

# Are There Testable Discrete Poincaré Invariant Physical Theories?

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In a model of physics taking place on a discrete set of points that approximates Minkowski space, one might perhaps expect there to be an empirically identifiable preferred frame. However, the work of Dowker, Bombelli, Henson, and Sorkin [1, 2] might be taken to suggest that random sprinklings of points in Minkowski space define a discrete model that is provably Poincaré invariant in a natural sense. We examine this possibility here.

We argue that a genuinely Poincaré invariant model requires a probability distribution on sprinklable sets – Poincaré orbits of sprinklings – rather than individual sprinklings. The corresponding  $\sigma$ -algebra contains only sets of measure zero or one. This makes testing the hypothesis of discrete Poincaré invariance problematic, since any local violation of Poincaré invariance, however gross and large scale, is possible, and cannot be said to be improbable.

We also note that the Bombelli-Henson-Sorkin [2] argument, which rules out constructions of preferred time-like directions for typical sprinklings, is not sufficient to establish full Lorentz invariance. For example, once a pair of timelike separated points is fixed, a preferred spacelike direction *can* be defined for a typical sprinkling, breaking the remaining rotational invariance.

## INTRODUCTION: DOWKER, BOMBELLI, HENSON AND SORKIN ON SPRINKLINGS

It seems very plausible that modelling space-time by a real manifold is an idealization. Indeed, much work on quantum gravity starts by hypothesizing that a fundamental theory of space-time physics involves discrete structures, whether spin networks, causal sets, or other mathematical objects.

It could be that this view is correct but only becomes interesting and coherent in the context of a theory that includes gravity, in which the manifold that approximately describes the large-scale structure of space-time turns out to be better described on small scales by a discrete point set with appropriate mathematical relations defined between pairs (or among subsets) of points. But one might also expect or hope that the theory makes sense without gravity, giving us a discrete version of special relativistic physics in Minkowski space. One reason for hoping this is that it is natural to look at discrete approximations to Minkowski space in order to understand whether discreteness is compatible with Lorentz and Poincaré invariance or whether these are necessarily broken. Defining the question precisely already raises subtle issues in this context; it seems much harder to give a meaningful definition for general Lorentzian manifolds. Another is that “discrete relativistic field theory in Minkowski space” might be much simpler to define than, and also a good stepping stone towards, “discrete quantum gravity”. Ideas in this direction have recently been presented by Bedingham [3]. Our discussion also applies to discrete approximations to more cosmologically relevant manifolds with continuous global symmetries.

Hossenfelder has shown that there are no Poincaré invariant networks with locally finite distributions of nodes and links in Minkowski space [4]. This leaves open the possibility that discrete theories might be defined by point sets approximating Minkowski space, either by requiring no network structure or by allowing nodes to have infinitely many links.[16]

So, are there Poincaré or Lorentz invariant discrete point set structures that approximate Minkowski space? For the Lorentz group, this is a special case of the general question – are there Lorentz invariant discrete structures that approximate Lorentzian manifolds? – considered by Dowker, Henson and Sorkin [1] (DHS) in a pioneering paper that sets out the foundations of causal set theory. As we understand it, the existence or otherwise of discrete approximations to Minkowski space is ultimately not very important for the causal set programme, which seeks rules for generating discrete structures that are approximately consistent with the phenomenology of general relativity and Big Bang cosmology. Our discussion here, which focusses on discretizations of Minkowski space, is indebted to work on causal sets, but not in any sense a critique of the main thrust of that programme.

As DHS note, there is a sense in which the answer to both questions is clearly negative, since no discrete structure can be invariant under the action of every element of the continuous Lorentz group:

Naturally, there can be no question of a literal action of the entire Lorentz group on an individual discrete structure.[1]

However, they argue that this is not the physically relevant sense:

Rather such a structure can only be Lorentz invariant in the same sense that a fluid is translation invariant. This should not detract from the fact that a fluid is indeed translation invariant in an important sense, whereas a crystalline solid is not.[1]

They go on to offer a physical criterion:

What does it mean to say that a discrete theory respects Lorentz invariance? It is difficult to give a precise answer, but intuitively the import is clear. Whenever a continuum is a good approximation to the underlying structure (and assuming specifically that the approximating continuum is a Lorentzian manifold  $M$ ), the underlying discreteness must not, in and of itself, suffice to distinguish a local Lorentz frame at any point of  $M$ . In consequence, no phenomenological theory in  $M$  derived from such a scheme can involve a local (or global) Lorentz frame either. [1]

This motivates the definition of a *sprinkling* (here taken to have Planck density) as:

a Poisson process. To see what this means, imagine dividing  $M$ , using any local coordinate systems, into small boxes of volume  $V$ , and then placing a “sprinkled point” independently into each box with probability  $V/V_{\text{fund}}$ , where  $V_{\text{fund}}$  is the fundamental volume (of order the Planck volume). The Poisson process is the limit of this procedure as  $V$  tends to zero. Because spacetime volume is an invariant, the limiting process is independent of the coordinate systems used to define the boxes. It follows that one cannot tell which frame was used to produce the sprinkling: the approximation is “equally good in all frames”. [1]

DHS make the following claim:

We want to emphasise that not only is the process of sprinkling Lorentz invariant but so also are almost all of the individual causets that are generated. [1]

As far as I understand it, their discussion of this point appeals to the intuition that the Lorentz invariance of the sprinkling process makes it seem plausible that almost all individual sprinklings are Lorentz invariant, together with the observation that some common objections (based, for example, on the existence of voids in sprinklings) can be refuted.

