-\frac{1}{2}}(x)\chi_{\frac{1}{2}}(\bar{x}) + s^4\chi_{j-1}(x) \quad (2.136)$$

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### 2.4.5 From Characters to Indices

In order to construct the index as defined in equation (2.49), one first needs to obtain the  $(-1)^F$  factor. This can be done by noticing that fermionic and bosonic states can be identified by comparing their  $SU(2)$  eigenvalues,

$$|j - \bar{j}| \in \mathbb{Z} + \frac{1}{2} \Leftrightarrow \text{Fermionic}, \quad (2.137)$$

$$|j - \bar{j}| \in \mathbb{Z} \Leftrightarrow \text{Bosonic}, \quad (2.138)$$

with  $(j, \bar{j})$  the  $(J_3, \bar{J}_3)$  eigenvalues of the highest weight state, hence, for the highest weight state,

$$(-1)^F = (-1)^{2J_3} (-1)^{2\bar{J}_3}, \quad (2.139)$$

Given that the action of  $J_-, \bar{J}_-$  change the eigenvalues of  $2J_3, 2\bar{J}_3$  by two units, one can use the same formula for the entire multiplet. Consequently, one can see that,

$$\chi^{f, \bar{f}}(s, u, -x, -\bar{x}) = \text{Tr}_{V^{f, \bar{f}}} (-1)^F s^{2H} u^R x^{2J_3} \bar{x}^{2\bar{J}_3} \quad (2.140)$$

In order to compute the single particle index (2.49), one can then take,

$$i^{f, \bar{f}}(t, x) = \chi^{f, \bar{f}}(y^{\frac{1}{2}}, y^{\frac{3}{2}} t^{-1}, -ty^{-1}, -x) \quad (2.141)$$

Note that we have added the  $(f, \bar{f})$  notation to the index definition to keep track of the contributions of the various multiplets. As was noted before, the index only depends on multiplets that are part of the  $Q_1$ -cohomology. This explains the fact that,

$$i^{0,0}(t, x) = i^{0, \frac{1}{2}}(t, x) = i^{0,1}(t, x) = 0, \quad (2.142)$$

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while the following nonzero results hold,

$$i_{j,\bar{j},r}^{\frac{1}{2},0}(t,x) = \frac{\chi_{\bar{j}}(x)}{(1-tx^{-1})(1-tx)} (-1)^{2(j+\bar{j})+1} t^{2j+r+2}, \quad (2.143)$$

$$i_{j,\bar{j},r}^{1,0}(t,x) = \frac{\chi_{\bar{j}}(x)}{(1-tx^{-1})(1-tx)} (-1)^{2\bar{j}} t^r, \quad (2.144)$$

$$i_{j,\bar{j}}^{\frac{1}{2},\frac{1}{2}}(t,x) = \frac{\chi_{\bar{j}}(x)}{(1-tx^{-1})(1-tx)} (-1)^{2(j+\bar{j})+1} t^{\frac{2}{3}(2j+\bar{j}+3)}, \quad (2.145)$$

$$i_j^{\frac{1}{2},1}(t,x) = \frac{1}{(1-tx^{-1})(1-tx)} (-1)^{2j+1} t^{\frac{4}{3}(j+1)}, \quad (2.146)$$

$$i_j^{1,\frac{1}{2}}(t,x) = \frac{\chi_{\bar{j}}(x) - t\chi_{\bar{j}-\frac{1}{2}}(x)}{(1-tx^{-1})(1-tx)} (-1)^{2\bar{j}} t^{\frac{2\bar{j}}{3}+\frac{2}{3}}, \quad (2.147)$$

formula (2.69) and (2.86) by using the above formulas,

$$i_\phi(t,x) = i_0^{\frac{1}{2},1}(t,x) + i_0^{1,\frac{1}{2}}(t,x), \quad (2.148)$$

$$i_\lambda(t,x) = i_{\frac{1}{2}}^{\frac{1}{2},1}(t,x) + i_{\frac{1}{2}}^{1,\frac{1}{2}}(t,x). \quad (2.149)$$

### 2.4.6 Characters, Indices and Gauge Symmetry

From the definition of the index as a limit of a character (2.141), one can see the need for the integration over the gauge group (2.52). Characters are useful in decomposing tensor product representations because of the orthogonality relations between characters of irreducible representations [60],

$$\int d\mu(g) \chi_{R_g}(g) \chi_{\bar{R}_g}(g) = \delta_{R_g \bar{R}_g}, \quad (2.150)$$

with  $d\mu(g)$  the normalised left invariant Haar measure on the gauge group seen as a manifold. This allows one to identify the number of of type  $R$  in a given tensor product representation  $R_1, \dots, R_n$ ,

$$\int d\mu(g) \chi_R(g) \prod_{i=1}^n \chi_{R_i}(g) = n_R, \quad (2.151)$$

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and in particular, one can project a given expression for an index onto gauge singlets by taking,

$$\int d\mu(g)\chi_R(g), \tag{2.152}$$

which leads to the expression for the index of gauged chiral multiplets and vector multiplets (2.52).

# Chapter 3

## Index and Arbitrary $R$ -Charges

In this chapter, we provide the appropriate setup to deal with indices for matter multiplets with arbitrary  $R$ -charges. After providing the relevant differential geometry generators to set up the theory on  $\mathbb{R} \times S^3$ , we define appropriate Killing spinors and vectors. We then expand the theory in  $S^3$  harmonics and canonically quantise it. Having obtained the appropriate quantum numbers, we take once again the approach provided in chapter (2) and compute the index in the form of the appropriate limits of the group characters.

### 3.1 Supersymmetry on $\mathbb{R} \times S^3$

#### 3.1.1 $\mathbb{R} \times S^3$ Group Conventions

We use standard conventions for  $\mathbb{R} \times S^3$  and take a line element to be given by,

$$ds^2 = -dt^2 + d\bar{s}^2, \quad d\bar{s}^2 = g_{\mu\nu} dx^\mu dx^\nu, \quad \mu, \nu = 1 \dots 3, \quad (3.1)$$

where the  $S^3$  bi-invariant metric  $g$  is,

$$g_{\mu\nu} = 2 e^m{}_\mu e_{m\nu} = 2 \bar{e}^m{}_\mu \bar{e}_{m\nu}, \quad (3.2)$$

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with greek spacetime indices and roman tangent space indices. The left or right  $SU(2)$  invariant one forms are then given by,

$$e = U dU^{-1}, \quad \bar{e} = U^{-1} dU, \quad (3.3)$$

for  $U \in SU(2)$ . Using the normalisation,

$$e = -\frac{i}{2} \sigma_m e^m, \quad \bar{e} = -\frac{i}{2} \sigma_m \bar{e}^m, \quad (3.4)$$

with  $\sigma_m$  the standard Pauli matrices, and denoting the Euler angles,  $x^\mu = (\theta^1, \theta^2, \theta^3)$ ,

$$0 \leq \theta^1 \leq \pi, \quad 0 \leq \theta^2 \leq 2\pi, \quad 0 \leq \theta^3 \leq 4\pi, \quad (3.5)$$

one can use the following standard parametrisation for  $SU(2)$ ,

$$U = \exp(-\frac{i}{2}\theta^2\sigma_3) \exp(-\frac{i}{2}\theta^1\sigma_2) \exp(-\frac{i}{2}\theta^3\sigma_3) = n_0 1 + i n_m \sigma_m, \quad (3.6)$$

with the following definition,

$$n_0 = \cos\left(\frac{1}{2}\theta^1\right) \cos\left(\frac{1}{2}\theta^2 + \frac{1}{2}\theta^3\right), \quad n_2 = -\sin\left(\frac{1}{2}\theta^1\right) \cos\left(\frac{1}{2}\theta^2 - \frac{1}{2}\theta^3\right), \quad (3.7)$$

$$n_1 = \sin\left(\frac{1}{2}\theta^1\right) \sin\left(\frac{1}{2}\theta^2 - \frac{1}{2}\theta^3\right), \quad n_3 = -\cos\left(\frac{1}{2}\theta^1\right) \sin\left(\frac{1}{2}\theta^2 + \frac{1}{2}\theta^3\right), \quad (3.8)$$

and,

$$\text{Det } U = n_0^2 + n_m n_m = 1, \quad (3.9)$$

then, the right invariant one form, which we will take as dreibein, is,

$$e^m = e^m{}_\mu dx^\mu. \quad (3.10)$$

Explicitly,

$$e^m = \begin{cases} e^1 = \sin \theta^3 d\theta^1 - \sin \theta^1 \cos \theta^3 d\theta^2, \\ e^2 = \cos \theta^3 d\theta^1 + \sin \theta^1 \sin \theta^3 d\theta^2, \\ e^3 = \cos \theta^1 d\theta^2 + d\theta^3. \end{cases} \quad (3.11)$$



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The left invariant one-form is given by,

$$\bar{e}^m = \begin{cases} \bar{e}^1 = \sin \theta^2 d\theta^1 - \sin \theta^1 \cos \theta^2 d\theta^3, \\ \bar{e}^2 = -\cos \theta^2 d\theta^1 - \sin \theta^1 \sin \theta^2 d\theta^3, \\ \bar{e}^3 = -\cos \theta^1 d\theta^3 - d\theta^2. \end{cases} \quad (3.12)$$

In this basis, the left invariant vector fields  $\nabla_m$  can be constructed from  $\omega^m(\nabla_n) = \delta_n^m$ , and denoting,

$$\nabla_m = e_m^\mu \partial_\mu, \quad (3.13)$$

one obtains,

$$\nabla_m = \begin{cases} \nabla_1 = \sin \theta^3 \partial_1 - \csc \theta^1 \cos \theta^3 \partial_2 + \cot \theta^1 \cos \theta^3 \partial_3, \\ \nabla_2 = \cos \theta^3 \partial_1 + \csc \theta^1 \sin \theta^3 \partial_2 - \cot \theta^1 \sin \theta^3 \partial_3, \\ \nabla_3 = \partial_3, \end{cases} \quad (3.14)$$

satisfying an  $SU(2)$  commutation relations as differential operators,

$$[\nabla_m, \nabla_n] = \varepsilon_{mnp} \nabla_p. \quad (3.15)$$

One can also construct the right invariant vector field  $\bar{\nabla}^m$  which also satisfies the above commutation relation, by imposing  $\bar{e}^m(\bar{\nabla}_m) = \delta_n^m$ . Note that both the definition (3.3) and the parametrisation have been chosen here so that they may be obtained by the parameter changes  $\theta^3 \leftrightarrow -\theta^2$

$$\bar{\nabla}_m = \begin{cases} \bar{\nabla}_1 = \sin \theta^2 \partial_1 + \cot \theta^1 \cos \theta^2 \partial_2 - \csc \theta^1 \cos \theta^2 \partial_3, \\ \bar{\nabla}_2 = -\cos \theta^3 \partial_1 + \cot \theta^1 \sin \theta^2 \partial_2 - \csc \theta^1 \sin \theta^2 \partial_3, \\ \bar{\nabla}_3 = -\partial_2, \end{cases} \quad (3.16)$$

These also satisfy,

$$[\bar{\nabla}_m, \bar{\nabla}_n] = \varepsilon_{mnp} \bar{\nabla}_p, \quad [\nabla_m, \bar{\nabla}_n] = 0, \quad (3.17)$$

For  $\mathbb{R} \times S^3$ , the left invariant one-forms and vector fields are supplemented by,

$$e^0 = dt, \quad \nabla_0 = \partial_0, \quad (3.18)$$

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determining the appropriate vierbein. The spin connection may then be determined by imposing the conservation of the dreibein. The torsion free Maurer-Cartan equations reads,

$$de_m = -\omega_{mn} \wedge e_n. \quad (3.19)$$

As there is no curvature in the time direction, one can focus on the spatial part,

$$\omega_{mnp} = e_m^\mu \nabla_n e_{p\mu} - e_m^\mu e_n^\nu \Gamma_{\mu\nu}^\rho e_{p\rho}, \quad (3.20)$$

where the non-zero Christoffel symbols are determined to be,

$$\Gamma_{23}^1 = \frac{1}{2} \sin \theta^1, \quad \Gamma_{13}^2 = \Gamma_{12}^3 = -\frac{1}{2} \csc \theta^1, \quad \Gamma_{12}^2 = \Gamma_{13}^3 = \frac{1}{2} \cot \theta^1, \quad (3.21)$$

gives non-zero components,

$$\omega_{mnp} = -\frac{1}{2} \varepsilon_{mnp}. \quad (3.22)$$

Denoting,

$$\sigma_{mn} = \frac{1}{2} [\sigma_m, \sigma_n] = i \varepsilon_{mnp} \sigma_p, \quad (3.23)$$

the spinor spacetime covariant derivative acting on spinors is defined as,

$$D_m \chi_\alpha = D_m \chi_\alpha = \nabla_m \chi_\alpha + \frac{1}{4} \omega_{mnp} (\sigma_{np} \chi)_\alpha \quad (3.24)$$

and after simplification,

$$D_0 \psi_\alpha = \partial_0 \psi_\alpha, \quad D_m \chi_\alpha = \nabla_m \chi_\alpha - \frac{i}{4} (\sigma_m \chi)_\alpha, \quad (3.25)$$

For vectors, the spacetime covariant derivative is given by,

$$D_0 A_m = \partial_0 A_m, \quad D_m A_n = \nabla_m A_n + \omega_{mnp} A_p, \quad (3.26)$$

which can be derived by imposing consistency with the spinor covariant derivative (3.25) and applying the Leibniz rule on a vector constructed in terms of two arbitrary spinors  $\bar{\psi} \sigma_m \psi$ .

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We will denote  $(D_0, D_m)$  the spacetime covariant derivatives for all kinds of fields. The gauge and spacetime covariant derivatives will be denoted  $(\mathcal{D}_0, \mathcal{D}_m)$ , and will be defined in due time.

#### 3.1.2 $SU(2)_r \times SU(2|1)_l$ Algebra and the Index

As was noticed in the previous section, superconformal invariance of a free field theory imposes strict restrictions on the  $R$ -charges. Of course, in the context of Seiberg duality, conformal invariance holds at the IR fixed-point for interacting theories. However, here, for calculational purposes, we break away from aforementioned restrictions, by considering theories invariant under a subgroup of the superconformal group. This subgroup should include the supercharge  $\mathcal{Q}$ . There are multiple possible choices,

1.  $SU(2)_r \times SU(2|1)_l$ , the supersymmetry algebra on  $\mathbb{R} \times S^3$ ,

$$Q_\alpha, \quad S^\beta, \quad H, \quad R, \quad J_m, \quad \bar{J}_m, \quad (3.27)$$

2.  $U(1)_r \times SU(2|1)_l$ , the three-sphere  $S^3$  is replaced by a squashed sphere [61],

$$Q_\alpha, \quad S^\beta, \quad H, \quad R, \quad J_m, \quad \bar{J}_3, \quad (3.28)$$

3.  $SU(2)_r \times U(1|1)_l$ , here the squashed sphere has a different orientation,

$$Q_1, \quad S^1, \quad H, \quad R, \quad J_3, \quad \bar{J}_m, \quad (3.29)$$

4.  $U(1)_r \times U(1|1)_l$ , which is the simplest possible algebra,

$$Q_1, \quad S^1, \quad H, \quad R, \quad J_3, \quad \bar{J}_3, \quad (3.30)$$

Here, we will consider theories defined on  $\mathbb{R} \times S^3$ , which is the appropriate space to canonically quantise a superconformal field theory in four dimensions. The reality condition on the action should be understood in the sense of the  $+$  operation. Also, the time translation generators corresponds to the dilation operator, and

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the rotation generators are the spatial isometry generators. The dilation operator for unitary, positive energy representations of the superconformal group, has a positive spectrum of real eigenvalues. Both the dilation operator and the rotation generators are hermitian in the sense of  $+$ . Considering theories invariant under  $SU(2)_r \times SU(2|1)_l$  only will allow us to write down free and interacting lagrangians for arbitrary  $R$ -charges matter fields.

#### 3.1.3 Differential Representation of $SU(2)_r \times SU(2|1)_l$

We here give the differential representation of the various operators  $H$ ,  $J_m$  generating the isometry of  $\mathbb{R} \times S^3$ . We closely follow a discussion in [62].

Following the usual prescription of radial quantisation, the dilation operator is identified with the time translation operator with action on any field  $X$  given by,

$$[H, X] = i\partial_0 X. \quad (3.31)$$

We now need to build the relevant differential operators needed to construct a supersymmetric action on  $\mathbb{R} \times S^3$ . Given an operator  $\mathcal{O}$ , we define the action of  $J_m$  on  $\mathcal{O}$ ,

$$[J_m, \mathcal{O}] = -\mathcal{J}_m \mathcal{O}, \quad (3.32)$$

with  $\mathcal{J}_m$  a differential operator acting on  $\mathcal{O}$  seen as a function, and the left hand side acting on  $\mathcal{O}$  seen as an operator. As emphasized in [62], this is still consistent with (2.31),

$$[\mathcal{J}_m, \mathcal{J}_n] = i\varepsilon_{mnp} \mathcal{J}_p, \quad (3.33)$$

because, given two operators  $A$ ,  $B$  with differential representation  $\mathcal{A}$ ,  $\mathcal{B}$  such that  $[A, B] = C$ , then,

$$AB(\mathcal{O}) = [A, [B, \mathcal{O}]] = -[A, \mathcal{B}\mathcal{O}] = -\mathcal{B}[A, \mathcal{O}] = \mathcal{B}\mathcal{A}\mathcal{O}, \quad (3.34)$$

which is then consistent with  $[\mathcal{A}, \mathcal{B}] = \mathcal{C}$ . Following the construction in section (3.1.1), we take the action of  $SU(2)_l$  generators acting on scalars, Weyl spinors,

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vectors, to be given by<sup>1</sup>,

$$i\mathcal{J}_m\phi = \nabla_m\phi, \quad (3.35)$$

$$i\mathcal{J}_m\psi_\alpha = D_m\psi_\alpha - \frac{i}{4}(\sigma_m)_\alpha{}^\beta\psi_\beta, \quad (3.36)$$

$$i\mathcal{J}_mA_n = D_mA_n - \frac{1}{2}\varepsilon_{mnp}A_p, \quad (3.37)$$

One can obtain these equations by considering equation (2.17) applied to a spinor operator, with the indices  $\mu, \nu$  restricted to spatial indices. Then, identifying,

$$M_{mn} = -\varepsilon_{mnp}J_p, \quad \mathcal{L}_{mn} = -\varepsilon_{mnp}\mathcal{J}_p, \quad S_{mn} = -\frac{1}{2}\varepsilon_{mnp}\sigma_p, \quad (3.38)$$

allows one to recover the spinor differential rotation generator (3.36), as the full angular momentum operator is the sum of the orbital and spin angular momenta operators,

$$\mathcal{J}_m = \mathcal{L}_m + \mathcal{S}_m. \quad (3.39)$$

One can then derive the equation for the vector by applying Leibniz rule on an object constructed out of two spinors whose indices are contracted with a Pauli matrix.

Let us show that this definition is consistent with the standard  $SU(2)$  commutation relations (2.31), by computing the commutation relations on a scalar operator  $\phi$ . First compute  $\mathcal{J}_m\mathcal{J}_n\phi$ . Note  $\mathcal{J}_n\phi$  is a vector, hence one should take the vector representation for  $\mathcal{J}_m$  given in (3.37), and the scalar representation for  $\mathcal{J}_n$  (3.35)

$$\mathcal{J}_m\mathcal{J}_n\phi = -i\mathcal{J}_m\nabla_n\phi = -\nabla_m\nabla_n\phi + \varepsilon_{mnp}\nabla_p\phi, \quad (3.40)$$

---

<sup>1</sup>One should note that these conventions are consistent with [32]. There, these equations are given in (3.1) and (4.3) in the form,

$$\delta_{K_i}^{J_i} = iK^i J_i, \quad \delta_{K_i}^{J_i}\phi = K^i\sigma_i^{(L)}\phi, \quad \delta_{K_i}^{J_i}\psi = K^i(\sigma_i^{(L)} + \frac{1}{4}\epsilon_{ijk}\gamma^{jk})\psi, \quad i = \{1, 2, 3\}$$

with  $\gamma^{jk}$  half the commutator of a four component spin matrix, and the metric convention mostly plus.

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Hence, using (3.14), the commutator of  $J_m$  is given by,

$$[\mathcal{J}_m, \mathcal{J}_n]\phi = \varepsilon_{mnp}\nabla_p\phi = i\varepsilon_{mnp}\mathcal{J}_p\phi, \quad (3.41)$$

as expected<sup>1</sup>.

#### 3.1.4 Supersymmetry on the Sphere, Killing Spinors

As was pointed out in [63], to define a supersymmetric theory on a curved background, one first needs to exhibit the Killing spinors of the background in question. For instance, the supersymmetry variations of a Yang-Mills theory put on a curved background is given by,

$$\delta\mathcal{L}_{\text{SYM}} = [(D_m\bar{\epsilon})\sigma^{np}\sigma^m\lambda + \bar{\lambda}\sigma^m\sigma^{np}(D_m\epsilon)] F_{np}, \quad (3.42)$$

with  $\epsilon, \bar{\epsilon}$  the supersymmetry variation parameters, or Killing spinors,  $\sigma$  the Pauli matrices and commutators thereof, and  $\lambda, \bar{\lambda}$  the Weyl spinor gaugino. In order for the theory to be invariant under such variations. one could obviously impose the vanishing of  $D_m\epsilon$ . However, one can take a less drastic restriction. The most general Killing spinor equation on a curved manifold is given by

$$D_m\epsilon = \sigma_m\epsilon', \quad (3.43)$$

where  $\epsilon'$  is not necessarily proportional to  $\epsilon$ . However, it was shown in [64] that, on spaces of constant curvature such as  $S^3$ , it is always possible to take  $\epsilon'$  proportional to  $\epsilon$ . Consequently, we will use Killing spinors such that,

$$\partial_0\epsilon^\alpha = -\frac{i}{2}\epsilon^\alpha, \quad D_m\epsilon^\alpha = \frac{i}{4}(\epsilon\sigma_m)^\alpha, \quad (3.44)$$

$$\partial_0\bar{\epsilon}_\alpha = \frac{i}{2}\bar{\epsilon}_\alpha, \quad D_m\bar{\epsilon}_\alpha = -\frac{i}{4}(\sigma_m\bar{\epsilon})_\alpha. \quad (3.45)$$

---

<sup>1</sup>Note that these conventions are unlike the ones used in [34]. The conventions in the latter paper, amount to, for any field  $C$  whether bosonic or fermionic,

$$[J_m, C] = -i\nabla_m C, \quad [J_m, J_n] = i\varepsilon_{mnp}J_p, \quad [\nabla_m, \nabla_n] = \varepsilon_{mnp}\nabla_p,$$

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Substituting the expression for the spinor spacetime covariant derivative shows that the Killing spinors are spatially constant

$$\nabla_m \epsilon = 0, \quad \nabla_m \bar{\epsilon} = 0. \quad (3.46)$$

One can actually derive the above equations from the  $SU(2, 1)$  algebra commutation relations. Following Wess and Bagger's conventions [65], one can write the supersymmetry transformation on any given field  $X$ , as<sup>1</sup>

$$\delta X = [\epsilon^\alpha Q_\alpha, X], \quad \bar{\delta} X = [\bar{\epsilon}_\alpha S^\alpha, X]. \quad (3.48)$$

Here, we do not give a differential operator representation of the supercharges, as we believe that a component field approach is simpler than a superspace approach. We now compute the following commutation relation,

$$[J_m, \epsilon^\alpha Q_\alpha]. \quad (3.49)$$

Following [66], one can see the operator  $\epsilon^\alpha Q_\alpha$  as a function, or as an operator,

$$[J_m, \epsilon^\alpha Q_\alpha] = -(\mathcal{J}_m \epsilon)^\alpha Q_\alpha = \epsilon^\alpha [J_m, Q_\alpha]. \quad (3.50)$$

Substituting the expression for the spinor rotation generators (3.36) and the rotation-supercharge commutation relation given in (A.8) yields the spatial Killing spinor equation (3.44)<sup>2</sup>. Similarly, the timelike Killing spinor equation is given

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<sup>1</sup>Note that in four dimensional flat space, the convention for hermitian conjugation is given by,

$$(\psi^\alpha \chi_\alpha)^\dagger = \bar{\chi}_{\dot{\alpha}} \bar{\psi}^{\dot{\alpha}},$$

and the supersymmetry variations are then given by  $\zeta^\alpha Q_\alpha + \bar{\zeta}_{\dot{\alpha}} \bar{Q}^{\dot{\alpha}}$  as in equation (3.3) in [65]. Here, our convention for hermitian conjugation is given by,

$$(\psi^\alpha \chi_\alpha)^+ = \bar{\chi}^\alpha \bar{\psi}_\alpha = -\bar{\psi}_\alpha \bar{\chi}^\alpha, \quad (3.47)$$

<sup>2</sup>One should note that such an approach also allows one to obtain the Killing spinor equation in [34]. With the conventions in this paper, the spatial Killing spinor equation can be obtained,

$$-(\mathcal{J}_m \epsilon_\alpha) Q_\alpha = \epsilon^\alpha [J_m, Q_\alpha] \Leftrightarrow \nabla_m \epsilon^\alpha = -\frac{i}{2} (\epsilon \sigma_m)^\alpha. \quad (3.51)$$

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by,

$$[H, \epsilon^\alpha Q_\alpha] = (i\partial_0 \epsilon^\alpha) Q_\alpha = \epsilon^\alpha [H, Q_\alpha]. \quad (3.52)$$

More recently in [67], it was shown how rigid supersymmetric theories on curved background can be constructed in a more systematic fashion by using background supergravity fields. The idea, which was already pointed out in [34] based on [68, 69] and in [70], is to start from  $\mathcal{N} = 1$  supergravity and freeze the gravitational background into a rigid manifold with the gravitino having a zero vacuum expectation value. The graviton supersymmetry variation will then be zero, as the gravitino VEV is set to zero. Then, if one can find a set of supersymmetry parameters – or Killing spinors – such that the gravitino supersymmetry variation is zero in this particular background, one is then assured be able to define a rigid supersymmetry theory in this background. This method is used in [67] to write down rigid supersymmetric theories on  $\mathbb{R} \times S^3$ , but also on  $\text{AdS}_4$ ,  $S^4$  and  $S^3$ .

These Killing spinors are also used in [53] for purposes of localisation of Wilson-loops and partition functions on the 3-sphere. As was also pointed out in the same paper, there exist some other Killing spinors which correspond to the opposite sign the spatial parts of equations (3.44) and (3.45), and can be most easily computed in a right invariant vielbein. The choice of coordinates we have made implies that the Killing spinors are constant in the left invariant frame on  $S^3$ . Ignoring time-like Killing spinor equations, there are four possible Killing spinors on  $S^3$ , half being constant in the left invariant frame which we can think of as corresponding to undotted supercharges, the other half so in the right invariant frame and which are associated with dotted supercharges of the superconformal group. The timelike Killing spinor equations double the number and the choice shown is consistent with  $\mathcal{Q}(\epsilon, \bar{\epsilon})$  commuting with  $\bar{J}_3$ . One should note that imposing the requirement of a the theory to be invariant under all supercharges dotted and undotted implies that the theory has to be fully superconformal. This then means that the shortening conditions relating scaling dimensions are restored, and the  $R$ -charge cannot take a set of continuous values. One then loses the freedom to adjust the  $R$ -charge of the multiplet and defeats the point of studying the theory on the sphere. Consequently we will only require invariance under  $Q_\alpha$ ,



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$S^\beta$  and hence the  $SU(2, 1)$  subalgebra of the superconformal group.

We give all left-invariant Killing spinors,

$$\epsilon_1^\alpha = e^{-\frac{it}{2}} \delta_1^\alpha, \quad \bar{\epsilon}_\alpha^1 = e^{\frac{it}{2}} \delta_\alpha^1, \quad (3.53)$$

$$\epsilon_2^\alpha = e^{-\frac{it}{2}} \delta_2^\alpha, \quad \bar{\epsilon}_\alpha^2 = e^{\frac{it}{2}} \delta_\alpha^2. \quad (3.54)$$

Note that the numerical sub/super-scripts are not spinor indices in this context. They only label the various spinors.

The Killing spinors can then be combined to get the Killing vector  $V_m$  satisfying,

$$V_0 = 0, \quad V_m = \epsilon \sigma_m \bar{\epsilon}, \quad D_m V_n = -\frac{1}{2} \varepsilon_{mnp} V_p. \quad (3.55)$$

$$V_0 = \epsilon \bar{\epsilon}, \quad V_m = 0. \quad (3.56)$$

It is clear that equation (3.55), (3.56) are a special case of the general Killing vector equation,

$$D_m V_n + D_n V_m = 0, \quad (3.57)$$

$$\partial_0 V_n + \nabla_n V_0 = 0, \quad (3.58)$$

One can check that this allows us to recover all Killing vectors on  $\mathbb{R} \times S^3$  as listed in [71, 72]. There are 3 Killing spatial vectors obtained from contracting  $\epsilon$ ,  $\bar{\epsilon}$  spinor with a Pauli matrix,  $\epsilon_1 \sigma_m \bar{\epsilon}^1 \sim \epsilon_2 \sigma_m \bar{\epsilon}^2$ , while the temporal Killing vector is obtained from contracting the spinors directly. Also, we have mentioned the Killing spinors corresponding to the dotted supercharges  $\bar{S}^{\dot{\alpha}}$ ,  $\bar{Q}_{\dot{\alpha}}$ . Repeating the same analysis with those Killing spinors provides us with an extra three Killing vectors. Hence, in total we get the expected 7 Killing spinors of  $\mathbb{R} \times S^3$ , which essentially correspond to  $H$ ,  $J_m$ ,  $\bar{J}_m$ .

For a conformal theory, we have an extra 8 conformal Killing vector, which correspond to the translation and inversion generators.

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#### 3.1.5 Chiral Multiplet

The primary fields in chiral and anti-chiral multiplets for ordinary  $\mathcal{N} = 1$  superconformal representation theory belong to conjugate representations  $(\phi, \psi_\alpha, F)$ ,  $(\bar{\phi}, \bar{\psi}^\alpha, \bar{F})$  of  $SU(2|1)_l$ , and are trivial representations of  $SU(2)_r$ .

A chiral multiplet is such that,

$$\bar{\delta}\phi = 0, \quad (3.59)$$

Having identified the supercharges on  $\mathbb{R} \times S^3$  as the  $Q$ ,  $S$  supercharges in the superconformal algebra allows one to write down the supersymmetry transformations. This is somehow more efficient than trial and error with generic transformations involving Killing spinors. One can raise, lower spinor indices with the antisymmetric tensor  $\varepsilon^{\alpha\beta}$ ,  $\varepsilon_{\alpha\beta}$

$$\chi_\alpha = \varepsilon_{\alpha\beta}\chi^\beta, \quad \chi^\alpha = \varepsilon^{\alpha\beta}\chi_\beta, \quad \varepsilon_{12} = -1, \quad \varepsilon^{12} = 1, \quad (3.60)$$

We now need to construct supersymmetry variations compatible with the following hermiticity requirements,

$$(\phi, \psi_\alpha, F)^+ = (\bar{\phi}, \bar{\psi}^\alpha, \bar{F}), \quad (3.61)$$

as well as the hermiticity requirements for the supercharges (2.33) and the supersymmetry variation equation (3.48). One gets the following transformations,

$$\frac{1}{\sqrt{2}} \delta\phi = \epsilon^\alpha \psi_\alpha, \quad (3.62)$$

$$\frac{1}{\sqrt{2}} \delta\psi_\alpha = \epsilon_\alpha F, \quad (3.63)$$

$$\frac{1}{\sqrt{2}} \bar{\delta}\psi_\alpha = -\bar{\epsilon}_\alpha (i\partial_0 + \frac{3}{2}r)\phi + 2i(\sigma_m \bar{\epsilon})_\alpha \nabla_m \phi, \quad (3.64)$$

$$\frac{1}{\sqrt{2}} \bar{\delta}F = -\bar{\epsilon}_\alpha (i\partial_0 + \frac{3}{2}r - 1)\psi^\alpha + 2i(\sigma_m \bar{\epsilon})_\alpha D_m \psi^\alpha, \quad (3.65)$$

Other transformation not indicated are zero. Similarly the anti-chiral multiplet

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is defined by,

$$\frac{1}{\sqrt{2}} \delta \bar{\phi} = 0, \quad (3.66)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} \bar{\phi} = \bar{\psi}^\alpha \bar{\epsilon}_\alpha, \quad (3.67)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} \bar{\psi}_\alpha = \bar{\epsilon}_\alpha \bar{F} \quad (3.68)$$

$$\frac{1}{\sqrt{2}} \delta \bar{\psi}^\alpha = \epsilon^\alpha (i\partial_0 - \frac{3}{2}r) \bar{\phi} - 2i(\epsilon\sigma_m)^\alpha \nabla_m \bar{\phi}, \quad (3.69)$$

$$\frac{1}{\sqrt{2}} \delta \bar{F} = -\epsilon^\alpha (i\partial_0 - \frac{3}{2}r + 1) \bar{\psi}_\alpha + 2i(\epsilon\sigma_m)^\alpha D_m \bar{\psi}_\alpha. \quad (3.70)$$

In contrast to flat space  $\mathcal{N} = 1$  superconformal case, however, for the  $SU(2)_r \times SU(2|1)_l$  algebra we have fewer (anti-)commutation relations to satisfy so that closure of the algebra of fields allows for the freedom of having arbitrary  $R$  charges, for free or interacting theories. For the purpose of constructing an invariant action we take  $\phi, \bar{\phi}$  to have opposite  $R$ -charge,

$$[R, \phi] = r\phi, \quad [R, \bar{\phi}] = -r\bar{\phi}, \quad (3.71)$$

so that,

$$[R, \psi_\alpha] = (r-1)\psi_\alpha, \quad [R, \bar{\psi}_\alpha] = (1-r)\bar{\psi}_\alpha, \quad (3.72)$$

$$[R, F] = (r-2)F, \quad [R, \bar{F}] = (2-r)\bar{F}. \quad (3.73)$$

An important remark to make is that, unless the theory is conformal in addition to being supersymmetric on  $\mathbb{R} \times S^3$ , the value of the  $R$ -charge for the scalar chiral multiplet is unconstrained in general, as the algebra closes consistently without any such requirement, unless conformality is restored.

A similar construction was published in [34] but seems to have gone unnoticed until very recently, and was rediscovered in [32, 33].

