Lévy State-space Models for Tracking and Intent Prediction of Highly Manoeuvrable Objects

Runze Gan, Bashar I. Ahmad, *Member, IEEE*, Simon J. Godsill *Senior Member, IEEE*

Abstract—In this paper, we present a Bayesian framework for manoeuvring object tracking and intent prediction using novel α -stable Lévy state-space models, expressed in continuous time as Lévy processes. In contrast to conventional (fully) Gaussian formulations, the proposed models are driven by heavy-tailed α -stable noise and are thus much more able to capture extreme values/behaviours. This can better characterise sharp changes in the state, which may be induced by sudden and frequent manoeuvres such as swift turns or abrupt accelerations. In particular, they are represented in a conditionally Gaussian series form which ensures the tractability of the applied inference algorithms. A corresponding estimation strategy with the Rao-Blackwellised particle filter is then proposed and an efficient intent inference procedure is introduced. Here, the underlying intent, driving the target's long-term behaviour (e.g. reaching its final destination), is modelled as a latent variable. Real vessel data from maritime surveillance and human computer interactions (e.g. cursor data from motor-impaired interface users) are utilised to demonstrate the effectiveness of the proposed approach. It is shown to deliver noticeable improvements in the tracking and intent prediction performance (whenever relevant) compared with a more conventional Gaussian dynamic model.

Index Terms—Lévy process, α -stable laws, Poisson series representation, manoeuvre tracking, intent inference, Rao-Blackwellised particle filtering

I. INTRODUCTION

In the object tracking field, there has been a growing interest in inferring, as rapidly as possible, a target's intent (e.g. its final destination and future motion patterns) from the available sensor measurements [12, 26, 3]. This aim is motivated by numerous applications, such as surveillance, sensor management, automation and robotics. Intent prediction can be considered as a *meta-level* tracking task, whose objective is to acquire a higher-level understanding of a scene by revealing the underlying intent driving the target's long-term behaviour (e.g. to reach an endpoint). This is in contrast with conventional *sensor-level* trackers [7, 41], aimed at estimating solely the object temporal kinematics (e.g. position, velocity, heading, etc.).

In this paper, we address the problem of estimating both the kinematic state and intent (typically the final destination), of highly manoeuvrable objects. Whilst the unknown endpoint cannot be directly observed, stochastic models can be devised that capture the influence of intent on the evolution of the observed target state over time. Constructing representative dynamic models for fast manoeuvring objects is generally

challenging, especially if intent-driven, since a target trajectory can feature large accelerations and rapid rotational motions, for example by vehicles [46] for obstacle avoidance, drones [10], vessels [17], insects undertaking erratic movements [48, 47], cursor pointing motion of users with severe motor impairments [1], to list a few. Here, a novel modelling approach for manoeuvring targets that can incorporate both kinematics and intent is proposed. The methods are a novel extension of continuous-time Lévy state-space models [5, 20], driven by heavy-tailed α -stable Lévy processes [44, 40]. These can better characterise sudden dramatic changes in the state induced by swift target manoeuvres, compared with the more commonly used Gaussian-driven dynamic model.

If the dynamic noise is considered as the combination of a large number of independent identically distributed perturbations, then it converges under appropriate scaling to a member of the adopted class of α -stable distribution as per the generalized central limit theorem [16]. Therefore, the α -stable dynamic noise, with parameterised skewness and heavy-tailedness, in the proposed Lévy state-space model offers a generic and flexible formulation that encompasses a wide range of possible system noise models as special cases, including the conventional Gaussian noise models [9, 15]. In the continuous-time setup, such a noise source can be treated as the time increments of an α -stable Lévy process, sharing several useful properties (e.g. self-similarity and infinite divisibility) with the more prevalent Brownian motion. Consequently, the α -stable process is studied in several areas to model extremal events or abrupt changes such as in telecommunications, econometrics and signal processing [6, 19, 43].

A. Contributions

The main contribution of this paper is introducing for the first time a stable Lévy modelling approach for multidimensional spatial tracking and intent prediction, which we propose to infer within a Bayesian framework. Detailed derivations of this novel generic formulation, with parameterised α -stable dynamic noise, are provided. We also show how it can be applied to estimate the kinematic state and/or destination of a manoeuvring target, presenting several different example models and inference strategies. The proposed stable Lévy model is devised here as a conditionally Gaussian infinite series [20, 29] that can be adapted to include the unknown intent of the tracked object via a mean reverting term. This conditionally Gaussian series structure is exact, but in practical applications requires the truncation of the infinite series and suitable approximation of the residual error terms. Owing to the employed conditionally Gaussian structure, both of the

R. Gan, B. I. Ahmad, and S. J. Godsill are with the Engineering Department, University of Cambridge, Cambridge CB2 1PZ, U.K. e-mails: {rg605, bia23, sjg30}@cam.ac.uk

B. I. Ahmad is with Aveillant, Thales Land and Air Systems, Evolution Business Park, Cambridge CB24 9NG, e-mail: bashar.ahmad@aveillant.com

manoeuvring target kinematic state and intent can be inferred with an efficient Rao-Blackwellised particle filter [11].

Additionally, we propose a latent destination model, where an extended latent state that incorporates destination is constructed. The intent prediction task with such a model then becomes a by-product of the state estimation (tracking) algorithm, thereby requiring minimal additional computations. It also allows a dynamically changing or static intended endpoint, which can be anywhere within the surveyed area and does not have to be confined to a predefined finite set of possible destinations, unlike prior work, e.g. in [3, 31, 14].

Results from real pointing and maritime surveillance data are presented to demonstrate that the competitive performance of the proposed modelling and inference approach, in terms of determining the kinematic state and final destination (when relevant) of a target whose trajectory can exhibit sharp manoeuvres. The utilised pointing data pertains to users with severe motor-impairment (namely cerebral palsy) pointing in 2-D on an interface with a mouse cursor and participants in a moving vehicle undertaking freehand pointing gestures in 3-D to select icons on the in-car display whilst experiencing severe perturbations due to the road/driving conditions (i.e. vibrations and accelerations). It is emphasised that the proposed stable Lévy models are also readily applicable for manoeuvring object tracking, i.e. without intent prediction, as in the reported maritime vessel tracking example below.

B. Related Work

Whilst the widely adopted linear Gaussian state-space model with additive Gaussian noise [30] for sensor or meta-level tracking produces simple closed form inference methods [45, 34, 3], they cannot accurately characterise abrupt changes in the target (kinematic) state caused by sudden manoeuvres. The Gaussian premise stipulates that such drastic changes are very unlikely, producing large estimation errors when they occur. More sophisticated models, which include a jump process in addition to the original Gaussian process such as the jump diffusion models or Markov/semi-Markov jump models, show improved inference results for manoeuvring targets [32, 14]. In contrast, we utilise in this paper an α -stable Lévy process as the driving noise, which consists of fewer parameters and leads to competitive inference performance. Other manoeuvring models, e.g. the constant turn[7] and intrinsic coordinates [17] models, rely on the object's speed and/or heading information. Whilst these are more fitting for curvilinear motion modelling, the model introduced in this paper is formulated in Cartesian coordinates with the aim of straightforward application to diverse tracking scenarios.

Despite their ability to capture highly manoeuvring behaviour through extreme changes in latent state values, stable Lévy models are usually highly intractable for inference due to the lack of a closed-form density function for the increments of the process. This has resulted in their limited use in the object tracking field, though see [21] for a discrete time tracking model that uses stable law innovations to achieve modelling of fast manoeuvres. To overcome this, a conditionally Gaussian Poisson series representation was introduced in [18, 28, 38, 27]

to represent the α -stable random variable and its stochastic integral, which can form the basis of a tractable inference framework for the stable Lévy process family [27, 29]. Sharp rates of convergence for these approaches have been recently proposed, see [37] and references therein. A more general Lévy state-space model was detailed in [20], however not as a spatial process applicable to object tracking. Here, we propose for the first time multidimensional Lévy state-space models in the conditionally Gaussian series form for spatial tracking and intent prediction, coupled with an efficient sequential Bayesian inference procedure based on Rao-Blackwellised particle filtering. On account of the typically isotropic properties in most tracking applications, we focus particularly on the symmetric stable Lévy process. Although skewed dynamic models may be appropriate in certain scenarios and can be included in our framework as in [20], we leave this as a topic for future work.

The modelling and inference approach proposed here builds on and extends our preliminary work in [13]. We include the detailed derivations of the proposed models and provide additional examples for both tracking and intent inference tasks. A different and novel inference procedure for the simultaneous estimation of the latent kinematic state and intent is further presented in this paper. Additional performance analysis, including using real data from two new real scenarios, is also conducted.

As a further motivation for our work, there is a growing interest in the application of non-Gaussian Lévy processes for animal behaviour modelling such as the 'Lévy flight' and 'Lévy walk' models [25]. In these approaches the searching or foraging patterns of various species (e.g. fruit fly, albatross, bumblebee, etc.) can be approximated as a heavy-tailed Lévy process, where occasional long distance steps in foraging animal trajectories correspond to the extreme noise in the Lévy process, see [35, 8, 24]. Nonetheless, this is often applied only informally for (hypothesis) validation purposes and not within a rigorous tracking formulation. For example, the 'Lévy flight' and 'Lévy walk' models in the biology field include only spatial position as a random variable, and this is usually time-discretised for ease of implementation. This renders them inappropriate for tracking with frequent (asynchronous) observations, in which the estimation of higher order kinematics such as velocity can provide valuable information, enhance accuracy, and potentially lead to deeper understanding of animal dynamics and the hypotheses that may apply. We note that our proposed modelling and inference frameworks can be readily applied to such animal tracking scenarios, which can be addressed in future work.

Finally, model-based intent predictors can be decoupled from the sensor-level tracker for computational efficiency or due to system requirements, e.g. legacy algorithms or system architectures (see [31, 12]). Alternatively, the intent information can be directly incorporated into the target dynamic model, with potentially simplified meta-level tracker and more accurate target state estimation. If determining the object endpoint is sought, the latter intent-driven models are usually destination-reverting such as the linear Gaussian models based on *Ornstein-Uhlenbeck* process [34, 4], bridging distributions [3, 31] or conditionally Markov process [36],

and jump diffusion models [14]. Another modelling approach, which can also detect anomalous trajectories or rendezvousing, employs context-free grammars [12, 26]. It discretises the state space where the target movements are assumed to be constrained (e.g. confined to a road map). Here, we tackle the intent inference task for manoeuvring objects or perturbed trajectories with stable Lévy state space models, capitalising on their heavy-tailed property to capture abrupt changes in the target motion, involving no discretisation of the state space or constraints on the object path. We also present a more efficient inference strategy compared with [4, 3, 14, 31] by modelling intent within an extended dynamical state that is inferred as part of the tracking algorithm. With the introduced latent destination models, the intended endpoint can be any spatial point/region within a surveyed area, i.e. not necessarily one of a finite set of predefined destinations as in [4, 3, 14, 31, 13].

C. Paper Layout

The remainder of the paper is organised as follows. The proposed stable Lévy modelling framework is described in Section II where exact and truncated series representations of the α -stable Lévy integral are provided. Section III then details the multidimensional state-space form of the stable Lévy model, with example models in Section IV. The inference algorithms for estimating the target state and intent are defined in Section V. Results from real data of various scenarios are shown in VI and conclusions are drawn in Section VII.

II. STABLE LÉVY MODELLING FRAMEWORK

We define a latent dynamical state in the spatial domain,

$$\mathbf{x}(t) = [\mathbf{x}_1(t)^\top, \mathbf{x}_2(t)^\top, \dots, \mathbf{x}_m(t)^\top]^\top, \quad (1)$$

where $\mathbf{x}_j(t) \in \mathbb{R}^{s_j}$, $j = 1, 2, \dots, m$, is the vector of the s_j hidden variables (e.g. position and velocity of an object) in the j^{th} spatial dimension, and m is the number of considered dimensions (e.g. $m = 2$ for planar and $m = 3$ for 3-D motion). Specific forms for $\mathbf{x}_j(t)$ are given in Section IV. The stable Lévy model is formulated as a continuous-time linear system driven by multivariate isotropic α -stable noise.