### The BHS theorem

Bombelli, Henson and Sorkin [2] (BHS) take the discussion further:

[ Ref. [1] ] presented strong evidence that causets produced by sprinkling into Minkowski spacetime meet this criterion, but a skeptic could still have found grounds for doubt. In this paper, we prove a theorem that we believe removes most of the remaining doubt. [2]

BHS underline that a Lorentz invariant definition of a probability distribution on point sets does *not*, per se, logically imply Lorentz invariance of a typical set:

The fact that the process of “causet sprinkling” in Minkowski space is Lorentz invariant is an important first step in the argument. (In this process we include both the Poisson sprinkling as such and the subsequent induction of the causal order. Both steps are manifestly Lorentz invariant since they depend only on the volume element and the causal structure of the spacetime, respectively). But Lorentz invariance of the resulting causal set in the above sense does not immediately follow. Consider by analogy a game of fortune in which a circular wheel is spun to a random orientation. While the distribution of final directions is indeed rotationally invariant, a particular outcome of the process is certainly not. (A form of “spontaneous symmetry breaking”, perhaps.) Likewise, a particular outcome of the Poisson process might be able to prefer a frame, even though the process itself does not.

So, the question becomes: Is it possible to use a sprinkling of Minkowski space to select a preferred frame? We will prove a theorem that answers “no” to this question. In fact, it answers the slightly more general question whether a sprinkling can pick out a preferred time-direction (which is certainly possible if an entire frame can be derived.) Below, we formalise the notion of deriving a direction from a sprinkling, and we prove a theorem showing that this cannot be done. In this sense, the situation with sprinklings of Minkowski space is even more comfortable than that with sprinklings of Euclidean space. It is possible to associate a direction from the rotation group to a point in such a sprinkling, as discussed later (although this will not stop anyone from maintaining that a gas behaves isotropically in the continuum approximation; these locally defined directions have little significance at that level), but the non-compactness of the Lorentz group makes the Lorentzian case different.

Based on the theorem, we can assert the following. Not only is the Poisson process in Minkowski space Lorentz invariant, but the individual realizations of the process are also Lorentz invariant in a definite and physically important sense.[2]

For their theorem, BHS consider  $n$ -dimensional Minkowski space  $M^n$ , with a fixed point  $O$ , the origin. They then consider the action of  $L_0$ , the connected component of the identity in  $O(n-1, 1)$ , on  $M^n$  with fixed point  $O$ . They define  $\Omega$  to be the set of possible sprinklings in  $M^n$ , denoting the sprinkling Poisson process by  $(\Omega, \Sigma, \mu)$ , where  $\mu$  is the probability measure on  $\Omega$  and  $\Sigma$  is the  $\sigma$ -algebra of all measurable subsets of  $\Omega$ . They state that the measure is invariant under  $L_0$ , i.e.

$$\mu = \mu \circ \Lambda, \quad \text{for all } \Lambda \in L_0, \quad (1)$$

adding the gloss that

the probability of a (measurable) set of possible sprinklings is the same as that of the set obtained applying a Lorentz transformation to it. [2]

They then consider hypothetical maps  $D : \Omega \rightarrow H$  from the set of sprinklings to the hyperboloid  $H$  of unit time-like vectors in  $M^n$ . They argue that if such a map defines a preferred timelike direction for each sprinkling in a way that genuinely depends only on the sprinkling (and so does not require a preferred frame or other data for its definition), then it must be equivariant under the Lorentz group:

$$D \circ \Lambda = \Lambda \circ D \quad \text{for all } \Lambda \in L_0. \quad (2)$$

But if  $D$  is measurable, then  $\mu \circ D^{-1}$  defines a probability measure on  $H$  that is invariant under  $L_0$ . Since  $H$  is non-compact and has infinite volume, they argue, no such measure can exist, and hence their theorem follows.

### SPRINKLINGS OR SPRINKLABLE SETS?

From here on we focus on the case of sprinklings in 4-dimensional Minkowski space, unless explicitly specified otherwise, i.e. we take the manifold  $M = M^4$ . Our discussion of sprinklable sets applies equally well to  $M^n$  for any  $n \geq 2$ ; our discussion of the lacuna in the BHS theorem applies equally well to  $M^n$  for any  $n \geq 3$ . We will discuss the definition and properties of sprinklings with respect to global coordinates defining an inertial frame, taking  $c = 1$ . [17]

Fixing coordinates  $(x, y, z, t)$  gives a useful way of visualizing the sprinkling process. One can imagine small boxes defined by coordinate increments  $dx, dy, dz, dt$ , with 4-volume  $V = dxdydzdt$ . Following DHS, we can take the probability of a point being “sprinkled” into each box to be  $V/V_{\text{fund}}$ , and define the sprinkling Poisson process to be the limit of this process as  $dx, dy, dz, dt$ , and hence  $V$ , tend to zero. However, DHS argue, because the 4-volume is Poincaré invariant, the limiting distribution is independent of the coordinate choice: it is defined by the property that the probability of finding a sprinkled point in any small 4-volume  $V$  is approximately proportional to  $V/V_{\text{fund}}$  and that these events are independent.

It is nonetheless difficult to define a notation for individual sprinklings without fixing coordinates. Given coordinates  $\{x, y, z, t\}$ , we can, for example describe a sprinkling  $S$  as an ordered list,  $S = \{P_1, P_2, \dots, P_n, \dots\}$ , where  $P_n = (x_n, y_n, z_n, t_n)$  is chosen so that  $l_1 \leq l_2 \leq \dots \leq l_n \leq \dots$ , where  $l_n = (x_n^2 + y_n^2 + z_n^2 + t_n^2)^{1/2}$ , with some tie-breaking condition if any of the  $l_k$  are equal. The “Euclidean distances from the origin”  $l_n$  have no fundamental geometric significance, but define a useful labelling.