One can also notice [34] that we can define the chiral and anti-chiral multiplets to be generated solely by the action of  $Q_\alpha$  on the respective highest weight states  $\phi, \bar{F}$ , both being annihilated by  $\bar{\delta}$ , but with the relations (3.70) holding for the anti-chiral case. This allows one to define  $\mathbb{R} \times S^3$  chiral superfields  $\Phi, \bar{\Phi}$  via

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Grassmann parameters  $\eta_\alpha$ , namely,

$$\Phi(t, \Omega, \eta) = \phi + \eta^\alpha \psi_\alpha + \eta^2 F, \quad (3.74)$$

$$\bar{\Phi}(t, \Omega, \eta) = \bar{F} + \eta^\alpha Q_\alpha \bar{F} + \eta^\alpha \eta^\beta Q_\alpha Q_\beta \bar{F}, \quad (3.75)$$

where  $\Omega$  denotes generic Euler angles for  $S^3$ . We will not attempt to base our analysis on this approach and will use component field instead.

#### 3.1.6 Vector Multiplet

Similarly, one can define the vector multiplet with the following shortening condition,

$$\bar{\delta}\lambda_\alpha = 0, \quad \delta\bar{\lambda}^\alpha = 0. \quad (3.76)$$

Hence the vector multiplet can be regarded as a spin- $\frac{1}{2}$  chiral multiplet. Defining the field strength on  $\mathbb{R} \times S^3$  in the usual fashion,

$$F_{mn} = D_m A_n - D_n A_m + [A_m, A_n], \quad (3.77)$$

$$F_{0m} = \partial_0 A_m - D_m A_0 + [A_0, A_m], \quad (3.78)$$

with  $D_m$  the spacetime covariant derivatives, we have,

$$\frac{1}{\sqrt{2}} \delta\lambda^\alpha = -iF_m^+(\epsilon\sigma_m)^\alpha + \frac{1}{2}\epsilon^\alpha D, \quad \frac{1}{\sqrt{2}} \bar{\delta}\bar{\lambda}_\alpha = iF_m^-(\sigma_m\bar{\epsilon})_\alpha + \frac{1}{2}\bar{\epsilon}_\alpha D, \quad (3.79)$$

where we have defined, for convenience,

$$F_m^\pm = F_{0m} \pm i\varepsilon_{mnp} F_{np}. \quad (3.80)$$

Denote  $\mathcal{D}_m$  the spacetime and gauge covariant derivatives, so that for any field  $X$ , in a gauge group representation generated by generators  $T^a$ ,

$$\mathcal{D}_m X = D_m X + [A_m, X], \quad [A_m, X] = A_m^a [T^a, X], \quad (3.81)$$

$$\mathcal{D}_0 X = \partial_0 X + [A_0, X]. \quad (3.82)$$

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The remaining supersymmetry transformations are,

$$\frac{1}{\sqrt{2}} \delta F_{mn} = \frac{1}{2} \epsilon \sigma_m \mathcal{D}_n \bar{\lambda} - \frac{1}{2} \epsilon \sigma_n \mathcal{D}_m \bar{\lambda} + \frac{1}{4} \epsilon_{mnp} \epsilon \sigma_p \bar{\lambda}, \quad (3.83)$$

$$\frac{1}{\sqrt{2}} \delta F_{0m} = -\frac{1}{2} \epsilon \sigma_m \mathcal{D}_0 \bar{\lambda} + \epsilon \mathcal{D}_m \bar{\lambda} - \frac{i}{2} \epsilon \sigma_m \bar{\lambda}, \quad (3.84)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} F_{mn} = \frac{1}{2} \mathcal{D}_n \lambda \sigma_m \bar{\epsilon} - \frac{1}{2} \mathcal{D}_m \lambda \sigma_n \bar{\epsilon} + \frac{1}{4} \epsilon_{mnp} \lambda \sigma_p \bar{\epsilon}, \quad (3.85)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} F_{0m} = -\frac{1}{2} \mathcal{D}_0 \lambda \sigma_m \bar{\epsilon} + \mathcal{D}_m \lambda \bar{\epsilon} + \frac{i}{2} \lambda \sigma_m \bar{\epsilon}, \quad (3.86)$$

$$\frac{1}{\sqrt{2}} \delta D = -i \epsilon \mathcal{D}_0 \bar{\lambda} + 2i \epsilon \sigma_m \mathcal{D}_m \bar{\lambda}, \quad (3.87)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} D = i \mathcal{D}_0 \lambda \bar{\epsilon} - 2i \mathcal{D}_m \lambda \sigma_m \bar{\epsilon}. \quad (3.88)$$

These transformations are consistent with,

$$\frac{1}{\sqrt{2}} \delta A_0 = \epsilon^\alpha \bar{\lambda}_\alpha, \quad \frac{1}{\sqrt{2}} \delta A_m = -\frac{1}{2} \epsilon \sigma_m \bar{\lambda}, \quad (3.89)$$

$$\frac{1}{\sqrt{2}} \bar{\delta} A_0 = \lambda^\alpha \bar{\epsilon}_\alpha, \quad \frac{1}{\sqrt{2}} \bar{\delta} A_m = -\frac{1}{2} \lambda \sigma_m \bar{\epsilon}, \quad (3.90)$$

along with the following  $R$ -charge assignments,

$$R\lambda_\alpha = -\lambda_\alpha, \quad R\bar{\lambda}^\alpha = \bar{\lambda}^\alpha, \quad (3.91)$$

$$RF_{mn} = 0, \quad RD = 0. \quad (3.92)$$

These charges are required for the algebra to close properly given the structure of the supersymmetry transformations on the gaugino given in equation (3.79) and thereafter. This is in contrast with the scalar chiral multiplet whose  $R$ -charges are unconstrained in non-conformal theories.

Useful to note, for closure of the Lagrangian under supersymmetry, is the Bianchi identity for the field strength,

$$\epsilon_{mnp} \mathcal{D}_m F_{np} = 0, \quad \mathcal{D}_m F_{n0} + \mathcal{D}_n F_{0m} + \mathcal{D}_0 F_{mn} = 0, \quad (3.93)$$

where the derivatives are spacetime and gauge covariant derivatives.

When coupling matter fields to the gauge sector, the supersymmetry transformations detailed in the previous sections get modified. All derivatives are replaced by gauge covariant derivatives, as defined above, and the  $F$  term transformations

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get modified to,

$$\frac{1}{\sqrt{2}} \delta \bar{F} = -\epsilon^\alpha (iD_0 - \frac{3}{2}r + 1) \bar{\psi}_\alpha + 2i(\epsilon \sigma_m)^\alpha D_m \bar{\psi}_\alpha - 2i \epsilon^\alpha [\lambda_\alpha, \bar{\phi}] , \quad (3.94)$$

$$\frac{1}{\sqrt{2}} \delta F = -\bar{\epsilon}_\alpha (iD_0 + \frac{3}{2}r - 1) \psi^\alpha + 2i(\sigma_m \bar{\epsilon})_\alpha D_m \psi^\alpha + 2i [\bar{\lambda}^\alpha, \phi] \bar{\epsilon}_\alpha . \quad (3.95)$$

while chiral multiplet gauge interaction terms arise from making the derivatives in (3.99) gauge covariant.

#### 3.1.7 Chiral and Vector Actions

The transformation formulae for chiral/anti-chiral fields written earlier allow us to write down an ungauged supersymmetric action on  $\mathbb{R} \times S^3$  as,

$$S_\phi = \int dt d^3\Omega (\mathcal{L}_\phi + \mathcal{L}_\psi + \mathcal{L}_F) , \quad (3.96)$$

$$\mathcal{L}_\phi = -(i\partial_0 - \frac{3}{2}r + 1) \bar{\phi} (i\partial_0 + \frac{3}{2}r - 1) \phi - 4\nabla_m \phi \nabla_m \bar{\phi} - \phi \bar{\phi} , \quad (3.97)$$

$$\mathcal{L}_\psi = -\bar{\psi}^\alpha (i\partial_0 + \frac{3}{2}r - 1) \psi_\alpha - 2i\bar{\psi}^\alpha (\sigma_m)_\alpha{}^\beta D_m \psi_\beta , \quad (3.98)$$

$$\mathcal{L}_F = \bar{F} F . \quad (3.99)$$

One should note that in the  $r = \frac{2}{3}$  case, one recovers the standard free field theory action for a conformal scalar field written in conformal coordinates, including the conformal mass term.

For the vector multiplet, we have the following action,

$$S_A = \frac{1}{g^2} \int dt d^3\Omega (\mathcal{L}_A + \mathcal{L}_\lambda + \mathcal{L}_D) , \quad (3.100)$$

$$\mathcal{L}_A = F_{0m} F_{0m} - 2F_{mn} F_{mn} , \quad (3.101)$$

$$\mathcal{L}_\lambda = -i\bar{\lambda}^\alpha D_0 \lambda_\alpha - 2i\bar{\lambda}^\alpha (\sigma_m)_\alpha{}^\beta D_m \lambda_\beta , \quad (3.102)$$

$$\mathcal{L}_D = \frac{1}{2} D^2 , \quad (3.103)$$

When gauging the scalar field multiplet action, one has to replace all derivatives in the chiral field action by gauge invariant derivatives  $D_m \rightarrow \mathcal{D}_m$  and also add

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an additional interaction term,

$$S_{\text{int.}} = \int dt d^3\Omega \mathcal{L}_{\text{int.}}, \quad (3.104)$$

$$\mathcal{L}_{\text{int.}} = i\phi [D, \bar{\phi}] - 2i\psi^\alpha [\lambda_\alpha, \bar{\phi}] + 2i\bar{\psi}^\alpha [\bar{\lambda}_\alpha, \phi], \quad (3.105)$$

to the overall action to maintain gauge symmetry and supersymmetry.

#### BRST Symmetry

In order to define the gauge localisation action properly we will make use of the gauge BRS operator below. The gauge invariant action (3.100) and corresponding path integral can be gauge fixed in the standard fashion, by adding to the vector Lagrangian in (3.101),

$$\mathcal{L}_{\text{gf}} = \frac{1}{g^2} s (\bar{c} \partial_0 A_0 - 2\bar{c} \nabla_m A_m), \quad (3.106)$$

with the BRS operator defined as,

$$\begin{aligned} sA_0 &= \mathcal{D}_0 c, & sc &= -\frac{1}{2}[c, c], \\ sA_m &= \mathcal{D}_m c, & s\bar{c} &= B, \\ sB &= 0. \end{aligned} \quad (3.107)$$

The ghost fields are taken to be invariant under supersymmetry,

$$Q_\alpha c = Q_\alpha \bar{c} = Q_\alpha B = 0, \quad (3.108)$$

and the same holds for  $\bar{Q}_\alpha$ .

#### A Note on Normalisation of Fields

One should note the two possible normalisation which can be taken to discuss the vector multiplet Lagrangian. So far we have use the so called ‘‘holomorphic’’ normalisation of [73, 74]. Alternatively, we could have taken the ‘‘canonical’’ normalisation. This distinction is important fo localisation purposes. To change

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from holomorphic normalisation equation (3.106) has to be rescaled as,

$$A \rightarrow gA \quad (3.109)$$

and leads to the canonically normalised Lagrangian,

$$\mathcal{L}_A^{\text{can.}} = F_{0m}F_{0m} - 2F_{mn}F_{mn}, \quad (3.110)$$

$$\mathcal{L}_{\text{gf}}^{\text{can.}} = s(\bar{c}\partial_0 A_0 - 2\bar{c}\nabla_m A_m), \quad (3.111)$$

with the field strength and the covariant derivative, then redefined as,

$$F_{mn} = \nabla_m A_n - \nabla_n A_m + g[A_m, A_n], \quad (3.112)$$

$$\mathcal{D}_m X = \nabla_m X + g[A_m, X], \quad (3.113)$$

Also, the interaction Lagrangian (3.105) is rescaled to

$$\mathcal{L}_{\text{int.}}^{\text{can.}} = ig\phi [D, \bar{\phi}] - 2ig\psi^\alpha [\lambda_\alpha, \bar{\phi}] + 2ig\bar{\psi}^\alpha [\bar{\lambda}_\alpha, \phi], \quad (3.114)$$

as given in [26], and taking the limit  $g \rightarrow 0$  with either normalisations is a free field theory limit.

## 3.2 Canonical Quantisation

In order to canonically quantise the theory and compute the partition function, we will expand all fields in  $S^3$  spherical harmonics which were first defined in [75]. A similar approach is taken in [76].

### 3.2.1 Spherical Harmonics on $S^3$

$S^3$  can be identified with a coset space  $G/H$  with  $G = SU(2)_l \times SU(2)_r$  the isometry group and  $H = SO(3)$  the Lorentz group. The generators of  $H$  are denoted as,

$$S_m = J_m + \bar{J}_m, \quad (3.115)$$



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and the representations of  $G$  are labelled by two spins  $(j, \bar{j})$  taking half-integer values. Denoting the states spanning these representations as  $|j m\rangle|\bar{j} \bar{m}\rangle$ , one can write the Clebsch-Gordan decomposition of representations of  $H$ ,

$$|s n; j \bar{j}\rangle = \sum_{m, \bar{m}} C_{jm, \bar{j}\bar{m}}^{sn} |j m\rangle|\bar{j} \bar{m}\rangle. \quad (3.116)$$

$C_{jm, \bar{j}\bar{m}}^{sn}$  denoting the Clebsch-Gordan coefficients. Also, the following restriction holds,

$$|j - \bar{j}| \leq s \leq j + \bar{j}. \quad (3.117)$$

One can then write down the following generic element of  $G/H$ , in terms of the Euler angles on the 3-sphere,

$$\Upsilon = e^{-i\theta^3 L_1} e^{-i\theta^2 L_3} e^{-i\theta^1 (J_1 - \bar{J}_1)}. \quad (3.118)$$

The spin- $s$  spherical harmonics are then given by the following formula,

$$\mathcal{Y}_{jm, \bar{j}\bar{m}}^{sn}(\Omega) = \sqrt{\frac{(2j+1)(2\bar{j}+1)}{2s+1}} \langle s n; j \bar{j} | \Upsilon(\Omega) | j m\rangle|\bar{j} \bar{m}\rangle. \quad (3.119)$$

With this normalisation, the spherical harmonics will satisfy the following orthonormality condition,

$$\sum_n \int d^3\Omega (\mathcal{Y}_{jm, \bar{j}\bar{m}}^{sn})^* \mathcal{Y}_{j'm', \bar{j}'\bar{m}'}^{sn} = \delta_{jj'} \delta_{\bar{j}\bar{j}'} \delta_{mm'} \delta_{\bar{m}\bar{m}'}, \quad (3.120)$$

with  $d^3\Omega$  the solid angle for  $S^3$ . Also, one can work out the complex conjugate of a given spherical harmonic using equations (3.118) and (3.119),

$$(\mathcal{Y}_{jm, \bar{j}\bar{m}}^{sn})^* = (-1)^{-j+\bar{j}-s+m-\bar{m}+n} \mathcal{Y}_{j-m, \bar{j}-\bar{m}}^{s-n} \quad (3.121)$$

This complex conjugation will on our context correspond to the hermitian conjugation  $^+$  defined in equations (2.21) and (2.33).

#### Scalar Harmonics

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Scalar spherical harmonics are such that:

$$s = 0 \Rightarrow j = \bar{j}. \quad (3.122)$$

They will be denoted as:

$$Y_{\ell m \bar{m}} = \mathcal{Y}_{\frac{\ell}{2} m, \frac{\ell}{2} \bar{m}}^{00} \quad (3.123)$$

with  $\ell$  an integer running from zero to infinity. The Laplacian acts on the scalar harmonics in the following fashion:

$$D^2 Y_{\ell m \bar{m}} = -\ell(\ell + 2) Y_{\ell m \bar{m}} \quad (3.124)$$

#### Spinor Harmonics

Spinor spherical harmonics are given by  $(j, \bar{j}) = (j \pm \frac{1}{2}, j)$ , and are denoted by:

$$Y_{\ell m \bar{m} \alpha}^{\kappa} = \begin{cases} Y_{\ell m \bar{m} \alpha}^{+1} = \mathcal{Y}_{\frac{\ell+1}{2} m, \frac{\ell}{2} \bar{m}}^{\frac{1}{2} \alpha} \\ Y_{\ell m \bar{m} \alpha}^{-1} = \mathcal{Y}_{\frac{\ell}{2} m, \frac{\ell+1}{2} \bar{m}}^{\frac{1}{2} \alpha} \end{cases} \quad (3.125)$$

where  $\alpha = \pm 1/2$  is the usual spinor index. The Dirac operator on  $S^3$  acts on the spinor harmonics in the following fashion:

$$\mathcal{D}_{S^3} Y_{\ell m \bar{m} \alpha}^{\kappa} = i(\sigma_n D_n Y_{\ell m \bar{m}}^{\kappa})_{\alpha} = \frac{\kappa}{2}(\ell + \frac{3}{2}) Y_{\ell m \bar{m} \alpha}^{\kappa}, \quad (3.126)$$

#### Vector Harmonics

We will use the following notation:

$$Y_{\ell m \bar{m} n}^{\rho} = \begin{cases} Y_{\ell m \bar{m} n}^1 = i \mathcal{Y}_{\frac{\ell+2}{2} m, \frac{\ell}{2} \bar{m}}^{1n} \\ Y_{\ell m \bar{m} n}^0 = \mathcal{Y}_{\frac{\ell}{2} m, \frac{\ell}{2} \bar{m}}^{1n} \\ Y_{\ell m \bar{m} n}^{-1} = -i \mathcal{Y}_{\frac{\ell}{2} m, \frac{\ell+2}{2} \bar{m}}^{1n} \end{cases} \quad (3.127)$$

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with  $n$  a spatial vector index running from 1 to 3. We will require the following Laplacian eigenvalues,

$$D^2 Y_{\ell m \bar{m} n}^{\rho=\pm 1} = -\frac{1}{4}((\ell + 2)^2 - 2) Y_{\ell m \bar{m} n}^{\kappa}. \quad (3.128)$$

Also, one should note the divergencelessness of the following harmonics,

$$D_n Y_{\ell m \bar{m} n}^{\rho=\pm 1} = 0, \quad (3.129)$$

#### 3.2.2 Chiral Multiplet

##### Scalar Component

Let us first expand the scalar field in spherical harmonics,

$$\phi(t, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m, \bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} \phi_{\ell m \bar{m}}(t) Y_{\ell m \bar{m}}(\Omega), \quad (3.130)$$

$$\bar{\phi}(t, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m, \bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} \bar{\phi}_{\ell m \bar{m}}(t) Y_{\ell m \bar{m}}^*(\Omega). \quad (3.131)$$

The Lagrangian density can then be expanded in terms of  $\bar{\phi}_{\ell m \bar{m}}$  modes and spatially integrated. Using the spherical harmonics orthonormality relation (3.120) as well as their laplacian eigenvalues (3.124) one gets the following Lagrangian,

$$L_{\phi} = \sum_{\ell m \bar{m}} (d_t + \frac{3}{2}ir - i) \bar{\phi}_{\ell m \bar{m}} (d_t - \frac{3}{2}ir + i) \phi_{\ell m \bar{m}} - (\ell + 1)^2 \bar{\phi}_{\ell m \bar{m}} \phi_{\ell m \bar{m}}, \quad (3.132)$$

One can write  $H$ ,  $J_3$ ,  $\bar{J}_3$ ,  $R$  in terms of the scalar modes. The conjugate momenta to the latter are given by,

$$\pi_{\ell m \bar{m}}(t) = (d_t + \frac{3}{2}ir - i) \bar{\phi}_{\ell m \bar{m}}, \quad \bar{\pi}_{\ell m \bar{m}}(t) = (d_t - \frac{3}{2}ir + i) \phi_{\ell m \bar{m}}. \quad (3.133)$$

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Define the following modes,

$$a_{\ell m \bar{m}}(t) = \frac{1}{\sqrt{2(\ell+1)}} (\bar{\pi}_{\ell m \bar{m}} + i(\ell+1)\phi_{\ell m \bar{m}}) , \quad (3.134)$$

$$a_{\ell m \bar{m}}^+(t) = \frac{1}{\sqrt{2(\ell+1)}} (\pi_{\ell m \bar{m}} - i(\ell+1)\bar{\phi}_{\ell m \bar{m}}) , \quad (3.135)$$

$$b_{\ell m \bar{m}}(t) = \frac{1}{\sqrt{2(\ell+1)}} (\pi_{\ell m \bar{m}} + i(\ell+1)\bar{\phi}_{\ell m \bar{m}}) , \quad (3.136)$$

$$b_{\ell m \bar{m}}^+(t) = \frac{1}{\sqrt{2(\ell+1)}} (\bar{\pi}_{\ell m \bar{m}} - i(\ell+1)\phi_{\ell m \bar{m}}) . \quad (3.137)$$

Canonically quantise this theory using the following commutation relations,

$$[a_{\ell m \bar{m}}(t), a_{\ell' m' \bar{m}'}^+(t)] = \delta_{\ell \ell'} \delta_{m m'} \delta_{\bar{m} \bar{m}'} , \quad (3.138)$$

One can then express the hamiltonian and the angular momentum generators as,

$$H = \sum_{\ell m \bar{m}} \left( \ell + \frac{3}{2}r \right) a_{\ell m \bar{m}}^+ a_{\ell m \bar{m}} + \left( \ell + 2 - \frac{3}{2}r \right) b_{\ell m \bar{m}}^+ b_{\ell m \bar{m}} , \quad (3.139)$$

$$J_3 = \sum_{\ell m \bar{m}} m (b_{\ell m \bar{m}}^+ b_{\ell m \bar{m}} - a_{\ell m \bar{m}}^+ a_{\ell m \bar{m}}) , \quad (3.140)$$

$$\bar{J}_3 = \sum_{\ell m \bar{m}} \bar{m} (b_{\ell m \bar{m}}^+ b_{\ell m \bar{m}} - a_{\ell m \bar{m}}^+ a_{\ell m \bar{m}}) , \quad (3.141)$$

$$R = \sum_{\ell m \bar{m}} r (b_{\ell m \bar{m}}^+ b_{\ell m \bar{m}} - a_{\ell m \bar{m}}^+ a_{\ell m \bar{m}}) , \quad (3.142)$$

which leads to the following expression for  $\mathcal{H}$  defined in (2.37),

$$\mathcal{H} = \sum_{\ell m \bar{m}} (\ell + 2m) a_{\ell m \bar{m}}^+ a_{\ell m \bar{m}} + (\ell - 2m + 2) b_{\ell m \bar{m}}^+ b_{\ell m \bar{m}} . \quad (3.143)$$

One can check that the index for the scalar field is obtained from the contribution of the zero modes for  $\mathcal{H}$ , which are created by  $a_{\ell - \frac{\ell}{2} \bar{m}}^+$ . Noting that the creation operator  $a_{\ell m \bar{m}}^+ \sim \bar{\phi}_{\ell m \bar{m}}$ , these correspond to  $SU(2)_l$  highest weight antichiral

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states. Summing over those modes gives,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} t^{r+\ell} x^{2\bar{m}} = \frac{t^r}{(1-tx)(1-tx^{-1})}. \quad (3.144)$$

Assume the  $R$ -charge now takes its superconformal value, as given by  $a$ -maximisation [51]. Those states are related to each other through the action of the translation generators  $P$  commuting with  $Q_1$ ,  $S^1$ . Following equation (2.39) these are given by  $P_{2\dot{\alpha}}$ . Consider the most general quantum states of the problem and denote them as,

$$|\Delta(\frac{\ell}{2}, m)(\frac{\bar{\ell}}{2}, \bar{m})\rangle_r, \quad (3.145)$$

where the various quantum numbers correspond to the action of  $H$ ,  $SU(2)_l$ ,  $SU(2)_r$  and  $R$  respectively. The scalar states contributing to the index are given by,

$$|\ell \bar{m}\rangle_r = |\ell + \frac{3}{2}r(\frac{\ell}{2}, \frac{\ell}{2})(\frac{\ell}{2}, \bar{m})\rangle_{-r}. \quad (3.146)$$

Note the minus sign on the eigenvalue of  $J_3$  here. Due to the complex conjugation properties of scalar spherical harmonics given in (3.121) one gets,

$$a_{\ell m \bar{m}}^+ |0\rangle = |\ell + \frac{3}{2}r(\frac{\ell}{2}, -m)(\frac{\ell}{2}, -\bar{m})\rangle_{-r}. \quad (3.147)$$

To generate all states  $|\ell \bar{m}\rangle_r$  contributing to the index, one acts with the  $\bar{J}_-$  generators, which gives us all possible values for  $\bar{m}$ , given a value for  $\ell$ . To generate all values of  $\ell$ , one can act on the following parent state,

$$|0\rangle_r = |\frac{3}{2}r(0, 0)(0, 0)\rangle_{-r}, \quad (3.148)$$

with  $P_{22}$ . Hence, one can generate all states  $|\ell, \bar{m}\rangle_r$  by acting with  $\bar{J}_-$ ,  $P_{22}$  and the corresponding module is generated by,

$$|\ell \bar{m}\rangle_r \propto \bar{J}_-^{\ell-\bar{m}} P_{22}^{\ell} |0\rangle_r. \quad \ell = 0 \dots \infty, \quad \bar{m} = -\frac{\ell}{2} \dots \frac{\ell}{2}. \quad (3.149)$$

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The character for this module can hence be written given the following correspondence,

$$P_{22} \rightarrow s^2 x \bar{x}, \quad \bar{J}_- \rightarrow \bar{x}^{-2}. \quad (3.150)$$

The character hence reads,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} s^{2\ell+3r} x^{\ell} \bar{x}^{2\bar{m}} u^{-r} = \frac{s^{3r} u^{-r}}{(1-s^2 x \bar{x})(1-s^2 x \bar{x}^{-1})} \quad (3.151)$$

Rescaling the variables as in equation (2.141), allows one to recover the expression for the scalar part of the index (3.144).

One can also obtain the index by considering the following states,

$$P_{21}^{\ell_1} P_{22}^{\ell_2} |0\rangle_r. \quad \ell_1, \ell_2 = 0 \dots \infty, \quad (3.152)$$

hence one can interpret the second factor in the denominator as coming from,

$$P_{21} \rightarrow s^2 x \bar{x}^{-1}, \quad (3.153)$$

#### Fermionic Component

Similarly, consider the fermion field part of the action in (3.96) and expand the fields in spinor harmonics. Using the shorthand,

$$\sum_{\ell m \bar{m} \kappa = +} = \sum_{\ell=0}^{\infty} \sum_{m=-\frac{\ell+1}{2}}^{\frac{\ell+1}{2}} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}}, \quad \sum_{\ell m \bar{m} \kappa = -} = \sum_{\ell=0}^{\infty} \sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{\bar{m}=-\frac{\ell+1}{2}}^{\frac{\ell+1}{2}}. \quad (3.154)$$

Note that we will use a similar shorthand for products in the following chapter. The spinors can be expanded in spherical harmonics (3.125),

$$\psi_{\alpha}(t, \Omega) = \sum_{\ell m \bar{m} \kappa} \psi_{\ell m \bar{m} \kappa}^{\alpha}(t) Y_{\ell m \bar{m} \alpha}^{\kappa}(\Omega), \quad \bar{\psi}^{\alpha} = \sum_{\ell m \bar{m} \kappa} \bar{\psi}_{\ell m \bar{m} \kappa}^{\alpha}(t) Y_{\ell m \bar{m} \alpha}^{\kappa}(\Omega)^*, \quad (3.155)$$

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Using the orthonormality relation (3.120) and the eigenvalues for the Dirac operator (3.126), the fermionic Lagrangian reads,

$$L_\psi = \sum_{\ell m \bar{m} \kappa} \bar{\psi}_{\ell m \bar{m}}^\kappa (i\partial_0 + \frac{3}{2}r - 1 - \kappa(\ell + \frac{3}{2})) \psi_{\ell m \bar{m}}^\kappa, \quad (3.156)$$

The conjugate momenta to the spinor modes read,

$$\pi_{\ell m \bar{m}}^+ = \frac{\partial L}{\partial \dot{\psi}_{\ell m \bar{m}}^+} = i\bar{\psi}_{\ell m \bar{m}}^+, \quad (3.157)$$

$$\pi_{\ell m \bar{m}}^- = \frac{\partial L}{\partial \dot{\psi}_{\ell m \bar{m}}^-} = i\bar{\psi}_{\ell m \bar{m}}^-. \quad (3.158)$$

This leads to the hamiltonian,  $R$ -charge and angular momentum generators,

$$H = \sum_{\ell m \bar{m} +} (\ell + \frac{5}{2} - \frac{3}{2}r) \psi_{\ell m \bar{m}}^+ \bar{\psi}_{\ell m \bar{m}}^+ + \sum_{\ell m \bar{m} -} (\ell + \frac{1}{2} + \frac{3}{2}r) \bar{\psi}_{\ell m \bar{m}}^- \psi_{\ell m \bar{m}}^-, \quad (3.159)$$

$$J_3 = \sum_{\ell m \bar{m} +} m \psi_{\ell m \bar{m}}^+ \bar{\psi}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} m \bar{\psi}_{\ell m \bar{m}}^- \psi_{\ell m \bar{m}}^-, \quad (3.159)$$

$$\bar{J}_3 = \sum_{\ell m \bar{m} +} \bar{m} \psi_{\ell m \bar{m}}^+ \bar{\psi}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} \bar{m} \bar{\psi}_{\ell m \bar{m}}^- \psi_{\ell m \bar{m}}^-, \quad (3.160)$$

$$R = \sum_{\ell m \bar{m} +} (r - 1) \psi_{\ell m \bar{m}}^+ \bar{\psi}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} (r - 1) \bar{\psi}_{\ell m \bar{m}}^- \psi_{\ell m \bar{m}}^-. \quad (3.161)$$

In all these expressions we interpret  $\psi_+$  and  $\bar{\psi}_-$  as creation operators, while  $\bar{\psi}_+$  and  $\psi_-$  are interpreted as annihilation operators. This leads to the expression for  $\mathcal{H}$ ,

$$\mathcal{H} = \sum_{\ell m \bar{m} +} (\ell - 2m + 1) \psi_{\ell m \bar{m}}^+ \bar{\psi}_{\ell m \bar{m}}^+ + \sum_{\ell m \bar{m} -} (\ell + 2m + 2) \bar{\psi}_{\ell m \bar{m}}^- \psi_{\ell m \bar{m}}^-, \quad (3.162)$$

and the zero modes are created by  $\psi_{\ell \frac{\ell+1}{2} \bar{m}}^+$ . Summing over those modes for the index, including the necessary minus sign, gives,

$$- \sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} t^{-r-\ell} x^{2\bar{m}} = - \frac{t^{2-r}}{(1-tx)(1-tx^{-1})} \dots \quad (3.163)$$

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All fermionic states contributing to the index are given by,

$$|\ell + \frac{1}{2}, \bar{m}\rangle_r = |\ell - \frac{3}{2}r + \frac{5}{2} (\frac{\ell+1}{2}, \frac{\ell+1}{2}) (\frac{\ell}{2}, \bar{m})\rangle_{r-1}, \quad (3.164)$$

One can then define the following parent fermionic state, which also turns out to be a chiral primary state,

$$|\frac{1}{2}\rangle_r = |-\frac{3}{2}r + \frac{5}{2} (\frac{1}{2}, \frac{1}{2}) (0, 0)\rangle_{r-1}, \quad (3.165)$$

The corresponding truncated module is generated by,

$$|\ell + \frac{1}{2}, \bar{m}\rangle_r \propto \bar{J}_-^{\ell - \frac{1}{2} - \bar{m}} P_{22}^\ell |\frac{1}{2}\rangle_r. \quad (3.166)$$

or equivalently,

$$P_{21}^{\ell_1} P_{22}^{\ell_2} |\frac{1}{2}\rangle_r. \quad \ell_1, \ell_2 = 0 \dots \infty, \quad (3.167)$$

This echoes the construction of the free field index leading up to (2.69) based on the states  $P_{12}^{\ell_1} P_{22}^{\ell_2} |0\rangle_r$  and  $P_{12}^{\ell_1} P_{22}^{\ell_2} |\frac{1}{2}\rangle_r$ . The character for the corresponding module or the spinors is given by,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} s^{2\ell+5-3r} x^{\ell+1} \bar{x}^{2\bar{m}} u^{r-1} = \frac{s^{5-3r} u^{r-1} x}{(1 - s^2 x \bar{x})(1 - s^2 x \bar{x}^{-1})}. \quad (3.168)$$

Rescaling the arguments of the above character as in (2.141), allows one to recover the expression for the fermion part of the index (3.163).

One can now define the supersymmetry generators for the chiral supermulti-



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plet,

$$Q_1 = \sum_{\ell m \bar{m}} \sqrt{2(\ell+1)} \psi_{\ell m + \frac{1}{2} \bar{m}}^+ b_{\ell m \bar{m}}, \quad (3.169)$$

$$Q_2 = \sum_{\ell m \bar{m}} \sqrt{2(\ell+1)} \psi_{\ell m - \frac{1}{2} \bar{m}}^+ b_{\ell m \bar{m}}, \quad (3.170)$$

$$S^1 = \sum_{\ell m \bar{m}} \sqrt{2(\ell+1)} \bar{\psi}_{\ell-1 m - \frac{1}{2} \bar{m}}^- a_{\ell m \bar{m}}, \quad (3.171)$$

$$S^2 = \sum_{\ell m \bar{m}} \sqrt{2(\ell+1)} \bar{\psi}_{\ell-1 m + \frac{1}{2} \bar{m}}^- a_{\ell m \bar{m}}, \quad (3.172)$$

The following diagrams summarises the previous paragraphs. The vertical axis  $z$  corresponds to  $H$ , the horizontal axis to  $J_3, \bar{J}_3$ . The blue, red states are scalar states, the green states are fermionic. The black upwards arrows are the  $Q$  supercharges, the  $S$  supercharges are the black downwards pointing arrows, the  $\bar{Q}$  supercharges are orange arrows. The upwards purple arrows are the  $P_{22}, P_{12}$  generators, while the horizontal purple arrow is a  $\bar{J}_-$  generator. The states contributing to the index are part of the yellow plane on all diagrams. The

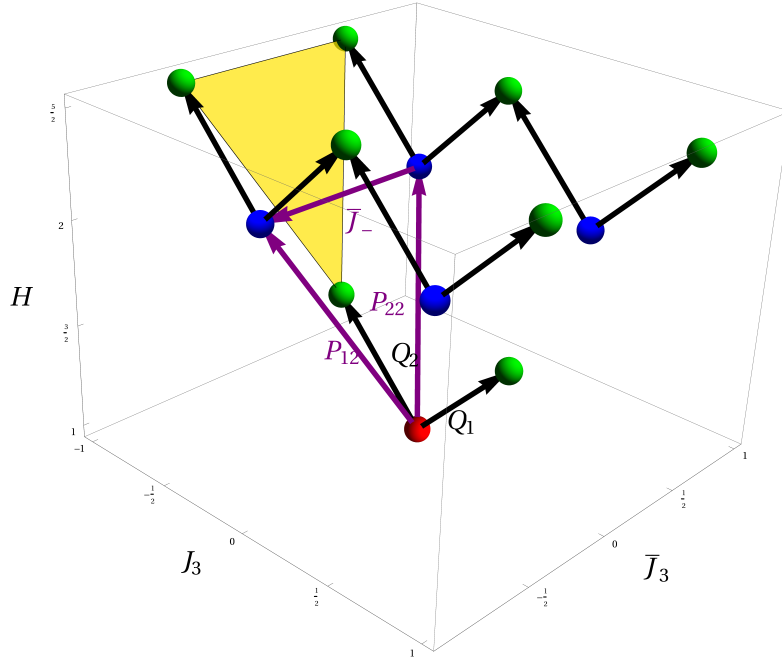


Figure 3.1: Chiral States in Conformal Theory

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previous diagram corresponds to chiral multiplet in a superconformal theory. The fermionic states are left-moving – the  $\kappa = +$  set of states of the previous sections – and are generated through the action of  $Q$  on the scalar states. The scalar chiral primary field  $|2 - \frac{3}{2}r, (0, 0), (0, 0), r\rangle$  is the lowest lying red state. One should note that this chiral primary is annihilated by the  $\bar{Q}$  as well as the  $S, \bar{S}$  generator, as expected. One generates all other blue scalar states through the action of the purple  $P$  generators. Because of the commutation relation  $[P, S] = \bar{Q}$ , these descendent blue scalar states are also annihilated by  $S$ . Also, because  $[P, \bar{Q}] = 0$  they are also annihilated by  $\bar{Q}$ . The point is that all scalar states are such that,

$$\bar{Q}|\phi_{\ell m \bar{m}}\rangle = S|\phi_{\ell m \bar{m}}\rangle = 0, \quad (3.173)$$

and are hence compatible with the usual definition of a chiral field as a field annihilated by  $\bar{Q}$ , but also compatible with our definition of a chiral field (3.59) as a field annihilated by the  $S$  supercharges. A supersymmetric chiral multiplet is unsurprisingly made of a scalar state and a left-moving spinor related through the action of a supercharge  $Q$ , which is illustrated in the diagram below. The dashed red line surrounds the chiral multiplet, where there are two multiplets surrounded by red and black tubes. Let us now focus on the antichiral multiplet.