For the canonical case, where there is only one multivariate isotropic noise on $\mathbb{R}^m$, the proposed stable Lévy model can be described by the following stochastic differential equation (SDE),

$$d\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}dt + \mathbf{H}d\mathbf{W}(t), \quad (2)$$

where $\mathbf{B}$ is a known input vector, e.g. serves as input in a mean reverting model [34, 3, 14], and,

$$\begin{aligned} \mathbf{A} &= \text{diag}(\mathbf{a}_1, \dots, \mathbf{a}_m), \quad \mathbf{B} = [\mathbf{b}_1^\top, \mathbf{b}_2^\top, \dots, \mathbf{b}_m^\top]^\top, \\ \mathbf{H} &= \begin{bmatrix} \mathbf{h}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{h}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{h}_m \end{bmatrix}, \end{aligned} \quad (3)$$

such that $\mathbf{a}_j \in \mathbb{R}^{s_j \times s_j}$, $\mathbf{b}_j \in \mathbb{R}^{s_j}$ and $\mathbf{h}_j \in \mathbb{R}^{s_j}$ characterise the transition function for the j^{th} dimension. $\mathbf{W}(t)$ is the isotropic multivariate α -stable Lévy process on $\mathbb{R}^m$ [40] which implies:

- 1) $\mathbf{W}(0) = \mathbf{0}$ almost surely,
- 2) Independent, isotropic stationary increments $\mathbf{W}(t) - \mathbf{W}(s)$, $t > s$, having characteristic function

$$\phi(\boldsymbol{\theta}) = \mathbb{E}[e^{i\boldsymbol{\theta}^\top(\mathbf{W}(t) - \mathbf{W}(s))}] = e^{-\sigma^\alpha(t-s)|\boldsymbol{\theta}|^\alpha}, \quad (4)$$

where α is the stability parameter on $(0, 2]$ and σ is the scale parameter. A special case is when $\alpha = 2$, and then $\mathbf{W}(t)$ is the m -dimensional isotropic Brownian motion. We focus in this paper on stable Lévy models with isotropic noise, as is common in much of the tracking literature [30, 41, 7]; however we note that the more general skewed case in [20] could also be extended to the current multidimensional setting in future work. The isotropic premise supports the typical lack of information about directionality of the randomness of dynamics in a tracking scenario.

The solution of the SDE in (2) can be written as [40]

$$\begin{aligned} \mathbf{x}(t + \Delta t) &= e^{\mathbf{A}\Delta t} \mathbf{x}(t) + \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{B} du \\ &\quad + \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{H} d\mathbf{W}(u). \end{aligned} \quad (5)$$

The first two terms are familiar from the solution of standard Brownian-driven models. The third term is a stable stochastic integral and requires special attention, see later development. The block-diagonal structure of the model matrices $\mathbf{H}$ and $\mathbf{A}$ in (3) is familiar from the structure of standard coordinate-uncoupled Gaussian models [30]. In the stable case however, the coordinates become coupled through certain latent random variables and are thus not treated independently of one another.

A more general stable Lévy system may be driven by multiple independent isotropic noises, i.e.

$$d\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}dt + \sum_{k=1}^{N_W} \mathbf{H}_k d\mathbf{W}_k(t), \quad (6)$$

where the $\mathbf{W}_k(t)$, $k = 1, 2, \dots, N_W$, are isotropic α -stable processes independent of each other and N_W is the number of such processes; $\mathbf{W}_k(t)$ can be of arbitrary dimension and will be mapped to the desired coordinate by $\mathbf{H}_k$. Such a SDE can describe the dynamics in more diverse scenarios, such as multiple interacting targets or single target with a linear combination of noise sources. The block definition for $\mathbf{x}(t)$, $\mathbf{A}$ and $\mathbf{B}$ remain the same and their specific elements should be defined as required. e.g. for multi-targets scenario, $\mathbf{x}_j$ should include kinematics from different objects, $\mathbf{a}_j$ should specify their interactions, and each $\mathbf{W}_k(t)$ for each object. A specific example of such a system will be given in Section III-B. Similar to (5), the solution of this SDE is

$$\begin{aligned} \mathbf{x}(t + \Delta t) &= e^{\mathbf{A}\Delta t} \mathbf{x}(t) + \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{B} du \\ &\quad + \sum_{k=1}^{N_W} \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{H}_k d\mathbf{W}_k(u). \end{aligned} \quad (7)$$

It can be noted from (5) and (7) that the only term that cannot be calculated explicitly in each case is the final α -stable stochastic integral. Without loss of generality, we consider this

integral with respect to a $\mathbf{W}(t)$ on $\mathbb{R}^m$ (it can be a $\mathbf{W}_k(t)$ in (7) with m of the appropriate size), and correspondingly define

$$\mathbf{I}(\Delta t) = \int_0^{\Delta t} \mathbf{f}(\Delta t, u) d\mathbf{W}(u), \quad (8)$$

with

$$\mathbf{f}(\Delta t, u) = e^{\mathbf{A}(\Delta t - u)} \mathbf{H}.$$

We next introduce the following two representations for this intractable integral: A) an exact Poisson series representation, which is in terms of an infinite series, and B) a truncated series representation, which approximates the exact representation accurately and is then utilised to devise tractable stable Lévy models in Sections III and IV.

A. Exact Series Representation of the α -stable Lévy Integral

With a direct multivariate extension of the Poisson series representation in [40, 20], the stochastic integral in (8) can be exactly expressed as

$$\mathbf{I}(\Delta t) \stackrel{D}{=} \Delta t^{1/\alpha} \sum_{i=1}^{\infty} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i, \quad (9)$$

where $\stackrel{D}{=}$ denotes both sides having the same distribution,

- $\{\Gamma_i\}$ are the unit rate Poisson process arrival times (epochs), i.e. $(\Gamma_{i+1} - \Gamma_i) \sim \exp(1)$ and $\exp(1)$ is an exponential distribution with mean 1;
- $\{V_i\}$ are independent and identically distributed (i.i.d.) random variables uniformly distributed in $(0, \Delta t]$, and
- $\{\mathbf{U}_i\}$ are any i.i.d. isotropic random vectors with finite α -th moment. Here, we choose to use $\mathbf{U}_i \sim \mathcal{N}(\mathbf{0}, \sigma_W^2 \mathbf{I}_m)$, where $\mathbf{I}_m \in \mathbb{R}^{m \times m}$ is the identity matrix, to construct a conditionally Gaussian form of $\mathbf{I}(\Delta t)$ for tractable inference.

In a typical single object tracking setup where $\mathbf{a}_j, \mathbf{h}_j$ in (3) are constants across different dimensions j , this series (9) represents an isotropic random variable. Nonetheless, the non-isotropic case can be easily included through a non-zero mean term and non-diagonal covariance matrix for $\{\mathbf{U}_i\}$. For discussion of this case, including certain required compensation terms, see [20] and references therein.

B. Truncated Series Representation

Note that the above series representation, although exact in terms of the distribution of $\mathbf{I}(\Delta t)$, has an infinite number of terms. For practical implementation we introduce a truncated series representation to approximate (9). Observe from (9) that the Poisson terms $\{\Gamma_i^{-1/\alpha}\}_{i=1}^{\infty}$ are strictly decreasing with i , and hence a reasonable approach is to perform a random truncation of the series at some value of i . Noting that the scale of series increases with Δt , we propose the truncation $\{i : \Gamma_i < c\Delta t\}$, $c \in (0, \infty)$, as in [20], and the expected number of terms in the truncated series thus grows linearly with the time interval Δt . Accordingly, the truncated series can be expressed as

$$\mathbf{I}_c(\Delta t) = \Delta t^{1/\alpha} \sum_{\Gamma_i < c\Delta t} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i. \quad (10)$$

To compensate for the error due to the series truncation in (10), we introduce an approximation to the resulting residual error series. Such a residual series can be expressed as

$$\begin{aligned} \mathbf{R}_c(\Delta t) &= \mathbf{I}(\Delta t) - \mathbf{I}_c(\Delta t) \\ &= \Delta t^{1/\alpha} \sum_{\Gamma_i \geq c\Delta t} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i. \end{aligned} \quad (11)$$

To characterise this residual series, we compute its first and second moments. Since $\{\mathbf{U}_i\}$ have zero mean and are independent from the other variables, its mean is given by

$$\begin{aligned} \mathbb{E}[\mathbf{R}_c(\Delta t)] &= \mathbb{E}[\mathbb{E}_{U_i}[\mathbf{R}_c(\Delta t)]] \\ &= \mathbb{E}[\Delta t^{1/\alpha} \sum_{\Gamma_i \geq c\Delta t} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbb{E}_{U_i}[\mathbf{U}_i]] = \mathbf{0}. \end{aligned}$$

The detailed derivation for the variance of the residual terms is in Appendix A. Specifically, we have

$$\text{Var}[\mathbf{R}_c(\Delta t)] = \sigma_W^2 \frac{\alpha}{2 - \alpha} c^{1 - \frac{2}{\alpha}} \mathbf{Q}(\Delta t),$$

$$\mathbf{Q}(\Delta t) = \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{H} \mathbf{H}^\top e^{\mathbf{A}(\Delta t - u)^\top} du. \quad (12)$$

Consequently, we can approximate this residual series with a Gaussian variable which exactly matches the first and second moments of $\mathbf{R}_c(\Delta t)$, similarly to the procedure in [20], i.e.

$$\tilde{\mathbf{R}}_c(\Delta t) \sim \mathcal{N}\left(\mathbf{0}, \sigma_W^2 \frac{\alpha}{2 - \alpha} c^{1 - \frac{2}{\alpha}} \mathbf{Q}(\Delta t)\right), \quad (13)$$

The fully truncated version for the isotropic stable Lévy integral is then conditionally Gaussian and expressed as

$$\mathbf{I}(\Delta t) \approx \mathbf{I}_c(\Delta t) + \tilde{\mathbf{R}}_c(\Delta t). \quad (14)$$

This accurate Gaussian approximation can be justified for large c by central limit theorem on $\mathbf{R}_c(\Delta t)$, see [20, 39, 37] and in particular the sharp convergence bounds for the symmetric scalar case reported in [37].

III. STABLE LÉVY STATE SPACE MODEL

The state-space form of the stable Lévy model can now be developed based on the truncated Poisson series representation in (14). For notational convenience, we define

$$\begin{aligned} \mathbf{F}(\Delta t) &= e^{\mathbf{A}\Delta t}, \\ \mathbf{M}(\Delta t) &= \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{B} du. \end{aligned} \quad (15)$$

Recall that our stable Lévy model can be driven by one or multiple α -stable noise(s) as specified in the SDEs in (2) and (6). Here, we first formulate the state-space form of the canonical single noise case described by (2). The multiple noises setting in (6) can then be similarly obtained by simply summing the multiple truncated series representations. However, in the example presented in this paper we will use a special-case multiple noises scenario where it consists of a non-Gaussian α -stable noise and a Gaussian noise, both on $\mathbb{R}^m$.

