On nonrelativistic 3D Spin-1 theories*

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ABSTRACT

We describe non-relativistic limits of the 3D Proca and $\sqrt{\text{Proca}}$ theories that yield spin-1 Schroedinger equations. Analogous results are found by generalized null reduction of the 4D Maxwell or complex self-dual Maxwell equations. We briefly discuss the extension to spin-2.

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1 Introduction

Relativistic field theories of massless particles, such as Maxwell's electrodynamics or Einstein's General Relativity (GR), have non-relativistic limits in which these particles disappear, in accordance with the instantaneous action at a distance of electro/magneto-statics or Newtonian gravity. Put differently, they disappear because their velocity of propagation has become infinite. This suggests that relativistic theories of *massive* particles, which move subluminally, should survive a non-relativistic limit, but this is not generally true. For example, one can take the speed of light to infinity in the Klein-Gordon (KG) equation if the particle's Compton wavelength is kept fixed, but the result is a Yukawa/Laplace equation that does *not* propagate any disturbance.

However, if the KG scalar field is *complex* then a different non-relativistic limit is possible, and this yields a Schroedinger equation for a massive particle of zero spin. A similar limit is possible for relativistic tensor field equations describing massive particles of non-zero integer spin, such as the Proca equations for spin-1 or the Fierz-Pauli (FP) equations for spin-2, but again only if the tensor fiield is complex. However, the 3D case (i.e. field theories in a spacetime of $2+1$ dimensions) is an exception to this rule. As we have recently shown [1], the 3D FP equations have a novel non-relativistic limit to a planar spin-2 Schroedinger equation proposed previously in the context of the "gapped" spin-2 GMP mode of fractional Quantum Hall states[1].

The 3D case is also special in another respect: one can take the "square-root" of the Proca equation [3] and of the spin-2 FP equation [4]. These first-order equations, which propagate a single mode rather than a parity doublet, are equivalent to linearizations of parity-violating "topologically massive" gauge theories, such as "topologically massive gravity" in the spin-2 case [5]; a systematic derivation of such equivalences may be found in [6]. For such theories a non-relativistic limit to a planar Schroedinger equation requires a complex field. Here we describe this limit for complex $\sqrt{\text{Proca}}$ and show that it yields the same planar spin-1 Schroedinger equation that one finds from an application of the "novel non-relativistic" limit of [1] to real-field Proca. Moreover, we do this at the level of the actions, not just the equations.

It was also shown in [1] that a generalized null reduction of the linearized 4D Einstein equations leads to the same planar spin-2 Schroedinger equation as found from the non-relativistic limit of the 3D real-field FP equations. This is a further illustration of the long-established connection between the Galilei and Lorentz groups in d and $d+1$ dimensions [7], and may be compared with the derivation of 4D Newtonian gravity from 5D GR [8]. Here we consider the same generalized null reduction for the 4D Maxwell equations (real, complex or complex self-dual) in Bargmann-Wigner form. We find a complete correspondence with the 3D non-relativistic limit results.

In our conclusions, we extract some general lessons and discuss briefly the extension of our results to spin-2 and beyond. For simplicity, we set $\hbar = 1$ throughout.

[1] See [2] for a recent discussion with references to the condensed matter literature.

2 The non-relativistic limit for 3D spin-1

Non-relativistic limits are most simply investigated at the level of field equations but it is also of interest to know whether, and if so how, the limit can be taken in the action. As the spin-1 cases considered here are relatively simple, we shall consider the non-relativistic limits of the complex Proca and $\sqrt{\text{Proca}}$ theories, in a 3D Minkowski vacuum, at the level of the action. In both cases the Proca field is a 3-vector A_μ ($\mu = 0, 1, 2$) but an additional auxilary vector field will be needed for the $\sqrt{\text{Proca}}$ case.

2.1 Complex 3D Proca

The Lagrangian density for a complex 3D Proca field A_μ ($\mu = 0, 1, 2$) is

$$\mathcal{L} = -\frac{1}{4}\sqrt{-\det\eta}\,\eta^{\mu\rho}\eta^{\nu\sigma}F^*_{\mu\nu}F_{\rho\sigma} - \frac{1}{2}(mc)^2\sqrt{-\det\eta}\,\eta^{\mu\nu}A^*_\mu A_\nu\,, \qquad (1)$$

where c is the speed of light and $\eta_{\mu\nu} = \text{diag.}(-c^2, 1, 1)$ is the 3D Minkowski metric. After a time-space split $\mu = (0, i)$ ($i = 1, 2$),