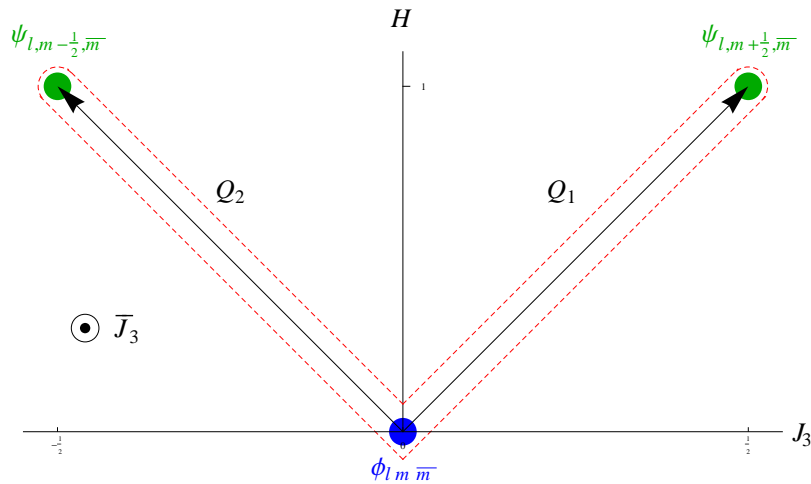


Figure 3.2: Basic Chiral Multiplet

This diagram corresponds to the antichiral supermultiplet mode expansion for

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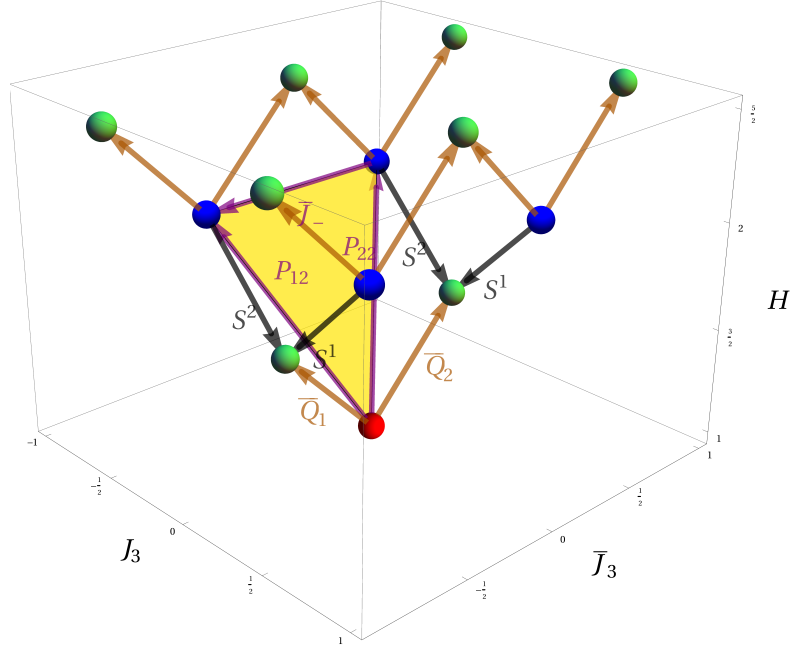


Figure 3.3: Antichiral States in Conformal Theory

a superconformal theory. The red antichiral primary state is annihilated by  $Q$ ,  $S$ ,  $\bar{S}$ . All other scalar states are generated through the action of the purple momentum generators  $P$ . Given that  $[P, Q] = 0$ , the blue descendent scalar states are annihilated by the  $Q$  supercharges. As expected, the antichiral states are unambiguously defined by the shortening condition,

$$Q|\phi_{\ell m \bar{m}}\rangle = 0. \quad (3.174)$$

One should then note that the fermionic states are right-moving – the  $\kappa = -$  set of states of the previous sections. The  $H = \ell + 1/2$ ,  $\ell = 0 \dots \infty$  set of fermionic states can be generated by acting with the orange  $\bar{Q}$  supercharges on the red & blue scalar states such that  $H = \ell$ . Alternatively one can act with the black  $S$  supercharges on the blue scalar states – not the red primary, which is annihilated by  $S$  charges – such that  $H = \ell + 1$ . This is illustrated in the diagram below. Here, we generate the fermionic states in the antichiral multiplet (3.66) by acting on scalar states with the  $S$  supercharge, which is indicated on

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the diagram below inside the black tube, while the standard definition of the antichiral multiplet is given by acting on the scalar states with  $\bar{Q}$  supercharges, as in the red tube below. The point is that the two definitions are equivalent, because the set of states contributing in a superconformal theory are the same, the only thing that varies is the way we arrange them in supermultiplets, either through the action of the  $\bar{Q}$  supercharges or the  $S$  supercharges. One can also

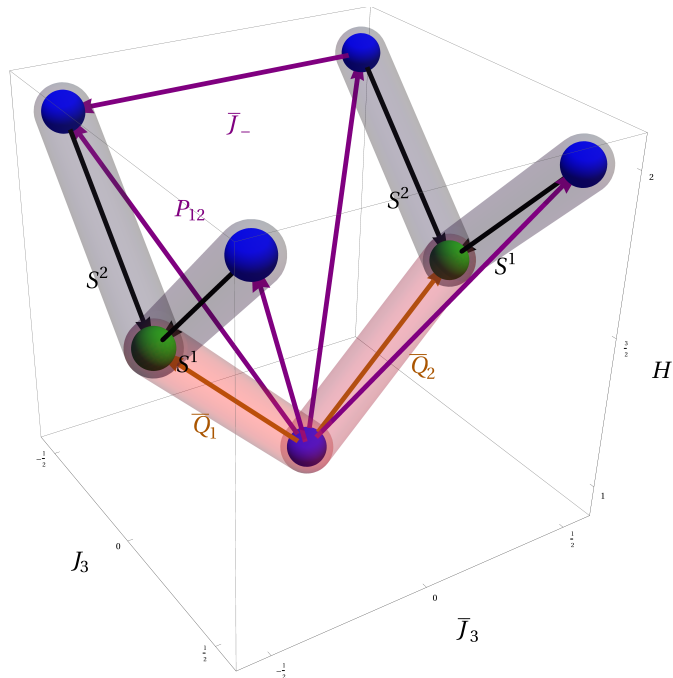


Figure 3.4: Antichiral Multiplet

construct the chiral multiplet for a non-conformal theory. In that case, the  $R$ -charge can take an arbitrary value above  $2/3$ . The only symmetry of the theory that remain are the supercharges  $Q$  and  $S$ , along with the rotation generators  $SU(2)_l \times SU(2)_r$ . The states with higher values of  $H$  are still present and arise from the higher dimension representations of  $SU(2)$ . One generates the fermionic states through action of  $Q$  upon scalar states. For the antichiral supermultiplet, the fermionic green states are generated by acting upon the scalar states with the  $S$  supercharges. This implies that the lowest lying state  $|\frac{3}{2}r, (0, 0), (0, 0), -r\rangle$  is annihilated by all supercharges of the non conformal theory on  $\mathbb{R} \times S^3$ , that is  $Q$  and  $S$ .

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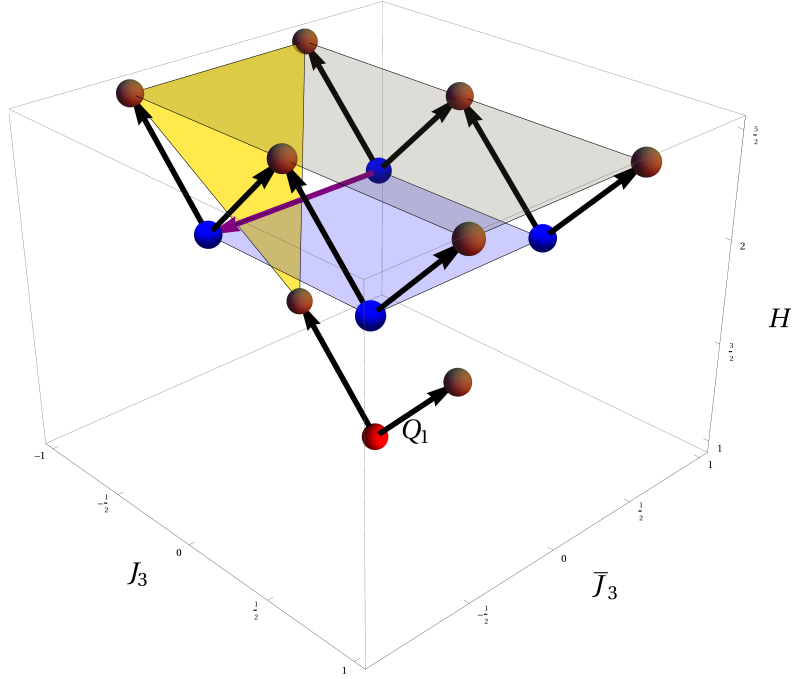


Figure 3.5: Chiral Multiplet in a non Conformal Theory

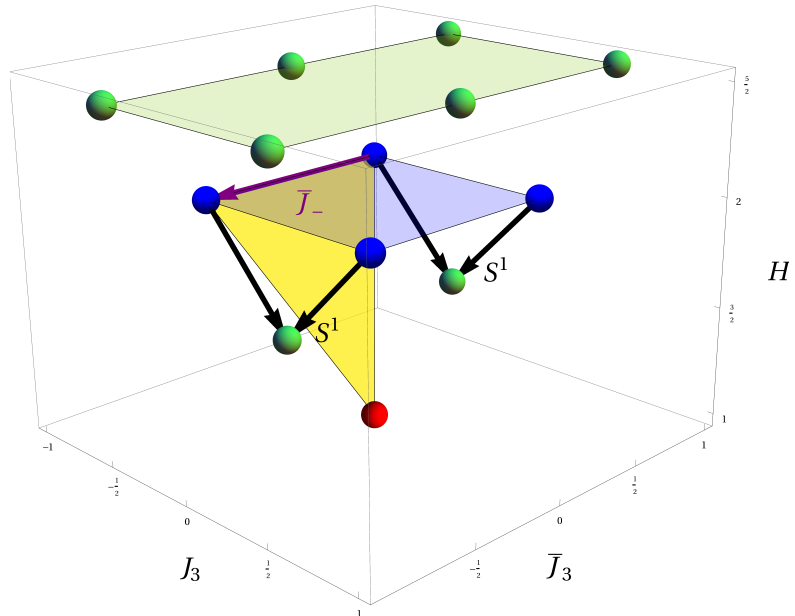


Figure 3.6: Antichiral Multiplet States in non-Conformal Theory

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#### 3.2.3 Vector Multiplet

##### Vector Field

In this section, we limit ourselves to a free abelian vector field. The non-abelian case will be treated in a functional approach in a later section, and we will also show that the free field theory index is exact, for both the scalar field and the vector field. In expanding the vector field action, we use the shorthand,

$$\sum_{\ell m \bar{m} \rho = +} = \sum_{\ell=0}^{\infty} \sum_{m=-\frac{\ell+2}{2}}^{\frac{\ell+2}{2}} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}}, \quad \sum_{\ell m \bar{m} \rho = -} = \sum_{\ell=0}^{\infty} \sum_{m=-\frac{\ell}{2}}^{\frac{\ell}{2}} \sum_{\bar{m}=-\frac{\ell+2}{2}}^{\frac{\ell+2}{2}}, \quad (3.175)$$

One can impose the Coulomb gauge for the spatial part of the vector field, and hence expand it in divergenceless  $S^3$  spherical harmonics. Also one can expand the timelike component of the vector in scalar harmonics,

$$A_0(t, \Omega) = \sum_{\ell m \bar{m}} A_{\ell m \bar{m}}(t) Y_{\ell m \bar{m}}(\Omega), \quad (3.176)$$

$$A_n(t, \Omega) = \sum_{\ell m \bar{m} \rho = \pm} A_{\ell m \bar{m}}^{\rho}(t) Y_{\ell m \bar{m} n}^{\rho}(\Omega). \quad (3.177)$$

Note that, the reality condition  $B_m = B_m^*$  imply the following relation between modes,

$$A_{\ell m \bar{m}}^* = A_{\ell -m -\bar{m}}. \quad (3.178)$$

The Lagrangian can the be obtained using (3.120) for the vector can then be rewritten, after expansion into harmonics, as,

$$L[A_m] = \sum_{\ell m \bar{m} \rho = \pm} A_{\ell -m -\bar{m}}^{\rho} (\partial_t - (\ell + 2)^2) A_{\ell m \bar{m}}^{\rho}, \quad (3.179)$$

$$L[A_0] = \sum_{\ell m \bar{m}} (-1)^{m-\bar{m}} \frac{\ell}{2} (\ell + 2) A_{\ell -m -\bar{m}} A_{\ell m \bar{m}}, \quad (3.180)$$

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is the free field value of the conformal dimension of vector field descendants. The conjugate momenta,

$$\pi_{\ell m \bar{m}}^+(t) = \partial_0 B_{n,-m,-\bar{m}}^+ + g[\alpha, B_{n,-m,-\bar{m}}^+], \quad (3.181)$$

$$\pi_{\ell m \bar{m}}^-(t) = \partial_0 B_{n,-m,-\bar{m}}^- + g[\alpha, B_{n,-m,-\bar{m}}^-]. \quad (3.182)$$

Define the following modes,

$$a_{\ell m \bar{m} \rho}(t) = \frac{1}{\sqrt{2(\ell+2)}} (\pi_{\ell-m-\bar{m}}^\rho + i(\ell+2)B_{\ell m \bar{m}}^\rho), \quad (3.183)$$

$$a_{\ell m \bar{m} \rho}^+(t) = \frac{1}{\sqrt{2(\ell+2)}} (\pi_{\ell m \bar{m}}^\rho - i(\ell+2)B_{\ell-m-\bar{m}}^\rho). \quad (3.184)$$

The  $R$ -charge being zero, the hamiltonian, angular momentum generator and  $\mathcal{H}$  are,

$$H = \sum_{\ell m \bar{m} \rho} (\ell+2) a_{\ell m \bar{m} \rho}^+ a_{\ell m \bar{m} \rho}, \quad (3.185)$$

$$J_3 = \sum_{\ell m \bar{m} \rho} m a_{\ell m \bar{m} \rho}^+ a_{\ell m \bar{m} \rho}, \quad (3.186)$$

$$\bar{J}_3 = \sum_{\ell m \bar{m} \rho} \bar{m} a_{\ell m \bar{m} \rho}^+ a_{\ell m \bar{m} \rho}, \quad (3.187)$$

$$\mathcal{H} = \sum_{\ell m \bar{m} \rho} (\ell+2-2m) a_{\ell m \bar{m} \rho}^+ a_{\ell m \bar{m} \rho}, \quad (3.188)$$

The index for the vector field is obtained from the contribution of the zero modes for  $\mathcal{H}$ , which are given by the  $a_{\ell-\frac{\ell+2}{2}, \bar{m}+}$  modes. Summing over those,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} t^{\ell+2} x^{2\bar{m}} = \frac{t^2}{(1-tx)(1-tx^{-1})}. \quad (3.189)$$

The vector states contributing to the vector index are given by,

$$|\ell+2, \bar{m}\rangle_0 = |\ell+2, (\frac{\ell+2}{2}, \frac{\ell+2}{2}) (\frac{\ell}{2}, \bar{m})\rangle_0. \quad (3.190)$$

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Define the following parent state,

$$|2\rangle_0 = |2 (1, 1) (0, 0)\rangle_0, \quad (3.191)$$

One can generate all states  $|\ell + 2, \bar{m}\rangle_0$  by acting with  $\bar{J}_-, P_{22}$ . The corresponding Verma module is spanned by,

$$|\ell + 2 \bar{m}\rangle_0 \propto \bar{J}_-^{\ell - \bar{m}} P_{22}^\ell |0\rangle_r. \quad (3.192)$$

or alternatively defined via the action of  $P_{2\dot{\alpha}}$ . The corresponding truncated character hence reads,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell}{2}}^{\frac{\ell}{2}} s^{2(\ell+2)} x^{\ell+2} \bar{x}^{2\bar{m}} = \frac{s^4 x^2}{(1 - s^2 x \bar{x})(1 - s^2 x \bar{x}^{-1})} \quad (3.193)$$

Rescaling the variables as in equation (2.141) allows one to recover the expression for the vector part of the index.

#### Gaugino

Finally for the gaugino, the Lagrangian reads,

$$L_\lambda = \sum_{\ell m \bar{m} \kappa} \bar{\lambda}_{\ell m \bar{m}}^\kappa (i\partial_0 - \kappa (\ell + \frac{3}{2})) \lambda_{\ell m \bar{m}}^\kappa, \quad (3.194)$$

which allows to write the conjugate momenta,

$$\pi_{\ell m \bar{m}}^+(t) = \frac{\partial L}{\partial \dot{\lambda}_{\ell m \bar{m}}^+} = i\bar{\lambda}_{\ell m \bar{m}}^+, \quad (3.195)$$

$$\pi_{\ell m \bar{m}}^-(t) = \frac{\partial L}{\partial \dot{\lambda}_{\ell m \bar{m}}^-} = i\bar{\lambda}_{\ell m \bar{m}}^-. \quad (3.196)$$



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This leads to the hamiltonian,  $R$ -charge and angular momentum generators,

$$H = \sum_{\ell m \bar{m} +} \left(\ell + \frac{3}{2}\right) \lambda_{\ell m \bar{m}}^+ \bar{\lambda}_{\ell m \bar{m}}^+ + \sum_{\ell m \bar{m} -} \left(\ell + \frac{3}{2}\right) \bar{\lambda}_{\ell m \bar{m}}^- \lambda_{\ell m \bar{m}}^-, \quad (3.197)$$

$$J_3 = \sum_{\ell m \bar{m} +} m \lambda_{\ell m \bar{m}}^+ \bar{\lambda}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} m \bar{\lambda}_{\ell m \bar{m}}^- \lambda_{\ell m \bar{m}}^-, \quad (3.198)$$

$$\bar{J}_3 = \sum_{\ell m \bar{m} +} \bar{m} \lambda_{\ell m \bar{m}}^+ \bar{\lambda}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} \bar{m} \bar{\lambda}_{\ell m \bar{m}}^- \lambda_{\ell m \bar{m}}^-, \quad (3.199)$$

$$R = \sum_{\ell m \bar{m} +} \lambda_{\ell m \bar{m}}^+ \bar{\lambda}_{\ell m \bar{m}}^+ - \sum_{\ell m \bar{m} -} \bar{\lambda}_{\ell m \bar{m}}^- \lambda_{\ell m \bar{m}}^-. \quad (3.200)$$

This leads to the expression for  $\mathcal{H}$ ,

$$\mathcal{H} = \sum_{\ell, m, \bar{m}, +} (\ell + 3 - 2m) \bar{\lambda}_{\ell m \bar{m}}^+ \lambda_{\ell m \bar{m}}^+ + \sum_{\ell, m, \bar{m}, -} (\ell - 2m) \bar{\lambda}_{\ell m \bar{m}}^- \lambda_{\ell m \bar{m}}^-, \quad (3.201)$$

and the zero modes are given by  $\bar{\lambda}_{\ell \frac{\ell}{2} \bar{m}}^-$ . Summing over those modes for the index, including the necessary minus sign, gives,

$$- \sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell+1}{2}}^{\frac{\ell+1}{2}} t^{\ell+1} x^{2\bar{m}} = \frac{t^2 - t(x + x^{-1})}{(1 - tx)(1 - tx^{-1})}. \quad (3.202)$$

All gaugino fermionic states contributing to the index are given by,

$$|\ell + \frac{3}{2}, \bar{m}\rangle_{-1} = |\ell + \frac{3}{2} (\frac{\ell}{2}, \frac{\ell}{2}) (\frac{\ell+1}{2}, \bar{m})\rangle_{-1}, \quad (3.203)$$

The character restricted to these states is given by,

$$\sum_{\ell=0}^{\infty} \sum_{\bar{m}=-\frac{\ell+1}{2}}^{\frac{\ell+1}{2}} s^{2(\ell+\frac{3}{2})} u^{-1} x^{\ell} \bar{x}^{2\bar{m}} = - \frac{s^3 x u^{-1} (s^2 - x^{-1}(\bar{x} + \bar{x}^{-1}))}{(1 - s^2 x \bar{x}^{-1})(1 - s^2 x \bar{x})} \quad (3.204)$$

From reading this expansion, one can see that all contributing states can be generated by acting with  $\{\bar{J}_-, P_{22}\}$  or  $P_{2\dot{\alpha}}$  on the following parent states,

$$|\frac{1}{2} \pm \frac{1}{2}\rangle_{-1} = |\frac{3}{2} (0, 0) (\frac{1}{2}, \pm \frac{1}{2})\rangle_{-1}, \quad (3.205)$$

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modulo the following constraint, which can most easily be explained in the context of a superconformal theory by the following equality,

$$P_{21}\bar{\lambda}_2 = P_{22}\bar{\lambda}_1. \quad (3.206)$$

To avoid double-counting any of those states, a compensation term proportional to  $s^5 u^{-1} x$  in the character expression, proportional to  $x^2$  in the index expression. As usual, the index can be recovered by rescaling the character variables following equation (2.141).

## 3.3 A Second Look at Characters

We now focus on repeating analysis from chapter (2) for the subalgebra  $SU(2, 1)$  which was defined in section (2.2).

### 3.3.1 Short, Long Representations of $SU(2,1)$ Subalgebra

As pointed earlier, the relevant subalgebra is (3.27). Let us first list the shortening conditions for this algebra. So called short representations, or chiral multiplets are now defined as,

$$f = 1, \quad Q_\alpha |\Delta, r, 0, \bar{j}\rangle^{\text{h.w.}} = 0, \quad (3.207)$$

$$\bar{f} = 1, \quad S^\alpha |\Delta, r, j, 0\rangle^{\text{h.w.}} = 0, \quad (3.208)$$

while semi-short representations are annihilated by the following supercharges,

$$\tilde{Q}_1 = Q_1 + \frac{1}{2j} Q_2 J_-, \quad j \neq 0, \quad (3.209)$$

$$\tilde{S}^1 = S^1 + \frac{1}{2j} J_+ S^2, \quad j \neq 0. \quad (3.210)$$

One can then define the following semi-short representations,

$$f = \frac{1}{2}, \quad \tilde{Q}_1 |\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0, \quad j \neq 0, \quad (3.211)$$

$$\bar{f} = \frac{1}{2}, \quad \tilde{S}^1 |\Delta, r, j, \bar{j}\rangle^{\text{h.w.}} = 0, \quad \bar{j} \neq 0, \quad (3.212)$$

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When  $j$  or  $\bar{j} = 0$ , the natural shortening condition is given by,

$$f = \frac{1}{2}, \quad Q^2|\Delta, r, 0, \bar{j}\rangle^{\text{h.w.}} = 0, \quad (3.213)$$

$$\bar{f} = \frac{1}{2}, \quad S^2|\Delta, r, j, 0\rangle^{\text{h.w.}} = 0, \quad (3.214)$$

Note that the shortening conditions impose much less severe restrictions on the  $R$ -charge and the scaling dimensions of the states part of the various representations.

These shortening conditions can be summarised with the following diagrams, In this diagram,  $\searrow$  corresponds to  $S^\beta$  while  $\swarrow$  corresponds to  $Q_\alpha$ . All states

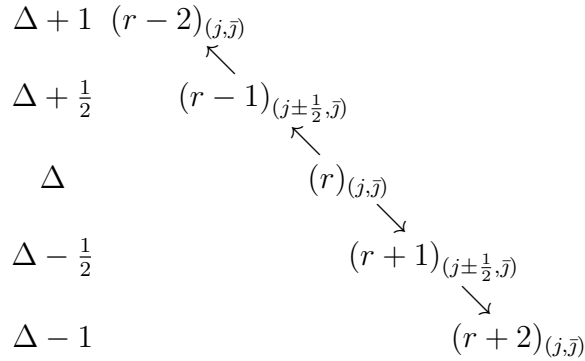


Figure 3.7: Long Representation  $\tilde{V}_{\Delta, r, j, \bar{j}}^{0,0}$

indicated corresponds to a  $(j, \bar{j})$  representation of  $SU(2)_l \times SU(2)_r$ . They also include an infinite tower of states which correspond to higher energy states, so, the seed state  $(r)_{j\bar{j}}$  also includes in fact all the following,

$$\begin{aligned}
 & (\Delta, r)_{(j, \bar{j})}, \\
 & (\Delta + 1, r)_{(j + \frac{1}{2}, \bar{j} + \frac{1}{2})}, \\
 & (\Delta + 2, r)_{(j + 1, \bar{j} + 1)}, \\
 & \dots
 \end{aligned}$$

One can then write down the corresponding diagram for what we will call chiral and antichiral representations. The shortening condition for the antichiral representations will involve the  $Q_\alpha$  supercharges, as usual, while the antichiral representations are defined as being annihilated by the  $S^\beta$  supercharges.

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$$\begin{array}{ccccccc}
 \Delta + 1 & (r - 2)_{(j, \bar{j})} & & \Delta & & (-r)_{(j, \bar{j})} & \\
 & \swarrow & & & & \searrow & \\
 \Delta + \frac{1}{2} & & (r - 1)_{(j \pm \frac{1}{2}, \bar{j})} & & \Delta - \frac{1}{2} & & (-r + 1)_{(j \pm \frac{1}{2}, \bar{j})} \\
 & & \swarrow & & & & \searrow \\
 \Delta & & (r)_{(j, \bar{j})} & & \Delta - 1 & & (-r + 2)_{(j, \bar{j})}
 \end{array}$$

Figure 3.8: Chiral, Antichiral Representations  $\tilde{V}_{\Delta, r, j, \bar{j}}^{0,1}$  and  $\tilde{V}_{\Delta, -r, j, \bar{j}}^{1,0}$  right

Finally, one has the short representations, In the last diagram, the remaining

$$\begin{array}{ccccccc}
 \Delta + \frac{1}{2} & (r - 1)_{(j \pm \frac{1}{2}, \bar{j})} & & \Delta & & (-r)_{(j, \bar{j})} & \\
 & \swarrow & & & & \searrow & \\
 \Delta & & (r)_{(j, \bar{j})} & & \Delta - \frac{1}{2} & & (-r + 1)_{(j - \frac{1}{2}, \bar{j})}
 \end{array}$$

Figure 3.9: Short Representations  $\tilde{V}_{\Delta, r, j, \bar{j}}^{\frac{1}{2},1}$  and  $\tilde{V}_{\Delta, -r, j, \bar{j}}^{1, \frac{1}{2}}$  right

supercharges are  $Q_2$  and  $S^2$ . The last representation is the only one whose quantum numbers are constrained by the shortening condition,

$$\{Q_1, S^\beta\} |\Delta, r, \bar{j}\rangle^{\text{h.w.}} \quad (3.215)$$

#### 3.3.2 Character and Indices Calculation

Based on the previous analysis, one can give the characters for the various representations. Define the supercharges factors,

$$Q(s, u, x) = \prod_{\epsilon=\pm 1} (1 + su^{-1}x^\epsilon), \quad (3.216)$$

$$S(s, u, x) = \prod_{\epsilon=\pm 1} (1 + s^{-1}ux^\epsilon), \quad (3.217)$$

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Hence, one can give the following characters for the various representations,

$$\chi_{\Delta,r,j,\bar{j}}^{0,0}(s, u, x, \bar{x}) = s^{2\Delta} u^r \chi_{\bar{j}}(\bar{x}) \chi_j(x) Q(s, u, x) S(s, u, x), \quad (3.218)$$

$$\chi_{\Delta,r,j,\bar{j}}^{0,1}(s, u, x, \bar{x}) = s^{2\Delta} u^r \chi_{\bar{j}}(\bar{x}) \chi_j(x) Q(s, u, x), \quad (3.219)$$

$$\chi_{\Delta,r,j,\bar{j}}^{1,0}(s, u, x, \bar{x}) = s^{2\Delta} u^{-r} \chi_{\bar{j}}(\bar{x}) \chi_j(x) S(s, u, x), \quad (3.220)$$

$$\chi_{\Delta,r,j,\bar{j}}^{\frac{1}{2},1}(s, u, x, \bar{x}) = s^{2\Delta} u^r \chi_{\bar{j}}(\bar{x}) (\chi_j(x) + s u^{-1} \chi_{j+\frac{1}{2}}(x)), \quad (3.221)$$

$$\chi_{\Delta,r,j,\bar{j}}^{1,\frac{1}{2}}(s, u, x, \bar{x}) = s^{2\Delta} u^{-r} \chi_{\bar{j}}(\bar{x}) (\chi_j(x) + s^{-1} u \chi_{j-\frac{1}{2}}(x)), \quad (3.222)$$

and finally,

$$\chi_{\Delta,r,j,\bar{j}}^{1,\frac{1}{2}}(s, u, x, \bar{x}) = \begin{cases} s^{2\Delta} u^{-r} \chi_{\bar{j}}(\bar{x}) (\chi_j(x) + s^{-1} u \chi_{j-\frac{1}{2}}(x)) & \text{if } j \neq 0, \\ s^{2\Delta} u^{-r} \chi_{\bar{j}}(\bar{x}) \chi_j(x) & \text{if } j = 0, \end{cases} \quad (3.223)$$

Based on the previous analysis, one can compute the index from by taking the appropriate limit as defined in equation (2.141). For the chiral field, one gets,

$$\sum_{l=0}^{\infty} \chi_{\Delta_{l,r},r,\frac{l}{2},\frac{l}{2}}^{\frac{1}{2},1}(y^{\frac{1}{2}}, y^{\frac{3}{2}} t^{-1}, -t y^{-1}, -x) = -\frac{t^{2-r}}{(1-tx)(1-tx^{-1})} \quad (3.224)$$

with,

$$\Delta_{l,r} = l + 2 - \frac{3}{2}r, \quad (3.225)$$

while for the antichiral field,

$$\sum_{l=0}^{\infty} \chi_{\bar{\Delta}_{l,r},r,-r,\frac{l}{2},\frac{l}{2}}^{1,\frac{1}{2}}(y^{\frac{1}{2}}, y^{\frac{3}{2}} t^{-1}, -t y^{-1}, -x) = \frac{t^r}{(1-tx)(1-tx^{-1})} \quad (3.226)$$

with the scaling dimension given by,

$$\bar{\Delta}_{l,r} = l + \frac{3}{2}r. \quad (3.227)$$

The values of the scaling dimensions can be obtained by radially quantising a theory invariant under the previous symmetry group. The states that are captured

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by the  $\chi^{\frac{1}{2},1}$  character limit in equation (3.224) are the following,

$$|l + 2 - \frac{3}{2}r, (\frac{l}{2}, m), (\frac{l}{2}, \bar{m})\rangle_r, \quad |l + \frac{5}{2} - \frac{3}{2}r, (\frac{l+1}{2}, m), (\frac{l}{2}, \bar{m})\rangle_{r-1}, \quad (3.228)$$

where we have used the following notation in the equation above,

$$|H, (j, m), (\bar{j}, \bar{m})\rangle_R, \quad (3.229)$$

which denotes an  $(m, \bar{m})$  eigenvalue state part of a  $(j, \bar{j})$  representation of the  $SU(2)_l \times SU(2)_r$ . In order to contribute to the index, these states have to exhibit a zero  $\mathcal{H}$  eigenvalue, and the only ones that actually fulfill this constraint are fermionic states such that,

$$|l + \frac{5}{2} - \frac{3}{2}r, (\frac{l+1}{2}, -\frac{l+1}{2}), (\frac{l}{2}, \bar{m})\rangle_{r-1}, \quad (3.230)$$

On the other hand, the states captured by the  $\chi^{1, \frac{1}{2}}$  character limit in equation (3.226) are the following,

$$|l + \frac{3}{2}r, (\frac{l}{2}, m), (\frac{l}{2}, \bar{m})\rangle_{-r}, \quad |l + \frac{1}{2} + \frac{3}{2}r, (\frac{l}{2}, m), (\frac{l+1}{2}, \bar{m})\rangle_{-r+1} \quad (3.231)$$

and the states contributing to the index are given by the following,

$$|l + \frac{3}{2}r, (\frac{l}{2}, -\frac{l}{2}), (\frac{l}{2}, \bar{m})\rangle_{-r} \quad (3.232)$$

Hence, one the antichiral scalar states contribute to the index, while the chiral spinor states contribute to the index.