A. Model Driven by a Single α -stable Noise Source

The stable Lévy model with a single α -stable noise has the SDE solution in (5). Substituting the series representations in (14) into (5) yields

$$\begin{aligned} \mathbf{x}(t + \Delta t) &= e^{A\Delta t} \mathbf{x}(t) + \int_0^{\Delta t} e^{A(\Delta t-u)} \mathbf{B} du + \int_0^{\Delta t} e^{A(\Delta t-u)} \mathbf{H} d\mathbf{W}(u) \\ &\approx \mathbf{F}(\Delta t) \mathbf{x}(t) + \mathbf{M}(\Delta t) + \mathbf{I}_c(\Delta t) + \tilde{\mathbf{R}}_c(\Delta t). \end{aligned} \quad (16)$$

Note that given all the required $\{\Gamma_i\}_{(0,c\Delta t]}$ and $\{V_i\}_{(0,\Delta t]}$, $\mathbf{I}_c(\Delta t)$ in (10) is the summation of independent Gaussian vectors with zero mean and covariance $\Delta t^{2/\alpha} \Gamma_i^{-2/\alpha} \sigma_W^2 \mathbf{f}(\Delta t, V_i) \mathbf{f}(\Delta t, V_i)^\top$, and thus $\mathbf{I}_c(\Delta t)$ also follows the conditional normal distribution, that is

$$\begin{aligned} p(\mathbf{I}_c(\Delta t) | \{\Gamma_i, V_i\}_{\Delta t}^c) &= \mathcal{N} \left(\mathbf{0}, \Delta t^{2/\alpha} \sigma_W^2 \sum_{\Gamma_i < c\Delta t} \Gamma_i^{-2/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{f}(\Delta t, V_i)^\top \right), \end{aligned} \quad (17)$$

where we define

$$\{\Gamma_i, V_i\}_{\Delta t}^c = \{\{\Gamma_i\}_{(0,c\Delta t]}, \{V_i\}_{(0,\Delta t]}\}.$$

Since the residual approximation $\tilde{\mathbf{R}}_c(\Delta t)$ is another independent Gaussian vector as given in (13), the transition density for the α -stable Lévy state-space model can be computed according to (16) as

$$\begin{aligned} p(\mathbf{x}(t + \Delta t) | \mathbf{x}(t), \{\Gamma_i, V_i\}_{\Delta t}^c) &= \mathcal{N}(\mathbf{F}(\Delta t) \mathbf{x}(t) + \mathbf{M}(\Delta t), \mathbf{S}(\Delta t)), \end{aligned}$$

where

$$\begin{aligned} \mathbf{S}(\Delta t) &= \Delta t^{2/\alpha} \sigma_W^2 \sum_{\Gamma_i < c\Delta t} \Gamma_i^{-2/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{f}(\Delta t, V_i)^\top \\ &\quad + \sigma_W^2 \frac{\alpha}{2-\alpha} c^{1-\frac{2}{\alpha}} \mathbf{Q}(\Delta t). \end{aligned} \quad (18)$$

The mean of this transition density, i.e. $\mathbf{F}(\Delta t) \mathbf{x}(t) + \mathbf{M}(\Delta t)$, reflects the overall motion trend of the target. By adjusting the parameters $\mathbf{A}$ and $\mathbf{B}$ in the SDE in (2), this model covers a wide range of linear dynamics for diverse applications, e.g. mean-reverting for intent inference or a constant velocity/acceleration behaviour for tracking. The noise parameters α and σ_W characterise the manoeuvrability (small α for more heavy-tailed) and scaling, respectively. Whilst such a model is only driven by a single α -stable noise, it is suitable for a single manoeuvring object, for both tracking and intent inference applications and examples are given in Section IV.

B. Model Driven by both Non-Gaussian α -stable and Gaussian Noises

The focus here is on multiple α -stable noises in (6), in particular two noises on $\mathbb{R}^m$ such that one is with $0 < \alpha < 2$ and the other has $\alpha = 2$ (i.e. Gaussian). Since the Gaussian stochastic integral has well-known closed-form solution, we

only employ the Poisson series representations for the non-Gaussian integral. Specifically, the SDE in (6) reduces to

$$d\mathbf{x}(t) = \mathbf{A}\mathbf{x}(t)dt + \mathbf{B}dt + \mathbf{H}d\mathbf{W}(t) + \mathbf{G}d\boldsymbol{\beta}(t), \quad (19)$$

where $\boldsymbol{\beta}(t)$ is the isotropic Brownian motion on $\mathbb{R}^m$, with the covariance matrix $\sigma_B^2 \mathbf{I}_m$; $\mathbf{G}$ has a block-diagonal structure similar to $\mathbf{H}$, i.e.

$$\mathbf{G} = \begin{bmatrix} \mathbf{g}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{g}_2 & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{g}_m \end{bmatrix}, \quad (20)$$

such that $\mathbf{g}_j \in \mathbb{R}^{s_j}$, with $j = 1, 2, \dots, m$, maps the j -th scalar components of $\boldsymbol{\beta}(t)$ to the desired latent variables in the j -th dimension. The stable Lévy state-space model for this setup is

$$\begin{aligned} \mathbf{x}(t + \Delta t) &\approx \mathbf{F}(\Delta t) \mathbf{x}(t) + \mathbf{M}(\Delta t) + \mathbf{I}_c(\Delta t) + \tilde{\mathbf{R}}_c(\Delta t) \\ &\quad + \int_0^{\Delta t} e^{A(\Delta t-u)} \mathbf{G} d\boldsymbol{\beta}(u). \end{aligned} \quad (21)$$

We still have the conditionally Gaussian form of $\mathbf{I}_c(\Delta t)$ as (17), and the following transition density can thus be attained

$$\begin{aligned} p(\mathbf{x}(t + \Delta t) | \mathbf{x}(t), \{\Gamma_i, V_i\}_{\Delta t}^c) &= \mathcal{N}(\mathbf{F}(\Delta t) \mathbf{x}(t) + \mathbf{M}(\Delta t), \mathbf{S}(\Delta t)), \end{aligned}$$

where

$$\begin{aligned} \mathbf{S}(\Delta t) &= \Delta t^{2/\alpha} \sigma_W^2 \sum_{\Gamma_i < c\Delta t} \Gamma_i^{-2/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{f}(\Delta t, V_i)^\top \\ &\quad + \sigma_W^2 \frac{\alpha}{2-\alpha} c^{1-\frac{2}{\alpha}} \mathbf{Q}(\Delta t) + \sigma_B^2 \mathbf{\Omega}(\Delta t), \end{aligned} \quad (22)$$

$$\mathbf{\Omega}(\Delta t) = \int_0^{\Delta t} e^{A(\Delta t-u)} \mathbf{G} \mathbf{G}^\top e^{A(\Delta t-u)} du. \quad (23)$$

A special case for this model is when $\sigma_B = 0$, i.e. the covariance matrix for Brownian motion $\boldsymbol{\beta}(t)$ is zero or negligible, the model reduces to the single α -stable noise driven case in the Section III-A. Typically, this additional Brownian motion aims to provide Gaussian randomness for multiple target dynamics modelling. Alternatively, it can, as well as the α -stable noise, apply to the same object. One example of this model for intent inference application is detailed in Section IV-B.

IV. MODEL EXAMPLES

In this section, we give concrete examples of models within the stable Lévy state space framework as defined in Section III. Due to the large number of models considered below, we do not provide their explicit discrete form, instead we introduce the general schemes to compute the required terms $\mathbf{M}(\Delta t)$, $\mathbf{Q}(\Delta t)$, and $\mathbf{\Omega}(\Delta t)$. These schemes are based on the assumption that the matrix exponential can be computed easily and accurately; hence by such an assumption $\mathbf{F}(\Delta t)$ in (15) is already easily calculated.

Let the square matrix $\mathbf{A}$ have dimension $s_A \times s_A$, i.e. $s_A = \sum_{j=1}^m s_j$. We start with the simpler integral $\mathbf{M}(\Delta t)$ in (15). It can be noticed that if $\mathbf{A}$ is invertible, we have a straightforward expression

$$\mathbf{M}(\Delta t) = \mathbf{A}^{-1} (e^{A\Delta t} - \mathbf{I}_{s_A}) \mathbf{B}. \quad (24)$$

Here, we propose a method to compute $M(\Delta t)$ regardless of the structure of $\mathbf{A}$ (including the singular case) such that

$$\mathbf{M}(\Delta t) = \mathbf{J}_M \mathbf{B},$$

$$\begin{bmatrix} \mathbf{J}_M \\ \mathbf{I}_{s_A} \end{bmatrix} = \expm \left(\begin{bmatrix} \mathbf{A} & \mathbf{I}_{s_A} \\ \mathbf{0}_{s_A} & \mathbf{0}_{s_A} \end{bmatrix} \Delta t \right) \begin{bmatrix} \mathbf{0}_{s_A} \\ \mathbf{I}_{s_A} \end{bmatrix}, \quad (25)$$

where $\expm(\cdot)$ computes the matrix exponential. The proof for (25) is in Appendix B. This approach is inspired by the *matrix fraction decomposition* technique, which was utilised for solving *matrix Riccati differential equation* in [22] and in [41] to compute the Gaussian covariance matrix. The latter matrix has the same form as $\mathbf{Q}(\Delta t)$ and $\mathbf{\Omega}(\Delta t)$ in our model. The scheme in [41] is used to compute $\mathbf{Q}(\Delta t)$, leading to

$$\mathbf{Q}(\Delta t) = \mathbf{J}_Q \mathbf{K}_Q^{-1},$$

$$\begin{bmatrix} \mathbf{J}_Q \\ \mathbf{K}_Q \end{bmatrix} = \expm \left(\begin{bmatrix} -\mathbf{A} & \mathbf{H}\mathbf{H}^\top \\ \mathbf{0}_{s_A} & \mathbf{A} \end{bmatrix} \Delta t \right) \begin{bmatrix} \mathbf{0}_{s_A} \\ \mathbf{I}_{s_A} \end{bmatrix}, \quad (26)$$

see [41] for the proof. Note that the $\mathbf{\Omega}(\Delta t)$ in (23) only differs from the $\mathbf{Q}(\Delta t)$ in (12) by the mapping matrix, thus we only need to replace $\mathbf{H}$ in (26) with $\mathbf{G}$ to obtain the $\mathbf{\Omega}(\Delta t)$.

However, we stress that in the case that these matrix exponentials cannot be computed, an explicit expression of $\mathbf{M}(\Delta t)$, $\mathbf{Q}(\Delta t)$, and $\mathbf{\Omega}(\Delta t)$ should be sought; this can be potentially simplified by the block-diagonal structure of $\mathbf{A}$, $\mathbf{H}$ and $\mathbf{G}$. For instance, with $\mathbf{Q}(\Delta t)$ the integral can be expanded as a block diagonal matrix with m integrals, i.e.

$$\mathbf{Q}(\Delta t) = \text{diag}(\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_m), \quad (27)$$

where $\mathbf{q}_j = \int_0^{\Delta t} e^{\mathbf{a}_j(\Delta t-u)} \mathbf{h}_j \mathbf{h}_j^\top e^{\mathbf{a}_j(\Delta t-u)^\top} du$, with $j = 1, 2, \dots, m$, is the integral on $\mathbb{R}^{s_j \times s_j}$ and thereby, we only require the explicit form of $\mathbf{q}_j$; this can be found in the literature for some of the models discussed below, e.g. for Singer model in [45] and Langevin model in [20].

Next, we present four examples of the proposed stable Lévy modelling approach. They are divided into two classes, those for manoeuvring object (sensor-level) tracking in Section IV-A and intent inference in Section IV-B. They all have isotropic linear setup such that the parameters $\mathbf{a}_j$ and $\mathbf{h}_j$ in (3) as well as $\mathbf{g}_j$ in (20) are constants across the different dimensions j , $j = 1, 2, \dots, m$. The j -th dimensional target state $\mathbf{x}_j$ also incorporates the same kind of kinematics for different dimensions. For notational brevity, we further define x_j as the position in the j -th dimension, $\dot{x}_j$ as its first derivative (i.e. velocity) and $\ddot{x}_j$ is the second derivative of x_j (i.e. acceleration). While we present just four exemplar model structures, it should be emphasised that any other linear state space models in continuous time can be adapted to the Lévy state space setting.

A. Models for Manoeuvring Object Tracking

Below, we consider two models for sensor-level tracking, both are driven by a single noise model as in Section III-A.

1) *Stable Lévy Langevin model*: This replaces the Brownian motion in the original planar Langevin dynamics with the isotropic stable Lévy motion. It can be described by the SDE in (2), with the parameters listed in Table I. Such parameters

TABLE I: Summary of the parameters in the SDEs (2) for Langevin, Singer and ERV models and (19) for the latent destination model, $j = 1, 2, \dots, m$.