$$c^{-1}\mathcal{L} = -\frac{1}{4}F^\star_{ij}F_{ij} + \frac{1}{2c^2}F^\star_{0i}F_{0i} + \frac{m^2}{2}A^\star_0 A_0 - \frac{(mc)^2}{2}A^\star_i A_i\,, \qquad (2)$$

where $F_{ij} = 2\partial_{[i}A_{j]}$ and $F_{0i} = \dot{A}_i - \partial_i A_0$, and there is an implied sum over repeated 2-space indices. We now define new complex fields (a_0, a_i) by setting

$$A_0 = e^{-imc^2t}a_0\,, \qquad A_i = e^{-imc^2t}a_i\,. \qquad (3)$$

After substitution into (2), one finds a cancellation of terms on the right hand side of (2) that diverge as $c \to \infty$. The subleading, and Galilean invariant, terms are

$$\mathcal{L}_{\text{non-rel}} = -\frac{1}{4}f^\star_{ij}f_{ij} + \frac{i}{2}ma^\star_i f_{0i} - \frac{i}{2}ma_i f^\star_{0i} + \frac{m^2}{2}a^\star_0 a_0\,, \qquad (4)$$

where $f_{ij} = 2\partial_{[i}a_{j]}$ and $f_{0i} = \dot{a}_i - \partial_i a_0$. The a_0 field is auxiliary and can be eliminated; omitting a total derivative, this yields

$$\mathcal{L}_{\text{non-rel}} = \frac{1}{2}a^*_i\nabla^2 a_i + \frac{i}{2}m\left(a^*_i\dot{a}_i - a_i\dot{a}^*_i\right)\,, \qquad \nabla^2 = \partial_1^2 + \partial_2^2\,. \qquad (5)$$

The field equation is

$$2mi\dot{a}_i + \nabla^2 a_i = 0\,, \qquad (6)$$

which is an $SO(2)$-rotation doublet of Schroedinger equations.

In terms of the following helicity eigenfunctions

$$\Psi[1] = a_1 + ia_2\,, \qquad \Psi[-1] = a^*_1 + ia^*_2\,, \qquad (7)$$

the equations (6) become

$$i\dot{\Psi}[\pm 1] = \mp\frac{1}{2m}\nabla^2\Psi[\pm 1]\,. \qquad (8)$$

The two equations are exchanged by parity, but we may impose the self-duality constraint

$$a_i = \mp i\epsilon_{ij} a_j \,, \tag{9}$$

which implies that $\Psi[\mp 1] = 0$, thereby leaving a single parity-violating planar spin-1 Schroedinger equation for $\Psi[\pm 1]$. Assuming that m is positive we must choose to retain the equation for $\Psi[1]$ in order to have a positive Hamiltonian.

2.2 Complex 3D $\sqrt{\text{Proca}}$

We now turn to the complex $\sqrt{\text{Proca}}$ theory. It turns out that to take the non-relativistic limit we must include a complex auxiliary vector field B_μ; the Lagrangian density is

$$\mathcal{L} = \varepsilon^{\mu\nu\rho} A^*_\mu \partial_\nu A_\rho - (mc)\sqrt{-\det \eta}\, \eta^{\mu\nu}(A^*_\mu A_\nu - 2B^*_\mu B_\nu) \tag{10}$$

or, after a time-space split,

$$\begin{aligned} \mathcal{L}_{\text{rel}} \;\; = \;\; & -\epsilon^{ij} A^\star_i \dot{A}_j - (mc^2)(A^*_i A_i - 2B^*_i B_i) \\ & + \epsilon^{ij}(A^\star_0 \partial_i A_j + c.c.) + m(A^*_0 A_0 - 2B^*_0 B_0) \,. \end{aligned} \tag{11}$$

We can eliminate the auxiliary fields A_0 and B_0 to get

$$\mathcal{L} = -\epsilon^{ij} A^\star_i \dot{A}_j - (mc^2)(A^*_i A_i - 2B^*_i B_i) - \frac{1}{m}\left|\epsilon^{ij}\partial_i A_j\right|^2 . \tag{12}$$

We now define new complex fields (a_i, b_i) by

$$A_i = e^{-imc^2 t} a_i \,, \qquad B_i = e^{-imc^2 t}\left([\mathcal{P}a]_i + \frac{1}{2mc^2} b_i\right) , \tag{13}$$

where $\mathcal{P}$ is a complex projector matrix:

$$\mathcal{P}_{ij} = \frac{1}{2}\left(\delta_{ij} - i\epsilon_{ij}\right) , \qquad \mathcal{P}^2 = \mathcal{P} \,, \quad \mathcal{P}\bar{\mathcal{P}} = \bar{\mathcal{P}}\mathcal{P} = 0 \,. \tag{14}$$