# Chapter 4

## Index and Localisation

In this chapter, we clarify the status of the topological nature of the index. In order to settle this issue, we compute the index as a partition function on  $\mathbb{R} \times S^3$  and then use localisation methods to prove the exactness of the free field theory result.

### 4.1 General Setup

#### 4.1.1 Index as a Path Integral

The usual definition of a path integral in bosonic quantum theory with a finite  $\beta$  time interval or  $\beta^{-1}$  temperature is given by,

$$\mathrm{Tr} e^{-\beta H} = \int_{\mathrm{PBC}} [dX] e^{iS[X]}, \quad (4.1)$$

with PBC denoting periodic boundary conditions on the bosonic quantum states, and  $H$  the time translation generator  $H = P_0$ . For fermionic quantum states, one then has the choice between periodic and antiperiodic boundary conditions. The former lead to an alternative definition of the index, while the latter leads to a regular path integral. Let us review the situation with the index, following closely a discussion in [34].

Consider a correlation function involving fermions exclusively. The number

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of fermions has to be even for the result to be nonzero. Let us take an operator approach, and consider the time ordered product of fermionic operators. Take one of the inserted fermionic operator and continuously move it accross the time interval. It will have to go past an odd number of fermionic operators, picking up a minus sign every time. It will then hit the periodic boundary and come back to its original position having gained one overall minus sign. In order for the correlation function to be invariant under such operation, one has to insert a factor of  $(-1)^F$  in front of all correlation functions. Going back to a path integral approach, the generalisation of the above formula (4.1) to a theory involving fermionic and bosonic states is given by,

$$\mathrm{Tr} (-1)^F e^{-\beta H} = \int_{\mathrm{PBC}} [dX] e^{iS[X]}, \quad \mathrm{Tr} e^{-\beta H} = \int_{\substack{\mathrm{Fer. aPBC} \\ \mathrm{Bos. PBC}}} [dX] e^{iS[X]}, \quad (4.2)$$

with the subscript aPBC denoting antiperiodic boundary conditions for fermions in the computation of the plain path integral. The index may be evaluated by employing radial quantisation, using a path integral representation, compactifying the time direction from  $\mathbb{R}$  to a circle  $S^1$  of period  $\beta$  so that, in (2.49),

$$s = e^{-\beta}, \quad (4.3)$$

To make the path integral well defined we will Wick rotate the time direction<sup>1</sup>,

$$\partial_0 = i\partial_\tau, \quad (4.4)$$

which then leads to the following definition for the index [77],

$$I(\beta) = \mathrm{Tr} (-1)^F e^{-\beta H} = \int_{\mathrm{PBC}} [dX] e^{-S_E[X]}, \quad (4.5)$$

where  $X$  denotes the fields contributing, and  $S_E$  denotes the Euclideanised action on  $S^1 \times S^3$ .

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<sup>1</sup>In the remainder of this thesis, we will always take the Minkowski time subscript to be zero, while the Euclidean time will be denoted b  $\tau$ .



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### 4.1.2 Topological Invariance of the Index

The above representation of the index is convenient to explore the topological properties of the index [77]. Define the expectation value of an operator  $\mathcal{O}$  as,

$$\langle \mathcal{O} \rangle_\beta = \frac{\int_{\text{PBC}} [dX] \mathcal{O} e^{-S_E[X]}}{\int_{\text{PBC}} [dX] e^{-S_E[X]}} \quad (4.6)$$

with periodic boundary conditions along a time interval of length  $\beta$ . One has the following properties of the index.

- The index is independent of  $\beta$ .
- The usual flat space vacuum expectation value of an operator is just the limit  $\langle \mathcal{O} \rangle_\infty$ .
- If  $\mathcal{O}$  is an operator such that,  $\delta \mathcal{O} = \epsilon^\alpha Q_\alpha \mathcal{O}$  then,

$$I(\beta) \langle \delta \mathcal{O} \rangle_\beta = 0 \quad (4.7)$$

Hence if  $I(\infty) \neq 0$  then supersymmetry is unbroken. A necessary condition for dynamical supersymmetry breaking is  $I(\beta) = 0$ .

- The index is invariant under continuous deformations of the theory. If one assumes,

$$\mathcal{L} = \sum_i \lambda_i \mathcal{L}_i, \quad (4.8)$$

then,

$$\frac{\partial I}{\partial \lambda_i} = -I(\beta) \int d^4x \langle \mathcal{L}_i \rangle_\beta \quad (4.9)$$

Assuming these Lagrangians are exact under supersymmetry, or equivalently that they can be recast as  $F$  or  $D$ -terms of certain superfields, one can write,

$$\langle \mathcal{L}_i \rangle \propto \langle \delta \mathcal{O}_i \rangle = 0, \quad (4.10)$$

and consequently equation (4.9) is zero. Hence, assuming all pieces of the Lagrangian to be  $\delta$ -exact implies the invariance of the Lagrangian under continuous deformation.

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### 4.1.3 Twisting the Boundary Conditions on $S^1 \times S^3$

When dealing with supersymmetry in the usual flat space context with any number of dimensions, the time translations are generated by the zeroth component of the vector of translation generators. When commuting this generator with the supercharges of the theory,

$$[P_0, Q_\alpha] = 0, \quad (4.11)$$

which then implies that the Killing spinors  $\epsilon$  does not have a time dependence. Consequently, the variation supersymmetry variation,

$$\phi \rightarrow \phi + \epsilon^\alpha \psi_\alpha, \quad (4.12)$$

is compatible with periodic boundary conditions for both the spinor  $\psi$  and the scalar  $\phi$ , as the left hand side and the right hand side are both invariant under,

$$\phi(\tau + \beta) = \phi(\tau), \quad \psi_\alpha(\tau + \beta) = \psi(\tau), \quad \frac{\partial \epsilon^\alpha}{\partial \tau} = 0. \quad (4.13)$$

In a radially quantised context, the situation is different. The time translation generator is identified with the dilation operator  $H$  which has nonzero commutator with all supercharges (A.7) which then leads to time dependence of the Killing spinor as shown in (3.53). This then implies that equation (4.12) is not compatible with plain periodic boundary conditions, because of the non-trivial time-dependence of the Killing spinor.

When computing an index as a path integral, the right definition for the time translation operator is not in term of dilation operator or in terms of translation generators, but rather in terms of supercharges. In the previous section, the following definition held for the time translation generator,

$$\{Q, Q^\dagger\} = 2P_0, \quad (4.14)$$

Here, we will do the same, having chosen one particular supercharge within our subalgebra  $\mathcal{Q} = Q_1$ , as well as having identified the appropriate hermitean conjugation  $^\dagger$ , the counterpart of equation (4.14) was given earlier in (2.34) or equiva-

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lently (2.34). Consequently, in order to maintain consistency with supersymmetry when compactifying the time direction from  $\mathbb{R}$  to  $S^1$  a twist of the boundary conditions is required, which implements the transformation,

$$H \xrightarrow{\text{twist}} \mathcal{H}, \quad (4.15)$$

or explicitly,

$$i\partial_0 \xrightarrow{\text{twist}} i\partial_0 + \frac{3}{2}\mathcal{R} - 2\mathcal{J}_3, \quad (4.16)$$

with  $\mathcal{R}$  the differential operator associated with the  $R$ -charge, as defined in equation (3.32). We will not need its expression<sup>1</sup>, however one can still use the approach in equation (3.50) to assign an  $R$ -charge to Killing spinors,

$$\mathcal{R}\epsilon^\alpha = \epsilon^\alpha, \quad \mathcal{R}\bar{\epsilon}_\alpha = -\bar{\epsilon}_\alpha. \quad (4.17)$$

In practice, the twisting is achieved by rescaling all fields  $X = \{\phi, \psi, F, A, \lambda, D\}$  and their conjugates as,

$$X = e^{-it(\frac{3}{2}\mathcal{R} - 2\mathcal{J}_3)} \tilde{X}, \quad (4.18)$$

and this also applies to Killing spinors [78], using the explicit  $\mathcal{R}$  charge we defined earlier 4.17. Then, using equation (4.18) applied to the Killing spinor, along with the eigenvalues we have assigned them allows, which leads to the following,

$$\epsilon_1^\alpha = e^{\frac{it}{2}} \tilde{\epsilon}_1^\alpha, \quad (4.19)$$

The periodicity boundary conditions are then imposed on the rescaled fields in the usual fashion for both fermionic and bosonic fields,

$$\tilde{X}(\tau + \beta) = \tilde{X}(\tau). \quad (4.20)$$

Then, imposing periodic boundary conditions on  $\{\tilde{\phi}, \tilde{\psi}\}$  is compatible with the times dependence of the Killing spinor, and a supersymmetric theory can be defined on a compact time direction, with the following supersymmetry transfor-

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<sup>1</sup>In a superspace formulation, with fermionic parameters  $(\eta_\alpha, \bar{\eta}^\alpha)$ , it would be  $\mathcal{R} = \eta^\alpha \frac{\partial}{\partial \eta^\alpha} + \bar{\eta}^\alpha \frac{\partial}{\partial \bar{\eta}^\alpha}$

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mation

$$\tilde{\phi} \rightarrow \tilde{\phi} + \tilde{\epsilon}^\alpha \tilde{\psi}_\alpha, \quad \text{with } \tilde{\epsilon}_1^\alpha = \delta_1^\alpha \quad (4.21)$$

If one wanted to write the supersymmetry transformations for the entire twisted chiral multiplet, one would just need to modify the time derivatives as in (4.16). For instance,

$$\tilde{\delta} \tilde{\phi} = \tilde{\epsilon}^\alpha \tilde{\psi}_\alpha, \quad (4.22)$$

$$\tilde{\delta} \tilde{\psi}_\alpha = \tilde{\epsilon}_\alpha (i\partial_0 - 2i\nabla_3) \phi + 2i(\sigma_m \tilde{\epsilon})_\alpha \nabla_m \phi, \quad (4.23)$$

One should note that the previous analysis is not valid for the Killing spinors defined in equation (3.54). This is because the  $\mathcal{J}_3$  eigenvalue of  $\psi_2$  is  $-\frac{1}{2}$  and does not allow for a proper cancellation of the prefactors to get to equation (4.21). The point is that the twisted theory is only invariant under the following supersymmetries,

$$\delta_1 X = [\epsilon_1^\alpha Q_\alpha, X], \quad \bar{\delta}^1 X = [\bar{\epsilon}_\alpha^1 S^\alpha, X]. \quad (4.24)$$

Once again, the 1 sub/super-scripts are not spinor indices, they simply label the various supersymmetry variations of the theory. This said, given the definition of  $\epsilon_1^\alpha, \bar{\epsilon}_\alpha^1$  given in equation (3.53), one can certainly think of  $\delta_1, \bar{\delta}^1$  as corresponding to  $Q_1, S^1$  respectively. The point is that the twisted theory is not invariant under  $Q_2, S^2$ , and so we have promoted  $Q_1, S_1$  to scalar supercharges through twisting.

This finally leads us to the path integral for the 4-dimensional index,

$$\text{Tr} (-1)^F e^{-\beta \mathcal{H}} = \int_{\text{PBC}} [dX] \exp(-S_E^{\text{twist}}[X]), \quad (4.25)$$

with the euclidean action twisted following equation (4.16). Overall, we have shown that we can compactify the time direction to a finite temperature interval with periodic boundary conditions for all fields and preserve 2 supersymmetries out of the 4 that we had started with in an uncompactified setting<sup>1</sup>.

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<sup>1</sup>The theory on  $\mathbb{R} \times S^3$  is an  $\mathcal{N} = 1$  supersymmetric theory with 4 supercharges

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We now want to compute the full superconformal index (2.49). In fact, we have no choice. As has been pointed out in the previous chapters, there is an infinite number of states in the  $\mathcal{Q}$ -cohomology contributing to the index. Hence we need to weigh these contributions by the remaining Cartans of  $SU(2, 1)$ . This can be done by twisting further the theory under consideration. In order to do that, we first write the fugacities  $s, x, y$  in terms of the corresponding chemical potentials,

$$s = e^{-\beta}, \quad x = e^{-\beta\gamma_x}, \quad y = e^{-\beta\gamma_y}, \quad (4.26)$$

and rewrite the index as,

$$\mathrm{Tr} (-1)^F s^{\mathcal{H} + \gamma_x(R - 2J_3) + 2\gamma_y\bar{J}_3}, \quad (4.27)$$

To compute this in the context we have laid out, we further twist the time derivative,

$$i\partial_0 \xrightarrow{\text{twist}} i\partial_0 + \frac{3}{2}\mathcal{R} - 2\mathcal{J}_3 + \gamma_x(\mathcal{R} - 2\mathcal{J}_3) - 2\gamma_y\bar{\mathcal{J}}_3 \quad (4.28)$$

One can straightforwardly understand the relevance of twisting in a generic, not necessarily supersymmetric theory, in modifying a given hamiltonian to include other generators. Consider the complex scalar field  $\phi$  with the Lagrangian:

$$L[\phi, \phi^\dagger] = -\phi^\dagger(d_\tau + a)^2\phi + b^2\phi^\dagger\phi \quad (4.29)$$

Note that this Lagrangian is euclidean. The conjugate momenta are given by the following:

$$p = (d_\tau - a)\phi^\dagger, \quad p^\dagger = (d_\tau + a)\phi. \quad (4.30)$$

Expanding in modes,

$$a_1^+ = \frac{1}{\sqrt{2b}}(p + b\phi^\dagger), \quad a_1 = \frac{1}{\sqrt{2b}}(p^\dagger - b\phi), \quad (4.31)$$

$$a_2^+ = \frac{1}{\sqrt{2b}}(p^\dagger + b\phi), \quad a_2 = \frac{1}{\sqrt{2b}}(p - b\phi^\dagger), \quad (4.32)$$

and imposing the following canonical commutation relations,

$$[a_1^+, a_1] = 1, \quad [a_2^+, a_2] = 1, \quad (4.33)$$

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the hamiltonian can then be written as,

$$H = (a + b)a_1^\dagger a_1 + (b - a)a_2^\dagger a_2 \quad (4.34)$$

Equation (4.34) implies that the eigenvalues of the hamiltonian are given by:

$$h = \pm a + b \quad (4.35)$$

This Lagrangian is invariant under  $U(1)$  transformations<sup>1</sup>:

$$[T, \phi] = \alpha \phi \quad (4.36)$$

One can write the charge operator in the following fashion:

$$T = -\alpha a_1^\dagger a_1 + \alpha a_2^\dagger a_2 \quad (4.37)$$

Hence if one defines the modified hamiltonian:

$$\tilde{H} = H + T = (a - \alpha + b)a_1^\dagger a_1 + (b - a + \alpha)a_2^\dagger a_2 \quad (4.38)$$

One can see that this modified Lagrangian is the same as the one in equation (4.34), where the following transformation has been performed,

$$a \xrightarrow{\text{twist}} a - \mathcal{T} \quad (4.39)$$

with  $\mathcal{T}$  the differential operator associated with  $\mathcal{T}$  as defined in equation (3.32). One can then compute the so called partition function associated with  $\tilde{H}$  by using the following twisted Lagrangian:

$$L[\phi, \phi^\dagger, \mathcal{T}] = -\phi^\dagger (d_\tau + a - \mathcal{T})^2 \phi + b^2 \phi^\dagger \phi \quad (4.40)$$

This prescription can be generalised to any set of fugacities.

To make contact with previous chapters, we will compute the generic partition

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<sup>1</sup>These considerations generalise to any type of global symmetries of the theory.

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function,

$$\mathrm{Tr}(-1)^F s^{2H} u^R x^{2J_3} \bar{x}^{2\bar{J}_3}, \quad (4.41)$$

using the twisting procedure previously discussed, the relevant fugacities are defined as,

$$s = e^{-\beta}, \quad u = e^{-\beta\gamma_u}, \quad x = e^{-\beta\gamma_x}, \quad \bar{x} = e^{-\beta\gamma_{\bar{x}}}, \quad (4.42)$$

which allows us to compute the partition function (4.41) by simply computing a free field theory path integral with the following twisted time derivatives,

$$\partial_\tau \xrightarrow{\text{twist}} \partial_\tau - \mathcal{T}, \quad \mathcal{T}(\gamma_u, \gamma_x, \gamma_{\bar{x}}) = \gamma_x \mathcal{J}_3 + \gamma_{\bar{x}} \bar{\mathcal{J}}_3 + \frac{1}{2} \gamma_u \mathcal{R}. \quad (4.43)$$

$$\partial_\tau X \xrightarrow{\text{twist}} \partial_\tau X + [T, X], \quad T(\gamma_u, \gamma_x, \gamma_{\bar{x}}) = \gamma_x J_3 + \gamma_{\bar{x}} \bar{J}_3 + \frac{1}{2} \gamma_u R. \quad (4.44)$$

We then take the limit (2.141) to obtain the free field value of the index, which we show to be the exact result in the following section.

### 4.1.4 Supersymmetry and Localisation

Here we review the localisation framework. Our aim is to explain the usual statement about localisation which is essentially the following [79],

*“Localisation allows the exact computation of correlation functions of  $\mathcal{Q}$ -closed operators, where  $\mathcal{Q}$  is a supercharge promoted to a scalar operator by twisting the boundary conditions of a finite time direction in a Euclidean field theory”.*

#### Localisation as a Handle on Strong Coupling

In the context of supersymmetric theories, localisation allows for exact computation of expectation values of certain operators invariant under a chosen scalar supercharge, regardless of the value of the coupling [80]. This makes it a complement to the strong coupling calculation methods based on dualities as the gauge-string duality in 4 and in 3 dimensions [6] and [7] as well as the electromagnetic duality. One of localisation’s main advantage is that it can be used with any theory which includes just one supercharge. However it is restricted to the computation of expectation values of operators closed under this supercharge.

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Obviously, calculations based on dualities do not have this restriction, however, the theories considered have more symmetries, and are hence more peculiar and less phenomenologically realistic.

Localisation has led to important progress in the understanding of supersymmetric theories over the last few years. There was first the proof by Pestun [81, 82] of the matrix model conjecture due Erickson, Semenoff, Zarembo and Drukker, Gross [83] and [84] for Wilson loops, which had first been postulated following AdS/CFT. Before that, the Seiberg Witten prepotential for 4 dimensional  $\mathcal{N} = 2$  theories had been computed exactly using localisation techniques [85, 86, 87]. More recently, Drukker, Mariño and Putrov computed the exact partition function for ABJM theory [88] as well as the planar free energy of the theory, from they were able to extract the correct  $N^{\frac{3}{2}}$  scaling for the number of degrees of freedom of M2 brane theories. Finally, the index for 3-dimensional theory was computed exactly for ABJM theory [12, 89], with a generalisation of this index to arbitrary  $R$ -charges given in [78]. This computation, just like Pestun's, is important because they correspond to field theory calculations which are exact to all orders in perturbation theory, which also incorporate all non-perturbative corrections. Localisation was also used [90] to test three dimensional Seiberg like dualities in three dimensions for Chern-Simons theory [91] [92]. Such a calculation was also performed for the index in 3 dimensional Chern-Simons matter theory for the index [93]. Also, a very similar approach to the one in this chapter is given in [78], where the index on  $S^2 \times S^1$  is computed for theories with general  $R$ -charge assignments for the matter fields.

### General Considerations

An exact computation of the path integral for the index can then be performed using localisation arguments [10]. This procedure relies on the existence of a fermionic supercharge  $\delta$ , which annihilates the action, and for which the square  $\delta^2 = \mathcal{H}$  is a bosonic symmetry of the action. This method relies on the invariance of the path integral under the addition of  $\delta$ -exact term to the action. The correlation function of any operator  $\delta$ -closed operator is invariant under such



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a transformation,

$$\frac{d}{d\alpha} \int [dX] \mathcal{O} e^{-S_E[X] - \alpha \delta \mathcal{V}} = 0, \quad \text{with } \delta \mathcal{O} = 0, \quad (4.45)$$

This can be shown in the following fashion [77], as equation (4.45) is proportional to,

$$\int [dX] (\delta V) \mathcal{O} e^{-S_E[X] - \alpha \delta \mathcal{V}} = \delta \left( \int [dX] \mathcal{O} V e^{-S_E[X] - \alpha \delta \mathcal{V}} \right) = 0, \quad (4.46)$$

The requirements on  $\mathcal{V}$  is invariance under the bosonic symmetry  $\mathcal{H}$ ,

$$\mathcal{H}\mathcal{V} = 0, \quad (4.47)$$

This property the allows one take the limit,

$$\alpha \rightarrow \infty, \quad (4.48)$$

provided  $\delta \mathcal{V}$  can be shown to be positive definite fo well-definedness of the limit. This then makes the saddle point approximation exact. Generically, the saddle point, or one loop approximation can be summarised as follows. Consider the bounded below function  $f(x)$ ,

$$\begin{aligned} \int dx e^{-\alpha f(x)} &= \int dx \exp\left(-\alpha f(x_c) - \frac{1}{2!} \alpha f^{(2)}(x_c)(x - x_c)^2 \right. \\ &\quad \left. - \frac{1}{3!} \alpha f^{(3)}(x_c)(x - x_c)^3 + \dots\right) \\ &= \sum_{\{x_c\}} \sqrt{\frac{2\pi}{\alpha f^{(2)}(x_c)}} e^{-\alpha f(x_c)} + \mathcal{O}(\alpha^{-1}), \end{aligned} \quad (4.49)$$

with  $x_c$  the set of critical points for  $f$  defined as,

$$\left. \frac{\partial f}{\partial x} \right|_{x_c} = 0. \quad (4.50)$$

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In the  $\alpha \rightarrow \infty$  limit, the approximation becomes more and more accurate. In the following we will show that for both the vector and the scalar multiplet,

$$\mathcal{L}_\phi = \delta\mathcal{V}_\phi, \quad \mathcal{L}_{\text{int.}} = \delta\mathcal{V}_{\text{int.}}, \quad \mathcal{L}_A = \frac{1}{g^2}(\delta + s)\mathcal{V}_A, \quad (4.51)$$

these actions have been defined in equations (3.96), (3.100), (3.104), and  $\mathcal{V}_A, \mathcal{V}_\phi$  are functionals invariant under  $\mathcal{H}$ . When dealing with the gauge theory, one has to localise the gauge fixed theory. This is achieved by using a standard BRS procedure as explained in [53] and by adding to the localisation supercharge the BRS charge,  $\delta \rightarrow \delta + s$ . One can then identify  $\alpha_A$  with the gauge coupling  $\frac{1}{g^2}$  and take the coupling  $g$  to zero which is a free field theory limit. We will need to put an  $\alpha_\phi$  parameter in front of  $S[\phi, \psi, F]$  as well, because the definition (3.96) of the chiral multiplet action does not include any superpotential terms. Should the theory under consideration have a superpotential, the  $\alpha_\phi \rightarrow \infty$  limit makes sure that the superpotential can be neglected in the sense that the associated coupling constant will not contribute, and the index can be computed in a free field theory approach. This is the input of the localisation approach.

### 4.1.5 A Remark on the Superpotential and the Index

One cannot however completely forget about the superpotential [32, 33] as it can break some of the global symmetries of the free theory. So the superpotential will in fact affect the very definition of the index, as one will associate fugacities to the symmetries that are preserved by the superpotential. In the case of a Wess-Zumino model with superpotential,

$$W(\phi) = \phi^n, \quad (4.52)$$

with the constraint,

$$nr = 2, \quad (4.53)$$

the global  $U(1)$  symmetry is broken down to  $\mathbb{Z}_n$  and the superconformal index for an ungauged chiral multiplet in equation (2.88) is modified to the following

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expression,

$$i_\phi(t, x, h) \rightarrow i_\phi(t, x, \omega) = \frac{t^r \omega - t^{2-r} \omega^{-1}}{(1-tx)(1-tx^{-1})}, \quad (4.54)$$

with  $\omega$  such that,

$$\omega^n = 1. \quad (4.55)$$

Given the constraints on the superpotential and the  $R$ -charge (4.53), the single particle index can be rewritten,

$$i_\phi(t, x, \omega) = \frac{t^r \omega - (t^r \omega)^{n-1}}{(1-tx)(1-tx^{-1})}. \quad (4.56)$$

This implies that for  $n = 2$ , i.e. for a massive multiplet, the contribution to the index drops out. This can be interpreted as the index being a quantity probing the infra-red of a given theory, which implies that massive multiplets are essentially integrated out from the index.

### 4.1.6 Practical Setup for Localisation

The localisation supercharge we will use will be a generic one,

$$[\mathcal{Q}_L, X] = (\delta + \bar{\delta})X, \quad (4.57)$$

which will be generated by mutually complex conjugate Killing spinor as defined in (3.53), so either  $\{\epsilon_1^\alpha, \bar{\epsilon}_\alpha^1\}$  or  $\{\epsilon_2^\alpha, \bar{\epsilon}_\alpha^2\}$ . Here, we will take  $\mathcal{Q}_L$  as the sum of  $\epsilon_1^\alpha Q_\alpha$  and  $\bar{\epsilon}_\alpha^1 S^\alpha$  for instance. Provided that  $\mathcal{Q}_L$  squares to a bosonic symmetry of the theory which leaves  $\mathcal{V}$  invariant, one can take  $\mathcal{Q}_L$  to be fermionic, hence the Killing spinors are commuting quantities. As defined, the supercharge squares to,

$$\mathcal{Q}_L^2 = H + \frac{3}{2}R - 2V_m J_m, \quad (4.58)$$

and if one makes the choice of Killing spinors  $\{\epsilon_1^\alpha, \bar{\epsilon}_\alpha^1\}$  then  $\mathcal{Q}_L$  squares to the modified hamiltonian  $\mathcal{H}$  defined in equation (2.37).

For practical calculations, we will need to take a finite size time direction. Consequently, to ensure compatibility with supersymmetry we will need to twist

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the theory and break half the supersymmetries. When taking the radius of the thermal circle  $S^1$  to infinity, we will recover the full supersymmetry.

One could then ask, why could we not take the radius of  $S^1$  to infinity, which would restore  $\mathcal{Q}_L$  as a preferred supercharge, as opposed to  $\mathcal{Q}$ , keep  $\mathcal{H}$  as a hamiltonian, include fugacities associated with  $R$ ,  $J_3$ ,  $\bar{J}_3$ , which commute with  $\mathcal{Q}_L$  in the untwisted infinite radius theory. We would then be able to define an index which would be topologically protected via localisation as well? It would be computed on an  $S^1$  and its radius would be taken to infinity at the end of the calculation. This would be its expression,

$$\text{Tr} (-1)^F s^{\mathcal{H}} u^R x^{J_3} \bar{x}^{\bar{J}_3}. \quad (4.59)$$

However this doesn't work [94]. In the infinite radius theory, the set of states to be considered becomes a continuum with an associated density of states. In particular, it might not be true that  $|\phi\rangle$  and  $\mathcal{Q}_L|\phi\rangle$  have the same density. The index expression would then look like,

$$(-1)^f \int d\omega (g_+(\omega) - g_-(\omega)) I_\omega(s, u, x, \bar{x}), \quad (4.60)$$

with  $g_+(\omega)$ ,  $g_-(\omega)$  the densities of states for  $|\phi\rangle$  and  $\mathcal{Q}_L|\phi\rangle$  respectively. If the density  $g_\pm$  are not identical at a given energy level, the previous analysis essentially breaks down. Some 2-dimensional examples of such phenomenon were given in [95] based on the Callias-Bott-Seeley index theorem [96, 97]. Also, in 4 dimensions, we were unable to exhibit any kind of electric-magnetic duality matching for indices defined as in (4.59).

In practice, we compute the localisation action  $[\mathcal{Q}_L, \mathcal{V}_\phi]$ ,  $[\mathcal{Q}_L, \mathcal{V}_A]$  using the standard supersymmetry transformations for instance (3.62) and apply the twisting at the end. As we have noted before, the twisting procedure modifies the the Killing spinors and makes them essentially time independent as was shown in equation (4.21). At the level of the supercharge, this then implies that under

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twisting, the supercharge  $\mathcal{Q}_L$ , when taken to be generated by  $\{\epsilon_1^\alpha, \epsilon_\alpha^1\}$ ,

$$\mathcal{Q}_L \xrightarrow{\text{twist}} \mathcal{Q} \quad (4.61)$$

where  $\mathcal{Q}$  had been defined in equation (2.36). So the twisting indeed promotes a spinorial supercharge to a scalar operator, and this can be traced back to the requirement of periodicity of all fields along the euclidean time direction. Now that we have established that the twisted path integral is invariant under  $\mathcal{Q}$  – and not  $\mathcal{Q}_L$  – it is clear from a path integral point of view, that the only index that can consistently be defined to satisfy the localisation requirements is the one given in equation (2.49). One should think of this quantity as the expectation value of an operator under twisted boundary conditions,

$$I(x, y) = \langle x^{-R+2J_3} y^{2\bar{J}_3} \rangle, \quad (4.62)$$

and the localisation argument can be used because,

$$[\mathcal{Q}, x^{-R+2J_3}] = [\mathcal{Q}, y^{2\bar{J}_3}] = 0. \quad (4.63)$$

In this setting, the computation of the index reduces to the computation of the twisted one loop determinant around trivial backgrounds for all fields.

### Geometric Interpretation

The twisting has the following impact on the supercharges of the theory. Consider a superconformal theory, and see how the supercharges decomposes under the isometries of  $\mathbb{R} \times S^3$  which are  $U(1) \times SU(2)_L \times SU(2)_R$ . In addition to these isometries, we have the  $R$ -charge.

$$\begin{array}{lll} SU(2, 2) \times U(1)_R & \rightarrow & SU(2)_L \times SU(2)_R \times U(1) \times U(1)_R \\ \bar{\mathcal{Q}}^\beta & 4_1 & \rightarrow (1, 2)_{-\frac{1}{2}, 1} \oplus (2, 1)_{\frac{1}{2}, 1} \\ \mathcal{Q}_A & \bar{4}_{-1} & \rightarrow (1, 2)_{\frac{1}{2}, -1} \oplus (2, 1)_{-\frac{1}{2}, -1} \end{array} \quad (4.64)$$

with  $\bar{\mathcal{Q}}^\beta$ ,  $\mathcal{Q}_A$  as defined in equation (2.29). When the theory is not conformal anymore, the only supercharges left are the the undotted supercharges  $Q_\alpha$ ,  $S^\beta$ .

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After twisting the theory, the theory is supersymmetric on  $S^1 \times S^3$  with periodic boundary conditions on the rescaled fields  $\tilde{X}$  defined, and the only supercharges left are  $Q_1$ ,  $S^1$  as scalar supercharges, and 0 charge under  $R$  and  $H$ ,

$$\begin{aligned} S^\beta & (2, 1)_{\frac{1}{2}, 1} \xrightarrow{\text{twist}} Q_1 (1, 1)_{0,0} \\ Q_\alpha & (2, 1)_{-\frac{1}{2}, -1} \xrightarrow{\text{twist}} S^1 (1, 1)_{0,0} \end{aligned} \quad (4.65)$$

This is a slightly different approach from the one taken in [80]. The theory under consideration there is a 4 dimensional  $\mathcal{N} = 2$  supersymmetric theory which is naturally constructed by extending a 3 dimensional Floer theory to a relativistic setting. One can also regard it as a spatially twisted version of 4 dimensional  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory.  $\mathbb{R}^4$  has an  $SU(2)_L \times SU(2)_R$  isometry. Also,  $\mathcal{N} = 2$  theories have the  $SU(2)_U \times U(1)_I$   $R$ -symmetry. The theory is twisted by considering an exotic action of the 4 dimensional rotation group. It is replaced with  $SU(2)_L \times SU(2)'_R$ , where  $SU(2)'_R$  is the diagonal sum of  $SU(2)_R$  and  $SU(2)_U$ . Under this twisting, the supercharges are modified as,

$$\begin{aligned} SU(2)_L \times SU(2)_R \times SU(2)_U \times U(1)_I & \xrightarrow{\text{twist}} SU(2)_L \times SU(2)'_R \times U(1)_I \\ (\frac{1}{2}, 0, \frac{1}{2})_{-1} \oplus (0, \frac{1}{2}, \frac{1}{2})_1 & \xrightarrow{\text{twist}} (\frac{1}{2}, \frac{1}{2})_{-1} \oplus (0, 1)_1 \oplus (0, 0)_1, \end{aligned} \quad (4.66)$$

hence the Lorentz scalar supercharge identified for localisation purposes is the  $(0, 0)_1$  component. Explicitly, this supercharge can be constructed by contracting  $SU(2)_R$  and  $SU(2)_I$  [98] with the usual antisymmetric tensor  $\epsilon$  used to raise and lower spinor indices,

$$\mathcal{Q} = \epsilon^{A\dot{\alpha}} Q_{A\dot{\alpha}}, \quad (4.67)$$

with  $\dot{\alpha}$  the usual  $SU(2)_R$  index and  $A$  the  $SU(2)_I$  index. In the approach we have taken, imposing the periodicity boundary conditions on the fields meant we had to redefine the degrees of freedom of the theory we are considering on  $S^1 \times S^3$ , and that this modified theory is only invariant under one preserved supercharge  $\mathcal{Q} = Q_1$  and its conjugate  $S^1$ . On the other hand, the twisted theory in [80] is really the standard  $\mathcal{N} = 2$  supersymmetric Yang-Mills theory written with an exotic action of the rotation group. Ultimately, Witten's twisting is a reformulation of the same theory in a different basis which allows for a

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scalar supercharge, whereas our approach is an actual modification of a given theory to make it fit on a compactified manifold, which only allows us to keep on supercharge, and the latter get promoted to a scalar operator.