This gives a concrete way of describing sprinklings related by Lorentz transformations, as in the BHS theorem. The sprinklings  $S = \{P_1, P_2, \dots, P_n, \dots\}$  and  $S' = \{P'_1, P'_2, \dots, P'_n, \dots\}$  are on the same Lorentz orbit,  $S' = \Lambda S$ , if and only if there is a bijection  $\rho : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$  such that  $P'_i = \Lambda P_{\rho(i)}$  for all  $i \in \mathbb{Z}^+$ .

But it also raises a concern: does DHS sprinkling in fact give a well-defined probability distribution on the class of countably discrete sets that are equipped with a Lorentzian distance function and a causal structure, and that could isometrically be embedded in  $M^4$ ? Or does it give something subtly different: a probability distribution on discrete subsets of  $M^4$  defined with respect to some set of coordinates, whose choice breaks Poincaré invariance?

It might be objected here that, even if labels for sprinkled points are necessary in order to distinguish sprinklings, they need not be defined by a coordinate system. One could, for example use a bijection between  $\mathbb{R}$  and  $M^4$  to label each point in Minkowski space by a real number, in a highly discontinuous way. [18] Even then, to exclude the possibility that Poincaré invariance is effectively broken by the labelling, one would need to show that there is no natural definition of coordinates implied by such a bijection. In any case, a question would still remain: does DHS sprinkling give a probability distribution on discrete subsets of  $M^4$  defined with respect to some labelling?

We can put these questions another way. Suppose that  $S \in \Omega$  is an outcome of a sprinkling process in  $M^4$ , defined with respect to a given origin and frame. Let  $L_0$  be the subgroup of the Lorentz group that preserves the causal ordering, and let  $P_0$  be the subgroup of the Poincaré group generated by  $L_0$  and space-time translations. Suppose that  $\Pi \in P_0$  is a Poincaré transformation with  $S \neq \Pi S = S'$ . [19] Then do we treat  $S$  and  $S'$  as two distinct possible outcomes of the sprinkling process,

because their points have different coordinates or labellings? Or do we treat them as identical, representing a single outcome, because they are isometric and have identical causal structures?

We need a separate terminology for this second option. Define a *sprinklable set*  $\tilde{S}$  to be a countably infinite set of points  $\{P_i\}$  on which a distance function  $d(P_i, P_j) \in \mathbb{R}$  is defined and a causal relation  $\prec$  is also defined, such that there exists a *causal isometry* – an isometry that also preserves the causal relation – between  $\tilde{S}$  and some sprinkling  $S$  in  $M^4$ .

### Sprinklable sets and sprinklable causal sets

A sprinklable set thus also defines a causal set, by retaining the causal ordering  $\prec$  but ignoring the quantitative distance function  $d$ . We could define a *sprinklable causal set*  $C$  to be a countably infinite set of points  $\{P_i\}$  on which a causal relation  $\prec$  is defined such that there exists a *causal map* – a map that preserves the causal relation – between  $C$  and some sprinkling  $S$  in  $M^4$ . This may be the more fundamentally relevant definition for standard causal set theory, which focusses on the causal ordering between points and does not assume a distance function. However, we are interested in general approaches to discretizing Minkowski space, which might also include a distance function between the discretized points. Since the distance function is defined on sprinklings, we will keep open the possibility that it might be physically significant, and focus on sprinklable sets rather than sprinklable causal sets. Every sprinklable set defines a unique sprinklable causal set, but not vice versa.[20]

We note below that the natural probability measure on sprinklable sets assigns only probability values 0 and 1. We argue that this is problematic, since it makes the hypothesis of discrete Poincaré invariance only very weakly testable. This issue would also arise if we took sprinklable causal sets to be the fundamental objects. The properties of the natural  $\sigma$ -algebra and probability measure on sprinklable causal sets would also need separate analysis. We thus focus on sprinklable sets from now on.

### Sprinklable sets, sprinklings and Poincaré invariance

If there is a causal isometry  $\phi : \tilde{S} \rightarrow S$  from a sprinklable set  $\tilde{S}$  to a sprinkling  $S$ , and  $\Lambda \in L_0$ , then  $\Lambda \circ \phi : \tilde{S} \rightarrow S'$  defines a causal isometry between  $\tilde{S}$  and the sprinkling  $S' = \Lambda S$ . The definition of sprinklable set is thus independent of any frame choice for  $M^4$ . Similarly,  $\tau \circ \phi : \tilde{S} \rightarrow S''$  defines a causal isometry between  $\tilde{S}$  and the sprinkling  $S'' = \tau S$ , where  $\tau$  is a space-time translation.

Treating sprinklings as the fundamental objects seems to undercut the case for sprinkling as a Poincaré invariant construction of discrete sets that could substitute for continuous space-time manifolds as a fundamental arena in which physics takes place. If sprinklings have to be understood as sets of points embedded in  $M^4$ , then our fundamental description of physics still involves a continuous space-time. If they also need to be defined with respect to fixed coordinates or a labelling, in order to distinguish causally isometric sprinklings, then our fundamental description of physics also involves either a preferred frame or a labelling. Even if these are undetectable by experiment, they remain present in the definition of the sample space  $\Omega$ .[21]

Treating sprinklable sets as the fundamental objects in a physical theory may thus look more promising. However, to define a theory based on randomly chosen sprinklable sets we need to define a probability space of these sets without referring to any preferred frame or origin. This requires us to define a  $\sigma$ -algebra and probability measure on the sample space of sprinklable sets, without using the frame-dependent definitions for  $\Sigma$  and  $\mu$ .