Substitution yields a Lagrangian in which the terms proportional to c^2 have cancelled. Taking the $c \to \infty$ limit, we arrive at

$$\mathcal{L}_{\text{non-rel}} = -\epsilon^{ij} a^*_i \dot{a}_j - \frac{1}{m}\left|\epsilon^{ij}\partial_i a_j\right|^2 + (b^*_i [Pa]_i + c.c.) \,. \tag{15}$$

The projection $\bar{\mathcal{P}}b$ of b no longer appears because it drops out in the $c \to \infty$ limit; the other projection of b is a Lagrange multiplier for the constraint $\mathcal{P}a = 0$, which is equivalent to

$$a_i = \left[\bar{\mathcal{P}}a\right]_i \,. \tag{16}$$

This implies that $\epsilon^{ij}\partial_i a_j = -i\partial_i a_i$, which can be used to show that

$$\left[\bar{\mathcal{P}}\partial\right]_i \left(\epsilon^{kl}\partial_k a_l\right)) = -\frac{i}{2}\nabla^2 a_i \,. \tag{17}$$

The field equation found from variation of a_i in (15) is

$$[\mathcal{P}b]_i = \epsilon^{ij}\dot{a}_j + m^{-1}\epsilon^{ij}\partial_j\left(\epsilon^{kl}\partial_k a_l\right) . \tag{18}$$

This determines $\mathcal{P}b$, which is therefore auxiliary, but it also implies that

$$m\left[\bar{\mathcal{P}}\dot{a}\right]_i + \left[\bar{\mathcal{P}}\partial\right]_i(\epsilon^{kl}\partial_k a_\ell) = 0\,. \tag{19}$$

Using (16) and (17), we may rewrite this as

$$2mi\dot{a}_i + \nabla^2 a_i = 0\,. \tag{20}$$

This is (6), but here a is constrained to satisfy $a_i = i\epsilon_{ij}a_j$; as we saw, this is equivalent to $\Psi[-1] = 0$, which leaves a single parity-violating Schroedinger equation for $\Psi[1]$, with positive Hamiltonian if m is positive. We saw previously that this truncation could be imposed 'by hand' but here it is the result of a field equation.

If the sign of m is changed then we can maintain the positivity of the Hamiltonian if we also replace $\mathcal{P}$ by $\bar{\mathcal{P}}$ in (13), but then we get a Schroedinger equation for $\Psi[-1]$ in place of $\Psi[1]$.

2.3 Real Proca

We are now going to see how the above result for the non-relativistic limit of $\sqrt{\text{Proca}}$ can also be found by taking a novel non-relativistic limit of *real* 3D Proca. To this end, we return to the Proca Lagrangian (1) but now for a *real* vector field $A_\mu = (A_0, \mathbf{A})$. A time-space decomposition then yields

$$\mathcal{L} = \frac{1}{2c^2}|\dot{\mathbf{A}}|^2 - \frac{1}{2}\mathbf{A}\cdot\Delta\mathbf{A} + \frac{1}{2}(\boldsymbol{\nabla}\cdot\mathbf{A})^2 + \frac{1}{2c^2}A_0\Delta A_0 + \frac{1}{c^2}A_0(\boldsymbol{\nabla}\cdot\dot{\mathbf{A}})\,, \tag{21}$$

where

$$\Delta = -\nabla^2 + (mc)^2 > 0\,. \tag{22}$$

Elimination of the auxiliary field A_0 yields the following Lagrangian

$$\mathcal{L} = \frac{1}{2c^2}\dot{A}_i K_{ij}\dot{A}_j - \frac{1}{2}A_i\Delta K_{ij}A_j \tag{23}$$

where A_i $(i = 1, 2)$ are the components of $\mathbf{A}$ and

$$K_{ij} = \delta_{ij} + \Delta^{-1}\partial_i\partial_j\,. \tag{24}$$

These are the entries of a matrix K, with inverse

$$K^{-1}_{ij} = \delta_{ij} - (mc)^{-2}\partial_i\partial_j\,. \tag{25}$$

Next, we set

$$A_i = [K^{-\frac{1}{2}}]_{ij}B_j \tag{26}$$

for a new 2-vector field $\mathbf{B}$. In terms of $\mathbf{B}$, the Lagrangian density is

$$\mathcal{L} = \frac{1}{2c^2}|\dot{\mathbf{B}}|^2 - \frac{1}{2}\mathbf{B}\cdot\Delta\mathbf{B}\,. \tag{27}$$