### 4.1.7 Localisation Actions

The scalar field action (3.96), with the derivatives gauge covariantised, including the gauge interaction terms (3.104) may be written in a  $\mathcal{Q}_L$ -exact fashion  $[\mathcal{Q}_L, \mathcal{V}_\phi]$  with  $\mathcal{V}_\phi$  defined as,

$$\begin{aligned} \mathcal{V}_\phi &= \frac{1}{2}(\psi^\beta \epsilon_\alpha [\mathcal{Q}_\beta, \bar{\psi}^\alpha] - \bar{\epsilon}_\alpha [\bar{\mathcal{Q}}_\beta, \bar{\psi}^\alpha]) - \frac{1}{2}(\bar{\psi}^\beta \epsilon_\alpha [\mathcal{Q}_\beta, \psi^\alpha] - \bar{\epsilon}_\alpha [\bar{\mathcal{Q}}_\beta, \psi^\alpha]) \\ &\quad + \phi [\mathcal{Q}_L, \bar{\phi}] - \bar{\phi} [\mathcal{Q}_L, \phi], \end{aligned} \quad (4.68)$$

$$\mathcal{V}_{\text{int.}} = -i\phi [\epsilon \bar{\lambda}, \bar{\phi}] + i\bar{\phi} [\bar{\epsilon} \lambda, \phi], \quad (4.69)$$

which can also be written,

$$\begin{aligned} \mathcal{V}_\phi &= -\frac{1}{2}(\epsilon\psi)(iD_0 - \frac{3r}{2} + 2)\bar{\phi} + i(\epsilon\sigma_m\psi)D_m\bar{\phi} \\ &\quad + \frac{1}{2}(\bar{\epsilon}\bar{\psi})(iD_0 + \frac{3r}{2} - 2)\phi + i(\bar{\epsilon}\sigma_m\bar{\psi})D_m\phi - \frac{1}{2}(\epsilon\bar{\psi}F + \bar{\epsilon}\psi\bar{F}) \end{aligned} \quad (4.70)$$

The vector multiplet action (3.100) may also be written in a  $(\mathcal{Q}_L + s)$  exact manner,

$$\mathcal{V}_A = \frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)F_m^+ + \frac{i}{2}(\epsilon\sigma_m\bar{\lambda})F_m^- - \frac{1}{4}(\epsilon\bar{\lambda} + \bar{\epsilon}\lambda)D. \quad (4.71)$$

### 4.1.8 Saddle Points & Fermionic Zero Modes

Having shown that the action for chiral and vector multiplets are localisation actions, in the sense that they are  $\mathcal{Q}_L$ -exact on  $\mathbb{R} \times S^3$  and  $\mathcal{Q}$ -exact on  $S^1 \times S^3$  when twisted, the remaining question is the identification of saddle points. As was shown in the saddle point approximation equation (4.49), the index has to be expanded around each critical point of the theory. This can be an involved task. In 2 dimensions, one has to expand around kink solutions, as in [94]. There, it is shown that a non compact spatial dimension, the boundary conditions at infinity can lead to kink vacuum solutions around which the index has to be expanded. Some of these solutions can also have non-integer fermionic charges [99, 100, 101].

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In 3 dimensions, monopole solutions play a crucial role in the computation of the index [12, 58] as a partition function on  $S^1 \times S^2$ . This is due to the existence of monopole operators and the absence of any zero modes on these background. On  $S^3$ , when computing partition functions [102] and expectations values of  $\mathcal{Q}$ -closed Wilson loops [53], the saddle points are trivial configurations. In 4 dimensions, the computation of Wilson loops expectation values localises on instanton configuration [81].

Here, we will show that, although instanton configurations can in principle be constructed on  $\mathbb{R} \times S^3$  [32], the existence of fermionic zero modes mean that there are no non-perturbative corrections to the index. After Wick rotation, the vector action can be rewritten as  $F_m^+ F_m^-$ , a positive definite quantity, hence the saddle points of this action can be written as,

$$F_m^\pm = F_{0m} \pm i\epsilon_{mnp} F_{np} = 0, \quad D = 0. \quad (4.72)$$

In the remainder of this section we will consider the  $F_m^- = 0$  configuration. In this background, the gaugino supersymmetry transformations result in lead to the definition of fermionic modes  $\lambda_0^\alpha$ ,

$$\delta\lambda^\alpha = -iF_m^+(\epsilon\sigma_m)^\alpha \equiv \lambda_0^\alpha, \quad \bar{\delta}\bar{\lambda}^\alpha = 0. \quad (4.73)$$

The first equation defines fermionic zero modes [103] living on the instanton background,

$$\mathcal{L}_{\lambda_0} = 0 \quad (4.74)$$

up to total derivatives. Consequently, the path-integral measure for the gaugino splits into non-zero modes  $\tilde{\lambda}$  and zero modes  $\lambda_0$ ,

$$[d\lambda] = [d\lambda_0][d\tilde{\lambda}] \quad (4.75)$$

and the Lagrangian for the gaugino obviously only depends on  $\tilde{\lambda}$ , this implies that,

$$\int [d\lambda_0] \int [d\tilde{\lambda}] \exp\left(-\int \mathcal{L}_{\tilde{\lambda}}\right) = 0, \quad (4.76)$$



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because of the rules of Grassman integration which essentially state that,

$$\int d\theta_1 d\theta_2 \dots d\theta_n \theta_n \theta_{n-1} \dots \theta_1, \quad (4.77)$$

is essentially the only non-zero integral one can construct with a set of Grassman variables  $\{\theta_i\}_1^n$ . This hence means that the only saddle point contributing to the index calculation of the index is the vacuum,

$$A_m = \lambda = D = 0, \quad \phi = \psi = F = 0. \quad (4.78)$$

So, the index does not received non-perturbative corrections in 4 dimensions.

## 4.2 A Calculation of the Index

### 4.2.1 Determinant Calculation

In the following we will need to compute the one-loop determinants for various fields. After reducing the theory to a quantum mechanical Lagrangian by expanding fields in spherical harmonics and performing a Wick rotation, all indices boil down to a product of determinants of operators of the following form:

$$\Delta_{a,b} = -(d_\tau + a)^2 + b^2, \quad (4.79)$$

where  $(a, b)$  are  $N \times N$  matrices, and the fields acted upon are periodic of period  $\beta$  in the euclidean time direction,  $s = e^{-\beta}$ . This allows to expand the fields acted upon by  $\Delta_{a,b}$  in Fourier modes of the form,

$$\phi_{a,b}(\tau) = \sum_{k \in \mathbb{Z}} e^{ik\omega\tau} \phi_{k,a,b}, \quad \omega = \frac{2\pi}{\beta}, \quad (4.80)$$

with  $\omega$  the fundamental Matsubara frequency. The determinant of the operator (4.79) is then,

$$\text{Det } \Delta_{a,b} = \text{Det} \prod_{k \in \mathbb{Z}} [-(ik\omega + a)^2 + b^2]. \quad (4.81)$$

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Then, use,

$$\text{Det } \Delta_{a,b} = \exp(\text{Tr } \ln \Delta_{a,b}) , \quad (4.82)$$

and following [104], the product can be expanded,

$$\begin{aligned} \text{Tr } \ln \Delta_{a,b} &= \text{Tr } \ln \prod_{k \in \mathbb{Z}} [-(ik\omega + a)^2 + b^2] , \\ &= \text{Tr } \ln (b^2 - a^2) \prod_{k \in \mathbb{N}^*} [-(ik\omega + a)^2 + b^2] [ -(-ik\omega + a)^2 + b^2 ] , \\ &= \text{Tr } \ln (b^2 - a^2) \prod_{k \in \mathbb{N}^*} k^4 \omega^4 \left( 1 + \frac{(a+b)^2}{k^2 \omega^2} \right) \left( 1 + \frac{(a-b)^2}{k^2 \omega^2} \right) , \\ &= \text{Tr } \ln \mathcal{M}^2 \sinh \left( \frac{\pi}{\omega} (a+b) \right) \sinh \left( \frac{\pi}{\omega} (-a+b) \right) , \\ &= \text{Tr } \ln \mathcal{M}^2 e^{\beta b} (1 - t^{a+b}) (1 - t^{-a+b}) , \end{aligned} \quad (4.83)$$

where we have used the following identity,

$$\prod_{n=1}^{\infty} \left( 1 - \frac{\beta^2}{n^2 \pi^2} \right) = \frac{\sin \beta}{\beta} , \quad (4.84)$$

with  $\mathcal{M}$  an infinite product factor independent of  $(a, b)$  which will cancel between fermionic and bosonic degrees of freedom given that we are considering  $\mathcal{N} = 1$  supersymmetric theories, and that we will hence ignore,

$$\mathcal{M} = \frac{\omega}{2\pi} \prod_{k \in \mathbb{N}^*} k^2 \omega^2 \rightarrow 1 . \quad (4.85)$$

Expanding the logarithm and Taylor expanding the various terms of the sum in (4.83) gives:

$$\text{Tr } \ln \Delta_{a,b} = \beta \text{Tr } b + \sum_{k=1}^{\infty} \frac{\text{Tr} (t^{k(a+b)} + t^{k(-a+b)})}{k} \quad (4.86)$$

where the first term corresponds to a Casimir energy term. Finally, we get:

$$\text{Det } \Delta_{a,b} = C_b \text{Pexp} [i_+(t) + i_-(t)] \quad (4.87)$$

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with  $C_b$  the Casimir energy factor and the plethystic exponential [40] defined as,

$$\text{Pexp}[g(t_1, \dots, t_r)] = \exp\left[\sum_{k=1}^r \frac{1}{k} g(t_1^k, \dots, t_r^k)\right] \quad (4.88)$$

and the single-particle determinants are given by:

$$i_+(t) = \text{Tr } t^{a+b}, \quad i_-(t) = \text{Tr } t^{-a+b}. \quad (4.89)$$

In general we will use the notation:

$$i(t) = i_+(t) + i_-(t) \quad (4.90)$$

Based on equation (4.35), one can hence write that:

$$i(t) = \text{Tr}_{\text{s.p.}} t^H \quad (4.91)$$

the trace over the single particle states. When dealing with a full theory, one needs to sum single particle contributions for all states labelled  $(\ell m \bar{m})$ .

### 4.2.2 Chiral Multiplet Calculation

In this section we compute the generic partition function (4.41) and take the appropriate limit (2.141) to compute the index (2.49) for the chiral and anti-chiral multiplets, ignoring for the moment gauge and flavour group symmetry.

#### Scalar Component

Following the generic twisting prescription (4.44) the twisted Lagrangian obtained from the scalar Lagrangian (3.132),

$$L_\phi^{\text{tw.}} = \sum_{\ell m \bar{m}} \left(d_0 + \frac{3}{2}ir - i + i\mathcal{T}\right) \bar{\phi}_{\ell m \bar{m}} \left(d_0 - \frac{3}{2}ir + i + i\mathcal{T}\right) \phi_{\ell m \bar{m}} - (\ell+1)^2 \bar{\phi}_{\ell m \bar{m}} \phi_{\ell m \bar{m}}, \quad (4.92)$$

with  $(\ell + 1)$  the mass of the  $(\ell, m, \bar{m})$  scalar mode on  $\mathbb{R} \times S^3$ , corresponding to usual conformal dimensions in the free, conformal limit, and  $\mathcal{T}$  the generic

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twisting defined in equation (4.44). Thus after Wick rotation (4.4), we have for the associated free path integral,

$$Z_\phi(s, u, x, \bar{x}) = \prod_{\ell m \bar{m}} \det^{-1} \left[ -\left(d_\tau - \frac{3}{2}r + 1 + T_{\ell m \bar{m}}^\phi\right)^2 + (\ell + 1)^2 \right], \quad (4.93)$$

with  $T_{\ell m \bar{m}}^\phi(s, u, x, \bar{x})$  the eigenvalue of the mode  $\phi_{\ell m \bar{m}}$  under the  $T$  operator defined in equation (4.44),

$$[T, \phi_{\ell m \bar{m}}] = T_{\ell m \bar{m}}^\phi \phi_{\ell m \bar{m}} \quad (4.94)$$

with the following conventions<sup>1</sup>,

$$[R, \phi_{\ell m \bar{m}}] = r \phi_{\ell m \bar{m}}, \quad [J_3, \phi_{\ell m \bar{m}}] = m \phi_{\ell m \bar{m}}, \quad [\bar{J}_3, \phi_{\ell m \bar{m}}] = \bar{m} \phi_{\ell m \bar{m}}, \quad (4.97)$$

which then leads to

$$T_{\ell m \bar{m}}^\phi(s, u, x, \bar{x}) = \gamma_x m + \gamma_{\bar{x}} \bar{m} + \frac{1}{2} \gamma_u r. \quad (4.98)$$

Evaluation results that the partition function is given by,

$$Z_\phi(s, u, x, \bar{x}) = C_\phi \text{Pexp}(-z_{0\sigma r}(s, x, \bar{x}, u) - z_{0-\sigma-r}(s, \bar{x}, x, u)), \quad (4.99)$$

where  $\sigma$  is here defined as,

$$\sigma = \frac{3}{2}r - 1, \quad (4.100)$$

where the normalisation  $C_\phi$  is the Casimir energy factor, and,

$$z_{j,\sigma,r}(s, u, x, \bar{x}) = u^r \sum_{\ell=0}^{\infty} \chi_{\frac{\ell}{2}+j}(x) \chi_{\frac{\ell}{2}}(\bar{x}) s^{2(j+\ell+1-\sigma)}, \quad (4.101)$$

---

<sup>1</sup>Note that the angular momentum conventions mean that the harmonics considered are slightly unusual in the sense that,

$$[J_3, \phi_{\ell m \bar{m}} Y_{\ell m \bar{m}}] = -\phi_{\ell m \bar{m}} \mathcal{J}_3 Y_{\ell m \bar{m}} = [J_3, \phi_{\ell m \bar{m}}] Y_{\ell m \bar{m}}, \quad (4.95)$$

hence,

$$\mathcal{J}_3 Y_{\ell m \bar{m}} = -m Y_{\ell m \bar{m}}. \quad (4.96)$$

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in terms of standard  $SU(2)$  group characters for spin  $j$  representations,

$$\chi_j(x) = \sum_{m=-j}^j x^{2m} = \frac{x^{2j+1} - x^{-2j-1}}{x - x^{-1}}, \quad (4.102)$$

allowing for more general unrestricted values of  $j, \sigma, r$  for later. The partition function in equation (4.101) can be evaluated explicitly,

$$z_{j,\sigma,r}(s, u, x, \bar{x}) = s^{2j+2-2\sigma} u^r P(s, x, \bar{x}) C_j(s, x, \bar{x}), \quad (4.103)$$

where,  $C_j(x)$ ,  $P(s, x, \bar{x})$  have been defined in equations (2.127), (2.136). Thus, for  $\sigma = 0$ , equation (4.99) is in accord with the contributions expected for free scalar fields with  $R$ -charge  $\pm \frac{2}{3}$ . Note that for a general conformal field theory, not assuming supersymmetry,  $z_{j0r}(s, u, x, \bar{x})$ ,  $z_{j0-r}(s, u, \bar{x}, x)$  are contributions to the single particle partition function for spin  $(j, 0)$ , respectively  $(0, j)$  free fields, with respective  $R$ -charges  $\pm r$

### Fermionic Component

The twisted Lagrangian for the spinor matter field is given by,

$$L_\psi^{\text{tw.}} = \sum_{\ell m \bar{m} \kappa} \bar{\psi}_{\ell m \bar{m}}^\kappa \left( i\partial_0 + \frac{3}{2}r - 1 - T_{\ell m \bar{m} \kappa}^\psi - \kappa \left( \ell + \frac{3}{2} \right) \right) \psi_{\ell m \bar{m}}^\kappa, \quad (4.104)$$

with  $T_{\ell m \bar{m} \kappa}^\psi(s, u, x, \bar{x})$  the eigenvalue of the mode  $\psi_{\ell m \bar{m} \kappa}$  under the twisting  $\mathcal{T}$  (4.44),

$$T_{\ell m \bar{m} \kappa}^\psi(s, u, x, \bar{x}) = \gamma_x m + \gamma_{\bar{x}} \bar{m} + \frac{1}{2} \gamma_u (r - 1). \quad (4.105)$$

The associated determinant is given by,

$$Z_\psi(s, u, x, \bar{x}) = \prod_{\ell m \bar{m} -} \det(-\partial_\tau + \frac{3}{2}r - T_{\ell m \bar{m} -}^\psi + \ell + \frac{1}{2}) \prod_{\ell m \bar{m} +} \det(\partial_\tau - \frac{3}{2}r + T_{\ell m \bar{m} +}^\psi + \ell + \frac{5}{2}), \quad (4.106)$$

which may be evaluated to give,

$$Z_\psi(s, u, x, \bar{x}) = C_\psi \text{Pexp} \left( z_{\frac{1}{2}, \sigma, r-1}(s, u, x, \bar{x}) + z_{\frac{1}{2}, -\sigma, -r+1}(s, u, \bar{x}, x) \right), \quad (4.107)$$

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with  $C_\psi$  the Casimir energy factor. Again, this is consistent with the ordinary conformal case,  $\sigma = 0$ , for  $j = \frac{1}{2}$ .

### The Chiral Multiplet Index

Since the  $F$  terms lead to trivial contribution in (3.96), the full modified partition function is therefore given by,

$$Z_\Phi(s, u, x, \bar{x}) = Z_\phi(s, u, x, \bar{x})Z_\psi(s, u, x, \bar{x}) = C_\Phi P \exp(z_\Phi(s, u, x, \bar{x}) + z_{\bar{\Phi}}(s, u, x, \bar{x})), \quad (4.108)$$

with the Casimir energy factor,

$$C_\Phi = C_\phi C_\psi, \quad (4.109)$$

and where the effective single particle partition functions for the chiral and anti-chiral scalar multiplet are given by,

$$\begin{aligned} z_\Phi(s, u, x, \bar{x}) &= (z_{\frac{1}{2}\sigma r-1} - z_{0\sigma r})(s, u, x, \bar{x}), & (4.110) \\ &= s^{4-3r} u^r P(s, x, \bar{x})(C_0 - su^{-1}C_{\frac{1}{2}})(s, x, \bar{x}), \end{aligned}$$

$$\begin{aligned} z_{\bar{\Phi}}(s, u, x, \bar{x}) &= (z_{\frac{1}{2}-\sigma 1-r} - z_{0-\sigma-r})(s, u, \bar{x}, x) & (4.111) \\ &= s^{3r} u^{-r} P(s, x, \bar{x})(C_0 - suC_{\frac{1}{2}})(s, x, \bar{x}). \end{aligned}$$

This is consistent with the ordinary superconformal case, for  $\sigma = 0$ ,  $r = \frac{2}{3}$ , with contributions to the generic partition function (4.41) from a multiplet with spin  $(j, 0)$  or  $(0, j)$  and conformal dimension  $j + 1$ . The conformal chiral multiplet is a the  $j = 0$  special case of the characters formula for the  $(1, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$  representation of the superconformal algebra given in equation (2.135). One should note however that the analysis performed in this chapter provides us with a more general result than the latter equation, as we have, in a sense, obtained a formula for the  $(1, \frac{1}{2})$  and  $(\frac{1}{2}, 1)$  character for a multiplet whose  $R$ -charge and scaling dimensions are not constrained by equation (2.117). This is due to the fact that the  $\bar{Q}$ ,  $\bar{S}$  supercharges do not generate symmetries of the theory considered here, so the multiplet is somewhat shortened, but the constraints associated with the

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superconformal algebra commutation relations do not apply.

To evaluate the index we make the variable change (2.141) and obtain,

$$i_{\Phi}(t, x) = z_{\Phi}(y^{\frac{1}{2}}, y^{\frac{3}{2}}t^{-1}, -ty^{-1}, -x) = \frac{t^r}{(1-tx)(1-tx^{-1})}, \quad (4.112)$$

$$i_{\bar{\Phi}}(t, x) = z_{\bar{\Phi}}(y^{\frac{1}{2}}, y^{\frac{3}{2}}t^{-1}, -ty^{-1}, -x) = -\frac{t^{2-r}}{(1-tx)(1-tx^{-1})}, \quad (4.113)$$

showing that, indeed, the  $y$  dependence drops out, as expected. Obviously, the full single particle index (1.8), modulo gauge and flavour symmetry, discussed shortly, is the sum of the two contributions above.

### 4.2.3 Vector Multiplet Calculation

#### Vector Component

Consider a non abelian gauge theory. The vector index will be computed by twisting the following free path integral,

$$Z_A = \int dA_0 dA_m \Delta \delta(\text{gauge fixing}) e^{-S_A} \quad (4.114)$$

with  $\Delta$  is the Fadeev-Popov determinant associated with the gauge-fixing choice,  $\delta$  a Dirac-delta function, and  $S_A$  the vector field action on  $S^1 \times S^3$  canonically normalised as in equation (3.110). Consider the  $A_0$  field and decompose it as,

$$A_0(\tau, \omega) = \alpha(\tau) + B(\tau, \Omega), \quad (4.115)$$

with  $\alpha(\tau)$  the constant holonomy mode over the sphere for  $A_0$ ,

$$\alpha(\tau) = \int d^3\Omega A_0(\tau, \Omega), \quad (4.116)$$

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and  $B$  the spatial fluctuations of  $A_0$ .  $\alpha(\tau)$  is a zero mode as its quadratic action vanishes and it can hence be independently rescaled,

$$\alpha(\tau) \rightarrow \frac{1}{g}\alpha(\tau). \quad (4.117)$$

Following this rescaling, in the free field theory limit, the action for the vector field is given by,

$$\mathcal{L}[B] = BD^2B, \quad (4.118)$$

$$\mathcal{L}[A_m, \alpha] = A_m \left( (\partial_\tau - i[\alpha, \cdot])^2 + 4\delta_{mp}D^2 - 4D_pD_m \right) A_p, \quad (4.119)$$

with  $D^2$  the scalar or vector laplacian on  $S^3$ . From now on, for notational purposes, we will denote  $-i[\alpha, \cdot]$  by  $-i\alpha_g$ , and  $\alpha_g$  is a matrix to be taken in the appropriate representation of the gauge group  $G$  for the field under consideration. The path integral associated with  $B$  hence yields the determinant of the scalar laplacian,

$$(\text{Det } D^2)^{-\frac{1}{2}} \quad (4.120)$$

Having integrated out the  $B$  fluctuations defined in (4.115) on the free field theory, one can impose the following gauge fixing on the remaining  $\alpha_g$  modes,

$$\partial_\tau \alpha_g = 0, \quad (4.121)$$

which leads to the following Fadeev-Popov determinant,

$$\Delta_0 = \text{Det}(d_\tau + i\alpha_g). \quad (4.122)$$

It can be computed in the same fashion as previously. Assume that the matrix  $\alpha$  has a set of eigenvalues  $\{\lambda_i\}_{i=1}^n$ , the determinant is given by,

$$\begin{aligned} \Delta_0 &= \prod_{k \in \mathbb{Z}^*} \prod_{i,j=1}^n (ik\omega + i(\lambda_i - \lambda_j)) = \mathcal{M}\beta \prod_{i,j=1}^n \left( 1 - \frac{1}{k^2\pi^2} \left( \frac{1}{2}\beta(\lambda_i - \lambda_j) \right)^2 \right), \\ &= \mathcal{M}\beta \prod_{i,j=1}^n \frac{2}{\beta(\lambda_i - \lambda_j)} \sin\left(\frac{1}{2}\beta(\lambda_i - \lambda_j)\right), \end{aligned}$$



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where we have used the infinite product formula (4.84) and the definition of the prefactor  $\mathcal{M}$  in equation (4.85). Given the integration measure for  $\alpha_g$ ,

$$d\alpha_g = \prod_{i=1}^n d\lambda_i \prod_{i<j} (\lambda_i - \lambda_j)^2 \quad (4.123)$$

one can identify  $d\alpha_g \Delta_0$  with the left-right invariant measure over matrices,

$$g = \exp(i\beta\alpha_g), \quad (4.124)$$

in the following fashion,

$$d\alpha_g \Delta_0 = d\mu(g) = \prod_{i=1}^n d\lambda_i \prod_{i<j} \sin^2\left(\frac{1}{2}\beta(\lambda_i - \lambda_j)\right), \quad (4.125)$$

We now decompose the vector field  $A_m$  into an exact and a divergenceless part,

$$A_m = B_m + \nabla_m \phi, \quad (4.126)$$

with  $B_m$  a divergenceless vector. Imposing the following gauge fixing condition on  $A_m$ ,

$$D_m A_m = 0, \quad (4.127)$$

implies no restriction on  $B_m$  and the following on  $\phi$ ,

$$D^2 \phi = 0, \quad (4.128)$$

Hence the associated Fadeev-Popov determinant is:

$$\Delta_1 = \text{Det} D^2, \quad (4.129)$$

Also, substituting the expression for  $A$  in terms of exact and closed contribution shows that  $\phi$  drops out of the action (4.119), as

$$\nabla_m \phi \delta_{mp} D^2 \nabla_p \phi - \nabla_m \phi D_p D_m \nabla_p \phi = 0 \quad (4.130)$$

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where we have used the no torsion condition,

$$[D_m, D_n]\phi = 0, \quad (4.131)$$

with  $D_m$  the spacetime covariant derivative. Acting on the scalar  $\phi$ , it is just the Killing vector  $\nabla_m$ , acting on the vector  $\nabla_m\phi$  it is the spacetime covariant derivative defined in equation (3.26). The term containing a time derivative involving  $\phi$  can just be integrated by parts away, thanks to the gauge fixing condition on  $\phi$  equation (4.128). Consequently, the overall contribution from  $\phi$  to the path integral is given by,

$$\int [d\phi]\delta(D^2\phi) = (\text{Det}D^2)^{-\frac{1}{2}}. \quad (4.132)$$

Overall, all laplacian determinant factors from fluctuations of  $A_0$  in equation (4.120), the FP determinant of the spatial gauge fixing-condition (4.129) and the integration of the the closed part of the vector  $A_m$  in equation (4.132) all cancel.

At this point, we have reduced the computation of the path integral to the following,

$$Z_A(s) = \int d\mu(g)\mathcal{Z}_A(s, g), \quad (4.133)$$

$$\mathcal{Z}_A(s, g) = \int [dB_m]\exp\left(-\int \mathcal{L}[B_m, \alpha_g]\right), \quad (4.134)$$

with  $g$  gauge group matrices defined in equation (4.124),  $d\mu(g)$  the gauge group integration measure, while  $B_m$  is a divergenceless 3-component vector and the dependence on  $s$  defined in (4.42) arising from the  $\beta$  periodicity in the time direction. The Lagrangian for the spatial vector only includes contributions from divergenceless  $B_m$ ,

$$\mathcal{L}[B_m, \alpha] = B_m \left( (\partial_\tau - i\alpha_g)^2 + 4(D^2 - \frac{1}{2}) \right) B_m, \quad (4.135)$$

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where we have used the following identity,

$$[D_m, D_n]B_p = \frac{1}{4}(\delta_{mp}B_n - \delta_{np}B_m), \quad (4.136)$$

to obtain the divergenceless vector action from the vector action in equation (4.119). Expanding this action in divergenceless vector spherical harmonics as in equation (3.179) and applying the generic twist leads to the following expression for the generic partition function,

$$\mathcal{Z}_A(s, u, x, \bar{x}, g) = \prod_{\ell m \bar{m} \rho} \det^{-1} [-(d_\tau + \mathcal{T}_{\ell m \bar{m} \rho}^A - i\alpha_g)^2 + (\ell + 2)^2], \quad (4.137)$$

with the following expression for the twisting,

$$\mathcal{T}_{\ell m \bar{m} \rho}^A = \gamma_x m + \gamma_{\bar{x}} \bar{m}, \quad (4.138)$$

Also  $\text{Tr} e^{\mp\beta\alpha_g}$  is the gauge character factor which one finds in equation (2.19) in [36]. Also when matter chiral fields are coupled to a gauge sector, an extra term  $-i\alpha_g$  has to be added to the twisting of the partition function (4.93), (4.106), with  $\alpha_g$  in the appropriate representation of the field considered. Also note that the integral measure  $d\mu(g)$  naturally arises from this path integral analysis, and that the integration over the holonomy zero-mode in the path integral corresponds to the projection onto gauge singlets in the approach based upon character calculations[36]. The holonomy around the time circle  $\text{Tr} g$  can also be thought of as a Wilson loop around the time circle [104]. One should note that, when setting the twisting to the appropriate value for the computation of the full superconformal index, one can proceed similarly as in equation (4.79) to show positive definiteness of the action.

Here it is useful to consider the  $\alpha_g$  term as an extra contribution to the twisting, and, using,

$$\text{Tr}_{\text{Adj.}} f(\alpha_g) = \text{Tr}_{\text{Adj.}} f(-\alpha_g), \quad (4.139)$$

evaluating the path integral in terms of determinants, similar to the scalar field

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case considered earlier, we obtain,

$$\mathcal{Z}_A(s, u, x, \bar{x}, g) = C_A \text{Pexp} \left( -(z_{100}(s, u, x, \bar{x}) + z_{100}(s, u, \bar{x}, x)) \chi_{\text{Adj.}}(g) \right), \quad (4.140)$$

for the contribution of the vector field to the overall path integral. Specifically, the single particle partition function  $z_{100}(s, u, x, \bar{x})$  (resp.  $z_{100}(s, u, \bar{x}, x)$ ) corresponds to  $\rho = +$  (resp.  $\rho = -$ ) harmonics as defined in equation (3.127). Also we have denoted,

$$\chi_R(g) = \text{Tr}_R \exp(i\beta\alpha_g), \quad (4.141)$$

the gauge group character in for representation  $R$ . The integration over the gauge group follows from the coupling of the chiral multiplet to the gauge field  $A_0$  and the associated zero-mode  $\alpha$ .

### Gaugino Component

The gaugino action can be written directly by essentially modifying the time derivative in the quark action considered earlier and taking a trace over the adjoint representation,

$$L_\lambda^{\text{tw.}} = \text{Tr}_{\text{Adj.}} \sum_{\ell m \bar{m} \kappa} \bar{\lambda}_{\ell m \bar{m}}^\kappa \left( d_\tau + T_{\ell m \bar{m}}^\lambda - i\alpha_g + \kappa \left( \ell + \frac{3}{2} \right) \right) \lambda_{\ell m \bar{m}}^\kappa. \quad (4.142)$$

The associated path integral may be evaluated similarly as before to give the following contribution, apart from an overall normalisation,

$$\mathcal{Z}_\lambda(s, u, x, \bar{x}, g) = C_\lambda \text{Pexp} \left( (z_{\frac{1}{2}, 0, 1}(s, u, x, \bar{x}) + z_{\frac{1}{2}, 0, -1}(s, u, \bar{x}, x)) \chi_{\text{Adj.}}(g) \right). \quad (4.143)$$

Here, the  $\kappa = +$  left-moving harmonics give rise to the first term  $z_{\frac{1}{2}, 0, 1}(s, u, x, \bar{x})$  in the single-particle partition function.

### The Vector Multiplet Index

The full modified partition function, in the absence of coupling of the gauge

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sector to matter, is given by,

$$Z_V(s, u, x, \bar{x}) = \int d\mu(g) \mathcal{Z}_A(s, u, x, \bar{x}, g) \mathcal{Z}_\lambda(s, u, x, \bar{x}, g), \quad (4.144)$$

and the single particle partition function is given by the following,

$$z_V(s, u, x, \bar{x}) = (z_{V,l} + z_{V,r})(s, u, x, \bar{x}), \quad (4.145)$$

$$z_{V,l}(s, u, x, \bar{x}) = (z_{\frac{1}{2}01} - z_{100})(s, u, x, \bar{x}), \quad (4.146)$$

$$z_{V,r}(s, u, x, \bar{x}) = (z_{\frac{1}{2}0-1} - z_{100})(s, u, \bar{x}, x), \quad (4.147)$$

where the two equations correspond to the contributions from left ( $\rho = -$ ) and right sectors ( $\rho = +$ ) with  $\rho$  defined in (3.127). To evaluate the index we make the variable change (2.141) and obtain the single particle index,

$$i_V(t, x) = (i_{V,l} + i_{V,r})(t, x), \quad (4.148)$$

$$i_{V,l}(t, x) = z_{V,l}(y^{\frac{1}{2}}, y^{\frac{3}{2}}t^{-1}, -ty^{-1}, -x) = \frac{t^2}{(1-tx)(1-tx^{-1})}, \quad (4.149)$$

$$i_{V,r}(t, x) = z_{V,r}(y^{\frac{1}{2}}, y^{\frac{3}{2}}t^{-1}, -ty^{-1}, -x) = \frac{t^2 - t(x + x^{-1})}{(1-tx)(1-tx^{-1})}, \quad (4.150)$$

with no  $y$  dependence.

### Gauged, Flavoured Chiral Multiplet

Here we assume that the chiral multiplet considered in the last subsection belongs to a reducible representation of the gauge group  $G$  and flavour symmetry group  $H$ . Each component we assume transforms in a representation  $R_{G,i}$  of  $G$  and  $R_{H,i}$  of  $H$  and has  $R$ -symmetry charge  $r_i$ . The effect of the flavour group on the twisting (4.44) is that

$$T \rightarrow T + i\alpha_h \quad (4.151)$$

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where  $\alpha_h$  is an element of the flavour algebra  $H$ . The chiral multiplet lagrangians (4.92) (3.156) then become,

$$\begin{aligned} L_\phi^{\text{tw.}} &= \sum_{\ell m \bar{m} i} \text{Tr} \bar{\phi}_{\ell m \bar{m}}^i \left( -\left(d_\tau - \frac{3}{2}r + 1 + T_{\ell m \bar{m}}^\phi - i\alpha_g + i\alpha_h\right)^2 + (\ell + 1)^2 \right) \phi_{\ell m \bar{m}}^i, \\ L_\psi^{\text{tw.}} &= \sum_{\ell m \bar{m} \kappa i} \text{Tr} \bar{\psi}_{\ell m \bar{m}}^{\kappa i} \left( d_\tau - \frac{3}{2}r + 1 + T_{\ell m \bar{m} \kappa}^\psi - i\alpha_g + i\alpha_h + \kappa \left(\ell + \frac{3}{2}\right) \right) \psi_{\ell m \bar{m}}^{\kappa i}. \end{aligned}$$

Evaluation of the path integrals associated with the previous lagrangians (4.152), (4.152) proceeds as before, using, for any analytic function  $f$ ,

$$\text{Tr}_{R_H} f(\alpha_h) = \text{Tr}_{\bar{R}_H} f(-\alpha_h), \quad (4.152)$$

and similarly for  $G$ . This leads to the contribution,

$$\begin{aligned} \mathcal{Z}_\Phi(s, u, x, \bar{x}, g, h) &= \prod_i C_{\Phi_i} \text{Pexp} \left( z_{\Phi_i}(s, u, x, \bar{x}) \chi_{R_{G,i}}(g) \chi_{R_{H,i}}(h) \right. \\ &\quad \left. + z_{\bar{\Phi}_i}(s, u, x, \bar{x}) \chi_{\bar{R}_{G,i}}(g) \chi_{\bar{R}_{H,i}}(h) \right) \end{aligned} \quad (4.153)$$

where we define the characters  $\chi_{R_G}(g)$ ,  $\chi_{R_F}(h)$  as in equation (4.141), the single particle partition functions defined in (4.110), and the Casimir energy factor (4.109).