Models	$\mathbf{x}_j$	$\mathbf{a}_j$	$\mathbf{b}_j$	$\mathbf{h}_j$	$\mathbf{g}_j$
Langevin	$\begin{bmatrix} x_j \\ \dot{x}_j \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ 0 & -\lambda \end{bmatrix}$	$\mathbf{0}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	N/A
Singer	$\begin{bmatrix} x_j \\ \dot{x}_j \\ \ddot{x}_j \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\lambda \end{bmatrix}$	$\mathbf{0}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	N/A
ERV	$\begin{bmatrix} x_j \\ \dot{x}_j \end{bmatrix}$	$\begin{bmatrix} 0 & 1 \\ -\eta & -\rho \end{bmatrix}$	$\eta \begin{bmatrix} 0 \\ p_j \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	N/A
Latent destination	$\begin{bmatrix} x_j \\ \dot{x}_j \\ l_j \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \\ -\eta & -\rho & \eta \\ 0 & 0 & 0 \end{bmatrix}$	$\mathbf{0}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

correspond to a SDE for the object speed of the j -th spatial dimension according to

$$d\dot{x}_j(t) = -\lambda \dot{x}_j(t) dt + dW_j(t), \quad (28)$$

where λ is the positive friction constant and $W_j(t)$ is the j -th component of $\mathbf{W}(t)$. This SDE describes $\dot{x}_j(t)$ as the Lévy Ornstein-Uhlenbeck process with the mean term 0; hence a target governed by this SDE always tends to slow down despite the fluctuations induced by the Lévy stable noise. It is noted that when the stable noise does not exhibit extreme values, the object displays small non-directional movements since the velocity is near zero. The trajectory is more likely to cluster in a small region during these periods. However, a large velocity can be generated due to an extreme noise value(s) producing a target motion along a straight-line, albeit the velocity decaying gradually, see [13] for example synthetic tracks. Hence, the target occasionally moves fast to other spatial regions, which can capture the efficient foraging/searching pattern of animals. Consequently, this Langevin model is expected to be able to model irregular manoeuvres of highly volatile objects, e.g. animals or insects [35, 8, 24].

2) *Stable Lévy Singer model*: whilst this is based on the well-known Gaussian Singer dynamic model in [45] for manoeuvring object tracking, in this paper we replace the Gaussian noise with an α -stable noise for more flexibility in representing increased object manoeuvrability. It follows the SDE in (2) and its parameters are listed in Table I. This extends the previous Langevin model to a higher order kinematics and its acceleration follows the Lévy Ornstein-Uhlenbeck process with mean term 0. Its j -th dimensional SDE is

$$d\ddot{x}_j(t) = -\lambda \ddot{x}_j(t) dt + dW_j(t). \quad (29)$$

The manoeuvres are introduced in this model by the non-zero acceleration and the positive parameter λ characterises the reversion ability from undertaking manoeuvring movements. For instance, a small λ is for a slower decaying of the manoeuvring action, i.e. the manoeuvring lasts longer. A special case of the stable Lévy Singer model is when $\lambda = 0$, which results in the stable Lévy constant acceleration model.

B. Models for Intent Inference

In this section, two models for the meta-level tracking task of destination prediction are presented.

1) *Stable Lévy equilibrium reverting velocity model*: this spatial model is a modified version of the *equilibrium reverting velocity* (ERV) model developed in [4] for 3-D pointing gesture movements. The single Gaussian noise of the ERV model in [4] is replaced by a stable Lévy noise to better handle severely perturbed motion. As described in Section III-A, it is a single noise driven model expressed by the SDE in (2). Its parameters are given in Table I such that η and ρ are positive reversion and damping coefficients, respectively. In this version of the model, vector $[p_1, p_2, \dots, p_m]^\top$ denotes the destination position. The j -th dimensional SDE is

$$d\dot{x}_j(t) = \eta(p_j - x_j(t))dt - \rho\dot{x}_j(t)dt + dW_j(t). \quad (30)$$

This SDE describes a target that reverts to the destination's position $[p_1, p_2, \dots, p_m]^\top$ under the impact of disturbances of a stable Lévy noise. The damping coefficient ρ ensures that the target speed does not become excessive high during its journey to/at the endpoint. This spatial stable Lévy ERV model is designed to capture the dependence of the destination and the target's movements, and hence it is suitable for destination-aware tracking and/or intent inference. It is emphasised that any other Ornstein-Uhlenbeck-based destination reverting models in [14, 4], can be utilised similarly to the ERV, with a stable Lévy dynamic noise in lieu of their original Gaussian noise.

2) *Latent destination model*: unlike the aforementioned ERV model which assumes that the intent/destination is a fixed (certain) parameter throughout the target journey, here we model the destination as a latent variable. This hidden endpoint can either be static or dynamic (i.e. follows a Brownian motion), while the tracked target (i.e. its kinematic state) still reverts to the latent endpoint with similar dynamic characteristics as the stable Lévy ERV model. Thereby, this Lévy dynamic/static latent destination model (abbreviated as DLD/SLD model) belongs to the family of models driven by a stable Lévy noise and Gaussian noise as in Section III-B, and follows the SDE in (19). Its parameters are defined in Table I, where l_j is the position of the intended destination in the j -th dimension. The SDE of the motion in the j -th dimension can be written as

$$\begin{aligned} d\dot{x}_j(t) &= \eta(l_j(t) - x_j(t))dt - \rho\dot{x}_j(t)dt + dW_j(t), \\ dl_j(t) &= d\beta_j(t), \end{aligned} \quad (31)$$

where $\beta_j(t)$ is the j -th component of the isotropic Brownian motion $\beta(t)$. By modelling the intent as a Brownian motion, the DLD model allows a change of destination with a moderate rate. Additionally, by setting the Brownian motion covariance matrix as zero, i.e. $\sigma_B = 0$, this SDE describes the SLD model, and characterises an object reverting to a fixed (*a priori* unknown) destination. However, this SLD model still allows us to estimate the intent with a Gaussian prior on a potential destination position. Most importantly, with this novel modelling approach, we can determine the probability of any point/region within the surveyed area being the target

intended endpoint, unlike the formulations in [3, 14, 31] that explicitly assume a finite set of nominal destinations. Practically, this model serves as the extension of the ERV or any other similar destination-reverting model to effectively tackle the problem of a dynamically changing intent or the number of possible destinations is too large. This is besides the α -stable dynamic noise aspect. We show in Section V-B that the intent prediction with this latent destination model is substantially more efficient than the stable Lévy ERV model.

V. ESTIMATION AND INTENT INFERENCE

Owing to the conditionally Gaussian structure of the stable Lévy model detailed above, standard state and parameters estimation procedures for Gaussian models can be employed within the introduced sequential Monte Carlo framework. Specifically, an efficient *Rao-Blackwellised* particle filter can be designed to carry out the state estimation, with a simple linear Gaussian observation function. Here, we first discuss computing the likelihoods and posteriors conditioned on the latent variables $\{\Gamma_i, V_i\}^c$, which will then be utilised in the *Rao-Blackwellised* particle filter for marginal state estimation and intent inference application.

Let $\mathbf{y}(t)$ be a vector of observations received at time t . We assume that the sensory observations are available at each of the successive discrete time points t_n ($n = 1, 2, \dots$). Additionally, we define $\mathbf{x}_n = \mathbf{x}(t_n)$, $\mathbf{y}_n = \mathbf{y}(t_n)$, and $\{\Gamma_i, V_i\}_{n:m}^c$ as all the latent variables required for state transition from t_n to t_m ($m > n$), i.e., $\{\Gamma_i, V_i\}_{n:m}^c = \bigcup_{k=n}^{m-1} \{\Gamma_i, V_i\}_{\Delta t_k}^c$, with $\Delta t_k = t_{k+1} - t_k$. Note that for different n , all $\{\Gamma_i, V_i\}_{n-1:n}^c$ s are defined in non-overlapping time intervals, and thus they are independent by construction. The transition density for $\{\Gamma_i, V_i\}^c$ can then be simplified as follows,

$$p(\{\Gamma_i, V_i\}_{n-1:n}^c | \{\Gamma_i, V_i\}_{1:n-1}^c) = p(\{\Gamma_i, V_i\}_{n-1:n}^c). \quad (32)$$

We aim to calculate the conditional likelihoods and posteriors, e.g. $p(\mathbf{x}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n})$, in closed forms. In a typical setup where the noise parameter σ_W^2 is a certain constant, this requires the following observation function,

$$p(\mathbf{y}_n | \mathbf{x}_n) = \mathcal{N}(\mathbf{C}\mathbf{x}_n + \mathbf{D}, \mathbf{R}), \quad (33)$$

where $\mathbf{R}$ is the covariance of the observation noise; $\mathbf{C}$ and $\mathbf{D}$ are parameters defining the transformation from state to the observations. Consequently, the sought conditional likelihoods and posteriors can be recursively computed by the standard Kalman filter with Gaussian priors, i.e.

$$p(\mathbf{x}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n}) = \mathcal{N}(\boldsymbol{\mu}_{n|n}, \mathbf{P}_{n|n}), \quad (34)$$

$$p(\mathbf{x}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n-1}) = \mathcal{N}(\boldsymbol{\mu}_{n|n-1}, \mathbf{P}_{n|n-1}), \quad (35)$$

$$p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n-1}) = \mathcal{N}(\hat{\mathbf{y}}_{n|n-1}, \boldsymbol{\Sigma}_{n|n-1}), \quad (36)$$

with means and covariances given by

$$\begin{aligned}
\Delta t &= t_n - t_{n-1}, \\
\boldsymbol{\mu}_{n|n-1} &= \mathbf{F}(\Delta t)\boldsymbol{\mu}_{n-1|n-1} + \mathbf{M}(\Delta t), \\
\mathbf{P}_{n|n-1} &= \mathbf{F}(\Delta t)\mathbf{P}_{n-1|n-1}\mathbf{F}(\Delta t)^\top + \mathbf{S}(\Delta t), \\
\hat{\mathbf{y}}_{n|n-1} &= \mathbf{C}\boldsymbol{\mu}_{n|n-1} + \mathbf{D}, \\
\boldsymbol{\Sigma}_{n|n-1} &= \mathbf{C}\mathbf{P}_{n|n-1}\mathbf{C}^\top + \mathbf{R}, \\
\mathbf{K} &= \mathbf{P}_{n|n-1}\mathbf{C}^\top\boldsymbol{\Sigma}_{n|n-1}^{-1}, \\
\boldsymbol{\mu}_{n|n} &= \boldsymbol{\mu}_{n|n-1} + \mathbf{K}(\mathbf{y}_n - \hat{\mathbf{y}}_{n|n-1}), \\
\mathbf{P}_{n|n} &= (\mathbf{I}_{s_A} - \mathbf{K}\mathbf{C})\mathbf{P}_{n|n-1},
\end{aligned} \tag{37}$$

where $\mathbf{S}(\Delta t)$ is defined in (18) for single noise driven model and in (22) for non-Gaussian α -stable and Gaussian noises driven case. It should be noted that without the linear Gaussian observation function in (33), the closed form results in (34)-(36) will be lost and one may have to sample both the state $\mathbf{x}$ as well as the latent variables $\{\Gamma_i, V_i\}$ in the particle filtering for state estimation.

Furthermore, it is possible to jointly and explicitly estimate the noise parameter σ_W^2 and the state conditioned on the latent variables, i.e. $p(\sigma_W^2, \mathbf{x}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n})$. Such a tractable estimation is only valid for the single noise driven stable Lévy models in Section III-A, and imposes two more assumptions, i.e. $\mathbf{R}$ is scaled with σ_W^2 and the prior $p(\sigma_W^2)$ is the inverted gamma distribution. Subsequently, the standard Kalman filter can still be employed to evaluate the joint posterior in a closed form. In this paper, we do not treat such a scenario and σ_W^2 is known as is common in object tracking, however some relevant details can be found in [20].