In terms of the complex field $B = (B_1 + iB_2)/\sqrt{2}$, this is

$$\mathcal{L} = \frac{1}{c^2}|\dot{B}|^2 + \bar{B}\nabla^2 B - (mc)^2|B|^2 \,. \tag{28}$$

If we now set

$$B = e^{-imc^2 t}\Psi[1] \,, \tag{29}$$

for new complex variable $\Psi[1]$, we may take the $c \to \infty$ limit to arrive at the Galilean invariant Lagrangian density

$$\mathcal{L}_{NR} = 2im\bar{\Psi}[1]\dot{\Psi}[1] + \bar{\Psi}[1]\nabla^2\Psi[1] \,. \tag{30}$$

The field equation is

$$-\frac{1}{2m}\nabla^2\Psi[1] = i\dot{\Psi}[1] \,, \tag{31}$$

which is the planar spin-1 Schroedinger equation.

3 Spin-1 Schroedinger from 4D Maxwell

We will now explain how all the spin-1 planar Schroedinger equations found above from non-relativistic limits of real or complex Proca, and complex $\sqrt{\text{Proca}}$, can also be found from a generalized null reduction of the real or complex 4D Maxwell equations, and the complex self-dual Maxwell equations. We work directly with equations, rather than the action, and we start from the Bargmann-Wigner (BW) form of Maxwell's equations for a complex symmetric $Sl(2;\mathbb{C})$ bi-spinor $F_{\alpha\beta}$ $(\alpha, \beta = 1, 2)$. In Fourier space, the BW equations are

$$p^{\alpha\dot{\alpha}}F_{\alpha\beta} = 0 \,, \tag{32}$$

where, in the $Sl(2;\mathbb{C})$ spinor conventions spelled out in [9],

$$p^{\alpha\dot{\alpha}} = \begin{pmatrix} -\sqrt{2}p_- & -p \\ -\bar{p} & \sqrt{2}p_+ \end{pmatrix} , \qquad p = p_1 + ip_2 \,. \tag{33}$$

Inspired by the Scherk-Schwarz dimensional reduction [10], we now effect a generalized null reduction by choosing[2]

$$p_- = m \tag{34}$$

for mass m. This choice is consistent with the fact that p_- is the Fourier-dual of the complex differential operator $-i\partial_-$ because $F_{\alpha\beta}$ is complex, and it gives us

$$p^{\alpha\dot{\alpha}} = \begin{pmatrix} -\sqrt{2}m & -p \\ -\bar{p} & -\sqrt{2}E \end{pmatrix} , \qquad E = -p_+ \,. \tag{35}$$

Using this in the BW equations, we have

$$\begin{pmatrix} -\sqrt{2}m & -p \\ -\bar{p} & -\sqrt{2}E \end{pmatrix} \begin{pmatrix} F_{1\beta} \\ F_{2\beta} \end{pmatrix} = 0 \,, \tag{36}$$

[2]The choice $m = 0$ corresponds to a standard null reduction, equivalent to supposing that all fields are independent of x^-, and does not lead to a Schroedinger equation.

which is equivalent to

$$\sqrt{2}\, mF_{1\beta} = -pF_{2\beta}\,, \qquad \left[2mE - |\mathbf{p}|^2\right] F_{2\beta} = 0\,. \tag{37}$$

In other words, $F_{1\beta}$ is auxiliary and $F_{2\beta}$ satisfies a Schroedinger equation. Moreover, because $F_{\alpha\beta} = F_{\beta\alpha}$ the same applies to $F_{\alpha 1}$ and $F_{\alpha 2}$, so the only component of $F_{\alpha\beta}$ that is not auxiliary is F_{22}, and this satisfies

$$\left[2mE - |\mathbf{p}|^2\right] F_{22} = 0\,, \tag{38}$$

which is a Schroedinger equation for a single complex wavefunction.

Taking the complex conjugate of this Schroedinger equation we find that

$$\left[2mE - |\mathbf{p}|^2\right] \bar{F}_{\dot{2}\dot{2}} = 0\,. \tag{39}$$

If we had started from the Maxwell equations for a *complex* vector potential then F_{22} and $\bar{F}_{\dot{2}\dot{2}}$ would be *independent* complex wavefunctions rather than complex conjugates of each other, and we would recover the parity-doublet of Schroedinger equations that we found from the non-relativistic limit of the complex 3D Proca equations. However, in this case it is consistent to impose $F_{\dot{\alpha}\dot{\beta}} = 0$, *without this implying* $F_{\alpha\beta} = 0$. This is equivalent to imposing a self-duality condition on the complex Maxwell field-strength 2-form. This effects the same truncation to the parity-violating spin-1 Schroedinger equation that we found from the non-relativistic limit of the complex $\sqrt{\text{Proca}}$ equations.