### The Full Index

As the  $D$  and  $F$  term contributions are trivial so that we finally obtain for the partition function,

$$\begin{aligned} \mathcal{Z}(s, u, x, \bar{x}, g, h) &= C_V \text{Pexp} \left( z_V(s, u, x, \bar{x}) \chi_{\text{Adj.}}(g) \right) \\ &\quad \left( \prod_i C_{\Phi_i} \text{Pexp} \left( z_{\Phi_i}(s, u, x, \bar{x}) \chi_{R_{G,i}}(g) \chi_{R_{H,i}}(h) \right. \right. \\ &\quad \left. \left. + z_{\bar{\Phi}_i}(s, u, x, \bar{x}) \chi_{\bar{R}_{G,i}}(g) \chi_{\bar{R}_{H,i}}(h) \right) \right), \end{aligned} \quad (4.154)$$

This is consistent with expectations for free field theory for chiral/anti-chiral

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multiplets with spin  $\frac{1}{2}$  primary fields. For the full index we obtain,

$$I(t, x, h) = \int d\mu(g) \text{Pexp}(i_V(t, x) \chi_{\text{Adj.}}(g)) \prod_i C_{\Phi, i} \text{Pexp}(i_{\Phi, i}(t, x) \chi_{R_{G, i}}(g) \chi_{R_{H, i}}(h) + i_{\bar{\Phi}, i}(t, x) \chi_{\bar{R}_{G, i}}(g) \chi_{\bar{R}_{H, i}}(h)) , \quad (4.155)$$

Obviously this leads to (1.8).

### 4.2.4 Casimir Energies

The scalar and fermionic Casimir energies are given by the following [104, 105],

$$\ln C_\phi = - \sum_{\ell=0}^{\infty} (\ell+1)^3, \quad \ln C_\psi = 2 \sum_{\ell=0}^{\infty} (\ell + \frac{3}{2})(\ell+1)(\ell+2), \quad (4.156)$$

as the scalar modes with energy  $(\ell+1)$  have a degeneracy  $(\ell+1)^2$ , as they fall into a  $(\frac{\ell}{2}, \frac{\ell}{2})$  representation of  $SU(2)_L \times SU(2)_R$ . The spinor modes with energy  $(\ell + \frac{3}{2})$  have degeneracy  $(\ell+1)(\ell+2)$  arising from their being in a  $(\frac{\ell\pm 1}{2}, \frac{\ell}{2})$  representation. This has to be regularised using zeta function regularisation [106],

$$\zeta(s) = \sum_{n \in \mathbb{N}^*} n^{-s} \quad (4.157)$$

which converges for  $s > 1$  and can be regularised to take finite values for  $s$  negative. This leads to,

$$\ln C_\phi = -\frac{1}{120}, \quad \ln C_\psi = \ln C_\lambda = -\frac{17}{480}, \quad (4.158)$$

with the Casimir energy for the gaugino the same as the one for the quark, and the vector contribution given by,

$$\ln C_A = - \sum_{\ell=0}^{\infty} (\ell+1)(\ell+2)(\ell+3) = -\frac{11}{120}. \quad (4.159)$$

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### 4.2.5 On Positive Definiteness and Localisation

One has to emphasize that the formula for the generic partition function (4.41) is not valid in an interacting theory and is only valid for free field theories. The index (4.155) is not modified in interacting theories, and, in four dimensions, does not receive any non-perturbative contributions. This observation relies on the positive definiteness of the action considered, so that the free field theory limit taken in the localisation procedure (4.48) is well defined. Consider the twisted action appropriate for the computation of the scalar field index. The propagator can be rewritten in the generic form (4.81), with,

$$a = a_{\ell m \bar{m}} = 2m + 1 + \gamma_t(-r + 2m) + \gamma_x 2\bar{m}, \quad b = b_{\ell m \bar{m}} = \ell + 1, \quad (4.160)$$

One can then note that the corresponding action amounts to a positive definite quantity, provided we assume  $\gamma_x, \gamma_t$  to be real, from the factorisation of the determinant given in the third line of equation (4.83). This then makes the limit (4.83) well defined as the localisation action is positive definite.

## 4.3 Further Generalisations of the Index

The index we have focused upon can be generalised in various ways, including the deformation of the 3-sphere to less symmetric spaces which preserves the symmetries relevant to the definition of the index [107]. Also, one can generalise the index by gauging the flavour symmetry and integrating over the flavour group in the same fashion as we have so far for gauge symmetry.

### 4.3.1 Deforming the 3-Sphere

#### Orbifolding

As was mentioned in the previous chapter, one can compute the index for spaces that preserve a subset of the superconformal algebra which includes at least (3.30). The sphere we have used to define the index can be deformed in a way that preserves the isometries in (3.30) and, most importantly, the super-



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charges  $Q_1, S^1$  at least. For instance, the index can be computed on lens spaces  $S^1 \times L(p, q)$ , with  $(p, q)$  two integers parameterizing an orbifold on the 3-sphere [107],

$$S^3 : \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}, \quad (z_1, z_2) \sim (e^{\frac{2\pi i q}{p}} z_1, e^{-\frac{2\pi i}{p}} z_2), \quad (4.161)$$

where the orbifold action is chosen so as to preserve the supercharge  $\mathcal{Q}$ . The important difference between the calculation on  $S^3$  and the one on the lens space is the degenerate set of vacuas labelled by the holonomy along the Hopf fibration direction of the underlying 3-sphere, which breaks the gauge group to product of  $p$  subgroups, and hence modifies the measure of integration appropriate for projection upon gauge singlets.

### Squashing the 3-Sphere

Also, bearing in mind the relation between  $\mathcal{N} = 2$  partition functions on 3-dimensional manifolds and the corresponding index once a temporal  $S^1$  is added to the space [108], a similar construction is given in [61]. The partition function for 3-dimensional squashed sphere is given and the partition function computed for squashed spheres preserving either  $SU(2)_l \times U(1)_r$  or  $U(1)_l \times U(1)_r$ . At a technical level, the reduction from 4 to 3 dimensions can be performed along following the prescription. Considering equation (4.83), the dimensional reduction from four to three dimensions can be performed as,

$$\prod_{k \in \mathbb{Z}} [-(ik\omega + a)^2 + b^2] \rightarrow -a^2 + b^2 \quad (4.162)$$

For instance, one can check that our result for the generic partition function agrees with the partition function given in [109] following a similar localisation prescriptions. This relationship between 3 dimensional partition functions and 4 dimensional index was emphasised in [108] [110] and lead to the conjecture of new supersymmetric dualities for  $d = 3, \mathcal{N} = 2$  theories based on the dualities for the parent  $d = 4, \mathcal{N} = 1$  supersymmetric dual theories.

# Chapter 5

## Index on Squashed-Sphere

Here we compute the index for  $\mathbb{R} \times S_e^3$  with  $S_e^3$  the squashed sphere, and  $e$  denotes the squashing parameter to be defined below. We first compute the following quantity,

$$\mathrm{Tr}(-1)^F e^{-\beta\mathcal{H}}, \quad (5.1)$$

where  $\mathcal{H}$  will be defined as the square of the appropriate supercharge.

Later on we show that the expression for the full index on the squashed sphere is the same as on the round sphere. Computation also shows that the final result is identical to the one on the round sphere.

### 5.1 Differential Geometry and Killing Spinors

#### 5.1.1 Differential Geometry

Following [61], consider the following right invariant one-form  $\mu^m = \mu^m{}_\mu dx^\mu$ , defined in terms of the round sphere right invariant one-form given in equations (3.10), (3.11),

$$\mu^m = \{\mu^1 = e^1, \quad \mu^2 = e^2, \quad \mu^3 = ee^3\}, \quad (5.2)$$

with  $e$  the squashing parameter. The left invariant one-form is the same as on the round sphere (3.12), and so is the right-invariant vector field. The left-invariant

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vector field  $\mu_m = \mu_m^\mu \partial_\mu$  is a deformed version of (3.14),

$$\mu_m = \{\mu_1 = \nabla_1, \quad \mu_2 = \nabla_2, \quad \mu_3 = e^{-1} \nabla_3\}, \quad (5.3)$$

the round sphere is given by  $e = 1$ . When  $e \neq 1$ , the sphere is squashed and has a  $U(1)_l \times SU(2)_r$  isometry. The spin connection is given by the following,

$$\omega_{mnp} = -\frac{1}{2}e\epsilon_{mnp} + \delta_{m3}(e - e^{-1})\epsilon_{np3}, \quad (5.4)$$

with  $\epsilon_{123} = 1$ . We will also need the Riemann tensor for the squashed sphere,

$$R_{mnpq} = \mu_m^\mu \mu_n^\nu (R_{\mu\nu})_{pq}, \quad (R_{\mu\nu})_{pq} = (\partial_\mu \omega_\nu - \partial_\nu \omega_\mu + [\omega_\mu, \omega_\nu])_{pq}, \quad (5.5)$$

as well as the Ricci tensor and scalar

$$R_{mn} = R_{mpnp}, \quad R = R_{mm}. \quad (5.6)$$

For the squashed sphere, the only non-zero component for the Riemann tensor  $R_{mnpq}$  are given by,

$$R_{1212} = \frac{1}{4}(4 - 3e^2), \quad R_{1313} = \frac{1}{4}e^2, \quad R_{2323} = \frac{1}{4}e^2. \quad (5.7)$$

and the Ricci tensor, scalar, are given by,

$$R_{mn} = \begin{pmatrix} 1 - \frac{1}{2}e^2 & 0 & 0 \\ 0 & 1 - \frac{1}{2}e^2 & 0 \\ 0 & 0 & \frac{1}{2}e^2 \end{pmatrix}, \quad R = 2 - \frac{1}{2}e^2. \quad (5.8)$$

One should note that  $R = \frac{3}{2}$  for the non squashed sphere. Usually one finds  $R = 6$  for the unit sphere. This would be the case had we defined the vielbein such that  $[\nabla_m, \nabla_n] = 2\epsilon_{mnq}\nabla_p$  on the round sphere. Instead, we have chose the normalisation of the dreibein such that  $[\nabla_m, \nabla_n] = \epsilon_{mnq}\nabla_p$ , which translates into the Ricci scalar definition by rescaling it by 4. Finally, the Bianchi identity reads,

$$\epsilon_{npq}R_{mnpq} = 0. \quad (5.9)$$

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### 5.1.2 Killing Spinors and Twisting

There are no Killing spinors on the squashed background [61]. The covariant spinor covariant derivative is given by,

$$D_m \chi^\alpha = \nabla_m \chi^\alpha + \frac{1}{4} i e (\chi \sigma_m)^\alpha - \frac{1}{2} i (\chi \sigma_3)^\alpha \delta_{m3} (e - e^{-1}) \quad (5.10)$$

Take a time dependent, spatially constant spinor  $\epsilon^\alpha \propto \delta_1^\alpha$ . In that case

$$(D_m + i V_m) \epsilon^\alpha = \frac{1}{4} i e (\epsilon \sigma_m)^\alpha \quad (5.11)$$

with the following definition,

$$V_m = \frac{1}{2} \delta_{m3} (e - e^{-1}). \quad (5.12)$$

Here, we have shown that by turning on a background gauge field associated with a  $U(1)$  global symmetry  $B$  under which the Killing spinor is charged, one can turn constant spinors into Killing spinors for the squashed sphere. Given that,

$$\mathcal{R} \epsilon^\alpha = \epsilon^\alpha, \quad (5.13)$$

one could be tempted to identify this global symmetry  $B$  with the  $R$ -charge. However, as was pointed in [51], at the classical level, the  $R$ -charge is indistinguishable from,

$$B = R + \sum_I s_I F_I, \quad (5.14)$$

with  $F_I$  all the non- $R$  flavour charges of the global symmetry group  $F$  and  $s_I$  arbitrary real parameters. Consequently, we define the spatially twisted covariant derivative as,

$$\mathcal{D}_m = D_m + i V_m B, \quad (5.15)$$

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with  $B$  a global symmetry for which a background gauge field will now be turned on.  $B$  also verifies equation (5.13), and  $D$  is the spacetime covariant derivative,

$$D_m A_n = \mu_m A_n + \omega_{mnp} A_p, \quad (5.16)$$

$$D_m \chi_\alpha = \mu_m \chi_\alpha + \frac{i}{4} \omega_{mnp} \epsilon_{npk} (\sigma_k \chi)_\alpha, \quad (5.17)$$

$$D_m \phi = \mu_m \phi. \quad (5.18)$$

We will also need the field strength associated with the background  $B$ -symmetry gauge field,

$$\mathcal{F}_{mn} = D_m V_n - D_n V_m, \quad (5.19)$$

By definition of  $V_m$  in equation (5.12), the field strength is given by,

$$\mathcal{F}_{mn} = -\frac{1}{2} e \epsilon_{mnp} V_p. \quad (5.20)$$

Having turned on the background  $R$ -charge gauge field, the Killing spinor equations are given by,

$$\mathcal{D}_m \epsilon^\alpha = \frac{1}{4} i e (\epsilon \sigma_m)^\alpha, \quad \partial_0 \epsilon^\alpha = \frac{1}{2} i e \epsilon^\alpha. \quad (5.21)$$

One can then give the commutator of the twisted covariant derivatives,

$$[\mathcal{D}_m, \mathcal{D}_n] A_p = i b_A \mathcal{F}_{mn} A_p + R_{mnpq} A_q, \quad (5.22)$$

$$[\mathcal{D}_m, \mathcal{D}_n] \psi_\alpha = i b_\psi \mathcal{F}_{mn} \psi_\alpha + \frac{i}{4} R_{mnpq} \epsilon_{pqs} (\sigma_s \psi)_\alpha, \quad (5.23)$$

$$[\mathcal{D}_m, \mathcal{D}_n] \phi = i b_\phi \mathcal{F}_{mn} \phi, \quad (5.24)$$

with  $b_X$  the eigenvalue of the field  $X$  under the symmetry  $B$ . Note that the last equation shows that, in this context, the twisting procedure is in fact a torsion procedure, as we have, from the geometric point of view, introduced torsion on this spacetime as seen from equation (5.24). Applying equation (5.23) along with (5.21) to the Killing spinors gives the following consistency condition,

$$\epsilon_{mpn} \mathcal{F}_{mp} \epsilon^\alpha = \left( \frac{1}{2} R - \frac{1}{4} e^2 \right) (\epsilon \sigma_n)^\alpha - R_{mn} (\epsilon \sigma_m)^\alpha, \quad (5.25)$$

## 5.2 Lagrangian and Free Partition Function

### 5.2.1 Matter Lagrangians

Consider the usual  $\mathbb{R} \times S^3$  scalar field Lagrangian and apply the twisting defined above which allows for Killing spinors on the squashed sphere. After some fiddling with mass parameters, one can then define the following Lagrangian which exhibits supersymmetry on  $\mathbb{R} \times S_e^3$ ,

$$\begin{aligned} \mathcal{L}_\phi &= -(i\partial_0 - \frac{3}{2}er + e)\bar{\phi}(i\partial_0 + \frac{3}{2}er - e)\phi - 4\mathcal{D}_m\bar{\phi}\mathcal{D}_m\phi + (\frac{3}{2}rR + e^2 - \frac{9}{4}re^2)\bar{\phi}\phi, \\ \mathcal{L}_\psi &= -\bar{\psi}^\alpha(i\partial_0 + \frac{3}{2}er - e)\psi_\alpha + 2i\bar{\psi}^\alpha(\sigma_m)_\alpha{}^\beta\mathcal{D}_m\psi_\beta, \end{aligned} \quad (5.26)$$

$$\mathcal{L}_F = F\bar{F}. \quad (5.27)$$

The supersymmetry transformations are given by,

$$\frac{1}{\sqrt{2}}\delta\phi = \epsilon^\alpha\psi_\alpha, \quad (5.28)$$

$$\frac{1}{\sqrt{2}}\delta\psi_\alpha = \epsilon_\alpha F, \quad (5.29)$$

$$\frac{1}{\sqrt{2}}\delta\bar{\psi}^\alpha = i\epsilon^\alpha\partial_0\bar{\phi} - 2i(\epsilon\sigma_m)^\alpha\mathcal{D}_m\bar{\phi} + 2ir\bar{\phi}(\mathcal{D}_m\epsilon\sigma_m)^\alpha, \quad (5.30)$$

$$\frac{1}{\sqrt{2}}\delta\bar{F} = -\epsilon^\alpha(i\partial_0 - \frac{3}{2}r + e)\bar{\psi}_\alpha + 2i(\epsilon\sigma_m)^\alpha\mathcal{D}_m\bar{\psi}_\alpha. \quad (5.31)$$

One can check that these supersymmetry transformations reduce to the ones given in equation (3.62) for the round sphere. The commutators of the supersymmetries on the fields are given by,

$$\frac{1}{2}[\delta, \bar{\delta}]\phi = i\partial_0\phi + \frac{3}{2}er\phi - 2iV_m\mathcal{D}_m\phi, \quad (5.32)$$

$$\frac{1}{2}[\delta, \bar{\delta}]\bar{\phi} = i\partial_0\bar{\phi} - \frac{3}{2}er\bar{\phi} - 2iV_m\mathcal{D}_m\bar{\phi}, \quad (5.33)$$

$$\frac{1}{2}[\delta, \bar{\delta}]\psi_\alpha = i\partial_0\psi_\alpha - 2iV_m(\mathcal{D}_m\psi - \frac{1}{4}ie\sigma_m\psi)_\alpha + \frac{3}{2}e(r-1)\psi_\alpha, \quad (5.34)$$

$$\frac{1}{2}[\delta, \bar{\delta}]\bar{\psi}^\alpha = i\partial_0\bar{\psi}^\alpha - 2iV_m(\mathcal{D}_m\bar{\psi} + \frac{1}{4}ie\sigma_m\bar{\psi})^\alpha - \frac{3}{2}e(r-1)\bar{\psi}^\alpha, \quad (5.35)$$

with  $V_m$  the Killing vector defined in equation (3.55) by contracting a Pauli matrix between two Killing spinors. Taking,

$$\epsilon^\alpha = e^{-\frac{it}{2}}\delta_1^\alpha, \quad \bar{\epsilon}_\alpha = e^{\frac{it}{2}}\delta_\alpha^1, \quad (5.36)$$

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and denote by  $\delta_1, \bar{\delta}^1$  the corresponding supersymmetry transformations. One can then compute the commutator of the supersymmetries acting upon the various fields of the theory. By definition of  $\mathcal{H}$ ,

$$\mathcal{H}X = \frac{1}{2} [\delta_1, \bar{\delta}^1] X, \quad (5.37)$$

for any field  $X$ . The relevant fields for the computation of the index are  $(\phi, \psi_\alpha)$  and the conjugates  $(\bar{\phi}, \bar{\psi}^\alpha)$ , hence,

$$\mathcal{H}\phi = i\partial_0\phi + e^{-1}\left(\frac{3}{2}r - 2i\nabla_3\right)\phi, \quad (5.38)$$

$$\mathcal{H}\bar{\phi} = i\partial_0\bar{\phi} + e^{-1}\left(\frac{3}{2}r - 2i\nabla_3\right)\bar{\phi}, \quad (5.39)$$

$$\mathcal{H}\psi_\alpha = i\partial_0\psi_\alpha + e^{-1}\left(-2i\nabla_3\psi_\alpha - (\sigma_3\psi)_\alpha + \frac{3}{2}r\psi_\alpha - \psi_\alpha\right) + \frac{1}{2}e\psi_\alpha, \quad (5.40)$$

$$\mathcal{H}\bar{\psi}^\alpha = i\partial_0\bar{\psi}^\alpha + e^{-1}\left(-2i\nabla_3\bar{\psi}^\alpha + (\bar{\psi}\sigma_3)^\alpha - \frac{3}{2}r\bar{\psi}^\alpha + \bar{\psi}^\alpha\right) - \frac{1}{2}e\bar{\psi}^\alpha, \quad (5.41)$$

One should note that contact can be made with the squashed sphere partition function of [61], whose Lagrangian is given by,

$$\tilde{\mathcal{L}}_\phi = \mathcal{D}_m\bar{\phi}\mathcal{D}_m\phi - \frac{1}{8}e^2r_{3D}(2r_{3D} - 1) + \frac{1}{4}r_{3D}R, \quad (5.42)$$

To match the three dimensional Lagrangian (5.42) with its four dimensional counterpart (5.27), one can shrink the radius of the temporal  $S^1$  to zero and hence set  $\partial_0$  to zero, and identify identify the 3 dimensional  $R$ -charge as,

$$r = \frac{3}{2}r_{3D}, \quad (5.43)$$

As the 3 dimensional  $R$ -charge is implicitly contained in the three dimensional Lagrangian in the spatial derivative term through the twisting, and that the  $R$ -charge is the  $U(1)$  symmetry which is used to construct the twisted Lagrangian in 3 dimensions, one cannot take  $B$  in to be the  $R$ -charge in 4 dimensions. We need to take  $B$  such that,

$$[B, \phi] = \frac{3}{2}r, \quad [B, \psi] = \frac{3}{2}r - 1, \quad (5.44)$$

We are now ready to compute the index as a free field partition function.

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### 5.2.2 Free Field Theory Partition Function

#### Scalar Component

One can then rewrite the Lagrangian after Wick rotation the time direction, which leads to the following Laplacian.

$$\Delta_\phi = -(\partial_\tau - \frac{3}{2}er + e)^2 - 4(\nabla_1\nabla_1 + \nabla_2\nabla_2 + e^{-2}(\nabla_3 - \frac{3}{4}ir(1-e^2))^2) + e^2 + 3r(1-e^2), \quad (5.45)$$

In order to compute the index, one needs to twist the time derivative appropriately,

$$H \rightarrow \mathcal{H}, \quad (5.46)$$

where  $\mathcal{H}$  can be read off from equation (5.38). The twisted laplacian is the such that the Wick rotated time derivative part is modified to,

$$\partial_\tau \rightarrow \partial_\tau + e^{-1}(\frac{3}{2}r + 2m), \quad (5.47)$$

#### Spinor Component

A similar analysis can be performed for the spinor Lagrangian. The action for the spinor can be rewritten as,

$$\mathcal{L}_\psi = \bar{\psi}(i\partial_0 - 2i(\sigma_1\nabla_1 + \sigma_2\nabla_2 + e^{-1}\sigma_3\nabla_3) + \frac{3}{2}e(r-1) - e^{-1} + (\frac{3}{2}r-1)(e-e^{-1})\sigma_3)\psi, \quad (5.48)$$

One can then rewrite the Wick rotated Dirac operator as,

$$\mathcal{D} = \begin{pmatrix} D_1^1 & D_1^2 \\ D_2^1 & D_2^2 \end{pmatrix} \quad (5.49)$$



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with,

$$D_1^1 = -\partial_\tau + \frac{3}{2}r(e - e^{-1}) + \frac{1}{2}e(3r - 5) - 2e^{-1}i\nabla_3, \quad (5.50)$$

$$D_1^2 = -2i(\nabla_1 - i\nabla_2), \quad (5.51)$$

$$D_2^1 = -2i(\nabla_1 + i\nabla_2), \quad (5.52)$$

$$D_2^2 = -\partial_\tau + \frac{3}{2}e^{-1}r - \frac{1}{2}e - 2e^{-1} + 2e^{-1}i\nabla_3, \quad (5.53)$$

Applying the twisting (5.46), modifies the following entries,

$$D_1^1 \rightarrow \tilde{D}_1^1 = -\partial_\tau + 3r(e - e^{-1}) - 4e^{-1}i\nabla_3, \quad (5.54)$$

$$D_2^2 \rightarrow \tilde{D}_2^2 = -\partial_\tau, \quad (5.55)$$

where one should note that the twisting is not the same for the first component of the spinor and the second.

### 5.2.3 Determinant Computation

#### Scalar Component

One can then expand the scalar Lagrangian in terms of  $S^3$  spherical harmonics, as these constitute a basis of three dimensional function, as well as a Fourier modes along the  $S^1$  time direction, which implies the correspondence between the time derivative and the eigenvalues  $\partial_\tau \sim ik\omega$  with  $k$  an integer, and  $\omega$  the fundamental Matsubara frequency of the time interval define in equation (4.80). One can then apply the twisting (5.46). This leads to the following twisted laplacian eigenvalues,

$$\begin{aligned} \Delta_{kjm} = & -(ik\omega - \frac{3}{2}r(e - e^{-1}) + 2me^{-1} + e)^2 + 4j(j + 1) - 4m^2 \\ & + (2me^{-1} - \frac{3}{2}r(e - e^{-1}))^2 + e^2 + 3r(1 - e^2) \end{aligned} \quad (5.56)$$

with the following quantum numbers,

$$k \in \mathbb{Z}, \quad j \in \frac{1}{2}\mathbb{Z}, \quad -j \leq m \leq j. \quad (5.57)$$

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This expression can be factorised when  $m = j$ ,

$$\Delta_{kjj} = -ik\omega (ik\omega + 4je^{-1} + e(2 - 3r) + 3e^{-1}r) \quad (5.58)$$

When  $m \neq j$ , the factorisation is,

$$\Delta_{kjm} = -(ik\omega - E_{jm}^+)(ik\omega - E_{jm}^-), \quad (5.59)$$

with,

$$E_{jm}^\pm = -(2m + \frac{3}{2}r)e^{-1} + e(\frac{3}{2}r - 1) \pm \sqrt{(2me^{-1} + e(1 - \frac{3}{2}r) + \frac{3}{2}e^{-1}r)^2 + 4(j - m)(j + m + 1)} \quad (5.60)$$

### Spinor Component

One can then expand expand the spinor Lagrangian in spherical harmonics as for the index on  $\mathbb{R} \times S^3$ , after Wick rotation. The Dirac operator can be block diagonalised by the following set of spinors,

$$e^{ik\omega\tau} \begin{pmatrix} |j, m\rangle \\ |j, m + 1\rangle \end{pmatrix}, \quad e^{ik\omega\tau} \begin{pmatrix} |j, j\rangle \\ 0 \end{pmatrix}, \quad (5.61)$$

with the following (half-)integers,

$$k \in \mathbb{Z}, \quad j \in \frac{1}{2}\mathbb{Z}, \quad -j \leq m \leq j \quad (5.62)$$

and the following representation for the rotation generators,

$$\mathcal{J}_3|j, m\rangle = i\nabla_3|j, m\rangle = m|j, m\rangle, \quad J_\pm|j, m\rangle = (j \mp 1)|j, m \pm 1\rangle \quad (5.63)$$

Note the expression for  $\mathcal{J}_3$ . Why did we change representation for the rotation generators? Here, the most convenient way of dealing with the Dirac operator is to consider harmonics of  $SU(2)_l \times SU(2)_r \times SU(2)_{\text{spin}}$ . Before, we considered harmonics  $SU(2)_l \times SU(2)_r$ , as the latter group was implicitly including the spin connection term when needed. In other words, we separate the action of what

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remains  $SU(2)_l$  into spin and orbital angular momentum. This leads to the following Dirac operator, after performing the twisting (5.55), when acting upon this basis of spinors,

$$\tilde{\mathcal{D}} = \begin{pmatrix} -ik\omega - 4e^{-1}m - 3e^{-1}r + (3r - 2)e & -2(j + m + 1) \\ -2(j - m) & -ik\omega \end{pmatrix}. \quad (5.64)$$

and the associated determinant is,

$$\mathcal{D}_{kjm} = ik\omega (ik\omega + 4e^{-1}m + 3e^{-1}r - e(3r - 2)) - 4(j - m)(j + m + 1), \quad (5.65)$$

which can be factorised once again as,

$$\mathcal{D}_{kjm} = (ik\omega - E_{jm}^+)(ik\omega - E_{jm}^-), \quad (5.66)$$

with  $E_{jm}^\pm$  defined in equation (5.61). One can then compute the determinant associated with the index defined in equation (5.1), which is just the ratio of the product given of eigenvalues for the scalar Laplacian (5.57) and the Dirac operator (5.64). As there is an infinite set of states which contribute to the index (5.1), we need to restrict our computation to a given value of  $j$  and  $k$  to compute both products. Given that there are  $2j + 1$  degenerate states contributing to the determinant for both the scalar and the spinor determinant, which are due to the  $SU(2)_r$  isometry of the squashed sphere, the following holds,

$$\mathrm{Tr}_{kj}(-1)^F e^{-\beta\mathcal{H}} = \prod_{m=-j}^j \frac{\mathcal{D}_{kjm}}{\Delta_{kjm}} = (-1)^{2j+1} \quad (5.67)$$

with the subscript  $k, j$  denoting the restriction to the subspace of spin  $j$  and wave number  $k$ , and where the appropriate determinants for the scalar and the spinor Lagrangian had been defined in equations (5.57) and (5.65). This expression is the one we expected, so we are doing something right. It shows that our supersymmetric theory is consistent, and that the object we have defined is truly an index. Hence, it receives only contributions from the kernel of the twisted Hamiltonian  $\mathcal{H}$ . By identifying those states, one can compute the full index on the squashed sphere, which we are now going to define.

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### 5.2.4 The Full Index on the Squashed-Sphere

We need to define the set of operators which commute with the supercharge associated with the supersymmetric variation  $\delta_1$ . In the flat space situation, the only such operators are  $\mathcal{H}$ ,  $-R+2J_3$ ,  $\bar{J}_3$ . They are defined as the operators which commute with the charge  $Q_1$  of the superconformal algebra, or equivalently, the corresponding Killing spinor. Here we do not have such algebra, but we have the Killing spinor in question. The commutation relations of the Killing spinor with  $R$ ,  $\bar{J}_3$  are unchanged compared with the round sphere. Let us focus on  $J_3$ , and given that the Killing spinor is constant  $\epsilon_1^\alpha \propto \delta_1^\alpha$ ,

$$\mathcal{J}_3 \epsilon_1^\alpha = \frac{1}{2} \epsilon_1^\alpha, \tag{5.68}$$

hence the definition of the full index on the squashed sphere is identical to the definition on the round sphere (2.47). To compute the index on the squashed sphere we then need to identify the states which contribute to the index as they are in the kernel of the twisted Hamiltonian  $\mathcal{H}$ . One can then canonically quantise the theory as set out in section (3.2). One can then recover the formulas (3.144) and (3.163).

# Chapter 6

## The Multiparticle Index

We have only focused on the single particle index so far. There are different approaches one can take to compute the full superconformal index. The generic approach is to compute take a large  $N$  limit in the number of colors considered. This allows one to compute an approximate result for the full index up to  $\mathcal{O}(\frac{1}{N})$  correction which can be compared with the corresponding result for the dual theory [36]. This approach is also used in routinely in 3 dimensional calculations [78, 12]. The second approach allows for an exact matching of indices and relies on the theory of elliptic hypergeometric functions [111, 36], and relies on recently discovered identities which from physicist's point of view relate indices on both sides of Seiberg dual theories [36, 42, 43, 112, 35, 113, 114]. It also yields exact results for 3 dimensional partition functions [108].

### 6.1 Matching Indices with Generic $R$ -charges

#### 6.1.1 Indices for Seiberg Dual Theories

In this section, we consider the large  $N_c$  limit of an  $SU(N_c)$  gauge theory and show how an exact expression for the index in this limit. We also take the number of flavours  $N_f$  to infinity, while keeping the ratio of the number of flavours to colors  $\frac{N_f}{N_c}$  constant. However, we depart slightly from the standard theories (1.1) and (1.2) and relax the constraints on the  $R$ -charges for the various fields. Imposing the matching of the index in the large  $N$  limit will then be shown to allow the

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recovery of the appropriate quantum numbers.

Field	$SU(N_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
$Q$	$f$	$f$	$1$	$1$	$r_Q$
$\tilde{Q}$	$\bar{f}$	$1$	$\bar{f}$	$-1$	$r_{\tilde{Q}}$
$\lambda$	adj.	$1$	$1$	$0$	$1$

Figure 6.1: Seiberg Electric Theory, Generic  $R$ -charges

Field	$SU(\tilde{N}_c)$	$SU(N_f)$	$SU(N_f)$	$U(1)_B$	$U(1)_R$
$q$	$f$	$\bar{f}$	$1$	$N_c/\tilde{N}_c$	$r_q$
$\tilde{q}$	$\bar{f}$	$1$	$f$	$-N_c/\tilde{N}_c$	$r_{\tilde{q}}$
$\tilde{\lambda}$	adj.	$1$	$1$	$0$	$1$
$M$	$1$	$f$	$\bar{f}$	$0$	$r_M$

Figure 6.2: Seiberg Magnetic Theory, Generic  $R$ -charges

Note that the  $R$ -charges for the vector multiplets  $V$  and  $\tilde{V}$  are fixed by the supersymmetry algebra on  $\mathbb{R} \times S^3$  given in equation (3.83).