A. Rao-Blackwellised Particle Filtering

Based on the above closed-form conditional densities, the efficient *Rao-Blackwellised* particle filter can be used to only sample the intractable latent variables $\{\Gamma_i, V_i\}^c$ and other marginal densities, e.g. $p(\mathbf{x}_n | \mathbf{y}_{1:n})$, can be estimated within the importance sampling framework. Therefore, we consider the posterior of $\{\Gamma_i, V_i\}^c$ as the particle filter target distribution, which can be factorised as per,

$$\begin{aligned}
p(\{\Gamma_i, V_i\}_{1:n}^c | \mathbf{y}_{1:n}) &\propto p(\{\Gamma_i, V_i\}_{n-1:n}^c | \{\Gamma_i, V_i\}_{1:n-1}^c) \\
&\times p(\{\Gamma_i, V_i\}_{1:n-1}^c | \mathbf{y}_{1:n-1}) p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^c, \mathbf{y}_{1:n-1}), \tag{38}
\end{aligned}$$

such that the transition for $\{\Gamma_i, V_i\}^c$ can be simplified as in (32). With N_P particles, then at time t_n each particle is a set of latent variables sequence $\{\Gamma_i, V_i\}_{1:n}^{c,(p)}$, where p is the index of particles. We now define the *unnormalized* importance weight for the p -th particle as $\omega_n^{(p)}$, i.e. $\omega_n^{(p)} = \frac{p(\{\Gamma_i, V_i\}_{1:n}^{c,(p)} | \mathbf{y}_{1:n})}{q(\{\Gamma_i, V_i\}_{1:n}^{c,(p)} | \mathbf{y}_{1:n})}$, and q is the proposal distribution. Let the *self-normalized* importance weight be $\omega_n^{*(p)} = \omega_n^{(p)} / \sum_{p=1}^{N_P} \omega_n^{(p)}$. Such *self-normalized* weights offer an empirical Dirac approximation of the target distribution, i.e.

$$p(\{\Gamma_i, V_i\}_{1:n}^c | \mathbf{y}_{1:n}) \approx \sum_{p=1}^{N_P} \omega_n^{*(p)} \delta(\{\Gamma_i, V_i\}_{1:n}^{c,(p)}). \tag{39}$$

To update the *self-normalized* importance weight, we expand it as follows,

$$\begin{aligned}
\omega_n^{*(p)} &\propto \omega_n^{(p)} = \frac{p(\{\Gamma_i, V_i\}_{1:n}^{c,(p)} | \mathbf{y}_{1:n})}{q(\{\Gamma_i, V_i\}_{1:n}^{c,(p)} | \mathbf{y}_{1:n})} \\
&\propto \frac{p(\{\Gamma_i, V_i\}_{1:n-1}^{c,(p)} | \mathbf{y}_{1:n-1})}{q(\{\Gamma_i, V_i\}_{1:n-1}^{c,(p)} | \mathbf{y}_{1:n-1})} p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^{c,(p)}, \mathbf{y}_{1:n-1}) \\
&\quad \times \frac{p(\{\Gamma_i, V_i\}_{n-1:n}^{c,(p)} | \{\Gamma_i, V_i\}_{1:n-1}^{c,(p)})}{q(\{\Gamma_i, V_i\}_{n-1:n}^{c,(p)} | \{\Gamma_i, V_i\}_{1:n-1}^{c,(p)}, \mathbf{y}_n)} \\
&\propto \omega_{n-1}^{*(p)} p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^{c,(p)}, \mathbf{y}_{1:n-1}), \tag{40}
\end{aligned}$$

where we choose the target transition density in (32) as the proposal, i.e. the *bootstrap* particle filter. The sampling scheme for such a proposal is summarized in Algorithm 1. The conditional predictive likelihood in (40) can be computed directly using the Kalman filter steps in (36).

Algorithm 1: Sampling Scheme for Latent Variables

Output: $\{\Gamma_i, V_i\}_{n-1:n}^c$.
 $\Delta t = t_n - t_{n-1}$.
 $i = 1$, sample $\Gamma_i \sim \exp(1)$.
while $\Gamma_i < c\Delta t$ **do**
 Sample $V_i \sim \text{Uniform}(0, \Delta t)$.
 Include Γ_i and V_i to $\{\Gamma_i, V_i\}_{n-1:n}^c$.
 $i = i + 1$, sample $\Gamma_i \sim \Gamma_{i-1} + \exp(1)$.
end

Consequently, we can recursively estimate the state and the nonlinear latent variables $\{\Gamma_i, V_i\}^c$ in an efficient *Rao-Blackwellisation* scheme, with each particle updating and storing the following information: the sample $\{\Gamma_i, V_i\}^c$, normalized weight ω^* , the conditional state mean $\boldsymbol{\mu}$ and the covariance $\mathbf{P}$. This filtering strategy is outlined in Algorithm 2. With this stored information, the estimation for other marginal densities can be extracted as required. Specifically,

$$\begin{aligned}
&p(\cdot | \mathbf{y}_{1:n}) \\
&= \int p(\cdot | \mathbf{y}_{1:n}, \{\Gamma_i, V_i\}_{1:n}^c) p(\{\Gamma_i, V_i\}_{1:n}^c | \mathbf{y}_{1:n}) d\{\Gamma_i, V_i\}_{1:n}^c \\
&\approx \sum_{p=1}^{N_P} \omega_n^{*(p)} p(\cdot | \mathbf{y}_{1:n}, \{\Gamma_i, V_i\}_{1:n}^{c,(p)}), \tag{41}
\end{aligned}$$

where we substitute (39) to obtain the third line. Note that ‘ $\cdot$ ’ can be any of the variables which have explicit conditional densities, such as $\mathbf{x}_n$ as given in (34)-(36). For example, the marginal posterior of the target state, which may be required in object tracking, can be approximated by (41) as,

$$p(\mathbf{x}_n | \mathbf{y}_{1:n}) \approx \sum_{p=1}^{N_P} \omega_n^{*(p)} p(\mathbf{x}_n | \mathbf{y}_{1:n}, \{\Gamma_i, V_i\}_{1:n}^{c,(p)}), \tag{42}$$

such that the conditional posterior $p(\mathbf{x}_n | \mathbf{y}_{1:n}, \{\Gamma_i, V_i\}_{1:n}^{c,(p)})$ is given in (34).

Algorithm 2: Single Iteration of Particle Filtering

Input: $\{\Gamma_i, V_i\}_{1:n-1}^{c,(p)}$, $\omega_{n-1}^{*(p)}$, and $\{\mu, P\}_{n-1|n-1}^{(p)}$.
Output: $\{\Gamma_i, V_i\}_{1:n}^{c,(p)}$, $\omega_n^{*(p)}$ and $\{\mu, P\}_{n|n}^{(p)}$.
if *Resample* **then**
 Resample particles and set *self-normalized* weights
 $\omega_{n-1}^{*(p)} = 1/N_{\mathcal{P}}$.
end
for particles $p = 1 : N_{\mathcal{P}}$ **do**
 Sample $\{\Gamma_i, V_i\}_{n-1:n}^{c,(p)}$ via Algorithm 1, and include
 it to $\{\Gamma_i, V_i\}_{1:n}^{c,(p)}$.
 Update stored information $\{\mu, P\}_{n|n}^{(p)}$ via (37).
 Compute the *self-normalized* weight $\omega_n^{*(p)}$ via (40).
end

B. Intent Inference

The objective here is to sequentially infer the target intent with recursive Bayesian filtering. For each of the two stable Lévy models in Section IV-B, an inference routine is introduced below with its pseudo-code.

1) *Intent inference for stable Lévy ERV model:* In this example the intent is modelled as a finite discrete grid of possible destinations, each with a prior probability, and the particle filter is used to estimate the probability of each destination from the sequentially evolving data; for the general approach see [4, 3, 14]. Let the total number of destination be $N_{\mathcal{D}}$ such that $\mathcal{D}_d$ is the d -th possible endpoint, whose position is known and defined by $\mathbf{p}_d = [p_1^d, p_2^d, \dots, p_m^d]^\top$. In particular, we seek determining the probability of each destination being the intended endpoint, which can be factorised as follows by the Bayesian formula,

$$p(\mathcal{D}_d | \mathbf{y}_{1:n}) \propto p(\mathbf{y}_{1:n} | \mathcal{D}_d) p(\mathcal{D}_d), \quad (43)$$

where the prior $p(\mathcal{D}_d)$ is chosen according to the contextual information (e.g. travel history, preferences, etc.). It can be uninformative, i.e. $p(\mathcal{D}_d) = 1/N_{\mathcal{D}}$, for all nominal destinations $\mathcal{D}_d$ if there is no such information is available. The most probable destination, in the *Maximum a Posteriori* sense, is

$$i = \max_{d=1,2,\dots,N_{\mathcal{D}}} \{d; p(\mathcal{D}_d | \mathbf{y}_{1:n})\}. \quad (44)$$

The destination likelihood in (43) can be recursively evaluated via the *prediction error decomposition* (PED) [23],

$$p(\mathbf{y}_{1:n} | \mathcal{D}_d) = p(\mathbf{y}_n | \mathbf{y}_{1:n-1}, \mathcal{D}_d) p(\mathbf{y}_{1:n-1} | \mathcal{D}_d). \quad (45)$$

Thus, the key operation is computing the predictive likelihood $p(\mathbf{y}_n | \mathbf{y}_{1:n-1}, \mathcal{D}_d)$. Note that this condition $\mathcal{D}_d$ can be naturally included in the stable Lévy ERV model by incorporating destination information in the parameters (as specified in Table I), so that the destination-conditioned model will revert to the position $\mathbf{p}_d$. Thereby, the sought predictive likelihood can be

evaluated via the approximation (41) within the importance sampling framework,

$$\begin{aligned} p(\mathbf{y}_n | \mathbf{y}_{1:n-1}, \mathcal{D}_d) &\approx \sum_{p=1}^{N_{\mathcal{P}}} \omega_{n-1}^{*(p,d)} p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^{c,(p,d)}, \mathbf{y}_{1:n-1}, \mathcal{D}_d) \\ &= \sum_{p=1}^{N_{\mathcal{P}}} \tilde{\omega}_n^{(p,d)}, \end{aligned} \quad (46)$$

where the superscript d stipulates that the variable/sample belongs to the d -th destination conditioned ERV model. In the last step of (46) the *update* weight is defined by

$$\tilde{\omega}_n^{(p,d)} = \omega_{n-1}^{*(p,d)} p(\mathbf{y}_n | \{\Gamma_i, V_i\}_{1:n}^{c,(p,d)}, \mathbf{y}_{1:n-1}, \mathcal{D}_d). \quad (47)$$

Due to the fact that the destination information is an implicit condition for a stable Lévy mean reverting model, the conditional predictive likelihood in (47) is in effect (36). Note that $\tilde{\omega}_n^{(p,d)}$ is also required in the *self-normalized* weight update equation (40). Subsequently, we have

$$\omega_n^{*(p,d)} \propto \tilde{\omega}_n^{(p,d)}. \quad (48)$$

For $N_{\mathcal{D}}$ possible destinations, then $N_{\mathcal{D}}$ *Rao-Blackwellised* particle filter filters, each with a particular destination information, would be needed to carry out the intent inference. By only sampling $\{\Gamma_i, V_i\}^c$ once, here all $N_{\mathcal{D}}$ particle filters share the same Lévy noise latent samples, which significantly reduces the computation time for a large number of destinations. The inference procedure for the stable Lévy mean reverting model is outlined in Algorithm 3. This pseudo-code only proceeds one time step from t_{n-1} to t_n up to the current observation $\mathbf{y}_n$. More details about this intent inference strategy can be found in [14], where a similar conditionally Gaussian intent inference framework is presented for jump models.

2) *Intent inference for latent destination models:* Since the position of the intended destination is modelled as a variable within the dynamical state (see Table I), the estimated intent can be directly extracted from the estimated “extended” dynamical state employing (41), i.e.,

$$\begin{aligned} p(\mathbf{l}_n | \mathbf{y}_{1:n}) &= p(\mathbf{G}^\top \mathbf{x}_n | \mathbf{y}_{1:n}) \\ &\approx \sum_{p=1}^{N_{\mathcal{P}}} \omega_n^{*(p)} p(\mathbf{G}^\top \mathbf{x}_n | \mathbf{y}_{1:n}, \{\Gamma_i, V_i\}_{1:n}^{c,(p)}) \\ &= \sum_{p=1}^{N_{\mathcal{P}}} \omega_n^{*(p)} \mathcal{N}(\mathbf{l}_n; \mathbf{G}^\top \boldsymbol{\mu}_{n|n}^{(p)}, \mathbf{G}^\top \mathbf{P}_{n|n}^{(p)} \mathbf{G}), \end{aligned} \quad (49)$$

where the last line is obtained by substituting the conditional state posterior in (34) and using change of variables for Gaussian vectors. Vector $\mathbf{l}_n$ denotes the position of the intent/endpoint at time t_n , i.e. $\mathbf{l}_n = [l_1(t_n), l_2(t_n), \dots, l_m(t_n)]^\top$. The mapping matrix $\mathbf{G}$ is given for the latent destination models in (20) as per Table I. The mean $\boldsymbol{\mu}_{n|n}^{(p)}$ and covariance $\mathbf{P}_{n|n}^{(p)}$, both defined in (34), are stored in the particle filter as described in Algorithm 2. One can see that the estimated intent position is a mixture of Gaussian densities, which has support $\mathbf{l}_n \in \mathbb{R}^m$ covering the entire area of interest (e.g. a surveyed region).