4 Conclusions

We have investigated the non-relativistic limits of free 3D field theories for spin-1 particles of non-zero mass m; specifically, 3D Proca and $\sqrt{\text{Proca}}$ and their complexifications. A common feature is that the initial relativistic 3D theory must propagate *pairs* of modes of equal mass, since two are needed for every complex wavefunction satisfying a Schroedinger equation in the limit. This condition is satisfied automatically for a complex vector field, and for the 3D Proca with real vector field but not for $\sqrt{\text{Proca}}$ with real vector field; in this last case the only non-relativistic limit is one to equations that do not propagate any non-relativistic particle.

Since the Proca theory preserves parity, one might expect its non-relativistic limit to also preserve parity and this is true if one starts from *complex* 3D Proca; its non-relativistic limit is a Schroedinger equation for two complex wavefunctions transforming under rotations as an $SO(2)$ doublet. This system of equations is equivalent to equations for a parity-doublet of complex helicity eigenstate wavefunctions $\Psi[\pm 1]$ with opposite sign Hamiltonians, and the non-relativistic limit of the parity-violating $\sqrt{\text{Proca}}$ theory yields just one of these Schroedinger equations, which one depending on the sign of m.

In general, there is no non-relativistic limit of the *real* Proca theory to a Schroedinger equation preserving rotational invariance but the 3D case is an exception to this rule. In this case there is such a limit, as we showed for the spin-2 FP equations in [1]. Here, for the spin-1 case, we have shown how this

novel non-relativistic limit may be taken in the action (and not merely for the field equations). Somewhat surprisingly, it leads to the same parity-violating single Schroedinger equation that one gets from complex $\sqrt{\text{Proca}}$.

We have further demonstrated a correspondence between these results and those obtained from a generalized null reduction [1] of 4D field theories for spin-1 particles of zero mass. Specifically, a generalized null reduction of the Bargmann-Wigner equations equivalent to complex 4D Maxwell, complex self-dual Maxwell and real 4D Maxwell leads to the same spin-1 planar Schroedinger equations as 4D complex Proca, complex $\sqrt{\text{Proca}}$ and real Proca, respectively. Furthermore, there is a straightforward generalization to any integer spin; for example, the linearized 4D Einstein equations are equivalent to the spin-2 Bargmann-Wigner equations

$$p^{\alpha\dot{\alpha}} R_{\alpha\beta\gamma\delta} = 0\,, \tag{40}$$

where $R_{\alpha\beta\gamma\delta}$ is totally symmetric in its four $Sl(2;\mathbb{C})$ indices. Following the example of the spin 1 case we deduce that only R_{2222} is independent, and that it satisfies

$$\left[2mE - |\mathbf{p}|^2\right] R_{2222} = 0\,. \tag{41}$$

This is equivalent to the generalized null-reduction of [1]. Complexifying the linearized Einstein equations leads to independent equations for R_{2222} and $\bar{R}_{\dot{2}\dot{2}\dot{2}\dot{2}}$, and the analog of Maxwell self-duality in this case is then $\bar{R}_{\dot{2}\dot{2}\dot{2}\dot{2}} = 0$.

Our results for non-relativistic limits of massive 3D spin-1 theories can also be generalized to any integer spin. For example, the $\sqrt{\text{FP}}$ action has a non-symmetric tensor field [4]. After the addition of an FP-type mass term for an auxiliary non-symmetric tensor field, redefinitions similar to those of (13) allow a non-relativistic limit to be taken. This yields the *same* parity-violating planar spin-2 Schroedinger equation as that found in [1] from a "novel non-relativistic limit" of the real spin-2 FP equations. However, as mentioned there, the need for a complex field in the $\sqrt{\text{FP}}$ case complicates the issue of interactions because a metric perturbation is naturally real. This problem does not arise for the real 3D FP equations because these result from linearization of the equations of "New Massive Gravity" [11].

Finally, we should mention that non-relativistic limits of the Jackiw-Nair equations for massive particles of any spin have been proposed in [12]; the limits considered there do not include the novel real-field limit of [1] that we have explained here for spin-1, but there is presumably an overlap with the non-relativistic limit of the complex $\sqrt{\text{Proca}}$ case considered here.

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