A natural choice seems to be to construct definitions that can be inferred from, but defined independently of, those of  $\Sigma$  and  $\mu$ , by considering an sprinklable set as a coset of the Poincaré group acting on  $\Omega$ . We will sketch such definitions below. They give us  $\sigma$ -algebra elements of measure zero – for example singleton sets  $\Sigma_S = \{\tilde{S}\}$  containing individual sprinklable sets  $\tilde{S}$ . They also give us  $\sigma$ -algebra elements of measure one – for example the set  $\Sigma_{x,y}$  of all sprinklable sets containing at least one pair of points  $P_i, P_j$  such that  $x < d(P_i, P_j) < y$  for any given pair of real numbers  $x < y$ . However, the  $\sigma$ -algebra contains no elements with measures between 0 and 1. We argue below that this leads to problems in understanding the physical implications of theories in which sprinklable sets are fundamental objects and in particular in testing such theories. First, we reconsider the BHS theorem and note a lacuna.

### A LACUNA IN THE BHS THEOREM

In Ref. [2], isometric sprinklings  $S$  and  $S' = \Lambda S$ , for nontrivial  $\Lambda \in L_0$ , are treated as distinct. This means that an algorithm for associating timelike directions to sprinklings should define an equivariant map  $D : \Omega \rightarrow H$  from the set of sprinklings to the hyperboloid  $H$  of unit time-like vectors in  $M^4$ , and it is argued that no measurable equivariant maps exist.

If we take sprinklable sets, rather than sprinklings, to be the fundamental physical objects, then this argument needs reconsidering. The  $\sigma$ -algebra and probability measure on sprinklable sets allow us to infer that, with probability one, a sprinklable set is

represented by sprinklings that have no Lorentz (or Poincaré) automorphism. Consider a hypothetical algorithm that is defined on sprinklable sets with no Poincaré automorphism and that with probability one produces a preferred timelike direction. This must be defined in terms of intrinsic properties of the sprinklable set: the positive or negative distances between its points and their causal relationships.

(An example of an algorithm defined in terms of intrinsic properties would be to choose the pair of points  $(P_1 \prec P_2)$  with smallest positive timelike separation, take  $P_1$  the origin, and take the vector  $P_1 P_2$  to define the preferred timelike direction. However, the probability of a sprinklable set having a pair of points with smallest positive timelike separation is zero, so this algorithm is almost never well defined.)

Such an algorithm would also define an algorithm that associates a preferred timelike direction to a generic sprinkling, since we can consider the sprinkling as a sprinklable set if we ignore its specific embedding in  $M^4$ . The BHS theorem shows that no such algorithm exists, and hence that no probability one algorithm for associating preferred timelike directions to sprinklable sets can exist. Any algorithm defined on a measurable set of sprinklable sets thus can only be defined on a set of measure zero, since (as we discuss below) the  $\sigma$ -algebra contains only sets of measure one and zero.

One might, though, query whether it is so reasonable to restrict attention to algorithms defined on measurable sets of sprinklable sets, precisely because the  $\sigma$ -algebra is so relatively sparse. In any case, there is another issue with the BHS theorem: it proves less than required. Showing that there is no algorithm defining a measurable map from sprinklings to timelike directions is necessary but not sufficient to establish that a typical sprinkling is effectively Lorentz invariant in all the physically relevant senses.

### Partial breaking of Lorentz invariance in sprinklings

Recall again BHS's comparison of sprinklings in Euclidean and Minkowski space. As BHS note, *given a fixed point  $P$* , there is a mathematically well-defined construction of a preferred direction from a Euclidean sprinkling, given by taking the nearest sprinkling point to  $P$ . It is important to be clear about the logic here: given some data, which would not break rotational invariance in the continuous manifold, there is a mathematical sense in which rotational invariance is broken in the discrete approximation. The point of BHS's theorem is to show that this does not happen in the Minkowski case: given a fixed point  $P$ , BHS argue, there is still not a mathematical construction that breaks Lorentz invariance.

But more is needed. One needs to show that, given any data that leave some continuous subgroup of the Lorentz group as a symmetry in the continuous case, there is no mathematical construction that breaks this symmetry in the discrete case. To see this is not the case, suppose we are given two timelike separated points,  $P \prec Q$ . In the continuous case, this breaks translation invariance, and partially breaks Lorentz invariance, but leaves invariance under the spatial rotation subgroup. In the discrete case, however, it allows mathematical constructions that break the spatial rotation invariance. For example, suppose the two given timelike separated points,  $P \prec Q$ , belong to a sprinkling  $S$ . Let the line  $PQ$  define the axis for a time coordinate. Now identify the point  $X \in S$  that has time coordinate between those of  $P$  and  $Q$  and attains the minimum spatial separation from the line  $PQ$  (measured at equal times) among all points in  $S$ . That is, if we denote the time coordinate by  $x_0$ , then  $x_0(P) < x_0(X) < x_0(Q)$ , and if  $X'(X)$  is the point on  $PQ$  with  $x_0(X') = x_0(X)$ , then  $X$  is chosen to minimize  $d(X, X'(X))$  over all  $X \in S$ . This defines  $X \in S$  uniquely except for a measure zero subset of the sprinklings.[22] We can (for all but a measure zero subset of sprinklings) then associate the spatial direction  $X'X$  to the combination  $(S, P, Q)$  — i.e. to the sprinkling together with two given timelike separated points.

Another way of constructing a set of preferred directions from timelike separated points  $P \prec Q$  is to consider the longest chain  $P \prec X_1 \prec \dots \prec X_n \prec Q$  in the set, assuming there is a nontrivial chain, and with some tiebreaking conditions[23] if more than one chain attains the maximal length. This defines preferred timelike vectors  $PX_1, X_1X_2, \dots, X_nQ$ , and a variety of constructions can be used to define preferred spacelike vectors from these.