One can then read off the single particle index for the electric theory with field content given in figure (1.1) combined with the formula for single particle index for chiral multiplet (2.88) and vector multiplet (2.87), the single particle index on the electric side is given by,

$$\begin{aligned}
& i_E(t, x, v, y, \tilde{y}, z) \\
&= \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})} \left( p_{N_c}(z)p_{N_c}(z^{-1}) - 1 \right) \\
&+ \frac{1}{(1-tx)(1-tx^{-1})} \left( t^{r_Q} v p_{N_f}(y) p_{N_c}(z) - t^{2-r_Q} v^{-1} p_{N_f}(y^{-1}) p_{N_c}(z^{-1}) \right. \\
&\quad \left. + t^{r_{\tilde{Q}}} v^{-1} p_{N_f}(\tilde{y}) p_{N_c}(z^{-1}) - t^{2-r_{\tilde{Q}}} v p_{N_f}(\tilde{y}^{-1}) p_{N_c}(z) \right) ,
\end{aligned} \tag{6.1}$$

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while on the magnetic side,

$$\begin{aligned}
& i_M(t, x, v, y, \tilde{y}, z) \\
&= \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})} (p_{\tilde{N}_c}(z)p_{\tilde{N}_c}(z^{-1}) - 1) \\
&+ \frac{1}{(1-tx)(1-tx^{-1})} (t^{r_q}v p_{N_f}(y)p_{\tilde{N}_c}(z) - t^{2-r_q}v^{-1}p_{N_f}(y^{-1})p_{\tilde{N}_c}(z^{-1}) \\
&\quad + t^{r_{\tilde{q}}}v^{-1}p_{N_f}(\tilde{y})p_{\tilde{N}_c}(z^{-1}) - t^{2-r_{\tilde{q}}}v p_{N_f}(\tilde{y}^{-1})p_{\tilde{N}_c}(z), \\
&\quad + t^{r_M}p_{N_f}(y)p_{N_f}(\tilde{y}^{-1}) - t^{2-r_M}p_{N_f}(y^{-1})p_{N_f}(\tilde{y})) ,
\end{aligned} \tag{6.2}$$

where we have introduced the following gauge variables,

$$z = (z_1, z_2, \dots, z_{N_c}), \tag{6.3}$$

a well as the flavour variables,

$$y = (y_1, y_2 \dots, y_{N_f}), \quad \tilde{y} = (\tilde{y}_1, \tilde{y}_2 \dots, \tilde{y}_{N_f}), \tag{6.4}$$

and the flavour and gauge characters  $\chi_R(z)$  and  $\chi_R(y)$  for  $z$  denoting elements of  $SU(N_c)$  and  $(y, \tilde{y})$  denoting elements of  $SU(N_f) \times SU(N_f)$ ,

$$\chi_{SU(N), f}(z) = \sum_{i=1}^n z_i = p_N(z), \tag{6.5}$$

$$\chi_{SU(N), \bar{f}}(z) = \sum_{i=1}^n z_i^{-1} = p_N(z^{-1}), \tag{6.6}$$

$$\chi_{SU(N), \text{Adj.}}(z) = \sum_{1 \leq i, j \leq n} \frac{z_i}{z_j} - 1 = p_N(z)p_N(z^{-1}) - 1, \tag{6.7}$$

while imposing the unitarity condition on  $z$  appropriate for  $SU(N)$ ,

$$\prod_{i=1}^N z_i = 1, \tag{6.8}$$

One can rewrite both single particle indices in the following generic form,

$$i(t, z) = f(t)(p_N(z)p_N(z^{-1}) - 1) + g(t)p_N(z) + \bar{g}(t)p_N(z^{-1}) + h(t), \tag{6.9}$$

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with,

$$\mathbf{t} = (t, x, y, \tilde{y}). \quad (6.10)$$

The full index associated with  $i(\mathbf{t}, z)$  is given by the plethystic integral,

$$I(\mathbf{t}) = \int_{SU(N)} d\mu(z) \exp \left( \sum_{n=1}^{\infty} \frac{1}{n} i(\mathbf{t}^n, z^n) \right) \quad (6.11)$$

with the left-invariant measure on  $SU(N)$  defined as,

$$\begin{aligned} \int_{SU(N)} d\mu(z) = \\ \frac{1}{(2\pi)^{N-1} N} \int_{-\pi \leq \theta_1 \leq \theta_2 \leq \dots \leq \theta_{N-1} \leq \pi} \prod_{i=1}^{N-1} d\theta_i \prod_{i < j} 4 \sin^2 \frac{1}{2}(\theta_i - \theta_j). \end{aligned} \quad (6.12)$$

The measure factor can be rewritten as,

$$\prod_{i < j} 4 \sin^2 \frac{1}{2}(\theta_i - \theta_j) = \prod_{i \neq j} \exp \left( - \sum_{n=1}^{\infty} \frac{1}{n} e^{in(\theta_i - \theta_j)} \right) \quad (6.13)$$

which leads to the following expression for the index,

$$I(\mathbf{t}) = \int \prod_{i=1}^{N-1} d\theta_i \text{Pexp}(-S(\mathbf{t}, \theta)) \quad (6.14)$$

with the so-called action  $S(\mathbf{t}, \theta)$  defined as,

$$S(\mathbf{t}, \theta) = (1 - f(\mathbf{t})) \sum_{i \neq j} e^{i(\theta_i - \theta_j)} - g(\mathbf{t}) \sum_i e^{-i\theta_i} - \bar{g}(\mathbf{t}) \sum_i e^{i\theta_i} - h(\mathbf{t}) + f(\mathbf{t}) \quad (6.15)$$

### 6.1.2 Infinite $N$ Limit

One can then take the large  $N$  limit, which has as a main consequence to replace the integration over  $\{\theta_i\}_{i=1}^{N-1}$  by a functional integral over a continuous variable  $[d\theta]$  ranging from  $-\pi$  to  $\pi$  such that,

$$\sum_{i=1}^{N-1} f(\theta_i) \rightarrow N \int_0^1 dx f(\theta_x). \quad (6.16)$$



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One can then evaluate the full index in the large  $N$  limit by considering the density function for  $\theta$  defined as,

$$\rho(\theta) = \frac{dx}{d\theta}. \quad (6.17)$$

The integration measure for the path integral (6.14) in terms of the Fourier modes of  $\rho$ ,

$$\rho_n = N \int_{-\pi}^{\pi} d\theta \rho(\theta) e^{in\theta}, \quad (6.18)$$

which leads to the following expression for the path integral measure,

$$\prod_{i=1}^{N-1} d\theta_i \rightarrow [d\theta] = [d\rho] = \prod_{n \geq 1} \frac{n}{\pi} d\rho_n d\rho_{-n}. \quad (6.19)$$

One can then rewrite the Plethystic exponential in (6.14) as,

$$\begin{aligned} & -\text{Log Pexp}(-S(t, \theta)) \rightarrow \\ & \sum_{n=1}^{\infty} \frac{1}{n} ((1 - f(t^n))\rho_n \rho_{-n} - g(t^n)\rho_{-n} - \bar{g}(t^n)\rho_n - h(t^n) + f(t^n)), \end{aligned} \quad (6.20)$$

and we are now left with a Gaussian integral which can be computed exactly, with the following saddle points,

$$\rho_n = \frac{g(t^n)}{1 - f(t^n)}, \quad \rho_{-n} = \frac{\bar{g}(t^n)}{1 - f(t^n)}, \quad n \in \mathbb{N}^* \quad (6.21)$$

and the expression for the index in the large  $N$  limit is then given by,

$$I_{N \rightarrow \infty}(t) = \text{Pexp} \left( \frac{g(t)\bar{g}(t)}{1 - f(t)} - f(t) + h(t) \right) \prod_{n=1}^{\infty} \frac{1}{1 - f(t^n)}. \quad (6.22)$$

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With the definition (6.10) for  $t$ , one can apply the previous analysis to Seiberg dual theories,

$$f_E(t) = \frac{2t^2 - t\chi_2(x)}{(1-tx)(1-tx^{-1})} = f_M(t), \quad (6.23)$$

$$g_E(t) = \frac{v}{(1-tx)(1-tx^{-1})} (t^{r_Q} p_{N_f}(y) - t^{2-r_{\tilde{Q}}} p_{N_f}(\tilde{y}^{-1})), \quad (6.24)$$

$$g_M(t) = \frac{v}{(1-tx)(1-tx^{-1})} (t^{r_q} p_{N_f}(y) - t^{2-r_{\tilde{q}}} p_{N_f}(\tilde{y}^{-1})), \quad (6.25)$$

$$\bar{g}_E(t) = \frac{v^{-1}}{(1-tx)(1-tx^{-1})} (t^{r_{\tilde{Q}}} p_{N_f}(\tilde{y}) - t^{2-r_Q} p_{N_f}(y^{-1})), \quad (6.26)$$

$$\bar{g}_M(t) = \frac{v^{-1}}{(1-tx)(1-tx^{-1})} (t^{r_{\tilde{q}}} p_{N_f}(\tilde{y}) - t^{2-r_q} p_{N_f}(y^{-1})), \quad (6.27)$$

$$h_E(t) = 0, \quad (6.28)$$

$$h_M(t) = \frac{1}{(1-tx)(1-tx^{-1})} (t^{r_M} p_{N_f}(y) p_{N_f}(\tilde{y}^{-1}) - t^{2-r_M} p_{N_f}(y^{-1}) p_{N_f}(\tilde{y})), \quad (6.29)$$

Equating the indices on both sides of the duality then imposes the following restrictions on the  $R$ -charges of the various fields in the theories,

$$r_Q + r_{\tilde{Q}} = r_M, \quad (6.30)$$

$$r_q + r_{\tilde{q}} = 2 - r_M, \quad (6.31)$$

which is compatible with the definition of the Seiberg dual theories under consideration, but a weaker condition. To get more constraints on the  $R$ -charges, one can compute the leading finite  $N$  correction to equation (6.22), and match the electric and magnetic results. This can be achieved by using Schur polynomials techniques as in [115, 36].

In the usual approach to Seiberg duality [26], the first constraint on the  $R$ -charges can be interpreted as the fact that the mesons  $M$  are composites of the electric quarks  $Q$ ,  $\tilde{Q}$ . The second equation corresponds to the  $R$ -charges constraints associated with to the magnetic theory superpotential,

$$W = \tilde{M}_i^j \phi^i \bar{\phi}_j, \quad i, j = 1 \dots N_f, \quad (6.32)$$

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with  $\phi, \bar{\phi}$  the magnetic squarks and  $\tilde{M}$  is the magnetic scalar mesino. It is interesting to note that, despite the nature of the index as a free field theory quantity, the matching of the full index on both sides of the duality contains implies the information regarding the superpotential. So, although the localisation argument implies that the superpotential does not impact the index<sup>1</sup>, the matching of the index seems, at least in this context, to provide the information contained in the superpotential.

### 6.1.3 Finite $N$ Corrections

To do this, one can recast the plethystic exponential which constitutes the integrand of the full multiparticle index expression given in equation (6.11) using,

$$\text{Pexp}(f(t)) = \sum_{\underline{a}} \frac{1}{\mathcal{N}_{\underline{a}}} f_{\underline{a}}(t). \quad (6.33)$$

where  $\underline{a}$  represents an infinite row vector  $\underline{a} = (a_1, a_2 \dots)$  and we have used the following notation,

$$p_{\underline{a}}(z) = \prod_{n=1}^{\infty} p_N(z^n)^{a_n}, \quad f_{\underline{a}}(t) = \prod_{n=1}^{\infty} f(t^n)^{a_n}, \quad (6.34)$$

as well as the normalisation constant,

$$\mathcal{N}_{\underline{a}} = \prod_{n=1}^{\infty} n^{a_n} a_n!, \quad (6.35)$$

which allows the index integrand to be recast in terms of power symmetric polynomials as,

$$\text{Pexp}(i(t, z)) = \sum_{\underline{a}, \underline{b}, \underline{\bar{b}}} \frac{1}{\mathcal{N}_{\underline{a}} \mathcal{N}_{\underline{b}} \mathcal{N}_{\underline{\bar{b}}}} f_{\underline{a}}(t) g_{\underline{b}}(t) \bar{g}_{\underline{\bar{b}}}(t) p_{\underline{a}+\underline{b}}(z) p_{\underline{a}+\underline{\bar{b}}}(z^{-1}). \quad (6.36)$$

The power symmetric polynomials can be rewritten in terms of Schur polynomials which satisfy standard orthogonality relations. Schur polynomials depend on the

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<sup>1</sup>See more refined statement in section (4.1.5)

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variables  $z = (z_1, z_2 \dots z_N)$  and the index  $\underline{\lambda} = (\lambda_1, \lambda_2 \dots \lambda_{\ell(\underline{\lambda})})$ ,

$$s_{\underline{\lambda}}(z) = s_{\lambda_1, \lambda_2 \dots \lambda_n}(z) = \frac{\det [z_i^{\lambda_j + n - j}]}{\det [z_i^{n - j}]} \quad (6.37)$$

which correspond to  $SU(N)$  characters, provided the unitarity constraint on the variables  $z$  given in (6.8), and the partition  $\underline{\lambda}$  is ordered,

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell(\underline{\lambda})} \geq 1, \quad \lambda_{\ell(\underline{\lambda})+1} = 0. \quad (6.38)$$

with  $\ell(\underline{\lambda})$  the length of the vector  $\underline{\lambda}$ . As a consequence of the unitarity constraint (6.8) on the product of  $x_i$ , the following relation holds,

$$s_{\underline{\lambda}}(z) = s_{\underline{\lambda} + n \underline{\rho}_N}(z), \quad \rho_N = (1, 1 \dots 1), \quad \ell(\underline{\rho}_N) = N, \quad (6.39)$$

which allows us to set  $\lambda_N = 0$ . One can then make contact with the previous definitions of  $SU(N)$  characters (6.7),

$$\chi_{SU(N), f}(z) = s_{1, 0 \dots 0, 0}(z), \quad (6.40)$$

$$\chi_{SU(N), \bar{f}}(z) = s_{1, 1 \dots 1, 0}(z), \quad (6.41)$$

$$\chi_{SU(N), \text{Adj.}}(z) = s_{2, 1 \dots 1, 0}(z). \quad (6.42)$$

One can then rewrite the power symmetric polynomials in terms of Schur polynomials,

$$p_{\underline{a}}(z) = \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \omega_{\underline{a}}^{\underline{\lambda}} s_{\underline{\lambda}}(z), \quad s_{\underline{\lambda}}(z) = \sum_{\underline{a}} \frac{1}{\mathcal{N}_{\underline{a}}} \omega_{\underline{a}}^{\underline{\lambda}} p_{\underline{a}}(z), \quad (6.43)$$

with,

$$|\underline{a}| = \sum_{n \geq 1} n a_n = |\underline{\lambda}| = \sum_{n \geq 1} \lambda_n. \quad (6.44)$$

We were able to invert equation (6.43) by using the following,

$$\sum_{\underline{\lambda}} \omega_{\underline{a}}^{\underline{\lambda}} \omega_{\underline{b}}^{\underline{\lambda}} = \mathcal{N}_{\underline{a}} \delta_{\underline{a} \underline{b}}. \quad (6.45)$$

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The orthogonality relations between Schur polynomials is given by,

$$\int_{SU(N)} d\mu(z) s_{\underline{\lambda}}(z) s_{\underline{\lambda}'}(z) = \delta_{\underline{\lambda}\underline{\lambda}'} + \sum_{n=1}^{\infty} (\delta_{\underline{\lambda}', \underline{\lambda} + n\rho_N} + \delta_{\underline{\lambda}, \underline{\lambda}' + n\rho_N}) \quad \ell(\underline{\lambda}), \ell(\underline{\lambda}') < N \quad (6.46)$$

which can then be translated to the corresponding result for the power symmetric polynomials,

$$\int_{SU(N)} d\mu(z) p_{\underline{a}}(z) p_{\underline{b}}(z) = \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \omega_{\underline{a}}^{\underline{\lambda}} \omega_{\underline{b}}^{\underline{\lambda}} + \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \sum_{n=1}^{\infty} (\omega_{\underline{a}}^{\underline{\lambda} + n\rho_N} \omega_{\underline{b}}^{\underline{\lambda}} + \omega_{\underline{a}}^{\underline{\lambda}} \omega_{\underline{b}}^{\underline{\lambda} + n\rho_N}).$$

This can then be used to give an exact, although rather involved, formula for the full superconformal index,

$$I(t) = \sum_{\underline{a}\underline{b}\bar{\underline{b}}} \frac{1}{\mathcal{N}_{\underline{a}} \mathcal{N}_{\underline{b}} \mathcal{N}_{\bar{\underline{b}}}} f_{\underline{a}}(t) g_{\underline{b}}(t) \bar{g}_{\bar{\underline{b}}}(t) \quad (6.47)$$

$$\left( \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \omega_{\underline{a}+\underline{b}}^{\underline{\lambda}} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda}} + \sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \sum_{n=1}^{\infty} (\omega_{\underline{a}+\underline{b}}^{\underline{\lambda} + n\rho_N} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda}} + \omega_{\underline{a}+\underline{b}}^{\underline{\lambda}} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda} + n\rho_N}) \right).$$

One can then use this to integrate equation (6.36) over  $SU(N)$ . The first term on the right hand side of (6.47) gives rise to the large  $N$  limit formula,

$$I_{N \rightarrow \infty}(t) = \sum_{\underline{a}\underline{b}} \frac{\mathcal{N}_{\underline{a}+\underline{b}}}{\mathcal{N}_{\underline{a}} \mathcal{N}_{\underline{b}}^2} f_{\underline{a}}(t) g_{\underline{b}}(t) \bar{g}_{\underline{b}}(t), \quad (6.48)$$

which is equivalent to equation (6.22). One can then compute the first finite  $N$  correction. The leading non-zero correction is the  $n = 1$ ,  $\underline{\lambda} = \underline{0}$  term, and is given by,

$$I_{N < \infty} = \sum_{\underline{b}} \frac{1}{\mathcal{N}_{\underline{b}}} (g_{\underline{b}}(t) + \bar{g}_{\underline{b}}(t)) \omega_{\underline{b}}^{\rho_N} + \dots, \quad |\underline{b}| = N, \quad (6.49)$$

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Applying this expression to the electric theory in figure (1.1), the first finite  $N$  corrections are proportional to,

$$I_{E, N < \infty} \sim \sum_{\underline{b}} \frac{1}{\mathcal{N}_{\underline{b}}} (t^{N_c r_Q} v^{N_c} p_{\underline{b}}(y) + t^{N_c r_{\tilde{Q}}} v^{-N_c} p_{\underline{b}}(\tilde{y})) \omega_{\underline{b}}^{\rho_{N_c}} + \dots \quad (6.50)$$

For the magnetic theory, this gives,

$$I_{M, N < \infty} \sim \sum_{\underline{b}} \frac{1}{\mathcal{N}_{\underline{b}}} (t^{\tilde{N}_c r_q} v^{\tilde{N}_c} p_{\underline{b}}(y) + t^{\tilde{N}_c r_{\tilde{q}}} v^{-\tilde{N}_c} p_{\underline{b}}(\tilde{y})) \omega_{\underline{b}}^{\rho_{\tilde{N}_c}} + \dots \quad (6.51)$$

Matching for  $t$  gives some extra relations on the  $R$ -charges,

$$N_c r_Q = \tilde{N}_c r_q, \quad N_c r_{\tilde{Q}} = \tilde{N}_c r_{\tilde{q}}, \quad (6.52)$$

When combined with the other relations on  $R$ -charges,

$$r_M = \frac{2\tilde{N}_c}{N_f}, \quad (6.53)$$

The quark charges are not fixed by this, as one parameter  $\alpha$  remains unfixed,

$$r_q = \frac{N_c}{N_f} + \alpha, \quad r_{\tilde{q}} = \frac{N_c}{N_f} - \alpha, \quad (6.54)$$

$$r_Q = \frac{\tilde{N}_c}{N_f} + \alpha \frac{\tilde{N}_c}{N_c}, \quad r_{\tilde{Q}} = \frac{\tilde{N}_c}{N_f} - \alpha \frac{\tilde{N}_c}{N_c}, \quad (6.55)$$

The large  $N$  limit of the index does not depend on the parameter  $\alpha$ , as can be seen from substituting the expression for the quark  $R$ -charges (6.55) into equation (6.22). However, one can impose the correct  $\alpha$  parameter by maximising the first finite  $N$  corrections to the index in the limit,

$$y = \tilde{y}, \quad v = 1, \quad (6.56)$$

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For instance in the Seiberg electric theory, the  $\alpha$  dependence for  $g_E$  is given by,

$$g_E(t) = \frac{v}{(1-tx)(1-tx^{-1})} t^{\alpha \frac{\tilde{N}_c}{N_f}} (t^{\frac{\tilde{N}_c}{N_f}} p_{N_f}(y) - t^{2-\frac{\tilde{N}_c}{N_f}} p_{N_f}(\tilde{y}^{-1})), \quad (6.57)$$

$$\bar{g}_E(t) = \frac{v^{-1}}{(1-tx)(1-tx^{-1})} t^{-\alpha \frac{\tilde{N}_c}{N_f}} (t^{\frac{\tilde{N}_c}{N_f}} p_{N_f}(\tilde{y}) - t^{2-\frac{\tilde{N}_c}{N_f}} p_{N_f}(y^{-1})). \quad (6.58)$$

Consequently, given that  $f_E(t)$  does not depend on any  $R$ -charge, and given that  $h_E(t)$  is zero, the large  $N$  limit of the index (6.22) does not depend on the  $\alpha$  parameter, as only the product  $g(t)\bar{g}(t)$  enters its definition. If one now focuses on the leading finite  $N$  correction to the index, one can see that,

$$\partial_\alpha I_{E, N < \infty}(y = \tilde{y}, v = 1) = \sum_{\underline{b}} \frac{2}{\mathcal{N}_{\underline{b}}} \tilde{N}_c \text{Log}(t) t^{\frac{N_c \tilde{N}_c}{N_f}} \sinh(\alpha \tilde{N}_c) p_{\underline{b}}(y) \omega_{\underline{b}}^{\rho_{N_c}} + \dots \quad (6.59)$$

One can see that in the limit (6.56), setting  $\alpha$  to zero extremises  $I_{E, N < \infty}$ . This analysis can be repeated on the magnetic side in order to recover the electromagnetic dual  $R$ -charges in figure (1.1) and figure (1.2).

### 6.2 Extremising the Index

The previous observation regarding index maximisation can in fact be expanded beyond finite  $N$  considerations. Consider the Seiberg electric theory. The expression for the full particle index given in equation (6.48) can be given emphasizing the dependence on parameter  $\alpha$  defined in equation (6.55),

$$I(t) = \sum_{\underline{a} \underline{b} \bar{\underline{b}}} \frac{1}{\mathcal{N}_{\underline{a}} \mathcal{N}_{\underline{b}} \mathcal{N}_{\bar{\underline{b}}}} f_{\underline{a}}(t) g_{\underline{b}, \alpha=0}(t) \bar{g}_{\bar{\underline{b}}, \alpha=0}(t) \quad (6.60)$$

$$\sum_{\substack{\underline{\lambda} \\ \ell(\underline{\lambda}) \leq N}} \left( \omega_{\underline{a}+\underline{b}}^{\underline{\lambda}} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda}} + \sum_{n=1}^{\infty} (t^{\alpha \tilde{N}_c n} \omega_{\underline{a}+\underline{b}}^{\underline{\lambda}+n\rho_N} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda}} + t^{-\alpha \tilde{N}_c n} \omega_{\underline{a}+\underline{b}}^{\underline{\lambda}} \omega_{\underline{a}+\bar{\underline{b}}}^{\underline{\lambda}+n\rho_N}) \right).$$

where we have made use of equation (6.44). In the limit (6.56), one can compute the derivative of the full index with respect to  $\alpha$ . In this limit, one should note that,

$$g_{\underline{b}}(t) = \bar{g}_{\bar{\underline{b}}}(t), \quad (6.61)$$

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which means that one can symmetrise the last terms in the sum (6.61) with respect to  $\underline{b} \leftrightarrow \bar{b}$ , so in the limit (6.56),

$$\begin{aligned} \partial_\alpha I(t) = & \sum_{\underline{a} \underline{b} \bar{b}} \frac{2}{\mathcal{N}_{\underline{a}} \mathcal{N}_{\underline{b}} \mathcal{N}_{\bar{b}}} f_{\underline{a}}(t) g_{\underline{b}, \alpha=0}(t) g_{\bar{b}, \alpha=0}(t) \\ & \sum_{\substack{\lambda \\ \ell(\lambda) \leq N}} \sum_{n=1}^{\infty} n \text{Log}(t) \tilde{N}_c \sinh(\alpha n \tilde{N}_c) \omega_{\underline{a}+\underline{b}}^{\lambda+n\rho_N} \omega_{\underline{a}+\bar{b}}^\lambda. \end{aligned} \quad (6.62)$$

Consequently, taking the parameter  $\alpha$  to zero minimises the index in the appropriate limit. This observation is reminiscent of  $a$ -maximisation [1] in 4 dimensions and  $Z$ -maximisation in 3 dimensions.

One should note that  $\alpha$  parameterises the difference in  $R$ -charge between the electric quarks  $Q$  and  $\tilde{Q}$ , and the same observation holds for the magnetic theory. Consequently, assuming chiral symmetry to be unbroken means that  $\alpha$  should be set to zero. However, it is still interesting to observe that the appropriate value for this parameter also happens to maximise the index.

### 6.3 Elliptic Hypergeometry of the Index

The full index can in fact be shown to agree on both sides of Seiberg dual theories [36, 42, 43, 112, 35, 113, 114]. This matching is due to non-trivial identities relating integrals of Gamma functions due to Rains [39] and Spiridonov. We here review the simplest matching of electric-magnetic indices [36] for Seiberg dual theories with a magnetic theory with  $(N_f, N_c) = (2, 3)$ .

The elliptic Gamma function will be used to rewrite the scalar multiplet single particle index, and is defined as,

$$\Gamma(y; p, q) = \prod_{j, k \geq 0} \frac{1 - y^{-1} p^{j+1} q^{k+1}}{1 - y p^j q^k}, \quad (6.63)$$

while the theta function and the  $(x; p)$  infinite product will be used to rewrite the



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single particle index for the vector multiplet,

$$\theta(z; p) = \prod_{j \geq 0} (1 - zp^j)(1 - z^{-1}p^{j+1}), \quad (x; p) = \prod_{j \geq 0} (1 - xp^j), \quad (6.64)$$

With these definitions and the following variable definition,

$$p = tx, \quad q = tx^{-1}, \quad y = t^r z \quad (6.65)$$

with  $z$  the  $U(1)_B$  fugacity. The single particle scalar index reads,

$$i_\phi(p, q, y) = \frac{t^r z - t^{2-r} z^{-1}}{(1 - tx)(1 - tx^{-1})} = \frac{y - pqy^{-1}}{(1 - p)(1 - q)}, \quad (6.66)$$

and its Plethystic exponential can be rewritten,

$$\text{Pexp}(i_\phi(p, q, y)) = \Gamma(y; p, q), \quad (6.67)$$

whereas the vector single particle index reads,

$$i_\lambda(p, q) = -\frac{p}{1 - p} - \frac{q}{1 - q} = 1 - \frac{1 - pq}{(1 - p)(1 - q)} \quad (6.68)$$

and its Plethystic reads,

$$\text{Pexp}(i_\lambda(y, p, q)(z + z^{-1})) = \frac{\theta(z; p)\theta(z; q)}{(1 - z)^2}, \quad (6.69)$$

$$= \frac{1}{(1 - z)(1 - z^{-1})\Gamma(z; p, q)\Gamma(z^{-1}; p, q)}, \quad (6.70)$$

$$\text{Pexp}(i_\lambda(y, p, q)) = (p; p)(q; q). \quad (6.71)$$

For the electric theory, the the flavour symmetry can be rewritten as an  $SU(6)$  global symmetry,  $U(1)_B \times SU(3) \times SU(3) \rightarrow SU(6)$ , with the electric quarks in the six dimensional fundamental representation and the magnetic dual quarks and mesons form the 15 dimensional antisymmetric representation  $\mathbb{A}$ . This leads

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to the expression for the electric single particle index,

$$i_E(p, q, \mathbf{u}, z) = - \left( \frac{p}{1-p} + \frac{q}{1-q} \right) \chi_3(z) + \frac{1}{(1-p)(1-q)} (p_6(u) - pqp_6(u^{-1})) \chi_2(z), \quad (6.72)$$

with  $u, z$  the flavour and gauge variables, and the polynomials  $p_N$  defined in equation (6.5). Similarly, one can write the single particle index for the

$$i_M(p, q, \mathbf{u}) = \frac{1}{(1-p)(1-q)} (\chi_{SU(6), \mathbb{A}}(\mathbf{u}) - pq \chi_{SU(6), \mathbb{A}}(\mathbf{u}^{-1})), \quad (6.73)$$

with the characters for the antisymmetric tensor representation  $\mathbb{A}$  of  $SU(6)$  given by,

$$\chi_{SU(N), \mathbb{A}}(\mathbf{u}) = \sum_{1 \leq i < j \leq N} u_i u_j, \quad (6.74)$$

One can then show that the full superconformal index for the electric theory can be recast as,

$$I_E(p, q, \mathbf{u}) = -(p; p)(q; q) \frac{1}{4\pi i} \oint \frac{dz}{z^3} \theta(z^2; p) \theta(z^2; q) \mathcal{I}(p, q, \mathbf{u}, z), \quad (6.75)$$

with the following definition,

$$\mathcal{I}(p, q, \mathbf{u}, z) = \prod_{a=1}^6 \Gamma(u_a z; p, q) \Gamma(u_a z^{-1}; p, q), \quad (6.76)$$

while on the magnetic side, there is no integration sign,

$$I_M(p, q, \mathbf{u}) = \prod_{1 \leq a < b \leq 6} \Gamma(u_a u_b; p, q). \quad (6.77)$$

The matching of the full indices (6.75) (6.77) then stems from an identity obtained by Spiridonov [37].

# Chapter 7

## Conclusions

In this thesis we have given a detailed analysis of the 4-dimensional index. After computing the index for a generic 4-dimensional free superconformal field theory, we were able to generalise the index to less symmetric theories by constructing  $\mathcal{N} = 1$  Lagrangians on  $\mathbb{R} \times S^3$  which reduce to radially quantised theories for conformal theories, but allow for more generic  $R$ -charges away from conformality. This then allowed us to write down the twisted action for a finite time interval and show exactness of the index formula (2.49). Finally, we focused on the index in the large  $N$  limit for Seiberg dual theories. Taking  $R$ -charges to be completely generic and imposing matching of the index and matching the infinite  $N$  limit as well as the first finite  $N$  corrections on both sides of the duality, we were able to recover the appropriate  $R$ -charges, and we also uncovered some sign that an extremisation principle similar to  $a$ ,  $Z$ -maximisation might be at play for the index. This in itself is no surprise, as the index can be dimensionally reduced to the three-dimensional partition function on the sphere. Also, the localisation principle makes explicit on loop nature of the index in 4 dimensions, which is also the case for the Weyl anomaly which defines the coefficient  $a$ .

This leaves many questions open, some of which are given below.

### Index and Non-Perturbative Contributions

The exactness of the matching of the index for Seiberg dual theories is an impressive result from a mathematical point of view, and provides a powerful new

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tool to test and postulate new dualities. However it only contains information about a limited portion of the spectrum of the theories under consideration. Also, it does not probe the non-perturbative regime of the theories under consideration. One could then ask whether non-perturbative physics can be probed while retaining exactness associated with localisation principles we have used here. Denoting the  $\mathcal{S}$  the appropriately twisted action which gives rise to the index, we have shown that,

$$I(t, x) = \int [dX] e^{-\mathcal{S}}, \quad (7.1)$$

is exact, with  $X$  the set of fields in the theory. In order to probe non perturbative physics, we somehow need to soak up the zero modes in the instanton background by having fermionic insertions. For localisation principles to be applicable, we can only  $\mathcal{Q}$ -closed operators. Keeping the same conventions as before such that  $\mathcal{Q} = Q_1 + S^1$  is twisted to become a scalar operator, one can consider the following two point correlation function,

$$\tilde{I}_\psi(t, x) = \int d\tau d^3\Omega \int [dX] (\bar{\psi}^1 \psi_1 + \bar{\lambda}^1 \lambda_1)(\tau, \Omega) e^{-\mathcal{S}}, \quad (7.2)$$

with insertions of the quarks and the gaugino. One can think of the insertion as the fermion counting operator  $F$  defined in equation (1.4). This makes the definition of the modified index (7.2) reminiscent of the modified index defined in [94]. For this quantity to be exact, the number of fermionic zero modes on the curved background should match the number of fermion insertions. Also, assuming this is true, one would need to define monopole spherical harmonics for  $\mathbb{R} \times S^3$  in the same spirit as [116, 117, 118].

### Further Geometric Deformations of the Sphere

As was pointed out in section (4.3.1) and developed in chapter (5), the index can be defined for more generic spaces with less than the superconformal symmetry which was initially required for the definition. In section (3.1.2), we have underlined the symmetries one has to preserve to define an index consistently, and we have investigated the case of the singly squashed sphere in chapter (5). One should extend this approach to the case of the doubly squashed sphere as in

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[61]. Another potential extension mentioned in (4.3.1) is the computation of the index on spaces with non-trivial holonomies such as the lens space [107], which would give rise to more complex structure for the index of gauge theories. Also, understanding better the relationship between four dimensional indices and three dimensional partition functions would be desirable. The interplay between those quantities could lead to a better understanding of  $a$ ,  $Z$ -maximisation as well as dualities.

# Appendix A

## Superconformal Algebra

In this appendix we give the explicit realisation of the algebra (2.26) for practical purposes. For  $Q_\alpha$ ,  $S^\alpha$ ,  $\bar{Q}_{\dot{\alpha}}$ ,  $\bar{S}^{\dot{\alpha}}$  denoting supercharges,

$$\{Q_\alpha, S^\beta\} = 2\delta_\alpha^\beta (H + \frac{3}{2}R) - 4J_m(\sigma_m)_\alpha{}^\beta, \quad (\text{A.1})$$

$$\{\bar{Q}_{\dot{\alpha}}, \bar{S}^{\dot{\beta}}\} = -2\delta_{\dot{\alpha}}^{\dot{\beta}} (H - \frac{3}{2}R) - 4\bar{J}_m(\sigma_m)^{\dot{\beta}}{}_{\dot{\alpha}}, \quad (\text{A.2})$$

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2P_{\alpha\dot{\alpha}}, \quad (\text{A.3})$$

$$\{S^\alpha, \bar{S}^{\dot{\alpha}}\} = 2K^{\dot{\alpha}\alpha}, \quad (\text{A.4})$$

where  $\sigma_m$  denote usual Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad (\text{A.5})$$

The dilation, translation and inversion operators act on the supercharges as,

$$[H, Q_\alpha] = \frac{1}{2}Q_\alpha, \quad [H, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2}\bar{Q}_{\dot{\alpha}}, \quad [H, P_{\alpha\dot{\alpha}}] = P_{\alpha\dot{\alpha}}, \quad (\text{A.6})$$

$$[H, S^\alpha] = -\frac{1}{2}S^\alpha, \quad [H, \bar{S}^{\dot{\alpha}}] = -\frac{1}{2}\bar{S}^{\dot{\alpha}}, \quad [H, K^{\dot{\alpha}\alpha}] = -K^{\dot{\alpha}\alpha}, \quad (\text{A.7})$$

The supercharges transform under rotation as,

$$[J_m, Q_\alpha] = -\frac{1}{2}(\sigma_m)_\alpha{}^\beta Q_\beta, \quad [J_m, S^\alpha] = \frac{1}{2}S^\beta(\sigma_m)_\beta{}^\alpha, \quad (\text{A.8})$$

$$[\bar{J}_m, \bar{Q}_{\dot{\alpha}}] = \frac{1}{2}\bar{Q}_{\dot{\beta}}(\sigma_m)^{\dot{\beta}}{}_{\dot{\alpha}}, \quad [\bar{J}_m, \bar{S}^{\dot{\alpha}}] = -\frac{1}{2}(\sigma_m)^{\dot{\alpha}}{}_{\dot{\beta}}\bar{S}^{\dot{\beta}}, \quad (\text{A.9})$$

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Or equivalently,

$$[J_3, Q_1] = -\frac{1}{2}Q_1, \quad [J_3, S^1] = \frac{1}{2}S^1, \quad (\text{A.10})$$

$$[J_+, Q_1] = -Q_2, \quad [J_-, Q_2] = -Q_1, \quad (\text{A.11})$$

$$[J_+, S^2] = S^1, \quad [J_-, S^1] = S^2, \quad (\text{A.12})$$

The  $R$ -charge action is given by,

$$[R, Q_\alpha] = -Q_\alpha, \quad [R, S^\alpha] = S^\alpha, \quad (\text{A.13})$$

$$[R, \bar{Q}_{\dot{\alpha}}] = \bar{Q}_{\dot{\alpha}}, \quad [R, \bar{S}^{\dot{\alpha}}] = -\bar{S}^{\dot{\alpha}}. \quad (\text{A.14})$$

The commutation relations between  $J$  rotation generators and the  $P$  translation and  $K$  inversion generators are given by,

$$[J_m, P_0] = \frac{1}{2}P_m, \quad [J_m, P_n] = \frac{1}{2}\delta_{mn}P_0 + \frac{i}{2}\epsilon_{mnp}P_p, \quad (\text{A.15})$$

$$[J_m, K_0] = \frac{1}{2}K_m, \quad [J_m, P_n] = \frac{1}{2}\delta_{mn}K_0 + \frac{i}{2}\epsilon_{mnp}K_p, \quad (\text{A.16})$$

$$[\bar{J}_m, P_0] = -\frac{1}{2}P_m, \quad [J_m, P_n] = -\frac{1}{2}\delta_{mn}P_0 + \frac{i}{2}\epsilon_{mnp}P_p, \quad (\text{A.17})$$

$$[\bar{J}_m, K_0] = -\frac{1}{2}K_m, \quad [J_m, P_n] = -\frac{1}{2}\delta_{mn}K_0 + \frac{i}{2}\epsilon_{mnp}K_p, \quad (\text{A.18})$$

In the spinorial basis,

$$[J_3, P_{11}] = -\frac{1}{2}P_{11}, \quad [J_3, P_{22}] = \frac{1}{2}P_{22}, \quad (\text{A.19})$$

$$[J_3, P_{12}] = -\frac{1}{2}P_{12}, \quad [J_3, P_{21}] = \frac{1}{2}P_{21}, \quad (\text{A.20})$$

Other commutators not shown are zero. Under radial hermitean conjugation these generators satisfy,

$$H^+ = H, \quad J_m^+ = J_m, \quad \bar{J}_m^+ = \bar{J}_m, \quad R^+ = R, \quad Q_\alpha^+ = S^\alpha, \quad \bar{Q}_{\dot{\alpha}}^+ = -\bar{S}^{\dot{\alpha}}, \quad (\text{A.21})$$

while under the “flat” hermitean conjugation,

$$H^\dagger = -H, \quad J_m^\dagger = \bar{J}_m, \quad R^\dagger = R, \quad Q_\alpha^\dagger = \bar{Q}_{\dot{\alpha}}, \quad S^{\alpha\dagger} = \bar{S}^{\dot{\alpha}} \quad (\text{A.22})$$

# Appendix B

## Localisation Actions

### B.1 Chiral Multiplet Action: Free Part

Here we determine  $\mathcal{Q}_L \mathcal{V}_\phi$  decomposing,

$$\mathcal{V}_\phi = \mathcal{V}_1^\phi + \mathcal{V}_2^\phi + \mathcal{V}_3^\phi, \quad (\text{B.1})$$

where we have defined,

$$\mathcal{V}_1^\phi = -\frac{1}{2}(\psi^\beta \bar{\epsilon}_\alpha S_\beta \bar{\psi}^\alpha + \bar{\psi}^\beta \epsilon_\alpha Q_\beta \psi^\alpha), \quad (\text{B.2})$$

$$\mathcal{V}_2^\phi = \frac{1}{2}(\psi^\beta \epsilon_\alpha Q_\beta \bar{\psi}^\alpha + \bar{\psi}^\beta \bar{\epsilon}_\alpha S_\beta \psi^\alpha), \quad (\text{B.3})$$

$$\mathcal{V}_3^\phi = \phi \mathcal{Q}_L \bar{\phi} - \bar{\phi} \mathcal{Q}_L \phi, \quad (\text{B.4})$$

excluding for the moment gauge interaction terms. We deal with the remaining terms of appearing in the localisation action arising from gauge interactions in the following subsection.