Algorithm 3: Single Iteration (t_{n-1} to t_n) of Intent Inference with the Stable Lévy ERV model

Input: $\omega_{n-1}^{*(p,d)}$, $p(\mathbf{y}_{1:n-1}|\mathcal{D}_d)$, and $\{\boldsymbol{\mu}, \mathbf{P}\}_{n-1|n-1}^{(p,d)}$.
Output: $\omega_n^{*(p,d)}$, $p(\mathbf{y}_{1:n}|\mathcal{D}_d)$, and $\{\boldsymbol{\mu}, \mathbf{P}\}_{n|n}^{(p,d)}$.
for particles $p = 1 : N_{\mathcal{P}}$ **do**
 Sample $\{\Gamma_i, V_i\}_{n-1:n}^{c,(p)}$ via Algorithm 1.
end
for destinations $d = 1 : N_{\mathcal{D}}$ **do**
 if Resample **then**
 Resample particles and set *self-normalized* weights $\omega_{n-1}^{*(p,d)} = 1/N_{\mathcal{P}}$.
 end
 for particles $p = 1 : N_{\mathcal{P}}$ **do**
 Set $\{\Gamma_i, V_i\}_{n-1:n}^{c,(p,d)} = \{\Gamma_i, V_i\}_{n-1:n}^{c,(p)}$.
 Update stored information $\{\boldsymbol{\mu}, \mathbf{P}\}_{n|n}^{(p,d)}$ via (37).
 Compute the *update* weight $\tilde{\omega}_n^{(p,d)}$ via (47).
 end
 Compute $p(\mathbf{y}_n|\mathbf{y}_{1:n-1}, \mathcal{D}_i)$ according to (46).
 Normalize particles weights $\omega_n^{*(p,d)}$ via (48).
 Evaluate destination likelihood $p(\mathbf{y}_{1:n}|\mathcal{D}_i)$ in (45).
end
Determine endpoint probability $p(\mathcal{D}_d|\mathbf{y}_{1:n})$ in (43).

The support of the intended destination position can be more restrictive in practice. For example, consider the intent inference task described above for stable Lévy ERV model, where the target's destination can only be one of the $N_{\mathcal{D}}$ nominal endpoints. Based on the estimated intent in (49), there are then several ways to decide which destination $\mathcal{D}_d$ is the intended one. Here, we present two methods to determine the most probable destination, whose index is denoted by i . Specifically, the most probable destination is the one that: a) achieves the smallest Euclidean distance to the mean of the estimated intent, i.e.

$$i = \min_{d=1,2,\dots,N_{\mathcal{D}}} \{d; \|\mathbf{E}(\mathbf{l}_n|\mathbf{y}_{1:n}) - \mathbf{p}_d\|\}, \quad (50)$$

where the mean can be computed as follows based on (49),

$$\mathbf{E}(\mathbf{l}_n|\mathbf{y}_{1:n}) \approx \sum_{p=1}^{N_{\mathcal{P}}} \omega_n^{*(p)} \mathbf{G}^{\top} \boldsymbol{\mu}_{n|n}^{(p)}.$$

or b) has the maximum marginal likelihood, i.e.

$$i = \max_{d=1,2,\dots,N_{\mathcal{D}}} \{d; p(\mathbf{l}_n = \mathbf{p}_d|\mathbf{y}_{1:n})\}, \quad (51)$$

where the likelihood can be directly evaluated according to (49). This intent inference strategy for latent destination models is summarised in Algorithm 4. Note that such an algorithm is a standard particle filter procedure with one additional destination-determining step at each time instant. Since these particle filtering steps are additionally necessary for the conventional tracking application, we can regard the estimated intent as a by-product of the (sensor-level) tracking algorithm.

Such a latent destination model intent inference Algorithm 4 has a complexity of $\mathcal{O}(N_{\mathcal{P}}N)$, where N is the number

of observations. Consequently, the complexity for ERV intent inference algorithm (i.e. a full iteration of Algorithm 3) should be less than $\mathcal{O}(N_{\mathcal{D}}N_{\mathcal{P}}N)$. This reduction follows from: i) the size of state vector is smaller compared with the latent destination model and ii) the $N_{\mathcal{D}}$ particle filters in the algorithm 3 share the same sample set $\{\Gamma_i, V_i\}_{n-1:n}^{c,(p)}$. Nonetheless, it should be noted that the computation complexity of the embedded Kalman filter calculations in Algorithm 3 are still relatively high for large number of nominal destinations. Therefore, the latent destination models, which only requires one particle filter, regardless of number of destinations, should be a more computationally efficient approach.

Algorithm 4: Intent Inference with the latent destination models

for particles $p = 1 : N_{\mathcal{P}}$ **do**
 Sample $\{\boldsymbol{\mu}_{1|1}^{(p)}, \mathbf{P}_{1|1}^{(p)}\}$ from prior $p(\boldsymbol{\mu}_1, \mathbf{P}_1)$.
 Compute the initial *self-normalized* weight $\omega_1^{*(p)} \propto p(\mathbf{y}_1|\boldsymbol{\mu}_{1|1}^{(p)}, \mathbf{P}_{1|1}^{(p)})$.
end
for each observations $n = 2 : N$ captured at t_n **do**
 Proceed one-step particle filtering via Algorithm 2, and store the output.
 Determine the most probable destination via (50) or (51).
end

VI. RESULTS

The performance of the introduced stable Lévy modelling is assessed here with real data from three scenarios. The first two consider the intent inference task for improving human-computer interaction (HCI), specifically when the user point-select motion in 2-D with a mouse cursor and 3-D free hand pointing gestures are subjected to severe perturbations. Knowing the interface item the user intends to select on the interface, early in the point-select task, can noticeably simplify and expedite the interactions, e.g. see [33, 1, 2]. In the third case, we use trajectory data from a fast manoeuvring maritime vessel to evaluate the tracking capability (i.e. only kinematic state estimation) of the proposed stable Lévy approach and compare it against other conventional Gaussian models. We also address practical implementation aspects of a stable Lévy model, including parameterisation and complexity.

A. Intent Inference for HCI with 2-D Pointing Motion

The employed dataset includes 16 2-D mouse cursor trajectories (each consists of 100-500 data points) recorded by a computer whilst participants suffering from motor-impairment, namely a mild-severe form of cerebral palsy, undertook interaction tasks with a mouse to select icons on experimental graphical user interface (GUI) displayed on the screen; this is the classical Fitt's law task, ISO 9241. The pointing trajectories in this case exhibit tremor and sharp jerks/jolts (i.e. sharp manoeuvres) induced by the motor-impairment. Four example tracks are depicted in Fig. 1 with the GUI which

comprises of 16 selectable icons in a circular formation. The participant always starts from the central icon prior to selecting a highlighted interface item.

Fig. 1: Example real mouse-cursor complete pointing tracks.

Here, we compare the intent prediction results of several Gaussian and stable-Lévy models from the above pointing dataset. They are the : 1) Gaussian ERV (G-ERV), 2) stable Lévy ERV (L-ERV), 3) Gaussian dynamic ($\sigma_B \neq 0$) latent destination (G-DLD), 4) Gaussian static ($\sigma_B = 0$) latent destination (G-SLD), 5) stable Lévy DLD (L-DLD), and 6) stable Lévy SLD (L-SLD) models. Performance is evaluated in terms of the intent inference success rate, defined as the fraction of the pointing motion duration (from the start of the task to selecting the highlighted interface item) for which the true destination is correctly predicted. The endpoint is determined according to the decision criteria (44) for ERV models and (51) for latent destination models. Whilst the standard Kalman filter is utilised for the Gaussian ERV model as in [4] and similar for the G-DLD/G-SLD, *Rao-Blackwellised* particle filter with 5000 particles is applied with the stable Lévy models and 10 Monte Carlo (MC) runs per trajectory are carried out. The overall average success rates (i.e. average from all 16 tracks and 10 MC runs per trajectory for the stable Lévy models) and corresponding standard deviations are shown in Fig. 2.

Fig. 2: Average intent inference success rate for 2-D pointing tasks; the red error bar indicates ± 1 standard deviation.

The parameters of the considered models are listed in Table II. They were manually tuned to achieve a good in-

TABLE II: Model parameters

Models	Parameters
G-ERV	$\eta = 0.4, \rho = 7, \sigma_{Gauss} = 50, \mathbf{R} = \mathbf{0}$.
L-ERV	$\alpha = 1.45, c = 1, \eta = 0.4, \rho = 1.2, \sigma_W = 20, \mathbf{R} = 25\mathbf{I}_2$.
G-SLD	$\eta = 0.4, \rho = 8, \sigma_{Gauss} = 100, \mathbf{R} = \mathbf{0}, \Sigma_d = 10000\mathbf{I}_2$.
L-SLD	$\alpha = 1.6, c = 1, \eta = 0.3, \rho = 1.5, \sigma_W = 25, \mathbf{R} = 25\mathbf{I}_2, \Sigma_d = 10000\mathbf{I}_2$.
G-DLD	$\eta = 0.3, \rho = 1.5, \sigma_{Gauss} = 35, \sigma_B = 10, \mathbf{R} = \mathbf{0}, \Sigma_d = 25\mathbf{I}_2$.
L-DLD	$\alpha = 1.5, c = 1, \eta = 0.3, \rho = 1.5, \sigma_W = 20, \sigma_B = 10, \mathbf{R} = 25\mathbf{I}_2, \Sigma_d = 25\mathbf{I}_2$.

dividual model performance, which entailed setting different measurement noise covariances, i.e. $\mathbf{R}$, albeit for the same (accurate) observations of the cursor location. We also specify the Gaussian prior of intent for the latent destination models, such that its mean is the first observed cursor's position (i.e. in the vicinity of the central GUI icon) and covariance (denoted by Σ_d) are stated in Table II. For the SLD model, it is necessary that Σ_d is sufficiently large to assign reasonable prior probabilities to all nominal endpoints.

It can be noticed from Fig. 2 that the proposed stable Lévy models deliver the highest intent prediction success, with the latent destination formulation achieving the best performance. This can be attributed to their ability to better capture the sharp swings and piece-wise straight lines present in the pointing motion (see Fig. 1). For the same type of dynamic noise, the success rates of the SLD and ERV models (i.e. G-ERV versus G-SLD or L-ERV versus L-SLD) are similar. This may be due to their common assumption of a fixed intent throughout the pointing task. However, we recall that the latent destination models require lower computational effort with the introduced efficient intent inference procedure. Additionally, the superior results with the DLD models can be due to their intrinsic capacity to quickly recover from a misjudgement on the true endpoint induced by a non-destination reverting (e.g. unintentional perturbation-originated) jump and/or capture the changing nature of intent during a pointing task. For instance, the intent at the beginning of the point-select motion may not be the precise location of the GUI, but rather its general on-screen region or even only its direction. This is supported by the common premise in HCI that a cursor pointing motion consists of two parts, a large ballistic movement to near the endpoint followed by several smaller homing ones [33].

The cursor trajectories in Fig. 1 exhibit abrupt jumps/jolts, clearly not following a Gaussian process, and hence more likely to be appropriate for our proposed Lévy models, which can represent more erratic (extreme) manoeuvres. Our models can deal with even more extreme patterns, for instance those induced by a more severe motor impairment condition, as was previously demonstrated in the 2-D synthetic tracks presented in [13].

B. Destination Prediction with Freehand Pointing Gestures

Predictive touch technology utilises observations from a gesture tracker, i.e. the coordinates of the pointing hand/finger

in 3-D, to predict (in real-time) the intended GUI icon on a display [4, 2]. In this section, we consider 10 complete 3-D pointing trajectories collected in a moving vehicle by a predictive touch system gesture tracker whilst participants interacted with the main in-car touchscreen. They feature severe perturbations in the form of abrupt manoeuvres and jolts caused by the present vibrations and accelerations due to aggressive driving on badly maintained roads. Fig. 3 depicts three of those tracks and the GUI with 37 selectable icons; a track recorded in a static car is included (in red) for reference. Fig. 4 also exhibits six “synthetic” trajectories generated by the developed spatial Lévy ERV model to illustrate its potential for representing perturbed intent-driven pointing movements in 3-D. They closely resemble the real data in Fig. 3.

Fig. 3: Real hand pointing data

Fig. 4: Synthetic pointing data from the spatial ERV model; they are not used in intent inference performance evaluation.