### Are these constructions physically relevant?

Whether these or other similar constructions might be physically significant in a fundamental theory is an interesting question, whose answer presumably depends on precisely which types of fundamental theory are considered. For example, it seems a priori conceivable that a dynamical theory on sprinklings might imply that particles propagating from  $P$  to  $Q$  would cause observable anisotropic effects associated with preferred directions selected by one of the rules above, thus giving empirical evidence of the violation of local Lorentz invariance. (Recall that DHS's criterion requires that the underlying discreteness in a physical theory whose approximating continuum is a Lorentzian manifold  $M$  should not suffice to distinguish even a *local* Lorentz frame at any point of  $M$ .) On the other hand, it might also be possible to characterise interesting classes of theory for which BHS's analogy with Euclidean sprinkling models of a gas holds good, in that such locally defined directions exist but can be shown to have no

global significance. Such a result would be very interesting, albeit weaker than BHS and DHS's claim that Lorentz invariance in *all* physically meaningful senses follows from the sprinkling construction.

Whatever the physical status of the constructions, we believe the logical point is clear. The BHS theorem is intended to give a mathematical proof that Lorentz invariance cannot be broken, as distinct from a physical argument that Lorentz invariance breaking is implausible. As BHS note, it is not possible to prove a version of their theorem in Euclidean space, because a generic point in a Euclidean sprinkling has a nearest neighbour, and so a choice of origin generically allows a preferred direction to be defined. Although the constructions above are not as simple, they break Lorentz invariance in a roughly analogous way, and so block the path to rigorously proving full Lorentz invariance by any argument like that of BHS. While the BHS theorem applies to sprinklings, the same argument applies for sprinklable sets, since our constructions use only intrinsic properties.

## POISSON PROCESSES ON THE REAL LINE

To illustrate the properties of sprinklings, sprinklable sets and their probability distributions, it is helpful to consider a simpler example than sprinklings in Minkowski space.

### Poisson processes on the real numbers

We first consider the real numbers  $\mathbb{R}$  with their standard structure: a preferred origin (0), a preferred direction (positive) and a metric ( $|x - y|$ ) that together allow us to define a signed distance function ( $d(x, y) = y - x$ ). This is the analogue of  $M^4$  with preferred coordinates  $(x, y, z, t)$ , a preferred orientation (distinguishing past and future timelike vectors), and the pseudo-Riemannian metric. (Note though that, while the pseudo-Riemannian metric does define positive and negative distances, it is the analogue of the metric  $|x - y|$ , not of the signed distance function  $d(x, y)$ . The latter has no good analogue in Minkowski space.)

We define sprinkling on  $\mathbb{R}$  as a random generalised Poisson process that selects a countable set  $S$  of points, with expected separation  $D$  between neighbouring points. One way to define the probability distribution on sprinklings  $S$  is as follows. First, we take a sprinkling to include all the points  $x_1 < x_2 < \dots$  generated by a Poisson process with mean  $D$  defined on  $\mathbb{R}^+$ . Then we define a second Poisson process with mean  $D$  starting at the point  $x_1$  and extending in the negative direction, giving us points  $\dots < x_{-1} < x_0 < x_1$ . Our sample space of sprinklings is then  $\Omega = \{\{x_i\}_{i \in \mathbb{Z}}\}$ , the set of ordered subsets of  $\mathbb{R}$  labelled by the integers.

For  $n \in \mathbb{N}$ , let  $F_n$  be the set of subsets of  $\Omega$  of the form

$$\{x_i \in [a_i, b_i] : -n \leq i \leq n\}, \quad (3)$$

where  $a_i < b_i$  for each  $i$ . Let  $F_\infty = \cup_{i \in \mathbb{N}} F_i$  be the collection of subsets of  $\Omega$  that can be defined by some finite number of statements about the location of points  $x_i$  in finite intervals, and let  $\mathcal{F} = \sigma(F_\infty)$  be the  $\sigma$ -algebra generated by  $F_\infty$ . Define the function  $\mu_0$  on  $F_\infty$  to be the probability that the generalized Poisson process assigns the relevant points to the relevant intervals. This extends to a probability measure  $\mu$  on  $(\Omega, \mathcal{F})$ .

We can treat this as a toy model of a physical universe, giving us a toy theory  $T_0$  that makes non-trivial probabilistic predictions. For example, given any point of a sprinkling  $S$  with coordinate  $x$ , the probability density for the next point in the positive direction having coordinate  $x + y$  is  $(\frac{1}{D}) \exp(-y/D)$ .

Given finite sets of data about sprinkling points, we can test theory  $T_0$  against others. For example, consider the deterministic theory  $T_1$  that predicts that points will be found precisely at the locations  $D\mathbb{Z}$ , and the theory  $T_2$  which predicts that the separations between points are uniformly distributed on the interval  $[0, 2D]$ . Suppose that we are somehow presented with a window onto the toy universe which exposes the first six points with positive coordinates,  $\{x_1, \dots, x_6\}$ , and gives us these coordinates to infinite precision. The theory  $T_1$  is excluded unless  $x_i = Di$  for  $i = 1, \dots, 6$ . On the other hand, if this condition does hold,  $T_1$  is effectively confirmed compared to the other theories, which assign probability zero to this precise configuration.

The relative probabilities of  $T_0$  and  $T_2$  are given respectively by

$$\left(\frac{1}{D}\right)^6 \exp(-x_1/D) \prod_{i=2}^6 \exp(-(x_i - x_{i-1})/D) \quad (4)$$

and

$$\left(\frac{1}{2D}\right)^6 \theta(2D - x_1) \prod_{i=2}^6 \theta(2D - x_i + x_{i-1}). \quad (5)$$

If the second of these is zero,  $T_2$  is also excluded. Otherwise, if we had non-dogmatic Bayesian priors for  $T_0$  and  $T_2$  before seeing the data, these are rescaled by the relative probabilities, but remain non-dogmatic.