In computing these expansions we use the following identities for the Killing spinors, namely,

$$\epsilon^\alpha \epsilon_\alpha = \bar{\epsilon}^\alpha \bar{\epsilon}_\alpha = 0, \quad \epsilon^\alpha \bar{\epsilon}_\beta = \frac{1}{2} \delta_\beta^\alpha + \frac{1}{2} V_m (\sigma_m)_\beta^\alpha, \quad \bar{\epsilon}^\alpha \epsilon_\beta = -\frac{1}{2} \delta_\beta^\alpha + \frac{1}{2} V_m (\sigma_m)_\beta^\alpha, \quad (\text{B.5})$$

where  $V_m$  is the Killing vector defined in equation (3.55) (3.56). Expanding  $\mathcal{Q}_L \mathcal{V}_1^\phi$



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we obtain,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_1^\phi = & -\frac{1}{2}(\epsilon^\alpha \epsilon^\gamma Q_\alpha \bar{\psi}_\beta Q^\beta \psi_\gamma + \epsilon^\alpha \bar{\epsilon}_\gamma Q_\alpha \psi^\beta S_\beta, \bar{\psi}^\gamma - \epsilon^\alpha \bar{\epsilon}_\gamma \psi^\beta Q_\alpha S_\beta \bar{\psi}^\gamma \\ & - \bar{\epsilon}^\alpha \bar{\epsilon}_\gamma S_\alpha \psi^\beta S_\beta, \bar{\psi}^\gamma - \bar{\epsilon}^\alpha \epsilon^\gamma S_\alpha \bar{\psi}_\beta Q^\beta \psi_\gamma + \bar{\epsilon}^\alpha \epsilon^\gamma \bar{\psi}_\beta S_\alpha Q^\beta \psi_\gamma). \end{aligned} \quad (\text{B.6})$$

The bosonic parts in this expansion are given by,

$$\epsilon^\alpha \epsilon^\gamma Q_\alpha \bar{\psi}_\beta Q^\beta \psi_\gamma = -2i(\epsilon \sigma_m \epsilon) \partial_m \bar{\phi} F, \quad (\text{B.7})$$

$$\epsilon^\alpha \bar{\epsilon}_\gamma Q_\alpha \psi^\beta S_\beta, \bar{\psi}^\gamma = F \bar{F}, \quad (\text{B.8})$$

$$-\bar{\epsilon}^\alpha \bar{\epsilon}_\gamma S_\alpha \psi^\beta S_\beta, \bar{\psi}^\gamma = 2i(\bar{\epsilon} \sigma_m \bar{\epsilon}) \partial_m \phi \bar{F}, \quad (\text{B.9})$$

$$-\bar{\epsilon}^\alpha \epsilon^\gamma S_\alpha \bar{\psi}_\beta Q^\beta \psi_\gamma = F \bar{F}, \quad (\text{B.10})$$

while the fermionic parts are,

$$\begin{aligned} -\epsilon^\alpha \bar{\epsilon}_\gamma \psi^\beta Q_\alpha S_\beta, \bar{\psi}^\gamma = & \frac{1}{2} \psi (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} + i\psi \sigma_m D_m \bar{\psi} \\ & + \frac{1}{2} V_m \psi \sigma_m (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} + iV_m \psi D_m \bar{\psi} + \varepsilon_{mnp} V_m \psi \sigma_n D_p \bar{\psi}, \end{aligned} \quad (\text{B.11})$$

$$\begin{aligned} \bar{\epsilon}^\alpha \epsilon^\gamma \bar{\psi}_\beta S_\alpha Q^\beta \psi_\gamma = & -\frac{1}{2} \bar{\psi} (i\partial_0 + \frac{3r-2}{2}) \psi + i\bar{\psi} \sigma_m D_m \psi \\ & + \frac{1}{2} V_m \bar{\psi} \sigma_m (i\partial_0 - \frac{3r-2}{2}) \psi - iV_m \bar{\psi} D_m \psi - \varepsilon_{mnp} V_m \bar{\psi} \sigma_n D_p \psi. \end{aligned} \quad (\text{B.12})$$

One can simplify the last equation by noting, up to boundary terms,

$$\varepsilon_{mnp} V_m \bar{\psi} \sigma_n D_p \psi = \varepsilon_{mnp} V_m \psi \sigma_n D_p \bar{\psi} - V_m \psi \sigma_m \bar{\psi}. \quad (\text{B.13})$$

Consequently, in total, the bosonic and fermionic parts are,

$$\mathcal{Q}_L \mathcal{V}_1^{\phi, \text{bos.}} = -F \bar{F} + i(\epsilon \sigma_m \epsilon) \partial_m \bar{\phi} F - i(\bar{\epsilon} \sigma_m \bar{\epsilon}) \partial_m \phi \bar{F}, \quad (\text{B.14})$$

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_1^{\phi, \text{ferm.}} = & -\frac{1}{2} \psi (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} - i\psi \sigma_m D_m \bar{\psi} \\ & - \frac{1}{2} V_m \psi \sigma_m (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} - iV_m \psi D_m \bar{\psi} - \frac{1}{2} V_m \psi \sigma_m \bar{\psi}. \end{aligned} \quad (\text{B.15})$$

Similarly,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_2^\phi = & \frac{1}{2}(\epsilon^\alpha \epsilon^\gamma Q_\alpha \psi_\beta Q^\beta \bar{\psi}^\gamma + \epsilon^\alpha \bar{\epsilon}_\gamma Q_\alpha \bar{\psi}^\beta S_\beta, \psi^\gamma - \epsilon^\alpha \bar{\epsilon}_\gamma \bar{\psi}^\beta Q_\alpha S_\beta \psi^\gamma \\ & - \bar{\epsilon}^\alpha \bar{\epsilon}_\gamma S_\alpha \bar{\psi}^\beta S_\beta, \psi^\gamma - \bar{\epsilon}^\alpha \epsilon^\gamma S_\alpha \psi_\beta Q^\beta \bar{\psi}^\gamma + \bar{\epsilon}^\alpha \epsilon^\gamma \psi_\beta S_\alpha Q^\beta \bar{\psi}^\gamma). \end{aligned} \quad (\text{B.16})$$

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The bosonic parts are,

$$\epsilon^\alpha \epsilon^\gamma Q_\alpha \psi_\beta Q^\beta \bar{\psi}_\gamma = -2i(\epsilon \sigma_m \epsilon) \partial_m \bar{\phi} F, \quad (\text{B.17})$$

$$-\bar{\epsilon}^\alpha \bar{\epsilon}^\gamma S_\alpha \bar{\psi}^\beta S_\beta \psi^\gamma = 2i(\bar{\epsilon} \sigma_m \bar{\epsilon}) \partial_m \phi \bar{F}, \quad (\text{B.18})$$

along with,

$$\begin{aligned} \epsilon^\alpha \bar{\epsilon}^\gamma Q_\alpha \bar{\psi}^\beta S_\beta, \psi^\gamma &= (\partial_0 + i\frac{3}{2}r) \bar{\phi} (\partial_0 - i\frac{3}{2}r) \phi - 4\partial_m \phi \partial_m \bar{\phi} \\ &\quad - 4i\epsilon_{mnp} V_p \partial_m \bar{\phi} \partial_n \phi + 2V_m \partial_m \bar{\phi} (\partial_0 - i\frac{3}{2}r) \phi \\ &\quad - 2V_m \partial_m \phi (\partial_0 + i\frac{3}{2}r) \bar{\phi}, \end{aligned} \quad (\text{B.19})$$

$$\begin{aligned} -\bar{\epsilon}^\alpha \epsilon^\gamma S_\alpha \psi_\beta Q^\beta \bar{\psi}_\gamma &= (\partial_0 + i\frac{3}{2}r) \bar{\phi} (\partial_0 - i\frac{3}{2}r) \phi - 4\partial_m \phi \partial_m \bar{\phi} \\ &\quad - 4i\epsilon_{mnp} V_p \partial_m \bar{\phi} \partial_n \phi + 2V_m \partial_m \phi (\partial_0 + i\frac{3}{2}r) \bar{\phi} \\ &\quad - 2V_m \partial_m \bar{\phi} (\partial_0 + i\frac{3}{2}r) \phi. \end{aligned} \quad (\text{B.20})$$

The last equation noticing may be simplified using, suppressing boundary terms,

$$\epsilon_{mnp} 4iV_p \partial_m \bar{\phi} \partial_n \phi = -4iV_m \phi \partial_m \bar{\phi}. \quad (\text{B.21})$$

Meanwhile, the fermionic parts are given by,

$$\begin{aligned} -\epsilon^\alpha \bar{\epsilon}^\gamma \bar{\psi}^\beta Q_\alpha S_\beta \psi^\gamma &= \frac{1}{2} \bar{\psi} (i\partial_0 + \frac{3r-2}{2}) \bar{\psi} - i\bar{\psi} \sigma_m D_m \psi + \psi \bar{\psi} \\ &\quad + \frac{1}{2} V_m \bar{\psi} \sigma_m (i\partial_0 + \frac{3r-2}{2}) \psi - iV_m \bar{\psi} D_m \psi + \epsilon_{mnp} V_m \bar{\psi} \sigma_n D_p \psi, \end{aligned} \quad (\text{B.22})$$

$$\begin{aligned} \bar{\epsilon}^\alpha \epsilon^\gamma \psi_\beta S_\alpha Q^\beta \bar{\psi}_\gamma &= -\frac{1}{2} \psi (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} - i\psi \sigma_m D_m \bar{\psi} + \psi \bar{\psi} \\ &\quad + \frac{1}{2} V_m \psi \sigma_m (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} + iV_m \psi D_m \bar{\psi} - \epsilon_{mnp} V_m \psi \sigma_n D_p \bar{\psi}. \end{aligned} \quad (\text{B.23})$$

Consequently,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_2^{\phi, \text{bos.}} &= -i(\epsilon \sigma_m \epsilon) \partial_m \bar{\phi} F + i(\bar{\epsilon} \sigma_m \bar{\epsilon}) \partial_m \phi \bar{F} \\ &\quad + (\partial_0 + i\frac{3}{2}r) \bar{\phi} (\partial_0 - i\frac{3}{2}r) \phi - 4\partial_m \phi \partial_m \bar{\phi} - 4iV_m \phi \partial_m \bar{\phi}, \end{aligned} \quad (\text{B.24})$$

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_2^{\phi, \text{fer.}} &= -\frac{1}{2} \psi (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} - i\psi \sigma_m D_m \bar{\psi} \\ &\quad + \frac{1}{2} V_m \psi \sigma_m (i\partial_0 - \frac{3r-2}{2}) \bar{\psi} + \psi \bar{\psi} - \frac{1}{2} V_m \psi \sigma_m \bar{\psi}, \end{aligned} \quad (\text{B.25})$$

gathering the bosonic and fermionic parts together. Finally, we have for the

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bosonic and fermionic parts of  $\mathcal{Q}_L \mathcal{V}_3^\phi$ ,

$$\mathcal{Q}_L \mathcal{V}_3^{\phi, \text{bos.}} = \phi(i\partial_0 - \frac{3}{2}r)\bar{\phi} - \bar{\phi}(i\partial_0 + \frac{3}{2}r)\phi + 2iV_m\phi\partial_m\bar{\phi} - 2i\bar{\phi}V_m\partial_m\phi, \quad (\text{B.26})$$

$$\mathcal{Q}_L \mathcal{V}_3^{\phi, \text{fer.}} = -\psi\bar{\psi} + V_m\psi\sigma_m\bar{\psi}. \quad (\text{B.27})$$

Collecting all bosonic and fermionic terms leads to (3.96) with (3.99).

### B.2 Chiral Multiplet Action: Interacting Part

The final piece of the action for an interacting scalar field, save for interaction terms from gauge covariant derivatives, is given by  $\mathcal{Q}_L \mathcal{V}_{\text{int.}}^\phi$ . where we have defined,

$$\mathcal{V}_{\text{int.}}^\phi = -i\phi[\epsilon\bar{\lambda}, \bar{\phi}] + i\bar{\phi}[\bar{\epsilon}\lambda, \phi]. \quad (\text{B.28})$$

Expanding this expression (B.28),

$$\begin{aligned} -i\mathcal{Q}_L\phi[\epsilon\bar{\lambda}, \bar{\phi}] &= -i(\epsilon\psi)[(\epsilon\bar{\lambda}), \bar{\phi}] + \frac{i}{2}\phi[D, \bar{\phi}] + \frac{i}{2}\bar{\psi}[\bar{\lambda}, \phi] \\ &\quad + \frac{i}{2}V_m\bar{\psi}\sigma_m[\bar{\lambda}, \phi] - V_m\phi[F_m^-, \bar{\phi}], \end{aligned} \quad (\text{B.29})$$

$$\begin{aligned} i\mathcal{Q}_L\bar{\phi}[\bar{\epsilon}\lambda, \phi] &= -i(\bar{\epsilon}\bar{\psi})[(\bar{\epsilon}\lambda), \phi] + \frac{i}{2}\phi[D, \bar{\phi}] - \frac{i}{2}\psi[\lambda, \bar{\phi}] \\ &\quad + \frac{i}{2}V_m\psi\sigma_m[\lambda, \bar{\phi}] - V_m\bar{\phi}[F_m^+, \phi]. \end{aligned} \quad (\text{B.30})$$

The sum of the latter may be simplified slightly using,

$$-V_m\phi[F_m^-, \bar{\phi}] - V_m\bar{\phi}[F_m^+, \phi] = -2iV_m\epsilon_{mnp}F_{np}[\phi, \bar{\phi}]. \quad (\text{B.31})$$

The analysis of the previous subsection also gets modified as all derivatives get covariantised and give rise to some extra terms resulting from the action of  $\mathcal{Q}_L$  on the connection  $A_0, A_m$  itself. These terms may be read off from (4.70), taking

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account only of  $A_0, A_m$  and  $F$ -term modifications, as,

$$-\frac{i}{2}\epsilon\psi[\mathcal{Q}_L A_0, \bar{\phi}] = \frac{i}{2}(\epsilon\psi)[(\epsilon\bar{\lambda}), \bar{\phi}] - \frac{i}{4}\psi[\lambda, \bar{\phi}] + \frac{i}{4}V_m\psi\sigma_m[\lambda, \bar{\phi}], \quad (\text{B.32})$$

$$\frac{i}{2}\bar{\epsilon}\bar{\psi}[\mathcal{Q}_L A_0, \phi] = \frac{i}{2}(\bar{\epsilon}\bar{\psi})[(\bar{\epsilon}\lambda), \phi] + \frac{i}{4}\bar{\psi}[\bar{\lambda}, \phi] + \frac{i}{4}V_m\bar{\psi}\sigma_m[\bar{\lambda}, \phi], \quad (\text{B.33})$$

$$i\epsilon\sigma_m\psi[\mathcal{Q}_L A_m, \bar{\phi}] = \frac{i}{2}(\epsilon\psi)[(\epsilon\bar{\lambda}), \bar{\phi}] - \frac{3i}{4}\psi[\lambda, \bar{\phi}] - \frac{i}{4}V_m\psi\sigma_m[\lambda, \bar{\phi}], \quad (\text{B.34})$$

$$i\bar{\epsilon}\sigma_m\bar{\psi}[\mathcal{Q}_L A_m, \phi] = \frac{i}{2}(\bar{\epsilon}\bar{\psi})[(\bar{\epsilon}\lambda), \phi] + \frac{3i}{4}\bar{\psi}[\bar{\lambda}, \phi] - \frac{i}{4}V_m\bar{\psi}\sigma_m[\bar{\lambda}, \phi], \quad (\text{B.35})$$

$$\frac{1}{2}(\bar{\epsilon}\psi)\mathcal{Q}_L\bar{F} = -\frac{i}{2}\psi[\lambda, \bar{\phi}] - \frac{i}{2}V_m\psi\sigma_m[\lambda, \bar{\phi}], \quad (\text{B.36})$$

$$\frac{1}{2}(\epsilon\bar{\psi})\mathcal{Q}_L F = \frac{i}{2}\bar{\psi}[\bar{\lambda}, \phi] - \frac{i}{2}V_m\bar{\psi}\sigma_m[\bar{\lambda}, \phi]. \quad (\text{B.37})$$

Furthermore, the equation (B.21) becomes modified to,

$$4i\varepsilon_{mnp}V_p D_m \bar{\phi} D_n \phi = -4iV_m \phi D_m \bar{\phi} + 2iV_m \varepsilon_{mnp} F_{np}[\phi, \bar{\phi}], \quad (\text{B.38})$$

giving rise to an extra term in (B.20), after derivatives are replaced by gauge covariant ones, cancelling the term arising from (B.31). Collecting all these terms leads to (3.104) with (3.105).

### B.3 Vector Multiplet Action

Here, defining,

$$\mathcal{V}_1^A = \frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)F_m^+, \quad (\text{B.39})$$

$$\mathcal{V}_2^A = \frac{i}{2}(\epsilon\sigma_m\bar{\lambda})F_m^-, \quad (\text{B.40})$$

$$\mathcal{V}_3^A = -\frac{1}{4}(\epsilon\bar{\lambda} + \bar{\epsilon}\lambda)D, \quad (\text{B.41})$$

suppressing integration symbols as before, the bosonic part of the action may be computed as,

$$\mathcal{Q}_L \mathcal{V}_1^{A,\text{bos.}} = \frac{i}{2}(\bar{\epsilon}\sigma_m \mathcal{Q}_L \lambda)F_m^+ = \frac{1}{2}F_{0m}F_{0m} - F_{mn}F_{mn} + \frac{i}{4}V_m F_m^+ D, \quad (\text{B.42})$$

$$\mathcal{Q}_L \mathcal{V}_2^{A,\text{bos.}} = \frac{i}{2}(\epsilon\sigma_m \mathcal{Q}_L \bar{\lambda})F_m^- = \frac{1}{2}F_{0m}F_{0m} - F_{mn}F_{mn} - \frac{i}{4}V_m F_m^- D, \quad (\text{B.43})$$

$$\mathcal{Q}_L \mathcal{V}_3^{A,\text{bos.}} = -\frac{1}{4}(\epsilon^\alpha \mathcal{Q}_L \bar{\lambda}_\alpha + \bar{\epsilon}^\alpha \mathcal{Q}_L \lambda_\alpha)D = \frac{i}{4}V_m(F_m^- - F_m^+)D + \frac{1}{4}D^2. \quad (\text{B.44})$$

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Collecting all those terms gives the bosonic part of the vector multiplet action (3.100). Determining the fermionic part of this expression, namely,

$$\mathcal{Q}_L \mathcal{V}_1^{A,\text{fer.}} = -\frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)\mathcal{Q}_L F_m^+, \quad (\text{B.45})$$

$$\mathcal{Q}_L \mathcal{V}_2^{A,\text{fer.}} = -\frac{i}{2}(\epsilon\sigma_m\bar{\lambda})\mathcal{Q}_L F_m^-, \quad (\text{B.46})$$

$$\mathcal{Q}_L \mathcal{V}_3^{A,\text{fer.}} = \frac{1}{4}(\epsilon\bar{\lambda} + \bar{\epsilon}\lambda)\mathcal{Q}_L D. \quad (\text{B.47})$$

Expansion gives in detail,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_1^{A,\text{fer.}} &= \frac{i}{4}(\bar{\epsilon}\sigma_m\lambda)(\epsilon\sigma_m\partial_0\bar{\lambda}) + \frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)(\epsilon D_m\bar{\lambda}) + \frac{1}{2}(\bar{\epsilon}\sigma_m\lambda)\varepsilon_{mnp}(\epsilon\sigma_n D_p\bar{\lambda}) \\ &\quad + \frac{i}{4}(\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}\sigma_m\partial_0\lambda) - \frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}D_m\lambda) + \frac{1}{2}(\bar{\epsilon}\sigma_m\lambda)\varepsilon_{mnp}(\bar{\epsilon}\sigma_n D_p\lambda) \\ &\quad + \frac{1}{2}(\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}\sigma_m\lambda), \end{aligned} \quad (\text{B.48})$$

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_2^{A,\text{fer.}} &= \frac{i}{4}(\epsilon\sigma_m\bar{\lambda})(\bar{\epsilon}\sigma_m\partial_0\lambda) - \frac{i}{2}(\epsilon\sigma_m\bar{\lambda})(\bar{\epsilon}D_m\lambda) - \frac{1}{2}(\epsilon\sigma_m\bar{\lambda})\varepsilon_{mnp}(\bar{\epsilon}\sigma_n D_p\lambda) \\ &\quad + \frac{i}{4}(\epsilon\sigma_m\bar{\lambda})(\epsilon\sigma_m\partial_0\bar{\lambda}) + \frac{i}{2}(\epsilon\sigma_m\bar{\lambda})(\epsilon D_m\bar{\lambda}) - \frac{1}{2}(\epsilon\sigma_m\bar{\lambda})\varepsilon_{mnp}(\epsilon\sigma_n D_p\bar{\lambda}) \\ &\quad + \frac{1}{2}(\epsilon\sigma_m\bar{\lambda})(\epsilon\sigma_m\bar{\lambda}), \end{aligned} \quad (\text{B.49})$$

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_3^{A,\text{fer.}} &= -\frac{i}{4}(\epsilon\bar{\lambda})(\bar{\epsilon}\partial_0\lambda) + \frac{i}{2}(\epsilon\bar{\lambda})(\bar{\epsilon}\sigma_m D_m\lambda) - \frac{i}{4}(\bar{\epsilon}\lambda)(\epsilon\partial_0\bar{\lambda}) - \frac{i}{2}(\bar{\epsilon}\lambda)(\epsilon\sigma_m D_m\bar{\lambda}) \\ &\quad - \frac{i}{4}(\epsilon\bar{\lambda})(\epsilon\partial_0\bar{\lambda}) - \frac{i}{2}(\epsilon\bar{\lambda})(\epsilon\sigma_m D_m\bar{\lambda}) - \frac{i}{4}(\bar{\epsilon}\lambda)(\bar{\epsilon}\partial_0\lambda) \\ &\quad + \frac{i}{2}(\bar{\epsilon}\lambda)(\bar{\epsilon}\sigma_m D_m\lambda), \end{aligned} \quad (\text{B.50})$$

To simplify, we use for  $\mathcal{Q}_L \mathcal{V}_1^{A,\text{fer.}}$  the relations, employing (3.53),

$$\frac{i}{4}(\bar{\epsilon}\sigma_m\lambda)(\epsilon\sigma_m\partial_0\bar{\lambda}) = \frac{3i}{8}\lambda\partial_0\bar{\lambda} - \frac{i}{8}V_m\lambda\sigma_m\partial_0\bar{\lambda}, \quad (\text{B.51})$$

$$\begin{aligned} \frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)(\epsilon D_m\bar{\lambda}) &= \frac{i}{4}\lambda\sigma_m D_m\bar{\lambda} + \frac{i}{4}V_m\lambda D_m\bar{\lambda} - \frac{1}{4}\varepsilon_{mnp}V_m\bar{\lambda}\sigma_n D_p\lambda \\ &\quad - \frac{1}{4}V_m\lambda\sigma_m\bar{\lambda}, \end{aligned} \quad (\text{B.52})$$

$$\frac{1}{2}(\bar{\epsilon}\sigma_m\lambda)\varepsilon_{mnp}(\epsilon\sigma_n D_p\bar{\lambda}) = \frac{i}{2}\lambda\sigma_m D_m\bar{\lambda} - \frac{i}{2}V_m\lambda D_m\bar{\lambda}, \quad (\text{B.53})$$

$$\frac{i}{4}(\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}\sigma_m\partial_0\lambda) = \frac{i}{4}(\bar{\epsilon}\lambda)(\bar{\epsilon}\partial_0\lambda), \quad (\text{B.54})$$

$$-\frac{i}{2}(\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}D_m\lambda) = -\frac{i}{2}(\bar{\epsilon}\lambda)(\bar{\epsilon}\sigma_m D_m\lambda), \quad (\text{B.55})$$

$$(\bar{\epsilon}\sigma_m\lambda)(\varepsilon_{mnp}\bar{\epsilon}\sigma_n D_p\lambda) = (\bar{\epsilon}\sigma_m\lambda)(\bar{\epsilon}\sigma_m\lambda) = 0, \quad (\text{B.56})$$

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and for  $\mathcal{Q}_L \mathcal{V}_2^{A,\text{fer.}}$  the relations,

$$\frac{i}{4}(\epsilon\sigma_m\bar{\lambda})(\bar{\epsilon}\sigma_m\partial_0\lambda) = -\frac{3i}{8}\bar{\lambda}\partial_0\lambda - \frac{i}{8}V_m\bar{\lambda}\sigma_m\partial_0\lambda, \quad (\text{B.57})$$

$$-\frac{i}{2}(\epsilon\sigma_m\bar{\lambda})(\bar{\epsilon}D_m\lambda) = \frac{i}{4}\bar{\lambda}\sigma_m D_m\lambda - \frac{i}{4}V_m\bar{\lambda}D_m\lambda + \frac{1}{4}\epsilon_{mnp}V_m\bar{\lambda}\sigma_n D_p\lambda \quad (\text{B.58})$$

$$-\frac{1}{2}(\epsilon\sigma_m\bar{\lambda})\epsilon_{mnp}(\bar{\epsilon}\sigma_n D_p\lambda) = \frac{i}{2}V_m\bar{\lambda}D_m\lambda + \frac{i}{2}\bar{\lambda}\sigma_m D_m\lambda, \quad (\text{B.59})$$

$$\frac{i}{4}(\epsilon\sigma_m\bar{\lambda})(\epsilon\sigma_m\partial_0\bar{\lambda}) = \frac{i}{4}(\epsilon\bar{\lambda})(\epsilon\partial_0\bar{\lambda}), \quad (\text{B.60})$$

$$\frac{i}{2}(\epsilon\sigma_m\bar{\lambda})(\epsilon D_m\bar{\lambda}) = \frac{i}{2}(\epsilon\bar{\lambda})(\epsilon\sigma_m D_m\bar{\lambda}), \quad (\text{B.61})$$

$$(\epsilon\sigma_m\bar{\lambda})(\epsilon_{mnp}\epsilon\sigma_n D_p\bar{\lambda}) = (\epsilon\sigma_m\bar{\lambda})(\epsilon\sigma_m\bar{\lambda}) = 0, \quad (\text{B.62})$$

to arrive at,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_1^A + \mathcal{Q}_L \mathcal{V}_2^{A,\text{fer.}} &= \frac{3i}{4}\lambda\partial_0\bar{\lambda} + \frac{3i}{2}\lambda\sigma_m D_m\bar{\lambda} - \frac{i}{4}V_m\lambda\sigma_m\partial_0\bar{\lambda} - \frac{i}{2}V_m\lambda D_m\bar{\lambda} \\ &\quad - \frac{1}{4}V_m\lambda\sigma_m\bar{\lambda} + \frac{i}{4}(\epsilon\bar{\lambda})(\epsilon\partial_0\bar{\lambda}) + \frac{i}{2}(\epsilon\sigma_m D_m\bar{\lambda}) \\ &\quad + \frac{i}{4}(\bar{\epsilon}\lambda)(\bar{\epsilon}\partial_0\lambda) - \frac{i}{2}(\bar{\epsilon}\lambda)(\bar{\epsilon}\sigma_m D_m\lambda). \end{aligned} \quad (\text{B.63})$$

Similarly, for  $\mathcal{Q}_L \mathcal{V}_1^{A,\text{fer.}}$  using the identities,

$$-\frac{i}{4}(\epsilon\bar{\lambda})(\bar{\epsilon}\partial_0\lambda) = -\frac{i}{8}\bar{\lambda}\partial_0\lambda + \frac{i}{8}V_m\bar{\lambda}\sigma_m\partial_0\lambda, \quad (\text{B.64})$$

$$\frac{i}{2}(\epsilon\bar{\lambda})(\bar{\epsilon}\sigma_m D_m\lambda) = \frac{i}{4}\bar{\lambda}\sigma_m D_m\lambda - \frac{i}{4}V_m\bar{\lambda}D_m\lambda - \frac{1}{4}\epsilon_{mnp}V_m\bar{\lambda}\sigma_n D_p\lambda, \quad (\text{B.65})$$

$$-\frac{i}{4}(\bar{\epsilon}\lambda)(\epsilon\partial_0\bar{\lambda}) = \frac{i}{8}\lambda\partial_0\bar{\lambda} + \frac{i}{8}V_m\lambda\sigma_m\partial_0\bar{\lambda}, \quad (\text{B.66})$$

$$\begin{aligned} -\frac{i}{2}(\bar{\epsilon}\lambda)(\epsilon\sigma_m D_m\bar{\lambda}) &= \frac{i}{4}\lambda\sigma_m D_m\bar{\lambda} + \frac{i}{4}V_m\lambda D_m\bar{\lambda} + \frac{1}{4}\epsilon_{mnp}V_m\bar{\lambda}\sigma_n D_p\lambda \\ &\quad + \frac{1}{4}V_m\lambda\sigma_m\bar{\lambda}, \end{aligned} \quad (\text{B.67})$$

we obtain,

$$\begin{aligned} \mathcal{Q}_L \mathcal{V}_3^{A,\text{fer.}} &= \frac{i}{4}\lambda\partial_0\bar{\lambda} + \frac{i}{2}\lambda\sigma_m D_m\bar{\lambda} + \frac{i}{4}V_m\lambda\sigma_m\partial_0\bar{\lambda} + \frac{i}{2}V_m\lambda D_m\bar{\lambda} + \frac{1}{4}V_m\lambda\sigma_m\bar{\lambda} \\ &\quad - \frac{i}{4}(\epsilon\bar{\lambda})(\epsilon\partial_0\bar{\lambda}) - \frac{i}{2}(\epsilon\sigma_m D_m\bar{\lambda}) - \frac{i}{4}(\bar{\epsilon}\lambda)(\bar{\epsilon}\partial_0\lambda) \\ &\quad + \frac{i}{2}(\bar{\epsilon}\lambda)(\bar{\epsilon}\sigma_m D_m\lambda). \end{aligned} \quad (\text{B.68})$$

Summing (B.63) and (B.68) the fermionic part of the action (3.100) follows.

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# Indices for Supersymmetric Quantum Field Theories in Four Dimensions

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In this thesis, we investigate four dimensional supersymmetric indices. The motivation for studying such objects lies in the physics of Seiberg's electric-magnetic duality in supersymmetric field theories. In the first chapter, we first define the index and underline its cohomological nature, before giving a first computation based on representation theory of free superconformal field theories. After listing all representations of the superconformal algebra based on shortening conditions, we compute the associated Verma module characters, from which we can extract the index in the appropriate limit. This approach only provides us with the free field theory limit for the index and does not account for the values of the  $R$ -charges away from free field theories. To circumvent this limitation, we then study a theory on  $\mathbb{R} \times S^3$  which allows for a computation of the superconformal index for multiplets with non-canonical  $R$ -charges. We expand the fields in harmonics and canonically quantise the theory to analyse the set of quantum states, identifying the ones that contribute to the index. To go beyond free field theory on  $\mathbb{R} \times S^3$ , we then use the localisation principle to compute the index exactly in an interacting theory, regardless of the value of the coupling constant. We then show that the index is independent of a particular geometric deformation of the underlying manifold, by squashing the sphere. In the final chapter, we show how the matching of the index can be used in the large  $N$  limit to identify the  $R$ -charges for all fields of the electric-magnetic theories of the canonical Seiberg duality. We then conclude by outlining potential further work.