Next, we evaluate the intent inference performance of the stable Lévy ERV model and compare it with the outcome of the jump and Gaussian ERV models in [14], which employed the same dataset of 10 real perturbed tracks. The parameters were manually tuned for successful endpoint predictions. For

the stable Lévy ERV model they are $\alpha = 1.4$, $\sigma_W = 300$, $c = 20$, $\eta = 60$ and $\rho = 15$; and for the Gaussian ERV model are $\eta = 55$, $\rho = 15$ and $\sigma = 3000$; the Jump-ERV model utilises the parameters stated in [14]. Similar to the cursor pointing data in Section VI-A, a Gaussian measurement model is assumed.

The prediction success rates for the three assessed models are shown in Fig. 5. It can be seen that the proposed Lévy state-space model achieves a significant performance improvement compared to the ERV model, and slightly outperforms the Jump-ERV model in [14]. The lower success rate and larger standard deviation with pointing gestures in 3-D compared to those in Section VI-A are due to the high number of nominal endpoints (37 instead of 16) with multi-circular layout.

Fig. 5: Average intent inference success rate for freehand pointing movement in 3-D; red error bar is ± 1 standard deviation.

C. Tracking a Manoeuvring Vessel

Lastly, we treat a conventional (sensor-level) tracking problem where the objective is to track a highly manoeuvring maritime target, as in [17], with the proposed stable Lévy model. The vessel trajectory¹ is depicted in Fig. 6. The voyage starts at time $t = 0$ and ends at $t = 2583$ such that the ground truth includes 118 position data points at irregular timesteps. To examine the presented Rao-Blackwellised particle filtering strategy, observations are obtained by adding Gaussian noise, with a standard deviation of 100, to the ground-truth data in each dimension. It should be noted that with such an observation model, the Gaussian Singer model is expected to track the vessel reasonably well. Thus, a significant improvement in the tracking performance with a stable Lévy model is not sought. Instead, below we compare the state estimation accuracy of the Gaussian Singer and stable Lévy Singer models, and outline the advantages of utilising the versatile parameters in the introduced stable Lévy modelling. We also use this real example to highlight key practical implementation aspects such as parameter sensitivity and computational requirements.

For evaluation purposes, we generate 100 measurement sets, each with $N = 118$ observations originating from the ground truth trajectory with an added Gaussian noise of covariance $R = 100^2 I_2$; a measurement set is displayed in Fig. 6. Here,

¹This dataset was kindly provided by QinetiQ Ltd., Winfrith.

Fig. 6: manoeuvring vessel truth trajectory from $t = 0$ to $t = 2583$ with 118 recorded locations and a measurement set.

the root mean square error (RMSE) is considered as the metric for tracking performance. It is defined, for one set, by

$$RMSE_{ds} = \frac{1}{N} \sqrt{\sum_{n=1}^N \|\hat{z}_n - z_n\|^2}, \quad (52)$$

where z_n is the ground truth of the vessel's position, $\hat{z}_n$ is its estimated value, ds is the index of the measurement set (i.e. $ds = 1, 2, \dots, 100$) and $N = 118$ is the total number of observations. The estimated position $\hat{z}_n$ is extracted from the mean of the estimated state in (42) as per

$$\hat{z}_n = C \hat{E}(x_n | y_{1:n}^{ds}) = \sum_{p=1}^{N_p} \omega_n^{*(p,ds)} C \mu_{n|n}^{(p,ds)}, \quad (53)$$

where C is the mapping matrix that extracts the spatial position from the state vector x_n (in this experiment C is the observation matrix in (33)), and $\mu_{n|n}^{(p,ds)}$ as well as $\omega_n^{*(p,ds)}$ are stored by the particle filter as in Algorithm 2. To quantify the overall tracking accuracy, we calculate the average RMSE over the total 100 measurement sets, dubbed M-RMSE, each with only one Monte Carlo (MC) run (e.g. for the Gaussian Singer model, it only involves running a Kalman-filter). To explore potential Monte Carlo implementation issues, we also define the P-RMSE as the average RMSE over 100 MC runs with the same measurement set.

In terms of the models parameterisation procedure, we first select the parameters for the well-known Gaussian Singer model that achieve a low M-RMSE. The model noise σ_{Gauss} for a fixed λ is then examined, noting that the tuned M-RMSE does not vary for $\lambda > 5$. Hence, we choose $\lambda = 10$ for the Gaussian Singer model, and commence tuning σ_W for the stable Lévy model with $\lambda = 10$. The M-RMSE for the stable Lévy Singer model and its standard deviation are listed in Table III for several values of the stability parameter α .

Table III shows that, for all presented α values, the proposed stable Lévy Singer models slightly outperform the Gaussian Singer model. Additionally, a noticeable trend is that the

TABLE III: Parameters and filtering performance of the stable Lévy (including Gaussian) Singer model. All models are with $\lambda = 10$, and for non-Gaussian ones $c = 0.1$, $N_p = 5000$.

Value of α	M-RMSE(m) and tuned model noise
2 (Gaussian)	95.2517 ± 7.5061 ($\sigma_{Gauss} = 1.565$)
1.6	94.1214 ± 8.0208 ($\sigma_W = 2.013 \times 10^{-1}$)
1.3	93.3269 ± 8.2627 ($\sigma_W = 8.944 \times 10^{-2}$)
1.05	92.7565 ± 8.2672 ($\sigma_W = 3.578 \times 10^{-2}$)
0.9	92.4250 ± 8.4911 ($\sigma_W = 1.118 \times 10^{-2}$)
0.7	92.0316 ± 8.5049 ($\sigma_W = 2.236 \times 10^{-3}$)
0.5	91.7698 ± 8.5211 ($\sigma_W = 1.118 \times 10^{-4}$)
0.3	91.7146 ± 8.5943 ($\sigma_W = 1.342 \times 10^{-7}$)

reduction of RMSE can be achieved with a lower α , which demonstrates the effectiveness of the α -stable extension of the conventional Gaussian noise in this simple linear Gaussian observation scenario. Here, we present only results for $\alpha > 0.3$ as our current implementations led to numerical issues for lower values. Additionally, we note that we also experimented with a stable Lévy Langevin model for this dataset (model as described in Section IV-A1 with a fixed $\lambda = 0.0001$) and this delivers similar performance to that of the analyzed Lévy Singer model, and the M-RMSE was marginally improved, e.g. 93.1955 for $\alpha = 1.3$, and 91.4907 for $\alpha = 0.3$. As for the Singer model, lower α also resulted in a lower RMSE, when σ_W is manually tuned, following a similar trend to Table III.

We next utilise the same stable Lévy Singer model setup with $\lambda = 10$, $\alpha = 1.2$ and tuned model noise specified as $\sigma_w = 0.06708$ to investigate the tracking accuracy and the computational requirement (measured in terms of execution run time) for different truncation parameters c of the Poisson series representation (10) and number of particles N_p . All algorithms are implemented in MATLAB, running on a laptop with Intel Core i9-9980HK at 2.4 GHz. The attained results are listed in Table IV such that the P-RMSE is computed with the measurement set displayed in the Fig. 6 and 100 MC runs are conducted.

It can be seen from Table IV that a higher N_p can lower the standard deviation of the P-RMSE as it reduces the MC error introduced by the particle filtering. It cannot, however, tangibly impact the standard deviation of the M-RMSE or the mean of both RMSE metrics. This is because the overall (average) tracking performance is sensitive to the measurement set, not to the filtering accuracy/uncertainty. Furthermore, the truncation parameter c only undermines the state estimation when its value is unreasonably low, e.g. 0.003, 0.001. This is due to the additional Gaussianity (and less heavy-tailedness) induced by the inaccurate integral representation since a lower c incorporates less Poisson terms in (10) with more residuals approximated by a Gaussian variable in (13). A higher c or N_p inevitably requires more computational effort. However, it can be noticed that for a wide range of c and N_p values, the M-RMSE for $\alpha = 1.2$ in Table IV is between that of $\alpha = 1.3$ and 1.05 in Table III as expected. Hence we can confidently anticipate that lower c and N_p values, for instance $c = 0.02$ and $N_p = 500$ which only demands a run time of 1.66

TABLE IV: Run time T , filtering P-RMSE and M-RMSE of the stable Lévy Singer model for different values of Poisson series truncation c values and particle numbers N_P .

c	N_P	T (s)	P-RMSE (m)	M-RMSE (m)
0.6	500	19.90	106.77 ± 0.296	93.15 ± 8.18
0.1	5000	41.1785	106.69 ± 0.101	93.09 ± 8.25
0.1	1500	12.6012	106.70 ± 0.200	93.10 ± 8.25
0.1	500	4.2055	106.74 ± 0.344	93.19 ± 8.29
0.02	1500	4.9956	106.71 ± 0.202	93.14 ± 8.24
0.02	500	1.6633	106.73 ± 0.323	93.09 ± 8.22
0.003	500	1.1446	106.93 ± 0.344	93.43 ± 8.26
0.001	500	1.0929	108.11 ± 0.333	94.53 ± 8.19

seconds per measurement set, can still deliver a comparable performance as in Table III.

We now illustrate the tracking performance by considering one particular set of observations, corresponding to a single generation of the observation noise process added to the ship trajectory data. The stable Lévy Singer ($\alpha = 0.3$) and Gaussian Singer models with the parameters listed in Table III are employed to perform the smoothing tasks. For the Gaussian model, the Rauch–Tung–Striebel (RTS) smoother [42] is applied to obtain the smoothed estimates. For the Lévy model, a straightforward smoothing strategy is carried out by running N_P RTS smoothers conditioned on the latent variable samples obtained from the particle filtering, i.e. the samples which provide empirical approximation in (39). The observations and smoothed trajectories for both models are depicted in Fig. 7, and the smoothing root squared error compared with the ground truth at each time-step is plotted in Fig. 8. It can be seen that the Lévy smoothed trajectory is improved compared to the tuned Gaussian model in two clear aspects: first, the Lévy model is better able to capture several of the sharp corners in the trajectory, see e.g. $t = 273$ where the Gaussian over-rounds the corner; and secondly, it is better able to capture the near straight-line trajectories in between the sharp manoeuvres, see e.g. times $t = 673$ to 1049 . The improved tracking performance of the Lévy Singer model can also be noticed in Fig. 8, where the Lévy Singer model achieves a lower root squared error for most of the voyage (including during the two periods mentioned above).

To illustrate the potential of a Lévy state space model to capture abrupt manoeuvres compared to the standard Gaussian formulation, including estimating the velocity vector, we consider a synthetic 2-D trajectory with sudden changes. Both, the Lévy and Gaussian Singer models, are examined for the smoothing task. The track, displayed in Fig. 9, is generated by the stable Lévy Singer model with $\alpha = 1.2$, $\lambda = 100$, and $\sigma_W = 14.14$. A measurement set is then generated by adding a Gaussian noise with a covariance of $R = 0.2^2 I_2$. The ground-truth, measurements set and the estimated smoothed trajectories are shown in Fig. 9. The Lévy model uses the same parameters as in the synthetic track, whilst the Gaussian model is manually fine tuned to give the lowest average RMSE over 100 measurement sets generated with the same observation noise covariance (including the testing measurement set showed in Fig. 9), specifically $\lambda = 100$ and $\sigma_{Gaus} = 93.3381$.

Fig. 7: Smoothed trajectories (mean of the estimate) from the Lévy ($\alpha = 0.3$) and Gaussian Singer models, and corresponding 95% confidence ellipse. The black squares are the true target positions corresponding to the error ellipse.

Fig. 8: Real time ($t = 0$ to $t = 2583$) root squared error compared with the ground truth for smoothing the observations in Fig. 7 with the Lévy ($\alpha = 0.3$) and Gaussian Singer models.