We could, of course, carry out similar calculations if we were given intervals for the  $x_i$  rather than precise coordinates. For example, if  $x_i \in [Di - \epsilon, Di + \epsilon]$  for  $i = 1, \dots, 6$ , where  $\epsilon \ll D$ , then  $T_1$  is strongly favoured compared to  $T_0$  and  $T_2$ , but neither of these are completely excluded.

So, if we, as local observers in this one-dimensional discrete toy universe, obtain evidence about the discretization in our local neighbourhood that makes the Poisson sprinkling hypothesis statistically improbable compared to others, we have *prima facie* justification for disfavouring it. As in any cosmological model, the inference could possibly be complicated by anthropic reasoning: if we had good reason to believe that observers tend to be located only in atypical regions of a Poisson sprinkling then we might readjust our inferred weights. But scientific inference works as well as one can hope in this type of toy model.

### Poisson processes on a line with no fixed origin

Now consider a translation invariant version of the previous Poisson sprinkling model in which  $\{x_i\}$  and  $\{x_i + x\}$  (for any real number  $x$ ) are identified as the same outcome. In other words, we have an action of the translation group  $\mathbb{R}$  given by  $t_x : \{x_i\} \rightarrow \{x_i + x\}$ , and all the sprinklings in each coset are identified as a single outcome. Our toy universe is now described by some countable ordered sequence of points on a one-dimensional Riemannian manifold which is isometrically isomorphic to  $\mathbb{R}$ , and has a preferred positive direction, but which has no fixed reference point. The sample space  $\Omega'$  consists of countably infinite unlabelled ordered sequences of points  $\underline{P}$ , between which a relative separation  $d(P, Q) \in \mathbb{R}$  is defined, reflecting the isometry with  $\mathbb{R}$  and the preferred positive direction. Thus we have  $d(P, Q) = -d(Q, P)$  and  $d(P, R) = d(P, Q) + d(Q, R)$ .

We now need a  $\sigma$ -algebra and probability measure defined on the cosets of Poisson sprinklings under the translation group. One approach is to define sets that have specified properties with respect to some (arbitrarily) chosen point  $P$  in the set. To define a notation to describe these properties we label the point  $P = P_0$ , and label other points by their proximity to  $P$ ; thus  $P_{\pm 1}$  are the closest points to  $P$  in the positive and negative directions, and so on.

For example, define  $F'_n$  to be the set of subsets of  $\Omega'$  that can be defined by the property that they contain a point  $P = P_0$  whose nearest  $n$  neighbouring points on either side,  $P_{-n} < \dots < P_0 < \dots < P_n$ , have separations lying in specified finite intervals. Thus an element of  $F'_n$  takes the form

$$F(a_{-n}, b_{-n}; \dots; a_{n-1}, b_{n-1}) = \{ \underline{P} : \exists P = P_0 \text{ such that } d(P_{-n}, P_{-n+1}) \in [a_{-n}, b_{-n}], \dots, d(P_{n-1}, P_n) \in [a_{n-1}, b_{n-1}] \}, \quad (6)$$

where  $0 \leq a_i < b_i$  for each  $i$ . Define

$$F'_\infty = \bigcup_{i \in \mathbb{N}} F'_i, \quad (7)$$

and let  $F' = \sigma(F'_\infty)$  be the  $\sigma$ -algebra generated by  $F'_\infty$ . Now the probability measure  $\mu'$  on  $(\Omega', F')$  has

$$\mu'(F(a_{-n}, b_{-n}; \dots; a_{n-1}, b_{n-1})) = 1, \quad (8)$$

since almost every sequence of points contains some finite subsequence of neighbouring points whose separations lie in any given sets of finite intervals.

We may enlarge the definition of  $F'$ , so as to include sets with well-defined asymptotic properties. For example, we may include sets

$$B_p = \{ \omega \in \Omega' : \exists P = P_0 \text{ such that } \lim_{n \rightarrow \infty} \frac{d(P_{-n}, P_n)}{2n} = \delta \}. \quad (9)$$

If this asymptotic property holds for one point  $P = P_0$  in the set then it holds for every point, so we could also write

$$B_p = \{ \omega \in \Omega' : \forall P \in \omega \text{ if } P = P_0 \text{ then } \lim_{n \rightarrow \infty} \frac{d(P_{-n}, P_n)}{2n} = \delta \}. \quad (10)$$

The probability measure has  $\mu'(B_p) = 0$  for  $\delta \neq D$  and  $\mu'(B_p) = 1$  for  $\delta = D$ .

We can also include sets with more complicated limiting properties, so that for example we can justify the statement that

If a sequence of points  $\underline{P}$  is partitioned into length  $(2n + 1)$  subsequences in any of the  $(2n + 1)$  possible ways, then, with probability one, the asymptotic proportion of subsequences  $X_1 \dots X_{2n+1}$  with separations  $d(X_1, X_2) \in [a_1, b_1], \dots, d(X_{2n}, X_{2n+1}) \in [a_{2n}, b_{2n}]$  is

$$\prod_{i=1}^{2n} (\exp(-a_i/D) - \exp(-b_i/D)). \quad (11)$$

These sets all have probability measure 0 or 1; it follows that their countable unions and intersections also have measure 0 or 1. Since no generating element of the translation invariant  $\sigma$ -algebra has non-trivial probability, it seems intuitively clear that the translation invariant probability measure assigns only trivial probabilities to all elements; Ref. [5] gives a formal proof.[24]

Suppose we are now somehow presented with a window onto this translation invariant toy universe, which exposes a length  $(2n + 1)$  subsequence of neighbouring points to us, without assigning any coordinate labels to the points. One would like to be able to say that there is probability  $\prod_{i=1}^{2n} (\exp(-a_i/D) - \exp(-b_i/D))$  that its separations obey  $d(X_1, X_2) \in [a_1, b_1], \dots, d(X_{2n}, X_{2n+1}) \in [a_{2n}, b_{2n}]$ . However, we now have an explanatory gap. The translation-invariant  $\sigma$ -algebra indeed contains a set of sequences defined by the property that they contain such a subsequence, but it has probability one. More generally, since the probability measure assigns only trivial probability values to sets in the  $\sigma$ -algebra, we cannot infer any non-trivial probability value about any proposition.

In comparing toy model theories, all we can thus do is exclude theories that assign probability zero to observed data. Consider a translation invariant version of our previous example, in which our window gives us seven neighbouring points  $P_0, P_1, \dots, P_6$ , with  $d(P_0, P_i) \in [Di - \epsilon, Di + \epsilon]$  for  $i = 1, \dots, 6$ , where  $\epsilon \ll D$ . Each of the theories  $T_0, T_1$  and  $T_2$ , in their translation invariant form, predicts that such sequences will arise with probability one. None is favoured over the others by our observation.

### ARE RELATIVE FREQUENCIES RELEVANT?

As we noted, although the translation invariant probability measure does not assign non-trivial probabilities to any event, it does assign probability one (or zero) to events defined by relative frequencies. If a sequence of points  $\underline{P}$  is partitioned into length  $(2n + 1)$  subsequences in any of the  $(2n + 1)$  possible ways, then, with probability one, the asymptotic proportion of subsequences  $X_1 \dots X_{2n+1}$  with separations  $d(X_1, X_2) \in [a_1, b_1], \dots, d(X_{2n}, X_{2n+1}) \in [a_{2n}, b_{2n}]$  is given by Eqn. (11).

There are certainly suggestions in the physics literature that showing with probability one that an event has relative frequency  $p$  in an infinite sequence implies that the individual events have probability  $p$ . Or, at least, that careful analysis of the statement “individual events have probability  $p$ ” leads to the conclusion that it means no more than that the relative frequency is (or would be)  $p$  in an infinite sequence. Discussions of many-worlds theories includes arguments along these lines by Hartle [6], Coleman [7] and Aguirre-Tegmark [8], among others.

These suggestions tend to be aligned with frequentist views of probability, which some find persuasive but which also have well known problems: a good summary of arguments and critiques can be found in Ref. [9]. Among the points we think worth highlighting are that in general relative frequencies do not respect countable additivity; moreover, the sets on which they are defined are not closed under countable union, nor under finite intersection [10–14]. Relative frequencies of Poisson processes also illustrate a version of Reichenbach’s machine-gun example and its challenge to frequentists. [10, 14]. To see this, consider an instance of a Poisson process on a line with no fixed origin, and then consider the set  $US$  of unrealised separations between neighbouring points in this instance, i.e. the positive real line minus the countable set  $\{\dots, d(X_1, X_2), d(X_2, X_3), \dots\}$ . By definition, the event that a separation between neighbouring points lies in  $US$  has relative frequency zero in the realised instance: none of the separations lies in  $US$ . However, the a priori probability that a separation will lie in  $US$ , according to the model, is one, since almost all real numbers belong to  $US$ . This and the other issues highlighted apply equally well, of course, to the probabilistic model of sprinklable sets described above.

Whatever view one takes on the problems of frequentism and probability in general, the fundamental issue here is simple. If we define the relevant Poisson processes for sprinklable sets in the standard way, in terms of a sample space,  $\sigma$ -algebra and measure, then we cannot derive non-trivial probabilistic statements, since the probability measure only allows trivial values. Some frequentists (among others) might take the view that *all* well-defined probabilistic statements ultimately refer to probabilities zero or one: that statements with intermediate probability values mean precisely that a relevant relative frequency takes a value with probability one. Even if defensible, this stance is not very helpful if our aim is to justify testing and relative confirmation of a sprinklable set model on the basis of finite data. If discrete space-times should properly be modelled by sprinklable sets, but theories of this type cannot be tested by any finite set of observations, it is cold comfort that models of discrete space-times as sprinklings cannot be finitely tested either.

## SUMMARY: PROBABILITY AND SPRINKLABLE SETS

Does there exist a well-defined probabilistic theory in which the fundamental physical objects are sprinklable sets with a particular dimensionality (for example,  $3 + 1$ )? An affirmative answer requires a rigorous definition of the probability space of sprinklable sets. So far as we are aware, no such definition has yet appeared in the causal set literature. However, the  $\sigma$ -algebra and measure on Poincaré orbits of sprinklings appear natural candidates.

A Poincaré invariant sigma algebra is a necessary precondition for a model of discrete space-times that might reasonably be said to respect Poincaré invariance, but it is not sufficient. To take an extreme example: the sigma-algebra

$$G = \{\emptyset, \Omega\}, \quad (12)$$

where  $\Omega$  is the set of all countable subsets of  $M^4$ , has a Poincaré invariant definition: both  $\emptyset$  and  $\Omega$  are Poincaré invariant sets. The measure  $\mu$  defined by  $\mu(\emptyset) = 0$  and  $\mu(\Omega) = 1$  gives us a well-defined probability space. But no one should claim that this defines a satisfactory physical model or that it gives good reason to be optimistic about the existence of discrete Poincaré invariance. The sigma algebra needs to have enough structure to allow us to derive whatever consequences are supposed to follow from the theory. These certainly should include the lack of an observable preferred frame.