Similar to the smoothing estimation for the real vessel data, it can be seen from Fig. 9 that the dynamic noise of a Gaussian model can be tuned to capture manoeuvres, despite still not following sharp turns and introducing over-smoothing effects (e.g. around $t = 9.15$ and $t = 11.55$). In contrast, the stable Lévy model offers more accurate tracking of abrupt turns and a lower overall RMSE (0.0809 and 0.0922 for the Lévy and Gaussian model respectively). To demonstrate its ability to capture sharp changes in the state, we examine the estimated velocity for the trajectory in Fig. 9. It can be clearly noticed from Fig. 10 that the stable Lévy Singer model can accurately follow the rapid changes in velocity (e.g. at $t = 8.3$, $t = 9.15$, and $t = 11.55$), whilst the Gaussian model fails to follow such sharp changes timely and instead over-smooths them. Furthermore, the Lévy model demonstrates a steady and accurate velocity estimation when the vessel exhibits a nearly constant velocity moving (e.g. from $t = 4$ to $t = 8$, from $t = 10$ to $t = 11.5$), and thus it provides useful information (e.g. direction of travel) for the position inference and intent inference tasks. For the Gaussian case, an increase in the dynamical noise is needed to (nearly) follow manoeuvres and reduce overall RMSE at the expense of higher noise during (nearly) straight trajectory sections

Fig. 9: Synthetic trajectories including 250 data points generated from the Lévy Singer ($\alpha = 1.2$) model and a measurement set. Smoothed trajectories with the Lévy and Gaussian Singer models are plotted with 95% confidence ellipse.

Fig. 10: Smoothed velocity estimates based on the observations in Fig. 9. Top and bottom panels are for x and y directions, respectively. The mean for 1) the Gaussian Singer model, 2) the Lévy Singer model, 3) each Monte Carlo conditioned Lévy Singer model estimate are also plotted.

where estimates are more exposed to noise in the available observations. Such an undesirable trade-off is not imposed by the proposed Lévy model, since its heavy-tailed behaviour can adapt automatically to both smooth/straight sections and to abrupt changes in state.

VII. CONCLUSION

A new probabilistic framework, based on novel stable Lévy models, is proposed in this paper for sensor-level (e.g. kinematic state estimation) and meta-level (e.g. destination pre-

diction) tracking of highly manoeuvring objects. An efficient intent inference strategy with the latent destination model, which can apply to Gaussian and/or stable Lévy dynamic noise(s) is also introduced. Results from real data illustrate the ability to deliver superior intent inference performance with the latent destination model, and demonstrate some useful performance improvements attained by the developed stable Lévy models compared with conventional Gaussian models.

From the classical sensor-level tracking example of a manoeuvring vessel and a synthetic highly manoeuvring trajectory, we can see that the proposed approach demonstrates its ability to track accurately the abrupt manoeuvres (e.g. sharp turns and/or accelerations). It also provides automatically a more steady kinematic state estimation on a near constant velocity movement. Although the overall positional estimates from a Lévy model are only slightly more accurate in our presented simulations, we anticipate that more significant tracking and intent improvements may be achieved in future work with other observation functions (e.g. non-linear and/or non-Gaussian), trajectories with more severe manoeuvres, and more general scenarios that may for example include clutter and data association issues. Other stable Lévy models, which have not been tested here (such as the constant velocity and constant acceleration models), may also provide performance enhancements for the general manoeuvring target tracking case. Whilst the stable Lévy model is formulated above as a continuous-time linear system driven by multivariate isotropic α -stable noise, a more general non-isotropic noise driven model can be explored in the future, see [20] for proposals in the univariate case.

APPENDIX A

Consider the residual terms in (11) as the summation of Poisson series on the fixed interval $[c\Delta t, d\Delta t)$, with $d \rightarrow \infty$, and their number $N_{[c,d]}$ is defined by the Poisson distribution

$$N_{[c,d]} \sim \text{Poisson}((d - c)\Delta t), \quad (54)$$

and consequently the residuals terms can be expressed by

$$\mathbf{R}_c(\Delta t) = \lim_{d \rightarrow \infty} \Delta t^{1/\alpha} \sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i, \quad (55)$$

with $\{\Gamma_i\}$ are now independently and uniformly distributed in the interval $[c\Delta t, d\Delta t)$. Additionally, the summands in (55) are independent and are zero mean. Subsequently, we have

$$\begin{aligned} & \text{Var} \left[\sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i \right] \\ &= \mathbb{E} \left[\left(\sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i \right) \left(\sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i \right)^\top \right] \\ &= \mathbb{E} \left[\sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-2/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i \mathbf{U}_i^\top \mathbf{f}(\Delta t, V_i)^\top \right] \\ &= \mathbb{E}[N_{[c,d]}] \mathbb{E}[\Gamma_i^{-2/\alpha}] \mathbb{E}[\mathbf{f}(\Delta t, V_i) \mathbf{U}_i \mathbf{U}_i^\top \mathbf{f}(\Delta t, V_i)^\top], \quad (56) \end{aligned}$$

where

$$\begin{aligned} E[N_{[c,d]}] &= (d - c)\Delta t, \\ E[\Gamma_i^{-2/\alpha}] &= \frac{1}{(d - c)\Delta t} \int_{c\Delta t}^{d\Delta t} \Gamma_i^{-2/\alpha} d\Gamma_i \\ &= \frac{d^{1-2/\alpha} - c^{1-2/\alpha}}{d - c} \Delta t^{-2/\alpha} \frac{\alpha}{\alpha - 2}, \end{aligned}$$

and

$$\begin{aligned} E[\mathbf{f}(\Delta t, V_i) \mathbf{U}_i \mathbf{U}_i^T \mathbf{f}(\Delta t, V_i)^T] \\ &= E_{V_i}[\mathbf{f}(\Delta t, V_i) E_{U_i}[\mathbf{U}_i \mathbf{U}_i^T] \mathbf{f}(\Delta t, V_i)^T] \\ &= \frac{\sigma_W^2}{\Delta t} \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{H} \mathbf{H}^T e^{\mathbf{A}(\Delta t - u)^T} du = \frac{\sigma_W^2}{\Delta t} \mathbf{Q}(\Delta t), \end{aligned}$$

Therefore, the sought variance reduces to

$$\begin{aligned} \text{Var}[\mathbf{R}_c(\Delta t)] &= \lim_{d \rightarrow \infty} \Delta t^{2/\alpha} \text{Var}\left[\sum_{i=1}^{N_{[c,d]}} \Gamma_i^{-1/\alpha} \mathbf{f}(\Delta t, V_i) \mathbf{U}_i\right] \\ &= \lim_{d \rightarrow \infty} \left(d^{1-2/\alpha} - c^{1-2/\alpha}\right) \sigma_W^2 \frac{\alpha}{\alpha - 2} \mathbf{Q}(\Delta t) \\ &= \sigma_W^2 \frac{\alpha}{2 - \alpha} c^{1-2/\alpha} \mathbf{Q}(\Delta t). \end{aligned}$$

APPENDIX B

The integral in (15) can be expressed as

$$\mathbf{M}(\Delta t) = \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} \mathbf{B} du = \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} du \mathbf{B}.$$

Let $\mathcal{J}(\Delta t) = \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} du$, and the aim is to prove that $\mathcal{J}(\Delta t)$ is equal to $\mathbf{J}_M$ in (25). By *Leibniz integral rule*, we have

$$\begin{aligned} \frac{d}{d\Delta t} \mathcal{J}(\Delta t) &= \frac{d}{d\Delta t} \int_0^{\Delta t} e^{\mathbf{A}(\Delta t - u)} du \\ &= \int_0^{\Delta t} \frac{d}{d\Delta t} e^{\mathbf{A}(\Delta t - u)} du + e^{\mathbf{A}(\Delta t - \Delta t)} = \mathbf{A} \mathcal{J}(\Delta t) + \mathbf{I}_{s_A}, \end{aligned}$$

recalling that $\mathbf{A}$ and the identity matrix $\mathbf{I}_{s_A}$ are of size $s_A \times s_A$. We can then construct the linear differential equation

$$\frac{d}{d\Delta t} \begin{bmatrix} \mathcal{J}(\Delta t) \\ \mathbf{I}_{s_A} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{I}_{s_A} \\ \mathbf{0}_{s_A} & \mathbf{0}_{s_A} \end{bmatrix} \begin{bmatrix} \mathcal{J}(\Delta t) \\ \mathbf{I}_{s_A} \end{bmatrix}. \quad (57)$$

It can be easily seen that equation (57) has the solution

$$\begin{aligned} \begin{bmatrix} \mathcal{J}(\Delta t) \\ \mathbf{I}_{s_A} \end{bmatrix} &= \expm \left(\begin{bmatrix} \mathbf{A} & \mathbf{I}_{s_A} \\ \mathbf{0}_{s_A} & \mathbf{0}_{s_A} \end{bmatrix} \Delta t \right) \begin{bmatrix} \mathcal{J}(0) \\ \mathbf{I}_{s_A} \end{bmatrix} \\ &= \expm \left(\begin{bmatrix} \mathbf{A} & \mathbf{I}_{s_A} \\ \mathbf{0}_{s_A} & \mathbf{0}_{s_A} \end{bmatrix} \Delta t \right) \begin{bmatrix} \mathbf{0}_{s_A} \\ \mathbf{I}_{s_A} \end{bmatrix}, \end{aligned} \quad (58)$$

which is exactly (25) so that we have $\mathcal{J}(\Delta t) = \mathbf{J}_M$.

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Runze Gan received B.Eng. degree in Electronic and Electrical Engineering from the University of Birmingham, U.K., and the Huazhong University of Science and Technology, China, in 2018. He is currently working towards the Ph.D. degree in information Engineering at the University of Cambridge, U.K. His research interests include stochastic modelling for time series, Bayesian approximate inference methods, and object tracking.

Bashar I. Ahmad, is a signal processing specialist at Aveillant, THALES Land and Air Systems (LAS) since early 2019 and a senior research associate at the Engineering Department, University of Cambridge since 2013; he was elected a Research Fellow of Wolfson College, Cambridge. Prior to joining Cambridge University, he was a postdoctoral research associate at Imperial College London, U.K., a project manager at DSPG Telecom (BT), U.K., and a research engineer at Philips/NXP Semiconductors, U.K. Bashar received the B.Eng. degree (Hons.) in electronic engineering and the Ph.D. degree from University of Westminster, U.K., in 2007 and 2011, respectively; he also undertook the Msc in Signal Processing and Communications (electives) at Imperial College London, U.K. His research areas of interest include the theory and applications of statistical signal processing and Bayesian inference for time-series data, such as in object tracking and data fusion, automatic target recognition, automation, sub-Nyquist data acquisition, intelligent vehicles and others.

Simon J. Godsill Simon Godsill is Professor of Statistical Signal Processing in the Engineering Department at Cambridge University. He is also a Professorial Fellow and tutor at Corpus Christi College Cambridge. He coordinates an active research group in Signal Inference and its Applications and is Head of the Signal Processing and Communications (SigProC) Laboratory at Cambridge. His group specialises in Bayesian computational methodology, multiple object tracking, audio and music processing, and financial time series modeling. A particular

methodological theme over recent years has been the development of novel techniques for optimal Bayesian filtering and smoothing, using Sequential Monte Carlo or Particle Filtering methods. Prof. Godsill has published extensively in journals, books and international conference proceedings, and has given a number of high profile invited and plenary addresses at conferences such as the Valencia conference on Bayesian Statistics (twice), the IEEE Statistical Signal Processing Workshop, the Conference on Bayesian Inference for Stochastic Processes (BISP), the IEEE Workshop on Machine Learning in Signal Processing (2013) and FUSION (2016). He co-authored a Springer text Digital Audio Restoration with Prof. Peter Rayner in 1998. He was technical chair of the IEEE NSSPW workshop in 2006 on sequential and nonlinear filtering methods, and has been on the conference panel for numerous other conferences/workshops. Prof. Godsill has served as Associate Editor for IEEE Tr. Signal Processing and the journal Bayesian Analysis. He was Theme Leader in Tracking and Reasoning over Time for the UK’s Data and Information Fusion Defence Technology Centre (DIF-DTC) and Principal Investigator on many grants funded by the EU, EPSRC, QinetiQ, General Dynamics, MOD, Microsoft UK, Citibank, Mastercard, Google, DSO Singapore, Huawei and Jaguar Landrover. In 2009–10 he was co-organiser of an 18 month research program in Sequential Monte Carlo Methods at the SAMSI Institute in North Carolina and in 2014 he co-organised a research programme at the Isaac Newton Institute on Sequential Monte Carlo methods. In 2018 he will be General Chair of the FUSION Conference in Cambridge. Two of his journal papers have recently received Best Paper awards from the IEEE and IET. He continues to be a Director of CEDAR Audio Ltd. (which has received numerous accolades over the years, including a technical Oscar), and for which he was a founding staff member in 1988. The company has commercialised many of the ideas from his research over the years.