One concern here is that, once we move from treating sprinklings as fundamental physical objects to considering sprinklable sets as fundamental, we lose the possibility of *proving* typical large-scale properties of the sample set from the properties of local probabilistic processes. Instead, we need to *postulate* the large-scale properties, and all of our postulates involve sets of measure zero or one. Essentially, we postulate that a large-scale property will almost always or almost never hold, by choosing to include in the sigma-algebra either the set of sprinklable sets that satisfy the property, or its complement. So long as we respect the closure axioms for the sigma algebra, we thus seem free to choose whether or not to adopt physically relevant postulates, such as those describing the asymptotic density of the sprinklable set. It might be argued that this is more of an aesthetic concern than a logical one, since all physical theories are based on some postulates. Still, we should at least be clear whether a proposed discrete theory has particular properties simply because we choose to impose them by fiat, or whether they are provable consequences of some simpler underlying structure.

A stronger concern also arises from the fact that the natural probability measure on sprinklable sets takes only values one and zero. It seems to follow from this that our credence in a theory can be altered only if observations produce data that the theory assigns probability zero, in which case our credence also becomes zero. If so, no apparent breakdown of Poincaré invariance (however dramatic) in any finite region (however large) of space-time gives any evidence against the hypothesis that the full sprinklable set derives from a Poincaré invariant model. It seems that proponents of a fundamentally indeterministic theory of sprinklable sets either have to accept that the hypothesis of discrete Poincaré invariance is effectively untestable, or perhaps try to justify a role for non-trivial probabilities in a theory of the matter distribution on sprinklable sets.

One possible fallback position is that the ultimate aim is to find a discrete cosmological theory of matter and gravity, which may include physically preferred space-time points (such as points modelling the initial singularity) or frames (such as the cosmological centre-of-mass frame). Whether any such theory can retain local Lorentz or Poincaré invariance in a physically meaningful and fundamentally significant sense is not obvious, though.

We also noted a lacuna in the BHS argument, which means that there is as yet no rigorous proof that the typical sprinkling or sprinklable set is fully Lorentz invariant. As BHS note, their argument also fails for sprinklings in Euclidean space, and this is not generally seen as a disaster for claims that physical systems modelled by Euclidean sprinklings are effectively isotropic. Still, it removes some of the comfort that a rigorous theorem would provide.

Some issues formally similar to those considered here arise in discussions of many-worlds theories. In particular, Aguirre–Tegmark’s “cosmological interpretation of quantum mechanics” [8] relies on propositions about relative frequencies in an infinite collection of unindexed universes. These raise questions about the definition of the probability space and the jump from propositions about relative frequencies to statements about probabilities of individual events, which appear very similar to those considered here. However, since Aguirre–Tegmark’s cosmological models are not perfectly analogous to sprinklable set models, and since both cosmological and Everettian many-worlds theories raise other issues that need separate discussion, we leave this for future work.

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  - [16] For example, we can define a network from a sprinkling by joining every pair of timelike separated points. In such a network, every point almost surely has infinitely many links.
  - [17] Of course, one *could* still consider local coordinate choices in Minkowski space, but this complicates the discussion and offers no obvious advantage.
  - [18] I thank Rafael Sorkin for this suggestion.
  - [19] It is not essential to the discussion at this point, but note that we should expect this last condition to hold almost surely for any given nontrivial  $\Pi$ , given a well-defined probability distribution on sprinklings that justifies the intuition that they are locally random. Any such distribution should assign measure zero to the set of sprinklings that have a non-trivial Lorentz automorphism.
  - [20] For example, a dilation  $D\tilde{S}$  of the sprinklable set  $\tilde{S}$ , defined with distance function  $Dd(P_i, P_j)$  for some fixed  $D > 0$ , corresponds to the same sprinklable causal set as does  $\tilde{S}$ . That said, if the sprinkling comes from a Poisson process, then the set of sprinklable sets with any density other than the given sprinkling density is measure zero. One can also find other constructions of distinct sprinklable sets corresponding to the same sprinklable causal set. Consider for example a sprinklable set of the form  $\tilde{S} = S^+ \cup S^-$ , where  $S^\pm$  are disjoint sets with the property that  $s_- \prec s_+$  for every  $s_- \in S^-$  and  $s_+ \in S^+$ . We can add a uniform timelike distance  $T$  between  $S^-$  and  $S^+$ , by defining  $d'(s_-, s_+) = d(s_-, s_+) + T$ , for some constant  $T > 0$  and all  $s_- \in S^-$  and  $s_+ \in S^+$ , and set  $d'(s_-, s'_-) = d(s_-, s'_-)$  for all  $s_- \in S^-$  and  $s'_- \in S^-$  and  $d'(s_+, s'_+) = d(s_+, s'_+)$  for all  $s_+ \in S^+$  and  $s'_+ \in S^+$ . The resulting sprinklable set  $\tilde{S}'$  corresponds to the same sprinklable causal set as does  $\tilde{S}$ . Again, though, this construction involves sprinklable sets belonging to a measure zero set of sprinklings. It is interesting to ask whether, in some sense to be rigorized, a sprinklable causal set arising from a Poisson sprinkling process defines a unique sprinklable set with probability one. I thank Fay Dowker for raising this question.
  - [21] As John Bell commented in a related context, this would seem an eccentric way to make a world.[15]
  - [22] We could impose further tie-breaking conditions to define  $X$  on a still larger measure one set.
  - [23] For example, we could choose the chain for which  $|d(P, X_1)|$  is largest; if more than one chains attain the maximum, then choose the chain amongst these for which  $|d(X_1, X_2)|$  is largest, and so on.
  - [24] I thank Rafael Sorkin for locating this reference.